# Mathematics 60 - Probability Theory

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## Homework 1

Chapter 1.2: 7, 9, 10, 12, 15, 23

### Chapter 1.2, Question 7

Let $A$ and $B$ be events such that $P(A \cap B) = \frac{1}{4}$, $P(\tilde{A}) = \frac{1}{3}$ and $P(B) = \frac{1}{2}$. What is $P(A \cup B)$?

By the property

$$P(\tilde{A}) = 1 - P(A) \qquad \text{for every } A \subset \Omega,$$

and when using $P(\tilde{A}) = \frac{1}{3}$, we know

$$P(A) = \frac{2}{3}.$$

By the inclusion-exclusion principle, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Inputting the values given into this equation, we find

$$P(A \cup B) = \frac{2}{3} + \frac{1}{2} - \frac{1}{4},$$

which simplifies to

$$P(A \cup B) = \frac{11}{12}.$$

### Chapter 1.2, Question 9

A student must choose exactly two out of three electives: art, French, and mathematics. He chooses art with probability $\frac{5}{8}$, French with probability $\frac{5}{8}$, and art and French together with probability $\frac{1}{4}$. What is the probability that he chooses mathematics? What is the probability that he chooses either art or French?

As the student must choose exactly two out of three electives, the probability that he/she does NOT choose mathematics is *equivalent to* the probability that he/she chooses art and French together.

By the property

$$P(\tilde{A}) = 1 - P(A) \qquad \text{for every } A \subset \Omega,$$

we know that the probability that he/she chooses mathematics and the probability that he/she does not choose mathematics must sum to one. Thus, the probability that he/she chooses mathematics is

$$P(\text{mathematics}) = 1 - \frac{1}{4},$$

which simplifies to

$$P(\text{mathematics}) = \frac{3}{4}.$$

To determine the probability that the student chooses either art or French, we use the inclusion-exclusion principles as before, which is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

In this situation, the expression is

$$P(\text{art} \cup \text{French}) = P(\text{art}) + P(\text{French}) - P(\text{art} \cap \text{French}).$$

Inputting the appropriate values, we find that

$$P(\text{art} \cup \text{French}) = \frac{5}{8} + \frac{5}{8} - \frac{1}{4},$$

which simplifies to

$$P(\text{art} \cup \text{French}) = 1.$$

Intuitively, this makes sense, as when a student chooses exactly two out of three electives, one (if not both) of them will always be art or French.

## Chapter 1.2, Question 10

For a bill to come before the President of the United States, it must be passed by both the House of Representatives and the Senate. Assume that, of the bills presented to these two bodies, 60 percent pass the House, 80 percent pass the Senate, and 90 percent pass at least one of the two. Calculate the probability that the next bill presented to the two groups will come before the President.

To determine the probability that the next bill presented to the two groups will come before the President, we must calculate the probability that it is passed by both the House of Representatives and the Senate. To do so with the given information, we use the inclusion-exclusion principle, which is stated as follows:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

In this situation, the expression is

$$\begin{aligned} P(\text{House of Representatives} \cup \text{Senate}) = {} & P(\text{House of Representatives}) + P(\text{Senate}) \\ & - P(\text{House of Representatives} \cap \text{Senate}) \end{aligned}$$

Inputting the appropriate values, we find that

$$\frac{9}{10} = \frac{6}{10} + \frac{8}{10} - P(\text{House of Representatives} \cap \text{Senate}).$$

Solving for the unknown value, we find

$$P(\text{House of Representatives} \cap \text{Senate}) = \frac{5}{10},$$

which simplifies to

$$P(\text{House of Representatives} \cap \text{Senate}) = \frac{1}{2}.$$

## Chapter 1.2, Question 12

You offer $3:1$ odds that your friend Smith will be elected mayor of your city. What probability are you assigning to the event that Smith wins?

The statement that the odds are $r$ to $s$ ($r : s$) in favor of an event $E$ occurring is equivalent to the statement that

$$P(E) = \frac{\frac{r}{s}}{\frac{r}{s} + 1} = \frac{r}{r + s}.$$

Applying this to the situation put forth in the problem, the probability you are assigning to the event that Smith wins with $3:1$ odds is

$$P(E) = \frac{3}{3 + 1},$$

which simplifies to

$$P(E) = \frac{3}{4}.$$

## Chapter 1.2, Question 15

John and Mary are taking a mathematics course. The course has only three grades: A, B, and C. The probability that John gets a B is 0.3. The probability that Mary gets a B is 0.4. The probability that neither gets an A but at least one gets a B is 0.1. What is the probability that at least one gets a B but neither gets a C?

Let $B_J$ represent the probability that John gets a B, so $B_J = 0.3$. Let $B_M$ represent the probability that Mary gets a B, so $B_M = 0.4$.

There are three cases in which neither John nor Mary gets an A, but at least one gets a B as follows: $(J, M) = (B, B), (B, C), (C, B)$

The combined probability of these cases is given to be 0.1.

There are three cases in which at least one (John and/or Mary) gets a B, but neither gets a C as follows: $(J, M) = (B, B), (B, A), (A, B)$

Considering that there is overlap $(B, B)$ between the cases listed, let us assign a combined probability $x$ to the cases: $(B, A), (A, B)$.

The probability that either John or Mary (or both) get a B is given by

$$B_{J\cup M} = 0.3 + 0.4 = 0.7.$$

The probability that both John and Mary get a B is given by

$$B_{J\cap M} = (0.3)(0.4) = 0.12.$$

Thus, when solving for $x$, we have

$$B_{J\cup M} = 0.7 = 0.1 + x + 0.12,$$

so $x = 0.48$.

Returning to the cases in which at least one (John and/or Mary) gets a B, but neither gets a C ($(J, M) = (B, B), (B, A), (A, B)$), we find that the combined probability is calculated by $P((B, B)) + x = 0.12 + 0.48 = 0.6$.

Thus, the probability that at least one gets a B but neither gets a C is 0.6.

## Chapter 1.2, Question 23

Let $\Omega$ be the sample space

$$\Omega = \{0, 1, 2, ...\}$$

and define a distribution function by

$$m(j) = (1 - r)^j \, r$$

for some fixed $r$, $0 < r < 1$, and for $j = 0, 1, 2, ....$ Show that this is a distribution function for $\Omega$.

Let $X$ be a random variable which denotes the value of the outcome of a certain experiment. Let $\Omega$ be the sample space of the experiment (i.e., the set of all possible values of $X$, or equivalently, the set of all possible outcomes of the experiment.) A distribution function for $X$ is a real-valued function $m$ whose domain is $\Omega$ and which satisfies the following two properties:

$$m(\omega) \geq 0, \forall \omega \in \Omega$$

$$\sum_{\omega\in\Omega} m(\omega) = 1$$

In the case described above, in which the sample space is countably infinite, the second property is as follows:

$$\int_{\Omega} m(j) = \int_{0}^{\infty} m(j) = 1$$

To prove that the second property holds true for $m(j) = (1-r)^j\, r$ for some fixed $r$, $0 < r < 1$ and for $j = 0, 1, 2, \ldots$, we consider the infinite sum with the first term $(a_1)$ equal to $r$ and the ratio equal to $(1-r)$ (the infinite sum is geometric). Knowing that the sum of an infinite geometric series (having a ratio $(r^*)$ with an absolute value less than one) is $S = \frac{a_1}{1-r^*}$, we find that

$$\int_{\Omega} m(j) = \frac{r}{1-(1-r)} = 1.$$

This is only known because the absolute value of the ratio $(1-r)$ is necessarily less than 1, as indicated by $0 < r < 1$. Thus, the second property for a distribution function is satisfied.

To finalize demonstrating that $m(j)$ is a distribution function for $\Omega$, we must show that for all $j \in \Omega = \{0, 1, 2, \ldots\}$, $m(j) \geq 0$ (the first property for a distribution function). This naturally follows from the fact that $r$ is fixed, $0 < r < 1$. The factors of $m(j)$ will always be positive, as $(1-r) > 0$ and $r > 0$ for $0 < r < 1$. Thus, we know that for all $j \in \Omega$, $m(j) \geq 0$ and so the first property for a distribution function is satisfied.

Thus, both properties for a distribution function have been satisfied; $m(j) = (1-r)^j\, r$ is a distribution function for the sample space $\Omega = \{0, 1, 2, \ldots\}$ for some fixed $r$, $0 < r < 1$ and for $j = 0, 1, 2, \ldots$.