# Stochastic Differential Equations

## Introduction

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# Outline

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# The Problem

Southern German water routes have had several drastic population changes concerning gammarids (Kinzler, 2008).

Much of this is due to canal construction.

**Native Species:**

*Gammarus pulex (Gp)*

**Invasive Species:**

*Dikerogammarus villosus (Dv)*
*Dikerogammarus haemobaphes (Dh)*
*Dikerogammarus bispinosus (Db)*
*Echinogammarus berilloni (Eb)*

# Killer Shrimp


http://www.arkive.org/killer-shrimp/dikerogammarus-villosus/image-G143154.html

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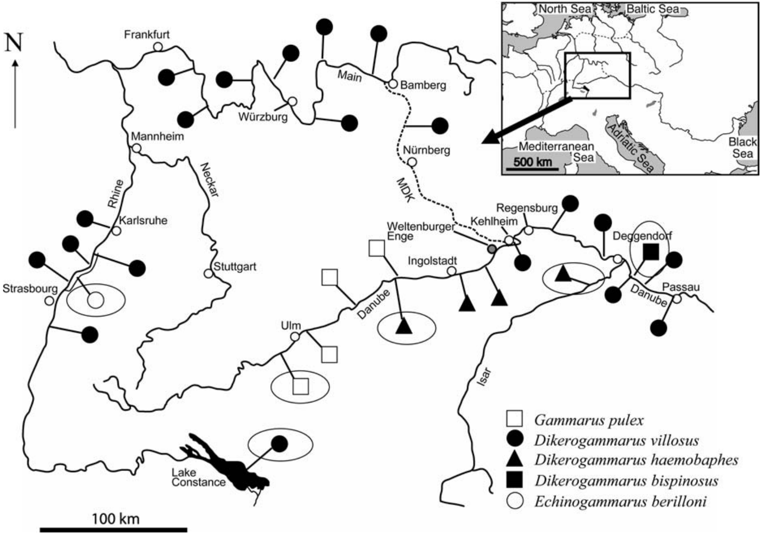



(Kinzler, 2008)

# Time Line

The time to prominence.

| Years | Species |
|---|---|
| $<$1976 | *Gp* native |
| 1976-1994 | *Dh* invades |
| 1992-1995 | *Dv* invades, *Dh* declines |
| $>$1995 | All but *Dv* coexist separate from *Dv* |

# Kinzler, 2008 Study

Isolated pairs of specimens in a controlled environment

- One freshly moulted (prey)
- One predator
- Total of 279 experiments

- Grouped by age, sex, and species*

Results

- Found *Dv* to be clear strongest predator
- Found *Dh* to have highest cannibalism rate

* Intraguild: predation between different species;
  Intraspecific: predation within species (cannibalism)

# Basic Goals

**Determine long term population trends!**

Does *Dv* totally dominate in the end?

Which species survive?

Is there an equilibrium?

# Outline

# Markov Process

(**Markov Process**) is a stochastic process with the following properties:

1. The number of possible outcomes or states is finite
2. The outcome at any stage depends only on the outcome of the previous stage.
3. The probabilities are constant over time.

# Brownian Motion

## Definition

(**Brownian Motion**) is a stochastic process that models random continuous motion. The stochastic process $B = \{B(t), t \geq 0\}$ is standard Brownian Motion if the following holds:

1. $B$ has independent increments.
2. For $0 \leq s < t$,

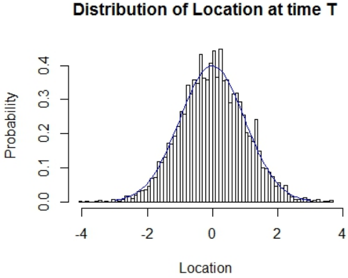
$$B(t) - B(s) \sim N(0, t - s).$$

3. The paths of $B$ are continuous with probability 1.
4. $B(0) = 0$

Random Walk
Random Walk [1]

# Monte Carlo Simulation - Brownian Motion

Recorded final step of 5000 Brownian Motions over [0,1]
We expect the probability to be $N(0,1)$.




$$\bar{x} = 0.019311, \quad s^2 = 0.9933, \quad t - test \;\; p - value = 0.16924$$

$$\text{CI:} \quad \mu \;\epsilon\; (-0.00822, 0.046844)$$

# Scaling a Brownian Motion

If $x(t)$ is a Brownian Motion then $\dfrac{x(\lambda t)}{\sqrt{\lambda}}$ is a Brownian motion.

$$
\begin{aligned}
& P\left(a \leq \frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}} \leq b\right) \\
&= P\left(a\sqrt{\lambda} \leq x(\lambda t) - x(\lambda s) \leq b\sqrt{\lambda}\right), \\
&= \frac{1}{\sqrt{2\pi(\lambda t - \lambda s)}} \int_{a\sqrt{\lambda}}^{b\sqrt{\lambda}} e^{\frac{-x^2}{2}(\lambda t - \lambda s)} dx, \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{\frac{-u^2}{t-s}} du.
\end{aligned}
$$

# Riemann-Stieltjes

(**Riemann-Stieltjes Integral**):

$$\int_a^b f(g)dg = \lim_{n\to\infty} \sum_{i=1}^{N} f(g(t_i)) \cdot (g(t_{i+1})) - g(t_i))$$

The main motivation for the Riemann-Stieltjes Integral comes from the concept of Cumulative Distribution Function (CDF) of a random variable.

# Wiener Process

A standard Wiener process (also called Brownian Motion) is a stochastic process $\{W_t\}_{t\geq 0}$, which has properties mutually consistent with those of Brownian motion.

**Wiener Integral** For a pair $(W_t, f(t))$ of a Wiener Process $W_t$, a random process $f(t)$, we define the Itô integral

$$I(f) = \int_0^\infty f(t)dW_t$$

# Itô's Formula

Itô's Formula is used in Itô Calculus to find the differential of a time-dependent function of a stochastic process.

**Differential Form**

$$\partial x_t = \left(\frac{\partial x}{\partial t} + \frac{1}{2}\left(\frac{\partial^2 x}{\partial B^2}\right)\right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

**Integral Form**

$$F(t, B(t)) - F(a, B(a)) = \int_a^t \frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial B^2} ds + \int_a^t \frac{\partial F}{\partial B} dB$$

# Itô's Formula

Let $F = tB^2$

$$\frac{dF}{dt} = B^2$$
$$\frac{dF}{dB} = 2t + B$$
$$\frac{d^2F}{dB^2} = 2t$$

$$tB^2(t) - aB^2(a) = \int_a^t B^s ds + \int_a^t 2sBdB + \int_a^t \frac{1}{2} 2sds$$
$$= \int_a^b B^2 + sds + \int_a^t 2sBdB$$

OR

$$tB^2(t) - aB^2(a) = \int_a^t B^s ds + \int_a^t 2sBdBt + \frac{1}{2}t^2 - \frac{1}{2}a^2$$

# Outline

# Shrimp Model

Our model is

$$\frac{dx}{dt} = rx^2\left(1 - \frac{x}{K}\right) - \alpha xy - \frac{x^2\gamma_\circ}{x + D},$$
$$\frac{dy}{dt} = \rho y^2\left(1 - \frac{y}{L}\right) - \beta xy - \frac{y^2\delta_\circ}{y + R}.$$

where $x$ is the population of $Dv$ and $y$ is the population of $Dh$, with parameters $[\ r, K, \alpha, \gamma_\circ, D, \rho, L, \beta, \delta_\circ, R\ ]$.

# Outline

# Nondimensionalization

The initial model is:

$$\frac{dx}{dt} = rx^2\left(1 - \frac{x}{K}\right) - \alpha xy,$$
$$\frac{dy}{dt} = \rho y^2\left(1 - \frac{y}{L}\right) - \beta xy.$$

Let

$$x \to A\hat{x}(s)$$
$$y \to B\hat{y}(s)$$
$$t \to \tau \cdot s$$

When you substitute and group terms the system becomes nondimensionalized.

After making the following substitutions:

$$A = K,$$
$$B = L,$$
$$\tau = \frac{1}{\rho}.$$

The nondimensionalized system is:

$$\frac{dx}{dt} = rx(1-x) - \alpha xy,$$
$$\frac{dy}{dt} = y(1-y) - \beta xy.$$

# Heun's Method

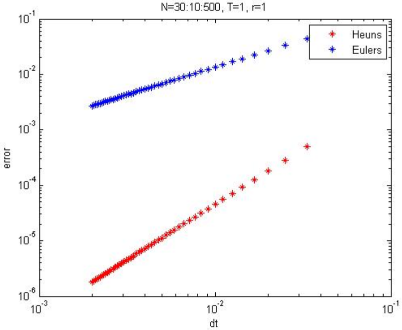
- Heun's method is a numerical procedure for approximating ordinary differential equations with a given initial value.
- First you calculate the intermediate value $\tilde{y}_{i+1}$ and then the final approximation $y_{i+1}$ at the next generation point.

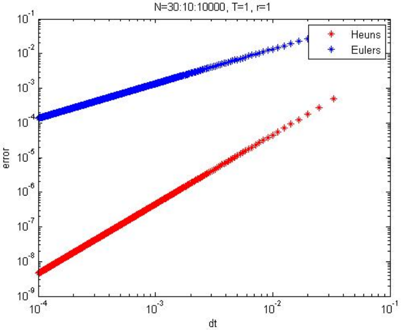
$$\tilde{y}_{i+1} = y_i + \Delta t \; f(t_i, y_i)$$
$$y_{i+1} = y_i + \frac{\Delta t}{2}\left[f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1})\right]$$

# Heun's Method vs. Euler's Method - Simulation

For the DE $y' = ry$ on $[0, T]$,

$$
\begin{aligned}
\text{Heun's:} \quad & \tilde{y}_{i+1} = y_i + \Delta t \ f(t_i, y_i) \\
\text{Euler's:} \quad & \tilde{y}_{i+1} = y_i + \Delta t \ f(r, y_i)
\end{aligned}
$$



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