Elliptical orbits

Kepler's laws state that a gravitational orbit should be an ellipse with the source of the gravitational field at one focus.

- Angular momentum is conserved
- Work in the CM frame
- chain rule to convert equations
- r as a function of angle ϕ.

CM-frame

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\ddot{\vec{R}} = \frac{1}{m_1 + m_2}\left[m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2 \right]$$

two objects $\vec{F}_{12} \wedge \vec{F}_{21}$

$$\vec{F}_{12} = -\vec{F}_{21}$$

$$\frac{1}{m_1 + m_2}\left[\vec{F}_{12} + \vec{F}_{21} \right] = 0$$

CM coordinate moves at a fixed velocity,

$$\mu \ddot{\vec{r}} = \vec{F}_{12}$$

$$\vec{P} = \vec{P}_1 + \vec{P}_2 = M\dot{\vec{R}} \quad \text{CM momentum}$$

$$\vec{q} = \mu \dot{\vec{r}}$$

identical objects $m_1 = m_2 = m$

$$\vec{P} = 2m \cdot \dot{\vec{R}} \qquad \vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$$

$$= m \cdot \dot{\vec{R}} \qquad \mu = \frac{1}{2} m$$

$$\vec{q} = \frac{1}{2} m \dot{\vec{r}}$$

$$K = \frac{P^2}{2M} + \frac{q^2}{2\mu}$$

In CM frame

$$L = \mu \cdot r^2 \dot{\phi} \qquad \begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

Equation of motion in r and ϕ

chain-rule application

$$\frac{d}{dt} r^2 = \frac{d}{dt}(x^2 + y^2)$$

$$= 2x \cdot \dot{x} + 2y \cdot \dot{y} = 2r \cdot \dot{r}$$

$$\dot{r} = \frac{dr}{d\ldots} = \frac{x}{\ldots}\dot{x} + \frac{y}{\ldots}\dot{y}$$

$$\ddot{r} = \frac{x}{r}\ddot{x} + \frac{y}{r}\ddot{y} + \frac{(\dot{x})^2+(\dot{y})^2}{r} - \frac{\dot{r}^2}{r}$$

$$\left(r = \sqrt{x^2+y^2}\right)$$

Friday March 13 (Taylor 8.1–8.3)

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

$$\mu \cdot \ddot{r} = \overbrace{\mu \cdot \cos\phi \cdot \ddot{x}}^{F_x} + \overbrace{\mu \sin\phi \cdot \ddot{y}}^{F_y} + \mu\left[\frac{r^2\dot{\phi}^2 + \dot{r}^2}{r} - \frac{\dot{r}^2}{r}\right]$$

$$F = F_x + F_y$$

$$\mu \cdot \ddot{r} = F + \mu \cdot r \cdot \dot{\phi}^2$$

$$L = \mu \cdot r^2 \cdot \dot{\phi}$$

$$\mu \cdot \ddot{r} = F + \frac{L^2}{\mu r^3} \Rightarrow$$

$$\ddot{r} = \frac{F}{\mu} + \frac{L^2}{\mu^2 r^3}$$

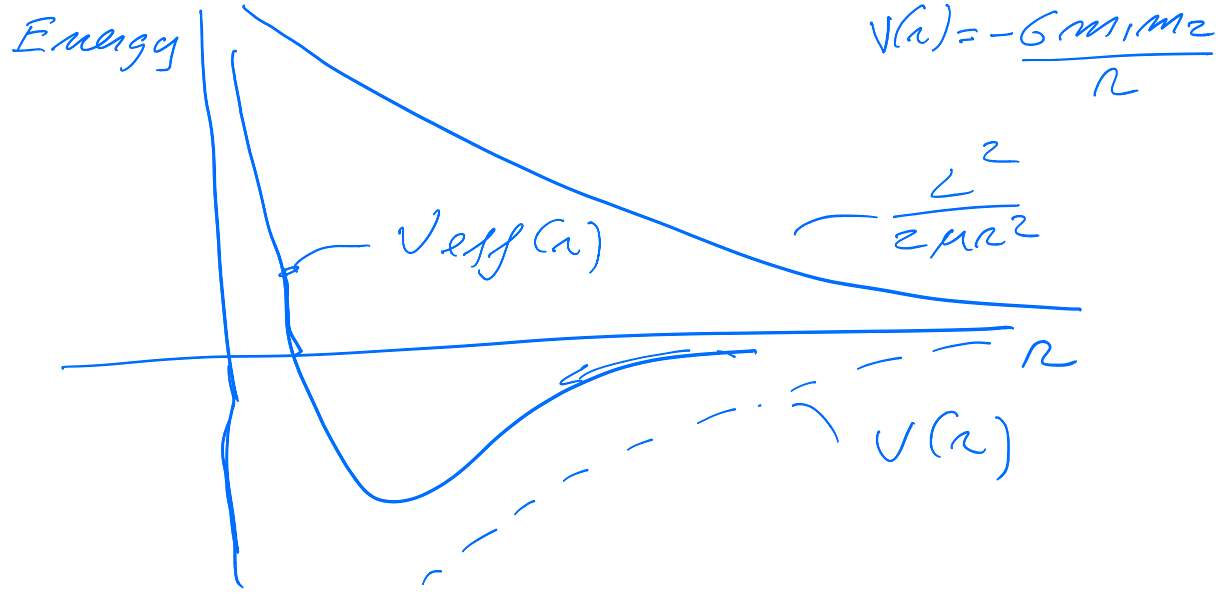
$$V_{eff} = V(r) + \frac{L^2}{2\mu r^2}$$

$$\boxed{\mu\ddot{r} = -\frac{dV_{eff}(r)}{dr}}$$

can be solved analytically.

$$\phi = \frac{L}{\mu r^2}$$

Energy diagrams (Taylor 8.4)



- r is large $\frac{L^2}{2\mu r^2}$ is small
  $V(r)$ dominates $V_{eff}(r) < 0$
  slopes up with increasing r.
  Think of comet moving towards the sun, $\ddot{r}$ is always inward.

- r is small $\frac{L^2}{2\mu r^2}$ which dominates, and near $r=0$
  $V_{eff}(r) > 0$ and slopes downward
  $\ddot{r}$ becomes outward and the planet/comet moves ~~out~~ away.

- $L=0 \Rightarrow \dot\phi = 0 \Rightarrow$ moves radially along a line with a constant $\phi \Rightarrow$ hits the sun.

Energy conservation

$$\mu \cdot \ddot r = -\frac{d}{dr} V_{eff}(r)$$

multiply with $\dot r$

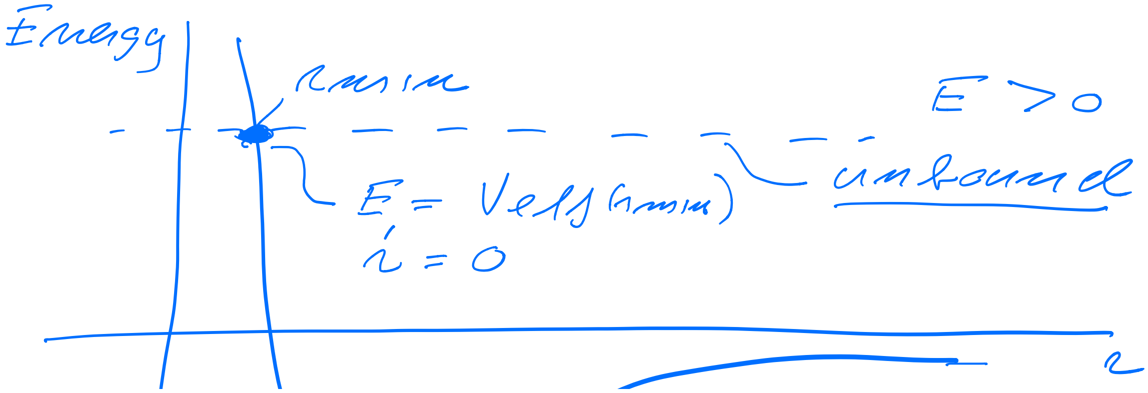
$$\underline{\mu \dot r \ddot r} = -\dot r \frac{d}{dr} V_{eff}(r)$$

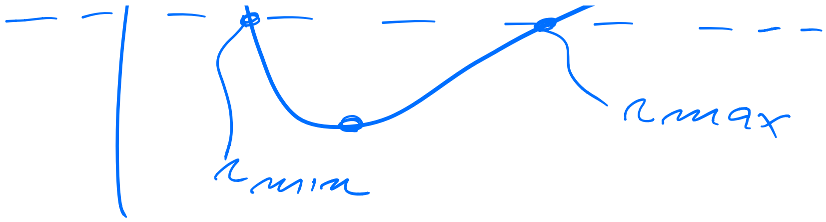
$$\frac{d}{dt}\left[\frac{1}{2}\mu \dot r^2\right] = -\frac{d}{dt}\left[V_{eff}(r)\right]$$

$$\frac{1}{2}\mu \dot r^2 + V_{eff}(r) = \text{const}$$

$$\frac{1}{2}\mu \dot r^2 + \frac{L^2}{2\mu r^2} + V(r) = E$$

one-dimensional equation.





$E < 0$ — bound motion

(ii) $E > 0$ $\quad \frac{1}{2}\mu \dot{r}^2 \geq 0 \Rightarrow$

$$E \geq V_{eff}$$

motion confined to regions where $E \geq V_{eff}$