MATH 38 - Graph Theory

Carter Kruse

Homework 3

Section 1.3 - Question 8

Which of the following are graphic sequences? Provide a construction or a proof of impossibility for each.

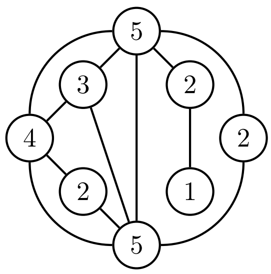
A graphic sequence is a degree sequence of some simple graph.

Theorem (Havel'55, Hakimi'62): A weakly decreasing degree sequence $(d_1, d_2, \ldots, d_n)$ is graphic if and only if the sequence $(d_2 - 1, d_3 - 1, \ldots, d_{d_1} - 1, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphic.

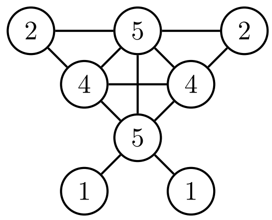
By the theorem, $(5, 5, 4, 3, 2, 2, 2, 1)$, $(5, 5, 4, 4, 2, 2, 1, 1)$, and $(5, 5, 5, 3, 2, 2, 1, 1)$ are graphic, though $(5, 5, 5, 4, 2, 1, 1, 1)$ is not.

Using the Havel-Hakimi Algorithm, we may construct the following graphs.

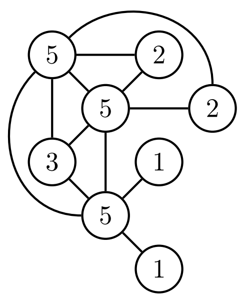
$(5, 5, 4, 3, 2, 2, 2, 1)$



$(5, 5, 4, 4, 2, 2, 1, 1)$



$(5, 5, 5, 3, 2, 2, 1, 1)$



Using the Havel-Hakimi Algorithm, we may determine that the list $(5,5,5,4,2,1,1,1)$ is not graphic, by providing a proof of impossibility.

$$(5,5,5,4,2,1,1,1)$$
$$(4,4,3,1,0,1,1)$$
$$(4,4,3,1,1,1,0)$$
$$(3,2,0,0,1,0)$$
$$(3,2,1,0,0,0)$$

The above is not the degree list of a simple graph (and thus is not graphic), as a vertex of degree 3 requires at least that many other vertices with non-zero degree.

Section 1.4 - Question 9

For each $n \geq 1$, prove or disprove that every simply digraph with n vertices has two vertices with the same out-degree or two vertices with the same in-degree.

For each $n \geq 1$, it is not the case that every simple digraph with n vertices has two vertices with the same out-degree or two vertices with the same in-degree.

That is, for each $n \geq 1$, there exists an n-vertex simple digraph in which the vertices have distinct in-degrees and distinct out-degrees. Consider the graph with vertices $v_1, \ldots, v_n$ and edges $\{v_i v_j : 1 \leq i < j \leq n\}$. That is, a single directed edge exists for each set of adjacent vertices. As a result, $d^-(v_i) = i - 1$ and $d^+(v_i) = n - i$. This implies that the in-degrees are distinct, and the out-degrees are distinct. $\square$

Section 1.4 - Question 10

Prove that a digraph is strongly connected if and only if for each partition of the vertex set into nonempty sets S and T, there is an edge from S to T.

A digraph is strongly connected if for every ordered pair of vertices u and v there is a path from u to v.

Let $x \in S$ and $y \in T$. Given that a digraph D is strongly connected, D contains a path from x to y. Thus, there must be an edge from S to T, such that the path from x to y leaves S and enters T.

Conversely, let S be the set of vertices "reachable" from an arbitrary vertex x in D. If there exists an edge between vertex sets for every partition of D, then there is an edge leaving S, which adds a vertex to the set S. Presuming that $S \neq V(D)$, we know that D is strongly connected, as x is an arbitrary choice, and thus, each vertex is "reachable" from every other. $\square$

Section 1.4 - Question 15

Let G be the simple digraph with vertex set $\left\{(i,j) \in \mathbb{Z}^2 : 0 \le i \le m \text{ and } 0 \le n\right\}$ and an edge from (i,j) to (i',j') if and only if (i',j') is obtained from (i,j) by adding 1 to one coordinate. Prove that the number of paths from $(0,0)$ to (m,n) in G is $\binom{m+n}{n}$.

In traversing each edge from $(0,0)$ to (m,n), the value of 1 is added to each coordinate, so that every path has $m+n$ edges.

Let us record each path as a binary string, i.e. a list of 0s and 1s, such that 0 represents increasing the first coordinate, while 1 represents increasing the second coordinate. Thus, each binary string containing m 0s and n 1s represents a unique path.

There are $\binom{m+n}{m} = \binom{m+n}{n}$ ways to create such a binary string by choosing positions for the 0s and 1s. Thus, the bijection implies that the number of paths from $(0,0)$ to (m,n) in G is $\binom{m+n}{m}$. $\square$