Sheet 3

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Exercise 9

i)

In this case we can use Lemma 1.2 and specify:

$$X(A) = \sum_{\omega \in A} X(\omega)$$

1. s.t. $X : \mathcal{A} \to [0, \infty]$

   Let $A \subset \mathbb{N}$ then there are two cases:

   Case 1: $2 \in A$

   $$X(A) \underset{L.1.2.}{=} \sum_{x \in A} X(x) = X(2) + \sum_{x \in A, x \neq 2} X(x) = 1 + 0 + 0 + ... = 1$$

   Case 2: $2 \notin A$:

   $$\sum_{x \in A} X(x) = 0 + 0 + ... = 0$$

   So $0 \leq X(A) < \infty$ does hold for every $A \subset \mathbb{N}$ and thus for every $A \in \mathcal{A}$
2. s.t. $X(\Omega) = 1$

   $$X(\Omega) = X(\mathbb{N}) = X(2) + \sum_{x \in A, x \neq 2} X(x) = 1 + 0 + 0 + ... = 1$$

3. s.t. $A_i \in \mathcal{A}$ pairwise disjoint implies $X\left(\biguplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{\infty} X(A_i)$

   Let $\mathcal{B} := \biguplus_{i=1}^{\infty} A_i$

   Then:

   $$X(\mathcal{B}) \underset{L.1.2}{=} \sum_{x \in \mathcal{B}} X(x) \underset{p.d.}{=} \sum_{i=1}^{\infty} \sum_{x \in A_i} X(x) = \sum_{i=1}^{\infty} X(A_i)$$

ii)

Let $B \in \mathcal{B}(\mathbb{R})$.

We know from sheet 2, that every single point set is contained in $\mathcal{B}$. So lets look at $\{1\} \in \mathcal{B}$:

$$X^{-1}(\{1\}) = \{2\} \notin \mathcal{A}$$

Since $\{2\}$ is not in $\mathcal{A}$ this contradicts the definition of $(\mathcal{A} - \mathcal{B})$ measurability.

Exercise 2

i)

Because $X_1(z)$ is strictly increasing maps to all real numbers, we can calculate the inverse $X_1^{-1}(z)$:

$$X_1(z) = 10z \quad \Leftrightarrow \quad X_1^{-1}(z) = \frac{z}{10}$$

And because $X_1(z)$ is also continuous, we can say:

$$a \leq x \leq b \implies X_1(a) \leq X_1(x) \leq X_1(b)$$

So:

$$X_1^{-1}((a,b]) = \left(\frac{a}{10}, \frac{b}{10}\right]$$

Therefore we can say:

$$P^{X_1}((a,b]) = P(X_1^{-1}((a,b])) = P\left(\left(\frac{a}{10}, \frac{b}{10}\right]\right) = \exp\left(-\frac{\lambda}{10}b\right) - \exp\left(-\frac{\lambda}{10}a\right)$$

And so this implies that $P^{X_1} = Exp\left(\frac{\lambda}{10}\right)$

ii)

Since $\exp(z)$ is strictly monotonically increasing $(\exp`(z) > 0 \ \ \forall z)$ and $\exp(0) = 1$, we can say that $X_2 : \mathbb{R}_+ \to \mathbb{R}_+ \setminus [0,1)$. If that is the case, we could argue:

$$X_2^{-1}(A) = \emptyset \quad \forall A \subset [0,1) \tag{1}$$

For every $z \in \mathbb{R}_+ \setminus [0,1)$ we can use the natural logarithm to find the inverse element:

$$X_2^{-1}(z) = \ln(z) \quad \forall z \in \mathbb{R}_+ \setminus [0,1) \tag{2}$$

Because $\exp(z)$ is also continuous it holds for every $a \geq 1$ that:

$$a \leq x \leq b \implies X_2(a) \leq X_2(x) \leq X_2(b) \tag{3}$$

Using these arguments we can now go through the different cases of the image measure of X_2.

On the last lecture we have shown, that $\{x\} \in \mathcal{B}(\mathbb{R}_+) x \in \mathbb{R}_+$. Using continuity from above $(c.a.)$ from **Theorem 1.9** and the set sequence $A_n = (x - \frac{1}{n}, x]$, we can show that $(s.p.)$:

$$\begin{aligned}
P(\{x\}) = P\left(\bigcap_{i=1}^{\infty} A_i\right) &\underset{(c.a.)}{=} \lim_{n\to\infty} P(A_i) \\
&= \lim_{n\to\infty} \exp(-\lambda x) - \exp\left(-\lambda\left(x - \frac{1}{n}\right)\right) \\
&= \exp(-\lambda x) - \lim_{n\to\infty} \exp\left(-\lambda\left(x - \frac{1}{n}\right)\right) \\
&= \exp(-\lambda x) - \exp(-\lambda x) \\
&= 0
\end{aligned}$$

Case 1: $a \geq 1$

$$\begin{aligned}
P^{X_2}((a,b]) &\underset{(3)}{=} P((X_2^{-1}(a), X_2^{-1}(b)]) \\
&\underset{(2)}{=} P((\ln(a), \ln(b)]) \\
&\underset{def.}{=} \exp(-\lambda \ln(b)) - \exp(-\lambda \ln(a)) \\
&= \exp(\ln(b^{-\lambda})) - \exp(\ln(a^{-\lambda})) \\
&= b^{-\lambda} - a^{-\lambda} \\
&= \left(\frac{1}{b}\right)^{\lambda} - \left(\frac{1}{a}\right)^{\lambda}
\end{aligned}$$

Case 2: $a < 1, b \geq 1 (*)$

We are also going to use **finite additivity** $(f.a)$ for this one (proven in lecture) and the statement

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$$
\begin{aligned}
P^{X_2}((a,b]) &\underset{(*)}{=} P^{X_2}((a,1)\cup\{1\}\cup(1,b]) \\
&\underset{(f.a)}{=} P^{X_2}((a,1)) + P^{X_2}(\{1\}) + P^{X_2}((1,b]) \\
&\underset{(1)+(2)}{=} P(\emptyset) + P(\{0\}) + P(X_2^{-1}((1,b]) \\
&\underset{(3)+(2)+(s.p.)}{=} 0 + 0 + P((\ln(1),\ln(b)]) \\
&\underset{def.}{=} \exp(-\lambda\cdot 0) - \exp(-\lambda\ln(b)) \\
&= 1 - \left(\frac{1}{b}\right)^{\lambda}
\end{aligned}
$$

Case 3: $b < 1$

In this case it holds that: $(a,b] \subset [0,1)(\#)$, therefore:

$$P^{X_2}((a,b]) = P(X_2^{-1}((a,b]) \underset{(1)+(\#)}{=} P(\emptyset) = 0$$

iii)

We can say $X_3 : \mathbb{R}_+ \to \{10\}$, since$X_3(z) = 10\forall z$. But this implies that: $X_3^{-1}(\{10\}) = \mathbb{R}_+$. And because $\Omega = \mathbb{R}_+$ and P is a probability-measure, the second defining property implies:

$$P^{X_3}(\{10\}) = P(\mathbb{R}_+) \underset{(ii)}{=} 1 \tag{4}$$

Since X_3 never maps to any other value but 10, it holds that:

$$P^{X_3}(A) = P(X_3^{-1}(A)) = P(\emptyset) \underset{T.1.9.}{=} 0 \qquad \forall A \subset \mathbb{R}_+ : 10 \notin A \tag{5}$$

Let's now take any set $A \in \mathcal{B}(\mathbb{R}_+)$. If $10 \in A$ we can say (6):

$$P^{X_3}(A) = P^{X_3}(\{10\}\cup(A\setminus\{10\})) \underset{(f.a)}{=} P^{X_3}(\{10\}) + P^{X_3}(A\setminus\{10\}) = 1 + 0 = 1$$

In total we could derive from (5) and (6):

$$P^{X_3}(A) = \begin{cases} 1, \text{if } \ 10 \in A \\ 0, \text{else} \end{cases}$$

And this is the exact behavior of the Dirac measure δ_{10}.

Exercise 11

Since we have shown, that the Lebesgue measure is a measure we only need to show $\lambda(\Omega) = 1$ in order to assume, that λ is a probability measure and therefore $(\Omega, \mathcal{A}, \lambda)$ is a probability space.

$$\lambda(\Omega) = \lambda([0,1]) = 1 - 0 = 1$$

i)

In order for A and B to be independent, it must hold that: $\lambda(A \cap B) = \lambda(A)\lambda(B)$
Because $A \subset B$:

$$\lambda(A \cap B) = \lambda(A) = \frac{3}{4} - \frac{2}{4} = \frac{1}{4}$$

And since $\lambda(B) < 1$ we can say:

$$\lambda(A \cap B) = \lambda(A) > \lambda(A)\lambda(B)$$

And therefore $\lambda(A \cap B) \neq \lambda(A)\lambda(B)$

So A and B are not stochasticly independent.

ii)

Let the same probability space as before $(\Omega, \mathcal{A}, \lambda)$ and $A = [0, 1/2]$, $B = [1/4, 3/4]$ and $C = [0, 1/4] \cup [1/2, 3/4]$. Then it holds that:

$$\lambda(A \cap B) = 1/4 = \lambda(A)\lambda(B) \tag{6}$$
$$\lambda(A \cap C) = 1/4 = \lambda(A)\lambda(C) \tag{7}$$
$$\lambda(C \cap B) = 1/4 = \lambda(C)\lambda(B) \tag{8}$$

But:

$$\lambda(A \cap B \cap C) = \lambda(\emptyset) = 0$$

And:

$$\lambda(A)\lambda(B)\lambda(C) = \left(\frac{1}{2}\right)^3 = \frac{1}{8} \neq 0$$

So there is no mutual independence, despite pairwise independence.

Exercise 12

i)

s.t. X is a random variable

From the definition of X it is directly implied, that:

$$X^{-1}(\{1\}) = (0, \frac{1}{3}]$$
$$X^{-1}(\{2\}) = (\frac{1}{3}, 1]$$
$$X^{-1}(\{0\}) = \{0\}$$

From sheet 2 we know that $X^{-1}(\{1\}), X^{-1}(\{2\}), X^{-1}(\{0\}) \in \mathcal{B}[0,1]$. And since these sets are pairwise disjoint and the third property of a σ-algebra (iii) holds, for any set $A \subset \Omega$ we can say that:

$$X^{-1}(A) = \biguplus_{\omega \in A} X^{-1}(\{\omega\}) \underset{(iii)}{\in} \mathcal{B}[0,1] \tag{9}$$

This implies:

$$X^{-1}(A) \in \mathcal{B}[0,1] \quad \forall A \in \mathcal{P}(\Omega)$$

making X a random variable by definition.

Probability distribution under $\lambda_{[0,1]}$.

We need to specify: $P^X(A) = P(X^{-1}(A)) = \lambda(X^{-1}(A))$,with $A \subset \Omega$. Using (9) and σ-additivity, we can now say:

$$\begin{aligned}\lambda(X^{-1}(A)) &= \lambda\left(\biguplus_{\omega \in A} X^{-1}(\{\omega\})\right) \\ &\underset{\sigma-a.}{=} \sum_{\omega \in A} \lambda(X^{-1}(\{\omega\}))\end{aligned}$$

This leads to:

$$\begin{aligned}P^X(\{0\}) &= 0 \\ P^X(\{1\}) &= P^X(\{1,0\}) = 1/3 \\ P^X(\{2\}) &= P^X(\{2,0\}) = 2/3 \\ P^X(\{1,2\}) &= P^X(\{1,2,0\}) = 1\end{aligned}$$

ii)

s.t. f and g measurable $\implies f^{-1}(g^{-1}(A)) \in \mathcal{A}_1 \quad \forall A \in \mathcal{A}_3$

Let $A \in \mathcal{A}_3$. Then $(\mathcal{A}_2 - \mathcal{A}_3)$-measurability of g implies that:

$$g^{-1}(A) \in \mathcal{A}_2$$

But since f is also $(\mathcal{A}_1 - \mathcal{A}_2)$-measurable, this implies, that:

$$f^{-1}(g^{-1}(A)) \in \mathcal{A}_1$$