

PHY321: Classical Mechanics 1

Homework 5, due Monday February 22

Jan 9, 2021

Practicalities about homeworks and projects.

1. You can work in groups (optimal groups are often 2-3 people) or by yourself. If you work as a group you can hand in one answer only if you wish. **Remember to write your name(s)!**
2. Homeworks are available ten days before the deadline.
3. How do I(we) hand in? You can hand in the paper and pencil exercises as a scanned document. For this homework this applies to exercises 1-5. Alternatively, you can hand in everything (if you are ok with typing mathematical formulae using say Latex) as a jupyter notebook at D2L. The numerical exercise(s) (exercise 6 here) should always be handed in as a jupyter notebook by the deadline at D2L.

Introduction to homework 5. This week's sets of classical pen and paper and computational exercises are a continuation of the topics from the previous homework set. We keep dealing with simple motion problems and conservation laws; energy, momentum and angular momentum. These conservation laws are central in Physics and understanding them properly lays the foundation for understanding and analyzing more complicated physics problems. The relevant reading background is

1. chapters 3 and 4 of Taylor (there are many good examples there) and
2. chapters 10-14 of Malthe-Sørenssen.

In both textbooks there are many nice worked out examples. Malthe-Sørenssen's text contains also several coding examples you may find useful.

The numerical homework focuses on another motion problem where you can use the code you developed in homework 4, almost entirely. Please take a look at the posted solution (jupyter-notebook) for homework 4. You need only to change the forces at play. The problem at hand is again a classic one, a block fastened to a spring moving back and forth along the x -axis. It is a simpler problem compared to the previous homework. It allows us however to introduce damping (friction) and study more realistic situations.

Exercise 1 (15 pt), Work-energy theorem and conservation laws. This exercise was partly discussed during the lectures. It has not yet been edited in the online notes. We will study a classical electron which moves in the x -direction along a surface. The force from the surface is

$$\mathbf{F}(x) = -F_0 \sin\left(\frac{2\pi x}{b}\right) \mathbf{e}_x.$$

The constant b represents the distance between atoms at the surface of the material, F_0 is a constant and x is the position of the electron.

- 1a (2pt) Is this a conservative force? And if so, what does that imply?
- 1b (4pt) Use the work-energy theorem to find the velocity $v(x)$.
- 1c (4pt) With the above expression for the force, find the potential energy.
- 1d (5pt) Make a plot of the potential energy and discuss the equilibrium points where the force on the electron is zero. Discuss the physical interpretation of stable and unstable equilibrium points. Use energy conservation. Recommended read here is Malthe-Sørensen chapter 11.3.2.

Exercise 2 (15pt), Rocket, Momentum and mass. Taylor exercise 3.11. It is an exercise meant to illustrate momentum conservation or not. This exercise was partly discussed during the lectures, see the notes on [Energy and Momentum etc](#), see the part on [Momentum conservation](#). Taylor's chapter 3.2 covers also this example.

Exercise 3 (10pt), More Rockets. Taylor exercises 3.13 (5pt) and 3.14 (5pt). This is a continuation of the previous exercise and most of the relevant background material can be found in Taylor chapter 3.2.

Exercise 4 (10pt), Center of mass. Taylor exercise 3.20. Here Taylor's chapter 3.3 can be of use. This relation will turn out to be very useful when we discuss systems of many classical particles.

Exercise 5 (10pt), Sliding Block and Spring (please not again), Scaling the Equations and getting started with numerical project. The relevant material with codes etc is covered by the [Lecture Notes on Oscillations](#). Taylor's chapter 5, in particular sections 5.1 and 5.2. 5.4 is relevant for the bonus exercise. We start now with our next numerical application with simple rewrites of our equations. The system we will look at is that of a block fastened to a spring (which in turn is tied to a wall). The force acting on the block from the spring in the x -direction only is

$$m \frac{d^2 x(t)}{dt^2} = F(x) = -k(x - b),$$

where k is a material specific constant and b is the equilibrium position. The block has mass m and t is time. Define the initial time as t_0 . We will for simplicity set the equilibrium position to zero, that is $b = 0$.

- 5a (3pt) Does this force conserve energy? If so, with given initial position and velocity x_0 and v_0 , respectively, find the expression for energy conservation in terms of the potential and kinetic energies.
- 5b (3pt) Define a constant $\omega_0 = \sqrt{k/m}$ and show that you can write the acceleration as $a(t) = -\omega_0^2 x(t)$. What is the dimensionality of ω_0 ? (it is normally called a natural frequency).
- 5c (4pt) Introduce now a dimensionless time $\tau = t\omega_0$. Show that you can rewrite the equation for the acceleration in terms of two first-order differential equations

$$\frac{dv}{d\tau} = -x(\tau),$$

and

$$\frac{dx}{d\tau} = v(\tau).$$

What are the dimensionalities of these two equations?

These are the equations which we will code in the next exercise.

Exercise 6 (40pt), Sliding Block and Spring, now the code. This exercise should be handed in as a jupyter-notebook at D2L. Remember to write your name(s).

Last week we:

1. Studied the Euler-Cromer and Velocity Verlet to find the position and the velocity of Earth orbiting around the sun.
2. We added the Euler-Cromer and Velocity-Verlet methods to our toolbag for solving ordinary differential equations.
3. We compared and analyzed the stability of the computational solutions.

This week we will reuse our code from homework 4 and study a block which slides back and forth without friction in the x -direction. The only thing we need to is to change the forces at play and since we limit motion only to the x -direction we can simplify our code to only that degree of freedom. The advantage with the block sliding back and forth is that we have analytical solutions for the velocity and the position.

- 6a (20pt) Find the analytical solutions for the velocity and position as function of the initial conditions.

- 6b (20 pt) Write then a program which solves the above differential equations for the sliding block using the Euler-Cromer method and the velocity Verlet method. Study the results for the position, velocity and energy conservation as function of time and the step size in time. Compare the numerical results with the analytical ones. Check that your results for position and velocity obey periodicity.

Is energy properly conserved? Discuss your results.

Here follows a simple code example which could be used to study the sliding block.

```
DeltaT = 0.001
#set up arrays
tfinal = 10 # in seconds
n = ceil(tfinal/DeltaT)
# set up arrays for time t, velocity v, and position r
t = np.zeros(n)
v = np.zeros(n)
x = np.zeros(n)
# Example of Initial conditions
x0 = 1.0
v0 = 0.0
r[0] = r0
v[0] = v0
# Start integrating using Euler's method
for i in range(n-1):
    # Set up the acceleration
    # Here you could have defined your own function for this
    a = -x[i]
    # update velocity, time and position using the Euler-Cromer method
    v[i+1] = v[i] + DeltaT*a
    x[i+1] = x[i] + DeltaT*v[i+1]
    t[i+1] = t[i] + DeltaT
```

Exercise 7 (30pt), Bonus exercise. You don't need to do this exercise, but it gives you a bonus score of 30 points. Taylor's section 5.4 and the lecture notes on damped oscillations cover most of the theoretical background.

In this exercise we will add to our program a first derivative of position (the velocity) which is meant to mimick the role of friction. Our differential equation is now

$$m \frac{d^2 x(t)}{dt^2} + \beta \frac{dx}{dt} + kx = ma(t) + \beta v(t) + kx(t) = 0,$$

where k is a material specific constant and β is a constant representing the resistance from the plane on which the block slides. We have thus assumed that friction depends on velocity in a linear way.

We introduce again the dimensionless time $\tau = t\omega_0$ with $\omega_0 = \sqrt{k/(m\omega_0)}$.

- 7a (10pt) Divide by m and show that you can rewrite your equation as two first-order coupled differential equations

$$\frac{dv}{d\tau} = -2v(t)\gamma - x(\tau),$$

and

$$\frac{dx}{d\tau} = v(\tau).$$

What are the dimensionalities of these two equations? We have defined $\gamma = \beta/2m$.

- 7b (20pt) Add the role of friction to your code from exercise 7 and study the cases where (i) $\gamma < 1$ (underdamping), (ii) $\gamma = 1$ (critical damping) and (iii) $\gamma > \omega_0$ (overdamping). Plot your resulting positions as functions of time and discuss the physical meaning of your results. Note that in the definition of γ we have used the dimensionless form of it. If you do not use that, then you would have $\gamma < \omega_0$, $\gamma = \omega_0$ or

$\gamma > \omega_0$, depending on whether you have underdamping, critical damping or overdamping, respectively.