Central forces + 2-body problem

$$V(r) = -\alpha/r \implies F(r) = -\alpha/r^2$$

Diff eq for r an ϕ $\quad r = 1/u$

$$\frac{d^2u}{d\phi^2} = -u - \frac{F\mu}{L^2u^2}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\dot{\phi} = \frac{L}{\mu r^2}$$

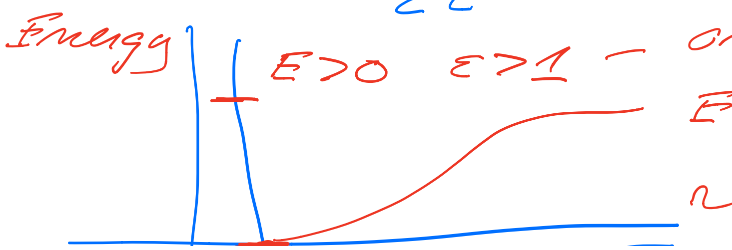
$$u(\phi) = \frac{\alpha\mu}{L^2} + A\cos(\phi)$$

$$= \frac{\alpha\mu}{L^2}\left(1 + \varepsilon\cos\phi\right)$$

$$c = \frac{L^2}{\alpha\mu}$$

$$r(\phi) = \frac{c}{1+\varepsilon\cos\phi}$$

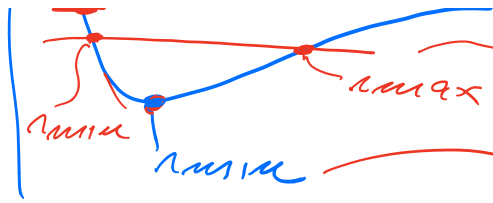
$$E = \frac{\alpha^2\mu}{2L^2}\left[\varepsilon^2 - 1\right]$$



$E>0$ $\varepsilon>1$ – orbit = hyperbola

$E=0$ $\varepsilon=1$

orbit = parabola



$0 < \varepsilon < 1$, $E < 0$
orbit = ellipse

$\varepsilon = 0$, $E < 0$
orbit = circle

bounded orbits

hw8 (Exercise 4 and 5)

$$r(\phi) = r = \frac{c}{1+\varepsilon\cos\phi}$$

$$x = r\cos\phi$$

$$\varepsilon = 1$$

$$r(1+\varepsilon\cos\phi) = c$$

$$r + \varepsilon x = c \quad \Rightarrow \quad r = c - \varepsilon x$$

$$\varepsilon = 1$$

$$r = c - x$$

$$r^2 = x^2 + y^2$$

$$x^2 + y^2 = (c-x)^2 = c^2 - 2xc + x^2$$

$$\underline{y^2 = c^2 - 2x \cdot c} \quad \text{parabola,}$$

$\underline{\varepsilon > 1}$

$$r = \frac{c}{1+\varepsilon\cos\phi}$$

$$r(1+\varepsilon\cos\phi) = c$$

$$r + \varepsilon x = c \quad \Rightarrow$$

$$r = c - \varepsilon x \qquad r^2 = x^2 + y^2$$

$$(\varepsilon^2-1)x^2 - 2c\varepsilon x - y^2 = -c^2$$

complete squares $\quad d = a\cdot\varepsilon$

$$a = \frac{c}{\varepsilon^2-1}$$

$$(\varepsilon^2-1)(x-d)^2 - y^2 = -c^2 + \frac{\varepsilon^2 c^2}{\varepsilon^2-1}$$

$$= \frac{c^2}{\varepsilon^2-1}$$

multiply both sides with $\frac{\varepsilon^2-1}{c^2}$

$$\boxed{\frac{(x-d)^2}{a^2} - \frac{y^2}{b^2} = 1}$$

eq for hyperbola.

$$b = \frac{c}{\sqrt{\varepsilon^2-1}}$$

Example 1

$$F_r = F = \alpha/r^2 \qquad \alpha > 0$$

repulsive potential

$$E = K + V = K + \alpha/r \qquad \gamma q_1 q_2$$

never negative

$$u'' = -u - \alpha\mu/L^2$$

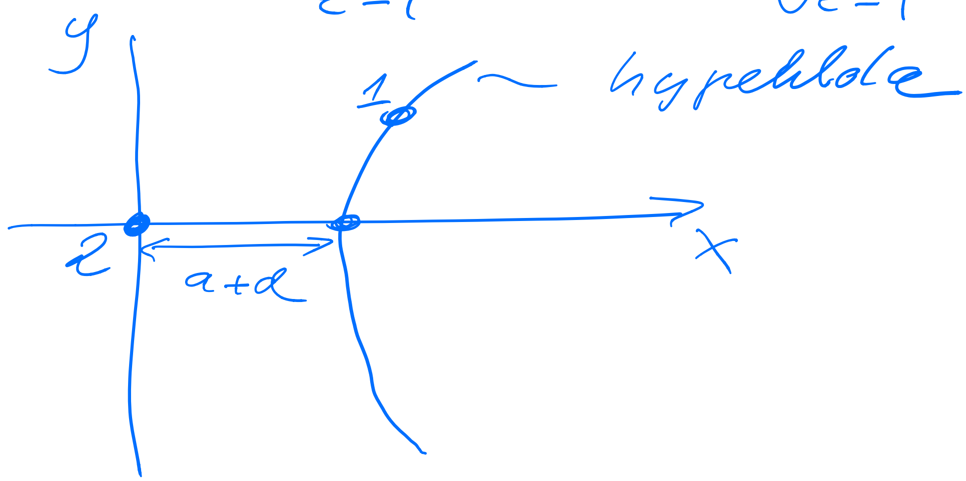
$$\left(\frac{1}{r} = u\right) \qquad r(\phi) = \frac{c}{\varepsilon\cos\phi - 1}$$

$$E = \frac{a^2 \mu}{2c^2}\left[\varepsilon^2 - 1\right]$$

$E \geq 0$ if $\varepsilon \geq 1$ $(E > 0)$

$$\frac{(x-d)^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$a = \frac{c}{\varepsilon^2 - 1} \qquad b = \frac{a}{\sqrt{\varepsilon^2 - 1}}$$



Example 2

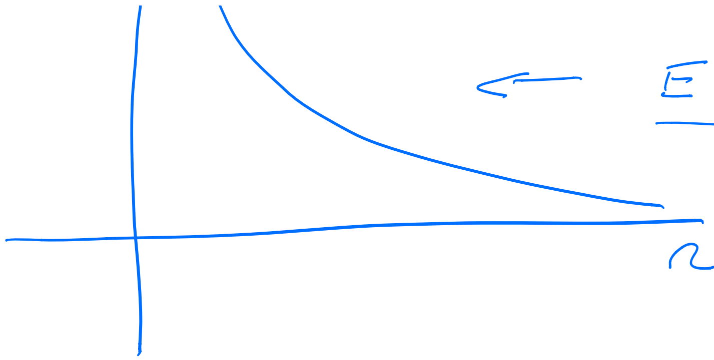
$$F = k/r^3 \qquad V = \frac{k}{2r^2}$$

$$F = -\frac{dV}{dr}$$

$$V_{eff} = V(r) + \frac{L^2}{2mr^2} = \frac{k + L^2/m}{2r^2}$$

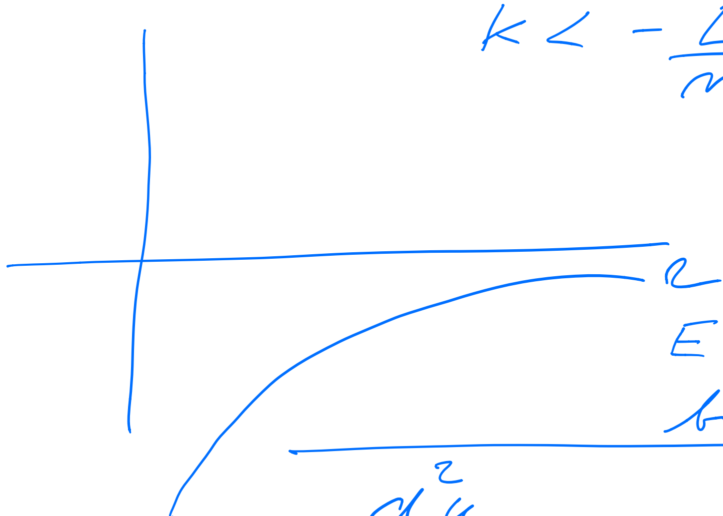
$k < 0$ or $k > 0$

i) $k > -L^2/m$



$\leftarrow E > 0$

$$k < -\frac{L^2}{m}$$



$E < 0$

bounded orbit

$r = \frac{1}{u}$

$$\boxed{\frac{d^2u}{d\phi^2} = -u - \frac{Fm}{L^2u^2}}$$

$F = k/r^3 = ku^3$

$$\frac{d^2u}{d\phi^2} = -\underbrace{\left(1+\frac{km}{L^2}\right)}_{\omega^2} u$$

$$= -\omega^2 u$$

$k > L^2/m$ then

$$1+\frac{km}{L^2} > 0$$

$$u(\phi) = \frac{1}{r(\phi)} = A\cdot\cos(\omega\cdot\phi - \delta)$$

$k < -L^2/m$

$$\frac{d^2u}{d\phi^2} = \lambda^2 u$$

$$u(\phi) = Ae^{\lambda \cdot \phi} + Be^{-\lambda\phi}$$

(Bounded type of orbit)

Example 3 (Ex 3)

$$V(r) = \beta \cdot r$$

$L \neq 0$ mass $= m$

$$\frac{dV_{eff}}{dr} = 0 = \frac{d}{dr}\left[\beta r + \frac{L^2}{2mr^2}\right]$$

$$\Rightarrow \quad \beta = \frac{L^2}{m r_{min}^3}$$

$$r_{min} = \left[\frac{L^2}{\beta m}\right]^{1/3}$$

$$\boxed{\dot\phi = \frac{L}{m r_{min}^2}} = \frac{\beta^{2/3}}{(mL)^{1/3}}$$

can find angular frequency of small perturbations around r_{min}

2

$$k_{eff} = \frac{d}{dr^2} V_{eff}(r) \Big|_{r_{min}}$$

$$= \frac{3L^2}{m r_{min}^4} \Rightarrow$$

$$\underline{\omega} = \sqrt{\frac{k_{eff}}{m}} = \frac{\beta^{2/3}}{(mL)^{1/3}} \sqrt{3}$$