MATH 053/126 - Partial Differential Equations

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Homework 2

Instructions/Notation

Please show all steps to get your answers. Specify the problems you discussed with other students (including names).

The starred problems are recommended but not required for undergraduate/non-math major graduate students and required for all math major graduate students.

Notation

- $\mathbb{R}$: The set of all real numbers.
- $\mathbb{R}^+$: The set all positive real numbers $\{x \in \mathbb{R} | x > 0\}$.
- $\frac{\partial u}{\partial t} = \partial_t u = u_t$
- $\frac{\partial u}{\partial x} = \partial_x u = u_x$

Questions

1) Let $(x, t) \in \mathbb{R}^2$ and $(\xi, \eta) \in \mathbb{R}^2$. The transformation from (x, t) to (ξ, η) is given by

$$\xi = x - t \qquad \eta = x + t$$

Show that

$$\partial x = \partial \xi + \partial \eta$$

$$\partial t = -\partial \xi + \partial \eta$$

$$\partial x + \partial t = 2\partial \eta$$

$$\partial x - \partial t = 2\partial \xi$$

To show the above, let us consider the different representations of the transformation, as follows:

$$x = \xi + t \qquad t = x - \xi$$

$$x = \eta - t \qquad t = \eta - x$$

Now, let us consider the following partial derivatives, which follow directly from the representations:

$$\frac{\partial x}{\partial \xi} = 1 \qquad \frac{\partial t}{\partial \xi} = -1$$

$$\frac{\partial x}{\partial \eta} = 1 \qquad \frac{\partial t}{\partial \eta} = 1$$

To show that $\partial x = \partial \xi + \partial \eta$, we aim to represent x strictly as a function of ξ and η, which we may do by adding $x = \xi + t$ and $x = \eta - t$, resulting in $2x = \xi + \eta$. Thus, we may use the following total derivative:

$$\partial x = \frac{\partial x}{\partial \xi}\partial \xi + \frac{\partial x}{\partial \eta}\partial \eta$$

which, according to the previously calculated partial derivatives, is equivalent to

$$\partial x = \partial \xi + \partial \eta$$

In a similar manner, to show that $\partial t = -\partial \xi + \partial \eta$, we aim to represent t strictly as a function of ξ and η, which we may do by adding $t = x - \xi$ and $t = \eta - x$, resulting in $2t = -\xi + \eta$. Thus, we may us the following total derivative:

$$\partial t = \frac{\partial t}{\partial \xi}\partial \xi + \frac{\partial t}{\partial \eta}\partial \eta$$

which, according to the previously calculated partial derivatives, is equivalent to

$$\partial t = -\partial \xi + \partial \eta$$

The key idea is to recognize that x and t may be represented as functions of ξ and η, and derive the forms from the respective partial derivatives.

To show that $\partial x + \partial t = 2\partial \eta$, we may simply add the expressions $\partial x = \partial \xi + \partial \eta$ and $\partial t = -\partial \xi + \partial \eta$. $\square$

Similarly, to show that $\partial x - \partial t = 2\partial \xi$, we may simply take the difference between the expressions $\partial x = \partial \xi + \partial \eta$ and $\partial t = -\partial \xi + \partial \eta$. $\square$

2) Use a similar transformation to solve the following transport equation. That is, derive the solution formula for the following problem

$$u_t + au_x = 0 \qquad x \in \mathbb{R} \qquad u(0,x) = \phi(x)$$

where a is constant and $\phi(x)$ is a given function. *Note that the point of this problem is not just the final formula.*

Using the following transformation:

$$\xi = x - at \qquad\qquad \eta = x + at$$

we are able to solve the transport equation, deriving a solution formula for the problem $u_t + au_x = 0, x \in \mathbb{R}, u(0,x) = \phi(x)$ where a is constant and $\phi(x)$ is a given function.

According to the transformation, $\partial x = \partial\xi + \partial\eta$ and $\partial t = -a\partial\xi + a\partial\eta$, which is analogous to the transformation performed previously.

Using this information, we may input the partial derivatives into the original partial differential equation

$$u_t + au_x = 0 \qquad x \in \mathbb{R} \qquad u(0,x) = \phi(x)$$

which results in

$$(-a\partial\xi + a\partial\eta) + a(\partial\xi + \partial\eta) = 0 \qquad x \in \mathbb{R} \qquad u(0,x) = \phi(x)$$

Cancelling and combining terms, this leads to

$$2a\partial\eta = 0$$

which implies that either $a = 0$ or $\partial\eta = 0$.

In the case where $a = 0$, the partial differential equation is $u_t = 0$, which is trivial, as it depends only on u_t. This indicates that the partial derivative of $u(t,x)$ with respect to t does not change (staying the constant value of 0), and so $u(t,x)$ is a solely a function of x. With the initial condition $u(0,x) = \phi(x)$, this implies that the solution to the partial differential equation is simply $u(t,x) = \phi(x)$.

However, in the case where $\partial\eta = 0$, there is slightly more work required to find the solution to the problem. The expression indicates that u is independent of η, that is, it does not depend on η. Given the transformation $\eta = x + at$ and $\xi = x - at$, this indicates that u depends only on $\xi = x - at$.

Now, let's say $u(t,x) = f(\xi) = f(x - at)$. By the initial condition $u(0,x) = f(x) = \phi(x)$. This gives us the appropriate form of the expression for $f(x)$ (which seems rather fortunate). So, we know that

$$u(t,x) = \phi(x - at)$$

3) Solve

$$u_{tt} - u_{xx} = 0 \qquad x \in \mathbb{R} \qquad u(0,x) = \cos(x) \qquad u_t(0,x) = \sin(x)$$

The general solution for the initial value problem is given as follows:

$$u(t,x) = \frac{1}{2}\left(\phi(x+t) + \phi(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\ ds$$

Thus, using the provided initial values of $u(0,x) = \phi(x) = \cos(x)$ and $u_t(0,x) = \psi(x) = \sin(x)$, we have

$$u(t,x) = \frac{1}{2}\left(\cos(x+t) + \cos(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \sin(s)\ ds$$

Evaluating this expression further (particularly the integral) produces

$$u(t,x) = \frac{1}{2}\left(\cos(x+t) + \cos(x-t)\right) + \frac{1}{2}\left[-\cos(s)\right]_{x-t}^{x+t}$$

which simplifies to

$$u(t,x) = \frac{1}{2}\cos(x-t) + \frac{1}{2}\cos(x-t)$$

Thus, the solution to the problem is

$$u(t,x) = \cos(x-t)$$

4) Show that the solution to the following problem is unique

$$-u_{xx} - u_{yy} = 1 \qquad (x,y) \in (0,1)^2 \qquad u(x,y) = \sin(2\pi(x+y)) \text{ for } x = 0,1 \text{ or } y = 0,1$$

To show that the solution to the given (2D Possion) problem is unique, we have the following framework.

$$-\Delta u = f \qquad (x,y) \in \Omega \subset \mathbb{R}^2 \qquad u(x,y) = g(x,y) \text{ on } \partial\Omega$$

Initially, we assume u_1 and u_2 are solutions satisfying

$$-\Delta u_1 = f \qquad -\Delta u_2 = f$$

Given that the Laplacian is a linear operator, this may be written as

$$-\Delta(u_1 - u_2) = 0$$

with the condition that

$$|u_1 - u_2|_{x \in \partial\Omega} = 0$$

Now, let $v = u_1 - u_@$, so

$$-\Delta v = 0 \qquad v(x,y) = 0 \text{ on } \partial\Omega$$

Now, we multiply by v, and compute the following:

$$\int_\Omega -(\Delta v)\, v\, dx = \int_\Omega 0$$

which may be written as

$$\int -(\nabla \cdot \nabla v)\, v\, dx$$

By the product rule, which states that

$$\nabla \cdot (u \nabla v) = \nabla v \cdot \nabla u + u \nabla v$$

we find that

$$\int_\Omega \nabla \cdot (v \nabla v) = \int_\Omega \nabla v \cdot \nabla v + \int_\Omega v \nabla v$$

By the divergence theorem, this is equivalent to

$$\int_{\partial\Omega} v \nabla v \cdot n\, dx$$

Thus, we have that

$$-\int_\Omega (\Delta v)\, v\, dx = -\int_{\partial\Omega} v \nabla v \cdot n\, ds + \int_\Omega \nabla v \cdot \nabla v\, dx = 0$$

On the boundary, the first term of the right-hand side goes to zero, as we consider $\frac{\partial v}{\partial n}$. Thus, the result of this process is

$$\int_\Omega |\nabla v|^2\, dx = 0$$

which demonstrates that $\nabla v = 0$. This is somewhat analogous to integration by parts.

Thus, we know that the gradient is zero in the domain, so v must be a constant, and according to the boundary value $v = 0$. Thus, we have that $u_1 - u_2 = 0$, and so $u_1 = u_2$, which proves the uniqueness of the solution to the given problem.

As indicated, this is the typical framework of proving the uniqueness of similar problems (involving the 2D Possion expression), though we go further to explain the reasoning as follows.

For the provided boundary conditions, it is relatively straightforward to demonstrate that

$$u(0,y) = u(1,y) = \sin(2\pi y)$$

$$u(x,0) = u(x,1) = \sin(2\pi x)$$

Now, as with the framework, suppose that we have two solutions u_1 and u_2 that satisfy the partial differential (wave) equation) on the domain, along with the boundary conditions. That is

$$-\Delta u_1 = 1 \qquad -\Delta u_2 = 1$$

for $(x, y) \in (0,1)$ with u_1 and u_2 satisfying the boundary conditions.

We know that $v = u_1 - u_2$ satisfies $-\Delta v = 0$ (by the linearity of the Laplacian operator) over the domain $(0,1)^2$. From this point, we may multiply by v as indicated, which results in

$$-v\Delta v = 0$$

Continuing to follow the steps of the problem, in taking the integral, applying the product rule, and applying the divergence theorem, we achieve the same result as we had previously.

$$-\int_{\Omega} (\Delta v)\, v\, dx = -\int_{\partial\Omega} v\nabla v \cdot n\, ds + \int_{\Omega} \nabla v \cdot \nabla v\, dx = 0$$

Essentially, the aim of this problem is to show that $-\int_{\partial\Omega} v\nabla v \cdot n\, ds$ is equal to zero on the boundary. By the given boundary conditions, as determined previously $u_1 = u_2 = \sin(2\pi(x+y))$ on $\partial\Omega$, so on $\partial\Omega$, $v = u_1 - u_2 = 0$. Hence, we have that $\nabla v = 0$ on $\partial\Omega$, and so the integrand $v\nabla v \cdot n = 0$ on $\partial\Omega$.

This achieves the result that the integral $-\int_{\partial\Omega} v\nabla v \cdot n\, ds = 0$ as required. To quickly summarize, since $v = 0$ on the boundary $\partial\Omega$, the integrand is zero and thus the integral itself is zero.

This allows us to determine (in line with the solution presented above), that $\boldsymbol{\nabla} v = 0$ on the domain, and so v must be a constant value k, which is equal to zero according to the boundary condition of $v = 0$. Thus, we have that $v = u_1 - u_2 = 0$, which implies $u_1 = u_2$. This demonstrates the uniqueness of the solution. $\square$

5) Show that the solution to the following problem is unique

$$-u_{xx} - u_{yy} = 1 \qquad (x, y) \in (0, 1)^2$$

$$u(x, y) = \sin(2\pi(x + y)) \text{ for } x = 0, 1 \text{ and } u_y(x, y) = \cos(2\pi(x + y)) \text{ for } y = 0, 1$$

To show that the solution to the given (2D Possion) problem is unique, we have the following framework.

$$-\Delta u = f \qquad (x, y) \in \Omega \subset \mathbb{R}^2 \qquad u(x, y) = g(x, y) \text{ on } \partial\Omega$$

Initially, we assume u_1 and u_2 are solutions satisfying

$$-\Delta u_1 = f \qquad -\Delta u_2 = f$$

Given that the Laplacian is a linear operator, this may be written as

$$-\Delta(u_1 - u_2) = 0$$

with the condition that

$$|u_1 - u_2|_{x \in \partial\Omega} = 0$$

Now, let $v = u_1 - u_@$, so

$$-\Delta v = 0 \qquad v(x, y) = 0 \text{ on } \partial\Omega$$

Now, we multiply by v, and compute the following:

$$\int_\Omega -(\Delta v) v \, dx = \int_\Omega 0$$

which may be written as

$$\int -(\nabla \cdot \nabla v) v \, dx$$

By the product rule, which states that

$$\nabla \cdot (u \nabla v) = \nabla v \cdot \nabla u + u \nabla v$$

we find that

$$\int_\Omega \nabla \cdot (v \nabla v) = \int_\Omega \nabla v \cdot \nabla v + \int_\Omega v \nabla v$$

By the divergence theorem, this is equivalent to

$$\int_{\partial\Omega} v \nabla v \cdot n \, dx$$

Thus, we have that

$$-\int_\Omega (\Delta v) v \, dx = -\int_{\partial\Omega} v \nabla v \cdot n \, ds + \int_\Omega \nabla v \cdot \nabla v \, dx = 0$$

On the boundary, the first term of the right-hand side goes to zero, as we consider $\frac{\partial v}{\partial n}$. Thus, the result of this process is

$$\int_\Omega |\nabla v|^2 \, dx = 0$$

which demonstrates that $\nabla v = 0$. This is somewhat analogous to integration by parts.

Thus, we know that the gradient is zero in the domain, so v must be a constant, and according to the boundary value $v = 0$. Thus, we have that $u_1 - u_2 = 0$, and so $u_1 = u_2$, which proves the uniqueness of the solution to the given problem.

As indicated, this is the typical framework of proving the uniqueness of similar problems (involving the 2D Possion expression), though we go further to explain the reasoning as follows.

Suppose that we have two solutions u_1 and u_2 such that $-\Delta u_1 = 1$ and $-\Delta u_2 = 1$ for $(x, y) \in (0,1)^2$, with the solutions u_1 and u_2 satisfying the boundary conditions. As highlighted we have that $v = u_1 - u_2$ satisfies $-\Delta v = 0$ over $(x, y) \in (0,1)^2$.

When we follow the method indicated above, we have

$$\int_\Omega |\nabla v|^2 \, dx = 0$$

which is equivalent to

$$\int_{\partial\Omega} v \, (\nabla v) \cdot n \, ds$$

In this particular case, our boundary is given by the boundary conditions, and as such $v_y = (\nabla u) \cdot y = (\nabla v) \cdot n$ on the boundary, and as such, either $v = 0$ (which is trivial) or $(\nabla v) \cdot n = v_y = 0$. Thus, from the above, we have

$$\int_\Omega |\nabla v|^2 \, dx = \int_{\partial\Omega} v \, (\nabla v) \cdot n \, ds = 0$$

which is exactly what we were hoping for. This is essentially how we determine the vanishing conditions on the boundary.

With this information, the rest of the proof follows exactly as above, as we know that $\int_\Omega |\nabla v|^2 \, dx = 0$, and so $\nabla v = 0$ in $(0,1)^2$. As the gradient is zero in the domain, v is simply a constant k, though we know $v = 0$ on the boundary of the domain. Thus, we know that $k = 0$, and so $v = u_1 - u_2 = 0$, so $u_1 = u_2$. This demonstrates the uniqueness of the solution. $\square$

6) Find the eigenvalues and eigenfunctions of the following problem

$$-u_{xx} - u_{yy} = \lambda u \qquad u(x,y) = 0 \text{ on } \partial(0,1)^2$$

where $(x,y) \in (0,1)^2$

This question asks us to find the eigenvalues and the eigenfunctions for Laplace's equation on a unit square, considering that $u(x,y) = 0$ on the boundary of such unit square ($\partial(0,1)^2$). Given this boundary condition, we may use the technique of separation of variables to propose the following:

$$u(x,y) = f(x)\,g(y)$$

When substituting this into the partial differential equation, we have the following:

$$-f''(x)\,g(y) - g''(y)\,f(x) = \lambda f(x)\,g(y)$$

This is equivalent to the following:

$$-\frac{f''(x)}{f(x)} - \frac{g''(y)}{g(x)} = \lambda$$

Now, let's consider a variable μ such that the following holds:

$$\frac{f''(x)}{f(x)} = -\mu^2$$

$$\frac{g''(x)}{g(x)} = -\lambda + \mu^2$$

This creates two separate second-order linear ordinary differential equations, with the solution forms as follows:

$$f(x) = \alpha_1 \sin(\mu x) + \beta_1 \cos(\mu x)$$
$$g(y) = \alpha_2 \sin\left(\sqrt{\lambda - \mu^2}y\right) + \beta_2 \cos\left(\sqrt{\lambda - \mu^2}y\right)$$

Thus, we may write the solution as follows:

$$u(x,y) = (\alpha_1 \sin(\mu x) + \beta_1 \cos(\mu x))\left(\alpha_2 \sin\left(\sqrt{\lambda - \mu^2}y\right) + \beta_2 \cos\left(\sqrt{\lambda - \mu^2}y\right)\right)$$

Applying the boundary conditions $u(x,y) = 0$ on $\partial(0,1)^2$, which is equivalent to $f(0) = f(1) = g(0) = g(1) = 0$, results in the following:

$$f(0) = \alpha_1 \sin(0) + \beta_1 \cos(0) = \beta_1 = 0$$

With this information, we have

$$f(1) = \alpha_1 \sin(\mu) = 0$$

so $\mu = n\pi$ for $n = 1, 2, \ldots, k$.

Similarly, we have

$$g(0) = \alpha_2 \sin(0) + \beta_2 \cos(0) = \beta_2 = 0$$

With this information, we have

$$g(1) = \alpha_2 \sin\left(\sqrt{\lambda - \mu^2}\right) = 0$$

so $\sqrt{\lambda - \mu^2} = m\pi$ for $m = 1, 2, \ldots, k$.

So, the eigenvalues λ are given as

$$\lambda = m^2\pi^2 + n^2\pi^2 \text{ for } n, m = 1, 2, \ldots, k$$

To determine the corresponding eigenfunctions, we use the eigenvalues, along with

$$u(x, y) = f(x) g(y)$$

with the functions $f(x)$ and $g(x)$ defined as follows

$$f(x) = \alpha_1 \sin(\mu x) + \beta_1 \cos(\mu x)$$
$$g(y) = \alpha_2 \sin\left(\sqrt{\lambda - \mu^2 y}\right) + \beta_2 \cos\left(\sqrt{\lambda - \mu^2 y}\right)$$

With this, we should consider the constraints imposed by the initial conditions, which simplifies the forms of $f(x)$ and $g(x)$ as follows:

$$f(x) = \alpha_1 \sin(\mu x)$$
$$g(y) = \alpha_2 \sin\left(\sqrt{\lambda - \mu^2 y}\right)$$

Further, we do not consider the coefficients α_1 and α_2 in front of the expressions, as the eigenfunction may be (trivially) adjusted to account for this. As such, the eigenfunctions of the expression are

$$u(x, y) = f(x) g(x) = \sin(n\pi x) \sin(m\pi y)$$

7) Use the energy conservation of the wave equation to prove that the only solution of

$$u_{tt} - u_{xx} = 0 \qquad u(0,x) = 0 \qquad u_t(0,x) = 0$$

is $u = 0$.

To use the energy conservation of the wave equation to prove that the only solution of $u_{tt} - u_{xx} = 0, u(0,x) = 0, u_t(0,x) = 0$ is $u = 0$, let us start by providing background with regard to the conservation of energy.

Let $u(t,x)$ be a solution to the wave equation $u_{tt} - u_{xx} = 0$ with the given initial conditions $u(0,x) = u_f(0,x) = 0$. We define the total energy as follows:

$$E = \frac{1}{2}\int_{-\infty}^{\infty} \left(u_t^2 + u_x^2\right) dx$$

From a physical standpoint, without an external force, the total energy is conserved/constant. Thus, given that the wave equation $u_{tt} - u_{xx} = 0$ is homogeneous, we are able to apply this method. We aim to show that $\frac{dE}{dt} = 0$. Thus, let us start by taking the derivative with respect to t:

$$\frac{d}{dt}E = \frac{d}{dt}\frac{1}{2}\int_{-\infty}^{\infty} \left(u_t^2 + u_x^2\right) dx$$

This is equivalent to the following (by chain rule):

$$\frac{dE}{dt} = \frac{1}{2}\int_{-\infty}^{\infty} \left(2u_t u_{tt} + 2u_x u_{xt}\right) dx$$

Now, to simplify this integral, we consider integration by parts on the second term of the integrand. Let's represent the following:

$$\frac{dE}{dt} = \frac{1}{2}\int_{-\infty}^{\infty} 2u_t u_{tt}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} 2u_x u_{xt}\, dx$$

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u_t u_{tt}\, dx + \int_{-\infty}^{\infty} u_x u_{xt}\, dx$$

The second term of the expression may be determined using integration by parts:

$$\int_{-\infty}^{\infty} u_x u_{xt}\, dx = \left[u_x u_t\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_{xx} u_t\, dx$$

Now, evaluating this, we consider the fact that $u \to 0$ as $x \to \pm\infty$. When this is not the case, $E = \infty$, which is not reasonable to consider within the scope of our interest. That is, we only care about the case where $u \to 0$, which simplifies the problem to the following:

$$\int_{-\infty}^{\infty} u_x u_{xt}\, dx = -\int_{-\infty}^{\infty} u_{xx} u_t\, dx$$

We may use this expression and input it into our expression for $\frac{dE}{dt}$.

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u_t u_{tt}\, dx - \int_{-\infty}^{\infty} u_{xx} u_t\, dx$$

By simplifying terms, we have the following:

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u_t u_{tt} - u_{xx} u_t\, dx$$

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u_t\left(u_{tt} - u_{xx}\right) dx$$

Considering that we already know $u_{tt} - u_{xx} = 0$, this indicates the following:

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u_t\left(u_{tt} - u_{xx}\right) dx = 0$$

To actually use the energy conservation of the wave equation to prove that the only solution of $u_{tt} - u_{xx} = 0, u(0,x) = 0, u_t(0,x) = 0$ is $u = 0$, let us start by stating that energy conservation gives us that $E(t) = E(0)$ for all time values t. When we consider $t = 0$,

$$\begin{aligned} E(0) &= \frac{1}{2}\int_{-\infty}^{\infty} \left((u_t(x,0))^2 + (u_x(x,0))^2 \right) \, dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \left((0)^2 + (0)^2 \right) \, dx = 0 \end{aligned}$$

Thus, for all t

$$\frac{1}{2}\int_{-\infty}^{\infty} \left((u_t(x,t))^2 + (u_x(x,t))^2 \right) \, dx = 0$$

which may be written more simply as

$$\frac{1}{2}\int_{-\infty}^{\infty} \left((u_t)^2 + (u_x)^2 \right) \, dx = 0$$

Therefore, we know that the integrand of this expression

$$(u_t)^2 + (u_x)^2 = 0$$

for all x and t, correspondingly. Thus, we know that

$$(u_t)^2 = 0 \qquad\qquad (u_x)^2 = 0$$

which implies that $u(t,x) = f(x)$ for some f and $u(t,x) = g(t)$ for some g. This further implies that $u(t,x)$ must be a constant k. Since we know that $u(0,x) = 0$, we know $k = 0$. Thus, this proves that the only solution to the partial differential equation with the corresponding boundary conditions is $u = 0$. $\square$

8) Use the result of the previous problem to show the uniqueness of the following problem

$$u_{tt} - u_{xx} = \sin(t + \cos(x)) \qquad u(0, x) = \sin(x) \qquad u_t(0, x) = \cos(x)$$

To show the uniqueness of $u_{tt} - u_{xx} = \sin(t + \cos(x))$ with the boundary values $u(0, x) = \sin(x)$ and $u_t(0, x) = \cos(x)$, we consider the general form of the problem, as follows:

$$u_{tt} - u_{xx} = f \qquad u(0, x) = \phi(x) \qquad u_t(0, x) = \psi(x)$$

Now, we assume that u_1 and u_2 are solutions that solve the problem, according to the partial differential equation and the boundary conditions. The following homogeneous wave equation may be written:

$$(u_1 - u_2)_{tt} - (u_1 - u_2)_{xx} = 0 \qquad u_1(0, x) = 0 - u_2(0, x) = 0 \qquad u_{1t}(0, x) - u_{2t}(0, x) = 0$$

Now, let v represent a solution to the homogeneous wave equation ($v = u_1 - u_2$). As demonstrated previously, we may use the conservation of energy to show that $v_{tt} - v_{xx} = 0$ with the boundary conditions $v(0, x) = 0$ and $v_t(0, x) = 0$ only has a solution of $v(t, x) = 0$.

Thus, since $v = u_1 - u_2 = 0$, we have that $u_1 = u_2$, and so we have demonstrated the uniqueness of the given problem. $\square$

9) Take the derivative of the following function $u(x)$ with respect to x

$$u(x) = \int_{x}^{x^2} e^{\sin(s)}\, ds$$

To take the derivative of the function $u(x)$ with respect to x, recall the following form of the fundamental theorem of calculus:

$$\frac{d}{dx}\int_{g(x)}^{f(x)} h(t)\, dt = h(f(x))\, f'(x) - h(g(x))\, g'(x)$$

Using this, $\frac{d}{dx}u(x)$ is equivalent to the following:

$$u_x(x) = 2xe^{\sin\left(x^2\right)} - e^{\sin(x)}$$

10) Find $\partial_x u$ where $u(t,x)$ is given by

$$u(t,x) = \int_{x-t}^{x+t} \sin(x+s)\ ds$$

To take the partial derivative of the function $u(t,x)$ with respect to x, we consider that the form $\partial_x u$ is equivalent (notation-wise).

In this case, due to the presence of x in the integrand (along with u being a function of t and x), the following form of the fundamental theorem of calculus works, though it is further complicated:

$$\frac{d}{dx}\int_{g(t,x)}^{f(t,x)} h(x,s)\ ds = h(f(x,s))\, f'(x,s) - h(g(x,s))\, g'(x,s)$$

This method only works because $h(x,s)$ may be represented as a function of $x+t$ when we take the integral with the given bounds. Using this, $\frac{\partial}{\partial x}u(t,x)$ is equivalent to the following:

$$u_x(t,x) = \sin(x+x+t)\, f'(x,s) - \sin(x+x-t)\, g'(x,s)$$

In this case, we consider $f'(x,s)$ and $g'(x,s)$ to indicate the derivative with respect to x, not s or t. Thus, the final solution is the following:

$$u_x(t,x) = 2\sin(2x+t) - 2\sin(2x-t)$$

Alternatively, the solution may be derived directly using integration and differentiation (with respect to x), in that particular order. The integral is not particularly difficult to calculate, unlike the previous problem. This process is relatively trivial, and thus is not covered in this section.

If the integrand was not a "simple" function in the sense that $\sin(x+s)\ ds$ is relatively easy to find an anti-derivative for, a method involving a change of the bounds of the integral would be necessary. This would allow us to take the derivative within the integral, thus further simplifying the problem.

Regardless, the output/results would be equivalent, which serves to indicate that any of these such methods is appropriate for this particular problem.

11) For smooth functions $u(x)$ and $v(x)$ in a simply connected smooth domain $D \in \mathbb{R}^d$, show that

$$\int_{\partial D} v \frac{\partial u}{\partial n}\, ds = \int_D \nabla v \cdot \nabla u\, dx + \int_D v \Delta u\, dx$$

The following provides the proof of the given identity, for smooth functions $u(x)$ and $v(x)$ in a simply connected smooth domain $D \in \mathbb{R}^d$. By the divergence theorem,

$$\int_D \nabla \cdot (v \nabla u)\, dx = \int_{\partial D} v \frac{\partial u}{\partial n}\, ds$$

Expanding the left-hand side (which represents the divergence), we have

$$\int_D \nabla v \cdot \nabla u + v \Delta u\, dx = \int_{\partial D} v \frac{\partial u}{\partial n}\, ds$$

Rearranging this expression gives the result, as follows:

$$\int_{\partial D} v \frac{\partial u}{\partial n}\, ds = \int_D \nabla v \cdot \nabla u\, dx + \int_D v \Delta u\, dx$$

This argument is equivalent to the following, which serves as a supplement. Let us start with $\int_{\partial D} v \frac{\partial u}{\partial n}\, ds$, knowing that $\frac{\partial u}{\partial n}\, ds = \nabla u \cdot ds$.

By the divergence theorem,

$$\int_{\partial D} v \nabla u \cdot ds = \int_{\partial D} v \nabla u \cdot ds = \int_D \nabla \cdot (v \nabla u)\, dx$$

Further, by the product rule, this is equivalent to

$$\int_D \nabla \cdot (v \nabla u)\, dx = \int_D \nabla v \cdot \nabla u\, dx + \int_D v \Delta u\, dx$$

This proves the result we aim to show. □

12) Also show that

$$\int_D (u\Delta v - v\Delta u)\ dx = \int_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right)\ ds$$

To demonstrate that $\int_D (u\Delta v - v\Delta u)\ dx = \int_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right)\ ds$, consider that we may represent the identity of the previous problem as follows:

$$\int_D v\Delta u\, dx = \int_{\partial D} v\frac{\partial u}{\partial n}\, ds - \int_D \nabla v \cdot \nabla u\, dx$$

By symmetry, we have that

$$\int_D u\Delta v\, dx = \int_{\partial D} u\frac{\partial v}{\partial n}\, ds - \int_D \nabla u \cdot \nabla v\, dx$$

Thus, to determine $\int_D (u\Delta v - v\Delta u)\ dx$, we may take the difference of these expressions, as follows:

$$\int_D (u\Delta v - v\Delta u)\ dx = \left(\int_{\partial D} u\frac{\partial v}{\partial n}\, ds - \int_D \nabla u \cdot \nabla v\, dx \right) - \left(\int_{\partial D} v\frac{\partial u}{\partial n}\, ds - \int_D \nabla v \cdot \nabla u\, dx \right)$$

This is equivalent to the following:

$$\int_D (u\Delta v - v\Delta u)\ dx = \int_{\partial D} u\frac{\partial v}{\partial n}\, ds - \int_{\partial D} v\frac{\partial u}{\partial n}\, ds$$

Thus, the following identity is true:

$$\int_D (u\Delta v - v\Delta u)\ dx = \int_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right)\ ds$$

This proves the result we aim to show. □

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