MATH 38 - Graph Theory

Carter Kruse

Homework 4

Section 2.1 - Question 15

Let G be a simple graph with diameter at least 4. Prove that $\overline{G}$ has diameter at most 2.

The contrapositive of the statement is as follows: if G is a simple graph and $\overline{G}$ has diameter at least 3, then G has diameter at most 3. This statement matches the following theorem.

Theorem: If G is a simple graph, then $diam\left(G\right) \geq 3 \implies diam\left(\overline{G}\right) \leq 3$.

Proof: When $diam\left(G\right) > 2$, there exist nonadjacent vertices $u, v \in V\left(G\right)$ with no common neighbor. Hence every $x \in V\left(G\right) - \{u, v\}$ has at least one of $\{u, v\}$ as a non-neighbor. This makes x adjacent in $\overline{G}$ to at least one of $\{u, v\}$ in $\overline{G}$. Since $uv \in E\left(\overline{G}\right)$, for every pair x, y there is an x, y-path of length at most 3 in $\overline{G}$ through $\{u, v\}$. Hence, $diam\left(\overline{G}\right) \leq 3$. □

Section 2.1 - Question 23

Let T be a tree in which every vertex has degree 1 or degree k. Determine the possible values of $n\left(T\right)$.

Given a tree T in which every vertex has degree 1 or k, let m represent the number of vertices of degree k. Using the degree-sum formula $\sum_{v \in V(G)} d\left(v\right) = 2e\left(G\right)$, we consider the following expression.

$$mk + \left(n\left(T\right) - m\right) = 2\left(n\left(T\right) - 1\right)$$

This is determined considering that the tree T has $n\left(T\right) - 1$ edges, by definition. The equation simplifies to the following:

$$n\left(T\right) = m\left(k - 1\right) + 2$$

Thus, as m is a non-negative integer, the possible values of $n\left(T\right)$ are the positive integers that are 2 more than a multiple of $k - 1$.

Section 2.1 - Question 27

Let $d_1, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n - 2$.

By definition, every n-vertex tree T is connected and has $n - 1$ edges. Using the degree-sum formula $\sum_{v \in V(G)} d\left(v\right) = 2e\left(G\right)$, this means that $\sum d_i = 2n - 2$. When $n \geq 2$, every vertex has degree at least 1.

To prove the existence of such a tree T with vertex degrees $d_1, \ldots, d_n$ for $n \geq 2$, we use an inductive method.

Base Case: Let $n = 2$. The only list of positive integers $d_1, \ldots, d_n$, such that there exists a tree T with the corresponding vertex degrees is $(1, 1)$. This represents the degree list of a tree T with only two vertices. Thus, $\sum d_i = 2 = 2\,(2) - 2$.

Inductive Step: Let $n > 2$. Consider a list of positive integers $d_1, \ldots, d_n$ satisfying the condition that there exists a tree T with the corresponding vertex degrees. Since $\sum d_i > n$, there must be an element exceeding 1. Since $\sum d_i < 2n$, there must be an element at most 1.

Let d' represent the list of positive integers given by subtracting 1 from the largest element of d and deleting an element that equals 1. $\sum d'_i = 2\,(n-2)$ Given that all of the elements in d' are positive, the induction hypothesis indicates there is a tree T' with $n-1$ vertices with vertex degrees d'.

Thus, by adding a new vertex u and an edge e (from u to the vertex whose degree is the value that was reduced by 1) to the tree T' produces a tree T with the appropriate vertex degrees $d_1, \ldots, d_n$. $\square$

Section 2.1 - Question 32

Prove that an edge e of a connected graph G is a cut-edge if and only if e belongs to every spanning tree. Prove that e is a loop if and only if e belongs to no spanning tree.

An edge e of a connected graph G is a cut-edge if an only if e belongs to every spanning tree.

Proof: By contraposition, if the connected graph G contains a spanning tree T that e does not belong to, then e is in a cycle in $T + e$. Thus, e is not a cut-edge in G. So, if an edge e of a connected graph G is a cut-edge, e belongs to every spanning tree.

Conversely, if e is not a cut-edge in G, then $G - e$ is connected and so contains a spanning tree T that is a spanning tree of G. Thus, there exists a spanning tree of G that e does not belong to. So, if an edge e of a connected graph G belongs to every spanning tree, e is a cut-edge. $\square$

An edge e of a connected graph G is a loop if and only if e belongs to no spanning tree.

Proof: If e is a loop in a connected graph G, then e is a cycle. Thus, by definition, e does not belong to a spanning tree (i.e. connected *acyclic* graph). So, if an edge e of a connected graph G is a loop, e belongs to no spanning tree.

Inversely, if e is not a loop and the connected graph G contains a spanning tree T that e does not belong to, then e is in a cycle in $T + e$. This cycle contains an edge e', such that $T + e - e'$ is a spanning tree that e belongs to, since it has no cycle and deleting an edge from a cycle of the connected graph $T + e$ does not disconnect it. Thus, we have a contradiction, so if an edge e of a connected graph G belongs to no spanning tree, e is a loop. $\square$

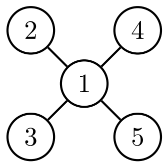
Section 2.2 - Question 1

Determine which trees have Prüfer codes that contain only one value, contain exactly two values, or have distinct values in all positions.

Consider that the degree of a vertex in the tree T is one more than the number of times it appears in the corresponding Prüfer code.

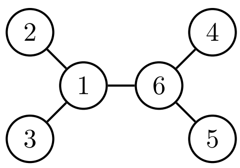
The trees with Prüfer codes that contain only one value are "stars". Given a tree T with n vertices, the $n-1$ labels that do not appear in the corresponding Prüfer code have degree 1 and the label that appears $n-2$ times in the Prüfer code has degree $n-1$.

Example



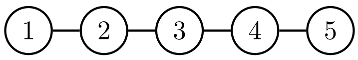
The trees with Prüfer codes that contain exactly two values are "double stars". Given a tree T with n vertices, there are $n-2$ labels that do not appear in the corresponding Prüfer code, and there are $n-2$ leaves in the tree.

Example



The trees with Prüfer codes that have distinct values in all positions (no repeated entries) are "paths". Given a tree T with n vertices, there are $n-2$ labels that appear once, whose vertices have degree 2. The 2 labels that are missing correspond to vertices that have degree 1 (leaves).

Example



Section 2.2 - Question 7

Use Cayley's formula to prove that the graph obtained from K_n by deleting an edge has $(n-2)\, n^{n-3}$ spanning trees.

Cayley's formula states that with a vertex set $[n]$, there are n^{n-2} trees. Thus, using the formula, there are n^{n-2} spanning trees of K_n.

Since each spanning tree has $n-1$ edges, there are $(n-1)\, n^{n-2}$ pairs (e, T) such that T is a spanning tree in K_n and $e \in E(T)$. Let us group the pairs according to the $\binom{n}{2}$ edges in K_n. By symmetry, each edge of K_n appears in the same number of spanning trees, so we may divide by $\binom{n}{2}$ to find the number of trees containing any given edge.

$$\frac{(n-1)\,n^{n-2}}{\binom{n}{2}} = \frac{(n-1)\,n^{n-2}}{n\,(n-1)\,/2} = 2n^{n-3}$$

To count the spanning trees in $K_n - e$, we may subtract the number that contain a particular edge e from the total number of spanning trees in K_n. This is equivalent to the following:

$$n^{n-2} - 2n^{n-3} = (n-2)\,n^{n-3}$$

Thus, Cayley's formula may be used to prove that the graph obtained from K_n by deleting an edge has $(n-2)\,n^{n-3}$ spanning trees. $\square$

Section 2.2 - Question 8

Count the following sets of trees with vertex set $[n]$, giving two proofs for each (one using the Prüfer correspondence and one using direct counting arguments).

Trees that have 2 leaves.

With vertex set $[n]$, there are $\frac{n!}{2}$ trees with 2 leaves. Every tree with 2 leaves is a path, as paths along distinct edges incident to a vertex of degree k leads to k distinct leaves, so having only 2 leaves in a tree implies maximum degree 2.

Direct: The vertices of a path in order form a permutation of the vertex set. Following the path from the other end produces another permutation. On the other hand, every permutation arises in this way. Hence, there are two permutations for every path, and the number of paths is $\frac{n!}{2}$. $\square$

Prüfer Correspondence: In the Prüfer code corresponding to a tree, the labels of the leaves are the labels that do not appear.

For paths (trees with 2 leaves), the other $n-2$ labels must each appear in the Prüfer code, so they must appear once each. Having chosen the leaf labels in $\binom{n}{2}$ ways, there are $(n-2)!$ ways to form a Prüfer code in which all the other labels appear. The product is $\frac{n!}{2}$. $\square$

Trees that have $n-2$ leaves.

With vertex set $[n]$, there are $\binom{n}{2}\left(2^{n-2}-2\right)$ trees with $n-2$ leaves. Every tree with $n-2$ leaves has exactly 2 non-leaves. Each leaf is adjacent to one of these two vertices, with at least one leaf neighbor for each of the two vertices.

Direct: We pick the two central vertices in one of $\binom{n}{2}$ ways and then pick the set of leaves adjacent to the lower of the two central vertices. This set is a subset of the $n-2$ remaining vertex labels, and it can be any subset other than the full set and the empty set. The number of ways to do this is the same no matter how the central vertices are chosen, so the number of trees is $\binom{n}{2}\left(2^{n-2}-2\right)$. $\square$

Prüfer Correspondence: In the Prüfer code corresponding to a tree, the labels of the leaves are the labels that do not appear.

For trees with $n-2$ leaves, exactly two labels appear in the Prüfer code. We can choose these two labels in $\binom{n}{2}$ ways. To form a Prüfer code (and thus a tree) with these two labels as non-leaves, we choose an arbitrary non-empty proper subset of the positions $1, \ldots, n-2$ for the appearances of the first label. There are $2^{n-2}-2$ ways to do this step. Hence there are $\binom{n}{2}\left(2^{n-2}-2\right)$ ways to form the Prüfer code.

Section 2.2 - Question 17

Use the Matrix Tree Theorem to prove Cayley's formula.

Cayley's formula states that with a vertex set $[n]$, there are n^{n-2} trees.

The Matrix Tree Theorem states the following: given a loopless graph G with vertex set $v_1, \ldots, v_n$, let $a_{i,j}$ be the number of edges with endpoints v_i and v_j. Let Q be the matrix in which entry (i, j) is $-a_{i,j}$ when $i \neq j$ and $d(v_i)$ when $i = j$. If Q^* is a matrix obtained by deleting row s and column t of Q, then $\tau(G) = (-1)^{s+t} \det Q^*$.

The number of labeled n-vertex trees is the number of spanning trees in K_n. Using the Matrix Tree Theorem, we compute this by subtracting the adjacency matrix from the diagonal matrix of degrees, deleting one row and column, and taking the determinant.

All degrees are $n-1$, so the initial matrix is $n-1$ on the diagonal and -1 elsewhere. Delete the last row and column. We compute the determinant of the resulting matrix.

By using row operations, we may add every row to the first row, which does not change the determinant, but makes every entry in the first row 1. Now add the first row to every other row. The determinant remains unchanged, but every row below the first is now 0 everywhere except on the diagonal, where the value is n. The matrix is now upper triangular, so the determinant is the product of the diagonal entries, which are 1 and $n-2$ copies of n. Hence the determinant is n^{n-2}, so this indicates that the Matrix Tree Theorem may be used to prove Cayley's formula. $\square$