Mathematics 60 - Probability Theory

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Homework 3

Chapter 3.1: 3, 7, 19(a)(b)(c)(d)

Chapter 3.2: 7, 22, 35

Chapter 3.1, Question 3

In a digital computer, a *bit* is one of the integers $\{0, 1\}$, and a *word* is any string of 32 bits. How many different words are possible?

There are 2^{32} different words possible, as this is the number of permutations for any given word, considering there are two choices for each bit.

Chapter 3.1, Question 7

Five people get on an elevator that stops at five floors. Assuming that each has an equal probability of going to any one floor, find the probability that they all get off at different floors.

There are 5! ways for the people to all get off at different floors (that is, for one person to be on each floor). Assuming that each person has an equal probability of going to any one floor, there are 5^5 total scenarios, considering that there are five people and each person has five choices.

Thus, the probability that the five people on the elevator all get off at different floors is given by

$$P = \frac{5!}{5^5}, \qquad \text{which simplifies to} \qquad P = \frac{24}{625} = 0.0384.$$

Chapter 3.1, Question 19

Suppose that on planet Zorg a year has n days, and that the lifeforms there are equally likely to have hatched on any day of the year. We would like to estimate d, which is the minimum number of lifeforms needed so that the probability of at least two sharing a birthday exceeds 1/2.

(a) In Example 3.3, it was shown that in a set of d lifeforms, the probability that no two life forms share a birthday is

$$\frac{(n)_d}{n^d},$$

where $(n)_d = (n)(n-1)\cdots(n-d+1)$. Thus, we would like to set this equal to 1/2 and solve for d.

(b) Using Stirling's Formula, show that

$$\frac{(n)_d}{n^d} \sim \left(1 + \frac{d}{n-d}\right)^{n-d+1/2} e^{-d}.$$

Let us start by representing $\frac{(n)_d}{n^d}$ in terms of factorials as

$$\frac{\frac{n!}{(n-d)!}}{n^d} = \frac{n!}{(n-d)!\, n^d}.$$

From this point, we apply Stirling's Formula, which is

$$n! \sim n^n e^{-n} \sqrt{2\pi n},$$

which results in

$$\frac{n^n e^{-n} \sqrt{2\pi n}}{(n-d)^{(n-d)} e^{-(n-d)} \sqrt{2\pi (n-d)} n^d}.$$

Simplifying this expression further by removing the $\sqrt{2\pi}$ yields

$$\frac{n^n e^{-n} n^{\frac{1}{2}}}{(n-d)^{(n-d)} e^{-(n-d)} (n-d)^{\frac{1}{2}} n^d}.$$

Simplifying exponents results in

$$\frac{n^{n-d+1/2} e^{-d}}{(n-d)^{(n-d+1/2)}},$$

which becomes

$$\left(\frac{n}{n-d}\right)^{n-d+1/2} e^{-d}.$$

By manipulating the expression, we find that

$$\left(1 - \frac{n-d}{n-d} + \frac{n}{n-d}\right)^{n-d+1/2} e^{-d} = \left(1 + \frac{d}{n-d}\right)^{n-d+1/2} e^{-d},$$

so

$$\frac{(n)_d}{n^d} \sim \left(1 + \frac{d}{n-d}\right)^{n-d+1/2} e^{-d}$$

which is what we were hoping to achieve. □

(c) Now take the logarithm of the right-hand expression, and use the fact that for small values of x, we have

$$\log(1+x) \sim x - \frac{x^2}{2}.$$

(We are implicitly using the fact that d is of smaller order of magnitude than n. We will also use this fact in part (d).)

Taking the logarithm of the right-hand expression, we have

$$(n-d+1/2)\left(\frac{d}{n-d} - \frac{\left(\frac{d}{n-d}\right)^2}{2}\right) + (-d)$$
$$= (n-d+1/2)\left(\frac{d}{n-d} - \frac{d^2}{2(n-d)^2}\right) + (-d)$$

Now, knowing that d is of smaller order of magnitude than n, we may simplify the expression using $n-d \sim n$ and $n-d+1/2 \sim n$. Thus, we have

$$n\left(\frac{d}{n} - \frac{d^2}{2n^2}\right) - d,$$

which simplifies further to

$$-\frac{d^2}{2n}.$$

(d) Set the expression found in part (c) equal to $-\log(2)$, and solve for d as a function of n, thereby showing that

$$d \sim \sqrt{2(\log 2)\, n}.$$

Setting the expression equal to $-\log(2)$, as this is equivalent to $\log(1/2)$, we find

$$-\frac{d^2}{2n} \sim -\log(2), \qquad \text{which simplifies to} \qquad d^2 \sim 2(\log 2)\, n.$$

Thus, we have

$$d \sim \sqrt{2(\log 2)\, n},$$

as expected. □

Chapter 3.2, Question 7

Show that

$$b(n,p,j) = \frac{p}{q}\left(\frac{n-j+1}{j}\right) b(n,p,j-1),$$

for $j \geq 1$. Use this fact to determine the value or values of j which give $b(n,p,j)$ its greatest value. *Hint:* Consider the successive ratios as j increases.

To show the above property, we use Theorem 3.6, which states that given n Bernoulli trials with probability p of success on each experiment, the probability of exactly j successes is

$$b(n,p,j) = \binom{n}{j} p^j q^{n-j},$$

where $q = 1-p$. Thus, we may write the initial expression as

$$\binom{n}{j} p^j q^{n-j} = \frac{p}{q}\left(\frac{n-j+1}{j}\right)\binom{n}{j-1} p^{j-1} q^{n-(j-1)}.$$

We simply need to show that this is true in order to prove the identity given. Simplifying exponents, we find that

$$\binom{n}{j} p^j q^{n-j} = \left(\frac{n-j+1}{j}\right)\binom{n}{j-1} p^j q^{n-j}.$$

Now, representing the binomial expression on the right hand side using factorials, we find

$$\binom{n}{j} p^j q^{n-j} = \left(\frac{n-j+1}{j}\right)\frac{n!}{(j-1)!\,(n-(j-1))!} p^j q^{n-j},$$

which simplifies to

$$\binom{n}{j} p^j q^{n-j} = \frac{n!}{j!\,(n-j)!} p^j q^{n-j}.$$

The right and left sides are clearly equivalent, thus we have shown what was to be demonstrated. □

To determine the value or values of j which give $b(n,p,j)$ its greatest value, we consider the successive ratios as j increases. The expression we use for $b(n,p,j)$ is recursive, so we reason that the value of the function will continue to grow until

$$\frac{p}{q}\left(\frac{n-j+1}{j}\right) \leq 1.$$

Thus, if we solve the above expression for j, we will be able to determine the value(s) of j which give $b(n,p,j)$ its greatest value. We start with

$$p(n-j+1) \leq qj,$$

which we may express as

$$pn - pj + p \leq qj.$$

Placing the j's on a single side, we have

$$pn + p \leq qj + pj,$$

which is equivalent to

$$pn + p \leq (q + p)j,$$

and because $p = 1 - q$, thus $p + q = 1$ by definition, we know that

$$pn + p \leq j.$$

This tells us that the value of j which gives $b(p, n, j)$ its greatest value is the maximum value of j such that the above inequality is false. That is, we take the maximum j such that $j < pn + p$.

This understanding is demonstrated by this Desmos graph. In the case where $p = 0.5$, there are two values of j for which $b(p, n, j)$ takes on its greatest value, the previously identified j and the value $j - 1$. This is due to the symmetry of the curve created when plotting j on the x-axis, with the value of $b(p, n, j)$ on the y-axis.

Chapter 3.2, Question 22

How many ways can six indistinguishable letters be put in three mail boxes? *Hint:* One representation of this is given by a sequence $|LL|L|LLL|$ where the $|$'s represent the partitions for the boxes and the L's the letters. Any possible way can be so described. Note that we need two bars at the ends and the remaining two bars and the six L's can be put in any order.

To solve this problem, we will focus only on the six L's and the two inner bars, as the two outer bars must remain on the outside. There are 8 possible slots for the six L's and two bars. Given these 8 slots, we must choose two of them to be bars. As such, the solution is

$$\binom{8}{2},$$

which is equivalent to $\binom{8}{6}$, which would be choosing six slots to be L's.

Thus, there are 228 ways that six indistinguishable letters can be put in three mail boxes.

Chapter 3.2, Question 35

Prove the following *binomial identity*

$$\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2.$$

Hint: Consider an urn with n red balls and n blue balls inside. Show that each side of the equation equals the number of ways to choose n balls from the urn.

As the hint indicates, let us consider an urn with n red balls and n blue balls inside. Counting the number of ways there are to select/choose n balls from the urn, we trivially see the answer is $\binom{2n}{n}$, as there are $2n$ total balls, and we are selecting n of them.

Counting in a different way, each selection of n balls contains a certain number (j) of red balls and $n - j$ blue balls. The number of red balls (j) ranges from 0 to n, so we sum from $j = 0$ to n. For any given value j, we can choose the red balls in $\binom{n}{j}$ ways and the blue balls in $\binom{n}{n-j}$ ways.

Thus, the number of ways to choose n balls with j red balls is $\binom{n}{j}\binom{n}{n-j}$. As we know $\binom{n}{n-j} = \binom{n}{j}$, this simplifies to $\binom{n}{j}^2$, which we hoped to achieve.

Thus, the *binomial identity* is proven by choosing n balls from a set of $2n$, with j of them being red and $n - j$ being blue. $\square$