# Stochastic Differential Equations

## Invasion of the Killer Shrimp

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# Outline

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# The Problem

Southern German water routes have had several drastic population changes concerning gammarids (Kinzler, 2008) [2].

Much of this is due to canal construction.

**Native Species:**

*Gammarus pulex (Gp)*

**Invasive Species:**

*Dikerogammarus villosus (Dv)*
*Dikerogammarus haemobaphes (Dh)*
*Dikerogammarus bispinosus (Db)*
*Echinogammarus berilloni (Eb)*

# Killer Shrimp


http://www.arkive.org/killer-shrimp/dikerogammarus-villosus/image-G143154.html

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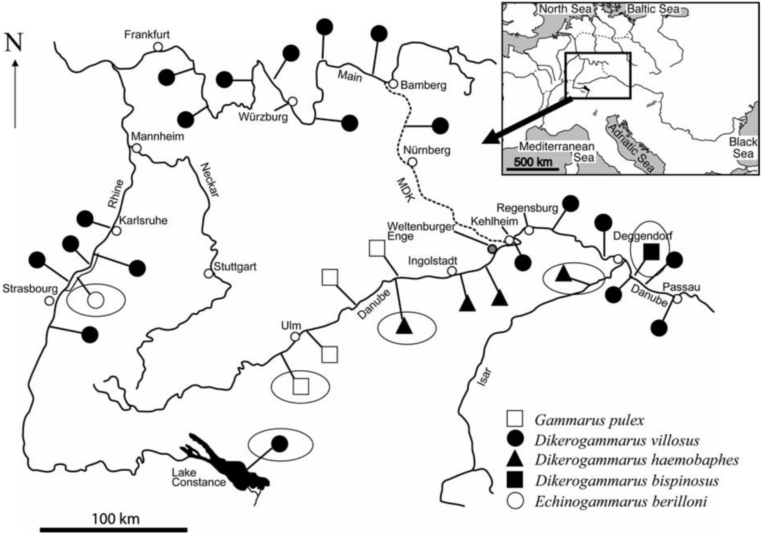



(Kinzler, 2008)

# Time Line

| Years | Species |
|---|---|
| $<1976$ | *Gp* native |
| 1976-1994 | *Dh* invades |
| 1992-1995 | *Dv* invades, *Dh* declines |
| $>1995$ | All but *Dv* coexist |

# Kinzler, 2008 Study

Isolated pairs of specimens in a controlled environment

- One freshly moulted (prey)
- One predator
- Total of 279 experiments
- Grouped by age, sex, and species *

Results

- Found *Dv* to be clear strongest predator
- Found *Dh* to have highest cannibalism rate

* Intraguild: predation between different species;
Intraspecific: predation within species (cannibalism) [3]

# Basic Goals

**Determine long term population trends!**

Does $Dv$ totally dominate in the end?

Why are we seeing this?

Why is $Dv$ dominating?

# Outline

# Markov Process

(**Markov Process**) is a stochastic process with the following properties:

1. The outcome at any stage depends only on the outcome of the previous stage.
2. The probabilities are constant over time.

# Brownian Motion

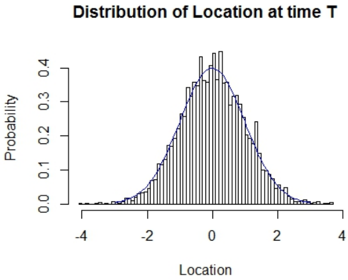
**Definition**

(**Brownian Motion**) is a stochastic process that models random continuous motion. The stochastic process $B = \{B(t), t \geq 0\}$ is standard Brownian Motion if the following holds:

1. $B$ has independent increments.
2. For $0 \leq s < t$,
$$B(t) - B(s) \sim N(0, t - s).$$
3. The paths of $B$ are continuous with probability 1.
4. $B(0) = 0$

# Monte Carlo Simulation - Brownian Motion

Recorded final step of 5000 Brownian Motions over [0,1]
We expect the distribution to be $N(0,1)$.




$$\bar{x} = 0.019311, \quad s^2 = 0.9933, \quad t - test \;\; p - value = 0.16924$$

$$\text{CI:} \quad \mu \;\epsilon\; (-0.00822, 0.046844)$$

# Scaling a Brownian Motion

If $x(t)$ is a Brownian Motion then $\dfrac{x(\lambda t)}{\sqrt{\lambda}}$ is a Brownian motion.

$$\begin{aligned}
& P\left(a \leq \frac{x(\lambda t)}{\sqrt{\lambda}} - \frac{x(\lambda s)}{\sqrt{\lambda}} \leq b\right) \\
&= P\left(a\sqrt{\lambda} \leq x(\lambda t) - x(\lambda s) \leq b\sqrt{\lambda}\right), \\
&= \frac{1}{\sqrt{2\pi(\lambda t - \lambda s)}} \int_{a\sqrt{\lambda}}^{b\sqrt{\lambda}} e^{\frac{-x^2}{2}(\lambda t - \lambda s)} dx, \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{\frac{-u^2}{t-s}} du.
\end{aligned}$$

# Wiener Process

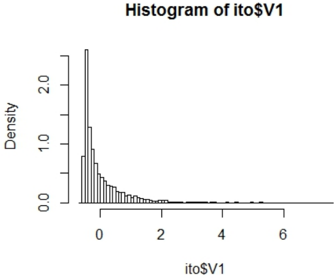
A standard Wiener process (also called Brownian Motion) is a stochastic process $\{W_t\}_{t\geq 0}$, which has properties mutually consistent with those of Brownian motion.

**Wiener Integral** For a pair $(W_t, f(t))$ of a Wiener Process $W_t$, a random process $f(t)$, we define the Itô integral

$$I(f) = \int_0^a f(t)dW_t$$

# Monte Carlo Simulation - Itô Integral

Recorded approximations of 5000 Itô Integrals for $\int_0^{1.5} W dW$
WHAT DO WE EXPECT???? I DON'T KNOW.




$$\bar{x} = 0.000555, \quad s^2 = 0.4925248$$

# Itô's Formula

Itô's Formula is used in Itô Calculus to find the differential of a time-dependent function of a stochastic process.

## Differential Form

$$dx_t = \left( \frac{\partial x}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 x}{\partial B^2} \right) \right) \cdot dt + \frac{\partial x}{\partial B} \cdot dB$$

## Integral Form

$$F(t, B(t)) - F(a, B(a)) = \int_a^t \frac{\partial F}{\partial s} + \frac{1}{2} \frac{\partial^2 F}{\partial B^2} ds + \int_a^t \frac{\partial F}{\partial B} dB$$

# Itô's Formula

Let $F = tB^2$

$$\frac{dF}{dt} = B^2$$
$$\frac{dF}{dB} = 2t \cdot B$$
$$\frac{d^2F}{dB^2} = 2t$$

$$tB^2(t) - aB^2(a) = \int_a^t B^2 \; ds + \int_a^t 2sB \; dB + \int_a^t \frac{1}{2} 2s \; ds$$

OR

$$tB^2(t) - aB^2(a) = \int_a^t B^2 \; ds + \int_a^t 2sB \; dB + \frac{1}{2}t^2 - \frac{1}{2}a^2$$

# Outline

# Shrimp Model

Our model is

$$\frac{dx}{dt} = rx^2\left(1 - \frac{x}{K}\right) - \alpha xy - \frac{x^2\gamma_\circ}{x + D},$$
$$\frac{dy}{dt} = \rho y^2\left(1 - \frac{y}{L}\right) - \beta xy - \frac{y^2\delta_\circ}{y + R},$$

where $x$ is the population of $Dv$ and $y$ is the population of $Dh$, with parameters [ $r, K, \alpha, \gamma_\circ, D, \rho, L, \beta, \delta_\circ, R$ ].

# Nondimensionalization

The initial model is

$$\frac{dx}{dt} = rx^2\left(1 - \frac{x}{K}\right) - \alpha xy - \frac{x^2\gamma_\circ}{x + D},$$
$$\frac{dy}{dt} = \rho y^2\left(1 - \frac{y}{L}\right) - \beta xy - \frac{y^2\delta_\circ}{y + R}.$$

Let

$$x \to A\hat{x}(s)$$
$$y \to B\hat{y}(s)$$
$$t \to \tau \cdot s$$

When you substitute and group terms the system becomes nondimensionalized.

After making the following substitutions:

$$A = K,$$
$$B = L,$$
$$\tau = \frac{1}{rK}.$$

The nondimensionalized system is:

$$\frac{dx}{dt} = x^2(1-x) - \alpha xy - \frac{\gamma_\circ x^2}{x+D},$$
$$\frac{dy}{dt} = \rho y^2(1-y) - \beta xy - \frac{\delta_\circ y^2}{y+R}$$

What happens if we add noise?

$$\frac{dx}{dt} = x^2(1-x) - \alpha xy - \frac{\gamma_\circ x^2}{x+D} + \text{“noise,”}$$
$$\frac{dy}{dt} = \rho y^2(1-y) - \beta xy - \frac{\delta_\circ y^2}{y+R} + \text{“noise.”}$$

If we add additive noise...

$$\frac{dx}{dt} = x^2(1-x) - \alpha xy - \frac{\gamma_\circ x^2}{x+D} + \upsilon\frac{dW}{dt},$$
$$\frac{dy}{dt} = \rho y^2(1-y) - \beta xy - \frac{\delta_\circ y^2}{y+R} + \kappa\frac{dB}{dt}.$$

If we add proportional noise...

$$\frac{dx}{dt} = x^2(1-x) - \alpha xy - \frac{\gamma_\circ x^2}{x+D} + \upsilon x \frac{dW}{dt},$$
$$\frac{dy}{dt} = \rho y^2(1-y) - \beta xy - \frac{\delta_\circ y^2}{y+R} + \kappa y \frac{dB}{dt}.$$

# Heun's Method

- Heun's method is a numerical procedure for approximating ordinary differential equations with a given initial value.
- First you calculate the intermediate value $\tilde{y}_{i+1}$ and then the final approximation $y_{i+1}$ at the next generation point.

$$\tilde{y}_{i+1} = y_i + \Delta t \ f(t_i, y_i)$$

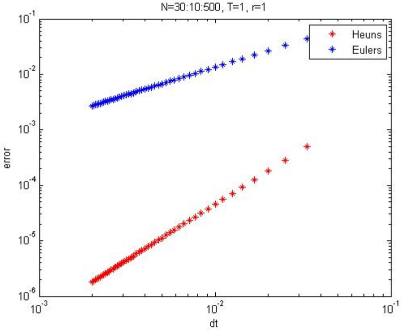
$$y_{i+1} = y_i + \frac{\Delta t}{2} \left[ f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1}) \right]$$

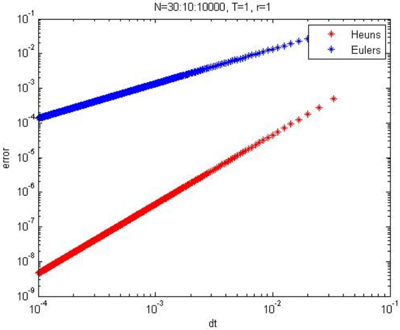
# Heun's Method vs. Euler's Method - Simulation

For the DE $y' = ry$ on $[0, T]$,

$$\text{Heun's:} \quad y_{i+1} = y_i + \frac{\Delta t}{2}\left[f(y_i, t_i) + f(\tilde{y}_{i+1}, t_{i+1})\right]$$

$$\text{Euler's:} \quad \tilde{y}_{i+1} = y_i + \Delta t \; f(y_i, t_i)$$



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