Sheet 4

Problem 1

We are going to additionally assume that:

$$X_n : \Omega \to \mathbb{R} \quad Y_n : \Omega \to \mathbb{R} \quad \forall n \in \mathbb{N}$$

a)

We know that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$. Levys c.t. then states that:

$$\varphi_{X_n}(t) \to \varphi_X(t) \quad \varphi_{Y_n}(t) \to \varphi_Y(t) \quad \forall t \in \mathbb{R}$$

Using Levy's c.t. we need to show that:

$$\varphi_{Y_n+X_n}(t) \to \varphi_{Y+X}(t)$$

So knowing this and using independence and the proposition from slide 93 S.T. we can state that:

$$\varphi_{X_n+Y_n}(t) = \varphi_{X_n}(t)\varphi_{Y_n}(t) = \mathbb{E}(\exp(itX_n))\mathbb{E}(\exp(itY_n)) \tag{1}$$

We can show that X and Y also must be independent in this case we can also state that:

$$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t) \tag{2}$$

It follows that:

$$\begin{aligned}
\lim_{n\to\infty} \varphi_{X_n+Y_n}(t) &= \lim_{n\to\infty} [\varphi_{X_n}(t)\varphi_{Y_n}(t)] \\
&= \lim_{n\to\infty} \varphi_{X_n}(t) \lim_{n\to\infty} \varphi_{Y_n}(t) \\
&= \varphi_X(t)\varphi_Y(t) \\
&\underset{(1)}{=} \varphi_{X+Y}(t)
\end{aligned}$$

It should be noted, that this is just a proof sketch. Some of the parts should be proven separately such as:

1): Let $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}$ and $(b_n)_{n\in\mathbb{N}} \in \mathbb{C}$ with $a_n \to a$ and $b_n \to b$ then:

$$a_n b_n \to ab$$

2) Let $X_n \to X$ and $Y_n \to Y$ be seq. of r.v. with X_n and Y_n independent for every $n \in \mathbb{N}$. Then it holds that X and Y are also independent.

b)

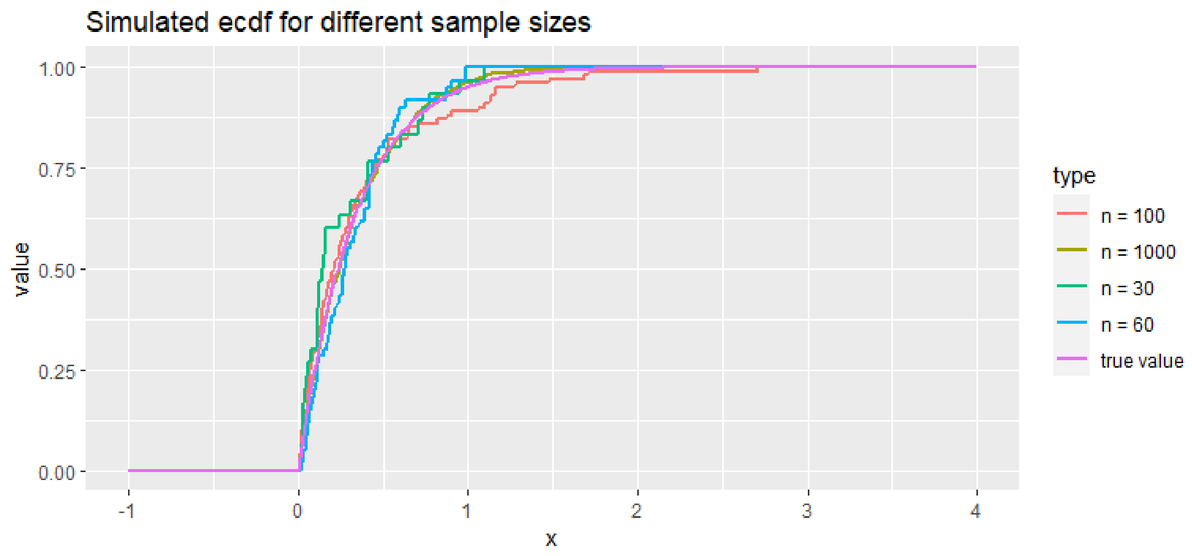
As independence plays a role in seperating the characteristic function into the product, the proof falls apart after dropping the assumption, that X_n and Y_n are independent.

Problem 2

```
## config ##################################################
n <- c(30, 60, 100, 1000)
lambda <- 3
estimate_points <- -100:400/100
set.seed(123)

## exercise ################################################
eval_ecdf <- function(data, estimate_points) {
  result <- rep(NA, length(estimate_points))
  i <- 1
  for (x in estimate_points) {
    result[i] <- sum(data <= x)/length(data)
    i = i+1
  }
  return(result)
}

sim_ecdf <- function(n, lambda, estimate_points) {
  data <- rexp(n, lambda)
  estimates <- eval_ecdf(data, estimate_points)
  return(estimates)
}

sim_ecdf_sup <- function(n, lambda, estimate_points){
  estimates <- sim_ecdf(n, lambda, estimate_points)
  true_values <- pexp(estimate_points, 3)
  return(max (abs(estimates-true_values)))
}
```

```
## ploting the curve of ecdf for different ns #################
result <- data.frame(type = rep("true value", length(estimate_po
                     value = pexp(estimate_points, lambda),
                     x = estimate_points)
for (sample_n in n) {
  current_sim <- data.frame(type= rep(paste("n =", sample_n), le
                            value = sim_ecdf(sample_n, lambda, e
                            x = estimate_points)
  result <- rbind(result, current_sim)
}

result %>% ggplot(aes(x = x,
                      y = value,
                      group = type,
                      color = type)) +
  geom_line(size = 1) +
  labs(title = "Simulated ecdf for different sample sizes")

## computing estimates for sup of abs-error ##################
estimates <- rep(NA, length(n))

j <- 1
for (sample_n in n){
  estimates[j] <- sim_ecdf_sup(sample_n, lambda, estimate_points
  j = j+1
}

estimates
# output:
# [1] 0.13123203 0.13377378 0.06567263 0.02018836
```

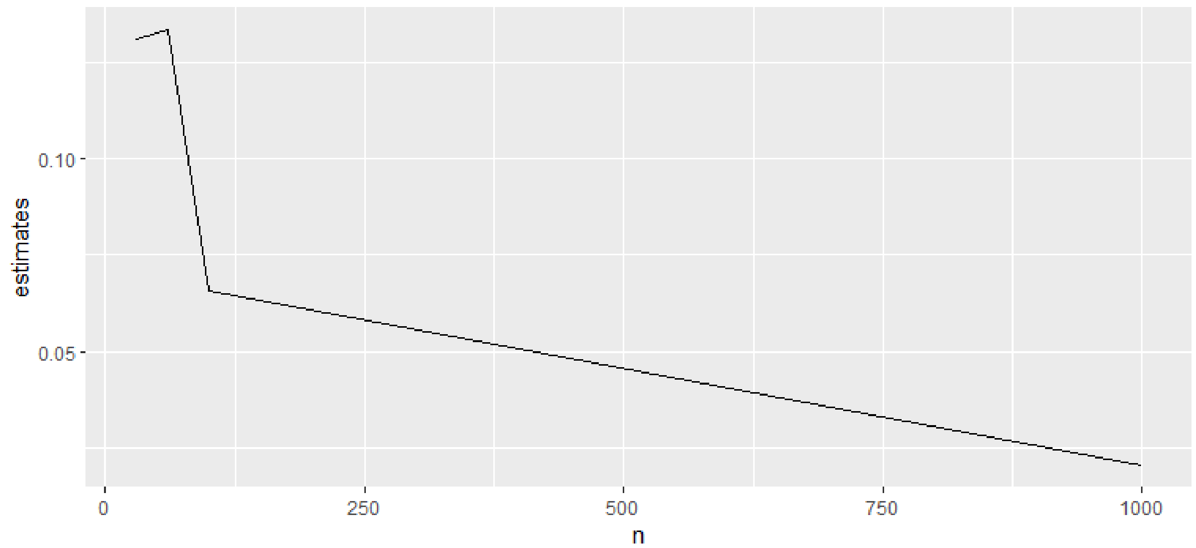
Plot of the ecdf



Here it is a bit hard to see but the estimate for the cdf gets better as the sample size increases.

The same can be seen in the plot of the supremum of the absolute error.

We get the behavior as expected. The maximal error decreases as n increases:



Problem 3

i)

In order for the statement to hold true it must hold that:

$$f^{(k)}(x) = (-1)^{k+1}(k-1)!(1+x)^{-k} \tag{3}$$

We can do this by way of **induction.** So starting at $k = 1$:

$$f^{(1)}(x) = \frac{1}{1+x} = (-1)^2(1-1)!(1+x)^{-1}$$

Then moving to I.S. $k \to k+1$:

$$\begin{aligned} f^{(k+1)}(x) = \frac{\partial f^{(k)}(x)}{\partial x} \\ &\underset{I.H.}{=} \frac{\partial}{\partial x}(-1)^{k+1}(k-1)!(1+x)^{-k} \\ &= (-1)^{k+1}(k-1)!(1+x)^{-(k+1)}(-k) \\ &= (-1)^{k+1+1}(k+1-1)!(1+x)^{-(k+1)} \end{aligned}$$

proving the statement.

We can now use this statement to show that the kth order polynomial is:

$$\begin{aligned} \frac{f^{(k)(0)}(0)}{k!}x^k &\underset{(3)}{=} (-1)^{k+1}(k-1)!(1+0)^{-k}\frac{1}{k!}x^k \\ &= (-1)^{k+1}(k-1)!\frac{1}{k!}x^k \\ &= (-1)^{k+1}\frac{x^k}{k} \end{aligned}$$

ii)

We will move from the RHS and prove by induction. Starting with $k = 0$ we get:

$$\begin{aligned} \int_a^{a+x} \frac{(a+x-t)^0}{0!} g^{0+1}(t)dt &= \int_a^{a+x} \frac{1}{1} g^{(1)}(t)dt \\ &= \int_a^{a+x} g^{(1)}(t)dt \\ &= g(a+x) - g(a) \\ &= g(a+x) - \sum_{j=0}^{0} \frac{g^{(j)}(a)x^j}{j!} \end{aligned}$$

Moving on to IS $k \to k+1$:

$$
\begin{aligned}
\int_a^{a+x} \frac{(a+x-t)^{k+1}}{(k+1)!} g^{(k+1+1)}(t)dt \underset{\text{int. by parts}}{=} & \left[\frac{(a+x-t)^{k+1}}{(k+1)!} g^{(k+1)}(t)\right]_a^{a+x} \\
& - \int_a^{a+x} (-1)(k+1)\frac{(a+x-t)^k}{(k+1)!} g^{(k+1)}(t)dt \\
& = \left[\frac{(a+x-t)^{k+1}}{(k+1)!} g^{(k+1)}(t)\right]_a^{a+x} \\
& + \int_a^{a+x} \frac{(a+x-t)^k}{k!} g^{(k+1)}(t)dt \\
& \underset{I.H.}{=} \left[\frac{(a+x-t)^{k+1}}{(k+1)!} g^{(k+1)}(t)\right]_a^{a+x} \\
& + g(a+x) - \sum_{j=0}^{k} \frac{g^{(j)}(a)x^j}{j!} \\
& = \left[\frac{(a+x-a-x)^{k+1}}{(k+1)!} g^{(k+1)}(a+x)\right] \\
& - \left[\frac{(a+x-a)^{k+1}}{(k+1)!} g^{(k+1)}(a)\right] \\
& + g(a+x) - \sum_{j=0}^{k} \frac{g^{(j)}(a)x^j}{j!} \\
& = - \left[\frac{(x)^{k+1}}{(k+1)!} g^{(k+1)}(a)\right] \\
& + g(a+x) - \sum_{j=0}^{k} \frac{g^{(j)}(a)x^j}{j!} \\
& = g(a+x) - \sum_{j=0}^{k+1} \frac{g^{(j)}(a)x^j}{j!}
\end{aligned}
$$

Which proves the second statement

iii)

Inserting the derivative into the result from ii) results in:

$$
\begin{aligned}
\int_a^{a+x} \frac{(a+x-t)^k}{k!} f^{(k+1)}(t)dt \underset{(3)}{=} & \int_a^{a+x} \frac{(a+x-t)^k}{k!} (-1)^{k+2} k!(1+t)^{-(k+1)}dt \\
= & \int_a^{a+x} \frac{(a+x-t)^k}{(1+t)^{k+1}} (-1)^{k+2}dt
\end{aligned}
$$

iv)

We now use the absolute value of this term and show that:

$$|R(f,0)(x)| \underset{n\to\infty}{\longrightarrow} 0$$

This means that:

$$\begin{aligned}\lim_{k\to\infty} |R_k(f,0)(x)| &= \lim_{k\to\infty} \left| \int_0^x \frac{(x-t)^k}{(1+t)^{k+1}} (-1)^{k+1} dt \right| \\ &= \lim_{k\to\infty} \left| \int_0^x \frac{(x-t)^k}{(1+t)^{k+1}} dt \right|\end{aligned}$$

substituting $z = 1 + t$ leads to:

$$\begin{aligned}\lim_{k\to\infty} \left| \int_0^x \frac{(x-t)^k}{(1+t)^{k+1}} dt \right| &= \lim_{k\to\infty} \left| \int_{-1}^{x-1} \frac{(x-(z-1))^k}{z^{k+1}} dz \right| \\ &\underset{x>z}{\leq} \lim_{k\to\infty} \left| \int_{-1}^{x-1} \left(\frac{(1+x-z)}{z} \right)^{k+1} dz \right| \\ &\underset{hint}{\leq} \lim_{k\to\infty} \left| \int_{-1}^{x-1} |x|^{k+1} dz \right| \\ &= \lim_{k\to\infty} \left[|x|^{k+1} t \right]_{-1}^{x-1} \\ &= \lim_{k\to\infty} \left(|x|^{k+1} (x - 1 - (-1)) \right) \\ &= \lim_{k\to\infty} |x|^{k+1} x \\ &\leq \lim_{k\to\infty} |x|^{k+2}\end{aligned}$$

And since $x \in (-1, 1)$ or $|x| < 1$ we can state that $|x|^{k+2} \to 0$.

Therefore proving that:

$$|R(f,0)(x)| \to 0 \quad \forall x \in (-1,1)$$

Problem 4

We can write the transformation as follows:

– –

$$
\begin{aligned}
Z_n = \sqrt{n}\frac{\bar{X}_n - \mathbb{E}(X_1)}{\sqrt{Var(X_1)}} \underset{iid}{=} \sqrt{n}\frac{\bar{X}_n - \mu}{\sigma} \\
&= \frac{\sqrt{n}}{\sigma}\frac{1}{n}\sum_{i=1}^{n} X_i - \sqrt{n}\frac{\mu}{\sigma} \\
&= \frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{n} X_i - \sqrt{n}\frac{\mu}{\sigma}
\end{aligned}
$$

Using **Theorem 4.4 iii),** we can think of the sum like $X_n \sim \mathcal{N}(\tilde{\mu}, \Sigma)$ with $\tilde{\mu} \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n \times n}$. Also $\tilde{\mu} = (\mu, ..., \mu)$ and $\Sigma = diag(\sigma^2, ..., \sigma^2)$. And $c = (1, ..., 1)' \in \mathbb{R}^n$. Then:

$$
\sum_{i=1}^{n} X_i = cX_n \underset{T.4.4iii)}{\sim} \mathcal{N}(c'\tilde{\mu}, c'\Sigma c) = \mathcal{N}(n\mu, n\sigma^2)
$$

You can also use **Theorem 4.4. v)** To get the same result.

Also using **Theorem 4.4 ii)** any linear transformation of a normally distributed r.v. is also normally distributed. Using $k = d = 1$ and $A = \frac{1}{\sqrt{n}\sigma}$ and $b = -\sqrt{n}\frac{\mu}{\sigma}$ we get:

$$
\begin{aligned}
\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n} X_i - \sqrt{n}\frac{\mu}{\sigma} \underset{T.4.4ii)}{\sim} \mathcal{N}\left(\frac{1}{\sigma\sqrt{n}}n\mu - \sqrt{n}\frac{\mu}{\sigma}, \left(\frac{1}{\sigma\sqrt{n}}\right)^2 n\sigma^2\right) \\
&= \mathcal{N}\left(\frac{\mu}{\sigma}\sqrt{n} - \sqrt{n}\frac{\mu}{\sigma}, \frac{n\sigma^2}{n\sigma^2}\right) \\
&= \mathcal{N}(0, 1)
\end{aligned}
$$

Since $Z_n \sim \mathcal{N}(0, 1)$ holds for every $n \in \mathbb{N}$, we can say that it also holds for $\lim_{n\to\infty} Z_n = Z$.