Mathematics 60 - Probability Theory

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Homework 2

Chapter 2.2: 2, 6, 8(d) and 8(e), 14, 16

Chapter 2.2, Question 2

Suppose you choose a real number X from the interval $[2, 10]$ with a density function of the form

$$f(x) = Cx$$

where C is a constant.

(a) Find C.

To determine the value of C, we note that for $f(x)$ to be a density function, the area under the curve in the interval must equal 1. This means

$$\int_2^{10} Cx \, dx = 1,$$

so when integrating, we find

$$\left[\frac{1}{2}Cx^2\right]_2^{10} = 1.$$

Solving for the value of C, we have as follows:

$$\frac{1}{2}C(100) - \frac{1}{2}C(4) = 1$$

$$\frac{1}{2}C(96) = 1$$

$$48C = 1$$

$$C = \frac{1}{48}$$

(b) Find $P(E)$, where $E = [a, b]$ is a subinterval of $[2, 10]$.

If $E = [a, b]$ is a subinterval of $[2, 10]$, we may use the density function to determine the probability associated with the subinterval by calculating the area of the region. This results in

$$\int_a^b \frac{1}{48} x \, dx,$$

which simplifies to

$$\frac{1}{96}\left(b^2 - a^2\right).$$

Thus, $P(E) = \frac{1}{96}\left(b^2 - a^2\right)$.

(c) Find $P(X > 5)$, $P(X < 7)$, and $P\left(X^2 - 12X + 35 > 0\right)$.

The value of $P(X > 5)$ is given by

$$\int_5^{10} \frac{1}{48} x \, dx,$$

which simplifies to

$$\frac{1}{96}\left(10^2 - 5^2\right).$$

Thus,

$$P(X > 5) = \frac{75}{96} = \frac{25}{32}$$

The value of $P(X < 7)$ is given by

$$\int_2^{7} \frac{1}{48} x \, dx,$$

which simplifies to

$$\frac{1}{96}\left(7^2 - 2^2\right).$$

Thus,

$$P(X < 7) = \frac{45}{96} = \frac{15}{32}$$

To calculate the value of $P(X^2 - 12X + 35 > 0)$, we first break down the inner expression as

$$(X - 7)(X - 5) > 0.$$

For this to be the case, we must have $X > 7$ or $X < 5$. Thus, the probability may be expressed as

$$P(X > 7) \cup P(X < 5).$$

Knowing the values for $P(X < 7)$ and $P(X > 5)$, we determine the following:

$$P(X > 7) \cup P(X < 5) = \left(1 - \frac{15}{32}\right) + \left(1 - \frac{25}{32}\right)$$

Therefore,

$$P\left(X^2 - 12X + 35 > 0\right) = \frac{17}{32} + \frac{7}{32} = \frac{24}{32},$$

so that

$$P\left(X^2 - 12X + 35 > 0\right) = \frac{3}{4}.$$

Chapter 2.2, Question 6

Assume that a new light bulb will burn out after t hours, where t is chosen from $[0, \infty)$ with an exponential density

$$f(t) = \lambda e^{-\lambda t}.$$

In this context, λ is often called the *failure rate* of the bulb.

(a) Assume that $\lambda = 0.01$, and find the probability that the bulb will *not* burn out before T hours. This probability is often called the *reliability* of the bulb.

Assuming that $\lambda = 0.01$, the probability that the bulb will *not* burn out before T hours is given by

$$1 - P(T),$$

where $P(T)$ denotes the probability that the bulb burns out before T hours as

$$\int_0^T f(t) \, dt.$$

To this end, the probability that the bulb will *not* burn out before T hours is

$$P(\tilde{T}) = 1 - \int_0^T f(t)\,dt = 1 - \left[-e^{-\lambda t}\right]_0^T ,$$

which evaluates to

$$P(\tilde{T}) = 1 - \left(-e^{-0.01T} + 1\right),$$

which simplifies to

$$P(\tilde{T}) = e^{-0.01T}.$$

(b) For what T is the reliability of the bulb $= \frac{1}{2}$?

To determine for what T the reliability of the bulb, $P(\tilde{T}) = \frac{1}{2}$, we simply use our previous answer as follows:

$$\frac{1}{2} = e^{-0.01T}$$

Solving for T, we have

$$\ln\left(\frac{1}{2}\right) = -0.01T$$

so

$$T = \frac{\ln\left(\frac{1}{2}\right)}{-0.01}.$$

This simplifies to

$$T = 100\ln(2).$$

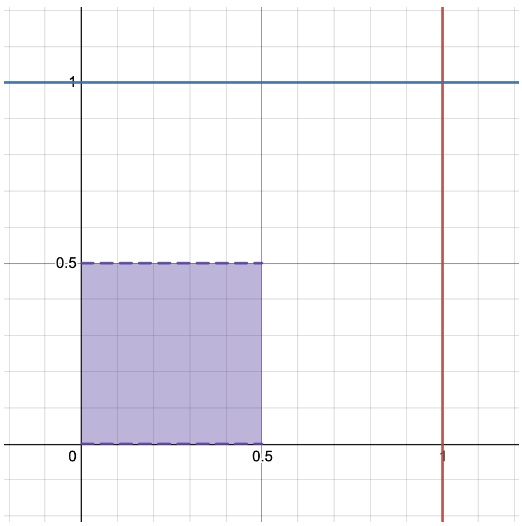
Chapter 2.2, Question 8(d)

Choose independently two numbers B and C at random from the interval $[0,1]$ with uniform density. Note that the point (B,C) is then chosen at random in the unit square. Find the probability that $\max\{B,C\} < \frac{1}{2}$.

If B and C are chosen at random from the interval $[0,1]$ with uniform density, the pair (B,C) represents a point chosen at random from the unit square with $x \in [0,1]$ and $y \in [0,1]$. Further, the probability that $\max\{B,C\} < \frac{1}{2}$ may be stated as

$$P\left(\max\{B,C\} < \frac{1}{2}\right) = P\left(B < \frac{1}{2} \cap C < \frac{1}{2}\right).$$

Geometrically, this may be shown as follows:



When considering the area of this region, we find that

$$P\left(\max\{B,C\} < \frac{1}{2}\right) = P\left(B < \frac{1}{2} \cap C < \frac{1}{2}\right) = \frac{1}{4}$$

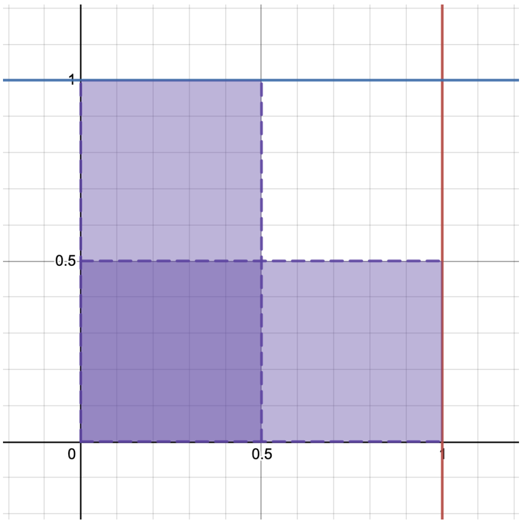
Chapter 2.2, Question 8(e)

Choose independently two numbers B and C at random from the interval $[0, 1]$ with uniform density. Note that the point (B, C) is then chosen at random in the unit square. Find the probability that $\max\{B, C\} < \frac{1}{2}$.

If B and C are chosen at random from the interval $[0, 1]$ with uniform density, the pair (B, C) represents a point chosen at random from the unit square with $x \in [0, 1]$ and $y \in [0, 1]$. Further, the probability that $\min\{B, C\} < \frac{1}{2}$ may be stated as

$$P\left(\min\{B, C\} < \frac{1}{2}\right) = P\left(B < \frac{1}{2} \cup C < \frac{1}{2}\right).$$

Geometrically, this may be shown as follows:



When considering the area of this region, we find that

$$P\left(\min\{B, C\} < \frac{1}{2}\right) = P\left(B < \frac{1}{2} \cup C < \frac{1}{2}\right) = \frac{3}{4}$$

Chapter 2.2, Question 14

Choose independently two numbers B and C at random from the interval $[-1, 1]$ with uniform distribution, and consider the quadratic equation

$$x^2 + Bx + C = 0.$$

Find the probability that the roots of this equation

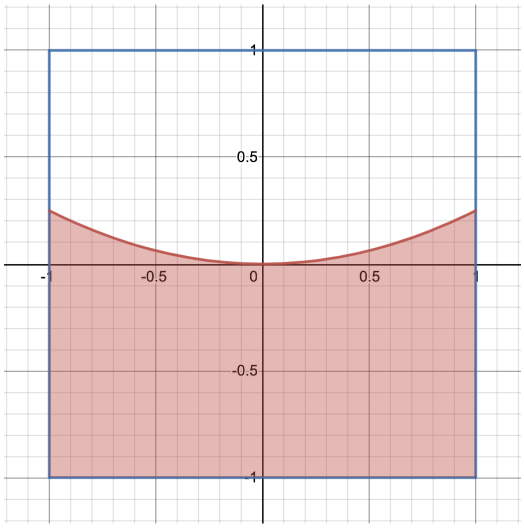
(a) are both real.

(b) are both positive.

Hint: (a) requires $0 \leq B^2 - 4C$, (b) requires $0 \leq B^2 - 4C$, $B \leq 0$, $0 \leq C$.

If B and C are chosen at random from the interval $[-1, 1]$ with uniform density, the pair (B, C) represents a point chosen at random from the square with $x \in [-1, 1]$ and $y \in [-1, 1]$.

To determine the probability that the roots of the equation are both real, we may geometrically represent the problem, with the restriction $0 \leq B^2 - 4C$ as follows:



To find the area of the shaded region, we take the following double integral

$$\int_{-1}^{1} \int_{-1}^{\frac{1}{4}B^2} 1 \, dC \, dB.$$

Note: I found this method to be easier than taking a single integral, as we must account for the area underneath the x-axis. Further, it is important to note that the upper bound of the inner integral is given from the equation $0 \leq B^2 - 4C$ when solving for C as $C \leq \frac{1}{4}B^2$.

Evaluating the integral, we find that

$$\int_{-1}^{1} \int_{-1}^{\frac{1}{4}B^2} 1 \, dC \, dB = \int_{-1}^{1} \frac{1}{4}B^2 + 1 \, dB,$$

which simplifies to

$$\left[\frac{1}{12}B^3 \right]_{-1}^{1} + 2.$$

Further simplifying this, we find that the shaded area is equal to

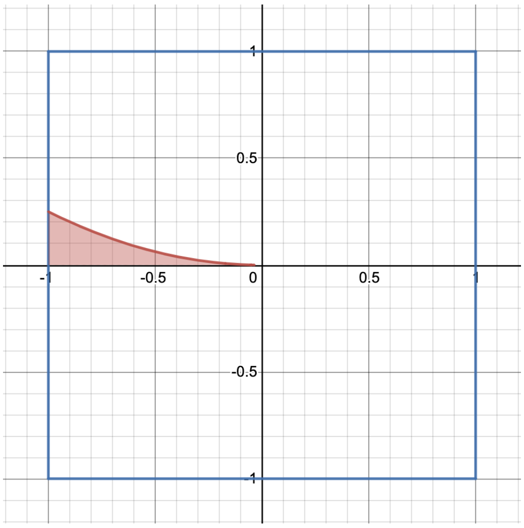
$$\frac{1}{6} + 2 = \frac{13}{6}.$$

Now given that the total area of the square bounded by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ is 4, we know that the probability (determined geometrically) is:

$$\left(\frac{13}{6} \right) / 4$$

Thus, the probability that the roots of the equation are both real is $\frac{13}{24}$.

To determine the probability that the roots of the equation are both positive, we may geometrically represent the problem, with the restrictions $0 \leq B^2 - 4C$, $B \leq 0$, $0 \leq C$ as follows:



To find the area of the shaded region, we take the following integral

$$\int_{-1}^{0} \frac{1}{4}B^2\, dB$$

Note: It is important to note that (once again) the upper bound of the inner integral is given from the equation $0 \leq B^2 - 4C$ when solving for C as $C \leq \frac{1}{4}B^2$.

Evaluating the integral, we find that

$$\left[\frac{1}{12}B^3\right]_{-1}^{0} = \frac{1}{12},$$

which is the area of the shaded region.
Now given that the total area of the square bounded by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ is 4, we know that the probability (determined geometrically) is:

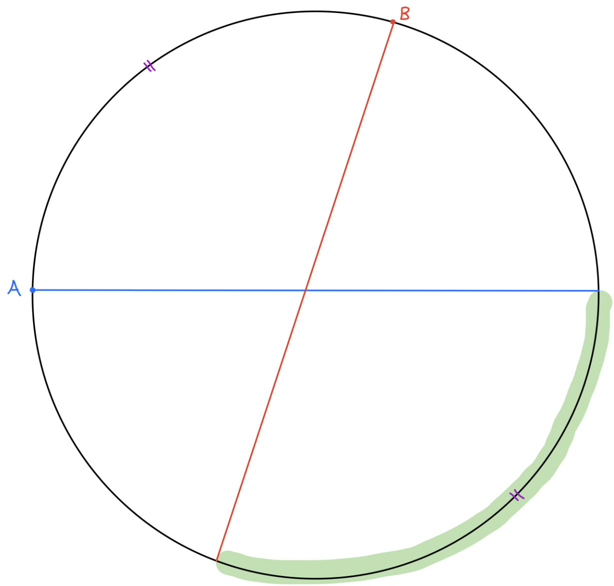
$$\left(\frac{1}{12}\right)/4$$

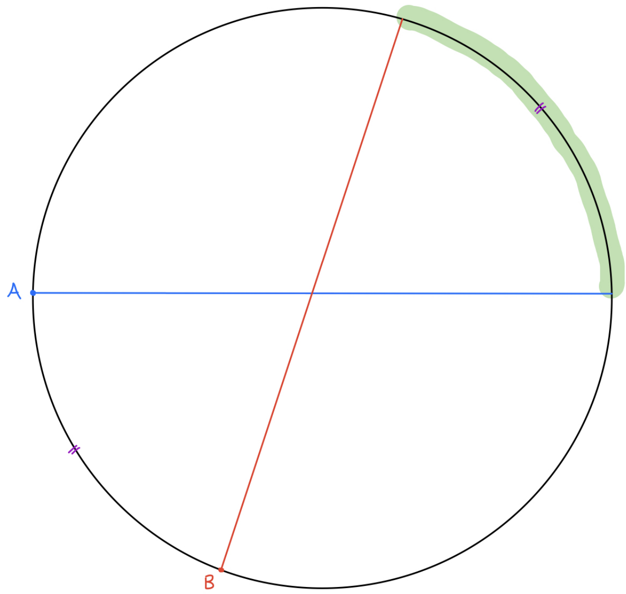
Thus, the probability that the roots of the equation are both positive is $\frac{1}{48}$.

Chapter 2.2, Question 16

Three points are chosen *at random* on a circle of *unit circumference*. What is the probability that the triangle defined by these points as vertices has three acute angles? *Hint:* One of the angles is obtuse if and only if all three points lie in the same semicircle. Take the circumference as the interval $[0, 1]$. Take one point at 0 and the others at B and C.

There are two situations for the placement of B, which are shown in the diagrams below (on the next page).





In the first diagram, the point B is placed on the upper semicircle. In the second diagram, the point B is placed on the lower semicircle.

By the hint that one of the angles is obtuse in a triangle formed by three points on a circle if and only if all three points lie in the same semicircle, the green shaded region of the semicircle represents the region where point C could be placed such that there would be three acute angles (no obtuse angles).

This green region was determined by considering the regions where all three points would lie in the same semicircle (the arc that is not shaded).

Note: In the case where B is placed directly opposite A, there are no such cases where a point C can be placed to create a triangle with three acute angles, so we disregard this situation.

For the first diagram, we note that the length of the shaded arc is equivalent to the length of the arc from A to B (clockwise), which is given by the value B. For the second diagram, we note that the length of the shaded arc is equivalent to the length of the arc from B to A (clockwise), which is given by the value $1 - B$. As the circle is a circle of unit circumference, we may set up the following expression to determine the probability that the triangle defined by the three points as vertices has three acute angles by considering the case where $B < \frac{1}{2}$ and the case where $B > \frac{1}{2}$ separately.

For the case $B < \frac{1}{2}$, we have the length of the shaded arc as

$$\int_0^{\frac{1}{2}} B \, dB.$$

For the case $B > \frac{1}{2}$, we have the length of the shaded arc as

$$\int_{\frac{1}{2}}^1 (1 - B) \, dB.$$

Taking the sum of these expressions results in

$$\int_0^{\frac{1}{2}} B \, dB + \int_{\frac{1}{2}}^1 (1 - B) \, dB.$$

When simplifying the expression, we find

$$\left[\frac{1}{2}B^2\right]_0^{\frac{1}{2}} + \left[B - \frac{1}{2}B^2\right]_{\frac{1}{2}}^1,$$

which evaluates to

$$\frac{1}{8} + \left(\left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{8}\right)\right).$$

This simplifies to

$$\frac{1}{8} + \frac{1}{8}.$$

Therefore, when three points are chosen at random on a circle of unit circumference, the probability that the triangle defined by these points as vertices has three acute angles is $\frac{1}{4}$.