MATH 053/126 - Partial Differential Equations

Carter Kruse, November 9, 2023

Homework 5

Instructions/Notation

Please show all steps to get your answers. Specify the problems you discussed with other students (including names).

The starred problems are recommended but not required for undergraduate/non-math major graduate students and required for all math major graduate students.

Notation

- $\mathbb{R}$: The set of all real numbers.
- $\mathbb{R}^+$: The set all positive real numbers $\{x \in \mathbb{R} | x > 0\}$.
- $\frac{\partial u}{\partial t} = \partial_t u = u_t$
- $\frac{\partial u}{\partial x} = \partial_x u = u_x$

Questions

Question 1

We consider an ordinary differential operator L in $(-1, 1)$

$$Lu = -\frac{d}{dx}\left(\left(1 - x^2\right)\frac{du}{dx}\right) + 2u$$

where $u(x) = 1$ for $x = \pm 1$. Assume that there exist eigenfunctions of L that are not constant functions. Show that eigenvalues of L are positive.

In this problem, we are given an ordinary differential operator L in $(-1, 1)$

$$Lu = -\frac{d}{dx}\left(\left(1 - x^2\right)\frac{du}{dx}\right) + 2u$$

where $u(x) = 1$ for $x = \pm 1$, which represents the boundary conditions. Now, let us assume that there exist eigenfunctions of L that are not constant functions. That is, there exist non-constant $u(x)$ such that $Lu = \lambda u$ for eigenvalues λ.

The goal is to demonstrate that the eigenvalues λ of the ordinary differential operator L are positive. To do so, let us multiply the expression $Lu = \lambda u$ by u and integrate over the domain $(-1, 1)$, as follows:

$$\int_{-1}^{1} uLu \, dx = \lambda \int_{-1}^{1} u^2 \, dx$$

This may be represented as follows:

$$\int_{-1}^{1} u\left(-\frac{d}{dx}\left(\left(1 - x^2\right)\frac{du}{dx}\right) + 2u\right) dx = \lambda \int_{-1}^{1} u^2 \, dx$$

Now, if we apply integration by parts $\left(\int u\,dv = uv - \int v\,du\right)$ to the left-hand side of the equation (given the boundary conditions), we have

$$\int_{-1}^{1} \left(1 - x^2\right) \frac{d^2u}{dx^2} + 2u^2\,dx = \lambda \int_{-1}^{1} u^2\,dx$$

Now, given that $\left(1 - x^2\right) \geq 0$ and $\frac{d^2u}{dx^2} \geq 0$ on $(-1, 1)$, we know that the value of the left-hand side of the equation is strictly positive. Considering $\int_{-1}^{1} u^2\,dx$ is positive as well, this implies that $\lambda > 0$. Therefore, the eigenvalues of L corresponding to non-constant eigenfunctions must be positive. $\square$

Question 2

Let L be a symmetric operator on a function space H. Show that the eigenfunctions corresponding to different eigenvalues are orthogonal.

To demonstrate that eigenfunctions corresponding to different eigenvalues (of a symmetric operator L on a function space H) are orthogonal, consider u and v as the eigenfunctions of L corresponding to distinct eigenvalues λ and μ, respectively.

$$Lu = \lambda u \qquad Lv = \mu v$$

Here L is considered to be a symmetric operator on a function space H, so for any $f, g \in H$,

$$\langle Lf, g \rangle = \langle f, Lg \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on H. In this case, let $f = u$ and $g = v$, so

$$\langle Lu, v \rangle = \langle u, Lv \rangle$$

If we substitute the eigenvalue expressions for Lu and Lv, respectively, this results in

$$\langle \lambda u, v \rangle = \langle u, \mu v \rangle$$

Now, we may pull λ and μ out of the inner products, since they are constants, which yields

$$\lambda \langle u, v \rangle = \mu \langle u, v \rangle$$

Since $\lambda \neq \mu$ (as the eigenvalues are distinct), we have $\langle u, v \rangle = 0$, indicating that the eigenfunctions u and v corresponding to distinct eigenvalues of the symmetric operator L (on a function space H) must be orthogonal. $\square$

Question 3 & 4

Show that $\Delta \ln |x| = C\delta(x)$ in 2D. Find the constant C.

The following is from Stanford. Define the function ϕ as follows. For $|x| \neq 0$, let

$$\phi(x) = -\frac{1}{2\pi} \ln |x|$$

Let $\alpha(n)$ be the volume of the unit ball in $\mathbb{R}^n$. We see that ϕ satisfies Laplace's equation on $\mathbb{R}^2 - \{0\}$. As we will show in the following claim, ϕ satisfies $-\Delta_x \phi = \delta_0$. For this reason, we call ϕ the *fundamental solution* of Laplace's equation.

Claim: For ϕ defined as above, ϕ satisfies

$$-\Delta_x \phi = \delta_0$$

in the sense of distributions. That is, for all $g \in D$,

$$-\int_{\mathbb{R}}^{2} \phi(x)\, \Delta_x g(x)\ dx = g(0)$$

This would imply that $\Delta \ln |x| = C\delta(x)$ in 2D, where $C = 2\pi$.

Proof: Let F_ϕ be the distribution associated with the fundamental solution ϕ. That is, let $F_\phi : D \to \mathbb{R}$ be defined such that

$$(F_\phi, g) = \int_{\mathbb{R}^2} \phi(x)\, g(x)\ dx$$

for all $g \in D$. Recall that the derivative of a distribution F is defined as the distribution G such that

$$(G, g) = -(F, g')$$

for all $g \in D$. Therefore, the *distributional Laplacian* of ϕ is defined as the distribution $F_{\Delta\phi}$ such that

$$(F_{\Delta\phi}, g) = (F_\phi, \Delta g)$$

for all $g \in D$. We will show that

$$(F_\phi, \Delta g) = -(\delta_0, g) = -g(0)$$

and therefore,

$$(F_{\Delta\phi}, g) = -g(0)$$

which means $-\Delta_x \phi = \delta_0$ in the sense of distributions. By definition

$$(F_\phi, \Delta g) = \int_{\mathbb{R}^2} \phi(x)\, \Delta g(x)\ dx$$

Now, we would like to apply the divergence theorem, but ϕ has a singularity at $x = 0$. We get around this, by breaking up the integral into two pieces: one piece consisting of the ball of radius δ about the origin, $B(0, \delta)$ and the other piece consisting of the complement of this ball in $\mathbb{R}^2$ (in this case it is a circle). Therefore, we have

$$\begin{aligned}
(F_\phi, \Delta g) &= \int_{\mathbb{R}^2} \phi(x)\, \Delta g(x)\ dx \\
&= \int_{B(0,\delta)} \phi(x)\, \Delta g(x)\ dx + \int_{\mathbb{R}^2 - B(0,\delta)} \phi(x)\, \Delta g(x)\ dx \\
&= I + J
\end{aligned}$$

We look first at term I, which is bounded as follows:

$$\begin{aligned}
\left|-\int_{B(0,\delta)} \frac{1}{2\pi} \ln|x| \, \Delta g(x) \, dx\right| &\leq C \left|\Delta g\right|_{L^{\infty}} \left|\int_{B(0,\delta)} \ln|x| \, dx\right| \\
&\leq C \left|\int_{0}^{2\pi} \int_{0}^{\delta} \ln|r| \, r \, dr \, d\theta\right| \\
&\leq C \left|\int_{0}^{\delta} \ln|r| \, r \, dr\right| \\
&\leq C \ln|\delta| \, \delta^{2}
\end{aligned}$$

Therefore, as $\delta \to 0^{+}$, $|I| \to 0$. Next, we look at term J. Applying the divergence theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^2 - B(0,\delta)} \phi(x) \, \Delta_x g(x) \, dx &= \int_{\mathbb{R}^2 - B(0,\delta)} \Delta_x \phi(x) \, g(x) \, dx - \int_{\partial(\mathbb{R}^2 - B(0,\delta))} \frac{\partial \phi}{\partial v} g(x) \, dS(x) + \int_{\partial(\mathbb{R}^2 - B(0,\delta))} \phi(x) \frac{\partial g}{\partial v} \, dS(x) \\
&= -\int_{\partial(\mathbb{R}^2 - B(0,\delta))} \frac{\partial \phi}{\partial v} g(x) \, dS(x) + \int_{\partial(\mathbb{R}^2 - B(0,\delta))} \phi(x) \frac{\partial g}{\partial v} dS(x) \\
&= J1 + J2
\end{aligned}$$

using the fact that $\Delta_x \phi(x) = 0$ for $x \in \mathbb{R}^2 - B(0,\delta)$. We first look at term $J1$. Now, by assumption, $g \in D$, and, therefore, g vanishes at ∞. Consequently, we only need to calculate the integral over $\partial B(0,\varepsilon)$ where the normal derivative v is the outer normal to $\mathbb{R}^2 - B(0,\delta)$. By a straightforward calculation, we see that for $n = 2$,

$$\nabla_x \phi(x) = -\frac{x}{n\alpha(n) |x|^{n}}$$

The outer unit normal to $\mathbb{R}^2 - B(0,\delta)$ on $B(0,\delta)$ is given by

$$v = -\frac{x}{|x|}$$

Therefore, the normal derivative of ϕ on $B(0,\delta)$ is given by

$$\frac{\partial \phi}{\partial v} = \left(-\frac{x}{n\alpha(n) |x|^{n}}\right) \cdot \left(-\frac{x}{|x|}\right) = \frac{1}{n\alpha(n) |x|^{n-1}}$$

Therefore, $J1$ can be written as

$$-\int_{\partial B(0,\delta)} \frac{1}{n\alpha(n) |x|^{n-1}} g(x) \, dS(x) = -\frac{1}{n\alpha(n) \delta^{n-1}} \int_{\partial B(0,\delta)} g(x) \, dS(x) = -\fint_{\partial B(0,\delta)} g(x) \, dS(x)$$

Now if g is a continuous function, then

$$-\fint g(x) \, dS(x) \to -g(0) \qquad \text{as} \qquad \delta \to 0$$

Lastly, we look at term $J2$. Now using the fact that g vanishes as $|x| \to +\infty$, we only need to integrate over $\partial B(0,\delta)$. Using the fact that $g \in D$, and, therefore, infinitely differentiable, we have

$$\begin{aligned}
\left|\int_{\partial B(0,\delta)} \phi(x) \frac{\partial g}{\partial v} \, dS(x)\right| &\leq \left|\frac{\partial g}{\partial v}\right|_{L^{\infty}(\partial B(0,\delta))} \int_{\partial B(0,\delta)} |\phi(x)| \, dS(x) \\
&\leq C \int_{\partial B(0,\delta)} |\phi(x)| \, dS(x)
\end{aligned}$$

For $n = 2$,

$$\begin{aligned}
\int_{\partial B(0,\delta)} |\phi(x)| \, dS(x) &= C \int_{\partial B(0,\delta)} |\ln|x|| \, dS(x) \\
&\leq C |\ln|\delta|| \int_{\partial B(0,\delta)} dS(x) \\
&= C |\ln|\delta|| (2\pi\delta) \leq C\delta |\ln|x||
\end{aligned}$$

Therefore, we conclude that term $J2$ is bounded in absolute value by $C\delta\left|\ln\delta\right|$. Therefore $\left|J2\right|\to 0$ as $\delta\to 0^{+}$. Combining these estimates, we see that

$$\int_{\mathbb{R}^2}\phi\left(x\right)\Delta_x g\left(x\right)\,dx=\lim_{\delta\to 0^{+}}I+J1+J2=-g\left(0\right)$$

Therefore, our claim is proven. $\square$

Question 5

In 1D, find a function $s(x)$ such that

$$s_{xx} = \delta(x)$$

Hint: We did this problem in class.

To find a function $s(x)$ such that $s_{xx} = \delta(x)$ in 1D, consider taking the integral of the differential function $s_{xx} = \delta(x)$, where $\delta(x)$ is the Dirac delta function.

$$s_x = H(x) + \alpha$$

where $H(x)$ is the Heaviside function and α is a constant of integration. Now, let us integrate the expression again, to find

$$s(x) = xH(x) + \alpha x + \beta$$

where β is a constant of integration. This provides the general form for the solution $s(x)$. This may be represented as follows:

$$s(x) = \frac{1}{2}|x| + \gamma x + \beta$$

where $\gamma = \alpha + \frac{1}{2}$. This provides the solution formula to the 1D differential equation $s_{xx} = \delta(x)$. $\square$

Question 6

Use the $s(x)$ in the previous problem to solve

$$u_{xx} = f(x) \qquad x \in \mathbb{R}$$

where $u(x) = 0$ at $x = \pm\infty$. Also, assume that $f(x)$ has a compact support and is smooth.

The following is from Stanford. We now turn to studying Laplace's equation

$$\Delta u = 0$$

and its in-homogeneous version, Poisson's equation,

$$-\Delta u = f$$

We say a function u satisfying Laplace's equation is a harmonic function.

Consider Laplace's equation in $\mathbb{R}^n$

$$\Delta u = 0 \qquad x \in \mathbb{R}^n$$

Clearly, there are a lot of functions u which satisfy this equation. In particular, any constant function is harmonic. In addition, any function of the form $u(x) = a_1 x_1 + \ldots + a_n x_n$ for constants a_n is also a solution. Of course, we can list a number of others. Here, however, we are interested in finding a particular solution of Laplace's equation which will allow us to solve Poisson's equation.

Given the symmetric nature of Laplace's equation, we look for a *radial* solution. That is, we look for a harmonic function u on $\mathbb{R}^n$ such that $u(x) = v(|x|)$. In addition to being a natural choice due to the symmetry of Laplace's equation, radial solutions are natural to look for because they reduce a PDE to an ODE, which is generally easier to solve. Therefore, we look for a radial solution.

Please see the reference for further information with respect to radial solutions.

From the calculations, we see that for any constants c_1, c_2, the function

$$u(x) = c_1 \ln|x| + c_2$$

for $x \in \mathbb{R}^n$, $|x| \neq 0$ is a solution of Laplace's equation in $\mathbb{R}^n - \{0\}$. We notice that the function u defined above satisfies $\Delta u(x) = 0$ for $x \neq 0$, but at $x = 0$, $\Delta u(x)$ is undefined. We claim that we can choose constants c_1 and c_2 appropriately so that

$$-\Delta_x u = \delta_0$$

in the sense of distributions. Recall that δ_0 is the distribution which is defined as follows. For all $\phi \in D$,

$$(\delta_0, \phi) = \phi(0)$$

In *Question 3 & 4*, we prove this claim. For now, though, let us assume we can find constants c_1, c_2 such that u defined previously satisfies

$$-\Delta_x u = \delta_0$$

Let ϕ denote the solution to the above. Then define

$$v(x) = \int_{\mathbb{R}^n} \phi(x-y) f(y)\, dy$$

Formally, we compute the Laplacian of v as follows.

$$\begin{aligned} -\Delta_x v &= -\int_{\mathbb{R}^n} \Delta_x \phi\left(x-y\right) f\left(y\right)\, dy \\ &= -\int_{\mathbb{R}^n} \Delta_y \phi\left(x-y\right) f\left(y\right)\, dy \\ &= \int_{\mathbb{R}^n} \delta_x f\left(y\right)\, dy = f\left(x\right) \end{aligned}$$

That is, v is a solution of Poisson's equation! Of course, this set of equalities above is not entirely formal. We have not proven anything yet. However, we have motivated a solution formula for Poisson's equation from a solution to the above.

We now return to solving Poisson's equation

$$-\Delta u = f \qquad x \in \mathbb{R}^n$$

From our discussion above, we *expect* the function

$$v\left(x\right) = \int_{\mathbb{R}^n} \phi\left(x-y\right) f\left(y\right)\, dy$$

to give us a solution of Poisson's equation. We now prove that this is in fact true. First, we make a remark.

Remark: If we hope that the function v defined above solves Poisson's equation, we must first verify that this integral actually converges. If we assume f has compact support on some bounded set K in $\mathbb{R}^n$, then we see that

$$\int_{\mathbb{R}^n} \phi\left(x-y\right) f\left(y\right)\, dy \le |f|_{L^\infty} \int_K \left|\phi\left(x-y\right)\right|\, dy$$

If we additionally assume that f is bounded, then $|f|_{L^\infty} \le C$. Thus, we may verify that

$$\int_K \left|\phi\left(x-y\right)\right|\, dy < +\infty$$

on any compact set K.

Assume $f \in C^2\left(\mathbb{R}^n\right)$ and has compact support. Let

$$u\left(x\right) = \int_{\mathbb{R}^n} \phi\left(x-y\right) f\left(y\right)\, dy$$

where ϕ is the fundamental solution of Laplace's equation. Then, $u \in C^2\left(\mathbb{R}^n\right)$ and $-\Delta u = f$ in $\mathbb{R}^n$.

Proof: By a change of variables, we write

$$u\left(x\right) = \int_{\mathbb{R}^n} \phi\left(x-y\right) f\left(y\right)\, dy = \int_{\mathbb{R}^n} \phi\left(y\right) f\left(x-y\right)\, dy$$

Let $e_i = \left(\ldots, 0, 1, 0, \ldots\right)$ be the unit vector in $\mathbb{R}^n$ with 1 in the i^{th} slot. Then

$$\frac{u\left(x+he_i\right) - u\left(x\right)}{h} = \int_{\mathbb{R}^n} \phi\left(y\right) \left[\frac{f\left(x+he_i-y\right) - f\left(x-y\right)}{h}\right] dy$$

Now $f \in C^2$ implies

$$\frac{f\left(x+he_i-y\right) - f\left(x-y\right)}{h} \to \frac{\partial f}{\partial x_i}\left(x-y\right) \text{ as } h \to 0$$

uniformly on $\mathbb{R}^n$. Therefore,

$$\frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} \phi(y) \frac{\partial f}{\partial x_i}(x-y)\, dy$$

Similarly,

$$\frac{\partial^2 u}{\partial x_i x_j}(x) = \int_{\mathbb{R}^n} \phi(y) \frac{\partial^2 f}{\partial x_i x_j}(x-y)\, dy$$

This function is continuous because the right-hand side is continuous. By the above calculations and the arguments presented in *Question 3 & 4*, we see that

$$\begin{aligned}
\Delta_x u(x) &= \int_{\mathbb{R}^n} \phi(y)\, \Delta_x f(x-y)\, dy \\
&= \int_{\mathbb{R}^n} \phi(y)\, \Delta_y f(x-y)\, dy \\
&= -f(x)
\end{aligned}$$

To solve the differential equation $u_{xx} = f(x)$ with $x \in \mathbb{R}$ and $u(x) = 0$ at $x = \pm\infty$, we may use the function $s(x)$ from the previous problem to construct the solution using the method of Green's functions. In this case, we assume that $f(x)$ has a compact support and is smooth.

The solution $u(x)$ may be expressed as the convolution of $s(x)$ with the source term $f(x)$, as follows:

$$u(x) = \int_{-\infty}^{\infty} s(x-y)\, f(y)\, dy$$

This is the case, considering that $s_{xx} = \delta(x)$. When we substitute the expression for $s(x)$ into the integral, we have the following.

$$u(x) = \int_{-\infty}^{\infty} \left(\frac{1}{2}|x-y| + \gamma(x-y) + \beta\right) f(y)\, dy$$

Since $f(x)$ has compact support, the integral may be limited to the region where $f(y)$ is non-zero. Therefore, we can write the integral as

$$u(x) = \int_{a}^{b} \left(\frac{1}{2}|x-y| + \gamma(x-y) + \beta\right) f(y)\, dy$$

This is equivalent to the evaluation of the integral over the non-zero support of $f(y)$, where $[a, b]$ is the compact support of $f(x)$. Since $f(x)$ has compact support, the integrals are over a finite region, so this solution satisfies the boundary solutions. The specific solution depends on the form of $f(x)$ and while the expression provides the general Green's function solution using $s(x)$. That is, the expression represents the solution $u(x)$ for the given differential equation with the specified boundary conditions and assumptions. The constants γ and β are determined by the constants of integration in the solution for $s(x)$.

Question 7

Summarize the Cauchy principle value (check Wikipedia) and explain the meaning of the following integral

$$\int_{\mathbb{R}} \frac{f(x)}{x}\, dx$$

where $f \in \mathcal{C}_0^\infty = \mathcal{C}_c^\infty$.

The following is according to Wikipedia.

Cauchy Principal Value The Cauchy principal value, named after Augustin Louis Cauchy, is a method for assigning values to certain improper integrals which would otherwise be undefined. In this method, a singularity on an integral interval is avoided by limiting the integral interval to the singularity (so the singularity is not covered by the integral).

Depending on the type of singularity in the integrand f, the Cauchy principal value is defined according to the following rules.

For a singularity at a finite number b,

$$\lim_{\varepsilon \to 0^+} \left[\int_a^{b-\varepsilon} f(x)\, dx + \int_{b+\varepsilon}^{c} f(x)\, dx \right]$$

with $a < b < c$ and where b is the difficult point, at which the behavior of the function f is such that $\int_a^b f(x)\, dx = \pm\infty$ for any $a < b$ and $\int_b^c f(x)\, dx = \pm\infty$ for any $b < c$.

For a singularity at infinity (∞),

$$\lim_{a \to \infty} \int_{-a}^{a} f(x)\, dx$$

where $\int_{-\infty}^{0} f(x)\, dx = \pm\infty$ and $\int_0^\infty f(x)\, dx = \pm\infty$.

In some cases it is necessary to deal simultaneously with singularities both at a finite number b and at infinity. This is usually done by a limit of the form

$$\lim_{\eta \to 0^+} \lim_{\varepsilon \to 0^+} \left[\int_{b-\frac{1}{\eta}}^{b-\varepsilon} f(x)\, dx + \int_{b+\varepsilon}^{b+\frac{1}{\eta}} f(x)\, dx \right]$$

In those cases where the integral may be split into two independent finite limits,

$$\lim_{\varepsilon \to 0^+} \left| \int_a^{b-\varepsilon} f(x)\, dx \right| < \infty$$

and

$$\lim_{\eta \to 0^+} \left| \int_{b+\eta}^{c} f(x)\, dx \right| < \infty$$

then the function is integrable in the ordinary sense. The result of the procedure for principal value is the same as the ordinary integral; since it no longer matches the definition, it is technically not a "principal value."

In a broader sense, the principal value can be defined for a wide class of singular integral kernels on the Euclidean space $\mathbb{R}^n$ If K has an isolated singularity at the origin, but is an otherwise "nice" function, then the principal-value distribution is defined on compactly supported smooth functions by

$$[\mathrm{PV}(K)](f) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n/B_\varepsilon(0)} f(x)\,K(x)\;dx$$

Such a limit may not be well defined, or, being well-defined, it may not necessarily define a distribution. It is, however, well-defined if K is a continuous homogeneous function of degree $-n$ whose integral over any sphere centered at the origin vanishes.

In summary, the Cauchy principal value is a method to assign values to certain improper integrals which would otherwise be undefined due to singularities. The idea is to evaluate the integral over the region excluding a small interval around the interval, and take the limit as the size of the excluded interval goes to zero.

For the given integral, $f \in C_0^\infty = C_c^\infty$, which indicates that $f(x)$ is a smooth function with compact support. This means that $f(x)$ is infinitely differentiable and zero outside of a bounded interval. The integrand $\frac{f(x)}{x}$ has a singularity at $x = 0$.

To compute the Cauchy principal value, we evaluate the integral over the region excluding the singularity, by excluding a small interval $[-\varepsilon, \varepsilon]$ around $x = 0$.

$$\mathrm{PV}\int_{\mathbb{R}} \frac{f(x)}{x}\,dx = \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\varepsilon} \frac{f(x)}{x}\,dx + \int_{\varepsilon}^{\infty} \frac{f(x)}{x}\,dx\right]$$

The meaning of this integral is that we are integrating the function $\frac{f(x)}{x}$ over the entire real line, while excluding the singularity at 0. By taking the limit as the exclude interval (around 0) shrinks to a point, the integral converges to a finite value, despite the integrand blowing up at $x = 0$. This captures the Cauchy principal value of the integral, as we assign a finite value to integrals with singularities by excluding the singularity.

For an explanation on the relation to the Hilbert transform, please see the following.

The integral $\int_{\mathbb{R}} \frac{f(x)}{x}\,dx$ relates to the Hilbert transform, which is an integral transform that takes a function $f(x)$ and produces a new function $Hf(x)$, defined as follows:

$$Hf(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}\,dt$$

Intuitively, the Hilbert transform reveals the "instantaneous frequency" content of a signal $f(x)$ by taking the convolution or weighted average of $f(x)$ with the function $\frac{1}{\pi(x-t)}$. The following represent key properties of the Hilbert transform:

- It applies to functions $f(x)$ that are in $C_0^\infty(\mathbb{R})$, the space of smooth, compactly-supported functions on the real line, which ensures that the integral converges.
- It is a linear operator, meaning $H(af(x) + bg(x)) = aHf(x) + bHg(x)$ for constants a, b.
- It is its own inverse, meaning $H(Hf(x)) = -f(x)$.
- It shifts the phase of $f(x)$ by 90 degrees (in the frequency domain), thus if $f(x)$ is a sinusoid, $Hf(x)$ is a cosine wave/function.

In summary, the given integral relates to the Hilbert transform of the function $f(x)$, which is a fundamental operator in mathematical analysis with applications in signal processing and physics (specifically for analyzing waveforms and wave propagation).

Question 8

Explain the meaning of the following integral

$$\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\, dy$$

where $f \in \mathcal{C}_0^\infty = \mathcal{C}_c^\infty$.

As previously, for the given integral, $f \in C_0^\infty = C_c^\infty$, which indicates that $f(x)$ is a smooth function with compact support. This means that $f(x)$ is infinitely differentiable and zero outside of a bounded region/interval.

The integral shown below represents the convolution of the function f with the kernel $\frac{1}{|x-y|}$:

$$\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\, dy$$

This is the integral representation of the Newton potential. The integration is over all of $\mathbb{R}^3$, hence, it computes a 3D potential by integrating over all possible values of y, for fixed x. The integrand $\frac{f(y)}{|x-y|}$ represents the fundamental solution of the Laplace equation in $\mathbb{R}^3$. In this sense, it gives the potential at point x generated by a point source at y, where $f(y)$ represents a volume density function (which is smooth with compact support).

In other words, the integrand $\frac{f(y)}{|x-y|}$ represents the kernel function $\frac{1}{|x-y|}$ centered at x, multiplied by the value of f at point y. By integrating the product of the density and the fundamental solution over all space $\mathbb{R}^3$, the total potential at x is calculated as a sum of the contributions from all the volume source points y.

The test function $f(y)$ in $C_0^\infty\left(\mathbb{R}^3\right)$, represents the space of smooth and compactly supported functions on $\mathbb{R}^3$. The kernel $\frac{1}{|x-y|}$ is what we are convolving f with and, as stated, represents the fundamental solution of the Laplace equation in 3D ($\mathbb{R}^3$). To this end, it satisfies $\Delta\left(\frac{1}{|x-y|}\right) = -4\pi\delta(x-y)$ in the distributional sense.

Physically, this integral may be interpreted as computing the potential at a point x due to charge distribution $f(y)$ in $\mathbb{R}^3$. The fundamental solution kernel represents the potential at x due to a point charge at y. Thus, in electromagnetism, this integral represents computing the electric potential at x due to a charge distribution $f(y)$, while in gravity, it represents computing the gravitational potential. Typically, solutions to the Laplace equation may be represented through integrals of this form.

Mathematically, this integral demonstrates the power of fundamental solutions and integral operator techniques for solving partial differential equations. The Laplace equation is translation invariant, so its fundamental solution only depends on $|x-y|$, which allows solutions for general $f(y)$ to be represented through convolution-type integrals.

In summary, this integral represents the convolution of f with the kernel $\frac{1}{|x-y|}$, which produces a smoothed version of f evaluated at each point x, representing the Newtonian potential generated by a charge distribution f at a point x. The volume density distribution f is smooth with compact support. The integral captures the average value of f in the neighborhood of x weighted by distance.

Question 9

Let $u(x)$ be the function representing the integral of the previous problem. Find Δu.

Let $u(x)$ be the function representing the convolution integral:

$$u(x) = \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\, dy$$

where $f \in C_0^\infty(\mathbb{R}^3)$. The goal is to find Δu, which is the Laplacian of $u(x)$. The Laplacian ($\Delta = \nabla^2$) in 3D is defined as:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Applying the Laplacian operator to $u(x)$, we find the following:

$$\begin{aligned}
\Delta u(x) &= \Delta \left(\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}, dy \right) \\
&= \frac{\partial^2}{\partial x^2} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\, dy + \frac{\partial^2}{\partial y^2} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\, dy + \frac{\partial^2}{\partial z^2} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\, dy \\
&= \int_{\mathbb{R}^3} \Delta_x \left(\frac{f(y)}{|x-y|} \right) dy \\
&= \int_{\mathbb{R}^3} f(y)\, \Delta_x \left(\frac{1}{|x-y|} \right) dy
\end{aligned}$$

In particular, the Laplacian is a linear operator and is able to pass under the integral sign. In this case, the Laplacian of the term in the second line only acts on x, which is why we are able to bring the Laplacian inside the integral.

Now, we know that $\Delta_x \left(\frac{1}{|x-y|} \right) = -4\pi\delta(x-y)$, where δ is the Dirac delta function. Thus, the solution, given as the Laplacian of $u(x)$, is given as follows:

$$\Delta u(x) = -4\pi f(x)$$

Question 10

Watch the lecture by former Dartmouth Professor Alex Barnett about boundary integral methods. Summarize the lecture.

The lecture by former Dartmouth Professor Alex Barnett covers boundary integral methods, which are numerical techniques for solving partial differential equations by reformulating them as boundary integral equations. The key idea is to represent the solution as "surface" integral over the boundary rather than a "volume" integral over the entire domain.

Boundary integral methods (as a class of numerical techniques) convert boundary value problems defined over a domain into integral equations defined on the boundary. This reduces the problem dimensionality and allows complex geometries to be more easily handled compared to finite element/difference methods. The key ideas/aspects are as follows.

- Boundary integral methods reformulate partial differential equations defined over a "volume" into integral equations defined only over the boundary "surface", thereby reducing the dimensionality of the problem, which reduces computational cost.
- To convert a partial differential equation into an integral formulation, Green's functions that satisfy the governing partial differential equation are used to construct a boundary-only framework.
- Boundary integral methods are well-suited to problems with unbounded (infinite or semi-infinite) domains, irregular geometries, and symmetry, as they avoid discretization of the full domain.
- The integral formulation naturally handles discontinuities in material properties across boundaries.

In summary, boundary integral methods are important for problems where an infinite domain can be reduced to a boundary formulation.

Typically, implementation of boundary integral methods requires discretizing boundaries into boundary elements and approximating geometry, unknown functions, and boundary conditions, which was the primary discussion of the lecture. While numerical integration is used to evaluate the boundary integrals, there are drawbacks, specifically that computationally dense matrix equations and singular integrals complicate the evaluation.

The goal is to enhance accuracy and guarantee stronger convergence for various boundary problems, such as $\left(\Delta + k^2\right) u = 0$ in Ω (interior) or $\mathbb{R}^d / \bar{\Omega}$ (exterior) with $u = f$ on $\partial\Omega$. There are various quadrature methods that approximate an integral using a sum of sampled points and weights, where the boundary is parameterized.

Aside: The fundamental solution follows from the theory behind differential operators.

$$\left(\Delta_x + k^2\right) \phi\left(x, y\right) = \delta\left(x - y\right)$$

As an example, consider the weak singularity that comes from a log kernel. This is just one of many examples of singularities that complicate numerical computation. The lecture discusses 5 ideas for computing the elements of the matrix A that is used in approximating the integral representation.

Idea #1: When applying numerical integration around the singularity, keep the same nodes, but change the weights of the nodes. There is a small region around the target point (with the singularity) where the weights should be modified.

Idea #2: Use the following analytic split for the kernel function. $K(t,s) = K_1(t,s)\log\left(4\sin\left(\frac{2t-s}{2}\right)\right) + K_2(t,s)$ In this case, the kernels are smooth, and the other expression is a known Fourier series, indicating we may use the trapezoid rule for approximation.

Idea #3: Write the function with the domain $(-1,1)$ in the complex plane, as a function of z. In this case, we are able to expand the function in the complex plane instead of with respect to s, which represents the arc length.

Idea #4: In the "dagger" integral that is used for approximation, use auxiliary nodes k, specifically a local high-order interpolation from the local solution of the density function.

Idea #5: Take a partial differential equation-based closed evaluation, as the limit of the potential.

Toward the end of the lecture, Professor Alex Barnett explained the following: If you know the density on a curve, the value of a function at quadrature nodes, and how to take the limit of a point (of a density) back on the boundary (quadrature nodes) approaching from the correct side, that tells you how to fill in the matrix elements for approximation.

The QBX method is dimension independent, and allows you to avoid dealing with difficult quadratures in 3D if you know how to express the local expansion and the addition function for a given problem.