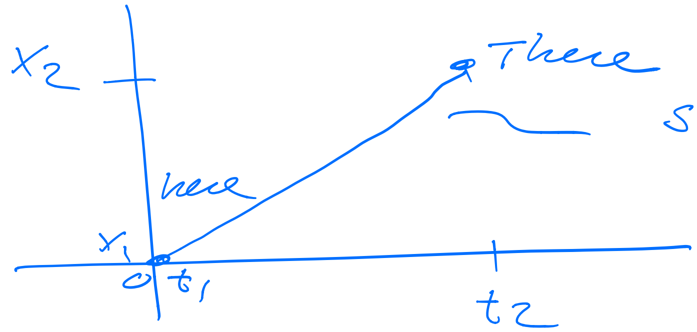
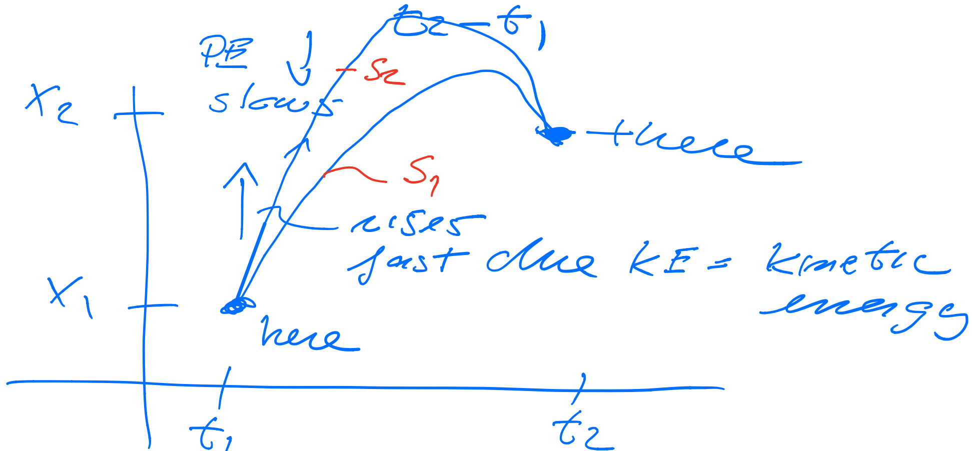
$$\text{Action} = S = \int_{t_1}^{t_2} \tfrac{1}{2} m \bar{v}^2 \, dt$$



smallest when we move with $\bar{v}$

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1}$$



$$S = \int_{t_1}^{t_2} (K \pm V)\, dt$$

$$S = \int_{t_1}^{t_2} \underbrace{(K - V)}_{\mathcal{L}(x,v,t)} dt$$

Properties of a minimum is that if Taylor expand around min $\mathcal{L}(x,v,t)$ has derivative equal to zero,

$$\mathcal{L}(x, v, t) \Rightarrow \mathcal{L}(x+\delta x, v+\delta v, t)$$

$$= \mathcal{L}(x,v,t) + \delta x \frac{\partial \mathcal{L}}{\partial x} + \delta v \frac{\partial \mathcal{L}}{\partial v}$$

+ higher-order terms

$$\delta x = \delta x(t) = \eta(t)$$

$$\delta v = \frac{d}{dt}\delta(x) = \frac{d\eta}{dt} = \eta'$$

$$\delta x(t_1) = \delta x(t_2) = 0$$

$$\eta'(t_1) = \eta'(t_2) = 0$$

$$\int_{t_1}^{t_2} \left[\frac{m}{2}\left(\dot{x}+\eta'\right)^2 - V(x+\eta) \right] dt$$

$$V(x+\eta) = V(x) + \eta V'(x) + \dots$$

$$= \int_{t_1}^{t_2} \left[\frac{m}{2}\dot{x}^2 - V(x) \right] dt$$

$$+ \int_{t_1}^{t_2} \left[m \cdot \dot{x}\eta' - \eta V'(x) \right] dt$$

(keep first-order terms in η' and η)

$$\delta S = \int_{t_1}^{t_2} \left[m\dot{x}\eta' - \eta V'(x) \right] dt$$

$$\frac{d}{dt}(\eta \cdot f) = \eta \frac{df}{dt} + f\eta' = \eta f' + f\eta'$$

$$\int f \eta' \, dt = \eta f - \int \eta f' \, dt$$

$$\delta S = \overbrace{m \eta \dot{x} \Big|_{t_1}^{t_2}}^{=0} - \int_{t_1}^{t_2} \frac{d}{dt}\left[m \dot{x}\right] \eta \, dt$$

$$- \int_{t_1}^{t_2} \underbrace{V'(x)}_{\frac{dV}{dx}} \eta(t) \, dt$$

$$0 = \delta S = \int_{t_1}^{t_2} \left[-m \ddot{x} - V'(x)\right] \eta(t) \, dt$$

$$-m\ddot{x} - V'(x) = 0$$

$$m \ddot{x} = -V'(x) = F(x)$$

$$-m\ddot{x} = V'(x) = -F(x)$$

$$\boxed{m \ddot{x} = F(x)}$$

Generalize coordinates

$$x(t) \rightarrow q(t)$$
$$v(t) \rightarrow \dot{q}(t)$$

$$\mathcal{L}(x, v, t) = \mathcal{L}(q, \dot{q}, t)$$

Euler-Lagrange eq:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$$

why this notation?

cartesian : $ds^2 = dx^2 + dy^2 + dz^2$

polar : $ds^2 = dr^2 + r^2 d\phi^2$

cylindrical : $ds^2 = \text{———} , \text{——}$
$+ dz^2$

spherical : $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$

$$v = \left(\frac{ds}{dt}\right)^2$$

kinetic energy ($ds \to s, x, y, z$
r, ϕ, θ)

cartesian : $\frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

polar : $\frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2)$ $(+\frac{\dot{z}^2}{2m})$

polar coordinates

$$\mathcal{L} = \mathcal{L}(r, \phi, \dot{r}, \dot{\phi}, t) = \mathcal{L}(\underline{q_1}, \underline{q_2}, \underline{\dot{q}_1}, \underline{\dot{q}_2}, \underline{t})$$

$$= \frac{m}{2}(\underline{\dot{r}}^2 + \underline{r}^2\underline{\dot{\phi}}^2) - \frac{\alpha}{\underline{r}}$$

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\phi}^2 + \alpha/r^2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\ddot{r}$$

$$m\ddot{r} = -\frac{\alpha}{r^2} - m r \dot{\phi}^2$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{d}{dt}\left(m r^2 \dot{\phi}\right)$$

$$= m r^2 \ddot{\phi} = 0$$

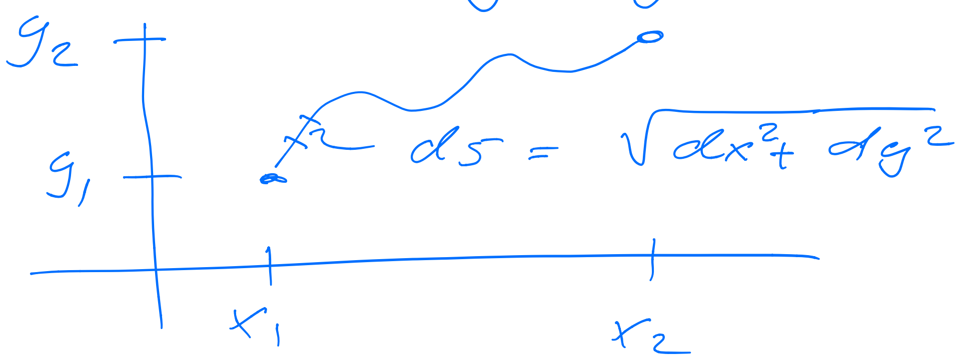
$$\frac{d\phi}{dt} = \dot{\phi} = \frac{L}{m r^2}$$

L is a constant of motion

Example

$\mathcal{L}$ and min of S to find the shortest distance between (x_1, y_1) (x_2, y_2)

want $y = y(x)$



$$ds = \sqrt{dx^2 + dy^2}$$

$$dy = \frac{dy}{dx}\, dx = y'(x)\, dx$$

$$ds = \sqrt{1+(y')^2}\, dx$$

$$S = L = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$$

$$= \int_{x_1}^{x_2} f(y(x), y'(x), x)\, dx$$

$$f = \sqrt{1+(y')^2}$$

Euler-Lagrange eq:

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \Rightarrow$$

$$\frac{\partial f}{\partial y'} = C \quad \text{(constant)}$$

$$y'^2 = C^2(1+y'^2) \quad \text{or}$$

$$y'(x) = D \Rightarrow y(x) = Dx + B$$

Straight line as shortest distance between two points.