HW 11

16.8

a. The δ is a measure of a player's patience; this means that a player with a high δ values the future more than one with a low δ. Somebody who values the future good payoffs more will be less inclined to take the one-stage deviation that will give them a lot of utility now and a lot less in the future.

16.9

1 / 2	C	N
C	0,0	7,-2
N	-2,7	5,5

$EU(n,n) = \frac{5}{1-\delta}$ $\quad$ $EU(\text{deviate}) = 7 + \delta 0 + \delta^2 0 + \dots + \delta^T \cdot 0 + \delta^{T+1} 5 + \delta^{T+2} + \dots$

$EU(\text{deviate}) = 7 + \frac{\delta^{T+1} 5}{1-\delta}$ $\quad$ $\frac{5}{1-\delta} > 7 + \frac{\delta^{T+1} 5}{1-\delta}$

$5 > 7 - 7\delta + \delta^{T+1} 5 \rightarrow \frac{5(1-\delta^{T+1})}{1-\delta} > 7$ $\quad$ $5 \lim_{\delta \to 1} \frac{1-\delta^{T+1}}{1-\delta}$

$= 5 \lim_{\delta \to 1} \frac{-(T+1)\delta^T}{-1} = 5(T+1)$ $\quad$ $5(T+1) > 7$

$\boxed{\xrightarrow[\delta \to 1]{\text{when}} \; 5(T+1) > 7}$

So, even with $T=1$, the forgiving trigger strategy makes n,n sustainable as a SPNE as long as δ is close to one. This is different than the Grim Trigger strategy, which had $\delta > 2/7$.

without taking limits

$-2 > -7\delta + \delta^{T+1} 5$

$2 < \delta(7 - \delta^T 5)$

This is really difficult algebra, probably not what the question is asking, so just see this paragraph.

16.15 $Q = 6 - p$ ~~___~~ $c = .01$ $c(q) = 0.01q$

recall $Q = a - bp$ $a = 6$ $b = 1$

Cartel: $q = \frac{6-0.01}{4} = \frac{a-c}{4b}$ $p = \frac{6+0.01}{2} = \frac{a+c}{2}$

Cournot: $q = \frac{a-c}{3b} = \frac{6-0.01}{3b}$ $p = \frac{1}{3}a + \frac{2}{3}c = \frac{1}{3}(6) + \frac{2}{3}(.01)$

Since c is so low compared to the other numbers, we will assume it is 0.

Utility with $p = \$2$ $Q = 4$ $q_i = 2$ $\pi_i = 4$

$E^u(\text{maintain}) = 4 + \delta 4 + \delta^2 4 = \frac{4}{1-\delta}$

$E^u(\text{deviate}) \Rightarrow p = \$1$ $Q = 5$ $q_i = 2.5$ $\pi_i = 2.5$ → after

$E^u(\text{deviate}) \Rightarrow p = \$1$ $Q = 5$ $q_i = 5$ $\pi_i = 5$ → on deviation

$= 5 + 2.5\delta + 2.5\delta^2 + 2.5\delta^3$

$= 5 + \frac{2.5\delta}{1-\delta} < \frac{4}{1-\delta} \rightarrow \frac{5}{1} < \frac{4 - 2.5\delta}{1-\delta}$

$5 - 5\delta < 4 - 2.5\delta$ $1 < 2.5\delta$ $\delta > 1/2.5$

$\boxed{\delta > 2/5}$ δ must be $> 2/5$. I feel like I explained it with the math above.

16.16 $E^u(\text{maintain}) = \frac{4}{1-\delta}$ $E^u(\text{deviate}) = 5 + 2.5\delta + \dots + 2.5\delta^T + 4\delta^{T+1} + 4\delta^{T+2} + \dots$

$Eu(\text{deviate}) = 5 + \frac{\delta^{T+1}4}{1-\delta} + \underbrace{2.5\delta + 2.5\delta^2 + 2.5\delta^3 + \dots + 2.5\delta^T}_{2.5\sum_{n=1}^{T}\delta^n}$

Explicit: the forgiving trigger must have δ close to one

$\frac{4}{1-\delta} > 5 + \frac{4\delta^{T+1}}{1-\delta} + 2.5\sum_{n=1}^{T}\delta^n$

$\frac{4(1-\delta^{T+1})}{1-\delta} - 2.5\sum_{n=1}^{T}\delta^n > 5$ $\lim_{\delta \to 1}$ $4(T+1) - 2.5 > 5$

$4(T+1) > 7.5$

↳ even with $T = 1$ $4(2) > 7.5$ ✓ so the forgiving trigger sustains it and is a SPNE

$\boxed{16.17}$ The maximum price that can arise in a subgame perfect equilibrium is

let's say $p = \$3$ $Q = 3$ $q_i = 1.5$ $\pi_i = 4.5$

EU(deviate) => $p = \$2$ $Q = 4$ $q_i = 4$ $\pi_i = 8$

EU(deviate) → after $\pi_i = 2.5$

$8 + \frac{2.5\delta}{1-\delta} < \frac{4.5}{1-\delta}$ $\quad 8 < \frac{4.5 - 2.5\delta}{1-\delta}$

$8 - 8\delta < 4.5 - 2.5\delta$ $\quad 3.5 < 5.5\delta$ $\quad \delta > \frac{7}{11}$

$p = \$4$ $Q = 2$ $q_i = 1$ $\pi_i = 4$

EU(deviate) => $p = \$3$ $Q = 3$ $q = 3$ $\pi_i = 9$

$9 + \frac{2.5\delta}{1-\delta} < \frac{4}{1-\delta}$ $\quad 9 < \frac{4 - 2.5\delta}{1-\delta}$ $\quad 9 - \delta 9 < 4 - 2.5\delta$

$5 < 6.5\delta$ $\quad \delta > 10/13$

$p = \$5$ $Q = 1$ $q_i = .5$ $\pi_i = 2.5$

EU(deviate) => $p = \$4$ $Q = 2$ $q = 2$ $\pi_i = 8$

$8 + \frac{2.5\delta}{1-\delta} < \frac{2.5}{1-\delta}$ $\quad 8 < \frac{2.5 - 2.5\delta}{1-\delta}$ $\quad 8 - 8\delta < 2.5 - 2.5\delta$

$5.5 < 5.5\delta$ $\quad \delta > 1$ → not sustainable for $\delta = .9$

$\boxed{\text{\$4 is the maximum price sustainable}}$

$\boxed{16.20}$ $\max\left(1 - \min\{p_1, p_2, p_3, p_4\}\right)p$ → assume we want to maximize like they are a monopoly

$\max (1-p)p$ → $Q = 1-p$ $\quad p = 1-Q$

$\max_q q(1-q) = q - q^2$ $\quad \frac{\partial \pi}{\partial q} = -2q + 1 = 0$

$2q = 1$

$p = 1/2$ ← $q = 1/2$

$\boxed{p = 1/2}$

$\pi = qp \to$ when $p = 1/2$ $Q = 1/2$ $qp = 1/4 \div 4 = 1/16$

16.21

P4 North - Maintain

P1 \ P2	Maintain	Deviate
Maintain	$0, 0, \frac{1}{4}, 0$	$0, \frac{1}{8}, \frac{1}{8}, 0$
Deviate	$\frac{1}{8}, 0, \frac{1}{8}, 0$	$\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, 0$

	M	D
M	$\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}$	$0, \frac{1}{4}, 0, 0$
D	$\frac{1}{4}, 0, 0, 0$	$\frac{1}{8}, \frac{1}{8}, 0, 0$

~~Maintain~~ West ~~Maintain~~ Deviate

P3 East - Maintain

	M	D
M	$0, 0, \frac{1}{8}, \frac{1}{8}$	$0, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}$
D	$\frac{1}{12}, 0, \frac{1}{12}, \frac{1}{12}$	$0, 0, 0, 0$

	M	D
M	$0, 0, 0, \frac{1}{4}$	$0, \frac{1}{8}, 0, \frac{1}{8}$
D	$\frac{1}{8}, 0, 0, \frac{1}{8}$	$\frac{1}{12}, \frac{1}{12}, 0, \frac{1}{12}$

South - Deviate

* Those who deviate get a larger fraction of the spoils.
* When they all maintain they are split evenly.
* When everyone deviates, the price is too low and there is not enough profit to go around
* The best response is usually to deviate.

16.22 $EU(\text{maintain}) = \frac{1}{16}\frac{1}{1-\delta}$

$EU(\text{deviate}) = \frac{1}{4} + 0 \to$ 1 deviates $\frac{1}{8} \to$ 2 deviates $\frac{1}{12} \to$ 3 deviates

$\frac{1}{16}\frac{1}{1-\delta} > \frac{1}{4}$ $\quad \frac{1}{1-\delta} > 4$ $\quad 1 > 4 - 4\delta$ $\quad 4\delta > 3$ $\quad \boxed{\delta > \frac{3}{4}}$ $\to$ if 1 deviates

$\frac{1}{16}\frac{1}{1-\delta} > \frac{1}{8}$ $\quad \frac{1}{1-\delta} > 2$ $\quad 1 > 2 - 2\delta$ $\quad 2\delta > 1$ $\quad \boxed{\delta > \frac{1}{2}}$ $\to$ if 2 deviate

$\frac{1}{16}\frac{1}{1-\delta} > \frac{1}{12}$ $\quad \frac{1}{1-\delta} > \frac{4}{3}$ $\quad 1 > \frac{4}{3} - \frac{4}{3}\delta$ $\quad \frac{4}{3}\delta > \frac{1}{3}$ $\quad \boxed{\delta > \frac{1}{4}}$ $\to$ if 3 deviate

16.32 $v \to$ life time payoff to conforming

$$v = 5 + \delta[(1-0.1)v + 0.1 \cdot \delta^T v]$$
$$v = 5 + v\delta(.9 + .1\delta^T)$$
$$v(1 - \delta(.9 + .1\delta^T)) = 5$$
$$v = \frac{5}{1 - \delta(.9 + .1\delta^T)}$$

$\to 2 < \frac{5\delta}{1-\delta(.9+.1\delta^T)}(1-\delta^T)(p_c - p_n)$ recall $T = 10$

16.5 $\to 2 < \delta \overset{7.39}{v}(1-\delta^T)(p_c - p_n)$ $\quad 2 < \frac{5(.9)(1-(.9)^{10})(p_c - p_n)}{1-.9(.9+.1(.9)^{10})}$

$\boxed{2 < 18.47\underset{(.5-.1)}{(p_c - p_n)} \checkmark \text{ Yes it is a sufficient deterrent}}$

16.32 continued

2.00017 > 2

$$5 \cdot \frac{.52675(1-.52675^{10})}{.52675(.9+.52675^{10})}(.5-.1) \approx 2.00017$$

Minimum $\delta = .52675$

16.1a $\boxed{1+3\delta+1\delta^2+3\delta^3+1\delta^4+\ldots}$

16.1b $1+1\delta+1\delta^2+1\delta^3+1\delta^4+1\delta^5+1\delta^6+1\delta^7+1\delta^8+1\delta^9$
$+3\delta^{10}+3\delta^{11}+3\delta^{12}+3\delta^{13}+3\delta^{14}+3\delta^{15}+3\delta^{16}+3\delta^{17}+3\delta^{18}+3\delta^{19}$

$$= \boxed{1 + 1\sum_{n=1}^{9}\delta^n + 3\sum_{n=10}^{19}\delta^n \ldots}$$

16.2a $\boxed{1+1\sum_{n=1}^{9}\delta^n + 3\sum_{n=10}^{29}\delta^n + \sum_{n=30}^{39}\delta^n + 3\sum_{n=40}^{59}\delta^n + \ldots}$

16.2b $\boxed{1+1\sum_{n=1}^{T_1-1}\delta^n + 3\sum_{n=T_1}^{T_2+T_1-1}\delta^n + \ldots}$