Problem 3.4

using Bayes rule:

$$f(\theta|x) \propto f(x|\theta)g(\theta)$$

Same with the whole sample:

$$\begin{aligned}\mathcal{L}(\theta|X) &\propto g(\theta)\prod_{i=1}^{n} f(x_i|\theta) \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}\exp(-\beta\theta)\theta^n \exp\left(-\theta\sum_{i=1}^{n} x_i\right) \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1+n}\exp\left(-\left(\beta+\sum_{i=1}^{n} x_i\right)\theta\right)\end{aligned}$$

So functionally it's a gamma distribution:

$$\theta|X \sim \text{Gamma}\left(\alpha+n, \beta+\sum_{i=1}^{n} x_i\right)$$

b)

$$\begin{aligned}\mathbb{E}((\theta-\delta(x))^2) &= \mathbb{E}(\theta^2 - 2\theta\delta(x) + \delta(x)^2) \\ &= \theta^2 - 2\theta\mathbb{E}(\delta(x)) + \mathbb{E}(\delta(x)^2)\end{aligned}$$

On starting on the other side:

$$\begin{aligned}Var(\delta(x)) + (\mathbb{E}(\delta(x)) - \theta^2)^2 &= \mathbb{E}(\delta(x)^2) - \mathbb{E}(\delta(x))^2 + (\mathbb{E}(\delta(x)) - \theta)^2 \\ &= \mathbb{E}(\delta(x)^2) - \mathbb{E}(\delta(x))^2 + \mathbb{E}(\delta(x))^2 - 2\theta\mathbb{E}(\delta(x)) - \theta^2 \\ &= \mathbb{E}(\delta(x)^2) - 2\theta\mathbb{E}(\delta(x)) - \theta^2\end{aligned}$$

Which is the same as above, showing equality.

c)

The conditional mean will always minimize the Bayes risk under quadratic loss:

$$\implies \delta_g(X) = \frac{\alpha+n}{\beta+\sum_{i=1}^{n} x_i}$$

d)
Yes, the rule is unique, because there is just one solution.

Only admissible, if $R(\theta, \delta_g)$ is continuous on Θ

Problem 3.5

a)

$$\begin{aligned} \Theta &= [0,1] \\ \mathscr{X} &= \{0,\dots,k\}^n = \{0,\dots,k\} \\ D &= \Theta \\ L(\theta,\delta) &= \frac{(\theta-d)^2}{\theta(1-\theta)} \end{aligned}$$

b)

$$\begin{aligned} f(\theta|X) &\propto f(X|\theta)g(\theta) \\ &= f(X|\theta) \end{aligned}$$

So posterior = likelihood because of uniformative prior. But also:

$$f(\theta|x_1) \propto \theta^{x_1}(1-\theta)^{k-x_1}1_{[0,1]}(\theta)$$

You can eliminate the binomial coefficient as well to keep the posterior risk simpler

c)
First we define the Bayes risk:

$$\begin{aligned} B(\delta,\pi) &= \int_0^1 L(\delta,\theta)f(\theta|x_1)d\theta \\ &\propto \int_0^1 \frac{(\delta-\theta)^2}{\theta(1-\theta)}\theta^{x_i}(1-\theta)^{k-x_1}d\theta \\ &= \int_0^1 (\delta-\theta)^2\theta^{x_1-1}(1-\theta)^{k-x_1-1}d\theta \end{aligned}$$

FOC:

$$
\begin{aligned}
\frac{\partial B(\delta,\pi)}{\partial \delta} &= 0 \\
\frac{\partial}{\partial \delta}\int_0^1 (\delta-\theta)^2\theta^{x_i-1}(1-\theta)^{k-x_i-1}d\theta &= 0 \\
\int_0^1 \frac{\partial}{\partial \delta}(\delta-\theta)^2\theta^{x_1-1}(1-\theta)^{k-x_1-1}d\theta &= 0 \\
\int_0^1 2(\delta-\theta)\theta^{x_1-1}(1-\theta)^{k-x_1-1}d\theta &= 0 \\
\delta\int_0^1 \theta^{x_1-1}(1-\theta)^{k-x_1-1}d\theta &= \int_0^1 \theta^{x_1}(1-\theta)^{k-x_1-1}d\theta \\
\delta(x_1) &= \frac{\int_0^1 \theta^{x_1}(1-\theta)^{k-x_1-1}d\theta}{\int_0^1 \theta^{x_1-1}(1-\theta)^{k-x_1-1}d\theta}
\end{aligned}
$$

d)

Is $\delta_B = \frac{X_1}{k}$ an equalizer rule?

$$
\begin{aligned}
R(\delta_B,\pi) &= \mathbb{E}(L(\delta_B,\theta)|\theta) \\
&= \mathbb{E}\left(\frac{(\delta-\theta)^2}{\theta(1-\theta)}|\theta\right) \\
&= \frac{1}{\theta(1-\theta)}\mathbb{E}\left(\left(\frac{X_1}{k}\right)^2 - 2\left(\frac{X_1}{k}\right)\theta - \theta^2|\theta\right) \\
&= \frac{1}{\theta(1-\theta)}\left[\frac{1}{k^2}\mathbb{E}(X_1^2|\theta) - \frac{2}{k}\theta E(X_1|\theta) + \theta^2\right] \\
&= \frac{1}{\theta(1-\theta)}\left[\frac{k(\theta(1-\theta)) + k^2\theta^2}{k^2} - \frac{2}{k}\theta k\theta + \theta^2\right] \\
&= \frac{1}{\theta(1-\theta)}\left[\frac{\theta(1-\theta)}{k} - \theta^2 + \theta^2\right] = \frac{1}{k}
\end{aligned}
$$

Which is invariant of θ. So it is a equalizer and Bayes rule making it minimax.

Problem 3.6

a)

$$
\begin{aligned}
\Theta &= \mathbb{R}_+ \\
\mathscr{X} &= \mathbb{R}^n \\
D = \{\delta_1,\delta_0\} &= \begin{cases} H_0 \text{ is true} \\ H_1 \text{ is true} \end{cases} \\
L &= \text{0-1 loss}
\end{aligned}
$$

b)
we have a linear combination of normal r.v. this means, that it's normal and we only have to find the mean and the variance:

$$
\begin{aligned}
\mathbb{E}(T) &= \frac{\mu}{\sqrt{\frac{8}{n}}} \\
&= \frac{\mu}{\sqrt{\frac{\sigma^2}{2n}}} \\
Var(T) &= \frac{\frac{4}{n^2}}{\frac{8}{n}} \sum_{i=1}^{n/2} Var(X_i) \\
&= \frac{32}{n^3} \sum_{i=1}^{n/2} \sigma^2 \\
&= \frac{16}{n^2}\sigma^2 = \frac{16^2}{16^2} = 1
\end{aligned}
$$

So:

$$
T \sim \mathcal{N}\left(\frac{\mu}{\sqrt{\frac{\sigma^2}{2n}}}, 1\right)
$$

And under H_0 it is standard normal.
c)

$$
\alpha = P_{H_0}(T \leq -1.645) = \Phi(-1.645) = 1 - \Phi(1.645) = 5\%
$$

so significance level is 5%
d)

$$
\lambda(X) = \frac{\mathcal{L}(X|0)}{\max_{\mu \in \mathbb{R}} \mathcal{L}(X|\mu)}
$$

The ML estimator for μ is the sample mean $\bar{x}_n$.
Now inserting this:

$$
\begin{aligned}
\lambda(X) &= \frac{\prod_{i=1}^n f(x_i|0, 4)}{\prod_{i=1}^n f(x_i|\bar{X}_n, 4)} \\
&= \frac{\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right)}{\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \bar{x}_n)^2\right)}
\end{aligned}
$$

And applying the transformation:

$$
\begin{aligned}
-2\ln\lambda(X) &= -2\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2 + \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2\right) \\
&= \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i^2 + 2\bar{x}_n \sum_{i=1}^{n} x_i - n\bar{x}_n\right) \\
&= \frac{1}{\sigma^2}\left(2n\bar{x}_n^2 - n\bar{x}_n^2\right) \\
&= \frac{n\bar{x}_n^2}{\sigma^2} \\
&= \frac{n\left(\frac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2\right)}{\sigma^2} \\
&= \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n\sigma^2} \\
&= \left(\frac{\sum_{i=1}^{n} x_i}{\sqrt{n}\sigma}\right)^2 \\
&\to \mathcal{X}_1^2 \quad \text{lindeberg levy clt}
\end{aligned}
$$