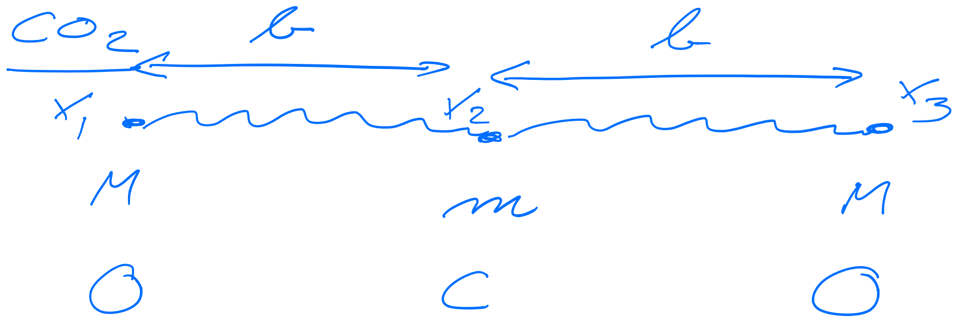
Classical case: linear chain



coordinate x_i $\quad i=1,2,3$

Define equilibrium positions around an equilibrium separation — b —

$$x_2^{(0)} - x_1^{(0)} = x_3^{(0)} - x_2^{(0)} = b$$

Define $u_i = x_i - x_i^{(0)}$ $\quad i=1,2,3$

kinetic energy

$$T = \frac{1}{2}\left[M(\dot{u}_1^2 + \dot{u}_3^2) + m\dot{u}_2^2\right]$$

$$V = \frac{1}{2}k\left[(u_2-u_1)^2 + (u_3-u_2)^2\right]$$

$$\mathcal{L} = T - V$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial u_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{u}_i} = 0}$$

$$\frac{\partial \mathcal{L}}{\partial u_1} = k(u_2-u_1)$$

$$\frac{\partial L}{\partial \eta_2} = -k(\eta_1 - \eta_2) + k(\eta_3 - \eta_2)$$

$$\frac{\partial L}{\partial \eta_3} = -k(\eta_3 - \eta_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_1} = M\ddot{\eta}_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_2} = m\ddot{\eta}_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_3} = M\ddot{\eta}_3$$

$$\boxed{\begin{aligned} \ddot{\eta}_1 &= -\frac{k}{M}(\eta_1 - \eta_2) \\ \ddot{\eta}_2 &= -\frac{k}{m}(2\eta_2 - \eta_1 - \eta_3) \\ \ddot{\eta}_3 &= -\frac{k}{M}(\eta_3 - \eta_2) \end{aligned}}$$

3 coupled 2nd-order equations

$$\frac{d\dot{\eta}_1}{dt} = \ddot{\eta}_1 \quad \wedge \quad \frac{d\eta_1}{dt} = \dot{\eta}_1$$

$\Rightarrow$ six coupled 1st-order diff equations.

$$\ddot{\eta}_i = \sum_{j=1}^{3} a_{ij}\,\eta_j \qquad i = 1, 2, 3$$

$$a_{11} = -\frac{k}{M} \qquad a_{12} = \frac{k}{M} \qquad a_{13} = 0$$

$$a_{21} = \frac{k}{m} \qquad a_{22} = -\frac{2k}{m} \qquad a_{23} = \frac{k}{m}$$

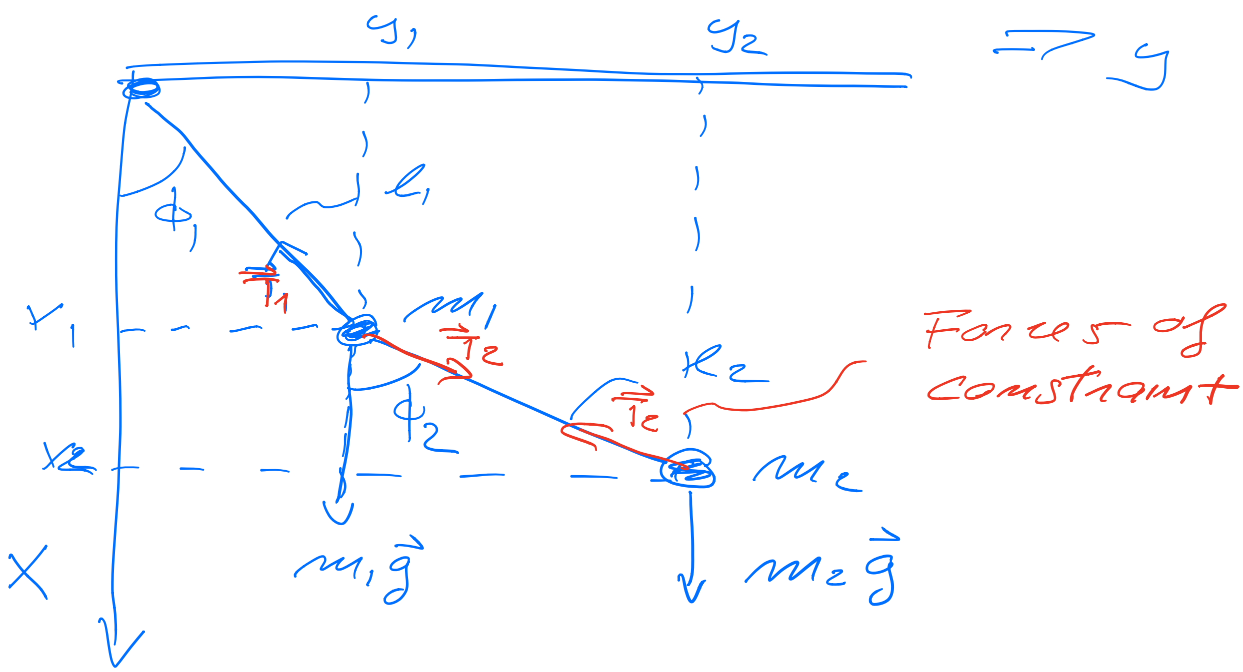
$$a_{31} = 0 \qquad a_{32} = \frac{k}{M} \qquad a_{33} = -\frac{k}{M}$$

$$\ddot{\eta} = \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \\ \ddot{\eta}_3 \end{bmatrix} \qquad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

$$\hat{A} = \begin{array}{c} \begin{array}{ccc} a_{11} & a_{12} & a_{13} \end{array} \\ \begin{bmatrix} -k/M & \frac{k}{M} & 0 \\ k/m & -\frac{2k}{m} & k/m \\ 0 & \frac{k}{M} & -k/M \end{bmatrix} \end{array}$$

$$\ddot{\eta} = \hat{A}\,\eta$$

Planar double pendulum



$$\vec{r}_1 = \ell_1 \cos\phi_1\,\hat{x} + \ell_1 \sin\phi_1\,\hat{y}$$

$$\vec{r}_2 = (l_1 \cos\phi_1 + l_2 \cos\phi_2)\,\hat{x}$$

$$+ (l_1 \sin\phi_1 + l_2 \sin\phi_2)\,\hat{y}$$

$m_1 = m_2 \quad \wedge \quad l_1 = l_2$

$x_1 = l\cos\phi_1 \qquad x_2 = l\cos\phi_1 + l\cos\phi_2$

$y_1 = l\sin\phi_1 \qquad y_2 = l\sin\phi_1 + \cancel{l\sin} + l\sin\phi_2$

kinetic energy

$$T = \frac{1}{2} m l^2 \dot\phi_1^2 + \frac{1}{2} m \left[(l\dot\phi_1 \cos\phi_1 + l\dot\phi_2 \cos\phi_2)^2 + (l\dot\phi_1 \sin\phi_1 + l\dot\phi_2 \sin\phi_2)^2 \right]$$

$$= \frac{1}{2} m l^2 \left[2\underline{\dot\phi_1}^2 + \dot\phi_2^2 + 2\underline{\dot\phi_1 \dot\phi_2} \, \underline{\cos}(\phi_1 - \phi_2) \right]$$

$$V = -mg(x_1 + x_2) = -mgl(2\cos\phi_1 + \cos\phi_2)$$

Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\phi_1} = \frac{\partial \mathcal{L}}{\partial \phi_1} \qquad \omega_0^2 = g/l$$

$$* \quad 2\ddot\phi_1 + \ddot\phi_2 \cos(\phi_1 - \phi_2) + \dot\phi_1 \dot\phi_2 \times \sin(\phi_1 - \phi_2) = -2\omega_0^2 \sin\phi_1$$

$$* \quad \ddot{\phi}_2 + \ddot{\phi}_1 \cos(\phi_1 - \phi_2) = -2\omega_0^2 \underline{\sin\phi_2}$$

small angles $\quad \sin\phi_2 \simeq \phi_2$

$\quad \sin\phi_1 \simeq \phi_1$

$$\cos(\phi_1 - \phi_2) \simeq \cos(0) = \underline{1}$$

$$\sin(\phi_1 - \phi_2) \simeq \sin(0) = \underline{0}$$

$$\boxed{\begin{aligned} 2\ddot{\phi}_1 + \ddot{\phi}_2 &= -2\omega_0^2 \phi_1 \\ \ddot{\phi}_1 + \ddot{\phi}_2 &= -\omega_0^2 \phi_2 \end{aligned}}$$

$\phi_2 = 0$

$$\ddot{\phi}_1 = -\omega_0^2 \phi_1 \Rightarrow A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$\phi_1 = A e^{i\omega t} \qquad \phi_2 = B e^{i\omega t}$$

$$\omega^2 = \frac{\omega_0^2}{1 \pm 1/\sqrt{2}}$$

$$\omega_\pm = \omega_0 \sqrt{\frac{1}{1 \pm 1/\sqrt{2}}}$$

$$\phi_1 = \frac{A_+}{\sqrt{2}} e^{i\omega_+ t}$$

$$\phi_2 = A_+ e^{i\omega_+ t}$$

$$\ldots \quad 1 \quad \ldots \quad e^{i\omega_- t}$$

$$\phi_1 = \frac{-A_-}{\sqrt{2}} e^{i\omega_- t}$$

$$\phi_2 = A_- e^{i\omega_- t}$$