MATH 053/126 - Partial Differential Equations

Carter Kruse, October 10, 2023

Homework 3

Instructions/Notation

Please show all steps to get your answers. Specify the problems you discussed with other students (including names).

The starred problems are recommended but not required for undergraduate/non-math major graduate students and required for all math major graduate students.

Notation

- $\mathbb{R}$: The set of all real numbers.
- $\mathbb{R}^+$: The set all positive real numbers $\{x \in \mathbb{R} | x > 0\}$.
- $\frac{\partial u}{\partial t} = \partial_t u = u_t$
- $\frac{\partial u}{\partial x} = \partial_x u = u_x$

Questions

1A) Let u be a function of x, y, z. Show that $u_{xx} + u_{yy} + u_{zz}$ is equal to the divergence of the gradient of u. That is $\nabla \cdot (\nabla u) = \Delta u$.

As indicated, let u be a function of x, y, z. The gradient of u is defined as follows:

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)$$

In taking the divergence of the gradient, we have the following:

$$\begin{aligned} \nabla \cdot (\nabla u) &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial z} \right) \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ &= u_{xx} + u_{yy} + u_{zz} \end{aligned}$$

Thus, we have shown that

$$\nabla \cdot (\nabla u) = \Delta u$$

where $\Delta u = u_{xx} + u_{yy} + u_{zz}$ represents the Laplacian of u.

1B) Let u be a function of x, y, z. Show that $\nabla \cdot (u\nabla u) = \nabla u \cdot \nabla u + u\Delta u$.

As indicated, let u be a function of x, y, z. The gradient of u is defined as follows:

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

The Laplacian of u is defined as follows:

$$\Delta u = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

In taking the divergence of the gradient (multiplied by u), we have the following:

$$\begin{aligned}
\nabla \cdot (u\nabla u) &= \nabla \cdot \left(u\frac{\partial u}{\partial x}, u\frac{\partial u}{\partial y}, u\frac{\partial u}{\partial z}\right) \\
&= u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + u\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial z} \\
&= u\Delta u + \nabla u \cdot \nabla u
\end{aligned}$$

Thus, we have shown that

$$\nabla \cdot (u\nabla u) = \nabla u \cdot \nabla u + u\Delta u$$

2) Let u be the solution to the following wave equation

$$u_{tt} - u_{xx} = 0 \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}$$

$$u(0, x) = \phi(x)$$

$$u_t(0, x) = \psi(x)$$

where $\phi(x)$ and $\psi(x)$ are given odd functions. That is, $\phi(-x) = -\phi(x)$ and $\psi(-x) = -\psi(x)$. Show that u is also an odd function in space. Do NOT use the solution formula.

Hint: Try $u(t, x) + u(t, -x)$ and use the uniqueness result.

To show that $u(t, x)$ is an odd function in space, we will use the hint, considering $v(t, x) = u(t, x) + u(t, -x)$. Using this expression, we may take the second derivative of v with respect to t and the second derivative of v with respect to x, which results in

$$v_{tt} - v_{xx} = 0 \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}$$

Thus, $v(t, x)$ satisfies the wave equation. This may further be considered as a result of the linearity of the wave equation, in that linear combinations of solutions to the wave equation will be a solution to the wave equation. (The differential operator is linear.)

The initial conditions for $v(t, x)$ are as follows. Consider that ϕ and ψ are odd functions.

$$\begin{aligned} v(0, x) &= u(0, x) + u(0, -x) \\ &= \phi(x) + \phi(-x) \\ &= \phi(x) - \phi(x) \\ &= 0 \\ v_t(0, x) &= u_t(0, x) + u_t(0, -x) \\ &= \psi(x) + \psi(-x) \\ &= \psi(x) - \psi(x) \\ &= 0 \end{aligned}$$

Hence, by the uniqueness of the solutions to the wave equation, the only solution to the wave equation $v_{tt} - v_{xx} = 0$ with the initial conditions $v(0, x) = 0$ and $v_t(0, x) = 0$ is the zero solution, so $v(t, x) = 0$ for all t and x.

By our definition of $v(t, x) = u(t, x) + u(t, -x)$, we find that $u(t, x) = -u(t, -x)$, which indicates that $u(t, x)$ is an odd function in space. $\square$

3) For the solution u of the following wave equation

$$u_{tt} - u_{xx} = 0 \qquad (t,x) \in \mathbb{R}^+ \times (-\infty, \infty)$$

$$u(0,x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$u_t(0,x) = 0$$

Consider the following wave equation

$$u_{tt} - u_{xx} = f(t,x) \qquad (t,x) \in (0,\infty) \times \mathbb{R} \tag{1}$$

$$u(0,x) = \phi(x)$$

$$u_t(0,x) = \psi(x)$$

This simply represents the general form of the wave equation. Consider the following solution:

$$u(t,x) = \frac{1}{2}\left(\phi(x+t) + \phi(x-t) + \int_{x-t}^{x+t} \psi(s)\, ds + \iint_{\Delta} f(t,x)\right)$$

where Δ is the characteristic triangle. The double integral is equal to the iterated integral

$$\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\, dy\, ds$$

To demonstrate that this is the solution, we use a method involving Green's theorem. In this method we integrate f over the past history triangle Δ:

$$\iint_{\Delta} f\, dx\, dt = \iint_{\Delta} (u_{tt} - u_{xx})\, dx\, dt$$

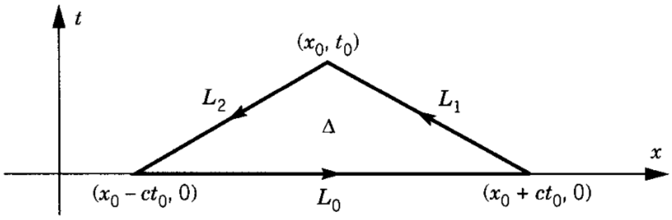
Green's theorem says that

$$\iint_{\Delta} (P_x - Q_t)\, dx\, dt = \int_{\partial\Delta} P\, dt + Q\, dx$$

for any functions P and Q, where the line integral on the boundary is taken counterclockwise. Thus,

$$\iint_{\Delta} f\, dx\, dt = \int_{L_0 + L_1 + L_2} (-u_x\, dt - u_t\, dx)$$

This is the sum of three line integrals over straight line segments, as shown by the figure.



We evaluate each piece separately. On L_0, $dt = 0$ and $u_t(x,0) = \psi(x)$, so

$$\int_{L_0} = -\int_{x_0 - t_0}^{x_0 + t_0} \psi(x)\, dx$$

On L_1, $x+t=x_0+t_0$, so that $dx+dt=0$, whence $-u_x\,dt-u_t\,dx=u_x\,dx+u_t\,dt=du$. This may be thought of in terms of the total derivative as well. Thus,

$$\int_{L_1} = \int_{L_1} du = u\left(t_0,x_0\right)-\phi\left(x_0+t_0\right)$$

In the same way,

$$\int_{L_2} = -\int_{L_2} du = -\phi\left(x_0-t_0\right)+u\left(t_0,x_0\right)$$

Adding these three results produces the following:

$$\iint_{\Delta} f\,dx\,dt = 2u\left(t_0,x_0\right)-\left[\phi\left(x_0+t_0\right)+\phi\left(x_0-t_0\right)\right]-\int_{x_0-t_0}^{x_0+t_0}\phi\left(x\right)\,dx$$

Thus,

$$u\left(t_0,x_0\right)=\frac{1}{2}\left(\iint_{\Delta} f\,dx\,dt+\phi\left(x_0+t_0\right)+\phi\left(x_0-t_0\right)+\int_{x_0-t_0}^{x_0+t_0}\psi\left(x\right)\,dx\right)$$

This proves the solution form that we previously determined. $\square$

As highlighted, the solution to the wave equation $u_{tt}-u_{xx}=f\left(t,x\right)$ for $\left(t,x\right)\in\mathbb{R}^{+}\times\left(-\infty,\infty\right)$ and $u\left(0,x\right)=\phi\left(x\right)$ and $u_t\left(0,x\right)=\psi\left(x\right)$ is given as follows:

$$u\left(t,x\right)=\frac{1}{2}\left(\phi\left(x+t\right)+\phi\left(x-t\right)+\int_{x-t}^{x+t}\psi\left(s\right)\,ds+\iint_{\Delta} f\left(t,x\right)\right)$$

where Δ is the characteristic triangle. The double integral is equal to the iterated integral

$$\int_0^t\int_{x-(t-s)}^{x+(t-s)} f\left(y,s\right)\,dy\,ds$$

For this particular problem, $f\left(t,x\right)=0$ and $u_t\left(0,x\right)=0$, so the solution simplifies to the following:

$$u\left(t,x\right)=\frac{1}{2}\left(\phi\left(x+t\right)+\phi\left(x-t\right)\right)$$

Further, $\phi\left(x\right)$ is given as the following piecewise function:

$$u\left(0,x\right)=\phi\left(x\right)=\begin{cases}1-x^2 & \text{if } |x|<1\\ 0 & \text{otherwise}\end{cases}$$

Thus, this should be taken into consideration when evaluating the solution.

A) Find $u\left(2,1\right)$.

Inputting this into the expression for the solution, we have

$$\begin{aligned}u\left(2,1\right)&=\frac{1}{2}\left(\phi\left(1+2\right)+\phi\left(1-2\right)\right)\\&=\frac{1}{2}\left(\phi\left(3\right)+\phi\left(-1\right)\right)\\&=\frac{1}{2}\left(0+0\right)\\&=0\end{aligned}$$

Find $u(1,2)$.

Inputting this into the expression for the solution, we have

$$\begin{aligned} u(1,2) &= \frac{1}{2}\left(\phi(2+1)+\phi(2-1)\right) \\ &= \frac{1}{2}\left(\phi(3)+\phi(1)\right) \\ &= \frac{1}{2}(0+0) \\ &= 0 \end{aligned}$$

Find $u(0,2)$.

Inputting this into the expression for the solution, we have

$$\begin{aligned} u(0,2) &= \frac{1}{2}\left(\phi(2+0)+\phi(2-0)\right) \\ &= \frac{1}{2}\left(\phi(2)+\phi(2)\right) \\ &= \frac{1}{2}(0+0) \\ &= 0 \end{aligned}$$

4) For a function $f(t,x)$, let $h(t,x) = \int_{\Delta} f = \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)\, dy\, ds$. Find $\partial_x h$ and $\partial_t h$.

The function $h(t,x)$ is defined as the following double integral:

$$h(t,x) = \int_{\Delta} f = \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)\, dy\, ds$$

This uses a function $f(t,x)$. To find $\partial_x h$, we take the partial derivative with respect to x, as follows:

$$\begin{aligned} \partial_x h &= \frac{\partial}{\partial x} \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)\, dy\, ds \\ &= \int_0^t \frac{\partial}{\partial x} \int_{x-(t-s)}^{x+(t-s)} f(s,y)\, dy\, ds \\ &= \int_0^t \left[f(s, x+(t-s)) - f(s, x-(t-s)) \right] ds \end{aligned}$$

Similarly, to find $\partial_t h$, we take the partial derivative with respect to t:

$$\begin{aligned} \partial_t h &= \frac{\partial}{\partial t} \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)\, dy\, ds \\ &= \int_0^t \frac{\partial}{\partial t} \int_{x-(t-s)}^{x+(t-s)} f(s,y)\, dy\, ds + \int_{x-(t-t)}^{x+(t-t)} f(s,y)\, dy \\ &= \int_0^t \left[f(s, x+(t-s)) + f(s, x-(t-s)) \right] ds \end{aligned}$$

Therefore, the partial derivatives are

$$\partial_x h = \int_0^t \left[f(s, x+(t-s)) - f(s, x-(t-s)) \right] ds$$

$$\partial_t h = \int_0^t \left[f(s, x+(t-s)) + f(s, x-(t-s)) \right] ds$$

5) Find $\partial_{xx}h$ and $\partial_{tt}h$. Note that $\partial_{tt}h - \partial_{xx}h$ must be in $2f$.

The partial derivatives are given as

$$\partial_x h = \int_0^t \left[f\left(s, x + (t-s)\right) - f\left(s, x - (t-s)\right)\right] ds$$

$$\partial_t h = \int_0^t \left[f\left(s, x + (t-s)\right) + f\left(s, x - (t-s)\right)\right] ds$$

To find $\partial_{xx}h$, we take the partial derivative of the first expression with respect to x, as follows:

$$\begin{aligned}\partial_{xx}h &= \frac{\partial}{\partial x}\int_0^t \left[f\left(s, x + (t-s)\right) - f\left(s, x - (t-s)\right)\right] ds \\ &= \int_0^t \left[\frac{\partial}{\partial x} f\left(s, x + (t-s)\right) - \frac{\partial}{\partial x} f\left(s, x - (t-s)\right)\right] ds\end{aligned}$$

Similarly, to find $\partial_{tt}h$, we take the partial derivative of the second expression with respect to t, as follows:

$$\begin{aligned}\partial_{tt}h &= \frac{\partial}{\partial t}\int_0^t \left[f\left(s, x + (t-s)\right) + f\left(s, x - (t-s)\right)\right] ds \\ &= \int_0^t \left[\frac{\partial}{\partial x} f\left(s, x + (t-s)\right) - \frac{\partial}{\partial x} f\left(s, x - (t-s)\right)\right] ds + f\left(t, x\right) + f\left(t, x\right)\end{aligned}$$

Thus, we know that $\partial_{tt}h - \partial_{xx}h = 2f\left(t, x\right)$. $\square$

6) Solve the following diffusion problem in $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$.

$$u_t - u_{xx} = 0 \qquad u(0, x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if otherwise} \end{cases}$$

Our purpose is to solve the problem

$$u_t - u_x x = 0 \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}$$

$$u(0, x) = \phi(x)$$

As with the wave equation, the problem on the infinite line has a certain "purity", which makes it easier to solve than the finite-interval problem. (The effects of boundaries will be discussed in the next several chapters.) Also as with the wave equation, we will end up with an explicit formula. But it will be derived by a method *very different* from the methods used before. (The characteristics for the diffusion equation are just the lines $t =$ constant and play no major role in the analysis.) Because the solution is not easy to derive, we first set the stage by making some general comments.

Our method is to solve it for a *particular* $\phi(x)$ and then build the general solution from this particular one. We'll use five basic *invariance properties* of the diffusion equation.

The *translate* $u(t, x - y)$ of any solution $u(t, x)$ is another solution for any fixed y.

Any *derivative* (u_x or u_t or u_{xx}, etc) of a solution is again a solution.

A *linear combination* of solutions is again a solution. (This is just linearity.)

An *integral* of solutions is again a solution. Thus, if $S(t, x)$ is a solution, then so is $S(t, x - y)$ and so is

$$v(t, x) = \int_{-\infty}^{\infty} S(t, x - y) g(y)\, dy$$

for any function $g(y)$, as long as this improper integral converges appropriately.

If $u(t, x)$ is a solution, so is the *dilated* function $u(at, \sqrt{a}x)$ for any $a > 0$. Prove this by the chain rule. Let $v(t, x) = u(at, \sqrt{a}x)$. Then $v_t = [\partial(at)/\partial t]\, u_t = au_t$ and $v_x = [\partial(\sqrt{a}x)/\partial x]\, u_x = \sqrt{a}u_x$ and $v_{xx} = \sqrt{a} \cdot \sqrt{a}u_{xx} = au_{xx}$.

Our goal is to find a particular solution and then construct all the other solutions using the integral property. The particular solution we will look for is the one, denoted $Q(t, x)$, which satisfies the *special initial condition*

$$Q(0, x) = 1 \text{ for } x > 0 \qquad Q(0, x) \text{ for } x < 0$$

The reason for this choice is that this initial condition does not change under dilation. We'll find Q in three steps.

Step 1 We'll look for $Q(t, x)$ of the special form

$$Q(t, x) = g(p) \qquad \text{where } p = \frac{x}{\sqrt{4t}}$$

and g is a function of only one variable (to be determined). (The $\sqrt{4}$ factor is included only to simplify a later formula.)

Why do we expect Q to have this special form? Because the dilation property says that the equation doesn't "see" the dilation $x \to \sqrt{a}x, t \to at$. Clearly, the expression for Q does not change at all under the dilation. So $Q(t, x)$, which is defined by the conditions, ought not to see the dilation either. How could that happen? In only one way. If Q depends on x and t solely through the combination $x/\sqrt{t}$. For the dilation takes $x/\sqrt{t}$ into $\sqrt{a}x/\sqrt{at} = x/\sqrt{t}$. Thus, let $p = x/\sqrt{4t}$ and look for Q which satisfies the conditions.

Step 2 Using this, we convert the diffusion equation into an ODe for g by use of the chain rule:

$$\begin{aligned} Q_t &= \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4t}}g'(p) \\ Q_x &= \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4t}}g'(p) \\ Q_{xx} &= \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4t}g''(p) \\ 0 = Q_t - Q_{xx} &= \frac{1}{t}\left[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right] \end{aligned}$$

Thus

$$g'' + 2pg' = 0$$

This ODE is easily solved using the integrating factor $\exp\int 2p\, dp = \exp\left(p^2\right)$. We get $g'(p) = c_1 \exp\left(-p^2\right)$ and

$$Q(t,x) = g(p) = c_1 \int e^{-p^2}\, dp + c_2$$

Step 3 We find a completely explicit formula for Q. We've just shown that

$$Q(t,x) = c_1 \int_0^{x/\sqrt{4t}} e^{-p^2}\, dp + c_2$$

This formula is valid only for $t > 0$. Now, we use the expression for Q, expressed as a limit as follows:

$$\text{if } x > 0, 1 = \lim_{t\longrightarrow 0} Q = c_1 \int_0^{+\infty} e^{-p^2}\, dp + c_2 = c_1\frac{\sqrt{\pi}}{2} + c_2$$

$$\text{if } x < 0, 0 = \lim_{t\longrightarrow 0} Q = c_1 \int_0^{-\infty} e^{-p^2}\, dp + c_2 = -c_1\frac{\sqrt{\pi}}{2} + c_2$$

Here $\lim_{t\longrightarrow 0}$ means limit from the right. This determines the coefficients $c_1 = 1/\sqrt{\pi}$ and $c_2 = \frac{1}{2}$. Therefore, Q is the function

$$Q(t,x) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4t}} e^{-p^2}\, dp$$

for $t > 0$.

Step 4 Having found Q, we now define $S = \partial Q/\partial x$. (The explicit formula for S will be written below.) By the property for derivatives, S is also a solution to the diffusion equation. Given any function ϕ, we also define

$$u(t,x) = \int_{-\infty}^{\infty} S(t, x-y)\,\phi(y)\, dy \text{ for } t > 0$$

By the properties, u is another solution of the diffusion equation. We claim that u is the unique solution. To verify the validity of the initial condition, we write

$$\begin{aligned} u(t,x) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(t, x-y)\,\phi(y)\, dy \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left[Q(t,x-y)\right]\phi(y)\, dy \\ &= +\int_{-\infty}^{\infty} Q(t,x-y)\,\phi'(y)\, dy - Q(t,x-y)\,\phi(y)\,\big|_{y=-\infty}^{y=\infty} \end{aligned}$$

upon integrating by parts. We assume these limits vanish. In particular, let's temporarily assume that $\phi(y)$ itself equals zero for $|y|$ large. Therefore,

$$\begin{aligned} u(0,x) &= \int_{-\infty}^{\infty} Q(0,x-y)\,\phi'(y)\, dy \\ &= \int_{-\infty}^{x} \phi'(y)\, dy = \phi|_{-\infty}^{x} = \phi(x) \end{aligned}$$

because of the initial condition for Q and the assumption that $\phi(-\infty) = 0$. This is the initial condition. We conclude that our solution formula is given as above, where

$$S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \text{ for } t > 0$$

That is,

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} \phi(y)\, dy$$

$S(t,x)$ is known as the source function, Green's function, fundamental solution Gaussian, or propagator of the diffusion equation, or simply the diffusion kernel. It gives the solution of the diffusion equation with any initial datum ϕ. The formula only gives the solution for $t > 0$. When $t = 0$, it makes no sense. $\square$

As highlighted, the solution to the diffusion equation $u_t - u_{xx} = 0$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ and $u(0,x) = \phi(x)$ is given as follows:

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} \phi(y)\, dy$$

For this particular problem, $\phi(x)$ is given as the following piecewise function:

$$u(0,x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if otherwise} \end{cases}$$

Thus, the solution simplifies to the following:

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-1}^{1} e^{-(x-y)^2/4t}\, dy$$

7) Show that the solution to the following diffusion problem is unique

$$u_t - u_{xx} = \exp\left(\sin\left(t + x^2\right)\right) \qquad x \in \mathbb{R} \qquad u(0, x) = \frac{x^2}{1 + x^4}$$

Consider the following diffusion equation

$$u_t - u_{xx} = f(t, x) \qquad (t, x) \in (0, \infty) \times \mathbb{R} \tag{2}$$

$$u(0, x) = \phi(x)$$

This simply represents the general form of the diffusion equation. To prove it's uniqueness, let $u_1(t, x)$ and $u_2(t, x)$ represent two solutions of the diffusion equation $u_t - u_{xx} = f(t, x)$ with $(t, x) \in (0, \infty) \times \mathbb{R}$, along with the initial condition $u(0, x) = \phi(x)$.

Now let $v(t, x) = u_1(t, x) - u_2(t, x)$. The following is true:

$$v_t - v_{xx} = 0 \qquad (t, x) \in (0, \infty) \times \mathbb{R}$$

$$v(0, x) = 0$$

This represents a homogeneous diffusion equation with the initial condition $v(0, x) = 0$. To solve for $v(t, x) : (0, \infty) \times \mathbb{R} \to \mathbb{R}$, we may use the following method.

As with the wave equation, the problem on the infinite line has a certain "purity", which makes it easier to solve than the finite-interval problem. (The effects of boundaries will be discussed in the next several chapters.) Also as with the wave equation, we will end up with an explicit formula. But it will be derived by a method *very different* from the methods used before. (The characteristics for the diffusion equation are just the lines $t =$ constant and play no major role in the analysis.) Because the solution is not easy to derive, we first set the stage by making some general comments.

Our method is to solve it for a *particular* $\phi(x)$ and then build the general solution from this particular one. We'll use five basic *invariance properties* of the diffusion equation.

The *translate* $u(t, x - y)$ of any solution $u(t, x)$ is another solution for any fixed y.

Any *derivative* (u_x or u_t or u_{xx}, etc) of a solution is again a solution.

A *linear combination* of solutions is again a solution. (This is just linearity.)

An *integral* of solutions is again a solution. Thus, if $S(t, x)$ is a solution, then so is $S(t, x - y)$ and so is

$$v(t, x) = \int_{-\infty}^{\infty} S(t, x - y)\, g(y)\; dy$$

for any function $g(y)$, as long as this improper integral converges appropriately.

If $u(t, x)$ is a solution, so is the *dilated* function $u(at, \sqrt{a}x)$ for any $a > 0$. Prove this by the chain rule. Let $v(t, x) = u(at, \sqrt{a}x)$. Then $v_t = [\partial(at)/\partial t]\, u_t = au_t$ and $v_x = [\partial(\sqrt{a}x)/\partial x]\, u_x = \sqrt{a}u_x$ and $v_{xx} = \sqrt{a} \cdot \sqrt{a}u_{xx} = au_{xx}$.

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$$Q(t,x) = g(p) \qquad \text{where } p = \frac{x}{\sqrt{4t}}$$

and g is a function of only one variable (to be determined). (The $\sqrt{4}$ factor is included only to simplify a later formula.)

Why do we expect Q to have this special form? Because the dilation property says that the equation doesn't "see" the dilation $x \to \sqrt{a}x, t \to at$. Clearly, the expression for Q does not change at all under the dilation. So $Q(t,x)$, which is defined by the conditions, ought not to see the dilation either. How could that happen? In only one way. If Q depends on x and t solely through the combination $x/\sqrt{t}$. For the dilation takes $x/\sqrt{t}$ into $\sqrt{a}x/\sqrt{at} = x/\sqrt{t}$. Thus, let $p = x/\sqrt{4t}$ and look for Q which satisfies the conditions.

Step 2 Using this, we convert the diffusion equation into an ODe for g by use of the chain rule:

$$Q_t = \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4t}}g'(p)$$

$$Q_x = \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4t}}g'(p)$$

$$Q_{xx} = \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4t}g''(p)$$

$$0 = Q_t - Q_{xx} = \frac{1}{t}\left[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right]$$

Thus

$$g'' + 2pg' = 0$$

This ODE is easily solved using the integrating factor $\exp\int 2p\,dp = \exp\left(p^2\right)$. We get $g'(p) = c_1 \exp\left(-p^2\right)$ and

$$Q(t,x) = g(p) = c_1 \int e^{-p^2}\,dp + c_2$$

Step 3 We find a completely explicit formula for Q. We've just shown that

$$Q(t,x) = c_1 \int_0^{x/\sqrt{4t}} e^{-p^2}\,dp + c_2$$

This formula is valid only for $t > 0$. Now, we use the expression for Q, expressed as a limit as follows:

$$\text{if } x > 0, 1 = \lim_{t \longrightarrow 0} Q = c_1 \int_0^{+\infty} e^{-p^2}\,dp + c_2 = c_1\frac{\sqrt{\pi}}{2} + c_2$$

$$\text{if } x < 0, 0 = \lim_{t \longrightarrow 0} Q = c_1 \int_0^{-\infty} e^{-p^2}\,dp + c_2 = -c_1\frac{\sqrt{\pi}}{2} + c_2$$

Here $\lim_{t \longrightarrow 0}$ means limit from the right. This determines the coefficients $c_1 = 1/\sqrt{\pi}$ and $c_2 = \frac{1}{2}$. Therefore, Q is the function

$$Q(t,x) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4t}} e^{-p^2}\,dp$$

for $t > 0$.

Step 4 Having found Q, we now define $S = \partial Q/\partial x$. (The explicit formula for S will be written below.) By the property for derivatives, S is also a solution to the diffusion equation. Given any function ϕ, we also define

$$u(t,x) = \int_{-\infty}^{\infty} S(t, x-y)\,\phi(y)\,dy \text{ for } t > 0$$

By the properties, u is another solution of the diffusion equation. We claim that u is the unique solution. To verify the validity of the initial condition, we write

$$\begin{aligned} u(t,x) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(t,x-y)\,\phi(y)\,dy \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left[Q(t,x-y)\right]\phi(y)\,dy \\ &= +\int_{-\infty}^{\infty} Q(t,x-y)\,\phi'(y)\,dy - Q(t,x-y)\,\phi(y)\,\Big|_{y=-\infty}^{y=\infty} \end{aligned}$$

upon integrating by parts. We assume these limits vanish. In particular, let's temporarily assume that $\phi(y)$ itself equals zero for $|y|$ large. Therefore,

$$\begin{aligned} u(0,x) &= \int_{-\infty}^{\infty} Q(0,x-y)\,\phi'(y)\,dy \\ &= \int_{-\infty}^{x} \phi'(y)\,dy = \phi|_{-\infty}^{x} = \phi(x) \end{aligned}$$

because of the initial condition for Q and the assumption that $\phi(-\infty) = 0$. This is the initial condition. We conclude that our solution formula is given as above, where

$$S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \text{ for } t > 0$$

That is,

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t}\phi(y)\,dy$$

$S(t,x)$ is known as the source function, Green's function, fundamental solution Gaussian, or propagator of the diffusion equation, or simply the diffusion kernel. It gives the solution of the diffusion equation with any initial datum ϕ. The formula only gives the solution for $t > 0$. When $t = 0$, it makes no sense. $\square$

Now, we may apply this formula to the diffusion equation

$$v_t - v_{xx} = 0 \qquad (t,x) \in (0,\infty) \times \mathbb{R}$$

$$v(0,x) = 0$$

With $\phi(x) = 0$, the solution formula is as follows, as the integral evaluates to zero:

$$v(t,x) = 0$$

Thus, we have determined that the *only* solution $v(t,x) : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is $v(t,x) = 0$. With $v(t,x) = u_1(t,x) - u_2(t,x)$, the following is true:

$$u_1(t,x) = u_2(t,x)$$

Thus, the solution to the diffusion equation $u_t - u_{xx} = f(t,x)$ for $(t,x) \in (0,\infty) \times \mathbb{R}$ with the initial condition $u(0,x) = \phi(x)$ is unique. $\square$

8) Find a solution of the following diffusion problem in $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$

$$u_t - u_{xx} = 1 \qquad u(0,x) = 0$$

Consider the in-homogeneous diffusion equation on the whole line

$$u_t - u_{xx} = f(t,x) \qquad (t,x) \in (0,\infty) \times \mathbb{R}$$

$$u(0,x) = \phi(x)$$

with $f(t,x)$ and $\phi(x)$ arbitrarily given functions. This simply represents the general form of the diffusion problem. We will show that the solution is

$$u(t,x) = \int_{-\infty}^{\infty} S(t,x-y)\,\phi(y)\;dy + \int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)\,f(s,y)\;dy\,ds$$

Notice that there is the usual term involving the initial data ϕ and another term involving the source f. Both terms involve the source function S.

Let's begin by explaining where this solution comes from. Later, we will actually prove the validity of the formula. (If a strictly mathematical proof is satisfactory to you, this paragraph and the next two can be skipped.) Our explanation is an analogy. The simplest analogy is the ODE

$$\frac{du}{dt} + Au(t) = f(t) \qquad u(0) = \phi$$

where A is a constant. Using the integrating factor e^{tA}, the solution is

$$u(t) = e^{-tA}\phi + \int_0^t e^{(s-t)Af(s)\,ds}$$

A more elaborate analogy is the following. Let's suppose that ϕ is an n-vector, $u(t)$ is an n-vector function of time, and A is a fixed $n \times n$ matrix. Then, the original ODE is a coupled system of n linear ODEs. In case $f(t) = 0$, the solution is given as $u(t) = S(t)\,\phi$, where $S(t)$ is the matrix $S(t)\,e^{-tA}$. So in case $f(t) \neq 0$, an integrating factor is $S(-t)\,e^{tA}$. Now we multiply the ODE on the left by this integrating factor to get

$$\frac{d}{dt}\left[S(-t)\,u(t)\right] = S(-t)\,\frac{du}{dt} + S(-t)\,Au(t) = S(-t)\,f(t)$$

Integrating from 0 to t, we get

$$S(-t)\,u(t) - \phi = \int_0^t S(-s)\,f(s)\;ds$$

Multiplying this by $S(t)$, we end up with the solution formula

$$u(t) = S(t)\,\phi + \int_0^t S(t-s)\,f(s)\;ds$$

The first term in represents the solution of the homogeneous equation, the second the effect of the source $f(t)$. For a single equation, of course, this reduces.

Now let's return to the original diffusion problem. There is an analogy which we now explain. The solution of the diffusion problem will have two terms. The first one will be the solution of the homogeneous problem, already solved previously, namely

$$\int_{-\infty}^{\infty} S(t,x-y)\,\phi(y)\;dy = (\mathcal{S}(t)\,\phi)(x)$$

$S(t, x-y)$ is the source function given by the previous formula. Here we are using $\mathcal{S}(t)$ to denote the *source operator*, which transforms any function ϕ to the new function given by the integral in the previous expression. (Remember: Operators transform functions into functions.) We can now guess what the whole solution to the diffusion problem must be. In analogy to the formula above, we guess that the solution of the diffusion problem is

$$u(t) = \mathcal{S}(t)\,\phi + \int_0^t \mathcal{S}(t-s)\, f(s)\; ds$$

This formula is exactly the same as before

$$u(t, x) = \int_{-\infty}^{\infty} S(t, x-y)\,\phi(y)\; dy + \int_0^t \int_{-\infty}^{\infty} S(t-s, x-y)\, f(s, y)\; dy\, ds$$

The method we have just used to find the formula is the operator method.

Proof All we have to do is verify that the function $u(t, x)$ which is defined above in fact satisfies the PDE and IC. Since the solution to the diffusion problem is unique, we would then know that $u(t, x)$ is that unique solution. For simplicity, we may as well let $\phi = 0$, since we understand the ϕ term already.

We first verify the PDE. Differentiating the solution, assuming $\phi = 0$ and using the rule for differentiating integrals, we have

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} \int_0^t \int_{-\infty}^{\infty} S(t-s, x-y)\, f(s, y)\; dy\, ds \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{\partial S}{\partial t}(t-s, x-y)\, f(s, y)\; dy\, ds + \lim_{s \longrightarrow t} \int_{-\infty}^{\infty} S(t-s, x-y)\, f(s, y)\; dy\end{aligned}$$

taking special care due to the singularity of $S(t-s, x-y)$ at $t - s = 0$. Using the fact that $S(t-s, x-y)$ satisfies the diffusion equation, we get

$$\frac{\partial u}{\partial t} = \int_0^t \int_{-\infty}^{\infty} \frac{\partial^2 S}{\partial x^2}(t-s, x-y)\, f(s, y)\; dy\, ds + \lim_{\varepsilon \longrightarrow 0} \int_{-\infty}^{\infty} S(\varepsilon, x-y)\, f(t, y)\; dy$$

Pulling the spatial derivative outside the integral and using the initial condition satisfied by S, we get

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2}{\partial x^2} \int_0^t \int_{-\infty}^{\infty} S(t-s, x-y)\, f(s, y)\; dy\, ds + f(t, x) \\ &= \frac{\partial^2 u}{\partial x^2} + f(t, x)\end{aligned}$$

This identity is exactly the PDe. Second, we verify the initial condition. Letting $t \to 0$, the first term in the solution tends to $\phi(x)$ because of the initial condition of S. The second term is an integral from 0 to 0. Therefore,

$$\lim_{t \longrightarrow 0} u(t, x) = \phi(x) + \int_0^0 \cdots = \phi(x)$$

This proves that the solution exists and is unique. □

Remembering that $S(t, x)$ is the Gaussian distribution, the solution formula takes the explicit form

$$\begin{aligned}u(t, x) &= \int_0^t \int_{-\infty}^{\infty} S(t-s, x-y)\, f(s, y)\; dy\, ds \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-(x-y)^2/4(t-s)} f(s, y)\; dy\, ds\end{aligned}$$

in the case that $\phi = 0$.

As highlighted, the solution to the in-homogeneous diffusion equation $u_t - u_{xx} = f(t,x)$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ and $u(0,x) = \phi(x)$ is given as follows:

$$u(t,x) = \int_{-\infty}^{\infty} S(t, x-y)\, \phi(y)\; dy + \int_0^t \int_{-\infty}^{\infty} S(t-s, x-y)\, f(s,y)\; dy\, ds$$

In this case, $S(t,x)$ is the source function, given as the following Gaussian distribution: $S(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$. When $u(0,x) = \phi(x) = 0$, the solution is as follows:

$$u(t,x) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-(x-y)^2/4(t-s)} f(s,y)\; dy\, ds$$

For this particular problem, $f(t,x) = 1$, so the solution simplifies to the following:

$$u(t,x) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-(x-y)^2/4(t-s)}\; dy\, ds$$

Further evaluation of this expression would simply involve computing the integral.

9) Solve the following diffusion problem in $(t,x) \in \mathbb{R}^+ \times (0,1)$

$$u_t - u_{xx} = \sin(2\pi x) \qquad u(0,x) = 0, u(t,0) = u(t,1) = 0$$

Prior to attempting to solve the diffusion problem in $(t,x) \in \mathbb{R}^+ \times (0,1)$, observe the following:

$$v(x) = \sin(2\pi x)$$

$$v(0) = 0 \qquad v(1) = 0$$

Thus, perhaps let us consider a solution to $u_t - u_{xx} = \sin(2\pi x)$ of the following form:

$$u(t,x) = T(t)\sin(2\pi x)$$

That is, with the bounded domain for the diffusion problem, we use separation of variables, assuming the form of the solution. In this case, however, we go further to state that the value of $X(x)$ is equal to the forcing term $\sin(2\pi x)$.

If we use this as input in the partial differential equation, we have the following:

$$T'(t)\sin(2\pi x) + 4\pi^2 T(t)\sin(2\pi x) = \sin(2\pi x)$$

Considering the case where $\sin(2\pi x) \neq 0$, that is, $x \neq \frac{1}{2}k$ where $k \in \mathbb{N}$, this simplifies to the following ordinary differential equation:

$$T'(t) + 4\pi^2 T(t) = 1$$

To solution to this ordinary differential equation (either by separating or the integrating factors method) is:

$$T(t) = Ce^{-4\pi^2 t} + \frac{1}{4\pi^2}$$

From the initial condition $u(0,x) = 0$, we are able to solve for the constant C, as follows:

$$T(0) = 0 = Ce^{-4\pi^2(0)} + \frac{1}{4\pi^2}$$

Thus, $C = -\frac{1}{4\pi^2}$, so we have a fully determined solution for the diffusion problem in $(t,x) \in \mathbb{R}^+ \times (0,1)$, which is as follows:

$$u(t,x) = \left(-\frac{1}{4\pi^2}e^{-4\pi^2 t} + \frac{1}{4\pi^2}\right)\sin(2\pi x)$$

This solution satisfies the initial/boundary conditions $u(0,x) = 0$ and $u(t,0) = u(t,1) = 0$, along with the partial differential equation $u_t - u_x x = \sin(2\pi x)$ for $(t,x) \in \mathbb{R}^+ \times (0,1)$. $\square$

10) Let A be a symmetric positive definite matrix. Show that the eigenvalues of A are positive. Also show that A is invertible.

To demonstrate that a symmetric positive definite matrix A has positive eigenvalues and is invertible, we consider the following. Since A is symmetric, it is diagonalizable, so there exists an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. Let $\lambda_1, \dots, \lambda_n$ be the diagonal entries of D, which represent the eigenvalues of A.

Since A is positive definite, for any non-zero vector $\boldsymbol{x}$, we have

$$\boldsymbol{x}^T A \boldsymbol{x} > 0$$

Let $\boldsymbol{x}$ be an eigenvector of A with eigenvalue λ. Then,

$$A\boldsymbol{x} = \lambda \boldsymbol{x}$$

$$\boldsymbol{x}^T A \boldsymbol{x} = \boldsymbol{x}^T \lambda \boldsymbol{x}$$

Now, given that $\boldsymbol{x}^T A \boldsymbol{x} > 0$, we have the following:

$$\lambda \left(\boldsymbol{x}^T \boldsymbol{x}\right) > 0$$

The value $\boldsymbol{x}^T \boldsymbol{x} > 0$, as it is a scalar representing the 'norm' of the vector $\boldsymbol{x}$ which is non-zero. Thus, $\lambda > 0$, which holds for all eigenvalues of A. Therefore, the eigenvalues of A are positive. □

To show that A is invertible, let us consider the form $A = PDP^T$ with $\lambda_1, \dots, \lambda_n$ as the diagonal entries of D, which represent the eigenvalues of A. Since the eigenvalues of A are non-zero (so the diagonal entries in D are non-zero), the matrix D is invertible.

Let D^{-1} be the inverse of D. Then, $A^{-1} = \left(PDP^T\right)^{-1} = PD^{-1}P^T$ exists, and so A is invertible. □

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