Sheet 4

Exercise 13

i)

We need to show the three properties for measures

i) s.t. $\rho : \mathcal{A} \to [0, \infty]$

Following the definition of simple function we can say:

$$\rho(E) = \int_E s d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E)$$

since $\alpha_i \geq 0 \forall i$ and $\mu : \mathcal{A} \to [0, \infty]$, we can say $\alpha_i \mu(A_i \cap E) \geq 0$ and therefore:

$$\rho(E) = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) \geq 0$$

Therefore showing, that $\rho(E)$ is a positive mapping

ii) s.t. $\rho(\emptyset) = 0$

$$\rho(E) = \int_\emptyset s d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap \emptyset) = \mu(\emptyset) \sum_{i=1}^{n} \alpha_i = 0 \sum_{i=1}^{n} \alpha_i = 0$$

iii) show σ-additivty

Let $\hat{A}_n \in \mathcal{A}$ be a set sequence pairwise disjoint. We can then say:

$$
\begin{aligned}
\rho\left(\biguplus_{i=1}^{\infty}\hat{A}_i\right) &= \int_{\uplus \hat{A}_i} s d\mu \\
&= \sum_{i=1}^{n} \alpha_i \mu\left(A_i \cap \biguplus_{j=1}^{\infty} \hat{A}_j\right) \\
&= \sum_{i=1}^{n} \alpha_i \mu\left(\biguplus_{j=1}^{\infty} (\hat{A}_j \cap A_i)\right) \\
&\underset{\mu \text{ measure}+\sigma\text{-add.}}{=} \sum_{i=1}^{n}\sum_{j=1}^{\infty} \alpha_i \mu(\hat{A}_j \cap A_i) \\
&= \sum_{j=1}^{\infty}\sum_{i=1}^{n} \alpha_i \mu(\hat{A}_j \cap A_i) \\
&= \sum_{j=1}^{\infty} \rho(\hat{A}_j)
\end{aligned}
$$

ii)

We will denote:

$$
s(\omega) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{A_i}(\omega) \quad t(\omega) = \sum_{i=1}^{\hat{n}} \beta_i \mathbf{1}_{B_i}(\omega)
$$

Our goal will be to construct a simple function from $\alpha s + \beta t$ so we can apply the definition of a measure integral. Afterwards we can deconstruct the resulting term to show the statement. Let's first take a look at the function $\alpha s + \beta t.$:

$$
(\alpha s + \beta t)(\omega) = \alpha \sum_{i=1}^{n} \alpha_i \mathbf{1}_{A_i}(\omega) + \beta \sum_{i=1}^{\hat{n}} \beta_i \mathbf{1}_{B_i}(\omega)
$$

From this term we can read that:

$$
\begin{aligned}
(\alpha s + \beta t)(\omega) = \alpha\alpha_i \quad \forall \omega \in A_i \setminus \biguplus_{j=1}^{n} B_j \\
(\alpha s + \beta t)(\omega) = \beta\beta_i \quad \forall \omega \in B_i \setminus \biguplus_{j=1}^{n} A_j \\
(\alpha s + \beta t)(\omega) = \alpha\alpha_i + \beta\beta_j \quad \forall \omega \in A_i \cap B_j
\end{aligned}
$$

By construction these sets are pairwise disjoint. So let C_k be defined as:

$$C_k := \begin{cases} A_k \setminus \biguplus_{j=1}^{n} B_j & \forall k \leq n \\ B_{k-n} \setminus \biguplus_{j=1}^{n} A_j & \forall n < k \leq n + \hat{n} \\ A_{\lfloor \frac{k-n}{\hat{n}} \rfloor} \cap B_{((k-n-\hat{n}) \mod \hat{n})+1} & \forall n + \hat{n} < k \leq n + \hat{n} + n\hat{n} \end{cases}$$

Basically C_k is just all sets from above listed after one another. We can now define the according coefficients c_k as:

$$c_k := \begin{cases} \alpha\alpha_k & \forall k \leq n \\ \beta\beta_{k-n} & \forall n < k \leq \hat{n} + n \\ \alpha\alpha_{\lfloor \frac{k-n}{\hat{n}} \rfloor} + \beta\beta_{((k-n-\hat{n}) \mod \hat{n})+1} & \forall n + \hat{n} < k \leq n + \hat{n} + n\hat{n} \end{cases}$$

Which is just the results listed after one another. With the set sequence C_k and the real sequence c_k in place we can finally say:

$$(\alpha s + \beta t)(\omega) = \sum_{k=1}^{n+\hat{n}+\hat{n}n} c_k 1_{C_k}(\omega)$$

With this simple function we can now finally apply the definition of a measure integral:

$$\begin{aligned} \int_E (\alpha s + \beta t) d\mu &= \sum_{k=1}^{n+\hat{n}+\hat{n}n} c_k \mu(C_k \cap E) \\ &= \sum_{i=1}^{n} \alpha\alpha_i \mu \left(\left(A_i \setminus \biguplus_{j=1}^{\hat{n}} B_j \right) \cap E \right) \\ &+ \sum_{i=1}^{\hat{n}} \beta\beta_i \mu \left(\left(B_i \setminus \biguplus_{j=1}^{n} A_j \right) \cap E \right) \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{\hat{n}} (\alpha\alpha_i + \beta\beta_j) \mu((A_i \cap B_j) \cap E) \\ &= \sum_{i=1}^{n} \alpha\alpha_i \mu \left(\left(A_i \setminus \biguplus_{j=1}^{\hat{n}} B_j \right) \cap E \right) \\ &+ \sum_{i=1}^{\hat{n}} \beta\beta_i \mu \left(\left(B_i \setminus \biguplus_{j=1}^{n} A_j \right) \cap E \right) \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{\hat{n}} \alpha\alpha_i \mu((A_i \cap B_j) \cap E) \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{\hat{n}} \beta\beta_j \mu((A_i \cap B_j) \cap E) \end{aligned}$$

Rearranging the sums leads to:

$$\begin{aligned}
\int_E (\alpha s + \beta t) d\mu = &\sum_{i=1}^{n} \alpha\alpha_i \mu\left(\left(A_i \setminus \biguplus_{j=1}^{\hat{n}} B_j\right) \cap E\right) \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{\hat{n}} \alpha\alpha_i \mu((A_i \cap B_j) \cap E) \\
&+ \sum_{i=1}^{\hat{n}} \beta\beta_i \mu\left(\left(B_i \setminus \biguplus_{j=1}^{n} A_j\right) \cap E\right) \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{\hat{n}} \beta\beta_j \mu((A_i \cap B_j) \cap E) \\
&= \sum_{i=1}^{n} \alpha\alpha_i \left(\mu\left(\left(A_i \setminus \biguplus_{j=1}^{\hat{n}} B_j\right) \cap E\right) + \sum_{j=1}^{\hat{n}} \mu((A_i \cap B_j) \cap E)\right) \\
&+ \sum_{i=1}^{\hat{n}} \beta\beta_i \left(\mu\left(\left(B_i \setminus \biguplus_{j=1}^{n} A_j\right) \cap E\right) + \sum_{j=1}^{n} \mu((B_i \cap A_j) \cap E)\right)
\end{aligned}$$

Now we can apply **finite additivity:**

$$\begin{aligned}
&= \sum_{i=1}^{n} \alpha\alpha_i \left(\mu\left(\left(A_i \setminus \biguplus_{j=1}^{\hat{n}} B_j\right) \cap E\right) + \mu\left(\left(A_i \cap \biguplus_{j=1}^{\hat{n}} B_j\right) \cap E\right)\right) \\
&+ \sum_{i=1}^{\hat{n}} \beta\beta_i \left(\mu\left(\left(B_i \setminus \biguplus_{j=1}^{n} A_j\right) \cap E\right) + \mu\left(\left(B_i \cap \biguplus_{j=1}^{n} A_j\right) \cap E\right)\right) \\
&= \sum_{i=1}^{n} \alpha\alpha_i \mu\left(\left(\left(A_i \setminus \biguplus_{j=1}^{\hat{n}} B_j\right) \cup \left(A_i \cap \biguplus_{j=1}^{\hat{n}} B_j\right)\right) \cap E\right) \\
&+ \sum_{i=1}^{\hat{n}} \beta\beta_i \mu\left(\left(\left(B_i \setminus \biguplus_{j=1}^{n} A_j\right) \cup \left(B_i \cap \biguplus_{j=1}^{n} A_j\right)\right) \cap E\right) \\
&= \sum_{i=1}^{n} \alpha\alpha_i \mu(A_i \cap E) + \sum_{i=1}^{\hat{n}} \beta\beta_i \mu(B_i \cap E) \\
&= \alpha \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) + \beta \sum_{i=1}^{\hat{n}} \beta_i \mu(B_i \cap E) \\
&= \alpha \int_E s d\mu + \beta \int_E t d\mu
\end{aligned}$$

q.e.d.

iii)

In this proof we are going to use **ii). We are going to additionally assume, that f, g are measurable.** We are going to approach this proof backwards, so starting on the LHS of the equation. But before starting, let's insert the definition of a measure integral on the RHS:

$$\int_\Omega (\alpha f + \beta g) d\mu = \sup \left\{ \int_\Omega h d\mu | 0 \leq h \leq \alpha f + \beta g, h \text{ simple} \right\}$$

Starting on the LHS we get:

$$\begin{aligned} \alpha \int_\Omega f d\mu + \beta \int_\Omega g d\mu = \alpha \sup \left\{ \int_\Omega s d\mu | 0 \leq s \leq f, s \text{ simple} \right\} \\ + \beta \sup \left\{ \int_\Omega t d\mu | 0 \leq t \leq g, t \text{ simple} \right\} \\ = \sup \left\{ \alpha \int_\Omega s d\mu | 0 \leq s \leq f, s \text{ simple} \right\} \\ + \sup \left\{ \beta \int_\Omega t d\mu | 0 \leq t \leq g, t \text{ simple} \right\} \\ = \sup \left\{ \alpha \int_\Omega s d\mu + \beta \int_\Omega t d\mu | 0 \leq s \leq f, 0 \leq t \leq g, s, t \text{ simple} \right\} \end{aligned}$$

Now we can apply **ii)** and say:

$$\sup \left\{ \alpha \int_\Omega s d\mu + \beta \int_\Omega t d\mu | 0 \leq s \leq f, 0 \leq t \leq g, s, t \text{ simple} \right\} \underset{(ii)}{=}$$
$$\sup \left\{ \int_\Omega (\alpha s + \beta t) d\mu | 0 \leq s \leq f, 0 \leq t \leq g, s, t \text{ simple} \right\}$$

We can now look at the inequalities and show that:

$$\begin{gathered} 0 \leq s \leq f \underset{\alpha \in \mathbb{R}_+}{\Leftrightarrow} 0 \leq \alpha s \leq \alpha f \\ 0 \leq t \leq g \underset{\beta \in \mathbb{R}_+}{\Leftrightarrow} 0 \leq \beta t \leq \beta g \\ \implies 0 \leq \alpha s + \beta t \leq \alpha f + \beta g \end{gathered}$$

This condition implies that:

$$\sup \left\{ \int_\Omega (\alpha s + \beta t) d\mu | 0 \leq s \leq f, 0 \leq t \leq g, s, t \text{ simple} \right\} =$$
$$\sup \left\{ \int_\Omega (\alpha s + \beta t) d\mu | 0 \leq \alpha s + \beta t \leq \alpha f + \beta g, t, s \text{ simple} \right\}$$

And because we have shown, that $\alpha s + \beta t$ is also a simple function, we can denote $h := \alpha s + \beta t$ and say:

$$\sup \left\{ \int_\Omega (\alpha s + \beta t) d\mu | 0 \leq \alpha s + \beta t \leq \alpha f + \beta g, t, s \text{ simple} \right\} =$$
$$\sup \left\{ \int_\Omega h d\mu | 0 \leq h \leq \alpha f + \beta g, h \text{ simple} \right\} = \int_\Omega (\alpha f + \beta g) d\mu$$

Exercise 14

i)

We firstly want to show, that the function f is **not** Lebesgue-integrable.

1. Show that f_n is $(\mathcal{B}([0,1]) - \mathcal{B}(\mathbb{R}_+))$-measurable
   The proof will show, that reverse images of intervals on f_n always map to intervals.

   Note, that it is possible to say: $f_n : [0,1] \to [0, 1/n) \cup [1, n]$ since:

$$\max_{x \in [0,1/n)} f_n(x) = \max_{x \in [0,1/n)} x < 1/n$$
$$\max_{x \in [1/n,1]} f_n(x) = \max_{x \in [1/n,1]} 1/x = n$$
$$\min_{x \in [0,1/n)} f_n(x) = ... = 0$$
$$\min_{x \in [1/n,1]} f_n(x) = ... = 1$$

   Hence we only need to consider Intervalls $[a,b] \subset [0,n]$. Three cases become possible. All other sets that lie not within this range will map to $\emptyset \in \mathcal{B}([0,1])$.

   Case 1: $a \geq 1$

$$f_n^{-1}([a,b]) = [f_n^{-1}(b), f_n^{-1}(a)] = [1/b, 1/a] \in \mathcal{B}([0,1]) \quad \forall n \in \mathbb{N}$$

   Case 2: $b \leq 1/n$

$$f_n^{-1}([a,b]) = [f_n^{-1}(a), f_n^{-1}(b)] = [a,b] \underset{b \leq 1}{\in} \mathcal{B}([0,1])$$

   Case 3: $a \leq 1/n, b \geq 1$

$$f_n^{-1}([a,b]) = f_n^{-1}([a,1)) \cup f_n^{-1}([1,b]) = [a, 1/n) \cup [1/b, 1]$$

   Since $[a,1)$ and $[1/b,1]$ are contained in $\mathcal{B}([0,1])$, the third property of σ-algebras implies, that the union also has to be contained in $\mathcal{B}([0,1])$.

   Since f_n is strictly increasing on $[0, 1/n)$ and strictly decreasing $[1/n, 1]$ and:

$$\max_{x\in[0,1/n)} f_n(x) < 1/n \leq \min_{x\in[1/n,1]} f_n(x)$$

We can say f_n^{-1} always exists and is well defined. If that is the case, then inverse images of single point sets will either be $\emptyset \in \mathcal{B}([0,1])$ or single point sets $\{x\} \in \mathcal{B}([0,1])$.

2. Show that $f_n \to f$

   Note that $f_n(0) = 0 = f(0) \;\; \forall n \in \mathbb{N}$.

   Let any $x \in (0,1]$ then it holds that:

$$\exists! \hat{n} \in \mathbb{N} : x > \frac{1}{\hat{n}} \implies f_{\hat{n}}(x) = \frac{1}{x} = f(x)$$

   Therefore we can say:

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in [0,1]$$

3. Apply hint

   We have now established, that f_n is measurable for all $n \in \mathbb{N}$. Now we can say:

$$\int_{[0,1]} f d\mu = \lim_{n\to\infty} \int_{[0,1]} f_n d\mu = \lim_{n\to\infty} \int_{[0,1/n)} f_n d\mu + \int_{[1/n,1]} f_n d\mu$$

   But:

$$\lim_{n\to\infty} \int_{[0,1/n)} f_n d\mu \to \int_{\{0\}} f_n d\mu = 0 \cdot 0 = 0$$

   So:

$$\begin{aligned}\lim_{n\to\infty} \int_{[0,1/n)} f_n d\mu + \int_{[1/n,1]} f_n d\mu &= \lim_{n\to\infty} \int_{[0,1/n)} f_n d\mu + \lim_{n\to\infty} \int_{[1/n,1]} f_n d\mu \\ &= 0 + \lim_{n\to\infty} \int_{[1/n,1]} f_n d\mu\end{aligned}$$

4. Apply **Definition 1.40** after partition

   We can now use the seq. of simple functions from **Definition 1.40** to approximate the lebesgue integral. So let $s_n : \mathbb{R} \to \mathbb{R}$ with $s_n : \mathbb{R} \to \mathbb{R}$ and $s_n := k/n$ for all $x \in \mathbb{R}$: $k/n \leq f(x) < k + 1/n$

   Now we can say:

$$
\begin{aligned}
\lim_{n\to\infty}\int_{[1/n,1]} f_n d\mu &= \lim_{n\to\infty}\int_{[1/n,1]} s_n d\mu \\
&= \lim_{n\to\infty}\sum_{i=1}^{\infty}\lambda(f_n^{-1}([i/n,(i+1)/n])i/n \\
&= \lim_{n\to\infty}\sum_{i=1}^{\infty}\lambda([n/(i+1),n/i])i/n \\
&= \lim_{n\to\infty}\sum_{i=1}^{\infty}(n/i-n/(i+1))i/n \\
&= \lim_{n\to\infty}\sum_{i=1}^{\infty}(1-i/(i+1)) \\
&= \lim_{n\to\infty}\sum_{i=1}^{\infty}\frac{1}{i}
\end{aligned}
$$

The last term is the famous **harmonic series.** This series is known for diverging and there are several proofs (for example on YouTube: (688) Proof: harmonic series diverges | Series | AP Calculus BC | Khan Academy - YouTube)

The rough idea of this proof is that:

$$\sum_{i=1}^{\infty}\frac{1}{i} \leq \sum_{i=1}^{\infty}\frac{1}{2} = \infty$$

Finally, since f is a non-negative function, we can say that:

$$\int_{[0,1]}|f|d\mu = \int_{[0,1]} fd\mu = \infty$$

Therefore proving, that indeed f is not Lebesgue-integrable.

Then we want to show, that $\int gd\mu$ is integrable

Because g_n is piecewise continuous and bounded by $[0,\sqrt{n}]$ we can say, that this function is Riemann-integrable (usually you would need a lemma for the second step.):

$$
\begin{aligned}
\int_{[0,1]} g_n d\lambda &= \int_{[0,1/n^2)} g_n d\lambda + \int_{[1/n^2,1]} g_n d\lambda \\
&= \int_0^{1/n^2} g_n(x)dx + \int_{1/n^2}^1 g_n(x)dx \\
&= \int_0^{1/n^2} xdx + \int_{1/n^2}^1 \frac{1}{\sqrt{x}}dx \\
&= \frac{1}{2}x^2|_0^{1/n^2} + 2\sqrt{x}|_{1/n^2}^1 \\
&= \frac{1}{2}\frac{1}{n^4} + 2\sqrt{1} - \frac{2}{n}
\end{aligned}
$$

And therefore we can argue:

$$
\begin{aligned}
\int_{[0,1]} gd\lambda \underset{(hint)}{=} \lim_{n\to\infty} \int_{[0,1]} g_n d\lambda &= \lim_{n\to\infty} \frac{1}{2}\frac{1}{n^4} + 2\sqrt{1} - \frac{2}{n} \\
&= 2 + \lim_{n\to\infty} \frac{1}{2}\frac{1}{n^4} + \frac{2}{n} \\
&= 2 + 0 = 2
\end{aligned}
$$

Because g is a non-negative function it holds that:

$$
\int_{[0,1]} gd\lambda = \int_{[0,1]} |g|d\lambda < \infty
$$

Therefore g is Lebesgue-integrable.

Exercise 15

a)

s.t. $f(x) > 0$ for all $x > 0$:

$$
\alpha, \beta, x > 0 \implies \Gamma(\alpha) > 0 \wedge x^{\alpha-1} > 0 \wedge \beta^\alpha > 0
$$

But this implies:

$$
f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} > 0 \quad \forall x > 0
$$

s.t. $\int f(x)dx = 1$

$$\begin{aligned}\int_0^\infty f(x)dx &= \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \beta\beta^{\alpha-1} x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \beta(\beta x)^{\alpha-1} e^{-\beta x} dx\end{aligned}$$

Now substituting $t = \beta x$ we get:

$$\begin{aligned}\frac{1}{\Gamma(\alpha)} \int_0^\infty \beta(\beta x)^{\alpha-1} e^{-\beta x} dx &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \beta(t)^{\alpha-1} e^{-t} d\frac{t}{\beta} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} dt \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1\end{aligned}$$

b)

We want to compute:

$$\mathbb{E}(X) = \int_0^\infty x f(x) dx$$

We Inserting the definition of f and simplifying the term leads to:

$$\begin{aligned}\int_0^\infty x f(x) dx &= \int_0^\infty x \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\ &= \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\beta x} dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)} (\beta x)^\alpha e^{-\beta x} dx\end{aligned}$$

Again substituting $t = \beta x$. And augmenting with β/β (1):

$$\begin{aligned}\int_0^\infty \frac{1}{\Gamma(\alpha)} (\beta x)^\alpha e^{-\beta x} dx &\underset{(1)}{=} \frac{1}{\Gamma(\alpha)} \frac{1}{\beta} \int_0^\infty t^\alpha e^{-t} dt \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\beta} \Gamma(\alpha+1) \\ &\underset{(hint)}{=} \frac{\Gamma(\alpha)\alpha}{\Gamma(\alpha)\beta} = \frac{\alpha}{\beta}\end{aligned}$$

c)

Let $g(x) = 1/x$.

We can use **Proposition 1.37** in order to compute the density of Y (We also use $g(\mathbb{R}) = \mathbb{R}$) Therefore we don't need to include the indicator and because $x > 0$ we can say $|1/x| = 1/x$:

$$\begin{aligned} f_Y(y) &= f_X(1/x)1/x \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{1-\alpha} e^{-\beta/x} x^{-1} \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha} e^{-\beta/x} \end{aligned}$$

For computing the expected value of Y we can use the extention of **Theorem 1.41:**

$$\mathbb{E}(Y) = \mathbb{E}(1/X) = \int_0^\infty \frac{1}{x} f(x) dx$$

Inserting the definition of $f(x)$ we get:

$$\begin{aligned} \int_0^\infty \frac{1}{x} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx &= \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-2} e^{-\beta x} dx \\ &= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \beta(\beta x)^{\alpha-2} e^{-\beta x} dx \end{aligned}$$

We can again substitute $t = \beta x$ and get:

$$\begin{aligned} \frac{\beta}{\Gamma(\alpha)} \int_0^\infty \beta(\beta x)^{\alpha-2} e^{-\beta x} dx &= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-2} e^{-t} dt \\ &= \beta \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} \\ &\underset{(hint)}{=} \frac{\beta\Gamma(\alpha-1)}{(\alpha-1)\Gamma(\alpha-1)} \\ &= \frac{\beta}{\alpha-1} \end{aligned}$$

d)

We can use a RNG, that will generate a sequence of random numbers, which will be uniformly distributed. Afterwards, we can transform the numbers with F_X^{-1}. According to the given formula, this will yield a sequence of random numbers, generated by F_X.

The distribution of interest is a exponential distribution:

$$X \sim Exp(\lambda)$$

This means the cdf of X is:

$$F(x) = 1 - e^{-\lambda x} \Leftrightarrow \frac{\ln(F(x) - 1)}{-\lambda} = x$$

Therefore would transform with:

$$F^{-1}(x) = \frac{\ln(x-1)}{-\lambda}$$