

# Stochastic Differential Equations

## Invasion of the Killer Shrimp

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# Outline

- 1 Background on Killer Shrimp
- 2 Stochastics
- 3 Modeling

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  - The Problem
- 2 Stochastics
  - Fundamental Concepts
  - Integration
  - Itô's Formula
- 3 Modeling
  - The Model
  - Nondimensionalization

# The Problem

Southern German water routes have had several drastic population changes concerning gammarids (Kinzler, 2008) [2].

Much of this is due to canal construction.

Native Species:

*Gammarus pulex* (Gp)

Invasive Species:

*Dikerogammarus villosus* (Dv)

*Dikerogammarus haemobaphes* (Dh)

*Dikerogammarus bispinosus* (Db)

*Echinogammarus berilloni* (Eb)

# Killer Shrimp



<http://www.arkive.org/killer-shrimp/dikerogammarus-villosus/image-G143154.html>

# Killer Shrimp

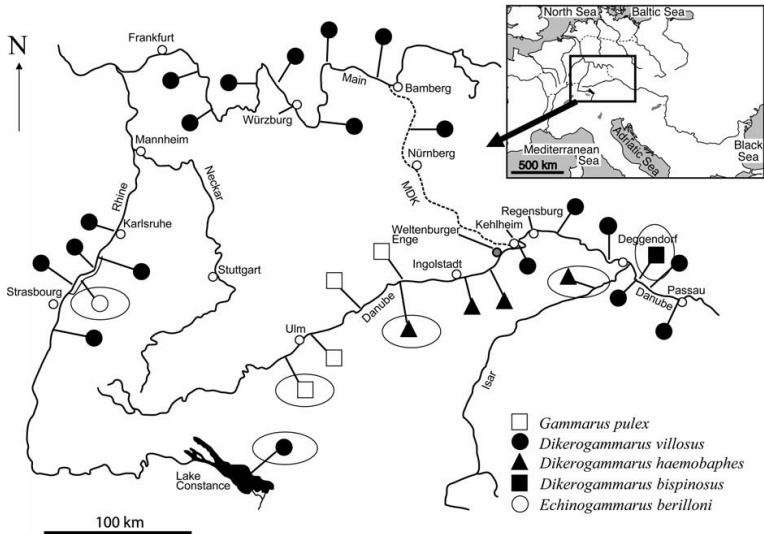


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# Killer Shrimp



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(Kinzler, 2008)



# Time Line

Years	Species
<1976	<i>Gp</i> native
1976-1994	<i>Dh</i> invades
1992-1995	<i>Dv</i> invades, <i>Dh</i> declines
>1995	All but <i>Dv</i> coexist

# Kinzler, 2008 Study

Isolated pairs of specimens in a controlled environment

- One freshly moulted (prey)
- One predator
- Total of 279 experiments
- Grouped by age, sex, and species \*

Results

- Found *Dv* to be clear strongest predator
- Found *Dh* to have highest cannibalism rate

\* Intraguild: predation between different species;  
Intraspecific: predation within species (cannibalism) [3]

# Basic Goals

**Determine long term population trends!**

Does  $Dv$  totally dominate in the end?

Why are we seeing this?

Why is  $Dv$  dominating?

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# Markov Process

(**Markov Process**) is a stochastic process with the following properties:

- ① The outcome at any stage depends only on the outcome of the previous stage.
- ② The probabilities are constant over time.

# Brownian Motion

## Definition

(**Brownian Motion**) is a stochastic process that models random continuous motion. The stochastic process  $B = \{B(t), t \geq 0\}$  is standard Brownian Motion if the following holds:

- ①  $B$  has independent increments.
- ② For  $0 \leq s < t$ ,

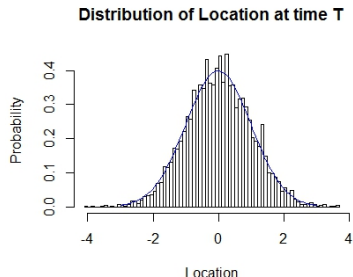
$$B(t) - B(s) \sim N(0, t - s).$$

- ③ The paths of  $B$  are continuous with probability 1.
- ④  $B(0) = 0$



# Monte Carlo Simulation - Brownian Motion

Recorded final step of 5000 Brownian Motions over  $[0,1]$   
We expect the distribution to be  $N(0, 1)$ .



$$\bar{x} = 0.019311, \quad s^2 = 0.9933, \quad t - test \quad p - value = 0.16924$$

$$CI: \quad \mu \in (-0.00822, 0.046844)$$



# Scaling a Brownian Motion

If  $x(t)$  is a Brownian Motion then  $\frac{x(\lambda t)}{\sqrt{\lambda}}$  is a Brownian motion.

$$\begin{aligned} & P\left(a \leq \frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}} \leq b\right) \\ &= P\left(a\sqrt{\lambda} \leq x(\lambda t) - x(\lambda s) \leq b\sqrt{\lambda}\right), \\ &= \frac{1}{\sqrt{2\pi(\lambda t - \lambda s)}} \int_{a\sqrt{\lambda}}^{b\sqrt{\lambda}} e^{\frac{-x^2}{2}(\lambda t - \lambda s)} dx, \\ &= \frac{1}{\sqrt{2\pi(t - s)}} \int_a^b e^{\frac{-u^2}{2}(t - s)} du. \end{aligned}$$

# Wiener Process

A standard Wiener process (also called Brownian Motion) is a stochastic process  $\{W_t\}_{t \geq 0}$ , which has properties mutually consistent with those of Brownian motion.

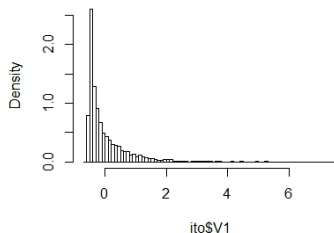
**Wiener Integral** For a pair  $(W_t, f(t))$  of a Wiener Process  $W_t$ , a random process  $f(t)$ , we define the Itô integral

$$I(f) = \int_0^a f(t) dW_t$$

# Monte Carlo Simulation - Itô Integral

Recorded approximations of 5000 Itô Integrals for  $\int_0^{1.5} WdW$   
 WHAT DO WE EXPECT???? I DON'T KNOW.

Histogram of ito\$V1



$$\bar{x} = 0.000555, \quad s^2 = 0.4925248$$

# Itô's Formula

Itô's Formula is used in Itô Calculus to find the differential of a time-dependent function of a stochastic process.

## Differential Form

$$dx_t = \left( \frac{\partial x}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 x}{\partial B^2} \right) \right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

## Integral Form

$$F(t, B(t)) - F(a, B(a)) = \int_a^t \frac{\partial F}{\partial s} ds + \frac{1}{2} \frac{\partial^2 F}{\partial B^2} ds + \int_a^t \frac{\partial F}{\partial B} dB$$

## Itô's Formula

Let  $F = tB^2$

$$\frac{dF}{dt} = B^2$$

$$\frac{dF}{dB} = 2t \cdot B$$

$$\frac{d^2F}{dB^2} = 2t$$

$$tB^2(t) - aB^2(a) = \int_a^t B^2 \, ds + \int_a^t 2sB \, dB + \int_a^t \frac{1}{2} 2s \, ds$$

OR

$$tB^2(t) - aB^2(a) = \int_a^t B^2 \, ds + \int_a^t 2sB \, dB + \frac{1}{2}t^2 - \frac{1}{2}a^2$$

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# Shrimp Model

Our model is

$$\begin{aligned}\frac{dx}{dt} &= rx^2 \left(1 - \frac{x}{K}\right) - \alpha xy - \frac{x^2 \gamma_o}{x + D}, \\ \frac{dy}{dt} &= \rho y^2 \left(1 - \frac{y}{L}\right) - \beta xy - \frac{y^2 \delta_o}{y + R},\end{aligned}$$

where  $x$  is the population of  $Dv$  and  $y$  is the population of  $Dh$ ,  
with parameters  $[r, K, \alpha, \gamma_o, D, \rho, L, \beta, \delta_o, R]$ .

# Nondimensionalization

The initial model is

$$\begin{aligned}\frac{dx}{dt} &= rx^2 \left(1 - \frac{x}{K}\right) - \alpha xy - \frac{x^2 \gamma_0}{x + D}, \\ \frac{dy}{dt} &= \rho y^2 \left(1 - \frac{y}{L}\right) - \beta xy - \frac{y^2 \delta_0}{y + R}.\end{aligned}$$

Let

$$x \rightarrow A\hat{x}(s)$$

$$y \rightarrow B\hat{y}(s)$$

$$t \rightarrow \tau \cdot s$$

When you substitute and group terms the system becomes nondimensionalized.



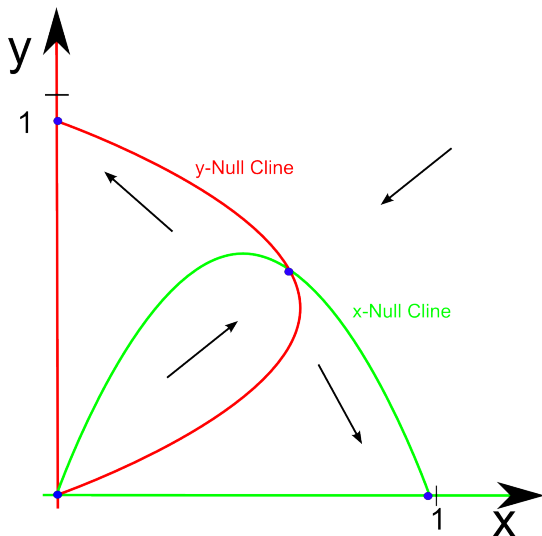
After making the following substitutions:

$$\begin{aligned}A &= K, \\ B &= L, \\ \tau &= \frac{1}{rK}.\end{aligned}$$

The nondimensionalized system is:

$$\begin{aligned}\frac{dx}{dt} &= x^2(1-x) - \alpha xy - \frac{\gamma_0 x^2}{x+D}, \\ \frac{dy}{dt} &= \rho y^2(1-y) - \beta xy - \frac{\delta_0 y^2}{y+R}\end{aligned}$$

# Phase Plane



- Four fixed points.
- Two are unstable.

What happens if we add noise?

$$\frac{dx}{dt} = x^2(1 - x) - \alpha xy - \frac{\gamma_0 x^2}{x + D} + \text{"noise,"}$$

$$\frac{dy}{dt} = \rho y^2(1 - y) - \beta xy - \frac{\delta_0 y^2}{y + R} + \text{"noise."}$$

If we add additive noise...

$$\begin{aligned}\frac{dx}{dt} &= x^2(1-x) - \alpha xy - \frac{\gamma_0 x^2}{x+D} + v \frac{dW}{dt}, \\ \frac{dy}{dt} &= \rho y^2(1-y) - \beta xy - \frac{\delta_0 y^2}{y+R} + \kappa \frac{dB}{dt}.\end{aligned}$$

If we add proportional noise...

$$\begin{aligned}\frac{dx}{dt} &= x^2(1-x) - \alpha xy - \frac{\gamma_0 x^2}{x+D} + vx \frac{dW}{dt}, \\ \frac{dy}{dt} &= \rho y^2(1-y) - \beta xy - \frac{\delta_0 y^2}{y+R} + \kappa y \frac{dB}{dt}.\end{aligned}$$

# Heun's Method

- Heun's method is a numerical procedure for approximating ordinary differential equations with a given initial value.
- First you calculate the intermediate value  $\tilde{y}_{i+1}$  and then the final approximation  $y_{i+1}$  at the next generation point.

$$\tilde{y}_{i+1} = y_i + \Delta t f(t_i, y_i)$$

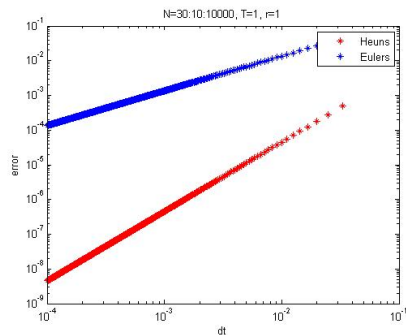
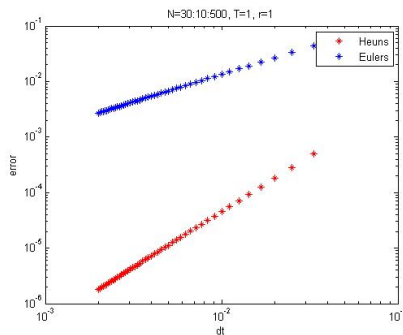
$$y_{i+1} = y_i + \frac{\Delta t}{2} [f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1})]$$

# Heun's Method vs. Euler's Method - Simulation

For the DE  $y' = ry$  on  $[0, T]$ ,

$$\text{Heun's: } y_{i+1} = y_i + \frac{\Delta t}{2} [f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1})]$$

$$\text{Euler's: } \tilde{y}_{i+1} = y_i + \Delta t f(y_i, t_i)$$



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