Summary

- Excourses
  - Taylor-Series

    Let f be a real valued function on an open interval $(a,b)=I$ and $c \in I$ then we call:

    $$T_n(f,c)(x) = \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

    A taylor polynomial of order n.

    If f is infinitely differentiable on I we call:

    $$T(f,c)(x) = \lim_{n\to\infty} T_n(f,c)(x)$$

    The Taylor series.

    - Remainder term

      Given a function f and the Taylor pol. of order n, $T_n(f,c)(x)$ we call the **remainder term:**

      $$R_n(f,c)(x) = f(x) - T_n(f,c)(x)$$

      We always find a ζ between x and c such that:

      $$R_n(f,c)(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-c)^{n+1}$$

      We can then express $f(x)$ such that:

      $$f(x) = T_n(f,c)(x) + R_n(f,c)(x) = \sum_{k=1}^{n} \frac{f^{(k)(c)}}{k!}(x-c)^k + \frac{f^{(n+1)(\zeta)}}{(n+1)!}(x-c)^{n+1}$$

      And it holds that $R_n(f,c)(x) = o((x-c)^n)$ for $x \to c$. And $R_n(f,c)(x) = O(|x-c|^{n+1})$

  - Deterministic convergence
    - Convergence in Analysis

      Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. a_n converges to $a \in \mathbb{R}$ iff:

      $$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : |a_n - a| < \varepsilon \quad \forall n \geq n_o$$

We also say: $a_n \to a$ or $a_n \xrightarrow[n\to\infty]{} a$

- Limit points (or accumulation points)

  Are the convergence points of subsequences.

- Divergence in Analysis

  Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. a_n is divergent iff:

$$\forall M \in \mathbb{R}, \exists n_0 \in \mathbb{N} : a_n > M \quad \forall n \geq n_0$$

- Bounded sequences

  A real sequence $(a_n)_{n\in\mathbb{N}}$ is bounded, iff there exists an $M \in \mathbb{R}$ such that:

$$|a_n| \leq M \quad \forall n \in \mathbb{N}$$

- Important statements
  - Every convergent sequence has a unique limit point
  - Every bounded and monotone sequence is convergent
  - Every convergent sequence is also bounded
  - Every monotone sequence either di- or converges

- Pointwise convergence of functions

  Let f_n be a real functional sequence and f. Then f_n converges pointwise to f iff:

$$\forall x \in \mathbb{R} : f_n(x) \to f(x)$$

- Uniform convergence

  We say a real functional sequence f_n converges uniformly to f iff:

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \to 0$$

- Landau Symbolds
  - For sequences of real numbers

    Let $(a_n)_{n\in\mathbb{N}} \in \mathbb{R}$ and $(b_n)_{n\in\mathbb{N}} \in \mathbb{R}$ then it holds that:

$$a_n \in \mathcal{O}(b_n) \iff \exists M \in \mathbb{R} : \left|\frac{a_n}{b_n}\right| \leq M \quad \forall n \in \mathbb{N}$$

    So the sequence is bounded. Meaning the sequence b_n **grows** roughly the same or faster.

    If:

$$\left|\frac{a_n}{b_n}\right| \to 0 \Leftrightarrow a_n = o(b_n)$$

Meaning the sequence b_n grows faster than a_n.

- For random variables

  Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be sequences of r.v. then we say $X_n = \mathcal{O}_p(Y_n)$ if $|X_n/Y_n|$ is **stochastically bounded.**

  This means that for every $\varepsilon > 0$ we find a boundary M_ε such that:

  $$\sup_{n\in\mathbb{N}} P\left(\left|\frac{X_n}{Y_n}\right| > M_\varepsilon\right) < \varepsilon$$

  And we say $X_n = o_p(Y_n)$ iff $|X_n/Y_n|$ converges in probability to zero:

  $$X_n = o_p(Y_n) \Longleftrightarrow \left|\frac{X_n}{Y_n}\right| \xrightarrow{P} 0$$

- For functions

  Let $f, g : \mathbb{R} \to \mathbb{R}$ and $S \subset \mathbb{R}$ then we say: $f(x) \in \mathcal{O}(g(x))$ for all $x \in S$ if $\exists C \in \mathbb{R}$ such that $|f(x)| \leq C|g(x)|$. So the function can be bounded by the function g and a constant value C.

  We also say f behaves like g.

  We also say $f(x) = o(g(x))$ as $x \to \infty$ or $x \to 0$ if $\frac{f(x)}{g(x)} \underset{x\to\infty}{\longrightarrow} 0$ or $\frac{f(x)}{g(x)} \underset{x\to 0}{\longrightarrow} 0$.

- Computing with landau symbols

  Let $a_n = \mathcal{O}(n^\alpha)$ and $b_n = \mathcal{O}(n^\beta)$ then it holds that:

  1. $a_n + b_n = \mathcal{O}(n^{\max\{\alpha,\beta\}})$
  2. $a_n b_n = \mathcal{O}(n^{\alpha+\beta})$
  3. $a_n^\beta = \mathcal{O}(n^{\beta\alpha})$

- Modes of convergence
  - modes of convergence
    - Pointwise vs uniformly

      We say a functional sequence f_n converges pointwise to some function f iff:

      $$\forall x \in \mathbb{R} : f_n(x) \longrightarrow f(x)$$

      But if f_n converges uniformly then:

      $$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} : |f_n(x) - f(x)| < \varepsilon \ \ \forall x \in \mathbb{R}$$

- In probability

  We say $X_n \xrightarrow{P} X_0$ iff:

  $$\forall \varepsilon > 0 : P(|X_n - X_0| > \varepsilon) \longrightarrow 0$$

- In distribution

  We say $X_n \xrightarrow{d} X_0$ if:

  $$\mathbb{E}(f(X_n)) \longrightarrow \mathbb{E}(f(X_0)) \quad \forall f \in C_b(\mathbb{R}^d)$$

  Or if the cdf of X_n converges pointwise to the cdf of X_0:

  $$\forall x \in \mathbb{R} : F_{X_n}(x) \longrightarrow F_{X_0}(x)$$

- Lindeberg-Levy

  We can show convergence in distribution by showing pointwise convergence for the characteristic funciton:

  $$X_n \xrightarrow{d} X \iff \varphi_{X_n}(t) \to \varphi_X(t) \forall t \in \mathbb{R}^k$$

- Almost sure convergence

  We say $X_n \xrightarrow{a.s.} X_0$ iff:

  $$P(\{\omega \in \Omega : X_n(\omega) \longrightarrow X_0(\omega)\}) = 1$$

- convergence for random vectors:
  - in probability

    For this we can just use the euclidian distance (because we defined X_n and X_0 to always map into $\mathbb{R}^d$:

    $$\forall \varepsilon > 0 : P(\|X_n - X_0\| > \varepsilon) \longrightarrow 0$$

  - almost sure convergence

    Here we take the intercept on all dimentions:

    $$X_n \xrightarrow{a.s.} X_0 \iff P(X_n \to X_0) = 1$$

    But this meaning:

    $$P(X_n \to X_0) = P(\{X_{n_1} \to X_{0_1}\} \cap \cdots \cap \{X_{n_d} \to X_{0_n}\}) = 1$$

  - Theorem 3.11 (continuous mapping theorem)

Let X_n and Y_n be $\mathbb{R}^d$ valued r.v. and $h : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ be a continuous transformation then:

$$\begin{aligned} X_n &\xrightarrow{d} X_0 \implies h(X_n) \xrightarrow{d} h(X_0) \\ X_n &\xrightarrow{p} X_0 \implies h(X_n) \xrightarrow{p} h(X_0) \\ X_n &\xrightarrow{a.s.} X_0 \implies h(X_n) \xrightarrow{a.s.} h(X_0) \end{aligned}$$

- proof

  i)

  Given

  $X_n \stackrel{d}{\to} X_0 \Leftrightarrow \mathbb{E}(f(X_n)) \to \mathbb{E}(f(X_0)) \forall f \in C_b(\mathbb{R}^d)$

  We want to show that:

  $$h(X_n) \stackrel{d}{\to} h(X_0) \quad h : \mathbb{R}^d \to \mathbb{R}^k$$

  This translates to:

  $$\mathbb{E}(f(h(X_n)) \to \mathbb{E}(f(h(X_0)) \quad \forall f \in C_b(\mathbb{R}^k)$$

  We can say $g := f \circ h$. Then it directly follows that $g \in C_b(\mathbb{R}^d)$ then we can use where we started.

  iii)

  We have $X_n \stackrel{a.s.}{\to} X_0$ which translates to:

  $$P(D := \{\omega \in \Omega : X_n(\omega) \to X_0(\omega)\}) = 1$$

  So lets look at:

  $$\tilde{D} := \{\omega \in \Omega : h(X_n(\omega)) \to h(X_0(\omega))\}$$

  Continuity of h implies:

  $$X_n(\omega) \to X_0(\omega) \implies h(X_n(\omega)) \to h(X_0(\omega))$$

  This means:

  $$P(D) \leq P(\tilde{D}) \Leftrightarrow 1 \geq P(\tilde{D}) \geq 1 \implies P(\tilde{D}) = 1$$

- Slutsky's lemma for $\mathbb{R}^d$-valued r. vectors:

  Let $X_n : \Omega \to \mathbb{R}^d$ and $A_n : \Omega \to \mathbb{R}^k$ and $B_n : \Omega \to \mathbb{R}^{d \times k}$ with $A_n \stackrel{d}{\to} A$ and $X_n \stackrel{d}{\to} X_0$ and $B_n \stackrel{d}{\to} B$ where B and A deterministic. Then it holds that:

$$A_n + B_n X_n \xrightarrow{d} A + BX_0$$

▼ addition of r.v.

Let X_n and Y_n be sequences of r.v. such that:

$$X_n \xrightarrow{P} X_0 \quad Y_n \xrightarrow{P} Y_0$$

Then it holds that:

$$X_n + Y_n \xrightarrow{P} X_0 + Y_0$$

Same holds for almost sure convergence. BUT not for convergence in distribution.

▼ Slutsky's lemma

Let X_n and Y_n be sequences of r.v. with $Y_n \xrightarrow{d} c \in \mathbb{R}$ and $X_n \xrightarrow{d} X_0$. Then

1. $X_n + Y_n \xrightarrow{d} X_0 + c$
2. $X_n Y_n \xrightarrow{d} X_0 c$
3. $X_n / Y_n \xrightarrow{d} X_0 / c \quad \forall c \neq 0$

▼ proof sketch

▼ implications of modes of convergence

We have been given the following map for implications of convergence

$$\begin{tikzcd}
X_n \xrightarrow{L_p} X \arrow[r] & X_n \xrightarrow{L_{p-1}} X \arrow[r] & X_n \xrightarrow{L_1} X \arrow[dr] & & \\
& & & X_n \xrightarrow{P} X \arrow[r] & X_n \xrightarrow{d} X \\
& & X_n \xrightarrow{a.s.} X \arrow[ur] & &
\end{tikzcd}$$

▼ Theorem 3.9

If $X_n \xrightarrow{P} X_0$ then for any subsequence X_{n_k} there exists a further subsequence of the subsequence $X_{n_{k_l}}$ that converges almost surely.

▼ Multi-variate Normals

▼ Variance-Covariance Matrix

Let $X = (X_1, ..., X_d)'$ be an $\mathbb{R}^d$ valued random vector. Then we define:

$$Cov(X) := \mathbb{E}((X - \mathbb{E}(X))(X - \mathbb{E}(X))')$$

The covariance matrix is **symmetric** and **positive semidefinite.:**

$$\forall x \in \mathbb{R}^d : x'Cov(X)x \geq 0$$

It holds that for any $\mu \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d \times d}$:

$$Cov(AX + \mu) = ACov(X)A'$$

▼ Multivariate normal distribution

Let $X = (X_1, ..., X_d)$. Then X is a multivariate standard normal distribution if:

$$\mathbb{E}(X) = (0, ..., 0) \quad Cov(X) = I_d$$

You can also say $X \sim \mathcal{N}(0, I_d)$.

Now let $A \in \mathbb{R}^{d \times d}$ **regular i.e.** $\det(A) \neq 0$ and $\mu \in \mathbb{R}^d$ then:

$$Y := AX + \mu \sim \mathcal{N}(\mu, AA')$$

The pdf for Y looks like:

$$f_Y(y_1, ..., y_d) = \frac{1}{\sqrt{\det(2\pi AA')}} \exp\left(-\frac{1}{2}(y - \mu)'(AA')^{-1}(y - \mu)\right)$$

▼ Borel Cantelli and strong LLN

▼ Borell Cantelli

Let $\omega \in \Omega$ and A_n a sequence of Events. And:

$$A := \{\omega \in \Omega : \omega \in A_n \text{ for infinitly many } n \in \mathbb{N}\}$$

You can show that:

$$A = \limsup_{n \to \infty} A_n$$

Borel cantelli lemma states:

$$\sum_{i=1}^{\infty} P(A_n) < \infty \implies P\left(\limsup_{n \to \infty} A_n\right) = 0$$

If however A_n is independent:

$$\sum_{i=1}^{\infty} P(A_n) = \infty \implies P\left(\limsup_{n \to \infty} A_n\right) = 1$$

▼ Proof

i)

$$\begin{aligned} P\left(\limsup_{n\to\infty} A_n\right) &= P\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) \\ &\leq P\left(\bigcup_{n=k_0}^{\infty} A_n\right) \\ &\underset{\sigma-\text{sub add.}}{=} \sum_{n=k_0}^{\infty} P(A_n) \underset{k_0\to\infty}{\longrightarrow} 0 \end{aligned}$$

This implies:

$$0 \underset{P \text{ measure}}{\leq} P\left(\limsup_{n\to\infty} A_n\right) \leq 0 \implies P\left(\limsup_{n\to\infty} A_n\right) = 0$$

ii)

We need the following inequality

$$1 - p \leq \exp(-p) \quad p \in [0,1]$$

Show that:

$$P\left(\liminf_{n\to\infty} A_n^c\right) = 0$$

Bounding from above leads to:

$$
\begin{aligned}
P\left(\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} A_n^c\right) &\underset{\text{cont. from above}}{=} \lim_{k\to\infty} P\left(\bigcap_{n=k}^{\infty} A_n^c\right) \\
&= \lim_{k\to\infty} P\left(\lim_{N\to\infty}\bigcap_{n=k}^{N} A_n^c\right) \\
&\underset{\text{cont. from below}}{=} \lim_{k\to\infty}\lim_{N\to\infty} P\left(\bigcap_{n=k}^{N} A_n^c\right) \\
&\underset{\text{ind.}}{=} \lim_{k\to\infty}\lim_{N\to\infty}\prod_{n=k}^{N} P(A_n^c) \\
&= \lim_{k\to\infty}\lim_{N\to\infty}\prod_{n=k}^{N}(1-P(A_n)) \\
&= \lim_{k\to\infty}\lim_{N\to\infty}\prod_{n=k}^{N}\exp(-P(A_n)) \\
&= \lim_{k\to\infty}\lim_{N\to\infty}\exp\left(\sum_{n=k}^{N}-P(A_n)\right)
\end{aligned}
$$

But we started with assuming $\sum_{n=1}^{\infty} P(A_n) = \infty$

So it holds that:

$$
\sum_{n=k}^{\infty} P(A_n) = \infty \quad \forall k \in \mathbb{N}
$$

So:

$$
\exp\left(-\sum_{n=k}^{\infty} P(A_n)\right) = 0
$$

And therefore:

$$
\lim_{k\to\infty}\lim_{N\to\infty}\exp\left(\sum_{n=k}^{N}-P(A_n)\right) = \lim_{k\to\infty} 0 = 0
$$

And therefore we have shown the statement that:

$$
P\left(\liminf_{n\to\infty} A_n^c\right) = 0 \iff P\left(\limsup_{n\to\infty} A_n\right) = 1
$$

▼ Example

A_n is urne n and white ball is drawn. $1 - n^2$ black balls and 1 white balls.

Then:

$$P(A_n) = \frac{1}{n^2}$$

And:

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

So (borel cantelli):

$$P\left(\limsup_{n\to\infty} A_n\right) = 0$$

▼ Glivenko Cantelli

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}$-valued r.v. Then we have:

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \xrightarrow{a.s.} 0$$

Where:

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{x_i \le x}(x)$$

Is the empirical cdf.

▼ Central limit theorem of Lindeberg Levy

Let $(X_n)_{n\in\mathbb{N}}$ be a seq. of iid r.v. with $\mathbb{E}(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Then:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

▼ proof

Only notes from the lecture

Use Lindeberg-Levy c.t. and show that:

$$\varphi_{\frac{S_n - n\mu}{\sigma\sqrt{n}}}(t) \to \exp(-t^2/2)$$

Looking at the transformation:

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)$$

We can define:

$$Y_i = \left(\frac{X_i - \mu}{\sigma} \right) \implies \mathbb{E}(Y_i) = 0, Var(Y_i) = 1$$

Every transformation can now be expressed as:

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i$$

We can look at this transformation or just set $\mu = 0$ and $\sigma = 1$ without losing generality!

At some point we use Taylor-approximation because we don't know the distribution of X_i. The development point is chosen by $c = 0$. This leads to:

$$T_2(\varphi_{\frac{S_n}{\sqrt{n}}}, 0)(t) = \varphi_{\frac{S_n}{\sqrt{n}}}(0) + \varphi'_{\frac{S_n}{\sqrt{n}}}(0)t + \varphi''_{\frac{S_n}{\sqrt{n}}}(0)t^2$$

We can do this because we assumed: $\mathbb{E}(X_i) = \mu < \infty$ and $\mathbb{E}(X_i^2) = \sigma^2 < \infty$

We can show that the remainder term is $o\left(t^2\right)$.

▼ Lindeberg-Feller CLT

Consider resampling when increasing the sample-size (like bootstrap). This yields a triangular array of r.v.:

$$\begin{array}{l}
X_{1,1} \\
X_{2,1}, X_{2,2} \\
X_{3,1}, X_{3,2}, X_{3,3} \\
\dots\dots\dots\dots\dots\dots \\
\dots\dots\dots\dots\dots\dots\dots \\
X_{n,1}, X_{n,2} \dots\dots\dots\dots\dots X_{n,n} \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{array}$$

We just assume independence among the r.v. but not identical distributions.

So to summarize:

- We assume a new prob. space $\{\Omega_n, \mathcal{A}_n, P_n\}$ for every new row

- We assume different means and variances $\mu_{n,i}$ and $\sigma_{n,i}$.
- We assume, that the sample size is a sequence K_n that $K_n \to \infty$.

The price:

We need to show that (Lindeberg Condition):

$$\forall \epsilon > 0 : \frac{1}{\sigma_n^2} \sum_{i=1}^{K_n} \mathbb{E}(X_{n,i}^2 1_{\{|X_{n,i}| > \epsilon\sigma_n^2\}}) \to 0$$

Or the slightly stronger: Lyapunov condition:

$$\forall \delta > 0 : \frac{1}{\sigma_n^{2+\delta}} \sum_{i=1}^{K_n} \mathbb{E}(|X_{n,i}|^{2+\delta}) \to 0$$

- This condition needs moments higher than 2 to exist.

Or (even more strong) Feller-condition:

$$\max_{i=\{1,\ldots,K_n\}} \left\{ \frac{\sigma_{n,i}}{\sigma_n^2} \right\} \to 0$$

We can say:

$$\text{Lyapunov} \implies \text{Lindeberg} \implies \text{Feller}$$

But note: The Feller condition is NOT sufficient for proving Lindenberg-Feller. But Feller becomes an important tool, when **DIS**poving, that an asymptotic distribution is normal.

▼ Proof (incomplete)

Pure sketch: we are trying to get rid of the indicator function

We can write the inequality of the indicator in Lindeberg as follows:

$$\frac{|X_{n,i}|}{\epsilon\sigma_n} > 1 \Leftrightarrow \left(\frac{|X_{n.i}|}{\epsilon\sigma_n}\right)^\delta > 1$$

We can then say:

$$\begin{aligned}\frac{1}{\sigma_n^2} \sum_{i=1}^{K_n} \mathbb{E}(X_{n,i}^2 1_{\{|X_{n,i}| > \epsilon\sigma_n\}}) &= \frac{1}{\sigma_n^2} \sum_{i=1}^{K_n} \mathbb{E}(X_{n,i}^2 1_{\left\{\left(\frac{|X_{n,i}|}{\epsilon\sigma_n}\right)^\delta > 1\right\}}) \\ &\leq \frac{1}{\sigma_n^2} \sum_{i=1}^{K_n} \mathbb{E}\left(X_{n.i}^2 \left(\frac{|X_{n,i}|}{\epsilon\sigma_n}\right)^\delta\right)\end{aligned}$$

▼ Application of Lyapunov condition

Often the interest lies in other asymptotic variances. So rates $(a_n)_{n\in\mathbb{N}}$ such that:

$$\sqrt{a_n}S_n \xrightarrow{d} \mathcal{N}(0,V)$$

holds.

In order to proof this rate, the Lyapunov-condition can be written as:

$$\sum_{i=1}^{K_n}\mathbb{E}((\sqrt{n}|X_{n,i}|)^{2+\delta}) = a_n^{\frac{2+\delta}{2}}\sum_{i=1}^{K_n}\mathbb{E}(|X_{n,i}^{2+\delta}|) \to 0$$

At the expense of showing that:

$$a_n\sigma_n^2 \to V \in \mathbb{R}$$

▼ Comparison to lindenberg-Levy

We know that Lindenberg-Feller is an abstraction of the traditional C.L.T. that we know from Lindenberg-Levy. We can choose:

$$\begin{aligned} K_n &= n \\ \sigma_{n,i} &= \sigma \\ \mu_{n,i} &= 0 \\ X_{n,i} &= X_i \quad iid \end{aligned}$$

Then the Lindenberg condition is basically for free.

We can show, that in this setup it generally holds.

▼ Delta Method

Let $X_1,..,X_n$ be iid r.v. with $\mathbb{E}(X_i)=\mu\forall i$ and $Var(X_i)=\sigma^2\forall i$. Further more let $S_n=\sum X_i$ then we have (according to Lindeberg-Levy C.L.T.:

$$\sqrt{n}\left(\frac{S_n}{n}-\mu\right) \xrightarrow{d} \mathcal{N}(0,\sigma^2)$$

Now let g : $\mathbb{R}\to\mathbb{R}$ be continuous and differentiable. Then we can say according to the delta method:

$$\sqrt{n}\left(g\left(\frac{S_n}{n}\right)-g(\mu)\right) \xrightarrow{d} \mathcal{N}(0,\sigma^2(g'(\mu))^2)$$

▼ Donsker's Theorem

The Donsker's Theorem tells us the asymptotic distribution of a scaled random walk.

So let $X_1, ..., X_n$ be iid with $\mathbb{E}(X_i) = 0 \forall i$ and $Var(X_i) = 1 \forall i$. Then for $S_n := \sum X_i$ we define $w_n(t) = S_{\lfloor nt \rfloor}$ with $t \in [0, 1]$. It then holds for $w_n(t)$ that:

$$(w_n(t), t \in [0, 1]) \Rightarrow w = (w(t), t \in [0, 1])$$

Where $w(t)$ is the brownian motion.