Lagrangian formalism

with constraints

holonomic $\quad g(q_1, q_2 \dots q_5) = 0$

nonholonomic $\quad g(q_1, q_2 \dots q_5) \neq 0$

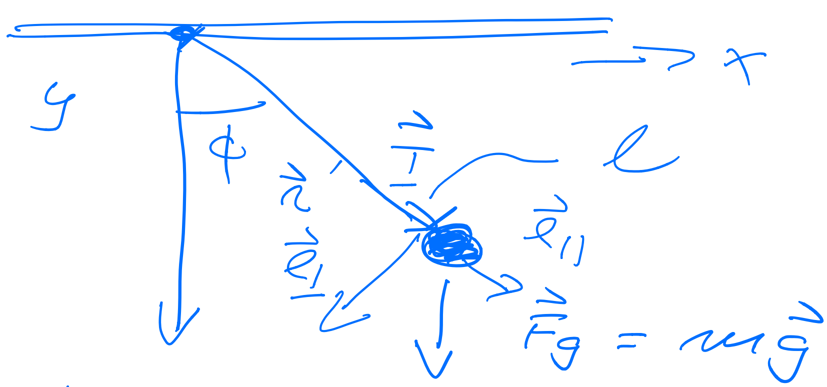
$$S = \int_{t_1}^{t_2} (\mathcal{L} + \lambda g)\, dt$$

$$\mathcal{L}(q, \dot{q}, t) \qquad [\mathcal{L}(q_1, q_2 \dots q_5, \dot{q}_1 \dots \dot{q}_5, t)]$$

$$\left[\frac{\partial}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]\left[\mathcal{L}(q, \dot{q}, t) + \lambda g(q, \dot{q}, t)\right]$$

Example: Mathematical pendulum

standard way:



$$m\ddot{\vec{r}} = \vec{F}_g + \vec{T}$$

$$\vec{r} = l \begin{bmatrix} \sin\phi \\ \cos\phi \end{bmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \end{matrix}$$

$$\dot{\vec{r}} = l\dot{\phi} \begin{bmatrix} \cos\phi \\ \dots \end{bmatrix} \qquad \frac{dl}{dt} = 0$$

$$\ddot{\vec{r}} = -\ell\ddot{\phi}\underbrace{\begin{bmatrix}-\cos\phi\\ \sin\phi\end{bmatrix}}_{\vec{e}_\perp} - \ell\dot{\phi}^2\underbrace{\begin{bmatrix}\sin\phi\\ \cos\phi\end{bmatrix}}_{\vec{e}_\parallel}$$

$$\vec{e}_\perp \, \vec{e}_\parallel = 0$$

obtain two separate equations

$$-m\ell\ddot{\phi} = mg\sin\phi \qquad \left(-\frac{k}{m}x = \ddot{x}\right)$$

$$\ddot{\phi} = -g/\ell \sin\phi \qquad \omega_0^2 = g/\ell$$

small angle approx

$$\ddot{\phi} \simeq -\omega_0^2\phi \qquad \sin\phi \simeq \phi$$

$$\phi = A\cos\omega_0 t + B\sin\omega_0 t$$

$$T = m\ell\dot{\phi}^2 + mg\cos\phi$$

Can we derive this using a Lagrangian + some constraint?

constraint $g(r,\phi) = r-\ell = 0$

$$\lambda g(r) = \lambda(r-\ell) = 0$$

$$\mathcal{L}(r,\dot{r},\phi,\dot{\phi},t) = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\phi}^2\right] - (-mg\underbrace{r\cos\phi}_{y})$$

$$\left[\frac{\partial}{\partial r} - \frac{d}{dt}\frac{\partial}{\partial \dot r}\right]\left[\mathcal{L} + \lambda(r-\ell)\right]$$

$$\frac{\partial(\mathcal{L}+\lambda(r-\ell))}{\partial r} = mr\dot\phi^2 + mg\cos\phi + \lambda$$

$$\frac{\partial}{\partial \dot r}(\mathcal{L}+\lambda(r-\ell)) = m\dot r \qquad \frac{d}{dt}(m\dot r) = m\ddot r$$

$$0 = m\ddot r = mr\dot\phi^2 + mg\cos\phi + \lambda$$

$$\frac{\partial}{\partial\phi}(\mathcal{L}+\lambda(r-\ell)) = -mgr\sin\phi$$

$$\frac{\partial\mathcal{L}}{\partial\dot\phi} = \underline{mr^2\dot\phi} \qquad \frac{d}{dt}(mr^2\dot\phi)$$

$$mr^2\ddot\phi = -mgr\sin\phi$$

$$\ddot\phi = -g/r\,\sin\phi$$

$$\underline{r = \ell \qquad \dot r = 0 \qquad \ddot r} = 0$$

$$\boxed{\ddot\phi = -g/\ell\,\sin\phi}$$

$$\lambda = m\ell\dot\phi^2 + mg\cos\phi$$

classical case

$$T = m\ell\dot\phi^2 + mg\cos\phi$$

λ = constraining force that be applied in order to keep the length of the pendulum constant.
Normally called a Tension force.

Conservation of Energy, linear Momentum and angular momentum

- conservation of Energy

$$\mathcal{L}(q, \dot{q}, t)$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}} = 0$$

isolated system decoupled from the outside.
Assumption: laws of motion are the same at different times.

$$-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial \dot{q}} \underbrace{\frac{d\dot{q}}{dt}}_{\ddot{q}} + \frac{\partial \mathcal{L}}{\partial q} \underbrace{\frac{dq}{dt}}_{\dot{q}} + \frac{\partial \mathcal{L}}{\partial t} \Rightarrow$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial \dot{q}} \ddot{q} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] \dot{q}$$

$$= \frac{d}{dt}\left[\dot{q} \frac{\partial L}{\partial \dot{q}}\right] \Rightarrow$$

$$\frac{d}{dt}\left[\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right] = 0$$

$$E = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$$

$$L = \frac{1}{2} m \dot{q}^2 - V(q)$$

$$E = m\dot{q}^2 - \frac{1}{2} m \dot{q}^2 + V(q)$$

$$= \frac{1}{2} m \dot{q}^2 + V(q)$$

Energy $\Rightarrow$

$$\frac{d}{dt} E = 0 \; ?$$

Generalize to more $q_1, q_2 \ldots q_s$

$$\boxed{E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L}$$

– Linear Momentum conservation

$\delta\vec{r}$ is a small uniform translation,

$$0 = \delta L = \sum_i \frac{\partial L}{\partial \vec{r}_i}\, \delta\vec{r} \implies$$

$$\sum_i \frac{\partial L}{\partial \vec{r}_i} = 0$$