MATH 38 - Graph Theory

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Homework 1

Question 10

Prove Or Disprove: The complement of a simple disconnected graph must be connected.

Definition: A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.

Definition: The complement $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E\left(\overline{G}\right)$ if and only if $uv \notin E(G)$.

Definition: A graph G is connected if each pair of vertices in G belongs to a path; otherwise, G is disconnected.

Thus, a disconnected graph G has vertices u, v that do not belong to a path, so u, v are adjacent in $\overline{G}$.

There is no vertex w in G such that u and v are adjacent to w, as that would create a path connecting u, v. Thus, every vertex in $\overline{G}$ that is not u or v is adjacent to u and/or v.

Thus, there is a path through $\{u, v\}$ for every vertex $w' \in \overline{G}$ to reach every other vertex, so the complement of a simple disconnected graph must be connected. $\square$

Question 13

Let G be the graph whose vertex set is the set of k-tuples with coordinates in $\{0, 1\}$, with x adjacent to y when x and y differ in exactly one position. Determine whether G is bipartite.

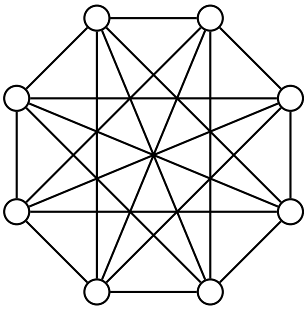
Definition: A graph G is bipartite if $V(G)$ is the union of two disjoint (possibly empty) independent sets called partite sets of G.

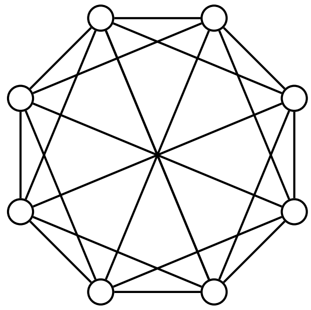
The graph G whose vertex set is the set of k-tuples with coordinates in $\{0, 1\}$ (i.e. $\{0, 1\}^k$), with x adjacent to y when x and y differ in exactly one position is bipartite.

To determine this, consider the 'parity' of 0s (or alternatively, 1s). When x and y differ in exactly one position (i.e. a 0 flips to a 1 or vice versa), the parity of 0s (or alternatively, 1s) flips. Thus, the partite sets are given by the parity, as adjacent vertices in the graph G have opposite parity.

Question 16

Determine whether the graphs below are isomorphic.





Definition: An isomorphism from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u) f(v) \in E(H)$. We say "G is isomorphic to H", written $G \cong H$, if there is an isomorphism from G to H.

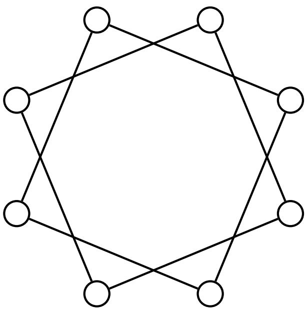
Definition: The complement $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E\left(\overline{G}\right)$ if and only if $uv \notin E(G)$.

The graphs are *not* isomorphic.

Remark: The structural properties of a graph are determined by its adjacency relation and hence are preserved by isomorphism. We can prove that G and H are not isomorphic by finding some structural property in which they differ (i.e. different complements).

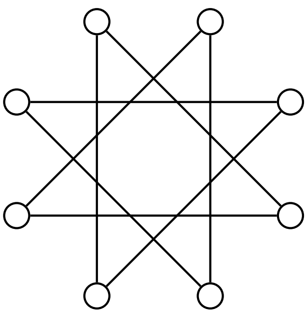
Definition: A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The (unlabeled) cycle with n vertices is denoted C_n, an n-cycle is a cycle with n vertices.

Let G be the first graph displayed. The complement $\overline{G}$ is as follows:



This is a graph of two 4-cycles, which are disconnected.

Let H be the second graph displayed. The complement $\overline{H}$ is as follows:



This is a graph of one 8-cycle, which is connected.

Thus, given the structural properties of $\overline{G}$ and $\overline{H}$, the graphs G and H are not isomorphic.

Question 29

Prove that every set of six people contains (at least) three mutual acquaintances or three mutual strangers.

Let G be a graph representing the relations of acquaintances. Let u represent a vertex in G ($u \in V(G)$), which is representative of an individual person.

Definition: When u and v are the endpoints of an edge, they are adjacent and are neighbors.

Since there are five vertices in G that u may be adjacent to, u must be adjacent to at least three vertices or non-adjacent to at least three vertices.

Thus, u is adjacent to at least three vertices in G or $\overline{G}$, by structural properties. The statement (to prove) remains the same regardless of whether we use G or $\overline{G}$, so we may simply assume that u is adjacent to at least three vertices in G (by symmetry).

If any two of these vertices are adjacent (indicating they are acquaintances), then there are three mutual acquaintances (with u included). Otherwise, if none of these vertices are adjacent, then there are three mutual strangers.

Thus, in every set of six people, there are (at least) three mutual acquaintances or three mutual strangers. $\square$

Question 30

Let G be a simple graph with adjacency matrix A and incidence matrix M. Prove that the degree of v_i is the ith diagonal entry in A^2 and in MM^T. What do the entries in position (i, j) of A^2 and MM^T say about G?

Definition: Let G be a loopless graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. The adjacency matrix of G, written $A(G)$, is the n-by-n matrix in which entry $a_{i,j}$ is the number of edges in G with endpoints $\{v_i, v_j\}$. The incidence matrix $M(G)$ is the n-by-m matrix in which entry $m_{i,j}$ is 1 if v_i is an endpoint of e_j and otherwise is 0. If vertex v is an endpoint of edge e, then v and e are incident.

Definition: The degree of vertex v (in a loopless graph) is the number of incident edges.

Remark: An adjacency matrix is determined by a vertex ordering. Every adjacency matrix is symmetric ($a_{i,j} = a_{j,i}$ for all i, j). An adjacency matrix of a simple graph G has entries 0 or 1, with 0s on the diagonal. The degree of v is the sum of the entries in the row for v in either $A(G)$ or $M(G)$.

The ith diagonal entry in A^2 and in MM^T is the sum of squares of the entries in the ith row. This follows from the definition of matrix multiplication. The adjacency matrix is symmetric, so $A = A^T$, thus, $A^2 = AA^T$.

Consider A^2. The graph G is simple, so the entries of the adjacency matrix are 0 or 1. Thus, the sum of squares of the entries in the ith row equals the number of 1s in the row, which denotes the adjacent vertices. This value is equal to the degree of the vertex.

Consider MM^T. The entries of the incidence matrix are 0 or 1. Thus, the sum of squares of the entries in the ith row equals the number of 1s in the row, which denotes the incident edges. This value is equal to the vertex of the degree.

Thus, the degree of v_i is the ith diagonal entry in A^2 and in MM^T. $\square$

When performing the matrix multiplication $A^2 = AA^T$, the entry in position (i, j) is given by the inner product of row i and column j ($\sum_{k=1}^{n} a_{i,k}a_{k,j}$). The graph G is simple, so the entries of the adjacency matrix are 0 or 1 (depending on whether the vertices are adjacent). Thus, $a_{i,k}a_{k,j} = 1$ if v_k is adjacent to v_i and v_j, 0 otherwise.

Thus, the entries in position (i, j) of A^2 indicate the number of vertices adjacent to v_i and v_j. $\square$

When performing the matrix multiplication MM^T, the entry in position (i, j) is given by the inner product of row i and column j ($\sum_{k=1}^{n} m_{i,k}m_{k,j}$). The entries of the incidence matrix are 0 or 1.

The ith row of M contains 1s corresponding to the edges incident to v_i. The jth column of M^T is equivalent to the jth row of M, which contains 1s corresponding to the edges incident to v_j. Thus, $m_{i,k}m_{k,j} = 1$ if a *given* edge is incident to v_i and v_j, 0 otherwise.

Thus, the entries in position (i, j) of MM^T indicate the number of edges between v_i and v_j. $\square$