EXAM COVER-SHEET

Name of the Exam:	**Statistical Theory (Probability and Decision Theory)**
Time and Date:	3:30-5:30pm 16.02.2023
Examiner:	Prof. Dr. C. Jentsch, Prof. Dr. M. Demetrescu

To be completed by the exam participant:

First name:	
Last name:	**Mat. Nr.:**
Degree:	
I hereby declare that I am capable of taking the exam!	
Date and signature of the exam participant	

Further information for the exam:

- Read the next page carefully!
- Good Luck with the Exam!

Examination in Statistical theory
(Winter Term 2022/2023)
Examination regulation

Feb. 16, 2023, 15:30

Premliminary remarks:

1. Please read the instructions carefully
2. You are not permitted to use any auxilliary tools
3. Write your name and enrollment (matriculation) number on every sheet of paper!
4. The exam consists of 6 pages (including title pages) and 6 problem sets, check for completeness!
5. Don't use a pencil
6. Round your solutions to 2 decimal places or use fractions
7. For all tests use a significance level of 5%, if nothing else is specified
8. You have 60 min. in total to answer the exam questions

Good Luck!

1 Probability Theory

Problem 1.1(30 pts) Let $\Omega = \{1, 2, 3, 4\}$ and $A = \{1\}$.

a) **3 pts.** State the general properties of a σ-Algebra

b) **2 pts.** Find $\mathcal{A} = \sigma(A)$ (the smallest σ-Algebra containing A)

c) **4 pts.** Let $\mathcal{A}_2 = \mathcal{P}(\Omega)$, is $\mathcal{A} \cup \mathcal{A}_2$ also a σ-Algebra ? If yes, show it.

d) **10 pts.** Let X_n such that:

$$P(X_n = 1) = \frac{n-1}{n} \quad P(X_n = 2) = \frac{1}{n}$$

Show that $X_n \xrightarrow{p} 1$

e) **11 pts.** Compute the characteristic function $\varphi_{X_n}(t) = \mathbb{E}[\exp(itX_n)]$ and use this to show that:

$$X_n \xrightarrow{d} 1$$

2 Decision Theory

Problem 2.1 (30 pts) Let $\{X_1, ..., X_n\} \overset{iid}{\sim} \text{Exp}(\theta)$ random sample such that:

$$f_{X_i}(x; \theta) = \theta \exp(-\theta x) 1_{\mathbb{R}_+}(x) \quad \forall i \in \{1, ..., n\}$$

It therefore holds that: $\mathbb{E}(X_i) = \frac{1}{\theta}, Var(X_i) = \frac{1}{\theta^2}$ for all $i \in \{1, ..., n\}$. Now consider we want to estimate $\mathbb{E}(X)$ with the quadratic loss $L(\delta, \theta) = \left(\delta - \frac{1}{\theta}\right)^2$

a) **4 pts.** Characterize this decision problem with $\mathscr{X}, \Theta, D, L$

b) **6 pts.** Find the maximum likelihood estimator for $\mathbb{E}(X) = \frac{1}{\theta}$

Hint: First find the ML-estimator for θ. It then holds, that: $\hat{\mathbb{E}}(X)_{ML} = \frac{1}{\hat{\theta}}$

c) **4 pts.** Find the risk of the ML-estimator $\hat{\theta}_{ML}$

d) **6 pts.** Compute the risk of the following estimator and compare it to the ML-estimator:

$$\hat{\theta}_2 = \frac{1}{n+1} \sum_{i=1}^{n} x_i$$