Sheet 5

Exercise 16

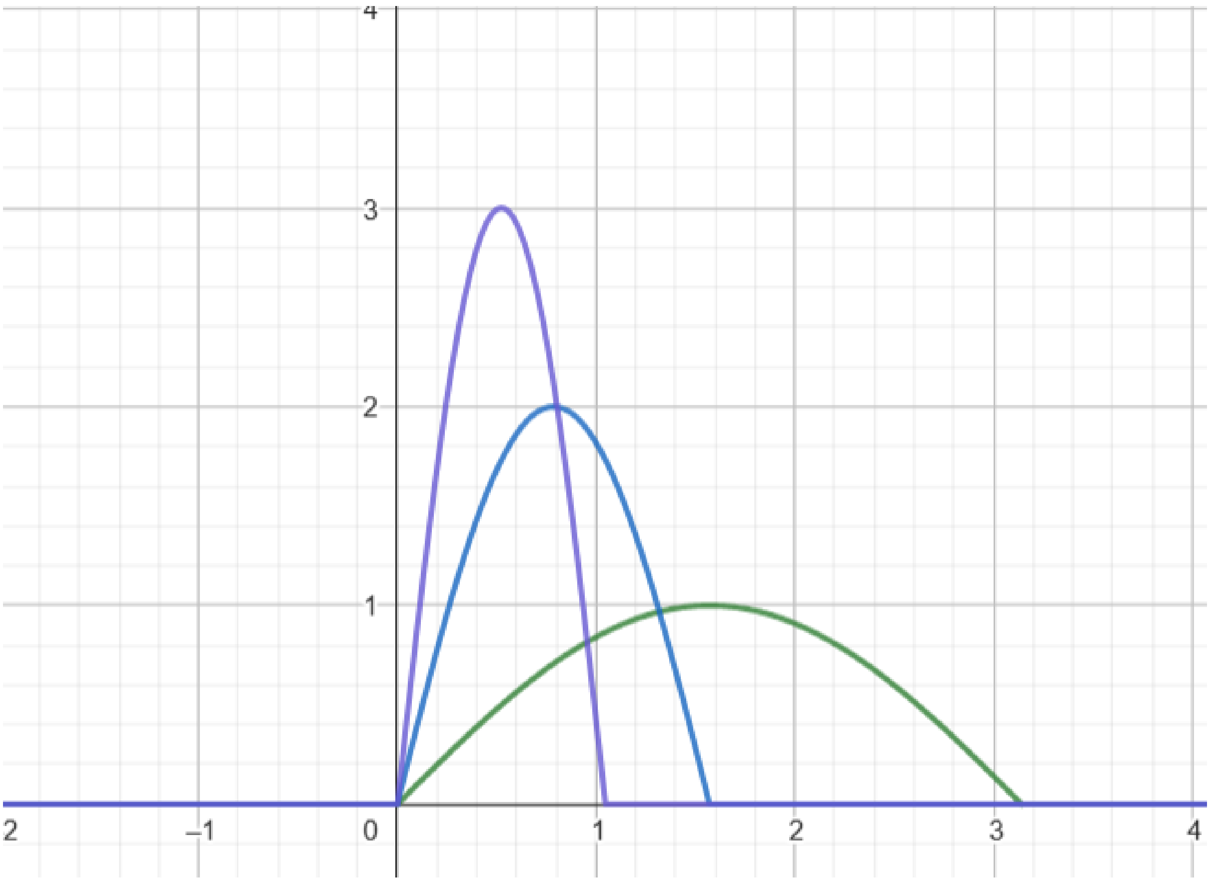
a)

In order to show that f_n is integrabale we need to show two things:

1. **f_n is measurable**
2. $\int |f_n| d\lambda < \infty$

Before starting the proof, a few properties of the sequence need to be established.

In essence the sequence f_n is compressing the first mode of the sinus function with every iteration. The following plot shows $f_1 \text{ (green)}, f_2 \text{ (blue)}, f_3 \text{ (purple)}$:



In the first mode, the sinus function reaches a maximum value of 1 at $\pi/2$. Therefore we can say:

$$\max_{x\in[0,\pi]} f_n(x) = \max_{x\in[0,\pi/n]} n\sin(nx) = \max_{x\in[0,\pi]} n\sin(x) = n \max_{x\in[0,\pi]} \sin(x) = n(1)$$

Furthermore $\sin$ is continuous and $\sin(0) = \sin(\pi) = 0$. Therefore f_n **is continuous** on its whole domain. Also f_n is a positive mapping. We can write:

$$f_n : [0,\pi] \to \mathbb{R} \Leftrightarrow f_n : [0,\pi] \to \mathbb{R}_+ \underset{(1)}{\Leftrightarrow} f_n : [0,\pi] \to [0,n]$$

s.t. f_n **is** $(\mathcal{B}([0,\pi]) - \mathcal{B}([0,n]))$**measurable**

We have already showed, that f_n is continuous, therefore f_n is measurable (Lecture slide 50).

s.t. $\int |f_n| d\lambda < \infty$

Using the same strategy from the last sheet, we will assume, that the Riemann and Lebesgue integral are equal. F

$$\int_{[0,\pi]} f_n d\lambda = \int_0^\pi n\sin(nx)1_{[0,\pi/n]}(x)dx = \int_0^{\pi/n} n\sin(nx)dx$$

Now we can substitute $t = nx$:

$$\begin{aligned}\int_0^{\pi/n} n\sin(nx)dx &= \int_0^\pi \sin(t)dt \\ &= (-\cos(t)|_0^\pi) \\ &= (-\cos(\pi) - (-\cos(0))) \\ &= (-(-1) - (-1)) = 2 < \infty\end{aligned}$$

And because we have established f_n as a positive mapping it we can argue:

$$\int |f_n| d\lambda = \int f_n d\lambda < \infty$$

b)

We will prove a more concrete statement: $f_n \to 0$.

Let any $x \in (0,\pi]$. Then we could argue:

$$\exists \hat{n} \in \mathbb{N} : x > \frac{\pi}{n} \quad \forall n \geq \hat{n}$$

But this implies that:

$$f_n(x) = 0 \quad \forall n \geq \hat{n}$$

So we can make the statement:

$$\forall \varepsilon > 0 \exists \hat{n} \in \mathbb{N} : |f_n(x) - 0| = 0 < \varepsilon \;\; \forall n \geq \hat{n}$$

Furthermore it holds for all $n \in \mathbb{N}$:

$$f_n(0) = n \sin(0) = 0$$

And therefore:

$$f_n(0) \underset{n\to\infty}{\longrightarrow} 0$$

Therefore showing that:

$$f_n(x) \underset{n\to\infty}{\longrightarrow} 0 \quad \forall x \in [0, \pi]$$

c)

The Lemma of Fatou only requires positive mapping, which is given at every element of the sequence f_n. Therefore we can say that:

$$\int_0^\pi \varliminf_{n\to\infty} f_n dx \leq \varliminf_{n\to\infty} \int_0^\pi f_n dx$$

As we will see in **d)** it does not hold, that $\mathbb{E}(\lim_{n\to\infty} f_n) \neq \lim_{n\to\infty} \mathbb{E}(f_n)$. So dominated covergence should not hold. We could define:

$$Y(x) = \sup\{f_n(x) | n \in \mathbb{N}\}$$

And this should lead to:

$$\int_{[0,\pi]} Y d\lambda = \infty$$

How exactly this proof works, has been unclear to me.

d)

Let’s first compute the LHS:

$$\begin{aligned}\lim_{n\to\infty}\int_0^\pi f_n d\mu &= \lim_{n\to\infty}\int_0^\pi f_n dx \\ &\underset{a)}{=} \lim_{n\to\infty} 2 = 2\end{aligned}$$

We have also established in **b)** that $f_n \to 0$. So

$$\int_0^\pi \lim_{n\to\infty} f_n d\mu = \int_0^\pi 0 d\mu = 0$$

So:

$$\lim_{n\to\infty}\int_{[0,\pi]} f_n d\mu \neq \int_{[0,\pi]} \lim_{n\to\infty} f_n d\mu$$

Exercise 17

a)

Before calculating the measure integral we are going to prove: $X_n \to 0$.

Let $x \in \mathbb{R}$. Then it holds that for any $\varepsilon > 0$:

$$\exists \hat{n} \in \mathbb{N} : f_n(x) = \frac{1}{n} = \left|\frac{1}{n} - 0\right| < \varepsilon \quad \forall n \geq \hat{n}$$

So $f_n(x) \to 0 \;\; \forall x \in \mathbb{R}$.

Therefore the measure integral on the LHS is:

$$\begin{aligned}\mathbb{E}(\lim_{n\to\infty} X_n) &= \int_{\mathbb{R}} \lim_{n\to\infty} X_n d\lambda \\ &= \int_{\mathbb{R}} 0 d\lambda = 0\end{aligned}$$

But the integral on the RHS is:

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{E}(X_n) &= \lim_{n\to\infty} \int_{\mathbb{R}} X_n d\lambda \\
&= \lim_{n\to\infty} \frac{1}{2n} \lambda([-n,n] \cap \mathbb{R}) \\
&= \lim_{n\to\infty} \frac{1}{2n} \lambda([-n,n]) \\
&= \lim_{n\to\infty} \frac{1}{2n} n - (-n) = \lim_{n\to\infty} \frac{2n}{2n} = 1
\end{aligned}
$$

b)

Because $\mathbb{E}(\lim_{n\to\infty} X_n) \neq \lim_{n\to\infty} \mathbb{E}(X_n)$ it is save to say that neither the monotone convergence or the dominated convergence theorem apply.

But for sake of completeness:

$X_n \leq X_{n+1}$ i.e. $X_n(x) \leq X_{n+1}(x) \;\; \forall x \in \mathbb{R}$ must hold for monotone convergence theorem must hold. But if we look at $x = 0$ and X_1 as well as X_2 then:

$$
X_1(0) = \frac{1}{2} > \frac{1}{4} = X_2(0)
$$

This acts as a counterexample where monotonicity does not hold.

c)

We can define Y in a way, where $Y(x) \geq X_n(x) \;\; \forall x \in \mathbb{R}, n \in \mathbb{N}$:

$$
Y(x) = \sup\{X_n(x) | n \in \mathbb{N}\} \Leftrightarrow Y(x) = \sum_{n=1}^{\infty} \frac{1}{2n} 1_{[n-1,n] \cup [-n,-(n-1)]}(x)
$$

Furthermore let:

$$
s_n = \sum_{i=1}^{n} \frac{1}{2i} 1_{[i-1,i] \cup [-i,-(i-1)]}(x)
$$

Then s_n is always simple and $s_n \to Y$ and $s_n \leq s_{n+1}$ (so monotone convergence applies):

$$\begin{aligned}
\int_{\mathbb{R}} Y d\lambda &= \int_{\mathbb{R}} \lim_{n\to\infty} s_n d\lambda \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} s_n d\lambda \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2i} \lambda(([i-1,i] \cup [-i,-(i-1)]) \cap \mathbb{R}) \\
&\underset{f.a.}{=} \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2i} \lambda([i-1,i]) + \lambda([-i,-(i-1)]) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2i} (i-(i-1)) + (-(i-1)+i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2i} * 2 = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} = \infty
\end{aligned}$$

Exercise 18

a)

We have to show that:

$$X_n \xrightarrow{P} X \not\Rightarrow \mathbb{E}(||X_n - X||^2) \to 0$$

The **counter example** for this proof was inspired by **exercise 16 but strongly simplified.** We will define:

$$f_n := \begin{cases} \sqrt{n} \text{ if } 0 \le x \le \frac{1}{n} \\ 0 \text{ if } \frac{1}{n} < x \le 1 \end{cases}$$

Then $f_n : [0,1] \to \mathbb{R}_+$ holds and we can consider the measuring space $([0,1], \mathcal{B}([0,1]), \lambda)$ for the domain. And because $\lambda([0,1]) = 1$ holds, this measuring space is also a probability space.

Now let's quickly prove measurablity:

Let $[a,b] \subset \mathbb{R}_+$ with $a,b \in \mathbb{R}_+$ then:

$$f_n^{-1}([a,b]) = \begin{cases} [0,1/n] \text{ if } \{0,\sqrt{n}\} \cap [a,b] = \{\sqrt{n}\} \\ [1/n,1] \text{ if } \{0,\sqrt{n}\} \cap [a,b] = \{0\} \\ [0,1] \text{ if } \{0,\sqrt{n}\} \cap [a,b] = \{0,\sqrt{n}\} \\ \emptyset \text{ else} \end{cases}$$

So $f_n^{-1}([a,b]) \in \mathcal{B}([0,1])$ holds for every $[a,b] \subset \mathbb{R}_+$. And therefore f_n is measurable and a real r.v..

Now we can argue $f_n \xrightarrow{P} 0$ because for any $\varepsilon > 0$ it holds that:

$$P(|f_n - 0| > \varepsilon) = P\left(\left[0, \frac{1}{n}\right]\right) = \lambda\left(\left[0, \frac{1}{n}\right]\right) = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

But trying to prove convergence in quadratic mean leads to:

$$\begin{aligned} \mathbb{E}(|f_n - 0|^2) = \mathbb{E}(|f_n|^2) = \int_0^1 f_n^2 d\lambda &= \int_0^1 f_n(x)^2 dx \\ &= \int_0^{\frac{1}{n}} n dx = \frac{n}{n} = 1 \end{aligned}$$

And so:

$$\lim_{n\to\infty} \mathbb{E}(|f_n - 0|^2) = \lim_{n\to\infty} 1 = 1 \neq 0$$

b)

According to the lecture we can say:

$$X_n \xrightarrow{P} b \implies X_n \xrightarrow{d} b$$

So let $f(x)$ be a continuous and bounded function. Specifically let this function be:

$$f(x) = \begin{cases} x - b \text{ if } x \in [b-1, b+1] \\ b - 1 \text{ if } x < b - 1 \\ b + 1 \text{ if } x > b + 1 \end{cases}$$

Then convergence in distribution implies that:

$$\mathbb{E}(f(X_n)) \xrightarrow[n\to\infty]{} \mathbb{E}(f(b)) = \mathbb{E}(b - b) = 0$$

So:

$$\mathbb{E}(X_n - b) \underset{n\to\infty}{\longrightarrow} 0 \Leftrightarrow \mathbb{E}(X_n) - b \underset{n\to\infty}{\longrightarrow} 0$$

But since b is a constant we can say:

$$\mathbb{E}(X_n) \underset{n\to\infty}{\longrightarrow} b$$

Exercise 19

a)

We start by saying for any $\varepsilon > 0$ it holds that:

$$P(|X_n - 1| > \varepsilon) = P(X_n \notin [1-\varepsilon, 1+\varepsilon]) = P(X_n < 1-\varepsilon)$$

We can continue this equation with:

$$\begin{aligned} P(X_n < 1-\varepsilon) &= P(\max_{0\le i\le n} X_i < 1-\varepsilon) \\ &\underset{iid}{=} P(X_1 < 1-\varepsilon)\cdot ... \cdot P(X_n < 1-\varepsilon) \\ &= \lambda([0, 1-\varepsilon))\cdot ... \cdot \lambda([0,1-\varepsilon)) \\ &= (1-\varepsilon)^n \end{aligned}$$

Now we can look at behavior against infinity

$$\lim_{n\to\infty} P(|X_n - 1| > \varepsilon) = \lim_{n\to\infty}(1-\varepsilon)^n = 0$$

So it is proven that:

$$X_n \xrightarrow{P} 1$$

b)

If $X_n \sim Exp(\lambda_n)$ then $P(X_n \le x) = 1 - e^{\lambda_n x}$. Also:

$$\begin{aligned} P(|X_n - 0| > \varepsilon) &= P(X_n > \varepsilon) \\ &= 1 - P(X_n \le \varepsilon) \\ &= 1 - 1 + e^{-\lambda_n x} = e^{-\lambda_n x} \end{aligned}$$

And so:

$$\lim_{n\to\infty} P(|X_n - 0| > \varepsilon) = \lim_{n\to\infty} e^{-\lambda_n x} \underset{\lambda_n\to\infty}{=} 0$$

c)

$$\begin{aligned}\lim_{n\to\infty} P(|X_n - 0| > \varepsilon) &= \lim_{n\to\infty} P(X_n > \varepsilon) \\ &\underset{\varepsilon>0}{=} \lim_{n\to\infty} P(X_n = n) \\ &= \lim_{n\to\infty} P(U \geq n) \\ &= \lim_{n\to\infty} 1 - P(U < n) \\ &= \lim_{n\to\infty} 1 - 1 + e^{-n} \\ &= \lim_{n\to\infty} e^{-n} = 0\end{aligned}$$

d)

$$P(|X_n - X| > \varepsilon) = P(|Y_n + X - X| > \varepsilon) = P(|Y_n| > \varepsilon)$$

In this case we can use **Chebychevs Inequality:**

$$\begin{aligned}\lim_{n\to\infty} P(|Y_n| > \varepsilon) &\leq \lim_{n\to\infty} P(|Y_n| \geq \varepsilon) \\ &\underset{\mathbb{E}(Y_n)\to 0}{=} \lim_{n\to\infty} P(|Y_n - \mathbb{E}(Y_n)| \geq \varepsilon) \\ &\underset{c.i.}{\leq} \lim_{n\to\infty} \frac{Var(Y_n)}{\varepsilon^2} \\ &= \lim_{n\to\infty} \frac{\sigma^2}{\varepsilon^2 n} = 0\end{aligned}$$

And since P is a positive mapping, we can state:

$$P(|Y_n| > \varepsilon) \underset{n\to\infty}{\longrightarrow} 0$$