Sheet 3

Problem 1

Let $A_n := S_n = 0$. Our overarching goal is to show that:

$$P\left(\limsup_{n\to\infty} A_n\right) = 0$$

1.) (Computing $P(A_n)$)

We firstly will start with $P(S_{2n} = 0)$. This is the case when $U_i = 1$ in n cases. Because of commutative law, the order in which these realizations exist does not matter. Then:

$$P(S_{2n} = 0) = \binom{2n}{n} p^n (1-p)^n \tag{1}$$

For the odd case ($P(S_{2n+1} = 0)$) The Probability is zero, because there is no combination of 1s and (-1)s that will add up to 0.

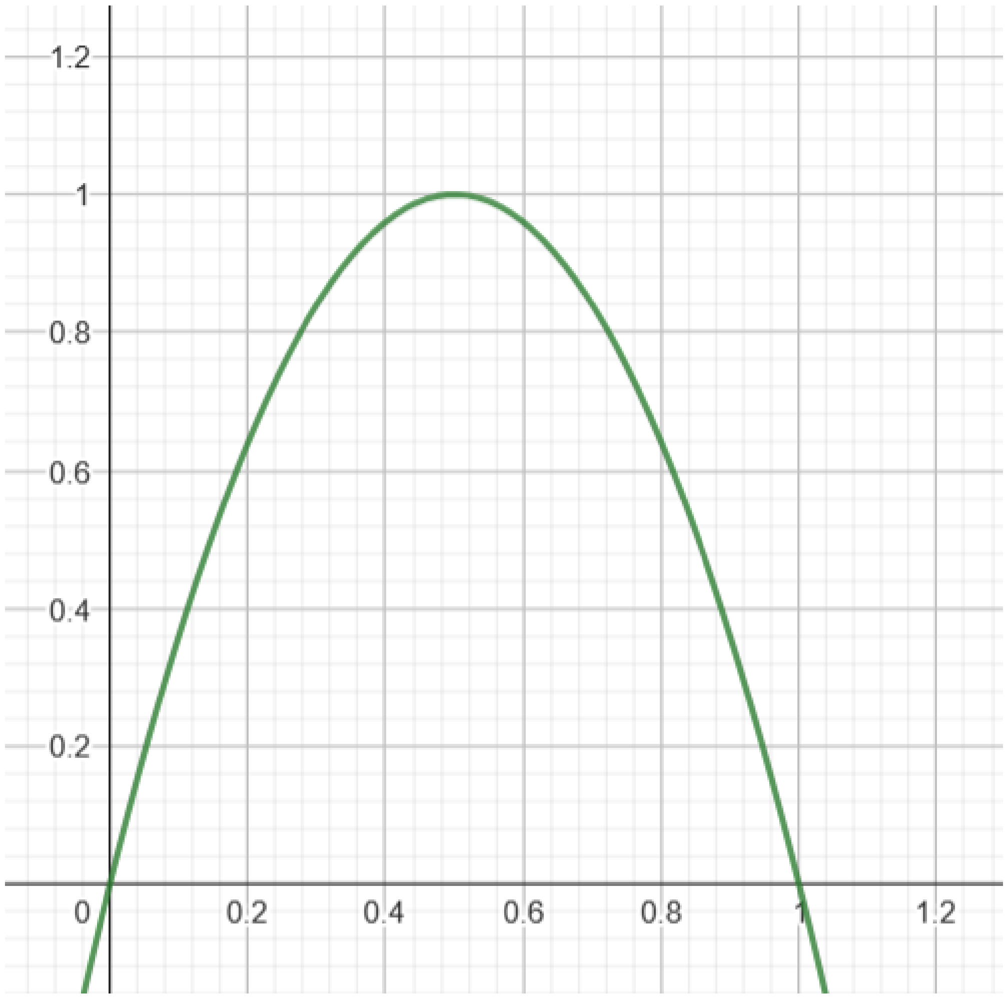
$$P(S_{2n+1} = 0) = 0 \quad \forall n \in \mathbb{N} \tag{2}$$

2.) Computing the infinite sum

For the sum we can use the inequality $\binom{2n}{n} \leq 4^n$. This means that:

$$
\begin{aligned}
\sum_{n=1}^{\infty} P(A_n) &= \sum_{n=1}^{\infty} P(S_n = 0) \\
&\underset{(2)}{=} \sum_{n=1}^{\infty} P(S_{2n} = 0) \\
&= \sum_{n=1}^{\infty} \binom{2n}{n} p^n (1-p)^n \\
&\underset{hint}{\leq} \sum_{n=1}^{\infty} 4^n p^n (1-p)^n \\
&= \sum_{n=1}^{\infty} (4p(1-p))^n
\end{aligned}
$$

Now we it is left to show that $|4p(1-p)| < 1$. Then the formula is the geometric series and converges.



This plot shows the function $f(x) = 4x(1-x)$. We see that the global maximum is reached at $x = \frac{1}{2}$ (can also be shown with FOC and SOC). Since we have assumed $x \neq \frac{1}{2}$, and $x \in [0,1]$ it holds, that $|4p(1-p)| < 1$.

Now we have shown that:

$$\sum_{n=1}^{\infty} P(A_n) = \frac{1}{1-4p(1-p)} < \infty$$

This is sufficient to apply Borel Cantellis theorem and directly implies that:

$$P\left(\limsup_{n\to\infty} A_n\right) = 0$$

Problem 2

In this problem the event of interest A_n can be defined as $A_n := Y_n = (1, ..., k)$. We can claim independence between $A_n \forall n \in \mathbb{N}$ because the permutations are totally random with every permutation being equally likely. Therefore we can say:

$$P(A_n) = P(Y_n = (1, ..., k)) = \frac{1}{k!} \quad \forall n \in \mathbb{N}$$

Where $k!$ is the number of possible permutations. Since $k < \infty$ we can also say that:

$$P(A_n) = \frac{1}{k!} > 0$$

But this means that:

$$\sum_{n=1}^{\infty} P(A_n) = \infty$$

Because independence between all A_n is given and the series of probabilites diverges Borel Cantellis Theorem directly implies that:

$$P\left(\limsup_{n\to\infty} A_n\right) = 1$$

Problem 3

1.)

We are going to show the statement indirectly by use of **theorem 3.8**.

For this firstly separate the sum into two parts:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i^2-2\bar{X}X_i+\bar{X}^2)\\
&= \frac{1}{n}\sum_{i=1}^{n}X_i^2-2\bar{X}\frac{1}{n}\sum_{i=1}^{n}X_i+\frac{1}{n}\sum_{i=1}^{n}\bar{X}^2\\
&= \frac{1}{n}\sum_{i=1}^{n}X_i^2-2\bar{X}^2+\frac{1}{n}n\bar{X}^2\\
&= \frac{1}{n}\sum_{i=1}^{n}X_i^2-2\bar{X}^2+\bar{X}^2\\
&= \frac{1}{n}\sum_{i=1}^{n}X_i^2-\bar{X}^2
\end{aligned}$$

Also note that that it holds that:

$$Var(X_1)=\mathbb{E}(X_1^2)-\mathbb{E}(X_1)^2$$

Meaning under **theorem 3.8** we only need to show that:

$$\frac{1}{n}\sum_{i=1}^{n}X_i^2 \xrightarrow{p} \mathbb{E}(X_1^2) \tag{3}$$

$$\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)^2 \xrightarrow{p} \mathbb{E}(X_1)^2 \tag{4}$$

Since $X_1, ..., X_n$ are iid the **weak law of large numbers 2** can be applied and this directly proves that (3) holds. For (4) this implies that:

$$\frac{1}{n}\sum_{i=1}^{n}X_i \xrightarrow{p} \mathbb{E}(X_1)$$

But we can apply **continuous mapping theorem** with $f:\mathbb{R}\to\mathbb{R}, f(x):=x^2$ continuous and show that:

$$\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)^2 = f\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)\xrightarrow{p} f(\mathbb{E}(X_1))=\mathbb{E}(X_1)^2$$

Now Theorem 3.8 can be applied and we can show that:

$$\frac{1}{n}\sum_{i=1}^{n}X_i^2-\bar{X}^2\xrightarrow{p}\mathbb{E}(X_1^2)-\mathbb{E}(X_1)^2=Var(X_1)\underset{iid}{=}\sigma^2$$

2.)

What if $\mu=\infty$?

The weak law of large numbers would still apply. But since ∞ is not a r.v. we could not apply theorem 3.8. causing the proof to fail.

What if $\sigma^2=\infty$?

Then convergence in probability could not be shown, since $\infty-\infty$ is not defined.

What if $\mathbb{E}(X_1^4)=\infty$?

The term is not relevant for the proof and therefore would not harm the proof and it’s validity in any way. If instead of iid we would have independence, we would need common means and variance as well as $\mathbb{E}(X_1^4)=Var(X_1^2)+\mathbb{E}(X_1^2)^2<\infty$.

3.)

We could say:

$$\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2=\frac{1}{n-1}\frac{n}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2=\frac{n}{n-1}\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2$$

We could then declare $(a_n)_{n\in\mathbb{N}}:=\frac{n}{n-1}\longrightarrow 1$. and express the estimator as:

$$s^2=a_nr^2$$

Then we can apply **Theorem 3.11 (ii)** and say:

$$r^2 \xrightarrow{p} \sigma^2 \implies s^2 = a_n r^2 \xrightarrow{p} 1 \cdot \sigma^2$$

And therefore:

$$s^2 \xrightarrow{p} \sigma^2$$

Problem 4

1.

Because $(U_n)_{n \in \mathbb{N}} \overset{iid}{\sim} U(0,1)$ every U_n has the same cdf:

$$\forall n \in \mathbb{N}: \quad F_{U_n}(x) := \begin{cases} 0 \text{ if } x < 0 \\ x \text{ if } x \in [0,1) \\ 1 \text{ if } x \geq 1 \end{cases}$$

Firstly let's look at the cdf of Y_n. We can compute $F_{Y_n}(x)$ by applying the definition:

$$\begin{aligned} F_{Y_n}(x) &= P(Y_n \leq x) \\ &= P(\max\{U_1, ..., U_n\} \leq x) \\ &= P(U_1 \leq x \wedge U_2 \leq x \wedge \cdots \wedge U_n \leq x) \\ &\underset{iid}{=} \prod_{i=1}^{n} P(U_i \leq x) \\ &= \prod_{i=1}^{n} F_{U_i}(x) \end{aligned}$$

Because we know the definition of $F_{U_i}(x) \forall i \in \mathbb{N}$ we have a definition for $F_{Y_n}(x)$:

$$F_{Y_n}(x) = \begin{cases} 0 \text{ if } x < 0 \\ x^n \text{ if } x \in [0,1) \\ 1 \text{ if } x \geq 1 \end{cases} \underset{n \to \infty}{\longrightarrow} F(x) = \begin{cases} 0 \text{ if } x < 1 \\ 1 \text{ if } x \geq 1 \end{cases}$$

But F is the cdf of the constant 1. So it holds that:

$$Y_n \xrightarrow{d} 1$$

2.)

Applying the alternative definition of convergence in probability, we now need to show that:

$$\varepsilon > 0: \quad P(|Y_n - 1| \leq \varepsilon) \to 1$$

Looking at $P(|Y_n - 1| \leq \varepsilon)$:

$$\begin{aligned} P(|Y_n - 1| \leq \varepsilon) &= P(Y_n \in [1-\varepsilon, 1+\varepsilon]) \\ &= P(Y_n \leq 1+\varepsilon) - P(Y_n \leq 1-\varepsilon) \\ &= F_{Y_n}(1+\varepsilon) - F_{Y_n}(1-\varepsilon) \end{aligned}$$

From 1.) we can now apply the definition of F_{Y_n}. Moving n to infinity this means that:

$$\begin{aligned} \lim_{n\to\infty} P(|Y_n - 1| \leq \varepsilon) &= \lim_{n\to\infty} F_{Y_n}(1+\varepsilon) - F(1-\varepsilon) \\ &= \lim_{n\to\infty} F_{Y_n}(1+\varepsilon) - \lim_{n\to\infty} F_{Y_n}(1-\varepsilon) \\ &\underset{\varepsilon>0}{=} 1 - 0 = 1 \end{aligned}$$

This now shows convergence in probability, according to the alternative definition.

3.)

We will start at the hint and look at for any $\varepsilon > 0$:

$$P\left(\limsup_{n\to\infty} A_n\right) \text{ with } A_n := Y_n \leq 1 - \varepsilon$$

If we want to apply Borel Cantelli we need to show that:

$$\sum_{n=1}^{\infty} P(A_n) < \infty$$

We can show for $P(A_n)$:

$$
\begin{aligned}
P(A_n) &= P(Y_n \leq 1-\varepsilon) \\
&\underset{iid}{=} \prod_{i=1}^{n} P(X_i \leq 1-\varepsilon) \\
&= \prod_{i=1}^{n} (1-\varepsilon) = (1-\varepsilon)^n
\end{aligned}
$$

From there the proof goes the same way as in **Problem 1:**

$$
\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} (1-\varepsilon)^n = (-1) + \sum_{n=0}^{\infty} (1-\epsilon)^n
$$

And because $\varepsilon > 0$ we know that $|1-\varepsilon| < 1$ and therefore the geometric series converges and:

$$
\begin{aligned}
\sum_{n=1}^{\infty} P(A_n) &= (-1) + \sum_{n=0}^{\infty} (1-\varepsilon)^n \\
&= (-1) + \frac{1}{1-(1-\varepsilon)} \\
&= (-1) + \frac{1}{\varepsilon} \\
&= \frac{1-\varepsilon}{\varepsilon} < \infty
\end{aligned}
$$