Problem 3.4

using Bayes rule:

$$f(\theta|x) \propto f(x|\theta)g(\theta)$$

Same with the whole sample:

$$egin{aligned} \mathcal{L}(heta|X) &\propto g(heta) \prod_{i=1}^n f(x_i| heta) \ &= rac{eta^lpha}{\Gamma(lpha)} heta^{lpha-1} \exp(-eta heta) heta^n \exp\left(- heta \sum_{i=1}^n x_i
ight) \ &= rac{eta^lpha}{\Gamma(lpha)} heta^{lpha-1+n} \exp\left(-\left(eta + \sum_{i=1}^n x_i
ight) heta
ight) \end{aligned}$$

So functionally it's a gamma distribution:

$$heta | X \sim \operatorname{Gamma} \left(lpha + n, eta + \sum_{i=1}^n x_i
ight)$$

b)

$$egin{aligned} \mathbb{E}((heta-\delta(x))^2) &= \mathbb{E}(heta^2 - 2 heta\delta(x) + \delta(x)^2) \ &= heta^2 - 2 heta\mathbb{E}(\delta(x)) + \mathbb{E}(\delta(x)^2) \end{aligned}$$

On starting on the other side:

$$egin{aligned} Var(\delta(x)) + (\mathbb{E}(\delta(x)) - heta^2)^2 &= \mathbb{E}(\delta(x)^2) - \mathbb{E}(\delta(x))^2 + (\mathbb{E}(\delta(x)) - heta)^2 \ &= \mathbb{E}(\delta(x)^2) - \mathbb{E}(\delta(x))^2 + \mathbb{E}(\delta(x))^2 - 2 heta\mathbb{E}(\delta(x)) - heta^2 \ &= \mathbb{E}(\delta(x)^2) - 2 heta\mathbb{E}(\delta(x)) - heta^2 \end{aligned}$$

Which is the same as above, showing equality.

c)

The conditional mean will always minimize the Bayes risk under quadratic loss:

$$\implies \delta_g(X) = rac{lpha + n}{eta + \sum_{i=1}^n x_i}$$

Yes, the rule is unique, because there is just one solution.

Only admissible, if $R(\theta, \delta_g)$ is continuous on Θ

Problem 3.5

a)

$$egin{aligned} \Theta &= [0,1] \ \mathscr{X} &= \{0,\dots,k\}^n = \{0,\dots,k\} \ D &= \Theta \end{aligned} \ L(heta,\delta) &= rac{(heta-d)^2}{ heta(1- heta)} \end{aligned}$$

b)

$$f(\theta|X) \propto f(X|\theta)g(\theta)$$
$$= f(X|\theta)$$

So posterior = likelihood because of uniformative prior. But also:

$$f(heta|x_1) \propto heta^{x_1}(1- heta)^{k-x_1} 1_{[0,1]}(heta)$$

You can eliminate the binomial coefficient as well to keep the posterior risk simpler **c)**

First we define the Bayes risk:

$$egin{align} B(\delta,\pi) &= \int_0^1 L(\delta, heta)f(heta|x_1)d heta \ &\propto \int_0^1 rac{(\delta- heta)^2}{ heta(1- heta)} heta^{x_i}(1- heta)^{k-x_1}d heta \ &= \int_0^1 (\delta- heta)^2 heta^{x_1-1}(1- heta)^{k-x_1-1}d heta \ \end{gathered}$$

FOC:

$$\begin{split} \frac{\partial B(\delta,\pi)}{\partial \delta} &= 0 \\ \frac{\partial}{\partial \delta} \int_0^1 (\delta - \theta)^2 \theta^{x_i - 1} (1 - \theta)^{k - x_i - 1} d\theta &= 0 \\ \int_0^1 \frac{\partial}{\partial \delta} (\delta - \theta)^2 \theta^{x_1 - 1} (1 - \theta)^{k - x_1 - 1} d\theta &= 0 \\ \int_0^1 2(\delta - \theta) \theta^{x_1 - 1} (1 - \theta)^{k - x_1 - 1} d\theta &= 0 \\ \delta \int_0^1 \theta^{x_1 - 1} (1 - \theta)^{k - x_1 - 1} d\theta &= \int_0^1 \theta^{x_1} (1 - \theta)^{k - x_1 - 1} d\theta \\ \delta(x_1) &= \frac{\int_0^1 \theta^{x_1} (1 - \theta)^{k - x_1 - 1} d\theta}{\int_0^1 \theta^{x_1 - 1} (1 - \theta)^{k - x_1 - 1} d\theta} \end{split}$$

d)

Is $\delta_B = \frac{X_1}{k}$ an equalizer rule?

$$egin{aligned} R(\delta_B,\pi) &= \mathbb{E}(L(\delta_B, heta)| heta) \ &= \mathbb{E}\left(rac{(\delta- heta)^2}{ heta(1- heta)}| heta
ight) \ &= rac{1}{ heta(1- heta)}\mathbb{E}\left(\left(rac{X_1}{k}
ight)^2 - 2\left(rac{X_1}{k}
ight) heta - heta^2| heta
ight) \ &= rac{1}{ heta(1- heta)}\left[rac{1}{k^2}\mathbb{E}(X_1^2| heta) - rac{2}{k} heta E(X_1| heta) + heta^2
ight] \ &= rac{1}{ heta(1- heta)}\left[rac{k(heta(1- heta)) + k^2 heta^2}{k^2} - rac{2}{k} heta k heta + heta^2
ight] \ &= rac{1}{ heta(1- heta)}\left[rac{ heta(1- heta)}{k} - heta^2 + heta^2
ight] = rac{1}{k} \end{aligned}$$

Which is invariant of θ . So it is a equalizer and Bayes rule making it minimax.

Problem 3.6

a)

$$egin{aligned} \Theta &= \mathbb{R}_+ \ \mathscr{X} &= \mathbb{R}^n \end{aligned} \ D &= \left\{ \delta_1, \delta_0
ight\} = \left\{ egin{aligned} H_0 ext{ is true} \ H_1 ext{ is true} \end{aligned}
ight. \ L &= 0 ext{-} 1 ext{ loss} \end{aligned}$$

b)

we have a linear combination of normal r.v. this means, that it's normal and we only have to find the mean and the variance:

$$egin{aligned} \mathbb{E}(T) &= rac{\mu}{\sqrt{rac{8}{n}}} \ &= rac{\mu}{\sqrt{rac{\sigma^2}{2n}}} \ Var(T) &= rac{rac{4}{n^2}}{rac{8}{n}} \sum_{i=1}^{n/2} Var(X_i) \ &= rac{32}{n^3} \sum_{i=1}^{n/2} \sigma^2 \ &= rac{16}{n^2} \sigma^2 = rac{16^2}{16^2} = 1 \end{aligned}$$

So:

$$T \sim \mathcal{N}\left(rac{\mu}{\sqrt{rac{\sigma^2}{2n}}}, 1
ight)$$

And under H_0 it is standard normal.

c)

$$lpha = P_{H_0}(T \le -1.645) = \Phi(-1.645) = 1 - \Phi(1.645) = 5\%$$

so significance level is 5%

d)

$$\lambda(X) = rac{\mathcal{L}(X|0)}{\max_{\mu \in \mathbb{R}} \mathcal{L}(X|\mu)}$$

The ML estimator for μ is the sample mean \bar{x}_n . Now inserting this:

$$\lambda(X) = rac{\prod_{i=1}^n f(x_i|0,4)}{\prod_{i=1}^n f(x_i|ar{X}_n,4)} \ = rac{\exp\left(-rac{1}{2\sigma^2}\sum_{i=1}^n x_i^2
ight)}{\exp\left(-rac{1}{2\sigma^2}\sum_{i=1}^n (x_i-ar{x}_n)^2
ight)}$$

And applying the transformation:

$$-2\ln\lambda(X) = -2\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2 + \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \bar{x}_n)^2\right)$$

$$= \frac{1}{\sigma^2}\left(\sum_{i=1}^n x_i^2 - \sum_{i=1}^n x_i^2 + 2\bar{x}_n\sum_{i=1}^n x_i - n\bar{x}_n\right)$$

$$= \frac{1}{\sigma^2}\left(2n\bar{x}_n^2 - n\bar{x}_n^2\right)$$

$$= \frac{n\bar{x}_n^2}{\sigma^2}$$

$$= \frac{n\left(\frac{1}{n^2}\left(\sum_{i=1}^n x_i\right)^2\right)}{\sigma^2}$$

$$= \frac{\left(\sum_{i=1}^n x_i\right)^2}{n\sigma^2}$$

$$= \left(\frac{\sum_{i=1}^n x_i}{\sqrt{n}\sigma}\right)^2$$

$$\to \mathcal{X}_1^2 \quad \text{lindeberg levy clt}$$