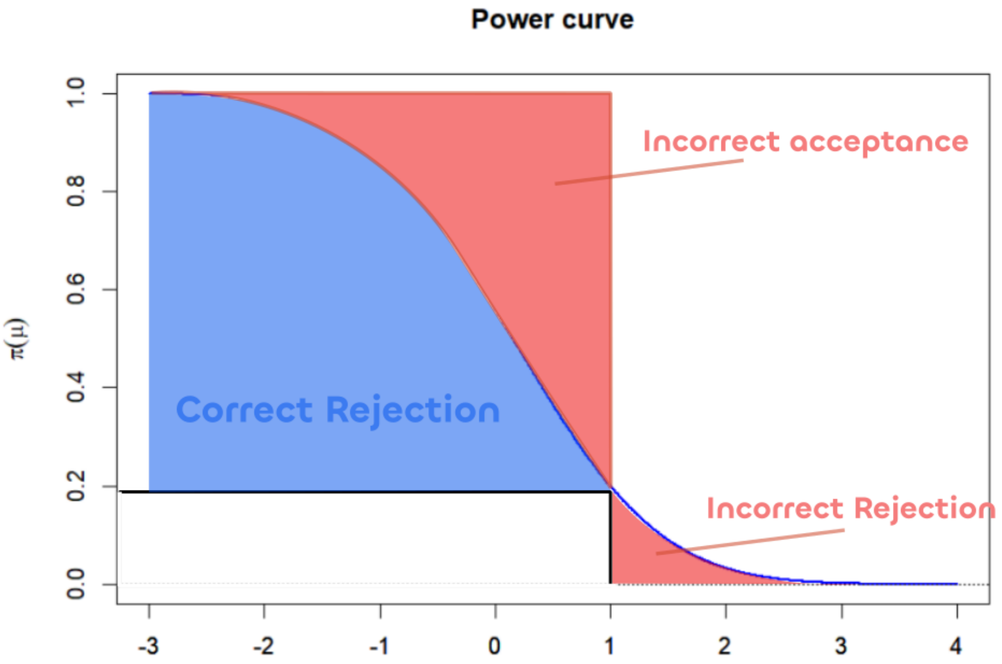
Sheet 2

Exercise 1

1.)



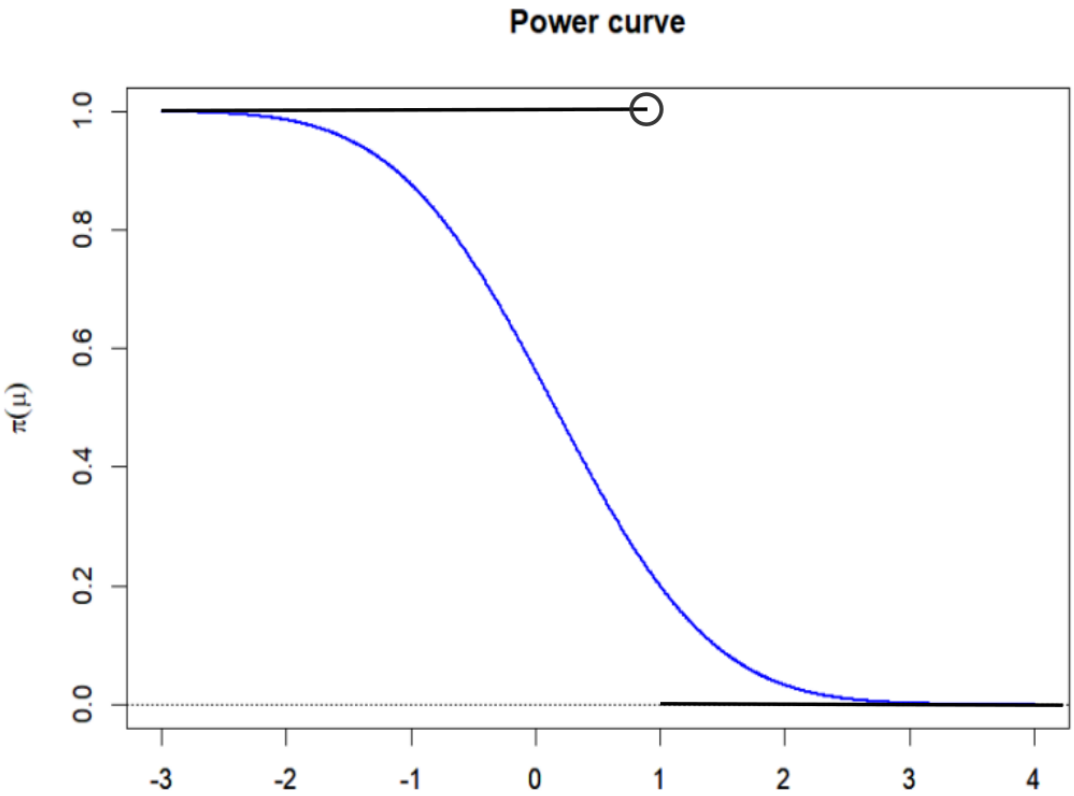
By definition the power-function $\pi(\mu)$ is the probability that H_0 gets rejected given the true parameter μ. To be exact:

$$\pi(\mu) = P(T \in C_r | \mu)$$

Where T is the test-statistic. The size α is defined as the supremum the power-function under the condition that H_1 is true. In the graph we can see α as the black line.

The type 2 error is therefore the probability that H_1 is incorrectly rejected. The space between 1 and the slope of $\pi(\mu)$.

2.)



Under optimal power, the test would always reject and accept correctly. When $\mu \geq 1$ then $P(T \in C_r|\mu) = 0$. And when $\mu < 1$ so H_1 is true, then $P(T \in C_r|\mu) = 1$. So the test always rejects given H_1 is true and never given H_0 is true.

We can attain such a Power-Function, if the test is consistent, the Samplesize goes to infinity a and we choose and arbitrarily small $\alpha > 0$. (Lecture slide 25). But in practice a sample size of infinity is not possible.

Exercise 2

1.)

We need to find:

$$
\begin{aligned}
\pi(\mu) = P(T \in C_r|\mu) &= P(T \notin [-1.282, 1.282]|\mu) \\
&= P\left(-1.282 \leq \frac{\frac{1}{n}\sum_{i=1}^n x_i - 1}{\sqrt{\frac{50}{n}}} \leq 1.282\right) \\
&= P\left(\frac{\sum_{i=1}^n x_i - 1}{\sqrt{n}\sigma} \notin [-1.282, 1.282]\right) \\
&= \left(\frac{\sum_{i=1}^n x_i - n\mu}{\sqrt{n}\sigma} \notin \left[-1.282 + \frac{n - n\mu}{\sqrt{n}\sigma}, 1.282 + \frac{n - n\mu}{\sqrt{n}\sigma}\right]\right) \\
&= P\left(\tilde{T} \notin \left[-1.282 + \frac{n - n\mu}{\sqrt{n}\sigma}, 1.282 + \frac{n - n\mu}{\sqrt{n}\sigma}\right]\right) \\
&= 1 - P\left(\tilde{T} \in \left[-1.282 + \frac{n(1-\mu)}{\sqrt{n}\sigma}, 1.282 + \frac{n(1-\mu)}{\sqrt{n}\sigma}\right]\right)
\end{aligned}
$$

Central limit theorem now states that $\tilde{T} \overset{approx}{\sim} \mathcal{N}(0,1)$.

So we get a somewhat concrete way of describing the power function of this test:

$$
\pi(\mu) \approx 1 - \Phi\left(1.282 + \frac{n(1-\mu)}{\sqrt{n}\sigma}\right) + \Phi\left(-1.282 + \frac{n(1-\mu)}{\sqrt{n}\sigma}\right)
$$

Where $\sigma = \sqrt{50}$.

The size is $\mu = 1$.

$$
\alpha = \pi(1) \approx 1 - \Phi(1.282) + \Phi(-1.282)
$$

2.)

To Show that the test is unbiased, we only need to look at the following term:

$$
f(c) = \Phi(1.282 + c) - \Phi(-1.282 + c)
$$

Now it holds that:

$$
\text{The test is unbiased} \Leftrightarrow \max_c f(c) = 0
$$

This equivalent because, $\mu = 1$ is the only time the term $\frac{n(1-\mu)}{\sqrt{n}\sigma}$ is zero and:

$$
\pi(\mu) = 1 - \Phi\left(1.282 + \frac{n(1-\mu)}{\sqrt{n}\sigma}\right) + \Phi\left(-1.282 + \frac{n(1-\mu)}{\sqrt{n}\sigma}\right) = 1 - f(c)
$$

Note that:

$$f(c) = \int_{-1,282+c}^{1.282+c} \phi(x)dx$$

So $f(c)$ is the integral of the pdf of the standard normal distribution on the interval $[-1.282+c, 1.282+c]$. Also note, that the length of the interval does not vary depending on c:

$$1.282 + c - (-1.282 + c) = 2 \cdot 1.282$$

We therefore are looking at the question: “How can we place an interval of the same length such that the functional values of $\phi(x)$ get maximal for each x in the interval”.

In order to prove that $c = 0$ is the solution to this problem, we have to look at the properties of the probability density function of the standard normal distribution ϕ. The following properties hold true:

$$\forall x \in \mathbb{R} : \phi(0) \geq \phi(x) \tag{1}$$
$$\forall x \in \mathbb{R} \setminus \{0\} : \phi(x) = \phi(-x) \tag{2}$$
$$\forall x_1, x_2 \in (0, \infty) \text{ with } x_1 \leq x_2 : \phi(x_1) \geq \phi(x_2) \tag{3}$$

So ϕ is uni-modal and symmetrical around the y-axsis. Because of this let (w.l.g.) $c > 0$. This means that $\exists x \in [-1.282+c, 1.282+c], x > 1.282$. But because of (3) it holds that:

$$\forall x' \in [0, 1.282] : \phi(x') \geq \phi(x)$$

(2) also implies that:

$$\forall x' \in [-1.282, 0] : \phi(x') \geq \phi(x)$$

But this means that:

$$f(c) = \int_{-1.282+c}^{1.282+c} \phi(x)dx \leq \int_{-1.282}^{1.282} \phi(x)dx = f(0)$$

proving that $c = 0$ is the solution for $\max_c f(c)$ and therefore proving, that the test is unbiased.

3.)

We want to show that $\mu \neq 1$ implies that:

$$\lim_{n \to \infty} \pi(\mu) = 1$$

We already discussed, that if $\mu \neq 1 \implies c := \frac{n(1-\mu)}{\sqrt{n}\sigma} \neq 0$.

Case 1: $c > 0$:

In this case c is increasing in n. So:

$$\begin{aligned}\lim_{n\to\infty} \pi(\mu) &= \lim_{n\to\infty} 1 - \Phi\left(1.282 + \frac{n(1-\mu)}{\sqrt{n}\sigma}\right) + \Phi\left(-1.282 + \frac{n(1-\mu)}{\sqrt{n}\sigma}\right) \\ &= \lim_{x\to\infty} 1 - \Phi(x) + \Phi(x) = 1 - 1 + 1 = 1\end{aligned}$$

Case 2: $c < 0$:

In this case c is decreasing in n. So:

$$\begin{aligned}\lim_{n\to\infty} \pi(\mu) &= \lim_{n\to\infty} 1 - \Phi\left(1.282 + \frac{n(1-\mu)}{\sqrt{n}\sigma}\right) + \Phi\left(-1.282 + \frac{n(1-\mu)}{\sqrt{n}\sigma}\right) \\ &= \lim_{x\to-\infty} 1 - \Phi(x) + \Phi(x) = 1 - 0 + 0 = 1\end{aligned}$$

Proving that the test is consistent.

Exercise 3

1.)

We are looking at the following optimization problem:

$$\min_{\beta_0 \in \mathbb{R}} RSS(\beta_0) = \min_{\beta_0 \in \mathbb{R}} \sum_{i=1}^{n} (x_i - \beta_0)^2$$

For this we will first derive the first order condition:

$$\begin{aligned}\frac{\partial RSS(\hat{\beta}_0)}{\partial \hat{\beta}_0} &\overset{!}{=} 0 \\ -2\sum_{i=1}^{n}(x_i - \hat{\beta}_0) &= 0 \\ \sum_{i=1}^{n} x_i - n\hat{\beta}_0 &= 0 \\ \frac{1}{n}\sum_{i=1}^{n} x_i &= \hat{\beta}_0\end{aligned}$$

2.)

Under this special case the Wald test simplifies to a one-sided gauß-test:

$$T = Z = \frac{\bar{X}_n - 1}{\frac{\sigma}{\sqrt{n}}}$$

It then holds that $Z \overset{H_0}{\sim} \mathcal{N}(0, 1)$. Then for the size of the test it must hold under H_0 that:

$$P_{H_0}(T \in (-\infty, c]) = \alpha \Longleftrightarrow \Phi(c) \overset{!}{=} \alpha$$

3.)

We use the same Test statistic as in Problem 2 but with $\sigma^2 = 50$ and a two-sided hypothesis.

4.)

It can be shown that:

$$Var(\bar{X}_n) = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{\sigma^2}{n}$$
$$Var(\bar{Y}_n) = \frac{1}{n^2} \sum_{i=1}^{n} Var(Y_i) = \frac{\sigma^2}{n}$$

But since $\sigma^2 = n$ we know that $Var(\bar{X}_n) = 1 = Var(\bar{Y}_n)$.

And since Y_i and X_i are pairwise independent, It holds that:

$$\Sigma_{\hat{\theta}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Sigma_{\hat{\theta}}^{-1}$$

We therefore get the Wald test-statistic:

$$W = (\hat{\theta} - \theta_0)'(\hat{\theta} - \theta_0) = (\bar{X}_n - \mu_x, \bar{Y}_n - \mu_y) \begin{pmatrix} \bar{X}_n - \mu_x \\ \bar{Y}_n - \mu_y \end{pmatrix}$$

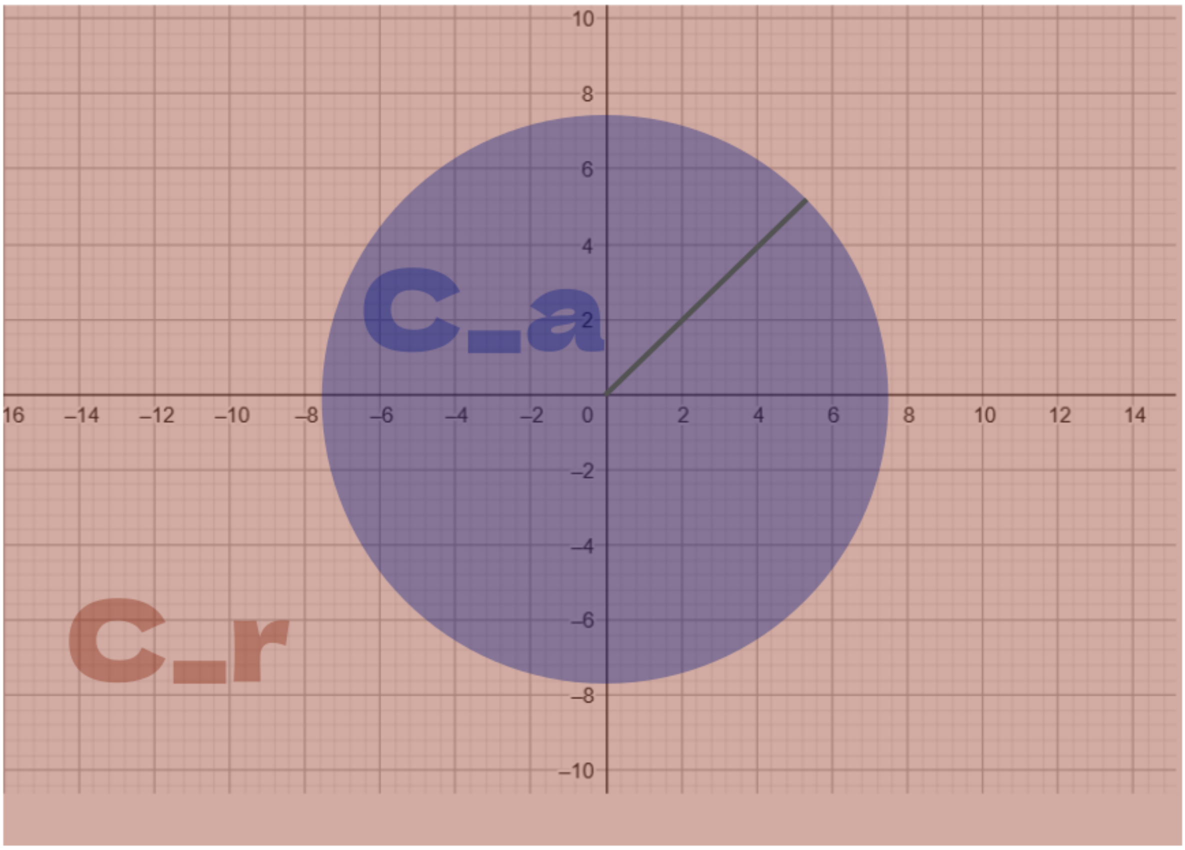
Also $W \overset{H_0}{\sim} \mathcal{X}_2^2$. If we compute the vector multiplication we get:

$$W = (\bar{X}_n - \mu_x)^2 + (\bar{Y}_n - \mu_Y)^2 \geq 0$$

We say **elipsoid,** because:

$$P_{H_0}(W \in C_r) = \alpha \Leftrightarrow W \geq \mathcal{X}^2_{2,(1-\alpha)} \Leftrightarrow \sqrt{(\bar{X}_n - \mu_x)^2 + (\bar{Y}_n - \mu_Y)^2} \geq \sqrt{\mathcal{X}^2_{2,(1-\alpha)}}$$

The last inequality is Pythagoras theorem. So if the realizations of $\bar{X}_n$ and $\bar{Y}_n$ lie outside of a circle with its middle in $(0,0)$ and the radius $\sqrt{\mathcal{X}^2_{2,(1-\alpha)}}$ then H_0 gets rejected.



Here a two graphical representation. The radius is marked as a gray line.