Fibonacci Boundary Value Problem

Carter Kruse, Arianna McCarty
Advisor: Dr. Kenneth Monks

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Abstract

Suppose a Fibonacci-like sequence, defined recursively by $U_{n+2} = U_{n+1} + U_n$ for $n \geq 0$, is bounded by $U_0 = U_N = 1$. For a fixed N, these boundary conditions determine a distinctive sequence. We define $U_1(N)$ to be the value of the u_1, the second term in the sequence, given N. Using generating functions, we calculate an expression for $U_1(N)$, as well as its limits as N approaches ∞, $-\infty$, and 0.

1 Introduction

Consider the following setup for a three-term boundary value problem with recursive characteristics. The recurrence

$$au_{n+1} + bu_n + cu_{n-1} = d_n \qquad (n = 1, 2, \ldots, N-1)\,, \tag{1}$$

where $N \in \mathbb{N}$, the constants a, b, and c, and the sequence $\{d_n\}_{n=1}^{N-1}$ are given in advance, determines a sequence completely, as long as the boundary conditions of u_0 and u_N are provided.

Now suppose we have such a situation, where a Fibonacci-like sequence, defined recursively by $U_{n+2} = U_{n+1} + U_n$ for $n \geq 0$, is bounded by $U_0 = U_N = 1$ for fixed N. We are unable to compute the sequence directly via recurrence, as we lack consecutive starting values of the recurrence. However, the method of generating functions is an effective way to approach such a problem.

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2 Main Result

For the situation proposed above, we have the following:

$$au_{n+1} + bu_n + cu_{n-1} = d_n$$

$$u_0 = u_N = 1 \qquad a = 1, b = c = -1$$

To start, we define our unknown and known generating functions, respectively:

$$U(x) = \sum_{j=0}^{N} u_j x^j \qquad \text{and} \qquad D(x) = \sum_{j=0}^{N-1} d_j x^j.$$

For reference, the expansion of $U(x)$ is $U(x) = u_0 + u_1 x + u_2 x^2 + \cdots + u_N x^N$. From the statement of the situation, we know that $D(x) = 0$, as no additional terms are included in the Fibonacci-like recurrence definition. Expressing (1) using our generating function for u_n results in

$$\sum_{n=1}^{N-1} u_{n+1} x^n - \sum_{n=1}^{N-1} u_n x^n - \sum_{n=1}^{N-1} u_{n-1} x^n = 0. \tag{2}$$

Using our expansion of $U(x)$, we deduce that the respective summations can be expressed in terms of $U(x)$ as such:

$$\sum_{n=1}^{N-1} u_{n+1} x^n = u_2 x + u_3 x^2 + \cdots + u_N x^{N-1} \longrightarrow \frac{U(x) - u_0 - u_1 x}{x}$$

$$\sum_{n=1}^{N-1} u_n x^n = u_1 x + u_2 x^2 + \cdots + u_{N-1} x^{N-1} \longrightarrow U(x) - u_0 - u_N x^N$$

$$\sum_{n=1}^{N-1} u_{n-1} x^n = u_0 x + u_1 x^2 + \cdots + u_{N-2} x^{N-1} \longrightarrow x \left(U(x) - u_{N-1} x^{N-1} - u_N x^N \right)$$

When we express (2) in terms of our generating function $U(x)$, it takes the form

$$\frac{U(x) - u_0 - u_1 x}{x} - \left(U(x) - u_0 - u_N x^N \right) - x \left(U(x) - u_{N-1} x^{N-1} - u_N x^N \right) = 0. \tag{3}$$

Solving for $U(x)$ through algebraic manipulation (namely multiplying by x and subtracting terms from both sides of the equation), the following becomes apparent:

$$\left(1 - x - x^2\right) U(x) = u_0 + u_1 x - u_0 x - u_N x^{N+1} - u_{N-1} x^{N+1} - u_N x^{N+2}. \tag{4}$$

Inputting our original boundary values ($u_0 = u_N = 1$) produces

$$\left(1 - x - x^2\right) U(x) = 1 + u_1 x - x - x^{N+1} - u_{N-1} x^{N+1} - x^{N+2}. \tag{5}$$

The unknown generating function $U(x)$ is then known except for constants u_1 and u_{N-1}. To find these constants, we recognize there are two values of x for wish the quadratic polynomial on the left side of (5) vanishes. In particular, these values are

$$\phi \text{ and } \psi = -\frac{1}{2}\left(1 \pm \sqrt{5}\right) \tag{6}$$

which are evidently the additive inverses of the roots corresponding to the golden ratio.

Considering that the quadratic polynomial on the left side of (5) vanishes for the given roots, we may express (5) as

$$u_1 x - u_{N-1} x^{N+1} = -\left(1 - x\right) + x^{N+1} + x^{N+2}. \tag{7}$$

The value of $1 - x$ reduces to x^2, in the context of this problem, due to the fact that $1 - x - x^2 = 0$ for the given roots, ϕ and ψ. Thus, our expression is rewritten as

$$u_1 x - u_{N-1} x^{N+1} = -x^2 + x^{N+1} + x^{N+2}, \tag{8}$$

from which we can factor out x^N on the right-side of (8), producing

$$u_1 x - u_{N-1} x^{N+1} = -x^2 + x^N \left(x + x^2\right). \tag{9}$$

As before, we use the knowledge that $1 - x - x^2 = 0$ to know that the expression $x + x^2$ reduces to 1, leading to the following expression:

$$u_1 x - u_{N-1} x^{N+1} = x^N - x^2. \tag{10}$$

Now say we define $U_1(N)$ to be the value of u_1, the second term in the sequence, given N. To solve for $U_1(N)$, we use Cramer's rule, with $U_1(N) = \frac{K_a}{K}$, where

$$K = \det \begin{bmatrix} \phi & -\phi^{N+1} \\ \psi & -\psi^{N+1} \end{bmatrix} = -\phi \left(\psi^{N+1}\right) + \psi \left(\phi^{N+1}\right) \tag{11}$$

and

$$K_a = \det \begin{bmatrix} \phi^N - \phi^2 & -\phi^{N+1} \\ \psi^N - \psi^2 & -\psi^{N+1} \end{bmatrix} = -\left(\phi^N - \phi^2\right)\left(\psi^{N+1}\right) + \left(\psi^N - \psi^2\right)\left(\phi^{N+1}\right). \tag{12}$$

Thus, we determine that

$$U_1(N) = \frac{-\left(\phi^N - \phi^2\right)\left(\psi^{N+1}\right) + \left(\psi^N - \psi^2\right)\left(\phi^{N+1}\right)}{-\phi\left(\psi^{N+1}\right) + \psi\left(\phi^{N+1}\right)}, \tag{13}$$

which gives us an explicit form for the value of the second term in the recurrence, given the value of N. By factorization and rearranging terms, this expression may be more concisely expressed as follows:

$$\begin{aligned} U_1(N) &= \frac{\phi\psi\left(-\left(\phi^{N-1} - \phi\right)\psi^N + \left(\psi^{N-1} - \psi\right)\phi^N\right)}{\phi\psi\left(-\psi^N + \phi^N\right)} \\ &= \frac{-\left(\phi^{N-1} - \phi\right)\psi^N + \left(\psi^{N-1} - \psi\right)\phi^N}{-\psi^N + \phi^N} \\ &= \frac{-\phi^{N-1}\psi^N + \phi\psi^N + \psi^{N-1}\phi^N - \psi\phi^N}{-\psi^N + \phi^N}. \end{aligned}$$

From the expression $1 - x - x^2 = 0$, we recall that $\phi\psi = -1$, thus $\phi^N \psi^N = (-1)^N$. From this, we can deduce the following:

- $\phi^{N-1}\psi^N = (-1)^{N-1}\psi$
- $\phi^N \psi^{N-1} = (-1)^{N-1}\phi$
- $\phi\psi^N = (-1)\psi^{N-1}$
- $\psi\phi^N = (-1)\phi^{N-1}$

Thus, we determine that the expression for $U_1(N)$ may be simplified to

$$U_1(N) = \frac{-(-1)^{(N-1)}\psi + (-1)\psi^{N-1} + (-1)^{(N-1)}\phi - (-1)\phi^{N-1}}{-\psi^N + \phi^N}, \tag{14}$$

which is more-concisely expressed as

$$U_1(N) = \frac{(-1)^{(N-1)}(\phi - \psi) + \left(\phi^{(N-1)} - \psi^{(N-1)}\right)}{\phi^N - \psi^N}. \tag{15}$$

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3 Limits Of $U_1(N)$

3.1 Limit As N Approaches ∞

Suppose we wish to take the limit of (15) as N approaches ∞. First, we note that ϕ and ψ are interchangeable, highlighting the symmetry of the expression previously found.

The first term in the numerator of (15), $\left((-1)^{(N-1)}(\phi - \psi)\right)$ oscillates, thus indicating it is insignificant as $N \to \pm\infty$. Thus, when evaluating such limit, we begin with

$$\lim_{N\to\infty} U_1(N) = \frac{\phi^{(N-1)} - \psi^{(N-1)}}{\phi^N - \psi^N}. \tag{16}$$

If we let $\phi = -\frac{1}{2}\left(1 + \sqrt{5}\right) \approx -1.618\ldots$ and $\psi = -\frac{1}{2}\left(1 - \sqrt{5}\right) \approx 0.618\ldots$, we may reduce the expression for $U_1(N)$ as such:

$$\lim_{N\to\infty} U_1(N) = \frac{\phi^{(N-1)}}{\phi^N}. \tag{17}$$

Evidently, this evaluates to

$$\lim_{N\to\infty} U_1(N) = \frac{1}{\phi} \tag{18}$$

which, when applying $\phi\psi = -1$, is equivalent to our final result:

$$\lim_{N\to\infty} U_1(N) = -\psi = \frac{1}{2}\left(1 - \sqrt{5}\right). \tag{19}$$

3.2 Limit As N Approaches $-\infty$

Now suppose we wish to take the limit of (15) as N approaches $-\infty$. As before, we start with

$$\lim_{N\to-\infty} U_1(N) = \frac{\phi^{(N-1)} - \psi^{(N-1)}}{\phi^N - \psi^N} \tag{20}$$

Letting $\phi = -\frac{1}{2}\left(1+\sqrt{5}\right) \approx -1.618\ldots$ and $\psi = -\frac{1}{2}\left(1-\sqrt{5}\right) \approx 0.618\ldots$, we reduce the expression for $U_1(N)$ as such:

$$\lim_{N\to-\infty} U_1(N) = \frac{\psi^{(N-1)}}{\psi^N}. \tag{21}$$

Evidently, this evaluates to

$$\lim_{N\to-\infty} U_1(N) = \frac{1}{\psi} \tag{22}$$

which, when applying $\phi\psi = -1$, is equivalent to our final result:

$$\lim_{N\to-\infty} U_1(N) = -\phi = \frac{1}{2}\left(1+\sqrt{5}\right). \tag{23}$$

The significance of this value can be thought of as such: if we have a Fibonacci-like sequence, starting with the $u_{-\infty} = 1$ and $u_0 = 1$, the term immediately following the u_0 term will be $-\phi = \frac{1}{2}\left(1+\sqrt{5}\right) \approx 1.618$.

3.3 Pseudo-Limit As N Approaches 0

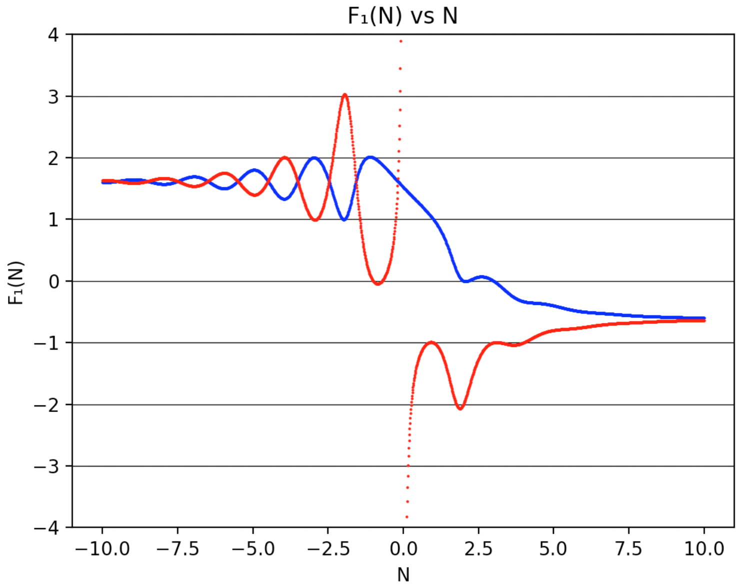
While the limit of (15) as N approaches 0 may not make sense in the context of sequences, it is still interesting to consider the result.

The reason such a value does not intuitively make sense in the context of sequences is that for $N = 0$, the boundary terms of the sequence are $u_0 = u_0 = 1$, indicating there is a repeated condition. As such, the sequence is not unique, as any value for u_1 would satisfy the original conditions in the problem statement.

At this time, we are still working on evaluating this limit analytically, which is to say without the use of a computer. That being said, a graphical representation is quite helpful in analyzing the following expression:

$$U_1(N) = \frac{(-1)^{(N-1)}(\phi-\psi) + \left(\phi^{(N-1)} - \psi^{(N-1)}\right)}{\phi^N - \psi^N}. \tag{24}$$

Using the *Matplotlib* library for Python, the following graph was developed:



4 Conclusion

As indicated by the contents of this paper, we consider a Fibonacci-like sequence, defined recursively by $U_{n+2} = U_{n+1} + U_n$ for $n \geq 0$, that bounded by $U_0 = U_N = 1$ for fixed N. Defining $U_1(N)$ to be the value of the U_1, the second term in the sequence, we calculate the following expression using generating functions:

$$U_1(N) = \frac{(-1)^{(N-1)}(\phi - \psi) + \left(\phi^{(N-1)} - \psi^{(N-1)}\right)}{\phi^N - \psi^N}. \tag{25}$$

Taking the limit as $N \to \pm\infty$ of this function, we have

$$\lim_{N\to\infty} U_1(N) = -\psi = \frac{1}{2}\left(1 - \sqrt{5}\right) \textbf{ and } \lim_{N\to-\infty} U_1(N) = -\phi = \frac{1}{2}\left(1 + \sqrt{5}\right). \tag{26}$$

The limit as $N \to 0$ should be considered a psuedo-limit, as in some sense, the function diverges, yet in another sense, the function converges to $\frac{1}{2}\left(1+\sqrt{5}\right)$, as evidenced by the graph displayed.

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6 References

[1] Wilf, Herbert S. “1.4 A Three Term Boundary Value Problem.” In *Generatingfunctionology*. Boston, MA: Academic Press, Inc., 1994.