Sheet 1

Exercise 1

a)

$$\left(\liminf_{n\to\infty} A_n\right)^c = \left(\bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} A_m\right)\right)^c$$
$$\underset{\text{De Morgan}}{=} \bigcap_{n=1}^{\infty}\left(\bigcap_{n=m}^{\infty} A_m\right)^c$$
$$\underset{\text{De Morgan}}{=} \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m^c$$
$$\underset{\text{def}}{=} \limsup_{n\to\infty} A_n^c$$

b)

Assuming $A_{n+1} \subset A_n$ we can say (starting from the RHS):

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{m=n}^{\infty} A_m\right)$$
$$\underset{A_{n+1}\subset A_n}{=} \bigcap_{n=1}^{\infty} A_n$$
$$= \lim_{n\to\infty} A_n$$

Also:

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} A_m\right)$$
$$\underset{A_{n+1}\subset A_n}{=} \bigcup_{n=1}^{\infty}\left(\lim_{m\to\infty} A_m\right)$$
$$= \lim_{m\to\infty} A_m$$

So both the limes inferior and superior are the convergence set of $(A_n)_{n\in\mathbb{N}}$.

Exercise 2

a)

We can transfer this statement to:

$$P(Y_n = 1) = \frac{1}{n+1} \wedge P(Y_n = 0) = \frac{n}{n+1}$$

Starting with $P(Y_n = 1)$:

$$P(Y_n = 1) = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n}{n+1} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

Then continuing with $P(Y_n = 0)$

$$P(Y_n = 0) = 1 - P(Y_n = 1) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

b)

Given any $\varepsilon > 0$ it holds that:

$$\begin{aligned} P(|Y_n| < \varepsilon) &\geq P(Y_n = 0) \\ &= \frac{n}{n+1} \underset{n\to\infty}{\longrightarrow} 1 \end{aligned}$$

Therefore we can say:

$$P(|Y_n - 0| \geq \varepsilon) = 1 - P(|Y_n| < \varepsilon) \underset{P(|Y_n|<\varepsilon)\to 1}{\longrightarrow} 0 \Leftrightarrow Y_n \xrightarrow{P} 0$$

For convergence in distribution we can look at the cdf of Y_n:

$$F_{Y_n}(x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{n}{n+1} \text{ if } 0 \leq x < 1 \\ 1 \text{ if } x \geq 1 \end{cases}$$

While a cdf of a constant is:

$$F_0(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \end{cases}$$

Now let $x \in \mathbb{R}$. Then it holds that:

$$F_{Y_n}(x) = F_0(x) \quad \forall x \in (-\infty, 0) \cup [1, \infty)$$

So it is left to show that:

$$F_{Y_n}(x) \to F_0(x) \quad \forall x \in [0, 1)$$

For this case $F_{Y_n}(x)$ is uniquely defined. So it holds that:

$$F_{Y_n}(x) = \frac{n}{n+1} \underset{n\to\infty}{\longrightarrow} 1 = F_0(x)$$

It therefore holds that: $F_{Y_n}(x) \to F_0(x) \forall x \in \mathbb{R}$. And therefore $Y_n \xrightarrow{d} 0$.

Exercise 3

a)

Given $X_n \xrightarrow{d} c$, this means that:

$$F_{X_n}(x) \to F(x) = \begin{cases} 0 \text{ if } x < c \\ 1 \text{ if } x \geq c \end{cases}$$

So we can say:

$$\begin{aligned} P(|X_n - c| \leq \varepsilon) &= P(X_n \in [c - \varepsilon, c + \varepsilon) \\ &= P(X_n \leq c + \varepsilon) - P(X_n \leq c - \varepsilon) \\ &= F_{X_n}(c + \varepsilon) - F_{X_n}(c - \varepsilon) \\ &\underset{n\to\infty}{\longrightarrow} F(c + \varepsilon) - F(c - \varepsilon) = 1 - 0 = 1 \end{aligned}$$

b)

Given $P(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = 1$ we need to show, that: $P(|X_n - X| \geq \varepsilon) \to 0$ for all $\varepsilon > 0$.

As stated in the exercise for any $\varepsilon > 0$ we define $Y_n := 1_{|X_n - X| > \varepsilon}(\omega)$. Then we know that:

$$\mathbb{E}(Y_n) = \int_\Omega Y_n dP = P(|X_n - X| > \varepsilon)$$

So:

$$P(|X_n - X| > \varepsilon) \to 0 \iff \mathbb{E}(Y_n) \to 0$$

Firstly let $Y : \Omega \to \mathbb{R}$ with $Y(\omega) = 1 \forall x \in \mathbb{R}$. Then it holds that $|Y_n| \leq Y$ and:

$$\mathbb{E}(Y) = \int Y dP = 1P(\Omega) = 1 < \infty$$

Therefore we can apply dominated convergence theorem and it holds that:

$$\underset{d.c.t.}{\Longrightarrow} \lim_{n \to \infty} \mathbb{E}(Y_n) = \mathbb{E}\left(\lim_{n \to \infty} Y_n\right)$$

And it holds that:

$$\begin{aligned}\mathbb{E}\left(\lim_{n \to \infty} Y_n\right) &\leq 1P(\{\omega \in \Omega : X_n(\omega) \not\to 1\}) + 0 \cdot P(\{\omega \in \Omega : X_n(\omega) \to 1\}) \\ &= 1 \cdot 0 + 0 \cdot 1 = 0\end{aligned}$$

So therefore we have shown that:

$$\mathbb{E}(Y_n) \to 0 \iff X_n \xrightarrow{p} 1$$

Exercise 4

We will show $A_n \xrightarrow{d} Y$ by showing $F_{A_n}(x) \to F_Y(x) \forall x \in \mathbb{R}$.

Looking at $F_{A_n}(x)$:

$$
\begin{aligned}
F_{A_n}(x) &= P(A_n \leq x) \\
&= P(\max\{X_1, ..., X_n\} \leq x + \log(n)) \\
&\underset{\text{iid}}{=} \prod_{i=1}^{n} P(X_i \leq x + \log(n)) \\
&= \prod_{i=1}^{n} 1 - \exp(-(x + \log(n)) \\
&= \prod_{i=1}^{n} (1 - \exp(-x - \log(n)) \\
&= \prod_{i=1}^{n} (1 - \exp(-x)/n) \\
&= (1 - \exp(-x)/n)^n \quad \forall x \in \mathbb{R}
\end{aligned}
$$

We know that:

$$\lim_{n\to\infty} (1 + x/n)^n = \exp(x)$$

So:

$$\lim_{n\to\infty} F_{A_n}(x) = \lim_{n\to\infty} (1 - \exp(-x)/n)^n = \exp(-\exp(-x)) = F_Y(x) \forall x \in \mathbb{R}$$