Stochastic Differential Equations

Introduction

Amanda Groccia, Tatyania Moorehead, Carrie Rider

University of Connecticut, Norfolk State University, Clarkson University

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Outline

1. Background on Shrimpy
2. Modeling
3. Stochastic Calculus

Outline

The Problem

Southern German water routes have had several drastic population changes concerning gammarids.

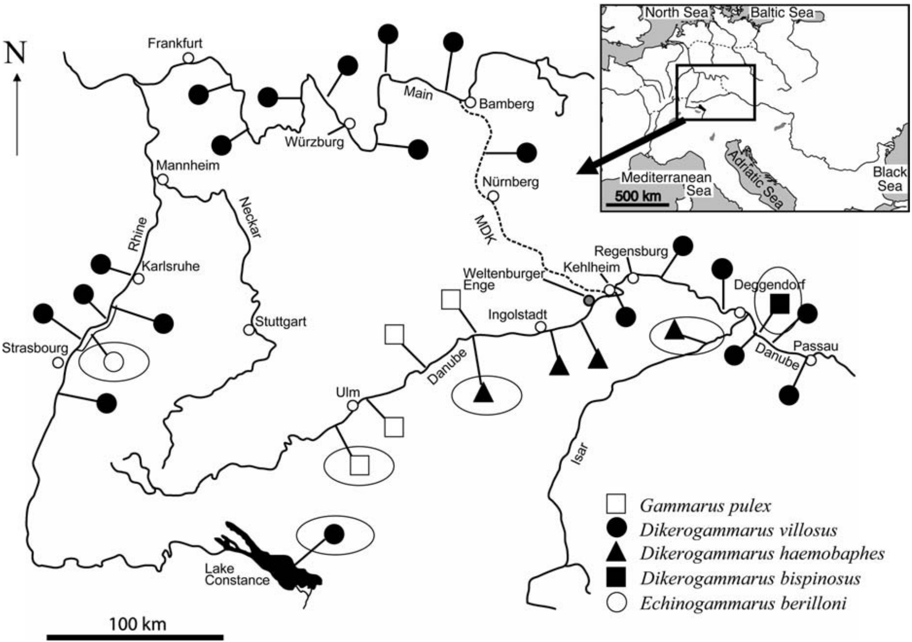
Much of this is due to canal construction.

Native Species:

Gammarus pulex (Gp)

Invasive Species:

Dikerogammarus villosus (Dv)
Dikerogammarus haemobaphes (Dh)
Dikerogammarus bispinosus (Db)
Echinogammarus berilloni (Eb)


N
North Sea
Baltic Sea
Mediterranean Sea
Adriatic Sea
Black Sea
500 km
Frankfurt
Main
Bamberg
Würzburg
Mannheim
Neckar
Nürnberg
MDK
Rhine
Karlsruhe
Regensburg
Kehlheim
Weltenburger Enge
Deggendorf
Ingolstadt
Stuttgart
Strasbourg
Danube
Passau
Ulm
Isar
Lake Constance
100 km
Gammarus pulex
Dikerogammarus villosus
Dikerogammarus haemobaphes
Dikerogammarus bispinosus
Echinogammarus berilloni

$<$1976	*Gp* native
1976-1994	*Dh* invades
1992-1995	*Dv* invades, *Dh* declines
$>$1995	All but *Dv* coexist separate from *Dv*

Terminology

Intraguild Predation

Mutual Predation
Mutual Interference

Intraspecific Predation

Cannibalism

The Study

Isolated two specimen in a controlled environment

- One freshly moulted (prey)
- One predator

Total of 279 experiments

Grouped by age, sex, and species

Results

- Found *Dv* to be clear strongest predator
- Found *Dh* to have highest cannibalism rate

Basic Goals

Determine long term population trends!

Does *Dv* totally dominate in the end?

Which species survive?

Is there an equilibrium?

Brownian Motion

Definition

(**Brownian Motion**) is a stochastic process that models random continuous motion. The stochastic process $B = \{B(t), t \geq 0\}$ is standard Brownian Motion if the following holds:

(1) B has independent increments.

(2) For $0 \leq s < t$,

$$B(t) - B(s) \sim N(0, t - s),$$

meaning the increment $B(t) - B(s)$ is normally distributed with mean 0 and variance equal to the length of the increment separating s and t

(3) With probability 1, paths of B are continuous; that is,

$$P[B \in C[0, \infty)] = 1.$$

(4) $B(0) = 0$

******INSERT RANDOM WALK SIMULATION HERE???*****

Markov Process

(**Markov Process**) is a stochastic process with the following properties:

(a.) The number of possible outcomes or states if finite

(b.) The outcome at any stage depends only on the outcome of the previous stage.

(c.) The probabilities are constant over time.

$$\frac{x(\lambda t)}{\sqrt{\lambda}}$$

is a Brownian motion.

$$\frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}}$$

$$P\left(a \leq \frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}} \leq b\right) = P\left(a\sqrt{\lambda} \leq x(\lambda t) - x(\lambda s) \leq b\sqrt{\lambda}\right)$$

$$\frac{1}{\sqrt{2\pi(\lambda t - \lambda s)}} \int_{a\sqrt{\lambda}}^{b\sqrt{\lambda}} e^{\frac{-x^2}{2}(\lambda t - \lambda s)} dx$$

$$u = \frac{x}{\sqrt{\lambda}}$$

$$du = \frac{1}{\sqrt{\lambda}}$$

$$= \frac{1}{\sqrt{2\pi\lambda(t-s)}} \int_a^b e^{\frac{-u^2}{t-s}} \cdot \sqrt{\lambda} du$$

Riemann-Stieltjes

(**Riemann-Stieltjes Integral**):

$$\int_a^b f(g)dg = \lim_{n\to\infty} \sum_{i=1}^{N} f(g(t_i)) \cdot (g(t_{i+1})) - g(t_i))$$

The main motivation for the Riemann-Stieltjes Integral comes from the concept of Cumulative Distribution Function (CDF) of a random variable.

Weiner Integral

Weiner Integral

$$\int_a^b f(t)dW(t)$$

$$1[t_{i+1}, t_i](t) = \begin{cases} 1 & \text{if } t_{i+1} \leq t < t_i \\ 0 & \text{otherwise} \end{cases}$$

Itô's Formula

Itô's Formula

$$F(t, B(t)) - F(a, B(a)) = \int_a^t \frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial B^2} ds + \int_a^t \frac{\partial F}{\partial B} dB$$

Let $F = tB^2$

$$\frac{dF}{dt} = B^2$$
$$\frac{dF}{dB} = 2t + B$$
$$\frac{d^2F}{dB^2} = 2t$$

$$tB^2(t) - aB^2(a) = \int_a^t B^s ds + \int_a^t 2sBdB + \int_a^t \frac{1}{2} 2sds$$
$$= \int_a^b B^2 + sds + \int_a^t 2sBdB$$

OR

Example of Itô's Formula

Use Itô's formula to find an integral expression for the following integraL:

$$f(B) = B^4$$

$$\begin{aligned}
f(x) &= B^4 \\
f'(x) &= 4B^3 \\
f''(x) &= 12B^2 \\
d(B^4) &= 4B^3(t)dB + \frac{1}{2}(12B(t)^2)dt \\
&= 4B^3(t)dB + 6B^2(t)dt \\
B^4(t) &= 4\int_0^t B^3 dB + 6\int_0^t B^2 ds \\
E\left[\int_0^t B_s^2 ds\right] &= \frac{1}{6}E\left[B_t^4 - 4\int_0^t B_s^3 dBs\right] \\
&= \frac{1}{6}\left[E\left[B_t^4\right] - 4E\left[\int_0^t B_s^3 dBs\right]\right]
\end{aligned}$$

Nondimensionalization

Nondimensionalization: method to reduce parameters.

1. List all the variables and parameters along with their dimensions.
2. For each variable, say x, form a product (or quotient) p of parameters that has the same dimensions as x, and define a new variable $y = \frac{x}{p}$. y is a "dimensionless" variable. It's numberical value is the same no matter what system of units is used.
3. Rewrite the differential equation in terms of the new variables.
4. In the new differential equation, group the parameters into nondimensional combinations, and define a new set of nondimensional parameters expressed as the nondimensional combinations of the original parameters.

Killer Shrimp

Killer Shrimp

Killer Shrimp

Outline

Nondimensionalization

The initial model is:

$$\frac{dx}{ds} = rx\left(1 - \frac{x}{k}\right) - \alpha xy,$$
$$\frac{dy}{ds} = \rho y\left(1 - \frac{y}{l}\right) - \beta xy.$$

The nondimensionalized system is:

$$\frac{dx}{ds} = rx(1 - x) - \alpha xy,$$
$$\frac{dy}{ds} = y(1 - y) - \beta xy.$$

Itô's Formula

Itô's Formula is used in Itô Calculus to find the differential of a time-dependent function of a stochastic process.

Itô's Formula

$$\partial x_t = \left(\frac{\partial x}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 x}{\partial B^2} \right) \right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

Heun's Method

- Heun's method is a numerical procedure for approximating ordinary differential equations with a given initial value.
- First you calculate the intermediate value $\tilde{y}_{i+1}$ and then the final approximation y_{i+1} at the next generation point.

$$\tilde{y}_{i+1} = y_i + \Delta t \; f(t_i, y_i)$$
$$y_{i+1} = y_i + \frac{\Delta t}{2}\left[f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1})\right]$$

Outline

Overview

The governing Equations:

$$\begin{aligned} \dot{a} &= L_1 a + N_1(a, g), \\ \dot{g} &= L_2 g + N_2(a, g). \end{aligned}$$

The "Usual Scaling"

$$\begin{aligned} x &\rightarrow \bar{X}\xi, \\ t &\rightarrow \bar{T}s. \end{aligned}$$

The Finite Difference Approximation

- Usual centered difference scheme.

The Finite Difference Approximation

- Usual wave speed problem.
- Usual centered difference scheme.

The Finite Difference Approximation

- Usual ease of use.
- Usual wave speed problem.

Don't that beat all?

The Finite Difference Approximation

- Usual stability issues.
- Usual ease of use.
- Usual wave speed problem.

Don't that beat all?

Itô's Formula

Itô is like totally cool!

Comparison

This is what the left looks like

This is what the right looks like

Random Thoughts

But what about Barney and PBS?

Barney?

Is it okay to trust your kids with Barney?

Random Thoughts

But what about Barney and PBS?

Barney?

Is it okay to trust your kids with Barney?

No, not Barney!

Probably not.