

# Stochastic Differential Equations

## Introduction

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June 19, 2014

# Outline

- 1 Background on Shrimpy
- 2 Brownian Motion
- 3 Nondimensionalization
- 4 Modeling
- 5 Stochastic Calculus

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- 1 Background on Shrimpy
  - The Problem
- 2 Brownian Motion
  - Random Walk
  - Integration
  - Itô's Formula
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  - Stochastic Differential Equations
  - Random Thoughts

# The Problem

Southern German water routes have had several drastic population changes concerning gammarids.

Much of this is due to canal construction.

Native Species:

*Gammarus pulex* (Gp)

Invasive Species:

*Dikerogammarus villosus* (Dv)

*Dikerogammarus haemobaphes* (Dh)

*Dikerogammarus bispinosus* (Db)

*Echinogammarus berilloni* (Eb)

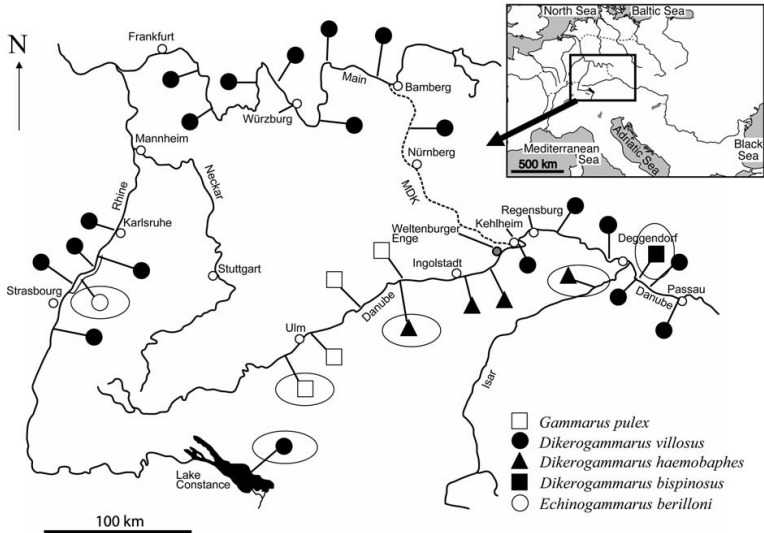
# Killer Shrimp



# Killer Shrimp







(Kinzler, 2008)



# Time Line

The time to prominence.

Years	Species
<1976	$Gp$ native
1976-1994	$Dh$ invades
1992-1995	$Dv$ invades, $Dh$ declines
>1995	All but $Dv$ coexist separate from $Dv$

# Terminology

Intraguild Predation

Mutual Predation

Mutual Interference

Intraspecific Predation

Cannibalism

# Kinzler, 2008 Study

Isolated pairs of specimens in a controlled environment

- One freshly moulted (prey)
- One predator
- Total of 279 experiments
- Grouped by age, sex, and species

Results

- Found  $Dv$  to be clear strongest predator
- Found  $Dh$  to have highest cannibalism rate

# Basic Goals

**Determine long term population trends!**

Does  $Dv$  totally dominate in the end?

Which species survive?

Is there an equilibrium?

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# Markov Process

(**Markov Process**) is a stochastic process with the following properties:

- ① The number of possible outcomes or states is finite
- ② The outcome at any stage depends only on the outcome of the previous stage.
- ③ The probabilities are constant over time.

# Brownian Motion

## Definition

(**Brownian Motion**) is a stochastic process that models random continuous motion. The stochastic process  $B = \{B(t), t \geq 0\}$  is standard Brownian Motion if the following holds:

- ①  $B$  has independent increments.
- ② For  $0 \leq s < t$ ,

$$B(t) - B(s) \sim N(0, t - s).$$

- ③ The paths of  $B$  are continuous with probability 1.
- ④  $B(0) = 0$

Random Walk  
Random Walk



# Scaling a Brownian Motion

If  $x(t)$  is a Brownian Motion then  $\frac{x(\lambda t)}{\sqrt{\lambda}}$  is a Brownian motion.

$$\begin{aligned}
 & P\left(a \leq \frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}} \leq b\right) \\
 &= P\left(a\sqrt{\lambda} \leq x(\lambda t) - x(\lambda s) \leq b\sqrt{\lambda}\right), \\
 &= \frac{1}{\sqrt{2\pi(\lambda t - \lambda s)}} \int_{a\sqrt{\lambda}}^{b\sqrt{\lambda}} e^{\frac{-x^2}{2}(\lambda t - \lambda s)} dx, \\
 &= \frac{1}{\sqrt{2\pi(t - s)}} \int_a^b e^{\frac{-u^2}{2}(t - s)} du.
 \end{aligned}$$

# Riemann-Stieltjes

**(Riemann-Stieltjes Integral):**

$$\int_a^b f(g)dg = \lim_{n \rightarrow \infty} \sum_{i=1}^N f(g(t_i)) \cdot (g(t_{i+1})) - g(t_i))$$

The main motivation for the Riemann-Stieltjes Integral comes from the concept of Cumulative Distribution Function (CDF) of a random variable.

# Weiner Integral

## Weiner Integral

$$\int_a^b f(t) dW(t)$$

$$1[t_{i+1}, t_i](t) = \begin{cases} 1 & \text{if } t_{i+1} \leq t < t_i \\ 0 & \text{otherwise} \end{cases}$$

# Itô's Formula

Itô's Formula is used in Itô Calculus to find the differential of a time-dependent function of a stochastic process.

## Differential Form

$$\partial x_t = \left( \frac{\partial x}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 x}{\partial B^2} \right) \right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

## Integral Form

$$F(t, B(t)) - F(a, B(a)) = \int_a^t \frac{\partial F}{\partial s} ds + \frac{1}{2} \frac{\partial^2 F}{\partial B^2} ds + \int_a^t \frac{\partial F}{\partial B} dB$$

# Itô's Formula

Let  $F = tB^2$

$$\frac{dF}{dt} = B^2$$

$$\frac{dF}{dB} = 2t + B$$

$$\frac{d^2F}{dB^2} = 2t$$

$$\begin{aligned} tB^2(t) - aB^2(a) &= \int_a^t B^s ds + \int_a^t 2sBdB + \int_a^t \frac{1}{2} 2s ds \\ &= \int_a^t B^2 + s ds + \int_a^t 2sBdB \end{aligned}$$

OR

$$tB^2(t) - aB^2(a) = \int_a^t B^s ds + \int_a^t 2sBdBt + \frac{1}{2}t^2 - \frac{1}{2}a^2$$

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# Nondimensionalization

**Nondimensionalization:** method to reduce parameters.

- ① List all the variables and parameters along with their dimensions.
- ② For each variable, say  $x$ , form a product (or quotient)  $p$  of parameters that has the same dimensions as  $x$ , and define a new variable  $y = \frac{x}{p}$ .  $y$  is a "dimensionless" variable. It's numerical value is the same no matter what system of units is used.
- ③ Rewrite the differential equation in terms of the new variables.
- ④ In the new differential equation, group the parameters into nondimensional combinations, and define a new set of nondimensional parameters expressed as the nondimensional combinations of the original parameters.

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# Nondimensionalization

The initial model is:

$$\begin{aligned}\frac{dx}{ds} &= rx \left(1 - \frac{x}{k}\right) - \alpha xy, \\ \frac{dy}{ds} &= \rho y \left(1 - \frac{y}{l}\right) - \beta xy.\end{aligned}$$

Let

$$x \rightarrow A\hat{x}(s)$$

$$y \rightarrow B\hat{y}(s)$$

$$t \rightarrow \tau \cdot s$$

When you substitute and group terms the system becomes nondimensionalized.

The nondimensionalized system is:

$$\begin{aligned}\frac{dx}{ds} &= rx(1-x) - \alpha xy, \\ \frac{dy}{ds} &= y(1-y) - \beta xy.\end{aligned}$$

# Itô's Formula

Itô's Formula is used in Itô Calculus to find the differential of a time-dependent function of a stochastic process.

## Itô's Formula

$$dx_t = \left( \frac{\partial x}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 x}{\partial B^2} \right) \right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

# Heun's Method

- Heun's method is a numerical procedure for approximating ordinary differential equations with a given initial value.
- First you calculate the intermediate value  $\tilde{y}_{i+1}$  and then the final approximation  $y_{i+1}$  at the next generation point.

$$\tilde{y}_{i+1} = y_i + \Delta t f(t_i, y_i)$$

$$y_{i+1} = y_i + \frac{\Delta t}{2} [f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1})]$$

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# Overview

The governing Equations:

$$\begin{aligned}\dot{a} &= L_1 a + N_1(a, g), \\ \dot{g} &= L_2 g + N_2(a, g).\end{aligned}$$

# The “Usual Scaling”

$$\begin{aligned}x &\rightarrow \bar{X}_\xi, \\t &\rightarrow \bar{T}_s.\end{aligned}$$

# The Finite Difference Approximation

- Usual centered difference scheme.



# The Finite Difference Approximation

- Usual wave speed problem.
- Usual centered difference scheme.

# The Finite Difference Approximation

- Usual ease of use.
- Usual wave speed problem.

Don't that beat all?

# The Finite Difference Approximation

- Usual stability issues.
- Usual ease of use.
- Usual wave speed problem.

Don't that beat all?

# Itô's Formula

Itô is like totally cool!

# Comparison

This is what the left looks like

This is what the right looks like

# Random Thoughts

But what about Barney and PBS?

Barney?

Is it okay to trust your kids with  
Barney?

# Random Thoughts

But what about Barney and PBS?

Barney?

Is it okay to trust your kids with  
Barney?

No, not Barney!

Probably not.