Summary

- Probability Theory
  - Measuring spaces

    Important terminology:

    - Sample space Ω
    - Event $A \subset \Omega$
    - Elementary event / outcome$\omega \in \Omega$

    Measuring space in the countable case:

    1. $P(\Omega) = 1$
    2. Sigma additivity for $A_1, ... \subset \Omega$ and $A_i \cup A_j = \emptyset \forall i \neq j$

    $$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

    Lemma 1.2

    If a sample space Ω is counable, you can specify a probability measure just by (while I is an index-set):

    $$P(\{\omega_i\}) = p_i \quad \forall i \in I$$

    For every set A it holds:

    $$P(A) = \sum_{\omega \in A} P(\{\omega\})$$

    σ-algebra:

    1. $\emptyset \in \mathcal{A}$
    2. if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$
    3. if $A_1, A_2, ... \in \mathcal{A} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{A}$

Definition of smallest σ-Algebra:

- If the smallest sigma algebra containing set A is called $\mathcal{A}$. Then for every sigma Alegbra $\mathcal{B}$on Ω it holds that:

$$A \subset \mathcal{B} \Rightarrow \mathcal{A} \subset \mathcal{B}$$

Theres also the smallest-σ-Algebra, that is denoted with the notation $\sigma(A)$

Lemma 1.5 → For set $A \subset \mathcal{P}(\Omega)$ $\sigma(A)$ **has a solution.**

▼ Measure

Defintion Measure:

1. $\mu : \mathcal{A} \to [0, \infty]$
2. $\mu(\emptyset) = 0$
3. $A_1, A_2, ... \in \mathcal{A}$ pairwise disjoint σ-additivity:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Definition Probability measure:

1. $\mu : \mathcal{A} \to [0, \infty]$
2. $A_1, A_2, ... \in \mathcal{A}$ pairwise disjoint σ-additivity:

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3. $P(\Omega) = 1$

▼ Borel sets

Let $A := \{(a, b) | a, b \in \mathbb{R}\}$ then the **Borel sigma field** is defined by:

$$\sigma(A) = \mathcal{B}$$

Each set $C \subset \mathbb{R}$ is called a **borel set** iff $C \in \mathcal{B}$

We will further define a **field** as a family of subsets $\mathcal{A}^* \subset \mathcal{P}(\Omega)$ if:

1. $\emptyset \in \mathcal{A}^*$
2. $A \in \mathcal{A}^* \implies A^\complement \in \mathcal{A}^*$
3. $A_1, A_2, ... \in \mathcal{A}^* \implies A_1 \cup A_2 \in \mathcal{A}^*$

▼ Pre- Measures

Definition: let $\mathcal{A}^*$ be a **field.** Then a function $P^* : \mathcal{A}^* \to [0, \infty)$ is called a **pre-measure** iff for every sequence $A_1, A_2, ... \in \mathcal{A}^*$ with $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}^*$ it holds that:

$$P^* \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} P^*(A_i)$$

Theorem of Carathéodory: let $\mathcal{A}^*$ be a field and $P^* : \mathcal{A}^* \to [0, \infty)$ be a **pre-measure.** Then there is one and only one **measure** $P : \sigma(\mathcal{A}^*) \to [0, \infty)$ such that:

$$P(A) = P^*(A)$$

▼ cdf and Lebesgue Stieltjes measure

Definition of cdf: Let $P : \mathcal{B} \to [0, \infty)$ be a probability measure on $(\mathbb{R}, \mathcal{B})$. Then the **cummulative distribution function** $F : \mathbb{R} \to [0, 1]$ is defined by:

$$F(a) = P((-\infty, b]) \quad \forall b \in \mathbb{R}$$

Properties of a destribution function:

1. $P((a, b]) = F(b) - F(a)$
2. $F(a) \leq F(b) \Leftrightarrow a \leq b$
3. For all sequences $(b_n \in \mathbb{R})_{n \in \mathbb{N}}$ monotnously decreasing with $b_n \to b$ it holds that: $F(b_n) \to F(b)$

4. $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$

We now have derived a **distribution function** from a probability measure. **Theorem 1.16** now states, that for every real function $F : \mathbb{R} \to [0, 1]$, that satisfies properties 2 -4 from above, there exists one and only one **probability measure** $P : \mathcal{B} \to [0, \infty)$ with: $F(b) = P((-\infty, b])$

Every probability measure, that is characterized by such a function is now called **Lebesgue-stieltjes-measure**

The **lebesque measure** $\lambda : \mathcal{B} \to [0, \infty)$ is defined by:

$$\lambda((a, b]) = b - a$$

▼ probability mass function and pdf

Definition of pmf: Let $f : \mathbb{R} \to \mathbb{R}_+$. Then f is called a pmf iff:

$$\sum_{x \in \mathcal{S}_f} f(x) = 1 \quad \text{with} \quad \mathcal{S}_f = \{x \in \mathbb{R} : f(x) > 0\}$$

$\mathcal{S}_f$ is called the **support** and must be **countable** in this definition**.** And we can define a corresponding **probability-measure** P **as:**

$$P(A) = \sum_{x \in (A \cap \mathcal{S}_f)} f(x)$$

▼ Discrete probability measures and pdfs

A probability measure on the measure space $(\mathbb{R}, \mathcal{B})$ is called **discrete iff:**

$$\exists A \subset \mathbb{R} | A \text{ countable} : P(A) = 1$$

Definition pdf: let $f : \mathbb{R} \to \mathbb{R}_+$ be a real and positve mapping. Then f is a pdf iff:

$$\int_{-\infty}^{\infty} f(x) = 1$$