Conservation of energy

$$\mathcal{L}(q, \dot{q}, t) = T - V$$

$$E = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}$$

$$= \frac{1}{2} m \dot{q}^2 + V(q)$$

Linear momentum

$\mathcal{L}$ should be invariant wrt choice of origin

$$\delta \mathcal{L} = \sum_i \frac{\partial \mathcal{L}}{\partial \vec{r}_i} \delta \vec{r} = 0$$

small uniform translation

$\delta \vec{r} \neq 0$

$$\boxed{\sum_i \frac{\partial \mathcal{L}}{\partial \vec{r}_i} = 0}$$

Euler-Lagrange equation

$$\underline{\frac{\partial \mathcal{L}}{\partial \vec{r}_i}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \vec{v}_i} = 0$$

$$\mathcal{L} = \frac{1}{2} m \sum_{i=1}^{n} \vec{v}_i^{\,2} - V$$

$$\frac{\partial \mathcal{L}}{\partial \vec{v}_i} = m \vec{v}_i = \vec{p}_i$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i \quad \text{(generalized linear momentum)}$$

Total momentum

$$\vec{P} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m \dot{\vec{r}}_i$$

$$\frac{\partial \mathcal{L}}{\partial \vec{r}_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} \quad \Rightarrow$$

$$\frac{d}{dt} \sum_i \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} = 0 = \frac{d}{dt} \sum_{i=1}^{n} \vec{p}_i$$

$$= \frac{d}{dt} \vec{P} = 0$$

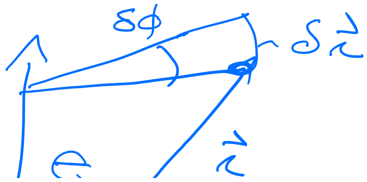
Most cases (isolated systems)

$$V = V(\vec{r}) \quad \Rightarrow$$

$$\frac{\partial \mathcal{L}}{\partial \vec{r}_i} = -\frac{\partial V}{\partial \vec{r}_i} = \vec{F}_i \quad \Rightarrow$$

$$\frac{d\vec{P}}{dt} = \vec{F} = \sum_i \vec{F}_i = 0$$

- Conservation of angular momentum



$\vec{V}$ isotropic rotation

Lagrangian unchanged under a rotation $\delta\phi$ wrt an arbitrary axis

$$\delta L = 0 \quad \sum_i \left(\frac{\partial L}{\partial \vec{r}_i} \underline{\delta \vec{r}_i} + \frac{\partial L}{\partial \vec{v}_i} \underline{\delta \vec{v}_i} \right)$$

$$= \sum_i \left(\dot{\vec{p}}_i \, \delta \vec{r}_i + \vec{p}_i \cdot \delta \vec{v}_i \right) = 0$$

$$\delta \vec{r} = \delta \vec{\phi} \times \vec{r}$$
$$\delta \vec{v} = \delta \vec{\phi} \times \vec{v}$$

vector subject to small rotations wrt inertial frame

$$\delta \vec{a} = \delta \vec{\phi} \times \vec{a}$$

rate of change

$$\frac{\delta \vec{a}}{\delta t} = \underline{\vec{\Omega} \times \vec{a}}$$

[non-inertial frames Taylor chap 9]

$$0 = \delta L = \sum_i \left[\dot{\vec{p}}_i (\delta \vec{\phi} \times \vec{r}_i) + \vec{p}_i (\delta \vec{\phi} \times \vec{v}_i) \right]$$

$$= \delta \vec{\phi} \sum_i \left[(\vec{r}_i \times \dot{\vec{p}}_i) + (\vec{v}_i \times \vec{p}_i) \right]$$

$$\Rightarrow \delta\vec{\phi}\, \frac{d}{dt} \sum_i \left[(r_i \times \vec{p}_i) \right]$$

$$= \delta\vec{\phi} \cdot \frac{d}{dt} \vec{L} \Rightarrow$$

$$\frac{d\vec{L}}{dt} = 0$$

Lagrangians in non-inertial frames:

- accelerating frame

$$\vec{r}_{s0} = \vec{r}_s + \vec{w} \cdot t$$

Newton's law in accelerating frame:

$$\ddot{r}_s = \frac{\vec{F}}{m} - \dot{\vec{w}} \Rightarrow$$

$$\mathcal{L}_s = \frac{m v_s^2}{2} - \underline{m \vec{r}_s \cdot \dot{\vec{w}}} - V(r_s)$$

Rotating frame

$$\vec{v}_{s0} = \vec{v}_s + \vec{\Omega} \times \vec{r}_s$$

$$\mathcal{L}_s = \frac{m}{2} v_s^2 + m\, \vec{v}_s (\vec{\Omega} \times \vec{r}_s)$$

$$+ \frac{m}{2}(\vec{\Omega} \times \vec{r}_s) - V(r_s)$$

$$\frac{\partial \mathcal{L}}{\partial \vec{v}_s} \quad , \quad \frac{\partial \mathcal{L}}{\partial \vec{r}_s}$$

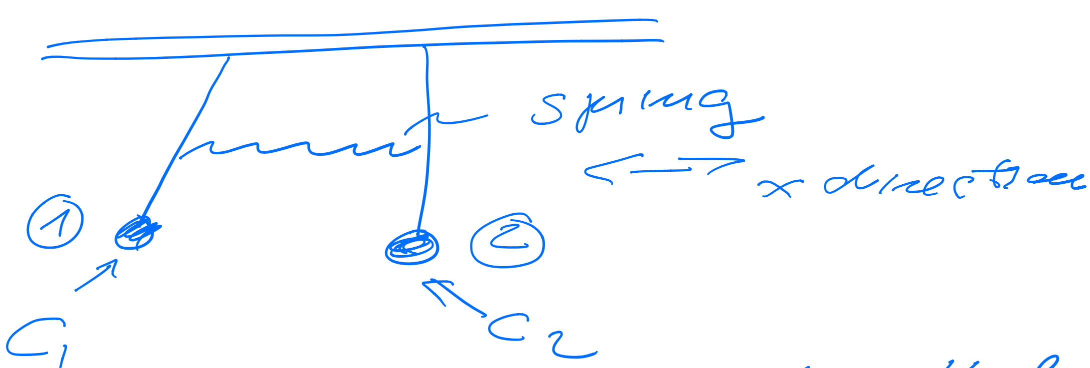
$$\vec{a}_s = -\frac{1}{m}\frac{\partial V}{\partial r_s} + (\vec{r}_s \times \dot{\vec{\Omega}})$$

$$+ \underbrace{2(\vec{v}_s \times \vec{\Omega})}_{\text{coriolis}} +$$

$$\underbrace{\vec{\Omega} \times (\vec{r}_s \times \vec{\Omega})}_{\text{centrifugal force,}}$$

Example

sympathetic pendulum



are defined by the length l_1 and l_2 and mg

$$m_1 = m_2 = m$$

$$m\ddot{x}_1 = -c_1 x_1 + c(x_2 - x_1)$$

$$m\ddot{x}_2 = -c_2 x_2 - c(x_2 - x_1)$$

$$l_1 \neq l_2 \qquad c_1 \neq c_2$$

Define

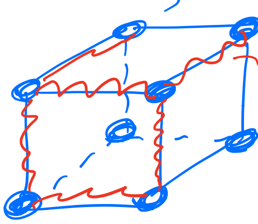
$$a_{11} = -(c_1+c)/m$$
$$a_{12} = c/m$$
$$a_{21} = c/m$$
$$a_{22} = -(c_2+c)/m$$

$$\ddot{x}_1 = a_{11}x_1 + a_{12}x_2$$
$$\ddot{x}_2 = a_{21}x_1 + a_{22}x_2$$

coupled-ordinary diff eqs,

$$\ddot{x}_i = \sum_{j=1}^{m} a_{ij}x_j \qquad i=1,2,\dots m$$

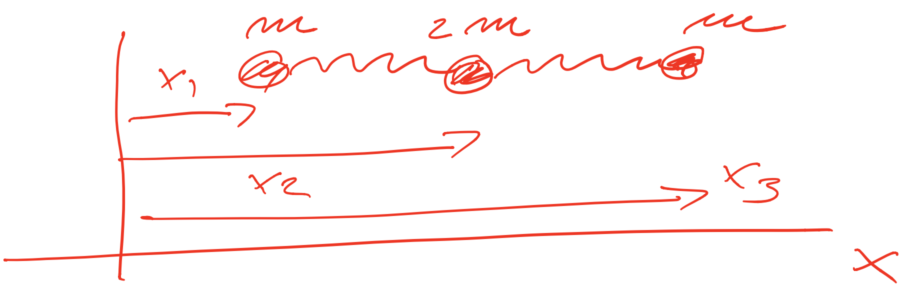
harmonic oscillator chain (linear)



ion/atom/molecule

spring force

linear chain



$$\mathcal{L} = ?$$