Statistical Theory

Sheet 1 Solutions

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1. Exercise 1

(i) To show this, we will find a contradictory example. Let $C \subset \Omega$ be a subset of the samplespace. Furthermore let $\Omega := \{1, 2, 3, 4\}$ and $A := \{1\}, B := \{2\}, C := \{4\}$. We will generate form $\{A, B\}$ and C:

$$\mathcal{A}_1 := \sigma(\{A, B\})$$
$$\mathcal{A}_2 := \sigma(C)$$

The following σ-fields will be generated:

$$\mathcal{A}_1 = \{\{1\}, \{2\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2\}, \{3, 4\}, \Omega, \emptyset\}$$
$$\mathcal{A}_2 = \{\{4\}, \{1, 2, 3\}, \Omega, \emptyset\}$$

Now $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra/field, because $\{1, 4\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$.

(ii) To show this, we will start with the trivial statement of $\{A, B\} \subset \mathcal{A}_1$ and work our way to the different sets through the properities $(ii)(A \in \mathcal{A}_1 \implies A^c \in \mathcal{A}_1)$ and $(iii)(A_1, A_2, ... \in \mathcal{A}_1 \implies \bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{A}_1)$:

$$\underset{(iii)}{\implies} A \cup B \in \mathcal{A}_1$$
$$\underset{(ii)}{\implies} \{A^c, B^c\} \subset \mathcal{A}_1$$
$$\underset{(iii)}{\implies} A^c \cup B^c \in \mathcal{A}_1$$

But since $A^c \cup B^c = (A \cap B)^c$ and property (ii) still holds we have shown that:

$$(A \cap B) \in \mathcal{A}_1$$

Now we only need to show, that $\{A\Delta B, A\setminus B\} \subset \mathcal{A}_1$:

$$\begin{aligned} &\{A, B, A^c, B^c\} \subset \mathcal{A}_1 \\ \underset{(iii)}{\Longrightarrow}\ & A\cup B^c = A^c \cup (A\cap B) \in \mathcal{A}_1 \\ \underset{(ii)}{\Longrightarrow}\ & (A^c \cup (A\cap B))^c = A\setminus B \in \mathcal{A}_1 \end{aligned}$$

Using the same procedure for $A^c \cup B$ it holds that:

$$\begin{aligned} &\{A\setminus B, B\setminus A\} \subset \mathcal{A}_1 \\ \underset{(iii)}{\Longrightarrow}\ & (A\setminus B)\cup(B\setminus A) = A\Delta B \in \mathcal{A}_1 \end{aligned}$$

2. Exercise 2

(i) To show, that a δ_{ω_0} is a measure, we need to proof the three properties that characterize a measure ((i): positive mapping, (ii): empty set is measured as zero and (iii): σ-additivity).

(i) Positive mapping is implied directly by the definition of the Dirac measure:

$$\delta_{\omega_0} : \mathcal{A} \to \{0,1\} \implies 0 \le \delta_{\omega_0}(A) < \infty \quad \forall A \in \mathcal{A}$$

(ii) Because ω_0 is not in the empty set, the measure maps to 0, satisfying the condition:

$$\omega_0 \notin \emptyset \implies \delta_{\omega_0}(\emptyset) = 0$$

(iii) We will look at **two cases**. **Fristly** consider: $\omega_0 \in \bigcup\limits_{i=1}^{\infty} A_i \quad (1)$. It follows that:

$$\delta_{\omega_0}\left(\bigcup_{i=1}^{\infty} A_i\right) \underset{defintion+(1)}{=} 1$$

Because $A_i \cap A_j = \emptyset \quad \forall i \neq j$, you could argue, that the Dirac measure only maps to 1 for one set of the sequence. Therefore the sum over all mappings would still be 1:

$$\begin{aligned} &\exists! i \in \mathbb{N} : \omega_o \in A_i \implies \delta_{\omega_0}(A_i) = 1 \wedge \delta_{\omega_0}(A_j) = 0 \quad \forall j \neq i \\ \implies & \sum_{j=0}^{\infty} \delta_{\omega_0}(A_j) = \delta_{\omega_0}(A_i) + \sum_{j=0, j\neq i}^{\infty} \delta_{\omega_0}(A_j) = 1 + 0 = 1 \end{aligned}$$

We have therefore shown the statement for the first case:

$$\delta_{\omega_0}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \delta_{\omega_0}(A_i)$$

The proof for the **second case:** $\omega_0 \notin \bigcup\limits_{i=1}^{\infty} A_i$ has the same structure and but is a bit more trivial. Thus it is spared in this solution.

(ii) We will now show that the **counting measure** $\nu(A) = |A|$ is a measure. Before starting the proof, we are going to additionally assume, that Ω is *at least countable*. Furthermore we need to proof some properties of the cardinality (since this wasn't proven in the lecture):

Defintion 1 Let set A be countable. Then the cardinality is definied by:

$$|A| := \sum_{x \in A} 1$$

(i) $|A|$ can be any integer (including zero) and ∞ (if the set is countable infinite):

$$\nu : \mathcal{A} \rightarrow (\mathbb{N}_0 \cup \infty) \subset [0, \infty]$$

Therefore the condition is satisfied by definition.

(ii) This condition is also trivial:

$$\nu(\emptyset) = |\emptyset| = 0$$

(iii) Firstly lets denote: $\mathcal{B} := \bigcup\limits_{i=1}^{n} A_i$. We can use the definition of cardinality and $A_i \cap A_j = \emptyset \quad \forall i \neq j (1)$, to show:

$$\begin{aligned}
\nu \left(\bigcup_{i=1}^{n} A_i \right) &= \sum_{x \in \mathcal{B}} 1 \\
&\underset{(1)}{=} \sum_{i=1}^{\infty} \sum_{x \in A_i} 1 \\
&\underset{D.1}{=} \sum_{i=1}^{\infty} \nu(A_i)
\end{aligned}$$

3. Exercise 3

(i) We again need to show the properties (i), (ii) and (iii). This time for probability measures. Before starting this proof, it should be noted, that **Lemma 1.2** applies to this measure. Therefore we can define for all sets A with $|A| > 1$, that:

$$P(A) := \sum_{x \in A} P(\{x\})$$

(i) Since $\omega \in \mathbb{N} \quad \forall \omega \in \Omega$, it holds, that:

$$0 < P(\{\omega\}) = \frac{\omega}{10} < \infty$$

And for any set A with $|A| > 1$ it holds that:

$$0 < P(\{\omega\}) \leq \sum_{x \in A} P(\{x\}) = P(A) \underset{\Omega finite}{<} \infty$$

In total:

$$0 \leq P(A) < \infty \qquad \forall A \in \mathcal{P}(\Omega)$$

Alternatively you could proof this property manually for every set from $\mathcal{P}(\Omega)$.

(ii) We will denote $\mathcal{B} := \bigcup\limits_{i=1}^{\infty} A_i$, with $A_i \cap A_j = \emptyset \quad \forall i \neq j \qquad (1)$. Then:

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i\right) = P(\mathcal{B}) &= \sum_{x \in \mathcal{B}} P(\{x\}) \\ &\underset{(1)}{=} \sum_{i=1}^{\infty} \sum_{x \in A_i} P(\{x\}) \\ &= \sum_{i=1}^{\infty} P(A_i) \end{aligned}$$

(iii) This proof is very straight forward:

$$P(\Omega) = P(\{1, 2, 3, 4\}) = \frac{1 + 2 + 3 + 4}{10} = \frac{10}{10} = 1$$

(ii) We can formulate a significantly easier definition for $\mathcal{E}$:

$$\mathcal{E} = \{A \in \mathcal{P}(\Omega) | \sum_{x \in A} x = 5\}$$

From this defintion we can derive:

$$\mathcal{E} = \{\{2, 3\}\{4, 1\}\}$$

It follows that (because the sets in $\mathcal{E}$ are already compliments of each other):

$$\sigma(\mathcal{E}) = \{\{2, 3\}\{4, 1\}, \Omega, \emptyset\}$$

4. Exercise 4

(i) To proof this statement, we will use a partition of $A \cup B$ and (ii) as well as (v) from **Theorem 1.9**:

$$\begin{aligned} P(A \cup B) &= P((A \setminus B) \cup (B \setminus A) \cup (A \cap B)) \\ &\underset{(ii)}{=} P(A \setminus B) + P(B \setminus A) + P(A \cap B) \\ &= P(A \setminus (A \cap B)) + P(B \setminus (A \cap B)) + P(A \cap B) \\ &\underset{(v)}{=} P(A) + P(B) - 2P(A \cap B) + P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

(ii) Let $n \in \mathbb{N}$. Then it holds that:

$$\begin{aligned} P(A_n) &\underset{A_n \subset A_{n-1}}{=} P(A_{n-1} \cap A_n) \\ &\underset{A_{n-1} \subset A_{n-2}}{=} P(A_{n-2} \cap A_{n-1} \cap A_n) \\ &= P(... \cap A_{n-2} \cap A_{n-1} \cap A_n) \\ &= P\left(\bigcap_{i=1}^{n} A_i\right) \qquad \forall n \in \mathbb{N} \end{aligned}$$

It follows that:

$$\lim_{n \to \infty} P(A_n) = \lim_{n \to \infty} P\left(\bigcap_{i=1}^{n} A_i\right) = P\left(\bigcap_{i=1}^{\infty} A_i\right)$$

(iii) This proof is similar to (i). We are going to use the properties (ii) (finite additivity) from **Theorem 1.9**:

$$\begin{aligned} & P(A \Delta B) && = 0 \\ \underset{(ii)+Hint}{\Leftrightarrow} \quad & P(A \setminus B) + P(B \setminus A) && = 0 \\ \Leftrightarrow \quad & P(A \setminus B) && = -P(B \setminus A) \\ \implies \quad & P(A \setminus B) = P(B \setminus A) = 0 && \end{aligned}$$

But this means, that:

$$\begin{aligned} P(A) = P((A \setminus B) \cup (A \cap B)) &\underset{(ii)}{=} P(A \setminus B) + P(A \cap B) = 0 + P(A \cap B) \\ &\Leftrightarrow P(A) = P(A \cap B) \end{aligned}$$

And also:

$$\begin{aligned} P(B) = P((B \setminus A) \cup (A \cap B)) &\underset{(ii)}{=} P(B \setminus A)) + P(A \cap B) = 0 + P(A \cap B) \\ &\Leftrightarrow P(B) = P(A \cap B) \end{aligned}$$

So:

$$P(A) = P(B) = P(A \cap B)$$