Econ 478 Game Theory

Key: H15 Harrington 15, DV22 Dutta Vergote 22, W13 Watson 13, O04 Osborne 04, AC11 Aliprantis Chakrabarti 11, R07 Rasmussen 07, DN08 Dixit Nalebuff 08, B92 Binmore 92, FT91 Fudenberg Tirole 91

L1 Introduction

Game Theory

1. Game Theory [slides]
2. Syllabus

Modeling

1. Models [slide]
2. Example of mechanism design: cake cutting

Games

1. Game components, examples [slides]
2. Utilities should reflect complete feelings
    a. Example: If scores of (57-56) and (57-12) both just count as wins, true payoffs (1,0) and (1,0) are equivalent.
    b. Example: If payoffs represent dollar amounts but different players (e.g. poor and rich) feel differently toward receiving money, should replace with utilities
        i. Example: Brother-in-law story, beating cute cheerleader at chess
        ii. Example: family as opponents
    c. Examples: dictator game, ultimatum game
        i. In lab experiments, sharing 50% is common, average dictator shares 1/3.
        ii. In ultimatum lab experiments shares below 1/3 are often rejected.
3. Play
    a. Game: 21
    b. Game: Nim/Marienbad
4. Analysis: simplify!
    a. 21: think backward
        i. Lose at 19 or 20, win at 18, lose at 17 … etc.

ii. Big idea: Think Celestial! (Nelson, Oct. 2023)
   1. Career prep
   2. Honesty
   3. Healing relationships

b. Nim
   i. Simplify: no sub-dividing piles (original Nim game)
   ii. Simplify: two piles
   iii. Simplest cases: (1,1), (2,1), (1,2), etc.
      1. Strategic equivalence
   iv. Analysis of Marienbad is similar.

c. Example: strategic legislative voting
   i. 3 legislators
   ii. Protocol: vote between bills A and B (or bill and amendment)
   iii. Preferences: $A \succ B \succ N, B \succ N \succ A, N \succ A \succ B$
   iv. Sincere/myopic/nonstrategic voting: A wins, then N; no bill is passed.
   v. Voter 1 strategic: vote B in first round, B wins

d. Tic tac toe: combine redundant strategies (homework)

e. Comment: checkers is solved, chess is not (H15 284)

f. Prisoner's dilemma [save for L2]

5. Normal form / Strategic form
   a. Player 1 on rows, Player 2 on columns; payoff pairs

6. Extensive form
   a. Game tree: root, branch, node, end node
   b. Bus/cab/subway example [Find better example]
   c. A strategy is an entire contingency plan
   d. Information sets
   e. Equivalence of extensive form / normal form

L2 Dominant/Dominated strategies

Dominant strategies

1. Dominant strategy

	Left	Center	Right
High	1,3	6,2	12,3
Med	4,7	6,3	13,9
Low	3,2	5,2	10,3

a. In general, optimal behavior depends on opponent behavior.

b. Dominant strategy s_i^* is best *no matter* what opponent does.

c. Here, Medium and Right are dominant strategies.

    i. Right is only *weakly* dominant (better or as good as Left and Center).

d. (M, R) is dominant strategy equilibrium.

e. Below, Medium is a dominant strategy but there is no dominant strategy equilibrium.

	Left	Center	Right
High	1,1	6,2	12,3
Med	4,7	6,3	13,**4**
Low	3,2	5,2	10,3

2. Formal definition: s_i^* such that $\pi_i(s_i^*, s_{-i}) \geq \pi(s_i', s_{-i})$ for all s_i', s_{-i}
    a. With more than two players, strategy notation gets complicated
    b. Strategy set S_i (e.g., $S_i = \{H, M, L\}$,$S_i = \{L, C, R\}$, $S_i = \{fight, flee\}$, $S_i = \{A, B, N\}$)
    c. Strategy $s_i \in S_i$ (specific strategy: s_i^* or $s_i^\#$ or s_i')
    d. Strategy vector $s = (s_1, s_2, \ldots, s_n)$ (strategy for each of n players)
    e. Opponent vector $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$
    f. Payoff/utility function $\pi(s) = \pi(s_1, s_2, \ldots, s_n)$
3. Application: auction
    a. Can bid \$1 to \$1,000 (trade off benefit of winning with probability of winning).
    b. Is bidding \$100 a dominant strategy? What if opponent bids higher/lower?
    c. Ebay: what if option to submit initial bid and then revise later?
    d. Mystery of revised Ebay bids (Dr. Platt research paper)
        i. Mistaken strategy?
        ii. Introspection?

iii. Increasing urgency

Prisoner's dilemma

1. Prisoner's dilemma
    a. Basic 2-player, 2-action games have all been characterized (and named). Prisoner's dilemma is most famous.
    b. Central features
        i. Dominant strategies
        ii. Pareto inferior equilibrium
    c. Applications: public goods (national defense, street lighting), common resources, arms race, steroids, lobbying/lawyers/marketing, cartels, trade wars
        i. Other applications?
    d. Enforcement: shoot military defectors, Cortez burning ships, Chinese task masters, campaign finance limits
    e. [If time] video: Golden Balls gameshow clips

Dominated strategies

1. Dominated strategy
    a. In example below, P2 has no dominant strategy but Center is dominated (by Right).
    b. Example:

	Left	Center	Right
High	3,3	4,4	3,5
Med	4,2	3,1	2,3
Low	2,4	3,3	1,2

    c. P1: Low is (weakly) dominated because it's never as good as Med.
    d. P2: No strategy is dominated. Center is never best strategy but is not dominated by any *specific* other strategy (sometimes better than Left, sometimes better than Right).
2. Formally, $s_i^{\#}$ such that $\pi_i(s_i^{\#}, s_{-i}) \leq \pi(s_i', s_{-i})$ for *some* s_i', for *all* s_{-i}
3. IEDS: Iterated Elimination of Dominated Strategies
    a. Sometimes, can iterate to find equilibrium.
    b. If P1 never plays Low then game reduces, Left becomes dominated by Right.

c. If P2 never plays Left then game reduces, Med becomes dominated.

d. If P1 never plays High then game reduces, Center becomes dominated.

e. If reach IEDS solution (as in this example, game is *dominance solvable*.

4. Commentary

   a. Layers of rationality

   b. Multiplicity: order of elimination matters

	Left	Right
Top	0,0	0,1
Bottom	1,0	0,0

i. Note: if we only eliminated strongly dominated strategies, never multiple IEDS solutions.

c. Nonexistence: not all games are dominance solvable.

	Left	Middle	Bad
Top	1,-1	-1,1	0,-2
Middle	-1,1	1,-1	0,-2
Bad	-2,0	-2,0	-2,-2

5. Life lesson: commandments play useful role of simplifying our decision-making by highlighting dominated strategies.

   e. Never lie.

   f. Never be unkind.

   g. Never violate Word of Wisdom / Law of Chastity.

   h. Pay tithing first.

   i. Other examples?

Bertrand price competition

1. Two firms, marginal cost 2, pricing strategy $p \in \{\$1, \$2, \dots, \$10\}$
2. Lower price firm meets full demand pd, profit $pd - 2$; equal prices share demined $\frac{p}{2}d - 2$
3. $p < \$2$ dominated by $p = \$2$

4. $p = \$10$ dominated by $p = \$9$, dominated by $p = \$8$, etc.
5. IEDS equilibrium: $p = \$2$. Bertrand punchline: even two competitors mimic perfect competition

L3 Nash equilibrium

Best response

1. Example

	Left	Center	Right
Up	3,3	4,1	2,0
Down	3,0	2,1	4,2

2. Best response: s_i^{br} is best response to s_i if $\pi_i(s_i^{br}, s_{-i}) \geq \pi_i(s_i', s_{-i})$ for all $s_i' \in S_i$
    a. If P1 believes P2 will play Left, best response is Down
    b. If P1 believes P2 will play Center, best response is Up
    c. If P1 believes P2 will play Right, best response is Down
    d. If P2 believes P1 will play Up, best response is Center
    e. If P2 believes P1 will play Down, best response is Right
    f. Left is never best response, but is not dominated by any other specific strategy (sometimes better than Center, sometimes better than Right).

Nash equilibrium

1. Nash equilibrium: each player plays best response to other players' strategies.
    a. Named for John Nash (main character in 2001 film Beautiful Mind, played by Russell Crowe)
    b. Correctly conjecture others' behavior
    c. In example above, (Up, Center) and (Down, Right) are equilibria.
2. Motivations
    a. Prescriptions
    Someone prescribes equilibrium to all game players. Follow if expect others to follow.
    b. Pre-play communication
    We talk it over, never agree to anything that is not Nash equilibrium. (Problematic in that "talking it over" should then be part of game description.)

c. Rational introspection

Never settle until conjecture equilibrium

d. Trial and error

Start with random strategy vector, change if unhappy, repeat game, …, only settle on Nash equilibrium.

3. Number of equilibria
   a. Equilibrium may not exist.
   b. May be multiple equilibria (as in example above).
   c. May be unique.
   d. Note: If dominant strategy solution exists, it is the unique Nash equilibrium.

Symmetric equilibrium

1. Play research grants game
   a. Each plays requests research grant between $0 and $100. Successful if both can be accommodated with $100 budget; otherwise, no grants awarded.
   b. What is Nash equilibrium? Any pair that sums to $100.
2. Equilibrium refinements: ways of choosing most reasonable equilibrium when there multiple equilibria exist
3. Symmetric equilibrium: identical players adopt identical strategies.
   a. Research grants game: ($50,$50).
   b. Outcomes like ($60,$40) and ($40,$60) require higher level of coordination.
   c. Symmetric equilibrium can be characterized by

Traveler's Paradox / Stag Hunt

	A	B
A	3,3	0,0 (-1,0)
B	0,0 (0,-1)	3,3 (1,1)

1. Central features: multiple symmetric equilibria, coordination problem
2. One equilibrium may be inferior
3. Superior equilibrium may be "focal"

4. But inferior equilibrium may be self-perpetuating (especially in Stag Hunt, where deviation from inferior equilibrium is costly unless synchronized)
5. [Introduce Mixed equilibria first: risks miscoordination (sidewalk collisions)]
6. Applications: social conventions/norms, language/standards, compatible technology, foot binding, bank runs, two party dominance/Duverger's law (with many players)
    a. Stag hunt: revolution, toeing party line

Battle of the Sexes / War and Peace in Congress

	A	B
A	3,1	0,0
B	0,0	1,3

1. Play: Battle of the Sexes
2. Central features: multiple equilibria, coordination problem
3. Asymmetric: P1 prefers one equilibrium, P2 favors the other
4. Neither equilibrium is focal (or is A? Experiment.)
5. Applications: choosing computer technology/accounting procedures for merging firms, housekeeping habits in marriage, property right norms, primary elections, bipartisan compromise

Chicken / Hawk-Dove

	Swerve	Straight
Swerve	0,0	-1,1
Straight	1,-1	-100,-100

1. Features: Two asymmetric equilibria, costly miscoordination
    a. Credibility device: disable steering
2. Applications: government shutdown, campaign promises, good Samaritan acts (e.g. call 911, traffic accident witness), web fights between desert spiders in New Mexico

L4 Cournot duopoly

Continuous actions

1. Setup: P1 chooses x, P2 chooses y, utilities $u_1(x, y)$, $u_2(x, y)$
2. Best responses
   a. Best response $x^{br}(y)$ solves FOC $\frac{\partial u_1(x,y)}{\partial x} = 0$
      i. x is endogenous (choice variable); y is exogenous.
   b. Best response $y^{br}(x)$ solves FOC $\frac{\partial u_2(x,y)}{\partial y} = 0$
      i. x is exogenous, y is endogenous.
3. Continuous actions: to find Nash equilibrium (x^*, y^*),
   a. Solve $\frac{\partial u_1(x,y)}{\partial x} = 0$ and $\frac{\partial u_2(x,y)}{\partial y} = 0$ simultaneously.
   b. Equivalently, solve $x^* = x^{br}\left(y^{br}(x^*)\right)$ and $y^* = y^{br}\left(x^{br}(y^*)\right)$.

Cournot duopoly

1. Cournot (1838) legacy
   a. Treatise on supply, demand, and monopolies, even predating Alfred Marshall (1890)
   b. First introduction of probability theory in economic models
   c. Earliest game theory model: pre-dates John Nash (1950) but effectively characterized best responses and Nash equilibrium.
2. Cournot duopoly model
   a. Two firms: middle ground between monopoly and perfect (infinite) competition
   b. Aggregate demand $P = a - bQ$ (where a and b both positive)
      i. Example ($a = 10, b = 1, c = 1$): $P = 10 - Q$
   c. Profit $\pi_i = Pq_i - c(q_i) = (P - c)q_i$
      i. Constant marginal cost $c(q_i) = cq_i$ for some positive c
   d. How much will each firm produce? What price will prevail? How high will Firm 1 profits be? How high will industry profits be?
3. Best response
   a. Suppose Firm 1 expects q_2, seeks to maximize profit π_1
   b. FOC: $\frac{\partial \pi_1}{\partial q_1} = q_1 \frac{\partial P}{\partial q_1} + P - c = -bq_1 + a - b(q_1 + q_2) - c = 0$

$\Leftrightarrow 2bq_1 = a - c - bq_2 \Leftrightarrow q_1^{br}(q_2) = \frac{a-c}{2b} - \frac{1}{2}q_2$

i. Remember, $P(q_1, q_2) = a - b(q_1 + q_2)$, so $\frac{\partial P}{\partial q_1} = -b$

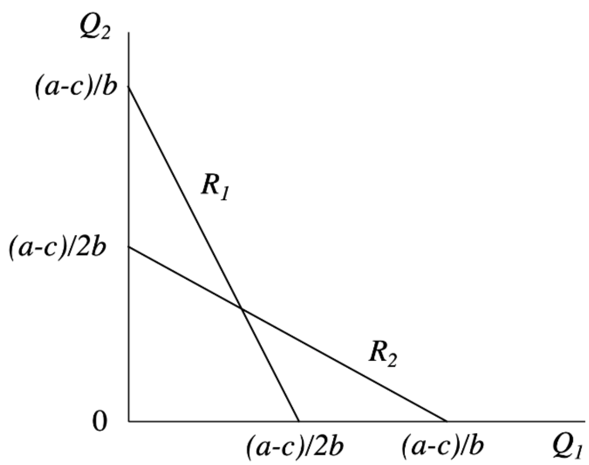
ii. SOC: $\frac{\partial^2 \pi_1}{\partial q_1^2} = -b < 0$

iii. Similarly, $q_2^{br}(q_1) = \frac{a-c}{2b} - \frac{1}{2}q_1$

iv. Example ($a = 10, b = 1, c = 1$): $q_1^{br}(q_2) = \frac{9}{2} - \frac{1}{2}q_2$

c. Firm 1 effectively faces its own private demand curve (different from the industry demand curve)

d. Does Firm 1 have a dominant strategy? No. What would $q_1^{br}(q_2)$ look like if it were a dominant strategy? Constant (flat slope).



4. Equilibrium

a. Quantities $(q_1^*, q_2^*) = \left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)$

i. $q_1^{br}\left(q_2^{br}(q_1)\right) = \frac{a-c}{2b} - \frac{1}{2}\left(\frac{a-c}{2b} - \frac{1}{2}q_1\right) = \frac{a-c}{4b} + \frac{1}{4}q_1$

ii. This is the quantity Firm 1 would produce if they knew Firm 2 expected them to produce q_1, and optimize accordingly.

iii. $q_1^* = \frac{a-c}{4b} + \frac{1}{4}q_1^* \Leftrightarrow \frac{3}{4}q_1^* = \frac{a-c}{4b} \Leftrightarrow q_1^* = \frac{a-c}{3b}$

iv. Similarly, $q_2^* = \frac{a-c}{3b}$; $Q^* = \frac{2(a-c)}{3b}$

v. Example ($a = 10, b = 1, c = 1$): $(q_1^*, q_2^*) = (3,3)$, $Q^* = 6$

b. Price

i. $P^* = a - bQ^* = a - \frac{2}{3}(a - c) = \frac{1}{3}a + \frac{2}{3}c$

ii. Example ($a = 10, b = 1, c = 1$): $P^* = \frac{10}{3} + \frac{2}{3} = 4$

c. Profit

i. $\pi_1^* = (P^* - c)q_1^* = \left(\frac{1}{3}a - \frac{1}{3}c\right)\left(\frac{a-c}{3b}\right) = \frac{(a-c)^2}{9b}$

1. Similarly, $\pi_2^* = \frac{(a-c)^2}{9b}$
2. Industry profit $\Pi^* = \pi_1^* + \pi_2^* = \frac{2(a-c)^2}{9b}$

ii. Example ($a = 10, b = 1, c = 1$): $\pi_1^* = \pi_2^* = 9, \Pi^* = 18$

5. Comparison with cartel/monopoly solution

a. What if Firms 1 and 2 colluded on Q (or if there were a monopoly)?

a. Any division of Q would produce same industry benefit PQ at same industry cost cQ

b. Cartel/monopoly quantities

i. $\frac{\partial \Pi}{\partial Q} = P'Q + P - c = -bQ + a - bQ - c = a - 2bQ - c = 0 \Leftrightarrow Q^m = \frac{a-c}{2b}$

ii. SOC: $\frac{\partial^2 \Pi}{\partial Q^2} = -2b < 0$

iii. Smaller than in duopoly

iv. Example ($a = 10, b = 1, c = 1$): $Q^m = \frac{9}{2}$

b. Cartel/monopoly price

v. $P^m = a - bQ^m = \frac{a}{2} + \frac{c}{2}$; lower than in duopoly (since $a > c$ if any units can be profitably produced)

vi. Example ($a = 10, b = 1, c = 1$): $P^m = \frac{10}{2} + \frac{1}{2} = 5.5$

c. Cartel/monopoly profits

vii. $\Pi^m = (P - c)Q^m = \left(\frac{a}{2} - \frac{c}{2}\right)\left(\frac{a-c}{2b}\right) = \frac{(a-c)^2}{4b}$; higher than in duopoly

i. Example ($a = 10, b = 1, c = 1$): $\Pi^m = \frac{81}{4} = 20.25$; $\pi_1^m = \pi_2^m = 10.125$; higher than in duopoly

ii. If $q_1^{br}(q_2^m) = q_1^m$ then cartel quantities would be equilibrium. That they are not ($q_1^m \neq q_1^*$) implies that $q_1^{br}(q_2^m) \neq q_1^m$. Since cartel quantities are *a possible* response to cartel quantities but not the *best* response, this implies that there is an even *better* response $q_1^{br}(q_2^m)$ available, with higher profits $\pi_1^{br}(q_2^m) > \pi_1^m$. In that sense, cartel is unstable: either firm would rather increase production to increase their own profit. But communal profits would be lower in that case, since $\Pi^m > \Pi^*$. In other words, the deviating firm would increase its own profits but decrease the rival firm's profit *even more*.

iii. The temptation to deviate from the cartel agreement creates a situation reminiscent of the Prisoner's Dilemma. (Strictly speaking, this is not a Prisoner's Dilemma because firms have more than two strategies and do not have a dominant strategy. However, restricting to specific "high" and "low" levels of production would indeed create a proper Prisoner's Dilemma.)

d. Application: OPEC

i. OPEC is a consortium of major oil-producing countries, formed in 1961.

ii. In 1970's, successfully restricted oil supply (stipulating quotas) and raised oil prices dramatically.

iii. In response, non-OPEC oil producers have increased production.

iv. Some OPEC nations have left or cheated on their quotas.

v. Large oil producers have largely stuck with cartel.

vi. OPEC would like all oil producers to join. Textbook cites estimate that profits would increase by over $1 million/day (*after* compensating non-OPEC nations).

vii. Since OPEC nations are low-cost producers, minimizing costs would require shutting down non-OPEC oil production completely. Mechanism for paying for non-production is unclear, and non-OPEC producers can free ride on high prices while still increasing their own production.

Stackelberg first-mover

1. Stackelberg (1934) variation of Cournot model: what if Firm 1 moves first? (Industry leader?)
2. Backward induction: Firm 2 merely plays best response $q_2^{br}(q_1) = \frac{a-c}{2b} - \frac{1}{2}q_1$, as before.
3. Price is now $P = a - bQ = a - b\left(q_1 + q_2^{br}(q_1)\right) = a - b\left(\frac{a-c}{2b} + \frac{1}{2}q_1\right)$, so $\frac{\partial P}{\partial q_1} = -\frac{b}{2}$

   a. q_1 now affects P both directly, and indirectly through affecting q_2^{br}.

4. Equilibrium

   a. Expecting Firm 2 to react optimally, Firms 1's FOC now solves:

   $0 = -\frac{\boldsymbol{b}}{\mathbf{2}}q_1 + a - b\left(\frac{a-c}{2b} + \frac{1}{2}q_1\right) - c$, solution $q_1^* = \frac{a-c}{2b}$ higher than before.

   b. Firm 1 produces higher quantity, knowing that producer 2 will react with a lower quantity (to avoid lowering the price). Firm 2 response: $q_2^* = \frac{a-c}{4b}$.

c. Total quantity $Q^* = \frac{3}{4}\frac{(a-c)}{b}$ higher than before, price $P^* = \frac{1}{4}a + \frac{3}{4}c$ lower than before

d. Firm 1 profits $\pi_1^* = \frac{(a-c)^2}{8b}$ higher than before, Firm 2 profits $\pi_2^* = \frac{(a-c)^2}{16b}$ lower than before. Industry profit $\Pi^* = \frac{3}{16}\frac{(a-c)^2}{b}$ lower than before.

e. Example ($a = 10, b = 1, c = 1$): $q_1^* = \frac{9}{2},\ q_2^* = \frac{9}{4},\ P^* = \frac{13}{4} = 3.25,\ \pi_1^* = \frac{81}{8} = 10.125,$ $\pi_2^* = \frac{81}{16} \approx 5, \Pi^* = 15$

Oligopoly

1. Example parameters: $P = 10 - Q, c = 1$
2. What if there are n firms instead of two? $Q = q_1 + q_2 + \cdots + q_n$
3. Best response
    a. $\pi_1 = (10 - q_1 + q_2 + \cdots + q_n - c)q_1$
    b. $\frac{\partial \pi_1}{\partial q_1} = (10 - q_1 + q_2 + \cdots + q_n - c) - q_1 = 0$
4. Symmetric equilibrium
    a. $q_1^* = q_2^* = \cdots = q_n^* = q^*$
    b. $10 - (n+1)q^* - c = 0$
    c. $q^* = \frac{10-c}{n+1}$
    d. $P^* = 10 - nq^* = \frac{10}{n+1} + \frac{cn}{n+1} \to c$ (approaches marginal cost pricing, as in perfect competition)
    e. $\pi_1 = (P^* - c)q^* \to 0$ (approaches zero profit, as in perfect competition)
5. Note: similar conclusions hold with nonlinear demand curve.

L5 Voting

Majority rule

1. Individuals make rational decisions: preferences are complete and transitive.
2. Often need group decisions. How can we do this rationally?
3. Majority rule works great with two candidates/alternatives
4. Three alternatives
    a. Voting on $\{x, y, z\}$, majority rule is transitive but incomplete.

20 voters prefer $x \succ y \succ z$

25 voters prefer $y \succ z \succ x$
30 voters prefer $z \succ y \succ x$

b. Condorcet (1785) winner: voting pairwise on $\{x, y, z\}$, one candidate/alternative that beats every other one-on-one.

   i. 55/75, $y \succ x$, 45/75 $y \succ z$

   ii. Need not exist

20 voters prefer $x \succ y \succ z$
25 voters prefer $y \succ z \succ x$
30 voters prefer $z \succ \boldsymbol{x} \succ \boldsymbol{y}$

c. Condorcet (1785) cycles: majority rule is complete but intransitive.

   i. 50/75 $x \succ y$, 45/75 $y \succ z$, 55/75 $z \succ x$

   ii. Subject to "money pump"

d. Incentive for strategic voting

5. Explore other voting rules

Voting rules

1. Plurality rule
   a. Candidate/alternative with most votes (not necessarily 51%) wins
   b. Subject to spoilers
      i. E.g. Nader/Gore/Bush, 2000; Clinton/Bush/Perot 1992
2. Binary agendas
   a. Condorcet winner always wins (if exists)
   b. If no Condorcet winner: subject to agenda, manipulable by agenda setter
   c. Extreme example: 10 voters prefer ABCDEF, 10 voters prefer BCDEFA, 10 voters prefer CDEFAB, agenda setter can elect F with agenda DE-C-B-A-F (start consulting business?)
   d. More extreme example: $(\$0, \$50, \$50) \rightarrow (\$49, \$51, 0) \rightarrow (\$99, \$0, \$1)$
3. Borda (1771) rule: give 0 points to least favorite candidate/alternative, 1 point to next least favorite, etc., then candidate/alternative with most votes wins. (Heisman trophy.)

4. Runoff: two plurality winners face off in second election (French presidential elections, California's non-partisan primaries)
5. Ranked choice voting: each voter ranks alternatives/candidates, candidate with fewest top votes is eliminated, that candidate's supporters are redistributed to their next favorite candidate, repeat until one candidate remains.
6. Arrow's (1950) "impossibility" theorem
    a. Seek to find a voting rule that satisfies good axioms.
    b. Axioms
        i. Transitive ranking
        ii. Preserves unanimity: if all voters prefer x to y then so does society.
        iii. Independent of irrelevant alternative (IIA): changing voter rankings of y vs. z doesn't affect society's ranking of x vs. y
        iv. Non-dictatorial (otherwise, following any one voter's preference would suffice)
    c. Impossibility
    d. This largely spawned fields of political economics, mechanism design
    e. Groups cannot make rational decisions?
    f. Gibbard-Satterthwaite theorem: no non-dictatorial strategy-proof mechanism exists.
    g. Score voting / approval voting

Hotelling-Downs spatial model

1. Based on Hotelling (1937) (model of firm locational decisions, mentioning politics only briefly) and Downs (1958) (whole book applying Hotelling model to politics).
2. Single-peaked preferences assumption: candidates/alternatives can be ordered left to right. Each voter has an ideal policy $\hat{x}_i$, with preferences left of $\hat{x}_i$ increasing and preferences right of $\hat{x}_i$ decreasing.
3. Median voter theorem: median ideal point is Condorcet winner.
4. No incentive for strategic voting.
5. Competition for votes
    a. Suppose policy space $[-1,1]$, median voter at $x_m = 0$.
    b. Best response to x_A is closer to center than x_B, vice versa.
    c. (Unique) equilibrium: $x_A = x_B = 0$.
        i. Conservative liberal against liberal conservative

ii. Good for voters (utilitarian)

1. Analogous to cake cutting

iii. Empirically, polarization seems high; lots of theories but remains somewhat mysterious.

1. Beholden to special interests
2. Voter participation

6. Other applications: cake cutting, geographic proximity (Hotelling's interpretation), product differentiation (e.g., phone memory storage / battery life)
   a. Convergence is desirable in political setting, because it minimizes the disutility voters experience from a policy outcome far from their preferred policy.
   b. In contrast, an agreement by firms to differentiate would make consumers better off.
   c. Key difference is that both firms service consumers but only one candidate services voters.

L6 Common resources

Tragedy of the commons [slides]

Applications:, assets of bankrupt firm by creditors, chandeliers on Provo temple

Pollution (public bad)

1. Greenhouse emissions $g = g_1 + g_2$ created by consumption $g_1 = c_1$, $g_2 = c_2$
2. Utility $u_1 = \log c_1 - g = \log c_1 - c_1 - c_2$
3. FOC: $\frac{\partial u_1}{\partial c_1} = \frac{1}{c_1} - 1 = 0$ if $c_1 = 1$; symmetrically, $c_2 = 1$ so utility $u_1 = u_2 = -2$
   a. Nash equilibrium is maybe more intuitive in continuous model: from arbitrary starting point, both players gradually optimize. E.g., status quo $(0.1, 1.8)$, player 2 cuts back on pollution, player 1 notices pollution but is so hungry he/she makes a campfire to cook on, player 2 cuts back on pollution even more, etc…
4. Planner: $W = \log c_1 + \log c_2 - 2(c_1 + c_2)$; FOC $\frac{1}{c_1} - 2 = 0 \Leftrightarrow c_1^{**} = \frac{1}{2}$, $u_1^{**} = \log\left(\frac{1}{2}\right) - 1 \approx -1.7$; everyone consumes less but is happier.
5. Reinterpret: litter, water pollution, noise pollution

Policy solutions

1. Ownership
    a. Most commons have been privatized; budget allocations in financial commons; cap and trade
2. Tax (or subsidy)
    a. Use fees (national parks), carbon tax, green appliance subsidies, cash for clunkers
3. Regulate number of users
    a. Cap and trade, fines for taking stalactites, National parks lottery
4. Persuade
    a. Negative shaming, positive preaching, preach environmentalism or preach love thy neighbor (requisite for eventual "all things common" regime in Zion)
5. Global cooperation
    a. In case of greenhouse gases, clean oceans, etc., collective actions within US insufficient: production merely moves foreign and continues. Need global cooperation. Many public goods are global, too (international peacekeeping, pandemic abatement)

Public good contributions

1. Volunteerism example (from earlier HW)
2. Utilities $u_1 = \sqrt{x+y} - x$ and $u_2 = \sqrt{x+y} - y$, where $x, y \in [0,4]$ (Homework was simpler: $\{0,1,2,3,4\}$)
3. $x^{br}(y) = \begin{cases} \frac{1}{4} - y \ \ if\ y \leq \frac{1}{4} \\ 0 \ \ \ if\ y > \frac{1}{4} \end{cases}$
    a. Given y, maximize u_1. FOC: $\frac{\partial u_1}{\partial x} = \frac{1}{2}(x+y)^{-\frac{1}{2}} - 1 = 0$

       $\Leftrightarrow (x+y)^{-\frac{1}{2}} = 2 \Leftrightarrow (x+y) = \frac{1}{4} \Leftrightarrow x^{br}(y) = \frac{1}{4} - y$
    b. SOC: $\frac{\partial^2 u_1}{\partial x^2} = -\frac{1}{4}(x+y)^{-\frac{3}{2}} < 0$
    c. Similarly, $y^{br}(x) = \begin{cases} \frac{1}{4} - x \ \ if\ x \leq \frac{1}{4} \\ 0 \ \ \ if\ x > \frac{1}{4} \end{cases}$
4. Equilibria

a. $x^{br}(y) \in \left[0, \frac{1}{4}\right]$ for any y and $y^{br}(x) \in \left[0, \frac{1}{4}\right]$ for any x, so (x^*, y^*) is equilibrium only if $x^*, y^* \in \left[0, \frac{1}{4}\right]$

b. Best response conditions then imply $x^* = \frac{1}{4} - y^*$ and $y^* = \frac{1}{4} - x^*$. But these are same equation.

c. Thus, any (x^*, y^*) pair with $x^* + y^* = .25$ is an equilibrium.

   i. Many equilibria! (But way more (x, y) pairs that are not equilibria.)

   ii. Examples: $(0, .25)$, $(.25, 0)$, $(.125, .125)$, $(.2, .05)$, but not $(0.3, 0.3)$ or $(0, 0)$.

d. Symmetric equilibrium: $(.125, .125)$

Exam 1 review

1. Modeling games
   a. Sequential games
   b. Payoffs = utilities (not dollars, not points)
   c. Solving sequential games: backward induction
   d. Information sets
   e. Simultaneous games, equivalence
2. Dominant / dominated strategies
   a. Dominant strategy
      i. Application: bidding in (second price) auctions
   b. Dominated strategy
   c. Iterated elimination of dominant strategies
      i. Application: Bertrand price competition
3. Nash equilibrium
   a. Best responses
   b. Symmetric equilibrium
   c. Continuous games
4. Simple games
   a. Prisoner's dilemma
   b. Traveler's paradox
   c. Battle of the sexes
   d. War and peace in Congress

    e. Stag hunt
    f. Chicken
5. Application: Cournot duopoly
    a. Firms choose quantity, market demand determines price.
    b. Equilibrium: higher quantity than cartel, lower profits
    c. More firms: price approaches marginal cost, profit approaches zero
    d. Stackelberg: first mover advantage
6. Application: Voting
    a. Majority rule: Condorcet winner (loser), Condorcet cycles
    b. Plurality rule, Borda rule, runoff, ranked choice voting: spoilers
    c. Agendas: manipulation by agenda setter
    d. Arrow's impossibility theorem
    e. Single-peaked preferences
    f. Median voter theorem
7. Application: Common resources
    a. Tragedy of the commons
    b. Public goods
    c. Public bads
    d. Large groups

L7 Mixed strategies

Matching pennies

1. Play matching pennies

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

    a. No equilibrium exists: $HH \to HT \to TT \to TH \to HH \to \cdots$
        i. Potential for eternal spiral renews appreciation of Nash equilibrium
    b. Applications: front-runner/underdog political campaigns, cybersecurity, soccer penalty kicks, rock-paper-scissors

Zero sum games

1. Zero sum / constant sum game: payoffs sum to zero (or some other constant) in every outcome, so can only win if someone else loses.
    a. In positive sum games like prisoner's dilemma, there is a way to make everyone better- (or worse-) off simultaneously.
    b. Examples: card games, board games, sports, elections, buyer/seller price negotiations

Mixed strategies

1. Mixed strategy = probability distribution over pure strategies
    a. Matching pennies equilibrium: play $P(H) = P(T) = .5$
2. Example game:

	L	C	R
U	1,0	4,3	4,1
M	2,3	2,0	2,1
D	4,0	1,3	4,1

    a. Computing expected utilities: $s_1 = (.2, .3, .5), s_2 = (.4, .6, 0)$
        i. $EU_1(s_1 = (.2, .3, .5), s_2 = (.4, .6, 0)) = (.2 \times .4)(1) + (.2 \times .6)(4) + 0 + (.3 \times .4)(2) + (.3 \times .6)(2) + 0 + (.5 \times .4)(4) + (.5 \times .6)(1) + 0 = 2.26$
        ii. $EU_2(s_1 = (.2, .3, .5), s_2 = (.4, .6, 0)) = (.2 \times .4)(0) + (.2 \times .6)(3) + 0 + (.3 \times .4)(3) + (.3 \times .6)(0) + 0 + (.5 \times .4)(0) + (.5 \times .6)(3) + 0 = 1.62$
    b. Mixed strategies can dominate some pure strategies
        i. P1 plays $(.5, .5, 0)$: $EU_1 \in \{2.5, 2.5, 4\}$
        ii. P1 plays $(0,1,0)$: $EU_1 = 2$
        iii. Intuition: insurance
        iv. P2: $(.5, .5, 0)$ dominates C
    c. IEDS may have no solution in pure strategies, but solution in mixed strategies
        i. Eliminate M (dominated by $(.5, 0, .5)$) and R (dominated by $(.5, .5, 0)$)
        ii. Then eliminate L, then D
3. Mixed strategies as best responses
    a. Expected payoff is weighted average of pure strategy expected payoffs

b. No reason to use mixture when one pure strategy higher than another

c. Useful equilibrium tip: therefore, only mix if indifferent.

4. Battle of sexes

   a. Example 1: payoffs if W plays 2/3 ballet; M plays ½ ballet

      i. Woman's $EU = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)(3) + \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)(0) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)(0) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)(1) = \frac{7}{6}$

      ii. Man's $EU = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)(1) + \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)(0) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)(0) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)(3) = \frac{5}{6}$

   b. Example 2: W plays 1/2 ballet, 1/2 boxing; how should M respond?

      i. $EU_m(ballet) = \frac{1}{2}(1) + \frac{1}{2}(0) = \frac{1}{2}$

      ii. $EU_m(boxing) = \frac{1}{2}(0) + \frac{1}{2}(3) = \frac{3}{2}$

   c. Example 3: $s_w = (p, 1-p)$; how should M respond?

      i. $EU_m(ballet) = p(1) + (1-p)(0) = p$

      ii. $EU_m(boxing) = p(0) + (1-p)(3) = 3 - 3p$

      iii. Ballet is best response if $p > 3 - 3p \Leftrightarrow p > \frac{3}{4}$

      iv. Boxing is best response if $p < 3 - 3p \Leftrightarrow p < \frac{3}{4}$

      v. Both are best responses if $p = \frac{3}{4}$

   d. Example 4: M plays ballet q; how should W respond?

      i. $EU_w(ballet) = q(3) + (1-q)(0) = 3q$

      ii. $EU_w(boxing) = q(0) + (1-q)(1) = 1 - q$

      iii. Both are best responses if $3q = 1 - q \Leftrightarrow q = \frac{1}{4}$

   e. (Third) equilibrium: $s_w = \left(\frac{3}{4}, \frac{1}{4}\right)$, $s_m = \left(\frac{1}{4}, \frac{3}{4}\right)$

      i. W only mixes if $s_m = \left(\frac{1}{4}, \frac{3}{4}\right)$

      ii. M only mixes if $s_w = \left(\frac{3}{4}, \frac{1}{4}\right)$

      iii. (In equilibrium, $\frac{3}{16}$ attend ballet, $\frac{3}{16}$ attend boxing, $\frac{10}{16}$ miscoordinate!)

5. Matching pennies example: P2 plays $P(H) = P(T) = .5$

   a. $EU_1(H) = EU_2(T) = .5(1) + .5(-1) = 0$

   b. $EU_1(T) = EU_2(H) = .5(-1) + .5(1) = 0$

   c. Intuition: value of mixed strategy is unpredictability

      i. Opponent who knows I'll play H can win

      ii. Opponent who knows I'll play T can win

iii. Mixed strategy keeps them guessing

6. Nash theorem: every (finite) game has at least one (possibly mixed) equilibrium
7. Application: random drug testing
    a. Incentive for steroids: s is dominant strategy

	Steroids	None
Steroids	0,0	1,-1
None	-1,1	0,0

    b. IOC: Test P1/Test P2

	P3: Test P1		P3: Test P2	
	Steroids	None	Steroids	None
Steroids	-2,1,1	-2,1,1	1,-2,1	1,-1,0
None	-1,1,0	0,0,0	1,-2,1	0,0,0

    c. Equilibrium: (None, None, (.5,.5)) (better than Test P1/Test P2 and cheaper than testing both)
8. Infinite games
    a. All of these examples are finite games with discrete probabilities. With infinite games, a mixed strategy is a density function. These are more complicated. I don't intend to ask you to solve mixed strategies in infinite games.
    b. Nash's theorem applies to some but not all infinite games. There can be infinite games even with no mixed equilibrium.
9. Summary
    a. Mixed strategies can be used to hedge against possible actions by opponent.
    b. Especially in zero-sum games, mixed strategies can deliberately avoid predictability.
    c. Can also reflect uncertainty regarding different types of players. (Are you a rock type, paper type, or scissors type?)
    d. Only play mixed strategies if indifferent between pure strategies.

Application: War of attrition

1. War of attrition
    a. Market should be natural monopoly (fixed costs high, demand low)
    b. Two or more existing firms decide whether to exit or not.
    c. If all remain in, negative profit for all.
    d. Any firm(s) that exit receive zero.
    e. If one exits, remaining firm(s) have greater profit.

	Date 0	Date 1	Date 2
Date 0	0,0	0,15	0,30
Date 1	15,0	-5,-5	-5,10
Date 2	30,0	10,-5	-10,-10

2. Symmetric mixed equilibrium
    a. Suppose opponent plays strategy (p_0, p_1, p_2)
    b. Expected response utility
        i. $EU(0) = 0$
        ii. $EU(1) = 15p_0 - 5p_1 - 5p_2$
        iii. $EU(2) = 30p_0 + 10p_1 - 10p_2$
    c. Indifference conditions
        i. Indifferent between 0 and 1 if $15p_0 - 5p_1 - 5p_2 = 0 \Leftrightarrow 3p_0 = p_1 + p_2$
        ii. Indifferent between 1 and 2 if $15p_0 - 5p_1 - 5p_2 = 30p_0 + 10p_1 - 10p_2 \Leftrightarrow$ $3p_0 + 3p_1 = p_2$
    d. Mixed equilibrium
        i. Indifferent between all three if $p_1 = -3p_1$; only true if $p_1 = 0$.
        ii. Indifference then also requires $3p_0 = p_2$, or $(p_0, p_1, p_2) = \left(\frac{1}{4}, 0, \frac{3}{4}\right)$.
3. Implications
    a. Both firms exit with probability $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$
    b. Both stay with probability $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$

Application: Good Samaritan service

1. Good Samaritan (e.g., 911 calls, traffic accident witness, etc.)
2. Hawk-dove / Chicken

	Call 911	Do nothing
Call 911	(-1,-1)	(-1,0)
Do nothing	(0,-1)	(-100,-100)

3. Equilibrium
    a. If P2 plays $(p, 1-p)$:
    b. $EU_1(911) = -1$
    c. $EU_1(not) = -100(1-p)$
    d. Symmetric equilibrium
        i. Indifferent if $p = .99$
        ii. This is symmetric equilibrium
        iii. Crashes occur with probability $.01^2 = 10^{-4}$
        iv. Both call with probability $.99^2 \approx .98$
4. Bystander's dilemma
    a. With $n+1$ players, receive 0 utility if someone else calls 911, -1 if you call 911, -100 if no one calls.
    b. Expected utility if all call 911 with probability p:
        i. $P(0) = (1-p)^n$
        ii. If i does not call: $EU_i = -100P(0) = -100(1-p)^n$
        iii. If i does call: $EU_i = -1$
    c. Equilibrium
        i. Indifferent if $.01 = (1-p)^n \Leftrightarrow p^* = 1 - e^{\frac{1}{n}\ln .01}$
    d. Large n
        i. Not surprising: p^* decreases with n, tends to $p^* \to 0$
        ii. Surprising: $P(0) = (1-p^*)^{n+1} = e^{\frac{n+1}{n}\ln .01}$ increases with n
        iii. (Don't worry: $P(0) \to e^{\ln .01} = .01$)

L8 Sequential games

Game trees

1. Decision nodes, root, branches, terminal nodes
2. Strategies specify behavior at every note (complete contingency plan)
3. Nature moves/chance nodes
4. Information sets
5. Perfect information: a game where all information sets are singletons
6. Examples: Amazon considers entering (organic) grocery market, where Walmart is dominant.
    a. Version 1 (Walmart response): if Amazon exits (0,5); if enters, Walmart plays tough (-2,-1) or accommodate (1,2).
    b. Version 2 (Amazon counter-response): Observing Walmart's tough response, Amazon plays tough (-2,-1) or accommodates (-3,1); observing accommodating response, Amazon plays tough (0,-3) or accommodates (1,2).
    c. Version 3: Amazon doesn't observe Walmart's response (info set / simultaneous).
7. Recall equivalence of strategic form versus extensive form.

Backward induction

1. Sequential rationality: assumes that choices at each node are rational, expecting future nodes to be rational.
2. Version 1
    a. Two equilibria: (E,A) and (O, T)
    b. Sequentially rational equilibrium (E,A): Walmart accommodates, Amazon enters
3. Version 2:
    a. Six equilibria (three types): (OTT, T), (OTA,T), (OAT, T), (OAA,T), (ETA, A), (EAA, A)
    b. Sequentially rational equilibrium: Amazon plays tough if Walmart plays tough, accommodates if Walmart accommodates, so Walmart accommodates, Amazon enters. (This happened in 2017 when Amazon purchased Whole Foods.)
4. Backward induction: method for identifying sequentially rational strategies
    a. Kuhn's (and Zermelo's) Theorem: with perfect information, every finite game has a backward induction solution. If no players are indifferent between outcomes, solution is unique.

    b. Equivalent to IEDS: any strategy with irrational final-round behavior is dominated with another strategy that is identical except in the last round. Eliminating this, the same holds for irrational second-to-last round strategies, etc.

5. Big idea: forward thinking
6. Life lesson
    a. Career prep: standard practice is choose courses > major > internships/work > jobs but better to think job < internships < major < courses
    b. World War 2 Nazi conquest.
    c. “First They Came” poem by Pastor Martin Niemoller, antisemitic Nazi sympathizer until opposed church takeovers, sent to concentration camp.

        “First they came for the Communists
        And I did not speak out
        Because I was not a Communist
        Then they came for the Socialists
        And I did not speak out
        Because I was not a Socialist
        Then they came for the trade unionists
        And I did not speak out
        Because I was not a trade unionist
        Then they came for the Jews
        And I did not speak out
        Because I was not a Jew
        Then they came for me
        And there was no one left
        To speak out for me”

    d. Current applications: Ukraine takeover, precedents in Constitutional interpretation/norms

Application: research and development

1. Scientific discoveries/technological innovations are public goods: costly to one firm ($2.6 billion to develop new drug), but can be used by all.
2. Patents convert public good to private good (by awarding monopoly power to innovator).

3. Patent race model (8K Television)
    a. LG (L) and Samsugn (S) race to develop patent. Winner: $20 billion royalties.
    b. Development requires n steps, in order.
    c. Costs $2 billion to take one step (per time period: year/quarter), $7 billion to take two steps, $15 billion to take two steps forward.
        i. Harder to move quickly: overtime pay/bonuses, hire personnel, equipment slack (e.g. overnight computing)
    d. Alternate making decisions (snoop on competitor).
    e. Let l and s denote steps remaining for L and S.
4. Cartel: how will research proceed if firms collude?
    a. One firm stays out, other proceeds one step per period, share profits.
5. Backward induction
    a. One step away: finish
    b. Two/three steps away: finish
        i. What if already spent $18 billion? Still finish! (Sunk costs will be lost if lose patent race.)
    c. Graph: trigger zone, where fewer than $l \leq 3$ or $s \leq 3$.
    d. Safety zone: where $l \leq 3 < s$ or $s \leq 3 < l$, one firm drops out (can't compete).
    e. Trigger zone 2
        i. If $(l, s) = (4,4)$ then either firm can enter safety zone for $2 billion. Competitor will drop out, then spend $6 to move incrementally toward completion.
        ii. Same logic if $l \leq 5$ or $s \leq 5$.
        iii. Smaller than trigger zone 1: if $l = 6$ then can reach safety zone for $15 billion, but completing patent will then require additional $6 billion, not worth $20 billion prize.
    f. Safety zone 2: where $l \leq 5 < s$ or $s \leq 5 < l$.
    g. Trigger zone 3: where $l \geq 7$ and $s \geq 7$.
    h. Safety zone 3: where $l \leq 7 < s$ or $s \leq 7 < l$.
    i. Trigger zone 4: where $l \geq 8$ or $s \geq 8$. Trigger zone 5: where $l \geq 9$ or $s \geq 9$. Trigger zone 6: where $l \geq 10$ or $s \geq 10$.
    j. Thus, predict: second mover gives up.
    k. Homework asks you to solve same game with different parameter values.

6. Extensions
    a. Asymmetric ability: lopsided trigger/safety zones: less efficient firm should get head start (if possible) or give up.
    b. More valuable patent creates larger trigger zones.
    c. Uncertainty: firm with slight (current) disadvantage may stay in, hoping for better luck.
    d. Public subsidies: compete with foreign R&D
        i. Unless foreign government subsidizes too: in that case, governments effectively play R&D game themselves.

Limits of backward induction

1. Play centipede game
    a. Players alternate take/grow "pie"
    b. Payoffs: (1,0), (0,2), (3,1), (2,4), (5,3)
    c. Analysis: better to keep growing, but take last round, so take second-to-last round, …, so take immediately.
    d. 100 rounds (source of "centipede" name): say at $20/pt., how far would you let it grow?
2. Difficult to guarantee that everyone will act rationally at every step.

L9 Credibility

Commitment

1. Decision theory: more options is always better (hence value of money, given free disposal).
2. Game theory: more options can be worse.
3. Version 1 example: if Walmart can commit to being tough, Amazon won't enter
4. Version 2 example: if Amazon can commit to being tough, Amazon will accommodate.

Commitment devices

1. Examples of commitment devices: contract, bank teller vault access, contracted trust, constitutional prohibition on terrorist negotiation, bet on weight loss (game against self), cultural norms of honesty (e.g. honor code, temple recommends).
2. Application: "poison pills" in corporate governance
    a. Sometimes mergers/acquisitions are friendly, sometimes hostile.
    b. Shareholder rights plan (or "poison pill") can deter hostile takeover.

i. Require costly management buyout payments.
ii. Allow board to dilute shares by issuing shares when buyout offer.
iii. Prohibit management from entertaining certain competitors' offers.
iv. Can also stagger board members' tenure.

c. Less common today, but common in 1990s.
d. In 2004, Anheuser-Busch (brewer with 30% market share) let poison pill provisions expire. In 2006, de-staggered board. InBev (Belgian-Brazilian brewer) seized the opportunity, bought out in 2008.

Subgame perfect equilibrium

1. Equilibrium *refinements*. When there are multiple Nash equilibria, we need way to choose. One possibility we've seen is symmetric equilibrium (only in symmetric games).
2. Subgame-perfect Nash equilibrium (SPNE): Nash equilibrium that is sequentially rational.
   a. Equivalently, Nash equilibrium in every subgame.
3. Example Version 3: Amazon doesn't observe Walmart's response (info set / simultaneous).
   a. Information set.
   b. Two equilibria in final subgame: (tough, tough) and (accommodate, accommodate).
   c. Two equilibria in game: (EA,A), (OA,T), (OT,T).
   d. Only (EA,A) and (OT,T) are subgame perfect.
4. With perfect information, SPNE are all backward induction solutions.
5. Comments
   a. Forward looking: costs are sunk, bygones are bygones. However we got here, we move forward rationally and expect others to do the same.
   b. Note this also implies an expectation that everyone will behave rationally *even* on paths that can only be reached irrationally.
   c. Subgame perfection slightly broader than backward induction because it applies to games that continue indefinitely, as we'll see in later sections.

L10 Finitely repeated games

1. Stage game
2. Finitely repeated games

a. Weekly Treasury bill auction: banks submit bids to buy T bills, then Treasury announces winners.
    i. Single-price auction: all bills sold at fixed price (2- and 5-year Treasury notes).
    ii. Multiprice auction: sell bills to highest bidders first, leftovers (if any) to lower bidders (13- and 26-week T bills, 3- and 10-year notes).
    iii. Ongoing (infinitely), but if managers paid annually then game (sort of) effectively ends after 52 weeks.
b. Drug patent races: same firms repeatedly produce drug patents, generic versions.

3. Infinitely repeated games
    a. NASDAQ market making: buyers and sellers trade stocks, then return next day (or next hour) to trade again.
    b. OPEC nations decide oil production, prices
    c. Pork-barrel spending and political favors: legislators allocate budget every year (forever).
    d. Trade agreements: trade barrier (e.g., import tariff) helps domestic industries but hurts foreign industries. Countries play prisoner's dilemma.
4. Once-repeated prisoner's dilemma
    a. Conjecture: by not confessing in first round, signal willingness to collude in second round.
    b. In last period, dominant strategy to confess.
    c. Thus, reward for collusion is not credible.
    d. Subgame perfect Nash equilibrium (unique): confess both periods.
5. Finitely repeated prisoner's dilemma
    a. Same logic: confess last period, therefore second-to-last, …, confess first period.
6. Modified repeated prisoner's dilemma: partial confession.

	Confess	Partial	Not
Confess	0,0	3,-1	7,-2
Partial	-1,3	3,3	6,0
Not	-2,7	0,6	5,5

    a. Utility of (P,P) between (C,C) and (N,N). If both partly confess, no benefit of full confession (police have sufficient evidence). If one does not confess, moderate punishment/reward for partial confession, larger punishment/reward for full confession.

b. Stage game equilibrium: (C,C) or (P,P). (Both are symmetric.)

c. How many strategies does a player have? 27!

    i. "Strategy" requires full contingency plan.

    ii. Example 1: play C in round 1, play C in round 2 if opponent played C, P in round 2 if opponent played P, and N in round 2 if opponent played N.

    iii. Example 2: Example 1: play C in round 1, play C in round 2 if opponent played C and P in round 2 if opponent played P or N.

d. Opportunity for rewards/punishments: condition period 2 behavior on opponent's period 1 behavior.

e. Subgame perfection: still must play equilibrium in period 2: C or P.

f. Better equilibrium / worse equilibrium still allow rewards / punishments.

g. Is (N, C/C/P) a symmetric SPE?

    i. No profitable period 2 deviation (since (C,C) and (P,P) are stage game equilibria).

    ii. Current utility U=5+3=8

    iii. Deviation U(P, C/C/P)=6+0=6.

    iv. Deviation U(C, C/C/P)=7+0=7.

h. Lesson: future rounds (can) change incentives in earlier rounds (e.g., give opportunities to incentivize good behavior).

i. Is (N, P/C/C), (C, C/C/P) an (asymmetric) SPE?

    i. No profitable period 2 deviation (since (C,C) and (P,P) are stage game equilibria).

    ii. P1 current utility U1=-2+3=1, P2 current utility U2=7+3=10

    iii. P1 deviation U(C, P/C/C)=0+0=0.

    iv. P1 deviation U(P, P/C/C)=-1+0=-1.

    v. P2 deviation U(P, C/C/P)=6+0=6.

    vi. P2 deviation U(N, C/C/P)=5+0=5.

    vii. Can sometimes sustain strange behavior by credibly threatening to punish anything else.

7. T-period version

    a. Is (N,C/C/N,C/C/N,…,C/C/N,C/C/P) a symmetric SPE?

        i. Yes: deviating gets even more severe punishment than in 2-period version.

b. Is (C, C/C/C) a symmetric SPE?

    i. Yes: deviating in any stage gets lower payoff in that stage.

8. General principles

    a. True or false: in *any* T-period repeated game (not just prisoner's dilemma), one SPNE is to play an equilibrium of the stage game in *every* stage.

        i. True: can't do better in any stage game (let alone multiple stage games) from deviating.

        ii. Don't need to choose the same equilibrium in every stage; could alternate or rotate with any frequency.

        iii. Since *any* sequence of stage game equilibria constitutes SPNE, multiplicity becomes more of a challenge in repeated games.

    b. True or false: if a stage game has a unique Nash equilibrium, there is a unique SPNE, repeating the stage game equilibrium.

        i. True: unique outcome in round T, and therefore round T-1, etc.

        ii. For example, repeating prisoner's dilemma T times never fosters cooperation.

    c. Future (even non-equilibrium) rewards can thus be used to motivate good behavior in earlier periods, but only credible when there are multiple equilibria (since only equilibria can credibly be offered as rewards/punishments).

    d. Note that, in equilibrium, punishments play important role but are never implemented. Still, credibility matters: if deviation really wouldn't be punished anyway, punishment threat is ineffective.

9. Parenting tip: only threaten what you're actually willing to implement (e.g. nothing outlandish like "you'll be grounded for a month").

Application: Treasury bill auctions

1. Single price or multi-price? Suppose Treasury wants to maximize collections.
2. Model assumptions

    a. Treasury has 100 bills to sell, repeated every week (for 52 weeks).

    b. Two banks can each offer to buy 50 or 75, at price h or price l, care only about profit (per unit, $\pi_l > \pi_h$).[Drop 50 option, since dominated?]

    c. If banks bid same price/quantity, sell 50 each at bid price.

    d. If banks bid same price but different quantities, sell 60 and 40 at bid price.

e. If banks bid different prices, sell full bid to high price bidder, remainder to low price bidder (at high price then low price in multi-price auction, at low price in single price auction).

3. Dominated strategies: In either auction, (75, h) dominates (50, h) and (75, l) dominates (50, l): no change in price, may sell more.
4. Payoffs
   a. Single price auction

	$75, h$	$75, l$
$75, h$	$50\pi_h, 50\pi_h$	$75\pi_l, 25\pi_l$
$75, l$	$25\pi_l, 75\pi_l$	$50\pi_l, 50\pi_l$

   b. Multi-price auction

	$75, h$	$75, l$
$75, h$	$50\pi_h, 50\pi_h$	$75\pi_h, 25\pi_l$
$75, l$	$25\pi_l, 75\pi_h$	$50\pi_l, 50\pi_l$

   c. Same unless bid different prices.
5. Competitive case: $50\pi_h > 25\pi_l$ (care more about quantity than price)
   a. Single price auction
      i. h is dominant strategy, (h,h) is unique stage game equilibrium, repeated in unique subgame perfect equilibrium.
      ii. Good for Treasury: banks can't make credible deals to keep prices low.
   b. Multi-price auction
      i. If $50\pi_l > 75\pi_h$ then (h,h) and (l,l) are both equilibria.
      ii. One subgame perfect equilibrium: (l,l) every period (with h enforcing deviations).
6. Collusive case: $50\pi_h < 25\pi_l$ (care more about price than quantity)
   a. Multi-price auction
      i. Lowballing: both banks offer l (dominant strategy)
   b. Single price auction
      i. Mixed equilibrium, $p^* = \frac{2\pi_l - 3\pi_h}{\pi_l - \pi_h}$
         1. $50\pi_h p + 75\pi_h(1-p) = 25\pi_l p + 50\pi_l(1-p)$

            2. Unique stage equilibrium, repeated in unique subgame perfect equilibrium.
            3. Better for Treasury, since banks sometimes pay high price.

7. Summary
    a. Treasury always prefers single price auction.
        i. In competitive case, always yields high price.
        ii. In collusive case, sometimes yields high price.
        iii. Ironic, since multi-price auction specifically designed to encourage high bidding.
            1. Low bidding in multi-price auction precisely because no one wants to be sole high bidder.
            2. In single price auction, willing to bid higher, since price only sometimes depends on bid. (Sometimes, both banks end up bidding higher.)