MATH 38 - Graph Theory

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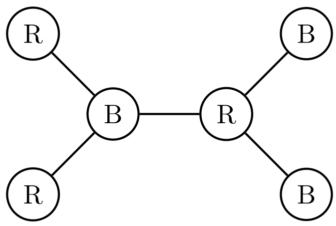
Homework 8

Section 5.1 - Question 12

Prove or Disprove: Every k-chromatic graph G has a proper k-coloring in which some color class has $\alpha(G)$ vertices.

False - Counterexample



In the graph G above, which is bipartite, every proper 2-coloring has 3 vertices in each color class, as indicated by the R and B values. Regardless, $\alpha(G) = 4$, as given by the leaf vertices.

Section 5.1 - Question 14

Prove or Disprove: For every graph G, $\chi(G) \leq n(G) - \alpha(G) + 1$.

True - If we give a color to a maximum independent set and give different colors to the remaining $n(G) - \alpha(G)$ vertices, we have constructed a proper coloring. $\square$

Section 5.1 - Question 33

Prove that every graph G has a vertex ordering relative to which greedy coloring uses $\chi(G)$ colors.

Let there be a coloring of G that is "optimal" (i.e. uses the least number of colors), say c. Let us represent the vertices of G $(v_1, \ldots, v_n)$ in terms of c (as numerals), so the vertices are labelled 1 if they correspond to the first color in c, 2 if they correspond to the second color, and so on.

We aim to prove that the greedy coloring for the vertex ordering of G gives vertex v_i a color (numeral) at most $c(v_i)$. To do so, we use induction on i.

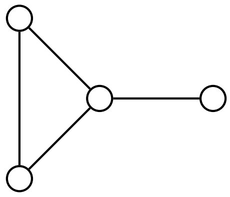
Base Case: Based on the construction, the vertex v_1 receives color $c(v_1) = 1$.

Induction Step: For $i > 1$, the induction hypothesis states that v_j receives color (at most) $c(v_j)$ for all $j < i$. Based on the construction, the only vertices v_j with $c(v_j) = c(v_i)$ are those in the same class (in terms of color) with v_i. In the "optimal" coloring, these are not adjacent to v_i.

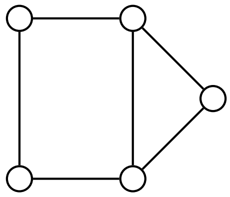
Thus, the colors used for earlier neighbors of v_i are in the set $\{1, \ldots, c(v_i) - 1\}$ and the construction assigns the color (at most) $c(v_i)$ to the vertex v_i. $\square$

Section 5.3 - Question 1

Compute the chromatic polynomials of the graphs below.



The chromatic polynomial of the graph is $k(k-1)^2(k-2)$. This follows from a simplicial elimination ordering.



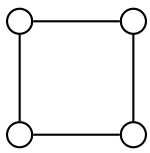
The chromatic polynomial of the graph is $k(k-1)\left(k^2-3k+3\right)(k-1)-k(k-1)\left(k^2-3k+3\right)$. This follows from the deletion-contraction method, using an edge e that connect to the rightmost vertex.

Section 5.3 - Question 3

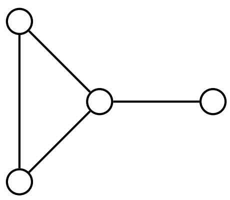
Prove that $k^4-4k^3+3k^2$ is not a chromatic polynomial.

According to Whitney's Theorem, the chromatic polynomial $\chi(G;k)$ has degree $n(G)$, with integer coefficients alternating in sign and beginning $1, -e(G), \ldots$

Thus, with $\chi(G;k) = k^4-4k^3+3k^2$, the graph G would have 4 vertices and 4 edges. The only such graphs are as follows:



This graph (C_4) has a chromatic polynomial of $k(k-1)\left(k^2-3k+3\right)$, which is equal to $k^4-4k^3+6k^2-3k$.

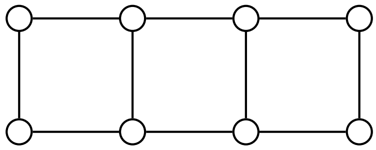


This graph has a chromatic polynomial of $k(k-1)^2(k-2)$, which is equal to $k^4-4k^3+5k^2-2k$.

As neither chromatic polynomial is equal to $k^4-4k^3+3k^2$, we have demonstrated that it is not a chromatic polynomial. $\square$

Section 5.3 - Question 5

For $n > 1$, let $G_n = P_n \Box K_2$; this is the graph with $2n$ vertices and $3n-2$ edges shown below. Prove that $\chi(G_n; k) = \left(k^2 - 3k + 3\right)^{n-1} k(k-1)$.



Proof - Induction on n

Base Case: As G_1 is a tree with 2 vertices, $\chi(G_1; k) = k(k-1)$.

Induction Step: For $n > 1$, let u_n and v_n be the two rightmost vertices of G_n. The proper colorings of G_n may be obtained from the proper colorings of G_{n-1} by assigning colors to u_n and v_n.

Given the construction of the graph, each proper coloring c of G reflects the following: $c(u_{n-1}) \neq c(v_{n-1})$. Thus, each coloring c extends to the same number of colorings of G_n.

There are $(k-1)^2$ ways to specify $c(u_n)$ and $c(v_n)$ so that $c(u_n) \neq c(u_{n-1})$ and $c(v_n) \neq c(v_{n-1})$. Of these possibilities, $k-2$ give u_n and v_n the same color, so we delete them.

Thus, as $(k-1)^2 - (k-2) = k^2 - 3k + 3$, we have (by the induction hypothesis, $\chi(G_n; k) = \left(k^2 - 3k + 3\right) \chi(G_{n-1}; k)$. This produces $\chi(G_n; k) = \left(k^2 - 3k + 3\right)^{n-1} k(k-1)$. $\square$