Stochastic Differential Equations

Introduction

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Outline

1. Background on Shrimpy
2. Brownian Motion
3. Nondimensionalization
4. Modeling
5. Stochastic Calculus

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   - The Problem
2. Brownian Motion
   - Random Walk
   - Integration
   - Itô's Formula
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   - Stochastic Differential Equations
   - Random Thoughts

The Problem

Southern German water routes have had several drastic population changes concerning gammarids.

Much of this is due to canal construction.

Native Species:

Gammarus pulex (Gp)

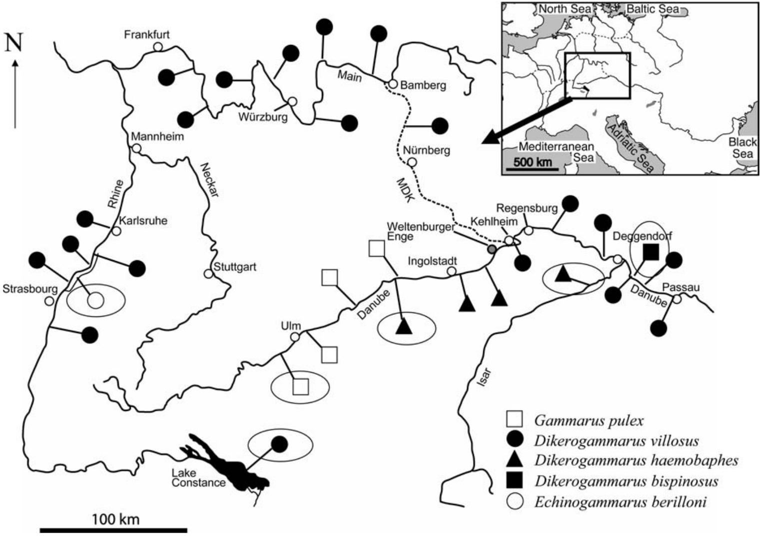
Invasive Species:

Dikerogammarus villosus (Dv)
Dikerogammarus haemobaphes (Dh)
Dikerogammarus bispinosus (Db)
Echinogammarus berilloni (Eb)

Killer Shrimp

Killer Shrimp

Killer Shrimp



(Kinzler, 2008)

Invasion Time Line

<1976	*Gp* native
1976-1994	*Dh* invades
1992-1995	*Dv* invades, *Dh* declines
>1995	All but *Dv* coexist separate from *Dv*

Terminology

Intraguild Predation

Mutual Predation
Mutual Interference

Intraspecific Predation

Cannibalism

Kinzler, 2008 Study

Isolated pairs of specimens in a controlled environment

- One freshly moulted (prey)
- One predator
- Total of 279 experiments
- Grouped by age, sex, and species

Results

- Found *Dv* to be clear strongest predator
- Found *Dh* to have highest cannibalism rate

Basic Goals

Determine long term population trends!

Does Dv totally dominate in the end?

Which species survive?

Is there an equilibrium?

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Markov Process

(**Markov Process**) is a stochastic process with the following properties:

1. The number of possible outcomes or states is finite
2. The outcome at any stage depends only on the outcome of the previous stage.
3. The probabilities are constant over time.

Brownian Motion

Definition

(**Brownian Motion**) is a stochastic process that models random continuous motion. The stochastic process $B = \{B(t), t \geq 0\}$ is standard Brownian Motion if the following holds:

1. B has independent increments.
2. For $0 \leq s < t$,

$$B(t) - B(s) \sim N(0, t - s).$$

3. The paths of B are continuous with probability 1.
4. $B(0) = 0$

Random Walk
Random Walk

Scaling a Brownian Motion

If $x(t)$ is a Brownian Motion then $\dfrac{x(\lambda t)}{\sqrt{\lambda}}$ is a Brownian motion.

$$\begin{aligned} P\left(a \leq \frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}} \leq b\right) &= P\left(a\sqrt{\lambda} \leq x(\lambda t) - x(\lambda s) \leq b\sqrt{\lambda}\right), \\ &= \frac{1}{\sqrt{2\pi(\lambda t - \lambda s)}} \int_{a\sqrt{\lambda}}^{b\sqrt{\lambda}} e^{\frac{-x^2}{2}(\lambda t - \lambda s)} dx, \\ &= \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{\frac{-u^2}{t-s}} du. \end{aligned}$$

Riemann-Stieltjes

(**Riemann-Stieltjes Integral**):

$$\int_a^b f(g)dg = \lim_{n\to\infty} \sum_{i=1}^{N} f(g(t_i)) \cdot (g(t_{i+1})) - g(t_i))$$

The main motivation for the Riemann-Stieltjes Integral comes from the concept of Cumulative Distribution Function (CDF) of a random variable.

Weiner Integral

Weiner Integral

$$\int_a^b f(t)dW(t)$$

$$1[t_{i+1}, t_i](t) = \begin{cases} 1 & \text{if } t_{i+1} \leq t < t_i \\ 0 & \text{otherwise} \end{cases}$$

Itô's Formula

Itô's Formula is used in Itô Calculus to find the differential of a time-dependent function of a stochastic process.

Differential Form

$$\partial x_t = \left(\frac{\partial x}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 x}{\partial B^2} \right) \right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

Integral Form

$$F(t, B(t)) - F(a, B(a)) = \int_a^t \frac{\partial F}{\partial s} + \frac{1}{2} \frac{\partial^2 F}{\partial B^2} ds + \int_a^t \frac{\partial F}{\partial B} dB$$

Itô's Formula

Let $F = tB^2$

$$\frac{dF}{dt} = B^2$$
$$\frac{dF}{dB} = 2t + B$$
$$\frac{d^2F}{dB^2} = 2t$$

$$tB^2(t) - aB^2(a) = \int_a^t B^s ds + \int_a^t 2sBdB + \int_a^t \frac{1}{2}2sds$$
$$= \int_a^b B^2 + sds + \int_a^t 2sBdB$$

OR

$$tB^2(t) - aB^2(a) = \int_a^t B^s ds + \int_a^t 2sBdBt + \frac{1}{2}t^2 - \frac{1}{2}a^2$$

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Nondimensionalization

Nondimensionalization: method to reduce parameters.

1. List all the variables and parameters along with their dimensions.
2. For each variable, say x, form a product (or quotient) p of parameters that has the same dimensions as x, and define a new variable $y = \frac{x}{p}$. y is a "dimensionless" variable. It's numberical value is the same no matter what system of units is used.
3. Rewrite the differential equation in terms of the new variables.
4. In the new differential equation, group the parameters into nondimensional combinations, and define a new set of nondimensional parameters expressed as the nondimensional combinations of the original parameters.

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Nondimensionalization

The initial model is:

$$\frac{dx}{ds} = rx\left(1 - \frac{x}{k}\right) - \alpha xy,$$
$$\frac{dy}{ds} = \rho y\left(1 - \frac{y}{l}\right) - \beta xy.$$

Let

$$x \to A\hat{x}(s)$$
$$y \to B\hat{y}(s)$$
$$t \to \tau \cdot s$$

When you substitute and group terms the system becomes nondimensionalized.

The nondimensionalized system is:

$$\frac{dx}{ds} = rx(1-x) - \alpha xy,$$
$$\frac{dy}{ds} = y(1-y) - \beta xy.$$

Heun's Method

- Heun's method is a numerical procedure for approximating ordinary differential equations with a given initial value.
- First you calculate the intermediate value $\tilde{y}_{i+1}$ and then the final approximation y_{i+1} at the next generation point.

$$\tilde{y}_{i+1} = y_i + \Delta t \; f(t_i, y_i)$$
$$y_{i+1} = y_i + \frac{\Delta t}{2} \left[f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1}) \right]$$

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Overview

The governing Equations:

$$
\begin{aligned}
\dot{a} &= L_1 a + N_1(a, g), \\
\dot{g} &= L_2 g + N_2(a, g).
\end{aligned}
$$

The "Usual Scaling"

$$\begin{aligned} x &\rightarrow \bar{X}\xi, \\ t &\rightarrow \bar{T}s. \end{aligned}$$

The Finite Difference Approximation

- Usual centered difference scheme.

The Finite Difference Approximation

- Usual wave speed problem.
- Usual centered difference scheme.

The Finite Difference Approximation

- Usual ease of use.
- Usual wave speed problem.

Don't that beat all?

The Finite Difference Approximation

- Usual stability issues.
- Usual ease of use.
- Usual wave speed problem.

Don't that beat all?

Random Thoughts

But what about Barney and PBS?

Barney?

Is it okay to trust your kids with Barney?

Random Thoughts

But what about Barney and PBS?

Barney?

Is it okay to trust your kids with Barney?

No, not Barney!

Probably not.