Fibonacci Boundary Value Problem

Carter Kruse & Arianna McCarty

Supervisor: Dr. Kenneth Monks

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Introduction

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What happens when we are given two **boundary** conditions? Let's say the sequence is F, and $F_0 = 1$ and $F_2 = 1$. What must F_1 be?

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It's actually the same sequence, so $F_1 = 0$. This is shown by the following:

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Unless you know how to work this out, the thought may be quite daunting.

Reference - Where Did This Come From?

"Suppose we imagine the Fibonacci recurrence, together with the additional data $F_0 = 1$ and $F_{735} = 1$. Well then, the sequence $\{F_n\}$ would be uniquely determined, but you wouldn't be able to compute it directly by recurrence because you would not be in possession of the two consecutive values that are needed to get the recurrence started."

–*Generatingfunctionology* by Herbert S. Wilf

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Main Results

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What we found was rather interesting. The expression for $F_1(N)$, where $N \in \mathbb{Z}$ was as follows:

$$F_1(N) = \frac{(-1)^{(N-1)} (\phi - \psi) + \left(\phi^{(N-1)} - \psi^{(N-1)}\right)}{\phi^N - \psi^N},$$

where $\phi = -\frac{1}{2}\left(1 + \sqrt{5}\right)$ and $\psi = -\frac{1}{2}\left(1 - \sqrt{5}\right)$.

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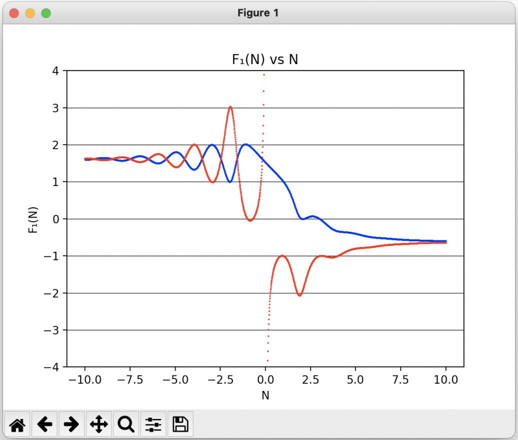
These values of ϕ and ψ should look familiar. They are the roots that satisfy the expression $1 - x - x^2 = 0$, and thus, one of them is the additive inverse of the golden ratio.

Limits As $N \to \pm\infty$

It's particularly interesting to consider the limits of the function $F_1(N)$ as $N \to \pm\infty$.



Limits As $N \to \pm\infty$

As we saw from the previous graph, as $N \to \infty$, the value of F_1 approaches $-0.618\ldots = \frac{1}{2}\left(1 - \sqrt{5}\right)$.

Additionally, as $N \to -\infty$, the value of F_1 approaches $1.618\ldots = \frac{1}{2}\left(1 + \sqrt{5}\right)$.

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For reference, these values are $-\psi$ and $-\phi$ from our definition, respectively. While this is certainly elegant, why is this the case?

Generating Functions

Consider the following setup for a three-term boundary value problem with recursive characteristics. The recurrence

$$af_{n+1} + bf_n + cf_{n-1} = d_n \qquad (n = 1, 2, \ldots, N-1)$$

where $N \in \mathbb{N}$, the constants a, b, and c, and the sequence $\{d_n\}_{n=1}^{N-1}$ are given in advance, defines a sequence F uniquely, as long as the boundary conditions f_0 and f_N are provided.

Generating Functions

For the situation presented, we have the following:

$$af_{n+1} + bf_n + cf_{n-1} = d_n \qquad (n = 1, 2, \ldots, N-1)$$

$$f_0 = f_N = 1 \qquad a = 1, b = c = -1$$

To start, we define our known and unknown generating functions, respectively:

$$F(x) = \sum_{j=0}^{N} f_j x^j \quad \text{and} \quad D(x) = \sum_{j=0}^{N-1} d_j x^j.$$

For reference, the expansion of $F(x)$ is

$$F(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_N x^N.$$

We know that $D(x) = 0$, as no additional terms are included in the Fibonacci-like recurrence definition. Using the generating function for $F(x)$ in our original definition results in

$$\sum_{n=1}^{N-1} f_{n+1}x^n - \sum_{n=1}^{N-1} f_n x^n - \sum_{n=1}^{N-1} f_{n-1}x^n = 0.$$

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After solving for $F(x)$ explicitly and inputting the boundary conditions $f_0 = f_N = 1$, our equation becomes

$$\left(1 - x - x^2\right) F(x) = 1 + f_1 x - x - x^{N+1} - f_{N-1}x^{N+1} - x^{N+2}.$$

Generating Functions

The unknown generating function $F(x)$ is then known except for constants f_1 and f_{N-1}.

To find these constants, we recognize there are two values of x for which the quadratic polynomial on the left side of our previous expression vanishes. In particular, these values are

$$\phi \text{ and } \psi = -\frac{1}{2}\left(1 \pm \sqrt{5}\right).$$

Considering that the quadratic polynomial on the left side vanishes for the given roots, we may simplify our original expression to

$$f_1 x - f_{N-1} x^{N+1} = x^N - x^2.$$

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Now say we define $F_1(N)$ to be the value of f_1, the second term in the sequence where $F_0 = F_N = 1$. To solve for $F_1(N)$, we use Cramer's rule, resulting in the equation

$$F_1(N) = \frac{-\left(\phi^N - \phi^2\right)\left(\psi^{N+1}\right) + \left(\psi^N - \psi^2\right)\left(\phi^{N+1}\right)}{-\phi\left(\psi^{N+1}\right) + \psi\left(\phi^{N+1}\right)}.$$

After performing quite a bit of algebraic manipulation, we're given the following result:

$$F_1(N) = \frac{(-1)^{(N-1)}(\phi - \psi) + \left(\phi^{(N-1)} - \psi^{(N-1)}\right)}{\phi^N - \psi^N}.$$

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Note that some of this manipulation involved recognizing that $\phi^N \psi^N = -1$, which allowed for exponents to be simplified.

Limits As $N \to \pm\infty$

Suppose we wish to take the limit as N approaches ∞. The first term in the numerator, $\left((-1)^{(N-1)}(\phi - \psi)\right)$ oscillates and is insignificant as $N \to \pm\infty$. Thus, when evaluating the limit as $N \to \infty$, we begin with the equation

$$\lim_{N \to \infty} F_1(N) = \frac{\phi^{(N-1)} - \psi^{(N-1)}}{\phi^N - \psi^N}.$$

If we let $\phi, \psi = -\frac{1}{2}\left(1 \pm \sqrt{5}\right) \approx -1.618\ldots$ and $0.618\ldots$, we may reduce the expression for $F_1(N)$ as such:

$$\lim_{N \to \infty} F_1(N) = \frac{\phi^{(N-1)}}{\phi^N}.$$

Evidently, this evaluates to

$$\lim_{N\to\infty} F_1(N) = \frac{1}{\phi},$$

which, when applying $\phi\psi = -1$, is equivalent to our final result (from the graph):

$$\lim_{N\to\infty} F_1(N) = -\psi = \frac{1}{2}\left(1 - \sqrt{5}\right).$$

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A similar method may be used to find that

$$\lim_{N\to-\infty} F_1(N) = -\phi = \frac{1}{2}\left(1+\sqrt{5}\right).$$

Applications

Generating functions are quite useful in interpolation applications in our modern world. Numerous boundary value problems arise in applications, such as interpolation by spline functions, which is used to store computer fonts.

In a computer, fonts are kept in memory by storing the parameters of spline functions that fit the letters in the font. Thus, generating functions can be used to store data in space-efficient ways.

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References

Wilf, Herbert S. "1.4 A Three Term Boundary Value Problem." In *Generatingfunctionology*. Boston, MA: Academic Press, Inc., 1994.