Oscillations

Morten Hjorth-Jensen[1,2]

Scott Pratt[1]

Carl Schmidt[3]

[1]Department of Physics and Astronomy and National Superconducting Cyclotron Laboratory, Michigan State University, USA
[2]Department of Physics, University of Oslo, Norway
[3]Department of Physics and Astronomy, Michigan State University, USA

Feb 11, 2020

Harmonic Oscillator

The harmonic oscillator is omnipresent in physics. Although you may think of this as being related to springs, it, or an equivalent mathematical representation, appears in just about any problem where a mode is sitting near its potential energy minimum. At that point, $\partial_x V(x) = 0$, and the first non-zero term (aside from a constant) in the potential energy is that of a harmonic oscillator. In a solid, sound modes (phonons) are built on a picture of coupled harmonic oscillators, and in relativistic field theory the fundamental interactions are also built on coupled oscillators positioned infinitesimally close to one another in space. The phenomena of a resonance of an oscillator driven at a fixed frequency plays out repeatedly in atomic, nuclear and high-energy physics, when quantum mechanically the evolution of a state oscillates according to e^{-iEt} and exciting discrete quantum states has very similar mathematics as exciting discrete states of an oscillator.

The potential energy for a single particle as a function of its position x can be written as a Taylor expansion about some point x_0

$$V(x) = V(x_0) + (x-x_0)\left.\partial_x V(x)\right|_{x_0} + \frac{1}{2}(x-x_0)^2\left.\partial_x^2 V(x)\right|_{x_0} + \frac{1}{3!}\left.\partial_x^3 V(x)\right|_{x_0} + \cdots \tag{1}$$

If the position x_0 is at the minimum of the resonance, the first two non-zero terms of the potential are

$$
\begin{aligned}
V(x) &\approx V(x_0) + \frac{1}{2}(x-x_0)^2 \, \partial_x^2 V(x)\big|_{x_0}, \qquad (2)\\
&= V(x_0) + \frac{1}{2}k(x-x_0)^2, \quad k \equiv \partial_x^2 V(x)\big|_{x_0},\\
F &= -\partial_x V(x) = -k(x-x_0).
\end{aligned}
$$

Put into Newton's 2nd law (assuming $x_0 = 0$),

$$
\begin{aligned}
m\ddot{x} &= -kx, \qquad (3)\\
x &= A\cos(\omega_0 t - \phi), \quad \omega_0 = \sqrt{k/m}. \qquad (4)
\end{aligned}
$$

Here A and ϕ are arbitrary. Equivalently, one could have written this as $A\cos(\omega_0 t) + B\sin(\omega_0 t)$, or as the real part of $Ae^{i\omega_0 t}$. In this last case A could be an arbitrary complex constant. Thus, there are 2 arbitrary constants (either A and B or A and ϕ, or the real and imaginary part of one complex constant. This is the expectation for a second order differential equation, and also agrees with the physical expectation that if you know a particle's initial velocity and position you should be able to define its future motion, and that those two arbitrary conditions should translate to two arbitrary constants.

A key feature of harmonic motion is that the system repeats itself after a time $T = 1/f$, where f is the frequency, and $\omega = 2\pi f$ is the angular frequency. The period of the motion is independent of the amplitude. However, this independence is only exact when one can neglect higher terms of the potential, $x^3, x^4 \cdots$. Once can neglect these terms for sufficiently small amplitudes, and for larger amplitudes the motion is no longer purely sinusoidal, and even though the motion repeats itself, the time for repeating the motion is no longer independent of the amplitude.

One can also calculate the velocity and the kinetic energy as a function of time,

$$
\begin{aligned}
\dot{x} &= -\omega_0 A\sin(\omega_0 t - \phi), \qquad (5)\\
K &= \frac{1}{2}m\dot{x}^2 = \frac{m\omega_0^2 A^2}{2}\sin^2(\omega_0 t - \phi),\\
&= \frac{k}{2}A^2\sin^2(\omega_0 t - \phi).
\end{aligned}
$$

The total energy is then

$$
E = K + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2. \qquad (6)
$$

The total energy then goes as the square of the amplitude.

A pendulum is an example of a harmonic oscillator. By expanding the kinetic and potential energies for small angles find the frequency for a pendulum of length L with all the mass m centered at the end by writing the eq.s of motion in the form of a harmonic oscillator.

The potential energy and kinetic energies are (for x being the displacement)

$$\begin{aligned} V &= mgL(1-\cos\theta) \approx mgL\frac{x^2}{2L^2}, \\ K &= \frac{1}{2}mL^2\dot{\theta}^2 \approx \frac{m}{2}\dot{x}^2. \end{aligned}$$

For small x Newton's 2nd law becomes

$$m\ddot{x} = -\frac{mg}{L}x,$$

and the spring constant would appear to be $k = mg/L$, which makes the frequency equal to $\omega_0 = \sqrt{g/L}$. Note that the frequency is independent of the mass.

Damped Oscillators

We consider only the case where the damping force is proportional to the velocity. This is counter to dragging friction, where the force is proportional in strength to the normal force and independent of velocity, and is also inconsistent with wind resistance, where the magnitude of the drag force is proportional the square of the velocity. Rolling resistance does seem to be mainly proportional to the velocity. However, the main motivation for considering damping forces proportional to the velocity is that the math is more friendly. This is because the differential equation is linear, i.e. each term is of order x, $\dot{x}$, $\ddot{x}\cdots$, or even terms with no mention of x, and there are no terms such as x^2 or $x\ddot{x}$. The equations of motion for a spring with damping force $-b\dot{x}$ are

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{7}$$

Just to make the solution a bit less messy, we rewrite this equation as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \quad \beta \equiv b/2m,\ \omega_0 \equiv \sqrt{k/m}. \tag{8}$$

Both β and ω have dimensions of inverse time. To find solutions (see appendix C in the text) you must make an educated guess at the form of the solution. To do this, first realize that the solution will need an arbitrary normalization A because the equation is linear. Secondly, realize that if the form is

$$x = Ae^{rt} \tag{9}$$

that each derivative simply brings out an extra power of r. This means that the Ae^{rt} factors out and one can simply solve for an equation for r. Plugging this form into Eq. (8),

$$r^2 + 2\beta r + \omega_0^2 = 0. \tag{10}$$

Because this is a quadratic equation there will be two solutions,

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}. \tag{11}$$

We refer to the two solutions as r_1 and r_2 corresponding to the $+$ and $-$ roots. As expected, there should be two arbitrary constants involved in the solution,

$$x = A_1 e^{r_1 t} + A_2 e^{r_2 t}, \tag{12}$$

where the coefficients A_1 and A_2 are determined by initial conditions.

The roots listed above, $\sqrt{\omega_0^2 - \beta_0^2}$, will be imaginary if the damping is small and $\beta < \omega_0$. In that case, r is complex and the factor ert will have some oscillatory behavior. If the roots are real, there will only be exponentially decaying solutions. There are three cases:

Underdamped: $\beta < \omega_0$**.**

$$\begin{aligned} x &= A_1 e^{-\beta t} e^{i\omega' t} + A_2 e^{-\beta t} e^{-i\omega' t}, \quad \omega' \equiv \sqrt{\omega_0^2 - \beta^2} \\ &= (A_1 + A_2) e^{-\beta t} \cos\omega' t + i(A_1 - A_2) e^{-\beta t} \sin\omega' t. \end{aligned} \tag{13}$$

Here we have made use of the identity $e^{i\omega' t} = \cos\omega' t + i \sin\omega' t$. Because the constants are arbitrary, and because the real and imaginary parts are both solutions individually, we can simply consider the real part of the solution alone:

$$\begin{aligned} x &= B_1 e^{-\beta t} \cos\omega' t + B_2 e^{-\beta t} \sin\omega' t, \\ \omega' &\equiv \sqrt{\omega_0^2 - \beta^2}. \end{aligned} \tag{14}$$

Critical dampling: $\beta = \omega_0$**.** In this case the two terms involving r_1 and r_2 are identical because $\omega' = 0$. Because we need to arbitrary constants, there needs to be another solution. This is found by simply guessing, or by taking the limit of $\omega' \to 0$ from the underdamped solution. The solution is then

$$x = Ae^{-\beta t} + Bte^{-\beta t}. \tag{15}$$

The critically damped solution is interesting because the solution approaches zero quickly, but does not oscillate. For a problem with zero initial velocity, the solution never crosses zero. This is a good choice for designing shock absorbers or swinging doors.

Overdamped: $\beta > \omega_0$**.**

$$x = A_1 \exp -(\beta + \sqrt{\beta^2 - \omega_0^2})t + A_2 \exp -(\beta - \sqrt{\beta^2 - \omega_0^2})t \tag{16}$$

This solution will also never pass the origin more than once, and then only if the initial velocity is strong and initially toward zero.

Given b, m and ω_0, find $x(t)$ for a particle whose initial position is $x = 0$ and has initial velocity v_0 (assuming an underdamped solution).

The solution is of the form,

$$
\begin{aligned}
x &= e^{-\beta t}\left[A_1\cos(\omega' t) + A_2\sin\omega' t\right], \\
\dot{x} &= -\beta x + \omega' e^{-\beta t}\left[-A_1\sin\omega' t + A_2\cos\omega' t\right]. \\
\omega' &\equiv \sqrt{\omega_0^2 - \beta^2}, \quad \beta \equiv b/2m.
\end{aligned}
$$

From the initial conditions, $A_1 = 0$ because $x(0) = 0$ and $\omega' A_2 = v_0$. So

$$x = \frac{v_0}{\omega'}e^{-\beta t}\sin\omega' t.$$

Our Sliding Block Code

Here we study first the case without additional friction term and scale our equation in terms of a dimensionless time τ.

Let us remind ourselves about the differential equation we want to solve (the general case with damping due to friction)

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx(t) = 0.$$

We divide by m and introduce $\omega_0^2 = \sqrt{k/m}$ and obtain

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_0^2 x(t) = 0.$$

Thereafter we introduce a dimensionless time $\tau = t\omega_0$ (check that the dimensionality is correct) and rewrite our equation as

$$\frac{d^2x}{d\tau^2} + \frac{b}{m\omega_0}\frac{dx}{d\tau} + x(\tau) = 0,$$

which gives us

$$\frac{d^2x}{d\tau^2} + \frac{b}{m\omega_0}\frac{dx}{d\tau} + x(\tau) = 0.$$

We then define $\gamma = b/(2m\omega_0)$ and rewrite our equations as

$$\frac{d^2x}{d\tau^2} + 2\gamma\frac{dx}{d\tau} + x(\tau) = 0.$$

This is the equation we will code below. The first version employs the Euler-Cromer method.

When setting up the value of γ we see that for $\gamma = 0$ we get the simple oscillatory motion with no damping. Choosing $\gamma < 1$ leads to the classical underdamped case with oscillatory motion, but where the motion comes to an end.

Choosing $\gamma = 1$ leads to what normally is called critical damping and $\gamma > 1$ leads to critical overdamping. Try it out and try also to change the initial position and velocity. Setting $\gamma = 1$ yields a situation, as discussed above, where the solution approaches quickly zero and does not oscillate. With zero initial velocity it will never cross zero.

Sinusoidally Driven Oscillators

Here, we consider the force

$$F = -kx - b\dot{x} + F_0 \cos\omega t, \tag{17}$$

which leads to the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = (F_0/m)\cos\omega t. \tag{18}$$

Consider a single solution with no arbitrary constants, which we will call a *particular solution*, $x_p(t)$. It should be emphasized that this is **A** particular solution, because there exists an infinite number of such solutions because the general solution should have two arbitrary constants. Now consider solutions to the same equation without the driving term, which include two arbitrary constants. These are called either *homogenous solutions* or *complementary solutions*, and were given in the previous section, e.g. Eq. (14) for the underdamped case. The homogenous solution already incorporates the two arbitrary constants, so any sum of a homogenous solution and a particular solution will represent the *general solution* of the equation. The general solution incorporates the two arbitrary constants A and B to accommodate the two initial conditions. One could have picked a different particular solution, i.e. the original particular solution plus any homogenous solution with the arbitrary constants A_p and B_p chosen at will. When one adds in the homogenous solution, which has adjustable constants with arbitrary constants A' and B', to the new particular solution, one can get the same general solution by simply adjusting the new constants such that $A' + A_p = A$ and $B' + B_p = B$. Thus, the choice of A_p and B_p are irrelevant, and when choosing the particular solution it is best to make the simplest choice possible.

To find a particular solution, one first guesses at the form,

$$x_p(t) = D\cos(\omega t - \delta), \tag{19}$$

and rewrite the differential equation as

$$D\left\{-\omega^2\cos(\omega t - \delta) - 2\beta\omega\sin(\omega t - \delta) + \omega_0^2\cos(\omega t - \delta)\right\} = \frac{F_0}{m}\cos(\omega t). \tag{20}$$

One can now use angle addition formulas to get

$$
\begin{aligned}
D\left\{(-\omega^2\cos\delta+2\beta\omega\sin\delta+\omega_0^2\cos\delta)\cos(\omega t)\right. && (21)\\
\left.+(-\omega^2\sin\delta-2\beta\omega\cos\delta+\omega_0^2\sin\delta)\sin(\omega t)\right\} &= \frac{F_0}{m}\cos(\omega t).
\end{aligned}
$$

Both the cos and sin terms need to equate if the expression is to hold at all times. Thus, this becomes two equations

$$
\begin{aligned}
D\left\{-\omega^2\cos\delta+2\beta\omega\sin\delta+\omega_0^2\cos\delta\right\} &= \frac{F_0}{m} \qquad (22)\\
-\omega^2\sin\delta-2\beta\omega\cos\delta+\omega_0^2\sin\delta &= 0.
\end{aligned}
$$

After dividing by $\cos\delta$, the lower expression leads to

$$
\tan\delta = \frac{2\beta\omega}{\omega_0^2-\omega^2}. \qquad (23)
$$

Using the identities $\tan^2+1=\csc^2$ and $\sin^2+\cos^2=1$, one can also express $\sin\delta$ and $\cos\delta$,

$$
\begin{aligned}
\sin\delta &= \frac{2\beta\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}, \qquad (24)\\
\cos\delta &= \frac{(\omega_0^2-\omega^2)}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}
\end{aligned}
$$

Inserting the expressions for $\cos\delta$ and $\sin\delta$ into the expression for D,

$$
D = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}. \qquad (25)
$$

For a given initial condition, e.g. initial displacement and velocity, one must add the homogenous solution then solve for the two arbitrary constants. However, because the homogenous solutions decay with time as $e^{-\beta t}$, the particular solution is all that remains at large times, and is therefore the steady state solution. Because the arbitrary constants are all in the homogenous solution, all memory of the initial conditions are lost at large times, $t >> 1/\beta$.

The amplitude of the motion, D, is linearly proportional to the driving force (F_0/m), but also depends on the driving frequency ω. For small β the maximum will occur at $\omega=\omega_0$. This is referred to as a resonance. In the limit $\beta\to 0$ the amplitude at resonance approaches infinity.

Alternative Derivation for Driven Oscillators

Here, we derive the same expressions as in Equations (19) and (25) but express the driving forces as

$$F(t) = F_0 e^{i\omega t}, \tag{26}$$

rather than as $F_0 \cos\omega t$. The real part of F is the same as before. For the differential equation,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}, \tag{27}$$

one can treat $x(t)$ as an imaginary function. Because the operations d^2/dt^2 and d/dt are real and thus do not mix the real and imaginary parts of $x(t)$, Eq. (27) is effectively 2 equations. Because $e^{\omega t} = \cos\omega t + i\sin\omega t$, the real part of the solution for $x(t)$ gives the solution for a driving force $F_0 \cos\omega t$, and the imaginary part of x corresponds to the case where the driving force is $F_0 \sin\omega t$. It is rather easy to solve for the complex x in this case, and by taking the real part of the solution, one finds the answer for the $\cos\omega t$ driving force.

We assume a simple form for the particular solution

$$x_p = De^{i\omega t}, \tag{28}$$

where D is a complex constant.

From Eq. (27) one inserts the form for x_p above to get

$$\begin{aligned} D\left\{-\omega^2 + 2i\beta\omega + \omega_0^2\right\} e^{i\omega t} &= (F_0/m)e^{i\omega t}, \\ D &= \frac{F_0/m}{(\omega_0^2 - \omega^2) + 2i\beta\omega}. \end{aligned} \tag{29}$$

The norm and phase for $D = |D|e^{-i\delta}$ can be read by inspection,

$$|D| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}, \qquad \tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}. \tag{30}$$

This is the same expression for δ as before. One then finds $x_p(t)$,

$$\begin{aligned} x_p(t) &= \Re \frac{(F_0/m)e^{i\omega t - i\delta}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \\ &= \frac{(F_0/m)\cos(\omega t - \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}. \end{aligned} \tag{31}$$

This is the same answer as before. If one wished to solve for the case where $F(t) = F_0 \sin\omega t$, the imaginary part of the solution would work

$$\begin{aligned} x_p(t) &= \Im \frac{(F_0/m)e^{i\omega t - i\delta}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \\ &= \frac{(F_0/m)\sin(\omega t - \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}. \end{aligned} \tag{32}$$

Consider the damped and driven harmonic oscillator worked out above. Given F_0, m, β and ω_0, solve for the complete solution $x(t)$ for the case where $F = F_0 \sin\omega t$ with initial conditions $x(t=0) = 0$ and $v(t=0) = 0$. Assume the underdamped case.

The general solution including the arbitrary constants includes both the homogenous and particular solutions,

$$x(t) \quad = \quad \frac{F_0}{m}\frac{\sin(\omega t - \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} + A\cos\omega' t e^{-\beta t} + B\sin\omega' t e^{-\beta t}.$$

The quantities δ and ω' are given earlier in the section, $\omega' = \sqrt{\omega_0^2 - \beta^2}, \delta = \tan^{-1}(2\beta\omega/(\omega_0^2 - \omega^2)$. Here, solving the problem means finding the arbitrary constants A and B. Satisfying the initial conditions for the initial position and velocity:

$$\begin{aligned} x(t=0) = 0 &= -\eta\sin\delta + A, \\ v(t=0) = 0 &= \omega\eta\cos\delta - \beta A + \omega' B, \\ \eta &\equiv \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}. \end{aligned}$$

The problem is now reduced to 2 equations and 2 unknowns, A and B. The solution is

$$A \quad = \quad \eta\sin\delta, \quad B = \frac{-\omega\eta\cos\delta + \beta\eta\sin\delta}{\omega'}. \tag{33}$$

Resonance Widths; the Q factor

From the previous two sections, the particular solution for a driving force, $F = F_0\cos\omega t$, is

$$\begin{aligned} x_p(t) &= \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}\cos(\omega_t - \delta), \\ \delta &= \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right). \end{aligned} \tag{34}$$

If one fixes the driving frequency ω and adjusts the fundamental frequency $\omega_0 = \sqrt{k/m}$, the maximum amplitude occurs when $\omega_0 = \omega$ because that is when the term from the denominator $(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2$ is at a minimum. This is akin to dialing into a radio station. However, if one fixes ω_0 and adjusts the driving frequency one minimize with respect to ω, e.g. set

$$\frac{d}{d\omega}\left[(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2\right] = 0, \tag{35}$$

and one finds that the maximum amplitude occurs when $\omega = \sqrt{\omega_0^2 - 2\beta^2}$. If β is small relative to ω_0, one can simply state that the maximum amplitude is

$$x_{\mathrm{max}} \approx \frac{F_0}{2m\beta\omega_0}. \tag{36}$$

$$\frac{4\omega^2\beta^2}{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2} = \frac{1}{2}. \tag{37}$$

For small damping this occurs when $\omega = \omega_0 \pm \beta$, so the $FWHM \approx 2\beta$. For the purposes of tuning to a specific frequency, one wants the width to be as small as possible. The ratio of ω_0 to $FWHM$ is known as the *quality* factor, or Q factor,

$$Q \equiv \frac{\omega_0}{2\beta}. \tag{38}$$

Numerical Studies of Driven Oscillations

Solving the problem of driven oscillations numerically gives us much more flexibility to study different types of driving forces. We can reuse our earlier code by simply adding a driving force. If we stay in the x-direction only this can be easily done by adding a term $F_{\mathrm{ext}}(x,t)$. Note that we have kept it rather general here, allowing for both a spatial and a temporal dependence.

Before we dive into the code, we need to briefly remind ourselves about the equations we started with for the case with damping, namely

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx(t) = 0,$$

with no external force applied to the system.

Let us now for simplicty assume that our external force is given by

$$F_{\mathrm{ext}}(t) = F_0\cos(\omega t),$$

where F_0 is a constant (what is its dimension?) and ω is the frequency of the applied external driving force. **Small question:** would you expect energy to be conserved now?

Introducing the external force into our lovely differential equation and dividing by m and introducing $\omega_0^2 = \sqrt{k/m}$ we have

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_0^2 x(t) = \frac{F_0}{m}\cos(\omega t),$$

Thereafter we introduce a dimensionless time $\tau = t\omega_0$ and a dimensionless frequency $\tilde{\omega} = \omega/\omega_0$. We have then

$$\frac{d^2x}{d\tau^2} + \frac{b}{m\omega_0}\frac{dx}{d\tau} + x(\tau) = \frac{F_0}{m\omega_0^2}\cos(\tilde{\omega}\tau),$$

Introducing a new amplitude $\tilde{F} = F_0/(m\omega_0^2)$ (check dimensionality again) we have

$$\frac{d^2x}{d\tau^2} + \frac{b}{m\omega_0}\frac{dx}{d\tau} + x(\tau) = \tilde{F}\cos(\tilde{\omega}\tau).$$

Our final step, as we did in the case of various types of damping, is to define $\gamma = b/(2m\omega_0)$ and rewrite our equations as

$$\frac{d^2x}{d\tau^2} + 2\gamma\frac{dx}{d\tau} + x(\tau) = \tilde{F}\cos(\tilde{\omega}\tau).$$

This is the equation we will code below using the Euler-Cromer method.

In the above example we have focused on the Euler-Cromer method. This method has a local truncation error which is proportional to Δt^2 and thereby a global error which is proportional to Δt. We can improve this by using the Runge-Kutta family of methods. The widely popular Runge-Kutta to fourth order or just **RK4** has indeed a much better truncation error. The RK4 method has a global error which is proportional to Δt.

Let us revisit this method and see how we can implement it for the above example.

The Runge-Kutta Family of Methods

Differential Equations, Runge-Kutta methods

Runge-Kutta (RK) methods are based on Taylor expansion formulae, but yield in general better algorithms for solutions of an ordinary differential equation. The basic philosophy is that it provides an intermediate step in the computation of y_{i+1}.

To see this, consider first the following definitions

$$\frac{dy}{dt} = f(t, y), \tag{39}$$

and

$$y(t) = \int f(t, y)dt, \tag{40}$$

and

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t, y)dt. \tag{41}$$

To demonstrate the philosophy behind RK methods, let us consider the second-order RK method, RK2. The first approximation consists in Taylor expanding $f(t, y)$ around the center of the integration interval t_i to t_{i+1}, that is, at $t_i + h/2$, h being the step. Using the midpoint formula for an integral, defining $y(t_i + h/2) = y_{i+1/2}$ and $t_i + h/2 = t_{i+1/2}$, we obtain

$$\int_{t_i}^{t_{i+1}} f(t, y)dt \approx hf(t_{i+1/2}, y_{i+1/2}) + O(h^3). \tag{42}$$

This means in turn that we have

$$y_{i+1} = y_i + hf(t_{i+1/2}, y_{i+1/2}) + O(h^3). \tag{43}$$

However, we do not know the value of $y_{i+1/2}$. Here comes thus the next approximation, namely, we use Euler's method to approximate $y_{i+1/2}$. We have then

$$y_{(i+1/2)} = y_i + \frac{h}{2}\frac{dy}{dt} = y(t_i) + \frac{h}{2}f(t_i, y_i). \tag{44}$$

This means that we can define the following algorithm for the second-order Runge-Kutta method, RK2.

$$k_1 = hf(t_i, y_i), \tag{45}$$

$$k_2 = hf(t_{i+1/2}, y_i + k_1/2), \tag{46}$$

with the final value

$$y_{i+i} \approx y_i + k_2 + O(h^3). \tag{47}$$

The difference between the previous one-step methods is that we now need an intermediate step in our evaluation, namely $t_i + h/2 = t_{(i+1/2)}$ where we evaluate the derivative f. This involves more operations, but the gain is a better stability in the solution.

The fourth-order Runge-Kutta, RK4, has the following algorithm

$$k_1 = hf(t_i, y_i) \qquad k_2 = hf(t_i + h/2, y_i + k_1/2)$$

$$k_3 = hf(t_i + h/2, y_i + k_2/2) \qquad k_4 = hf(t_i + h, y_i + k_3)$$

with the final result

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right).$$

Thus, the algorithm consists in first calculating k_1 with t_i, y_1 and f as inputs. Thereafter, we increase the step size by $h/2$ and calculate k_2, then k_3 and finally k_4. The global error goes as $O(h^4)$.

Principle of Superposition and Periodic Forces (Fourier Transforms)

If one has several driving forces, $F(t) = \sum_n F_n(t)$, one can find the particular solution to each F_n, $x_{pn}(t)$, and the particular solution for the entire driving force is

$$x_p(t) = \sum_n x_{pn}(t). \tag{48}$$

This is known as the principal of superposition. It only applies when the homogenous equation is linear. If there were an anharmonic term such as x^3 in the homogenous equation, then when one summed various solutions, $x = (\sum_n x_n)^2$, one would get cross terms. Superposition is especially useful when $F(t)$ can be written as a sum of sinusoidal terms, because the solutions for each sinusoidal term is analytic, and are given in the previous two subsections.

Driving forces are often periodic, even when they are not sinusoidal. Periodicity implies that for some time τ

$$F(t+\tau) = F(t). \tag{49}$$

One example of a non-sinusoidal periodic force is a square wave. Many components in electric circuits are non-linear, e.g. diodes, which makes many wave forms non-sinusoidal even when the circuits are being driven by purely sinusoidal sources.

For the sinusoidal example studied in the previous subsections the period is $\tau = 2\pi/\omega$. However, higher harmonics can also satisfy the periodicity requirement. In general, any force that satisfies the periodicity requirement can be expressed as a sum over harmonics,

$$F(t) = \frac{f_0}{2} + \sum_{n>0} f_n \cos(2n\pi t/\tau) + g_n \sin(2n\pi t/\tau). \tag{50}$$

From the previous subsection, one can write down the answer for $x_{pn}(t)$, by substituting f_n/m or g_n/m for F_0/m into Eq.s (31) or (32) respectively. By writing each factor $2n\pi t/\tau$ as $n\omega t$, with $\omega \equiv 2\pi/\tau$,

$$F(t) = \frac{f_0}{2} + \sum_{n>0} f_n \cos(n\omega t) + g_n \sin(n\omega t). \tag{51}$$

The solutions for $x(t)$ then come from replacing ω with $n\omega$ for each term in the particular solution in Equations (19) and (25),

$$\begin{aligned}
x_p(t) &= \frac{f_0}{2k} + \sum_{n>0} \alpha_n \cos(n\omega t - \delta_n) + \beta_n \sin(n\omega t - \delta_n), \\
\alpha_n &= \frac{f_n/m}{\sqrt{((n\omega)^2 - \omega_0^2) + 4\beta^2 n^2 \omega^2}}, \\
\beta_n &= \frac{g_n/m}{\sqrt{((n\omega)^2 - \omega_0^2) + 4\beta^2 n^2 \omega^2}}, \\
\delta_n &= \tan^{-1}\left(\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}\right).
\end{aligned} \tag{52}$$

Because the forces have been applied for a long time, any non-zero damping eliminates the homogenous parts of the solution, so one need only consider the particular solution for each n.

The problem will considered solved if one can find expressions for the coefficients f_n and g_n, even though the solutions are expressed as an infinite sum. The coefficients can be extracted from the function $F(t)$ by

$$\begin{aligned} f_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt\; F(t)\cos(2n\pi t/\tau), \\ g_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt\; F(t)\sin(2n\pi t/\tau). \end{aligned} \tag{53}$$

To check the consistency of these expressions and to verify Eq. (53), one can insert the expansion of $F(t)$ in Eq. (51) into the expression for the coefficients in Eq. (53) and see whether

$$f_n \;\;=?\;\; \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt\;\left\{\frac{f_0}{2} + \sum_{m>0} f_m\cos(m\omega t) + g_m\sin(m\omega t)\right\}\cos(n\omega t). \tag{54}$$

Immediately, one can throw away all the terms with g_m because they convolute an even and an odd function. The term with $f_0/2$ disappears because $\cos(n\omega t)$ is equally positive and negative over the interval and will integrate to zero. For all the terms $f_m\cos(m\omega t)$ appearing in the sum, one can use angle addition formulas to see that $\cos(m\omega t)\cos(n\omega t) = (1/2)(\cos[(m+n)\omega t] + \cos[(m-n)\omega t]$. This will integrate to zero unless $m = n$. In that case the $m = n$ term gives

$$\int_{-\tau/2}^{\tau/2} dt\;\;\cos^2(m\omega t) = \frac{\tau}{2}, \tag{55}$$

and

$$\begin{aligned} f_n \;\;&=?\;\; \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt\; f_n/2 \\ &= \;\; f_n \checkmark. \end{aligned} \tag{56}$$

The same method can be used to check for the consistency of g_n.

Consider the driving force:

$$F(t) = At/\tau, \quad -\tau/2 < t < \tau/2, \quad F(t+\tau) = F(t). \tag{57}$$

Find the Fourier coefficients f_n and g_n for all n using Eq. (53).

Only the odd coefficients enter by symmetry, i.e. $f_n = 0$. One can find g_n integrating by parts,

$$
\begin{aligned}
g_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt\ \sin(n\omega t)\frac{At}{\tau} \qquad (58)\\
u &= t,\ dv = \sin(n\omega t)dt,\ v = -\cos(n\omega t)/(n\omega),\\
g_n &= \frac{-2A}{n\omega\tau^2}\int_{-\tau/2}^{\tau/2} dt\ \cos(n\omega t) + 2A\frac{-t\cos(n\omega t)}{n\omega\tau^2}\Bigg|_{-\tau/2}^{\tau/2}.
\end{aligned}
$$

The first term is zero because $\cos(n\omega t)$ will be equally positive and negative over the interval. Using the fact that $\omega\tau = 2\pi$,

$$
\begin{aligned}
g_n &= -\frac{2A}{2n\pi}\cos(n\omega\tau/2) \qquad (59)\\
&= -\frac{A}{n\pi}\cos(n\pi)\\
&= \frac{A}{n\pi}(-1)^{n+1}.
\end{aligned}
$$

Response to Transient Force

Consider a particle at rest in the bottom of an underdamped harmonic oscillator, that then feels a sudden impulse, or change in momentum, $I = F\Delta t$ at $t = 0$. This increases the velocity immediately by an amount $v_0 = I/m$ while not changing the position. One can then solve the trajectory by solving Eq. (14) with initial conditions $v_0 = I/m$ and $x_0 = 0$. This gives

$$
x(t) = \frac{I}{m\omega'}e^{-\beta t}\sin\omega' t, \quad t > 0. \qquad (60)
$$

Here, $\omega' = \sqrt{\omega_0^2 - \beta^2}$. For an impulse I_i that occurs at time t_i the trajectory would be

$$
x(t) = \frac{I_i}{m\omega'}e^{-\beta(t-t_i)}\sin[\omega'(t-t_i)]\Theta(t-t_i), \qquad (61)
$$

where $\Theta(t - t_i)$ is a step function, i.e. $\Theta(x)$ is zero for $x < 0$ and unity for $x > 0$. If there were several impulses linear superposition tells us that we can sum over each contribution,

$$
x(t) = \sum_i \frac{I_i}{m\omega'}e^{-\beta(t-t_i)}\sin[\omega'(t-t_i)]\Theta(t-t_i) \qquad (62)
$$

Now one can consider a series of impulses at times separated by Δt, where each impulse is given by $F_i\Delta t$. The sum above now becomes an integral,

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} dt'\ F(t') \frac{e^{-\beta(t-t')}\sin[\omega'(t-t')]}{m\omega'}\Theta(t-t') \qquad (63)\\
&= \int_{-\infty}^{\infty} dt'\ F(t')G(t-t'),\\
G(\Delta t) &= \frac{e^{-\beta\Delta t}\sin[\omega'\Delta t]}{m\omega'}\Theta(\Delta t)
\end{aligned}
$$

The quantity $e^{-\beta(t-t')}\sin[\omega'(t-t')]/m\omega'\Theta(t-t')$ is called a Green's function, $G(t-t')$. It describes the response at t due to a force applied at a time t', and is a function of $t-t'$. The step function ensures that the response does not occur before the force is applied. One should remember that the form for G would change if the oscillator were either critically- or over-damped.

When performing the integral in Eq. (63) one can use angle addition formulas to factor out the part with the t' dependence in the integrand,

$$
\begin{aligned}
x(t) &= \frac{1}{m\omega'} e^{-\beta t}\left[I_c(t)\sin(\omega' t) - I_s(t)\cos(\omega' t)\right], \qquad (64)\\
I_c(t) &\equiv \int_{-\infty}^{t} dt'\ F(t')e^{\beta t'}\cos(\omega' t'),\\
I_s(t) &\equiv \int_{-\infty}^{t} dt'\ F(t')e^{\beta t'}\sin(\omega' t').
\end{aligned}
$$

If the time t is beyond any time at which the force acts, $F(t' > t) = 0$, the coefficients I_c and I_s become independent of t.

Consider an undamped oscillator ($\beta \to 0$), with characteristic frequency ω_0 and mass m, that is at rest until it feels a force described by a Gaussian form,

$$
F(t) = F_0 \exp\left\{\frac{-t^2}{2\tau^2}\right\}.
$$

For large times ($t >> \tau$), where the force has died off, find $x(t)$.
Solve for the coefficients I_c and I_s in Eq. (64). Because the Gaussian is an even function, $I_s = 0$, and one need only solve for I_c,

$$
\begin{aligned}
I_c &= F_0 \int_{-\infty}^{\infty} dt'\ e^{-t'^2/(2\tau^2)}\cos(\omega_0 t')\\
&= \Re F_0 \int_{-\infty}^{\infty} dt'\ e^{-t'^2/(2\tau^2)} e^{i\omega_0 t'}\\
&= \Re F_0 \int_{-\infty}^{\infty} dt'\ e^{-(t'-i\omega_0\tau^2)^2/(2\tau^2)} e^{-\omega_0^2\tau^2/2}\\
&= F_0 \tau\sqrt{2\pi} e^{-\omega_0^2\tau^2/2}.
\end{aligned}
$$

The third step involved completing the square, and the final step used the fact that the integral

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/2} \quad = \quad \sqrt{2\pi}.$$

To see that this integral is true, consider the square of the integral, which you can change to polar coordinates,

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} dx\ e^{-x^2/2} \\
I^2 &= \int_{-\infty}^{\infty} dxdy\ e^{-(x^2+y^2)/2} \\
&= 2\pi \int_{0}^{\infty} rdr\ e^{-r^2/2} \\
&= 2\pi.
\end{aligned}$$

Finally, the expression for x from Eq. (64) is

$$x(t >> \tau) \quad = \quad \frac{F_0\tau}{m\omega_0}\sqrt{2\pi}e^{-\omega_0^2\tau^2/2}\sin(\omega_0 t).$$

The classical pendulum and scaling the equations

Let us end our discussion of oscillations with another classical case, the pendulum.

The angular equation of motion of the pendulum is given by Newton's equation and with no external force it reads

$$ml\frac{d^2\theta}{dt^2} + mgsin(\theta) = 0, \tag{65}$$

with an angular velocity and acceleration given by

$$v = l\frac{d\theta}{dt}, \tag{66}$$

and

$$a = l\frac{d^2\theta}{dt^2}. \tag{67}$$

We do however expect that the motion will gradually come to an end due a viscous drag torque acting on the pendulum. In the presence of the drag, the above equation becomes

$$ml\frac{d^2\theta}{dt^2} + \nu\frac{d\theta}{dt} + mgsin(\theta) = 0, \tag{68}$$

where ν is now a positive constant parameterizing the viscosity of the medium in question. In order to maintain the motion against viscosity, it is necessary to add some external driving force. We choose here a periodic driving force. The last equation becomes then

$$ml\frac{d^2\theta}{dt^2} + \nu\frac{d\theta}{dt} + mgsin(\theta) = Asin(\omega t), \tag{69}$$

with A and ω two constants representing the amplitude and the angular frequency respectively. The latter is called the driving frequency.

We define

$$\omega_0 = \sqrt{g/l},$$

the so-called natural frequency and the new dimensionless quantities

$$\hat{t} = \omega_0 t,$$

with the dimensionless driving frequency

$$\hat{\omega} = \frac{\omega}{\omega_0},$$

and introducing the quantity Q, called the *quality factor*,

$$Q = \frac{mg}{\omega_0 \nu},$$

and the dimensionless amplitude

$$\hat{A} = \frac{A}{mg}$$

More on the Pendulum

We have

$$\frac{d^2\theta}{d\hat{t}^2} + \frac{1}{Q}\frac{d\theta}{d\hat{t}} + sin(\theta) = \hat{A}cos(\hat{\omega}\hat{t}).$$

This equation can in turn be recast in terms of two coupled first-order differential equations as follows

$$\frac{d\theta}{d\hat{t}} = \hat{v},$$

and

$$\frac{d\hat{v}}{d\hat{t}} = -\frac{\hat{v}}{Q} - sin(\theta) + \hat{A}cos(\hat{\omega}\hat{t}).$$

These are the equations to be solved. The factor Q represents the number of oscillations of the undriven system that must occur before its energy is significantly reduced due to the viscous drag. The amplitude $\hat{A}$ is measured in units of the maximum possible gravitational torque while $\hat{\omega}$ is the angular frequency of the external torque measured in units of the pendulum's natural frequency.