Special analytical solutions of the damped-anharmonic-oscillator equation

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The anharmonic-oscillator equation with dissipation has been widely used in various areas of science, and surprisingly little has been known about its analytical solutions. In this paper, we survey the known classes of solutions and the methods employed to obtain them including the Painlevé Test and its results. We propose several approaches, one of which is based on compatibility with a Bernoulli equation. Another method investigated explores the many solutions that take the form of a rational fraction of linear combinations of exponentials. In addition to elementary function solutions, those involving elliptic functions are examined. The way the restriction on the parameters of the equation may be relaxed somewhat is demonstrated, leading to an extension of elliptic-type solutions.

I. INTRODUCTION

The anharmonic-oscillator equation (AOE) [1]

$$\ddot{u}+\gamma\dot{u}=Au+Bu^2+Cu^3 , \tag{1.1}$$

where the dots denote differentiation with respect to time, arises in many situations in physics apart from the obvious application to a classical anharmonic oscillator with a frictional term, proportional to γ. The addition of an additional forcing term [2] to Eq. (1.1) results in the Duffing oscillator equation with its fascinating chaotic properties [3]. A particularly important application of the AOE occurs in the study of the kinetics of phase transitions [4]. In the case of a nonconserved order parameter η, the equation describing the most probable evolution of the system towards equilibrium is the celebrated time-dependent Landau-Ginzburg equation (TDLGE)

$$\frac{1}{\Gamma}\frac{\partial n}{\partial t}=D\nabla^2\eta+P(\eta) , \tag{1.2}$$

where $P(\eta)$ is a polynomial whose form and degree depend on the particular application, i.e, the type of transition and symmetry requirements. Examples of this type of modeling arise in liquid crystals [5], ferroelectrics [6], structural phase transitions [7] and uniaxial ferromagnets [8], to mention but a few. In the kinetics of chemical reactions, a similar type of equation is referred to as the reaction-diffusion equation, but it almost always requires a multicomponent order parameter [9] in order to describe the reactants present. Numerous other applications to a broad range of areas in science and engineering have been reviewed elsewhere [10]. Obviously, only a special type of reduction (for example, for traveling waves) is capable of bringing the partial differential equation (PDE) in (1.2) to an ordinary differential equation (ODE) form exemplified by (1.1). Following Skierski, Grundland, and Tuszyński [11], who published a systematic study of symmetry reductions for (1.2), it was found that the latter type of reductions fall into two different categories, i.e., first, traveling plane-wave solutions where the symmetry variable is $\xi=x+vt$, and second, spiral patterns of the order parameter for which $\xi=\phi-b\ln r$, where ϕ is an azimuthal angle and r is the radius in the x-y plane. In this method, the symmetry variable ξ is found as a new independent coordinate such that $\eta=\rho(x,y,z)W(\xi)$ and $W(\xi)$ satisfies an ODE of the AOE type.

It is perhaps more surprising to see reductions to a one-dimensional damped anharmonic oscillator form taking place for nonlinear wave equations. An example of this phenomenon is the nonlinear Klein-Gordon equation (NLKGE)

$$\frac{\partial^2\eta}{\partial t^2}=D\nabla^2\eta+P(\eta) . \tag{1.3}$$

This equation is derived for nondissipative critical systems in 3+1-dimensional space time. Again, reductions involving spiral waves and also some other more complicated surfaces of constant amplitude for the order parameter lead to an envelope equation of type (1.1). A very detailed account of the symmetry reduction analysis for NLKGE can be found in Ref. [12].

Finally, there is another important reason for studying the AOE. In a recent paper by two of the present authors, a connection was provided between the AOE and the generalized Emden equation [13]

$$\frac{d^2\eta}{d\xi^2}+\frac{k}{\xi}\frac{d\eta}{d\xi}=P(\eta) . \tag{1.4}$$

In the simplest case, when the standard type of Emden equation is considered with $P(\eta)=\eta^n$, with n an integer, it is well known that this type of connection can be simply found [14] by putting

$$\eta=\xi^sW(-\ln\xi) . \tag{1.5}$$

This yields

$$W''+W'(1-2s-k)+W[s(s-1)+ks]+W^n=0 , \quad (1.6)$$

provided s is chosen so that $s(1-n)=2$, which is obviously a form of the AOE. Henceforth W' will denote differentiation with respect to ξ.

In the next section we will briefly survey the elementary solutions of the AOE that are known to exist from the literature. More complicated types of solutions will be discussed in later sections together with a number of new approaches where we provide the form of solution.

II. SURVEY OF ELEMENTARY FUNCTION SOLUTIONS

In the past a number of papers were published where special solutions of Eq. (1.1) were found. In this secton we shall discuss these special forms of the AOE equation that allow exact analytical solutions in terms of elementary functions, and we shall also briefly comment on the physical contexts in which they arose. It should be noted that very frequently the solutions found were for a PDE whose special reduction results in an ODE in the form of (1.1). For the sake of consistency in our paper we will appropriately relabel the variables and parameters of the original articles.

The simplest solutions of Eq. (1.1) are obviously constant solutions of $P(W)=0$, or specifically

$$W(A+BW+CW^2)=0 , \quad (2.1)$$

which often play the role of attractors for inhomogeneous solutions of the equation

$$W''+\gamma W'=AW+BW^2+CW^3 . \quad (2.2)$$

In physical language these constant solutions, which are attractors, act as stable phase equilibria, and for particular values of A, B, and C, the system may in fact bifurcate to yield new characteristic behavior.

Chan [15] considered (2.2) and presented his particular solution as

$$W=W_3/[1+\exp(W_3\sqrt{C/2}\xi)] , \quad (2.3)$$

where W_1, W_2, and W_3 are the roots of the polynomial on the right-hand side of (2.2), namely

$$W_1=0, \quad W_2,W_3=-\frac{1}{2C}[B\mp(B^2-4AC)^{1/2}] . \quad (2.4)$$

The solution in (2.3) represents a kink that interpolates asymptotically between W_1 and W_3, and it exists only provided

$$\gamma=-\tfrac{1}{4}\sqrt{2/C}\,[3(B^2-4AC)^{1/2}-B^2] .$$

The physical application that stimulated Chan's study [15] was concerned with the propagation of interfacial profiles in the kinetics of diffusionless first-order phase transitions.

A particular example of this type of modeling was given by Metiu [16] in connection with liquid vapor interfaces. It involved a slight generalization of Eq. (2.2), namely

$$W''+\gamma W'=a(W-W_1)(W-W_2)(W-W_3) . \quad (2.5)$$

The solution describing the interface profile was first obtained by Montroll [17] and may be written as

$$W=W_1+(W_3-W_1)[1+\exp(\mu\xi)]^{-1} , \quad (2.6)$$

where $\mu=\pm(W_3-W_1)\sqrt{a/2}$. This again is only allowed when the dissipation factor γ is given by

$$\gamma=\pm\sqrt{2a}\,(W_3+W_1-2W_2) .$$

In the special case of the equation

$$W''+\gamma W'=W^3-W , \quad (2.7)$$

it was independently demonstrated [18] that for $\gamma=3/\sqrt{2}$ a kink solution can be found in the form

$$W=\mp\frac{1}{2}\left[1-\tanh\left[\frac{1}{2\sqrt{2}}(\xi-\xi_0)\right]\right] , \quad (2.8)$$

which is the limiting case of a damped elliptic function, which will be discussed later. In fact, $W(\xi)$ in (2.8) had been obtained earlier by Lal [19].

Extending nonlinearity in Eq. (2.2) beyond the cubic term leads to new sets of analytical solutions. In his study of the kinetics of first-order ferroelectric phase transitions, Gordon [20] derived the following kinetic equation:

$$W''+\gamma W'=AW+CW^3+EW^5 , \quad (2.9)$$

and found a particular kinklike solution

$$W=W_1[1+\exp(-\mu\xi)]^{-1/2} , \quad (2.10)$$

where W_1 is the root of the quintic polynomial on the right-hand side of (2.9) that corresponds to the global minimum of the underlying potential energy. The constants A, C, and E in (2.9) are not arbitrary for solution (2.10) to exist but must be related in a special way. Details can be found in Ref. [20]. In addition, the parameter μ is a complicated function of A, C, E, and γ, as shown in Ref. [20].

In the context of structural phase transitions, Parliński and Zieliński [7] arrived at an equation of the type

$$W''+\gamma W'=AW+BW^{n+1}+CW^{2n+1} , \quad (2.11)$$

where n is an integer. The exact kinklike solution

$$W=W_0\{1+\exp[\gamma(\xi-\xi_0)]\}^{-1/n} \quad (2.12)$$

satisfies (2.11) provided the parameters involved satisfy a self-consistency condition. In addition, they found an approximate solution composed of a pair of solutions like (2.12) of the form

$$W\simeq W_0^2\{1+\exp[\gamma(\xi-\xi_0)]\}^{-1/n} \times\{1+\exp[-\gamma(\xi-\xi_1)]\}^{-1/n} . \quad (2.13)$$

This is exact as $\xi\rightarrow\mp\infty$.

Hereman and Takaoka [21] have recently studied a generalization of the Fisher equation that is of fundamental importance in the population dynamics of biological species. For traveling-wave reductions this takes the form

$$W''-\gamma W'=W(W^c-1)\ , \tag{2.14}$$

where c is an arbitrary real number. A kinklike solution was found by these authors and independently by Wang [22] that can be written as

$$W=2^{-2/c}\left\{1-\tanh\left[\left[\frac{c}{2\sqrt{2c+4}}\right](\xi-\xi_0)\right]\right\}^{2/c}, \tag{2.15}$$

provided $\gamma=(c+4)/\sqrt{2c+4}$. It is worth noting that for $c=2$ we retrieve Eq. (2.8) discussed earlier.

Finally, Otwinowski, Paul, and Laidlaw [23] proposed a systematic method of finding localized solutions for a class of AOE's represented by

$$W''+\gamma W'=P(W)\ , \tag{2.16}$$

where $P(W)$ is an arbitrary polynomial in W The method is based on the assumption that such special solutions of (2.16) simultaneously satisfy the first-order equation

$$W'=R(W)\ , \tag{2.17}$$

where $R(W)$ is another polynomial that has to be skillfully constructed so that is satisfies compatibility conditions between (2.17) and (2.16). Three separate cases were calculated explicitly with

(i) $P(W)=A_1+A_2W+A_3W^2+A_4W^3$,

(ii) $P(W)=A_2W+A_4W^3+A_6W^5$,

and

(iii) $P(W)=A_2W+A_3W^2$

where the A_i's ($i=1-6$) are constants to be adjusted. In the first case, i.e., (i), one recovers Eq. (2.8) again, which is shifted by a constant due to the emergence of a new stable equilibrium in the mean-field potential. In case (ii) a solution is found in the form of (2.12) for $n=2$. In the last case it appears that the solution of the resultant Fisher equation takes the form

$$W=-\frac{2A_2}{A_3}\left\{1+\exp\left[\mp\sqrt{-A_2/6}(\xi-\xi_0)\right]\right\}^{-2}, \tag{2.18}$$

with $\gamma=\mp5\sqrt{-A_2/6}$. This agrees exactly with the solution in (2.15) for $c=+1$.

We have applied the above method to the equation

$$W''+W'=W^{n-1}-AW\ . \tag{2.19}$$

When $A=\frac{2}{9}$ and $n=4$ we obtain the kink solution

$$W=\mp\sqrt{A}\,/\{1+\exp[\pm\sqrt{A}\,(\xi-\xi_0)]\}\ . \tag{2.20}$$

On the other hand, when $A=\frac{3}{16}$ and $n=6$ we find

$$W=-\left[\frac{4}{\sqrt{3}}\{1+\exp[\pm\tfrac{1}{2}(\xi-\xi_0)]\}\right]^{-1/2}. \tag{2.21}$$

To summarize, we have presented a number of special types of solution for the AOE. First of all, they all tend to conform to the ratio of exponential structures or the root of such an entity. This particular representation will be exploited as a basis for a systematic search for solutions in a later section. Secondly, there is an obvious relationship between the ability to generate kinklike solutions of the AOE and the condition to satisfy the Painlevé test by the equation. This latter question will be addressed in Sec. V.

Lastly, with respect to the method based on the ansatz (2.17), two general comments should be made. This is indeed a general way of reducing the order of an autonomous equation by putting $W'=Q$ and then $W''=Q(dQ/dW)$, so the only limitation is being able to solve the first-order ODE that results, which is often not an easy task at all. Looked upon in this general way, the form of the nonlinearity $P(W)$ is not restricted to polynomials, and the method can be seen to work for other cases as well (e.g., the damped sine-Gordon equation).

III. RATIO OF LINEAR COMBINATIONS OF EXPONENTIALS

In this section we consider a special functional form that we wish to be a solution for Eq. (2.2) with $\gamma=-1$ involving a number of adjustable constants. By imposing compatibility relations among the constants, we find a particular class of solutions. To this end we suppose a solution takes the form

$$W=[\lambda_1\exp(\alpha\xi)+\lambda_2\exp(-\alpha\xi)+\lambda_3]\times[\lambda_4\exp(\alpha\xi)+\lambda_5\exp(-\alpha\xi)+\lambda_6]^{-1}\ , \tag{3.1}$$

where the λ_i ($i=1-6$) and α are constants to be determined. We note that the six λ_i are not independently capable of varying, as the numerator may be divided through by one λ_i from the denominator, assumed nonzero. It is also clear that if this is to be a solution, namely $W(\xi)$, then so is $W(\xi+\xi_0)$ for ξ_0 an arbitrary constant, since Eq. (2.2) is invariant in form to such a translation. Inserting (3.1) into (2.2), canceling the common $[\lambda_4\exp(\alpha\xi)+\lambda_5\exp(-\alpha\xi)+\lambda_6]^{-3}$, assuming its argument does not vanish, and equating the coefficients of $e^{3\alpha\xi}$, $e^{2\alpha\xi}$, $e^{\alpha\xi}$, a constant, $e^{-\alpha\xi}$, $e^{-2\alpha\xi}$, and $e^{-3\alpha\xi}$ on both sides, respectively, we find the following:

$$e^{+3\alpha\xi}:\ C\lambda_1^3+B\lambda_1^2\lambda_4+A\lambda_1\lambda_4^2=0\ , \tag{3.2}$$

$$e^{+2\alpha\xi}:\ 3C\lambda_1^2\lambda_3+B(\lambda_1^2\lambda_6+2\lambda_1\lambda_3\lambda_4)+A(2\lambda_4\lambda_6\lambda_1+\lambda_3\lambda_4^2)=-\lambda_1\lambda_4\lambda_6\alpha(\alpha+1)+\lambda_3\lambda_4^2\alpha(\alpha+1)\ , \tag{3.3}$$

$$\begin{aligned}e^{+\alpha\xi}:\ &C(3\lambda_3^2\lambda_1+3\lambda_1^2\lambda_2)+B(\lambda_3^2\lambda_4+\lambda_1^2\lambda_5+2\lambda_1\lambda_2\lambda_4+2\lambda_1\lambda_3\lambda_6)+A(\lambda_1\lambda_6^2+\lambda_2\lambda_4^2+2\lambda_4\lambda_5\lambda_1+2\lambda_3\lambda_4\lambda_6)\\&=\lambda_1\lambda_6^2\alpha(\alpha-1)-\lambda_1\lambda_4\lambda_5\alpha(2+4\alpha)+\lambda_2\lambda_4^2\alpha(2+4\alpha)-\lambda_3\lambda_4\lambda_6\alpha(\alpha-1)\ ,\end{aligned} \tag{3.4}$$

$$\text{const: } C(\lambda_3^3+6\lambda_1\lambda_2\lambda_3)+B(2\lambda_2\lambda_3\lambda_4+2\lambda_1\lambda_3\lambda_5+2\lambda_1\lambda_2\lambda_6+\lambda_6\lambda_3^2] + A(2\lambda_1\lambda_5\lambda_6+2\lambda_2\lambda_4\lambda_6+2\lambda_3\lambda_4\lambda_5+\lambda_3\lambda_6^2)=\lambda_1\lambda_5\lambda_6\alpha(3\alpha-3)+\lambda_2\lambda_4\lambda_6\alpha(3\alpha+3)-6\lambda_3\lambda_4\lambda_5\alpha^2 , \tag{3.5}$$

$$e^{-\alpha\xi}\text{: } C(+3\lambda_1\lambda_2^2+3\lambda_2\lambda_3^2)+B(\lambda_2^2\lambda_4+\lambda_3^2\lambda_5+2\lambda_1\lambda_2\lambda_5+2\lambda_2\lambda_3\lambda_6)+A(\lambda_1\lambda_5^2+\lambda_2\lambda_6^2+2\lambda_2\lambda_4\lambda_5+2\lambda_3\lambda_5\lambda_6) =\lambda_1\lambda_5^2\alpha(-2+4\alpha)+\lambda_2\lambda_6^2\alpha(\alpha+1)+\lambda_2\lambda_4\lambda_5\alpha(2-4\alpha)-\lambda_3\lambda_5\lambda_6\alpha(\alpha+1) , \tag{3.6}$$

$$e^{-2\alpha\xi}\text{: } 3C\lambda_2^2\lambda_3+B(\lambda_2^2\lambda_6+2\lambda_2\lambda_3\lambda_5)+A(2\lambda_2\lambda_5\lambda_6+\lambda_3\lambda_5^2)=-\lambda_2\lambda_5\lambda_6\alpha(\alpha-1)+\lambda_3\lambda_5^2\alpha(\alpha-1) , \tag{3.7}$$

$$e^{-3\alpha\xi}\text{: } C\lambda_2^3+B\lambda_2^2\lambda_5+A\lambda_5^2\lambda_2=0 . \tag{3.8}$$

Equations (3.2)–(3.8) appear to be seven equations in, effectively, six unknowns (five from the λ_i's and α). However, if we assume that

$$\lambda_4/\lambda_1=\lambda_5/\lambda_2 , \tag{3.9}$$

or its equivalent, no λ_i $(i=1,2,4,5)$ vanishing, then because of the symmetries between (3.8) and (3.2), this pair of equations become equivalent. That is, (3.8) may be written, using (3.9), as

$$\left[\frac{\lambda_5}{\lambda_4}\right]^3(C\lambda_1^3+B\lambda_1^2\lambda_4+A\lambda_1\lambda_4^2)=0 . \tag{3.10}$$

Similarly, (3.7) becomes, on substituting λ_2 from (3.9),

$$\left[\frac{\lambda_5}{\lambda_4}\right]^2=\left[\frac{\lambda_5}{\lambda_4}\right]^2[-\lambda_1\lambda_4\lambda_6\alpha(\alpha-1)+\lambda_3\lambda_4^2\alpha(\alpha-1)] , \tag{3.11}$$

where L represents the left-hand side of Eq. (3.3). The entity in the square brackets on the right-hand side of (3.11) is *not* the right-hand side of (3.3). If we subtract $(\lambda_5/\lambda_4)^2$ multiplied by Eqs. (3.3), assuming $\lambda_5/\lambda_4\neq0$ and $\alpha\neq0$, we find that

$$\lambda_1\lambda_6=\lambda_3\lambda_4 , \tag{3.12}$$

i.e., (3.12) is deduced from the seven equations using only (3.9) and results in Eqs. (3.7) and (3.3) becoming linear combinations of one another. We also find that

$$\mathcal{L}=\left[\frac{\lambda_2}{\lambda_1}\right]\mathcal{L}'=\left[\frac{\lambda_2}{\lambda_1}\right]R =\lambda_2\lambda_6^2\alpha(\alpha-1)-\lambda_3\lambda_6\lambda_5\alpha(\alpha-1) , \tag{3.13}$$

where $\mathcal{L}$ represents the left-hand side of Eq. (3.6), $\mathcal{L}'$ represents the left-hand side of Eq. (3.4), and R represents the right-hand side of Eq. (3.4). The function on the far right of (3.13) is *not* the right-hand side of (3.6), but if we subtract (3.13) from it, we obtain again, using (3.12), an identity because of (3.9). Making the assumption in (3.9) has the consequence described by the relation in (3.12). These two conditions imply that Eqs. (3.2), (3.3), (3.7), and (3.8) are all multiples of one equation. Hence, in this case, there are only three independent equations and five unknowns, in which case it would appear we have a general solution *provided* $\lambda_4/\lambda_1=\lambda_5/\lambda_2$.

We should perhaps also comment on another important observation, namely that if the numerator of (3.1) is a constant, i.e., $\lambda_1=\lambda_2=0$, then Eqs. (3.3) and (3.7) reduce to

$$A\lambda_3\lambda_4^2=\lambda_3\lambda_4^2\alpha(\alpha+1) \tag{3.14}$$

and

$$A\lambda_3\lambda_5^2=\lambda_3\lambda_5^2\alpha(\alpha-1) . \tag{3.15}$$

However, under our assumption that the numerator of (3.1) is a constant, $\lambda_3\neq0$, otherwise $W=0$. If both λ_4 and λ_5 are nonzero, then the only solution is $\alpha=0$, leading to a constant value for W. When $\lambda_4=0$ and $\lambda_5\neq0$, then a nonzero value of α is allowed. Similarly, when $\lambda_5=0$ but $\lambda_4\neq0$ we can again have $\alpha\neq0$. Thus, if the numerator of (3.1) is a constant, only one exponential is allowed with a nonzero α in the denominator, leading to a kink solution described by a Fermi type of function.

IV. REDUCTION OF THE AOE TO A BERNOULLI EQUATION

In Sec. II we discussed a method of solving the AOE that relies on an ansatz where compatibility is postulated with a first-order autonomous equation; see Eq. (2.17). In the present section we propose a new method that is somewhat similar in spirit. However, the main difference will be in requiring compatibility with a nonautonomous first-order ODE, namely a Bernoulli equation.

We take as our starting point Eq. (2.2) with $\gamma=-1$, without loss of generality. Writing W as a product,

$$W=uV , \tag{4.1}$$

and choosing V so that no first derivative in u appears we find

$$u''=(\tfrac{1}{4}+A)u+\bar{B}e^{\xi/2}u^2+\bar{C}e^{\xi}u^3 , \tag{4.2}$$

where $\bar{C}=Ce^{2\kappa}$ and $\bar{B}=Be^{\kappa}$, κ being an arbitrary integration constant.

The next transformation we perform on (4.2) is to write u as a function of e^{ξ}, i.e.,

$$u=G(e^{\xi}) . \tag{4.3}$$

In terms of the new independent variables $s=e^{\xi}$ we find

$$sG''+G'=(\tfrac{1}{4}+A)Gs^{-1}+\bar{B}s^{-1/2}G^2+\bar{C}G^3 , \tag{4.4}$$

where in (4.4) the primes denote differentiation with respect to s. By choosing to modify the independent variable s so that

$$\beta^2 s = y^2 , \tag{4.5}$$

where β is an adjustable constant, we rid (4.4) of square roots to give

$$\beta^2 y^2 \frac{d^2G}{dy^2} + \beta^2 y \frac{dG}{dy} = \beta^2(1+4A)G + \tilde{B}\beta y G^2 + \tilde{C} y^2 G^3, \tag{4.6}$$

where $\tilde{B}=4\bar{B}$ and $\tilde{C}=4\bar{C}$. This nonautonomous nonlinear equation may appear, at first sight, to be worse than our starting point (2.2), but the simple "Bernoulli ansatz"

$$\frac{dG}{dy} = \mu G^3 + \lambda G y^{-1} \tag{4.7}$$

may be substituted into (4.6). Compatibility conditions for the constants μ, λ, and β may then be found as

$$2\beta^2\mu^2 = \tilde{C} , \tag{4.8}$$

$$\beta^2\lambda^2 = (1+4A)\beta^2 , \tag{4.9}$$

and

$$3\beta^2\lambda\mu + \mu\beta^2 = \beta\tilde{B} . \tag{4.10}$$

In view of the fact that $\tilde{C}$ and $\tilde{B}$ contain the arbitrary integration constant κ, one might be tempted to think that (4.8), (4.9), and (4.10) are three equations in the three unknowns that may be solved uniquely for λ, μ, and κ (or β). However, from (4.9), provided $\beta \neq 0$,

$$\lambda = \mp\sqrt{1+4A} , \tag{4.11}$$

and in (4.8),

$$\mu = \mp\sqrt{2\tilde{C}}/\beta = \mp\sqrt{2C}e^{\kappa}/\beta , \tag{4.12}$$

so that when (4.12) and (4.11) are inserted into (4.10) we find

$$4B = \epsilon_1\sqrt{2C}(3\epsilon_2\sqrt{1+4A}+1) , \tag{4.13}$$

where ϵ_1, $\epsilon_2 = \mp 1$, i.e., a relationship among the fixed constants A, B, and C with β and the remaining arbitrary constant absent. Thus the postulated Bernoulli ansatz is expected to work only when Eq. (4.13) is satisfied by the parameters A, B, and C of the AOE equation. Assuming that this is the case we may solve the first-order Bernoulli equation (4.7), which yields

$$G = (\lambda+1)y^{\lambda}/[(\lambda+1)\kappa_1 - \mu y^{\lambda+1}] \tag{4.14}$$

for $\lambda \neq -1$ and

$$G = (\kappa_1 y - \mu y \ln y)^{-1} \tag{4.15}$$

for $\lambda = -1$. These two equations may be transformed back to give the solution W in (2.2) as

$$W = \frac{(\lambda+1)[\epsilon_2\beta\exp(\xi/2)]^{\lambda}\exp[(\xi/2)+\kappa]}{(\lambda+1)\kappa_1 - \mu[\epsilon_2\beta\exp(\xi/2)]^{\lambda+1}} \tag{4.16}$$

for $\lambda \neq -1$ and

$$W = e^{\kappa}/[(-\mu\epsilon_2\beta\ln|\beta| + \kappa_2\epsilon_2\beta) - (\mu\epsilon_2\beta/2)\xi] \tag{4.17}$$

for $\lambda = -1$, where κ_1 and κ_2 in the above equations are arbitrary constants. The first of the two solutions obtained here is again in the rational exponential form discussed at length in the previous section. It is worth noting that the first solution in (4.16) is, in general, nonsingular, and interpolates between zero and nonzero asymptotic limits. The other solutions, on the other hand, has a simple pole. Finally, as (2.2) is autonomous, we may put $W' = Q^{-1}$ and obtain a standard Abel equation of the form

$$Q' = -Q^2 - P(W)Q^3 , \tag{4.18}$$

but this is nevertheless possibly more difficult to solve.

V. PAINLEVÉ CASES AND THEIR SOLUTIONS

It has been stated [24] that an ODE, of second order, has the Painlevé property if its solutions have no movable critical points, i.e., branch points or essential singularities whose positions are determined by integration constants, excluding poles. The importance of this classification is to the so-called Painlevé conjecture that equations passing this test are integrable and thus have solutions in the form of elementary functions, elliptic functions, or the six Painlevé transcendents. The procedure to apply the test to an arbitrary second-order ODE is algorithmic, and for most of the cases one can, in fact, make use of standard sources in the literature for verification [25]. For example, according to Murphy [14] the class of equations in which Eq. (2.2) falls has the form

$$W''(\xi) = (f_0 W + f_1)W' + g_0 W^3 + g_1 W^2 + g_2 W + g_3 , \tag{5.1}$$

where f_i's and g_i's are constants or functions of ξ. If the general solution of (5.1) is to be free of movable critical points, then f_0 and g_0 are restricted to five pairs of constants tabulated by Murphy [14]. To make the connection with Eq. (2.2) we clearly must have $f_0=0$, which reduces the cases to $f_0=0$, $g_0=0$ or $f_0=0$, $g_0=2$. However, if $g_0=0$, no cubic term appears on the right-hand side of (5.1), or in (2.2) $C=0$, but this appears to be unphysical due to the fact that it will lead to a structurally unstable cubic potential in the effective Hamiltonian. The solutions for this case may be found [14,26], and lead to elliptic functions for $b=0$ below or Painlevé transcendents otherwise, the standard equation reducing to the form

$$W''(\xi) = a + b\xi + cW^2 . \tag{5.2}$$

When $f_0=0$ and $g_0=2$, $f_1 \neq 0$, the only possibility that satisfies the Painlevé test must have $g_1=g_3=0$ [in our equation (2.2) the quadratic nonlinearity is missing $B=0$] and

$$f_1 = -3q \quad \text{with } g_2 = -(2q^2+q') , \tag{5.3}$$

where q is an arbitrary function of ξ.

Thus, the standard equation becomes

$$W''(\xi) + 3q(\xi)W' = 2W^3 - (2q^2+q')W . \tag{5.4}$$

By making the substitutions [14]

$$W=u(s)V(\xi)\ ,$$
$$\ln V=-\int^{\xi}q\,d\xi\ , \tag{5.5}$$
$$s=\int^{\xi}V\,d\xi\ ,$$

Eq. (5.4) is converted to the standard form

$$u''(s)=2u^3\ , \tag{5.6}$$

whose first integrâl is

$$(u')^2=u^4+c_0\ , \tag{5.7}$$

where c_0 is an integration constant. Real solutions of (5.7) exist only for $c_0<0$ and are given by [8(b)]

$$u=\mp\alpha\left[\mathrm{cn}\left[\sqrt{2}\alpha(s-s_0),\frac{1}{\sqrt{2}}\right]\right]^{-1}\ , \tag{5.8}$$

where $\alpha=(|c_0|)^{1/4}$ and s_0 is an arbitrary constant. These are obviously periodic divergent solutions. The only nonsingular solution of Eq. (5.7) is α itself.

It is well known that for a special form of (2.2), the equation satisfies the Painlevé criterion, namely, when $B=0$, $C=\mp 1$ and $A=-\frac{2}{9}$ [8(b)]. The general class of nonsingular solutions take the form

$$W(\xi)=\mp\tfrac{1}{2}c_1\exp(+\tfrac{1}{3}\xi)\times sd\left[\sqrt{2}[c_1\exp(+\tfrac{1}{3}\xi)+c_2],\frac{1}{\sqrt{2}}\right]\ , \tag{5.9}$$

where c_1 and c_2 are arbitrary integration constants. A parallel class of singular solutions has been discovered by Cerveró and Estévez [18], and the main difference is in the use of a ds functon as opposed to an sd function in (5.9), where these functions are defined by

$$sd=sn/dn,\quad ds=dn/sn\ , \tag{5.10}$$

where arguments and moduli have been omitted for simplicity. In the case where the nonlinear part of the right-hand side of (2.2) is quintic, the equation is again only integrable for a particular value of the parameter premultiplying the linear term [11].

The method used by Cerveró and Estévez [18] is quite general and can be extended to the class of equations below

$$W''+W'+aW+\epsilon W^{n-1}=0\ . \tag{5.11}$$

The method consists in postulating solutions to be of the form

$$W=\lambda(\sqrt{a}\,\xi)u(s),\quad s=s(\sqrt{a}\,\xi)\ . \tag{5.12}$$

The above equation can be substituted into (5.11) to give

$$[a\lambda(s')^2]u''+(2\lambda' as'+\lambda as''+\lambda\sqrt{a}\,s')u'+(a\lambda''+a\lambda+\sqrt{a}\,\lambda')u+\lambda^{n-1}u^{n-1}=0\ . \tag{5.13}$$

The next step is to divide Eq. (5.13) by $a\lambda(s')^2$ and demand that the coefficients premultiplying u' and u are zero, with the prefactor of u^{n-1} being a constant. This yields the conditions

$$\frac{\lambda'}{\lambda}=\left[\frac{2}{n-2}\right]\frac{s''}{s'}\ , \tag{5.14a}$$

$$\frac{\lambda'}{\lambda}=\frac{-1}{\sqrt{a}}\frac{2}{n+2}\ , \tag{5.14b}$$

$$a=\frac{2n}{(n+2)^2}\ , \tag{5.14c}$$

$$u''+c_1u^{n-1}=0\ , \tag{5.15}$$

where c_1 is an arbitrary constant. Simultaneously one finds that

$$\lambda=c_2\exp\left[-\frac{1}{\sqrt{a}}\frac{2}{(n+2)}\xi\right] \tag{5.16}$$

and

$$s=-\sqrt{a}\left[\frac{n+2}{n-2}\right]\left[\frac{c_2}{c_1}\right]^{(n-2)/2}\exp\left[-\frac{1}{\sqrt{a}}\frac{(n-2)}{(n+2)}\xi\right]\ , \tag{5.17}$$

where c_2 is another arbitrary integration constant.

Note that for each value of n, only one particular choice of a leads to an integrable second-order ODE. This can be independently confirmed using the Panlevé analysis [8(b),12]. In fact, when $n=4$, the corresponding value of $a=\frac{2}{9}$ and when $n=6$, $a=\frac{3}{16}$. Through formulas (5.16) and (5.17) we can readily see that increasing the nonlinearity of our equation, i.e., increasing n, has a twofold effect. First, it increases the amplitude of W through the exponential term in λ. Secondly, it contracts the domain of u through the form of s for positive ξ. The remaining ODE, i.e., Eq. (5.15), is easily integrable once to give

$$(u')^2+\frac{2c_1}{n}u^n=c_3\ , \tag{5.18}$$

where c_3 is another integration constant. This equation possesses nonsingular real solutions only when $c_1>0$ and it is easily integrable when $n=4$ or 6. Using standard tables of elliptic integrals [27], the result in the former case can be found as

$$u(s)=\left[\frac{2c_3}{c_1}\right]^{1/4}cn\left[2(c_1c_3)^{1/4}(s-s_0),\frac{1}{\sqrt{2}}\right]\ , \tag{5.19}$$

where s_0 is an integration constant.

When $n=6$ we first substitute $u=\sqrt{H}$ into (5.15) and then use a standard elliptic integral [27] to find

$$u=\left[\frac{3c_3}{c_1}\right]^{1/6}\left[\frac{cn(z,k)-1}{(1-\sqrt{3})cn(z,k)+(1+\sqrt{3})}\right]^{1/2}\ , \tag{5.20}$$

where

$$z=3^{1/4}\sqrt{4c_3}\left[\frac{c}{3c_3}\right]^{1/6}(s-s_0)\ , \tag{5.21}$$

and the Jacobi modulus k is given by

$$k=\sqrt{2-\sqrt{3}/4}\ . \tag{5.22}$$

In this connection we wish to bring the reader's attention to an earlier paper by Lakshmanan and Kaliappan [28], who considered our equation (5.11) in a slightly rescaled form, namely,

$$\beta^2 W''+\alpha W'+W-W^{n-1}=0\ . \tag{5.23}$$

The method used by these authors is virtually identical, as can be seen by comparing their case for $n=4$ with ours, i.e., Eq. (5.15). However, in their equation (C.10) a misprint has occurred, and the power on the right-hand side should be 3. They have also solved the case for $n=3$, i.e., the special case of the Fisher equation.

VI. LINEAR SUPERPOSITION ANSATZ FOR THE CASE $B\neq 0$ IN EQ. (2.2)

Here we begin our analysis with Eq. (5.4) and substitute

$$W=u+V\ , \tag{6.1}$$

where u and V are functions of ξ, with u having the form

$$u=u_1F(u_2)\ . \tag{6.2}$$

This leads to the following equation for the functon F:

$$\begin{aligned} F''+F'\{[2u_1'u_2'+u_1u_2''+3qu_1u_2')/u_1(u_2')^2]\} \\ =F^3[2u_1^3/u_1(u_2')^2]+F^2[6Vu_1^2/u_1(u_2')^2] \\ +F\{u_1[6V^2-(2q+q')]-u_1''-3qu_1'\}/u_1(u_2')^2 \\ +[2V^3-(2q^2+q')V-V''-3qV']/u_1(u_2')^2\ . \end{aligned} \tag{6.3}$$

In order to make Eq. (6.3) both autonomous and of the same form as (2.2), we must clearly demand

$$2u_1'u_2'+u_1u_2''+3qu_1u_2'=-u_1(u_2')^2\ , \tag{6.4}$$

$$2u_1^2=C(u_2')^2\ , \tag{6.5}$$

$$6Vu_1=B(u_2')^2\ , \tag{6.6}$$

$$u_1[6V^2-(2q^2+q')]-u_1''-3qu_1'=Au_1(u_2')^2\ , \tag{6.7}$$

and

$$2V^3-(2q^2+q')V-V''-3qV'=0\ . \tag{6.8}$$

From (6.5), $1/(u_2')^2=C/(2u_1^2)$, so that in (6.6) we have

$$u_1=(3VC)/B\ . \tag{6.9}$$

Thus, from (6.9) and (6.5), (6.7) becomes

$$V^3\left[6-\frac{18CA}{B^2}\right]-V(2q+q')-V''-3qV'=0\ . \tag{6.10}$$

Equation (6.10) is obviously compatible with (6.8) provided

$$CA/B^2=\tfrac{2}{9}\ . \tag{6.11}$$

The function V may be obtained directly from (6.8), having the same form of soluton as (5.4) but with possibly different arbitrary constants. Hence u_1 is obtained from (6.9) and thence u_2 from (6.5), giving

$$F\left[\mp\int^{\xi}\frac{3C}{B}\sqrt{2/C}\,V d\xi\right]=\frac{B}{3C}[(W/V)-1]\ . \tag{6.12}$$

Having found u_1, u_2, and V, it is easy to demonstrate that the function q required, obtained from (6.4), is given by

$$q=(\pm\sqrt{2C}\,V^2/B-V')/V\quad (V\neq 0)\ . \tag{6.13}$$

To summarize, this approach allows a solution F given by (6.12) of the original AOE in (2.2) with $\gamma=-1$, and A, B, and C linked only through (6.11) with $B\neq 0$. This is achieved by first calculating a form of W that satisfies Eq. (5.4), and then it transpires that the other function required for the calculation of F, namely V, also satisfies a similar type of equation as that of W, i.e., Eq. (6.8). Finally, then, the knowledge of both W and V is sufficient to obtain F using (6.12). We wish to emphasize here that, unlike in many previous instances, the present approach allows B to be nonzero and the "magic" number $\frac{2}{9}$ appears in a different guise.

VII. CONCLUSIONS

This paper has been concerned with finding analytical solutions of the classical anharmonic-oscillator equation with damping. In the Introduction, an overview was given of the various areas of physics where this equation plays an important role. The most notable examples occur in the kinetics of phase transitions. In chemical and biological applications, this type of equaton arises in connection with reaction-diffusion models. We have then surveyed a variety of elementary function solutions to some special cases of the equation, all of which take the form of kinks or bumps, i.e., describe solitary wave propagation. It is also worth noting that the mathematical form exhibited by these types of solutions is a rational function of linear combinations of exponentials. This links with our own method of postulating a new ansatz of this form and showing consistency among the parameters of the solution but not on the fixed constants of the differential equation. A particular class of solutions is found by making the equation compatible with a Bernoulli equation after first transforming the original nonautonomous equation to a particular form. This results in two forms of solution, once again falling into the class of a ratio of linear combinations of exponential functions, and the other a simple pole with a movable singularity. Following earlier attempts elsewhere in the literature, we have examined the general form of the AOE from the point of view of Painlevé analysis. Under a strict set of conditions on the parameters in the original

ODE, the Painlevé test is satisfied, and classes of general solutions are found involving elliptic functions, with exponential amplitudes and arguments. Finally, we have presented a section that is an extension of the previous discussion and makes use of the method for solving the Painlevé cases through a particular type of substitution. This results in a more general set of possibilities.

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