Sheet 6 Solutions

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Ex 22

a) $X \sim \delta_0 \iff \varphi_X(t) = 1 \quad \forall t \in \mathbb{R}$

$$\varphi_X(t) \overset{def}{=} \mathbb{E}(\exp(itX)) \underset{\text{discreet}}{=} \sum_{k=1}^{\infty} e^{itk} f_X(k)$$

$X \mapsto \{0,1\}$

$$f(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 \end{cases} \Rightarrow \mathbb{E}(\exp(itX)) = \exp(it0)\cdot 1 + 0$$

$$= 1$$

b) $N(0,\frac{1}{n}) \xrightarrow{d} \delta_0$ Note "weakly converges" $\iff$ "converges in distribution"

using the hint:

$$\varphi_{N(0,\frac{1}{n})}(t) = e^{-\underbrace{\frac{1}{n^2}}_{0}\frac{t}{2}} \underset{n\to\infty}{\longrightarrow} 1 = \varphi_{\delta_0}(t) \quad \forall t \in \mathbb{R}$$

c) $X_n \sim N(0,n) \qquad X_n \xrightarrow{d} X\,?$

using hint

$$\varphi_{X_n}(t) = e^{-n\frac{t^2}{2}} \underset{n\to\infty}{\longrightarrow} 0 \quad \forall t \neq 0$$

$$\text{if } t=0 \qquad \varphi_{N(0,n)}(0) = 1$$

d) $X \sim U\{a;b\}$, $a<b$

$$\varphi_X(t) \overset{def}{=} \mathbb{E}(\exp(itX)) = \int \exp(itx) f_X(x)\,dx$$

$$= \int_a^b \exp(itx)\frac{1}{b-a}\,dx$$

$$= \frac{1}{it}\exp(itx)\frac{1}{b-a}\Big|_a^b$$

$$= \frac{1}{it(b-a)}(\exp(itb) - \exp(ita))$$

e) $U[0,\frac{1}{n}] \xrightarrow{d} \delta_0$

$$\lim_{n\to\infty} \frac{\exp(it\frac{1}{n}) - \exp(it0)}{it(\frac{1}{n}-0)} = \lim_{n\to\infty} \frac{-\frac{1}{n^2}it\exp(it\frac{1}{n})}{-it\frac{1}{n^2}} = \lim_{n\to\infty} \exp(it\frac{1}{n}) = 1 \quad \forall t \in \mathbb{R}$$

f) $X_n \sim U\{0,n\}$

$$\varphi_{X_n}(t) = \frac{\exp(itn) - \exp(it0)}{itn} = \frac{\cos(tn) + i\sin(tn) - 1}{itn}$$

$$= \frac{i}{tn} + \frac{i\cos(tn)}{tn} + \frac{\sin(tn)}{tn}$$

$$= i\left(\frac{1}{t}(1-\cos(tn))\right) + \frac{\overset{0}{\sin(tn)}}{}$$

$$= i\underbrace{\left(\frac{1}{tn}\left(1-\cos(tn)\right)\right)}_{\text{Imaginary}} + \underbrace{\frac{\sin(tn)}{tn}}_{\text{Real}}$$

$= 0 \quad \forall t \neq 0$

if $t = 0$ $\quad \lim_{\substack{n\to\infty \\ t\to 0}} \frac{e^{itn}-1}{itn} = \dots = 1$ not cont. around 0

$\Rightarrow$ not a valid r.v.

<u>Ex 23</u> $\quad X \sim Bin(n,p)$

$$\varphi_X(t) = E(\exp(itn)) = \sum_{k=0}^{\infty} e^{itk} p^k (1-p)^{n-k} \binom{n}{k}$$

$$= \sum_{k=0}^{\infty} (e^{it}p)^k (1-p)^{n-k} \binom{n}{k}$$

$$= \underline{\left(e^{it}p + (1-p)\right)^n} \quad \forall t \in \mathbb{R}$$

$$E(X) = \frac{\partial \varphi_X(t)}{\partial t} \frac{1}{i}\Big|_{t=0} = \frac{1}{i} n\left(pe^{it} + (1-p)\right)^{n-1} p\, i\, e^{it}\Big|_{t=0} = np$$

$$E(X^2) = \frac{\partial \varphi_X(t)}{\partial t^2}\frac{1}{i^2} = \dots = n(n-1)p^2 + np$$

$$Var(X) = E(X^2) - E(X)^2 = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np(1-p)$$

b) same wie im Blatt.

<u>Ex 20</u>:

s.t. $\lim_{n\to\infty} P(|X_n - 1| > \varepsilon) = 0 \quad \forall \varepsilon > 0$

$\Rightarrow \exists n_0 \geq \varepsilon - 1 : \quad P(|X_n - 1| > \varepsilon) = P(X_n = n) \quad \forall n \geq n_0$
$n \in \mathbb{N}$

$$\lim_{n\to\infty} P(|X_n - 1| > \varepsilon) = \lim_{n\to\infty} \frac{1}{n^\alpha} = 0 \quad \underline{\forall \alpha > 0}$$

b)

Same wie in der Lösung

c)

$X_n \xrightarrow{d} 1 \iff E(f(X_n)) - E(f(1)) \longmapsto 0 \quad \forall f \in C_b$

So let $f \in C_b$

$$\Rightarrow E(f(X_n)) = f(1)\left(1 - \frac{1}{n^\alpha}\right) + f(n)\frac{1}{n^\alpha}$$

$$\Rightarrow \lim_{n\to\infty} E(f(X_n)) - E(f(1)) = f(1)\left(1-\frac{1}{n^\alpha}\right) + f(n)\frac{1}{n^\alpha} - f(1)$$

$$= \frac{1}{n^\alpha}\left(f(n) - f(1)\right) \longmapsto 0$$

$$\Rightarrow \quad n \to \infty \quad \mathbb{E}(f(X_n)) - \mathbb{E}(f(1)) = f(1)(1 - n^\alpha) + f(n)\, n^\alpha - f(1)$$

$$= \frac{1}{n^\alpha}\big(\underset{\hat{\infty}}{f(n)} - f(1)\big) \xrightarrow[n\to\infty]{} 0$$

f bounded

Ex. 21

s.t. $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \xrightarrow{p} \sigma^2$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} ((X_i - \mu) - (\bar{X}_n - \mu))^2 = \dots = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2 - 2\bar{X}_n + 2\mu \bar{X}_n - 2\mu^2 + \bar{X}_n - 2\mu \bar{X}_n + \mu^2$$

$$= \underbrace{\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)}_{\sigma^2} - \underbrace{(\bar{X}_n - \mu)^2}_{\downarrow p \; 0}$$

LLN: $\bar{X}_n \xrightarrow{p} \mu \Leftrightarrow \bar{X}_n - \mu \xrightarrow{p} 0$ = const.

$\Rightarrow \bar{X}_n - \mu \xrightarrow{d} 0 \Rightarrow (\bar{X}_n - \mu)^2 \xrightarrow{d} 0$

CMT: $y = x^2$ - const

$\Rightarrow (\bar{X}_n - \mu)^2 \xrightarrow{p} 0$

$$\begin{pmatrix} \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2 \\ (\bar{X}_n - \mu)^2 \end{pmatrix} \xrightarrow[d]{p} \begin{pmatrix} \sigma^2 \\ 0 \end{pmatrix}$$

CMT $f\begin{pmatrix} x \\ y \end{pmatrix} = x - y$ - const