MATH 38 - Graph Theory

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Homework 6

Section 3.1 - Question 5

Prove that $\alpha(G) \geq \frac{n(G)}{\Delta(G)+1}$ for every graph G.

Initially, recall that an independent set is a set of mutually non-adjacent vertices. The expression $\alpha(G)$ denotes the maximum size of an independent set in G.

To form such independent set S, iteratively select a remaining vertex (in the graph G) for S and delete the vertex and all of its neighbors.

In doing so, each iterative step adds a single vertex to S and deletes at most $\Delta(G)+1$ vertices from G, as $\Delta(G)$ denotes the maximum degree of any vertex in the graph G. Thus, we must perform at least $\frac{n(G)}{\Delta(G)+1}$ steps to create an independent set at least the size of $\alpha(G)$.

Thus, for every graph G, $\alpha(G) \geq \frac{n(G)}{\Delta(G)+1}$. $\square$

Section 3.1 - Question 9

Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

Let M be a maximal matching in a graph G. The vertices "saturated" by M form a vertex cover necessarily, as if an edge did not have a vertex in the set, it could be added to M, as a maximal matching.

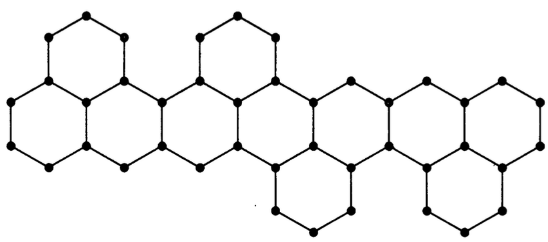
Every vertex cover has size of at least $\alpha'(G)$, which denotes the maximum size of a matching in G.

Given that $\beta(G) \geq \alpha'(G)$, as demonstrated in class, where $\beta(G)$ denotes the minimum size of a vertex cover in G, we have that $2|M| \geq \beta(G) \geq \alpha'(G)$.

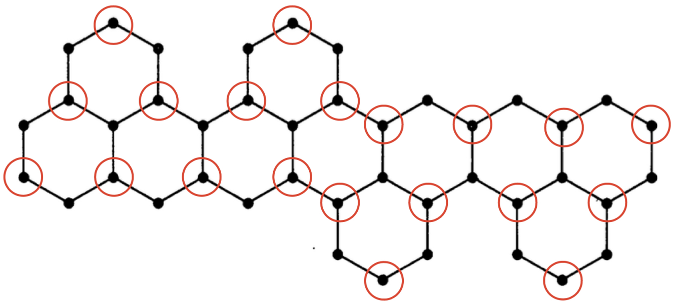
Thus, we know that every maximal matching in a graph G has at least $\geq \alpha'(G)$ edges. $\square$

Section 3.1 - Question 28

Exhibit a perfect matching in the graph below or give a short proof that it has none. (Lovász-Plummer (1986)



The graph displayed above has 42 vertices, thus, a perfect matching would have 21 edges. Given that the edges of a matching must be covered by distinct vertices in a vertex cover, consider that the marked vertices form a vertex cover of size 20. Thus, we may conclude that there is no matching with more than 20 edges, so the graph does not have a perfect matching.



Section 4.1 - Question 5

Let G be a connected graph with at least three vertices. Form G' from G by adding an edge with endpoints x, y whenever $d_G(x, y) = 2$. Prove that G' is 2-connected.

Let u and v be vertices of G, where G is a connected graph with at least three vertices. Since G is connected, there is an u, v path, let us say $x_0, x_1, \ldots, x_n$.

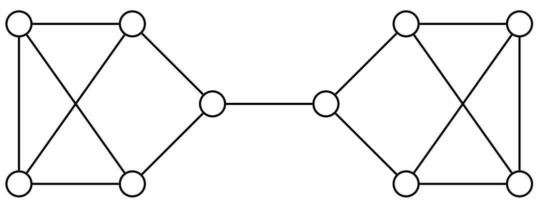
Given the construction of G' from G, in that we add an edge with endpoints x, y whenever $d_G(x, y) = 2$, there is a u, v path given as $x_0, x_2, x_4, \ldots, x_k$ and an (internally) disjoint u, v path given as $x_0, x_1, x_3, \ldots, x_k$.

Thus, at least two vertices must be deleted to separate u and v in the graph G', so we have demonstrated that G' is 2-connected. $\square$

Section 4.1 - Question 10

Find (with proof) the smallest 3-regular simple graph having connectivity 1.

The following graph G is the smallest 3-regular simple graph having connectivity 1.



As the graph above is 3-regular and has connectivity 1, it suffices to show that every 3-regular simple graph with connectivity 1 has at least 10 vertices.

Since $\kappa = \kappa'$ for 3-regular graphs, we aim to find the smallest 3-regular connected graph G with cut-edge e. The graph $G - e$ has two components; each component has a single vertex of degree 2 and the rest of degree 3.

Given that the components have a vertex of degree 3, each component must have at least 4 vertices. Given that there are an even number of vertices of degree 3, each component must have at least 5 vertices.

Thus, we have found (and demonstrated via proof) the smallest 3-regular simple graph having connectivity 1. □

Section 4.1 - Question 11

Prove that $\kappa'(G) = \kappa(G)$ when G is a simple graph with $\Delta(G) \leq 3$.

The argument follows similarly to the proof of the following (which was done in class):

If G is 3-regular, then $\kappa(G) = \kappa'(G)$.

Let S be a minimum vertex cut, so that $|S| = \kappa(G)$. As we know that $\kappa(G) \leq \kappa'(G)$ always, we simply need to construct an edge cut of size $|S|$.

Let P and Q be two components of $G - S$. As S is a minimum vertex cut, each $v \in S$ has a neighbor in P and a neighbor in Q. Since $\Delta(G) \leq 3$, v cannot have two neighbors in P and two neighbors in Q.

For each $v \in S$, delete the edge to the member of P or Q that has only a single edge between v and P or Q. The count of the edges broken is exactly $\kappa(G)$.

This breaks all of the paths from P to Q, except in the case where there is an edge between v_1 and v_2 in S, so a path can come into S via v_1 and leave via v_2. In this special case, simply choose to delete the edges going to P for each v_1 and v_2. The result remains the same (in terms of counting).

Thus, we have demonstrated that if G is 3-regular, then $\kappa(G) = \kappa'(G)$, as we found an edge cut of size $|S| = \kappa(G)$. □

Section 4.1 - Question 15

Use Proposition 4.1.12 and Theorem 4.1.11 to prove that the Petersen graph is 3-connected.

Proposition 4.1.12: If S is a set of vertices in a graph G, then $|\left[S, \overline{S}\right]| = \left[\sum_{v \in S} d(v)\right] - 2e(G[S])$.

Theorem 4.1.11: If G is a 3-regular graph, then $\kappa(G) = \kappa'(G)$.

To demonstrate that the Petersen graph is 3-connected, first consider that the Petersen graph G is 3-regular. Thus, by **Theorem 4.1.11**, we may demonstrate that G is 3-edge connected, that is $\kappa(G) = \kappa'(G) = 3$.

Let $\left[S, \overline{S}\right]$ be a minimum edge cut. If $\left[S, \overline{S}\right] < 3$, then, by **Proposition 4.1.12**, $\left[\sum_{v \in S} d(v)\right] - 2e(G[S]) \leq 2$. This value may be determine from either side of the edge cut, so we naturally assume $|S| \leq |\overline{S}|$.

The Petersen graph G does not have a cycle of length less than 5. Thus, when $|S| < 5$, $e(G[S]) \leq |S| - 1$. From the above, this produces $3|S| - 2(|S| - 1) \leq 2$, which is not possible for non-empty S, as it simplifies to $|S| \leq 0$.

For $|S| = 5$, we have $3|S| - 2|S| \leq 2$, which gives $|S| \leq 2$, which is false. Thus, we have demonstrated that there does not exist an edge cut of size less than 3, so we have illustrated that the Petersen graph is 3-connected. □