Sheet 2

Exercise 1

a)

Let $A_n := [0, 1/n]$ then $A_n \to [0, 0] = \{0\}$. And continuity from above implies that:

$$\lambda(A_n) \longrightarrow \lambda(\{0\})$$

So:

$$\lim_{n\to\infty} \lambda(A_n) = \lim_{n\to\infty} \frac{1}{n} = 0$$

Thus:

$$\lambda(\{0\}) = \lim_{n\to\infty} \lambda(A_n) = 0$$

b)

The cdf $F_{X_n}(x)$ is defined by:

$$F_{X_n}(x) := P(X_n \leq x)$$

Because $X_n : \Omega \to \{0, n\}$, we can define F_{X_n} as a step-function. For this we need to compute the probability of $X_n = 0$ and $X_n = n$:

$$P(X_n = 0) = P((1/n, 1]) = \lambda((1/n, 1]) = \frac{n-1}{n}$$
$$P(X_n = n) = P([0, 1/n]) = \lambda([0, 1/n]) = \frac{1}{n}$$

This yields the cdf

$$F_{X_n}(x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{n-1}{n} \text{ if } 0 \leq x < n \\ 1 \text{ if } x > n \end{cases}$$

c)

We could say $X_n \xrightarrow{p} 0$:

$$\forall \varepsilon > 0 : P(|X_n| \leq \varepsilon) \geq P(X_n = 0) = \frac{n-1}{n} \longrightarrow 1 (*)$$

But this means that:

$$\forall \varepsilon > 0 : P(|X_n| > \varepsilon) = 1 - P(|X_n| \leq \varepsilon) \underset{(*)}{\longrightarrow} 0$$

Exercise 2

By definition of multivariate normals, it holds that:

$$X := \tilde{\Sigma}\tilde{X} + \mu$$

Where $\tilde{X} \sim \mathcal{N}(0, I_n)$ and $\tilde{\Sigma} \in \mathbb{R}^{n \times n}$ regular and $\Sigma = \tilde{\Sigma}\tilde{\Sigma}'$ and $\mu \in \mathbb{R}^n$.

So we can say:

$$\begin{aligned} Y &:= AX + b \\ &= A(\tilde{\Sigma}\tilde{X} + \mu) + b \\ &= A\tilde{\Sigma}\tilde{X} + A\mu + b \end{aligned}$$

Denoting $\tilde{A} := A\tilde{\Sigma}$ we know know that: $Y \sim \mathcal{N}(A\mu + b, \tilde{A}\tilde{A}')$.

$$\tilde{A}\tilde{A}' = A\tilde{\Sigma}(A\tilde{\Sigma})' = A\tilde{\Sigma}\tilde{\Sigma}'A' = A\Sigma A'$$

Therefore proving that:

$$Y \sim \mathcal{N}(A\mu + b, A\Sigma A')$$

Exercise 3

a)

Let $X_n \sim Bernoulli(0.5 + 1/n)$ and $X_0 \sim Bernoulli(1/2)$

Then X_n has the cdf:

$$F_{X_n}(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 - (0.5 + 1/n) \text{ if } 0 \leq x < 1 \\ 1 \text{ if } x \geq 1 \end{cases}$$

And X_0 has the cdf:

$$F_{X_0}(x) = \begin{cases} 0 \text{ if } x < 0 \\ 0.5 \text{ if } 0 \leq x < 1 \\ 1 \text{ if } x \geq 1 \end{cases}$$

Now let $x \in \mathbb{R}$ be any real number. Then for $x \in (-\infty, 0) \cup [1, \infty)$ it holds that $F_{X_n}(x) = F_{X_0}(x)$ and for $x \in [0, 1)$ we have:

$$\begin{aligned} \lim_{n \to \infty} F_{X_n}(x) &= \lim_{n \to \infty} 0.5 - 1/n \\ &= 0.5 - \lim_{n \to \infty} 1/n \\ &= 0.5 - 0 = 0.5 = F_{X_0}(x) \end{aligned}$$

So we can say:

$$\forall x \in \mathbb{R} : F_{X_n}(x) \longrightarrow F_{X_0}(x)$$

And therefore $X_n \xrightarrow{d} X_0$.

b)

The following example was inspired by exercise 1 but with a recursive definition.

Let X_n be defined such that $P(X_1 = 1) = 0.5$ and $P(X_n = 1) = P(X_{n-1} = 1) + P(X_{n-1} \neq 1)/2$ then we will show that $X_n \xrightarrow{p} 1$.

Generally you can now show by induction that:

$$P(X_n = 1) = \sum_{i=1}^{n} \frac{1}{2^i}$$

IH: $n = 1$ holds by construction:

$$P(X_1 = 1) = \frac{1}{2} = \sum_{i=1}^{1} \frac{1}{2^i}$$

Before starting the induction step. Lets show:

$$\sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2^n} \sum_{i=0}^{n-1} 2^i \tag{1}$$

We can show this by:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{2^i} &= \sum_{i=1}^{n} \frac{2^{n-i}}{2n} \\
&= \frac{1}{2^n} + \sum_{i=1}^{n} 2^{n-i} \\
&= \frac{1}{2^n} + (2^{n-1} + 2^{n-2} + \cdots 2^{n-n}) \\
&= \frac{1}{2^n} + (2^0 + \cdots 2^{n-1}) \\
&= \frac{1}{2^n} + \sum_{i=0}^{n-1} 2^i
\end{aligned}$$

IS: $n \to n+1$

$$
\begin{aligned}
P(X_{n+1}=1) &= P(X_n=1) + P(X_n \neq 1)/2 \\
&= P(X_n=1) + (1 - P(X_n=1))/2 \\
&\underset{I.H.}{=} \sum_{i=1}^{n} \frac{1}{2^i} + \left(1 - \sum_{i=1}^{n} \frac{1}{2^i}\right)\frac{1}{2} \\
&\underset{(1)}{=} \sum_{i=1}^{n} \frac{1}{2^i} + \left(1 - \frac{1}{2^n}\sum_{i=0}^{n-1} 2^i\right)\frac{1}{2} \\
&\underset{\text{geom. Series}}{=} \sum_{i=1}^{n} \frac{1}{2^i} + \left(1 - \frac{1}{2^n}\frac{1-2^n}{1-2}\right)\frac{1}{2} \\
&= \sum_{i=1}^{n} \frac{1}{2^i} + \left(1 - \frac{1-2^n}{2^n}\right)\frac{1}{2} \\
&= \sum_{i=1}^{n} \frac{1}{2^i} + \frac{1}{2^n}\frac{1}{2} \\
&= \sum_{i=1}^{n+1} \frac{1}{2^i}
\end{aligned}
$$

Now we can state:

$$
\begin{aligned}
P(|X_n - 1| < \varepsilon) &\geq P(X_n = 1) \\
&= \sum_{i=1}^{n} \frac{1}{2^i} \\
&= \sum_{i=0}^{n} \frac{1}{2^i} - 1 \underset{n\to\infty}{\longrightarrow} \frac{1}{1-\frac{1}{2}} - 1 \\
&= 2 - 1 = 1
\end{aligned}
$$

So:

$$
P(|X_n - 1| \geq \varepsilon) = 1 - P(|X_n - 1| < \varepsilon) \to 1 - 1 = 0
$$

By definition this means:

$$
X_n \xrightarrow{p} 1
$$

c)

Let $([0,1], \mathcal{B}([0,1]), \lambda)$ be a prob. space. We define $X_n : [0,1] \to \mathbb{R}$:

$$X_n := \begin{cases} 0 \text{ if } x \in \mathbb{Q} \\ (1 + x/n)^n \text{ else} \end{cases}$$

Furthermore let $X_0 = \exp(x)$. Then:

$$\forall x \in [0,1] \setminus \mathbb{Q} : X_n(x) = (1 + x/n)^n \longrightarrow \exp(x) = X_0(x)$$

We can then say:

$$\begin{aligned} P(X_n \to X_0) &= P([0,1] \setminus \mathbb{Q}) \\ &= \lambda([0,1] \setminus \mathbb{Q}) \\ &= \lambda([0,1]) - \lambda([0,1] \cap \mathbb{Q}) \\ &= 1 - \sum_{x \in [0,1] \cap \mathbb{Q}} \lambda(\{x\}) \\ &= 1 - \sum_{x \in [0,1] \cap \mathbb{Q}} 0 \\ &= 1 \end{aligned}$$

This means that:

$$X_n \xrightarrow{a.s.} X_0$$

d)

Let $X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ then and $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Then we can show $\bar{X}_n \xrightarrow{L_2} \mu$.

$$
\begin{aligned}
\mathbb{E}(|\bar{X}_n - \mu|^2) &= \mathbb{E}((\bar{X}_n - \mu)^2) \\
&= Var(\bar{X}_n) \\
&= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right) \\
&\underset{iid}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \\
&= \frac{1}{n^2} n\sigma^2 \\
&= \frac{\sigma}{n} \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

This means that:

$$\mathbb{E}(|\bar{X}_n - \mu|^2) \to 0$$

Showing L_2 convergence for $\bar{X}_n$.

Exercise 4

a)

Firstly we know that:

$$f_{U_1}(x) = f_{V_1}(x) = \frac{1}{1-0} = 1$$

Also because U_1 and V_1 are stochastically independent:

$$f_{U_1,V_1}(x,y) = f_{U_1}(x) f_{V_1}(y) = 1 \cdot 1 = 1 \tag{2}$$

Using what we have been given we can derive:

$$
\begin{aligned}
\mathbb{E}(H(U_1, V_1)) &= \int_0^1 \int_0^1 H(x,y) f_{U_1,V_1}(x,y) dx dy \\
&\underset{(2)}{=} \int_0^1 \int_0^1 H(x,y) dx dy \\
&= \int_0^1 \int_0^{f(x)} 1 dy dx \\
&= \int_0^1 f(x) dx
\end{aligned}
$$

b)

I'm supposing a typo and will show that:

$$\frac{1}{n}\sum_{i=1}^{n} H(U_i, Y_i) \xrightarrow{p} I$$

We know that $\mathbb{E}(H(U_i, V_i)) = I \ \ \forall i$. Furthermore knowing that U_i and V_i are iid distributed, we can apply the law of large numbers and state:

$$\frac{1}{n}\sum_{i=1}^{n} H(U_i, V_i) \xrightarrow{P} \mathbb{E}(H(U_i, V_i)) = I$$

c)

```
## Setup ##############################################################
set.seed(123)
## config #############################################################
n <- 1000

## implementation #####################################################

U <- runif(n)
V <- runif(n)

H <- function(u, v){
  return(v <= u^2)
}
```

```
sum(H(U,V))/n

# output: [1] 0.343
```

True result:

$$\int_0^1 x^2 dx = \frac{1}{3}x^3|_0^1 = \frac{1}{3}$$

For a thousand iterations an error higher than 1% is pretty bad.