Lagrangian formalism:

generalized coordinates

$\mathcal{L}(q, \dot{q}, t)$ $\qquad [\, \mathcal{L}(x, v, t) \,]$

action $S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt$

many generalized coordinates

$\mathcal{L}(q, \dot{q}, t) \to \mathcal{L}(q_1, q_2, \ldots q_s, \dot{q}_1 \ldots \dot{q}_s, t)$

s = # degrees of freedom

Example in polar coordinates $\quad q_1 = r\,, \quad \dot{q}_1 = \dot{r}$

$q_2 = \phi\,, \quad \dot{q}_2 = \dot{\phi}$

<u>Euler-Lagrange equations</u>

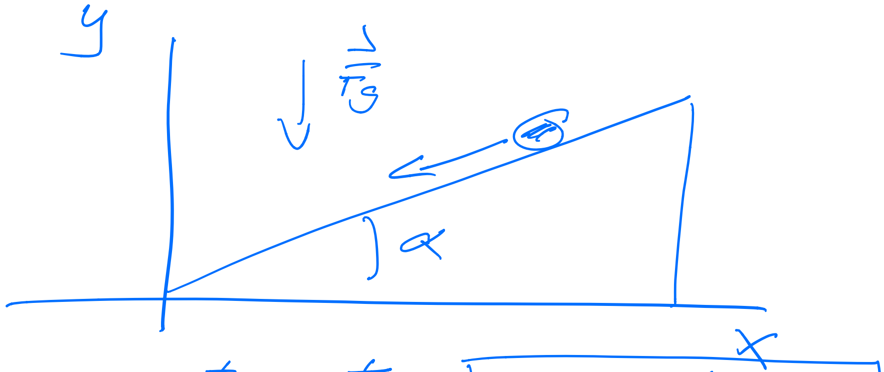
$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$$

$$\mathcal{L} = K - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2) - \alpha/r$$

$$\Rightarrow \boxed{\begin{aligned} m\ddot{r} &= -\frac{\alpha}{r^2} - mr\dot{\phi}^2 \\ mr\ddot{\phi} &= 0 \end{aligned}}$$

what about constraints?

Example: Motion on incline



constraint $\boxed{y = x \cdot \tan\alpha}$

$$\underline{\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy}$$

y is not independent of x

How do we solve the equations of motion with a constraint?

Lagrange multipliers

$$\mathcal{L} + \lambda(y - x\tan\alpha)$$

$$\boxed{\mathcal{L} + \lambda\, g(x,y)} \quad ; \quad g(x,y) = y - x\tan\alpha = 0$$

Holonomic constraint.

$$\left[\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right]\left[\mathcal{L} + \lambda(y - x\tan\alpha)\right] = 0$$

$$\left[\frac{\partial}{\ } - \frac{d}{\ }\frac{\partial}{\ }\right]\left[\mathcal{L} + \lambda(y - x\tan\alpha)\right] = 0$$

$$\left[\frac{\partial}{\partial y} - \frac{d}{dt}\frac{\partial}{\partial \dot y}\right]\left[\ldots\right] = 0$$

$$\times \quad m\ddot{x} - \lambda \tan\alpha = 0 \qquad \leftarrow \tan\alpha$$

$$\ddot{x}\tan\alpha = \ddot{y}$$

$$\times \quad m\ddot{y} - m\cdot g + \lambda = 0$$

subtract

$$- m\ddot{y} - \lambda \tan^2\alpha = 0$$

$$-\lambda\tan^2\alpha - \lambda + mg = 0 \quad / \cos^2\alpha$$

$$-\lambda\sin^2\alpha - \lambda\cos^2\alpha + mg\cdot\cos^2\alpha$$

$$\lambda = m\cdot g\cos^2\alpha$$

$$\ddot{x} = -g\sin\alpha\cdot\cos\alpha$$

$$\ddot{y} = -g\sin^2\alpha$$

$$x(t) = x_0 + \dot{x}_0 - \tfrac{1}{2} g t^2 \sin\alpha\cos\alpha \qquad (t_0 = 0)$$

$$y(t) = y_0 + \dot{y}_0 - \tfrac{1}{2} g t^2 \sin^2\alpha$$

What is the mathematics behind the Lagrangian multiplier?

Want to minimize (or maximize) $f(x_1, x_2)$ (min at $(\tilde{x}_1, \tilde{x}_2)$) subject to the constraint $g(\tilde{x}_1, \tilde{x}_2) = 0$

Minimize $f(x_1 x_2) = \underline{-3x_1^2 - 6x_1 x_2}$

$\underline{-5x_2^2 + 7x_1 + 5x_2}$

Subject to $\underline{x_1 + x_2 = 5}$

$g(x_1, x_2) = 0 = x_1 + x_2 - 5$

$f(x_1 x_2) \wedge g(x_1 x_2) = 0$

$g(\tilde{x}_1 + dx_1, \tilde{x}_2 + dx_2) = 0$

Taylor expand

$$g(\tilde{x}_1 + dx_1, \tilde{x}_2 + dx_2) = g(\tilde{x}_1, \tilde{x}_2) + \frac{\partial g}{\partial x_1}\Big|_{\tilde{x}_1 \tilde{x}_2} dx_1 + \frac{\partial g}{\partial x_2}\Big|_{\tilde{x}_1 \tilde{x}_2} dx_2 = 0$$

with no constraint

[1] $$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} \underline{dx_2} = 0$$

$$dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 = 0$$

at $(\tilde{x}_1 \tilde{x}_2)$

$$\frac{\partial g}{\partial x_2} \neq 0$$

$$\underline{dx_2} = -\frac{\partial g / \partial x_1}{\partial g / \partial x_2} dx_1$$

arbitrary variation.

substitute dx_2 in [1]

[2] $$df = \left[\frac{\partial f}{\partial x_1} - \frac{\partial g/\partial x_1}{\partial g/\partial x_2}\frac{\partial f}{\partial x_2}\right]_{\tilde{x}_1 \tilde{x}_2} dx_1$$

$= 0$

constrained equation for f. Valid for all dx_1

Def λ (Lagrange multiplier)

$$\lambda = \left[\frac{\partial f/\partial x_2}{\partial g/\partial x_2}\right]_{(\tilde{x}_1 \tilde{x}_2)}$$

plug into [2]

$$\left[\frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1}\right]_{(\tilde{x}_1 \tilde{x}_2)} = 0$$

$$g(\tilde{x}_1, \tilde{x}_2) = 0$$

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1 x_2) + \lambda g(x_1 x_2)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g}{\partial x_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g}{\partial x_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = g(x_1 x_2) = 0 = x_1 + x_2 - 5$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -6x_1 - 6x_2 + 7 + \underline{\lambda_1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -6x_1 - 10x_2 + 5 + \lambda_1 = 0$$

$$\lambda_1 = 23 \qquad x_1 = 11/2 \qquad x_2 = -1/2$$

Euler - Lagrange - equations

$$\left[\frac{\partial}{\partial q} - \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\right]\left[\mathcal{L}(q, \dot{q}, t) + \sum_{i=1}^{d} \lambda_i \, g_i(q, \dot{q}, t)\right]$$

generalize to $q \to q_j(t)$

$\dot{q} \to \dot{q}_j(t)$

Pendulum