Sheet 5

Problem 1

a)

$$\begin{aligned} MSE(T_n) &= \mathbb{E}_\theta((T_n-\tau)^2) \\ &= \mathbb{E}_\theta(T_n^2 - 2T_n\tau + \tau^2) \\ &= \mathbb{E}_\theta(T_n^2) - 2\tau\mathbb{E}_\theta(T_n) + \tau^2 \\ &= \mathbb{E}_\theta(T_n^2) - \mathbb{E}_\theta(T_n)^2 + \mathbb{E}_\theta(T_n)^2 - 2\tau\mathbb{E}_\theta(T_n) + \tau^2 \\ &= Var(T_n) + \mathbb{E}_\theta(T_n)^2 - 2\tau\mathbb{E}_\theta(T_n) + \tau^2 \\ &= Var(T_n) + (\mathbb{E}_\theta(T_n) - \tau)^2 \\ &= Var(T_n) + Bias(T_n)^2 \quad \blacksquare \end{aligned}$$

b)

We need to show that:

$$MSE(T_n) \to 0 \implies \forall \varepsilon > 0: \quad P(|T_n - \tau| \leq \varepsilon) \to 1$$

So we can state that (using markov's inequality with $g(x) = x^2$):

$$\begin{aligned} P(|T_n - \tau| \geq \varepsilon) &\underset{m.i.}{\leq} \frac{\mathbb{E}(|T_n-\tau|^2)}{\varepsilon^2} \\ &= \frac{\mathbb{E}((T_n-\tau)^2)}{\varepsilon^2} \\ &= \frac{MSE(T_n)}{\varepsilon^2} \to 0 \end{aligned}$$

And because P is a non-negative mapping we can state that:

$$0 \leq \lim_{n\to\infty} P(|T_n - \tau|) \leq 0 \implies \lim_{n\to\infty} P(|T_n-\tau|) = 0 \quad \blacksquare$$

c)

We are going to define:

$$Y_n := \sqrt{n}(T_n - \tau) \quad B_n := \frac{1}{\sqrt{n}}$$

Because you could say that $B_n(\omega) = 1/\sqrt{n}$ for every $\omega \in \Omega$. Then it holds that B_n is measureable (because the reverse image contains Ω and $\emptyset$, which is contained in any sigma algebra) and therefore a r.v.. It also holds that (exercise):

$$Y_n \xrightarrow{d} Y \sim \mathcal{N}(0, \sigma^2) \quad B_n \xrightarrow{d} 0$$

Slutsky now implies that:

$$B_n Y_n \xrightarrow{d} 0Y = 0$$

We have shown on Problem 3 of sheet 1 a) that this implies:

$$B_n Y_n \xrightarrow{p} 0 \Leftrightarrow (T_n - \tau) \xrightarrow{p} 0$$

Continuous mapping theorem then implies that (for $h(x) = x + \tau$:

$$(T_n - \tau) \xrightarrow{p} 0 \implies T_n = T_n - \tau + \tau = h(B_n Y_n) \xrightarrow{p} h(0) = \tau$$

So finally we get the result that:

$$T_n \xrightarrow{p} \tau \quad \blacksquare$$

Problem 2

a)

We can say (using dominated convergence theorem):

$$\begin{aligned}
\frac{\partial}{\partial t}\varphi_{X_1}(t)|_{t=0} &= \frac{\partial}{\partial t}\mathbb{E}(\exp(itX_1)|_{t=0} \\
&\underset{d.c.t.}{=} \mathbb{E}\left(\frac{\partial}{\partial t}\exp(itX_1)\right)|_{t=0} \\
&= \mathbb{E}(iX_1 \exp(itX_1))|_{t=0} \\
&= \mathbb{E}(iX_1) \\
&= i\mathbb{E}(X_1) = i\mu \quad \blacksquare
\end{aligned}$$

We have seen the property of characteristic functions to derive moments:

$$\varphi^{(k)}(0) = i^k \mathbb{E}(X^k)$$

Using $k = 1$ results in:

$$\varphi'_{X_i}(0) = i\mathbb{E}(X_i) \underset{iid}{=} i\mu$$

Also:

$$\varphi_{X_i}(0) = \mathbb{E}(\exp(i0X_i)) = \mathbb{E}(\exp(0)) = 1 \quad \blacksquare$$

b)

Using the definition of a characteristic function and the iid property of the sample, we can show that:

$$\begin{aligned}
\varphi_{\bar{X}_n}(t) &= \mathbb{E}\left(\exp\left(it\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right) \\
&= \mathbb{E}\left(\exp\left(\sum_{i=1}^{n} it\frac{1}{n}X_i\right)\right) \\
&= \mathbb{E}\left(\prod_{i=1}^{n} \exp\left(it\frac{1}{n}X_i\right)\right) \\
&\underset{\text{independence}}{=} \prod_{i=1}^{n} \mathbb{E}(\exp\left(it\frac{1}{n}X_i\right) \\
&\underset{iid}{=} \prod_{i=1}^{n} \varphi_{X_1}\left(\frac{1}{n}t\right) \\
&= \left(\varphi_{X_1}\left(\frac{1}{n}t\right)\right)^n \quad \blacksquare
\end{aligned}$$

c)

Using taylor expansion with $c = 0$ for the characteristic function of X_1 it holds that:

$$\varphi_{X_1}(t) = \varphi_{X_1}(0) + \varphi'_{X_1}(0)t + R_1(\varphi_{X_1}, 0)(t)$$

We can then insert the results of **a):**

$$\varphi_{X_1}(t) = 1 + i\mu t + R_1(\varphi_{X_1}, 0)(t) \quad \forall t \in \mathbb{R}$$

Then to continue, we use the result of the lecture, that:

$$R_n(f, c)(x) = o(|x - c|^n) \text{ as } x \to c$$

If we look at $\varphi_{X_1}(t/n)$ we have $t/n \to 0 \forall t$, so in this case we can say that:

$$\varphi_{X_1}\left(\frac{t}{n}\right) = 1 + i\mu\frac{t}{n} + o\left(\left|\frac{t}{n} - 0\right|\right) \quad \forall t \in \mathbb{R}$$

Then we can insert this result into the result of **b):**

$$\varphi_{\bar{X}_n}(t) = \left(\varphi_{X_1}\left(\frac{t}{n}\right)\right)^n = \left(1 + i\mu\frac{t}{n} + o\left(\left|\frac{t}{n}\right|\right)\right)^n \quad \forall t \in \mathbb{R}$$

As $n \to \infty$ we can use the hint and prove that:

$$\lim_{n\to\infty} \varphi_{\bar{X}_n}(t) = \lim_{n\to\infty} \left(1 + i\mu\frac{t}{n} + o\left(\left|\frac{t}{n}\right|\right)\right)^n = \exp(it\mu) \quad \blacksquare$$

d)

Note, that the characteristic function of a constant $\mu < \infty$ is defined as:

$$\varphi_\mu(t) = \mathbb{E}(\exp(it\mu)) = \exp(it\mu)$$

And so we know that:

$$\varphi_{\bar{X}_n}(t) \underset{n\to\infty}{\longrightarrow} \varphi_\mu(t) \quad \forall t \in \mathbb{R} \iff \bar{X}_n \xrightarrow{d} \mu$$

But then we can use the same statement, we have used in problem 1 and directly implicate that:

$$\bar{X}_n \xrightarrow{p} \mu \quad \blacksquare$$

Problem 3

a)

If we assume $\alpha = 2$ and use the formula for the moments of the gamma distribution, the first 4 moments are given by:

$$\mathbb{E}(X^1) = \frac{1}{\beta}\frac{\Gamma(2+1)}{\Gamma(2)} = \frac{2\Gamma(2)}{\beta\Gamma(2)} = \frac{2}{\beta} \tag{1}$$
$$\mathbb{E}(X^2) = \frac{1}{\beta^2}\frac{\Gamma(2+2)}{\Gamma(2)} = \frac{3\cdot 2}{\beta^2} = \frac{6}{\beta^2} \tag{2}$$
$$\mathbb{E}(X^3) = \frac{4\cdot 3\cdot 2}{\beta^3} = \frac{24}{\beta^3} \tag{3}$$
$$\mathbb{E}(X^4) = \frac{5!}{\beta^4} = \frac{120}{\beta^4} \tag{4}$$

b)

We already have iid for X_i^2 for all $i \in 1, ..., n$. In order to apply Lindeberg-Levy CLT we need to ensure $\mathbb{E}(X_i^2) < \infty$ and $Var(X_i^2) < \infty$. The first we have already shown in **a)**:

$$\mathbb{E}(X_i^2) \underset{a)}{=} \frac{6}{\beta^2} < \infty \quad \forall i$$

For the Variance we just have:

$$\begin{aligned} Var(X_i^2) &= \mathbb{E}\left((X_i^2)^2\right) - \mathbb{E}(X_i^2)^2 \\ &= \mathbb{E}(X_i^4) - \mathbb{E}(X_i^2)^2 \\ &\underset{a)}{=} \frac{120}{\beta^4} - \frac{6^2}{\beta^4} = \frac{84}{\beta^4} < \infty \end{aligned}$$

We can now apply the CLT of Lindeberg-Levy and say:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i^2 - \frac{6}{\beta^2}}{\frac{\sqrt{84}}{\beta^2}}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

But this is still not $\overline{X_n^2}$. So we need some transformations:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i^2-\frac{6}{\beta^2}}{\frac{\sqrt{84}}{\beta^2}}\right)=\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i^2-\frac{6}{\beta^2})}{\frac{\sqrt{84}}{\beta^2}}$$
$$=\frac{\frac{1}{n}\sum_{i=1}^{n}X_i^2-\frac{6}{\beta^2}}{\frac{\sqrt{84}}{\beta^2}\frac{1}{\sqrt{n}}}$$

It now holds that:

$$\frac{\overline{X_n^2}-\frac{6}{\beta^2}}{\frac{1}{\sqrt{n}}\frac{\sqrt{84}}{\beta^2}}\xrightarrow{d}\mathcal{N}(0,1)$$

Problem 4

We know from iid that (using the formula from the exercise):

$$\mathbb{E}(X_i)=\frac{\alpha}{\beta}$$

Also for the Variance for every X_i is:

$$\begin{aligned}Var(X_i)&=\mathbb{E}(X_1^2)-\mathbb{E}(X_1)^2\\&=\frac{\alpha(1+\alpha)}{\beta^2}-\frac{\alpha^2}{\beta^2}\\&=\frac{\alpha}{\beta^2}\end{aligned}$$

Furthermore we have given $\beta=\sqrt{2}$ and $\alpha=2$. So:

$$\mathbb{E}(X_i)=\frac{2}{\sqrt{2}}=\sqrt{2}<\infty\quad\forall i \tag{5}$$
$$Var(X_i)=\frac{2}{2}=1<\infty\quad\forall i \tag{6}$$

Now we can apply Lindeberg-Levy CLT and know that:

$$Z_n:=\frac{\frac{1}{n}\sum_{i=1}^{n}X_i-\sqrt{2}}{\frac{1}{\sqrt{n}}}\xrightarrow{d}\mathcal{N}(0,1)$$

Using this we can approximate the cdf of Z_n with the standard normal cdf:

$$F_{Z_n}(x) \approx \Phi(x)$$

We want our estimation to not deviate by 0.01 with prob. of 95%. This is why:

$$\begin{aligned} P(|\bar{X}_n - \sqrt{2}| \leq 0.01) &\overset{!}{\geq} 0.95 \\ \iff P\left(-\sqrt{n}0.01 \leq \frac{\bar{X}_n - \sqrt{2}}{\frac{1}{\sqrt{n}}} \leq \sqrt{n}0.01\right) &\geq 0.95 \\ \iff P(-\sqrt{n}0.01 \leq Z_n \leq \sqrt{n}0.01) &\geq 0.95 \end{aligned}$$

Approximating the LHS with $\Phi(x)$ we get:

$$\Phi(\sqrt{n}0.01) - \Phi(-\sqrt{n}0.01) \geq 0.95$$

But now using symmetry of Φ around 0 this inequality is the same as:

$$\begin{aligned} \Phi(\sqrt{n}0.01) &\geq 0.975 \\ \iff \sqrt{n}0.01 &\geq \Phi^{-1}(0.975) \\ \iff \sqrt{n}0.01 &\geq 1.96 \end{aligned}$$

So we are trying to solve:

$$\begin{aligned} & \sqrt{n}0.01 \geq 1.96 \\ \iff \quad & n \geq (1.96 \cdot 100)^2 = 38416 \end{aligned}$$

In summary we get that around 38.5 k samples are required for 95% confidence intervals to be smaller than 0.02.