## Problem 2.1

a)

We first compute the likelihood function:

$$egin{aligned} \mathcal{L}(X| heta) &= \prod_{i=1}^n f(x_i, heta) \ &= \prod_{i=1}^n rac{x_i}{ heta^2} \mathrm{exp}\left(-rac{x_i}{ heta}
ight) \ &= rac{1}{ heta^{2n}} \mathrm{exp}\left(-rac{1}{ heta}\sum_{i=1}^n x_i
ight) \prod_{i=1}^n x_i \end{aligned}$$

Then compute the log likelihood:

$$\ln \mathcal{L}(X| heta) = -2n \ln( heta) - rac{1}{ heta} \sum_{i=1}^n x_i + \sum_{i=1}^n \ln(x_i)$$

Then FOC:

$$egin{align} \ln \mathcal{L}'(X| heta) &= -rac{2n}{ heta} + rac{1}{ heta^2} \sum_{i=1}^n x_i = 0 \ &-2n + rac{1}{ heta} \sum_{i=1}^n x_i = 0 \ &rac{1}{ heta} \sum_{i=1}^n x_i = 2n \ &rac{1}{2n} \sum_{i=1}^n x_i = heta \ \end{aligned}$$

b)

We already have been given the lower boundary:

$$\frac{1}{n\mathbb{E}\left[\left(\frac{\partial}{\partial \theta}l(X;\theta)\right)^2\right]}$$

So let's first derive the score:

$$egin{aligned} rac{\partial}{\partial heta} l(x; heta) &= rac{\partial}{\partial heta} \ln \left( rac{x}{ heta^2} \exp \left( -rac{x}{ heta} 
ight) 
ight) \ &= rac{\partial}{\partial heta} \ln x - 2 \ln heta - rac{x}{ heta} \ &= -rac{2}{ heta} + rac{x}{ heta^2} \end{aligned}$$

second derivative:

$$rac{2}{ heta^2} - rac{2x}{ heta^3}$$

And apply the mean:

$$egin{split} -\mathbb{E}\left(rac{2}{ heta^2}-rac{2X}{ heta^3}
ight) &= -rac{2}{ heta^2}+rac{2}{ heta^3}2 heta \ &= -rac{2}{ heta^2}+rac{4}{ heta^2} \ &= rac{2}{ heta^2} \end{split}$$

So we get the lower bound:

$$Var(T) \geq rac{1}{rac{2n}{ heta^2}} = rac{ heta^2}{2n}$$

c)

Computing the variance of the maximum likelihood estimator:

$$egin{split} Var(\hat{ heta}_{ML}) &= rac{1}{4n^2} Var\left(\sum_{i=1}^n X_i
ight) \ &= rac{1}{4n^2} \sum_{i=1}^n Var(X_i) \ &= rac{1}{4n^2} \sum_{i=1}^n 6 heta^2 - 4 heta^2 \ &= rac{1}{4n^2} n2 heta^2 \ &= rac{ heta^2}{2n} \end{split}$$

So the lower bound is reached but is the estimator unbiased?

$$egin{aligned} \mathbb{E}(\hat{ heta}_{ML}) &= rac{1}{2n} \sum_{i=1}^n \mathbb{E}(X_i) \ &= rac{2n}{2n} heta = heta \end{aligned}$$

So yes and it is admissible because of bias variance decomposition.

d)

Bias variance decomposition:

$$egin{split} Var(\hat{ heta}_2) &= rac{1}{4(n+1)^2} \sum_{i=1}^n Var(X_i) \ &= rac{n6 heta^2}{4(n+1)^2} \end{split}$$

And the bias:

$$egin{align} \mathbb{E}(\hat{ heta}_2) &= rac{1}{2(n+1)} \sum_{i=1}^n \mathbb{E}(X_i) \ &= rac{n}{n+1} heta \ \mathrm{Bias}(\hat{ heta}_2)^2 &= rac{ heta^2}{(n+1)^2} \ \end{aligned}$$

Comming to the risk:

$$egin{split} R(\hat{ heta}_2, heta) &= rac{6n heta^2}{4(n+1)^2} + rac{ heta^2}{(n+1)^2} \ &= rac{(6n-4) heta^2}{4(n+1)^2} \leq rac{ heta^2}{2n} orall n > 0 \end{split}$$

## Problem 2.2

a)

$$egin{aligned} R(\delta_{a,b},\mu) &= \mathbb{E}((\mu-a-bX)^2) \ &= Var(\delta_{a,b}) + \mathrm{Bias}(\delta_{a,b})^2 \ &= b^2 + (a+b\mu-\mu)^2 \ &= b^2 + (a+(b-1)\mu)^2 \end{aligned}$$

b)

Inserting into the formula from a)

c)

No because the risk increases in b

## Problem 2.3

a)

NP-lemma is applied and we look at the likelihood ratio:

$$egin{aligned} LQ &= rac{\mathcal{L}(ec{X}|p_1)}{\mathcal{L}(ec{X}|p_0)} > k \quad orall x \in C \ &rac{\prod_{i=1}^{10} p_1^{x_i} (1-p_1)^{1-x_i}}{\prod_{i=1}^{10} p_0^{x_i} (1-p_0)^{1-x_i}} > k \quad orall x \in C \ &rac{\left(p_1^{\sum_{i=1}^{10} x_i} (1-p_1)^{10-\sum_{i=1}^{10} x_i}
ight)}{\left(p_0^{\sum_{i=1}^{10} x_i} (1-p_0)^{10-\sum_{i=1}^{10} x_i}
ight)} > k \quad orall x \in C \ &\left(rac{p_1}{p_0}
ight)^{\sum_{i=1}^{10} x_i} \left(rac{1-p_1}{1-p_0}
ight)^{\sum_{i=1}^{10} x_i} > k \left(rac{1-p_0}{1-p_1}
ight)^{10} & orall x \in C \ &\left(rac{p_1}{p_0} rac{1-p_0}{1-p_1}
ight)^{\sum_{i=1}^{10} x_i} > k \left(rac{1-p_0}{1-p_1}
ight)^{10} & orall x \in C \end{aligned}$$

Because  $\log_a(x)$  is a decreasing function for all a < 1, the inequality shifts in the other direction.

b)

The size is (in this case) the probability of making a type 1 error:

$$egin{aligned} lpha &= P(ec{X} \in C) \ &= P_{p_0}\left(\sum_{i=1}^{10} x_i \leq c_r - 1
ight) + P_{p_0}\left(\sum_{i=1}^{10} x_i = c_r
ight) \gamma \end{aligned}$$

we can choose  $c_r = 6$ , then:

$$P\left(\sum_{i=1}^{10} x_i \leq c_r - 1
ight) = P\left(\sum_{i=1}^{10} x_i \leq 5
ight) = 0.150$$

And choose  $\gamma$  such that:

$$egin{aligned} P\left(\sum_{i=1}^{10}x_i=6
ight)\gamma &= lpha - P\left(\sum_{i=1}^{10}x_i \leq 5
ight) \ 0.2\gamma &= 0.2 - 0.15 \ rac{1}{5}\gamma &= 5\% \ \implies \gamma &= 5*5\% = 25\% \end{aligned}$$

c)

Assuming 0-1 loss we can say:

$$egin{aligned} R(\phi,p) &= egin{cases} P_{p_0}(ec{X} \in C) & p = p_0 \ P_{p_1}(ec{X} 
otin C) & p = p_1 \end{cases} \ &= egin{cases} P_{p_0}\left(\sum_{i=1}^{10} x_i \leq 6\right) + 0.56 \cdot P_{p_0}\left(\sum_{i=1}^{10} x_i = 7\right) & p = 0.7 \ P_{p_1}\left(\sum_{i=1}^{10} x_i \geq 8
ight) & p = 0.3 \end{cases} \ &= egin{cases} pprox 0.5 & p = 0.7 \ pprox 0 & p = 0.3 \end{cases} \end{aligned}$$

No this is not minimax, because the risk is not invariant of p.