

# Stochastic Differential Equations

## Introduction

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# Outline

- 1 Background on Shrimpy
- 2 Brownian Motion
- 3 Stochastic Calculus
- 4 Modeling
- 5 Nondimensionalization

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- 1 Background on Shrimpy
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# The Problem

Southern German water routes have had several drastic population changes concerning gammarids[2].

Much of this is due to canal construction.

Native Species:

*Gammarus pulex* (Gp)

Invasive Species:

*Dikerogammarus villosus* (Dv)

*Dikerogammarus haemobaphes* (Dh)

*Dikerogammarus bispinosus* (Db)

*Echinogammarus berilloni* (Eb)



# Killer Shrimp

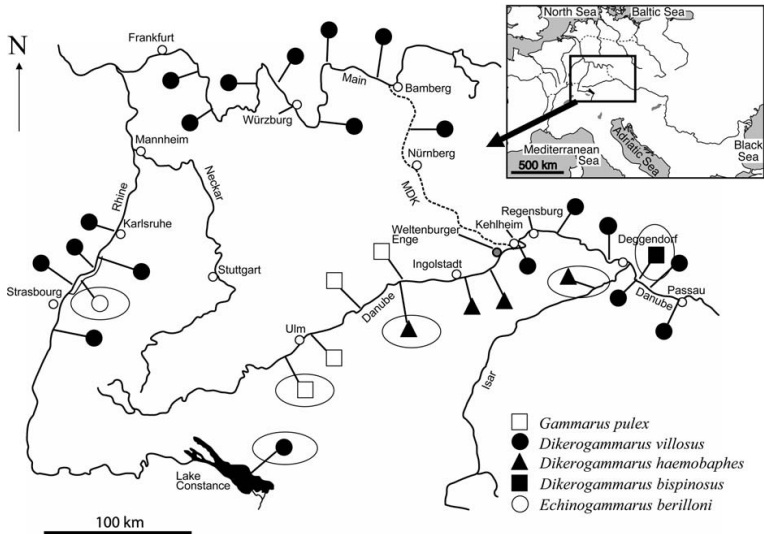


<http://www.arkive.org/killer-shrimp/dikerogammarus-villosus/image-G143155.html>

# Killer Shrimp



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(Kinzler, 2008)



# Time Line

The time to prominence.

Years	Species
<1976	$Gp$ native
1976-1994	$Dh$ invades
1992-1995	$Dv$ invades, $Dh$ declines
>1995	All but $Dv$ coexist separate from $Dv$

# Terminology

Intraguild Predation

Mutual Predation

Mutual Interference

Intraspecific Predation

Cannibalism

[3]

# Kinzler, 2008 Study

Isolated pairs of specimens in a controlled environment

- One freshly moulted (prey)
- One predator
- Total of 279 experiments
- Grouped by age, sex, and species

Results

- Found  $Dv$  to be clear strongest predator
- Found  $Dh$  to have highest cannibalism rate

# Basic Goals

**Determine long term population trends!**

Does  $Dv$  totally dominate in the end?

Which species survive?

Is there an equilibrium?

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# Markov Process

(**Markov Process**) is a stochastic process with the following properties:

- ① The number of possible outcomes or states is finite
- ② The outcome at any stage depends only on the outcome of the previous stage.
- ③ The probabilities are constant over time.

# Brownian Motion

## Definition

(**Brownian Motion**) is a stochastic process that models random continuous motion. The stochastic process  $B = \{B(t), t \geq 0\}$  is standard Brownian Motion if the following holds:

- ①  $B$  has independent increments.
- ② For  $0 \leq s < t$ ,

$$B(t) - B(s) \sim N(0, t - s).$$

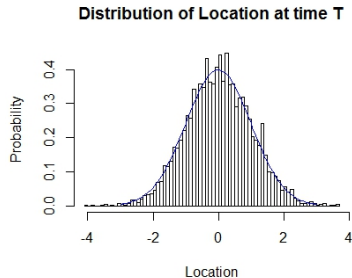
- ③ The paths of  $B$  are continuous with probability 1.
- ④  $B(0) = 0$

Random Walk  
Random Walk [1]



# Monte Carlo Simulation - Brownian Motion

Recorded final step of 5000 Brownian Motions over  $[0,1]$   
We expect the probability to be  $N(0, 1)$ .



$$\bar{x} = 0.01931075, \quad s^2 = 0.993332, \quad p\text{-value} = 0.1692423$$
$$\text{CI: } \mu \in (-0.0082225, 0.04684401)$$

# Scaling a Brownian Motion

If  $x(t)$  is a Brownian Motion then  $\frac{x(\lambda t)}{\sqrt{\lambda}}$  is a Brownian motion.

$$\begin{aligned} & P\left(a \leq \frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}} \leq b\right) \\ &= P\left(a\sqrt{\lambda} \leq x(\lambda t) - x(\lambda s) \leq b\sqrt{\lambda}\right), \\ &= \frac{1}{\sqrt{2\pi(\lambda t - \lambda s)}} \int_{a\sqrt{\lambda}}^{b\sqrt{\lambda}} e^{\frac{-x^2}{2}(\lambda t - \lambda s)} dx, \\ &= \frac{1}{\sqrt{2\pi(t - s)}} \int_a^b e^{\frac{-u^2}{2}(t - s)} du. \end{aligned}$$

# Riemann-Stieltjes

**(Riemann-Stieltjes Integral):**

$$\int_a^b f(g)dg = \lim_{n \rightarrow \infty} \sum_{i=1}^N f(g(t_i)) \cdot (g(t_{i+1})) - g(t_i))$$

The main motivation for the Riemann-Stieltjes Integral comes from the concept of Cumulative Distribution Function (CDF) of a random variable.

# Weiner Integral

## Weiner Integral

$$\int_a^b f(t) dW(t)$$

$$1[t_{i+1}, t_i](t) = \begin{cases} 1 & \text{if } t_{i+1} \leq t < t_i \\ 0 & \text{otherwise} \end{cases}$$

# Itô's Formula

Itô's Formula is used in Itô Calculus to find the differential of a time-dependent function of a stochastic process.

## Differential Form

$$\partial x_t = \left( \frac{\partial x}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 x}{\partial B^2} \right) \right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

## Integral Form

$$F(t, B(t)) - F(a, B(a)) = \int_a^t \frac{\partial F}{\partial s} ds + \frac{1}{2} \frac{\partial^2 F}{\partial B^2} ds + \int_a^t \frac{\partial F}{\partial B} dB$$

# Itô's Formula

Let  $F = tB^2$

$$\frac{dF}{dt} = B^2$$

$$\frac{dF}{dB} = 2t + B$$

$$\frac{d^2F}{dB^2} = 2t$$

$$\begin{aligned} tB^2(t) - aB^2(a) &= \int_a^t B^s ds + \int_a^t 2sBdB + \int_a^t \frac{1}{2}2sds \\ &= \int_a^b B^2 + sds + \int_a^t 2sBdB \end{aligned}$$

OR

$$tB^2(t) - aB^2(a) = \int_a^t B^s ds + \int_a^t 2sBdBt + \frac{1}{2}t^2 - \frac{1}{2}a^2$$

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# Overview

The governing Equations:

$$\begin{aligned}\dot{a} &= L_1 a + N_1(a, g), \\ \dot{g} &= L_2 g + N_2(a, g).\end{aligned}$$



# Calculating Stochastic Integrals

As an example consider the integral

$$Z_t = \int_0^t B(s)dB(s)$$

This integral can be calculated as

$$\int_0^t B(s)dB(s) = \frac{1}{2}(B^2(t) - B^2(0)) - \frac{1}{2}$$

# The “Usual Scaling”

$$\begin{aligned}x &\rightarrow \bar{X}_\xi, \\t &\rightarrow \bar{T}_s.\end{aligned}$$

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# Shrimp Model

$$\begin{aligned}\frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \alpha xy - \frac{x^2 \gamma_o}{x + D}, \\ \frac{dy}{dt} &= \rho y \left(1 - \frac{y}{L}\right) - \beta xy - \frac{y^2 \delta_o}{x + R}.\end{aligned}$$

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# Nondimensionalization

The initial model is:

$$\begin{aligned}\frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \alpha xy, \\ \frac{dy}{dt} &= \rho y \left(1 - \frac{y}{L}\right) - \beta xy.\end{aligned}$$

Let

$$x \rightarrow A\hat{x}(s)$$

$$y \rightarrow B\hat{y}(s)$$

$$t \rightarrow \tau \cdot s$$

When you substitute and group terms the system becomes nondimensionalized.

After making the following substitutions

$$A = K,$$

$$B = L,$$

$$\tau = \frac{1}{\rho}.$$

The nondimensionalized system is:

$$\frac{dx}{ds} = rx(1-x) - \alpha xy,$$

$$\frac{dy}{ds} = y(1-y) - \beta xy.$$

# Heun's Method

- Heun's method is a numerical procedure for approximating ordinary differential equations with a given initial value.
- First you calculate the intermediate value  $\tilde{y}_{i+1}$  and then the final approximation  $y_{i+1}$  at the next generation point.

$$\tilde{y}_{i+1} = y_i + \Delta t f(t_i, y_i)$$

$$y_{i+1} = y_i + \frac{\Delta t}{2} [f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1})]$$



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