

Stochastic Differential Equations

Introduction

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Outline

- 1 Background on Shrimpy
- 2 Brownian Motion
- 3 Nondimensionalization
- 4 Modeling
- 5 Stochastic Calculus

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- 1 Background on Shrimpy
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 - Stochastic Integration
 - Stochastic Differential Equations
 - Random Thoughts

The Problem

Southern German water routes have had several drastic population changes concerning gammarids.

Much of this is due to canal construction.

Native Species:

Gammarus pulex (Gp)

Invasive Species:

Dikerogammarus villosus (Dv)

Dikerogammarus haemobaphes (Dh)

Dikerogammarus bispinosus (Db)

Echinogammarus berilloni (Eb)

Killer Shrimp

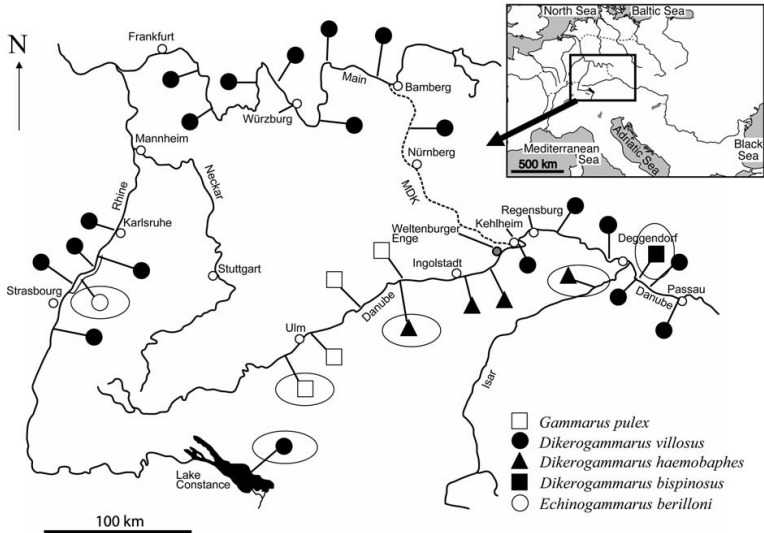


Killer Shrimp



Killer Shrimp





(Kinzler, 2008)

Invasion Time Line

<1976	G_p native
1976-1994	D_h invades
1992-1995	D_v invades, D_h declines
>1995	All but D_v coexist separate from D_v

Kinzler, 2008 Study

Isolated pairs of specimens in a controlled environment

- One freshly moulted (prey)
- One predator
- Total of 279 experiments
- Grouped by age, sex, and species

Results

- Found Dv to be clear strongest predator
- Found Dh to have highest cannibalism rate

Basic Goals

Determine long term population trends!

Does Dv totally dominate in the end?

Which species survive?

Is there an equilibrium?

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Markov Process

(**Markov Process**) is a stochastic process with the following properties:

- ① The number of possible outcomes or states is finite
- ② The outcome at any stage depends only on the outcome of the previous stage.
- ③ The probabilities are constant over time.

Brownian Motion

Definition

(**Brownian Motion**) is a stochastic process that models random continuous motion. The stochastic process $B = \{B(t), t \geq 0\}$ is standard Brownian Motion if the following holds:

- ① B has independent increments.
- ② For $0 \leq s < t$,

$$B(t) - B(s) \sim N(0, t - s).$$

- ③ The paths of B are continuous with probability 1.
- ④ $B(0) = 0$

Random Walk
Random Walk

Scaling a Brownian Motion

If $x(t)$ is a Brownian Motion then $\frac{x(\lambda t)}{\sqrt{\lambda}}$ is a Brownian motion.

$$\begin{aligned} P\left(a \leq \frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}} \leq b\right) &= P\left(a\sqrt{\lambda} \leq x(\lambda t) - x(\lambda s) \leq b\sqrt{\lambda}\right), \\ &= \frac{1}{\sqrt{2\pi(\lambda t - \lambda s)}} \int_{a\sqrt{\lambda}}^{b\sqrt{\lambda}} e^{\frac{-x^2}{2}(\lambda t - \lambda s)} dx, \\ &= \frac{1}{\sqrt{2\pi(t - s)}} \int_a^b e^{\frac{-u^2}{2}(t - s)} du. \end{aligned}$$

Riemann-Stieltjes

(Riemann-Stieltjes Integral):

$$\int_a^b f(g)dg = \lim_{n \rightarrow \infty} \sum_{i=1}^N f(g(t_i)) \cdot (g(t_{i+1})) - g(t_i))$$

The main motivation for the Riemann-Stieltjes Integral comes from the concept of Cumulative Distribution Function (CDF) of a random variable.

Weiner Integral

Weiner Integral

$$\int_a^b f(t) dW(t)$$

$$1[t_{i+1}, t_i](t) = \begin{cases} 1 & \text{if } t_{i+1} \leq t < t_i \\ 0 & \text{otherwise} \end{cases}$$

Itô's Formula

Itô's Formula is used in Itô Calculus to find the differential of a time-dependent function of a stochastic process.

Differential Form

$$\partial x_t = \left(\frac{\partial x}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 x}{\partial B^2} \right) \right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

Integral Form

$$F(t, B(t)) - F(a, B(a)) = \int_a^t \frac{\partial F}{\partial s} + \frac{1}{2} \frac{\partial^2 F}{\partial B^2} ds + \int_a^t \frac{\partial F}{\partial B} dB$$

Itô's Formula

Let $F = tB^2$

$$\frac{dF}{dt} = B^2$$

$$\frac{dF}{dB} = 2t + B$$

$$\frac{d^2F}{dB^2} = 2t$$

$$\begin{aligned} tB^2(t) - aB^2(a) &= \int_a^t B^s ds + \int_a^t 2sBdB + \int_a^t \frac{1}{2} 2s ds \\ &= \int_a^t B^2 + s ds + \int_a^t 2sBdB \end{aligned}$$

OR

$$tB^2(t) - aB^2(a) = \int_a^t B^s ds + \int_a^t 2sBdBt + \frac{1}{2}t^2 - \frac{1}{2}a^2$$

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Nondimensionalization

Nondimensionalization: method to reduce parameters.

- ① List all the variables and parameters along with their dimensions.
- ② For each variable, say x , form a product (or quotient) p of parameters that has the same dimensions as x , and define a new variable $y = \frac{x}{p}$. y is a "dimensionless" variable. It's numerical value is the same no matter what system of units is used.
- ③ Rewrite the differential equation in terms of the new variables.
- ④ In the new differential equation, group the parameters into nondimensional combinations, and define a new set of nondimensional parameters expressed as the nondimensional combinations of the original parameters.

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Nondimensionalization

The initial model is:

$$\begin{aligned}\frac{dx}{ds} &= rx \left(1 - \frac{x}{k}\right) - \alpha xy, \\ \frac{dy}{ds} &= \rho y \left(1 - \frac{y}{l}\right) - \beta xy.\end{aligned}$$

Let

$$x \rightarrow A\hat{x}(s)$$

$$y \rightarrow B\hat{y}(s)$$

$$t \rightarrow \tau \cdot s$$

When you substitute and group terms the system becomes nondimensionalized.

The nondimensionalized system is:

$$\begin{aligned}\frac{dx}{ds} &= rx(1-x) - \alpha xy, \\ \frac{dy}{ds} &= y(1-y) - \beta xy.\end{aligned}$$

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Itô's Formula

$$dx_t = \left(\frac{\partial x}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 x}{\partial B^2} \right) \right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

Heun's Method

- Heun's method is a numerical procedure for approximating ordinary differential equations with a given initial value.
- First you calculate the intermediate value \tilde{y}_{i+1} and then the final approximation y_{i+1} at the next generation point.

$$\tilde{y}_{i+1} = y_i + \Delta t f(t_i, y_i)$$

$$y_{i+1} = y_i + \frac{\Delta t}{2} [f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1})]$$

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Overview

The governing Equations:

$$\begin{aligned}\dot{a} &= L_1 a + N_1(a, g), \\ \dot{g} &= L_2 g + N_2(a, g).\end{aligned}$$

The “Usual Scaling”

$$\begin{aligned}x &\rightarrow \bar{X}_\xi, \\t &\rightarrow \bar{T}_s.\end{aligned}$$

The Finite Difference Approximation

- Usual centered difference scheme.

The Finite Difference Approximation

- Usual wave speed problem.
- Usual centered difference scheme.

The Finite Difference Approximation

- Usual ease of use.
- Usual wave speed problem.

Don't that beat all?

The Finite Difference Approximation

- Usual stability issues.
- Usual ease of use.
- Usual wave speed problem.

Don't that beat all?

Itô's Formula

Itô is like totally cool!

Comparison

This is what the left looks like

This is what the right looks like

Random Thoughts

But what about Barney and PBS?

Barney?

Is it okay to trust your kids with
Barney?

Random Thoughts

But what about Barney and PBS?

Barney?

Is it okay to trust your kids with
Barney?

No, not Barney!

Probably not.