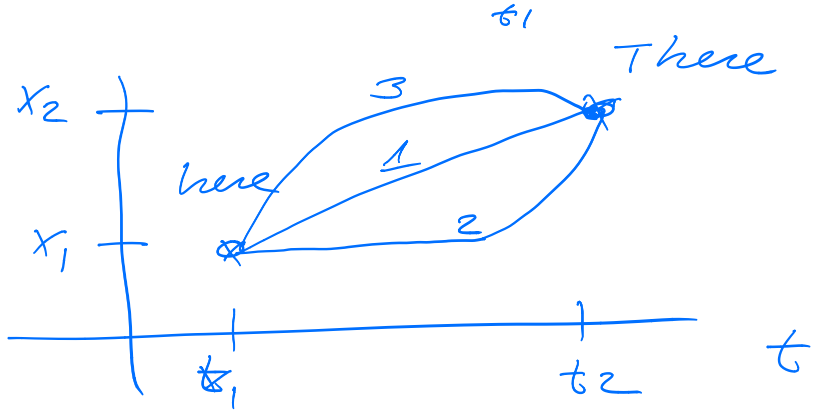
# Variational calculus

Min or Max (1-dim)

$$\text{action} = S = \int_{t_1}^{t_2} \mathcal{L}(x, v, t)\, dt$$

$$\delta S = \int_{t_1}^{t_2} \mathcal{L}(x+\delta x, v+\delta v, t)\, dt - \int_{t_1}^{t_2} \mathcal{L}(x, v, t)\, dt \to 0$$




$$\delta x(t_1) = \delta x(t_2) = 0$$

$$\delta v(t_1) = \delta v(t_2) = 0$$

$$\mathcal{L}(x+\delta x, v+\delta v, t) = \mathcal{L}(x, v, t) + \frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial v}\delta v + \cancel{\text{higher order terms}}$$

$$\delta S \simeq \int_{t_1}^{t_2} \left[ \frac{\partial \mathcal{L}}{\partial x}\underline{\delta x} + \underline{\frac{\partial \mathcal{L}}{\partial v}\delta v} \right] dt$$

$$\delta v = \delta v(t) = \frac{d}{dt}\delta x(t)$$

$$\delta v = \delta x(t) - \frac{d}{dx}\,\delta x(t)$$

Integrate by parts:

$$\underline{\delta S} = \frac{\partial \mathcal{L}}{\partial v}\,\delta v\,\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left[\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v}\right]\delta x\,dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} = 0$$

Euler-Lagrange Equations

$$\mathcal{L} = K - V$$

<u>Example</u>

$$K = \frac{1}{2} m v^2$$

$$V(x) = \frac{1}{2} k x^2$$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx \qquad \mathcal{L} = K - V$$

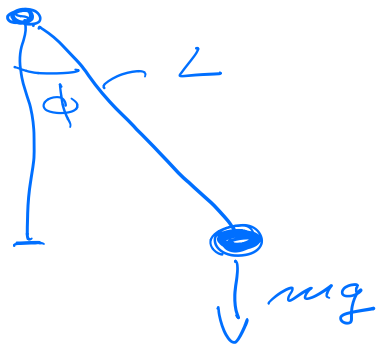
$$-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} = -\frac{d}{dt}\, m \cdot v = -m\frac{dv}{dt} = -m \cdot a$$

$$-ma - kx = 0 \implies$$

$$\boxed{ma = -kx} \quad \text{Equation}$$

<u>Pendulum</u>

$$\mathcal{L} = \frac{1}{2} m v^2 - V$$




$$V(\phi) = -m g L \cos\phi$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\begin{bmatrix} L \sin\phi \\ L \cos\phi \end{bmatrix}$$

$$v = |\vec{v}| = l\dot{\phi}$$

$$\frac{1}{2} m v^2 = \frac{1}{2} m L^2 \dot{\phi}^2$$

$$\mathcal{L} = \frac{1}{2} m L^2 \dot{\phi}^2 + m g L \cos\phi$$

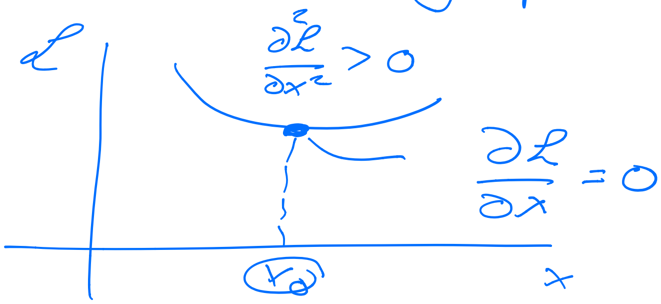
$$\frac{\partial \mathcal{L}}{\partial \phi} = -m g l \sin\phi$$

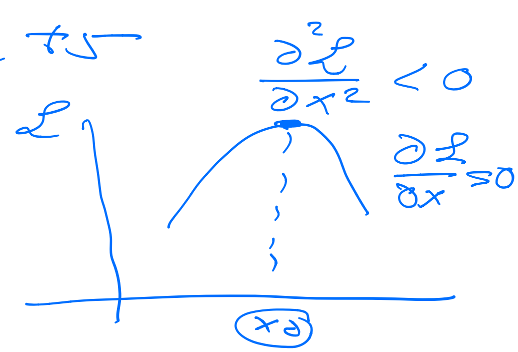
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{d}{dt} m l^2 \dot{\phi} = m l^2 \ddot{\phi}$$

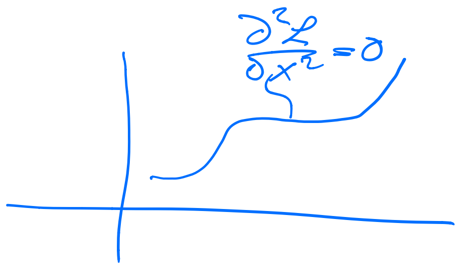
$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0 \implies$$

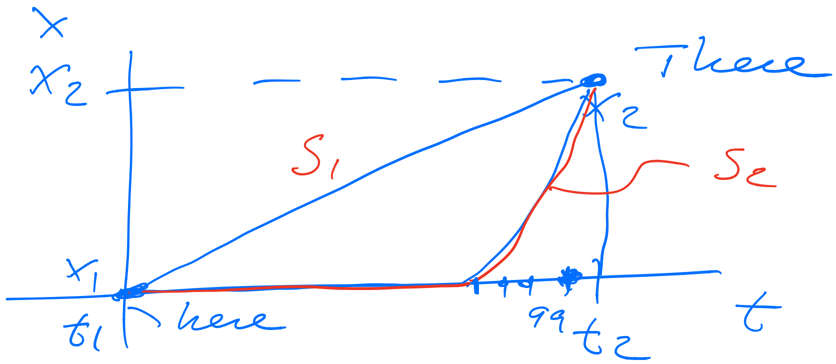
$$\boxed{m\ddot{\phi} = g/L \sin\phi}$$

stationary points






$S_1 < S_2$?

$x_1 = 0 \qquad x_2 = 100\,m$ | no external force

$t_1 = 0 \qquad t_2 = 100\,s$

$$S = \int_{t_1}^{t_2} \frac{1}{2} m v^2 \, dt$$

straight line leads to smallest

average velocity $\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = 1\,m/s$

$$S = \int_{t_1}^{t_2} \frac{1}{2} m \bar{v}^2 \, dt = \frac{1}{2} m \bar{v}^2 (t_2 - t_1)$$

$m = 1\,kg$

$$= \frac{1}{2} m \cdot 100\, m^2/$$

$$= 50\, J\,s$$

$$S_2 = \int_{t_1}^{t_2} \frac{1}{2} m v^2 \, dt \simeq \Delta t \cdot \sum_{i=1}^{n=100} \frac{1}{2} m v_i^2$$

$\Delta t = 1\,s$

$$v_i = [0, 0, 0, \ldots\ldots, 100] \quad m/s$$

$$S_2 = \Delta t \cdot \frac{1}{2}(100)^2 = 5000\,JS \gg 50\,JS$$

$$v_i' = [0, 0, \ldots\, 0, 25, 25, 25, 25]$$

$$S_3 = \frac{1}{2}(25)^2 \cdot 4 = 1250\,JS$$

$$v_i' = [0, 1.1, 1.1, \ldots\ldots\, 1.1]$$

$$S_4 = 50.5\,JS$$

Mean squared error = MSE =

$$\frac{1}{n}\sum_{i=1}^{n}(v_i - \bar{v})^2$$

$$v_i = \bar{v} \implies \boxed{MSE \equiv 0}$$

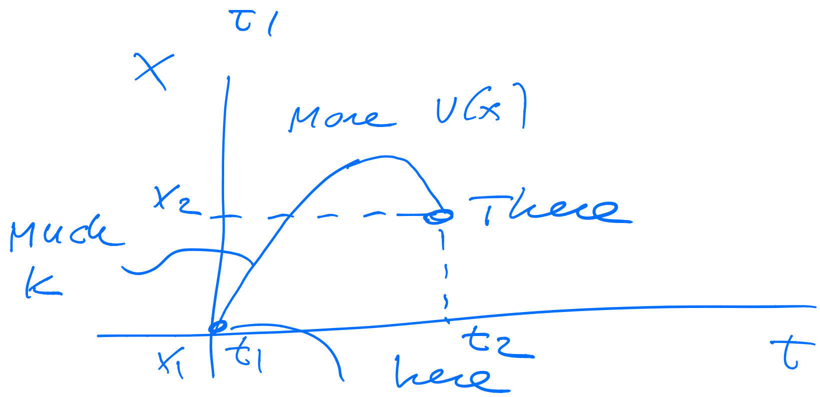
$$v = [0, 0, 0, \ldots\ldots, 100] \quad i = 1, 2, \ldots 100$$

$$MSE = \frac{99}{100}\underset{\substack{\uparrow \\ 1\,m/s}}{\bar{v}^2} + \frac{99^2}{100} = 99 \neq 0$$

$$v = [0, 0, 0, \ldots\, 25, 25, 25, 25]$$

$$MSE = 24$$

$$S = \int^{t_2}_{t_1}\left(\frac{1}{2}mv^2 - V(x)\right)dt$$




Feynman lecture on the least action principle.