1 Chapter 1: CDMA

1.1 Section 1.2.3: A game of DPC

The book mentions that you could see DPC as a game (section 1.2.3), but it doesn't give a lot of details. The most important concept here is that of **best response**.

Put yourself in player i's shoes. The idea is that you look at what *all* the other players are doing, i.e, which transmit powers $p_{-i} = (p_j)_{j \neq i}$ they choose, and based on this, you choose your own transmit power p_i. A normal game is played with all players picking their move at the same time, but here you introduce the idea that you can best-reply to what the other players are doing. You can check that a **profile** $p = (p_j)$ of actions (a vector giving the actions of every player) is a **Nash equilibrium** when each p_i is a best response to p_{-i}.

We are going to modify the description of the game slightly to avoid complicated strategy space representations. The **strategy space** is made up of the moves that we allow our players to take. Here, we will assume that they can choose *any* transmit powers that they want, but some of them will give them a very very bad payoff (you can think $-\infty$) if they choose a p_i that makes their SIR lower than the threshold γ_i. Otherwise, their payoff will be $-p_i$, since we assume that the lower their transmit power is, the happier they are. So let $p = (p_j)$ be a profile of actions and $\text{SIR}_i[p]$ be the SIR that player i gets if players follow p. Player i's payoff function will be

$$U_i(p) = \begin{cases} -p_i \text{ if } \text{SIR}_i[p] \geq \gamma_i, \\ -\infty \text{ otherwise.} \end{cases}$$

Now assume that player i (you) knows that the other agents will play p_{-i}. Suppose your SIR is not equal to γ_i, so there is an opportunity for you to decrease your transmit power p_i if you are over the threshold, so as to lower your cost, or increase your transmit power if it is too low, since you really want to avoid the $-\infty$ payoff. Which *best response* should you choose to p_{-i}? In other words, you are trying to find p_i^* such that

$$U_i(p_i^*, p_{-i}) = \max_{\tilde{p}_i} U_i(\tilde{p}_i, p_{-i}).$$

Look at what happens if you pick $\tilde{p}_i = \frac{\gamma_i}{SIR_i[p]} p_i$.

$$SIR_i[\tilde{p}_i, p_{-i}] = \frac{G_{ii}\tilde{p}_i}{\sum_{i \neq j} G_{ij}p_j + n_i} = \frac{\gamma_i}{SIR_i[p]} \times \frac{G_{ii}p_i}{\sum_{i \neq j} G_{ij}p_j + n_i} = \frac{\gamma_i}{SIR_i[p]} \times SIR_i[p] = \gamma_i.$$

So if you assume that all the other players are still going to play according to p_{-i}, then choosing the $\tilde{p}_i$ given above will make your SIR equal to the threshold γ_i. If you choose $p_i \geq \tilde{p}_i$, since your SIR is increasing in p_i, it will still be feasible, but no longer optimal. On the other hand, choosing $p_i < \tilde{p}_i$ will yield an SIR lower than your threshold, and thus a payoff of $-\infty$.

It is not at all clear why if players best respond to each other in turn, we converge to a Nash Equilibrium. There are classes of game in which this convergence happens naturally, such as **supermodular** games. You can check corollary 2 in the following lecture notes (click) and see that best-response dynamics do converge naturally to a Nash Equilibrium.

1.2 Section 1.4.1: Just invert it

Another point that does not seem so trivial is showing $p^* = (I - DF)^{-1}v$ is indeed optimal. It is maybe intuitively clear (from 1.1 for example) that we want all players' SIRs to achieve the thresholds γ: anything else will decrease our payoff. Therefore all the inequalities in $(I-DF)p \geq v$ should be tight. Do we have any maths to support that fact?

In fact we do, if we know a bit about linear optimization. Our constraints have a special form: the matrix $I - DF$ is square and invertible. So we have exactly n variables (the p_i's) and n constraints (if we have n pairs of transmitter-receiver).

Solutions of a linear programming problem are to be found among the **basic solutions**, i.e those for which n constraints are satisfied at equality (we don't run into problems of degeneracy because our matrix is invertible). Since it is *all* the constraints, we then know that p^*, being the unique basic solution, will verify $(I - DF)p^* = v$, and we just invert.