MATH 053/126 - Partial Differential Equations

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Final

Instructions/Notation

This is a take-home exam. There are 7 problems, do all of them. Show all the steps to get your answers. You are okay to use computers/internet/books. If you have received help from other resources, please specify them.

Notation

- $\frac{\partial u}{\partial t} = \partial_t u = u_t$
- $\frac{\partial u}{\partial x} = \partial_x u = u_x$

Question 1

Let $u(x)$ be the solution of the weak formulation

$$\int_D \nabla u(x) \cdot \nabla \phi(x)\ dx - \int_{\partial\Omega} h(x)\,\phi(x)\ dS = \int_D f(x)\,\phi(x)\ dx \qquad \phi(x) \in C^\infty(D)$$

where D is a bounded domain with a smooth boundary in $\mathbb{R}^d$.

Part A

By assuming that u is smooth, show that u is actually the solution of the Poisson equation. (Do not forget the boundary value!)

$$-\Delta u = f \text{ in } D \qquad \frac{\partial u}{\partial n} = h(x) \text{ on } \partial D$$

To show that u is the solution of the Poisson equation, we must demonstrate that u satisfies $-\Delta u = f$ in D and $\frac{\partial u}{\partial n} = h(x)$ on ∂D, where Δ is the Laplacian operator and $\frac{\partial}{\partial n}$ is the outward normal derivative. We are given the weak formulation

$$\int_D \nabla u \cdot \nabla \phi\, dx - \int_{\partial\Omega} h(x)\,\phi\, dS = \int_D f\phi\, dx \qquad \phi(x) \in C^\infty(D)$$

where D is a bounded domain with smooth boundary in $\mathbb{R}^d$. By assuming u is smooth, we may use integration by parts (divergence theorem) on the left-hand side of the weak formulation. For a smooth function u and test function ϕ, the integration by parts formula is

$$\int_D \nabla u \cdot \nabla \phi\, dx = -\int_D \Delta u\phi\, dx + \int_{\partial D} \frac{\partial u}{\partial n}\phi\, dS$$

Substituting this into the weak formulation, we have

$$-\int_D \Delta u\phi\, dx + \int_{\partial D} \frac{\partial u}{\partial n}\phi\, dS - \int_{\partial\Omega} h(x)\,\phi\, dS = \int_D f\phi\, dx \qquad \phi(x) \in C^\infty(D)$$

This may be expressed as follows:

$$-\int_D \Delta u\phi\, dx - \int_{\partial\Omega} h(x)\,\phi\, dS = \int_D f\phi\, dx - \int_{\partial D} \frac{\partial u}{\partial n}\phi\, dS \qquad \phi(x) \in C^\infty(D)$$

Now, according to the following lemma (shown in the appendix), we are able to determine the result. Let

$$\int_{\Omega} f(x)\,\phi(x)\,dx + \int_{\partial\Omega} g(x)\,\phi(x)\,dx = \int_{\Omega} h(x)\,\phi(x)\,dx + \int_{\partial\Omega} m(x)\,\phi(x)\,dx$$

for all $\phi(x) \in \mathbb{C}^{\infty}(\Omega)$. This indicates $f(x) = h(x)$ and $g(x) = m(x)$.

Thus, given that the equation

$$-\int_{D} \Delta u \phi\, dx - \int_{\partial\Omega} h(x)\,\phi\, dS = \int_{D} f\phi\, dx - \int_{\partial D} \frac{\partial u}{\partial n}\phi\, dS$$

holds for all test functions $\phi(x) \in C^{\infty}(D)$, we may use the lemma (based on the fundamental lemma of calculus of variations) to find that

$$-\Delta u = f \text{ in } D \qquad \frac{\partial u}{\partial n} = h(x) \text{ on } \partial D$$

Part B

Is the weak solution unique?

The uniqueness of the weak solution depends on the regularity of the solution space and the compatibility of the data f and h (the function spaces involved). Typically, weak solutions to the Poisson equation may not be unique unless further conditions are imposed.

To determine the uniqueness in this particular instance, we would need to consider the regularity of the domain D, the regularity of the solution u and the compatibility of the data f and h. In other words, if the domain D is sufficiently smooth, the solution space is appropriately chosen, and the data f and h satisfy certain compatibility conditions, then the uniqueness of the weak solution may be established.

In this case, the weak solution is unique, as the $u(x)$ as a solution of the weak formulation is smooth and D is a bounded domain with a smooth boundary in $\mathbb{R}^d$. This indicates that the solution u and domain D meet the regularity requirements ("smoothness"), assuming that f and h meet the required compatibility requirements.

Part C

What is n? Find n at $x = (x_1, 0, x_3)$ when $D = \{(x_1, x_2, x_3)\,|x_2 \le 0\}$.

In the context of boundary value problems and the weak formulation provided, n typically represents/denotes the outward unit normal vector to the boundary ∂D of the domain D. At a point (x_1, x_2, x_3) on the boundary, the outward unit normal vector n is a vector that points perpendicular to the boundary and has unit length.

For the given domain $D = \{(x_1, x_2, x_3)\,|x_2 \le 0\}$, the boundary is defined by the plane $x_2 = 0$. At a point on this boundary, such as $x = (x_1, 0, x_3)$, the outward unit normal vector n points in the positive y-direction, since the boundary is in the $x_2 \le 0$ half-space.

That is, the outward unit normal vector n may be found by normalizing the gradient of the function defining the boundary. The boundary in this case is given by $x_2 = 0$, so the gradient is $(0, 1, 0)$, thus the outward unit normal vector at the point $x = (x_1, 0, x_3)$ is

$$n = (0, 1, 0)$$

Question 2

Let $D = (0,1)^2$. For the solution of the following PDE

$$u_t - \Delta u = 0 \text{ in } D \qquad u(t,x) = h(x) \text{ on } \partial D \qquad u(0,x) = \phi(x)$$

show that the energy $\int_D u^2 \, dx$ decays in time.

To show that the energy decays in time, let us consider the energy functional associated with the given PDE. The energy functional is defined as

$$E(t) = \int_D u^2(t,x) \, dx$$

for $D = (0,1)^2$, where $u(t,x)$ is the solution to the following PDE

$$u_t - \Delta u = 0 \text{ in } D \qquad u(t,x) = h(x) \text{ on } \partial D \qquad u(0,x) = \phi(x)$$

Now, differentiate this with respect to time, to find

$$\frac{dE}{dt} = \frac{d}{dt}\int_D u^2 \, dx = 2\int_D uu_t \, dx$$

Since u satisfies the partial differential equation $u_t - \Delta u = 0$, we may use this to replace u_t in the expression, as follows:

$$\frac{dE}{dt} = 2\int_D u\Delta u \, dx$$

Now, we may apply integration by parts (divergence theorem) to the term $u\Delta u$, which allows us to determine the following:

$$\int_D u\Delta u \, dx = \int_{\partial D} u\nabla u \cdot n \, dS - \int_D \nabla u \cdot \nabla u \, dx$$

where n is the outward unit normal vector on the boundary ∂D and dS is the surface area element. Now, since u is prescribed on the boundary ∂D as $u(t,x) = h(x)$, the first integral on the right-hand side becomes zero (vanishes):

$$\int_{\partial D} u\nabla u \cdot n \, dS = 0$$

Therefore, we have

$$\frac{dE}{dt} = -2\int_D \nabla u \cdot \nabla u \, dx$$

Now, notice that the right-hand side is negative, which implies that $E(t)$ is decreasing. This shows that the energy $\int_D u^2 \, dx$ decays in time for the given PDE. □

Now, $h(x) = x_1$ (where $x = (x_1, x_2)$). Find the limit of the energy when $t \to \infty$.

Now, we aim to find the limit of the energy as t approaches infinity. Since the energy is decreasing (and bounded below), it must converge to a limit. Let E_∞ be the limit

$$\lim_{t\to\infty} E(t) = E_\infty$$

As t approaches infinity, the energy functional approaches a minimum value, and we have

$$\lim_{t\to\infty} \int_D u^2 \, dx = E_\infty$$

This shows that the energy converges to a limit E_∞. Now, let's compute the limit of the energy at $t \to \infty$, taking into account the specific form of $h(x) = x_1$. We have the energy functional

$$E(t) = \int_D (u(t,x))^2$$

where $u(t,x)$ is the solution to the given heat equation with initial condition $u(0,x) = \phi(x)$ and boundary condition $u(t,x) = h(x) = x_1$ on ∂D. As we showed before, the rate of change of the energy is given by

$$\frac{dE}{dt} = -2\int_D |\nabla u|^2 \, dx$$

Now, let us consider the limit as $t \to \infty$. As $t \to \infty$, the solution $u(t,x)$ approaches a steady-state solution. IN the steady state, the time derivative is zero ($u_t = 0$) and the Laplacian of u is also zero ($\Delta u = 0$). Therefore, in the steady-state, u is a harmonic function in D.

Now since u is harmonic in D and $u(t,x) = x_1$ on ∂D, the solution u is the harmonic extension of $h(x) = x_1$ into D. The harmonic extension of a function $h(x)$ in a domain D is given by the solution to the Laplace equation $\Delta u = 0$ in D with boundary condition $u(x) = h(x)$ on ∂D. This all serves to formalize what follows.

For $h(x) = x_1$, the harmonic extension $u(x)$ is given by $u(x) = x_1$. Now, let's compute the limit of the energy as $t \to \infty$.

$$\lim_{t\to\infty} E(t) = \lim_{t\to\infty} \int_D (u(t,x))^2 \, dx$$

Since $u(x) = x_1$ is the steady-state solution, we may write

$$\lim_{t\to\infty} E(t) = \int_D (x_1)^2 \, dx$$

Now, we integrate over D with respect to x, as follows:

$$\begin{aligned}\lim_{t\to\infty} E(t) &= \int_0^1 \int_0^1 x_1^2 \, dx_1 \, dx_2 \\ &= \int_0^1 \left[\frac{1}{3}x_1^3\right]_0^1 dx_2 \\ &= \int_0^1 \frac{1}{3} \, dx_2 \\ &= \frac{1}{3}\end{aligned}$$

Therefore, the limit of the energy as $t \to \infty$ is $\frac{1}{3}$. $\square$

Please see the appendix for a further discussion of the calculation of the limit of the energy.

Question 3

Find $s(t,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}$, when s satisfies

$$s_{tt} - s_{xx} = 0 \qquad s(0,x) = 0 \qquad s_t(0,x) = \delta(x)$$

The following provides the solution $s(t,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}$ when s satisfies

$$s_{tt} - s_{xx} = 0 \qquad s(0,x) = 0 \qquad s_t(0,x) = \delta(x)$$

by re-framing s as u, and following the method outlined in *Partial Differential Equations* by Rustum Choksi.

We consider the 1D wave equation $u_{tt} - u_{xx} = 0$ and its IVP

$$u_{tt} - u_{xx} = 0 \qquad -\infty < x < \infty, t > 0$$

$$u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x) \qquad -\infty < x < \infty$$

Following the approach outlined in the textbook for the diffusion equation, we will first place the (1D) delta function in the data. For the wave equation, there are two data functions. Let us start with the velocity and consider the following:

$$u_{tt} - u_{xx} = 0 \qquad -\infty < x < \infty, t > 0$$

$$u(x,0) = 0 \qquad -\infty < x < \infty \qquad u_t(x,0) = \delta_0$$

One can easily derive a potential solution candidate by informally treating the delta function as a true function and simply "placing" it into D'Alembert's formula. In doing so, one obtains

$$\frac{1}{2}\int_{x-t}^{x+t} \delta_0(y)\, dy$$

Intuitively, the above equals $\frac{1}{2}$ if the region of integration contains $y = 0$, and 0 otherwise. This suggests the solution, denoted by $\Phi(x,t)$ should be

$$\Phi(x,t) = \frac{1}{2}H(t - |x|) = \begin{cases} \frac{1}{2} & \text{if } |x| < t, t > 0 \\ 0 & \text{if } |x| \geq t, t > 0 \end{cases}$$

where H is the Heaviside function. Once can derive the same formula by taking the Fourier transform (in the sense of tempered distributions) with respect to x, though that method is not demonstrated here.

But how should we interpret this candidate solution? Unlike the fundamental solution for the diffusion equation, $\Phi(x,t)$ is not a smooth function. One can show that the function $\Phi(x,t)$ is a solution to the wave equation in the sense of distributions, and its time derivative converges to δ_0 in the sense of distributions.

Please see the appendix for a discussion of $\Phi(x,t)$ as the fundamental solution for the 1D wave equation.

Question 4

Using the solution formula $u(t) = \int_0^t e^{\alpha(s-t)} f(s)\, ds$ of an ODE $u' + \alpha u = f$ with $u(0) = 0$, **guess** the fundamental solution of the differential operator $Lu = u' + \alpha u$. By taking derivatives, show that your answer satisfies $u' + \alpha u = \delta(t)$.

According to the solution formula

$$u(t) = \int_0^t e^{\alpha(s-t)} f(s)\, ds$$

of an ODE $u' + \alpha u = f$ with $u(0) = 0$, the fundamental solution of the differential operator $Lu = u' + \alpha u$ is

$$\tilde{u}(t) = e^{-\alpha t} H(t)$$

This is evident considering that the solution formula may be re-written as follows:

$$u(t) = \int_0^t e^{-\alpha(t-s)} f(s)\, ds$$

which displays the convolution of $e^{-\alpha t} H(t)$ and the function f. In particular, the Heaviside function is included to ensure that the solution formula has the appropriate bounds of 0 and t, as convolution typically takes place over the whole real line $(-\infty, \infty)$. This does not change the underlying structure of the fundamental solution, however.

To verify that the fundamental solution of the differential operator $Lu = u' + \alpha u$ is $\tilde{u}(t) = e^{-\alpha t} H(t)$, we may take the the derivative of $\tilde{u}$ and use this in the differential operator. The derivative of $\tilde{u} = e^{-\alpha t} H(t)$ with respect to t is

$$\tilde{u'} = \frac{d}{dt}\left(e^{-\alpha t} H(t)\right)$$

Using the product rule and considering the derivative of the Heaviside step function,

$$\tilde{u'} = -\alpha e^{-\alpha t} H(t) + e^{-\alpha t} \delta(t)$$

This is essentially equivalent to the following, up to a scaling value on the delta function, which is not relevant for the purposes of demonstrating that the fundamental solution satisfies $u' + \alpha u = \delta u$:

$$\tilde{u'} = -\alpha e^{-at} H(t) + \delta(t)$$

Now, let us find $\alpha \tilde{u}$:

$$\alpha \tilde{u} = \alpha \cdot e^{-\alpha t} H(t)$$

Thus, we may input this into the differential operator, to find

$$L\tilde{u} = \tilde{u'} + \alpha \tilde{u} = -\alpha e^{-\alpha t} H(t) + \delta(t) + \alpha e^{-\alpha t} H(t)$$

By combining like terms, this is equivalent to

$$L\tilde{u} = \tilde{u'} + \alpha \tilde{u} = \delta(t)$$

where $\delta(t)$ is the Dirac delta function. Recall that the delta function takes on the value of 0 for all values $t \neq 0$, while the integral of $\int \delta = 1$.

Thus, by taking the derivative and inputting it into the differential operator, we have demonstrated that the guessed fundamental solution $\tilde{u} = e^{-\alpha t} H(t)$ does indeed satisfy the differential operator $L\tilde{u} = \tilde{u'} + \alpha \tilde{u} = \delta(t)$, which verifies that it is the correct fundamental solution. $\square$

The following paper (Section 1.2, Example 3) covers the process of "discovering" the fundamental solution, though uses a different form for the final solution formula, specifically the other convolution.

Please see the appendix for a discussion of the convolution of the fundamental solution for the general case.

Question 5

In $\mathbb{R}^2$, show that $\Delta \ln |x| = 0$ when $|x| \neq 0$.

We have previously demonstrated that in $\mathbb{R}^2$ (2D), $\Delta \ln |x| = C\delta(x)$, with the constant $C = 2\pi$.

Let us define the function ϕ as follows. For $|x| \neq 0$, let

$$\phi(x) = -\frac{1}{2\pi} \ln |x|$$

Let $\alpha(n)$ be the volume of the unit ball in $\mathbb{R}^n$. (In this case, we consider the unit circle in $\mathbb{R}^2$, though the proof is generalized for any dimension, though the value of C changes.)

We see that ϕ satisfies Laplace's equation on $\mathbb{R}^2 - \{0\}$. As indicated by the following claim, ϕ satisfies $-\Delta_x \phi = \delta_0$. For this reason, we call ϕ the *fundamental solution* of Laplace's equation.

Claim: For ϕ defined as above, ϕ satisfies

$$-\Delta_x \phi = \delta_0$$

in the sense of distributions. That is, for all $g \in D$,

$$-\int_{\mathbb{R}^2} \phi(x)\, \Delta_x g(x)\ dx = g(0)$$

This claim implies that $\Delta \ln |x| = C\delta(x)$ in $\mathbb{R}^2$ (2D), where $C = 2\pi$. The rigorous proof of the claim, which is rather extensive, is provided in the appendix.

In this case, we do not particularly care about the value of C, as we aim to show that $\Delta \ln |x| = 0$ when $|x| \neq 0$. This is equivalent, in practice, to showing that $\Delta \ln |x| = C\delta(x)$, as the Dirac delta function takes on the value of zero for all points x, except that at $x = 0$. In other words, the C is just a scaling value of the delta function, but we are not considering the case where $x = 0$, according to the problem specifications.

At this point, the solution is essentially complete, as we have determined that in $\mathbb{R}^2$, $\Delta \ln |x| = 0$ when $|x| \neq 0$, which is equivalent to the Dirac delta function. $\square$

To further emphasize the result and provide additional clarity, let's delve into the implications of the claim and how it establishes the desired property of the Laplacian of the logarithm.

The claim essentially asserts that the function $\phi(x) = -\frac{1}{2\pi} \ln |x|$ acts as a distributional solution to the equation $-\Delta_x \phi = \delta_0$ in $\mathbb{R}^2 - \{0\}$. This means that when ϕ is integrated against a test function g with compact support in $\mathbb{R}^2$, the resulting expression behaves as if ϕ were the Dirac delta function centered at the origin.

The importance of this claim lies in its connection to the behavior of the Laplacian of the logarithm function. By showing that ϕ satisfies the distributional equation $-\Delta_x \phi = \delta_0$, we are establishing a link between the logarithm and the Dirac delta function.

In essence, we are asserting that the Laplacian of $\ln |x|$ behaves like a scaled Dirac delta function centered at the origin, where the scaling factor is $C = 2\pi$. This means that $\Delta \ln |x| = 0$ when $|x| \neq 0$, according to the definition of the Dirac delta function.

The fact that $\Delta \ln |x| = C\delta(x)$ in $\mathbb{R}^2$ implies that the Laplacian of the logarithm has a singular behavior at the origin. However, as mentioned earlier, our main interest lies in the case where $|x| \neq 0$. In this regime, $\Delta \ln |x| = 0$, reaffirming that the Laplacian of the logarithm is indeed a well-behaved function away from the origin in $\mathbb{R}^2$.

In summary, the claim establishes a fundamental connection between the Laplacian of the logarithm and the Dirac delta function, providing a rigorous foundation for understanding the behavior of $\ln |x|$ in $\mathbb{R}^2$. The detailed proof of the claim, available in the appendix, ensures the validity of the distributional equality and, by extension, the vanishing Laplacian away from the origin.

Question 6

Solve for $u(t,x)$ when u satisfies

$$u_t - u_{xx} = 0 \qquad t > 0, x \in (0,\infty)$$

$$u(0,x) = e^{-x}$$

$$u(t,0) = 1 \text{ and } u(t,x) \to 0 \text{ as } x \to \infty$$

To solve for $u(t,x)$, let $v(t,x) = u(t,x) - 1$. Thus, we have the following

$$v_t - v_{xx} = 0 \qquad t > 0, x \in (0,\infty)$$

$$v(0,x) = e^{-x} - 1$$

$$v(t,0) = 0$$

In this case, we will worry about the boundary condition $u(t,x) \to 0$ as $x \to \infty$ at the end. This transformation allows us to consider diffusion on the half-line $(0,\infty)$ with the Dirichlet boundary condition at the single endpoint $x = 0$. The partial differential equation is supposed to be satisfied in the open region $\{0 < x < \infty, 0 < t < \infty\}$. If it exists, we know the solution $v(t,x)$ of this problem is unique (due to the discussion from the textbook).

The method uses the idea of an odd function. Any function $\psi(x)$ that satisfies $\psi(-x) = -\psi(x)$ is called an odd function. Its graph $y = \psi(x)$ is symmetric with respect to the origin. Automatically, by putting $x = 0$ in the definition, $\psi(0) = 0$.

Now the initial datum $\phi(x)$ of our problem is defined only for $x \geq 0$. Let ϕ_{odd} be the unique odd extension of ϕ to the whole line. That is,

$$\phi_{\text{odd}} = \begin{cases} \phi(x) & \text{for } x > 0 \\ -\phi(-x) & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Let $w(t,x)$ be the solution of

$$w_t - w_{xx} = 0$$

$$w(0,x) = \phi_{\text{odd}}$$

for the *whole line* $-\infty < x < \infty, 0 < t < \infty$. According to the textbook, it is given by the formula

$$w(t,x) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi_{\text{odd}}(y)\,dy$$

Its "restriction"

$$v(t,x) = w(t,x) \qquad \text{for} x > 0$$

will be the unique solution of our new problem. There is no difference at all between v and w except that the negative values of x are not considered when discussing v.

Why is $v(t,x)$ the solution? Notice that $w(t,x)$ must also be an odd function of x. That is, $w(t,-x) = -w(t,x)$. Putting $x = 0$, it is clear that $w(t,0) = 0$. So the boundary condition $v(t,0) = 0$ is automatically satisfied. Furthermore, v solves the partial differential equation as well as the initial condition for $x > 0$, simply because it is equal to w for $x > 0$ and w satisfies the same partial differential equation for all x and the same initial condition for $x > 0$.

The explicit formula for $v(t,x)$ is easily deduced as follows:

$$w(t,x) = \int_0^\infty S(x-y,t)\,\phi(y)\;dy - \int_{-\infty}^0 S(x-y,t)\,\phi(-y)\;dy$$

Changing the variable $-y$ to $+y$ in the second integral, we get

$$w(t,x) = \int_0^\infty [S(x-y,t) - S(x+y,t)]\,\phi(y)\;dy$$

(Notice the change in the limits of integration.) Hence for $0 < x < \infty, 0 < t < \infty$, we have the solution to the partial differential equation for $v(t,x)$, given as

$$v(t,x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right] \phi(y)\;dy$$

This is the complete solution formula, which was attained using the method of odd extensions (reflection method), as the graph of $\phi_{\text{odd}}(x)$ is the reflection of the graph $\phi(x)$ across the origin.

With the initial condition/datum, we are able to write $v(t,x)$ as follows:

$$v(t,x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right] \left(e^{-y} - 1\right)\;dy$$

Now, consider the previous substitution for $v(t,x)$ in place of $u(t,x)$. The *final* solution $u(t,x)$ may be written as follows:

$$u(t,x) = 1 + \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right] \left(e^{-y} - 1\right)\;dy$$

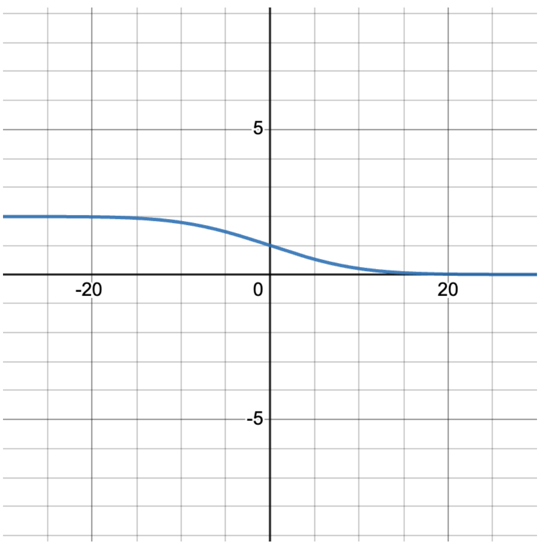
An overview of the approach is presented in the appendix.

Now, we return to the boundary condition $u(t,x) \to 0$ as $x \to \infty$, which is imposed on the problem. As indicated previously, if we know a solution $u(t,x)$ exists, it should be unique. To verify that the boundary condition is satisfied, we consider the limit as $x \to \infty$.

The key idea is to consider the value of t as fixed as $x \to \infty$. That is, t, which represents time, does extend to infinity, though we consider the boundary condition at $x \to \infty$ keeping t constant. It turns out that the solution

$$u(t,x) = 1 + \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right] \left(e^{-y} - 1\right)\;dy$$

will converge to $u(t,x) \to 0$ for $x \to \infty$, which is demonstrated by the following Desmos graph.



In particular, we consider that the upper bound of the integral term is ∞ "pushes" the solution down toward 0 as $x \to \infty$. This is demonstrated by varying the z value of the Desmos graph. In the case where the z value is not ∞, this would reflect a non-zero boundary condition, but that would require a different method of solving the PDE.

A student may be tempted to think that the expression

$$\frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right] \left(e^{-y} - 1\right) \, dy$$

approaches 0 as $x \to \infty$, however, this fails to consider the interaction between the terms $\left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right]$ and $(e^{-y} - 1)$ for fixed t and values y as the integral is calculated. In other words, as $x \to \infty$, both $e^{-(x-y)^2/4t}$ and $e^{-(x+y)^2/4t}$ terms will approach zero, though the integral is still influenced by the behavior of $\phi(y) = e^{-y} - 1$.

This is demonstrated by the red line of the Desmos graph, which highlights the case where there is no interaction term, i.e. $\phi(y) = 1$.

In this case $\phi(y) = e^{-y} - 1$ is such that the integral converges and the contribution from $\phi(y)$ is significant, so the limit of $u(t, x) - 1$ as $x \to \infty$ will not necessarily be zero, it actually ends up being -1, which indicates that the limit of $u(t, x)$ as $x \to \infty$ is zero.

This satisfies the Dirichlet "boundary" condition, considering that the integral converges ($\phi(y)$ does not "blow up" the integral).

Essentially, we may consider that, given that the solution to the diffusion problem is unique, the boundary condition is predetermined, in that it cannot be anything else. In this way, extending this problem beyond that presented in the midterm is of little consequence.

Question 7

True or False - Just answers, you **do not** need to explain.

(**False**) For $u_t + (x+u)\,u_x = 0$ in $(0,\infty)\times\mathbb{R}$, the solution of the characteristic equation is a straight line.

(**True**) For the solution $-\Delta u = 0$ in $D = (0,1)^2$ with $u = 0$ on ∂D, the normal derivative of u is zero, that is, $\frac{\partial u}{\partial n} = 0$ on ∂D.

(**False**) For the wave equation $u_{tt} - u_{xx} = 0$ in $(0,\infty)\times\mathbb{R}$, all information propagate at a constant speed.

(**True**) Let $f(x)$ be a function with a compact support in $(0,1)$. Then

$$\frac{1}{2}\partial f/\partial n + 3\partial^2 f/\partial n^2 = 0$$

at $x = 0$.

(**False**) Let u_1 and u_2 be the solutions of the following PDEs

$$(u_1)_t - \Delta u_1 = x \qquad u_1(0,x) = \phi(x) \qquad u_1(t,x) = g(x) \text{ on } \partial D$$

$$(u_2)_t - \Delta u_2 = u_2 \qquad u_2(0,x) = \psi(x) \qquad u_2(t,x) = h(x) \text{ on } \partial D$$

Then $u = u_1 + u_2$ is the solution to

$$u_t - \Delta u = u + x \qquad u(0,x) = \phi(x) + \psi(x) \qquad u(t,x) = g(x) + h(x) \text{ on } \partial D$$

(**True**) Math 053/126: Partial Differential Equations (PDEs) is difficult. (Please be honest. Your answer will not affect grading.)

Resources

- Partial Differential Equations (Walter A. Strauss)
- Partial Differential Equations (Rustum Choksi)
- Wave Equation Overview (Stanford)
- Method Of Characteristics (Stanford)
- Heat Equation Overview (Stanford)

Appendix

Question 1

The following provides a necessary proof for the exercise, which was previously covered.

Let

$$\int_{\Omega} f(x)\,\phi(x)\;dx + \int_{\partial\Omega} g(x)\,\phi(x)\;dx = \int_{\Omega} h(x)\,\phi(x)\;dx + \int_{\partial\Omega} m(x)\,\phi(x)\;dx$$

for all $\phi(x) \in \mathbb{C}^{\infty}(\Omega)$. Show that $f(x) = h(x)$ and $g(x) = m(x)$.

Similarly to the previous, the following is a proof that functions $f(x)$ and $h(x)$ are equal and functions $g(x)$ and $m(x)$ are equal if the integral over the domain Ω and $\partial\Omega$, respectively, are equal for all smooth test functions $\phi(x) \in C^{\infty}(\Omega)$. We consider an arbitrary open domain Ω in $\mathbb{R}^n$.

Proof: Let $f(x)$, $g(x)$, $h(x)$, and $m(x)$ be functions defined on Ω and $\partial\Omega$, respectively. Suppose

$$\int_{\Omega} f(x)\,\phi(x)\;dx + \int_{\partial\Omega} g(x)\,\phi(x)\;dx = \int_{\Omega} h(x)\,\phi(x)\;dx + \int_{\partial\Omega} m(x)\,\phi(x)\;dx$$

for any smooth test function $\phi(x)$ defined on Ω, including those that have compact support.

We aim to show that $f(x) = h(x)$ for all $x \in \Omega$ and $g(x) = m(x)$ for all $x \in \partial\Omega$. Thus, consider a smooth function $\phi(x)$ that is 1 in a small ball $B(c, \varepsilon)$ and 0 outside a slightly larger ball $B(c, 2\varepsilon)$ for some $c \in \Omega$. The integral condition becomes

$$\int_{B(c,\varepsilon)} f(x)\;dx = \int_{B(c,\varepsilon)} h(x)\;dx$$

As $\varepsilon \to 0$, this implies $f(c) = h(c)$ by the fundamental theorem of calculus on $\mathbb{R}^n$. Since our choice of c was arbitrary, this implies that $f(x) = h(x)$ for all $x \in \Omega$.

Similarly, consider a smooth function $\psi(x)$ that is 1 on an ε-neighborhood of a point $x_0 \in \partial\Omega$ on the boundary, and 0 elsewhere. The integral condition becomes

$$\int_{\partial\Omega} g(x)\,\psi(x)\;dS = \int_{\partial\Omega} m(x)\,\psi(x)\;dS$$

As indicated by the previous example, as $\varepsilon \to 0$, this implies $g(x_0) = m(x_0)$ by the fundamental theorem of calculus on $\mathbb{R}^n$. Since our choice of x_0 was arbitrary, this implies that $g(x) = m(x)$ for all $x \in \partial\Omega$.

Thus, if functions have equal integrals against all smooth test functions (including those with compact support) on Ω and $\partial\Omega$, they must be equal point-wise and represent the same functions. The test functions $\phi(x)$ and $\psi(x)$ localize the equality to an arbitrary point c in the domain (interior) and x_0 on the boundary (boundary), respectively. This completes the proof. $\square$

Further arguments are presented as follows, as they enhance the rigor of the argument presented above.

To show that $f(x) = h(x)$ and $g(x) = m(x)$ based on the given equation, we can use the fundamental lemma of calculus of variations, which is useful in functional analysis and variational calculus in deducing the equality of functions under certain conditions.

The fundamental lemma states that if for all smooth test functions $\phi(x)$, the following equation holds:

$$\int_{\Omega} (f(x) - h(x))\, \phi(x)\ dx + \int_{\partial\Omega} (g(x) - m(x))\ dx = 0$$

Then, it implies that $f(x) = h(x)$ and $g(x) = m(x)$. To prove this, we need to choose a suitable test function $\phi(x)$ and show that the equation above implies $f(x) = h(x)$ and $g(x) = m(x)$.

Let's start with $f(x) = h(x)$. Suppose $f(x) \neq h(x)$ for some x. Then, we can construct a test function $\phi(x)$ that is non-zero at the point where $f(x) \neq h(x)$ and zero elsewhere. Then, the left-hand side of the equation becomes:

$$\int_{\Omega} (f(x) - h(x))\, \phi(x)\ dx = \int_{\Omega} (f(x) - h(x))\ dx \neq 0$$

The last step is non-zero because $f(x) \neq h(x)$ at some point in Ω. However, this cannot be the case, because $\phi(x)$ is zero everywhere except where $f(x)$ and $h(x)$ are integrated. This contradiction shows that $f(x) = h(x)$.

Now, let's show that $g(x) = m(x)$. Similarly, suppose $g(x) \neq m(x)$ for some x. Then, we can construct a test function $\phi(x)$ that is non-zero at the point where $g(x) \neq m(x)$ and zero elsewhere. Then, the left-hand side of the equation becomes:

$$\int_{\partial\Omega} (g(x) - m(x))\, \phi(x)\ dx = \int_{\partial\Omega} (g(x) - m(x))\ dx \neq 0$$

The last step is non-zero because $g(x) \neq m(x)$ at some point in $\partial\Omega$. However, this cannot be case, because $\phi(x)$ is zero everywhere except where $g(x)$ and $m(x)$ are integrated. This contradiction shows that $g(x) = m(x)$.

While the structure follows similarly, there are nuances in the argument for the choice of the test function, particularly the way in which it may impact the other integrals (Ω or $\partial\Omega$). This is outlined in the initial argument, thus, we are able to demonstrate that if the given equation holds for all smooth test functions $\phi(x)$, then $f(x) = h(x)$ and $g(x) = m(x)$.

The results of this example may be used for the previous, by simply considering the case where the test function vanishes on the boundary $\partial\Omega$, which gives the exact form of the previous example.

The following demonstrates the derivation of the weak formulation of the Poisson equation with a non-zero Neumann boundary condition, i.e. the other direction.

Derive the weak formulation of the following Poisson equation with a non-zero Neumann boundary condition

$$-\Delta u = f \text{ in } \Omega \qquad \frac{\partial u}{\partial n} = h\left(x\right) \text{ on } \partial\Omega$$

To derive the weak formulation of the following Poisson equation with a non-zero Neumann boundary condition, we use a test function space C_0^∞, which represents compact support. The motivation behind this lies in integration by parts, which is given as follows: $\int_\Omega u\,dv = \int_{\partial\Omega} uv - \int_\Omega v\,du$. Now, let us start with the strong formulation:

$$-\Delta u = f \text{ in } \Omega \qquad \frac{\partial u}{\partial n} = h\left(x\right) \text{ on } \partial\Omega$$

As implied, we multiply by a test function $v \in H^1\left(\Omega\right) \in C_0^\infty$, and integrate over the domain Ω:

$$\int_\Omega -\left(\Delta u\right) v\,dx = \int_\Omega f v\,dx$$

The Laplacian term may be re-written to determine the following:

$$-\int_\Omega \nabla \cdot \left(\nabla u\right) v\,dx = \int_\Omega f v\,dx$$

Now, we apply the following form: $\nabla\left(v\nabla u\right) = \nabla v \cdot \nabla u + v\Delta u$, which implies $\Delta u\, v = \nabla \cdot \left(v\nabla u\right) - \nabla v \cdot \nabla u$. This leads to the following:

$$\int_\Omega \nabla u \cdot \nabla v\,dx - \int_\Omega \nabla \cdot \left(v\nabla u\right)\,dx = \int_\Omega f v\,dx$$

From integration by parts (divergence theorem), this results in

$$\int_\Omega \nabla u \cdot \nabla v\,dx - \int_{\partial\Omega} v\nabla u \cdot n\,ds = \int_\Omega f v\,dx$$

which is equivalent to

$$\int_\Omega \nabla u \cdot \nabla v\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v\,ds = \int_\Omega f v\,dx$$

Now, we may substitute the Neumann boundary condition to find the weak formulation

$$\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f v\,dx + \int_{\partial\Omega} h\left(x\right) v\,ds$$

for all $v \in V$, where V is the appropriate function space for the solution u.

Question 2

The following provides a summary/overview of the steps.

The domain is $D = (0,1)^2$, which is the unit square. The PDE is the heat equation

$$u_t - \Delta u = 0$$

in D with the boundary condition $u(t,x) = h(x)$ on ∂D and initial condition $u(0,x) = \phi(x)$. We take the energy defined as

$$E(t) = \int_D (u(t,x))^2 \, dx$$

and take the time derivative of the energy, as follows:

$$\frac{dE}{dt} = \frac{d}{dt} \int_D u^2 \, dx$$

By the Leibniz integral rule,

$$\frac{dE}{dt} = 2 \int_D u u_t \, dx$$

Now, substituting the PDE, we have

$$\frac{dE}{dt} = 2 \int_D u \Delta \, dx$$

Following this, we may apply integration by parts (divergence theorem), to find

$$\frac{dE}{dt} = -2 \int_D |\nabla u|^2 \, dx$$

This skips over the steps taken to handle the boundary term, but the key idea is that the boundary term vanishes, which allows us to determine the following. Since $|\nabla u|^2 \geq 0$, we have

$$\frac{dE}{dt} \leq 0$$

This shows that the energy is decreasing over time. This makes sense physically as the heat equation models heat diffusion, which causes smoothing and decay of energy over time.

In summary, we showed the energy decay by taking the time derivative of the energy integral, substituting the PDE, applying integration by parts, and using the non-negativity of the gradient squared term, which proves the desired result. $\square$

The following provides an overview of the method to find the limit of the energy, along with the specific case.

To find the limit of the energy as $t \to \infty$, let us examine the expression for the rate of change of energy

$$\frac{DE}{dt} = -2 \int_D \nabla u \cdot \nabla u \, dx$$

Since the right-hand side is negative, it implies that the energy $E(t)$ is decreasing over time. As t approaches infinity, we are interested in the limit of $E(t)$. If the energy is decreasing and bounded below, then it must converge to a limit. Let us denote the limit as $\lim_{t\to\infty} E(t) = E_\infty$. Therefore, we have

$$\lim_{t\to\infty} E(t) = \lim_{t\to\infty} \int_D u^2 \, dx = E_\infty$$

To proceed further, we need information about the behavior of the solution $u(t,x)$ as $t \to infty$. Assuming that the solution $u(t,x)$ approaches a steady state or becomes periodic as $t \to \infty$, it is possible that the limit of the energy E_∞ is determined by the steady-state or periodic behavior.

With the boundary condition $h(x) = x_1$, let us find the limit of the energy as $t \to \infty$. The solution $u(t,x)$ converges to the steady state solution $u_\infty(x)$ as $t \to \infty$. This steady state solution satisfies

$$\Delta u_\infty = 0 \text{ in } D$$

$$u_\infty = x_1 \text{ on } \partial D$$

Thus, we can show that

$$u_\infty(x) = x_1$$

so the limit of the energy is

$$\begin{aligned}
\lim_{t\to\infty} E(t) &= \lim_{t\to\infty} \int_D (u(t,x))^2 \, dx \\
&= \int_D (u_\infty(x))^2 \, dx \\
&= \int_D x_1^2 \, dx \\
&= \int_0^1 \int_0^1 x_1^2 \, dx_2 \, dx_1 \\
&= \int_0^1 x_1^2 \, dx_1 \\
&= \frac{1}{3}
\end{aligned}$$

So with the given boundary condition, the energy decays over time to the limit $\frac{1}{3}$ as $t \to \infty$. This makes sense, since the steady state solution is linear in x_1 which gives a finite integral over the domain.

Question 3

The following provides an overview of $\Phi(x,t)$ as the fundamental solution of the 1D wave equation, which is relevant to the problem.

We call $\Phi(x,t)$, defined in the problem, the fundamental solution of the 1D wave equation. Why? For starters, it can immediately be used to solve the IVP

$$u_{tt} - u_{xx} = 0 \qquad -\infty < x < \infty, t > 0$$

$$u(x,0) = 0 \qquad u_t(x,0) = \psi(x) \qquad \infty < x < \infty$$

for any $\psi(x)$. Indeed, just as for the diffusion equation, the solution to the IVP problem is given by

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)\,\psi(y)\;dy$$

While this is hardly surprising given that Φ is a (distributional) solution to the given problem, we may also simply place the solution according to the integral formula into the above to find

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)\,\psi(y)\;dy = \frac{1}{2}\int_{x-t}^{x+t} \psi(y)\;dy$$

which (full circle) takes us back to D'Alembert's solution.

Now suppose we have nonzero data for ϕ (the initial displacement) and wish to solve

$$u_{tt} - u_{xx} = 0 \qquad -\infty < x < \infty, t > 0$$

$$u(x,0) = \phi(x) \qquad u_t(x,0) = 0 \qquad -\infty < x < \infty$$

The solution can also be obtained from Φ and is given by

$$u(x,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \Phi(x-y,t)\,\phi(y)\;dy$$

To this end, we find

$$\begin{aligned}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \Phi(x-y,t)\,\phi(y)\;dy &= \frac{\partial}{\partial t}\frac{1}{2}\int_{x-t}^{x+t} \phi(y)\;dy \\ &= \frac{1}{2}\left[\phi(x+t) + \phi(x-t)\right]\end{aligned}$$

which, again, takes us back to D'Alembert's solution.

Lastly, suppose we put the delta function as an in-homogeneous term in the wave equation and consider

$$u_{tt} - u_{xx} = \delta_0$$

in the sense of distributions on $\mathbb{R}^2$ where δ_0 is the 2D delta function, informally written as "$\delta_0(x)\,\delta_0(t)$". The solution to this PDE is $\Phi(x,t)$, trivially extended to all $t \in \mathbb{R}$:

$$\Phi(x,t) = \begin{cases} \frac{1}{2} & \text{if } |x| < t, t > 0 \\ 0 & \text{if } |x| \geq t, t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

With this in hand, we can solve the in-homogeneous IVP

$$u_{tt} - u_{xx} = f(x,t) \qquad -\infty < x < \infty, t > 0$$

$$u(x,0) = 0 \qquad u_t(x,0) = 0 \qquad -\infty < x < \infty$$

via the spatio-temporal convolution of Φ with f:

$$u(x,t) = \int_0^\infty \int_{-\infty}^\infty \Phi(x-y, t-\tau) f(y,\tau)\, dy\, d\tau$$

Indeed, using the explicit form of Φ, we find

$$u(x,t) = \frac{1}{2} \iint_D f(y,\tau)\, dy\, d\tau$$

where D is precisely the domain of dependence associated with (x,t), i.e. the triangle in the xt-plane with top point (x,t) and base points $(x-t,0)$ and $(x+t,0)$. This is exactly what we found before via Duhamel's Principle.

Conclusion: The key (in fact only) object required to solve the *full* IVP

$$u_{tt} - u_{xx} = f(x,t) \qquad -\infty < x < \infty, t > 0$$

$$u(x,0) = \phi \qquad u_t(x,0) = \psi \qquad -\infty < x < \infty$$

is the fundamental solution Φ defined previously.

Question 4

The following provides an overview of the proof as to why convolution with the fundamental solution works, according to Wikipedia.

Denote the convolution of functions F and g as $F * g$. Say we are trying to find the solution of $Lf = g(x)$. We want to prove that $F * f$ is a solution of the previous equation, i.e. we want to prove that $L(F * g) = g$. When applying the differential operator, L, to the convolution, it is known that

$$L(F * g) = (LF) * g$$

provided L has constant coefficients. If F is the fundamental solution, the right side of the equation reduces to

$$\delta * g$$

Since the delta function is an identity element for convolution, this is simply $g(x)$. In summary,

$$L(F * g) = (LF) * g = \delta(x) * g(x) = \int_{-\infty}^{\infty} \delta(x-y)\, g(y)\; dy = g(x)$$

Therefore, if F is the fundamental solution, the convolution $F * g$ is one solution of $Lf = g(x)$. This does not mean that it is the only soultion. Several solutiosn for different inital conditions can be found.

Question 5

The following provides a necessary proof for the exercise, which was previously covered.

Show that $\Delta \ln |x| = C\delta(x)$ in 2D. Find the constant C.

The following is from Stanford. Define the function ϕ as follows. For $|x| \neq 0$, let

$$\phi(x) = -\frac{1}{2\pi} \ln |x|$$

Let $\alpha(n)$ be the volume of the unit ball in $\mathbb{R}^n$. We see that ϕ satisfies Laplace's equation on $\mathbb{R}^2 - \{0\}$. As we will show in the following claim, ϕ satisfies $-\Delta_x \phi = \delta_0$. For this reason, we call ϕ the *fundamental solution* of Laplace's equation.

Claim: For ϕ defined as above, ϕ satisfies

$$-\Delta_x \phi = \delta_0$$

in the sense of distributions. That is, for all $g \in D$,

$$-\int_{\mathbb{R}^2} \phi(x) \Delta_x g(x)\, dx = g(0)$$

This would imply that $\Delta \ln |x| = C\delta(x)$ in 2D, where $C = 2\pi$.

Proof: Let F_ϕ be the distribution associated with the fundamental solution ϕ. That is, let $F_\phi : D \to \mathbb{R}$ be defined such that

$$(F_\phi, g) = \int_{\mathbb{R}^2} \phi(x) g(x)\, dx$$

for all $g \in D$. Recall that the derivative of a distribution F is defined as the distribution G such that

$$(G, g) = -(F, g')$$

for all $g \in D$. Therefore, the *distributional Laplacian* of ϕ is defined as the distribution $F_{\Delta\phi}$ such that

$$(F_{\Delta\phi}, g) = (F_\phi, \Delta g)$$

for all $g \in D$. We will show that

$$(F_\phi, \Delta g) = -(\delta_0, g) = -g(0)$$

and therefore,

$$(F_{\Delta\phi}, g) = -g(0)$$

which means $-\Delta_x \phi = \delta_0$ in the sense of distributions. By definition

$$(F_\phi, \Delta g) = \int_{\mathbb{R}^2} \phi(x) \Delta g(x)\, dx$$

Now, we would like to apply the divergence theorem, but ϕ has a singularity at $x = 0$. We get around this, by breaking up the integral into two pieces: one piece consisting of the ball of radius δ about the origin, $B(0,\delta)$ and the other piece consisting of the complement of this ball in $\mathbb{R}^2$ (in this case it is a circle). Therefore, we have

$$\begin{aligned}(F_\phi, \Delta g) &= \int_{\mathbb{R}^2} \phi(x)\,\Delta g(x)\; dx \\ &= \int_{B(0,\delta)} \phi(x)\,\Delta g(x)\; dx + \int_{\mathbb{R}^2 - B(0,\delta)} \phi(x)\,\Delta g(x)\; dx \\ &= I + J\end{aligned}$$

We look first at term I, which is bounded as follows:

$$\begin{aligned}\left| -\int_{B(0,\delta)} \frac{1}{2\pi} \ln|x|\,\Delta g(x)\; dx \right| &\le C\,|\Delta g|_{L^\infty} \left| \int_{B(0,\delta)} \ln|x|\; dx \right| \\ &\le C \left| \int_0^{2\pi} \int_0^\delta \ln|r|\, r\, dr\, d\theta \right| \\ &\le C \left| \int_0^\delta \ln|r|\, r\, dr \right| \\ &\le C \ln|\delta|\,\delta^2\end{aligned}$$

Therefore, as $\delta \to 0^+$, $|I| \to 0$. Next, we look at term J. Applying the divergence theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^2 - B(0,\delta)} \phi(x)\,\Delta_x g(x)\; dx &= \int_{\mathbb{R}^2 - B(0,\delta)} \Delta_x \phi(x)\, g(x)\; dx - \int_{\partial(\mathbb{R}^2 - B(0,\delta))} \frac{\partial \phi}{\partial v} g(x)\; dS(x) \\ &\qquad + \int_{\partial(\mathbb{R}^2 - B(0,\delta))} \phi(x)\,\frac{\partial g}{\partial v}\; dS(x) \\ &= -\int_{\partial(\mathbb{R}^2 - B(0,\delta))} \frac{\partial \phi}{\partial v} g(x)\; dS(x) + \int_{\partial(\mathbb{R}^2 - B(0,\delta))} \phi(x)\,\frac{\partial g}{\partial v} dS(x) \\ &= J1 + J2\end{aligned}$$

using the fact that $\Delta_x \phi(x) = 0$ for $x \in \mathbb{R}^2 - B(0,\delta)$. We first look at term $J1$. Now, by assumption, $g \in D$, and, therefore, g vanishes at ∞. Consequently, we only need to calculate the integral over $\partial B(0,\varepsilon)$ where the normal derivative v is the outer normal to $\mathbb{R}^2 - B(0,\delta)$. By a straightforward calculation, we see that for $n = 2$,

$$\nabla_x \phi(x) = -\frac{x}{n\alpha(n)\,|x|^n}$$

The outer unit normal to $\mathbb{R}^2 - B(0,\delta)$ on $B(0,\delta)$ is given by

$$v = -\frac{x}{|x|}$$

Therefore, the normal derivative of ϕ on $B(0,\delta)$ is given by

$$\frac{\partial \phi}{\partial v} = \left(-\frac{x}{n\alpha(n)\,|x|^n}\right) \cdot \left(-\frac{x}{|x|}\right) = \frac{1}{n\alpha(n)\,|x|^{n-1}}$$

Therefore, $J1$ can be written as

$$-\int_{\partial B(0,\delta)} \frac{1}{n\alpha(n)\,|x|^{n-1}} g(x)\; dS(x) = -\frac{1}{n\alpha(n)\,\delta^{n-1}} \int_{\partial B(0,\delta)} g(x)\; dS(x) = -\fint_{\partial B(0,\delta)} g(x)\; dS(x)$$

Now if g is a continuous function, then

$$-\int g(x)\ dS(x) \to -g(0) \qquad \text{as} \qquad \delta \to 0$$

Lastly, we look at term $J2$. Now using the fact that g vanishes as $|x| \to +\infty$, we only need to integrate over $\partial B(0,\delta)$. Using the fact that $g \in D$, and, therefore, infinitely differentiable, we have

$$\begin{aligned}
\left| \int_{\partial B(0,\delta)} \phi(x) \frac{\partial g}{\partial v}\, dS(x) \right| &\leq \left| \frac{\partial g}{\partial v} \right|_{L^\infty(\partial B(0,\delta))} \int_{\partial B(0,\delta)} |\phi(x)|\ dS(x) \\
&\leq C \int_{\partial B(0,\delta)} |\phi(x)|\ dS(x)
\end{aligned}$$

For $n = 2$,

$$\begin{aligned}
\int_{\partial B(0,\delta)} |\phi(x)|\ dS(x) &= C \int_{\partial B(0,\delta)} |\ln|x||\ dS(x) \\
&\leq C\,|\ln|\delta|| \int_{\partial B(0,\delta)} dS(x) \\
&= C\,|\ln|\delta||\,(2\pi\delta) \leq C\delta\,|\ln|x||
\end{aligned}$$

Therefore, we conclude that term $J2$ is bounded in absolute value by $C\delta\,|\ln\delta|$. Therefore $|J2| \to 0$ as $\delta \to 0^+$. Combining these estimates, we see that

$$\int_{\mathbb{R}^2} \phi(x)\,\Delta_x g(x)\ dx = \lim_{\delta \to 0^+} I + J1 + J2 = -g(0)$$

Therefore, our claim is proven. $\square$

Question 6

The following provides an overview of the approach of this problem.

The given problem involves solving a heat equation on the half-line with a Dirichlet boundary condition at the single endpoint. The solution involves the use of the odd extension method, which extends the initial condition to an odd function defined on the whole line. Let's summarize the solution.

Transformation

Define $v(t,x) = u(t,x) - 1$. The transformed function v satisfies the following partial differential equation:

$$v_t - v_{xx} = 0 \qquad t > 0, x \in (0, \infty)$$

with initial condition $v(0,x) = e^{-x} - 1$ and boundary condition $v(t,0)$.

Odd Extension

Extend the initial condition $v(0,x) = e^{-x} - 1$ to an odd function $\phi_{\text{odd}}(x)$ defined on the whole line:

$$\phi_{\text{odd}} = \begin{cases} \phi(x) & \text{for } x > 0 \\ -\phi(-x) & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Solution For The Whole Line

Find the solution $w(t,x)$ for the heat equation on the whole line $(-\infty, \infty)$ with the initial condition ϕ_{odd}:

$$w_t - w_x x = 0 \qquad w(0,x) = \phi_{\text{odd}}(x)$$

The solution is given by

$$w(t,x) = \int_{-\infty}^{\infty} S(x-y,t)\, \phi_{\text{odd}}(y)\, dy$$

Restriction To Half-Line

Define $v(t,x) = w(t,x)$ for $x > 0$. The function $v(t,x)$ is the unique solution for the problem on the half line.

Explicit Formula

The explicit for $v(t,x)$ is given as follows, after simplification:

$$v(t,x) = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t} \right] \phi(y)\, dy$$

Final Solution

The final solution for $u(t,x)$ is obtained by adding 1 to $v(t,x)$, and applying the initial datum:

$$u(t,x) = 1 + \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} \left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t} \right] \left(e^{-y} - 1 \right) dy$$

This completes the proof for the given heat/diffusion equation problem on the half-line with the specified initial and boundary conditions.