MATH 053/126 - Partial Differential Equations

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Midterm

Instructions/Notation

This is a take-home exam. There are 12 problems, and do all of them. Show all the steps to get your answers. If you have received help from other resources, please specify them.

Notation

- $\frac{\partial u}{\partial t} = \partial_t u = u_t$
- $\frac{\partial u}{\partial x} = \partial_x u = u_x$

Questions

For problems 1-2, we consider the following wave equation

$$u_{tt} - u_{xx} = f\left(t, x\right) \qquad \left(t, x\right) \in \left(0, \infty\right) \times \mathbb{R} \tag{1}$$

$$u\left(0, x\right) = \phi\left(x\right)$$

$$u_t\left(0, x\right) = \psi\left(x\right)$$

Question 1

(25 pts) Prove the uniqueness of (1).

Let $u_1\left(t, x\right)$ and $u_2\left(t, x\right)$ represent two solutions of the wave equation $u_{tt} - u_{xx} = f\left(t, x\right)$ with $\left(t, x\right) \in \left(0, \infty\right) \times \mathbb{R}$, along with the initial conditions $u\left(0, x\right) = \phi\left(x\right)$ and $u_t\left(0, x\right) = \psi\left(x\right)$.

Now let $v\left(t, x\right) = u_1\left(t, x\right) - u_2\left(t, x\right)$. The following is true:

$$v_{tt} - v_{xx} = 0 \qquad \left(t, x\right) \in \left(0, \infty\right) \times \mathbb{R}$$

$$v\left(0, x\right) = 0$$

$$v_t\left(0, x\right) = 0$$

This represents a homogeneous wave equation with the initial conditions $v\left(0, x\right) = 0$ and $v_t\left(0, x\right) = 0$. To solve for $v\left(t, x\right) : \left(0, \infty\right) \times \mathbb{R} \to \mathbb{R}$, we may use the energy conservation of the wave equation to prove that the only solution is $v = 0$.

To use the energy conservation of the wave equation to prove that the only solution of $v_{tt} - v_{xx} = 0, v\left(0, x\right) = 0, v_t\left(0, x\right) = 0$ is $v = 0$, let us start by providing background with regard to the conservation of energy.

Let $v(t,x)$ be a solution to the wave equation $v_{tt} - v_{xx} = 0$ with the given initial conditions $v(0,x) = v_t(0,x) = 0$. We define the total energy as follows:

$$E = \frac{1}{2}\int_{-\infty}^{\infty} \left(v_t^2 + v_x^2\right)\, dx$$

From a physical standpoint, without an external force, the total energy is conserved/constant. Thus, given that the wave equation $v_{tt} - v_{xx} = 0$ is homogeneous, we are able to apply this method. We aim to show that $\frac{dE}{dt} = 0$. Thus, let us start by taking the derivative with respect to t:

$$\frac{d}{dt}E = \frac{d}{dt}\frac{1}{2}\int_{-\infty}^{\infty} \left(v_t^2 + v_x^2\right)\, dx$$

This is equivalent to the following (by chain rule):

$$\frac{dE}{dt} = \frac{1}{2}\int_{-\infty}^{\infty} \left(2v_t v_{tt} + 2v_x v_{xt}\right)\, dx$$

Now, to simplify this integral, we consider integration by parts on the second term of the integrand. Let's represent the following:

$$\frac{dE}{dt} = \frac{1}{2}\int_{-\infty}^{\infty} 2v_t v_{tt}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} 2v_x v_{xt}\, dx$$

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} v_t v_{tt}\, dx + \int_{-\infty}^{\infty} v_x v_{xt}\, dx$$

The second term of the expression may be determined using integration by parts:

$$\int_{-\infty}^{\infty} v_x v_{xt}\, dx = \left[v_x v_t\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v_{xx} v_t\, dx$$

Now, evaluating this, we consider the fact that $v \to 0$ as $x \to \pm\infty$. When this is not the case, $E = \infty$, which is not reasonable to consider within the scope of our interest. That is, we only care about the case where $v \to 0$, which simplifies the problem to the following:

$$\int_{-\infty}^{\infty} v_x v_{xt}\, dx = -\int_{-\infty}^{\infty} v_{xx} v_t\, dx$$

We may use this expression and input it into our expression for $\frac{dE}{dt}$.

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} v_t v_{tt}\, dx - \int_{-\infty}^{\infty} v_{xx} v_t\, dx$$

By simplifying terms, we have the following:

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} v_t v_{tt} - v_{xx} v_t\, dx$$

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} v_t\left(v_{tt} - v_{xx}\right)\, dx$$

Considering that we already know $v_{tt} - v_{xx} = 0$, this indicates the following:

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} v_t\left(v_{tt} - v_{xx}\right)\, dx = 0$$

To actually use the energy conservation of the wave equation to prove that the only solution of $v_{tt} - v_{xx} = 0, v(0,x) = 0, v_t(0,x) = 0$ is $v = 0$, let us start by stating that energy conservation gives us that $E(t) = E(0)$ for all time values t. When we consider $t = 0$,

$$\begin{aligned} E(0) &= \frac{1}{2}\int_{-\infty}^{\infty} \left((v_t(0,x))^2 + (v_x(0,x))^2\right)\, dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \left((0)^2 + (0)^2\right)\, dx = 0 \end{aligned}$$

Thus, for all t

$$\frac{1}{2}\int_{-\infty}^{\infty}\left((v_t(x,t))^2+(v_x(x,t))^2\right)\,dx=0$$

which may be written more simply as

$$\frac{1}{2}\int_{-\infty}^{\infty}\left((v_t)^2+(v_x)^2\right)\,dx=0$$

Therefore, we know that the integrand of this expression

$$(v_t)^2+(v_x)^2=0$$

for all x and t, correspondingly. Thus, we know that

$$(v_t)^2=0 \qquad (v_x)^2=0$$

which implies that $v(t,x)=f(x)$ for some f and $v(t,x)=g(t)$ for some g. This further implies that $v(t,x)$ must be a constant k. Since we know that $v(0,x)=0$, we know $k=0$. Thus, this proves that the only solution to the partial differential equation with the corresponding boundary conditions is $v=0$. $\square$

Thus, we have determined that the *only* solution $v(t,x):(0,\infty)\times\mathbb{R}\to\mathbb{R}$ is $v(t,x)=0$. With $v(t,x)=u_1(t,x)-u_2(t,x)$, the following is true:

$$u_1(t,x)=u_2(t,x)$$

Thus, the solution to the wave equation $u_{tt}-u_{xx}=f(t,x)$ for $(t,x)\in(0,\infty)\times\mathbb{R}$ with the initial conditions $u(0,x)=\phi(x)$ and $u_t(0,x)=\psi(x)$ is unique. $\square$

Question 2

(25 pts) Prove the existence of (1).

Consider the following solution:

$$u(t,x) = \frac{1}{2}\left(\phi(x+t) + \phi(x-t) + \int_{x-t}^{x+t} \psi(s)\, ds + \iint_{\Delta} f(t,x)\right)$$

where Δ is the characteristic triangle. The double integral is equal to the iterated integral

$$\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\, dy\, ds$$

To demonstrate that this is the solution, we use a method involving Green's theorem. In this method we integrate f over the past history triangle Δ:

$$\iint_{\Delta} f\, dx\, dt = \iint_{\Delta} (u_{tt} - u_{xx})\, dx\, dt$$

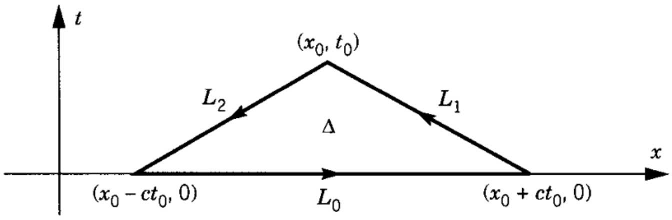
Green's theorem says that

$$\iint_{\Delta} (P_x - Q_t)\, dx\, dt = \int_{\partial\Delta} P\, dt + Q\, dx$$

for any functions P and Q, where the line integral on the boundary is taken counterclockwise. Thus,

$$\iint_{\Delta} f\, dx\, dt = \int_{L_0+L_1+L_2} (-u_x\, dt - u_t\, dx)$$

This is the sum of three line integrals over straight line segments, as shown by the figure.



We evaluate each piece separately. On L_0, $dt = 0$ and $u_t(x,0) = \psi(x)$, so

$$\int_{L_0} = -\int_{x_0-t_0}^{x_0+t_0} \psi(x)\, dx$$

On L_1, $x + t = x_0 + t_0$, so that $dx + dt = 0$, whence $-u_x\, dt - u_t\, dx = u_x\, dx + u_t\, dt = du$. This may be thought of in terms of the total derivative as well. Thus,

$$\int_{L_1} = \int_{L_1} du = u(t_0, x_0) - \phi(x_0 + t_0)$$

In the same way,

$$\int_{L_2} = -\int_{L_2} du = -\phi(x_0 - t_0) + u(t_0, x_0)$$

Adding these three results produces the following:

$$\iint_{\Delta} f\, dx\, dt = 2u(t_0, x_0) - [\phi(x_0+t_0) + \phi(x_0-t_0)] - \int_{x_0-t_0}^{x_0+t_0} \phi(x)\, dx$$

Thus,

$$u(t_0, x_0) = \frac{1}{2}\left(\iint_{\Delta} f\, dx\, dt + \phi(x_0+t_0) + \phi(x_0-t_0) + \int_{x_0-t_0}^{x_0+t_0} \psi(x)\, dx\right)$$

This proves the solution form that we previously determined. $\square$

For problems 3-4, we consider the following diffusion equation

$$u_t - u_{xx} = f(t,x) \qquad (t,x) \in (0,\infty) \times \mathbb{R} \tag{2}$$

$$u(0,x) = \phi(x)$$

Question 3

(25 pts) Prove the uniqueness of (2).

Let $u_1(t,x)$ and $u_2(t,x)$ represent two solutions of the diffusion equation $u_t - u_{xx} = f(t,x)$ with $(t,x) \in (0,\infty) \times \mathbb{R}$, along with the initial condition $u(0,x) = \phi(x)$.

Now let $v(t,x) = u_1(t,x) - u_2(t,x)$. The following is true:

$$v_t - v_{xx} = 0 \qquad (t,x) \in (0,\infty) \times \mathbb{R}$$

$$v(0,x) = 0$$

This represents a homogeneous diffusion equation with the initial condition $v(0,x) = 0$. To solve for $v(t,x) : (0,\infty) \times \mathbb{R} \to \mathbb{R}$, we may use the following method.

As with the wave equation, the problem on the infinite line has a certain "purity", which makes it easier to solve than the finite-interval problem. (The effects of boundaries will be discussed in the next several chapters.) Also as with the wave equation, we will end up with an explicit formula. But it will be derived by a method *very different* from the methods used before. (The characteristics for the diffusion equation are just the lines $t =$ constant and play no major role in the analysis.) Because the solution is not easy to derive, we first set the stage by making some general comments.

Our method is to solve it for a *particular* $\phi(x)$ and then build the general solution from this particular one. We'll use five basic *invariance properties* of the diffusion equation.

The *translate* $u(t, x-y)$ of any solution $u(t,x)$ is another solution for any fixed y.

Any *derivative* (u_x or u_t or u_{xx}, etc) of a solution is again a solution.

A *linear combination* of solutions is again a solution. (This is just linearity.)

An *integral* of solutions is again a solution. Thus, if $S(t,x)$ is a solution, then so is $S(t,x-y)$ and so is

$$v(t,x) = \int_{-\infty}^{\infty} S(t,x-y)\, g(y)\, dy$$

for any function $g(y)$, as long as this improper integral converges appropriately.

If $u(t,x)$ is a solution, so is the *dilated* function $u(at, \sqrt{a}x)$ for any $a > 0$. Prove this by the chain rule. Let $v(t,x) = u(at, \sqrt{a}x)$. Then $v_t = [\partial(at)/\partial t]\, u_t = a u_t$ and $v_x = [\partial(\sqrt{a}x)/\partial x]\, u_x = \sqrt{a} u_x$ and $v_{xx} = \sqrt{a} \cdot \sqrt{a} u_{xx} = a u_{xx}$.

Our goal is to find a particular solution and then construct all the other solutions using the integral property. The particular solution we will look for is the one, denoted $Q(t,x)$, which satisfies the *special initial condition*

$$Q(0,x) = 1 \text{ for } x > 0 \qquad Q(0,x) \text{ for } x < 0$$

The reason for this choice is that this initial condition does not change under dilation. We'll find Q in three steps.

Step 1 We'll look for $Q(t,x)$ of the special form

$$Q(t,x) = g(p) \qquad \text{where } p = \frac{x}{\sqrt{4t}}$$

and g is a function of only one variable (to be determined). (The $\sqrt{4}$ factor is included only to simplify a later formula.)

Why do we expect Q to have this special form? Because the dilation property says that the equation doesn't "see" the dilation $x \to \sqrt{a}x, t \to at$. Clearly, the expression for Q does not change at all under the dilation. So $Q(t,x)$, which is defined by the conditions, ought not to see the dilation either. How could that happen? In only one way. If Q depends on x and t solely through the combination $x/\sqrt{t}$. For the dilation takes $x/\sqrt{t}$ into $\sqrt{a}x/\sqrt{at} = x/\sqrt{t}$. Thus, let $p = x/\sqrt{4t}$ and look for Q which satisfies the conditions.

Step 2 Using this, we convert the diffusion equation into an ODe for g by use of the chain rule:

$$Q_t = \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4t}}g'(p)$$

$$Q_x = \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4t}}g'(p)$$

$$Q_{xx} = \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4t}g''(p)$$

$$0 = Q_t - Q_{xx} = \frac{1}{t}\left[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right]$$

Thus

$$g'' + 2pg' = 0$$

This ODE is easily solved using the integrating factor $\exp\int 2p\,dp = \exp\left(p^2\right)$. We get $g'(p) = c_1\exp\left(-p^2\right)$ and

$$Q(t,x) = g(p) = c_1\int e^{-p^2}\,dp + c_2$$

Step 3 We find a completely explicit formula for Q. We've just shown that

$$Q(t,x) = c_1\int_0^{x/\sqrt{4t}} e^{-p^2}\,dp + c_2$$

This formula is valid only for $t > 0$. Now, we use the expression for Q, expressed as a limit as follows:

$$\text{if } x > 0, 1 = \lim_{t\longrightarrow 0} Q = c_1\int_0^{+\infty} e^{-p^2}\,dp + c_2 = c_1\frac{\sqrt{\pi}}{2} + c_2$$

$$\text{if } x < 0, 0 = \lim_{t\longrightarrow 0} Q = c_1\int_0^{-\infty} e^{-p^2}\,dp + c_2 = -c_1\frac{\sqrt{\pi}}{2} + c_2$$

Here $\lim_{t\longrightarrow 0}$ means limit from the right. This determines the coefficients $c_1 = 1/\sqrt{\pi}$ and $c_2 = \frac{1}{2}$. Therefore, Q is the function

$$Q(t,x) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4t}} e^{-p^2}\,dp$$

for $t > 0$.

Step 4 Having found Q, we now define $S = \partial Q/\partial x$. (The explicit formula for S will be written below.) By the property for derivatives, S is also a solution to the diffusion equation. Given any function ϕ, we also define

$$u(t,x) = \int_{-\infty}^{\infty} S(t,x-y)\,\phi(y)\,dy \text{ for } t > 0$$

By the properties, u is another solution of the diffusion equation. We claim that u is the unique solution. To verify the validity of the initial condition, we write

$$\begin{aligned} u(t,x) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(t, x-y)\,\phi(y)\;dy \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left[Q(t,x-y)\right]\phi(y)\;dy \\ &= +\int_{-\infty}^{\infty} Q(t,x-y)\,\phi'(y)\;dy - Q(t,x-y)\,\phi(y)\,\Big|_{y=-\infty}^{y=\infty} \end{aligned}$$

upon integrating by parts. We assume these limits vanish. In particular, let's temporarily assume that $\phi(y)$ itself equals zero for $|y|$ large. Therefore,

$$\begin{aligned} u(0,x) &= \int_{-\infty}^{\infty} Q(0,x-y)\,\phi'(y)\;dy \\ &= \int_{-\infty}^{x} \phi'(y)\;dy = \phi|_{-\infty}^{x} = \phi(x) \end{aligned}$$

because of the initial condition for Q and the assumption that $\phi(-\infty) = 0$. This is the initial condition. We conclude that our solution formula is given as above, where

$$S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \text{ for } t > 0$$

That is,

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t}\phi(y)\;dy$$

$S(t,x)$ is known as the source function, Green's function, fundamental solution Gaussian, or propagator of the diffusion equation, or simply the diffusion kernel. It gives the solution of the diffusion equation with any initial datum ϕ. The formula only gives the solution for $t > 0$. When $t = 0$, it makes no sense. □

Now, we may apply this formula to the diffusion equation

$$v_t - v_{xx} = 0 \qquad (t,x) \in (0,\infty) \times \mathbb{R}$$

$$v(0,x) = 0$$

With $\phi(x) = 0$, the solution formula is as follows, as the integral evaluates to zero:

$$v(t,x) = 0$$

Thus, we have determined that the *only* solution $v(t,x) : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is $v(t,x) = 0$. With $v(t,x) = u_1(t,x) - u_2(t,x)$, the following is true:

$$u_1(t,x) = u_2(t,x)$$

Thus, the solution to the diffusion equation $u_t - u_{xx} = f(t,x)$ for $(t,x) \in (0,\infty) \times \mathbb{R}$ with the initial condition $u(0,x) = \phi(x)$ is unique. □

Question 4

(25 pts) Prove the existence of (2).

Consider the in-homogeneous diffusion equation on the whole line

$$u_t - u_{xx} = f(t, x) \qquad (t, x) \in (0, \infty) \times \mathbb{R}$$

$$u(0, x) = \phi(x)$$

with $f(t, x)$ and $\phi(x)$ arbitrarily given functions. We will show that the solution is

$$u(t, x) = \int_{-\infty}^{\infty} S(t, x - y)\, \phi(y)\, dy + \int_0^t \int_{-\infty}^{\infty} S(t - s, x - y)\, f(s, y)\, dy\, ds$$

Notice that there is the usual term involving the initial data ϕ and another term involving the source f. Both terms involve the source function S.

Let's begin by explaining where this solution comes from. Later, we will actually prove the validity of the formula. (If a strictly mathematical proof is satisfactory to you, this paragraph and the next two can be skipped.) Our explanation is an analogy. The simplest analogy is the ODE

$$\frac{du}{dt} + Au(t) = f(t) \qquad u(0) = \phi$$

where A is a constant. Using the integrating factor e^{tA}, the solution is

$$u(t) = e^{-tA}\phi + \int_0^t e^{(s-t)Af(s)\, ds}$$

A more elaborate analogy is the following. Let's suppose that ϕ is an n-vector, $u(t)$ is an n-vector function of time, and A is a fixed $n \times n$ matrix. Then, the original ODE is a coupled system of n linear ODEs. In case $f(t) = 0$, the solution is given as $u(t) = S(t)\,\phi$, where $S(t)$ is the matrix $S(t)\, e^{-tA}$. So in case $f(t) \neq 0$, an integrating factor is $S(-t)\, e^{tA}$. Now we multiply the ODE on the left by this integrating factor to get

$$\frac{d}{dt}\left[S(-t)\, u(t)\right] = S(-t)\frac{du}{dt} + S(-t)\, Au(t) = S(-t)\, f(t)$$

Integrating from 0 to t, we get

$$S(-t)\, u(t) - \phi = \int_0^t S(-s)\, f(s)\, ds$$

Multiplying this by $S(t)$, we end up with the solution formula

$$u(t) = S(t)\,\phi + \int_0^t S(t - s)\, f(s)\, ds$$

The first term in represents the solution of the homogeneous equation, the second the effect of the source $f(t)$. For a single equation, of course, this reduces.

Now let's return to the original diffusion problem. There is an analogy which we now explain. The solution of the diffusion problem will have two terms. The first one will be the solution of the homogeneous problem, already solved previously, namely

$$\int_{-\infty}^{\infty} S(t, x - y)\, \phi(y)\, dy = (\mathcal{S}(t)\,\phi)(x)$$

$S(t, x - y)$ is the source function given by the previous formula. Here we are using $\mathcal{S}(t)$ to denote the *source operator*, which transforms any function ϕ to the new function given by the integral in the previous expression. (Remember: Operators transform functions into functions.) We can now guess what the whole solution to the diffusion problem must be. In analogy to the formula above, we guess that the solution of the diffusion problem is

$$u(t) = \mathcal{S}(t)\,\phi + \int_0^t \mathcal{S}(t-s)\, f(s)\; ds$$

This formula is exactly the same as before

$$u(t,x) = \int_{-\infty}^{\infty} S(t,x-y)\,\phi(y)\; dy + \int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)\, f(s,y)\; dy\, ds$$

The method we have just used to find the formula is the operator method.

Proof All we have to do is verify that the function $u(t,x)$ which is defined above in fact satisfies the PDE and IC. Since the solution to the diffusion problem is unique, we would then know that $u(t,x)$ is that unique solution. For simplicity, we may as well let $\phi = 0$, since we understand the ϕ term already.

We first verify the PDE. Differentiating the solution, assuming $\phi = 0$ and using the rule for differentiating integrals, we have

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} \int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)\, f(s,y)\; dy\, ds \\
&= \int_0^t \int_{-\infty}^{\infty} \frac{\partial S}{\partial t}(t-s,x-y)\, f(s,y)\; dy\, ds + \lim_{s\longrightarrow t} \int_{-\infty}^{\infty} S(t-s,x-y)\, f(s,y)\; dy
\end{aligned}$$

taking special care due to the singularity of $S(t-s,x-y)$ at $t-s=0$. Using the fact that $S(t-s,x-y)$ satisfies the diffusion equation, we get

$$\frac{\partial u}{\partial t} = \int_0^t \int_{-\infty}^{\infty} \frac{\partial^2 S}{\partial x^2}(t-s,x-y)\, f(s,y)\; dy\, ds + \lim_{\varepsilon\longrightarrow 0} \int_{-\infty}^{\infty} S(\varepsilon,x-y)\, f(t,y)\; dy$$

Pulling the spatial derivative outside the integral and using the initial condition satisfied by S, we get

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2}{\partial x^2} \int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)\, f(s,y)\; dy\, ds + f(t,x) \\
&= \frac{\partial^2 u}{\partial x^2} + f(t,x)
\end{aligned}$$

This identity is exactly the PDe. Second, we verify the initial condition. Letting $t \to 0$, the first term in the solution tends to $\phi(x)$ because of the initial condition of S. The second term is an integral from 0 to 0. Therefore,

$$\lim_{t\longrightarrow 0} u(t,x) = \phi(x) + \int_0^0 \cdots = \phi(x)$$

This proves that the solution exists and is unique. □

Remembering that $S(t,x)$ is the Gaussian distribution, the solution formula takes the explicit form

$$\begin{aligned}
u(t,x) &= \int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)\, f(s,y)\; dy\, ds \\
&= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-(x-y)^2/4(t-s)} f(s,y)\; dy\, ds
\end{aligned}$$

in the case that $\phi = 0$.

For problems 5-6, we consider the following transport equation

$$u_t + au_x = f(t, x) \qquad (t, x) \in (0, \infty) \times \mathbb{R} \tag{3}$$

$$u(0, x) = \phi(x)$$

Question 5

(25 pts) Prove the uniqueness of (3).

Let $u_1(t, x)$ and $u_2(t, x)$ represent two solutions of the transport equation $u_t + au_x = f(t, x)$ with $(t, x) \in (0, \infty) \times \mathbb{R}$, along with the initial condition $u(0, x) = \phi(x)$.

Now let $v(t, x) = u_1(t, x) - u_2(t, x)$. The following is true:

$$v_t + av_x = 0 \qquad (t, x) \in (0, \infty) \times \mathbb{R}$$

$$v(0, x) = 0$$

This represents a homogeneous transport equation with an initial condition of $v(0, x) = 0$. To solve for $v(t, x) : (0, \infty) \times \mathbb{R} \to \mathbb{R}$, we may use the method of characteristics. According to the partial differential equation

$$v_t + av_x = 0$$

we have the following characteristic curves, which are represented by ordinary differential equations:

$$\dot{x} = a \qquad \dot{z} = 0$$

In this case, we are using the variable t to parameterize the curves, which is evident, considering that $\dot{t} = 1$. The initial conditions for the respective ordinary differential equations are as follows:

$$x(0) = x_0 \qquad z(0) = 0$$

Let us solve the expressions as follows:

$$x = x_0 + at$$

$$z = 0$$

Thus, we have determined that the *only* solution $v(t, x) : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is $v(t, x) = 0$. With $v(t, x) = u_1(t, x) - u_2(t, x)$, the following is true:

$$u_1(t, x) = u_2(t, x)$$

Thus, the solution to the transport equation $u_t + au_x = f(t, x)$ for $(t, x) \in (0, \infty) \times \mathbb{R}$ with the initial condition $u(0, x) = \phi(x)$ is unique. $\square$

Question 6

(25 pts) Prove the existence of (3).

This transport equation has a forcing term $f(t,x)$, along with a (potentially) non-homogeneous initial condition $u(0,x) = \phi(x)$. To solve for $u(t,x) : (0,\infty) \times \mathbb{R} \to \mathbb{R}$, we may use the method of characteristics. According to the partial differential equation

$$u_t + a u_x = f(t,x)$$

we have the following characteristic curves, which are represented by ordinary differential equations:

$$\dot{x} = a \qquad \dot{z} = f(t,x)$$

In this case, we are using the variable t to parameterize the curves, which is evident, considering that $\dot{t} = 1$. The initial conditions for the respective ordinary differential equations are as follows:

$$x(0) = x_0 \qquad z(0) = \phi(x_0)$$

Let us solve the expressions as follows:

$$x = x_0 + at$$

$$z = \phi(x_0) + \int_0^t f(s, x_0 + as)\, ds$$

Using the first equation, we may represent the second equation strictly in terms of x and t, as follows:

$$z = \phi(x - at) + \int_0^t f(s, x - at + as)\, ds$$

Thus, we have determined a solution $u(t,x) : (0,\infty) \times \mathbb{R} \to \mathbb{R}$, which is given by

$$u(t,x) = \phi(x - at) + \int_0^t f(s, x - at + as)\, ds$$

This integral exists and is finite for each $t \geq 0$ and $x \in \mathbb{R}$. Therefore, the solution $u(t,x)$ exists for all $(t,x) : (0,\infty) \times \mathbb{R}$. $\square$

Question 7

(25 pts) For $u(t, \boldsymbol{x}) : (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$, solve for u when u satisfies

$$u_t + u_{x_1} + 2u_{x_2} = \sin(x_1 + x_2)$$

where $u(0, \boldsymbol{x}) = \exp\left(-x_1^2 - x_2^2\right)$. Note that $\boldsymbol{x} = (x_1, x_2)$.

To solve for $u(t, x) : (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$, we may use the method of characteristics. According to the partial differential equation

$$u_t + u_{x_1} + 2u_{x_2} = \sin(x_1 + x_2)$$

we have the following characteristic curves, which are represented by ordinary differential equations:

$$\dot{x_1} = 1 \qquad \dot{x_2} = 2 \qquad \dot{z} = \sin(x_1 + x_2)$$

In this case, we are using the variable t to parameterize the curves, which is evident, considering that $\dot{t}(t) = 1$. The initial conditions for the respective ordinary differential equations are as follows:

$$x_1(0) = x_1^0 \qquad x_2(0) = x_2^0 \qquad z(0) = e^{-\left(x_1^0\right)^2 - \left(x_2^0\right)^2}$$

Let us solve the expressions as follows:

$$x_1 = t + x_1^0$$

$$x_2 = 2t + x_2^0$$

$$z = e^{-\left(x_1^0\right)^2 - \left(x_2^0\right)^2} + \int_0^t \sin\left(s + x_1^0 + 2s + x_2^0\right)\, ds$$

Now, we simply need to determine the values of x_1^0 and x_2^0 in terms of known quantities/variables, including x, y, and t. In this case, the process is relatively straightforward, as we simply use the first two equations to write

$$x_1^0 = x_1 - t \text{ and } x_2^0 = x_2 - 2t$$

Thus, we have determined the solution $u(t, \boldsymbol{x}) : (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$, which is given as follows:

$$u(t, \boldsymbol{x}) = e^{-(x_1 - t)^2 - (x_2 - 2t)^2} + \int_0^t \sin(x_1 + x_2 + 3(s - t))\, ds$$

This satisfies the partial differential equation $u_t + u_{x_1} + 2u_{x_2} = \sin(x_1 + x_2)$ with the initial condition $u(0, \boldsymbol{x}) = \exp\left(-x_1^2 - x_2^2\right)$ where $\boldsymbol{x} = (x_1, x_2)$. $\square$

Question 8

(25 pts) Let u be the solution of a PDE

$$u_t - 2u_x = 0 \qquad x \in (0,1)$$

$$u(0,x) = x$$

$$u(t,0) = h(t) \qquad u(t,1) = \cos(t)$$

Find $h(t)$ so that the solution u is continuous everywhere (including the boundary).

The problem represents a transport equation, with boundary conditions at $x = 0$ and $x = 1$, along with an initial condition for $t = 0$.

In this case, the quantity $u_t - 2u_x$ is the directional derivative of u in the direction of the vector $\boldsymbol{V} = (1, -2)$. It must always be zero. This means that $u(t,x)$ must be constant in the direction of $\boldsymbol{V}$. The vector $(2,1)$ is orthogonal to $\boldsymbol{V}$. The lines parallel to $\boldsymbol{V}$ have the equations $2t + x = k$, where k is a constant.

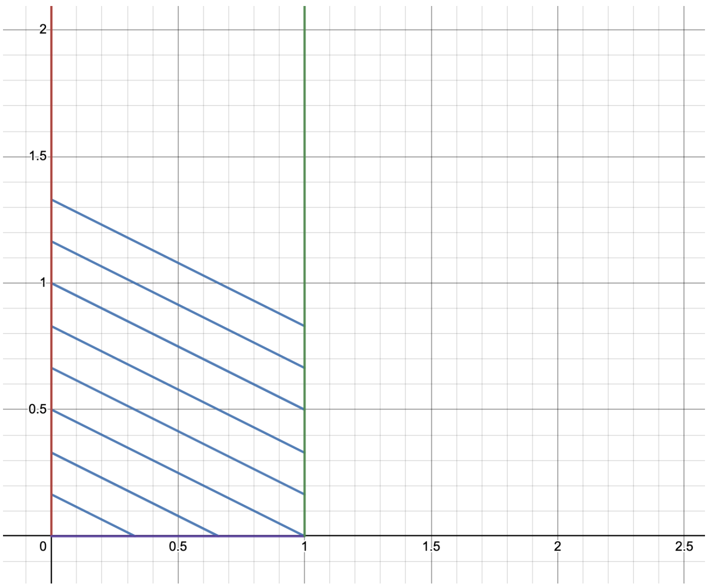
Using the characteristic lines, we may express the solution as constant on each line. Thus, $u(t,x)$ depends on $2t + x$ only, so the solution is

$$u(t,x) = f(2t + x)$$

where f is any function of one variable. Now, let us use the initial condition to find that $u(0,x) = f(x) = x$. This allows us to represent the solution not as an unknown function f, but explicitly, as follows:

$$u(t,x) = 2t + x$$

This satisfies the partial differential equation $u_t - 2u_x = 0$ for $x \in (0,1)$ and the initial condition $u(0,x) = x$. Now, we must consider the boundary conditions for $x = 0$ and $x = 1$. The best way to do so is to visualize the problem, as follows:



The depiction above shows the characteristic lines, on which the value of $u(t,x)$ is consistent. The y-axis is representative of t, while the x-axis is representative of x.

In this case, we are given that $u(t,1) = \cos(t)$, which is the boundary condition for the green line (at $x = 1$). Further, we are given that $u(0,x) = x$, which is the initial condition for the purple line (at $t = 0$). Using this information, we are asked to find the boundary condition $u(t, 0) = h(t)$ such that the solution u is continuous everywhere (including the boundary).

That is, we need to determine the appropriate function for the red line in terms of t that gives the value of $u(t, x)$ which follows from the other boundary/initial conditions. From inspection, we see that there is a disconnect at $t = \frac{1}{2}$, where the characteristic lines switch over from being dependent on the initial condition ($t = 0$) to being dependent on the boundary condition ($x = 1$). Thus, it is natural to assume that $h(x)$ will be piecewise.

For $0 \leq t \leq 0.5$, the boundary condition $h(t)$ is only dependent on the initial condition, so we may use this as we typically do. That is $u(t, 0) = 2t = h(t)$. This is determined using the expression $u(t, x) = 2t + x$ for the characteristic lines. (Essentially, we transform the variable x to the variable t to represent the value of $u(t, x)$ in terms of $h(t)$ for the red boundary condition $x = 0$.)

Now, for $t \geq 0.5$, the boundary condition is no longer dependent on the initial condition, as it is dependent on the other boundary condition. (This may be thought of as "transporting" the value from the green boundary to the red boundary along the characteristic lines.)

Given that the boundary condition for $x = 1$ is already given in terms of t, we simply must apply a shift factor to determine the boundary condition for $x = 0$ which makes $u(t, x)$ continuous. From $x = 1$ to $x = 0$, the value of t is shifted up by $\frac{1}{2}$, so we correct for this with the following: $h(t) = \cos\left(t - \frac{1}{2}\right)$

Now that we have determined the piecewise components of $h(t)$ so that the solution u is continuous everywhere (including the boundary), we may represent the following:

$$h(t) = \begin{cases} 2t & \text{if } 0 < t \leq 0.5 \\ \cos\left(t - \frac{1}{2}\right) & \text{if } t \geq 0.5 \end{cases}$$

In particular, the piecewise function is continuous at $t = 0.5$, as $h(0.5) = 1$, regardless of the component used.

Question 9

(25 pts) Solve $u_{tt} - u_{xx} = e^{ax}, u(0, x) = 0, u_t(0, x) = 0$ for $x \in \mathbb{R}$.

To solve the wave equation $u_{tt} - u_{xx} = f(t, x)$ on the whole line ($x \in \mathbb{R}$) with the initial conditions $u(0, x) = \phi(x)$ and $u_t(0, x) = \psi(x)$, we may use the following formula:

$$u(t, x) = \frac{1}{2} \left(\phi(x + t) + \phi(x - t) + \int_{x-t}^{x+t} \psi(s) \, ds + \iint_{\Delta} f(t, x) \right)$$

where Δ is the characteristic triangle. The double integral is equal to the iterated integral

$$\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y, s) \, dy \, ds$$

Thus, with the initial conditions $\phi(x) = \psi(x) = 0$, and the forcing term $f(t, x) = e^{ax}$, this simplifies to

$$u(t, x) = \frac{1}{2} \iint_{\Delta} f(t, x) = \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{ay} \, dy \, ds$$

This is computed as follows:

$$\begin{aligned}
u(t, x) &= \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{ay} \, dy \, ds \\
&= \frac{1}{2a} \int_0^t \left[e^{ay} \right]_{x-(t-s)}^{x+(t-s)} \, ds \\
&= \frac{1}{2a} \int_0^t \left(e^{a(x+(t-s))} - e^{a(x-(t-s))} \right) \, ds \\
&= \frac{1}{2a^2} \left[-e^{a(x+(t-s))} - e^{a(x-(t-s))} \right]_0^t \\
&= \frac{1}{2a^2} \left(-e^{ax} - e^{ax} + e^{a(x+t)} + e^{a(x-t)} \right)
\end{aligned}$$

This is equivalent to the following, which is the solution to $u_{tt} - u_{xx} = e^{ax}$ with initial conditions $u(0, x) = u_t(0, x) = 0$ for $x \in \mathbb{R}$:

$$u(t, x) = \frac{1}{2a^2} \left(e^{a(x+t)} + e^{a(x-t)} - 2e^{ax} \right)$$

Question 10

(25 pts) Solve for u

$$\begin{gathered} u_{tt} - u_{xx} = 0 \qquad (t,x) \in (0,\infty) \times (0,1) \\ u(0,x) = \sin\left(\frac{\pi x}{2}\right) \\ u_t(0,x) = 0 \\ u(t,0) = 0 \qquad u(t,1) = 1 \end{gathered} \tag{4}$$

To solve for $u(t,x)$, let $v(t,x) = u(t,x) - x$. Thus, we have the following

$$\begin{gathered} v_{tt} - v_{xx} = 0 \qquad (t,x) \in (0,\infty) \times (0,1) \\ v(0,x) = \sin\left(\frac{\pi x}{2}\right) - x \\ v_t(0,x) = 0 \\ v(t,0) = 0 \qquad v(t,1) = 0 \end{gathered}$$

This transformation allows us to consider the wave equation on a finite interval $(0,1)$ with the homogeneous Dirichlet boundary conditions at the endpoints $x = 0$ and $x = 1$. The partial differential equation is supposed to be satisfied in the interval $\{0 < x < 1, 0 < t < \infty\}$. If it exists, we know the solution $v(t,x)$ of this problem is unique.

The method consists of building up the general solution as a linear combination of special ones that are easy to find, as described by the textbook.

A *separated solution* is a solution of the wave equation of the form

$$v(t,x) = X(x)\,T(t)$$

(It is important to distinguish between the independent variable written as a lowercase letter and the function written as a capital letter.) Our first goal is to look for as many separated solutions as possible.

Plugging the form of the separated solution into the wave equation, we get

$$X(x)\,T''(t) = X''(x)\,T(t)$$

or, dividing by $X(x)\,T(t)$

$$-\frac{T''}{T} = -\frac{X''}{X} = \lambda$$

This defines a quantity λ, which must be a constant. (Proof: $\partial\lambda/\partial x = 0$ and $\partial\lambda/\partial t = 0$, so λ is a constant. Alternatively, we can argue that λ doesn't depend on x because of the first expression and doesn't depend on t because of the second expression, so that it doesn't depend on any variable.) We will show at the end of this section that $\lambda > 0$. (This is the reason for introducing the minus signs the way we did.)

So let $\lambda = \beta^2$, where $\beta > 0$. Then the equations are a pair of *separate* ordinary differential equations for $X(x)$ and $T(t)$:

$$X'' + \beta^2 X = 0 \qquad T'' + \beta^2 T = 0$$

These ODEs are easy to solve. The solutions have the form

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

$$T(t) = A\cos(\beta t) + B\sin(\beta t)$$

where A, B, C, D are constants.

The second step is to impose the boundary conditions on the separated solution. They simply require that $X(0) = 0 = X(1)$. Thus

$$0 = X(0) = C \qquad 0 = X(1) = D\sin(\beta)$$

Surely, we are not interested in the obvious solution $C = D = 0$. So we must have $\beta = n\pi$, a root of the sine function. That is,

$$\lambda_n = (n\pi)^2 \qquad X_n(x) = \sin(n\pi x) \qquad n = 1, 2, 3, \ldots$$

are distinct solutions. Each sine function may be multiplied by an arbitrary constant.

Therefore, there are an *infinite* number of separated solutions, one for each n. They are

$$v_n(t, x) = (A_n\cos(n\pi t) + B_n\sin(n\pi t))\sin(n\pi x)$$

$n = 1, 2, 3, \ldots$, where A_n and B_n are arbitrary constants. The sum of solutions is again a solution, so *any finite sum* (as follows) is also a solution of the wave equation.

$$v(t, x) = \sum_n (A_n\cos(n\pi t) + B_n\sin(n\pi t))\sin(n\pi x)$$

This formula solves the initial conditions, provided that

$$\phi(x) = \sum_n A_n\sin(n\pi x)$$

and

$$\psi(x) = \sum_n n\pi B_n\sin(n\pi x)$$

Thus, for any initial data of this form, the problem has a simple explicit solution. Practically any function $\phi(x)$ on the interval $(0, 1)$ can be expanded in an infinite series.

To determine the coefficients of the Fourier series, let us start with the Fourier sine series

$$\phi(x) = \sum_{n=1}^{\infty} A_n\sin(n\pi x)$$

in the interval $(0, 1)$. The first problem we tackle is to try to find the coefficients A_n if $\phi(x)$ is a given function. The key observation is that the sine functions have the wonderful property that

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\ dx = 0 \qquad \text{if } m \neq n$$

The far-reaching implications of this observation are astounding. Let's *fix* m, multiply the original expression by $\sin(m\pi x)$ and integrate the series term by term to get

$$\int_0^1 \phi(x)\sin(m\pi x)\ dx = \int_0^1 \sum_{n=1}^{\infty} A_n\sin(n\pi x)\sin(m\pi x)\ dx = \sum_{n=1}^{\infty} A_n \int_0^1 \sin(n\pi x)\sin(m\pi x)\ dx$$

All but one term in this sum vanishes, namely the one with $n = m$ (n just being a "dummy" index that takes on all integer values ≥ 1). Therefore, we are left with the single term

$$A_n \int_0^1 \sin^2(m\pi x)\ dx$$

which equals $\frac{1}{2}A_n$ by explicit integration. Therefore,

$$A_n = 2\int_0^1 \phi(x)\sin(n\pi x)\ dx$$

This is the famous *formula for the Fourier coefficients* in the series. That is, if $\phi(x)$ has the expansion

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

then the coefficients must be given by

$$A_n = 2\int_0^1 \phi(x)\sin(n\pi x)\ dx$$

Thus, we aim to solve the problem

$$\begin{gathered} v_{tt} - v_{xx} = 0 \qquad (t,x) \in (0,\infty)\times(0,1) \\ v(0,x) = \sin\left(\frac{\pi x}{2}\right) - x \\ v_t(0,x) = 0 \\ v(t,0) = 0 \qquad v(t,1) = 0 \end{gathered}$$

From the above, we know that $v(t,x)$ has an expansion

$$v(t,x) = \sum_n (A_n\cos(n\pi t) + B_n\sin(n\pi t))\sin(n\pi x)$$

Differentiating with respect to time yields

$$v_t(t,x) = \sum_{n=1}^{\infty} n\pi(-A_n\sin(n\pi t) + B_n\cos(n\pi t))\sin(n\pi x)$$

Setting $t = 0$, we have

$$0 = \sum_{n=1}^{\infty} n\pi B_n \sin(n\pi x)$$

so that all the $B_n = 0$.
Setting $t = 0$ in the expansion of $v(t,x)$, we have

$$\sin\left(\frac{\pi x}{2}\right) - x = \sum_n A_n \sin(n\pi x)$$

To determine the coefficients of A_n, we may use the expression

$$A_n = 2\int_0^1 \left(\sin\left(\frac{\pi x}{2}\right) - x\right)\sin(n\pi x)\ dx$$

Thus, the solution $v(t,x)$ is given as follows:

$$v(t,x) = \sum_n A_n\cos(n\pi t)\sin(n\pi x)$$

where

$$A_n = 2\int_0^1 \left(\sin\left(\frac{\pi x}{2}\right) - x\right)\sin(n\pi x)\ dx$$

Wait! Remember, that the solution we are looking for is not $v(t,x)$, it is $u(t,x) = v(t,x) + x$, as follows:

$$u(t,x) = x + \sum_n A_n\cos(n\pi t)\sin(n\pi x)$$

where

$$A_n = 2\int_0^1 \left(\sin\left(\frac{\pi x}{2}\right) - x\right)\sin(n\pi x)\ dx$$

This expression is difficult to compute analytically, thus, we will leave it in its current form.

Question 11

(25 pts) Solve for $u(t,x)$ when u satisfies

$$u_t - u_{xx} = 0 \qquad t > 0, x \in (0,\infty)$$

$$u(0,x) = e^{-x}$$

$$u(t,0) = 1$$

To solve for $u(t,x)$, let $v(t,x) = u(t,x) - 1$. Thus, we have the following

$$v_t - v_{xx} = 0 \qquad t > 0, x \in (0,\infty)$$

$$v(0,x) = e^{-x} - 1$$

$$v(t,0) = 0$$

This transformation allows us to consider diffusion on the half-line $(0,\infty)$ with the Dirichlet boundary condition at the single endpoint $x = 0$. The partial differential equation is supposed to be satisfied in the open region $\{0 < x < \infty, 0 < t < \infty\}$. If it exists, we know the solution $v(t,x)$ of this problem is unique (due to the discussion from the textbook).

The method uses the idea of an odd function. Any function $\psi(x)$ that satisfies $\psi(-x) = -\psi(x)$ is called an odd function. Its graph $y = \psi(x)$ is symmetric with respect to the origin. Automatically, by putting $x = 0$ in the definition, $\psi(0) = 0$.

Now the initial datum $\phi(x)$ of our problem is defined only for $x \geq 0$. Let ϕ_{odd} be the unique odd extension of ϕ to the whole line. That is,

$$\phi_{\text{odd}} = \begin{cases} \phi(x) & \text{for} x > 0 \\ -\phi(-x) & \text{for} x < 0 \\ 0 & \text{for} x = 0 \end{cases}$$

Let $w(t,x)$ be the solution of

$$w_t - w_{xx} = 0$$

$$w(0,x) = \phi_{\text{odd}}$$

for the *whole line* $-\infty < x < \infty, 0 < t < \infty$. According to the textbook, it is given by the formula

$$w(t,x) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi_{\text{odd}}(y)\,dy$$

Its "restriction"

$$v(t,x) = w(t,x) \qquad \text{for} x > 0$$

will be the unique solution of our new problem. There is no difference at all between v and w except that the negative values of x are not considered when discussing v.

Why is $v(t,x)$ the solution? Notice that $w(t,x)$ must also be an odd function of x. That is, $w(t,-x) = -w(t,x)$. Putting $x = 0$, it is clear that $w(t,0) = 0$. So the boundary condition $v(t,0) = 0$ is automatically satisfied. Furthermore, v solves the partial differential equation as well as the initial condition for $x > 0$, simply because it is equal to w for $x > 0$ and w satisfies the same partial differential equation for all x and the same initial condition for $x > 0$.

The explicit formula for $v(t,x)$ is easily deduced as follows:

$$w(t,x) = \int_0^\infty S(x-y,t)\,\phi(y)\;dy - \int_{-\infty}^0 S(x-y,t)\,\phi(-y)\;dy$$

Changing the variable $-y$ to $+y$ in the second integral, we get

$$w(t,x) = \int_0^\infty [S(x-y,t) - S(x+y,t)]\,\phi(y)\;dy$$

(Notice the change in the limits of integration.) Hence for $0 < x < \infty, 0 < t < \infty$, we have the solution to the partial differential equation for $v(t,x)$, given as

$$v(t,x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right] \phi(y)\;dy$$

This is the complete solution formula, which was attained using the method of odd extensions (reflection method), as the graph of $\phi_{\text{odd}}(x)$ is the reflection of the graph $\phi(x)$ across the origin.

With the initial condition/datum, we are able to write $v(t,x)$ as follows:

$$v(t,x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right] \left(e^{-y} - 1\right)\;dy$$

Now, consider the previous substitution for $v(t,x)$ in place of $u(t,x)$. The *final* solution $u(t,x)$ may be written as follows:

$$u(t,x) = 1 + \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right] \left(e^{-y} - 1\right)\;dy$$

Question 12

(25 pts) Check the Wikipedia page for "Fokker-Planck" equation. Write down the equation and summarize/discuss its applications (using at most ten sentences). Note that this equation has applications in many areas.

According to Wikipedia, the "Fokker-Plank" equation is a partial differential equation in statistical mechanics and information theory that "describes the time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces."

In one spatial dimension x, for an Itô process driven by the standard Wiener process W_t and described by the stochastic differential equation (SDE)

$$dX_t = \mu\left(X_t, t\right)\, dt + \sigma\left(X_t, t\right)\, dW_t$$

with drift $\mu\left(X_t, t\right)$ and diffusion coefficient $D\left(X_t, t\right) = \sigma^2\left(X_t, t\right)/2$, the Fokker-Planck equation for the probability density $p\left(x, t\right)$ of the random variable X_t is

$$\frac{\partial}{\partial t} p\left(x, t\right) = -\frac{\partial}{\partial x}\left[\mu\left(x, t\right) p\left(x, t\right)\right] + \frac{\partial^2}{\partial x^2}\left[D\left(x, t\right) p\left(x, t\right)\right]$$

In higher dimensions, if

$$d\boldsymbol{X}_t = \boldsymbol{\mu}\left(\boldsymbol{X}_t, t\right)\, dt + \boldsymbol{\sigma}\left(\boldsymbol{X}_t, t\right)\, d\boldsymbol{W}_t$$

where $\boldsymbol{X}_t$ and $\boldsymbol{\mu}\left(\boldsymbol{X}_t, t\right)$ are N-dimensional vectors, $\boldsymbol{\sigma}\left(\boldsymbol{X}_t, t\right)$ is an $N \times N$ matrix and $\boldsymbol{W}_t$ is an M-dimensional standard Weiner process, the probability density $p\left(\boldsymbol{x}, t\right)$ for $\boldsymbol{X}_t$ satisfies the Fokker-Planck equation

$$\frac{\partial p\left(\boldsymbol{x}, t\right)}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\mu_i\left(\boldsymbol{x}, t\right) p\left(\boldsymbol{x}, t\right)\right] + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j}\left[D_{ij}\left(\boldsymbol{x}, t\right) p\left(\boldsymbol{x}, t\right)\right]$$

with drift vector $\boldsymbol{\mu} = \left(\mu_1, \ldots, \mu_N\right)$ and diffusion tensor $\boldsymbol{D} = \frac{1}{2}\boldsymbol{\sigma}\boldsymbol{\sigma}^T$, i.e.

$$D_{ij}\left(\boldsymbol{x}, t\right) = \frac{1}{2}\sum_{k=1}^{M} \sigma_{ik}\left(\boldsymbol{x}, t\right) \sigma_{jk}\left(\boldsymbol{x}, t\right)$$

Stochastic Processes - The Fokker-Planck equation is used to model the evolution of the probability density function of a particle undergoing Brownian motion or other stochastic processes, predicting the probability distribution at later times based on the initial distribution.

Particle Transport Phenomena - The Fokker-Planck equation may be used in plasma physics (or astrophysics) to model the collective motion of particles, considering the influence of external forces and random collisions.

Economics/Finance - The Fokker-Planck equation may be used in quantitative finance to price options by modeling the stochastic evolution of underlying asset prices.

Genetics/Biology - The Fokker-Planck equation may be used to model to dynamics of population densities over time, to gain an understanding of competition and genetic drift.

Neuroscience - The Fokker-Planck equation may be used to model the behavior of neurons, particularly with regard to evolution in response to input.

In summary, the Fokker-Planck equation is a versatile partial differential equation used in information theory, physics, economics, genetics, etc, which allows for a description of the time evolution of probability distributions in stochastic systems.

Resources

- Partial Differential Equations (Walter A. Strauss)
- Partial Differential Equations (Rustum Choksi)
- Wave Equation By Separation Of Variables (UBC Mathematics)
- Wave Equations In Various Settings (M. Vajiac & J. Tolosa)
- Wave Equation Overview (Stanford)
- Method Of Characteristics (Stanford)