Lorenz Equations and Chaotic Orbits

MATH-3550 - Differential Equations

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Introduction

Chaotic behavior was demonstrated using simple differential equations in the 1960's paper by MIT meteorologist Edward Lorenz titled "Deterministic Nonperiodic Flow." Based off of his work, Lorenz coined the term "butterfly effect" for non-periodic systems, suggesting that a butterfly flapping its wings in Beijing could affect the weather in the United States the next day. Lorenz simplified a nonlinear system of differential equations that described flow in a layer of uniform depth with a constant temperature difference between the two layers.

$$\begin{pmatrix} \frac{\mathrm{d}X}{\mathrm{d}t} \\ \frac{\mathrm{d}Y}{\mathrm{d}t} \\ \frac{\mathrm{d}Z}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} -\sigma X + \sigma Y \\ -XZ + rX - Y \\ XY - bZ \end{pmatrix}$$

X(t) is convective motion, or the transfer of heat and t is a dimensionless representation of time. Y(t) is the temperature difference between the two layers, and Z(t) is the deviation from a linear temperature across the planes. Additionally, σ, *r*, and *b* are constants, with σ representing the dimensionless Prandtl number defined as the ratio of momentum diffusivity to thermal diffusivity.

Lorenz' work is important for better understanding non-periodic systems such as the weather. Apart from weather systems, Lorenz was able to represent small changes causing large consequences. His work was also the foundation for future studies of chaos theory and forever changed the classical understanding of the natural world.

In this project, our team linearized Lorenz' equations so that his results could be reproduced at the same critical points. The system was numerically solved using Matlab, and the model was used to examine the system in varying parameters by changing the values of the constants.

Linearization

We began by observing Lorenz's simplified set of ordinary differential equations used to describe the flow in a layer of uniform depth with a constant temperature difference imposed between the upper and lower surfaces of the layer.

$$\begin{pmatrix} \frac{\mathrm{d}X}{\mathrm{d}t} \\ \frac{\mathrm{d}Y}{\mathrm{d}t} \\ \frac{\mathrm{d}Z}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} -\sigma X + \sigma Y \\ -XZ + rX - Y \\ XY - bZ \end{pmatrix}$$

In order to identify the equilibria of the nonlinear system, we set each derivative equal to zero and we found the nullclines. The difference from a system with only two variables is that with three dimensions the nullclines will be two dimensional instead of one-dimensional.

- $\frac{\mathrm{d}X}{\mathrm{d}t} = -\sigma X + \sigma Y = 0$

$$\sigma X = \sigma Y$$

$$X = Y \text{ (null plane)}$$

- $\frac{\mathrm{d}Y}{\mathrm{d}t} = -XZ + rX - Y = 0$

$$rX = XZ + Y$$
$$(r-1)X - XZ = 0$$

$$r - 1 = Z$$

- $\frac{\mathrm{d}Z}{\mathrm{d}t} = XY - bZ = 0$

$$XY = bZ$$

$$Z = \frac{X^2}{b}$$

Assuming $r \geq 1$, and working with these equations, we were able to find three equilibria:

$$(0, 0, 0)$$

$$(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$$

$$(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$$

When $0 < r < 1$, there is only one equilibrium point at (0, 0, 0) because $\sqrt{b(r-1)}$ is imaginary so the other two points, $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$, are not equilibrium points when $0 < r < 1$.

Next we linearized the system by creating the Jacobian matrix as follows. It is important to note that linearization only works near each equilibrium point and that the linearized system may not describe the behavior of the system further away from the equilibrium point.

$$J = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z & -1 & -X \\ Y & X & -b \end{pmatrix}$$

In order to characterize each equilibria, we set

$$det(J - \lambda I) = 0$$

For the equilibrium point at (0,0,0) we found

$$det(J-\lambda I) = \begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{vmatrix}$$

$$= (-b-\lambda)\begin{vmatrix} -\sigma-\lambda & \sigma \\ r & -1-\lambda \end{vmatrix}$$

$$= (-b-\lambda)(\lambda^2 + (\sigma+1)\lambda + \sigma(1-r))$$

This allows us to find the eigenvalues and characterize the equilibrium. There are three eigenvalues in this case:

$$\lambda = -b$$

or

$$\lambda = -\tfrac{\sigma+1}{2} \pm \tfrac{1}{2}\sqrt{(\sigma+1)^2 - 4\sigma(1-r)}$$

$$T = -(\sigma+1) \qquad D = \sigma(1-r)$$

Thus we found, at $r = 1$, T = -2, and $D = 0$ for a couple of the eigenvalues and the third eigenvalue is -b which is always negative, which shows that the point is a sink when r = 1. At $r > 1$, $D = -k\sigma$, for some constant k where k = (r-1). This makes the point a saddle for all values of r > 1 (assuming $\sigma > 0$). For any values of 0 < r < 1 the equilibrium point (0, 0, 0) will be a sink because all the eigenvalues are negative. This equilibrium point can only be a spiral sink if $r < -\frac{(\sigma-1)^2}{4\sigma}$, but we are only considering r values greater than 0. The conclusion is that for the equilibrium point (0, 0, 0), there is a bifurcation when r = 1.

For the remaining equilibria, both determinants are the same. So we will provide the work of one of the points.

$$det(J-\lambda I) = \begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ r-(r-1) & -1-\lambda & \sqrt{b(r-1)} \\ -\sqrt{b(r-1)} & -\sqrt{b(r-1)} & -b-\lambda \end{vmatrix}$$

$$= (-\sqrt{b(r-1)})\begin{vmatrix} \sigma & 0 \\ -1-\lambda & \sqrt{b(r-1)} \end{vmatrix}$$
$$+ (\sqrt{b(r-1)})\begin{vmatrix} -\sigma-\lambda & 0 \\ 1 & \sqrt{b(r-1)} \end{vmatrix}$$
$$- (b+\lambda)\begin{vmatrix} -\sigma-\lambda & \sigma \\ 1 & -1-\lambda \end{vmatrix}$$

$$= \lambda^3 + (\sigma+b+1)\lambda^2 + b(\sigma+r)\lambda + 2\sigma b(r-1)$$

To deal with this difficult equation, we defined
$f(\lambda) = \lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r - 1)$. We assumed $r \geq 1$ and that σ and b are positive fixed constants.

If $\lambda > 0$, then $f(\lambda)$ must be greater than zero, because each part of the equation is positive or zero.

If $\lambda = -(\sigma + b + r)$ then $f(\lambda)$ becomes $f(-(\sigma + b + r))$ which equals:

$$(-(\sigma+b+r))^3+(\sigma+b+1)(-(\sigma+b+r))^2+b(\sigma+r)(-(\sigma+b+r))+2b\sigma(r-1).$$

We assume that $r \geq 1$. This means that $(\sigma + b + r) \geq (\sigma + b + 1)$.

So, $(-(\sigma + b + r))^3 \leq (\sigma + b + 1)(-(\sigma + b + r))^2$.

If $b(\sigma + r)(-(\sigma + b + r)) + 2b\sigma(r - 1) < 0$, then it would prove that $f(-(\sigma + b + r)) < 0$.

$$b(\sigma + r)(-(\sigma + b + r)) + 2b\sigma(r - 1)$$

$$= -b\sigma^2 - b^2\sigma - 2rb\sigma - b^2r - br^2 + 2rb\sigma - 2b\sigma$$

$$= -b(\sigma^2 + b\sigma + br + r^2 + 2\sigma) \leq 0$$

$$f(-(\sigma + b + r)) < 0.$$

From these two proofs, we set boundaries for some eigenvalue λ_1 which must exist, such that, $-(\sigma + b + r) \leq \lambda_1 \leq 0$. This is because of the intermediate value theorem. f is continuous and so if $f(\lambda) > 0$ when $\lambda \geq 0$ and if $f(\lambda) < 0$ when $\lambda = -(\sigma + b + r)$ then that has to mean that there is some value of λ that exists between 0 and $-(\sigma + b + r)$ such that $f(\lambda) = 0$ because 0 is between positive numbers and negative numbers.

For a generic 3x3 matrix A,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The characteristic equation is given by two equations:

- $\lambda^3 - trace(A)\,\lambda^2 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{13}a_{31} - a_{12}a_{21} - a_{23}a_{32})\,\lambda - detA = 0$
- $\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\,\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\,\lambda - \lambda_1\lambda_2\lambda_3 = 0$
  - Assuming eigenvalues λ_1, λ_2, λ_3 (possibly complex)

We then let λ_2 and λ_3 be purely imaginary. For example, $\lambda_2 = \beta i$ and $\lambda_3 = -\beta i$. It follows that $\lambda_2 + \lambda_3 = 0$. We also knew that $\lambda_1 = trace(A) = -(\sigma + b + 1)$ from an earlier calculation. We used the previous assumptions to find a formula for the critical value of r, r_c:

$$\lambda^3 - \lambda_1\lambda^2 + \beta^2\lambda - \beta^2\lambda_1 = \lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r - 1)$$

$$\beta^2 = b(\sigma + r) \text{ and } -\beta^2 = \frac{2\sigma b(r-1)}{\sigma + b + 1}$$

$$b(\sigma + r)(\sigma + b + 1) = 2\sigma b(r-1)$$

$$\sigma^2 + \sigma b + \sigma + r\sigma + rb + r = 2\sigma r - 2\sigma$$

$$\sigma^2 + \sigma b + 3\sigma = (\sigma - b - 1)r$$

$$\text{Thus we found } r_c = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

Note that in order for $r_c > 0$ one must assume that $\sigma > b + 1$ due to the formula found for r_c.

Finally, we considered $g(r) = f(-(\sigma + b + 1))$ as a function of r.

$$g(r) = f(-(\sigma + b + 1)) = -b(\sigma + r)(\sigma + b + 1) + 2\sigma b(r-1)$$

$$\text{Thus, } g'(r) = 2\sigma b - b(\sigma + b + 1) = \sigma b - b^2 - b = b(\sigma - b - 1)$$

So $g'(r) > 0$ if $\sigma > b + 1$. This gives us the condition that $g(r_c) = 0$ <u>if and only if</u> $\lambda_1 = -(\sigma + b + 1)$. So when $r > r_c$ then $g(r) > 0$ as the point is moving to the right on the graph. Recall that $f(-(\sigma + b + r)) < 0$. However, if $r > r_c$ then $f(-(\sigma + b + 1)) > 0$. So by the intermediate value theorem, $-(\sigma + b + r) < \lambda_1 < -(\sigma + b + 1)$, and the eigenvalue λ_1 becomes more negative.

Under previous conditions, we had $\lambda_1 + \lambda_2 + \lambda_3 = -(\sigma + b + 1)$. This old scenario assumed $r = r_c$ and $\lambda_2 + \lambda_3 = 0$ and $\lambda_1 = -(\sigma + b + 1)$. The new scenario assumes $r > r_c$, thus $\lambda_1 < -(\sigma + b + 1)$. As a result, $\lambda_2 + \lambda_3 > 0$ which means that one of these eigenvalues must have a positive real part to make this true. Since we cannot have a positive real number, we must have a complex number with a positive real part. This results in a spiral source.

In conclusion, the system is not stable when $r > r_c$ and $\sigma > b + 1$.

Reproduction of Lorenz' Results

For the next part of our project, we were able to reproduce Lorenz' results by assuming $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$. Under these parameters, the critical value of r was found as follows:

$$r_c = \frac{\sigma(\sigma+b+3)}{\sigma-b-1} = \frac{10(10+8/3+3)}{10-8/3-1} = \frac{470}{19} \approx 24.74$$

The eigenvalues at (0,0,0) were found using the equations we found for λ previously. Thus, $\lambda = -b = -\frac{8}{3}$ is one eigenvalue. The other two eigenvalues were found from the equation

$$\lambda = -\tfrac{\sigma+1}{2} \pm \tfrac{1}{2}\sqrt{(\sigma+1)^2 - 4\sigma(1-r)}$$

After plugging in the values for the parameters and solving for λ, we found that the remaining eigenvalues are $-\frac{11}{2} \pm \frac{1}{2}\sqrt{1309} \approx 12.59,\ -23.59$. The trace was given by $-(\sigma+1) = -11$ and the determinant was given by $\sigma(1-r) = -270$. Thus confirming we have a saddle at these values.

For the other equilibria points, $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$, $\lambda_1 = -(\sigma + b + 1) = -(10 + \frac{8}{3} + 1) = -\frac{41}{3}$.

Using MatLab were were able to solve the system with the given parameter values and initial condition $(X(0), Y(0), Z(0)) = (0, 1, 0)$. Using the improved-Euler's method, it was possible to reproduce the graphs that Lorenz created.

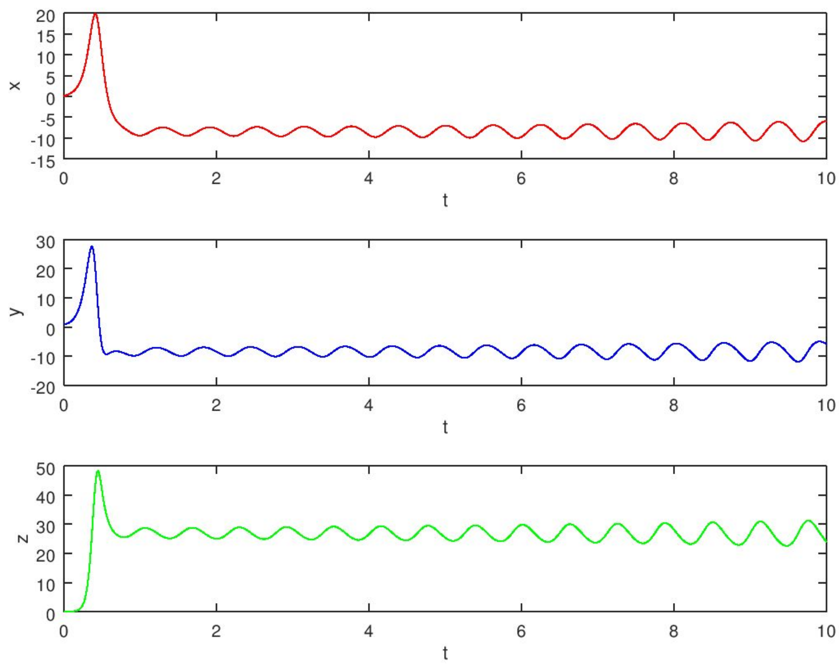

Figure 1: Plots of X, Y, and Z versus t. For the parameters σ = 10.0, b = 8/3, r = 28.

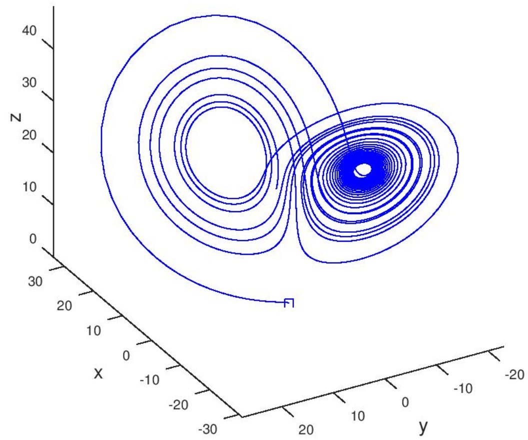

Figure 2: 3D Plot of X,Y,Z for the parameters σ = 10.0, b = 8/3, r = 28 with initial conditions of X(0) = 0, Y(0) = 1, Z(0) = 0, in the time interval from t = 0 to t = 25.

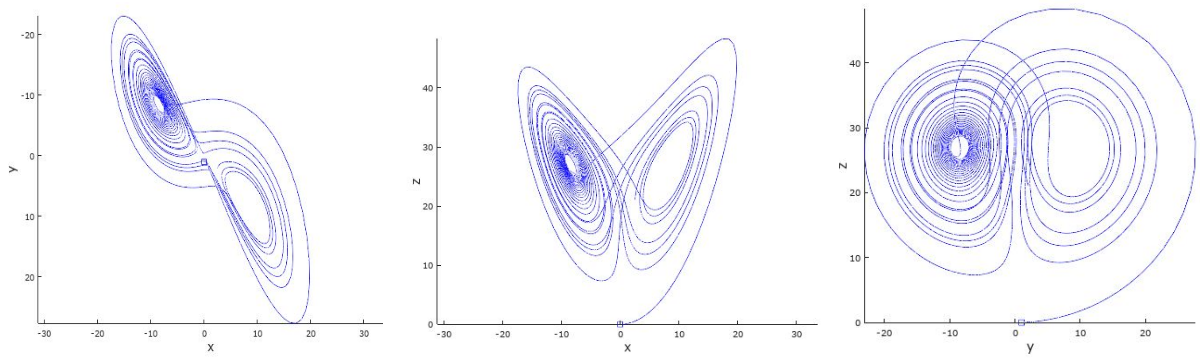

Figure 3: Numerical solution of the convection equations for $0 \le t \le 25$ and parameters σ = 10.0, b = 8/3, r = 28, with initial conditions of X(0) = 0, Y(0) = 1, Z(0) = 0. Graphs in order of top left, top right, and bottom are Y vs. X, Z vs. X, and Z vs. Y.

The solution curve goes between the two equilibrium points as predicted, but it is interesting to see that for a while the solution curve moves toward the unstable equilibrium points before it gets too close and starts to move outwards towards the other equilibrium point.

Sub-critical and Super-critical Conditions

For the last part of the project, we examined the difference in the system behavior for sub-critical ($r < r_c$) and super-critical ($r > r_c$) conditions. Thus, we changed some of the parameters in our code.

If we keep the same parameters as before, we can adjust r to observe changes in system behavior. Since $r_c \approx 24.74$, and our previous example had r at 28, we observed a super-critical condition. To observe the sub-critical condition we set $r < r_c$. The following plots were created by keeping the previous parameters and setting $r = 20$.

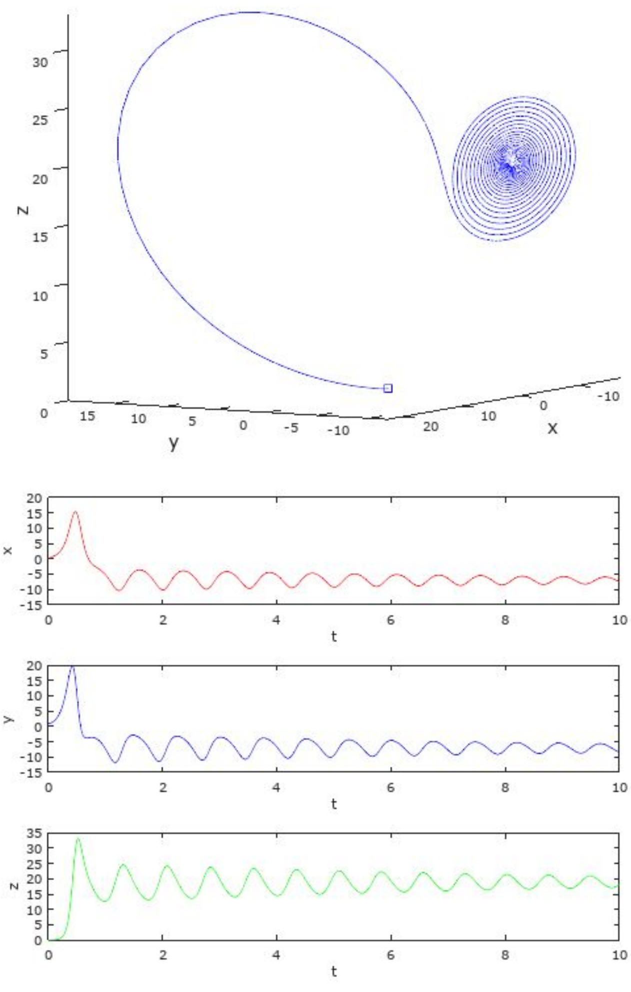

Figure 4: Numerical solution of the convection equations for $0 \leq t \leq 25$ and r=20.

The results above show how at a subcritical value of r, the two equilibrium points are spiral sinks which causes the solution curve to spiral into one of the equilibrium points. One can see the solution converging onto the equilibrium point in the graphs with X, Y, and Z vs. t.

While reproducing Lorenz' results, the values of the parameters were $\sigma = 10.0$, $b = 8/3$, $r = 28$. We also altered the values of σ and b to see how it would affect the system. By setting $\sigma = 10$ and b=10, we satisfy the condition $\sigma < b+1$. Thus we would expect a stable system of spiral sources. The following plots were created under these new parameters.

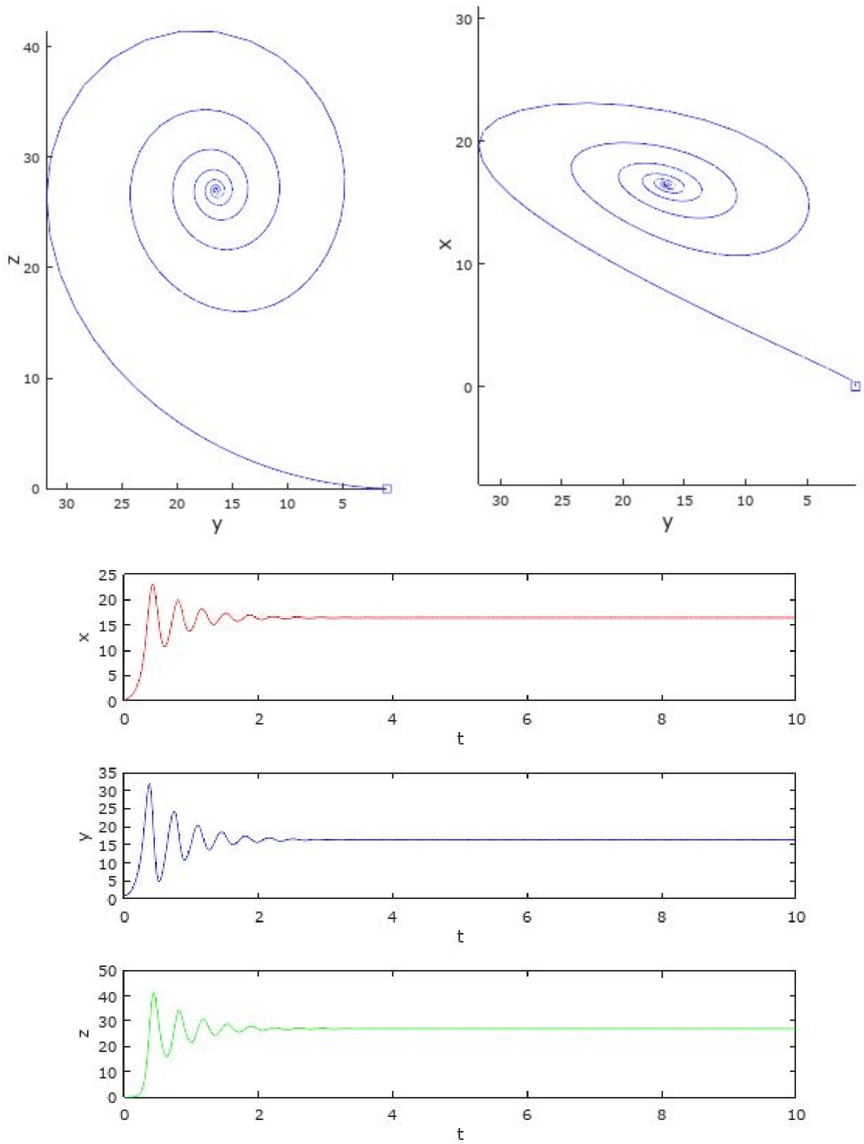

Figure 5: Numerical solution of the convection equations for $0 \leq t \leq 25$ and $\sigma = 10$, b=10, and r=28.

As expected, the result is a spiral sink which converges on the equilibrium point very quickly. This can be seen on the graphs that have X, Y, and Z versus time (t).

The second to last case that we thought would be interesting to explore is the one in which the value of r is negative. The parameters used are $\sigma = 10$, $b = (8/3)$, and $r = -1$. The time interval used is from t = 0 to t = 100. The same initial conditions were used as before.

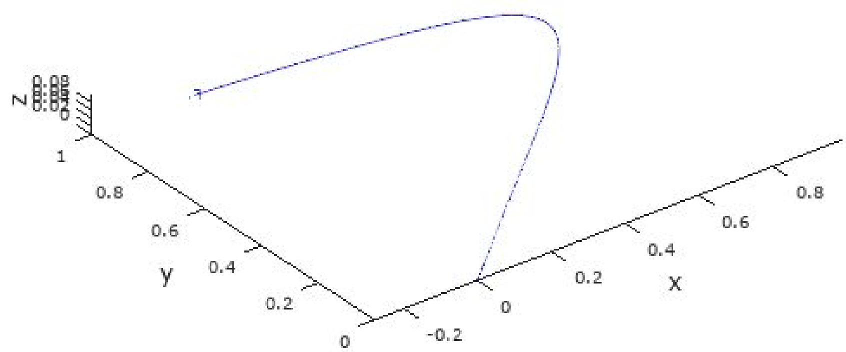

Figure 6: 3D Plot of X,Y,Z for the parameters σ = 10.0, b = 8/3, r = -1 with initial conditions of X(0) = 0, Y(0) = 1, Z(0) = 0, in the time interval from t = 0 to t = 100.

The plot above shows the equilibrium point (0, 0, 0) as a sink (it happens to be the only equilibrium point because r < 1) which is what the calculations would have us expect. The only logical place to go from here is to try changing the value of r so that $r < -\frac{(\sigma-1)^2}{4\sigma}$. Keeping the other parameters the same, the values of r that are necessary to make (0, 0, 0) a spiral sink are $r < -2.025$. So the parameters that we decided on are $\sigma = 10$, $b = (8/3)$, and $r = -5$. The time interval used is from t = 0 to t = 100. In the figure below, there is a clear spiral which is what was expected with calculations. This means that there is a bifurcation when $r < -\frac{(\sigma-1)^2}{4\sigma}$.

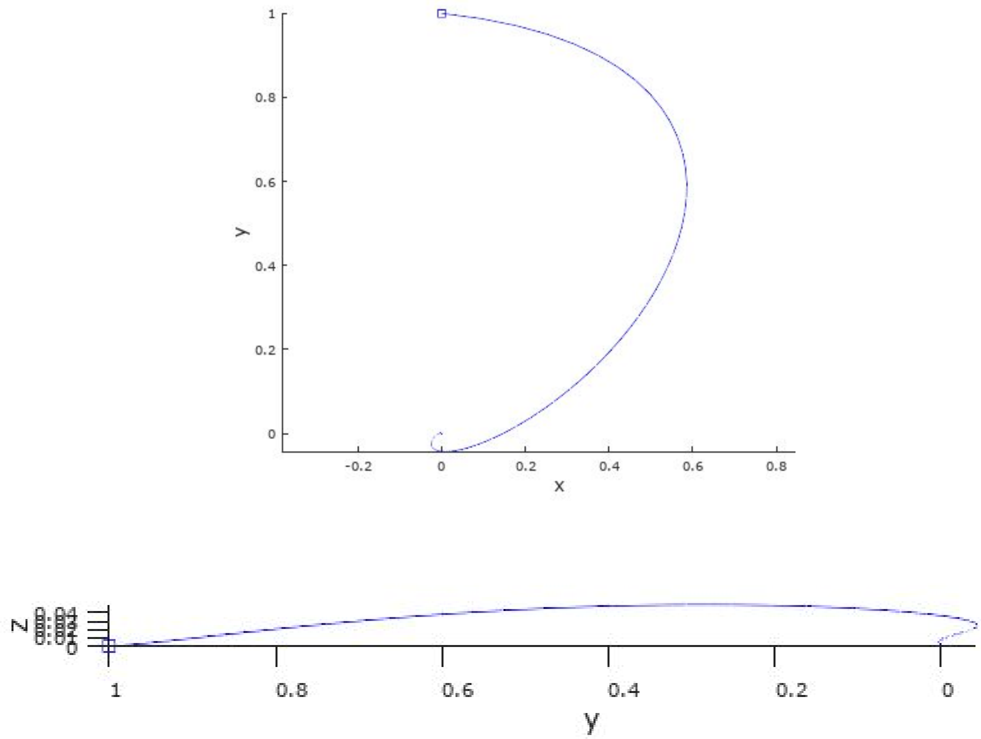

Figure 7: Numerical solution of the convection equations for $0 \leq t \leq 100$ and $\sigma = 10$, *b*=8/3, and *r* = -5. Top: graph of Y vs. X, bottom: graph of Z vs. Y

Conclusion

Lorenz' system was examined in three main parts. The system was linearized, critical points were classified, original data was reproduced, and the parameters were tested at subcritical and supercritical points. By finding nullclines, Lorenz' nonlinear system was manipulated so that three critical points could be found. The only equilibrium point was at (0,0,0). The other two points, $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$, and $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$ were imaginary when $0 < r < 1$. Lorenz' results were then linearized using the Jacobian, and each critical point was classified. The equilibrium point was a sink at 0 < r < 1 and a spiral sink at $r < -\frac{(\sigma-1)^2}{4\sigma}$ (This was not a concern as only values of r greater than 0 were considered). It was determined that when r=1, the system begins to branch towards the other two critical points. The intermediate value theorem was used to classify the other critical points. Because f is continuous and $f(\lambda) > 0$ when $\lambda \geq 0$, or $f(\lambda) < 0$ when $\lambda = -(\sigma + b + r)$, then there is some value of λ that exists between 0 and $-(\sigma + b + r)$ such that $f(\lambda) = 0$. When drafting a characteristic equation, λ_2 and λ_3 were imaginary and $\lambda_1 = trace(A) = -(\sigma + b + 1)$. This meant that an equation:

$$\lambda^3 - \lambda_1 \lambda^2 + \beta^2 \lambda - \beta^2 \lambda_1 = \lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r-1)$$

could be used to determine a critical r value where real numbers and results could be reproduced. We found that $r_c = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$ so long as one of the eigenvalues for λ_2 *or* λ_3 must have a positive real component which would result in a spiral source. Otherwise, the system would be unstable for $r > r_c$ and $\sigma > b+1$. Using Matlab and Improved Euler's Method, Lorenz' results were reproduced under the assumption $\sigma = 10$, $b = \frac{8}{3}$, $r = 28$, and $r_c \approx 24.74$. The other two eigenvalues were 12.59 *and* -23.59. Of course there is some error in the graphs that we developed because of the fact that improved Euler's method is an approximation method, so the results are not exact, but they serve the purpose of giving a general idea of what the solution might look like. Finally, the difference in the system behavior for sub-critical ($r < r_c$) and super-critical ($r > r_c$) conditions was examined by changing various parameters in the Matlab code. We observed the system behavior as it transitioned from unstable to stable and from regular sink to a spiral sink. The skills we have learned this past semester were influential in our ability to to find critical points of a non-linear differential system, linearize the system so that the points could be classified, and plot using matlab various conditions that changed the behavior of a system with a bifurcation point.

References

[1] E. Lorenz, Deterministic nonperiodic flow. J. Atmos. Sci., 20 (1963) 130-141.

[2] Barenghi, Carlo F. "Chaos with Matlab." www.mas.ncl.ac.uk/~ncfb/mat3.pdf.

[3] Johnson, Brody. Office Hours and Appointments.

Appendix

Matlab Codes:

Code 1:

```
% chaos2.m
% Aim: to solve the Lorenz equations and display a three dimensional plot of the solutions.
% dx/dt=sigma*(y-x); dy/dt=-x*z+r*x-y; dz/dt=x*y-b*z
%       Improved Euler's Method
% values of the parameters
```

```
  sig=10.0; b=(8/3); r= 28;
% t0 - initial value for t
  t0 = 0;
% x0 = x(t0) - initial value for x(t0)
  x0 = 0;
% y0 = y(t0) - initial value for y(t0)
  y0 = 1;
% z0 = z(t0) - initial value for z(t0)
  z0 = 0;
% dt - step size
  dt = 0.01;
% tu - upper limit of interval for solution
  tu = 20;
% fstring defines the function dy/dt = f(t,y)
  fstring1 = 'sig*(y-x)';
  eval(['f1 = @(t,x,y,z) ',fstring1,';']);

  fstring2 = '(-x*z)+(r*x)-y';
  eval(['f2 = @(t,x,y,z) ',fstring2,';']);

  fstring3 = '(x*y)-(b*z)';
  eval(['f3 = @(t,x,y,z) ',fstring3,';']);

% vectors for t and y
  t = [t0:dt:tu];
  x = zeros(size(t));
  y = zeros(size(t));
  z = zeros(size(t));
  x(1) = x0;
  y(1) = y0;
  z(1) = z0;

  for k=2:length(t)

        m1 = f1(t(k-1),x(k-1),y(k-1),z(k-1));
        n1 = f2(t(k-1),x(k-1),y(k-1),z(k-1));
        p1 = f3(t(k-1),x(k-1),y(k-1),z(k-1));

        u = x(k-1) + dt*m1;
```

```
        v = y(k-1) + dt*n1;
        w = z(k-1) + dt*p1;

        m2 = f1(t(k),u,v,w);
        n2 = f2(t(k),u,v,w);
        p2 = f3(t(k),u,v,w);

        x(k) = x(k-1) + dt*0.5*(m1+m2);
        y(k) = y(k-1) + dt*0.5*(n1+n2);
        z(k) = z(k-1) + dt*0.5*(p1+p2);
 end

 plot3(x,y,z,'b',x(1),y(1),z(1),'sb')
 xlabel('x','FontSize',14)
 ylabel('y','FontSize',14)
 zlabel('z','FontSize',14)
 axis('equal')
```

Code 2:

```
% Aim: to solve the Lorenz equations and display graphs: X, Y, and Z vs. t.
% dx/dt=sigma*(y-x); dy/dt=-x*z+r*x-y; dz/dt=x*y-b*z
%       Improved Euler's Method
% values of the parameters
 sig=10.0; b=(8/3); r=28;
% to - initial value for t
 to = 0;
% xo = x(to) - initial value for x(to)
 xo = 0;
% yo = y(to) - initial value for y(to)
 yo = 1;
% zo = z(to) - initial value for z(to)
 zo = 0;
% dt - step size
 dt = 0.01;
% tu - upper limit of interval for solution
 tu = 20;
% fstring defines the function dy/dt = f(t,y)
 fstring1 = 'sig*(y-x)';
 eval(['f1 = @(t,x,y,z) ',fstring1,';']);
```

```
  fstring2 = '(-x*z)+(r*x)-y';
  eval(['f2 = @(t,x,y,z) ',fstring2,';']);

  fstring3 = '(x*y)-(b*z)';
  eval(['f3 = @(t,x,y,z) ',fstring3,';']);
% vectors for t and y
  t = [t0:dt:tu];
  x = zeros(size(t));
  y = zeros(size(t));
  z = zeros(size(t));
  x(1) = x0;
  y(1) = y0;
  z(1) = z0;
  for k=2:length(t)

        m1 = f1(t(k-1),x(k-1),y(k-1),z(k-1));
        n1 = f2(t(k-1),x(k-1),y(k-1),z(k-1));
        p1 = f3(t(k-1),x(k-1),y(k-1),z(k-1));

        u = x(k-1) + dt*m1;
        v = y(k-1) + dt*n1;
        w = z(k-1) + dt*p1;

        m2 = f1(t(k),u,v,w);
        n2 = f2(t(k),u,v,w);
        p2 = f3(t(k),u,v,w);

        x(k) = x(k-1) + dt*0.5*(m1+m2);
        y(k) = y(k-1) + dt*0.5*(n1+n2);
        z(k) = z(k-1) + dt*0.5*(p1+p2);
  end

  subplot(3,1,1);
  plot(t,x,'r');
  xlabel('t');
  ylabel('x');
  subplot(3,1,2);
  plot(t,y,'b');
```

```
xlabel('t');
ylabel('y');
subplot(3,1,3);
plot(t,z,'g');
xlabel('t');
ylabel('z');
```