MATH 038 - Graph Theory

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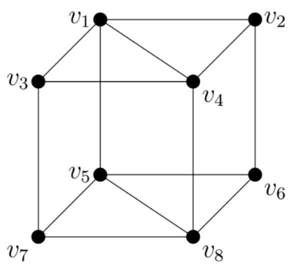
Final Exam - Spring 2023 (Wednesday, May 31)

Question 1

Determine if the following statements are true or false, and prove why.

(a) The following graph is self-complementary.



True $\rightarrow$ A graph is *self-complementary* if it is isomorphic to its complement. **(Definition 1.1.32)** That is, an n-vertex graph H is *self-complementary* if and only if K_n has a decomposition consisting of two copies of H.

In this case, K_8 decomposes into two copies of H, the graph shown above. To further highlight that H and $\overline{H}$ are isomorphic, we consider the following:

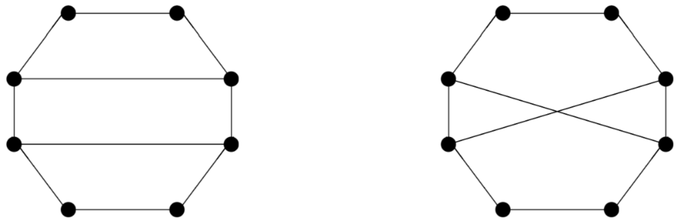
- H and $\overline{H}$ are not bipartite.
- H and $\overline{H}$ have 8 vertices and 14 edges $(\frac{1}{2}\binom{8}{2})$.
- H and $\overline{H}$ have 4 vertices of degree 4 and 4 vertices of degree 3.
- H and $\overline{H}$ have 4 cycles of length 3 and 4 cycles of length 4.

While this alone is not enough to prove that the graph is *self-complementary*, we may consider the adjacency matrix given for H and $\overline{H}$ (with permutations) is as follows:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Thus, the graph is *self-complementary*, as it is isomorphic to its complement (it is possible to decompose K_8 into two copies of H). $\square$

(b) The graphs G_1 and G_2 below are isomorphic.



G_1 G_2

False $\rightarrow$ An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u) f(v) \in E(H)$. **(Definition 1.1.20)**

To show that two graphs are not isomorphic, we may find a "structural property" (preserved by isomorphisms) on which they differ. In this case, the graph G_1 consists of 3 cycles of length 4, 2 cycles of length 6, and a cycle of length 8. The graph G_2 consists of 2 cycles of length 5, 1 cycle of length 4, 2 cycles of length 7, and a cycle of length 8.

Thus, the graphs G_1 and G_2 are not isomorphic, as they contain different cycles. $\square$

Alternatively, consider the following. G_1 is bipartite, while G_2 is not. (This follows from the idea that a graph is bipartite if and only if it has no odd cycle.) **(Theorem 1.2.18)** This indicates that the graphs G_1 and G_2 are not isomorphic. $\square$

(c) The sequence $(7, 7, 7, 6, 6, 6, 5, 5, 5, 4, 4, 4)$ is a graphic sequence.

True $\rightarrow$ The sequence $(7, 7, 7, 6, 6, 6, 5, 5, 5, 4, 4, 4)$ is a graphic sequence, as determined by the Havel & Hakimi Theorem.

The *degree sequence* of a graph is a list of vertex degrees, usually written in non-increasing order, as $d_1 \geq \cdots \geq d_n$ **(Definition 1.3.27)**

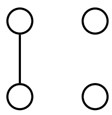
A *graphic sequence* is a list of non-negative numbers that is the degree sequence of some simple graph. A simple graph with degree sequence d "realizes" d.

Havel & Hakimi Theorem: For $n > 1$, an integer list d of size n is graphic if and only if d' is graphic, where d' is obtained from d by deleting its largest element Δ and subtracting 1 from its Δ next largest elements. The only 1-element graphic sequence is $d_1 = 0$. **(Theorem 1.3.31)**

If we apply the Havel & Hakimi Theorem to the given sequence, we may determine if the sequence is a graphic sequence, as follows:

$$\begin{aligned}
(7,7,7,6,6,6,5,5,5,4,4,4) & \\
(6,6,5,5,5,4,4,5,4,4,4) & \\
(6,6,5,5,5,5,4,4,4,4,4) & \qquad \text{(rearranging)} \\
(5,4,4,4,4,3,4,4,4,4) & \\
(5,4,4,4,4,4,4,4,4,3) & \qquad \text{(rearranging)} \\
(3,3,3,3,3,4,4,4,3) & \\
(4,4,4,3,3,3,3,3,3) & \qquad \text{(rearranging)} \\
(3,3,2,2,3,3,3,3) & \\
(3,3,3,3,3,3,2,2) & \qquad \text{(rearranging)} \\
(2,2,2,3,3,2,2) & \\
(3,3,2,2,2,2,2) & \qquad \text{(rearranging)} \\
(2,1,1,2,2,2) & \\
(2,2,2,2,1,1) & \qquad \text{(rearranging)} \\
(1,1,2,1,1) & \\
(2,1,1,1,1) & \qquad \text{(rearranging)} \\
(0,0,1,1) & \\
(1,1,0,0) & \qquad \text{(rearranging)}
\end{aligned}$$

The final degree sequence $(0,0,1,1)$ is a graphic sequence, as evidenced by the following graph.



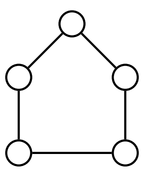
Thus, the initial degree sequence $(7,7,7,6,6,6,5,5,5,4,4,4)$ is a graphic sequence, by the Havel & Hakimi Theorem. $\square$

(d) For every graph G, all the roots of its chromatic polynomial $\chi(G;k)$ are integers.

False $\rightarrow$ It is not the case that for every graph G, all the roots of its chromatic polynomial $\chi(G;k)$ are integers.

Given $k \in \mathbb{N}$ and a graph G, the value $\chi(G;k)$ is the number of proper colorings $f : V(G) \rightarrow [k]$. The set of available colors is $[k] = \{1, \ldots, k\}$; the k colors need not all be used in a coloring f. Changing the names of the colors that are used produces a different coloring. **(Definition 5.3.1)**

Consider the following graph (C_5).



The chromatic polynomial of the graph is $\chi(G;k) = x^5 - 5x^4 + 10x^3 - 10x^2 + 4x$. This may be determined by the chromatic recurrence, as follows. If G is a simple graph and $e \in E(G)$, then $\chi(G;k) = \chi(G-e;k) - \chi(G \cdot e;k)$. **(Theorem 5.3.6)**

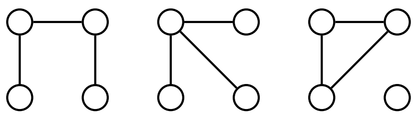
While the *real* roots of the chromatic polynomial $\chi(G;k) = x^5 - 5x^4 + 10x^3 - 10x^2 + 4x$ are integers (specifically 0, 1, and 2), there are complex roots as well (specifically $1+i$ and $1-i$).

Thus, it is not the case that for every graph G, all the roots of its chromatic polynomial $\chi(G;k)$ are integers. $\square$

(e) There exists a graph G with chromatic polynomial $\chi(G;k) = k^4 - 3k^3 + 4k$.

False $\rightarrow$ The chromatic polynomial $\chi(G;k)$ has degree $n(G)$, with integer coefficients alternating in sign and beginning $1, -e(G), \ldots$. **(Theorem 5.3.8)**

Thus, the chromatic polynomial $\chi(G;k) = k^4 - 3k^3 + 4k$ corresponds to a graph G with $n(G) = 4$ vertices and $e(G) = 3$ edges. The only such "unlabled" graphs (grouped by isomorphism class) are as follows:

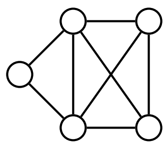


The chromatic polynomial for the first two graphs is $\chi(G;k) = k(k-1)^3 = k^4 - 3k^3 + 3k^2 - k$. The chromatic polynomial for the third graph is $\chi(G;k) = k^2(k-1)(k-2) = k^4 - 3k^3 + 2k^2$.

Thus, there is no graph G with chromatic polynomial $\chi(G;k) = k^4 - 3k^3 + 4k$. $\square$

(f) There exists a graph G with chromatic polynomial $\chi(G;k) = k(k-1)(k-2)^2(k-3)$.

True $\rightarrow$ There exists a graph G with chromatic polynomial $\chi(G;k) = k(k-1)(k-2)^2(k-3)$, as displayed below.



To determine the chromatic polynomial of the graph G, let us start by assigning k colors to the vertex on the far left. That leaves $k-1$ colors for the vertex at the top left of the box, which leaves $k-2$ colors for the vertex at the bottom left of the box.

Considering the adjacency of vertices, there are $k-2$ colors for the vertex at the bottom right of the box and $k-3$ colors for the vertex at the top right of the box.

This method of assigning colors comes from a simplicial elimination ordering. A vertex of G is *simplicial* if its neighborhood in G induces a clique. A *simplicial elimination ordering* is an ordering $v_n, \ldots, v_1$ for deletion of vertices so that each vertex v_i is a simplicial vertex of the remaining graph induced by $\{v_1, \ldots, v_i\}$. **(Definition 5.3.12)**

Question 2

Let G be a simple graph of order 5 or more. Show that at most one of G and $\overline{G}$ is bipartite (i.e. they cannot both be bipartite at the same time).

To demonstrate that at most one of G and $\overline{G}$ is bipartite, let us first consider the case where G is not bipartite, in which case the proof is complete. Regardless of whether or not $\overline{G}$ is bipartite, the statement holds. $\square$.

Now, let us consider the case where G is bipartite. We aim to demonstrate that if G is a simple graph of order 5 or more, $\overline{G}$ is not bipartite.

The *complement* $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E\left(\overline{G}\right)$ if and only if $uv \notin E(G)$. A *clique* in a graph is a set of pairwise adjacent vertices. An *independent set* in a graph is a set of pairwise non-adjacent vertices. **(Definition 1.1.8)**

A graph G is *bipartite* if $V(G)$ is the union of two disjoint (possibly empty) independent sets called *partite sets* of G. **(Definition 1.1.10)**

A *bipartition* of G is a specification of two disjoint independent sets in G whose union is $V(G)$. **(Definition 1.2.17)**

Suppose $X \cup Y = V(G)$ is a bipartition of G, into two disjoint independent sets, X and Y. Thus, there are no edges between vertices in X, and no edges between vertices in Y. That is, the only edges in the graph G are those that span X and Y.

As G is a simple graph of order 5 or more, there are 5 vertices distributed between X and Y. Thus, at least one of X and Y has 3 or more vertices. Let us select 3 of such vertices, u, v, w.

In G, u, v, and w are not pairwise adjacent, as they are within the same independent set (X or Y). Thus, in $\overline{G}$, uv, uw, and vw are edges of the graph. There is a cycle u, v, w of length 3 in the graph $\overline{G}$.

König: A graph is bipartite if and only if it has no odd cycle. **(Theorem 1.2.18)**

Thus, given that there is a cycle of odd length in the graph $\overline{G}$, the graph $\overline{G}$ is not bipartite. Hence, we have demonstrated that if G is a simple *bipartite* graph of order 5 or more, $\overline{G}$ is not bipartite.

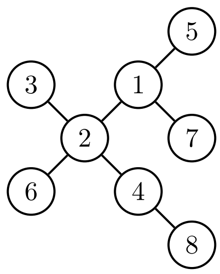
The argument starting with a bipartite graph $\overline{G}$ and concluding with a graph G that is not bipartite follows similarly.

Thus, if G is a simple graph of order 5 or more, at most one of G and $\overline{G}$ is bipartite (i.e. they cannot both be bipartite at the same time). $\square$

Question 3

Find the tree whose Prüfer code is $(2, 1, 2, 1, 2, 4)$.

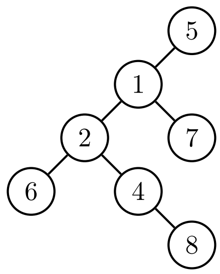
The tree whose Prüfer code is $(2, 1, 2, 1, 2, 4)$ is given as follows:



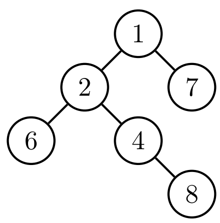
(Algorithm 2.2.1)
Prüfer Code: Production of $f(T) = (a_1, \ldots, a_{n-2})$.
Input: A tree T with vertex set $S \subseteq \mathbb{N}$.
Iteration: At the ith step, delete the least remaining leaf, and let a_i be the *neighbor* of this leaf.

To demonstrate that this tree corresponds to the Prüfer code $(2, 1, 2, 1, 2, 4)$, consider the following order of the removal of leaves.

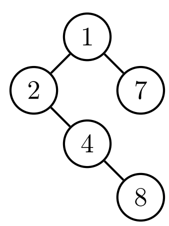
Prüfer Code: (2)



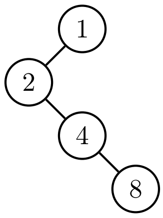
Prüfer Code: $(2, 1)$



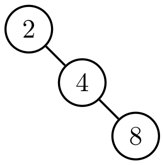
Prüfer Code: $(2, 1, 2)$



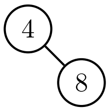
Prüfer Code: $(2, 1, 2, 1)$



Prüfer Code: $(2, 1, 2, 1, 2)$



Prüfer Code: $(2, 1, 2, 1, 2, 4)$



The method outlined above indicates that the tree shown corresponds to the Prüfer code $(2, 1, 2, 1, 2, 4)$. The vertices that are *not* in the Prüfer code are the leaves of the graph G.

Question 4

(a) Let $n \geq 2$. What is the number of trees with vertex set $\{1, 2, \dots, n\}$ where vertex 1 is *not* a leaf?

The number of trees with vertex set $\{1, 2, \dots, n\}$ where vertex 1 is *not* a leaf is equal to the number of trees with vertex set $\{1, 2, \dots, n\}$, minus the number of trees with vertex set $\{1, 2, \dots, n\}$ where vertex 1 is a leaf.

Cayley's Formula: For a set $S \subseteq \mathbb{N}$ of size n, there are n^{n-2} trees with vertex set S. **(Theorem 2.2.3)**

This indicates that there are n^{n-2} trees with vertex set $[n]$. The corresponding Prüfer codes are $(a_1, a_2, \dots, a_{n-2})$ with $a_i \in \{1, 2, \dots, n\}$.

There are $(n-1)^{n-2}$ trees with vertex set $[n]$ where vertex 1 is a leaf. The corresponding Prüfer codes are $(a_1, a_2, \dots, a_{n-2})$ with $a_i \in \{2, 3, \dots, n\}$. (The tree may have other leaves as well.)

Thus, there are $n^{n-2} - (n-1)^{n-2}$ trees with vertex set $\{1, 2, \dots, n\}$ where vertex 1 is *not* a leaf. $\square$

(b) Let $n \geq 4$. What is the number of trees with vertex set $\{1, 2, \dots, n\}$ where vertex 1 has degree 3?

Cayley's Formula: For a set $S \subseteq \mathbb{N}$ of size n, there are n^{n-2} trees with vertex set S. **(Theorem 2.2.3)**

Given positive integers $d_1, \dots, d_n$ summing to $2n-2$, there are exactly $\frac{(n-2)!}{\prod(d_i-1)!}$ trees with vertex set $[n]$ such that vertex i has degree d_i for each i. **(Corollary 2.2.4)**

To determine the number of trees with vertex set $\{1, 2, \dots, n\}$, where vertex 1 has degree 3, we may consider the Prüfer code, which encodes trees.

Given that the vertex 1 has a degree 3, it must appear exactly twice in the Prüfer code. The number of ways for this to occur is given as $\binom{n-2}{2}$, as there are $n-2$ "slots" in the Prüfer code. (In this case, we do not care about the order that the vertex 1 is included in the Prüfer code, hence why a combination, as opposed to a permutation, is used.)

For the other $n-4$ slots that are not filled by the vertex 1, there are $n-1$ options for the vertex label to be chosen, as that represents the vertices that are not vertex 1.

Thus, the number of trees with vertex set $\{1, 2, \dots, n\}$, where vertex 1 has degree 3 is given by $\binom{n-2}{2}(n-1)^{n-4}$.

The first few results are given as follows: $n = 4 \rightarrow 1$, $n = 5 \rightarrow 12$, $n = 6 \rightarrow 150$. Now it's time to be honest... I spent so incredibly long trying to develop an intelligent combinatorial way to count the number of trees where a given vertex has a degree of 3. Thus, I did end up confirming the validity of the results above, counting the number of trees with a vertex set [4], [5], and [6] for which vertex 1 has degree 3... it was not super fun. Anyways, just thought I would share that. I now understand the significance of finding the simple solution, rather than the complex one. :)

Question 5

Let G be a graph with no isolated vertices, and let M be a maximum matching of G. For each vertex v not saturated by M, choose an edge incident to v. Let T be the set of all the chosen edges, and let $L = M \cup T$. Determine if each of the following statements is true or false, and explain why.

(a) L is always an edge cover of G.

True $\rightarrow$ To demonstrate that L is always an edge cover of G, let us start with the following definitions.

A *matching* in a graph G is a set of non-loop edges with no shared endpoints. The vertices. The vertices incident to the edges of a matching M are *saturated* by M; the others are *unsaturated*. **(Definition 3.1.1)**

A *maximum matching* is a matching of maximum size among all matching in the graph. **(Definition 3.1.4)**

An *edge cover* of G is a set L of edges such that every vertex of G is incident to some edge of L. **(Definition 3.1.19)**

Thus, to demonstrate the L is always an edge cover of G, we must show that every vertex in G is incident to at least one edge of L.

The construction of L is as follows. We start with G, a graph with no isolated vertices and M, a maximum matching of G, which saturates as many vertices as possible (among all matchings). Every vertex saturated by M is incident to an edge in M.

Now, for each vertex v not saturated (unsaturated) by M, we choose an edge incident to v. T is the set of all the chosen edges. As v was not saturated by M, there is at least one edge incident to v that is not in the maximum matching M of G. Therefore, choosing an edge incident to v ensures that there is at least one edge incident to v in T.

By taking the union $L = M \cup T$, the edges incident to the vertices in M are added to the edges incident to the vertices in T. Thus, L covers every vertex in G, as every vertex in G is incident to some edge of L.

Hence, L is always an edge cover of G. $\square$

(b) L is always a minimum edge cover of G.

True $\rightarrow$ To demonstrate that L is always a minimum edge cover of G, we aim to show that L has the minimum possible number of edges among all edge covers of G. That is, we have already demonstrated that L is an edge cover of G, so now we aim to show that it is a minimum edge cover.

Suppose there exists an edge cover Q of G with fewer edges than L. This would indicate that L is not a minimum edge cover. If e is an edge of Q, e is an edge of L, according to the construction of L. This creates a contradiction, so the edge cover Q cannot have fewer edges than L.

To demonstrate this, let v be a vertex in G that is covered by Q. Since Q is an edge cover, there must be an edge e in Q incident to v. We aim to show that e is within M or T. If e is within M, v is saturated by M, and so there is an edge in M (and so L) incident to v. This contradicts the assumption that there was no such edge in L, and thus, completes the proof.

If e is not within M (the maximum matching of the graph G), then e was selected when creating the set T of edges. This set represents the edges incident to vertices that were not saturated by M in G. In the construction of L, we chose an an edge incident to v to be included in T, indicating that e would have already been included in L. Thus, Q is not an edge cover with fewer edges than L.

As there is no edge cover of G with fewer edges than L, L is always a minimum edge cover of G. $\square$

Gallai's Theorem: If G is a graph without isolated vertices, then $\alpha'(G)+\beta'(G)=n(G)$ **(Theorem 3.1.22)** The maximum size of a matching is given as $\alpha'(G)$ and the minimum size of an edge cover is given as $\beta'(G)$.

The proof of Gallai's Theorem indicates the relationship between a maximum matching and a minimum edge cover.

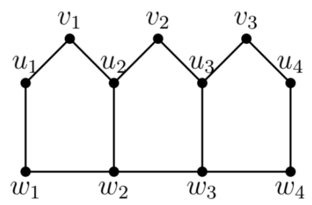
From a maximum matching M, we will construct an edge cover of size $n(G)-|M|$. Since the smallest edge cover is no bigger than this cover, this will imply that $\beta'(G) \leq n(G)-\alpha'(G)$. Also, from a minimum edge cover L, we will construct a matching of size $n(G)-|L|$. Since a largest matching is no smaller than this matching, this will imply that $\alpha'(G) \geq n(G)-\beta'(G)$. These two inequalities complete the proof.

Let M be a maximum matching in G. We construct an edge cover of G by adding to M one edge incident to each unsaturated vertex. We have used one edge for each vertex, except that each edge of M takes care of two vertices, so the total size of this edge cover is $n(G)-|M|$, as desired.

Now let L be a minimum edge cover. If both endpoints of an edge e belong to edges in L other than e, then $e \notin L$, since $L-\{e\}$ is also an edge cover. Hence each component formed by edges of L has at most one vertex of degree exceeding 1 and is a star (a tree with at most one non-leaf). Let k be the number of these components. Since L has one edge for each non-central vertex in each star, we have $|L|=n(G)-k$. We form a matching M of size $k=n(G)-|L|$ by choosoing one edge from each star in L.

Question 6

For $n \geq 1$, let H_n be the graph with $3n-1$ vertices consisting of a path $u_1, v_1, u_2, v_2, \ldots, u_{n-1}, v_{n-1}, u_n$, another path $w_1, w_2, \ldots, w_n$, and the edges $u_i w_i$ for $1 \leq i \leq n$. As an example, the graph H_4 is drawn below. In general, you can think of H_n as a block of $n-1$ "houses".



(a) Find a formula for the chromatic polynomial $\chi(H_n; k)$. *Hint: Express* $\chi(H_n; k)$ *in terms of* $\chi(H_{n-1}; k)$ *and use induction on* n.

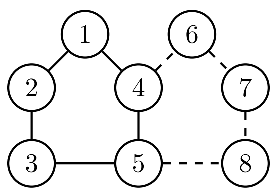
Given $k \in \mathbb{N}$ and a graph G, the value $\chi(G; k)$ is the number of proper colorings $f : V(G) \to [k]$. The set of available colors is $[k] = \{1, \ldots, k\}$; the k colors need not all be used in a coloring f. Changing the names of the colors that are used produces a different coloring. **(Definition 5.3.1)**

If T is a tree with n vertices, then $\chi(T; k) = k(k-1)^{n-1}$. **(Proposition 5.3.3)**

Chromatic Recurrence: If G is a simple graph and $e \in E(G)$, then $\chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k)$. **(Theorem 5.3.6)**

To find a formula for the chromatic polynomial $\chi(H_n; k)$, the value will be expressed in terms of $\chi(H_{n-1}; k)$, and induction on n will allow us to find a (non-recursive) closed form.

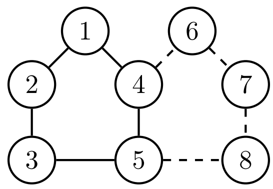
Let's say we have a block of $n-1$ "houses" with the connection as follows.



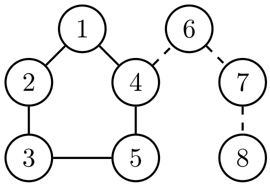
The graph only displays the "end" of the row of "houses", to simplify the construction. Using the chromatic recurrence (with edge deletion and contraction), we may determine the chromatic polynomial $\chi(H_n; k)$ in terms of $\chi(H_{n-1}; k)$.

Let us give the following graphs labels, for clarity. The solid lines represent parts of the graph that are already in place, while the dotted lines represent the parts that we look to iteratively compute using the chromatic recurrence.

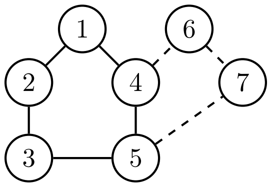
G



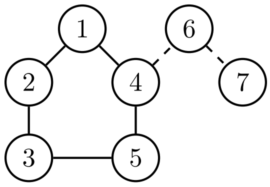
$G - e_1$



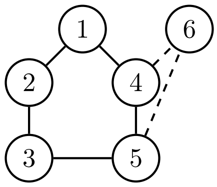
$G \cdot e_1$



$(G \cdot e_1) - e_2$



$(G \cdot e_1) \cdot e_2$



Thus, we may determine that the chromatic polynomial $\chi(G;k) = \chi(G - e_1;k) - \chi(G \cdot e_1;k)$, which is equivalent to $\chi(G;k) = \chi(G - e_1;k) - (\chi((G \cdot e_1) - e_2;k) - \chi((G \cdot e_1) \cdot e_2;k))$.

This simplifies to

$$\chi(G;k) = \chi(G - e_1;k) - \chi((G \cdot e_1) - e_2;k) + \chi((G \cdot e_1) \cdot e_2;k)$$

In this case, the edge deletion and contraction required two iterations, to simplify the graph in such a way that would allow for the chromatic polynomial to be easily computed.

Now, let us consider $G = H_n$, with the graphs displayed above, to determine the recurrence relation. Using the appropriate coloring of respective vertices, the following is the result:

$$\chi(H_n;k) = (k-1)^3 \chi(H_{n-1};k) - (k-1)^2 \chi(H_{n-1};k) + (k-2)\chi(H_{n-1};k)$$

This expression simplifies to the following:

$$\chi(H_n;k) = (k-2)\left(k^2 - 2k + 2\right)\chi(H_{n-1};k)$$

Now that we have the recurrence relation, we may use induction on n to determine the final result. The base case is the graph H_2, which is a block of a single "house". The chromatic polynomial for the graph (C_5) is given as:

$$\chi(C_5;k) = k(k-1)^4 - k(k-1)\left(k^2 - 3k + 3\right)$$

To determine this, the chromatic recurrence may be used (reducing to a P_5 and C_4), though this is not shown, as it is relatively trivial.

As for $n \geq 1$, H_n is the graph with $3n-1$ vertices consisting of a path $u_1, v_1, u_2, v_2, \ldots, u_{n-1}, v_{n-1}, u_n$, another path $w_1, w_2, \ldots, w_n$, and the edges $u_i w_i$ for $1 \leq i \leq n$, the inductive method is relatively straightforward.

By induction on n, the recurrence relation may be re-written as the following:

$$\chi(H_n; k) = (k-2)^{n-2} \left(k^2 - 2k + 2\right)^{n-2} \left(k(k-1)^4 - k(k-1)\left(k^2 - 3k + 3\right)\right)$$

In particular, the exponent on the multiplying factor of the chromatic polynomial is $n-2$, rather than $n-1$, as H_n represents a block of $n-1$ "houses". Thus, the formula is adjusted, so that H_2 reflects the base case (C_5), just a single "house" for which the chromatic polynomial was previously determined. This, in turn, corrects the formula for all values of n.

Reflecting on the effort demonstrated, it is now evident that perhaps starting with the base case of $n = 1$ (which is simply two adjacent vertices) would have been easier.

In this case, the formula (which remains the same) reduces to a simple format, as follows:

$$\chi(H_n; k) = (k-2)^{n-1} \left(k^2 - 2k + 2\right)^{n-1} (k(k-1))$$

In this manner, the shifting of the n value does not need to be accounted for, and the base case of $n = 1$ is relatively easy to determine the chromatic polynomial for, as it is simply two adjacent vertices. $\square$

(b) Use the formula from part (a) to compute $\chi(H_3; 3)$.

Using the formula from part (a), we may determine that $\chi(H_3; 3)$ is equal to the following.

$$\chi(H_n; k) = (k-2)^{n-1} \left(k^2 - 2k + 2\right)^{n-1} (k(k-1))$$

When the values $n = 3$ and $k = 3$ are inputted, the formula is as follows:

$$\chi(H_3; 3) = (3-2)^{3-1} \left(3^2 - 2(3) + 2\right)^{3-1} (3(3-1)) = 150$$

That is, there are 150 proper k-colorings (where $k = 3$) of the graph H_3, the block of 2 "houses".

Question 7

Let G be a connected planar graph with $n \geq 5$ and e edges, and suppose that the length of the smallest cycle in G is 5.

(a) Prove that $e \leq \frac{5}{3}(n-2)$.

Euler's Formula: If a connected plane graph G has exactly n vertices, e edges, and f faces, then $n - e + f = 2$. **(Theorem 6.1.21)** Thus, Euler's Formula relates n and e if we replace f.

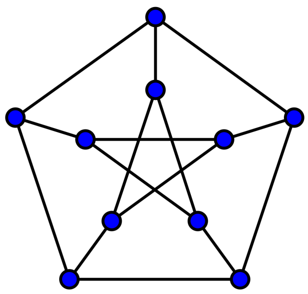
The *length* of a face in a plane graph G is the total length of the closed walk(s) in G bounding the face. **(Definition 6.1.11)**

If $l(F_i)$ denotes the length of face F_i in a plane graph G, then $2e(G) = \sum l(F_i)$. **(Proposition 6.1.13)**

By the proposition, there is an inequality between e and f. If G is a connected planar graph and the length of the smallest cycle in G is 5, the faces have length at least 5. That is, every face boundary in the graph contains at least 5 edges (if $n \geq 5$, which is the case).

Letting $\{f_i\}$ be the list of face lengths, this yields $2e = \sum f_i \geq 5f$ or $\frac{2}{5}e \geq f$. Substituting this into $n - e + f = 2$ yields $n - e + \frac{2}{5}e \geq 2$. Solving for e produces $e \leq \frac{5}{3}(n-2)$. $\square$

(b) Use part (a) to show that the Petersen graph is not planar.



The Petersen graph, shown above, is a connected graph with $n = 10$ and $e = 15$. The length of the smallest cycle in the Petersen graph is 5. Thus, the Petersen graph matches the requirements necessary to check if it is planar.

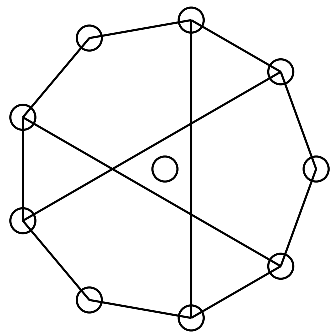
The non-planarity of the Petersen graph follows directly from part (a). For the graph (with $n = 10$), $e = 15 > \frac{40}{3} = \frac{5}{3}(n-2)$. This directly opposes the expression $e \leq \frac{5}{3}(n-2)$. The Petersen graph has too many edges to be planar. $\square$

(c) Use Kuratowski's Theorem (Theorem 6.2.2) to show that the Petersen graph is not planar.

In a graph G, a *subdivision* of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w. **(Definition 4.2.5)**

Kuratowski's Theorem states that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. **(Theorem 6.2.2)**

The Petersen graph contains no vertices of degree at least 4, thus, it does not contain a K_5 subdivision. Regardless, the graph does contain the following subdivision of $K_{3,3}$.



Therefore, by Kuratowski's Theorem, the Petersen graph is non-planar as it contains a subdivision of $K_{3,3}$. $\square$
