PHY321: Classical Mechanics 1

Homework 4, due Friday February 7

Feb 3, 2020

Practicalities about homeworks and projects.

1. You can work in groups (optimal groups are often 2-3 people) or by yourself. If you work as a group you can hand in one answer only if you wish. **Remember to write your name(s)**!

2. Homeworks are available Wednesday/Thursday the week before the deadline. The deadline is at the Friday lecture.

3. How do I(we) hand in? You can hand in the paper and pencil exercises as a hand-written document. For this homework this applies to exercises 1-5. Alternatively, you can hand in everyhting (if you are ok with typing mathematical formulae using say Latex) as a jupyter notebook at D2L. The numerical exercise(s) (exercise 6 here) should always be handed in as a jupyter notebook by the deadline at D2L.

Introduction to homework 4.   This week's sets of classical pen and paper and computational exercises deal with simple motion problems and conservation laws; energy, momentum and angular momentum. These conservation laws are central in Physics and understanding them properly lays the foundation for understanding and analyzing more complicated physics problems. The relevant reading background is

1. chapters 3, 4.1, 4.2 and 4.3 of Taylor (there are many good examples there) and

2. chapters 10-13 of Malthe-Sørenssen.

In both textbooks there are many nice worked out examples. Malthe-Sørenssen's text contains also several coding examples you may find useful.

The numerical homework focuses on another motion problem where you can use the code you developed in homework 3, almost entirely. Please take a look at the posted solution (jupyter-notebook) for homework 3. You need only to change the forces at play. The problem at hand is a classic, the gravitational force acting

between the Sun and the Earth. Here you will notice also that the standard Euler-integration algorithm is not the best choice and we will introduce the so-called Euler-Cromer method and the Velocity-Verlet method. These methods will give much more stable numerical results with only few additions to your code.

The code you develop here will also be reused when we analyze energy conservation in homework set 5. And for those of you doing the honors project, it serves as a starting point for the solar system variant.

Exercise 1 (10 pt), Conservation laws, Energy and momentum.

- 1a (2pt) How do we define a conservative force?

- 1b (4pt) Use the work-energy theorem to show that energy is conserved with a conservative force.

- 1c (4pt) Assume that you have only internal two-body forces acting on N objects in an isolated system. The force from object i on object j is $\boldsymbol{F}_{ij}$. Show that the linear momentum is conserved.

Exercise 2 (10 pt), Conservation of angular momentum.

- 2a (2pt) Define angular momentum and the torque for a single object with external forces only.

- 2b (4pt) Define angular momentum and the torque for a system with N objects/particles with external and internal forces. The force from object i on object j is $\boldsymbol{F}_{ij}$.

- 2c (4pt) With internal forces only, what is the mathematical form of the forces that allows for angular momentum to be conserved?

Exsercise 3 (10pt), Example of potential. Consider a particle of mass m moving according to the potential

$$ V(x, y, z) = A \exp\left\{ -\frac{x^2 + z^2}{2a^2} \right\}. $$

- 3a (2pt) Is energy conserved? If so, why?

- 3b (4pt) Which of the quantities, p_x, p_y, p_z are conserved?

- 3c (4pt) Which of the quantities, L_x, L_y, L_z are conserved?

Exercise 4 (10pt), Angular momentum case. At $t = 0$ we have a single object with position $\boldsymbol{r}_0 = x_0 \boldsymbol{e}_x + y_0 \boldsymbol{e}_y$. We add also a force in the x-direction at $t = 0$. We assume that the object is at rest at $t = 0$.

$$\boldsymbol{F} = F\boldsymbol{e}_x.$$

- 4a (3pt) Find the velocity and momentum at a given time t by integrating over time with the above initial conditions.

- 4b (3pt) Find also the position at a time t.

- 4c (4pt) Use the position and the momentum to find the angular momentum and the torque. Is angular momentum conserved?

Exercise 5 (10pt), forces and potentials. A particle of mass m has velocity $v = \alpha/x$, where x is its displacement.

- 5a (3pt) Find the force $F(x)$ responsible for the motion.

A particle is thereafter under the influence of a force $F = -kx + kx^3/\alpha^2$, where k and α are constants and k is positive.

- 5b (3pt) Determine $U(x)$ and discuss the motion. It can be convenient here to make a sketch/plot of the potential as function of x.

- 5c (4pt) What happens when the energy of the particle is $E = (1/4)k\alpha^2$? Hint: what is the maximum value of the potential energy?

Exercise 6 (10pt), Warming up for the Earth-Sun system, Scaling the Equations. The aim of this exercise (as well as the next) is to study the motion of objects under the influence of the gravitational force. We will limit ourselves to the Earth-Sun system. Here we will scale the equations and sketch our first algorithm for solving the equations, namely using Euler's method again, as we did in homework 3. This part together with the numerical part forms also the entry point for the solar system honors project. Furthermore, we will reuse parts of these results when analyzing energy conservation in homework 5.

We will limit ourselves (in order to test the algorithm) to a hypothetical solar system with the Earth only orbiting around the sun. The only force in the problem is gravity. Newton's law of gravitation is given by a force F_G

$$F_G = \frac{GM_\odot M_{\text{Earth}}}{r^2},$$

where $M_\odot$ is the mass of the Sun and M_{Earth} is the mass of the Earth. The gravitational constant is G and r is the distance between the Earth and the Sun. The Sun has a mass which is much larger than that of the Earth. We can therefore safely neglect the motion of the Sun in this problem.

We assume that the orbit of the Earth around the Sun is co-planar, and we take this to be the xy-plane. Using Newton's second law of motion we get the following equations

$$\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_{\text{Earth}}},$$

and

$$\frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_{\text{Earth}}},$$

where $F_{G,x}$ and $F_{G,y}$ are the x and y components of the gravitational force.

We will use so-called astronomical units when rewriting our equations. Using astronomical units (AU as abbreviation)it means that one astronomical unit of length, known as 1 AU, is the average distance between the Sun and Earth, that is 1 AU $= 1.5 \times 10^{11}$ m. It can also be convenient to use years instead of seconds since years match better the time evolution of the solar system. The mass of the Sun is $M_{\text{sun}} = M_{\odot} = 2 \times 10^{30}$ kg. The masses of all relevant planets and their distances from the sun are listed in the table here in kg and AU.

Planet	Mass in kg	Distance to sun in AU
Earth	$M_{\text{Earth}} = 6 \times 10^{24}$ kg	1AU
Jupiter	$M_{\text{Jupiter}} = 1.9 \times 10^{27}$ kg	5.20 AU
Mars	$M_{\text{Mars}} = 6.6 \times 10^{23}$ kg	1.52 AU
Venus	$M_{\text{Venus}} = 4.9 \times 10^{24}$ kg	0.72 AU
Saturn	$M_{\text{Saturn}} = 5.5 \times 10^{26}$ kg	9.54 AU
Mercury	$M_{\text{Mercury}} = 3.3 \times 10^{23}$ kg	0.39 AU
Uranus	$M_{\text{Uranus}} = 8.8 \times 10^{25}$ kg	19.19 AU
Neptun	$M_{\text{Neptun}} = 1.03 \times 10^{26}$ kg	30.06 AU
Pluto	$M_{\text{Pluto}} = 1.31 \times 10^{22}$ kg	39.53 AU

In setting up the equations we limit ourselves to a co-planar motion and use only the x and y coordinates. But you should feel free to extend your equations to three dimensions, it is not very difficult and the data from NASA are all in three dimensions.

NASA has an excellent site at http://ssd.jpl.nasa.gov/horizons.cgi#top. From there you can extract initial conditions in order to start your differential equation solver. At the above website you need to change from **OBSERVER** to **VECTOR** and then write in the planet you are interested in. The generated data contain the x, y and z values as well as their corresponding velocities. The velocities are in units of AU per day. Alternatively they can be obtained in terms of km and km/s.

For the system below involving only the Earth and the Sun, you could just initialize the position with say $x = 1$ AU and $y = 0$ AU.

We assume that mass units can be obtained by using the fact that Earth's orbit is almost circular around the Sun.

For circular motion we know that the force must obey the following relation

$$F_G = \frac{M_{\text{Earth}} v^2}{r} = \frac{GM_{\odot}M_{\text{Earth}}}{r^2},$$

where v is the velocity of Earth. The latter equation can be used to show that

$$v^2 r = GM_\odot = 4\pi^2 \text{AU}^3/\text{yr}^2.$$

- 6a (5pt) Show how to derive the last equation and use this to scale the differential equations, getting thus rid of the constant G and the two masses. Split the differential equations for the motion in the x and y directions in terms of four coupled differential equations.

- 6b (5pt) Discretize the above differential equations and set up an algorithm for solving these equations using Euler's forward algorithm and the so-called velocity Verlet method discussed in the lecture notes. Here you can reuse what you did in homework 3, exercises 6 and 7.

Exercise 7 (40pt), Numerical elements, solving exercise 6 numerically. This exercise should be handed in as a jupyter-notebook at D2L. Remember to write your name(s).

Last week we:

1. Studied Euler's Method to find the position and the velocity of a falling object, including air resistance and gravity

2. Compared analytical and computational solutions

3. Added additional forces to our model

This week we will reuse our code from homework 3 (exercises 6 and 7, see also the code examples at the course homepage) and replace the air resistance and force from the ground with the gravitational force. Then we will study the stability of the system as function of initial conditions and the time step length.

We start by importing some standard packages

```
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# let's start by importing useful packages we are familiar with
import numpy as np
from math import *
import matplotlib.pyplot as plt
%matplotlib inline
```

- 7a (5 pt) Write then a program which solves the above differential equations for the Earth-Sun system using Euler's method and the velocity Verlet method. Find out which initial value for the velocity that gives a circular orbit and test the stability of your algorithm as function of different time steps Δt. Make a plot of the results you obtain for the position of the Earth (plot the x and y values and/or if you prefer to use three dimensions the z-value as well) orbiting the Sun. Discuss eventual differences between the Verlet algorithm and the Euler algorithm.

- 7b (5pt) Consider then a planet which begins at a distance of 1 AU from the sun. Find out by trial and error what the initial velocity must be in order for the planet to escape from the sun. Can you find an exact answer? How does that match your numerical results?

Here we add a code-example which may aid in the above studies using Euler's forward method.

```
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DeltaT = 0.001
#set up arrays
tfinal = 10 # in years
n = ceil(tfinal/DeltaT)
# set up arrays for time t, velocity v, and position r
t = np.zeros(n)
v = np.zeros((n,2))
r = np.zeros((n,2))
# Initial conditions as compact 2-dimensional arrays
r0 = np.array([1.0,0.0])
v0 = np.array([0.0,2*pi])
r[0] = r0
v[0] = v0
Fourpi2 = 4*pi*pi
# Start integrating using Euler's method
for i in range(n-1):
    # Set up the acceleration
    # Here you could have defined your own function for this
    rabs = sqrt(sum(r[i]*r[i]))
    a =  -Fourpi2*r[i]/(rabs**3)
    # update velocity, time and position using Euler's forward method
    v[i+1] = v[i] + DeltaT*a
    r[i+1] = r[i] + DeltaT*v[i]
    t[i+1] = t[i] + DeltaT
```

Exercise 8 (30pt), Bonus exercise. You don't need to do this exercise, but it gives you a bonus score of 30 points.

This time the bonus exercise is a simple survey. We are now moving into our fifth week and we would very much have your feedback on how things are functioning. In particular, since this is (together with fall 2019) the first time we have numerical exercises every week, we need to see if the pace is ok, if there is a good link between lectures and the homework exrcises etc etc.

- Is the weekly load with paper and pencil exercises and the numerical exercises reasonable?

- Is there enough material (lectures and lecture material) to get you started with the exercises? We are thinking of both the paper and pencil and the numerical exercises?

- Is the pace during the lectures reasonable? And do the lectures link well with the exercises?

- In the beginning there will always be some elements of repetition of material many of you have seen before. Do you find the choice of material the first weeks adequate? Too easy? Too difficult?

- For those of you who have taken CMSE 201 Introduction to Computational Modeling, do you feel the material taught there links well with the exercises you have done in this course? Is there is something we are missing?

- Any other topic you would like to comment on?