# Inverse Kinematics

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^0_fT = \begin{bmatrix} c_2c_{34} & -c_2s_{34} & -s_2 & a_4c_2c_{34} + a_3c_2c_3 \\ s_{34} & c_{34} & 0 & a_4s_{34} + a_3s_3 - d_2 \\ s_2c_{34} & -s_2s_{34} & c_2 & a_4s_2c_{34} + d_1 + a_3s_2c_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Goal ${}^0_fT$

Want to find: $d_1, \theta_2, \theta_3, \theta_4$ as functions of known values

Known values:
- Goals: $r_{11}, r_{12}, \ldots r_{33}, P_x, P_y, P_z$
- Link lengths: $d_2, a_3, a_4$

$$[{}^0_1T(d_1)]^{-1} \underbrace{{}^0_fT}_{\text{Goal}} = [{}^0_1T(d_1)]^{-1} \; {}^0_fT = {}^1_fT$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_2c_{34} & -c_2s_{34} & -s_2 & a_4c_2c_{34} + a_3c_2c_3 \\ s_{34} & c_{34} & 0 & a_4s_{34} + a_3s_3 - d_2 \\ s_2c_{34} & -s_2s_{34} & c_2 & a_4s_2c_{34} + a_3s_2c_3 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z - d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_fT = \begin{bmatrix} c_2c_{34} & -c_2s_{34} & -s_2 & a_4c_2c_{34} + a_3c_2c_3 \\ s_{34} & c_{34} & 0 & a_4s_{34} + a_3s_3 - d_2 \\ s_2c_{34} & -s_2s_{34} & c_2 & a_4s_2c_{34} + a_3s_2c_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_z - d_1 = a_4s_2c_{34} + a_3s_2c_3$$

$$d_1 = P_z - a_4s_2c_{34} - a_3s_2c_3 \quad \ldots (1)$$

$$r_{23} = 0$$

$$\left.\begin{aligned} r_{33} &= c_2 \\ r_{13} &= -s_2 \rightarrow s_2 = -r_{13} \end{aligned}\right\} \tan\theta_2 = \frac{-r_{13}}{r_{33}} \rightarrow \boxed{\theta_2 = \operatorname{Atan2}(-r_{13}, r_{33})}$$

$$\left.\begin{aligned} r_{21} &= s_{34} \\ r_{22} &= c_{34} \end{aligned}\right\} \tan(\theta_3+\theta_4) = \frac{r_{21}}{r_{22}} \qquad \theta_3+\theta_4 = \operatorname{Atan2}(r_{21}, r_{22}) \ldots (2)$$

$$d_1 = P_z - a_4(-r_{13})r_{22} - a_3(-r_{13})c_3$$

$$d_1 = P_z + a_4r_{13}r_{22} + a_3r_{13}c_3 \quad \ldots (3)$$

Need to find $\theta_3$ to get $d_1$ and $\theta_4$, then everything will be solved.

$$P_x = a_4 c_2 c_{34} + a_3 c_2 c_3 \rightarrow c_3 = \frac{P_x - a_4 c_2 c_{34}}{a_3 c_2} \quad \ldots (4)$$

$$P_y = a_4 s_{34} + a_3 s_3 - d_2 \rightarrow s_3 = \frac{P_y - a_4 s_{34} + d_2}{a_3}$$

$$\tan\theta_3 = \frac{(P_y - a_4 s_{34} + d_2)\, \cancel{a_3}\, c_2}{(P_x - a_4 c_2 c_{34})\, \cancel{a_3}}$$

$$\theta_3 = \mathrm{Atan2}\Big((P_y - a_4 \underset{\substack{\uparrow \\ r_{21}}}{s_{34}} + d_2)\, \underset{\substack{\downarrow \\ r_{33}}}{c_2}\ ,\ P_x - a_4 \underset{\substack{\downarrow \\ r_{33}}}{c_2} \underset{\substack{\downarrow \\ r_{22}}}{c_{34}}\Big)$$

$$\boxed{\theta_3 = \mathrm{Atan2}\big((P_y - a_4 r_{21} + d_2)\, r_{33},\ P_x - a_4 r_{33} r_{22}\big)} \quad \ldots (5)$$

From (2) & (5):

$$\theta_3 + \theta_4 = \mathrm{Atan2}(r_{21}, r_{22})$$

$$\theta_4 = \mathrm{Atan2}(r_{21}, r_{22}) - \theta_3$$

$$\boxed{\theta_4 = \mathrm{Atan2}(r_{21}, r_{22}) - \mathrm{Atan2}\big((P_y - a_4 r_{21} + d_2)\, r_{33},\ P_x - a_4 r_{33} r_{22}\big)}$$

From (3) & (4):

$$d_1 = P_z + a_4 r_{13} r_{22} + a_3 r_{13} c_3$$

$$d_1 = P_z + a_4 r_{13} r_{22} + \cancel{a_3} r_{13} \left(\frac{P_x - a_4 c_2 c_{34}}{\cancel{a_3} c_2}\right)$$

$$d_1 = P_z + a_4 r_{13} r_{22} + \frac{P_x r_{13}}{\underset{\substack{\downarrow \\ r_{33}}}{c_2}} - a_4 \underset{\substack{\downarrow \\ r_{22}}}{c_{34}} r_{13}$$

$$d_1 = P_z + \cancel{a_4 r_{13} r_{22}} + \frac{P_x r_{13}}{r_{33}} - \cancel{a_4 r_{22} r_{13}}$$

$$\boxed{d_1 = P_z + \frac{P_x r_{13}}{r_{33}}}$$