Mathematics 60 - Probability Theory

Carter Kruse

Homework 4

Chapter 4.1: 7, 16, 26, 38

Chapter 4.2: 3, 5(c)

Chapter 4.1, Question 7

A coin is tossed twice. Consider the following events.

A: Heads on the first toss.

B: Heads on the second toss.

C: The two tosses come out the same.

(a) Show that A, B, C are pairwise independent but not independent.

Let us consider the following pairs $(A, B), (A, C), (B, C)$:

$$P(A \cap B) = P(HH) = \frac{1}{4}$$
$$P(A)P(B) = P(HH \text{ or } HT)\,P(HH \text{ or } TH) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

$$P(A \cap C) = P(HH) = \frac{1}{4}$$
$$P(A)P(C) = P(HH \text{ or } HT)\,P(HH \text{ or } TT) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

$$P(B \cap C) = P(HH) = \frac{1}{4}$$
$$P(B)P(C) = P(HH \text{ or } TH)\,P(HH \text{ or } TT) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

As $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, and $P(B \cap C) = P(B)P(C)$, the events A, B, C are pairwise independent.

For A, B, C, we also have the following:

$$P(A \cap B \cap C) = P(HH) = \frac{1}{4}$$
$$P(A)P(B)P(C) = P(HH \text{ or } HT)\,P(HH \text{ or } TH)\,P(HH \text{ or } TT) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

As $P(A \cap B \cap C) \neq P(A)P(B)P(C)$, the events A, B, C are not independent.

(b) Show that C is independent of A and B but not of $A \cap B$.

Let us consider the following pairs $(A, C), (B, C)$:

$$P(A \cap C) = P(HH) = \frac{1}{4}$$

$$P(A)P(C) = P(HH \text{ or } HT)\, P(HH \text{ or } TT) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

$$P(B \cap C) = P(HH) = \frac{1}{4}$$
$$P(B)P(C) = P(HH \text{ or } TH)\, P(HH \text{ or } TT) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

As $P(A \cap C) = P(A)P(C)$ and $P(B \cap C) = P(B)P(C)$, the events (A and C) and (B and C) are independent. Thus, C is independent of A and B.

To demonstrate that C is not independent of $A \cap B$, consider the following:

$$P(C \cap (A \cap B)) = P(HH) = \frac{1}{4}$$
$$P(C)\, P(A \cap B) = P(HH \text{ or } TT)\, P(HH) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8}$$

As $P(C \cap (A \cap B)) \neq P(C)\, P(A \cap B)$, the event C is not independent of $A \cap B$.

Chapter 4.1, Question 16

Prove that for any three events A, B, C, each having positive probability, and with the property that $P(A \cap B) > 0$,

$$P(A \cap B \cap C) = P(A)\, P(B|A)\, P(C|A \cap B)\,.$$

For the solution, we simply need to consider the following property: we call $P(F|E)$ the conditional probability of F occurring given that E occurs, and compute it using the formula

$$P(F|E) = \frac{P(F \cap E)}{P(E)}.$$

This may alternatively be expressed as

$$P(E)P(F|E) = P(F \cap E)\,.$$

Using this, we may simplify the right-hand side of the initial expression as follows:

$$P(A \cap B \cap C) = P(A \cap B)\, P(C|A \cap B)\,.$$

Knowing that $P(A \cap B) > 0$, we may further simplify this expression to be

$$P(A \cap B \cap C) = P(A \cap B \cap C)\,,$$

which proves what we hoped to achieve. $\square$

Chapter 4.1, Question 26

Suppose that A and B are events such that $P(A|B) = P(B|A)$ and $P(A \cup B) = 1$ and $P(A \cap B) > 0$. Prove that $P(A) > \frac{1}{2}$.

By the inclusion-exclusion principle, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)\,.$$

This property will be fundamental to the main steps in the proof.

Further, $P(F|E)$ is the the conditional probability of F occurring given that E occurs, computed using the formula

$$P(F|E) = \frac{P(F \cap E)}{P(E)}.$$

Thus, we know that the statement

$$P(A|B) = P(B|A)$$

implies

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} = P(B|A),$$

so $P(A) = P(B)$.

Therefore, using the expression from the inclusion-exclusion principle, with the knowledge that $P(A) = P(B)$ and $P(A \cup B) = 1$, we have

$$1 = P(A) + P(A) - P(A \cap B).$$

Simplifying this expression by combining terms and adding $P(A \cap B)$ to both sides, we have

$$2P(A) = 1 + P(A \cap B),$$

where $P(A \cap B) > 0$. Thus, we deduce that $2P(A) > 1$, and so

$$P(A) > \frac{1}{2},$$

as we hoped to achieve. □

Chapter 4.1, Question 38

A fair coin is tossed three times. Let X be the number of heads that turn up on the first two tosses and Y the number of heads that turn up on the third toss. Give the distribution of the following:

(a) The random variables X and Y.

The distribution function of the random variables X and Y is given by

$$m_X(\omega) = \begin{cases} \frac{1}{4} & \text{for } \omega = 0 \\ \frac{1}{2} & \text{for } \omega = 1 \\ \frac{1}{4} & \text{for } \omega = 2 \end{cases}$$

$$m_Y(\omega) = \begin{cases} \frac{1}{2} & \text{for } \omega = 0 \\ \frac{1}{2} & \text{for } \omega = 1 \end{cases}$$

(b) The random variable $Z = X + Y$.

The distribution function of the random variable $Z = X + Y$ is given by

$$m_Z(\omega) = \begin{cases} \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) & \text{for } \omega = 0 \\ \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) & \text{for } \omega = 1 \\ \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) & \text{for } \omega = 2 \\ \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) & \text{for } \omega = 3 \end{cases}$$

This results in the following distribution function

$$m_Z(\omega) = \begin{cases} \frac{1}{8} & \text{for } \omega = 0 \\ \frac{3}{8} & \text{for } \omega = 1 \\ \frac{3}{8} & \text{for } \omega = 2 \\ \frac{1}{8} & \text{for } \omega = 3 \end{cases}$$

(c) The random variable $W = X - Y$.

The distribution function of the random variable $W = X + Y$ is given by

$$m_W(\omega) = \begin{cases} \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) & \text{for } \omega = -1 \\ \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) & \text{for } \omega = 0 \\ \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) & \text{for } \omega = 1 \\ \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) & \text{for } \omega = 2 \end{cases}$$

This results in the following distribution function

$$m_W(\omega) = \begin{cases} \frac{1}{8} & \text{for } \omega = -1 \\ \frac{3}{8} & \text{for } \omega = 0 \\ \frac{3}{8} & \text{for } \omega = 1 \\ \frac{1}{8} & \text{for } \omega = 2 \end{cases}$$

Chapter 4.2, Question 3

The Acme Super light bulb is known to have a useful life described by the density function

$$f(t) = 0.01e^{-0.01t},$$

where t is time measured in hours.

(a) Find the *failure rate* of this bulb (see Exercise 2.2.6).

According to Exercise 2.2.6, when we assume that a new light bulb will burn out after t hours, where t is chosen from $[0, \infty)$ with an exponential density $f(t) = \lambda e^{-\lambda t}$, the *failure rate* of the bulb is λ. This means that for the Acme Super light bulb, the failure rate is 0.01.

(b) Find the *reliability* of this bulb after 20 hours.

According to Exercise 2.2.6, the probability that the bulb will *not* burn out before T hours is called the *reliability* of the bulb.

Assuming that $\lambda = 0.01$, the probability that the bulb will *not* burn out before T hours is given by

$$1 - P(T),$$

where $P(T)$ denotes the probability that the bulb burns out before T hours as

$$\int_0^T f(t)\,dt.$$

To this end, the probability that the bulb will *not* burn out before T hours is

$$P(\tilde{T}) = 1 - \int_0^T f(t)\,dt = 1 - \left[-e^{-\lambda t}\right]_0^T,$$

which evaluates to

$$P(\tilde{T}) = 1 - \left(-e^{-0.01T} + 1\right),$$

which simplifies to

$$P(\tilde{T}) = e^{-0.01T}.$$

Inputting the value of 20 hours for T, the reliability of the bulb is $e^{-0.2}$.

(c) Given that it lasts 20 hours, find the probability that the bulb lasts another 20 hours.

To express the probability that the bulb lasts another 20 hours, given that it lasts 20 hours, we have the following:

$$P(t > 40 | t > 20) = \frac{P(t > 40)}{P(t > 20)}$$

We have already determined that the reliability of the bulb for 20 hours is $e^{-0.2}$, so the expression above evaluates to

$$e^{0.2}\int_{40}^{\infty} 0.01e^{-0.01t}\,dt.$$

Solving this expression by evaluating the integral yields

$$e^{0.2}\left[-e^{-0.01t}\right]_{40}^{\infty},$$

which is equivalent to

$$e^{0.2}\left(e^{-0.4}\right).$$

Thus, given that it lasts 20 hours, the probability that the bulb lasts another 20 hours is $e^{-0.2}$.

(d) Find the probability that the bulb burns out in the forty-first hour, given that it lasts 40 hours.

To express the probability that the bulb burns out in the forty-first hour, given that it lasts 40 hours, we have the following:

$$P\left(40 < t < 41 | t > 40\right) = \frac{P\left(40 < t < 41\right)}{\left(t > 40\right)}$$

Using our definition of probability (with integration), we set up the following expression:

$$\frac{\int_{40}^{41} 0.01e^{-0.01t}\,dt}{\int_{40}^{\infty} 0.01e^{-0.01t}\,dt}$$

Solving this expression by evaluating the integrals yields

$$\frac{\left[-e^{-0.01t}\right]_{40}^{41}}{\left[-e^{-0.01t}\right]_{40}^{\infty}}$$

which is equivalent to

$$\frac{e^{-0.40} - e^{-0.41}}{e^{-0.40}}.$$

Thus, given that it lasts 40 hours, the probability that the bulb burns out in the forty-first hour is $1 - e^{-0.01}$

Chapter 4.2, Question 5(c)

Suppose you choose two numbers x and y, independently at random from the interval $[0, 1]$. Given that their sum lies in the interval $[0, 1]$, find the probability that $\max\{x, y\} < \frac{1}{2}$.

In the context of continuous conditional probability, if X is a continuous random variable with density function $f(x)$, and if E is an event with positive probability, we define a conditional density function by the formula

$$f\left(x|E\right) = \begin{cases} f(x)/P(E) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Then for any event F, we have

$$P\left(F|E\right) = \int_F f\left(x|E\right)\,dx.$$

The expression $P\left(F|E\right)$ is called the conditional probability of F given E. As in the previous section, it is easy to obtain an alternative expression for this probability:

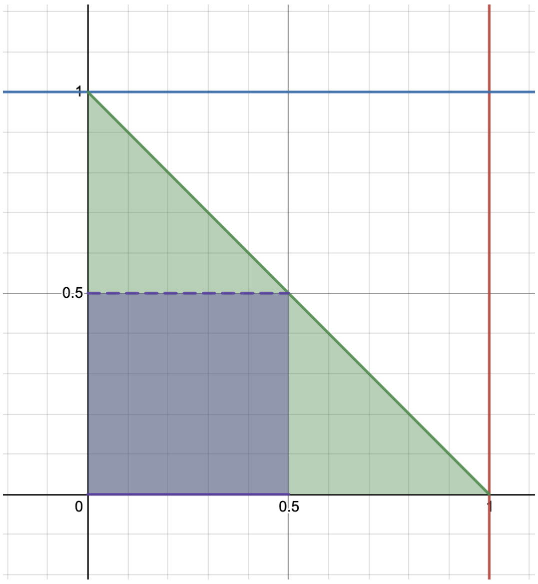
$$P\left(F|E\right) = \int_F f\left(x|E\right)\,dx = \int_{E\cap F} \frac{f(x)}{P(E)}\,dx = \frac{P\left(E\cap F\right)}{P\left(E\right)}$$

Applying this understanding, we start by recognizing that if x and y are chosen at random from the interval $[0, 1]$ with uniform density, the pair (x, y) represents a point chosen at random from the unit square with $x \in [0, 1]$ and $y \in [0, 1]$.

Further, the probability that $\max\{x, x\} < \frac{1}{2}$ may be stated as

$$P\left(\max\{x, y\} < \frac{1}{2}\right) = P\left(x < \frac{1}{2} \cap y < \frac{1}{2}\right).$$

To determine the probability that $\max\{x, y\} < \frac{1}{2}$ *conditional on* $x + y \in [0, 1]$, we consider the following geometry.



Let us assign the event F to be $\max\{x, y\} < \frac{1}{2}$ and the event E to be $x + y \in [0, 1]$, such that $x + y \leq 1$. Then the probability associated with event F is represented by the purple-shaded region and the probability associated with event E is represented by the green (and purple)-shaded region.

The probability that $\max\{x, y\} < \frac{1}{2}$, given that their sum lies in the interval $[0, 1]$ is thus equivalent to

$$P(F|E) = \frac{P(E \cap F)}{P(E)},$$

as we know from the description of continuous conditional probability.

When considering the areas of the shaded regions, we find that

$$P(E \cap F) = \frac{1}{4} \text{ and } P(E) = \frac{1}{2},$$

so

$$\frac{P(E \cap F)}{P(E)} = \frac{1/4}{1/2} = 1/2.$$

Thus, we find that when choosing two numbers x and y, independently at random from the interval $[0, 1]$, given that their sum lies in the interval $[0, 1]$, the probability that $\max\{x, y\} < \frac{1}{2}$ is equal to $\frac{1}{2}$.