MATH 38 - Graph Theory

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Homework 2

Section 1.2 - Question 8

Determine the values of m and n such that $K_{m,n}$ is Eulerian.

The complete bipartite graph $K_{m,n}$ is Eulerian if and only if $m, n \in 2\mathbb{N}$ or $m = 0$ or $n = 0$.

If $m = 0$ or $n = 0$, the graph contains no edges, thus, it is Eulerian. Otherwise, the graph is connected (as it is $K_{m,n}$) with vertices of degree m or n.

By Theorem 1.2.26, a graph G is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree.

Thus, if $K_{m,n}$ is Eulerian, all of its vertices must be of even degree, which is only the case if (and only if) m and n are even.

Section 1.2 - Question 10

Every Eulerian bipartite graph has an even number of edges.

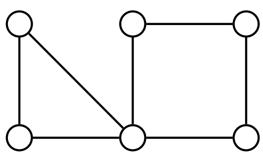
True - By Theorem 1.2.26, a graph G is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree.

To count the edges of a bipartite graph, we may consider a single partite set. The sum of the degrees of the vertices in a partite set will be equal to the number of edges in the bipartite graph, as each edge is counted exactly once.

Thus, as each of the elements in the sum are even (as the degree of each vertex is even), the sum is even, which indicates an even number of edges.

Every Eulerian simple graph with an even number of vertices has an even number of edges.

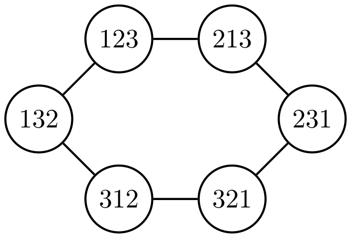
False - As a counterexample, suppose we have the simple cycles C_3 and C_4. A composition of these cycles is given as follows:



The above graph is Eulerian, and has an even number of vertices, but has an odd number of edges.

Section 1.2 - Question 17

Let G_n be the graph whose vertices are the permutations of $\{1, \ldots, n\}$, with two permutations $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ adjacent if they differ by interchanging a pair of adjacent entries (G_3 shown below). Prove that G is connected.



To demonstrate that G is connected, we aim to show that every "node" which represents a permutation is connected to the permutation $\{1, \ldots, n\}$. This implies that for all $u, v \in V(G)$, there is a u, v-path in G, which indicates G is connected.

To demonstrate that every node is connected to the permutation $\{1, \ldots, n\}$, consider an algorithm where the element 1 is shifted to the front by adjacent "swaps" with other elements. Then, do the same for element 2, moving it to the second position, and so on.

This creates a walk from any given vertex in the graph to $\{1, \ldots, n\}$, which indicates there is a path from any given vertex to $\{1, \ldots, n\}$. Thus, G is connected.

Section 1.2 - Question 29

Let G be a connected simple graph not having P_4 or C_3 as an induced subgraph. Prove that G is a biclique (complete bipartite graph).

Suppose we choose a vertex $v \in V(G)$. As there is no C_3 in G, the elements of $N(v)$ are independent of each other (where $N(v)$ represents the adjacent vertices of v, or the "neighbors").

If we remove the original vertex and the neighbors from the set of vertices, we have the following set: $S = V(G) - \{v\} - N(v)$. Every vertex $u \in S$ is adjacent to a vertex in $N(v)$, otherwise, there would be a path from v to u with an induced P_4. That is, if $u \in S$ is adjacent to $v' \in N(v)$ but not $v'' \in N(v)$, the path P_4 may be given as u, v', v, v'' by revisiting the neighbors of v.

Thus, every vertex of S must be adjacent to every vertex of $N(v)$, to satisfy the requirements. As G does not contain C_3, $\{v\}$ and S are independent. Thus, we may construct a complete bipartite graph with the following partition: $N(v)$, $S \cup \{v\}$.

Section 1.2 - Question 38

Prove that every n-vertex graph with at least n edges contains a cycle.

Suppose there exists a n-vertex graph with at least n edges that does not contain a cycle. By Theorem 1.2.14, an edge is a cut-edge if and only if it belongs to no cycle. Thus, every edge of the graph is a cut-edge.

By Definition 1.2.12, a cut-edge of a graph is an edge whose deletion increases the number of components. Thus, as cut-edges are removed the number of components increases; removing all cut-edges produces $n + 1$ components. This contradicts the statement that the graph contains n vertices.

Section 1.3 - Question 1

If u and v are the only vertices of odd degree in a graph G, then G contains a u, v-path.

Given that the graph G may be broken into components, consider that the degree of a vertex in a given component of G is equivalent its degree in G. If two vertices u and v are the only vertices of odd degree in a graph G, they must be in the same component, as otherwise the sum of the degree in each component would be odd.

Thus, as u and v are in the same component of a graph G, there is a walk from u to v, thus, there is a u, v-path.

Section 1.3 - Question 12

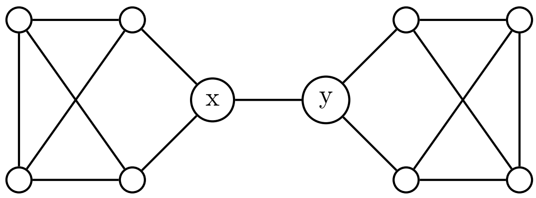
Prove that an even graph has no cut-edge. For each $k \geq 1$, construct a $2k + 1$-regular simple graph having a cut-edge.

Let us assume (by contradiction) that a graph G has a cut-edge. In removing the edge from the graph, there are two (induced) subgraphs that individual have an odd sum of degrees. This establishes a contradiction as the sum of degrees of every graph is even.

To construct a $2k + 1$-regular simple graph with a cut-edge, start with K_{2k+2} and remove k pairwise disjoint edges. Following this, add a vertex adjacent to all of the vertices that lost an edge.

This creates a graph with $2k + 2$ vertices of degree $2k + 1$ and a single vertex of degree $2k$. The final $2k + 1$-regular simple graph may be formed by adding an edge between disjoint copies of the graph, specifically by joining the vertices of degree $2k$.

The following graph is given as an example:



The graph is 3-regular with 10 vertices and a cut-edge xy.

Section 1.3 - Question 32

Prove that the number of simple even graphs with vertex set $[n]$ is $2^{\binom{n-1}{2}}$. (Hint: Establish a bijection to the set of all simple graphs with vertex set $[n - 1]$.)

Let A be the set of simple even graphs with vertex set $[n]$ and B be the set of *all* simple graphs with vertex set $[n - 1]$. The size of B is $2^{\binom{n-1}{2}}$, thus, we aim to establish a bijection from A to B.

Given a graph $G \in A$, we may get a graph $G' \in B$ by simply deleting a vertex of G. To illustrate that each graph G' arises only once, consider that we may construct G by adding the vertex back and making it adjacent to each vertex with odd degree in G'.

Thus, the vertices with odd degree in G' must have even degree in G. The removed/added vertex is of even degree, as there are an even number of vertices in G' with odd degree. Thus, we reaffirm that $G \in A$.

G' is the graph obtained by deleting a vertex from G and every simple even graph in which deleting such a vertex yields G' must have that vertex adjacent to the same vertices as in G.

Thus, this creates a bijection from A to B, so the sets have the same size.