Central forces & Two-body problems

overarching view:

2-body problem:

$$m_1\ddot{\vec{r}}_1 = \vec{F}_1(x,y) \qquad m_2\ddot{\vec{r}}_2 = F_2(x,y)$$

$$a_{x_1} = \frac{F_{x_1}}{m_1} \qquad a_{x_2} = \frac{F_{x_2}}{m_2}$$

$$a_{y_1} = \frac{F_{y_1}}{m_1} = \frac{dv_{y_1}}{dt} \qquad a_{y_2} = \frac{F_{y_2}}{m_2}$$

$$\frac{dy_1}{dt} = v_{y_1}$$ + 3 more equations

<u>hw 4</u> 4 coupled ODE, sun at rest

$$x,y \to r,\phi \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

CM-frame

$$\vec{R} = 0$$

$$K = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) \qquad \mu = \frac{m_1 m_2}{m_1+m_2}$$

$$L = \mu r^2 \dot{\phi}$$

$$K = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{\mu^2 r^3}$$

$$\underline{\dot{\phi} = \frac{L}{\mu r^2}} \quad \wedge \quad \underline{\ddot{r} = -\frac{dV_{eff}}{dr}\frac{1}{\mu}}$$

$$V_{eff}(r) = V(r) + \frac{L^2}{\ldots}$$

$2\mu r^2$

$V(r) = -\alpha / r \qquad \alpha = G m_1 m_2$

$$\boxed{r(\phi) = \frac{c}{1+\varepsilon\cos\phi}} \qquad \varepsilon = \frac{A L^2}{\alpha \cdot \mu}$$

$$c = \frac{L^2}{\alpha \mu}$$

$$r_{min} = \frac{c}{1+\varepsilon}, \quad \phi = 0$$

can connect it to the energy of the system.

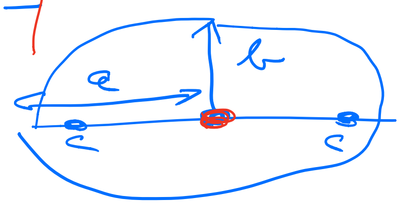
$$r_{max} = \frac{c}{1-\varepsilon}, \quad \phi = \pm\pi$$

elliptical orbits?

$$(1+\varepsilon\cos\phi)\, r = c \qquad x = r\cos\phi, \quad y = r\sin\phi$$

Ellipse

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$



$a \geq b \Rightarrow$

focal points $c = \sqrt{a^2 - b^2}$ $\quad \pm c$

$r + \varepsilon x = c$ or $r = c - \varepsilon x$

$(x^2 + y^2 = r^2)$

$$r^2 = c^2 + \varepsilon^2 x^2 - 2\varepsilon x c = x^2 + y^2$$

$$x^2(1-\varepsilon^2)+2c\varepsilon x+y^2=c^2$$

Divide both sides with $1-\varepsilon^2$

$$d=\frac{c\varepsilon}{1-\varepsilon^2}$$

$$\frac{(x^2+2\cdot dx)}{}+\frac{y^2}{1-\varepsilon^2}=\frac{c^2}{1-\varepsilon^2}$$

add d^2 to both sides

$$(x+d)^2+\frac{y^2}{1-\varepsilon^2}=\frac{c^2}{1-\varepsilon^2}+d^2$$

$$=\frac{c^2}{1-\varepsilon^2}\left[1+\frac{\varepsilon^2}{1-\varepsilon^2}\right]$$

$$=\frac{c^2}{(1-\varepsilon^2)^2}=a^2$$

$$\boxed{\frac{(x+d)^2}{a^2}+\frac{y^2}{b^2}=1}$$

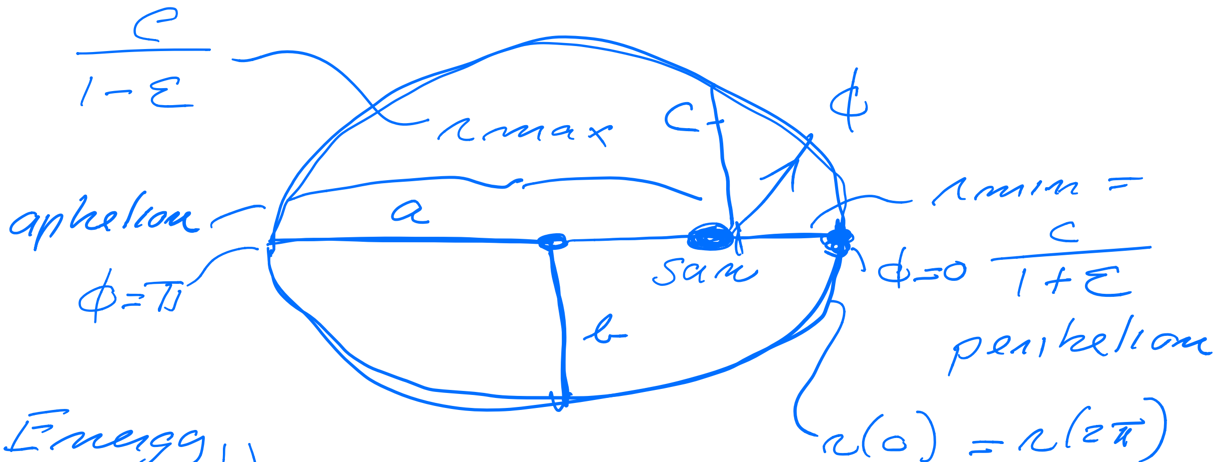
$$b=a\sqrt{(1-\varepsilon^2)}$$

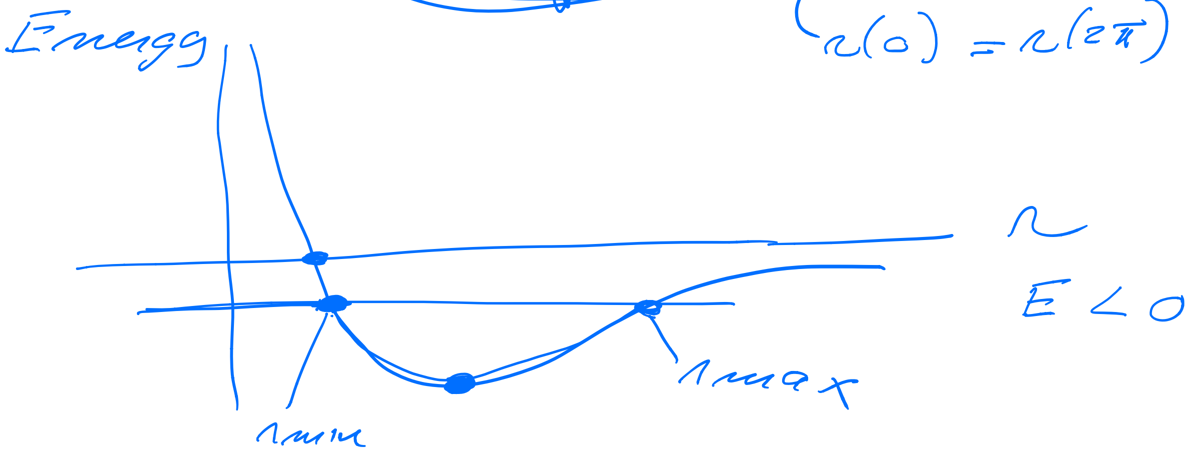
$$a=\frac{c}{1-\varepsilon^2}$$

$$b=\frac{c}{\sqrt{1-\varepsilon^2}}$$

$$d=a\varepsilon$$

$$c=\frac{L^2}{\alpha\mu}$$





$\underline{E<0}$ $\rightarrow$ ellipse $\qquad 0 < \varepsilon < 1$

$\varepsilon = 0$ circle

$$E = V_{eff}(r_{min})$$

$$= -\frac{\alpha}{r_{min}} + \frac{L^2}{2\mu r_{min}^2}$$

$$= \frac{1}{2 r_{min}}\left[\frac{L^2}{\mu r_{min}} - 2\alpha\right]$$

$$r_{min} = \frac{c}{1+\varepsilon} \qquad c = \frac{L^2}{\alpha\mu}$$

$$E = \frac{\alpha^2\mu}{2L^2}\left[\underline{\varepsilon^2 - 1}\right]$$

$E < 0 \Rightarrow \varepsilon < 1$

$E = 0 \Rightarrow \varepsilon = \pm 1 \qquad \varepsilon \geq 0$

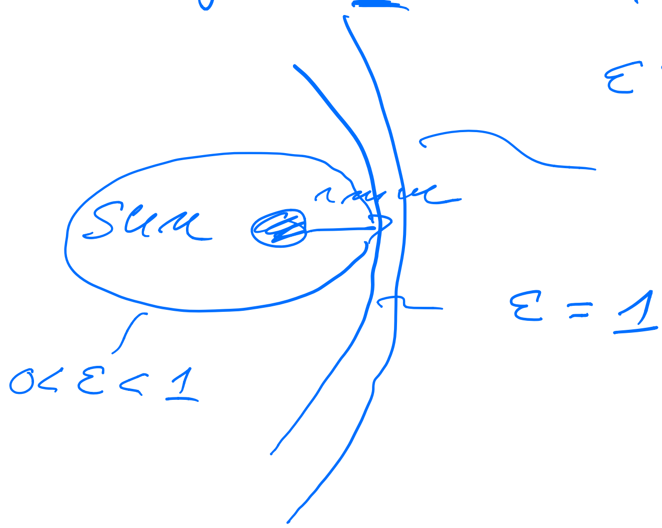
$\varepsilon = +1$

$r(\phi) \to \infty$ at $\phi \to \pm\pi$

$$r(\phi) = \frac{c}{1 + \varepsilon \cos\phi}$$

$r(1 + \varepsilon\cos\phi) = c \qquad \varepsilon = 1 \Rightarrow$

$y^2 = c^2 - 2cx$ parabola.



$$c = \frac{l^2}{\mu\alpha}$$

ε	Energy	orbit
$\varepsilon = 0$	$E < 0$	circle
$0 < \varepsilon < 1$	$E < 0$	ellipse
$\varepsilon = 1$	$E = 0$	parabola
$\varepsilon > 1$	$E > 0$	hyperbola.