CS 81 - Principles of Robot Design & Programming

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Homework 2 - Kinematics & Transformations

Purpose

In this individual assignment, you will put in practice forward, inverse kinematics, and transformations we saw in class, for a wheeled robot.

General Instructions

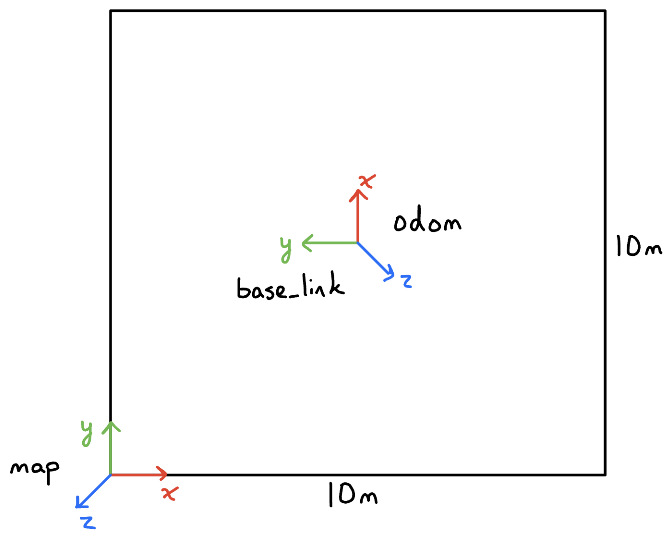
Answer to the following questions, showing all the drawings and intermediate math work by uploading a PDF file containing the work. Feel free to add comments in the text box. You can assume that each question consider the robot starting point as the initial one, i.e., it does not depend on the answer to previous questions.

Question 1

Assume that you have the Turtlebot 3 Burger modeled as a differential drive robot, in a rectangular warehouse ($10m \times 10m$). The warehouse has a flat floor and no obstacles. Its walls are aligned with respect to the compass points (north and east). The 'map' reference frame is attached to the bottom left corner of the warehouse, at position $0, 0, 0$, and with roll, pitch, yaw of $0, 0, 0$. The robot is turned on starting at the center of the warehouse, pointing towards the north.

Draw the reference frames of the 'map', 'odom', and 'base_link', specifying the conventions followed.

According to the conventions, the 'map' reference frame is oriented with respect to the north and east compass points (with $x = east$ and $y = north$). The 'odom' and 'base_link' reference frames are oriented following the right-hand rule, with x representing the 'forward' direction of the robot.



Write the homogeneous transformation matrices between 'map' and 'odom', and between 'odom' and 'base_link'.

To create the homogeneous transformation matrices between 'map' and 'odom', we consider the rotation and translation for each case.

$$^{map}T_{odom} = \begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) & 0 & 5 \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow {}^{map}\ T_{odom} = \begin{bmatrix} 0 & -1 & 0 & 5 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{odom}T_{map} = \begin{bmatrix} \cos\left(-\frac{\pi}{2}\right) & -\sin\left(-\frac{\pi}{2}\right) & 0 & -5 \\ \sin\left(-\frac{\pi}{2}\right) & \cos\left(-\frac{\pi}{2}\right) & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow {}^{odom}T_{map} = \begin{bmatrix} 0 & 1 & 0 & -5 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The homogeneous transformation matrices between 'odom' and 'base_link' are each I_4, as 'odom' and 'base_link' are similarly oriented (with respect to rotation) and start at the same location (no translation).

$$^{odom}T_{base\_link} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{base\_link}T_{odom} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 2

Looking at the information you collected, consider the distance between the wheels of the Turtlebot 3 Burger.

Given a velocity on the left wheel of $0.45\pi\, m/s$ and velocity on the right wheel of $0.55\pi\, m/s$ for 0.5 seconds, write in a document what the new pose of the robot in the map reference frame is.

Please show also the ICC in the map reference frame and update the homogeneous transformation matrix between 'odom' and 'base_link'.

There is a single point, the instantaneous center of rotation or instantaneous center of curvature (ICR or ICC), around which each wheel moves in a circular motion.

The rate of rotation around ICC must be the same for both wheels $w = \frac{v_r}{R+\frac{l}{2}}$, where R is the distance from the ICC to the center of the axis between the wheels and l is the distance between the wheels.

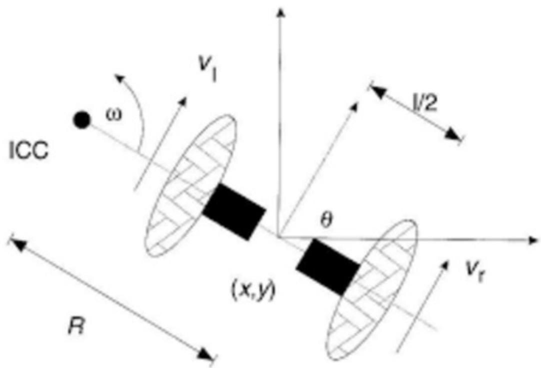
Thus, given that the distance between wheels is $l = 0.16\, m$ and $v_l = 0.45\pi\, m/s$ and $v_r = 0.55\pi\, m/s$, we may solve for angular velocity ω using the following formula: $\omega = \frac{v_r - v_l}{l}$

$$\omega = \frac{0.1\pi\, m/s}{0.16\, m} = \frac{5}{8}\pi\, rad/s \approx 1.963\, rad/s$$

Similarly, we may solve for the distance to the ICC R using the following formula: $R = \frac{l}{2}\left(\frac{v_r+v_l}{v_r-v_l}\right)$

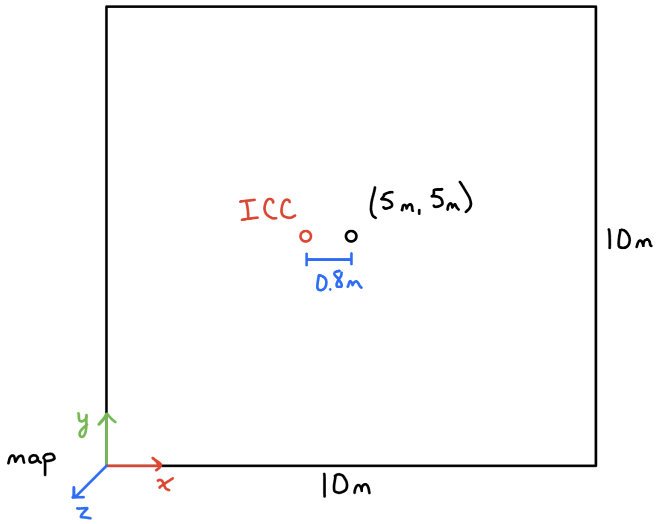
$$R = \frac{0.16\, m}{2}\left(\frac{\pi\, m/s}{0.1\pi\, m/s}\right) = \frac{4}{5}\, m = 0.8\, m$$

In the map reference frame, we may solve for the ICC using the following formula:
$ICC = [x - R\sin(\theta), y + R\cos(\theta)]$, with $\theta = \frac{pi}{2}$, according to the following image.



$$ICC = \left[x - R\sin\left(\frac{\pi}{2}\right), y + R\cos\left(\frac{\pi}{2}\right)\right] \longrightarrow ICC = [x - R, y]$$

Thus, the ICC is given as $ICC = [4.2\, m, 5.0\, m]$, as $x = y = 5\, m$ and $R = 0.8\, m$, in the map reference frame. This is shown as follows (on the next page).



To determine the new pose of the robot in the map reference frame, we use the following formula:

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - ICC_x \\ y - ICC_y \\ \theta \end{bmatrix} + \begin{bmatrix} ICC_x \\ ICC_y \\ \omega t \end{bmatrix}$$

Applying this to the given situation, with $t = 0.5\,s$, produces the following:

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} 0.556 & -0.831 & 0 \\ 0.831 & 0.556 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8\,m \\ 0\,m \\ \frac{\pi}{2}\,rad \end{bmatrix} + \begin{bmatrix} 4.2\,m \\ 5.0\,m \\ 0.982\,rad \end{bmatrix}$$

This gives us the pose of the robot in the map reference frame.

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} 4.645\,m \\ 5.665\,m \\ 2.553\,rad \end{bmatrix}$$

That is, in the map reference frame, the robot is located at $(x, y, z) = (4.645\,m, 5.665\,m, 0\,m)$, with $(roll, pitch, yaw) = (0\,rad, 0\,rad, 2.553\,rad)$.

To update the homogeneous transformation matrix between 'odom' and 'base_link', we consider the rotation and translation. The rotation is $0.982\,rad$, as determined previously via $\theta = \omega t$. The translation is $(x, y) = (0.665\,m, 0.355\,m)$, based on the location (pose) of the robot.

$${}^{odom}T_{base\_link} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0.665 \\ \sin(\theta) & \cos(\theta) & 0 & 0.355 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow {}^{odom}\,T_{base\_link} = \begin{bmatrix} 0.556 & -0.831 & 0 & 0.665 \\ 0.831 & 0.556 & 0 & 0.355 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For the transformation from 'odom' to 'base_link', we may take the inverse of the above transformation, either conceptually or mathematically.

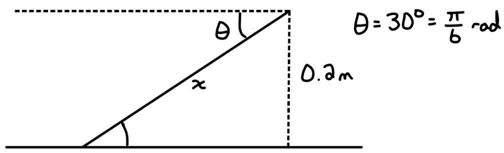
$${}^{base\_link}T_{odom} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & -0.665 \\ -\sin(\theta) & \cos(\theta) & 0 & 0.355 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow {}^{base\_link}\,T_{odom} = \begin{bmatrix} 0.556 & 0.831 & 0 & -0.665 \\ -0.831 & 0.556 & 0 & 0.355 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 3

Suppose a sonar sensor is in front of the robot at 30 degrees pitch rotation and translated of $0.2\,m$ along the z-axis of 'base_link'.

Write what the expected measurement from the sensor is, given that the floor is flat.

Given that the sonar sensor is in front of the robot at 30 degrees pitch rotation and translated $0.2\,m$ along the z-axis of 'base_link', the following diagram may be used to determine the expected measurement from the sensor, given that the floor is flat.



Thus, $x = \frac{0.2\,m}{\sin\left(\frac{\pi}{6}\right)} \longrightarrow x = 0.4\,m$, which is the expected measurement from the sensor.

Write such a point in the 'map' reference frame by applying the appropriate transformation.

To write such a point in the 'map' reference frame, we use the following series of transformations:

$$^{map}p = ^{map}T_{odom}\ ^{odom}T_{base\_link}\ ^{base\_link}T_{sensor}\ ^{sensor}p$$

That is, we aim to convert the point $(x, y, z) = (0.4\,m, 0\,m, 0\,m)$ to a point in the 'map' reference frame. The following matrices are known.

$$^{map}T_{odom} = \begin{bmatrix} 0 & -1 & 0 & 5 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^{odom}T_{base\_link} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^{sensor}p = \begin{bmatrix} 0.4\,m \\ 0\,m \\ 0\,m \\ 1 \end{bmatrix}$$

Thus, we simply need to determine the transformation $^{base\_link}T_{sensor}$. By the according translation $(\Delta x, \Delta y, \Delta z) = (0\,m, 0\,m, 0.2\,m)$ and rotation $(\Delta roll, \Delta pitch, \Delta yaw) = \left(0\,rad, \frac{\pi}{6}\,rad, 0\,rad\right)$, the transformation is the following (with $\theta = \frac{\pi}{6}$).

$$^{base\_link}T_{sensor} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.866 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ -0.5 & 0 & 0.866 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we may determine the point in the 'map' reference frame using the series of transformations.

$$^{map}p = \begin{bmatrix} 0 & -1 & 0 & 5 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ -0.5 & 0 & 0.866 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4\,m \\ 0\,m \\ 0\,m \\ 1 \end{bmatrix} \longrightarrow {}^{map}p = \begin{bmatrix} 5.0\,m \\ 5.346\,m \\ 0\,m \\ 1 \end{bmatrix}$$

Thus, the point that would result in the sensor's measurement (as given) is $(x, y, z) = (5.0\,m, 5.346\,m, 0\,m)$ in the map reference frame.

Question 4

An operator gives the robot a goal destination in the 'map' reference frame, $4.8\,m, 4.8\,m, -\frac{\pi}{4}$ (x, y, yaw) – the rest (z, roll, pitch) are 0.

Write such a point in the 'base_link' reference frame.

To write such a point in the 'base_link' reference frame, we use the following series of transformations.

$$^{base\_link}p = ^{base\_link}T_{odom}\; ^{odom}T_{map}\; ^{map}p$$

That is, we aim to convert the point $(x, y, z) = (4.8\,m, 4.8\,m, 0\,m)$ to a point in the 'base_link' reference frame. The following matrices are known.

$$^{base\_link}T_{odom} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^{map}T_{odom} = \begin{bmatrix} 0 & 1 & 0 & -5 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^{map}p = \begin{bmatrix} 4.8\,m \\ 4.8\,m \\ 0\,m \\ 1 \end{bmatrix}$$

Now, we may determine the point in the 'base_link' reference frame using the series of transformations.

$$^{base\_link}p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -5 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4.8\,m \\ 4.8\,m \\ 0\,m \\ 1 \end{bmatrix} \longrightarrow \; ^{base\_link}p = \begin{bmatrix} -0.2\,m \\ 0.2\,m \\ 0 \\ 1 \end{bmatrix}$$

Thus, the point is $(x, y, z) = (-0.2\,m, 0.2\,m, 0)$ in the 'base_link' reference frame.

In the 'map' reference frame, the robot at it's goal destination has an angle (yaw) of $-\frac{\pi}{4}$. To convert this to the 'base_link' reference frame, consider the rotation of the 'map' reference frame to the 'base_link' reference frame (90 degrees ($\frac{\pi}{2}$ rad)).

Thus, in the 'base_link' reference frame, the robot at it's goal destination has an angle (yaw) of $-\frac{3\pi}{4}\,rad$, while the roll and pitch remain $0\,rad$.

Question 5

Solve the inverse kinematics to get to the goal destination specified in Question 4, writing the velocities that need to be sent to the wheels and the time duration, with a rotation-translation-rotation motion, and write the updated the homogeneous transformation matrix between 'odom' and 'base_link'.

To solve the inverse kinematics to get to the goal destination specified, with a rotation-translation-rotation motion, we use the following inverse kinematics equations (in the 'map' reference frame).

$$(Translation) \longrightarrow \begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} x + vt\cos(\theta) \\ y + vt\sin(\theta) \\ \theta \end{bmatrix}$$

$$(Rotation) \longrightarrow \begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta + 2vt/l \end{bmatrix}$$

Rotation $\longrightarrow$ The initial location is $(x, y) = (5.0\,m, 5.0\,m)$, the initial rotation is $\theta = \frac{\pi}{2}$, the final location is $(x, y) = (5.0\,m, 5.0\,m)$, the distance between wheels is $l = 0.16\,m$, and the angle to the goal destination is $\frac{5\pi}{4}$.

Thus, if we set $v = 0.1\,m/s$ (indicating that the right wheel moves at velocity $0.1\,m/s$ and the left wheel moves at velocity $-0.1\,m/s$, we can solve for the time t.

$$x' = x \longrightarrow 5.0\,m = 5.0\,m$$

$$y' = y \longrightarrow 5.0\,m = 5.0\,m$$

$$\theta' = \theta + 2vt/l \longrightarrow \frac{5\pi}{4} = \frac{\pi}{2} + 2\,(0.1\,m/s)\,(t)\,/\,(0.16\,m) \longrightarrow t \approx 1.885\,s$$

Translation $\longrightarrow$ The initial location is $(x, y) = (5.0\,m, 5.0\,m)$, the initial rotation is $\theta = \frac{5\pi}{4}$, the final location is $(x, y) = (4.8\,m, 4.8\,m)$, and the angle to the goal destination is $\frac{5\pi}{4}$.

Thus, if we set $v = 0.1\,m/s$ (indicating that the right and left wheels move at velocity $0.1\,m/s$, we can solve for the time t.

$$x' = x + vt\cos(\theta) \longrightarrow 4.8\,m = (5.0\,m) + (0.1\,m/s)\,(t)\cos\left(\frac{5\pi}{4}\right) \longrightarrow t \approx 2.828\,s$$

$$y' = y + vt\sin(\theta) \longrightarrow 4.8\,m = (5.0\,m) + (0.1\,m/s)\,(t)\sin\left(\frac{5\pi}{4}\right) \longrightarrow t \approx 2.828\,s$$

$$\theta' = \theta \longrightarrow \frac{5\pi}{4}\,rad = \frac{5\pi}{4}\,rad$$

Rotation $\longrightarrow$ The initial location is $(x, y) = (4.8\,m, 4.8\,m)$, the initial rotation is $\theta = \frac{5\pi}{4}$, the final location is $(x, y) = (4.8\,m, 4.8\,m)$, the distance between wheels is $l = 0.16\,m$, and the angle to the goal destination is $-\frac{\pi}{4}$.

Thus, if we set $v = 0.1\,m/s$ (indicating that the right wheel moves at velocity $0.1\,m/s$ and the left wheel moves at velocity $-0.1\,m/s$, we can solve for the time t.

$$x' = x \longrightarrow 4.8\,m = 4.8\,m$$

$$y' = y \longrightarrow 4.8\,m = 4.8\,m$$

$$\theta' = \theta + 2vt/l \longrightarrow -\frac{\pi}{4} + (2\pi) = \frac{5\pi}{4} + 2\,(0.1\,m/s)\,(t)\,/\,(0.16\,m) \longrightarrow t \approx 1.257\,s$$

Note: The (2π) is used to maintain the direction of rotation (counterclockwise), alongside ensuring the rotation is along the shortest chord (and thus takes the minimal time).

To determine the updated homogeneous transformation matrix between 'odom' and 'base_link', we consider the rotation and translation. For the transformation from 'base_link' to 'odom', the rotation is $\frac{5\pi}{4}\, rad$ and the translation is $(x, y) = (-0.2\, m, 0.2\, m)$, based on the location (pose) of the robot.

$$^{odom}T_{base\_link} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & -0.2 \\ \sin(\theta) & \cos(\theta) & 0 & 0.2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow {^{odom}}\, T_{base\_link} = \begin{bmatrix} -0.707 & 0.707 & 0 & -0.2 \\ -0.707 & -0.707 & 0 & 0.2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For the transformation from 'odom' to 'base_link', we may take the inverse of the above transformation, either conceptually or mathematically.

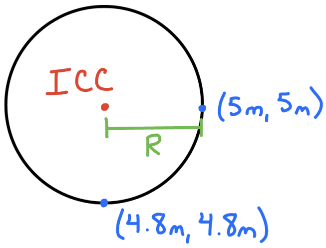
$$^{base\_link}T_{odom} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & \frac{\sqrt{2}}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow {^{base\_link}}\, T_{odom} = \begin{bmatrix} -0.707 & 0.707 & 0 & 0 \\ -0.707 & -0.707 & 0 & 0.283 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Extra Credit

Write what the wheel velocities should be if we wanted the robot to reach the goal location following only an arc.

If we simply care about the robot reaching the goal location, without consideration of the angle, it is relatively straightforward to determine the appropriate wheel velocities so that the robot follows an arc.

The following image depicts a way to set up the problem, so that the robot follows an arc (i.e. there are two locations given on a circle).



From this set up, it is relatively clear that $R = 0.2\,m$. Thus, we may solve for the velocities according to the distance to the ICC R using the following formula: $R = \frac{l}{2}\left(\frac{v_r+v_l}{v_r-v_l}\right)$

$$0.2\,m = \frac{0.16\,m}{2}\left(\frac{v_r + v_l}{v_r - v_l}\right)$$

$$\frac{5}{2} = \left(\frac{v_r + v_l}{v_r - v_l}\right) \longrightarrow 5v_r - 5v_l = 2v_r + 2v_l \longrightarrow 3v_r = 7v_l$$

Thus, $v_r = 0.07\,m/s$ and $v_l = 0.03\,m/s$ is a possibility for the wheel velocities that should be set if we wanted the robot to reach the goal location following only an arc.