HW 26

P1

a) $G(\varepsilon)$ has to be the octant volume divided by Δp^3. Thus

$$G_i = \frac{1}{8}\frac{4\pi}{3}\left(\frac{p_i}{\Delta p}\right)^3 = \frac{p_i^3 a^3}{6\pi^2\hbar^3} = \frac{(2m\varepsilon_i)^{3/2}V}{6\pi^2\hbar^3}$$

the momentum states and energy levels are so closely related that $\varepsilon_i \to \varepsilon$ $G_i \to G(\varepsilon)$. This is for macroscopic systems

b) $dG(\varepsilon) = g(\varepsilon)\,d\varepsilon$

$$\boxed{g(\varepsilon) = \frac{\partial G(\varepsilon)}{d\varepsilon} = \frac{d}{d\varepsilon}\frac{(2m\varepsilon_i)^{3/2}V}{6\pi^2\hbar^3} = \frac{mV}{2\pi^2\hbar^3}(2m\varepsilon)^{1/2}}$$

c) $Z_{sp} = \sum g_i e^{-\varepsilon_i/k_BT} \to \int g(\varepsilon) e^{-\varepsilon/k_BT}\,d\varepsilon$

$\xi = \varepsilon/k_BT$

$$\int g(\varepsilon) e^{-\varepsilon/k_BT}\,d\varepsilon = \int \left({mV(2m\varepsilon)^{1/2}}/{2\pi^2\hbar^3}\right) e^{-\varepsilon/k_BT}\,d\varepsilon$$

$$= {2}/{\sqrt{\pi}}\, V\left({mk_BT}/{2\pi\hbar^2}\right)^{3/2} \int_0^\infty \xi^{1/2} e^{-\xi}\,d\xi$$

$$= {2}/{\sqrt{\pi}}\, V\left({mk_BT}/{2\pi\hbar^2}\right)^{3/2} \cdot {\sqrt{\pi}}/{2} = Vn_Q$$

$$\boxed{Z_{sp} = Vn_Q \quad \text{where } n_Q = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}}$$

Z_{sp} is the continuous form of the single-particle partition function. We take it's integral to get rid of the Σ and then prove that $Z_{sp} = Vn_Q$

P2

a) $P_i = \frac{g_i e^{-\varepsilon_i/k_BT}}{Z_{sp}} \rightarrow P(\varepsilon)\,d\varepsilon = \frac{g(\varepsilon)}{Z_{sp}(\varepsilon)} e^{-\varepsilon/k_BT}$

continuous probability that a particle has an energy between ε and $\varepsilon + d\varepsilon$

$$P(\varepsilon)d\varepsilon = \frac{g(\varepsilon) e^{-\varepsilon/k_BT}}{Z_{sp}(\varepsilon)} d\varepsilon = \left\{ \frac{mV(2m\varepsilon)^{1/2}}{2\pi^2\hbar^3} \right\} \frac{e^{-\varepsilon/k_BT}}{V n_Q(T)} d\varepsilon$$

$$\boxed{= \frac{2}{\sqrt{\pi}} \frac{\varepsilon^{1/2}}{(k_BT)^{3/2}} e^{-\varepsilon/k_BT}}$$

b) $P(v)dv = P(\varepsilon)\,d\varepsilon = P(\varepsilon(v)) \, \partial\varepsilon/\partial v \, dv$

$$= P(\varepsilon = mv^2/2)\, mv\, dv = 2\left(\frac{mv^2}{2\pi(k_BT)^3}\right)^{1/2} e^{-mv^2/2k_BT} mv\,dv$$

$$\boxed{= 4\pi v^2 \left(\frac{m}{2\pi k_BT}\right)^{3/2} e^{-mv^2/2k_BT}}$$

c) $P(\varepsilon)$ use $\int_0^\infty \sqrt{x}\, e^{-x} dx = \sqrt{\pi}/2$

$$\int_0^\infty P(\varepsilon) d\varepsilon = \int_0^\infty \frac{2}{\sqrt{\pi}} \frac{\varepsilon^{1/2}}{(k_BT)^{3/2}} e^{-\varepsilon/k_BT} d\varepsilon$$

$$= \frac{2}{\sqrt{\pi}} \underbrace{\left\{ \int_0^\infty \frac{\varepsilon^{1/2}}{(k_BT)^{3/2}} e^{-\varepsilon/k_BT} \right\}}_{\sqrt{\pi}/2} = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \boxed{1}$$

$P(v)$ use $\int_0^\infty x^2 e^{-x^2/2} dx = \sqrt{\pi}/\sqrt{2}$

$$\int_0^\infty P(v) dv = \int_0^\infty 4\pi v^2 \left(\frac{m}{2\pi k_BT}\right)^{3/2} e^{-mv^2/2k_BT} dv$$

$$\frac{4}{\sqrt{2^3}} = \sqrt{2} \qquad \frac{\pi}{\sqrt{\pi^3}} = \frac{1}{\sqrt{\pi}} \qquad \frac{\sqrt{2}}{\sqrt{\pi}} \underbrace{\int_0^\infty v^2 \left(\frac{m}{k_BT}\right)^{3/2} e^{-mv^2/2k_BT}}_{\sqrt{\pi}/\sqrt{2}}$$

$$\frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{2}} = \boxed{1}$$