CS 70 - Foundations Of Applied Computer Science

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Reading Assignment 1

Vectors

CS70 - Chapter 1

Definition: A *vector space* V is any set of objects (*vectors*) on which two operations are defined: addition, and multiplication by real numbers called *scalars*. For all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and scalars α, β, the following properties must hold:

- Commutativity: $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ *and* $\alpha \boldsymbol{u} = \boldsymbol{u} \alpha$
- Associativity: $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$ *and* $\alpha (\beta \boldsymbol{u}) = (\alpha \beta) \boldsymbol{u}$
- Distributivity: $\alpha (\boldsymbol{u} + \boldsymbol{v}) = \alpha \boldsymbol{u} + \alpha \boldsymbol{v}$ *and* $(\alpha + \beta) \boldsymbol{u} = \alpha \boldsymbol{u} + \beta \boldsymbol{u}$
- Multiplicative Identity: $1 \boldsymbol{u} = \boldsymbol{u}$
- Additive Identity: *There is a zero vector* $\mathbf{0}$ *such that* $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$
- Additive Inverse: *There is a vector* $-\boldsymbol{u}$ *such that* $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$

Any collection of objects satisfying all of these properties is a vector space.

Definition: A *norm* is any function that assigns a number to each vector and satisfies these properties (for all vectors $\boldsymbol{u}, \boldsymbol{v}$ and scalars α):

- Non-Negativity: $|\boldsymbol{u}| \geq 0$
- Norm Of Zero Vector: $|\boldsymbol{u}| = 0 \iff \boldsymbol{u} = \mathbf{0}$
- Distributivity Of Multiplication: $|\alpha \boldsymbol{u}| = |\alpha| \, |\boldsymbol{u}|$
- Triangle Inequality: $|\boldsymbol{u}| + |\boldsymbol{v}| \geq |\boldsymbol{u} + \boldsymbol{v}|$

Definition: An *inner product* is any function that assigns to any two vectors $\boldsymbol{u}, \boldsymbol{v}$ a number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ satisfying the following properties:

- Symmetry: $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
- Non-Negativity Of Inner Product With Self: $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0$
- Inner Product Of Zero Vector: $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = \mathbf{0} \iff \boldsymbol{u} = \mathbf{0}$
- Distributivity Of Multiplication: $\langle \alpha \boldsymbol{u}, \boldsymbol{v} \rangle = \alpha \langle \boldsymbol{u}, \boldsymbol{v} \rangle$
- Distributivity Of Addition: $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$

Definition: A set of vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ is *linearly independent* if the equation

$$\alpha_1 \boldsymbol{u}_1 + \cdots + \alpha_n \boldsymbol{u}_n = \sum_{i=1}^{n} \alpha_i \boldsymbol{u}_i = \mathbf{0}$$

can only be satisfied by setting all the coefficients $\alpha_1 = \cdots = \alpha_n = 0$.

Numerical Methods

Preliminaries: Numbers & Sets

- Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
- Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Rational Numbers: $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$
- Real Numbers: Encompasses $\mathbb{Q}$ as well as *irrational* numbers like π and $\sqrt{2}$.
- Complex Numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$, where $i = \sqrt{-1}$

Definition: The *span* of a set $S \subseteq V$ of vectors is the set

$$\text{span } S = \{a_1 \boldsymbol{v}_1 + \cdots a_k \boldsymbol{v}_k : \boldsymbol{v}_i \in S \text{ and } a_i \in \mathbb{R} \text{ for all } i\}$$

Definition: The *dimension* of V is the maximal size $|S|$ of a linearly independent set $S \subset V$ such that span $S = V$. Any set S satisfying this property is called a *basis* for V.

Definition: Two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ are perpendicular, or *orthogonal*, when $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Definition: The *transpose* of a matrix $A \in \mathbb{R}^{n \times m}$ is a matrix $A^T \in \gtrdot \times \ltimes$ with elements $\left(A^T\right)_{ij} = A_{ji}$.

Geometry Of Linear Equations

CS70 - Chapter 2

Matrix Formulation - Put all of the coefficients into a box of numbers A and all of the constants into a column of numbers $\boldsymbol{b}$.

The original system of equations is represented as $A\boldsymbol{x} = \boldsymbol{b}$.

The "Row" View: The row view considers each equation (row of A and corresponding element in $\boldsymbol{b}$) of the linear system one by one.

The "Column" View: The column view considers the system of equations to be a single vector equation, producing a linear combination.

Do solutions always exist? No