Energy, Momentum and Conservation Laws

Morten Hjorth-Jensen[1,2]

Scott Pratt[1]

Carl Schmidt[3]

[1]Department of Physics and Astronomy and National Superconducting Cyclotron Laboratory, Michigan State University, USA
[2]Department of Physics, University of Oslo, Norway
[3]Department of Physics and Astronomy, Michigan State University, USA

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Work, Energy, Momentum and Conservation laws

Energy conservation is most convenient as a strategy for addressing problems where time does not appear. For example, a particle goes from position x_0 with speed v_0, to position x_f; what is its new speed? However, it can also be applied to problems where time does appear, such as in solving for the trajectory $x(t)$, or equivalently $t(x)$.

Before we start formulating a strategy for energy conservation, we need to discuss integration methods and the concept of work and how it relates to energy.

Work and Energy

Till our own material is placed here, we recommend reading chapters 10 and 11 of Malthe-Sørenssen and "Taylor chapters 4.1-4.3":" `https://www.uscibooks.com/taylor2.htm`."

On work, chapter 10 of Malthe-Sørenssen is a good read.

Energy Conservation

Energy is conserved in the case where the potential energy, $V(\boldsymbol{r})$, depends only on position, and not on time. The force is determined by V,

$$\boldsymbol{F}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r}). \quad (1)$$

The net energy, $E = V + K$ where K is the kinetic energy, is then conserved,

$$\begin{aligned} \frac{d}{dt}(K+V) &= \frac{d}{dt}\left(\frac{m}{2}(v_x^2+v_y^2+v_z^2)+V(\boldsymbol{r})\right) \\ &= m\left(v_x\frac{dv_x}{dt}+v_y\frac{dv_y}{dt}+v_z\frac{dv_z}{dt}\right)+\partial_x V\frac{dx}{dt}+\partial_y V\frac{dy}{dt}+\partial_z V\frac{dz}{dt} \\ &= v_xF_x+v_yF_y+v_zF_z-F_xv_x-F_yv_y-F_zv_z=0. \end{aligned} \tag{2}$$

The same proof can be written more compactly with vector notation,

$$\begin{aligned} \frac{d}{dt}\left(\frac{m}{2}v^2+V(\boldsymbol{r})\right) &= m\boldsymbol{v}\cdot\dot{\boldsymbol{v}}+\nabla V(\boldsymbol{r})\cdot\dot{\boldsymbol{r}} \\ &= \boldsymbol{v}\cdot\boldsymbol{F}-\boldsymbol{F}\cdot\boldsymbol{v}=0. \end{aligned} \tag{3}$$

Inverting the expression for kinetic energy,

$$v=\sqrt{2K/m}=\sqrt{2(E-V)/m}, \tag{4}$$

allows one to solve for the one-dimensional trajectory $x(t)$, by finding $t(x)$,

$$t=\int_{x_0}^{x}\frac{dx'}{v(x')}=\int_{x_0}^{x}\frac{dx'}{\sqrt{2(E-V(x'))/m}}. \tag{5}$$

Note this would be much more difficult in higher dimensions, because you would have to determine which points, x, y, z, the particles might reach in the trajectory, whereas in one dimension you can typically tell by simply seeing whether the kinetic energy is positive at every point between the old position and the new position.

Consider a simple harmonic oscillator potential, $V(x)=kx^2/2$, with a particle emitted from $x=0$ with velocity v_0. Solve for the trajectory $t(x)$,

$$\begin{aligned} t &= \int_0^x\frac{dx'}{\sqrt{2(E-kx^2/2)/m}} \\ &= \sqrt{m/k}\int_0^x\frac{dx'}{\sqrt{x_{\max}^2-x'^2}}, \quad x_{\max}^2=2E/k. \end{aligned} \tag{6}$$

Here $E=mv_0^2/2$ and $x_{\max}$ is defined as the maximum displacement before the particle turns around. This integral is done by the substitution $\sin\theta=x/x_{\max}$.

$$\begin{aligned} (k/m)^{1/2}t &= \sin^{-1}(x/x_{\max}), \\ x &= x_{\max}\sin\omega t, \quad \omega=\sqrt{k/m}. \end{aligned} \tag{7}$$

Numerical Integration

As an example of an integral to solve numerically, consider the following integral. First, rewrite the integral as a sum,

$$t = \sum_{n=1}^{N} \Delta x [2(E - V(x_n)/m]^{-1/2}, \tag{8}$$

where $\Delta x = (x - x_0)/N$ and $x_n = x_0 + (n - 1/2)\Delta x$. Note that for best accuracy the value of x_n has been placed in the center of the n^{th} interval. The accuracy will improve for higher values of N, or equivalently, smaller step size Δx.

More material weill be added shortly.

Conservation of Momentum

Newton's third law which we met earlier states that **For every action there is an equal and opposite reaction**, is more accurately stated as **If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction**.

This means that for two bodies i and j, if the force on i due to j is called $\boldsymbol{F}_{ij}$, then

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}. \tag{9}$$

Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$, can be written for a particle i as

$$\boldsymbol{F}_i = \sum_{j \neq i} \boldsymbol{F}_{ij} = m_i \boldsymbol{a}_i, \tag{10}$$

where $\boldsymbol{F}_i$ (a single subscript) denotes the net force acting on i. Because the mass of i is fixed, one can see that

$$\boldsymbol{F}_i = \frac{d}{dt} m_i \boldsymbol{v}_i = \sum_{j \neq i} \boldsymbol{F}_{ij}. \tag{11}$$

Now, one can sum over all the particles and obtain

$$\begin{aligned} \frac{d}{dt} \sum_i m_i v_i &= \sum_{ij, i \neq j} \boldsymbol{F}_{ij} \\ &= 0. \end{aligned} \tag{12}$$

The last step made use of the fact that for every term ij, there is an equivalent term ji with opposite force. Because the momentum is defined as $m\boldsymbol{v}$, for a system of particles,

$$\frac{d}{dt} \sum_i m_i \boldsymbol{v}_i = 0, \quad \text{for isolated particles.} \tag{13}$$

By "isolated" one means that the only force acting on any particle i are those originating from other particles in the sum, i.e. “no external” forces. Thus, Newton’s third law leads to the conservation of total momentum,

$$\begin{aligned}\boldsymbol{P} &= \sum_i m_i \boldsymbol{v}_i, \qquad (14)\\ \frac{d}{dt}\boldsymbol{P} &= 0.\end{aligned}$$

Consider the rocket of mass M moving with velocity v. After a brief instant, the velocity of the rocket is $v + \Delta v$ and the mass is $M - \Delta M$. Momentum conservation gives

$$\begin{aligned}Mv &= (M - \Delta M)(v + \Delta v) + \Delta M(v - v_e)\\ 0 &= -\Delta M v + M\Delta v + \Delta M(v - v_e),\\ 0 &= M\Delta v - \Delta M v_e.\end{aligned}$$

In the second step we ignored the term $\Delta M \Delta v$ because it is doubly small. The last equation gives

$$\begin{aligned}\Delta v &= \frac{v_e}{M}\Delta M, \qquad (15)\\ \frac{dv}{dt} &= \frac{v_e}{M}\frac{dM}{dt}.\end{aligned}$$

Integrating the expression with lower limits $v_0 = 0$ and M_0, one finds

$$\begin{aligned}v &= v_e \int_{M_0}^{M} \frac{dM'}{M'}\\ v &= -v_e \ln(M/M_0)\\ &= -v_e \ln[(M_0 - \alpha t)/M_0].\end{aligned}$$

Because the total momentum of an isolated system is constant, one can also quickly see that the center of mass of an isolated system is also constant. The center of mass is the average position of a set of masses weighted by the mass,

$$\bar{x} = \frac{\sum_i m_i x_i}{\sum_i m_i}. \qquad (16)$$

The rate of change of $\bar{x}$ is

$$\dot{\bar{x}} = \frac{1}{M}\sum_i m_i \dot{x}_i = \frac{1}{M}P_x. \qquad (17)$$

Thus if the total momentum is constant the center of mass moves at a constant velocity, and if the total momentum is zero the center of mass is fixed.

Conservation of Angular Momentum

Consider a case where the force always points radially,

$$\boldsymbol{F}(\boldsymbol{r}) = F(r)\hat{r}, \tag{18}$$

where $\hat{r}$ is a unit vector pointing outward from the origin. The angular momentum is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times \boldsymbol{v}. \tag{19}$$

The rate of change of the angular momentum is

$$\begin{aligned} \frac{d\boldsymbol{L}}{dt} &= m\boldsymbol{v} \times \boldsymbol{v} + m\boldsymbol{r} \times \dot{\boldsymbol{v}} \\ &= m\boldsymbol{v} \times \boldsymbol{v} + \boldsymbol{r} \times \boldsymbol{F} = 0. \end{aligned} \tag{20}$$

The first term is zero because $\boldsymbol{v}$ is parallel to itself, and the second term is zero because $\boldsymbol{F}$ is parallel to $\boldsymbol{r}$.

As an aside, one can see from the Levi-Civita symbol that the cross product of a vector with itself is zero. Here, we consider a vector

$$\begin{aligned} \boldsymbol{V} &= \boldsymbol{A} \times \boldsymbol{A}, \\ V_i &= (\boldsymbol{A} \times \boldsymbol{A})_i = \sum_{jk} \epsilon_{ijk} A_j A_k. \end{aligned} \tag{21}$$

For any term i, there are two contributions. For example, for i denoting the x direction, either j denotes the y direction and k denotes the z direction, or vice versa, so

$$V_1 = \epsilon_{123} A_2 A_3 + \epsilon_{132} A_3 A_2. \tag{22}$$

This is zero by the antisymmetry of ϵ under permutations.

If the force is not radial, $\boldsymbol{r} \times \boldsymbol{F} \neq 0$ as above, and angular momentum is no longer conserved,

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F} \equiv \boldsymbol{\tau}, \tag{23}$$

where $\boldsymbol{\tau}$ is the torque.

For a system of isolated particles, one can write

$$\begin{aligned} \frac{d}{dt} \sum_i \boldsymbol{L}_i &= \sum_{i \neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} \\ &= \frac{1}{2} \sum_{i \neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} + \boldsymbol{r}_j \times \boldsymbol{F}_{ji} \\ &= \frac{1}{2} \sum_{i \neq j} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij} = 0, \end{aligned} \tag{24}$$

where the last step used Newton's third law, $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. If the forces between the particles are radial, i.e. $\boldsymbol{F}_{ij} \,||\, (\boldsymbol{r}_i - \boldsymbol{r}_j)$, then each term in the sum is zero and the net angular momentum is fixed. Otherwise, you could imagine an isolated system that would start spinning spontaneously.

One can write the torque about a given axis, which we will denote as $\hat{z}$, in polar coordinates, where

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\cos\phi, \quad z = r\cos\theta, \tag{25}$$

to find the z component of the torque,

$$\begin{aligned} \tau_z &= xF_y - yF_x \\ &= -r\sin\theta\left\{\cos\phi\partial_y - \sin\phi\partial_x\right\}V(x,y,z). \end{aligned} \tag{26}$$

One can use the chain rule to write the partial derivative w.r.t. ϕ (keeping r and θ fixed),

$$\begin{aligned} \partial_\phi &= \frac{\partial x}{\partial\phi}\partial_x + \frac{\partial_y}{\partial\phi}\partial_y + \frac{\partial z}{\partial\phi}\partial_z \\ &= -r\sin\theta\sin\phi\partial_x + \sin\theta\cos\phi\partial_y. \end{aligned} \tag{27}$$

Combining the two equations,

$$\tau_z = -\partial_\phi V(r,\theta,\phi). \tag{28}$$

Thus, if the potential is independent of the azimuthal angle ϕ, there is no torque about the z axis and L_z is conserved.

Symmetries and Conservation Laws

When we derived the conservation of energy, we assumed that the potential depended only on position, not on time. If it depended explicitly on time, one can quickly see that the energy would have changed at a rate $\partial_t V(x,y,z,t)$. Note that if there is no explicit dependence on time, i.e. $V(x,y,z)$, the potential energy can depend on time through the variations of x,y,z with time. However, that variation does not lead to energy non-conservation. Further, we just saw that if a potential does not depend on the azimuthal angle about some axis, ϕ, that the angular momentum about that axis is conserved.

Now, we relate momentum conservation to translational invariance. Considering a system of particles with positions, $\boldsymbol{r}_i$, if one changed the coordinate system by a translation by a differential distance $\boldsymbol{\epsilon}$, the net potential would change by

$$
\begin{aligned}
\delta V(\boldsymbol{r}_1, \boldsymbol{r}_2 \cdots) &= \sum_i \boldsymbol{\epsilon} \cdot \nabla_i V(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots) \\
&= -\sum_i \boldsymbol{\epsilon} \cdot \boldsymbol{F}_i \\
&= -\frac{d}{dt} \sum_i \boldsymbol{\epsilon} \cdot \boldsymbol{p}_i.
\end{aligned} \tag{29}
$$

Thus, if the potential is unchanged by a translation of the coordinate system, the total momentum is conserved. If the potential is translationally invariant in a given direction, defined by a unit vector, $\hat{\epsilon}$ in the $\boldsymbol{\epsilon}$ direction, one can see that

$$
\hat{\epsilon} \cdot \nabla_i V(\boldsymbol{r}_i) = 0. \tag{30}
$$

The component of the total momentum along that axis is conserved. This is rather obvious for a single particle. If $V(\boldsymbol{r})$ does not depend on some coordinate x, then the force in the x direction is $F_x = -\partial_x V = 0$, and momentum along the x direction is constant.

We showed how the total momentum of an isolated system of particle was conserved, even if the particles feel internal forces in all directions. In that case the potential energy could be written

$$
V = \sum_{i,j \leq i} V_{ij}(\boldsymbol{r}_i - \boldsymbol{r}_j). \tag{31}
$$

In this case, a translation leads to $\boldsymbol{r}_i \to \boldsymbol{r}_i + \boldsymbol{\epsilon}$, with the translation equally affecting the coordinates of each particle. Because the potential depends only on the relative coordinates, δV is manifestly zero. If one were to go through the exercise of calculating δV for small $\boldsymbol{\epsilon}$, one would find that the term $\nabla_i V(\boldsymbol{r}_i - \boldsymbol{r}_j)$ would be canceled by the term $\nabla_j V(\boldsymbol{r}_i - \boldsymbol{r}_j)$.

The relation between symmetries of the potential and conserved quantities (also called constants of motion) is one of the most profound concepts one should gain from this course. It plays a critical role in all fields of physics. This is especially true in quantum mechanics, where a quantity A is conserved if its operator commutes with the Hamiltonian. For example if the momentum operator $-i\hbar\partial_x$ commutes with the Hamiltonian, momentum is conserved, and clearly this operator commutes if the Hamiltonian (which represents the total energy, not just the potential) does not depend on x. Also in quantum mechanics the angular momentum operator is $L_z = -i\hbar\partial_\phi$. In fact, if the potential is unchanged by rotations about some axis, angular momentum about that axis is conserved. We return to this concept, from a more formal perspective, later in the course when Lagrangian mechanics is presented.

Consider a particle of mass m moving according to the potential

$$
V(x, y, z) = A \exp\left\{-\frac{x^2 + z^2}{2a^2}\right\}.
$$

Of the quantities, $E, p_x, p_y, p_z, L_x, L_y, L_z$ are conserved?

Solution: One has both rotational and translational invariance about the y axis, so L_y and p_y are conserved. Because the potential does not depend explicitly on time, energy is also conserved.