technische universität dortmund

From the Ashes to Consistency:
Phoenix T_3

A Self Normalizing Periodogram Test for Stationary Lattice Processes

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1 Introduction

Jentsch and Pauly 2015 motivate their use of L_2 tests by the inconsistency of raw periodogram tests. Frankly, periodogram tests have been well known for their inconsistent estimate of the spectral density and using them for teststatistics seems to prove very unfavourable at least looking towards asymptotic behaviour.

However the problem induced with L_2 teststatistics, defined in a more general way in Eichler 2008, is the slow convergence twoards the very favourable and easy to use asymptotic distribution, justifying the use of randomization techniques and giving way to the paper Jentsch and Pauly 2015 and my Master's thesis. In this context it was suprising to find the work of Scaccia and Martin 2005 in which they propose a test for axial symmetry on lattice processes using the periodogram. Two observations stood out from their work, particularly regarding their third teststatistic T_3:

1. The speed of convergence to the normal distribution, needing merely a 11×11 lattice to be near enough to the normal distribution to properly hold it's size
2. Outstanding results from powersimulation regarding T_3

Convinced by their results, I attempted to apply T_3 to the equality of spectraldensities testingproblem. This document serves as a quick and consise presentation of the results of my efforts.

2 The Test

Given two stationary random fields $\{X_s|s \in \mathbb{Z}^2\}$ and $\{Y_s|s \in \mathbb{Z}^2\}$ with:

$$Y_s, X_s : (\Omega, \mathcal{A}, P) \to \mathbb{R}$$

We want to test whether:

$$\begin{aligned} &H_0 : f_X(\vec{\omega}) = f_Y(\vec{\omega}) = f(\vec{\omega}) \\ &H_1 : f_X(\vec{\omega}) \neq f_Y(\vec{\omega}) \quad \forall \vec{\omega} \in A|\lambda(A) > 0 \end{aligned}$$

where $\vec{\omega} \in [-\pi, \pi]^2$

2.1 Formalization

This teststatistic leans on the well known fact that:

$$\frac{I(\vec{\omega}_{kl})}{f(\vec{\omega}_{kl})} \xrightarrow{d} \text{Exp}(1) \quad n_x, n_y \to \infty$$

Where as n_x, n_y are the width and height of the 2D-lattice and $\vec{\omega}_{kl}$ is a pair of Fouririer frequencies along these axes.

Scaccia and Martin 2005 now define G_s, which is used to compare $I(\omega_k, \omega_l)$ with $I(\omega_k, -\omega_l)$. In our case, we use it to compare $I_X(\omega_k, \omega_l)$ with $I_Y(\omega_k, \omega_l)$:

$$\begin{aligned} G_s(\vec{\omega}_{kl}) &= \frac{\frac{I_X(\vec{\omega}_{kl}) - I_Y(\vec{\omega}_{kl})}{f(\vec{\omega}_{kl})}}{\frac{I_X(\vec{\omega}_{kl}) + I_Y(\vec{\omega}_{kl})}{f(\vec{\omega}_{kl})}} \\ &= \frac{I_X(\vec{\omega}_{kl}) - I_Y(\vec{\omega}_{kl})}{I_X(\vec{\omega}_{kl}) + I_Y(\vec{\omega}_{kl})} \sim_{H_0} \mathcal{U}(-1, 1) \end{aligned}$$

Leaving out duplicate values (because they perfectly correlate, caused by the symmetry property of the periodogram) and frequencies unequal to 0 and π (because they do not converge to an exponetial distribution). Note how we divide both the numerator and denominator by $f(\vec{\omega})$. Under H_0, G_s becomes self normalizing, meaning the values are scaled by the value of the true spectral density. Asymptotically this is just a transformation of exponentially distributed random variables without any unknown rates. Now aggregating and normalizing should yield at least in the asymptotic sense a Z-statstic:

$$\begin{aligned} PT_3 &:= \sqrt{n_x n_y} \frac{\overline{|G_s|} - \mathbb{E}_{H_0}(|G_s|)}{\sqrt{Var_{H_0}(|G_s|)}} \\ &= \sqrt{n_x n_y}\sqrt{12}(\overline{|G_s|} - 1/2) \xrightarrow[clt]{} \mathcal{N}(0, 1) \quad \text{as } n_x, n_y \to \infty \end{aligned}$$

Where $\overline{|G_s|}$ is the average of all G_s.

2.2 Consistency: Proof Sketch

In the event, that H_1 is true, $|G_s|$ will have an expectation strictly larger than $1/2$:

$$
\begin{aligned}
\mathbb{E}(|G_s|) &= \mathbb{E}\left(\left|\frac{I_X - I_Y}{I_X + I_Y}\right|\right) \\
&= \mathbb{E}\left(\left|\frac{aU - bV}{aU + bV}\right|\right), \quad U, V \overset{\text{i.i.d.}}{\sim} \text{Exp}(1),\ a = f_X,\ b = f_Y \\
&= \int_0^1 |2t-1| \frac{ab}{(tb + (1-t)a)^2}\, dt \\
&= 1 + \frac{2r}{(r-1)^2} \log\left(\frac{4r}{(1+r)^2}\right), \quad r = \frac{a}{b} \\
&= \frac{1}{2} + \delta(r), \quad \delta(r) = \frac{2r}{(r-1)^2} \log\left(\frac{4r}{(1+r)^2}\right) + \frac{1}{2} - 1 > 0 \text{ for } r \neq 1.
\end{aligned}
$$

We can then partition the average $\overline{|G_s|}$ into parts that violate the null hypothesis ($B := \{\vec{\omega}_{kl} \in A\}$) and parts that don't. For set B we choose r to be a constant, sucht that $r = \min_{\vec{\omega}_{kl} \in B} r(\vec{\omega}_{kl})$.:

$$
\overline{|G_s|} = \frac{1}{n - \#B} \sum_{\vec{\omega}_{kl} \notin B} |G_s(\vec{\omega}_{kl})| + \frac{1}{\#B} \sum_{\vec{\omega}_{kl} \in B} |G_s(\vec{\omega}_{kl})|
$$

A being positively measured, will ensure, that $\#B \to \infty$ as $n_x, n_y \to \infty$. And asymptotic independence will ensure, that we can apply LLN, such that:

$$
\begin{aligned}
\frac{1}{n - \#B} \sum_{\vec{\omega}_{kl} \notin B} |G_s(\vec{\omega}_{kl})| &\xrightarrow{p} 1/2 \\
\frac{1}{\#B} \sum_{\vec{\omega}_{kl} \in B} |G_s(\vec{\omega}_{kl})| &\xrightarrow{p} 1/2 + \delta(r)
\end{aligned}
$$

Then following Slutsky:

$$
\overline{|G_s|} \xrightarrow{p} 1/2 + \delta(r)
$$

Implying that:

$$
PT_3 = \mathcal{O}_p(1) + \sqrt{n_x\, n_y}\sqrt{12}\delta(r) \xrightarrow{p} \infty
$$

Proving that:

$$
P(|PT_3| > z_{1-\alpha/2}) \to 1 \quad n_x, n_y \to \infty
$$

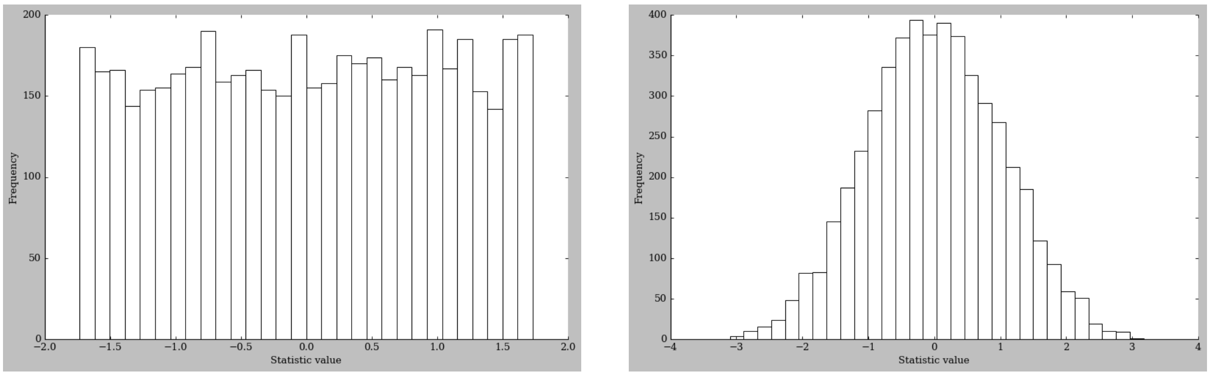
3 Power and Size Simulations

Size Simulations After implementation, the speed of convergence for this teststatistic is quite astonishing. Figure 1 compares two Monte Carlo Simulations done on the new test for a sample size of 3×3 and 5×5 with $N = 5000$ resamples. When moving from a 3×3 lattice to a 5×5 i.e. moving from an effective sample size of 2 to 8 (becaue duplicate values are left out as well as frequency pairs containing 0 and Nyquist), we instantly get a normal looking distribution from the Monte Carlo simulations. Approximated size jumps from 0 in the uniform distribution to $\approx 5.2\%$. Not only is the test holding it's size, it is instantly jumping to it's asymptotic distribution for a single digit effective samplesize.

Furthermore the test is extremely computationally efficient, doing the whole Monte Carlo Simulation without any optimization attempts within 1.5s.

Power Simulations The powersimulations show that only looking at certain parts of the periodogram comes at a price. Figures 2 and 3 compare the estimated powerfunctions between PT_3 and φ_n^*. It becomes very apparent that φ_n^* outperforms PT_3 at every grid size under the proposed alternatives.

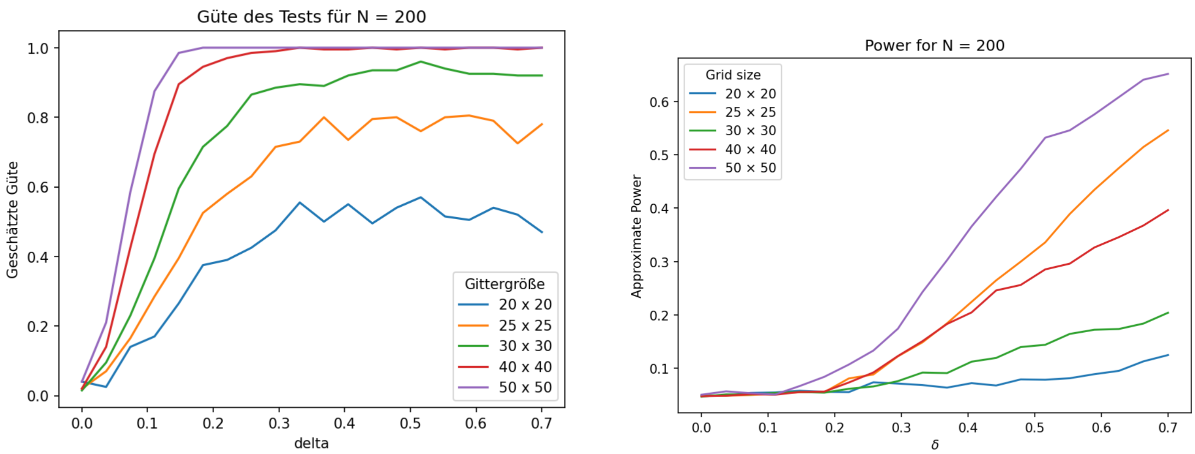
These results may challange the way we look at randomization, not as a device to merely hold the size under H_0 but also increase efficiency under the alternative.



(a) Distribution for 3×3 samples

(b) Distribution for 5×5 samples

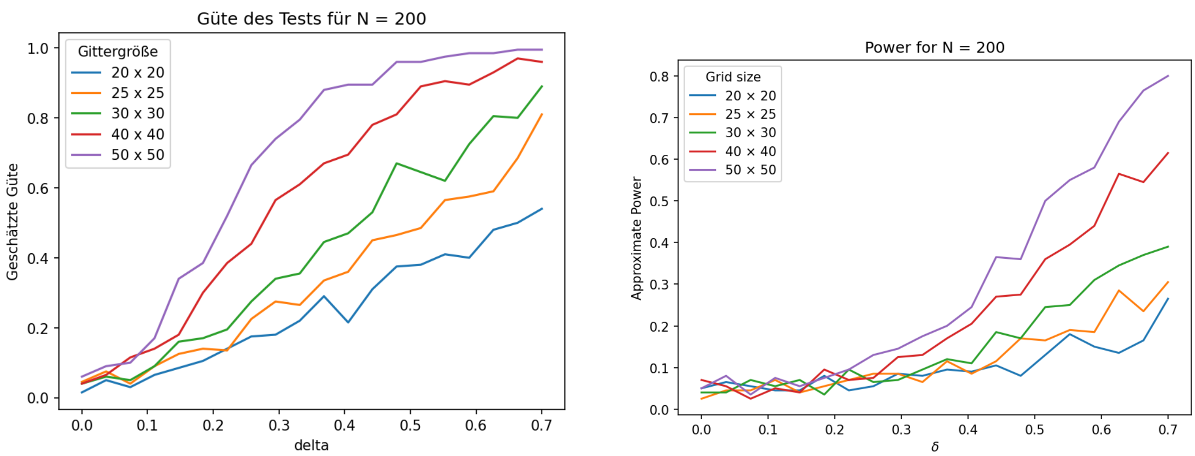
Figure 1: Comparison of test statistic distributions for different sample sizes.



(a) φ_n^* for different grid sizes.

(b) PT_3 for different grid sizes.

Figure 2: Comparison of test statistic power for MA blur alternative.



(a) φ_n^* for different grid sizes.

(b) PT_3 for different grid sizes.

Figure 3: Comparison of test statistic power for $AR(1)$ rotation alternative.

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