QSS/Mathematics 30.04 - Evolutionary Game Theory

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Homework 3

Prompt/Instructions

Cooperation is not limited to human groups, but also widely observed in microbial populations. For example, in order for the budding yeast *Saccharomyces cerevisiae* to grow on sucrose, the disaccharide must be hydrolysed by the enzyme invertase. However, the monosaccharides created by this sucrose hydrolysis reaction can diffuse away before they can be imported into the cell itself. This fact makes invertase production and secretion a cooperative behavior, thereby creating a public good that defectors who forgo invertase secretion can exploit. Let us assume the invertase production has a cost c, and generates total benefits of unity that are captured with efficiency ϵ to the cell itself whereas the remaining $1 - \epsilon$ diffuses away and benefits everyone else in the population. Assume the frequency of cooperators in the yeast population is x and the frequency of defectors is y. We have $x + y = 1$.

Question 1

It follows that the payoff of a cooperator, π_C, can be written as $\pi_C = \epsilon + x(1 - \epsilon) - c$. The payoff of a defector, π_D, can be written as $\pi_D = x(1 - \epsilon)$. Write down the equations for the replicator dynamics in terms of the time-evolution frequencies of cooperators x and of defectors y.

The replicator dynamics in terms of the time-evolution frequencies of cooperators x and of defectors y are given by the following formulas:

$$\dot{x} = x(1 - x)(f(x) - f(y))$$

$$\dot{y} = y(1 - y)(f(y) - f(x))$$

where $[x, y]$ is the composition of the population, $f(x)$ is the fitness of (payoff to) cooperators, and $f(y)$ is the fitness of (payoff to) defectors. Further note that $\dot{x}$ represents $\frac{dx}{dt}$, while $\dot{y}$ represents $\frac{dy}{dt}$.

Using this information (inputting $f(x) = \pi_C = \epsilon + x(1 - \epsilon) - c$ and $f(y) = \pi_D = x(1 - \epsilon)$), the replicator dynamics are given as

$$\dot{x} = x(1 - x)\left[(\epsilon + x(1 - \epsilon) - c) - (x(1 - \epsilon))\right]$$

$$\dot{y} = y(1 - y)\left[(x(1 - \epsilon)) - (\epsilon + x(1 - \epsilon) - c)\right],$$

which simplify to

$$\dot{x} = x(1 - x)(\epsilon - c)$$

$$\dot{y} = y(1 - y)(c - \epsilon)$$

Question 2

Under what condition, cooperators are not vanishing in the population?

Assuming cooperators already exist in the population, that is $x > 0$, the condition that assures cooperators do not vanish in the population is given by $\epsilon - c \geq 0$, which evaluates to $\epsilon \geq c$.

To determine this, we consider the selection/replicator dynamics, specifically the expression $f(x) - f(y) = \pi_C - \pi_D$, which evaluates to $\epsilon - c$, as indicated. Given that this expression is greater than or equal to zero, the composition of the population will NOT move toward defection (cooperators will not vanish from the population).

Intuitively, this result makes sense, given the payoffs (fitness) of cooperators and defectors. The replicator dynamics of the time-evolution frequencies of cooperators x and defectors y reveals the stable and unstable equilibria for given values of ϵ and c.

Question 3

Let us now assume nonlinear effect of invertase production on individual benefits as follows:

$$\pi_C = [\epsilon + x(1-\epsilon)]^2 - c \qquad \pi_D = [x(1-\epsilon)]^2$$

Let $\epsilon = 0.5$ and $c = 0.5$. What is the equilibrium frequency of cooperators in the population? Which type of game interactions are the yeasts now playing (Prisoner's Dilemma, Snowdrift, or Coordination game)?

To determine the equilibrium frequency of cooperators in the population, we write down the equation for the replicator dynamics in terms of the time-evolution frequencies of cooperators x, which is given by

$$\dot{x} = x(1-x)(f(x) - f(y)),$$

where $[x, y]$ is the composition of the population, $f(x)$ is the fitness of (payoff to) cooperators, and $f(y)$ is the fitness of (payoff to) defectors. Further note that $\dot{x}$ represents $\frac{dx}{dt}$.

Using this information (inputting $f(x) = \pi_C = [\epsilon + x(1-\epsilon)]^2 - c$ and $f(y) = \pi_D = [x(1-\epsilon)]^2$), the replicator dynamics are given as

$$\dot{x} = x(1-x)\left[\left([\epsilon + x(1-\epsilon)]^2 - c\right) - \left([x(1-\epsilon)]^2\right)\right],$$

which simplifies to

$$\dot{x} = x(1-x)\left(\epsilon^2 + 2\epsilon x(1-\epsilon) - c\right).$$

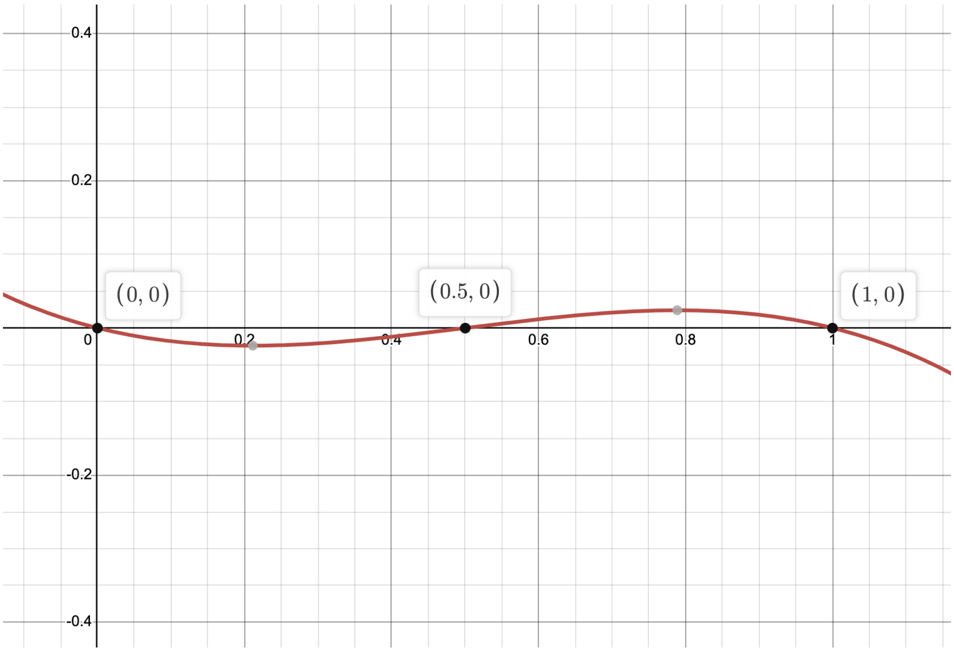
Setting $\epsilon = 0.5$ and $c = 0.5$ (as given by the problem statement), this expression evaluates to

$$\dot{x} = x(1-x)(0.25 + 2(0.5)x(1-0.5) - 0.5),$$

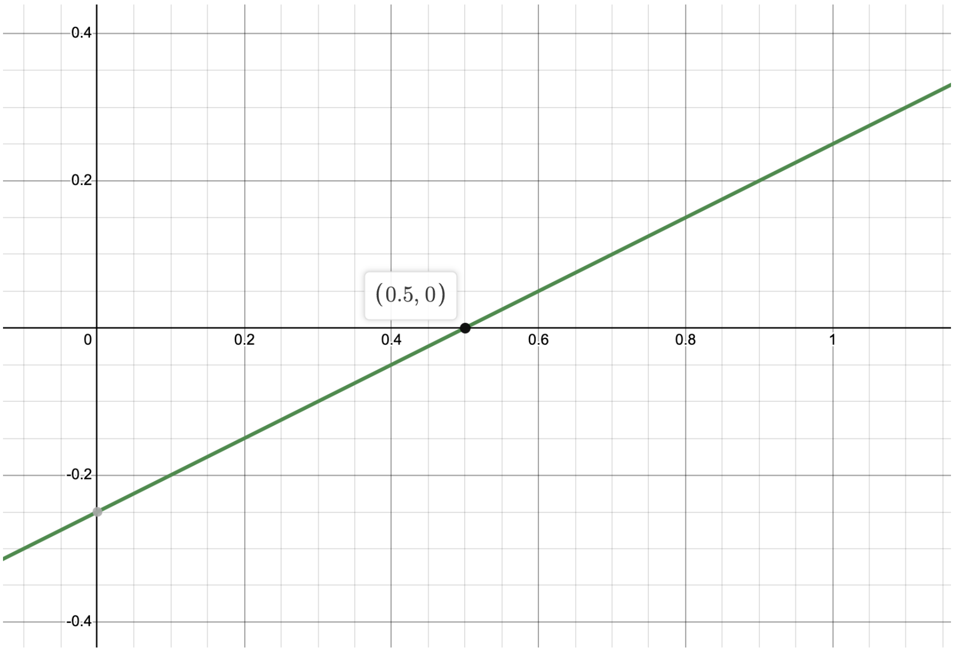
which simplifies to

$$\dot{x} = x(1-x)(-0.25 + 0.5x).$$

Graphically, this expression is shown as follows:



The selection dynamics are shown as follows:



The above indicate that there are stable equilibria at $x = 0$ and $x = 1$ and an unstable equilibrium at $x = 0.5$. This represents the bi-stable case of replicator dynamics in two-player games.

To mathematically solve for this solution, we simply set $-0.25 + 0.5x$ equal to zero, and solve for x, yielding $x = 0.5$.

Thus, the equilibrium frequency of cooperators in the population is 0.5, indicating the equilibrium frequency of defectors in the population is 0.5. This assumes that neither strategy dominates the other.

In this set-up, the yeasts would be playing a "Coordination" game, as the replicator dynamics are bi-stable.