Sheet 4

Problem 1

1.)

The investor is presented with two opportunities, a project for natural resources or a risk-free coupon:

$$D := \{\text{n.r. project, coupon}\}$$

The state of the world can be either success or failure:

$$\theta \in \{\text{success}, \text{failure}\}$$

The functions are (assuming inflation = 0%):

$$L(\theta|\text{n.r. project}) := \begin{cases} -4000€ \text{ if } \theta = \text{success} \\ 0€ \text{ if } \theta = \text{failure} \end{cases}$$
$$L(\theta|\text{coupon}) := -(1000 + 3.5\% \cdot 1000€ + 3.5\% \cdot 1000€)$$

Because the coupon pays regardless of the state of the world, we can ignore the cases of θ. Furhermore because we are looking at the costs, profits are negative.

2.)

The risk function is the expected loss depending on the decision:

$$R(L(\theta|d)) = L(\text{success}|d)P(\theta = \text{success}) + L(\text{failure}|d)P(\theta = \text{failure})$$

3.)

θ is binary. Therefore the uncertainty of θ can be modeled in a bernoulli distribution. We know from the exercise that $P(\theta = \text{success}) = 0.1$. In this case we could say:

$$\theta \sim Bernoulli(0.1)$$

4.)

The posterior risk for the decision for the natural resources is much lower than the starting capital:

$$R(L(\theta|\text{n.r. project}) = 0.1 \cdot (-4000€) + 0.9 \cdot 0€ = -400€$$

The coupon option yields:

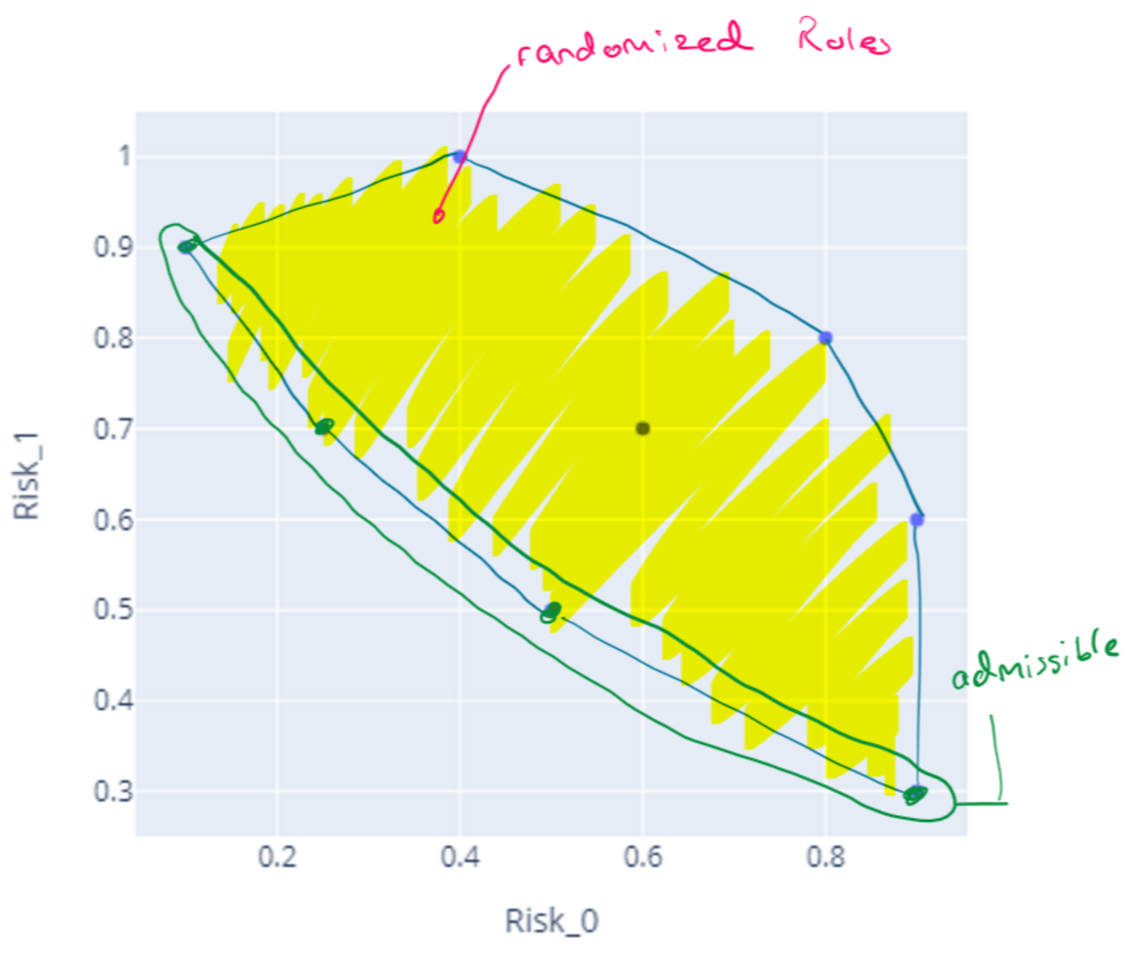
$$R(L(\theta|\text{coupon}) = 1 \cdot (-(1000€ + 7\% \cdot 1000€)) = -1070€$$

Problem 2

1.)

The table shows the sample realization and risk of every decision. The column on the very left shows every available decision RULE $\delta \in D = \{\delta_1, ..., \delta_8\}$. Every decision ruel δ is a mapping: $\delta : \Omega \to \{0, 1\}$, from every possible value of X to a decision. 0 and 1 represent the decision for θ_0 or θ_1 respectively. For example δ_1 decides θ_0 for every possible realization of X. The values for $X = 0, X = 1, X = 2$ of the decision rule are shown in the next 3 columns. Next to the values of δ are the Risk values dependent of the true parameter θ and the decision δ. Because $\Theta = \{\theta_0, \theta_1\}$ so only two parameters are possible, we get two Risk values for each decision rule, which take up the last two columns of the table.

2.)



The convex shape encapsulated, by all rules are the randomized and non-randomized rules (Because the risk points resulting from randomized rules are a linear combination of randomized rules (where the sum of the coefficients needs to add up to 1).

The admissible rules are per definition admissible, if there exists no R-better rule in the decision-space. An easy way to find R-better rules is to draw a rectangle from the risk point of the decision and seeing if there is any risk point of another decision within this rectangle. If this is the case, the rule is not admissible.

For this problem the for non-randomized rules, that are admissible are: 1,2,5,7

3.)

The Baysian risk with prior $\pi_p(\theta)$ is:

$$\begin{aligned} B(\pi,\delta) &= \pi_p(\theta_0)R(\theta_0,\delta) + \pi_p(\theta_1)R(\theta_1,\delta) \\ &= pR(\theta_0,\delta) + (1-p)R(\theta_1,\delta) \\ &= \frac{1}{4}R(\theta_0,\delta) + \frac{3}{4}R(\theta_1,\delta) \end{aligned}$$

The Bayes rule is now the rule that minimizes the Baysian-Risk:

$$B(\pi,\delta_B) = \inf_{\delta\in D} B(\pi,\delta)$$

Computing this risk for every non-random rule gets:

	Risk_0	Risk_1	name	baysian_risk
0	0.10	0.9	1	0.7000
1	0.25	0.7	2	0.5875
2	0.40	1.0	3	0.8500
3	0.80	0.8	4	0.8000
4	0.50	0.5	5	0.5000
5	0.60	0.7	6	0.6750
6	0.90	0.3	7	0.4500
7	0.90	0.6	8	0.6750

As we can see, rule 7 gets the minimal baysian risk.

Problem 3

1.)

We need to show that:

$$\exists\theta_1,\theta_2\in\Theta : R(\theta_1,C) < R(\theta_1,T) \land R(\theta_2,C) > R(\theta_2,T)$$

If $d = C \in \mathbb{R}$ then:

$$\begin{aligned} R(\theta,d) = R(\theta,C) &= \int_{\mathbb{R}} (\theta - C)^2 f(x|\theta)dx \\ &= (\theta - C)^2 \int_{\mathbb{R}} f(x|\theta)dx \\ &= (\theta - C)^2 \end{aligned}$$

On the other hand $t:\mathbb{R}\to\mathbb{R}$ yields:

$$R(\theta,T)=\int_{\mathbb{R}}(\theta-t(x))^2 f(x|\theta)dx$$

Because T is non-constant we know that there exists a $x\in\mathbb{R}$ such that $t(x)\neq\theta$ this means that $(\theta-t(x))^2>0$ and therefore (assuming the support of $f(x|\theta)$ is $\mathbb{R}$):

$$\forall\theta\in\mathbb{R}:R(\theta,T)=\int_{\mathbb{R}}(\theta-t(x))^2 f(x|\theta)dx>0$$

However for every $d=C$ there exists a θ_1 where $\theta_1=C$. In this case the risk becomes zero:

$$\forall C\in\mathbb{R}\exists\theta_1\in\Theta:R(\theta_1,C)=R(C,C)=(C-C)^2=0$$

Meaning we will always (regardless of C and T) find a θ_1 such that

$$R(\theta_1,C)<R(\theta_1,T)$$

holds, proving the first part R-incomparability. It furthermore proves, that $d=C$ is an admissible rule (because the Risk can not be negative if the loss is always positive). From here we can use the proposition from the lecture:

$$d=C\text{ and }d=T\text{ both admissible}\implies d=C\text{ and }d=T\text{ R-equivalent or R-incomparable}$$

But since we have found a $\theta_1\in\Theta$ for which $R(\theta_1,C)\neq R(\theta_2,T)$, we can say, that they are not R-equivalent and therefore must be R-incomparable.

2.)

We need to show that for any $a\neq 0$ and $T=t(x)$ an unbiased estimator it holds that:

$$\forall\theta\in\Theta:R(\theta,T)<R(\theta,T+a)$$

This statement is equivalent to:

$$\forall\theta\in\Theta\exists C>0:R(\theta,T+a)=R(\theta,T)+C$$

Our goal will be, to find the C. Firstly we “unbiased” means that:

$$\int_{\mathbb{R}}t(x)f(x|\theta)dx=\mathbb{E}(t(X))=\theta$$

Continuing with finding C, let’s firstly only look at the loss term:

$$\begin{aligned}(\theta-(t(x)+a))^2&=\theta^2-2\theta(t(x)+a)+(t(x)+a)^2\\&=\theta^2-2\theta t(x)-2\theta a+t(x)^2+2t(x)a+a^2\\&=\theta^2-2\theta t(x)+t(x)^2-2\theta a+2t(x)a+a^2\\&=(\theta-t(x))^2-2\theta a+2t(x)a+a^2\end{aligned}$$

From this we can say for the loss of $T + a$:

$$
\begin{aligned}
R(\theta, T + a) &= \int_{\mathbb{R}} (\theta - (t(x) + a))^2 f(x|\theta) dx \\
&= \int_{\mathbb{R}} (\theta - t(x))^2 f(x|\theta) dx \\
&- 2\theta a \int_{\mathbb{R}} f(x|\theta) dx + 2a \int_{\mathbb{R}} t(x) f(x|\theta) dx \\
&+ a^2 \int_{\mathbb{R}} f(x|\theta) dx \\
&\underset{\mathbb{E}(t(X))=\theta}{=} R(\theta, T) - 2\theta a + 2a\theta + a^2 \\
&= R(\theta, T) + a^2
\end{aligned}
$$

So $C = a^2$ and since $a \neq 0$ it holds that $C > 0$. Prooving that $d = T + a$ is inadmissible.