Announcements

Announcements

- Sign up for exam time if you haven't yet done so!
- Exam questions will be posted on Sunday
  - Some material from today/Tuesday will be included, so you may not be able to do all of it immediately
- Same procedure as last time
- Any questions?

greedy, DP, network flows (LP)

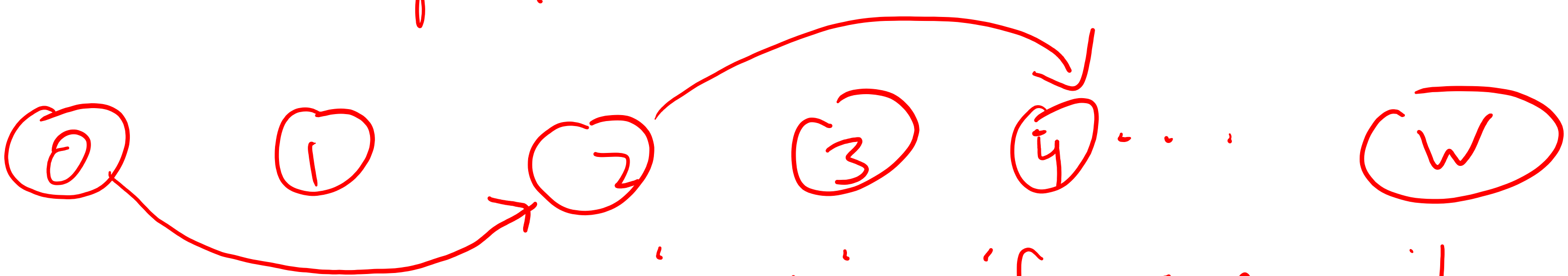
KP w/out rep
All-pairs s.p.

2-dimensional DAG / 2-d array

Dynamic Programming

KP w/repetition

longest path from 0



$i \to j$ if some item had weight $= j - i$, weight of edge = value of item

Knapsack without Repetition

- Same as before, except now you can't have multiple copies of an item.
- Are subproblems from before still useful?

what is max val I can get with a total weight of exactly i?

not useful now, because it doesn't track which items already used

Knapsack without Repetition

- Same as before, except now you can't have multiple copies of an item.
- Are subproblems from before still useful? total weight
- Define $K(w, j)$ = maximum value achievable with capacity of w, ~~using~~ choosing only from items $1, \ldots, j$
- What is the value of $K(w, j)$?

Knapsack without Repetition

- Same as before, except now you can't have multiple copies of an item.
- Are subproblems from before still useful?
- Define *K*(*w*, *j*) = maximum achievable with capacity of *w*, using only items 1, …, *j*
- What is the value of *K*(*w*, *j*)?

choose to include item j

$$K(w,j) = \max\{K(w-w_j, j-1) + v_j, K(w, j-1)\}$$

do not include item j

Knapsack without Repetition

$w_1 = 1$ $K(w,3)$ $K(1,1)$ depends on

$K(w,4)$ $K(w-w_4,3)$ $K(1,0)$ and $K(1-w_1,0)$

Initialize all $K(0,j)=0$ and all $K(w,0)=0$

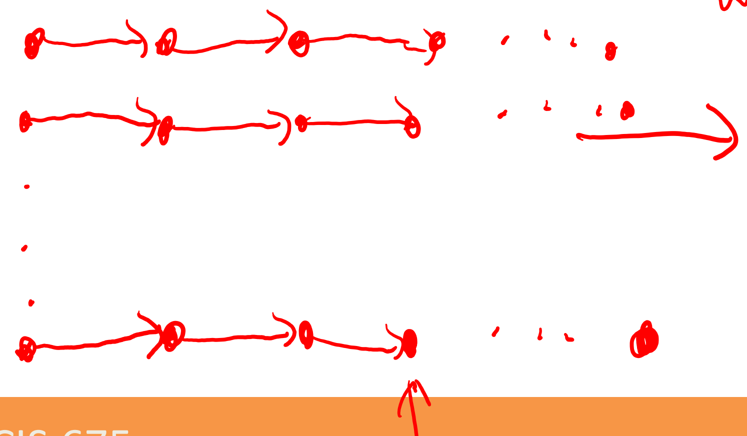
for $j=1$ to n: n iterations

  for $w=1$ to W: W iterations $O(n \cdot W)$

    if $w_j > w$: $K(w,j) = K(w,j-1)$

    else: $K(w,j) = \max\{K(w,j-1), K(w-w_j,j-1)+v_j\}$

return $K(W,n)$

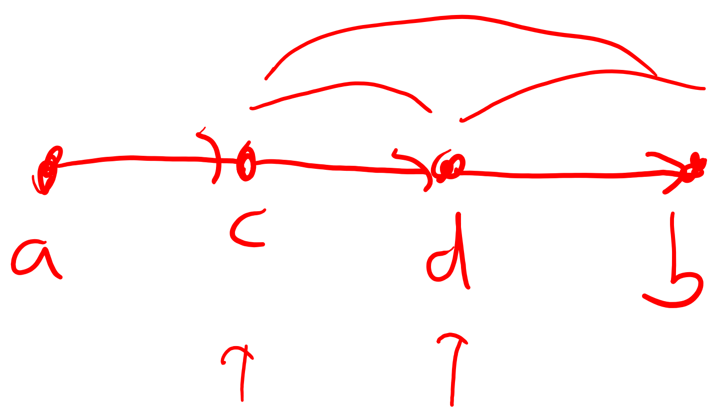


$w\downarrow$ $j\rightarrow$	0	1	2	3	...	N
0	0	0	0	0	...	0
1	0	•	•			
2	0					
⋮	⋮					
W	0					

- Given the adjacency matrix of a weighted, directed graph *G*
- We want to find the shortest path length between *each pair* of nodes

S_{ij} = length of s.p. $i \to j$

- One solution: run Dijkstra's from each node as source
  - Too slow!



All-Pairs Shortest Paths: the Floyd-Warshall Algorithm

- Idea: use a shortest path matrix M to keep track of all-pairs shortest path lengths (APSPLs)
- M^i corresponds to the state of matrix M at the beginning of step i
- Define an *intermediate vertex* on a shortest path as a node that is not the source or destination
- Subproblem: In each ~~step~~ iteration i, we want to find the APSPLs that use only nodes $1, \ldots, i$ as intermediate nodes

All-Pairs Shortest Paths: the Floyd-Warshall Algorithm

- Subproblem: In each step *i*, we want to find the APSPLs that use only nodes *1, …, i*
- What is the initial value of *M*?
- What is the update equation for M^i_{jk}?

M^0 represents cases where we are not allowed to use any int. verts.

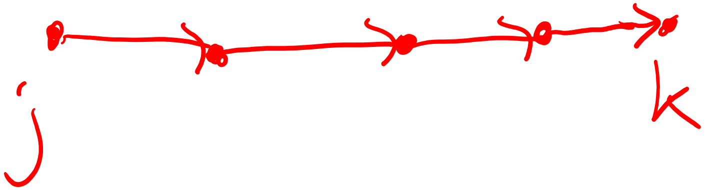
M^0_{ij} = weight of edge (i,j) if such an edge exists, ∞ otherwise

All-Pairs Shortest Paths: the Floyd-Warshall Algorithm

M^x for $x < i$

- What is the update equation for M^i_{jk}?

s.p.l. from $j \to k$ using only nodes from $\{1, \ldots, i\}$ as inter. verts.

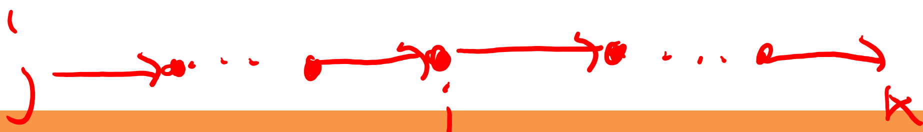


N iterations
each iter is
$O(N^2)$ time,
total
$O(N^3)$

2 possibilities:

1. don't use node i, then $M^i_{jk} = M^{i-1}_{jk}$

2. do use node i. The path looks like

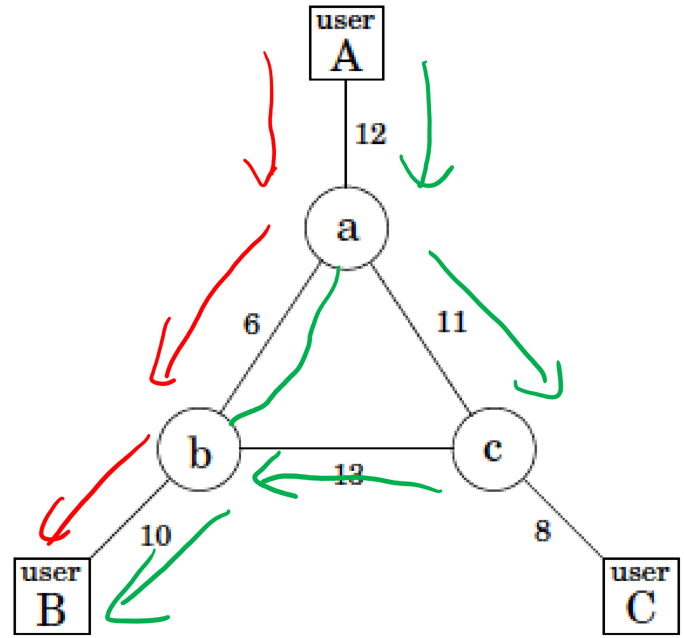


$M^i_{jk} = M^{i-1}_{ji} + M^{i-1}_{ik}$

Network Flows

Allocating Bandwidth

- Need to allocate bandwidth to connect A-B, B-C, and A-C
- A-B pays $3 per unit of bandwidth, B-C pays $2, and A-C pays $4
- Can connect via long path or short path
- Each ~~connection~~ pair of users must have at least 2 units of bandwidth
- Each edge has a maximum capacity



Allocating Bandwidth

- Each connection can use the long path or short path: consider each possibility

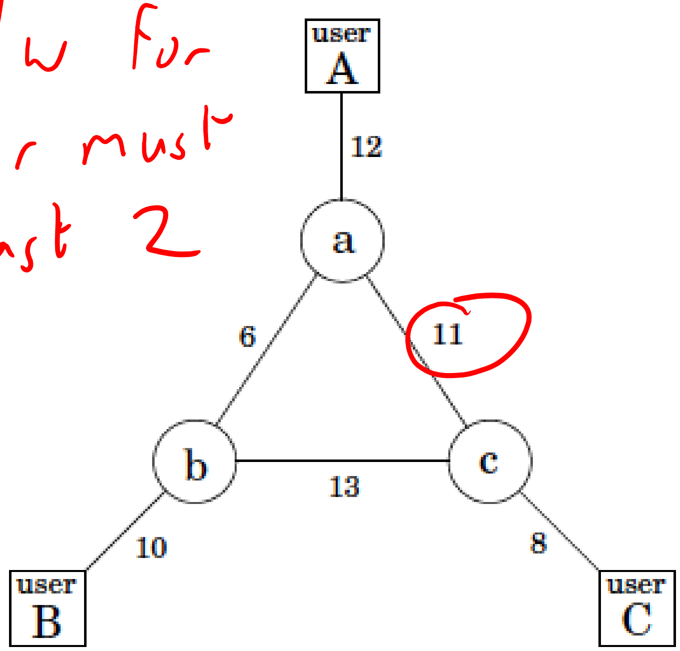
x_{AB} = amount of b/w alloc. to A/B along short path

obj. function

$$\max\ 3x_{AB} + 3x'_{AB} + 2x_{BC} + 2x'_{BC} + 4x_{AC} + 4x'_{AC}$$

$$x_{AB} + x'_{AB} \geq 2$$
$$x_{BC} + x'_{BC} \geq 2$$
$$x_{AC} + x'_{AC} \geq 2$$

total b/w for each pair must be at least 2



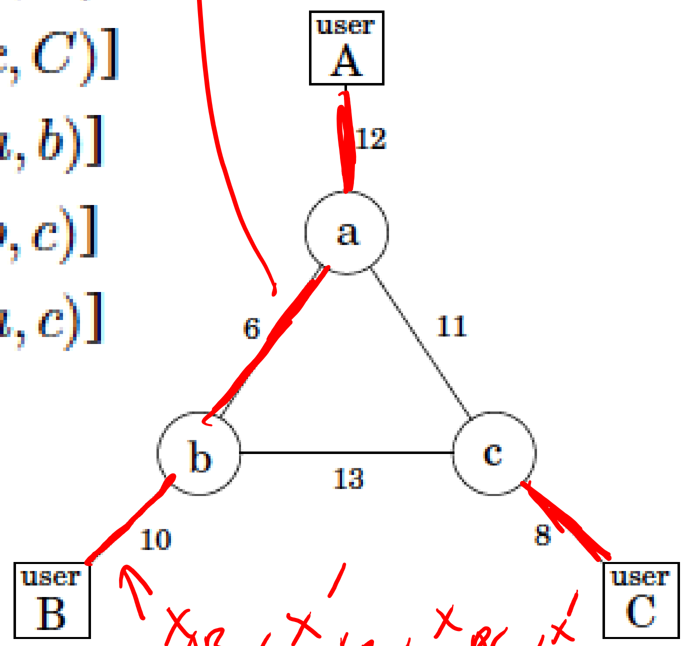
x'_{AB} = amount allocated to A/B along long path

Allocating Bandwidth

- There are bandwidth constraints on each edge

$$
\begin{aligned}
x_{AB} + x'_{AB} + x_{BC} + x'_{BC} &\le 10 && \text{[edge } (b, B)] \\
x_{AB} + x'_{AB} + x_{AC} + x'_{AC} &\le 12 && \text{[edge } (a, A)] \\
x_{BC} + x'_{BC} + x_{AC} + x'_{AC} &\le 8 && \text{[edge } (c, C)] \\
x_{AB} + x'_{BC} + x'_{AC} &\le 6 && \text{[edge } (a, b)] \\
x'_{AB} + x_{BC} + x'_{AC} &\le 13 && \text{[edge } (b, c)] \\
x'_{AB} + x'_{BC} + x_{AC} &\le 11 && \text{[edge } (a, c)]
\end{aligned}
$$

x_{AB}
x'_{AC}
x'_{BC}



$x_{AB}, x'_{AB}, x_{BC}, x'_{BC}$

Allocating Bandwidth: Putting it Together

$$\max \quad 3x_{AB} + 3x'_{AB} + 2x_{BC} + 2x'_{BC} + 4x_{AC} + 4x'_{AC}$$

$$\begin{aligned}
x_{AB} + x'_{AB} + x_{BC} + x'_{BC} &\le 10 \\
x_{AB} + x'_{AB} + x_{AC} + x'_{AC} &\le 12 \\
x_{BC} + x'_{BC} + x_{AC} + x'_{AC} &\le 8 \\
x_{AB} + x'_{BC} + x'_{AC} &\le 6 \\
x'_{AB} + x_{BC} + x'_{AC} &\le 13 \\
x'_{AB} + x'_{BC} + x_{AC} &\le 11 \\
x_{AB} + x'_{AB} &\ge 2 \\
x_{BC} + x'_{BC} &\ge 2 \\
x_{AC} + x'_{AC} &\ge 2 \\
x_{AB}, x'_{AB}, x_{BC}, x'_{BC}, x_{AC}, x'_{AC} &\ge 0
\end{aligned}$$

capacity constraints

Network Flows

- What happens as the number of nodes in the previous problem becomes large?

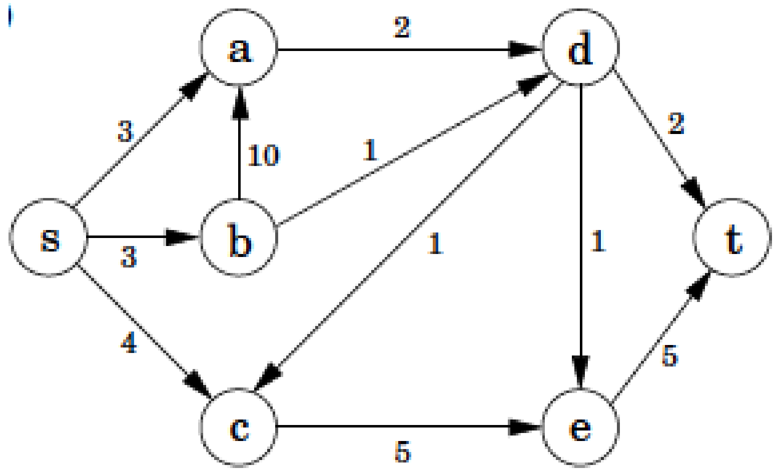
number of constraints ↑

not an efficient solution!

- Better solution: network flow algorithms

Network Flows

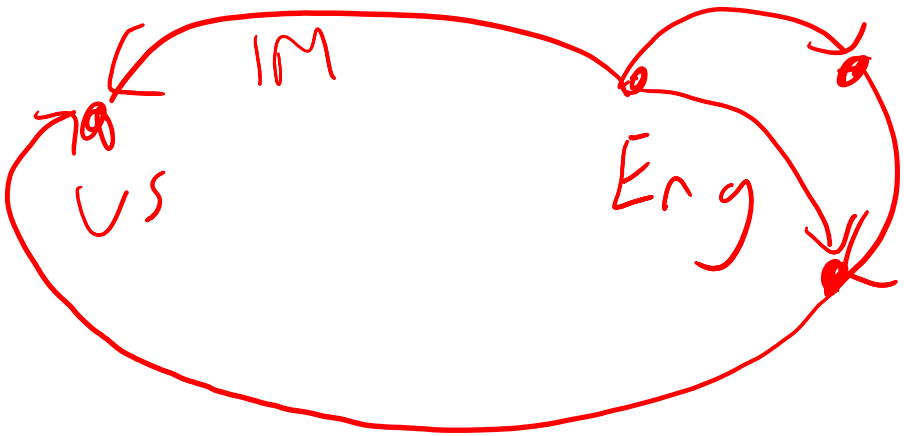
- You are given a network with capacities on the edges

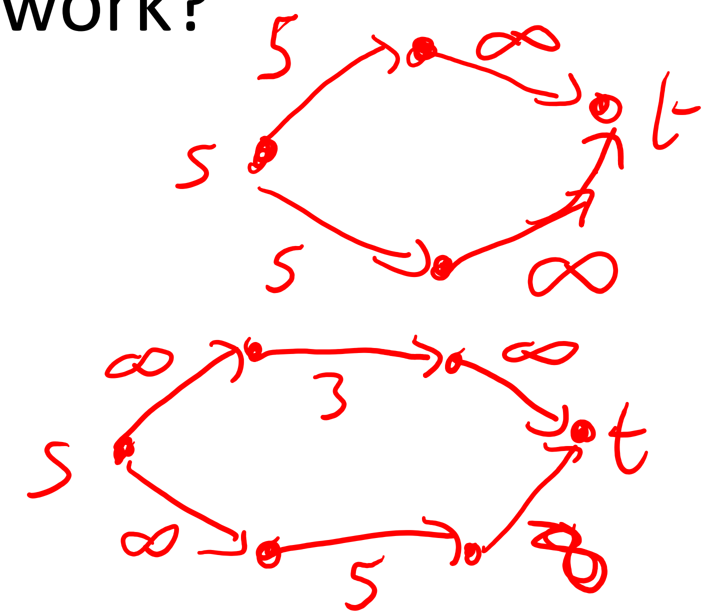


- Goal is to decide how much flow to send along each edge, to maximize total amount from s to t

Examples of Network Flows

- How many products can we ship from England to the US, given the current set of flights?
- How much data can we send from one computer to another, given current network?





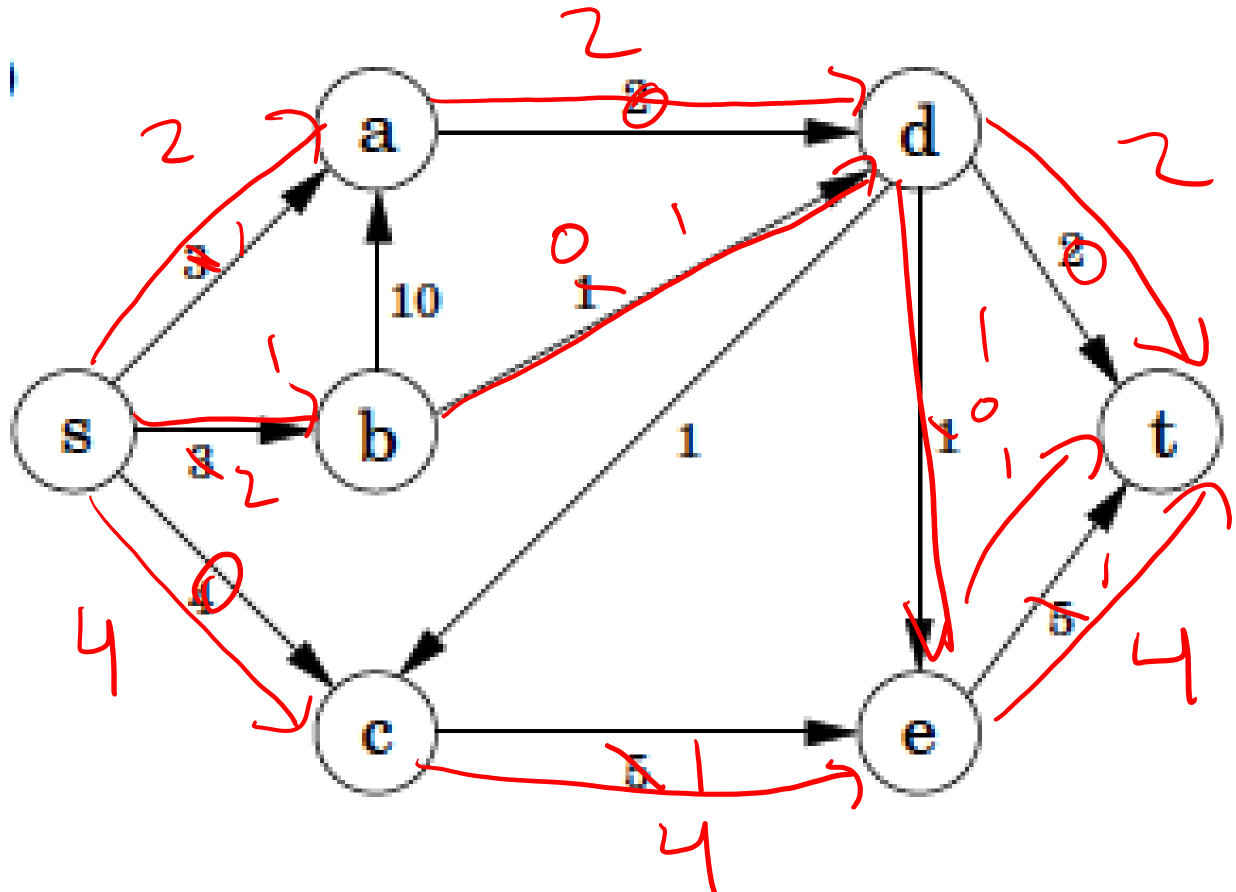
Rules of Network Flows

- The flow sent along an edge cannot exceed the total capacity of that edge constraint
- The total amount of flow entering a node must equal the amount of flow leaving a node, excepting source + target/sink
- This is a linear programming problem!

not efficient to view it like this!

In-Class Exercise

Find the maximum amount of flow from s to t



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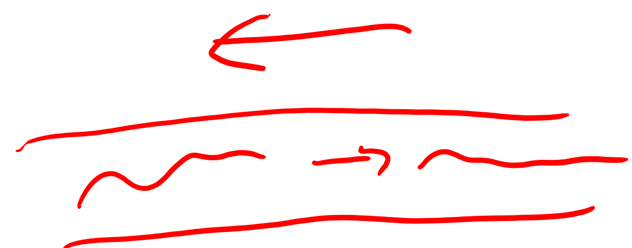
The Ford-Fulkerson Network Flow Algorithm

- Choose a path from S to T – doesn't matter which path (except for efficiency)
- Send as much flow as possible along that path – min capacity of an edge along path
- Rules of path selection:
  - You can use an edge if there is capacity left
  - You can *reverse* flow sent along an edge



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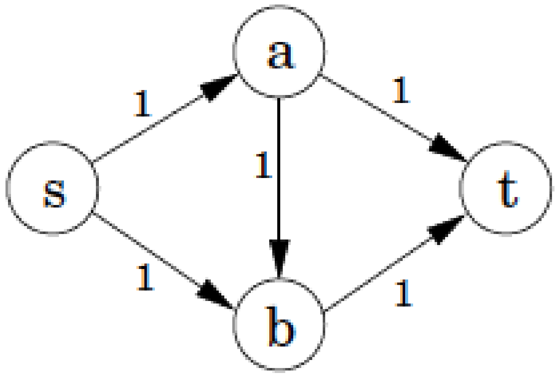
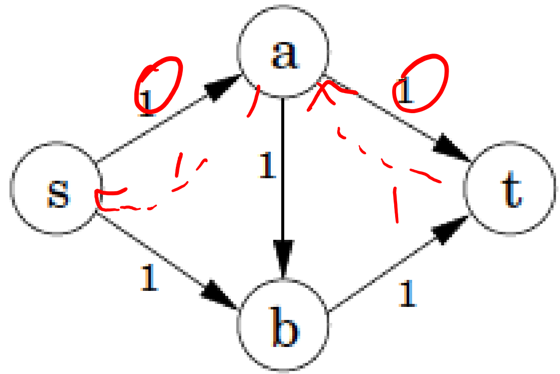
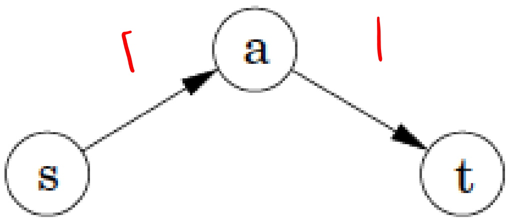
Sketch of Network Flow Algorithm



Sketch of Network Flow Algorithm





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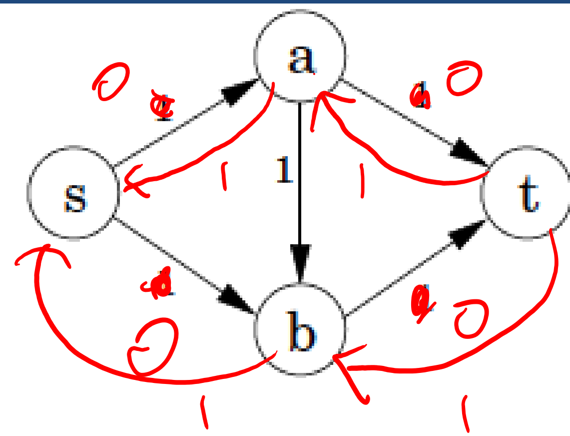

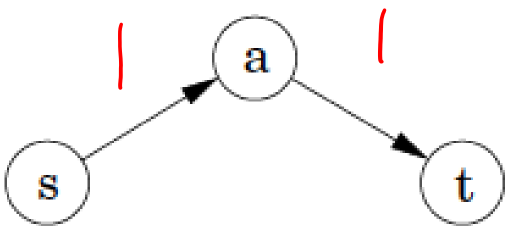

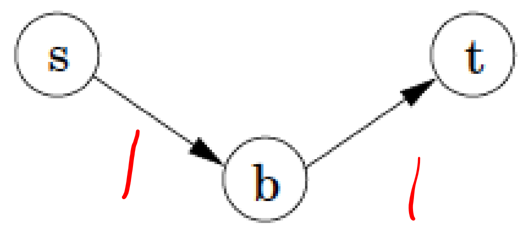

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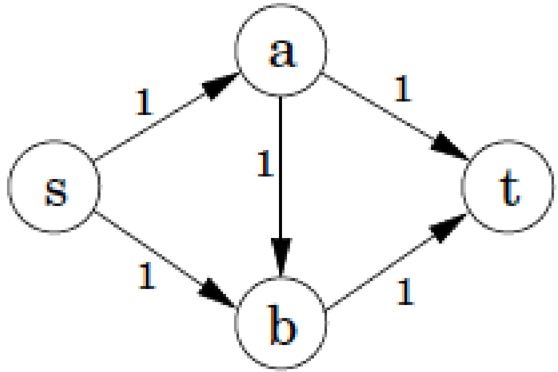
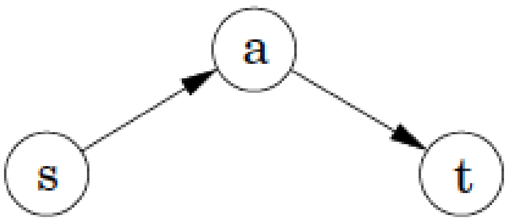
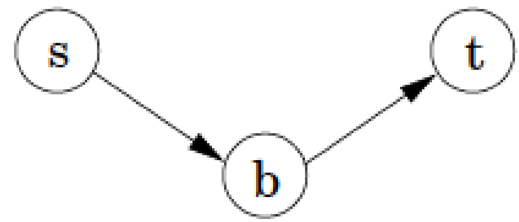
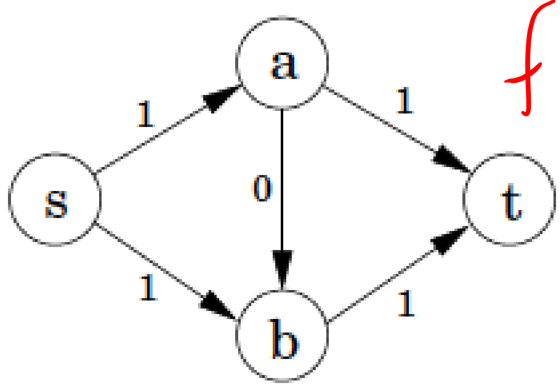


capacity graph

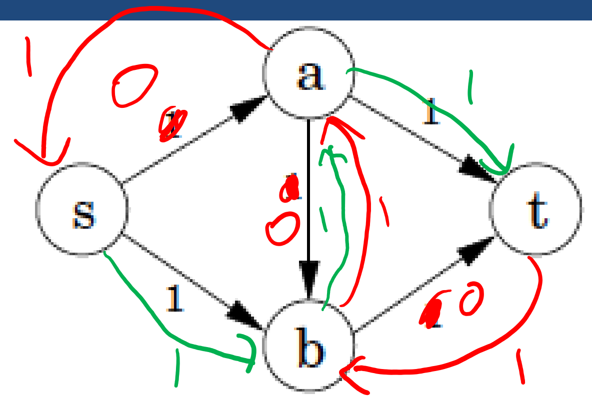


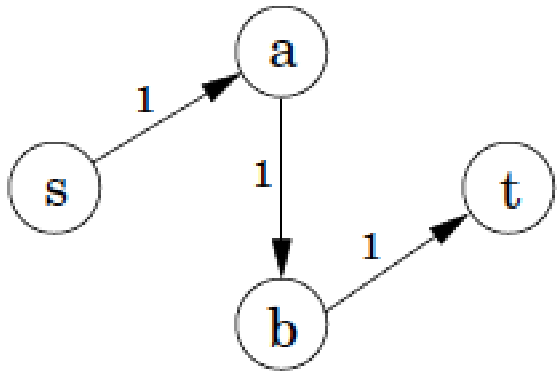




flow graph

Sketch of Network Flow Algorithm





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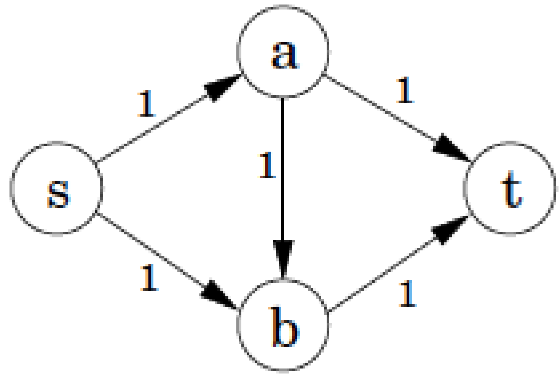

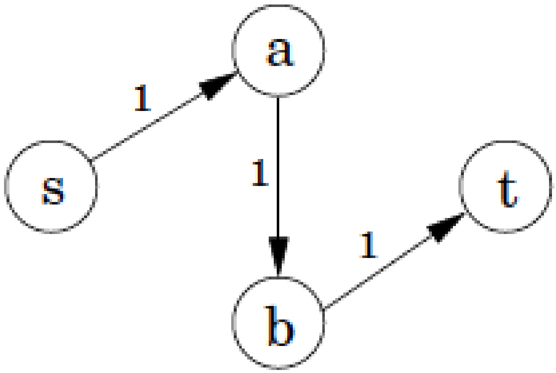

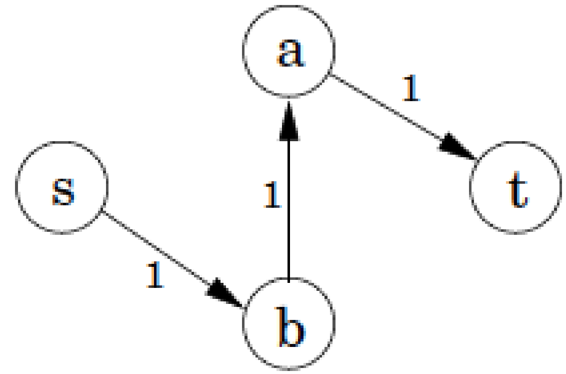

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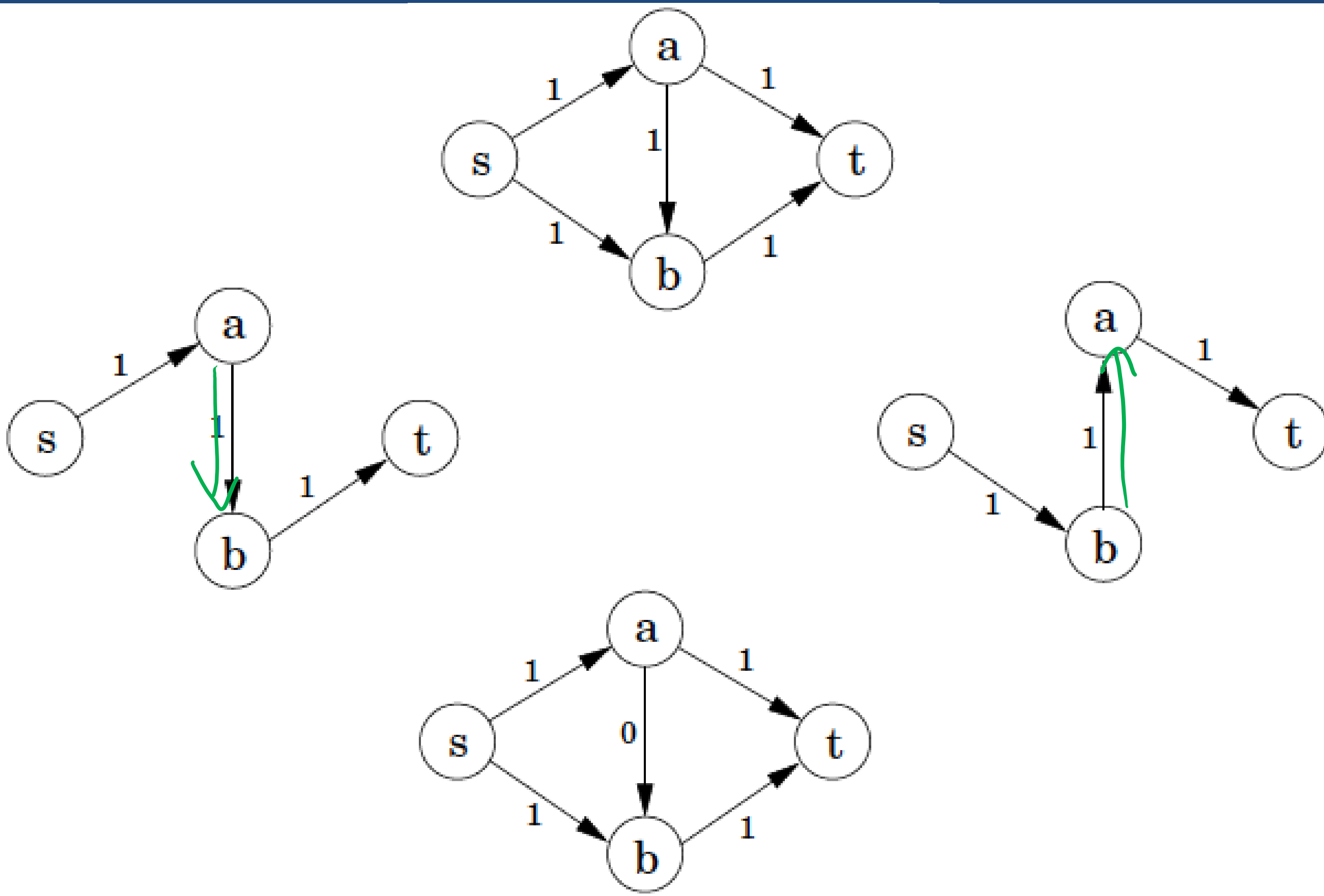

Network Flow Example

Find the maximum amount of flow from s to t