MATH 053/126 - Partial Differential Equations

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Homework 4

Instructions/Notation

Please show all steps to get your answers. Specify the problems you discussed with other students (including names).

The starred problems are recommended but not required for undergraduate/non-math major graduate students and required for all math major graduate students.

Notation

- $\mathbb{R}$: The set of all real numbers.
- $\mathbb{R}^+$: The set all positive real numbers $\{x \in \mathbb{R} | x > 0\}$.
- $\frac{\partial u}{\partial t} = \partial_t u = u_t$
- $\frac{\partial u}{\partial x} = \partial_x u = u_x$

Questions

Question 1

For harmonic functions, prove the maximum principle using the mean value property. (Try a proof by contradiction).

While we present the maximum principle in dimension 3, it holds in any space dimension. Loosely speaking, the maximum principle states that *harmonic functions do not tolerate local extrema (maxima and minima).* More precisely, the principle asserts that if u is a $\mathcal{C}^2$ harmonic function on a **bounded** domain $\Omega \subset \mathbb{R}^3$ which is continuous up to the boundary, i.e. $u \in \mathcal{C}\left(\overline{\Omega}\right)$, then the maximum value of u must occur on the boundary.

This means that if we denote M as the maximum value of u over $\overline{\Omega} = \Omega \cup \partial\Omega$, then there must be a point on the boundary for which u takes on M. In fact, a stronger statement is true since, by assumption, our domain Ω is connected. It states that the maximum of u can **only** occur on the boundary unless u is identically constant.

The Maximum Principle: Let u be a $\mathcal{C}^2$ harmonic function on a bounded domain $\Omega \subset \mathbb{R}^3$ which is continuous up to the boundary, i.e., $u \in \mathcal{C}\left(\overline{\Omega}\right)$. If u attains its maximum over $\overline{\Omega} = \Omega \cup \partial\Omega$ at a point in Ω, then u must be identically constant inside Ω.

We give a proof which ends in an informal, yet convincing, step that can readily be made precise with a bit of point set topology (open and closed sets).

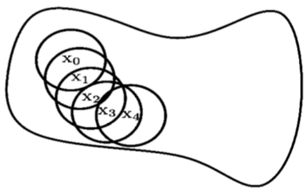
Proof: Since u is continuous on $\overline{\Omega}$ (a compact set), u must attain its maximum somewhere in $\overline{\Omega}$. Let M be the maximum value of u over $\overline{\Omega}$. By assumption, $u(\boldsymbol{x}_0) = M$ for some $\boldsymbol{x}_0 \in \Omega$. Choose a sphere centered at $\boldsymbol{x}_0$ which lies entirely inside Ω. Then, by the mean value property, the value of the center (i.e. $u(\boldsymbol{x}_0)$) equals the average of the values on the sphere. Quickly, we arrive at the punch line:

The only way for M to be **both** the maximum value of u and the average value of u over the sphere is for u to be identically equal to M everywhere on the sphere.

Since the above is true for **any** sphere with center $\boldsymbol{x}_0$ contained in Ω, we have

$$u(\boldsymbol{x}) = M \qquad \text{on } \textbf{any} \text{ solid ball with center } \boldsymbol{x}_0 \text{ which is contained in } \Omega$$

Fix such a ball and now repeat this argument by choosing a new center point (say, $\boldsymbol{x}_1$) in the ball. Since $u(\boldsymbol{x}_1) = M$, we conclude $u(\boldsymbol{x}) = M$ on any solid ball with center $\boldsymbol{x}_1$ which lies in Ω. Repeating on successive balls (perhaps of very small radii), we can “fill up” all the connected domain Ω. We conclude that $u(\boldsymbol{x}) = M$ for **all** $\boldsymbol{x} \in \Omega$. One only need a bit of point set topology (open and closed sets) to make this last point rigorous. Here one shows that the set of all points upon which u is M is both an open and a closed subset of Ω. The only such subset is Ω itself. $\square$



The key idea is to prove the maximum principle. by successively applying the mean value property, specifically using a proof by contradiction.

Question 2

Prove the mean value property for the harmonic function.

The mean value property states that the value of a harmonic function at a given point equals its average on any sphere centered at the point (its **spherical mean**). More precisely, we have the following theorem stated in dimension 3.

Mean Value Property: Let u be a $\mathcal{C}^2$ harmonic function on a domain $\Omega \subset \mathbb{R}^3$. Let $\boldsymbol{x}_0 \in \Omega$ and $r > 0$ such that $B(\boldsymbol{x}_0, r) \subset \Omega$. Then

$$u(\boldsymbol{x}_0) = \frac{1}{4\pi r^2} \iint_{\partial B(\boldsymbol{x}_0, r)} u(\boldsymbol{x})\, dS_{\boldsymbol{x}}$$

The analogous statement holds in 2D (in fact, in any space dimension). In 2D, we would replace $4\pi r^2$ with the corresponding "size" of $\partial B(\boldsymbol{x}_0, r)$, which, in 2D, is $2\pi r$. While we state the proof in 3D, the same proof with minor modifications works in any space dimension. The key step in the proof is the differentiation of the spherical mean.

Proof: Fix $\boldsymbol{x}_0 \in \Omega$. For any $r > 0$ such that $B(\boldsymbol{x}_0, r) \subset \Omega$, we define the spherical mean

$$\phi(r) = \frac{1}{4\pi r^2} \iint_{\partial B(\boldsymbol{x}_0, r) u(\boldsymbol{x})\, dS_{\boldsymbol{x}}}$$

Our goal is to show that ϕ is a constant, i.e. $\phi'(r) = 0$. Once we have established this, we can invoke the averaging lemma to conclude that this constant must be equal to

$$\lim_{r \longrightarrow 0^+} \frac{1}{4\pi r^2} \iint_{\partial B(\boldsymbol{x}_0, r)} u(\boldsymbol{x})\, dS_{\boldsymbol{x}} = u(\boldsymbol{x}_0)$$

This will yield the result, which proves the mean value property.

How do we calculate $\phi'(r)$? We change variables with a translation and dilation in such a way that the new domain of integration becomes $\partial B(\mathbf{0}, 1)$, which is independent of r. To this end, let

$$\boldsymbol{y} = \frac{\boldsymbol{x} - \boldsymbol{x}_0}{r} \quad \text{or} \quad \boldsymbol{x} = \boldsymbol{x}_0 + r\boldsymbol{y}$$

Recall, from spherical coordinates, that in the new variable $\boldsymbol{y} \in \partial B(\mathbf{0}, 1)$, we have

$$r^2 dS_{\boldsymbol{y}} = dS_{\boldsymbol{x}}$$

Thus,

$$\phi(r) = \frac{1}{4\pi r^2} \iint_{\partial B(\boldsymbol{x}_0, r)} u(\boldsymbol{x})\, dS_{\boldsymbol{x}} = \frac{1}{4\pi} \iint_{\partial B(\mathbf{0}, 1)} u(\boldsymbol{x}_0 + r\boldsymbol{y})\, dS_{\boldsymbol{y}}$$

Note that changing to the new variable $\boldsymbol{y}$ still yield an average (a spherical mean). Now we can take the derivative with respect to r by bringing it into the integral and applying the chain rule:

$$\phi'(r) = \frac{1}{4\pi} \iint_{\partial B(\mathbf{0}, 1)} \frac{d}{dr} u(\boldsymbol{x}_0 + r\boldsymbol{y})\, dS_{\boldsymbol{y}} = \frac{1}{4\pi} \iint_{\partial B(\mathbf{0}, 1)} \nabla u(\boldsymbol{x}_0 + r\boldsymbol{y}) \cdot \boldsymbol{y}\, dS_{\boldsymbol{y}}$$

Reverting back to our original coordinates of $\boldsymbol{x}$, we find

$$\phi'(r) = \frac{1}{4\pi r^2} \iint_{\partial B(\boldsymbol{x}_0, r)} \nabla u(\boldsymbol{x}) \cdot \frac{\boldsymbol{x} - \boldsymbol{x}_0}{r}\, dS_{\boldsymbol{x}}$$

For $\boldsymbol{x} \in \partial B(\boldsymbol{x}_0, r)$, $\frac{\boldsymbol{x}-\boldsymbol{x}_0}{r}$ denotes the outer unit normal to $\partial B(\boldsymbol{x}_0, r)$. Thus, we may apply the divergence theorem to conclude that

$$\phi'(r) = \frac{1}{4\pi r^2} \iint_{\partial B(\boldsymbol{x}_0, r)} \nabla u(\boldsymbol{x}) \cdot \frac{\boldsymbol{x}-\boldsymbol{x}_0}{r} \, dS_{\boldsymbol{x}} = \frac{1}{4\pi r^2} \iint_{B(\boldsymbol{x}_0, r)} \Delta u \, d\boldsymbol{x}$$

Since u is harmonic, i.e. $\Delta u = 0$, we have $\phi'(r) = 0$. □

Not surprisingly, we would obtain the same result if we were to average the values over the solid ball, that is

$$u(\boldsymbol{x}_0) = \frac{1}{\frac{4\pi}{3} r^3} \iiint_{B(\boldsymbol{x}_0, r)} u(\boldsymbol{x}) \, d\boldsymbol{x}$$

This directly follows by first integrating over spheres of radius ρ and then integrating from $\rho = 0$ to $\rho = r$. On each spherical integral, we apply the mean value property.

It turns out that the mean value property is **equivalent** to being harmonic in the sense that the converse of the mean value theorem holds true. If $u \in \mathcal{C}^2(\Omega)$ and satisfies the mean value property for all $B(\boldsymbol{x}_0, r) \subset \Omega$, then $\Delta u = 0$ in Ω. Hence, one could say the mean value property is the **essence** of being harmonic.

Averaging Lemma: This result is based upon the notion of the average of a function. It applies in any space dimension, but for convenience let us fix the dimension to be 3. Given a continuous function ϕ on $\mathbb{R}^3$ and a ball $B(\mathbf{0}, r)$ with spherical boundary $\partial B(\mathbf{0}, r)$, the average values of ϕ over the ball and the sphere are given by

$$\frac{3}{4\pi r^3} \iiint_{B(\mathbf{0}, r)} \phi(\boldsymbol{x}) \, d\boldsymbol{x} \quad \text{and} \quad \frac{1}{4\pi r^2} \iint_{\partial B(\mathbf{0}, r)} \phi(\boldsymbol{x}) \, dS \text{ respectively}$$

Note that in the former case we have divided the volume integral by $\frac{4}{3}\pi r^3$, the volumne of the ball $B(\mathbf{0}, r)$, whereas in the latter we have divided the surface integral by $4\pi r^2$, the surface area of the sphere $\partial B(\mathbf{0}, r)$. What happens if we let r tend to 0, i.e. if we take averages over smaller and smaller sets?

Lemma: Suppose that ϕ is continuous on $\mathbb{R}^3$. Then we have

$$\phi(\mathbf{0}) = \lim_{r \to 0^+} \frac{3}{4\pi r^3} \iiint_{B(\mathbf{0}, r)} \phi(\boldsymbol{x}) \, d\boldsymbol{x} = \lim_{r \to 0^+} \frac{1}{4\pi r^2} \iint_{\partial B(\mathbf{0}, r)} \phi(\boldsymbol{x}) \, dS$$

The averaging lemma holds for any point $\boldsymbol{x}_0$ (not just the origin) by taking balls and spheres centered at $\boldsymbol{x}_0$. Let us provide an intuitive explanation for the second equality followed by a proof. The proof of the first equality is almost identical.

Intuitive Explanation: In either case we are considering an integral over a region (either a ball of radius r or a sphere of radius r) which is small and encompasses the origin. Since ϕ is a continuous function, on any such small region, its variation from $\phi(\mathbf{0})$ is small. In other words, for r small, $\phi(\boldsymbol{x})$ is "close to" $\phi(\mathbf{0})$ for x in either $B(\mathbf{0}, r)$ or $\partial B(\mathbf{0}, r)$. Hence, focusing on the sphere, we have some *very small* fixed r_0:

$$\begin{aligned}
\lim_{r \to 0^+} \frac{1}{4\pi r^2} \iint_{\partial B(\mathbf{0}, r)} \phi(\boldsymbol{x}) \, dS \sim \frac{1}{4\pi r_0^2} \iint_{\partial B(\mathbf{0}, r)} \phi(\boldsymbol{x}) \, dS \sim \frac{1}{4\pi r_0^2} \iint_{\partial B(\mathbf{0}, r)} \phi(\mathbf{0}) \, dS \\
= \phi(\mathbf{0}) \frac{1}{4\pi r_0^2} \iint_{\partial B(\mathbf{0}, r)} 1 \, d\boldsymbol{x} \\
= \phi(\mathbf{0}) \frac{1}{4\pi r_0^2} 4\pi r_0^2 = \phi(\mathbf{0})
\end{aligned}$$

where we loosely use the symbol $\sim$ to denote "approximately equals".

Now let us turn this intuition into a proper proof by using the continuity of our function to effectively control the function's fluctuations from $\phi(\mathbf{0})$ on a very small sphere.

Proof: Let $\varepsilon > 0$ and note that $\phi(\mathbf{0}) = \frac{1}{4\pi r^2} \iint_{\partial B(\mathbf{0},r)} \phi(\mathbf{0})\, dS$. We want to show that there exists $\delta > 0$ such that if $r < \delta$, then

$$\left| \frac{1}{4\pi r^2} \iint_{\partial B(\mathbf{0},r)} \phi(\boldsymbol{x})\, dS - \phi(\mathbf{0}) \right| = \left| \frac{1}{4\pi r^2} \iint_{\partial B(\mathbf{0},r)} (\phi(\boldsymbol{x}) - \phi(\mathbf{0}))\, dS \right| < \varepsilon$$

To this end, by the continuity of ϕ there exists a $\delta > 0$ such that if $|x| < \delta$, then $|\phi(\boldsymbol{x}) - \phi(\mathbf{0})| < \varepsilon$. Hence, if $r < \delta$, we have

$$\begin{aligned}
\left| \frac{1}{4\pi r^2} \iint_{\partial B(\mathbf{0},r)} (\phi(\boldsymbol{x}) - \phi(\mathbf{0}))\, dS \right| &\leq \frac{1}{4\pi r^2} \iint_{\partial B(\mathbf{0},r)} |\phi(\boldsymbol{x}) - \phi(\mathbf{0})|\, dS \\
&< \frac{1}{4\pi r^2} \iint_{\partial B(\mathbf{0},r)} \varepsilon\, dS \\
&= \varepsilon \frac{1}{4\pi r^2} \iint_{\partial B(\mathbf{0},r)} 1\, dS \\
&= \varepsilon \frac{1}{4\pi r^2} 4\pi r^2 \\
&= \varepsilon
\end{aligned}$$

We make a few remarks.

- The averaging lemma applies in any space dimension. In each case the average entails division by the appropriate "size" of the (hyper)ball or (hyper)sphere.
- One might wonder if there is something special about balls and spheres. The answer is no. For example, the analogous result holds for averaging over smaller and smaller cubes.
- It is instructive to recall the fundamental theorem of calculus and to think of the averaging lemma as a result about differentiation. Indeed, one can view these limits as "derivatives" of an integral.
- The averaging lemma is very robust in that there is a version which applies to "pretty much every" function (not only continuous functions).

Question 3

Derive the weak formulation of the following Poisson equation with a non-zero Neumann boundary condition

$$-\Delta u = f \text{ in } \Omega \qquad \frac{\partial u}{\partial n} = h(x) \text{ on } \partial\Omega$$

To derive the weak formulation of the following Poisson equation with a non-zero Neumann boundary condition, we use a test function space C_0^∞, which represents compact support. The motivation behind this lies in integration by parts, which is given as follows: $\int_\Omega u\, dv = \int_{\partial\Omega} uv - \int_\Omega v\, du$. Now, let us start with the strong formulation:

$$-\Delta u = f \text{ in } \Omega \qquad \frac{\partial u}{\partial n} = h(x) \text{ on } \partial\Omega$$

As implied, we multiply by a test function $v \in H^1(\Omega) \in C_0^\infty$, and integrate over the domain Ω:

$$\int_\Omega -(\Delta u)\, v\, dx = \int_\Omega f v\, dx$$

The Laplacian term may be re-written to determine the following:

$$-\int_\Omega \nabla \cdot (\nabla u)\, v\, dx = \int_\Omega f v\, dx$$

Now, we apply the following form: $\nabla(v\nabla u) = \nabla v \cdot \nabla u + v\Delta u$, which implies $\Delta u\, v = \nabla \cdot (v\nabla u) - \nabla v \cdot \nabla u$. This leads to the following:

$$\int_\Omega \nabla u \cdot \nabla v\, dx - \int_\Omega \nabla \cdot (v\nabla u)\; dx = \int_\Omega f v\, dx$$

From integration by parts (divergence theorem), this results in

$$\int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\partial\Omega} v\nabla u \cdot n\, ds = \int_\Omega f v\, dx$$

which is equivalent to

$$\int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds = \int_\Omega f v\, dx$$

Now, we may substitute the Neumann boundary condition to find the weak formulation

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega f v\, dx + \int_{\partial\Omega} h(x)\, v\, ds$$

for all $v \in V$, where V is the appropriate function space for the solution u.

Question 4

Let

$$\int_{\Omega} f(x)\,\phi(x)\;dx = \int_{\Omega} g(x)\,\phi(x)\;dx$$

for all $\phi(x) \in \mathbb{C}^{\infty}(\Omega)$. Show that $f(x) = g(x)$.

The following is a proof that two functions $f(x)$ and $g(x)$ are equal if their integrals over some domain Ω are equal for all smooth test functions $\phi(x) \in C^{\infty}(\Omega)$. We consider an arbitrary open domain Ω in $\mathbb{R}^n$.

Proof: Let $f(x)$ and $g(x)$ be two functions defined on Ω. Suppose

$$\int_{\Omega} f(x)\,\phi(x)\;dx = \int_{\Omega} g(x)\,\phi(x)\;dx$$

for any smooth test function $\phi(x)$ defined on Ω, including those that have compact support.

We aim to show that $f(x) = g(x)$ for all $x \in \Omega$. Thus, consider a smooth function $\phi(x)$ that is 0 outside a small ball $B(c, \varepsilon)$ centered at c and 1 in a smaller ball $B(c, \varepsilon/2)$ for some $c \in \Omega$. The integral condition becomes

$$\int_{B(c,\varepsilon)} f(x)\;dx = \int_{B(c,\varepsilon)} g(x)\;dx$$

As $\varepsilon \to 0$, this implies $f(c) = g(c)$ by the fundamental theorem of calculus on $\mathbb{R}^n$. Since our choice of c was arbitrary, this implies that $f(x) = g(x)$ for all $x \in \Omega$.

Thus, if two functions have equal integrals against all smooth test functions (including those with compact support) on Ω, they must be equal point-wise and represent the same function. The test function $\phi(x)$ localizes the equality to an arbitrary point c in the domain. This completes the proof. $\square$

Further arguments (beyond what is demonstrated here) are presented in the next example, which serves as a stronger statement.

Question 5

Let

$$\int_{\Omega} f(x)\,\phi(x)\,dx + \int_{\partial\Omega} g(x)\,\phi(x)\,dx = \int_{\Omega} h(x)\,\phi(x)\,dx + \int_{\partial\Omega} m(x)\,\phi(x)\,dx$$

for all $\phi(x) \in \mathbb{C}^{\infty}(\Omega)$. Show that $f(x) = h(x)$ and $g(x) = m(x)$.

Similarly to the previous, the following is a proof that functions $f(x)$ and $h(x)$ are equal and functions $g(x)$ and $m(x)$ are equal if the integral over the domain Ω and $\partial\Omega$, respectively, are equal for all smooth test functions $\phi(x) \in C^{\infty}(\Omega)$. We consider an arbitrary open domain Ω in $\mathbb{R}^n$.

Proof: Let $f(x)$, $g(x)$, $h(x)$, and $m(x)$ be functions defined on Ω and $\partial\Omega$, respectively. Suppose

$$\int_{\Omega} f(x)\,\phi(x)\,dx + \int_{\partial\Omega} g(x)\,\phi(x)\,dx = \int_{\Omega} h(x)\,\phi(x)\,dx + \int_{\partial\Omega} m(x)\,\phi(x)\,dx$$

for any smooth test function $\phi(x)$ defined on Ω, including those that have compact support.

We aim to show that $f(x) = h(x)$ for all $x \in \Omega$ and $g(x) = m(x)$ for all $x \in \partial\Omega$. Thus, consider a smooth function $\phi(x)$ that is 1 in a small ball $B(c, \varepsilon)$ and 0 outside a slightly larger ball $B(c, 2\varepsilon)$ for some $c \in \Omega$. The integral condition becomes

$$\int_{B(c,\varepsilon)} f(x)\,dx = \int_{B(c,\varepsilon)} h(x)\,dx$$

As $\varepsilon \to 0$, this implies $f(c) = h(c)$ by the fundamental theorem of calculus on $\mathbb{R}^n$. Since our choice of c was arbitrary, this implies that $f(x) = h(x)$ for all $x \in \Omega$.

Similarly, consider a smooth function $\psi(x)$ that is 1 on an ε-neighborhood of a point $x_0 \in \partial\Omega$ on the boundary, and 0 elsewhere. The integral condition becomes

$$\int_{\partial\Omega} g(x)\,\psi(x)\,dS = \int_{\partial\Omega} m(x)\,\psi(x)\,dS$$

As indicated by the previous example, as $\varepsilon \to 0$, this implies $g(x_0) = m(x_0)$ by the fundamental theorem of calculus on $\mathbb{R}^n$. Since our choice of x_0 was arbitrary, this implies that $g(x) = m(x)$ for all $x \in \partial\Omega$.

Thus, if functions have equal integrals against all smooth test functions (including those with compact support) on Ω and $\partial\Omega$, they must be equal point-wise and represent the same functions. The test functions $\phi(x)$ and $\psi(x)$ localize the equality to an arbitrary point c in the domain (interior) and x_0 on the boundary (boundary), respectively. This completes the proof. $\square$

Further arguments are presented as follows, as they enhance the rigor of the argument presented above.

To show that $f(x) = h(x)$ and $g(x) = m(x)$ based on the given equation, we can use the fundamental lemma of calculus of variations, which is useful in functional analysis and variational calculus in deducing the equality of functions under certain conditions.

The fundamental lemma states that if for all smooth test functions $\phi(x)$, the following equation holds:

$$\int_{\Omega} (f(x) - h(x))\,\phi(x)\,dx + \int_{\partial\Omega} (g(x) - m(x))\,dx = 0$$

Then, it implies that $f(x) = h(x)$ and $g(x) = m(x)$. To prove this, we need to choose a suitable test function $\phi(x)$ and show that the equation above implies $f(x) = h(x)$ and $g(x) = m(x)$.

Let's start with $f(x) = h(x)$. Suppose $f(x) \neq h(x)$ for some x. Then, we can construct a test function $\phi(x)$ that is non-zero at the point where $f(x) \neq h(x)$ and zero elsewhere. Then, the left-hand side of the equation becomes:

$$\int_{\Omega} (f(x) - h(x))\, \phi(x)\, dx = \int_{\Omega} (f(x) - h(x))\, dx \neq 0$$

The last step is non-zero because $f(x) \neq h(x)$ at some point in Ω. However, this cannot be the case, because $\phi(x)$ is zero everywhere except where $f(x)$ and $h(x)$ are integrated. This contradiction shows that $f(x) = h(x)$.

Now, let's show that $g(x) = m(x)$. Similarly, suppose $g(x) \neq m(x)$ for some x. Then, we can construct a test function $\phi(x)$ that is non-zero at the point where $g(x) \neq m(x)$ and zero elsewhere. Then, the left-hand side of the equation becomes:

$$\int_{\partial\Omega} (g(x) - m(x))\, \phi(x)\, dx = \int_{\partial\Omega} (g(x) - m(x))\, dx \neq 0$$

The last step is non-zero because $g(x) \neq m(x)$ at some point in $\partial\Omega$. However, this cannot be case, because $\phi(x)$ is zero everywhere except where $g(x)$ and $m(x)$ are integrated. This contradiction shows that $g(x) = m(x)$.

While the structure follows similarly, there are nuances in the argument for the choice of the test function, particularly the way in which it may impact the other integrals (Ω or $\partial\Omega$). This is outlined in the initial argument, thus, we are able to demonstrate that if the given equation holds for all smooth test functions $\phi(x)$, then $f(x) = h(x)$ and $g(x) = m(x)$.

The results of this example may be used for the previous, by simply considering the case where the test function vanishes on the boundary $\partial\Omega$, which gives the exact form of the previous example.

The following arguments relate to the previous exercise, though they certainly extend to this quite naturally.

We have that

$$\int \phi\,(f - g) = 0 \qquad \text{for all } \phi \in C_0^\infty$$

Every integrable function can be approximated, in the L^2 norm, by functions C_0^∞. In particular, there exists a sequence $\{\phi_n\} \subset C_0^\infty$ such that $||f - g - \phi_n||_{L^2(\Omega)} \to 0$. Then

$$\begin{aligned}\int (f-g)^2 &= \int (f-g)\,\phi_n + \int (f-g)\,(f-g-\phi_n) \\ &= \int (f-g)\,(f-g-\phi_n) \\ &\leq ||f-g||\,||f-g-\phi_n|| \to 0\end{aligned}$$

Thus, $\int (f-g)^2 = 0$, which implies that $f(x) - g(x) = 0$.

Since f and g are continuous, $f - g$ is also continuous. By linearity, this means that we should verify whether

$$\int_{\Omega} h(x)\,\phi(x) = 0$$

for all continuous (compactly supported) functions implies $h(x) = 0$, where $h(x) = f(x) - g(x)$. This would prove $f(x) = g(x)$.

Assume $h(x_0) > 0$, so then by continuity there exists a ball $B_\delta(x_0)$ such that $f(x) > \frac{1}{2} f(x_0)$ for all x in said ball. There is also a ball $B_{\delta'}(x_0)$ such that $B_\delta(x_0) \nsubseteq B_{\delta'}(x_0)$ for which $h(x) > 0$. Take a continuous function ϕ such that $\phi_{B_\delta(x_0)} = 1$ and $\phi_{B_{\delta'}(x_0)} = 0$.

Now, we check the integral

$$\int_\Omega h(x)\,\phi(x)\;dx = \int_{B_{\delta'}(x_0)} h(x)\,\phi(x)\;dx > \int_{B_\delta(x_0)} h(x)\;dx > \frac{1}{2} h(x_0)\, m(B_\delta(x_0)) > 0$$

This contradicts the assumption, so we may conclude that $h(x) = 0$, and so $f(x) = g(x)$ everywhere on Ω.

Question 6

State the Lax-Milgram theorem. (Check Wikipedia.)

The following is according to Wikipedia.

Lax-Milgram Theorem The Lax-Milgram theorem gives conditions under which a bilinear function can be "inverted" to show the existence and uniqueness of a weak solution to a given boundary value problem. The result is named after the mathematicians Peter Lax and Arthur Milgram.

Formulation The following is a formulation of the Lax-Milgram theorem which relies on properties of the symmetric part of the bilinear form. It is not the most general form.

Let V be a Hilbert space and $a(\cdot,\cdot)$ a bilinear form on V, which is bounded $(|a(u,v)| \leq C\,||u||\,||v||)$ and coercive $\left(a(u,u) \geq c\,||u||^2\right)$. Then, for any $f' \in V'$, there is a unique solution $u \in V$ to the equation

$$a(u,v) = f(v) \qquad \forall v \in V$$

and it holds

$$||u|| \leq \frac{1}{c}\,||f||_{V'}$$

The following is according to Wolfram MathWorld.

Lax-Milgram Theorem In functional analysis, the Lax-Milgram theorem is a sort of representation theorem for bounded linear functionals on a Hilbert space H. The result is of tantamount significance in the study of function spaces and partial differential equations.

Let ϕ be a bounded coercive bilinear form on a Hilbert space H. The Lax-Milgram theorem states that, for every bounded linear functional f on H, there exists a unique $x_f \in H$ such that

$$f(x) = \phi(x, x_f) \qquad \forall x \in H$$

Question 7

Prove that the weak formulation of the following problem (Poisson equation with a non-zero Neumann boundary condition) has a solution.

$$-\Delta u = f \text{ in } \Omega \qquad \frac{\partial u}{\partial n} = h\left(x\right) \text{ on } \partial\Omega$$

The weak formulation of the Poisson equation with a non-zero Neumann boundary condition is as follows:

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} fv - \int_{\partial\Omega} h\left(x\right) v$$

To prove that this weak formulation has a solution, we consider the following.

Proof: Let V be the space of functions in $H^1\left(\Omega\right)$. This is a Hilbert space with the inner product

$$\left(u, v\right)_V = \int_{\Omega} \nabla u \cdot \nabla v$$

Define the bilinear form $a\left(u, v\right)$ and linear functional $L\left(v\right)$ as

$$a\left(u, v\right) = \int_{\Omega} \nabla u \cdot \nabla v \qquad L\left(v\right) = \int_{\Omega} fv - \int_{\partial\Omega} h\left(x\right) v$$

The bilinear form $a\left(u, v\right)$ is bounded and coercive on V, since

$$\left|a\left(u, v\right)\right| \leq \left|\left|u\right|\right|_V \left|\left|v\right|\right|_V$$

$$a\left(v, v\right) = \left|\left|v\right|\right|_V^2$$

To show $a\left(u, v\right)$ is bounded on V, we use the Cauchy-Schwarz inequality:

$$\begin{aligned}
\left|a\left(u, v\right)\right| &= \left|\int_{\Omega} \nabla u \cdot \nabla v\right| \\
&\leq \int_{\Omega} \left|\nabla u\right| \left|\nabla v\right| \\
&\leq \left(\int_{\Omega} \left|\nabla u\right|^2\right)^{\frac{1}{2}} \left(\int_{\Omega} \left|\nabla v\right|^2\right)^{\frac{1}{2}} \\
&= \left|\left|u\right|\right|_V \left|\left|v\right|\right|_V
\end{aligned}$$

This is the case where the norms are defined as

$$\left|\left|u\right|\right|_V = \left(\int_{\Omega} \left|\nabla u\right|^2\right)^{\frac{1}{2}} \qquad \left|\left|v\right|\right|_V = \left(\int_{\Omega} \left|\nabla v\right|^2\right)^{\frac{1}{2}}$$

The Cauchy-Schwarz inequality allows us to pull the integrals inside the absolute value and bound the term. This shows that $\left|a\left(u, v\right)\right|$ is bounded above by the product of the V norms of u and v.

Thus, for any $u, v \in V$, we have demonstrated there exists a constant C such that

$$\left|a\left(u, v\right)\right| \leq C \left|\left|u\right|\right|_V \left|\left|v\right|\right|_V$$

which proves the bilinear form $a\left(u, v\right)$ is bounded on V.

A bilinear form $a(u,v)$ is said to be coercive on a Hilbert space H if there exists a constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha ||v||^2$$

for all v in H. In our case with $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$, we have

$$a(v,v) = \int_\Omega |\nabla v|^2$$

Since the integral of $|\nabla v|^2$ is non-negative, we have $a(v,v) \geq 0$. Setting $\alpha = 1$ gives

$$a(v,v) \geq ||v||_V^2$$

for all $v \in V$. Thus, $a(u,v)$ is coercive with the condition $\alpha = 1$. This shows the bilinear form is coercive on the space V with the H^1 norm. The key idea is that coercivity follows directly from the fact that the bilinear form represents the H^1 norm, which is always non-negative.

To show $L(v)$ is bounded on V, we use the Cauchy-Schwarz inequality. First, let us recall the definition of $L(v)$.

$$L(v) = \int_\Omega fv - \int_{\partial\Omega} h(x)\, v$$

To show that $L(v)$ is bounded, we must should that there exists a constant M such that

$$|L(v)| \leq M ||v||_V$$

for all $v \in V$. Using the triangle inequality:

$$|L(v)| \leq \left|\int_\Omega fv\right| + \left|\int_{\partial\Omega} h(x)\, v\right|$$

Applying Cauchy-Schwarz to each term:

$$\left|\int_\Omega fv\right| \leq \left(\int_\Omega |f|^2\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2\right)^{\frac{1}{2}}$$

$$\left|\int_{\partial\Omega} h(x)\, v\right| \leq \left(\int_{\partial\Omega} |h(x)|^2\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |v|^2\right)^{\frac{1}{2}}$$

This implies the following:

$$|L(v)| \leq \left(\int_\Omega |f|^2\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2\right)^{\frac{1}{2}} + \left(\int_{\partial\Omega} |h(x)|^2\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |v|^2\right)^{\frac{1}{2}}$$

Now, using the norm definition for the terms, along with the inequality

$$\int_\Omega |v|^2 \leq C \int_\Omega |\nabla v|^2 = C ||v||_V^2$$

we may further bound the above:

$$|L(v)| \leq \left(\int_\Omega |f|^2\right)^{\frac{1}{2}} (C ||v||_V) + \left(\int_{\partial\Omega} |h(x)|^2\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |v|^2\right)^{\frac{1}{2}}$$

A complete description of the inequality used (Poincare's) is provided below, according to the justification provided during class.

To further strengthen the bound, we must use trace theorem to demonstrate that there exists a constant $C_T > 0$ such that

$$||v||_{L^2(\partial\Omega)} \leq C_T\, ||v||_{H^1(\Omega)}$$

for all $v \in H^1(\Omega)$. This is used to demonstrate that $L(v)$ is bounded on V.

Using this information, we have

$$\left(\int_{\partial\Omega} |v|^2\right)^{\frac{1}{2}} \leq C_T\, ||v||_{H^1(\Omega)} = C_T\, ||v||_V$$

When we input this into the previous bound, we find that

$$|L(v)| \leq \left(\int_{\Omega} |f|^2\right)^{\frac{1}{2}} (C\, ||v||_V) + \left(\int_{\partial\Omega} |h(x)|^2\right)^{\frac{1}{2}} (C_T\, ||v||_V)$$

This is equivalent to stating the following, for some constant $\tilde{C}$:

$$|L(v)| \leq \tilde{C}\, ||v||_V$$

This rigorously shows that $L(v)$ is bounded, as $L(v)$ is bounded on V by the V norm of v and the L^2 norm of f. The Cauchy-Schwarz inequality provided the key result to prove the bound.

In summary, the Lax-Milgram theorem requires that $a(u,v)$ is bounded on V, $a(u,v)$ is coercive on V, and $L(v)$ is bounded on V, as follows:

$$|a(u,v)| \leq C\, ||u||_{H^1}\, ||v||_{H^1}$$

$$a(v,v) \geq \alpha\, ||v||_{H^1}^2$$

$$|L(v)| \leq C\, ||v||_{H^1}$$

By the Lax-Milgram theorem, there exists a unique $u \in V$ (where $V = H^1(\Omega)$ such that

$$a(u,v) = L(v) \qquad \forall v \in V$$

Thus, the weak formulation exists and has a unique solution u in V. The Lax-Milgram theorem guarantees existence and uniqueness of a solution to the weak formulation, proving that it is well-posed.
$\square$

Further justification of a similar problem was provided in class, and thus, is given here. (The notation may vary slightly.)

$$B(u,v) = L(v) \text{ where } B(u,v) = \int \nabla u \cdot \nabla v \, dx \text{ and } L(v) = \int fv \, dx$$

The following results are trivial: $|B(u,v)| \le C\,||u||$ and $B(u,v) \ge C\,|u|^2$ To prove it, we simply use Cauchy-Schwarz, which is stated as

$$||u|| = \left(\int u^2 \, dx\right)^{\frac{1}{2}}$$

To consider this, we must convert to the Hilbert space $H = \{q | q = \nabla Q\}$. This means that $w = \nabla u$ and $z = \nabla v$, so that $B(w,z) = \langle w, z\rangle$. Now, we have

$$\int \nabla u \cdot \nabla v \, dx = \int w \cdot z \, dz = \langle w, z\rangle \le ||w||\,||z||$$

Naturally, this implies

$$B(w,w) = \langle w, w\rangle = ||w||^2$$

Now, we must show $L(v) \le C\,||\nabla v||$, which is the challenging part. The question becomes the following: How do you show $\int fv\,dx \le ||f||\,||\nabla v||$? In this case, we consider that $L(v) = \int fv$. We know that $\int fv\,dx \le C\,||f||\,||v||$, but we do not know if $\int fv\,dx \le C\,||f||\,||\nabla v||$. In other words, all we aim to show is that

$$||v||^2 \le C\,||\nabla v||^2$$

This is equivalent to saying

$$\int v^2 \, dx \le C \int |\nabla v|^2 \, dx$$

which is the final step. This happens to be Poincare's inequality, which has a complete proof elsewhere, though we will provide the broad strokes, using the fundamental theorem of calculus.

$$|v(x)| = \left| \int_a^x v'(s) \, ds \right| = ||v'||$$

This implies that

$$\int v^2 \, dx \le \int ||v'||^2 \, dx \le C\,||v'||^2$$

The idea behind all of this is that the norm of v is controlled by the norm of the gradient, which allows us to make the statement about the boundedness of $L(v)$.

Question 8

State the elliptic regularity theorem.

The elliptic regularity theorem is a fundamental result in the theory of partial differential equations, specifically for elliptic equations. It provides information about the smoothness of solutions to elliptic PDEs.

Elliptic Regularity Theorem: Let L be a linear elliptic partial differential operator with smooth coefficients defined on an open domain $\Omega \subset \mathbb{R}^n$, and consider the equation $Lu = f$, where u is the unknown function and f is a given function. If u is a weak solution to this equation, i.e. $u \in H^1(\Omega)$ and $Lu \in L^2(\Omega)$, then u is a classical solution, which means $u \in \mathbb{C}^\infty(\Omega)$.

In simpler terms, the elliptic regularity theorem asserts that if a solution u to an elliptic partial differential equation is sufficiently smooth in a certain function space, then it is, in fact, infinitely differentiable throughout the domain Ω. This result is crucial in the analysis of elliptic equations, as it ensures that solutions have the desired regularity properties needed for various applications.

In other words, if f is smooth enough and u is a weak solution to the elliptic partial differential equation $Lu = f$, then u is smooth. This highlights that the "regularity" of f is transferred to the solution u.

The following is according to *Partial Differential Equations* by Rustum Choksi.

> **Smoothness (Regularity)** To speak about a classical solution to $\Delta u = 0$, one would need the particular second derivatives $u_{x_i x_i}$, $i = 1, \ldots, n$ to exist. A natural question to ask is how smooth is this solution? Mathematicians refer to smoothness properties as *regularity.* The particular combination of the derivatives in the Laplacian have a tremendous power in "controlling" all derivatives. In fact, a harmonic function is not only $\mathbb{C}^2$ (in other words, all second derivatives exist and are continuous) but, also, $\mathbb{C}^\infty$. Precisely, if $u \in \mathbb{C}^2(\Omega)$ solves $\Delta u = 0$ in Ω, then $u \in \mathbb{C}^\infty(\Omega)$. This should be intuitive by either the equilibrium interpretation or the mean value property. The averaging out of values of u prohibits any singularities in any derivative.
>
> In fact, even the assumption that $u \in \mathbb{C}^2$ turns out to be redundant in the following sense. Following, we will extend distributions to several independent variables and see how to interpret Laplace's equation in the sense of distributions. With this in hand, the following assertion holds true. If F is *any* distribution such that $\Delta u = 0$ in the sense of distributions, then $F = F_u$, a distribution generated by a $\mathbb{C}^\infty$ function u. This result is known as *Weyl's Lemma.*
>
> A broad class of PDEs which have a similar smoothness property are called *elliptic.* **Solutions to elliptic PDEs have an increased measure of smoothness, often referred to as *elliptic regularity*, beyond the derivatives which appear in the PDE.** The Laplacian is the canonical example of an elliptic PDE.

The following is according to Wikipedia.

Elliptic Operator In the theory of partial differential equations, *elliptic operators* are differential operators that generalize the Laplace operator. They are defined by the condition that the coefficients of the highest-order derivatives be positive, which implies the key property that the principal symbol is invertible, or equivalently that there are no real characteristic directions.

Elliptic operators are typical of potential theory, and they appear frequently in electrostatics and continuum mechanics. ***Elliptic regularity*** **implies that their solutions tend to be smooth functions (if the coefficients in the operator are smooth).** Steady-state solutions to hyperbolic and parabolic equations generally solve elliptic equations.

Elliptic Regularity Theorem Let L be an elliptic operator of order $2k$ with coefficients having $2k$ continuous derivatives. The Dirichlet problem for L is to find a function u, given a function f and some appropriate boundary values, such that $Lu = f$ and such that u has the appropriate boundary values and normal derivatives. The existence theory for elliptic operators, using Gårding's inequality and the Lax–Milgram lemma, only guarantees that a weak solution u exists in the Sobolev space H^k.

This situation is ultimately unsatisfactory, as the weak solution u might not have enough derivatives for the expression Lu to be well-defined in the classical sense.

The elliptic regularity theorem guarantees that, provided f is square-integrable, u will in fact have $2k$ square-integrable weak derivatives. In particular, if f is infinitely-often differentiable, then so is u.

Any differential operator exhibiting this property is called a hypoelliptic operator; thus, every elliptic operator is hypoelliptic. The property also means that every fundamental solution of an elliptic operator is infinitely differentiable in any neighborhood not containing 0.

As an application, suppose a function f satisfies the Cauchy–Riemann equations. Since the Cauchy-Riemann equations form an elliptic operator, it follows that f is smooth.

Question 9

Find the weak derivative of $f(x) = |x|$

Let $f(x) = |x|$. First, let us ask what $f'(x)$ is. Pointwise, we would say

$$f'(x) = \begin{cases} -1 & x < 0 \\ \text{undefined} & x = 0 \\ 1 & x > 0 \end{cases}$$

Let us do this differentiation in the sense of distributions. We have

$$\left\langle (F_f)', \phi \right\rangle = -\langle F_f, \phi' \rangle = -\int_{-\infty}^{\infty} |x| \, \phi'(x) \, dx = -\int_{-\infty}^{0} -x\phi'(x) \, dx - \int_{0}^{\infty} x\phi'(x) \, dx$$

and integrating by parts in each integral, we find that

$$\begin{aligned} \left\langle (F_f)', \phi \right\rangle &= -\int_{-\infty}^{0} \phi(x) \, dx + [x\phi(x)]_{-\infty}^{0} + \int_{0}^{\infty} \phi(x) \, dx - [x\phi(x)]_{0}^{\infty} \\ &= -\int_{-\infty}^{0} \phi(x) \, dx + \int_{0}^{\infty} \phi(x) \, dx \\ &= \int_{-\infty}^{\infty} g(x) \, \phi(x) \, dx = \langle F_g, \phi \rangle \end{aligned}$$

where g is the function

$$g(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Note here how we dispensed with the boundary terms. They were trivially zero at $x = 0$ and were zero at $\pm\infty$ because ϕ has compact support. For example

$$[x\phi(x)]_{-\infty}^{0} = \lim_{L \longrightarrow \infty} [x\phi(x)]_{-L}^{0} = \lim_{L \longrightarrow 0} L\phi(-L) = 0$$

since $\phi = 0$ outside some fixed interval. In conclusion, f' is simply g *in the sense of distributions.* The function g is known as the **signum function** (usually abbreviated by *sgn*) since it effectively gives the sign (either ± 1) of x. That is,

$$\operatorname{sgn}(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \\ 0 & x = 0 \end{cases}$$

Note that in this instance we took the value at $x = 0$ to be 0 but could just as well have taken it to be a different value. The value at one point has no effect on *sgn* as a distribution.

Question 10

Find $\partial_{xx}|x|$ in the sense of distributions. (Use the result of the previous problem.)

Now, suppose we want to find f''. This is equivalent to $\partial_{xx}|x|$ in the sense of distributions. From the example of the Heaviside function (provided below), we know that doing this point-wise will not give us the right answer. So let us work in the sense of distributions. We have

$$\begin{aligned}\left\langle (F_f)'', \phi\right\rangle &= \langle F_f, \phi''\rangle \\ &= \int_{-\infty}^{\infty} |x|\,\phi''(x)\ dx \\ &= \int_{-\infty}^{0} (-x)\,\phi''(x)\ dx + \int_{0}^{\infty} x\phi''(x)\ dx\end{aligned}$$

By integration by parts, this is equivalent to the following:

$$\begin{aligned}\left\langle (F_f)'', \phi\right\rangle &= \int_{-\infty}^{0} \phi'(x)\ dx - \left[x\phi'(x)\right]_{-\infty}^{0} - \int_{0}^{\infty} \phi'(x)\ dx + \left[x\phi(x)\right]_{0}^{\infty} \\ &= \int_{-\infty}^{0} \phi'(x)\ dx - \int_{0}^{\infty} \phi'(x)\ dx \\ &= 2\phi(0)\end{aligned}$$

Thus, $f'' = g' = 2\delta_0$ in the sense of distributions.

Alternatively, let us consider $\partial_{xx}|x|$ in the sense of distributions, using the result of the previous problem. If we know that the weak derivative of $f(x) = |x|$ is g, where g is

$$g(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

we may express this in terms of the Heaviside function as follows:

$$g(x) = 2H(x) - 1$$

In this case, the Heaviside function is given as

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

The value of $H(0)$ does not particularly matter, considering that the value of $g(0)$ is not important for the weak derivative. To compute $\partial_{xx}|x|$ in the sense of distributions, we simply consider the weak derivative of $g(x) = 2H(x) - 1$, with the understanding that $H'(x) = \delta(x)$. Thus, we find that

$$\partial_{xx}|x| = 2\delta(x)$$

Heaviside Function: Let us return to the Heaviside function and differentiate it as a distribution, i.e. differentiate in the sense of distributions. We have for any test function ϕ

$$\begin{aligned}
\left\langle (F_H)', \phi \right\rangle &= -\left\langle F_H, \phi' \right\rangle = -\int_{-\infty}^{\infty} H(x)\,\phi'(x)\,dx \\
&= -\int_{0}^{\infty} \phi'(x)\,dx \\
&= -\left(\phi(+\infty) - \phi(0)\right) \\
&= \phi(0) \\
&= \left\langle \delta_0, \phi \right\rangle
\end{aligned}$$

Note that the term $\phi(+\infty)$ vanished because of compact support. Thus, $(F_H)' = \delta_0$ or, in order words, the derivative of the Heaviside function in the sense of distributions is the Dirac delta function.

Acknowledgements

In working on this homework assignment, independent solutions were discussed with Tunmay Gerg, specifically Questions 4, 5, and 7.

Further, the following references are helpful:

- Weak Derivative - Wikipedia
- Weak Formulation - Wikipedia
- Dirac Delta Function - Wikipedia
- Poincare Inequality - Wikipedia
- Trace Theorem - Wikipedia

Appendix

The following represent the relevant course notes to this assignment.

Maximum Principle Let D be a connected bounded open set. Let u be a harmonic function in D that is continuous on $\overline{D} = D \cup \partial D$. Then, the max/min of u is on the boundary, and nowhere inside. If $|\nabla u|^2 = 0$, then u is constant, which is only true if D is connected.

Mean Value Property Let u be a harmonic function ($\Delta u = 0$) in a disk D. The mean value property states that the value of u at the center of D equals the average of u on its circumference.

$$u(x^*) = \frac{1}{|B(r)|} \int_{B(r)} u \, ds$$

Maximum Principle The maximum principle may be proven by contradiction, according to the mean value property. If there is a point on the inside of D that is greater than the maximum, this forces the boundary to contain a greater value.

That is, assume the maximum is at a point x^*, which is the center of a small disk $B(\varepsilon)$. This disk may be arbitrarily small. In this case, we have $u(x^*) > u(x)$ and $u(x^*) =$ average of $u(x)$ on $B(\varepsilon)$. This creates a contradiction.

Harmonic Functions For a harmonic function ($\Delta u = 0$),

$$\frac{1}{|B|} \int_B u \, ds = u(x^*)$$

where B is a disk centered at x^*. From the mean value property, we proved the maximum principle. The main idea was proof by contradiction.

$$\frac{1}{|B|} \int_B u \, ds = \frac{1}{2\pi r} \int_0^{2\pi} u(r, \theta) \, r \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) \, d\theta$$

In this case, B is a disk of radius r centered around x^*. Now, let

$$Q(r) = \frac{1}{|B|} \int_B u \, ds = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) \, d\theta$$

The goal is to show that $\frac{\partial Q}{\partial r} = 0$. We highlight that $\lim_{r \longrightarrow 0} Q(r) = u(x^*)$. That is, we aim to demonstrate that Q is a constant, specifically $u(x^*)$.

$$Q = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) \, d\theta$$

$$\frac{dQ}{dr} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial r} \, d\theta = \frac{1}{2\pi r} \int_0^{2\pi} \frac{\partial u}{\partial r} r \, d\theta = \frac{1}{2\pi r} \int_B \frac{\partial u}{\partial r} r \, d\theta = \frac{1}{2\pi r} \int \nabla u \cdot n \, ds$$

In this case, n is the outward normal vector, so $\nabla u \cdot n = \frac{\partial u}{\partial n} = \frac{\partial u}{\partial r}$ and ds is the circumference of the circle. With a circle D and disk B, we have $\partial D = B$, thus, we have

$$\frac{1}{2\pi r} \int_{\partial D} \nabla u \cdot n \, ds$$

Using the divergence theorem, we may conclude the following:

$$\frac{dQ}{dr} = \frac{1}{2\pi r}\int_{\partial D} \nabla u \cdot n \, ds = \frac{1}{2\pi r}\int_D \nabla \cdot \nabla u \, dx = \frac{1}{2\pi r}\int_D \Delta u \, dx = 0$$

This follows from the idea that the function is harmonic.

Weak Solutions We have the following:

$$-\Delta u = f \text{ in } D$$

with $u = 0$ on ∂D. Let us consider the test function space $\mathbb{C}_0^\infty$, which represents compact support, such that $u \neq 0$. That is, the support must belong inside of the domain, as a proper subset.

The motivation behind compact support lies in integration by parts, as follows:

$$\int_\Omega u \, dv = \int_{\partial\Omega} uv - \int_\Omega v \, du$$

We want the boundary condition to disappear, which we must take into consideration when using derivatives.

Using integration by parts, and considering $\phi \in \mathbb{C}_0^\infty$, we have

$$\int_D -(\Delta u) \cdot \phi \, dx = \int_D f\phi \, dx$$

$$-\int \nabla \cdot (\nabla u) \cdot \phi \, dx = \int_D f\phi \, dx$$

Now, let us consider the following:

$$\nabla (\phi \nabla u) = \nabla\phi \cdot \nabla u + \phi \Delta u$$

$$\Delta u \phi = \nabla \cdot (\phi \nabla u) - \nabla \phi \cdot \nabla u$$

This allows us to conclude the following, with divergence theorem:

$$-\int_D \nabla \cdot (\phi \nabla u) \, dx + \int_D \nabla u \cdot \nabla \phi \, dx = \int_D f\phi \, dx$$

$$-\int_{\partial D} \phi \nabla u \cdot n \, dx + \int_D \nabla u + \nabla \phi \, dx = \int_D f\phi \, dx$$

The boundary term goes to zero with compact support, so we have the weak formulation of the Poisson, as follows:

$$\int_D \nabla u \cdot \nabla \phi \, dx = \int_D f\phi \, dx$$

The strong form requires that the u is twice differentiable, though the integral form allows us to consider once differentiable forms.

Bilinear Form $B(u, v)$ is bilinear if it is linear with respect to u (fixed v) and v (fixed u). The Lax-Milgram theorem states $B(u, v) = L(v)$, which guarantees that there exists a solution.

Elliptic regularity theory indicates that the weak solution is always smooth, so we do not have to worry about extra derivatives.

Distributions A "distribution" is a linear map (linear functional) from $\mathbb{C}_0^\infty$ to $\mathbb{R}$. A distribution is always differentiable in the sense of distributions. That is, a distribution always has a weak derivative.

Let T be a distribution. Then, the weak derivative of T is given by $T'(\phi) = -T(\phi')$ for all $\phi \in \mathbb{C}_0^\infty$. If T is a standard function, which is differentiable, the weak derivative must be equal to the standard derivative.

Consider the following:

$$
\begin{aligned}
T'(\phi(x)) &= \int_{-\infty}^{\infty} t'(x)\,\phi(x)\;dx \\
&= [t(x)\,\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} t(x)\,\phi'(x)\;dx \\
&= -T(\phi'(x))
\end{aligned}
$$

In this case, the boundary value vanishes with compact support. Using integration by parts with this vanishing boundary values allows us to define the derivative of a distribution (weak derivative) in terms of a the derivative of a different variable.