Two-body problem & central forces

– Taylor chapter 8.1 – 8.7

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}$$

$$M = m_1 + m_2$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad m_1 = m_2 = m$$

$$M = 2m$$

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$$

$$\vec{r}_2 = \vec{R} - \frac{m_1 \vec{r}}{M}$$

$$K = \frac{1}{2}\left[m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2\right]$$

$$\frac{d\vec{r}_1}{dt}$$

$$= \frac{1}{2}\left[M \cdot \dot{R}^2 + \mu \dot{r}^2\right]$$

$$\mu = \frac{m_1 m_2}{M} \; ; \quad m_1 = m_2 = m$$

$$\mu = \frac{1}{2} m$$

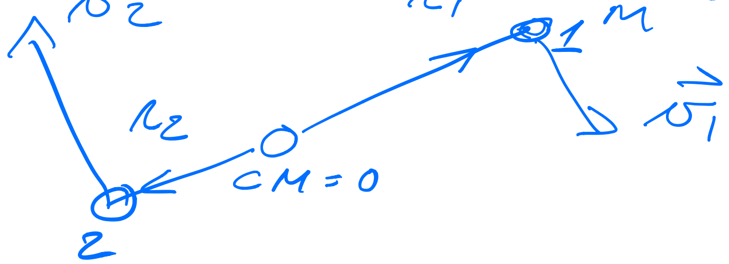
$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

$$= \mu(\vec{r} \times \dot{\vec{r}}) \Longleftarrow \text{CM Frame}$$

CM-frame $\vec{R} = 0$

$$\vec{r}_1 = \frac{m_2}{M}\vec{r} \quad \wedge \quad \vec{r}_2 = -\frac{m_1}{M}\vec{r}$$

$$\vec{r}_1 = \frac{m_2}{M}\vec{r}$$




CM-frame:

$$K = \frac{1}{2}\mu \dot{r}^2 = \frac{1}{2}\mu v_r^2$$

$$= \frac{1}{2}\mu\left(v_x^2 + v_y^2\right)$$

$$E = K + V(r)$$

$$r = \sqrt{x^2+y^2+z^2}$$

2-dim

$$x = r\cos\phi \qquad r \in [0,\infty)$$

$$y = r\sin\phi \qquad \phi \in [0, 2\pi]$$

$$\boxed{\vec{R} = 0}$$

hw 4 : $\frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = v_y$

$$\frac{dv_x}{dt} = a_x \qquad \frac{dv_y}{dt} = a_y$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\cos\phi\,\vec{i}$$

$$+ \frac{dr}{dt}\sin\phi\,\vec{j}$$

$$+ \; r \frac{d\phi}{dt} (-\sin\phi) \, \vec{i}$$

$$+ \; r \frac{d\phi}{dt} \cos\phi \, \vec{j}$$

$$K = \frac{1}{2} \mu \dot{r}^2 = \frac{1}{2} \mu \left| \frac{d\vec{r}}{dt} \right|^2$$

$$\left[ \vec{i} \cdot \vec{j} = 0 \right]$$

$$= \frac{1}{2} \mu \left[ \left( \frac{dr}{dt} \right)^2 (\cos^2\phi + \sin^2\phi) + r^2 \left( \frac{d\phi}{dt} \right)^2 (\cos^2\phi + \sin^2\phi) \right]$$

$$\dot{r} = \frac{dr}{dt} \qquad \dot{\phi} = \frac{d\phi}{dt}$$

$$= \frac{1}{2} \mu \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right)$$

$$E = \underline{\qquad\qquad} \; , \; \underline{\qquad\qquad} + V(r)$$

$$\vec{L} = \mu \left( \vec{r} \times \frac{d\vec{r}}{dt} \right) =$$

$$\mu \left( r\cos\phi \, \vec{i} + r\sin\phi \, \vec{j} \right) \times$$

$$\left( \frac{dr}{dt} \cos\phi \, \vec{i} + \frac{dr}{dt} \sin\phi \, \vec{j} - r\dot{\phi} \sin\phi \, \vec{i} + r\dot{\phi} \cos\phi \, \vec{j} \right)$$

$$= \mu \cdot r^2 \dot{\phi}^2 \, \vec{k}$$

Lagrangian formalism + Euler-Lagrange equations
Equation of motion for radial degrees of freedom:

$$\mu \cdot \ddot{r} - \mu r \cdot \dot{\phi}^2 + \frac{dV}{dr} = 0$$

$$\mu \cdot \ddot{r} = \underbrace{-\frac{dV}{dr}}_{F(r)} + \mu r \dot{\phi}^2$$

$$L = \mu r^2 \dot{\phi}$$

$$\mu \cdot \ddot{r} = -\frac{dV}{dr} + \frac{L^2}{\mu r^3}$$

$$-\frac{L^2}{\mu r^3} = \frac{d}{dr}\left(\frac{L^2}{2\mu r^2}\right)$$

$$V_{eff}(r) = V(r) + \frac{L^2}{2\mu r^2}$$

$$\vec{F} = -\vec{\nabla} V(r)$$

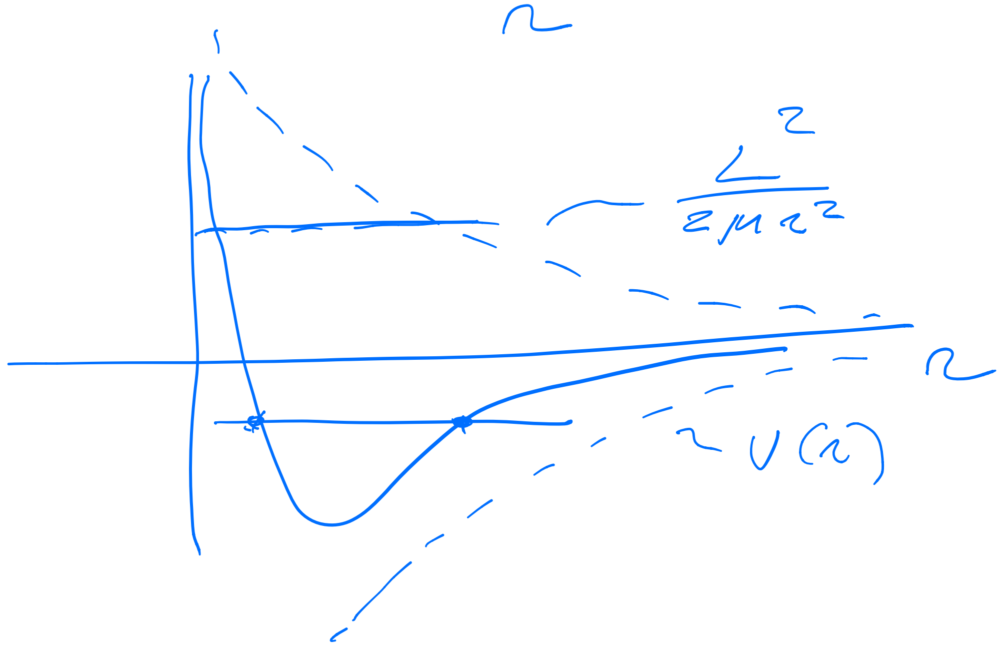
$$\mu \ddot{r} = -\frac{d V_{eff}(r)}{dr}$$

$$E = \frac{1}{2}\mu \dot{r}^2 + V(r) + \frac{L^2}{2\mu r^2}$$

$$= \frac{1}{2}\mu \dot{r}^2 + V_{eff}(r)$$

Kinetic energy part

$$V(r) = -\frac{G m_1 m_2}{r}$$




Differential equations the brute force way:

$$\frac{d}{dt} r^2 = \frac{d}{dt}(x^2+y^2) =$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2\cdot r\cdot \dot{r} = 2r\frac{dr}{dt}$$

$$\frac{dr}{dt} = \dot{r} = \frac{x}{r}\frac{dx}{dt} + \frac{y}{r}\frac{dy}{dt}$$

$$= \frac{x}{r}\dot{x} + \frac{y}{r}\dot{y}$$

$$\frac{d^2 r}{dt^2} = \ddot{r} = \frac{x}{r}\frac{d^2x}{dt^2} + \frac{y}{r}\frac{d^2y}{dt^2} + \underbrace{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}_{\dot{r}^2 + r^2\dot{\phi}^2} - \frac{\dot{r}^2}{r}$$

$$\frac{x}{r}\frac{d^2x}{dt^2} \qquad x = r\cos\phi$$

$$\frac{y}{r}\frac{d^2y}{dt^2} \qquad y = r\sin\phi$$

$$\ddot{r} = \cos\phi\frac{d^2x}{dt^2} + \sin\phi\frac{d^2y}{dt^2} + \frac{\dot{r}^2 + r^2\dot{\phi}^2}{r} - \frac{\dot{r}^2}{r}$$