Gravity and central forces:

$$V_{ab} = -\frac{G m_a m_b}{|\vec{r}_a - \vec{r}_b|} = V(\vec{r}_{ab})$$

$$\vec{r}_{ab} = \vec{r}_a - \vec{r}_b$$

$$F(\vec{r}_{ab}) = -\left[\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right]$$

$$\times \left\{ V(\vec{r}_{ab}) \right\}$$

$$\vec{r}_{ab} = \vec{r}$$

$$-\underline{\vec{\nabla}} V(\vec{r}) = -\frac{\partial}{\partial r}\left[\frac{\partial r}{\partial x_a}\vec{i} + \frac{\partial r}{\partial y_a}\vec{j} + \frac{\partial r}{\partial z_a}\vec{k}\right] V(\vec{r})$$

$$r = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2}$$

$$\frac{\partial r}{\partial x_a} = \frac{2x_a}{2r} = -\frac{(x_a - x_b)}{r}$$

$$\Rightarrow -\vec{i}\frac{\partial}{\partial r}\frac{\partial r}{\partial x}, V(\vec{r}) = +\frac{x_b \vec{i}}{r}\frac{\partial V(\vec{r})}{\partial r}$$

$$V(\vec{r}) = -\frac{G m_a m_b}{|\vec{r}_a - \vec{r}_b|} = -\frac{G m_a m_b}{r}$$

$$= V(r) \Rightarrow \frac{\partial V(r)}{\partial r} = +\frac{G m_a m_b}{r^2}$$

F force from b to a

$$F_{ba}(r) = -\frac{G m_a m_b}{r^3}\vec{r}$$

Internal forces only and more objects (N)

$$F_a = \sum_{\substack{i=1 \\ i\neq a}}^{N} F(\vec{r}_{ia}) =$$

Total force

$$F^{Total} = \sum_{\substack{a i \\ i\neq a}}^{N} F(\vec{r}_{ia})$$

$$\vec{\nabla}\times\vec{F} = 0$$

$$= \left[\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right]\vec{i}$$

$$- \left[\frac{\partial}{\partial x}F_z - \frac{\partial}{\partial z}F_x\right]\vec{j}$$

$$+ \left[\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right]\vec{k} = 0$$

— Two-body problems

center of mass position

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

$$= \frac{m_1 r_1 + m_2 r_2}{M}$$

$$M = \sum_{i=1}^{2} m_i$$

Momentum $\vec{P} = M \cdot \frac{d\vec{R}}{dt}$

$$= M\dot{\vec{R}}$$

simple case ; $m_1 = m_2$

$$\vec{R} = \frac{m(\vec{r}_1 + \vec{r}_2)}{2m} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$$

Relative distance

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\Rightarrow \quad \vec{r}_1 = \vec{R} + \frac{1}{2}\vec{r} \qquad \Bigg| \quad m_1 \neq m_2 \quad \vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$$

$$\vec{r}_2 = \vec{R} - \frac{1}{2}\vec{r} \qquad \Bigg| \quad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

Total kinetic energy ;

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \qquad \Bigg| \quad m_1 = m_2$$

$$= \frac{1}{2}\left[m_1\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + m_2\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)^2\right]$$

$$= \frac{1}{2}\left[M\dot{R}^2 + \frac{m_1 m_2}{M}\dot{r}^2\right]$$

Define relative mass

$$\mu = \frac{m_1 m_2}{M}$$

$$K = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2$$

$m_1 = m_2 = m$ $\quad \mu = \frac{1}{2} m$

$M = 2m$

$$K = m \dot{R}^2 + \frac{1}{4} m \dot{r}^2$$

CM Reference Frame

$F(\vec{r})$ depends only on relative position.
Equations of motion can be decoupled

$$M \cdot \ddot{\vec{R}} = M \cdot \frac{d^2 \vec{R}}{dt^2} = M \cdot \vec{A} = 0$$

$$\mu \cdot \ddot{\vec{r}} = \mu \frac{d^2 \vec{r}}{dt^2} = \mu \vec{a} = -\vec{\nabla} V(r)$$

In the CM frame, the total CM position $\underline{\vec{R} = 0}$

$$\vec{r}_1 = \frac{m_2}{M} \vec{r} \quad \wedge \quad \vec{r}_2 = -\frac{m_1}{M} \vec{r}$$

Total angular momentum

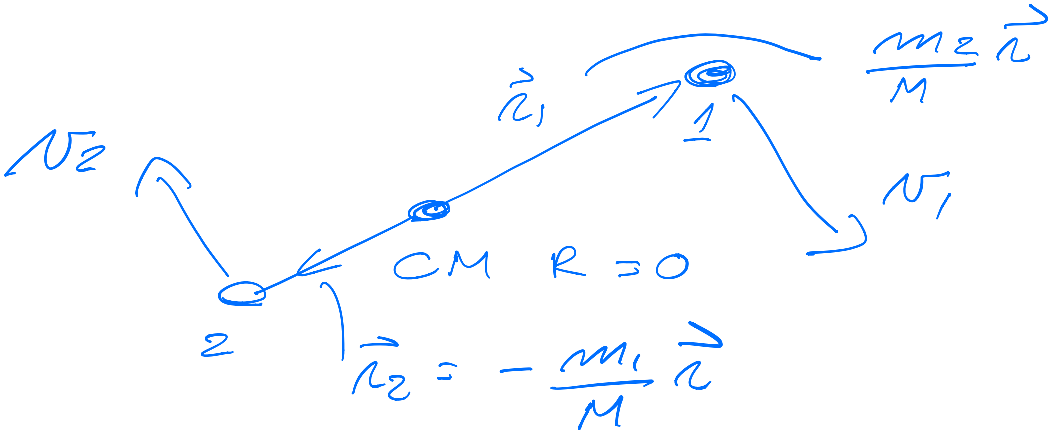
$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

$$= m_1 \left(\vec{r}_1 \times \dot{\vec{r}}_1 \right) + m_2 \left(\vec{r}_2 \times \dot{\vec{r}}_2 \right)$$

$\vec{p}_i = m_i \vec{v}_i$ $\qquad \frac{d\vec{r}}{dt}$

$$\vec{L} = \frac{m_1 m_2}{M^2} \left(m_2 \vec{r} \times \dot{\vec{r}} + m_1 \left(\vec{r} \times \dot{\vec{r}} \right) \right)$$

$$= \underline{\mu} \cdot \left(\underline{\vec{r}} \times \underline{\dot{\vec{r}}} \right)$$



Total ang mom. in CM frame is reduced to a "onebody"-like momentum

Angular momentum is conserved (central force)

$\Rightarrow$ $\vec{r} \times \dot{\vec{r}}$ is a constant of the motion