Sheet 1

Exercise 1

a)

$$egin{aligned} \left(\liminf_{n o \infty} A_n
ight)^c &= \left(igcup_{n=1}^\infty \left(igcap_{m=n}^\infty A_m
ight)
ight)^c \ &= \prod_{\mathrm{De\ Morgan}} igcap_{n=1}^\infty \left(igcap_{n=m}^\infty A_m
ight)^c \ &= \prod_{\mathrm{De\ Morgan}} igcap_{n=1}^\infty igcup_{m=n}^\infty A_m^c \ &= \limsup_{\mathrm{def\ } n o \infty} A_n^c \end{aligned}$$

b)

Assuming $A_{n+1}\subset A_n$ we can say (starting from the RHS):

$$egin{aligned} \limsup_{n o \infty} A_n &= igcap_{n=1}^\infty \left(igcup_{m=n}^\infty A_m
ight) \ &= igcap_{A_{n+1} \subset A_n} igcap_{n=1}^\infty A_n \ &= \lim_{n o \infty} A_n \end{aligned}$$

Also:

$$egin{aligned} \liminf_{n o\infty} A_n &= igcup_{n=1}^\infty \left(igcap_{m=n}^\infty A_m
ight) \ &= igcup_{A_{n+1}\subset A_n} igcup_{n=1}^\infty \left(\lim_{m o\infty} A_m
ight) \ &= \lim_{m o\infty} A_m \end{aligned}$$

So both the limes inferior and superior are the convergence set of $(A_n)_{n\in\mathbb{N}}$.

Exercise 2

a)

We can transfer this statement to:

$$P(Y_n=1)=rac{1}{n+1}\wedge P(Y_n=0)=rac{n}{n+1}$$

Starting with $P(Y_n = 1)$:

$$P(Y_n=1)=rac{1}{2}\cdotrac{2}{3}\cdotsrac{n}{n+1}=rac{n!}{(n+1)!}=rac{1}{n+1}$$

Then continuing with $P(Y_n = 0)$

$$P(Y_n=0)=1-P(Y_n=1)=1-rac{1}{n+1}=rac{n}{n+1}$$

b)

Given any $\varepsilon>0$ it holds that:

$$egin{aligned} P(|Y_n| < arepsilon) &\geq P(Y_n = 0) \ &= rac{n}{n+1} & \mathop{\longrightarrow}\limits_{n o \infty} 1 \end{aligned}$$

Therefore we can say:

$$P(|Y_n-0|\geq arepsilon)=1-P(|Y_n|$$

For convergence in distribution we can look at the cdf of Y_n :

$$F_{Y_n}(x) = egin{cases} 0 ext{ if } x < 0 \ rac{n}{n+1} ext{ if } 0 \leq x < 1 \ 1 ext{ if } x \geq 1 \end{cases}$$

While a cdf of a constant is:

$$F_0(x) = egin{cases} 0 ext{ if } x < 0 \ 1 ext{ if } x \geq 0 \end{cases}$$

Now let $x \in \mathbb{R}$. Then it holds that:

$$F_{Y_n}(x) = F_0(x) \quad \forall x \in (-\infty, 0) \cup [1, \infty)$$

So it is left to show that:

$$F_{Y_n}(x) o F_0(x) \quad orall x \in [0,1)$$

For this case $F_{Y_n}(x)$ is uniquely defined. So it holds that:

$$F_{Y_n}(x)=rac{n}{n+1} \mathop{\longrightarrow}\limits_{n o\infty} 1=F_0(x)$$

It therefore holds that: $F_{Y_n}(x) o F_0(x)orall x\in\mathbb{R}.$ And therefore $Y_n\stackrel{d}{\longrightarrow} 0.$

Exercise 3

a)

Given $X_n \stackrel{d}{\longrightarrow} c_i$ this means that:

$$F_{X_n}(x) o F(x) = egin{cases} 0 ext{ if } x < c \ 1 ext{ if } x \geq c \end{cases}$$

So we can say:

$$egin{aligned} P(|X_n-c| \leq arepsilon) &= P(X_n \in [c-arepsilon, c+arepsilon) \ &= P(X_n \leq c+arepsilon) - P(X_n \leq c-arepsilon) \ &= F_{X_n}(c+arepsilon) - F_{X_n}(c-arepsilon) \ & \stackrel{\longrightarrow}{\longrightarrow} F(c+arepsilon) - F(c-arepsilon) = 1 - 0 = 1 \end{aligned}$$

b)

Given $P(\{\omega\in\Omega:X_n(\omega) o X(\omega)\})=1$ we need to show, that: $P(|X_n-X|\geq arepsilon) o 0$ for all arepsilon>0.

As stated in the exercise for any $\varepsilon>0$ we define $Y_n:=1_{|X_n-X|>\varepsilon}(\omega)$. Then we know that:

$$\mathbb{E}(Y_n) = \int_{\Omega} Y_n dP = P(|X_n - X| > arepsilon)$$

So:

$$P(|X_n - X| > \varepsilon) \to 0 \Longleftrightarrow \mathbb{E}(Y_n) \to 0$$

Firstly let $Y:\Omega o\mathbb{R}$ with $Y(\omega)=1 orall x\in\mathbb{R}.$ Then it holds that $|Y_n|\leq Y$ and:

$$\mathbb{E}(Y) = \int Y dP = 1 P(\Omega) = 1 < \infty$$

Therefore we can apply dominated convergence theorem and it holds that:

$$\Longrightarrow \lim_{d.c.t.} \mathbb{E}(Y_n) = \mathbb{E}\left(\lim_{n o\infty} Y_n
ight)$$

And it holds that:

$$egin{aligned} \mathbb{E}\left(\lim_{n o\infty}Y_n
ight) &\leq 1P(\{\omega\in\Omega:X_n(\omega)
egtrightarrow 1\}) + 0\cdot P(\{\omega\in\Omega:X_n(\omega) o 1\}) \ &= 1\cdot 0 + 0\cdot 1 = 0 \end{aligned}$$

So therefore we have shown that:

$$\mathbb{E}(Y_n) o 0 \Longleftrightarrow X_n \stackrel{p}{\longrightarrow} 1$$

Exercise 4

We will show $A_n \stackrel{d}{\longrightarrow} Y$ by showing $F_{A_n}(x) o F_Y(x) orall x \in \mathbb{R}.$ Looking at $F_{A_n}(x)$:

$$egin{aligned} F_{A_n}(x) &= P(A_n \leq x) \ &= P(\max\{X_1,...,X_n\} \leq x + \log(n)) \ &= \prod_{i = 1}^n P(X_i \leq x + \log(n)) \ &= \prod_{i = 1}^n 1 - \exp(-(x + \log(n))) \ &= \prod_{i = 1}^n (1 - \exp(-x - \log(n))) \ &= \prod_{i = 1}^n (1 - \exp(-x)/n) \ &= (1 - \exp(-x)/n)^n \quad orall x \in \mathbb{R} \end{aligned}$$

We know that:

$$\lim_{n\to\infty} (1+x/n)^n = \exp(x)$$

So:

$$\lim_{n o\infty}F_{A_n}(x)=\lim_{n o\infty}(1-\exp(-x)/n)^n=\exp(-\exp(-x))=F_Y(x)orall x\in\mathbb{R}$$