QSS/Mathematics 30.04 - Evolutionary Game Theory

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Homework 2

Prompt/Instructions

Typical Prisoner's Dilemma games assume compulsory participation with the two strategies Cooperation (C) and Defection (D). In this game, cooperation incurs a cost c to bring a benefit b to the co-player, whereas defection does nothing. Let us now consider an optional Prisoner's Dilemma, where players can choose to unilaterally opt out the prisoner's dilemma interaction with his co-player. In this case, he and his co-player both receive a payoff δ, regardless of their strategies. Let us name this third "opt out" strategy as "Loner" (L), in addition to C and D.

Question 1

Write down the 3×3 payoff matrix for the three strategies, C, D, and L.

The 3×3 payoff matrix for the three strategies is given as follows, where the entries in the table represent the payoff to the player whose strategy is given by the leftmost column.

	C	D	L
C	$b-c$	$-c$	δ
D	b	0	δ
L	δ	δ	δ

Question 2

Write down the replicator dynamics for the evolution of these three strategies in a infinitely-large well-mixed population.

Hint: Use the formula $\dot{x}_i = x_i \left(f_i\left(\vec{x}\right) - \bar{x}\right)$, where x_i is the frequency of strategy i in the population, $f_i\left(\vec{x}\right)$ is the fitness of strategy i, and $\bar{x}$ is the average fitness of the population.

Let us begin by determining the fitness of each strategy $i \in \{C, D, L\}$ as follows:

$$f_C\left(\vec{x}\right) = \left(b-c\right) x_C + \left(c\right) x_D + \left(\delta\right) x_L$$

$$f_D\left(\vec{x}\right) = \left(b\right) x_C + \left(\delta\right) x_L$$

$$f_L\left(\vec{x}\right) = \left(\delta\right) x_C + \left(\delta\right) x_D + \left(\delta\right) x_L$$

Now, to determine the average fitness of the population, we have the following:

$$\bar{x} = x_C f_C\left(\vec{x}\right) + x_D f_D\left(\vec{x}\right) + x_L f_L\left(\vec{x}\right)$$

Using the equations for the fitness of each strategy, the average fitness may be expressed alternatively as

$$\bar{x} = \left(\left(b-c\right) x_C^2 + \left(c\right) x_D x_C + \left(\delta\right) x_L x_C\right) + \left(\left(b\right) x_C x_D + \left(\delta\right) x_L x_D\right) + \left(\left(\delta\right) x_C x_L + \left(\delta\right) x_D x_L + \left(\delta\right) x_L^2\right),$$

which may be simplified to

$$\bar{x} = (b-c)\, x_C^2 + (c+b)\, x_D x_C + (2\delta)\, x_L x_C + (2\delta)\, x_L x_D + (\delta)\, x_L^2.$$

Now, we use the formula $\dot{x}_i = x_i \left(f_i\left(\vec{x}\right) - \bar{x}\right)$ for each strategy $i \in \{C, D, L\}$, where x_i is the frequency of strategy i in the population.:

$$\dot{x}_C = x_C \left(f_C\left(\vec{x}\right) - \bar{x}\right)$$
$$\dot{x}_D = x_D \left(f_D\left(\vec{x}\right) - \bar{x}\right)$$
$$\dot{x}_L = x_L \left(f_L\left(\vec{x}\right) - \bar{x}\right)$$

Making the appropriate substitutions, this yields

$$\dot{x}_C = x_C \left((b-c)\, x_C + (c)\, x_D + (\delta)\, x_L - \left((b-c)\, x_C^2 + (c+b)\, x_D x_C + (2\delta)\, x_L x_C + (2\delta)\, x_L x_D + (\delta)\, x_L^2\right)\right)$$

$$\dot{x}_D = x_D \left((b)\, x_C + (\delta)\, x_L - \left((b-c)\, x_C^2 + (c+b)\, x_D x_C + (2\delta)\, x_L x_C + (2\delta)\, x_L x_D + (\delta)\, x_L^2\right)\right)$$

$$\dot{x}_L = x_L \left((\delta)\, x_C + (\delta)\, x_D + (\delta)\, x_L - \left((b-c)\, x_C^2 + (c+b)\, x_D x_C + (2\delta)\, x_L x_C + (2\delta)\, x_L x_D + (\delta)\, x_L^2\right)\right)$$

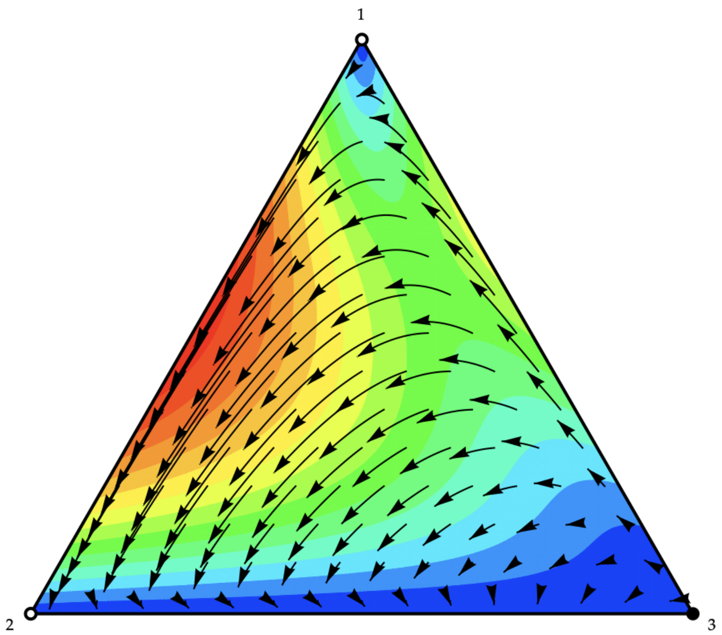
These are the replicator dynamics for the evolution of the three strategies in an infinitely-large well-mixed population. Further simplification would follow if using $x_C + x_D + x_L = 1$, though this does not seem to provide meaningful insight.

Question 3

Download the Dynamo package from https://www.ssc.wisc.edu/∼whs/dynamo/. Use it to plot the phase diagram in the simplex S3 with the following parameter values (i) $b = 2, c = 1, \delta = 0.1$ and (ii) $b = 2, c = 1, \delta = -0.01$. What do you find? Please argue your findings using simple selection dynamics on the edges of S3.

The phase diagrams in the simplex S3 with the given parameters are shown below:

(i) $b = 2, c = 1, \delta = 0.1$



For this scenario, there is a Nash equilibrium at $(0, 0, 1)$, representative of the third "opt out" strategy as a Loner (L). There is no ESS.

From the diagram, we find the following:

- If most of the population uses a strategy of cooperation (C), the population will shift to a strategy of defection (D).
- If most of the population uses a strategy of opting out (L), the population will shift to a strategy of cooperation (C), followed by a shift to a strategy of defection (D).
- If most of the population uses a strategy of defection (D), the population will shift to a strategy of opting out (L).

Using simple selection dynamics on the edges of the Dynamo S3 simplex, we analyze the following matrices:

	C	D
C	$b-c$	$-c$
D	b	0

	C	L
C	$b-c$	δ
L	δ	δ

	D	L
D	0	δ
L	δ	δ

When inputting the appropriate values, these matrices are as follows:

	C	D
C	1	-1
D	2	0

	C	L
C	1	0.1
L	0.1	0.1

	D	L
D	0	0.1
L	0.1	0.1

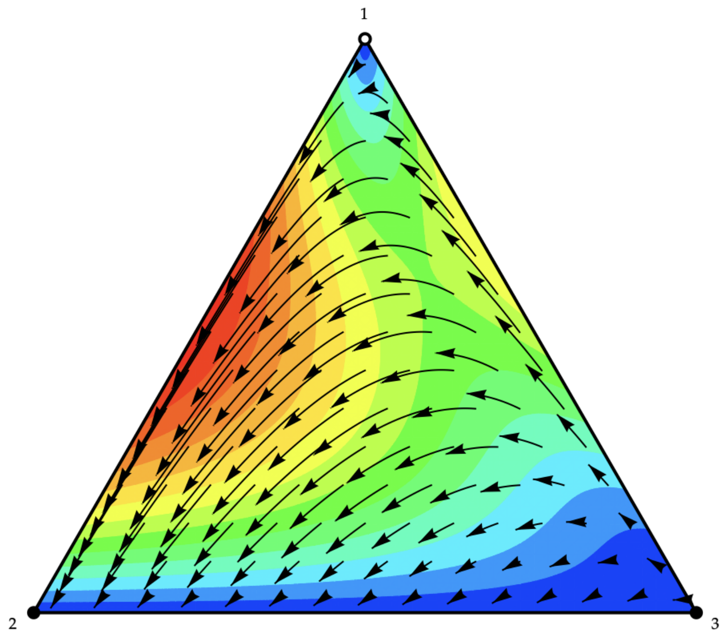
On the edge of cooperation (C) - defection (D), players will choose the optimal strategy of defection, as it yields a higher payoff.

On the edge of cooperation (C) - opting out (L), players will choose the optimal strategy of cooperation. Though the payoff is the same as opting out if the opponent chooses to opt out, cooperation yields a higher payoff if the opponent chooses cooperation.

On the edge of defection (D) - opting out (L), players will choose the optimal strategy of opting out. Though the payoff is the same as defection if the opponent chooses to opt out, opting out yields a higher payoff if the opponent chooses to defect.

Thus, the simple selection dynamics on the edges confirm the qualitative findings from the Dynamo S3 simplex.

(ii) $b = 2, c = 1, \delta = -0.01$



For this scenario, there are two Nash equilibria at $(0, 0, 1)$ and $(0, 1, 0)$, representative of the third "opt out" strategy as a Loner (L) and of the second "defect" strategy as a Defector (D), respectively. There is an ESS at $(0, 1, 0)$, representative of the second "defect" strategy as a Defector (D).

From the diagram, we find the following:

- If most of the population uses a strategy of cooperation (C), the population will shift to a strategy of defection (D).
- If most of the population uses a strategy of opting out (L), the population will shift to a strategy of cooperation (C), followed by a shift to a strategy of defection (D)
- If most of the population uses a strategy of defection (D), the population will maintain a strategy of defection (D).

Using simple selection dynamics on the edges of the Dynamo S3 simplex, we analyze the following matrices:

	C	D
C	$b-c$	$-c$
D	b	0

	C	L
C	$b-c$	δ
L	δ	δ

	D	L
D	0	δ
L	δ	δ

When inputting the appropriate values, these matrices are as follows:

	C	D
C	1	-1
D	2	0

	C	L
C	1	-0.01
L	-0.01	-0.01

	D	L
D	0	-0.01
L	-0.01	-0.01

On the edge of cooperation (C) - defection (D), players will choose the optimal strategy of defection, as it yields a higher payoff.

On the edge of cooperation (C) - opting out (L), players will choose the optimal strategy of cooperation. Though the payoff is the same as opting out if the opponent chooses to opt out, cooperation yields a higher payoff if the opponent chooses cooperation.

On the edge of defection (D) - opting out (L), players will choose the optimal strategy of defection. Though the payoff is the same as opting out if the opponent chooses to opt out, defection yields a higher payoff if the opponent chooses to defect.

Thus, the simple selection dynamics on the edges confirm the qualitative findings from the Dynamo S3 simplex.