

PHY321: Forces, Newton's Laws and Motion Example

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Newton's Laws

This will be inserted soon.

Motion of a Single Object

Here we consider the motion of a single particle moving under the influence of some set of forces. We will consider some problems where the force does not depend on the position. In that case Newton's law $m\dot{\mathbf{v}} = \mathbf{F}(\mathbf{v})$ is a first-order differential equation and one solves for $\mathbf{v}(t)$, then moves on to integrate \mathbf{v} to get the position. In essentially all of these cases we can find an analytical solution.

Air Resistance in One Dimension

Air resistance tends to scale as the square of the velocity. This is in contrast to many problems chosen for textbooks, where it is linear in the velocity. The choice of a linear dependence is motivated by mathematical simplicity (it keeps the differential equation linear) rather than by physics. One can see that the force should be quadratic in velocity by considering the momentum imparted on the air molecules. If an object sweeps through a volume dV of air in time dt , the momentum imparted on the air is

$$dP = \rho_m dV v, \quad (1)$$

where v is the velocity of the object and ρ_m is the mass density of the air. If the molecules bounce back as opposed to stop you would double the size of the term. The opposite value of the momentum is imparted onto the object itself. Geometrically, the differential volume is

$$dV = Avdt, \quad (2)$$

where A is the cross-sectional area and vdt is the distance the object moved in time dt .

Resulting Acceleration

Plugging this into the expression above,

$$\frac{dP}{dt} = -\rho_m Av^2. \quad (3)$$

This is the force felt by the particle, and is opposite to its direction of motion. Now, because air doesn't stop when it hits an object, but flows around the best it can, the actual force is reduced by a dimensionless factor c_W , called the drag coefficient.

$$F_{\text{drag}} = -c_W \rho_m Av^2, \quad (4)$$

and the acceleration is

$$\frac{dv}{dt} = -\frac{c_W \rho_m A}{m} v^2. \quad (5)$$

For a particle with initial velocity v_0 , one can separate the dt to one side of the equation, and move everything with vs to the other side. We did this in our discussion of simple motion and will not repeat it here.

On more general terms, for many systems, e.g. an automobile, there are multiple sources of resistance. In addition to wind resistance, where the force is proportional to v^2 , there are dissipative effects of the tires on the pavement, and in the axel and drive train. These other forces can have components that scale proportional to v , and components that are independent of v . Those independent of v , e.g. the usual $f = \mu_K N$ frictional force you consider in your first Physics courses, only set in once the object is actually moving. As speeds become higher, the v^2 components begin to dominate relative to the others. For automobiles at freeway speeds, the v^2 terms are largely responsible for the loss of efficiency. To travel a distance L at fixed speed v , the energy/work required to overcome the dissipative forces are fL , which for a force of the form $f = \alpha v^n$ becomes

$$W = \int dx f = \alpha v^n L. \quad (6)$$

For $n = 0$ the work is independent of speed, but for the wind resistance, where $n = 2$, slowing down is essential if one wishes to reduce fuel consumption. It

is also important to consider that engines are designed to be most efficient at a chosen range of power output. Thus, some cars will get better mileage at higher speeds (They perform better at 50 mph than at 5 mph) despite the considerations mentioned above.

Going Ballistic, Projectile Motion or a Softer Approach, Falling Raindrops

As an example of Newton's Laws we consider projectile motion (or a falling raindrop) with a drag force. Even though air resistance is largely proportional to the square of the velocity, we will consider the drag force to be linear to the velocity, $\mathbf{F} = -m\gamma\mathbf{v}$, for the purposes of this exercise. The acceleration, $\mathbf{a} = \mathbf{F}/m$, becomes

$$\begin{aligned}\frac{dv_x}{dt} &= -\gamma v_x, \\ \frac{dv_y}{dt} &= -\gamma v_y - g,\end{aligned}\tag{7}$$

and γ has dimensions of inverse time.

Ways of solving these equations

We will go over two different ways to solve this equation. The first by direct integration, and the second as a differential equation. To do this by direct integration, one simply multiplies both sides of the equations above by dt , then divide by the appropriate factors so that the v s are all on one side of the equation and the dt is on the other. For the x motion one finds an easily integrable equation,

$$\begin{aligned}\frac{dv_x}{v_x} &= -\gamma dt, \\ \int_{v_{0x}}^{v_{fx}} \frac{dv_x}{v_x} &= -\gamma \int_0^{t_f} dt, \\ \ln\left(\frac{v_{fx}}{v_{0x}}\right) &= -\gamma t_f, \\ v_{fx} &= v_{0x}e^{-\gamma t}.\end{aligned}\tag{8}$$

Here, we leave the subscript off the final time in the last expression. This is very much the result you would have written down by inspection. For the y -component of the velocity,

$$\begin{aligned}
\frac{dv_y}{v_y + g/\gamma} &= -\gamma dt \\
\ln\left(\frac{v_{fy} + g/\gamma}{v_{0y} + g/\gamma}\right) &= -\gamma t_f, \\
v_{fy} &= -\frac{g}{\gamma} + \left(v_{0y} + \frac{g}{\gamma}\right)e^{-\gamma t}.
\end{aligned} \tag{9}$$

Whereas v_x starts at some value and decays exponentially to zero, v_y decays exponentially to the terminal velocity, $v_t = -g/\gamma$.

Solving as differential equations

Although this direct integration is simpler than the method we invoke below, the method below will come in useful for some slightly more difficult differential equations in the future. The differential equation for v_x is straight-forward to solve. Because it is first order there is one arbitrary constant, A , and by inspection the solution is

$$v_x = Ae^{-\gamma t}. \tag{10}$$

The arbitrary constants for equations of motion are usually determined by the initial conditions, or more generally boundary conditions. By inspection $A = v_{0x}$, the initial x component of the velocity.

Differential Equations, contn

The differential equation for v_y is a bit more complicated due to the presence of g . Differential equations where all the terms are linearly proportional to a function, in this case v_y , or to derivatives of the function, e.g., v_y , dv_y/dt , $d^2v_y/dt^2 \dots$, are called linear differential equations. If there are terms proportional to v^2 , as would happen if the drag force were proportional to the square of the velocity, the differential equation is not longer linear. Because this expression has only one derivative in v it is a first-order linear differential equation. If a term were added proportional to d^2v/dt^2 it would be a second-order differential equation. In this case we have a term completely independent of v , the gravitational acceleration g , and the usual strategy is to first rewrite the equation with all the linear terms on one side of the equal sign,

$$\frac{dv_y}{dt} + \gamma v_y = -g. \tag{11}$$

Splitting into two parts

Now, the solution to the equation can be broken into two parts. Because this is a first-order differential equation we know that there will be one arbitrary

constant. Physically, the arbitrary constant will be determined by setting the initial velocity, though it could be determined by setting the velocity at any given time. Like most differential equations, solutions are not “solved”. Instead, one guesses at a form, then shows the guess is correct. For these types of equations, one first tries to find a single solution, i.e. one with no arbitrary constants. This is called the *particular* solution, $y_p(t)$, though it should really be called “a” particular solution because there are an infinite number of such solutions. One then finds a solution to the *homogenous* equation, which is the equation with zero on the right-hand side,

$$\frac{dv_{y,h}}{dt} + \gamma v_{y,h} = 0. \quad (12)$$

Homogenous solutions will have arbitrary constants.

The particular solution will solve the same equation as the original general equation

$$\frac{dv_{y,p}}{dt} + \gamma v_{y,p} = -g. \quad (13)$$

However, we don’t need find one with arbitrary constants. Hence, it is called a **particular** solution.

The sum of the two,

$$v_y = v_{y,p} + v_{y,h}, \quad (14)$$

is a solution of the total equation because of the linear nature of the differential equation. One has now found a *general* solution encompassing all solutions, because it both satisfies the general equation (like the particular solution), and has an arbitrary constant that can be adjusted to fit any initial condition (like the homogenous solution). If the equation were not linear, e.g if there were a term such as v_y^2 or $v_y \dot{v}_y$, this technique would not work.

More details

Returning to the example above, the homogenous solution is the same as that for v_x , because there was no gravitational acceleration in that case,

$$v_{y,h} = Be^{-\gamma t}. \quad (15)$$

In this case a particular solution is one with constant velocity,

$$v_{y,p} = -g/\gamma. \quad (16)$$

Note that this is the terminal velocity of a particle falling from a great height. The general solution is thus,

$$v_y = Be^{-\gamma t} - g/\gamma, \quad (17)$$

and one can find B from the initial velocity,

$$v_{0y} = B - g/\gamma, \quad B = v_{0y} + g/\gamma. \quad (18)$$

Plugging in the expression for B gives the y motion given the initial velocity,

$$v_y = (v_{0y} + g/\gamma)e^{-\gamma t} - g/\gamma. \quad (19)$$

It is easy to see that this solution has $v_y = v_{0y}$ when $t = 0$ and $v_y = -g/\gamma$ when $t \rightarrow \infty$.

One can also integrate the two equations to find the coordinates x and y as functions of t ,

$$\begin{aligned} x &= \int_0^t dt' v_{0x}(t') = \frac{v_{0x}}{\gamma} (1 - e^{-\gamma t}), \\ y &= \int_0^t dt' v_{0y}(t') = -\frac{gt}{\gamma} + \frac{v_{0y} + g/\gamma}{\gamma} (1 - e^{-\gamma t}). \end{aligned} \quad (20)$$

If the question was to find the position at a time t , we would be finished. However, the more common goal in a projectile equation problem is to find the range, i.e. the distance x at which y returns to zero. For the case without a drag force this was much simpler. The solution for the y coordinate would have been $y = v_{0y}t - gt^2/2$. One would solve for t to make $y = 0$, which would be $t = 2v_{0y}/g$, then plug that value for t into $x = v_{0x}t$ to find $x = 2v_{0x}v_{0y}/g = v_0 \sin(2\theta_0)/g$. One follows the same steps here, except that the expression for $y(t)$ is more complicated. Searching for the time where $y = 0$, and we get

$$0 = -\frac{gt}{\gamma} + \frac{v_{0y} + g/\gamma}{\gamma} (1 - e^{-\gamma t}). \quad (21)$$

This cannot be inverted into a simple expression $t = \dots$. Such expressions are known as “transcendental equations”, and are not the rare instance, but are the norm. In the days before computers, one might plot the right-hand side of the above graphically as a function of time, then find the point where it crosses zero.

Now, the most common way to solve for an equation of the above type would be to apply Newton’s method numerically. This involves the following algorithm for finding solutions of some equation $F(t) = 0$.

1. First guess a value for the time, t_{guess} .
2. Calculate F and its derivative, $F(t_{\text{guess}})$ and $F'(t_{\text{guess}})$.
3. Unless you guessed perfectly, $F \neq 0$, and assuming that $\Delta F \approx F' \Delta t$, one would choose
4. $\Delta t = -F(t_{\text{guess}})/F'(t_{\text{guess}})$.
5. Now repeat step 1, but with $t_{\text{guess}} \rightarrow t_{\text{guess}} + \Delta t$.

If the $F(t)$ were perfectly linear in t , one would find t in one step. Instead, one typically finds a value of t that is closer to the final answer than t_{guess} . One breaks the loop once one finds F within some acceptable tolerance of zero. A program to do this might look like:

Motion in a Magnetic Field

Another example of a velocity-dependent force is magnetism,

$$\begin{aligned}\mathbf{F} &= q\mathbf{v} \times \mathbf{B}, \\ F_i &= q \sum_{jk} \epsilon_{ijk} v_j B_k.\end{aligned}\tag{22}$$

For a uniform field in the z direction $\mathbf{B} = B\hat{z}$, the force can only have x and y components,

$$\begin{aligned}F_x &= qBv_y \\ F_y &= -qBv_x.\end{aligned}\tag{23}$$

The differential equations are

$$\begin{aligned}\dot{v}_x &= \omega_c v_y, & \omega_c &\equiv qB/m \\ \dot{v}_y &= -\omega_c v_x.\end{aligned}\tag{24}$$

One can solve the equations by taking time derivatives of either equation, then substituting into the other equation,

$$\begin{aligned}\ddot{v}_x &= \omega_c \dot{v}_y = -\omega_c^2 v_x, \\ \ddot{v}_y &= -\omega_c \dot{v}_x = -\omega_c^2 v_y.\end{aligned}\tag{25}$$

The solution to these equations can be seen by inspection,

$$\begin{aligned}v_x &= A \sin(\omega_c t + \phi), \\ v_y &= A \cos(\omega_c t + \phi).\end{aligned}\tag{26}$$

One can integrate the equations to find the positions as a function of time,

$$\begin{aligned}x - x_0 &= \int_{x_0}^x dx = \int_0^t dt v(t) \\ &= \frac{-A}{\omega_c} \cos(\omega_c t + \phi), \\ y - y_0 &= \frac{A}{\omega_c} \sin(\omega_c t + \phi).\end{aligned}\tag{27}$$

The trajectory is a circle centered at x_0, y_0 with amplitude A rotating in the clockwise direction.

The equations of motion for the z motion are

$$\dot{v}_z = 0, \quad (28)$$

which leads to

$$z - z_0 = V_z t. \quad (29)$$

Added onto the circle, the motion is helical.

Note that the kinetic energy,

$$T = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}m(\omega_c^2 A^2 + V_z^2), \quad (30)$$

is constant. This is because the force is perpendicular to the velocity, so that in any differential time element dt the work done on the particle $\mathbf{F} \cdot d\mathbf{r} = dt \mathbf{F} \cdot \mathbf{v} = 0$.

One should think about the implications of a velocity dependent force. Suppose one had a constant magnetic field in deep space. If a particle came through with velocity v_0 , it would undergo cyclotron motion with radius $R = v_0/\omega_c$. However, if it were still its motion would remain fixed. Now, suppose an observer looked at the particle in one reference frame where the particle was moving, then changed their velocity so that the particle's velocity appeared to be zero. The motion would change from circular to fixed. Is this possible?

The solution to the puzzle above relies on understanding relativity. Imagine that the first observer believes $\mathbf{B} \neq 0$ and that the electric field $\mathbf{E} = 0$. If the observer then changes reference frames by accelerating to a velocity \mathbf{v} , in the new frame \mathbf{B} and \mathbf{E} both change. If the observer moved to the frame where the charge, originally moving with a small velocity v , is now at rest, the new electric field is indeed $\mathbf{v} \times \mathbf{B}$, which then leads to the same acceleration as one had before. If the velocity is not small compared to the speed of light, additional γ factors come into play, $\gamma = 1/\sqrt{1 - (v/c)^2}$. Relativistic motion will not be considered in this course.