# MATH 38 - Graph Theory

Carter Kruse

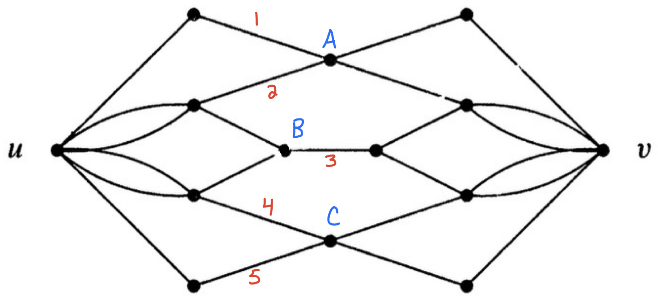
## Homework 7

### Section 4.2 - Question 1

Determine $\kappa(u,v)$ and $\kappa'(u,v)$ in the graph drawn below. (Hint: Use the dual problems to give short proofs of optimality.)

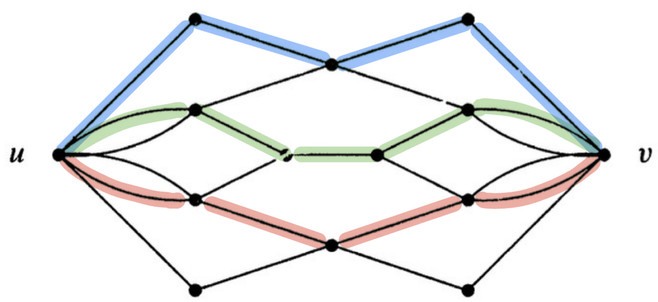
---

In the graph, $\kappa(u,v) = 3$ and $\kappa'(u,v) = 5$. To demonstrate this, consider deleting the vertices labeled $\{A,B,C\}$ or the edges labeled $\{1,2,3,4,5\}$. This removes all potential $u,v$ paths, and so makes $v$ unreachable from $v$ (and vice versa). This highlights the upper bounds for the values of $\kappa(u,v)$ and $\kappa'(u,v)$.

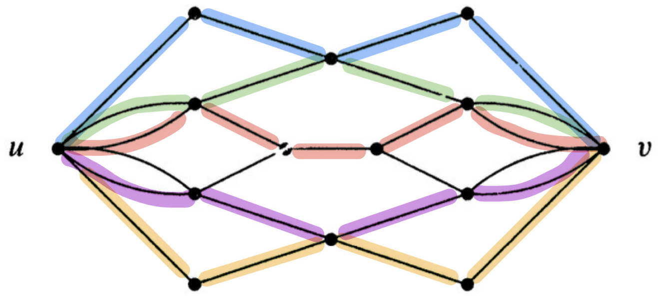



To highlight the lower bounds for the values of $\kappa(u,v)$ and $\kappa'(u,v)$, we consider the following.

Showing a set of 3 pairwise internally disjoint $u,v$ paths proves $\kappa(u,v) \geq 3$, as distinct vertices (within the path) must be deleted to remove the paths. This is shown as follows.



Showing a set of 5 pairwise edge disjoint $u, v$ paths proves $\kappa'(u, v) \geq 5$, as distinct edges (within the path) must be deleted to remove the paths. This is shown as follows.



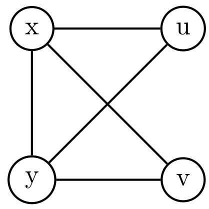

## Section 4.2 - Question 4

**Prove Or Disprove:** If $P$ is a $u, v$ path in a 2-connected graph $G$, then there is a $u, v$ path $Q$ that is internally disjoint from $P$.

---

False - If $P$ is a $u, v$ path in a 2-connected graph $G$, then there does not have to be a $u, v$ path that is internally disjoint from $P$.

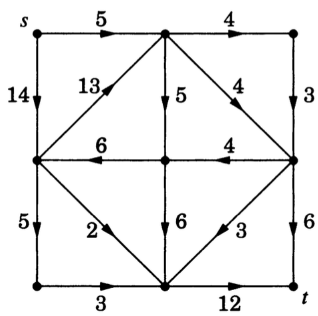
Consider the graph $G = K_4 - uv$, with $V(G) = \{u, x, y, v\}$ as shown below.




This graph is 2-connected (connected with no cut vertex), yet does not contain a $u, v$ path internally disjoint from the $u, v$ path $P$ that is $u \to x \to y \to v$. $\square$

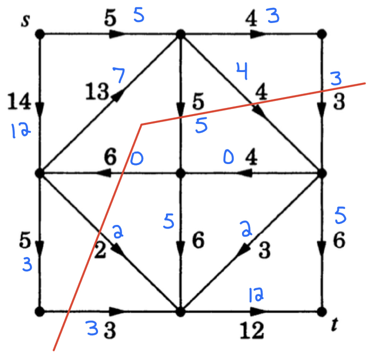
## Section 4.3 - Question 2

In the network below, find a maximum flow from $s$ to $t$. Prove that your answer is optimal by using the dual problem, and explain why this proves optimality.




---

In the following network, with edge capacities as indicated, the flow values shown in blue form a maximum feasible flow.




By inspection, the flow values satisfy the capacity and conservation constraints. The value of the flow is 17. To prove that the flow is optimal, let us demonstrate a solution to the dual (min cut) problem that has the same value. The optimality follows because the value of every feasible flow is at most the capacity of this cut.

The cut (shown in red) has a source set of the vertices located above the red line and a sink set of the vertices located below the red line. The edges of the cut are those that the red line passes through, with the exception of any edge with a flow of zero. The total capacity of this cut is 17, which is equivalent to the value of the flow previously determined. □

## Section 4.3 - Question 10

Use network flows to prove the König-Egerváry Theorem ($\alpha'(G) = \beta(G)$ if $G$ is bipartite).

---

Let $G$ be a bipartite graph with $X$ and $Y$ as the bipartite sets. Let $N$ represent a network with a source $s$ and a sink $t$, such that there are edges (with capacity 1) from $s$ to each $x \in X$ and from $t$ to each $y \in Y$. Within $N$, orient each (relevant) edge of $G$ from $X$ to $Y$, with infinite capacity.

According to the *Integrality Theorem*, if all capacities are integers, then there exists a maximum flow assigning integer values to all edges. Thus, there is a maximum flow $f$ with an integer value at each edge for the network $N$.

The edges of capacity one thus force the edges between $X$ and $Y$ with no-zero flow in $f$ to be a matching. The expression $val(f)$ is the number of these edges, as conservation constraints require the flow along each edge to extend by edges of capacity one from $s$ to $t$. Thus, we have constructed a matching of size $val(f)$, so $\alpha'(G) \geq val(f)$.

Let $[S, T]$ be a minimum cut in $N$ (which has finite capacity, since $[s, V(N) - s]$ is a cut of finite capacity). Further, let $X' = S \cap X$ and $Y' = T \cap Y$. A cut of finite capacity has no edge of infinite capacity from $S$ to $T$. Thus, $G$ has no edge from $X'$ to $Y'$.

This indicates that $(X - S) \cup (Y - T)$ is a set of vertices that covers every edge in $G$. The cut $[S, T]$ consists of the edges from $s$ to $X - S = X \cap T$ and the edges from $Y - T = Y \cap S$ to $T$. The capacity of the cut is equivalent to the number of these edges, which is $|(X - S) \cup (Y - T)|$. Thus, there is a vertex cover of size $\bigcap(S, T)$, so $\beta(G) \leq \bigcap(S, T)$.

By the *Max Flow-Min Cut Theorem*, $\beta(G) \leq \bigcap(S, T) = val(f) \leq \alpha'(G)$. Given that $\alpha'(G) \leq \beta(G)$, for any graph $G$, we have $\alpha'(G) = \beta(G)$ for every bipartite graph $G$. □

## Section 4.3 - Question 14

In a large university with $k$ academic departments, we must appoint an important committee. One professor will be chosen from each department. Some professors have joint appointments in two or more departments, but each must be the designated representative of at most one department. We must use equally many assistant professors, associate professors, and full professors among the chosen representatives (assume that $k$ is divisible by 3). How can the committee be found?

(Hint: Build a network in which units of flow correspond to professors chosen for the committee and capacities enforce the various constraints. Explain how to use the network to test whether such a committee exists and find it if it does.) (Hall [1956])

---

Let us consider the network flow corresponding to choosing $\frac{k}{3}$ assistant professors, $\frac{k}{3}$ associate professors, and $\frac{k}{3}$ full professors to represent each department.

Let there be a node for each department, each professor, and each rank in this maximum flow network. The source node $s$ sends an edge of capacity one to each department node. Each department node sens an edge to each of its professors' nodes, which may have infinite capacity. Each professor node sends an edge of capacity one to the node corresponding to that professor's rank. There is an edge of capacity $\frac{k}{3}$ from each rank to the sink $t$.

By setting the scenario up as indicated, each unit of flow in the network selects a professor on the committee, with the following constraints. The edges from the source to the departments ensure that each department is represented at most once.

Since the capacity of an edge leaving each professor is one, the professor can represent only one (a single) department. The capacities on the three edges into the sink ensure that there is a balanced representation across ranks.

To determine if the desired committee exists, we consider when the network has a feasible flow of value $k$. This is only the case if every source/sink cut has a capacity of at least $k$.

Using edges with infinite capacity simplifies the analysis of finite cuts, as cuts $[S, T]$ cannot have an edge of infinite capacity from $S$ to $T$. Any capacity of at least one on the edges from a department to its professors yields the same feasible flows. $\square$