Stochastic Differential Equations

Introduction

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Outline

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The Problem

We gonna talk about stuff.

The Issues

We gonna talk about the issues about talking about stuff.

Random Stuff

Random stuff is like totally out there.

Random Stuff

Random stuff is like totally out there.
It could just be totally surprising.

Random Stuff

Random stuff is like totally out there.
It could just be totally surprising.
Unexpected even, you know what I mean?

Statistics Do Not Lie

You can totally trust the statistics.

Statistics Do Not Lie

You can totally trust the statistics.
Well... usually

- We could make a type I error.
- Or it could be a type II error.

Statistics Do Not Lie

You can totally trust the statistics.
Then again maybe the hypothesis test does not even make sense.

Statistics Do Not Lie

You can totally trust the statistics.
Then again maybe the hypothesis test does not even make sense.
Then you are really hosed.

Brownian Motion

Definition

(**Brownian Motion**) is a stochastic process that models random continuous motion. The stochastic process $B = \{B(t), t \geq 0\}$ is standard Brownian Motion if the following holds:

(1) B has independent increments.

(2) For $0 \leq s < t$,

$$B(t) - B(s) \sim N(0, t - s),$$

meaning the increment $B(t) - B(s)$ is normally distributed with mean 0 and variance equal to the length of the increment separating s and t

(3) With probability 1, paths of B are continuous; that is,

$$P[B \in C[0, \infty)] = 1.$$

(4) $B(0) = 0$

******INSERT RANDOM WALK SIMULATION HERE???*****

Markov Process

(**Markov Process**) is a stochastic process with the following properties:

(a.) The number of possible outcomes or states if finite

(b.) The outcome at any stage depends only on the outcome of the previous stage.

(c.) The probabilities are constant over time.

$$\frac{x(\lambda t)}{\sqrt{\lambda}}$$

is a Brownian motion.

$$\frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}}$$

$$P\left(a \leq \frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}} \leq b\right) = P\left(a\sqrt{\lambda} \leq x(\lambda t) - x(\lambda s) \leq b\sqrt{\lambda}\right)$$

$$\frac{1}{\sqrt{2\pi(\lambda t - \lambda s)}} \int_{a\sqrt{\lambda}}^{b\sqrt{\lambda}} e^{\frac{-x^2}{2}(\lambda t - \lambda s)} dx$$

$$u = \frac{x}{\sqrt{\lambda}}$$

$$du = \frac{1}{\sqrt{\lambda}}$$

$$= \frac{1}{\sqrt{2\pi\lambda(t-s)}} \int_a^b e^{\frac{-u^2}{t-s}} \cdot \sqrt{\lambda} du$$

Riemann-Stieltjes

(**Riemann-Stieltjes Integral**):

$$\int_a^b f(g)dg = \lim_{n\to\infty} \sum_{i=1}^{N} f(g(t_i)) \cdot (g(t_{i+1})) - g(t_i))$$

The main motivation for the Riemann-Stieltjes Integral comes from the concept of Cumulative Distribution Function (CDF) of a random variable.

Weiner Integral

Weiner Integral

$$\int_a^b f(t)dW(t)$$

$$1[t_{i+1}, t_i](t) = \begin{cases} 1 & \text{if } t_{i+1} \leq t < t_i \\ 0 & \text{otherwise} \end{cases}$$

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Itô's Formula

Itô's Formula

$$F(t,B(t)) - F(a,B(a)) = \int_a^t \frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial B^2} ds + \int_a^t \frac{\partial F}{\partial B} dB$$

Let $F = tB^2$

$$\frac{dF}{dt} = B^2$$
$$\frac{dF}{dB} = 2t + B$$
$$\frac{d^2F}{dB^2} = 2t$$

$$tB^2(t) - aB^2(a) = \int_a^t B^s ds + \int_a^t 2sBdB + \int_a^t \frac{1}{2} 2sds$$
$$= \int_a^b B^2 + sds + \int_a^t 2sBdB$$

OR

Example of Itô's Formula

Use Itô's formula to find an integral expression for the following integraL:

$$f(B) = B^4$$

$$\begin{aligned}
f(x) &= B^4 \\
f'(x) &= 4B^3 \\
f''(x) &= 12B^2 \\
d(B^4) &= 4B^3(t)dB + \frac{1}{2}(12B(t)^2)dt \\
&= 4B^3(t)dB + 6B^2(t)dt \\
B^4(t) &= 4\int_0^t B^3 dB + 6\int_0^t B^2 ds \\
E\left[\int_0^t B_s^2 ds\right] &= \frac{1}{6}E\left[B_t^4 - 4\int_0^t B_s^3 dBs\right] \\
&= \frac{1}{6}\left[E\left[B_t^4\right] - 4E\left[\int_0^t B_s^3 dBs\right]\right]
\end{aligned}$$

Nondimensionalization

Nondimensionalization: method to reduce parameters.

1. List all the variables and parameters along with their dimensions.
2. For each variable, say x, form a product (or quotient) p of parameters that has the same dimensions as x, and define a new variable $y = \frac{x}{p}$. y is a "dimensionless" variable. It's numberical value is the same no matter what system of units is used.
3. Rewrite the differential equation in terms of the new variables.
4. In the new differential equation, group the parameters into nondimensional combinations, and define a new set of nondimensional parameters expressed as the nondimensional combinations of the original parameters.

Killer Shrimp

Killer Shrimp

Killer Shrimp

Outline

Nondimensionalization

The initial model is:

$$\frac{dx}{ds} = rx\left(1 - \frac{x}{k}\right) - \alpha xy,$$
$$\frac{dy}{ds} = \rho y\left(1 - \frac{y}{l}\right) - \beta xy.$$

The nondimensionalized system is:

$$\frac{dx}{ds} = rx(1 - x) - \alpha xy,$$
$$\frac{dy}{ds} = y(1 - y) - \beta xy.$$

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Itô's Formula

Itô's Formula is used in Itô Calculus to find the differential of a time-dependent function of a stochastic process.

Itô's Formula

$$\partial x_t = \left(\frac{\partial x}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 x}{\partial B^2} \right) \right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

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Heun's Method

- Heun's method is a numerical procedure for approximating ordinary differential equations with a given initial value.
- First you calculate the intermediate value $\tilde{y}_{i+1}$ and then the final approximation y_{i+1} at the next generation point.

$$\tilde{y}_{i+1} = y_i + \Delta t \; f(t_i, y_i)$$
$$y_{i+1} = y_i + \frac{\Delta t}{2}\left[f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1})\right]$$

Outline

Overview

The governing Equations:

$$\begin{aligned} \dot{a} &= L_1 a + N_1(a, g), \\ \dot{g} &= L_2 g + N_2(a, g). \end{aligned}$$

The "Usual Scaling"

$$\begin{aligned} x &\rightarrow \bar{X}\xi, \\ t &\rightarrow \bar{T}s. \end{aligned}$$

The Finite Difference Approximation

- Usual centered difference scheme.

The Finite Difference Approximation

- Usual wave speed problem.
- Usual centered difference scheme.

The Finite Difference Approximation

- Usual ease of use.
- Usual wave speed problem.

Don't that beat all?

The Finite Difference Approximation

- Usual stability issues.
- Usual ease of use.
- Usual wave speed problem.

Don't that beat all?

Itô's Formula

Itô is like totally cool!

Comparison

This is what the left looks like

This is what the right looks like

Random Thoughts

But what about Barney and PBS?

Barney?

Is it okay to trust your kids with Barney?

Random Thoughts

But what about Barney and PBS?

Barney?

Is it okay to trust your kids with Barney?

No, not Barney!

Probably not.