MATH 053/126 - Partial Differential Equations

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Homework 1

Instructions/Notation

Please show all steps to get your answers. Specify the problems you discussed with other students (including names).

The starred problems are recommended but not required for undergraduate/non-math major graduate students and required for all math major graduate students.

Notation

- $\mathbb{R}$: The set of all real numbers.
- $\mathbb{R}^+$: The set all positive real numbers $\{x \in \mathbb{R} | x > 0\}$.
- $\frac{\partial u}{\partial t} = \partial_t u = u_t$
- $\frac{\partial u}{\partial x} = \partial_x u = u_x$

Questions

1a) Solve the following eigenvalue problem (ODE)

$$-\partial_{xx} u_n = \lambda_n u_n \qquad x \in (0, L) \qquad u_n(0) = u_n(L) = 0$$

Here L is a constant. You need to find the eigenfunctions and eigenvalues.

To solve the eigenvalue problem $-\partial_{xx} u_n = \lambda_n u_n, x \in (0, L)$ with boundary conditions $u_n(0) = u_n(L) = 0$, we consider various cases.

First, let us consider the case where $\lambda_n < 0$. In this case, the general solution is given by the following:

$$u_n(x) = \alpha \cos\left(\sqrt{-\lambda_n} x\right) + \beta \sin\left(\sqrt{-\lambda_n} x\right)$$

If we apply the first boundary condition, we have the following:

$$u_n(0) = \alpha \cos(0) + \beta \sin(0) = \alpha = 0$$

Using this information, we apply the second boundary condition to find:

$$u_n(L) = \beta \sin\left(\sqrt{-\lambda_n} L\right) = 0$$

For nontrivial solutions $\beta \neq 0$, we must have

$$\sin\left(\sqrt{-\lambda_n} L\right) = 0$$

which gives values of λ_n.

Next, let us consider the case where $\lambda_n = 0$. In this case, the general solution is given by the following:

$$u_n(x) = \alpha x + \beta$$

If we apply the first boundary condition, we have the following:

$$u_n(0) = \alpha(0) + \beta = \beta = 0$$

Using this information, we apply the second boundary condition to find:

$$u_n(L) = \alpha(L) = 0$$

For nontrivial solutions $\alpha \neq 0$, we must have $L = 0$ though this cannot be the case, as $L \neq 0$, thus we only have trivial solutions.

Finally, let us consider the case where $\lambda_n > 0$. In this case, the general solution is given by the following:

$$u_n(x) = \alpha \cos\left(\sqrt{\lambda_n} x\right) + \beta \sin\left(\sqrt{\lambda_n} x\right)$$

If we apply the first boundary condition, we have the following:

$$u_n(0) = \alpha \cos(0) + \beta(0) = \alpha = 0$$

Using this information, we apply the second boundary condition to find:

$$u_n(L) = \beta \sin\left(\sqrt{\lambda_n} L\right) = 0$$

For nontrivial solutions $\beta \neq 0$, we must have

$$\sin\left(\sqrt{\lambda_n} L\right) = 0$$

Thus, we know that

$$\sqrt{\lambda_n} L = n\pi \qquad n \in \mathbb{Z}$$

This allows us to determine the values of λ_n as follows:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

These values serve as the eigenvalues. To determine the eigenfunctions, we consider

$$u_n(x) = \sin\left(\sqrt{\lambda_n} x\right)$$

which is equivalent to

$$u_n(x) = \sin\left(\frac{n\pi}{L} x\right)$$

1b) Solve the following eigenvalue problem (ODE)

$$-\partial_{xx} u_n = \lambda_n u_n \qquad x \in (0, L) \qquad u_n(0) = \partial_x u_n(L) = 0$$

Here L is a constant. You need to find the eigenfunctions and eigenvalues.

To solve the eigenvalue problem $-\partial_{xx} u_n = \lambda_n u_n, x \in (0, L)$ with boundary conditions $u_n(0) = \partial_x u_n(L) = 0$, we consider various cases, as with the previous problem. In this instance, however, the details are skipped, as they follow from the previous.

Specifically, in the case where $\lambda_n = 0$, the general solution is given by the following:

$$u_n(x) = \alpha x + \beta$$

If we apply the first boundary condition, we have the following:

$$u_n(0) = \alpha(0) + \beta = \beta = 0$$

Using this information, we apply the second boundary condition to find:

$$\partial_x u_n(L) = \alpha = 0$$

This produces a trivial solution, as in the case previously.

Let us consider the case where $\lambda_n > 0$. In this case, the general solution is given by the following:

$$u_n(x) = \alpha \cos\left(\sqrt{\lambda_n} x\right) + \beta \sin\left(\sqrt{\lambda_n} x\right)$$

If we apply the first boundary condition, we have the following:

$$u_n(0) = \alpha \cos(0) + \beta(0) = \alpha = 0$$

Using this information, we apply the second boundary condition to find:

$$\partial_x u_n(L) = \beta \sqrt{\lambda_n} \cos\left(\sqrt{\lambda_n} L\right) = 0$$

For nontrivial solutions $\beta \neq 0$, we must have

$$\cos\left(\sqrt{\lambda_n} L\right) = 0$$

Thus, we know that

$$\sqrt{\lambda_n} L = \frac{\pi}{2} + n\pi \qquad n \in \mathbb{Z}$$

This allows us to determine the values of λ_n as follows:

$$\lambda_n = \left(\frac{(1 + 2n)\pi}{2L}\right)^2$$

These values serve as the eigenvalues. To determine the eigenfunctions, we consider

$$u_n(x) = \sin\left(\sqrt{\lambda_n} x\right)$$

which is equivalent to

$$u_n(x) = \sin\left(\frac{(1 + 2n)\pi}{2L} x\right)$$

2) We consider the vector space $\mathbb{R}^2$. Let $\boldsymbol{v} = (2, -1)$. Find the representation of $\boldsymbol{v}$ using the basis $\{\boldsymbol{b_1}, \boldsymbol{b_2}\}$ where $\boldsymbol{b_1} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\boldsymbol{b_2} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. That is, find α_1 and α_2 where $\boldsymbol{v} = \alpha_1 \boldsymbol{b_1} + \alpha_2 \boldsymbol{b_2}$.

This may be written as follows:

$$\begin{pmatrix} \boldsymbol{b_1} & \boldsymbol{b_2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \boldsymbol{v} \longrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

To determine the values of α_1 and α_2, we may either solve the system of equations, or compute the inverse of the matrix. Given the simplicity of the system, let's just solve it directly.

$$\alpha_1 \frac{1}{\sqrt{2}} - \alpha_2 \frac{1}{\sqrt{2}} = 2 \qquad \alpha_1 \frac{1}{\sqrt{2}} + \alpha_2 \frac{1}{\sqrt{2}} = -1$$

Adding these equations, we find that

$$2\alpha_1 \frac{1}{\sqrt{2}} = 1 \longrightarrow \alpha_1 = \frac{1}{\sqrt{2}}$$

Similarly, subtracting these equations, we find that

$$2\alpha_2 \frac{1}{\sqrt{2}} = -3 \longrightarrow \alpha_2 = -\frac{3}{\sqrt{2}}$$

This gives us a representation of $\boldsymbol{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ using the provided basis. $\square$

3) Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

That is, find the $\boldsymbol{v}$ and λ such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$.

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

To do so, we consider $A\boldsymbol{v} - \lambda\boldsymbol{v} = 0$ or $(A - \lambda I)\,\boldsymbol{v} = 0$. Considering $\boldsymbol{v} \neq 0$, we may use the determinant $\det(A - \lambda I) = 0$. For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = ad - bc$, so

$$\det \begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = (-2-\lambda)^2 - 1 = 0$$

Thus,

$$4 + 4\lambda + \lambda^2 - 1 = 0 \longrightarrow \lambda^2 + 4\lambda + 3 = 0 \longrightarrow (\lambda + 1)(\lambda + 3) = 0$$

so the eigenvalues are given as follows

$$\lambda = -1, -3$$

Given $(A - \lambda I)\,\boldsymbol{v} = \boldsymbol{0}$ with $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, for the eigenvalue $\lambda = -1$, we have

$$\left(\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Given $(A - \lambda I)\,\boldsymbol{v} = \boldsymbol{0}$ with $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, for the eigenvalue $\lambda = -3$, we have

$$\left(\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

4) Solve the following system of ODEs for $X \in \mathbb{R}^2$

$$\boldsymbol{X_t} = A\boldsymbol{X} \qquad \boldsymbol{X}(0) = (1,2)^T$$

where A is the matrix from the previous problem.

To solve the system of ODEs for $\boldsymbol{X} \in \mathbb{R}^2$ with A as the matrix from the previous problem, suppose $\boldsymbol{X} = \alpha e^{\lambda t}$. Thus,

$$\boldsymbol{X_t} = \alpha\lambda e^{\lambda t} = A\alpha e^{\lambda t} = A\boldsymbol{X}$$

This implies that $A\alpha = \lambda\alpha$, so α is an eigenvector with the corresponding eigenvalue λ. Thus, using the eigenvalues/eigenvectors determined previously, we propose a solution of the following form:

$$\boldsymbol{X}(t) = C_1\boldsymbol{\beta_1}e^{\lambda_1 t} + C_2\boldsymbol{\beta_1}e^{\lambda_2 t}$$

where $\boldsymbol{\beta}$ represents the eigenvectors and λ represents the eigenvalues. Thus, we have

$$\boldsymbol{X} = C_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t} + C_2\begin{pmatrix}-1\\1\end{pmatrix}e^{-3t}$$

Aside: Given that the matrix is diagonalizable (and symmetric), we know that the vectors $\boldsymbol{\beta}$ span the entire solution. set, and thus, this form represents our complete general solution.

Now, we solve for the values C_1 and C_2 using the initial value $\boldsymbol{X}(0) = \begin{pmatrix}1\\2\end{pmatrix}$, as follows:

$$C_1 - C_2 = 1 \longrightarrow C_1 = C_2 + 1$$

We may input this into the following condition:

$$C_1 + C_2 = 1 \longrightarrow C_2 + 1 + C_2 = 2$$

This implies that $C_2 = \frac{1}{2}$, so $C_1 = \frac{3}{2}$. Thus, the particular solution to the system of ODEs is given as

$$X = \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}e^{-t} + \frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}e^{-3t}$$

5) Find the series representation of the following vector (in $\mathbb{R}^2$) in terms of t. e^{At} Note that A is the matrix from the previous problem.

We may represent the matrix exponential e^{At} via the power series expansion of the exponential function.

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n$$

In other words, the appropriate representation is as follows:

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^nt^n}{n!}$$

To take repeated powers of A, we should diagonalize the matrix. Using the calculated eigenvalues/eigenvectors, we have

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Thus, we have

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & (-3)^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} t^n$$

This may be simplified, using the exponential form

$$e^{At} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

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6) Solve the following Poisson problem

$$-u_{xx} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x) \qquad x \in (0,1), u(0) = u(1) = 0$$

Using integration, we have the following:

$$\iint -u_{xx}\, dx\, dx = \iint \sum_{n=1}^{\infty} \left[\frac{1}{n^2} \sin(n\pi x)\right] dx\, dx$$

The left-hand side of the equation simplifies to $-u$, while the right-hand side is given as follows:

$$\begin{aligned} -u &= \iint \sum_{n=1}^{\infty} \left[\frac{1}{n^2} \sin(n\pi x)\right] dx\, dx \\ &= \int \sum_{n=1}^{\infty} \left[-\frac{1}{n^2} \frac{\cos(n\pi x)}{n\pi}\, dx\right] + C_1 \\ &= \sum_{n=1}^{\infty} \left[-\frac{1}{n^2} \frac{\sin(n\pi x)}{n^2\pi^2}\right] + C_1 x + C_2 \end{aligned}$$

Simplifying the expression (by removing the negative signs on either side and combining terms), this results in the following:

$$u(x) = \sum_{n=1}^{\infty} \left[\frac{1}{n^4\pi^2} \sin(n\pi x)\right] + C_1 x + C_2$$

Now, we are tasked with solving for the values C_1 and C_2, which we may do using the initial boundary values. Using $u(0) = 0$, we find

$$u(0) = \sum_{n=1}^{\infty} \left[\frac{1}{n^4\pi^2} \sin(0)\right] + C_1(0) + C_2 = C_2$$

Thus, we know $C_2 = 0$. Now, to find the value of C_1, we use the other boundary condition $u(1) = 0$.

$$u(1) = \sum_{n=1}^{\infty} \left[\frac{1}{n^4\pi^2} \sin(n\pi)\right] + C_1 = C_1$$

Thus, we know $C_1 = 0$ as well. With this information, we have the final solution.

$$u(x) = \sum_{n=1}^{\infty} \left[\frac{1}{n^4\pi^2} \sin(n\pi x)\right]$$

7) Solve the following problem

$$-u_{xx} = 1 \qquad x \in (0,1)\,, u(0) = u(1) = 0$$

This problem shows that the eigenfunction expansion does not always work. Can you explain why?

Using integration, we have the following:

$$\iint u_{xx}\,dx\,dx = \iint -1\,dx\,dx$$

The left-hand side of the equation simplifies to u, while the right-hand side is given as follows:

$$\begin{aligned}
u &= \iint_x -1\,dx\,dx \\
&= \int -x + C_1\,dx = \int -(x + C_1)\;dx \\
&= -\frac{1}{2}(x + C_1)^2 + C_2
\end{aligned}$$

Now, we are tasked with solving for the values C_1 and C_2, which we may do using the initial boundary values. This results in the following:

$$u(0) = -\frac{1}{2}(0 + C_1)^2 + C_2 = 0$$

$$u(1) = -\frac{1}{2}(1 + C_1)^2 + C_2 = 0$$

From the first expression, we find that $C_2 = \frac{1}{2}C_1^2$, which we input into the second expression, as follows:

$$\begin{aligned}
-\frac{1}{2}(1 + C_1)^2 + \frac{1}{2}C_1^2 &= 0 \\
-\frac{1}{2}\left(1 + 2C_1 + C_1^2\right) + \frac{1}{2}C_1^2 &= 0 \\
-\frac{1}{2} - C_1 - \frac{1}{2}C_1^2 + \frac{1}{2}C_1^2 &= 0 \\
-\frac{1}{2} - C_1 &= 0
\end{aligned}$$

Thus, we know that the value of $C_1 = -\frac{1}{2}$, which means the value of $C_2 = \frac{1}{8}$. These values are inputted into the expression for u as follows:

$$\begin{aligned}
u &= -\frac{1}{2}(x + C_1)^2 + C_2 \\
&= -\frac{1}{2}\left(x - \frac{1}{2}\right)^2 + \frac{1}{8} \\
&= -\frac{1}{2}\left(x^2 - x + \frac{1}{4}\right) + \frac{1}{8} \\
&= -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{8} + \frac{1}{8} \\
&= -\frac{1}{2}x^2 + \frac{1}{2}x
\end{aligned}$$

Thus, we have solved the problem $-u_{xx} = 1$ with the given initial boundary conditions, resulting in the solution $u(x) = -\frac{1}{2}x^2 + \frac{1}{2}x$. $\square$

Looking back on this, it may have been easier to consider the value C_1 outside of the integral, with an x term after the second evaluation of the integral. Regardless, this produces the same result.

To solve this problem using the eigenfunction expansion method, one would typically assume that the solution $u(x)$ can be written as an infinite sum of eigenfunctions multiplied by coefficients, as follows:

$$u(x) = \sum_{n=1}^{\infty} \alpha_n \phi(x)$$

where $\phi(x)$ are the eigenfunctions associated with the given boundary conditions, and α_n are the coefficients to be determined.

However, this problem illustrates why the eigenfunction expansion method does not always work. In this case, the issue arises from the nature of the differential equation and the boundary conditions. The given boundary conditions $u(0) = u(1) = 0$ specify that the solution $u(x)$ must be zero at both ends of the interval $(0, 1)$ and the differential equation is non-homogeneous, which creates an issue.

Typically, eigenfunction expansion involves expressing the solution as a series of eigenfunctions, usually sine or cosine functions, and finding the coefficients that satisfy the given boundary conditions and the PDE. The method is most effective when the PDE is linear and homogeneous, indicating that the right-hand side is zero.

When you substitute the expression $u(x) = \sum_{n=1}^{\infty} \alpha_n \phi(x)$ with the appropriate eigenfunctions, there may not be terms that cancel out, making it challenging to satisfy the boundary conditions.

Even in the case where we have

$$u(x) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x)$$

as determined previously, this yields

$$\partial_{xx} u(x) = \sum_{n=1}^{\infty} \alpha_n n^2 \pi^2 \sin(n\pi x) = 1$$

which does not necessarily hold true for the case where $x = 0$ or $x = 1$. This seems to imply that there is interesting behavior at the boundary of the PDE with eigenfunction expansion, considering that the boundary values evaluate to zero, while the PDE does not.

Thus, finding eigenfunctions (and the corresponding coefficients) would prove difficult, indicating why this method does not always work. In this case, separation of variables is an effective alternative.

8) Solve the following problem

$$u_t - u_{xx} = 0 \qquad (t,x) \in (0,\infty) \times (0,1)\,, u(t,0) = u(t,1) = 0$$

The initial value is $u(0,x) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin(n\pi x)$. *Hint:* Assume $u(t,x) = \sum_{n=1}^{\infty} \alpha_n(t)\, u_n(x)$.

To solve this problem, we may use the method of separation of variables, as implied by the hint. Thus, we assume the solution has the form

$$u(t,x) = \sum_{n=1}^{\infty} \alpha_n(t)\, u_n(x)$$

Let us consider the case where

$$u_n(x) = \frac{1}{n^2}\sin(n\pi x)$$

This form is chosen based on the conditions that $u(t,0) = u(t,1) = 0$, alongside the initial value $u(0,x) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin(n\pi x)$. As we will demonstrate, it satisfies them appropriately.

If we use this as input in the partial differential equation, we have the following:

$$\begin{aligned} u_t - u_{xx} &= \sum_{n=1}^{\infty} \alpha_n'(t)\, u_n(x) - \sum_{n=1}^{\infty} \alpha_n(t)\, u_n''(x) \\ &= \sum_{n=1}^{\infty} \left[\alpha_n'(t)\frac{1}{n^2}\sin(n\pi x)\right] - \sum_{n=1}^{\infty} \left[\alpha_n(t)\left(-\frac{n^2\pi^2}{n^2}\right)\sin(n\pi x)\right] \\ &= \sum_{n=1}^{\infty} \left[\alpha_n'(t)\frac{1}{n^2}\sin(n\pi x)\right] + \sum_{n=1}^{\infty} \left[\alpha_n(t)\,\pi^2\sin(n\pi x)\right] \end{aligned}$$

Now, given that $u_t - u_{xx} = 0$, we may equate the coefficients of $\sin(n\pi x)$ within the equation, as follows:

$$\alpha_n'(t)\frac{1}{n^2} + \alpha_n(t)\,\pi^2 = 0$$

This is an ordinary differential equation (for $\alpha_n(t)$), which is separable, so we may solve it as follows:

$$\frac{d\alpha_n(t)}{dt}\frac{1}{n^2} + \alpha_n(t)\,\pi^2 = 0 \longrightarrow \frac{d\alpha_n(t)}{dt}\frac{1}{n^2} = -\alpha_n(t)\,\pi^2 \longrightarrow \frac{d\alpha_n(t)}{dt} = -\alpha_n(t)\,\pi^2 n^2$$

$$\frac{1}{\alpha_n(t)}\,d\alpha_n(t) = -\pi^2 n^2\,dt$$

Now, we take the integral of both sides, which results in

$$\int \frac{1}{\alpha_n(t)}\,d\alpha_n(t) = \int -\pi^2 n^2\,dt$$

$$\ln|\alpha_n(t)| = -\pi^2 n^2 t + C$$

When we take the exponential of both sides to solve for $\alpha_n(t)$, we have the following:

$$\alpha_n(t) = e^{-\pi^2 n^2 t + C}$$

Since e^C is just a constant value, we may simplify this expression as follows:

$$\alpha_n(t) = Ce^{-\pi^2 n^2 t}$$

Now, the solution $u(t,x)$ may be written as

$$u(t,x) = \sum_{n=1}^{\infty} Ce^{-\pi^2 n^2 t}\frac{1}{n^2}\sin(n\pi x)$$

To determine the value of C, we use the initial condition $u\left(0,x\right)=\frac{1}{n^2}\sin\left(n\pi x\right)$.

$$u\left(0,x\right)=\sum_{n=1}^{\infty}C\frac{1}{n^2}\sin\left(n\pi x\right)$$

Thus, we know that the appropriate value is $C=1$. So, the solution to the given partial differential equation with the initial condition is

$$u\left(t,x\right)=\sum_{n=1}^{\infty}e^{-\pi^2n^2t}\frac{1}{n^2}\sin\left(n\pi x\right)$$

9) For $u(t,x): \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, solve for $u(t,x), t > 0$, when u satisfies

$$u_t + 2u_x = 1$$

where $u(0,x) = \exp\left(-x^2\right)$

To solve the partial differential equation, let us start by assuming $\frac{\partial x}{\partial t} = 2$, according to the set-up of the equation. This is called the characteristic equation. This implies that $\frac{du}{dt} = 1$, such that u is dependent on t and a constant.

Solving the differential equation $\frac{dx}{dt} = 2$ yields $x = 2t + c \to c = -2t + x$. This implies that u is constant along this line (not considering the previous t accounting for the non-homogeneous solution), for various c values. That is, the general solution of the partial differential equation is

$$u(t,x) = t + f(-2t + x)$$

Setting $t = 0$ yields the equation $u(0,x) = \exp\left(-x^2\right) = f(x)$. Therefore,

$$u(t,x) = t + \exp\left(-(-2t + x)^2\right)$$

Now, we must check to ensure that the solution we found works with the partial differential equation and the initial condition. For the initial condition $u(0,x) = \exp\left(-x^2\right)$ holds. The partial differential equation is determined as follows:

$$u_t + 2u_x = \left(1 + 4\exp\left(-(-2t + x)^2\right)\right) + 2\left(-2\left(\exp\left(-(-2t + x)^2\right)\right)\right) = 1$$

Thus, we have verified the partial differential equation, indicating that the solution $u(t,x) = t + \exp\left(-(-2t + x)^2\right)$ is valid for the given case. $\square$

10) (Strauss 1.3.9) This is an exercise on the divergence theorem

$$\iiint_D \boldsymbol{\nabla} \cdot \boldsymbol{F} d\boldsymbol{x} = \iint_{\text{boundary of } D} \boldsymbol{F} \cdot \boldsymbol{n} dS$$

valid for any bounded domain D in space with boundary and unit outward normal vector $\boldsymbol{n}$. Verify it in the following case by calculating both sides separately.

$$\boldsymbol{F} = r^2 \boldsymbol{x}$$

where $x = (x_1, x_2, x_3)$, $r^2 = x_1^2 + x_2^2 + x_3^2$, and D is the ball of radius a centered at the origin.

Let us calculate the left-hand side first, as follows. Observe $\frac{\partial r}{\partial x_1} = 2x_1^2$, $\frac{\partial r}{\partial x_2} = 2x_2^2$, $\frac{\partial r}{\partial x_3} = 2x_3^2$. Thus,

$$\begin{aligned}\iiint_D \boldsymbol{\nabla} \cdot \boldsymbol{F} d\boldsymbol{x} &= \iiint_D \boldsymbol{\nabla} \cdot \left(r^2 x_1, r^2 x_2, r^2 x_3\right) d\boldsymbol{x} \\ &= \iiint_D \left(r^2 + 2x_1^2 + r^2 + 2x_2^2 + r^2 + 2x_3^2\right) d\boldsymbol{x}\end{aligned}$$

This is equivalent to

$$\begin{aligned}\iiint_D \boldsymbol{\nabla} \cdot \boldsymbol{F} d\boldsymbol{x} &= \iiint_D \left(3r^2 + 2\left(x_1^2 + x_2^2 + x_3^2\right)\right) d\boldsymbol{x} \\ &= \iiint_D \left(3r^2 + 2r^2\right) d\boldsymbol{x} \\ &= \iiint_D \left(5r^2\right) d\boldsymbol{x}\end{aligned}$$

Converting to spherical coordinates, we have

$$\iiint_D \boldsymbol{\nabla} \cdot \boldsymbol{F} d\boldsymbol{x} = \iiint_D \left(5r^2\right) r^2 \sin(\phi)\, dr\, d\phi\, d\theta$$

Using the bounds of $a \in [0, a]$, $\phi \in [0, \pi]$, and $\theta \in [0, 2\pi]$ for the sphere, this is equivalent to

$$\begin{aligned}\iiint_D \boldsymbol{\nabla} \cdot \boldsymbol{F} d\boldsymbol{x} &= 2\pi \int_0^\pi \int_0^a 5r^4 \sin(\phi)\, dr\, d\phi \\ &= 4\pi \int_0^a 5r^4 \sin(\phi)\, dr \\ &= 4\pi a^5\end{aligned}$$

Now, let us calculate the right-hand side, as follows.

$$\begin{aligned}\iint_{\text{boundary of } D} \boldsymbol{F} \cdot \boldsymbol{n} dS &= \iint_{\text{boundary of } D} r^2 \left(x_1, x_2, x_3\right) \cdot \frac{\left(x_1, x_2, x_3\right)}{r} dS \\ &= \iint_{\text{boundary of } D} r\left(x_1^2 + x_2^2 + x_3^2\right) dS \\ &= \iint_{\text{boundary of } D} r\left(r^2\right) dS \\ &= \iint_{\text{boundary of } D} r^3 dS\end{aligned}$$

Converting to spherical coordinates, we have

$$\begin{aligned}\iint_{\text{boundary of } D} \boldsymbol{F} \cdot \boldsymbol{n} dS &= \iint_{\text{boundary of } D} r^3 r^2 \sin(\phi)\, d\phi\, d\theta \\ &= \iint_{\text{boundary of } D} a^3 a^2 \sin(\phi)\, d\phi\, d\theta\end{aligned}$$

Using the bounds of $\phi \in [0, \pi]$, and $\theta \in [0, 2\pi]$ for the sphere, this is equivalent to

$$\begin{aligned}\iint_{\text{boundary of } D} \boldsymbol{F} \cdot \boldsymbol{n} dS &= \int_0^{2\pi} \int_0^{\pi} a^5 \sin(\phi) \, d\phi \, d\theta \\ &= 2\pi \int_0^{\pi} a^5 \sin(\phi) \, d\phi \\ &= 4\pi a^5\end{aligned}$$

Thus, we have demonstrated that the divergence theorem

$$\iiint_D \boldsymbol{\nabla} \cdot \boldsymbol{F} d\boldsymbol{x} = \iint_{\text{boundary of } D} \boldsymbol{F} \cdot \boldsymbol{n} dS$$

is valid for the bounded domain D in space as a ball of radius a centered at the origin, with unit outward normal vector $\boldsymbol{n}$, alongside the given specifications. $\square$

11) Let $f(x)$ be a continuous function in a finite closed interval $[a,b]$. Assume that $f(x) \geq 0$ in the interval and that $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$. *(This is mostly for future reference.)*

To prove this, we prove the contrapositive. Suppose $\exists x_0 \in [a,b]$ such that $f(x) \neq 0$. Since $f(x_0) \geq 0$, $f(x_0) > 0$. We now show that $\int_a^b f(x)\,dx \neq 0$ (specifically, $\int_a^b f(x)\,dx > 0$).

Let us consider the case where $x_0 \in (a,b)$. Since f is continuous at x_0, $\exists \delta_1$ such that if $x \in [a,b]$ and $|x - x_0| < \delta_1$, then $|f(x) - f(x_0)| < \frac{f(x_0)}{2}$. Since x_0 is in an open interval, $\exists \delta_2$ such that if $x \in \mathbb{R}$ with $|x - x_0| < \delta_2$, then $x \in (a,b)$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\forall x$ such that $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \frac{f(x_0)}{2}$. This inequality implies

$$-\frac{f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0)}{2}$$

$$\frac{f(x_0)}{2} < f(x) < \frac{3f(x_0)}{2}$$

We now define the step function $g(x)$ such that $g(x) = 0$ if $x \in [a, x_0 - \delta] \cup [x_0 + \delta, b]$ and $g(x) = \frac{f(x_0)}{2}$ if $x \in (x_0 - \delta, x_0 + \delta)$. Step functions are integrable and furthermore, $g(x) \leq f(x)\,\forall x \in [a,b]$. Also, $0 < \delta f(x_0) = \int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$. This proves this case.

We now have to consider the case where $x_0 = a$ or $x_0 = b$. We still have continuity at these points, except that now we have to consider it on the half interval $[a, \delta')$ or $(\delta', b]$ for some δ'. We can then use the same f inequality to reach the previous highlighted equation for points in the half interval. We could then define the step function to be zero outside this interval and $\frac{f(x_0)}{2}$ inside it, and conclude $0 < \frac{\delta' f(x_0)}{2} = \int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$. This proves this case. $\square$

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