Statistical Theory
Sheet 2 Solutions

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1. Lemmas

Lemma A let $(a_n)_{n\in\mathbb{N}} \in \mathbb{R}$ monotonically increasing with $a_n \to a < \infty$. Furthermore let $(A_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence of sets with $A_n := (a_n, a] \quad \forall n \in \mathbb{N}$. Then it holds that:

$$\bigcap_{i=1}^{\infty} A_i = \{a\}$$

proof by contradiction let's assume the following:

$$\exists x \in \mathbb{R} : x \in \bigcap_{i=1}^{\infty} A_i \wedge x \neq a$$

Then it holds that:

$$x \in A_i = (a_i, a] \quad \forall i \in \mathbb{N} \tag{1.1}$$

Furthermore since $x \neq a$ it must be that $x < a$:

$$x < a \Leftrightarrow 0 < a - x$$

When we denote $\varepsilon := a - x$ then the definition of convergence implies that:

$$\exists n_0 \in \mathbb{N} : |a - a_n| < \varepsilon = a - x \quad \forall n \geq n_0$$

But since $a_n \leq a \quad \forall n \in \mathbb{N}$ we can say that:

$$\exists n \in \mathbb{N} : a - a_n < a - x \Leftrightarrow a_n > x \quad \forall n \geq n_0$$

But $a_n > x \quad \forall n \geq n_0$ implies that $x \notin (a_n, a] \quad \forall n \geq n_0$, contradicting (1.1).

Lemma B Let samplespace Ω and sequence $(A_n)_{n\in\mathbb{N}} \subset \Omega$ with $A_{n+1} \subset A_n \quad \forall n \in \mathbb{N}$. Also let $\mathcal{A} := \sigma(\{A_1, A_2, ...\})$. Then it holds that:

$$\lim_{n\to\infty} A_n \in \mathcal{A}$$

Proof Since $A_n \in \mathcal{A} \quad \forall n \in \mathbb{N}$ property (ii) of σ-algebras implies that $A_n^c \in \mathcal{A}$.

$$\underset{(iii)}{\Longrightarrow} \bigcup_{i=1}^{\infty} A_i^c \in \mathcal{A} \underset{(ii)}{\Longrightarrow} \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathcal{A}$$

But if we apply de morgans law, we can show that:

$$\begin{aligned}\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c &\underset{\text{de morgan}}{=} \bigcap_{i=1}^{\infty} (A_i^c)^c \\ &= \bigcap_{i=1}^{\infty} A_i \\ &\underset{A_{n+1} \subset A_n}{=} A\end{aligned}$$

So:

$$\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathcal{A} \Leftrightarrow A \in \mathcal{A}$$

2. Exercise 5

a) With the help of Lemma A and B, the proof is much easier. Let $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}$ monotonically increasing with $x_n \to x \in \mathbb{R}$. Also let $A_n := (x_n, x]$ be a sequence of subsets of $\mathbb{R}$. Then Lemma A implies, that:

$$\bigcap_{i=1}^{\infty} A_i = \{x\}$$

And since $A_n \in \mathcal{B}(\mathbb{R})$ and $A_{n+1} \subset A_n \quad \forall n \in \mathbb{N}$ and $\mathcal{B}(\mathbb{R})$ is a σ-algebra, Lemma B can be applied:

$$\bigcap_{i=1}^{\infty} A_i = \lim_{n\to\infty} A_n \in \mathcal{B}(\mathbb{R}) \Leftrightarrow \{x\} \in \mathcal{B}(\mathbb{R})$$

b) We can proof this property by using the statement we showed on *Sheet 1*. Since $(a, b], \{x\} \in \mathcal{B}(\mathbb{R}) \forall a, b, x \in \mathbb{R}$, we can use the statement from exercise 1.ii) if $x = b$:

$$(a, b] \setminus \{b\} \in \mathcal{B}(\mathbb{R})$$

But:

$$(a, b] \setminus \{b\} = (a, b)$$

So every open interval (a, b) is contained in $\mathcal{B}(\mathbb{R})$.

c) Let $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}$ be a real monotonically decreasing sequence with $x_n \to -\infty$ and $(A_n)_{n\in\mathbb{N}} = (x_n, b]$ be a set sequence of left open intervalls (just like in a)). Since $A_n \in \mathcal{B}(\mathbb{R}) \;\; \forall n \in \mathbb{N}$:

$$\underset{(iii)}{\Longrightarrow} \bigcup_{i=1}^{\infty} A_i \in \mathcal{B}(\mathbb{R})$$

But since $A_n \subset A_{n+1} \forall n$ we can say:

$$\bigcup_{i=1}^{\infty} A_i = \lim_{n\to\infty} A_n = \lim_{n\to\infty} (x_n, b] = (-\infty, b]$$

So $(-\infty, b] \in \mathcal{B}(\mathbb{R}) \;\; \forall b \in \mathbb{R}$. Using the same procedure from b) we can state, that:

$$\underset{b)}{\Longrightarrow} (-\infty, b] \setminus \{b\} \in \mathcal{B}(\mathbb{R})$$

This leads directly to:

$$(-\infty, b) \in \mathcal{B}(\mathbb{R})$$

3. Exercise 6

a) We are going to use a simple example, where this does not hold. Let $\omega_1, \omega_2 \in \Omega$ two observations and $\nu_1 := \delta_{\omega_1}, \nu_2 := \delta_{\omega_2}$ their corresponding Dirac-measures. Furthermore let $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}$ a pairwise disjoint sequence of sets such that: $\omega_1 \in A_1 \wedge \omega_2 \in A_2$. Then:

$$\begin{aligned}
\mu_1\left(\biguplus_{i=1}^{\infty} A_i\right) &= \nu_1\left(\biguplus_{i=1}^{\infty} A_i\right)\nu_2\left(\biguplus_{i=1}^{\infty} A_i\right) \\
&= \delta_{\omega_1}\left(\biguplus_{i=1}^{\infty} A_i\right)\delta_{\omega_2}\left(\biguplus_{i=1}^{\infty} A_i\right) \\
&= 1 \cdot 1 = 1
\end{aligned}$$

However:

$$\begin{aligned}
\sum_{i=1}^{\infty} \mu_1(A_i) &= \sum_{i=1}^{\infty} \nu_1(A_i)\nu_2(A_i) \\
&= \delta_{\omega_1}(A_1)\delta_{\omega_2}(A_1) + \delta_{\omega_1}(A_2)\delta_{\omega_2}(A_2) + \sum_{i=3}^{\infty} \nu_1(A_i)\nu_2(A_i) \\
&= 1 \cdot 0 + 0 \cdot 1 + 0 + 0 + ... = 0
\end{aligned}$$

This contradicts the σ-additivity of measures. So $\nu_1 \cdot \nu_2$ is not a measure.

b) We take the same example as above (not redefining it) and try to prove σ-additivity:

$$\mu_2\left(\biguplus_{i=1}^{\infty} A_i\right) = \min\left(\delta_{\omega_1}\left(\biguplus_{i=1}^{\infty} A_i\right), \delta_{\omega_2}\left(\biguplus_{i=1}^{\infty} A_i\right)\right)$$
$$= \min(1,1) = 1$$

However if we look at the sum:

$$\sum_{i=1}^{\infty} \mu_2(A_i) = \sum_{i=1}^{\infty} \min(\delta_{\omega_1}(A_i), \delta_{\omega_2}(A_i))$$
$$= 0 + 0 + 0 + ... = 0$$

c) Same setup as in the other two. This time the first set A_1 of the sequence $(A_n)_{n\in\mathbb{N}}$ maps to -1:

$$\mu_3(A_1) = \nu_2(A_1) - \nu_1(A_1) = 0 - 1 = -1$$

This hurts the first property of measures (positive mapping).

4. Exercise 7

Summary We assume following setup. Let $(\Omega, \mathcal{A})$ be a measuring space and $\mu : \mathcal{A} \to [0, \infty]$ be a measure. Then every set $A \in \mathcal{A}$ is considered measurable. Therefore in this setup, if we want to show, that a set A is Lebesgue measurable, we only need to show: $A \in \mathcal{B}([a, b])$. Furthermore we are using the definition for a Borel-σ-field of exercise 5.

proof 1 s.t. $x \in (a, b] \implies x$ is Lebesque measureable and $\lambda(\{x\}) = 0$

Let $(a_n)_{n\in\mathbb{N}} \in (a, b]$ be a real montonuously increasing sequence with $a_n \to c \in (a, b]$. Furthermore let $(A_n)_{n\in\mathbb{N}} := (a_n, c]$ be the same sequence as in Lemma A, but only with upper boundaries in $(a, b]$. Then Lemma A implies that:

$$\bigcap_{i=1}^{\infty} A_i = \{c\}$$

Because $A_n \in \mathcal{B}([a, b])$ is monotonically decreasing and $\mathcal{B}([a, b])$ is assumed to be a σ-algebra, Lemma B implies that:

$$\bigcap_{i=1}^{\infty} A_i = \lim_{n\to\infty} A_n = \{c\} \in \mathcal{B}([a, b])$$

Therefore we can say, that $\{c\}$ is Lebesque measureable. And because A_n is monotonically decreasing ($A_{n+1} \subset A_n$) we can apply the hint and say:

$$\lambda(A_n) \underset{n\to\infty}{\longrightarrow} \lambda\left(\bigcap_{i=1}^{\infty} A_i\right) \underset{\text{Lemma A}}{=} \lambda(\{c\})$$

And:

$$\lambda(\{c\}) = \lim_{n\to\infty} \lambda(A_n) = \lim_{n\to\infty} c - a_n = c - \lim_{n\to\infty} a_n = c - c = 0 \quad \forall c \in (a, b]$$

It therefore must follow that (using finite additivity):

$$\lambda((a, b]) = \lambda((a, b) \cup \{b\}) = \lambda((a, b)) + \lambda(\{b\}) = \lambda((a, b)) + 0 = \lambda((a, b))$$

proof 2 s.t. $\{a\}$ is Lebesgue measurable and $\lambda(\{a\}) = y$ with $y \in \mathbb{R}_+$

Let $(a_n)_{n\in\mathbb{N}} \in (a, b]$ be a monotonically decreasing real sequence with $a_n \to a$ and $(A_n)_{n\in\mathbb{N}} := (a_n, b] \in \mathcal{B}([a, b])$ a sequence of intervalls. It follows that:

$$A_n \in \mathcal{B}([a, b]) \underset{(iii)}{\implies} A_n \cup \{a_n\} \in \mathcal{B}([a, b]) \underset{(ii)}{\implies} (A_n \cup \{a_n\})^c = [a, a_n) \in \mathcal{B}([a, b]) \quad \forall n$$

Let's now denote $\hat{A}_n = [a, a_n)$. With the same procedure as in Lemma A we can now prove that[1]:

$$\bigcap_{i=1}^{\infty} \hat{A}_i = \{a\}$$

Because $\hat{A}_n$ is monotically decreasing we can say (using Lemma B):

$$\lim_{n\to\infty} \hat{A}_n = \bigcap_{i=1}^{\infty} \hat{A}_i = \{a\} \in \mathcal{B}([a, b])$$

Therefore proving that $\{a\}$ is Lebesque measurable. However the exact value of $\lambda(\{a\})$ is unclear. $\lambda(\{a\})$ could map to every $y \in \mathbb{R}_+$ and λ would still be a Lebesgue-measure[2]. From this we get the properties:

$$\begin{aligned}
&\lambda([a, c]) = y + c - a \quad \forall c \in [a, b] \\
&\lambda([a, b]) = y + b - a \neq \lambda((a, b]) \\
&\lambda([d, e]) = d - e \quad \forall d \leq e \in (a, b] \\
&\lambda[\{x | x \in [a, b]\}] = \{0, y\}
\end{aligned}$$

For $y = 0$ This implies $\lambda((a, b)) = \lambda((a, b]) = \lambda([a, b])$.

proof 3 s.t. $\mathbb{Q} \cap [a, b]$ is Lebesgue measurable.

Let a sequence $(b_n)_{n\in\mathbb{N}} \in \mathbb{Q} \cap [a, b]$ with $b_i \neq b_j \;\; \forall i \neq j$. Because $\mathbb{Q} \cap [a, b]$ is countable it holds that[3]:

$$\biguplus_{i=1}^{\infty} \{b_i\} = \mathbb{Q} \cap [a, b]$$

[1]looking at the properties of $x \in \cap_{i=1}^{\infty} A_i \wedge x \neq a$ will lead to $\exists n_0 \in \mathbb{N} : x > a_n \;\; \forall n \geq n_0$ and will yield the same contradiciton

[2](i) and (ii) are trivial. Also I was unable to find a sequence where σ-addivity falls apart (iff λ is only defined for leftopen intervals).

[3]If this wasn't true, $\mathbb{Q}$ would be uncountably infinite.

Furthermore, we can denote $(B_n)_{n\in\mathbb{N}} = \{b_n\}$. Using proof 1 & 2 and property (iii) of σ-fields, we can argue:

$$B_n \in \mathcal{B}([a,b]) \ \forall n \underset{(iii)}{\implies} \biguplus_{i=1}^{\infty} B_n \in \mathcal{B}([a,b])$$

And because λ is a measure on $\mathcal{B}([a,b])$, we can say, that $\mathbb{Q} \cap [a,b]$ is Lebesque measurable.

5. Exercise 8

a) We are going to prove the three properties of a field:

(i) s.t. $\emptyset \in \mathcal{A}$

$$|\emptyset| = 0 < \infty \implies \emptyset \in \mathcal{A}$$

(ii) s.t. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$

Let $A \subset \mathbb{N}$ be a set with $A \in \mathcal{A}$. Then two cases can be possible:
Case 1 $|A| < \infty$: Per defintion of $\mathcal{A}$ it follows that:

$$|A| < \infty \Leftrightarrow |(A^c)^c| < \infty \implies A^c \in \mathcal{A}$$

Case 2 $|A^c| < \infty$: This implies directly per definition:

$$|A^c| < \infty \implies A^c \in \mathcal{A}$$

(iii) s.t. $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

Case 1 $|A| < \infty \wedge |B| < \infty$: In that case we can say:

$$|A \cup B| \leq |A| + |B| < \infty$$

So $|A \cup B| < \infty$ and therefore $A \cup B \in \mathcal{A}$.
Case 2 (symmetric case) $|A| < \infty \wedge |B^c| < \infty$:

$$|A \cup B| \underset{\text{de morgan}}{=} |A^c \cap B^c| \leq |B^c| < \infty$$

So $|A \cup B| < \infty$ and therefore $A \cup B \in \mathcal{A}$[4].
Case 3 $|B^c| < \infty \wedge |A^c| < \infty$:

$$|A \cup B| \underset{\text{de Morgan}}{=} |A^c \cap B^c| \leq |B^c| < \infty$$

So $|A \cup B| < \infty$ and therefore $A \cup B \in \mathcal{A}$

[4] If $|A^c| < \infty \wedge |B| < \infty$ you could just denote $\hat{A} := B$ and $\hat{B} := A$. This statement then holds for $\hat{A} \cup \hat{B} = A \cup B$.

b) Let $(A_n)_{n\in\mathbb{N}} := \{2n\} \subset \mathbb{N}$ be a setsequence of all even integers. Then since $|\{A_n\}| = 1 < \infty\ \forall n$ it holds that $A_n \in \mathcal{A}$. So in order for $\mathcal{A}$ to be a σ-algebra, it must hold that:

$$\biguplus_{i=1}^{\infty} A_i \in \mathcal{A}$$

But the cardinality of all even numbers is ∞ and the cardinality of all odd numbers is ∞:

$$\left|\biguplus_{i=1}^{\infty} A_i\right| = \left|\left(\biguplus_{i=1}^{\infty} A_i\right)^c\right| = \infty$$

So:

$$\left|\biguplus_{i=1}^{\infty} A_i\right| \notin \mathcal{A}$$

Therfore $\mathcal{A}$ is not a σ-algebra.

c) We are going to generate a σ-algebra from a subset $\mathcal{B}$ of $\mathcal{A}$ and prove that: $\sigma(\mathcal{B}) = \mathcal{P}(\mathbb{N})$. Then this implies, that $\sigma(\mathcal{A}) = \mathcal{P}(\mathbb{N})$. Let $\mathcal{B} := \{\{x\}|x \in \mathbb{N}\}$ a familiy of all single element sets in $\mathbb{N}$. It follows that $\mathcal{B} \subset \mathcal{A}$. So it also must follow that: $\sigma(\mathcal{B}) \subset \sigma(\mathcal{A})$. Since $\mathbb{N}$ is countable, we can define $(a_n)_{n\in\mathbb{N}} := n$ a sequence of integers and state:

$$\biguplus_{i=1}^{\infty}\{a_i\} = \biguplus_{i=1}^{\infty}\{i\} = \mathbb{N}$$

Now let set $A \subset \mathbb{N}$ with $A \neq \emptyset$, $b \in A$ and function $f_A : \mathbb{N} \to \mathbb{N}$ such that:

$$f_A(x) := \begin{cases} x & \text{if} \quad x \in A \\ b & \text{else} \end{cases}$$

Furthermore let subsequence of a_n be a_{n_k} with $n_k = f_A(n) : (a_n)$ it holds that:

$$\biguplus_{i=1}^{\infty}\{a_{n_i}\} = \{f_A(1), f_A(2), ...\} = A$$

But because $\{a_{n_k}\} \in \mathcal{B}$ it must also hold, that $\{a_{n_k}\} \in \sigma(\mathcal{B})$. Property (iii) then implies:

$$\biguplus_{i=1}^{\infty}\{a_{n_i}\} \in \sigma(\mathcal{B}) \Leftrightarrow A \in \sigma(\mathcal{B})$$

Since A is any nonempty subset of $\mathbb{N}$. We can say that:

$$\mathcal{P}(\mathbb{N}) = \{A|A \subset \mathbb{N} \wedge A \neq \emptyset\} \cup \emptyset \subset \sigma(\mathcal{B}) \subset \mathcal{P}(\mathbb{N}) \Leftrightarrow \sigma(\mathcal{B}) = \mathcal{P}(\mathbb{N})$$

But since $\mathcal{B} \subset \mathcal{A}$ we could argue:

$$\mathcal{P}(\mathbb{N}) = \sigma(\mathcal{B}) \subset \sigma(\mathcal{A}) \subset \mathcal{P}(\mathbb{N})$$

And this gives us: $\sigma(\mathcal{A}) = \mathcal{P}(\mathbb{N})$

d) Since we have shown σ-additivity for the counting measure in the last sheet, we can say, that the counting measure is also a pre-measure.