PHY321: Forces, Newton's Laws and Motion Example

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Basic Steps of Scientific Investigations

An overarching aim in this course is to give you a deeper understanding of the scientific method. The problems we study will all involve cases where we can apply classical mechanics. In our previous material we already assumed that we had a model for the motion of an object. Alternatively we could have data from experiment (like Usain Bolt's 100m world record run in 2008). Or we could have performed ourselves an experiment and we want to understand which forces are at play and whether these forces can be understood in terms of fundamental forces.

Our first step consists in identifying the problem. What we sketch here may include a mix of experiment and theoretical simulations, or just experiment or only theory.

Identifying our System

Here we can ask questions like

1. What kind of object is moving

2. What kind of data do we have

3. How do we measure position, velocity, acceleration etc

4. Which initial conditions influence our system

5. Other aspects which allow us to identify the system

Defining a Model

With our eventual data and observations we would now like to develop a model for the system. In the end we want obviously to be able to understand which forces are at play and how they influence our specific system. That is, can we extract some deeper insights about a system?

We need then to

1. Find the forces that act on our system

2. Introduce models for the forces

3. Identify the equations which can govern the system (Newton's second law for example)

4. More elements we deem important for defining our model

Solving the Equations

With the model at hand, we can then solve the equations. In classical mechanics we normally end up with solving sets of coupled ordinary differential equations or partial differential equations.

1. Using Newton's second law we have equations of the type $\boldsymbol{F} = m\boldsymbol{a} = md\boldsymbol{v}/dt$

2. We need to define the initial conditions (typically the initial velocity and position as functions of time) and/or initial conditions and boundary conditions

3. The solution of the equations give us then the position, the velocity and other time-dependent quantities which may specify the motion of a given object.

We are not yet done. With our lovely solvers, we need to start thinking.

Analyze

Now it is time to ask the big questions. What do our results mean? Can we give a simple interpretation in terms of fundamental laws? What do our results mean? Are they correct? Thus, typical questions we may ask are

1. Are our results for say $\boldsymbol{r}(t)$ valid? Do we trust what we did? Can you validate and verify the correctness of your results?

2. Evaluate the answers and their implications

3. Compare with experimental data if possible. Does our model make sense?

4. and obviously many other questions.

The analysis stage feeds back to the first stage. It may happen that the data we had were not good enough, there could be large statistical uncertainties. We may need to collect more data or perhaps we did a sloppy job in identifying the degrees of freedom.

All these steps are essential elements in a scientific enquiry. Hopefully, through a mix of numerical simulations, analytical calculations and experiments we may gain a deeper insight about the physics of a specific system.

Let us now remind ourselves of Newton's laws, since these are the laws of motion we will study in this course.

Newton's Laws

When analyzing a physical system we normally start with distinguishing between the object we are studying (we will label this in more general terms as our **system**) and how this system interacts with the environment (which often means everything else!)

In our investigations we will thus analyze a specific physics problem in terms of the system and the environment. In doing so we need to identify the forces that act on the system and assume that the forces acting on the system must have a source, an identifiable cause in the environment.

A force acting on for example a falling object must be related to an interaction with something in the environment. This also means that we do not consider internal forces. The latter are forces between one part of the object and another part. In this course we will mainly focus on external forces.

Forces are either contact forces or long-range forces.

Contact forces, as evident from the name, are forces that occur at the contact between the system and the environment. Well-known long-range forces are the gravitional force and the electromagnetic force.

Setting up a model for forces acting on an object

In order to set up the forces which act on an object, the following steps may be useful

1. Divide the problem into system and environment.

2. Draw a figure of the object and everything in contact with the object.

3. Draw a closed curve around the system.

4. Find contact points—these are the points where contact forces may act.

5. Give names and symbols to all the contact forces.

6. Identify the long-range forces.

7. Make a drawing of the object. Draw the forces as arrows, vectors, starting from where the force is acting. The direction of the vector(s) indicates

the (positive) direction of the force. Try to make the length of the arrow indicate the relative magnitude of the forces.

8. Draw in the axes of the coordinate system. It is often convenient to make one axis parallel to the direction of motion. When you choose the direction of the axis you also choose the positive direction for the axis.

Newton's Laws, the Second one first

Newton's second law of motion: The force $\boldsymbol{F}$ on an object of inertial mass m is related to the acceleration a of the object through

$$\boldsymbol{F} = m\boldsymbol{a},$$

where $\boldsymbol{a}$ is the acceleration.

Newton's laws of motion are laws of nature that have been found by experimental investigations and have been shown to hold up to continued experimental investigations. Newton's laws are valid over a wide range of length- and time-scales. We use Newton's laws of motion to describe everything from the motion of atoms to the motion of galaxies.

The second law is a vector equation with the acceleration having the same direction as the force. The acceleration is proportional to the force via the mass m of the system under study.

Newton's second law introduces a new property of an object, the so-called inertial mass m. We determine the inertial mass of an object by measuring the acceleration for a given applied force.

Then the First Law

What happens if the net external force on a body is zero? Applying Newton's second law, we find:

$$\boldsymbol{F} = 0 = m\boldsymbol{a},$$

which gives using the definition of the acceleration

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = 0.$$

The acceleration is zero, which means that the velocity of the object is constant. This is often referred to as Newton's first law. An object in a state of uniform motion tends to remain in that state unless an external force changes its state of motion. Why do we need a separate law for this? Is it not simply a special case of Newton's second law? Yes, Newton's first law can be deduced from the second law as we have illustrated. However, the first law is often used for a different purpose: Newton's First Law tells us about the limit of applicability of Newton's Second law. Newton's Second law can only be used in reference systems where the First law is obeyed. But is not the First law always valid?

No! The First law is only valid in reference systems that are not accelerated. If you observe the motion of a ball from an accelerating car, the ball will appear to accelerate even if there are no forces acting on it. We call systems that are not accelerating inertial systems, and Newton's first law is often called the law of inertia. Newton's first and second laws of motion are only valid in inertial systems.

A system is an inertial system if it is not accelerated. It means that the reference system must not be accelerating linearly or rotating. Unfortunately, this means that most systems we know are not really inertial systems. For example, the surface of the Earth is clearly not an inertial system, because the Earth is rotating. The Earth is also not an inertial system, because it ismoving in a curved path around the Sun. However, even if the surface of the Earth is not strictly an inertial system, it may be considered to be approximately an inertial system for many laboratory-size experiments.

And finally the Third Law

If there is a force from object A on object B, there is also a force from object B on object A. This fundamental principle of interactions is called Newton's third law. We do not know of any force that do not obey this law: All forces appear in pairs. Newton's third law is usually formulated as: For every action there is an equal and opposite reaction.

Motion of a Single Object

Here we consider the motion of a single particle moving under the influence of some set of forces. We will consider some problems where the force does not depend on the position. In that case Newton's law $m\dot{\boldsymbol{v}} = \boldsymbol{F}(\boldsymbol{v})$ is a first-order differential equation and one solves for $\boldsymbol{v}(t)$, then moves on to integrate $\boldsymbol{v}$ to get the position. In essentially all of these cases we cna find an analytical solution.

Air Resistance in One Dimension

Air resistance tends to scale as the square of the velocity. This is in contrast to many problems chosen for textbooks, where it is linear in the velocity. The choice of a linear dependence is motivated by mathematical simplicity (it keeps the differential equation linear) rather than by physics. One can see that the force should be quadratic in velocity by considering the momentum imparted on the air molecules. If an object sweeps through a volume dV of air in time dt, the momentum imparted on the air is

$$dP = \rho_m dV v, \tag{1}$$

where v is the velocity of the object and ρ_m is the mass density of the air. If the molecules bounce back as opposed to stop you would double the size of the

term. The opposite value of the momentum is imparted onto the object itself. Geometrically, the differential volume is

$$dV = Av\,dt, \tag{2}$$

where A is the cross-sectional area and $v\,dt$ is the distance the object moved in time dt.

Resulting Acceleration

Plugging this into the expression above,

$$\frac{dP}{dt} = -\rho_m A v^2. \tag{3}$$

This is the force felt by the particle, and is opposite to its direction of motion. Now, because air doesn't stop when it hits an object, but flows around the best it can, the actual force is reduced by a dimensionless factor c_W, called the drag coefficient.

$$F_{\text{drag}} = -c_W \rho_m A v^2, \tag{4}$$

and the acceleration is

$$\frac{dv}{dt} = -\frac{c_W \rho_m A}{m} v^2. \tag{5}$$

For a particle with initial velocity v_0, one can separate the dt to one side of the equation, and move everything with vs to the other side. We did this in our discussion of simple motion and will not repeat it here.

On more general terms, for many systems, e.g. an automobile, there are multiple sources of resistance. In addition to wind resistance, where the force is proportional to v^2, there are dissipative effects of the tires on the pavement, and in the axel and drive train. These other forces can have components that scale proportional to v, and components that are independent of v. Those independent of v, e.g. the usual $f = \mu_K N$ frictional force you consider in your first Physics courses, only set in once the object is actually moving. As speeds become higher, the v^2 components begin to dominate relative to the others. For automobiles at freeway speeds, the v^2 terms are largely responsible for the loss of efficiency. To travel a distance L at fixed speed v, the energy/work required to overcome the dissipative forces are fL, which for a force of the form $f = \alpha v^n$ becomes

$$W = \int dx \ f = \alpha v^n L. \tag{6}$$

For $n = 0$ the work is independent of speed, but for the wind resistance, where $n = 2$, slowing down is essential if one wishes to reduce fuel consumption. It is also important to consider that engines are designed to be most efficient at a chosen range of power output. Thus, some cars will get better mileage at higher speeds (They perform better at 50 mph than at 5 mph) despite the considerations mentioned above.

Going Ballistic, Projectile Motion or a Softer Approach, Falling Raindrops

As an example of Newton's Laws we consider projectile motion (or a falling raindrop) with a drag force. Even though air resistance is largely proportional to the square of the velocity, we will consider the drag force to be linear to the velocity, $\boldsymbol{F} = -m\gamma\boldsymbol{v}$, for the purposes of this exercise. The acceleration, $\boldsymbol{a} = \boldsymbol{F}/m$, becomes

$$\frac{dv_x}{dt} = -\gamma v_x, \tag{7}$$
$$\frac{dv_y}{dt} = -\gamma v_y - g,$$

and γ has dimensions of inverse time.

Ways of solving these equations

We will go over two different ways to solve this equation. The first by direct integration, and the second as a differential equation. To do this by direct integration, one simply multiplies both sides of the equations above by dt, then divide by the appropriate factors so that the vs are all on one side of the equation and the dt is on the other. For the x motion one finds an easily integrable equation,

$$\begin{align} \frac{dv_x}{v_x} &= -\gamma dt, \tag{8}\\ \int_{v_{0x}}^{v_x} \frac{dv_x}{v_x} &= -\gamma \int_0^t dt,\\ \ln\left(\frac{v_x}{v_{0x}}\right) &= -\gamma t,\\ v_x(t) &= v_{0x} e^{-\gamma t}. \end{align}$$

This is very much the result you would have written down by inspection. For the y-component of the velocity,

$$\begin{align} \frac{dv_y}{v_y + g/\gamma} &= -\gamma dt \tag{9}\\ \ln\left(\frac{v_y + g/\gamma}{v_{0y} - g/\gamma}\right) &= -\gamma t_f,\\ v_{fy} &= -\frac{g}{\gamma} + \left(v_{0y} + \frac{g}{\gamma}\right) e^{-\gamma t}. \end{align}$$

Whereas v_x starts at some value and decays exponentially to zero, v_y decays exponentially to the terminal velocity, $v_t = -g/\gamma$.

Solving as differential equations

Although this direct integration is simpler than the method we invoke below, the method below will come in useful for some slightly more difficult differential equations in the future. The differential equation for v_x is straight-forward to solve. Because it is first order there is one arbitrary constant, A, and by inspection the solution is

$$v_x = Ae^{-\gamma t}. \tag{10}$$

The arbitrary constants for equations of motion are usually determined by the initial conditions, or more generally boundary conditions. By inspection $A = v_{0x}$, the initial x component of the velocity.

Differential Equations, contn

The differential equation for v_y is a bit more complicated due to the presence of g. Differential equations where all the terms are linearly proportional to a function, in this case v_y, or to derivatives of the function, e.g., v_y, dv_y/dt, $d^2v_y/dt^2 \cdots$, are called linear differential equations. If there are terms proportional to v^2, as would happen if the drag force were proportional to the square of the velocity, the differential equation is not longer linear. Because this expression has only one derivative in v it is a first-order linear differential equation. If a term were added proportional to d^2v/dt^2 it would be a second-order differential equation. In this case we have a term completely independent of v, the gravitational acceleration g, and the usual strategy is to first rewrite the equation with all the linear terms on one side of the equal sign,

$$\frac{dv_y}{dt} + \gamma v_y = -g. \tag{11}$$

Splitting into two parts

Now, the solution to the equation can be broken into two parts. Because this is a first-order differential equation we know that there will be one arbitrary constant. Physically, the arbitrary constant will be determined by setting the initial velocity, though it could be determined by setting the velocity at any given time. Like most differential equations, solutions are not "solved". Instead, one guesses at a form, then shows the guess is correct. For these types of equations, one first tries to find a single solution, i.e. one with no arbitrary constants. This is called the *particular* solution, $y_p(t)$, though it should really be called "a" particular solution because there are an infinite number of such solutions. One then finds a solution to the *homogenous* equation, which is the equation with zero on the right-hand side,

$$\frac{dv_{y,h}}{dt} + \gamma v_{y,h} = 0. \tag{12}$$

Homogenous solutions will have arbitrary constants.

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The particular solution will solve the same equation as the original general equation

$$\frac{dv_{y,p}}{dt} + \gamma v_{y,p} = -g. \tag{13}$$

However, we don't need find one with arbitrary constants. Hence, it is called a **particular** solution.

The sum of the two,

$$v_y = v_{y,p} + v_{y,h}, \tag{14}$$

is a solution of the total equation because of the linear nature of the differential equation. One has now found a *general* solution encompassing all solutions, because it both satisfies the general equation (like the particular solution), and has an arbitrary constant that can be adjusted to fit any initial condition (like the homogneous solution). If the equation were not linear, e.g if there were a term such as v_y^2 or $v_y \dot{v}_y$, this technique would not work.

More details

Returning to the example above, the homogenous solution is the same as that for v_x, because there was no gravitational acceleration in that case,

$$v_{y,h} = B e^{-\gamma t}. \tag{15}$$

In this case a particular solution is one with constant velocity,

$$v_{y,p} = -g/\gamma. \tag{16}$$

Note that this is the terminal velocity of a particle falling from a great height. The general solution is thus,

$$v_y = B e^{-\gamma t} - g/\gamma, \tag{17}$$

and one can find B from the initial velocity,

$$v_{0y} = B - g/\gamma, \quad B = v_{0y} + g/\gamma. \tag{18}$$

Plugging in the expression for B gives the y motion given the initial velocity,

$$v_y = (v_{0y} + g/\gamma) e^{-\gamma t} - g/\gamma. \tag{19}$$

It is easy to see that this solution has $v_y = v_{0y}$ when $t = 0$ and $v_y = -g/\gamma$ when $t \to \infty$.

One can also integrate the two equations to find the coordinates x and y as functions of t,

$$x = \int_0^t dt'\, v_{0x}(t') = \frac{v_{0x}}{\gamma}\left(1 - e^{-\gamma t}\right), \tag{20}$$

$$y = \int_0^t dt'\, v_{0y}(t') = -\frac{gt}{\gamma} + \frac{v_{0y} + g/\gamma}{\gamma}\left(1 - e^{-\gamma t}\right).$$

If the question was to find the position at a time t, we would be finished. However, the more common goal in a projectile equation problem is to find the range, i.e. the distance x at which y returns to zero. For the case without a drag force this was much simpler. The solution for the y coordinate would have been $y = v_{0y}t - gt^2/2$. One would solve for t to make $y = 0$, which would be $t = 2v_{0y}/g$, then plug that value for t into $x = v_{0x}t$ to find $x = 2v_{0x}v_{0y}/g = v_0\sin(2\theta_0)/g$. One follows the same steps here, except that the expression for $y(t)$ is more complicated. Searching for the time where $y = 0$, and we get

$$0 = -\frac{gt}{\gamma} + \frac{v_{0y} + g/\gamma}{\gamma}\left(1 - e^{-\gamma t}\right). \tag{21}$$

This cannot be inverted into a simple expression $t = \cdots$. Such expressions are known as "transcendental equations", and are not the rare instance, but are the norm. In the days before computers, one might plot the right-hand side of the above graphically as a function of time, then find the point where it crosses zero.

Now, the most common way to solve for an equation of the above type would be to apply Newton's method numerically. This involves the following algorithm for finding solutions of some equation $F(t) = 0$.

1. First guess a value for the time, t_{guess}.

2. Calculate F and its derivative, $F(t_{\text{guess}})$ and $F'(t_{\text{guess}})$.

3. Unless you guessed perfectly, $F \neq 0$, and assuming that $\Delta F \approx F'\Delta t$, one would choose

4. $\Delta t = -F(t_{\text{guess}})/F'(t_{\text{guess}})$.

5. Now repeat step 1, but with $t_{\text{guess}} \to t_{\text{guess}} + \Delta t$.

If the $F(t)$ were perfectly linear in t, one would find t in one step. Instead, one typically finds a value of t that is closer to the final answer than t_{guess}. One breaks the loop once one finds F within some acceptable tolerance of zero. A program to do this might look like:

Motion in a Magnetic Field

Another example of a velocity-dependent force is magnetism,

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}, \tag{22}$$

$$F_i = q\sum_{jk} \epsilon_{ijk} v_j B_k.$$

For a uniform field in the z direction $\boldsymbol{B} = B\hat{z}$, the force can only have x and y components,

$$
\begin{aligned}
F_x &= qBv_y \\
F_y &= -qBv_x.
\end{aligned}
\tag{23}
$$

The differential equations are

$$
\begin{aligned}
\dot{v}_x &= \omega_c v_y, \omega_c = qB/m \\
\dot{v}_y &= -\omega_c v_x.
\end{aligned}
\tag{24}
$$

One can solve the equations by taking time derivatives of either equation, then substituting into the other equation,

$$
\ddot{v}_x = \omega_c \dot{v}_y = -\omega_c^2 v_x,
\tag{25}
$$
$$
\ddot{v}_y = -\omega_c \dot{v}_x = -\omega_c v_y.
$$

The solution to these equations can be seen by inspection,

$$
\begin{aligned}
v_x &= A\sin(\omega_c t + \phi), \\
v_y &= A\cos(\omega_c t + \phi).
\end{aligned}
\tag{26}
$$

One can integrate the equations to find the positions as a function of time,

$$
\begin{aligned}
x - x_0 &= \int_{x_0}^{x} dx = \int_0^t dt\, v(t) \\
&= \frac{-A}{\omega_c}\cos(\omega_c t + \phi), \\
y - y_0 &= \frac{A}{\omega_c}\sin(\omega_c t + \phi).
\end{aligned}
\tag{27}
$$

The trajectory is a circle centered at x_0, y_0 with amplitude A rotating in the clockwise direction.

The equations of motion for the z motion are

$$
\dot{v}_z = 0,
\tag{28}
$$

which leads to

$$
z - z_0 = V_z t.
\tag{29}
$$

Added onto the circle, the motion is helical.

Note that the kinetic energy,

$$
T = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}m(\omega_c^2 A^2 + V_z^2),
\tag{30}
$$

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is constant. This is because the force is perpendicular to the velocity, so that in any differential time element dt the work done on the particle $\boldsymbol{F} \cdot dr = dt \boldsymbol{F} \cdot v = 0$.

One should think about the implications of a velocity dependent force. Suppose one had a constant magnetic field in deep space. If a particle came through with velocity v_0, it would undergo cyclotron motion with radius $R = v_0/\omega_c$. However, if it were still its motion would remain fixed. Now, suppose an observer looked at the particle in one reference frame where the particle was moving, then changed their velocity so that the particle's velocity appeared to be zero. The motion would change from circular to fixed. Is this possible?

The solution to the puzzle above relies on understanding relativity. Imagine that the first observer believes $\boldsymbol{B} \neq 0$ and that the electric field $\boldsymbol{E} = 0$. If the observer then changes reference frames by accelerating to a velocity $\boldsymbol{v}$, in the new frame $\boldsymbol{B}$ and $\boldsymbol{E}$ both change. If the observer moved to the frame where the charge, originally moving with a small velocity v, is now at rest, the new electric field is indeed $\boldsymbol{v} \times \boldsymbol{B}$, which then leads to the same acceleration as one had before. If the velocity is not small compared to the speed of light, additional γ factors come into play, $\gamma = 1/\sqrt{1 - (v/c)^2}$. Relativistic motion will not be considered in this course.

Sliding Block tied to a Wall

Another classical case is that of simple harmonic oscillations, here represented by a block sliding on a horizontal frictionless surface. The block is tied to a wall with a spring. If the spring is not compressed or stretched too far, the force on the block at a given position x is

$$F = -kx.$$

The negative sign means that the force acts to restore the object to an equilibrium position. Newton's equation of motion for this idealized system is then

$$m\frac{d^2x}{dt^2} = -kx,$$

or we could rephrase it as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega_0^2 x,$$

with the angular frequency $\omega_0^2 = k/m$.

The above differential equation has the advantage that it can be solved analytically with solutions on the form

$$x(t) = A cos(\omega_0 t + \nu),$$

where A is the amplitude and ν the phase constant. This provides in turn an important test for the numerical solution and the development of a program for more complicated cases which cannot be solved analytically.

Simple Example, Block tied to a Wall

With the position $x(t)$ and the velocity $v(t) = dx/dt$ we can reformulate Newton's equation in the following way

$$\frac{dx(t)}{dt} = v(t),$$

and

$$\frac{dv(t)}{dt} = -\omega_0^2 x(t).$$

We are now going to solve these equations using the Runge-Kutta method to fourth order discussed previously.

Simple Example, Block tied to a Wall

Before proceeding however, it is important to note that in addition to the exact solution, we have at least two further tests which can be used to check our solution.

Since functions like *cos* are periodic with a period 2π, then the solution $x(t)$ has also to be periodic. This means that

$$x(t + T) = x(t),$$

with T the period defined as

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{k/m}}.$$

Observe that T depends only on k/m and not on the amplitude of the solution.

Simple Example, Block tied to a Wall

In addition to the periodicity test, the total energy has also to be conserved. Suppose we choose the initial conditions

$$x(t = 0) = 1 \text{ m} \qquad v(t = 0) = 0 \text{ m/s},$$

meaning that block is at rest at $t = 0$ but with a potential energy

$$E_0 = \frac{1}{2} k x(t = 0)^2 = \frac{1}{2} k.$$

The total energy at any time t has however to be conserved, meaning that our solution has to fulfil the condition

$$E_0 = \frac{1}{2} k x(t)^2 + \frac{1}{2} m v(t)^2.$$

Simple Example, Block tied to a Wall

An algorithm which implements these equations is included below.

- Choose the initial position and speed, with the most common choice $v(t = 0) = 0$ and some fixed value for the position.

- Choose the method you wish to employ in solving the problem.

- Subdivide the time interval $[t_i, t_f]$ into a grid with step size

$$h = \frac{t_f - t_i}{N},$$

where N is the number of mesh points.

- Calculate now the total energy given by

$$E_0 = \frac{1}{2}kx(t = 0)^2 = \frac{1}{2}k.$$

- Choose ODE solver to obtain x_{i+1} and v_{i+1} starting from the previous values x_i and v_i.

- When we have computed $x(v)_{i+1}$ we upgrade $t_{i+1} = t_i + h$.

- This iterative process continues till we reach the maximum time t_f.

- The results are checked against the exact solution. Furthermore, one has to check the stability of the numerical solution against the chosen number of mesh points N.

Simple Example, Block tied to a Wall, python code

The following python program performs essentially the same calculations as the previous c++ code.

The classical pendulum and scaling the equations

The angular equation of motion of the pendulum is given by Newton's equation and with no external force it reads

$$ml\frac{d^2\theta}{dt^2} + mgsin(\theta) = 0, \tag{31}$$

with an angular velocity and acceleration given by

$$v = l\frac{d\theta}{dt}, \tag{32}$$

and

$$a = l\frac{d^2\theta}{dt^2}. \tag{33}$$

More on the Pendulum

We do however expect that the motion will gradually come to an end due a viscous drag torque acting on the pendulum. In the presence of the drag, the above equation becomes

$$ml\frac{d^2\theta}{dt^2} + \nu\frac{d\theta}{dt} + mgsin(\theta) = 0, \tag{34}$$

where ν is now a positive constant parameterizing the viscosity of the medium in question. In order to maintain the motion against viscosity, it is necessary to add some external driving force. We choose here a periodic driving force. The last equation becomes then

$$ml\frac{d^2\theta}{dt^2} + \nu\frac{d\theta}{dt} + mgsin(\theta) = Asin(\omega t), \tag{35}$$

with A and ω two constants representing the amplitude and the angular frequency respectively. The latter is called the driving frequency.

More on the Pendulum

We define

$$\omega_0 = \sqrt{g/l},$$

the so-called natural frequency and the new dimensionless quantities

$$\hat{t} = \omega_0 t,$$

with the dimensionless driving frequency

$$\hat{\omega} = \frac{\omega}{\omega_0},$$

and introducing the quantity Q, called the *quality factor*,

$$Q = \frac{mg}{\omega_0\nu},$$

and the dimensionless amplitude

$$\hat{A} = \frac{A}{mg}$$

More on the Pendulum

We have

$$\frac{d^2\theta}{d\hat{t}^2} + \frac{1}{Q}\frac{d\theta}{d\hat{t}} + sin(\theta) = \hat{A}cos(\hat{\omega}\hat{t}).$$

This equation can in turn be recast in terms of two coupled first-order differential equations as follows

$$\frac{d\theta}{d\hat{t}} = \hat{v},$$

and

$$\frac{d\hat{v}}{d\hat{t}} = -\frac{\hat{v}}{Q} - sin(\theta) + \hat{A}cos(\hat{\omega}\hat{t}).$$

These are the equations to be solved. The factor Q represents the number of oscillations of the undriven system that must occur before its energy is significantly reduced due to the viscous drag. The amplitude $\hat{A}$ is measured in units of the maximum possible gravitational torque while $\hat{\omega}$ is the angular frequency of the external torque measured in units of the pendulum's natural frequency.

Gravitational Force

We start with a simpler case first, the Earth-Sun system in two dimensions only. The gravitational force F_G is

$$F = \frac{GM_\odot M_E}{r^2},$$

where G is the gravitational constant,

$$M_E = 6 \times 10^{24}\text{Kg},$$

the mass of Earth,

$$M_\odot = 2 \times 10^{30}\text{Kg},$$

the mass of the Sun and

$$r = 1.5 \times 10^{11}\text{m},$$

is the distance between Earth and the Sun. The latter defines what we call an astronomical unit **AU**. From Newton's second law we have then for the x direction

$$\frac{d^2x}{dt^2} = \frac{F_x}{M_E},$$

and

$$\frac{d^2y}{dt^2} = \frac{F_y}{M_E},$$

for the y direction.

Building a code for the solar system, force equations

Introducing $x = r\cos(\theta)$, $y = r\sin(\theta)$ and

$$r = \sqrt{x^2 + y^2},$$

we can rewrite

$$F_x = -\frac{GM_\odot M_E}{r^2}\cos(\theta) = -\frac{GM_\odot M_E}{r^3}x,$$

and

$$F_y = -\frac{GM_\odot M_E}{r^2}\sin(\theta) = -\frac{GM_\odot M_E}{r^3}y,$$

for the y direction.

Building a code for the solar system, coupled equations

We can rewrite these two equations

$$F_x = -\frac{GM_\odot M_E}{r^2}\cos(\theta) = -\frac{GM_\odot M_E}{r^3}x,$$

and

$$F_y = -\frac{GM_\odot M_E}{r^2}\sin(\theta) = -\frac{GM_\odot M_E}{r^3}y,$$

as four first-order coupled differential equations

$$\frac{dv_x}{dt} = -\frac{GM_\odot}{r^3}x,$$

$$\frac{dx}{dt} = v_x,$$

$$\frac{dv_y}{dt} = -\frac{GM_\odot}{r^3}y,$$

$$\frac{dy}{dt} = v_y.$$

Building a code for the solar system, final coupled equations

The four coupled differential equations

$$\frac{dv_x}{dt} = -\frac{GM_\odot}{r^3}x,$$

$$\frac{dx}{dt} = v_x,$$

$$\frac{dv_y}{dt} = -\frac{GM_\odot}{r^3}y,$$

$$\frac{dy}{dt} = v_y,$$

can be turned into dimensionless equations or we can introduce astronomical units with $1\,\text{AU} = 1.5 \times 10^{11}$.

Using the equations from circular motion (with $r = 1\text{AU}$)

$$\frac{M_E v^2}{r} = F = \frac{GM_\odot M_E}{r^2},$$

we have

$$GM_\odot = v^2 r,$$

and using that the velocity of Earth (assuming circular motion) is $v = 2\pi r/\text{yr} = 2\pi\text{AU}/\text{yr}$, we have

$$GM_\odot = v^2 r = 4\pi^2 \frac{(\text{AU})^3}{\text{yr}^2}.$$

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Building a code for the solar system, discretized equations

The four coupled differential equations can then be discretized using Euler's method as (with step length h)

$$v_{x,i+1} = v_{x,i} - h\frac{4\pi^2}{r_i^3}x_i,$$

$$x_{i+1} = x_i + hv_{x,i},$$

$$v_{y,i+1} = v_{y,i} - h\frac{4\pi^2}{r_i^3}y_i,$$

$$y_{i+1} = y_i + hv_{y,i},$$

Building a code for the solar system, adding Jupiter

It is rather straightforward to add a new planet, say Jupiter. Jupiter has mass

$$M_J = 1.9 \times 10^{27}\text{kg},$$

and distance to the Sun of 5.2 AU. The additional gravitational force the Earth feels from Jupiter in the x-direction is

$$F_x^{EJ} = -\frac{GM_J M_E}{r_{EJ}^3}(x_E - x_J),$$

where E stands for Earth, J for Jupiter, r_{EJ} is distance between Earth and Jupiter

$$r_{EJ} = \sqrt{(x_E - x_J)^2 + (y_E - y_J)^2},$$

and x_E and y_E are the x and y coordinates of Earth, respectively, and x_J and y_J are the x and y coordinates of Jupiter, respectively. The x-component of the velocity of Earth changes thus to

$$\frac{dv_x^E}{dt} = -\frac{GM_\odot}{r^3}x_E - \frac{GM_J}{r_{EJ}^3}(x_E - x_J).$$

Building a code for the solar system, adding Jupiter

We can rewrite

$$\frac{dv_x^E}{dt} = -\frac{GM_\odot}{r^3}x_E - \frac{GM_J}{r_{EJ}^3}(x_E - x_J).$$

to

$$\frac{dv_x^E}{dt} = -\frac{4\pi^2}{r^3}x_E - \frac{4\pi^2 M_J/M_\odot}{r_{EJ}^3}(x_E - x_J),$$

where we used

$$GM_J = GM_\odot\left(\frac{M_J}{M_\odot}\right) = 4\pi^2\frac{M_J}{M_\odot}.$$

Similarly, for the velocity in y-direction we have

$$\frac{dv_y^E}{dt} = -\frac{4\pi^2}{r^3}y_E - \frac{4\pi^2 M_J/M_\odot}{r_{EJ}^3}(y_E - y_J).$$

Similar expressions apply for Jupiter. The equations for x and y derivatives are unchanged. This equations are similar for all other planets and as we will see later, it will be convenient to object orient this part when we program the full solar system.

How can I get the initial velocities and positions of the planets

NASA has an excellent site at `http://ssd.jpl.nasa.gov/horizons.cgi#top`. From there you can extract initial conditions in order to start your differential equation solver. At the above website you need to change from **OBSERVER** to **VECTOR** and then write in the planet you are interested in. The generated data contain the x, y and z values as well as their corresponding velocities. The velocities are in units of AU per day. Alternatively they can be obtained in terms of km and km/s.

For the first simple system involving the Earth and the Sun, you could just initialize the position with say $x = 1$ AU and $y = 0$ AU.

How do we develop a larger numerical project

When building up a numerical project there are several elements you should think of, amongst these we take the liberty of mentioning the following.

1. How to structure a code in terms of functions and modules

2. How to read input data flexibly from the command line

3. Find tests and how to write unit tests (test functions). A very good example is Catch, a modern, C++-native, header-only, framework for unit-tests. Try to find suitable tests of the mathematical algorithms as well as tests which reflect the physics of the system.

4. It takes one quick command to let all your code undergo heavy testing

5. How to refactor code in terms of classes (instead of functions only)

6. How to conduct and automate large-scale numerical experiments

More on large numerical projects

The conventions and techniques outlined here will save you a lot of time when you incrementally extend software over time from simpler to more complicated problems.

1. Scale your equations in order to simplify, make for example the equations dimensionless or scale them in convenient units

2. New code is added in a modular fashion to a library (modules)

3. Programs are run through convenient user interfaces

4. Tedious manual work with running programs is automated,

5. Your scientific investigations are reproducible, scientific reports with top quality typesetting are produced both for paper and electronic devices.

How to start our solar system project

The simplest possible step is to code the Earth-Sun system using Euler's method in two dimensions. The four coupled differential equations can then be discretized using Euler's method as (with step length h) are

$$v_{x,i+1} = v_{x,i} - h \frac{4\pi^2}{r_i^3} x_i,$$

$$x_{i+1} = x_i + h v_{x,i},$$

$$v_{y,i+1} = v_{y,i} - h \frac{4\pi^2}{r_i^3} y_i,$$

$$y_{i+1} = y_i + h v_{y,i},$$