Notes From Chapters 12 and 13

Extensive Form Games, Backward Induction, R&D Application

Game Trees

Games trees must satisfy three properties.

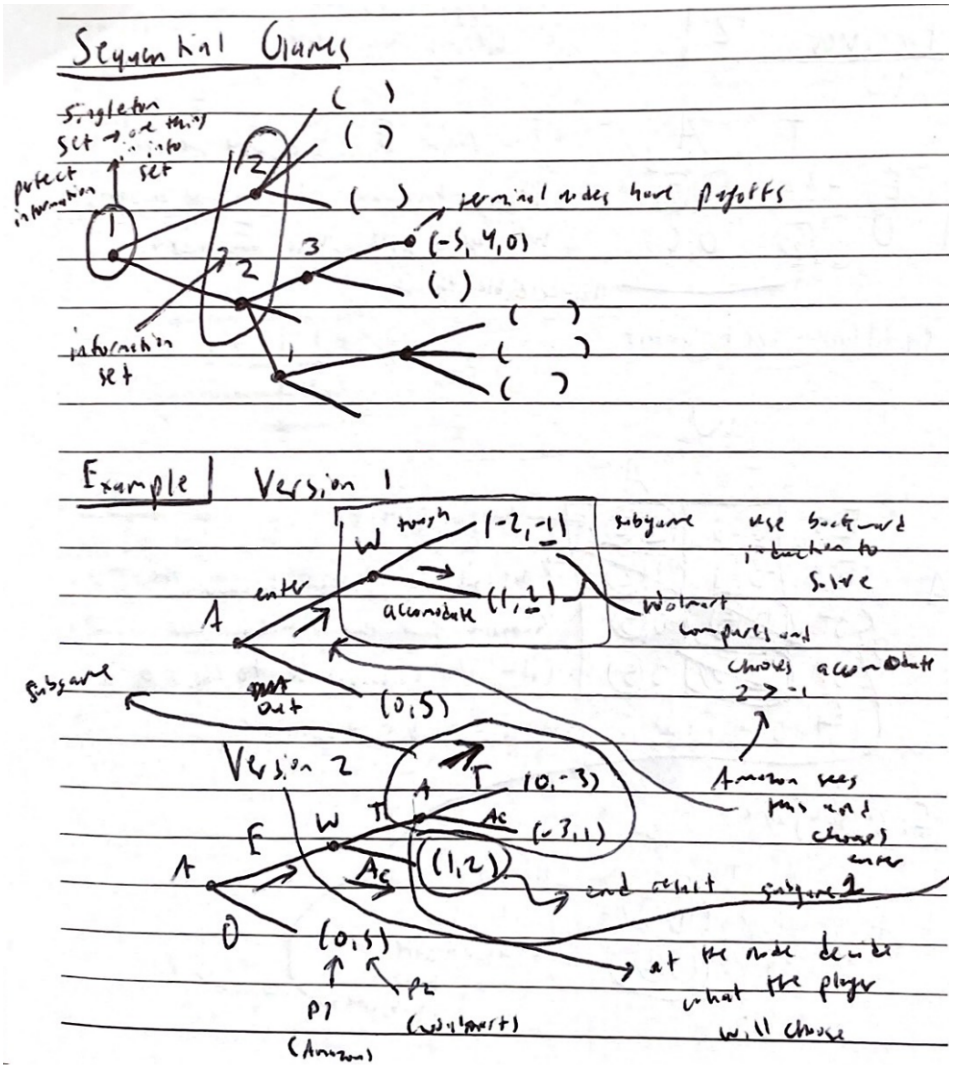
1. Single starting point
2. No cycles
3. One way to proceed

The **predecessor** of a node, say x, are those nodes from which you can go (through a sequence of branches) to x. Four guarantees of a consistent (the three points above) game tree that predecessors must follow.

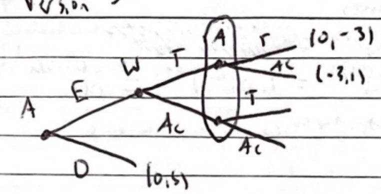
1. A node cannot be a predecessor of itself.
2. A predecessor's predecessor is also a predecessor.
3. Predecessors can be ranked.
4. There must be a common predecessor.

Perfect Information Games and Backward Induction

A **game of perfect information** is an extensive form game with the property that there is exactly one node in every information set.


Sequential Games
singleton set → are thier own info set
perfect information
terminal nodes have payoffs
(-5, 4, 0)
information set
Example 1 Version 1
tough (-2, -1)
accomodate (1, 2)
enter
out (0, 5)
subgame
use backward induction to solve
Walmart compares and chooses accomodate 2 > -1
Amazon sees this and chooses enter
subgame
Version 2
T (0, -3)
Ac (-3, 1)
(1, 2)
end result
subgame 2
(0, 5)
at the node decide what the player will choose
P1 (Amazon)
P2 (walmart)

Version 3



Use backward induction

For version 2!

Subgame perfect (Nash) equilibrium

		W	
		T	A
A	E	-2,-1	1,2
	O	0,5	0,5

Two Nash Equilibria. One we found through backward induction and another we found with Nash Equilibria

Non-credible threat

equilibrium refinement

		W	
		T	A
A	ET	0,-3	1,2
	EA	-3,1	~~1,2~~
O	OT	~~0,5~~	0,5
	OA	~~0,5~~	0,5

Subgame perfect

not subgame perfect because the subgame shows that Amazon's equilibrium (their choices) would be to choose 0,-3.

Staying out but if I had gone in I'd accommodate

First subgame

		W	
		T	A
A	T	0,-3	1,2
	A	-3,1	1,2

Walmart will always accommodate in first subgame so Cross out

Kuhn' s (and Zermelo' s Theorem) : Every game of perfect information with a finite number of nodes has a solution to backward induction.

An Application: Research and Development

13.3 BACKWARD INDUCTION: ANALYSIS OF THE MODEL

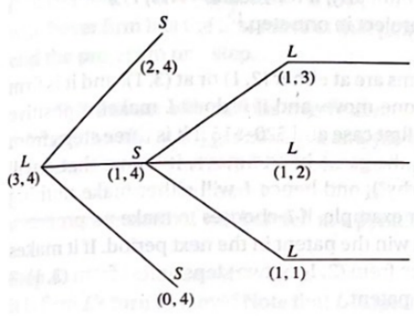
What would be the duopoly outcome (with the two firms competing in their pursuit of the patent)? Somewhat surprisingly, it turns out that we get a sharp answer to that question if we make one last assumption:

5. The two firms take turns deciding how much to spend on R&D; if LG makes an R&D decision this period, it waits to make any further decisions until it learns of Samsung's next R&D commitment. Furthermore, Samsung makes its announcement in the period following LG's announcement.

One way to think about this assumption is that each firm's management makes periodic reviews of the project. These reviews are conducted every few months, and at each such review a decision is taken about R&D spending until the next review. The decision might be to step up spending levels, to hold them at the current level, or to cut back. Firms alternate in their decisions if the review dates are different, although the length of the review period is the same for the two firms.

Assumption 5 turns the patent race into a game of perfect information; let us look at its extensive form. In Figure 13.1, LG has the first R&D decision, and LG and Samsung are, respectively, three and four steps from completing the project.

A somewhat more transparent depiction of this same situation can be given in a *location space* picture. By that we mean a picture in which the "coordinates" of the two firms—that is, how far they are from finishing—are graphed. In this



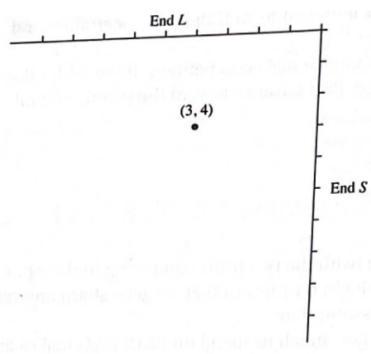


FIGURE 13.2.

location space picture, the northeast point refers to the joint finish line for both firms; that is, successful completion of R&D by both firms. The finish line for S is the vertical terminal line, whereas the finish line for L is the horizontal terminal line; see Figure 13.2.

The following notation will be useful for the location space. If L is l steps from completion while S is s steps away, we will denote their location as (l, s). The game will be solved by backward induction on the location space. In other words, we will show that when either firm is near completion there is a best way for them to make their R&D choice. In turn, these eventual decisions will affect the choices the firms make at more formative stages of R&D.

To illustrate these ideas, we will proceed in a stepwise fashion:

Step 1. Suppose that the game is at $(1, s)$, and it is firm L's turn to move. Its optimal decision evidently is to finish the game in one move. That will yield a patent of value \$20 (billion) and will cost \$2 (billion). Similarly, if the location is $(l, 1)$ and it is firm S's turn to move, S will complete the project in one step.[19]

Step 2. Now suppose that the two firms are at either (2, 1) or at (3, 1), and it is firm L's turn to move. It can complete in one move, and if it does L makes a positive profit at both locations: \$20–\$7 in the first case and \$20–\$15 if it is three steps from finishing. Indeed, if L does not finish the game in one move, it knows that S will do so at the very next opportunity (why?), and hence L will either make nothing from that point on or suffer a loss. For example, if L chooses to make no progress, which is costless, it will find that S will win the patent in the next period. If it makes incomplete progress—one step starting from (2, 1) or two steps or less from (3, 1)—it will incur a cost but will not win the patent.

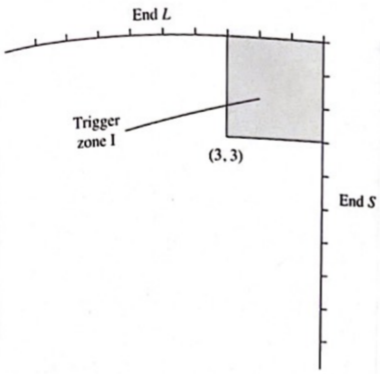


FIGURE 13.3.

Hence, it is best for L to complete in one step if it has a move at $(2, 1)$ or at $(3, 1)$. The same result holds if the firms are at $(1, 2)$ or at $(1, 3)$, and it is firm S's turn to move; S will complete in one step.

In turn this result has the following implication:

Step 3.

(a) Use the previous analysis to show that if the game is at $(2, 2)$, whichever firm has the first move should invest for two steps and finish the game.

(b) Can you then show that if the game is at $(3, 2)$ and it is firm L's turn to move, it should finish in one step. What if the game is at $(2, 3)$ and S has the first move?[20]

In fact what we have shown via steps 1–3 is the following:

Proposition. **(Trigger Zone I).** If the game is at any location (l, s), $l \leq 3$ and $s \leq 3$, whichever firm has the first move at that point will trigger a completion, that is, will end the project in one step.

Call this set of locations *trigger zone I*, as seen in Figure 13.3. Let us use the information about trigger zone I to analyze what happens at other locations. What we are really going to do is fold the location space back; that is, we will do backward induction on it. Because we know what is going to happen at the "end" of the space, we can now ask what will happen at a penultimate zone of that same space.

Step 4. For instance, what can we conclude about a location such as $(4, 3)$ when it is firm L's turn to move? Note that L cannot finish the game in one step. The most

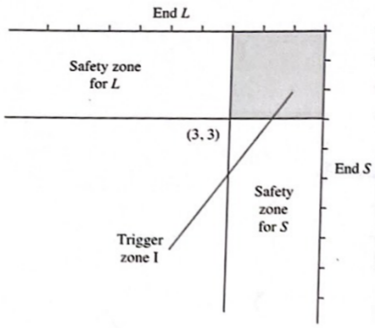


FIGURE 13.4.

that it can do is move its project forward by three steps to (1, 3). Or it can move two steps to (2, 3) or one step to (3, 3). Or it can remain where it is by stopping R&D. In the first three of these cases, *L* knows that *S* will finish the game at the next step. (Why?) So the best response for *L* is to pick the fourth option, make no progress. This is equivalent to dropping out of the race.

If firm *L* finds it in its best interest to drop out of the patent race at (4, 3), what should firm *S* do subsequently? Well, firm *S*, as the sole survivor, will get the patent eventually. Because rapid R&D is more costly than slow R&D, the best approach for *S* is to move in the least costly fashion, one step at a time, toward the patent.

Step 5. Show that the same conclusion, *L* should drop out of the race and *S* should then advance slowly, is true also for locations (4, 2) and (4, 1). What about locations (5, 3), (5, 2), and (5, 1)?

Iterating, we can conclude the following:

Proposition. For all locations (l, s), whenever $l > 3$ and $s \leq 3$, the best thing that firm *L* as a first mover can do is drop out. After this, firm *S* can take a step forward at a time.

This set of locations is therefore called *safety zone I* for *S*; in these locations, *S* can coast and *L* will drop out. Because the game is symmetric, we can also conclude that all locations (l, s), $l \leq 3$ and $s > 3$, are part of safety zone I for firm *L*. The two safety zones are pictured in Figure 13.4.

Let us continue with the backward induction argument. We know that in trigger zone I there will be a preemptive move while in safety zone I, the "war" is over.

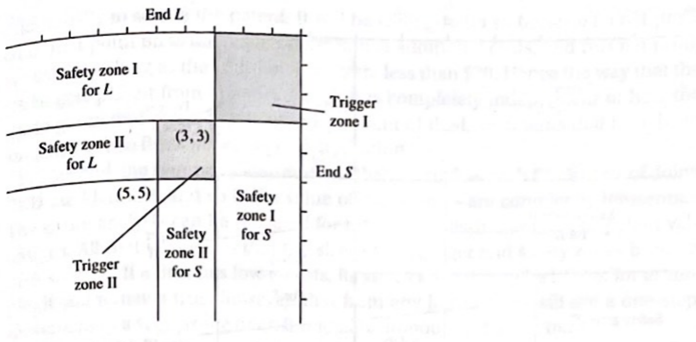


FIGURE 13.5.

The next question to ask is: Is there an incentive for either firm to try and get into its own safety zone even if doing so means doing rapid R&D?

Step 6. Consider a location such as (4, 4). Suppose it is firm L's turn to move. Firm L can take the game into its safety zone I in one step—at a cost of \$2. Thereafter it knows that S will drop out, and hence it can move a step at a time toward eventual completion; those three steps will cost a further \$6. The total costs then will be \$8, and that is less than the value of the patent.

More is true; as long as L has a way to get into its safety zone—and thereafter move a step at a time—while incurring costs that are no more than \$20, the value of the patent, it is worth L's while to do so. The argument applies symmetrically to firm S.

Step 7. Show that from locations (l, s), if $l, s = 4, 5$, the first mover will find it profitable to take the game in to its safety zone I. Show that if L has to move at (5, 4), its consequent net profit is \$7.

However, it is not worth firm L's while to move into its safety zone if it is at (6, 5). That would cost \$21 in total (why?), which exceeds the value of the patent. This result implies that, from that location, the best response for L, if it is the first mover, is to drop out of the race. (Why?) These arguments give us the following:

Proposition. There is a second trigger zone between (3, 3) and (5, 5); the first mover in this zone should move the game into its own safety zone I. There is also a second set of safety zones. The one for L is $3 \leq l \leq 5$ and $s > 5$ (and symmetrically for S). In firm L's safety zone II, S should immediately drop out. See Figure 13.5.

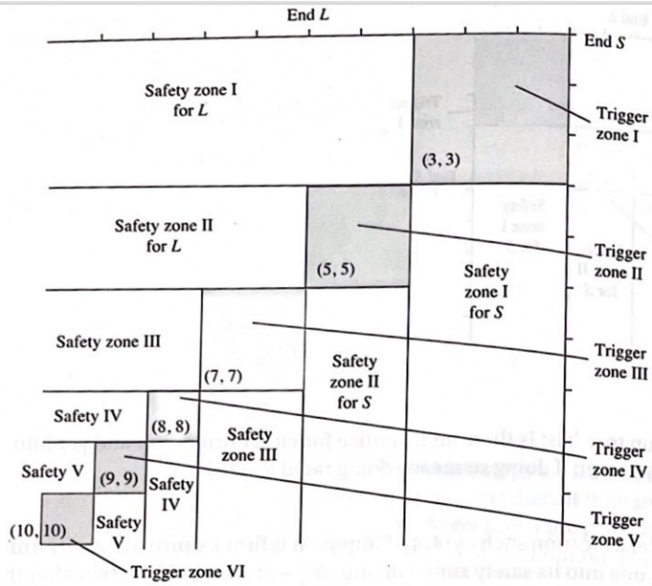


FIGURE 13.6.

Continuing in this fashion, we get the picture shown in Figure 13.6 for the solution in location space.

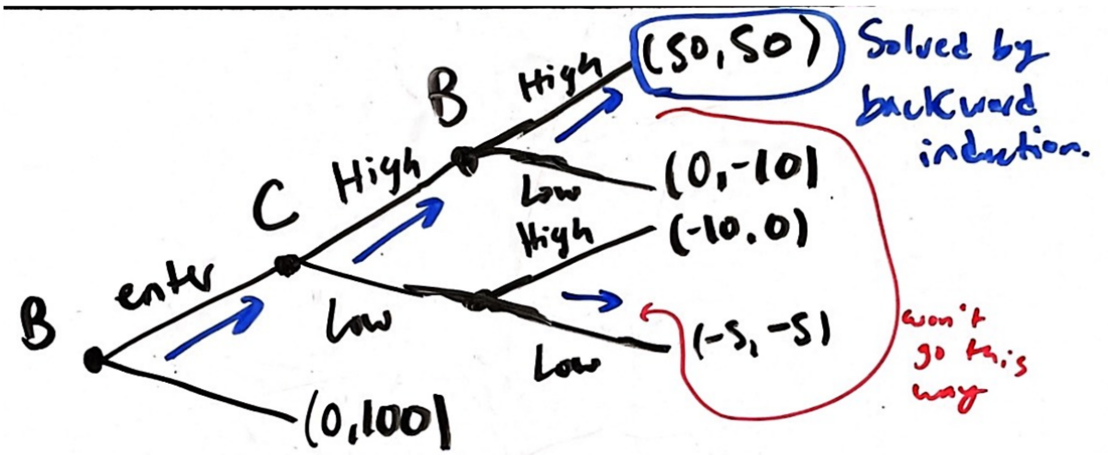
To summarize, the associated strategies are the following: If S is in L's safety zone—whatever the zone number—the best thing it can do is drop out of the race. Firm S in its own safety zone spends the minimum amount on R&D, moves a step at a time, and coasts to win the patent. In trigger zone n, each firm spends what it needs to—profitably—get an invincible advantage for itself and move the game into safety zone $n - 1$.[21]

For these numbers on costs and patent value, there are six trigger zones and five corresponding safety zones. Different numbers on the cost and patent value variables will change the size and number of these zones, but the numbers will not change the qualitative feature of the solution. Indeed, you will establish the truth of this assertion in the Exercises.

Problems From Chapters 12 and 13

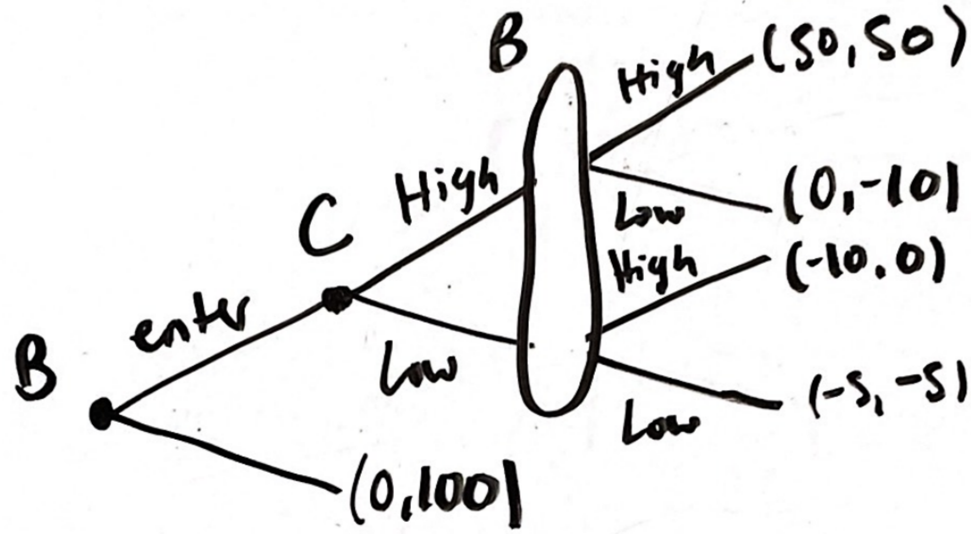
Problems 12.4 - 12.13, 13.6

Problems 12.4 - 12.6



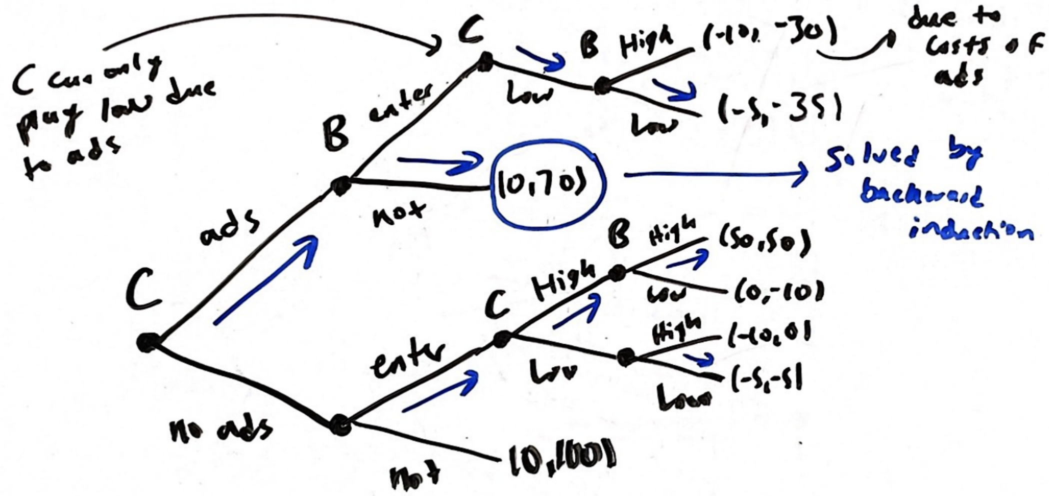
It is a game of perfect information. There are no simultaneous moves. All nodes are singletons.

12.6



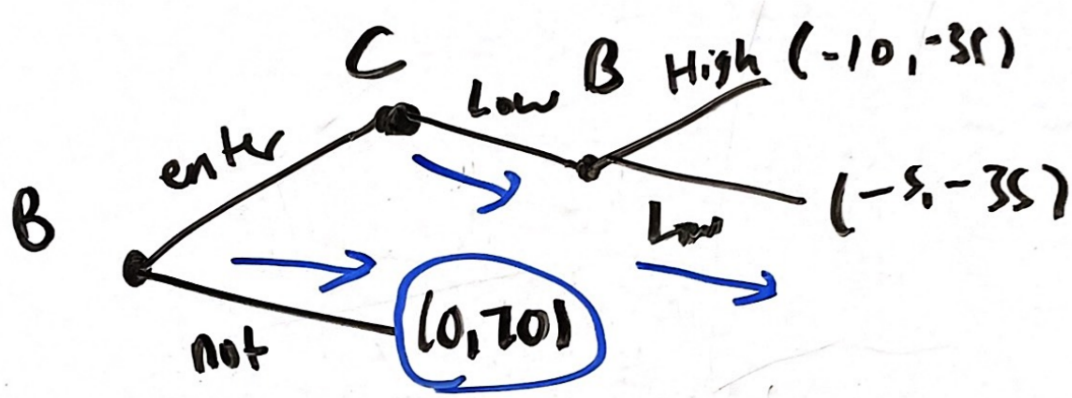
Not a game of perfect information. Perfect information games cannot have simultaneous moves.

Problems 12.7 - 12.9



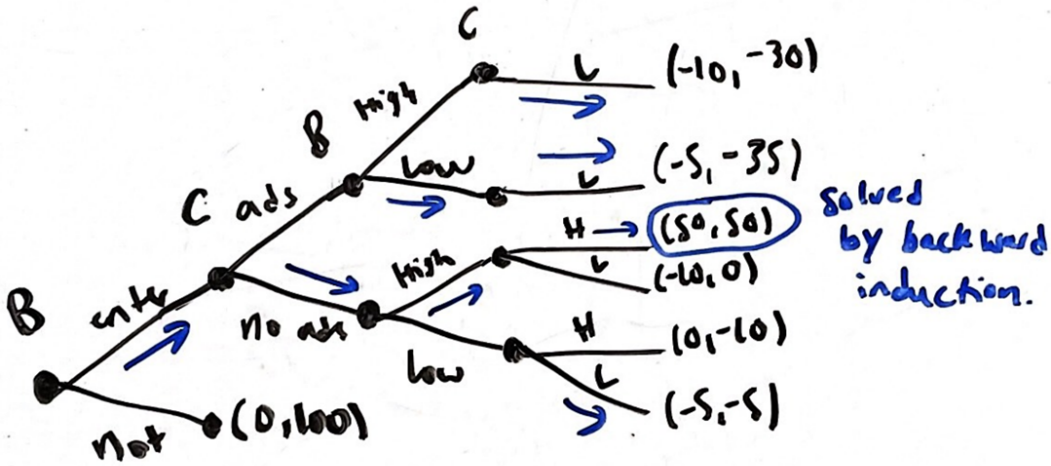
The Ad Company can charge up to 50 for its ad campaign. At 50, C will be indifferent to the ads and no ads (payoffs for both would be 50).

Problems 12.10 - 12.11



Same answer as 12.8

Problems 12.12 - 12.13



Different from 12.8 & 12.11.
This time, B made the commitment
to enter first. Instead of C
getting 70.60m get 50.

Problem 13.6