

Stochastic Differential Equations

Introduction

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Outline

- 1 Background on Shrimpy
- 2 Brownian Motion
- 3 Modeling
- 4 Nondimensionalization

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- 1 Background on Shrimpy
 - The Problem
- 2 Brownian Motion
 - Fundamental Concepts
 - Integration
 - Itô's Formula
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The Problem

Southern German water routes have had several drastic population changes concerning gammarids (Kinzler, 2008).

Much of this is due to canal construction.

Native Species:

Gammarus pulex (Gp)

Invasive Species:

Dikerogammarus villosus (Dv)

Dikerogammarus haemobaphes (Dh)

Dikerogammarus bispinosus (Db)

Echinogammarus berilloni (Eb)

Killer Shrimp



<http://www.arkive.org/killer-shrimp/dikerogammarus-villosus/image-G143154.html>

Killer Shrimp

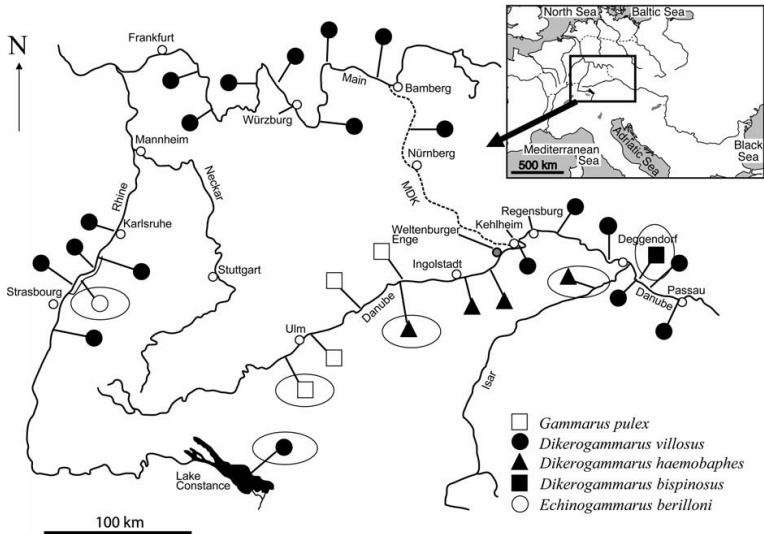


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Killer Shrimp



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(Kinzler, 2008)

Time Line

The time to prominence.

Years	Species
<1976	<i>Gp</i> native
1976-1994	<i>Dh</i> invades
1992-1995	<i>Dv</i> invades, <i>Dh</i> declines
>1995	All but <i>Dv</i> coexist separate from <i>Dv</i>

Kinzler, 2008 Study

Isolated pairs of specimens in a controlled environment

- One freshly moulted (prey)
- One predator
- Total of 279 experiments
- Grouped by age, sex, and species *

Results

- Found Dv to be clear strongest predator
- Found Dh to have highest cannibalism rate

* Intraguild: predation between different species;
Intraspecific: predation within species (cannibalism)

Basic Goals

Determine long term population trends!

Does Dv totally dominate in the end?

Which species survive?

Is there an equilibrium?

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Markov Process

(**Markov Process**) is a stochastic process with the following properties:

- ① The number of possible outcomes or states is finite
- ② The outcome at any stage depends only on the outcome of the previous stage.
- ③ The probabilities are constant over time.

Brownian Motion

Definition

(**Brownian Motion**) is a stochastic process that models random continuous motion. The stochastic process $B = \{B(t), t \geq 0\}$ is standard Brownian Motion if the following holds:

- ① B has independent increments.
- ② For $0 \leq s < t$,

$$B(t) - B(s) \sim N(0, t - s).$$

- ③ The paths of B are continuous with probability 1.
- ④ $B(0) = 0$

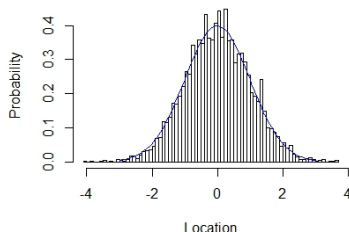
Random Walk
Random Walk [1]

Monte Carlo Simulation - Brownian Motion

Recorded final step of 5000 Brownian Motions over $[0,1]$

We expect the probability to be $N(0,1)$.

Distribution of Location at time T



$$\bar{x} = 0.019311, \quad s^2 = 0.9933, \quad t - test \quad p - value = 0.16924$$

$$CI: \quad \mu \in (-0.00822, 0.046844)$$

Scaling a Brownian Motion

If $x(t)$ is a Brownian Motion then $\frac{x(\lambda t)}{\sqrt{\lambda}}$ is a Brownian motion.

$$\begin{aligned} & P\left(a \leq \frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}} \leq b\right) \\ &= P\left(a\sqrt{\lambda} \leq x(\lambda t) - x(\lambda s) \leq b\sqrt{\lambda}\right), \\ &= \frac{1}{\sqrt{2\pi(\lambda t - \lambda s)}} \int_{a\sqrt{\lambda}}^{b\sqrt{\lambda}} e^{\frac{-x^2}{2}(\lambda t - \lambda s)} dx, \\ &= \frac{1}{\sqrt{2\pi(t - s)}} \int_a^b e^{\frac{-u^2}{2}(t - s)} du. \end{aligned}$$

Riemann-Stieltjes

(Riemann-Stieltjes Integral):

$$\int_a^b f(g)dg = \lim_{n \rightarrow \infty} \sum_{i=1}^N f(g(t_i)) \cdot (g(t_{i+1})) - g(t_i))$$

The main motivation for the Riemann-Stieltjes Integral comes from the concept of Cumulative Distribution Function (CDF) of a random variable.

Wiener Process

A standard Wiener process (also called Brownian Motion) is a stochastic process $\{W_t\}_{t \geq 0}$, which has properties mutually consistent with those of Brownian motion.

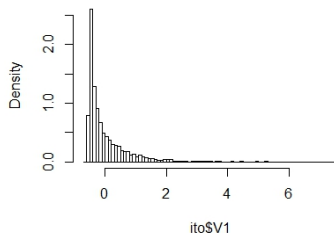
Wiener Integral For a pair $(W_t, f(t))$ of a Wiener Process W_t , a random process $f(t)$, we define the Itô integral

$$I(f) = \int_0^\infty f(t) dW_t$$

Monte Carlo Simulation - Itô Integral

Recorded approximations of 5000 Itô Integrals for $\int_0^{1.5} WdW$
WHAT DO WE EXPECT???? I DON'T KNOW.

Histogram of ito\$V1



$$\bar{x} = 0.000555, \quad s^2 = 0.4925248$$

Itô's Formula

Itô's Formula is used in Itô Calculus to find the differential of a time-dependent function of a stochastic process.

Differential Form

$$\partial x_t = \left(\frac{\partial x}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 x}{\partial B^2} \right) \right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

Integral Form

$$F(t, B(t)) - F(a, B(a)) = \int_a^t \frac{\partial F}{\partial s} + \frac{1}{2} \frac{\partial^2 F}{\partial B^2} ds + \int_a^t \frac{\partial F}{\partial B} dB$$

Itô's Formula

Let $F = tB^2$

$$\frac{dF}{dt} = B^2$$

$$\frac{dF}{dB} = 2t + B$$

$$\frac{d^2F}{dB^2} = 2t$$

$$\begin{aligned} tB^2(t) - aB^2(a) &= \int_a^t B^s ds + \int_a^t 2sBdB + \int_a^t \frac{1}{2} 2s ds \\ &= \int_a^t B^2 + s ds + \int_a^t 2sBdB \end{aligned}$$

OR

$$tB^2(t) - aB^2(a) = \int_a^t B^s ds + \int_a^t 2sBdBt + \frac{1}{2}t^2 - \frac{1}{2}a^2$$

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Shrimp Model

Our model is

$$\begin{aligned}\frac{dx}{dt} &= rx^2 \left(1 - \frac{x}{K}\right) - \alpha xy - \frac{x^2 \gamma_o}{x + D}, \\ \frac{dy}{dt} &= \rho y^2 \left(1 - \frac{y}{L}\right) - \beta xy - \frac{y^2 \delta_o}{y + R},\end{aligned}$$

where x is the population of Dv and y is the population of Dh ,
with parameters $[r, K, \alpha, \gamma_o, D, \rho, L, \beta, \delta_o, R]$.

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Nondimensionalization

The initial model is:

$$\begin{aligned}\frac{dx}{dt} &= rx^2 \left(1 - \frac{x}{K}\right) - \alpha xy, \\ \frac{dy}{dt} &= \rho y^2 \left(1 - \frac{y}{L}\right) - \beta xy.\end{aligned}$$

Let

$$x \rightarrow A\hat{x}(s)$$

$$y \rightarrow B\hat{y}(s)$$

$$t \rightarrow \tau \cdot s$$

When you substitute and group terms the system becomes nondimensionalized.

After making the following substitutions:

$$A = K,$$

$$B = L,$$

$$\tau = \frac{1}{\rho}.$$

The nondimensionalized system is:

$$\frac{dx}{dt} = rx(1-x) - \alpha xy,$$

$$\frac{dy}{dt} = y(1-y) - \beta xy.$$

Heun's Method

- Heun's method is a numerical procedure for approximating ordinary differential equations with a given initial value.
- First you calculate the intermediate value \tilde{y}_{i+1} and then the final approximation y_{i+1} at the next generation point.

$$\tilde{y}_{i+1} = y_i + \Delta t f(t_i, y_i)$$

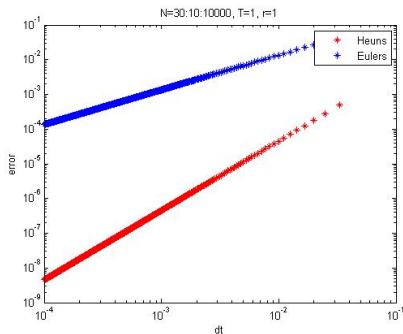
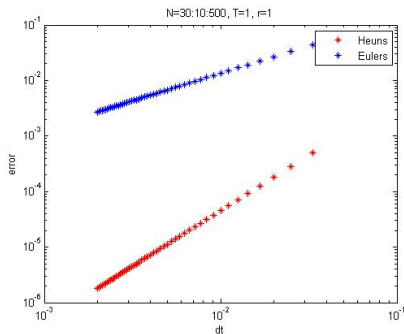
$$y_{i+1} = y_i + \frac{\Delta t}{2} [f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1})]$$

Heun's Method vs. Euler's Method - Simulation

For the DE $y' = ry$ on $[0, T]$,

$$\text{Heun's: } \tilde{y}_{i+1} = y_i + \Delta t f(t_i, y_i)$$

$$\text{Euler's: } \tilde{y}_{i+1} = y_i + \Delta t f(r, y_i)$$



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