Problem 2.1

a)

We first compute the likelihood function:

$$
\begin{aligned}
\mathcal{L}(X|\theta) &= \prod_{i=1}^{n} f(x_i, \theta) \\
&= \prod_{i=1}^{n} \frac{x_i}{\theta^2} \exp\left(-\frac{x_i}{\theta}\right) \\
&= \frac{1}{\theta^{2n}} \exp\left(-\frac{1}{\theta} \sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} x_i
\end{aligned}
$$

Then compute the log likelihood:

$$
\ln \mathcal{L}(X|\theta) = -2n \ln(\theta) - \frac{1}{\theta} \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln(x_i)
$$

Then FOC:

$$
\begin{aligned}
\ln \mathcal{L}'(X|\theta) &= -\frac{2n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i = 0 \\
-2n + \frac{1}{\theta} \sum_{i=1}^{n} x_i &= 0 \\
\frac{1}{\theta} \sum_{i=1}^{n} x_i &= 2n \\
\frac{1}{2n} \sum_{i=1}^{n} x_i &= \theta
\end{aligned}
$$

b)

We already have been given the lower boundary:

$$
\frac{1}{n\mathbb{E}\left[\left(\frac{\partial}{\partial \theta} l(X;\theta)\right)^2\right]}
$$

So let's first derive the score:

$$\begin{aligned}\frac{\partial}{\partial\theta}l(x;\theta) &= \frac{\partial}{\partial\theta}\ln\left(\frac{x}{\theta^2}\exp\left(-\frac{x}{\theta}\right)\right)\\ &= \frac{\partial}{\partial\theta}\ln x - 2\ln\theta - \frac{x}{\theta}\\ &= -\frac{2}{\theta} + \frac{x}{\theta^2}\end{aligned}$$

second derivative:

$$\frac{2}{\theta^2} - \frac{2x}{\theta^3}$$

And apply the mean:

$$\begin{aligned}-\mathbb{E}\left(\frac{2}{\theta^2} - \frac{2X}{\theta^3}\right) &= -\frac{2}{\theta^2} + \frac{2}{\theta^3}2\theta\\ &= -\frac{2}{\theta^2} + \frac{4}{\theta^2}\\ &= \frac{2}{\theta^2}\end{aligned}$$

So we get the lower bound:

$$Var(T) \geq \frac{1}{\frac{2n}{\theta^2}} = \frac{\theta^2}{2n}$$

c)

Computing the variance of the maximum likelihood estimator:

$$\begin{aligned}Var(\hat{\theta}_{ML}) &= \frac{1}{4n^2}Var\left(\sum_{i=1}^{n}X_i\right)\\ &= \frac{1}{4n^2}\sum_{i=1}^{n}Var(X_i)\\ &= \frac{1}{4n^2}\sum_{i=1}^{n}6\theta^2 - 4\theta^2\\ &= \frac{1}{4n^2}n2\theta^2\\ &= \frac{\theta^2}{2n}\end{aligned}$$

So the lower bound is reached but is the estimator unbiased?

$$
\begin{aligned}
\mathbb{E}(\hat{\theta}_{ML}) &= \frac{1}{2n}\sum_{i=1}^{n}\mathbb{E}(X_i) \\
&= \frac{2n}{2n}\theta = \theta
\end{aligned}
$$

So yes and it is admissible because of bias variance decomposition.

d)

Bias variance decomposition:

$$
\begin{aligned}
Var(\hat{\theta}_2) &= \frac{1}{4(n+1)^2}\sum_{i=1}^{n}Var(X_i) \\
&= \frac{n6\theta^2}{4(n+1)^2}
\end{aligned}
$$

And the bias:

$$
\begin{aligned}
\mathbb{E}(\hat{\theta}_2) &= \frac{1}{2(n+1)}\sum_{i=1}^{n}\mathbb{E}(X_i) \\
&= \frac{n}{n+1}\theta \\
\text{Bias}(\hat{\theta}_2)^2 &= \frac{\theta^2}{(n+1)^2}
\end{aligned}
$$

Comming to the risk:

$$
\begin{aligned}
R(\hat{\theta}_2, \theta) &= \frac{6n\theta^2}{4(n+1)^2} + \frac{\theta^2}{(n+1)^2} \\
&= \frac{(6n-4)\theta^2}{4(n+1)^2} \leq \frac{\theta^2}{2n}\forall n > 0
\end{aligned}
$$

Problem 2.2

a)

$$
\begin{aligned}
R(\delta_{a,b}, \mu) &= \mathbb{E}((\mu - a - bX)^2) \\
&= Var(\delta_{a,b}) + \text{Bias}(\delta_{a,b})^2 \\
&= b^2 + (a + b\mu - \mu)^2 \\
&= b^2 + (a + (b-1)\mu)^2
\end{aligned}
$$

b)

Inserting into the formula from a)

$$R(\delta_{a,b}, \mu) = 1$$

c)

No because the risk increases in b

Problem 2.3

a)

NP-lemma is applied and we look at the likelihood ratio:

$$\begin{aligned}
LQ = \frac{\mathcal{L}(\vec{X}|p_1)}{\mathcal{L}(\vec{X}|p_0)} &> k \quad \forall x \in C \\
\frac{\prod_{i=1}^{10} p_1^{x_i}(1-p_1)^{1-x_i}}{\prod_{i=1}^{10} p_0^{x_i}(1-p_0)^{1-x_i}} &> k \quad \forall x \in C \\
\frac{\left(p_1^{\sum_{i=1}^{10} x_i}(1-p_1)^{10-\sum_{i=1}^{10} x_i}\right)}{\left(p_0^{\sum_{i=1}^{10} x_i}(1-p_0)^{10-\sum_{i=1}^{10} x_i}\right)} &> k \quad \forall x \in C \\
\left(\frac{p_1}{p_0}\right)^{\sum_{i=1}^{10} x_i}\left(\frac{1-p_1}{1-p_0}\right)^{10-\sum_{i=1}^{10} x_i} &> k \quad \forall x \in C \\
\left(\frac{p_1}{p_0}\frac{1-p_0}{1-p_1}\right)^{\sum_{i=1}^{10} x_i} &> k\left(\frac{1-p_0}{1-p_1}\right)^{10} \quad \forall x \in C \\
\sum_{i=1}^{10} x_i &< k^* \quad \forall x \in C
\end{aligned}$$

Because $\log_a(x)$ is a decreasing function for all $a < 1$, the inequality shifts in the other direction.

b)

The size is (in this case) the probability of making a type 1 error:

$$\begin{aligned}
\alpha &= P(\vec{X} \in C) \\
&= P_{p_0}\left(\sum_{i=1}^{10} x_i \leq c_r - 1\right) + P_{p_0}\left(\sum_{i=1}^{10} x_i = c_r\right)\gamma
\end{aligned}$$

we can choose $c_r = 6$, then:

$$P\left(\sum_{i=1}^{10} x_i \leq c_r - 1\right) = P\left(\sum_{i=1}^{10} x_i \leq 5\right) = 0.150$$

And choose γ such that:

$$P\left(\sum_{i=1}^{10} x_i = 6\right)\gamma = \alpha - P\left(\sum_{i=1}^{10} x_i \leq 5\right)$$
$$0.2\gamma = 0.2 - 0.15$$
$$\frac{1}{5}\gamma = 5\%$$
$$\implies \gamma = 5 * 5\% = 25\%$$

c)

Assuming 0-1 loss we can say:

$$\begin{aligned} R(\phi, p) &= \begin{cases} P_{p_0}(\vec{X} \in C) & p = p_0 \\ P_{p_1}(\vec{X} \notin C) & p = p_1 \end{cases} \\ &= \begin{cases} P_{p_0}\left(\sum_{i=1}^{10} x_i \leq 6\right) + 0.56 \cdot P_{p_0}\left(\sum_{i=1}^{10} x_i = 7\right) & p = 0.7 \\ P_{p_1}\left(\sum_{i=1}^{10} x_i \geq 8\right) & p = 0.3 \end{cases} \\ &= \begin{cases} \approx 0.5 & p = 0.7 \\ \approx 0 & p = 0.3 \end{cases} \end{aligned}$$

No this is not minimax, because the risk is not invariant of p.