Intermediate International Trade

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Version: September 23, 2025

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1 Introduction

The formalization of trade theory has very old origins. For example, Aristotle (ca. 350 B.C./2009, *Nicomachean Ethics*, V.5, 1133a) employed the concept of proportional modeling developed by the mathematician Euclid (ca. 300 B.C./1956, *Elements*, Book V), and formalized an exchange equation that already included assumptions commonly used by economists. If A is a builder, B a shoemaker, C a house, and D a pair of shoes, the relationship is established as $A : B = xD : C$. Two aspects are fundamental: determining x (the number of pairs of shoes equivalent to one house) and interpreting the ratio builder/shoemaker. The theory of international trade has examined this problem from different perspectives. Even the "new" theoretical approaches that have emerged in recent decades frequently, though not explicitly, rely on Schumpeterian concepts, in which entrepreneurs, "new markets," innovation, and market power play a central role, or draw on concepts from Newtonian physics, which itself was influenced by Aristotelian propositions.

It is therefore no coincidence that "Nothing exists in the world, except the blind forces of nature, that is not Greek in its origin" (Maine, cited in Livingstone, 1921). International trade theory is no exception. These are the conceptual foundations that later evolved into the propositions of the labor theory of value, which enabled the theories of absolute and comparative advantage. These theories can be extended to a larger number of goods and countries and explain trade patterns and the international organization of labor primarily through factor endowments, relative size, and technological differences.

The extension to two factors through the Heckscher–Ohlin (H–O) model, and its generalization in the Heckscher–Ohlin–Vanek (H–O–V) model, was enriched by a virtuous and intense cycle between empirical research and conceptual advances. This has been the evolution of trade theory—like many other fields of science—through empirical verification and the search for explanations of trade between relatively similar countries or of flows of goods in industries where close substitutes compete. As in other areas, increasing efforts with a stronger microeconomic foundation have emerged, seeking to explain how firms enter "new" markets and how global companies organize production processes across different locations and countries. These explanatory efforts are conventionally presented in textbooks and even in specialized journals as if isolated from entrepreneurial concepts, although in reality they are closely linked.

The relative availability of international trade data (values, volumes, identification of buyers and sellers, locations, among others) has also enabled the development of empirical explanations. These explanations do not necessarily follow models of welfare theory and are often ad hoc. Services pose an even greater challenge for the empirical verification of explanatory approaches. Similarly, Porter's concepts, product life-cycle theory, and the incorporation of more dynamic aspects—despite the absence of equations derived from microeconomic optimization models—have contributed detail and improved characterization of observed trade structures. Much of this description remains fundamentally linked to material transformation (production functions, technological relationships such as increasing returns, productivity, and technological change).

Trade relations have also acquired a strategic dimension, with the imposition of tariffs and trade barriers that modify the global equilibrium. This demands preparation to interpret and act upon regulatory and economic changes in this context. The concepts of “new markets” and the fundamental role of innovation, entrepreneurship, and market power—widely discussed since Schumpeter and influenced by Newtonian physics—are applied today in the analysis of international trade flows.

This document is the result of teaching international trade theory at the School of Economics of the University of Costa Rica. It addresses different topics, aiming to provide reference material with solid formal grounding, as well as exercises at an intermediate level of depth.

2 Microeconomic Overview

This chapter is a briefly review of: (i) Consumer Theory, (ii) Firm Theory and (iii) General Equilibrium in pure exchange.

2.1 Consumer Theory

Consumer theory analyzes how an individual chooses a consumption bundle to maximize utility (or satisfaction) given income and market prices. Dually, for a given utility level and given prices, the individual can choose the bundle that attains that utility at the minimum possible expenditure.
The individual's utility-maximization problem can be stated as follows:

$$\begin{aligned} \max_{x_1,\ldots,x_n} \quad & U(x_1,\ldots,x_n) && (2.1.1)\\ \text{s.t.} \quad & \sum_{i=1}^{n} p_i x_i \leq m, && (2.1.2)\\ & x_i \geq 0 \quad \forall i = 1,\ldots,n \end{aligned}$$

Under non-satiation and an interior solution, the associated Lagrangian is

$$\mathcal{L}(x_1,\ldots,x_n,\lambda) = U(x_1,\ldots,x_n) + \lambda\left(m - \sum_{i=1}^{n} p_i x_i\right) \tag{2.1.3}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial U}{\partial x_i} - \lambda p_i = 0 \quad \forall i = 1,\ldots,n, \tag{2.1.4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - \sum_{i=1}^{n} p_i x_i = 0 \tag{2.1.5}$$

Combining (2.1.4) for any two goods i and j yields the optimality (Marginal Rate of Substitution = price ratio) condition:

$$\frac{\partial U/\partial x_i}{\partial U/\partial x_j} = \frac{p_i}{p_j} \tag{2.1.6}$$

Substituting the optimal condition into the equation (2.1.2) (budget constraint) gives the *Marshallian (ordinary) demand* for each good i:

$$x_i = x_i(m, p_1,\ldots,p_n) \tag{2.1.7}$$

Equation (2.1.7) states that the optimal quantity of good i depends on income and all prices. Plugging these demands into the utility function defines the *indirect utility function*:

$$v(m, p_1,\ldots,p_n) = U\big(x_1^M(m,\mathbf{p}),\ldots,x_n^M(m,\mathbf{p})\big) \tag{2.1.8}$$

The indirect utility $v(m, \mathbf{p})$ gives the maximum attainable utility at income m and price vector p and it is useful for comparing scenarios in which income and prices change simultaneously. By the envelope theorem, the Lagrange multiplier satisfies

$$\lambda^* = \frac{\partial v(m, \mathbf{p})}{\partial m}$$

so λ^* is the marginal utility of income.
Consider the following utility function

$$U(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}, \qquad \alpha_i > 0 \tag{2.1.9}$$

From (2.1.6),

$$\frac{x_i}{x_j} = \frac{\alpha_i p_j}{\alpha_j p_i} \tag{2.1.10}$$

Substituting into (2.1.2) yields the Marshallian demand

$$x_i(m, \mathbf{p}) = \frac{\alpha_i m}{p_i \sum_{k=1}^{n} \alpha_k} \tag{2.1.11}$$

Plugging (2.1.11) into (2.1.9) gives the indirect utility:

$$v(m, p_1, \ldots, p_n) = \left(\frac{m}{\sum_{k=1}^{n} \alpha_k}\right)^{\sum_{k=1}^{n} \alpha_k} \prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} \tag{2.1.12}$$

For a target utility level $\bar{u}$, the dual problem is

$$\min_{x_1, \ldots, x_n} \quad \sum_{i=1}^{n} p_i x_i \tag{2.1.13}$$

$$\text{s.t.} \quad U(x_1, \ldots, x_n) \geq \bar{u}, \tag{2.1.14}$$

$$x_i \geq 0 \quad \forall i = 1, \ldots, n$$

With an interior solution, the Lagrangian is

$$\mathcal{L}(x_1, \ldots, x_n, \lambda) = \sum_{i=1}^{n} p_i x_i + \lambda\big(\bar{u} - U(x_1, \ldots, x_n)\big) \tag{2.1.15}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_i} = p_i - \lambda \frac{\partial U}{\partial x_i} = 0 \quad \forall i = 1, \ldots, n \tag{2.1.16}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{u} - U(x_1, \ldots, x_n) = 0 \tag{2.1.17}$$

Combining the two-good first-order conditions in (2.1.16) once again produces the optimality condition found in (2.1.6). Substituting this condition into (2.1.14) yields the *Hicksian (compensated) demand*.

$$h_i = h_i(\bar{u}, p_1, \ldots, p_n) \tag{2.1.18}$$

Substituting the Hicksian demand into (2.1.13) defines the resulting *expenditure function*:

$$e(\bar{u}, p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i h_i(\bar{u}, \mathbf{p}) \tag{2.1.19}$$

The Hicksian demand tells how much of each good is needed to achieve utility $\bar{u}$ at minimum cost, while the expenditure function gives that minimum cost. By the envelope theorem, the multiplier λ measures how much the minimum expenditure must increase to raise utility by one unit.

In the previous example, substituting (2.1.10) into the utility function yields to the Hicksian demand:

$$h_i(\bar{u}, \mathbf{p}) = \frac{\alpha_i \bar{u}^{\frac{1}{\sum_{i=1}^{n} \alpha_i}}}{p_i} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{\sum_{i=1}^{n} \alpha_i}}$$

Inserting this result into the expenditure function gives:

$$e(\bar{u}, \mathbf{p}) = \left[\bar{u}^{\frac{1}{\sum_{i=1}^{n} \alpha_i}} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{\sum_{i=1}^{n} \alpha_i}}\right] \sum_{i=1}^{n} \alpha_i$$

2.1.1 Duality Properties

Let $v(m, \mathbf{p})$ denote the indirect utility function, $e(\bar{u}, \mathbf{p})$ the expenditure function, $x_i(m, \mathbf{p})$ the Marshallian (ordinary) demand, and $h_i(\bar{u}, \mathbf{p})$ the Hicksian (compensated) demand. Prices are $\mathbf{p} = (p_1, \ldots, p_n)$, income is m, and $\bar{u}$ is a target utility level.

Roy's identity

$$x_i(m, \mathbf{p}) = -\frac{\frac{\partial v(m, \mathbf{p})}{\partial p_i}}{\frac{\partial v(m, \mathbf{p})}{\partial m}} \quad \forall i = 1, \ldots, n \tag{2.1.20}$$

Shephard's lemma

$$h_i(\bar{u}, \mathbf{p}) = \frac{\partial e(\bar{u}, \mathbf{p})}{\partial p_i} \quad \forall i = 1, \ldots, n \tag{2.1.21}$$

Indirect utility and expenditure functions are inverses (duality)

$$e\big(v(m, \mathbf{p}), \mathbf{p}\big) = m, \tag{2.1.22}$$

$$v\big(e(\bar{u}, \mathbf{p}), \mathbf{p}\big) = \bar{u} \tag{2.1.23}$$

Marshallian and Hicksian demands relationship

$$h_i(\bar{u}, \mathbf{p}) = x_i\big(e(\bar{u}, \mathbf{p}), \mathbf{p}\big), \tag{2.1.24}$$
$$x_i(m, \mathbf{p}) = h_i\big(v(m, \mathbf{p}), \mathbf{p}\big) \tag{2.1.25}$$

Given the Cobb–Douglas indirect utility function in equation (2.1.12), Roy's identity delivers the Marshallian demand:

$$x_i(m, \mathbf{p}) = \frac{\alpha_i\, m}{p_i \sum_{k=1}^{n} \alpha_k} \tag{2.1.26}$$

Equation (2.1.23) asserts that the indirect utility evaluated at the minimum expenditure equals the target utility. Substituting the form (2.1.12) and solving for $e(u, \mathbf{p})$ yields the minimum expenditure function

$$e(u, \mathbf{p}) = \Big[u \prod_{i=1}^{n} \Big(\tfrac{p_i}{\alpha_i}\Big)^{\alpha_i}\Big]^{\frac{1}{\sum_{k=1}^{n} \alpha_k}} \sum_{k=1}^{n} \alpha_k \tag{2.1.27}$$

Applying Shephard's lemma to the minimum expenditure function gives the Hicksian demand:

$$h_i(u, \mathbf{p}) = \frac{\alpha_i}{p_i}\Big[u \prod_{j=1}^{n} \Big(\tfrac{p_j}{\alpha_j}\Big)^{\alpha_j}\Big]^{\frac{1}{\sum_{k=1}^{n} \alpha_k}} \tag{2.1.28}$$

Solving the Marshallian demand for good i for p_i and substituting it into the indirect utility function restores the original utility function.

2.2 Firm Theory

In the theory of the firm, input choice is viewed either as profit maximization or, equivalently, as minimizing the cost of producing a given output level; the analysis first adopts the cost-minimization perspective.

$$\begin{aligned} \min_{z_1, z_2, \ldots, z_n} \quad & \sum_{i=1}^{n} w_i z_i \\ \text{s.t.} \quad & q(z_1, z_2, \ldots, z_n) \geq \bar{q} \\ & z_i \geq 0 \quad \forall i = 1, \ldots, n \end{aligned}$$

Assuming an interior solution, the Lagrangian for the cost-minimization problem is:

$$\mathcal{L}(z_1, \ldots, z_n, \lambda) \;=\; \sum_{i=1}^{n} w_i z_i \;+\; \lambda\big(\bar{q} - q(z_1, \ldots, z_n)\big)$$

By the envelope theorem, the Lagrange multiplier λ equals the marginal cost—the increase in minimum total cost required to produce one additional unit of output.

$$\frac{\partial \mathcal{L}}{\partial z_i} = w_i - \lambda \frac{\partial q(z_1, \ldots, z_n)}{\partial z_i} = 0 \quad \forall i = 1, \ldots, n,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{q} - q(z_1, \ldots, z_n) = 0$$

Combining the first-order conditions for any two inputs, i and j, implies that the marginal rate of technical substitution between them equals the ratio of their input prices.

$$\frac{\frac{\partial q(z_1,\ldots,z_n)}{\partial z_i}}{\frac{\partial q(z_1,\ldots,z_n)}{\partial z_j}} = \frac{w_i}{w_j} \quad \forall i, j = 1, \ldots, n \tag{2.2.1}$$

Substituting the optimality condition (2.2.1) back into the production constraint yields the *conditional input demand functions*, denoted by

$$z_i = z(\bar{q}, w_1, \ldots, w_n) \tag{2.2.2}$$

where $\bar{q}$ is the fixed output target and $\mathbf{w} = (w_1, \ldots, w_n)$ is the vector of input prices. Each function $z_i(\bar{q}, \mathbf{w})$ gives the amount of input that minimizes costs i required to produce units of output $\bar{q}$ at the prevailing prices, thus completing the solution to the firm's cost-minimization problem.

Substituting (2.2.2) into the cost objective yields the *minimum cost function*

$$C(\bar{q}, \mathbf{w}) = \sum_{i=1}^{n} w_i \, z_i(\bar{q}, \mathbf{w}), \tag{2.2.3}$$

which gives the least expenditure required to produce the target output $\bar{q}$ at input prices $\mathbf{w}$. Define the *scale elasticity* as

$$\varepsilon_S = \sum_{i=1}^{n} \frac{\partial q(z_1, \ldots, z_n)}{\partial z_i} \frac{z_i}{q}$$

Classification follows immediately:

$$\varepsilon_S \begin{cases} > 1 & \text{Increasing Returns to Scale (IRS)}, \\ = 1 & \text{Constant Returns to Scale (CRS)}, \\ < 1 & \text{Decreasing Returns to Scale (DRS)}. \end{cases}$$

Additionally, the *cost elasticity* with respect to output is

$$\varepsilon_C(q) = \frac{\partial C(\bar{q}, \mathbf{w})}{\partial q} \frac{q}{C(\bar{q}, \mathbf{w})} = \frac{MC(q)}{AC(q)},$$

Hence

$$\varepsilon_C(q) \begin{cases} < 1 & \text{Economies of scale } (AC \downarrow), \\ = 1 & \text{Constant returns to scale,} \\ > 1 & \text{Diseconomies of scale } (AC \uparrow). \end{cases}$$

To illustrate these concepts, consider the following production function:

$$q(z_1, \ldots, z_n) \; = \; \left(\sum_{i=1}^{n} a_i \, z_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \qquad \sigma > 1 \tag{2.2.4}$$

The marginal product of input z_i is defined as

$$\frac{\partial q(z_1, \ldots, z_n)}{\partial z_i} = \alpha_i z_i^{\frac{-1}{\sigma}} \, q \left(\sum_{i=1}^{n} a_i \, z_i^{\frac{\sigma-1}{\sigma}} \right)^{-1}$$

The (2.2.1) would yield to:

$$\left(\frac{z_k}{z_i} \right)^{\frac{1}{\sigma}} = \left(\frac{\alpha_k w_i}{\alpha_i w_k} \right)$$

Note that evaluating the *elasticity of substitution* between inputs i and j under the optimality condition yields

$$\sigma_{ij} \; = \; \frac{\partial \big(z_j / z_i \big)}{\partial \big(w_i / w_j \big)} \frac{\big(w_i / w_j \big)}{\big(z_j / z_i \big)} = \sigma$$

Because the elasticity of substitution, σ, remains constant for every input combination, the production function is called the *Constant Elasticity of Substitution (CES)* function.
Defining the Dixit–Stiglitz CES price index as

$$W(\mathbf{w}) \; = \; \left(\sum_{i=1}^{n} \alpha_i^{\sigma} p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{2.2.5}$$

The *conditional input demand* associated to this production function can be written as

$$z_i(\bar{q}, w) = \left(\frac{\alpha_i}{w_i} \right)^{\sigma} W^{\sigma} \bar{q}$$

Minimum cost function is

$$C(\bar{q}, \mathbf{w}) = q \sum_{i=1}^{n} w_i \left(\frac{\alpha_i}{w_i} \right)^{\sigma} W^{\sigma}$$

Note that $\varepsilon_S = 1$ and $\varepsilon_C(q) = 1$.
The supply curve for the firm if the good market is competitive is as follows

$$P = \sum_{i=1}^{n} w_i \left(\frac{\alpha_i}{w_i} \right)^{\sigma} W^{\sigma}$$

Now if we were to maximize the firms profit

$$\max_{z_1,\ldots,z_n} \pi \;=\; p\, q(z_1,\ldots,z_n) - \sum_{i=1}^{n} w_i z_i$$

Assuming an interior solution; the *First Order Conditions (FOC)*

$$p\,\frac{\partial q(z_1,\ldots,z_n)}{\partial z_i} \;=\; w_i\,, \qquad \forall i = 1,\ldots,n$$

Solving the system yields the *unconditional* (profit-maximizing) input demands

$$z_i^* \;=\; z_i(p,\mathbf{w}), \qquad i = 1,\ldots,n,$$

If the technology exhibits *decreasing returns to scale* (i.e. diseconomies of scale), marginal cost is increasing so supply curve is upward-sloping in the output price. Substituting the *unconditional* factor demands into the production function therefore yields the firm's supply function:

$$q = q(z_1(p,\mathbf{w}),\ldots,z_i(p,\mathbf{w}))$$

2.3 General Equilibrium in pure exchange

Suppose an economy with I consumers and n goods. Consumer j is endowed with $\boldsymbol{\omega}_j = (\omega_{1j},\ldots,\omega_{nj}) \in \mathbb{R}^n_+$. Through trade, every consumer tries to become *more satisfied* (i.e. reach a higher utility level) than at the initial endowment.
Considering a *Decentralized equilibrium*, consumer j solves

$$\max_{x_{1j},\ldots,x_{nj}} u_j(x_{1j},\ldots,x_{nj}) \tag{2.3.1}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} p_i x_{ij} \;=\; \sum_{i=1}^{n} p_i \omega_{ij} \tag{2.3.2}$$

The first-order (interior) optimality condition reads

$$\frac{\partial u_j/\partial x_{ij}}{\partial u_j/\partial x_{kj}} \;=\; \frac{p_i}{p_k}, \qquad \forall\, i \neq k.$$

The optimality condition, once substituted into equation (2.3.2) (the budget constraint) generates individual j's demand for each good. Repeating this procedure for every individual yields the complete system of demand functions—one for each of the n goods for each of the I individuals, i.e. $n \times I$ demand functions. Note that this is the same as getting the Marshallian demand for each good and replacing income m with the value of individual j's endowment, $\sum_{i=1}^{n} p_i \omega_{ij}$.

In equilibrium, by market clearing for every good i total demand equals total supply:

$$\sum_{j=1}^{I} x_{ij} \;=\; \sum_{j=1}^{I} \omega_{ij}, \qquad i = 1, \ldots, n.$$

The resulting system of n equations determines the equilibrium price vector $\left(p_1^*, \ldots, p_n^*\right)$ (defined up to a positive scalar normalization).

The equilibrium allocation is a *Pareto equilibrium*: no individual's utility can be increased without lowering someone else's. Varying individual endowments while maintaining the aggregate endowment fixed traces out the *Pareto set*. This set can be characterized by imposing, for each individual j, the optimality condition that equates the marginal substitution rate with the equilibrium price ratio, that is,

$$\frac{\partial u_j / \partial x_{ij}}{\partial u_j / \partial x_{kj}} = \frac{p_i^*}{p_k^*}$$

Note that some allocations make one or both individuals better off relative to their initial endowments while still allowing an increase in one person's utility without reducing the other's. This collection of allocations is called the *lens of trade*. Within this lens lies that segment of the *Pareto set* where no further utility gains are possible for anyone without harming someone else. The final allocation must therefore lie inside the lens of trade and on the Pareto set.

Take the following example with 2 individual and 2 goods; each individual's utility function is given by:

$$u_j(x_{1j}, x_{2j}) = \sqrt{x_{1j} x_{2j}} \qquad j = 1, 2$$

The resulting demand functions are:

$$x_{1j} = \frac{1}{2} \cdot \frac{p_1 \omega_{1j} + p_2 \omega_{2j}}{p_1} \qquad \wedge \qquad x_{2j} = \frac{1}{2} \cdot \frac{p_1 \omega_{1j} + p_2 \omega_{2j}}{p_2} \qquad j = 1, 2$$

Market clearing in equilibrium requires that

$$\begin{cases} \frac{1}{2} \cdot \frac{p_1 \omega_{11} + p_2 \omega_{21}}{p_1} + \frac{1}{2} \cdot \frac{p_1 \omega_{12} + p_2 \omega_{22}}{p_1} = \omega_{11} + \omega_{12} \\ \frac{1}{2} \cdot \frac{p_1 \omega_{11} + p_2 \omega_{21}}{p_2} + \frac{1}{2} \cdot \frac{p_1 \omega_{12} + p_2 \omega_{22}}{p_2} = \omega_{21} + \omega_{22} \end{cases}$$

Solving the first equation for the relative price yields:

$$\frac{p_2^*}{p_1^*} = \frac{(\omega_{11} + \omega_{12})}{(\omega_{21} + \omega_{22})} \tag{2.3.3}$$

The equilibrium *relative price* is unique, whereas the absolute price vector is determined only up to a positive scalar. For instance, in (2.3.3) one may set

$$p_2 = \lambda\big(\omega_{11} + \omega_{12}\big), \qquad p_1 = \lambda\big(\omega_{21} + \omega_{22}\big),$$

for any $\lambda > 0$, leaving the ratio p_1/p_2 unchanged.
Equation (2.3.3) states that the equilibrium *relative price* equals the ratio of total endowments. In words, the price of good 1 relative to good 2 equals the economy-wide endowment of good 2 relative to that of good 1, and vice-versa.
This illustrates *Walras's Law*: if a price vector clears total demand and supply in one market, it necessarily clears the remaining market. Concretely, choose any $\lambda > 0$ and set

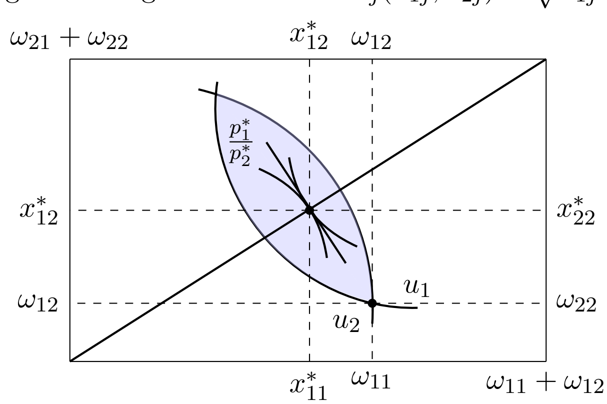
$$p_2 = \lambda\big(\omega_{11} + \omega_{12}\big), \qquad p_1 = \lambda\big(\omega_{21} + \omega_{22}\big).$$

These prices equate aggregate demand and supply in the first market; by Walras's Law, they also clear the second market. More generally, if a price vector $(p_1^*, \dots, p_n^*)$ clears $n-1$ markets, it clears all n markets.
The Pareto-equilibrium allocation is as follows

$$x_{1j}^* = \frac{\omega_{1j}}{2} + \frac{\omega_{2j}}{2}\,\frac{\omega_{21} + \omega_{22}}{\omega_{11} + \omega_{12}}, \qquad x_{2j}^* = \frac{\omega_{1j}}{2}\,\frac{\omega_{21} + \omega_{22}}{\omega_{11} + \omega_{12}} + \frac{\omega_{2j}}{2}, \qquad j = 1, 2.$$

The Pareto-equilibrium allocation shown above depends on the *individual* endowments. If these endowments are redistributed—keeping the *aggregate* endowment constant—the Pareto allocation changes, yet the equilibrium relative price remains the one in (2.3.3). Hence a redistribution can raise one person's utility while lowering the other's.
The constancy of the relative price here is a special case: both agents share identical, symmetric preferences. In general, each Pareto equilibrium that arises from a given initial allocation is supported by its own relative-price vector, as stated by the Second Welfare Theorem. In this symmetric setting, however, every Pareto allocation is backed by the same price ratio in (2.3.3). Figure 1 illustrates the Edgeworth box for this example.

Figure 1: Edgeworth box for $u_j(x_{1j}, x_{2j}) = \sqrt{x_{1j}x_{2j}}$



Social–planner problem and the First Welfare Theorem Finally, note that a benevolent social planner who reallocates each good so that the aggregate assignment equals the aggregate endowment would implement exactly the same Pareto–efficient allocation that arises in the decentralized competitive equilibrium derived in (2.3.3). Formally, the planner solves

$$\begin{aligned} &\max_{x_{ij}\ \forall i,j} \sum_{j=1}^{I} \lambda_j\, u_j\big(x_{1j}, \dots, x_{nj}\big) \\ &\text{s.t.} \qquad \sum_{j=1}^{I} x_{ij} \;=\; \sum_{j=1}^{I} \omega_{ij}, \qquad \forall\, i = 1, \dots, n, \end{aligned} \tag{2.3.4}$$

where the Pareto weights λ_j in (2.3.4) are proportional to the marginal utility of the value of household j's original endowment. Because competitive markets already equate marginal rates of substitution across agents while respecting the resource constraints, the solution to (2.3.4) coincides with the decentralized equilibrium allocation—an illustration of the First Welfare Theorem.

2.4 Exercises

1. Consider a consumer whose preferences are represented by

$$U(x_1, x_2, \ldots, x_n) = x_k \prod_{i \neq k}^{n} (x_i - \theta_i), \qquad x_i > \theta_i \quad \forall i \neq k$$

   1. Derive the Marshallian demand functions.
   2. Obtain the indirect utility function.
   3. Derive the Hicksian (compensated) demand functions.
   4. Determine the expenditure function.

2. Let the expenditure function be

$$e(\bar{u}, \mathbf{p}) = \bar{u} p_1 - \frac{p_1^2}{4} \sum_{i=2}^{n} \frac{1}{p_i}, \qquad \bar{u} > \frac{p_1}{2} \sum_{i=2}^{n} \frac{1}{p_i}$$

   1. Derive the Marshallian demand functions.
   2. Obtain the indirect utility function.
   3. Derive the Hicksian (compensated) demand functions.
   4. Recover the underlying utility function.

3. Consider a firm with the production function

$$q(L, K) = \left[\max\left\{\min\{2L, K\}, \min\{L, 2K\}\right\}\right]^{\rho}, \qquad 0 < \rho < 1$$

   1. Derive the conditional factor demand functions.
   2. Determine the cost function.
   3. Derive the firm's supply function.
   4. Obtain the unconditional factor demands[1].

4. Consider a firm with the production function

$$q(L, K) = [\min\{\max\{2L, K\}, \max\{L, 2K\}\}]^{\rho}, \qquad 0 < \rho < 1$$

   1. Derive the conditional factor demand functions.
   2. Determine the cost function.
   3. Derive the firm's supply function.
   4. Obtain the unconditional factor demands.

[1]Consider $z_i(p, \mathbf{w}) = z_i(q(p, z_i(p, \mathbf{w})), \mathbf{w})$

5. Consider a pure-exchange economy with two consumers. Consumer A's utility is

$$u^A(x_{1A}, x_{2A}) = 2x_{1A} + x_{2A}$$

while consumer B's utility is

$$u^B(x_{1B}, x_{2B}) = \min\{x_{1B}, x_{2B}\}$$

Their endowments are $w^A = (0, \bar{w})$ and $w^B = (\bar{w}, 0)$

    1. State the initial endowment point.
    2. Determine the lens of trade.
    3. Characterize the Pareto set.
    4. Compute the equilibrium relative price.
    5. Identify the set of equilibrium allocations.

6. A small town has n residents. Resident i is endowed with $\bar{w}$ units of good i and none of the other goods. Every resident's preferences are

$$u_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} \ln x_j$$

    1. Derive the equilibrium relative prices.
    2. Characterize the Pareto set.
    3. Identify the equilibrium allocation set.
    4. Show that equilibrium prices are independent of the endowments and explain the intuition.
    5. Verify that the centralized allocation matches the competitive equilibrium.

3 The Export Condition and Ricardian Model

3.1 The Export Condition

Suppose there is Home (H) and (F) and n goods. Each good i requires a_i units of labor per unit of output, and a_i^* units abroad. The wage rate in Home is w, and in Foreign it is w^*. If e denotes the nominal exchange rate (units of Foreign currency per unit of Home currency), the unit cost of producing good i is

$$c_i^H = e \cdot w \cdot a_i, \qquad c_i^F = w^* \cdot a_i^*$$

Here, c_i^H is measured in Foreign currency by multiplying by e, while c_i^F is expressed in Foreign currency. This adjustment ensures cost to be measured in the same currency.

The fundamental export condition is that Home exports good i if it can supply the good at lower cost than Foreign:

$$c_i^H < c_i^F.$$

Substituting from above:

$$e \cdot w \cdot a_i < w^* \cdot a_i^* \tag{3.1.1}$$

Equation (3.1.1) can be written to state that Home exports good i if

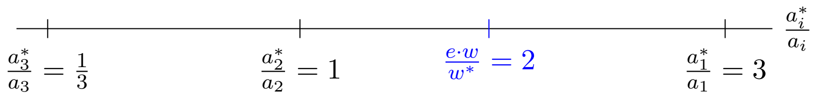
$$\frac{e \cdot w}{w^*} < \frac{a_i^*}{a_i} \tag{3.1.2}$$

Equation (3.1.2) can be interpreted as follows: Home exports good i whenever the relative wage—that is, the wage in Home expressed in terms of the Foreign wage—is lower than the relative cost of producing the good abroad, expressed in terms of Home's cost. The right-hand side of the inequality thus represents the relative unit labor requirements, capturing the notion of comparative efficiency.

To illustrate, consider three goods x_1, x_2, and x_3. The unit labor requirements in Home are $(1, 2, 3)$, while in Foreign they are $(3, 2, 1)$. Suppose the Home wage (in Foreign currency) is 2, and the Foreign wage (in Foreign currency) is 1. According to the export condition, Home will have a cost advantage in goods x_1, thereby producing and exporting them, while it will import x_2 and x_3 from Foreign.

Figure 2 illustrates this scenario. The goods are ordered from lowest to highest according to Home's relative unit requirements. Relative wage adjusted by exchange rate is in blue. All goods positioned to the right of relative wage are produced and exported by Home (and imported by Foreign), while those to the left are produced and exported by Foreign (and imported by Home).

Figure 2: Export condition in example given



3.1.1 Transport Costs

Consider the presence of transport costs, denoted by τ, which are assumed to be identical across countries and goods. The cost τ is measured in units of the good itself. Such costs are commonly referred to as *iceberg costs*, since τ represents the additional quantity of the good that must be shipped for one unit to arrive in the destination country.

The export condition for Home to supply good i under iceberg transport costs is given by

$$(1+\tau)e \cdot w \cdot a_i < w^* \cdot a_i^*, \tag{3.1.3}$$

which can be rewritten as

$$\frac{e \cdot w}{w^*} < \frac{a_i^*}{(1+\tau) \cdot a_i}, \tag{3.1.4}$$

where equation (3.1.4) expresses the condition in terms of relative wages.

Suppose that in the absence of transport costs, Home exports good i and Foreign exports good j. Once transport costs are introduced, the two goods may become nontraded if

$$(1+\tau)e \cdot w \cdot a_i > w^* \cdot a_i^*, \tag{3.1.5}$$

$$e \cdot w \cdot a_j < w^* \cdot a_j^*(1+\tau). \tag{3.1.6}$$

Although Home is assumed to have a comparative advantage in good i, a sufficiently high transport cost may lead condition (3.1.5) to hold, implying that Home no longer exports good i because the effective cost of exporting exceeds the cost of production in Foreign. Analogously, the same reasoning applies to Foreign with respect to good j in condition (3.1.6).

Equations (3.1.5) and (3.1.6) can be equivalently expressed as

$$\frac{(1+\tau)e \cdot w}{w^*} > \frac{a_i^*}{a_i}, \tag{3.1.7}$$

$$\frac{e \cdot w}{(1+\tau)w^*} < \frac{a_j^*}{a_j}. \tag{3.1.8}$$

Considering multiple goods, these inequalities define the range of nontraded goods. The relative requirements that satisfy equations (3.1.7) and (3.1.8) with equality determine the boundaries of the nontraded sector. Any good k whose relative requirement lies within these bounds will not be traded internationally.

3.2 The Ricardian Model of International Trade

Consider a world economy with two countries: Home (denoted by H) and Foreign (denoted by F). The economy produces two goods, indexed by q_1^i and q_2^i, where $i \in \{H, F\}$. Labor is the only factor of production, and each country is endowed with a fixed labor supply $L^i > 0$.
Technology is characterized by constant unit labor requirements: producing one unit of good q_1 in country i requires a_1^i units of labor, while producing one unit of good q_2 requires a_2^i units of labor. We assume $a_1^i, a_2^i > 0$ and constant returns to scale.

3.2.1 Production

The production function for good $i \in \{1, 2\}$ in country $j \in \{H, F\}$ is given by:

$$q_i^j = \frac{L_i^j}{a_i^j} \tag{3.2.1}$$

where L_i^j denotes the amount of labor allocated to sector i in country j.
This functional form reflects a fixed-coefficient technology: each unit of output q_i^j requires exactly a_i^j units of labor. Equivalently, $1/a_i^j$ is the marginal product of labor in sector i of country j.
Labor is perfectly mobile across sectors within a country but immobile across countries. The labor resource constraint is therefore:

$$L_1^j + L_2^j = L^j, \qquad j \in \{H, F\}, \tag{3.2.2}$$

which reflects the fact that total labor demand must equal the exogenously supply labor.

3.2.2 Production Possibility Frontier

To characterize the set of feasible output combinations, substitute $L_i^j = a_i^j q_i^j$ into the labor constraint:

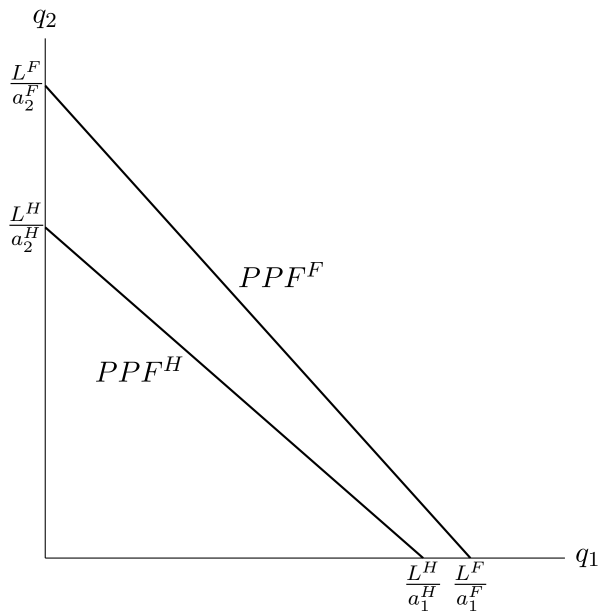
$$a_1^j q_1^j + a_2^j q_2^j = L^j, \qquad j \in \{H, F\}. \tag{3.2.3}$$

Equation (3.2.3) is a linear relationship between q_1^j and q_2^j with slope $-\frac{a_1^j}{a_2^j}$, which we refer to as the **relative requirements**. The intercepts are $\frac{L^j}{a_1^j}$ on the q_1^j axis and $\frac{L^j}{a_2^j}$ on the q_2^j axis. These represent the maximum quantity of each good that country j could produce if it devoted all its labor to that sector. Equation (3.2.3) thus defines the Production Possibility Frontier (PPF).
The linearity of the PPF follows directly from the assumption of constant unit labor requirements. Its slope, $-\frac{a_1^j}{a_2^j}$, measures the opportunity cost of producing one unit of q_1^j in terms of forgone units of q_2^j. Since this opportunity cost is constant, the model does not feature diminishing returns to specialization. As a result, **full specialization naturally emerges**

under trade.

Figure 3 illustrates the PPF in a case where Home has a comparative advantage in good q_1. The intercepts depend on each country's labor supply and labor requirements. Assuming $L^H = L^F$, the figure suggests that Foreign enjoys an absolute advantage in both goods, even though Home maintains a comparative advantage in good q_1.

Figure 3: Production Possibility Frontier



3.2.3 Autarky

In autarky, competitive equilibrium requires that the relative price of the two goods equals their opportunity cost in production. Let p_1^j and p_2^j denote the prices of goods q_1^j and q_2^j in country j, respectively. Under perfect competition and zero profits, the unit cost of producing good i must equal its price:

$$p_1^j = w^j a_1^j, \tag{3.2.4}$$
$$p_2^j = w^j a_2^j \tag{3.2.5}$$

where w^j is the wage in country j. Dividing equation (3.2.4) by equation (3.2.5) yields:

$$\frac{p_1^j}{p_2^j} = \frac{a_1^j}{a_2^j}. \tag{3.2.6}$$

Thus, in autarky, the relative price equals the constant marginal rate of transformation implied by the PPF.

3.2.4 Opening to Trade

When the economy opens to trade, the relevant relative price is the *world* relative price, $\frac{p_1^W}{p_2^W}$. Suppose that Home has a comparative advantage[2] in good q_1, meaning:

$$\frac{a_1^H}{a_2^H} < \frac{a_1^F}{a_2^F}, \tag{3.2.7}$$

where a_i^F are the unit labor requirements in Foreign.
Comparative advantage is therefore determined entirely by the ratio of unit labor requirements across goods and countries. If Home has a comparative advantage in q_1, then by construction, Foreign must have a comparative advantage in q_2.
It is also possible for one country to have an *absolute advantage*[3] in both goods. Absolute advantage is defined by direct productivity levels, while comparative advantage arises from relative productivity differences and ultimately governs trade patterns.
Figure 4 illustrates the relative offer curve of the model once the economy opens to trade. The derivation, assuming that Home has a comparative advantage in good q_1, proceeds as follows:

- If the world relative price $\frac{p_1^W}{p_2^W}$ is lower than both Home's autarky price ratio $\frac{a_1^H}{a_2^H}$ and Foreign's autarky price ratio $\frac{a_1^F}{a_2^F}$, then both Home and Foreign fully specialize in the production of q_2. In this case:

$$q_2^W = \frac{L^H}{a_2^H} + \frac{L^F}{a_2^F}, \qquad q_1^W = 0$$

  Thus, $\frac{q_1^W}{q_2^W} = 0$, since both countries specialize in q_2 (whose relative price exceeds its relative cost in both economies). This corresponds to the segment A–B in Figure 4.

- If the world relative price $\frac{p_1^W}{p_2^W}$ lies between Home's autarky price ratio $\frac{a_1^H}{a_2^H}$ and Foreign's autarky price ratio $\frac{a_1^F}{a_2^F}$, then Home fully specializes in q_1 and Foreign in q_2. In this case:

$$q_1^W = \frac{L^H}{a_1^H}, \qquad q_2^W = \frac{L^F}{a_2^F}$$

  Therefore,

$$\frac{q_1^W}{q_2^W} = \frac{L^H/a_1^H}{L^F/a_2^F}$$

  This scenario is the most economically relevant: each country specializes in the good for which it has comparative advantage. It corresponds to the segment C–D in Figure 4.

[2] A country has a comparative advantage in good i if its relative cost of producing i is lower than that of the other country. In other words, Home sacrifices less of good j to produce one unit of i compared to Foreign.

[3] A country has an absolute advantage in good i if, with the same resources, it can produce more of i than the other country. Formally, country j has absolute advantage in good i if $a_i^j < a_i^{-j}$, where $-j$ denotes the other country.

- If the world relative price $\frac{p_1^W}{p_2^W}$ is higher than both Home's and Foreign's autarky price ratios, then both countries fully specialize in q_1. In this case:

$$q_1^W = \frac{L^H}{a_1^H} + \frac{L^F}{a_1^F}, \qquad q_2^W = 0$$

Hence, $\frac{q_1^W}{q_2^W} = \infty$, as both countries allocate all resources to q_1. This corresponds to the segment D–∞ on the horizontal axis in Figure 4.

- If the world relative price equals Home's autarky price ratio, $\frac{p_1^W}{p_2^W} = \frac{a_1^H}{a_2^H}$, then Home is indifferent between producing q_1, q_2, or any combination of both. If it participates in trade, it may choose any production in its PPF, while Foreign specializes in q_2. In this case:

$$\frac{q_1^W}{q_2^W} \in \left[0,\ \frac{L^H/a_1^H}{L^F/a_2^F}\right]$$

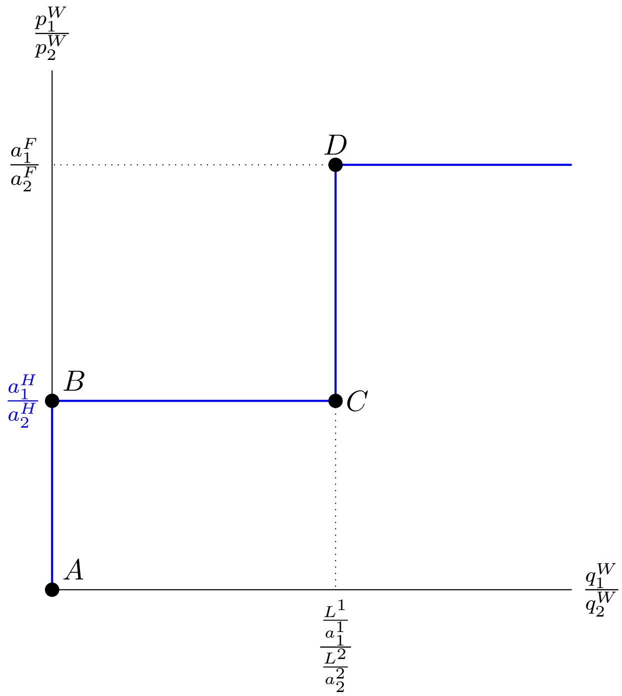
This corresponds to the segment B–C in Figure 4.

- If the world relative price equals Foreign's autarky price ratio, $\frac{p_1^W}{p_2^W} = \frac{a_1^F}{a_2^F}$, then Foreign is indifferent between producing q_1, q_2, or any combination of both. If it participates in trade, it may choose any production in its PPF, while Home specializes in q_1. In this case:

$$\frac{q_1^W}{q_2^W} \in \left[\frac{L^H/a_1^H}{L^F/a_2^F},\ \infty\right]$$

This corresponds to the segment D–∞ in Figure 4.

Figure 4: Relative Market in the Ricardian model



As discussed earlier, the equilibrium arises when each country specializes in a different good. Figure 5 illustrates this equilibrium outcome.
Assume that both countries share a homothetic utility function. For example, let preferences be represented by

$$U_j(q_1, q_2) = q_1 q_2, \qquad j \in H, F$$

The corresponding optimality condition is:

$$\frac{q_2}{q_1} = \frac{p_1}{p_2} \tag{3.2.8}$$

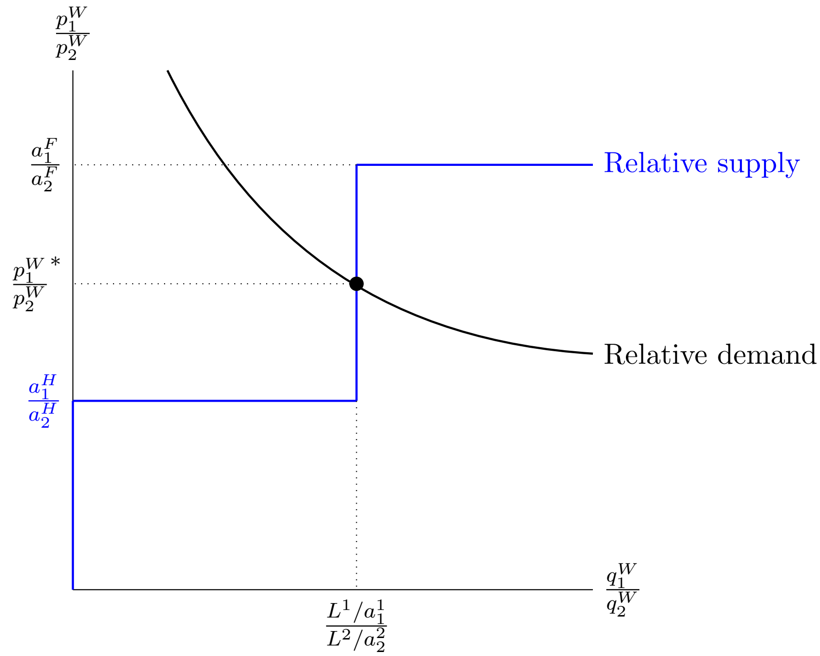
Equation (3.2.8) can be expressed as

$$\frac{p_1}{p_2} = \frac{1}{q_1/q_2} \tag{3.2.9}$$

Equation (3.2.9) represents the relative demand. This curve has a negative slope and is convex in relative quantities. To determine the equilibrium world relative price, the ratio $\frac{q_1^W}{q_2^W}$ can be substituted for the case in which each country specializes in the good for which it holds a comparative advantage. The resulting expression yields the equilibrium world relative price:

$$\frac{p_1^W}{p_2^W} = \frac{L^2/a_2^2}{L^1/a_1^1} \tag{3.2.10}$$

Figure 5: Relative Market in the Ricardian model



For the relative demand curve to intersect the vertical segment of the relative supply curve at the relative quantity $\frac{L^H/a_1^H}{L^F/a_2^F}$ in Figure 5, the equilibrium world relative price must satisfy

$$\left(\frac{p_1^W}{p_2^W}\right)^* \in \left]\frac{a_1^H}{a_2^H},\ \frac{a_1^F}{a_2^F}\right[$$

3.3 Exercises

1. Consider the Export Condition studied here (Section 3.1) where there are three goods and three countries. Table 1. reports the unit labor requirements for each good in each country.

Requirements	**A**	**B**	**C**
a_1	2	3	4
a_2	4	3	1
a_3	1	2	4

    1. Suppose it is known that country A produces and exports good x_3, country B produces and exports good x_1, and country C produces and exports good x_2. What must be true about the relative wages, expressed in the currency of country C?

2. Based on the export condition (Section 3.1), consider an economy with n goods, where wages are identical across countries, expressed in a common currency. The labor requirements are specified as

$$(a_1, a_2, \dots, a_n) = (1, 2, \dots, n), \quad (a_1^*, a_2^*, \dots, a_n^*) = (n, n-1, \dots, 1).$$

    1. Determine which goods are produced and exported by Home, and which goods are produced and exported by Foreign.
    2. Assume a transport cost of $\frac{1}{4}$ and $n = 10$. Identify the set of goods produced and exported by Home and those produced and exported by Foreign. Additionally, determine whether any goods become nontraded.
    3. Consider a transport cost equal to τ and n goods. Characterize the goods produced and exported by Home and Foreign. Furthermore, assess the existence of nontraded goods as a function of τ. Demonstrate that the range of nontraded goods expands as the transport cost increases.

3. Consider the Ricardian model (Section 3.2) where country A requires 2 units of labor to produce one unit of q_1 and 1 unit of labor to produce one unit of q_2. Country B requires 4 units of labor to produce one unit of q_1 and 3 units of labor to produce one unit of q_2. The labor endowment in each country is not specified.

    1. Identify which country has absolute advantage and which has comparative advantage.
    2. Derive the Production Possibility Frontier (PPF) for each country.
    3. Obtain the world relative supply.
    4. Assuming preferences are represented by $U(q_1, q_2) = q_1^2 q_2$ in both countries, determine equilibrium production, relative prices, and wages under autarky.

5. Using the same utility function, determine equilibrium production, relative prices, and wages under international trade.
6. Propose a method to evaluate whether each country is better off under trade compared to autarky.

4. Consider the Ricardian model (Section 3.2) where Home has a comparative advantage in q_2. Suppose the economy is open to trade and equilibrium occurs at a point where each country fully specializes in the good for which it has comparative advantage. Answer the following questions and explain the underlying intuition:

    1. Why can Home still benefit from trade even if it has an absolute advantage in both goods?
    2. What happens if the population in Home increases? Is Home better off? Is Foreign better off?
    3. What happens if the population in Foreign decreases? Is Home better off? Is Foreign better off?
    4. What happens if technology improves in Home at the same proportional rate for both goods? Is Home better off? Is Foreign worse off?
    5. What happens if technology improves in Foreign for good q_1 only? Is Home better off? Is Foreign worse off?

4 Comparative Advantage, Trade, and Payments in a Ricardian Model with a Continuum of Goods (Dornbusch et al., 1977)

Dornbusch et al. (1977) extend the basic Ricardian framework to a continuum of goods. Consider two countries, Home (H) and Foreign (F), and a single factor of production, labor, with endowments L^H and L^F, respectively. Labor is assumed to be perfectly mobile across sectors within each country but immobile across countries. Markets operate under perfect competition and exhibit constant returns to scale. Wages are denoted by w^H in Home and w^F in Foreign. The exchange rate, defined as the price of one unit of Foreign's currency in terms of Home's currency, is represented by e^H.

4.1 Supply

Relative efficiency between the two countries for any good z is defined as

$$A(z) = \frac{a^F(z)}{a^H(z)}, \qquad A'(z) < 0 \tag{4.1.1}$$

where $a^H(z)$ denotes the unit labor requirement (units of labor necessary to produce one unit of good z) in Home, and $a^F(z)$ represents the corresponding requirement in Foreign.

Goods are represented by a continuum indexed by $z \in [0, 1]$, ordered such that Home's comparative advantage decreases with z. This implies that goods located near $z = 0$ are those in which the relative productive efficiency of Home with respect to Foreign is most evident, whereas goods located near $z = 1$ are those in which the relative productive efficiency of Foreign is most pronounced. This ordering justifies the assumption $A'(z) < 0$ in equation (4.1.1).

Under free trade, each good z is produced in the country with the lower unit cost. The export condition requires that Home produces z if

$$a^H(z)\, e^H w^H \le a^F(z)\, w^F \quad \Leftrightarrow \quad \frac{e^H w^H}{w^F} \le \frac{a^F(z)}{a^H(z)}$$

The cutoff good $\tilde{z}$ is defined at the point where the unit costs are exactly equal across countries. This condition determines the supply schedule for the model:

$$\frac{e^H w^H}{w^F} = \frac{a^F(z)}{a^H(z)} \tag{4.1.2}$$

All goods with $z < \tilde{z}$ are produced by Home (the range in which it holds comparative advantage), while all goods with $z > \tilde{z}$ are produced by Foreign.

Given constant returns to scale and perfect competition, the price of good z equals its marginal cost. Hence, the relative price of two goods produced in Home is

$$\frac{p^H(z)}{p^H(z')} = \frac{w^H \cdot a^H(z)}{w^H \cdot a^H(z')} = \frac{a^H(z)}{a^H(z')}$$

which corresponds to the relative labor requirements between the two goods.
By contrast, the relative price of a Home-produced good z in terms of a Foreign-produced good z'' is given by

$$\frac{p^H(z)}{p^F(z'')} = \frac{e^H w^H \cdot a^H(z)}{w^F \cdot a^F(z'')}$$

4.2 Demand

Preferences are assumed to be identical and homothetic across countries. Let $b^H(z)$ denote the *expenditure-share density function* for good z in Home, which must satisfy

$$\int_0^1 b^H(z)\,dz = 1, \qquad b^H(z) = \frac{p^H(z)\,x^H(z)}{y^H} > 0 \tag{4.2.1}$$

where $p^H(z)$ is the price of good z in Home, $x^H(z)$ is the quantity consumed in Home, and y^H denotes Home's income. Hence, $b^H(z)$ represents the fraction of home's income spent on good z. Because preferences are identical across countries, it follows that $b^H(z) = b^F(z)$.
Define

$$B(\tilde{z}) = \int_0^{\tilde{z}} b^H(z)\,dz \tag{4.2.2}$$

where $B(\tilde{z})$ is the share of global spending on Home-produced goods.
World income is given by

$$Y^W = e^H w^H L^H + w^F L^F \tag{4.2.3}$$

In equilibrium, the value of Home's exports must equal the value of its production:

$$B(\tilde{z})\,Y^W = e^H w^H L^H \tag{4.2.4}$$

Substituting (4.2.2) and (4.2.3) into (4.2.4) and solving for relative factor prices yields the demand side of the model:

$$\frac{e^H w^H}{w^F} = \left(\frac{\int_0^{\tilde{z}} b^H(z)\,dz}{1 - \int_0^{\tilde{z}} b^H(z)\,dz}\right)\frac{L^F}{L^H} \tag{4.2.5}$$

Note that $1 - \int_0^{\tilde{z}} b^H(z)\,dz$ represents the share of global expenditure on Foreign-produced goods.
Equation (4.2.5) can be interpreted as follows: if the range of domestically produced goods were to expand at constant relative wages, demand for domestic labor (and goods) would rise while demand for Foreign labor would decline. To restore equilibrium, an increase in the

domestic relative wage is required. Consequently, equation (4.2.5) is upward-sloping in relative wages.

Another way to understand why equation (4.2.5) is upward-sloping is that, as when z increases, the share of income allocated to home goods rises, while the share allocated to foreign goods declines. Consequently, $\left(\frac{\int_0^{\tilde{z}} b^H(z)\,dz}{1-\int_0^{\tilde{z}} b^H(z)\,dz}\right)$ increases.

An alternative formulation of the trade balance condition is

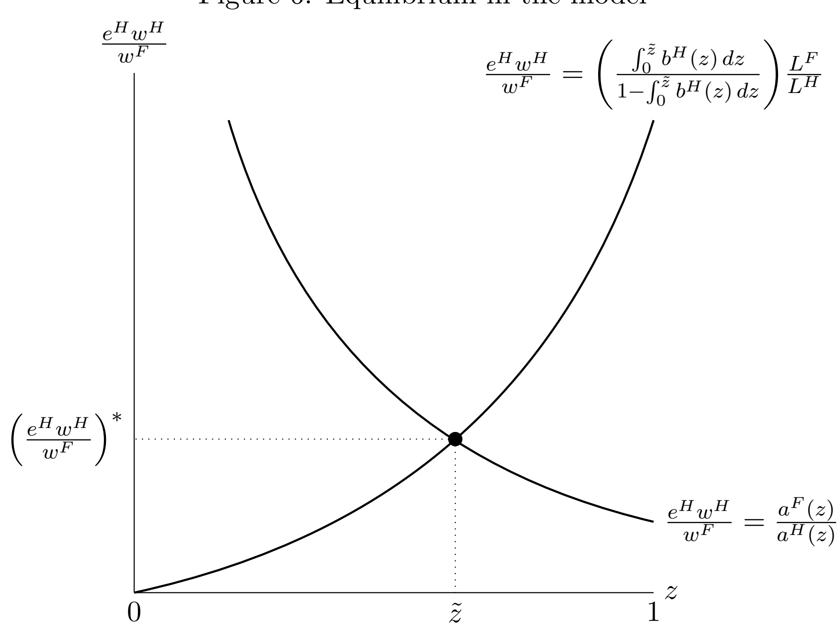
$$e^H w^H L^H \left(1 - \int_0^{\tilde{z}} b^H(z)\,dz\right) = w^F L^F \int_0^{\tilde{z}} b^H(z)\,dz \tag{4.2.6}$$

Equation (4.2.6) states that equilibrium in trade requires imports (the left-hand side) to equal exports (the right-hand side). The schedule implied by (4.2.6) is upward-sloping because an expansion in the range of goods produced at Home, at constant relative wages, reduces imports and increases exports. The resulting imbalance must be corrected through an increase in Home's relative wage.

4.3 Equilibrium

Equilibrium is attained at the relative wage that equalizes the right-hand sides of equations (4.1.2) and (4.2.5). This condition determines $\tilde{z}$, the equilibrium cutoff that delineates the range of goods produced and exported by Home from those produced and exported by Foreign. Figure 6 illustrates the equilibrium in this framework.

Figure 6: Equilibrium in the model



Note that, given a home wage w^H, all goods produced and exported by Home (i.e., $z \in [0, \tilde{z}]$) have a price equal to the Home autarky price for that z, expressed in foreign currency.

Meanwhile, all goods produced and exported by Foreign are priced at the foreign autarky price. Formally, this can be written as

$$p(z) = \begin{cases} e^H w^H a^H(z), & z \in [0, \tilde{z}], \\ w^F a^F(z), & z \in [\tilde{z}, 1]. \end{cases} \tag{4.3.1}$$

If welfare in Home is measured by its real wage, observe that for those goods exported by Home, the real wage in equilibrium remains the same as under autarky. However, for goods imported by Home, the real wage is higher, since the foreign price of these goods is lower than the autarky price, given the export condition. This implies that Home is better off when trade is opened. The same intuition applies symmetrically to the foreign country.
To illustrate these mechanisms more clearly, consider a discrete setting with n goods, where the two countries are characterized by the following production functions:

$$x_i^H = \frac{1}{i} L_i^H, \qquad x_i^F = \frac{1}{(n-i+1)} L_i^F$$

The utility function is specified as

$$U^j(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^i, \qquad j = H, F$$

The supply schedule is given by

$$\frac{e^H w^H}{w^F} = \frac{(n-i+1)}{i} \tag{4.3.2}$$

The fraction of national income spent on good i is

$$b_i = \frac{2i}{n(n+1)}$$

Accordingly, the fraction of world income devoted to goods produced by Home is

$$\sum_{i=1}^{i^*} \frac{2i}{n(n+1)} = \frac{i^*(i^*+1)}{n(n+1)}$$

where i^* denotes the cutoff good separating production between Home and Foreign. This condition implies the following demand schedule:

$$\frac{e^H w^H}{w^F} = \left(\frac{\frac{i^*(i^*+1)}{n(n+1)}}{1 - \frac{i^*(i^*+1)}{n(n+1)}} \right) \frac{L^F}{L^H} \tag{4.3.3}$$

The cutoff i^* is determined by combining (4.3.2) and (4.3.3) and solving for i^*:

$$\frac{(n - i^* + 1)}{i^*} = \left(\frac{\frac{i^*(i^*+1)}{n(n+1)}}{1 - \frac{i^*(i^*+1)}{n(n+1)}} \right) \frac{L^F}{L^H} \tag{4.3.4}$$

Expression (4.3.4) implies that i^* increases with either n or L^H, whereas i^* decreases with an increase in L^F.

Note expression (4.3.4) is increasing in i^* if n or L^H increases and i^* decreases if L^F increases.

4.4 The Price-Specie Flow Mechanism

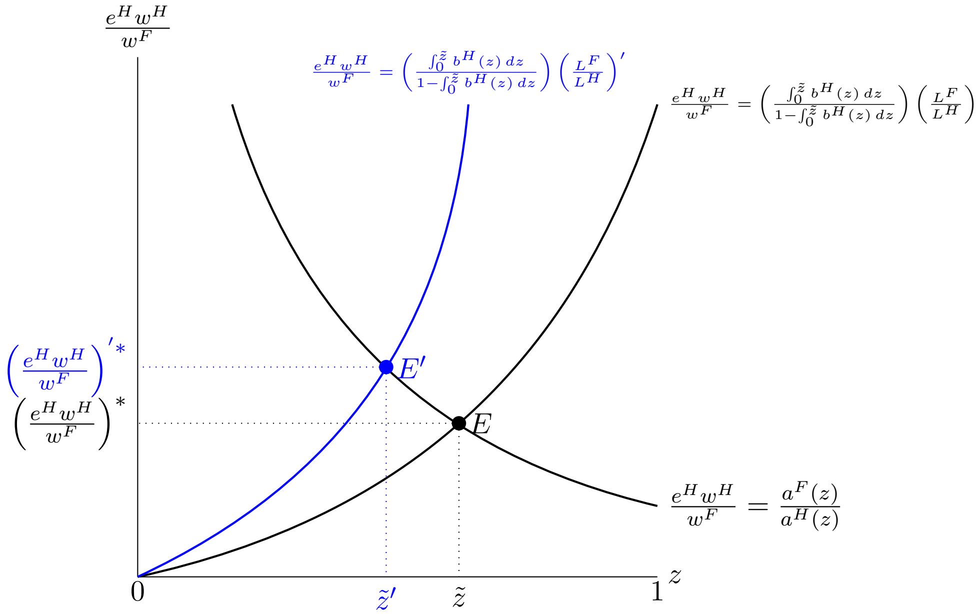
This mechanism was first articulated by Hume (1752) in which gold or silver served as the means of international settlement. Suppose that Home initially runs a trade surplus. The excess demand for Home goods implies that foreign buyers must pay in specie, leading to an inflow of gold or silver into Home. As specie enters the economy, the domestic money supply expands, generating an increase in the price level of Home goods. This raises Home's wage measured in specie, thereby eroding the set of goods for which Home has a comparative cost advantage. Goods that were previously competitive for Home shift toward the Foreign production margin. Consequently, Home's exports contract and imports expand until there is a balance in trade.

4.5 Comparative statics in the model

Equilibrium in Figure 6 is determined by preferences, technology in both Home and Foreign, and the labor endowments of each country. Suppose there is an increase in the relative labor endowment $\frac{L^F}{L^H}$. This shift raises the right-hand side of equation (4.2.5), thereby shifting the demand schedule upward.

At the initial equilibrium, this adjustment generates an imbalance in equation (4.2.5) (or equivalently in equation (4.2.6)), as the value of Home's exports exceeds the value of its imports. According to the Price–Specie Flow Mechanism, this imbalance induces an increase in wages at Home. The resulting rise in relative wages reduces Home's competitiveness, causing the cutoff $\bar{z}$ to decline until a new equilibrium is reached at point E', as illustrated in Figure 7.

Figure 7: Effect of an increase in relative labor endowment $\frac{L^F}{L^H}$



Suppose that w^H remains constant while w^F decreases, thereby raising Home's relative wage in the new equilibrium shown in Figure 7. The welfare analysis can then be summarized as follows:

1. For $z \in [0, \tilde{z}']$, the price of z remains unchanged after the increase in the foreign relative labor endowment. This is because neither w^H nor $a^H(z)$ is affected.

2. For $z \in (\tilde{z}, 1]$, the price of z is lower after the increase in foreign labor endowment because w^F falls while $a^F(z)$ remains unchanged.

3. For $z \in (\tilde{z}, \tilde{z}')$, the price of z becomes

$$p'(z) = w'^F a^F(z),$$

which is strictly lower than its equilibrium price before the change in the foreign relative labor endowment, that is,

$$p(z) = e^H w^H a^H(z),$$

due to the export condition.

From this analysis, and measuring welfare by the relative wage, it follows that Home is better off after the increase in the foreign relative labor endowment, while Foreign is worse off. Home benefits because the price of z is lower for $z > \tilde{z}'$ while its wage remains constant. In contrast, the foreign relative wage does not change for $z \in [\tilde{z}, 1]$, but it decreases for $z < \tilde{z}$.

4.6 Exercises

1. Examine the behavior of the exchange rate in recent years for Costa Rica, measured in colones per U.S. dollar.

Table 1: Average exchange rate: colones per U.S. dollar

Year	Bid	Ask
2022	₡644	₡651
2023	₡541	₡547
2024	₡512	₡518
2025[4]	₡502	₡508
Average 2022 to 2025	**571**	**577**

   1. Based on the model developed in this section, analyze the implications of this evolution for Costa Rica. Illustrate your explanation with a graph.

2. Using the model studied in this section, analyze the effects of the following changes (support your answer with graphs):

   1. An increase in the population (and therefore the labor force) in Home.
   2. A decrease in the population (and therefore the labor force) in Foreign.
   3. An improvement in technology in Home.
   4. An improvement in technology in Foreign.
   5. A stronger preference for goods produced in Foreign.

3. Consider a continuum of goods indexed by $z \in [0, 1]$, with the following production functions for Home and Foreign:

$$x^H(z) = \frac{L^H(z)}{(\alpha\sqrt{z} - b\sqrt[3]{z^2})}, \qquad x^F(z) = \frac{L^F(z)}{\sqrt{z}}$$

Preferences in both countries are represented by the utility function

$$U = \int_0^1 \ln(x)\, dx$$

Initially, the labor endowment is the same in both countries.

   1. Derive the supply curve implied by the model.
   2. Derive the demand curve implied by the model.

3. Determine the cutoff (borderline) between goods produced by Home and those produced by Foreign. Also, compute relative wages in equilibrium.
4. Compute the production of each good.
5. Suppose Foreign's labor endowment increases by 5%. Analyze how this change affects the equilibrium and provide the economic intuition behind the result.

5 General Equilibrium in production

Two equilibrium concepts are distinguished: (i) Equilibrium with only production, and (ii) Equilibrium with consumption and production. Together, these concepts provide the theoretical foundation for the *Heckscher–Ohlin–Vanek* model.

5.1 Equilibrium with only production

Consider an economy that produces two goods, q_1 and q_2, using two factors of production—labour (L) and capital (K). The aggregate endowments of these factors are $\bar{L}$ and $\bar{K}$, respectively. Each sector is described by a production function exhibiting constant returns to scale (i.e. a function homogeneous of degree 1).

Assume that the two sectors differ in factor intensities: good q_i is *capital-intensive* relative to good q_j, meaning

$$\frac{K_i^*(q_i,w,r)}{L_i^*(q_i,w,r)} > \frac{K_j^*(q_j,w,r)}{L_j^*(q_j,w,r)}$$

This ordering is invariant for all admissible factor-price pairs (w,r); that is, there is *no factor-intensity reversal*.

Solving the problem:

$$\begin{aligned} \min_{L_i,K_i} \quad & wL_i + rK_i \\ \text{s.t.} \quad & q_i = q_i(L_i,K_i)\,, \qquad \forall i = 1,2 \end{aligned}$$

The optimization delivers the conditional factor demands $L_i^*(q_i,w,r)$ and $K_i^*(q_i,w,r)$. Substituting these expressions into the total cost function generates the minimum total cost (same as unit cost as per constant returns to scale) for the good i, denoted $C_i(q_i,w,r)$. Differentiating C_i with respect to the output yields the marginal cost and because the production function exhibits constant returns to scale, profits are zero, and under perfect competition the good's price equals its (constant) marginal cost

$$p_i \;=\; \frac{\partial C_i(q_i,w,r)}{\partial q_i} = c_i(w,r)$$

Let a_{Li} and a_{Ki} denote the labour and capital required to produce one unit of good i ($q_i = 1$):

$$a_{Li} \equiv L_i^*(1,w,r), \qquad a_{Ki} \equiv K_i^*(1,w,r)$$

Consequently, the zero-profit conditions can be expressed as

$$\begin{cases} p_1 = w\,a_{L1} + r\,a_{K1}, \\ p_2 = w\,a_{L2} + r\,a_{K2} \end{cases} \tag{5.1.1}$$

where prices equal unit costs for each sector.

Factor–market clearing (full utilization of labor and capital) requires

$$\begin{cases} \bar{L} = L_i^*(q_i, w, r) + L_j^*(q_j, w, r), \\ \bar{K} = K_i^*(q_i, w, r) + K_j^*(q_j, w, r) \end{cases} \tag{5.1.2}$$

which is equivalent to

$$\begin{cases} \bar{L} = a_{Li}q_i + a_{Lj}q_j, \\ \bar{K} = a_{Ki}q_i + a_{Kj}q_j \end{cases}$$

Solving this linear system yields the equilibrium outputs (q_i^*, q_j^*).

Sector i is *labor-intensive* relative to sector j if

$$\frac{a_{Li}}{a_{Ki}} > \frac{a_{Lj}}{a_{Kj}} \quad \iff \quad \frac{L_i^*}{K_i^*} > \frac{L_j^*}{K_j^*}$$

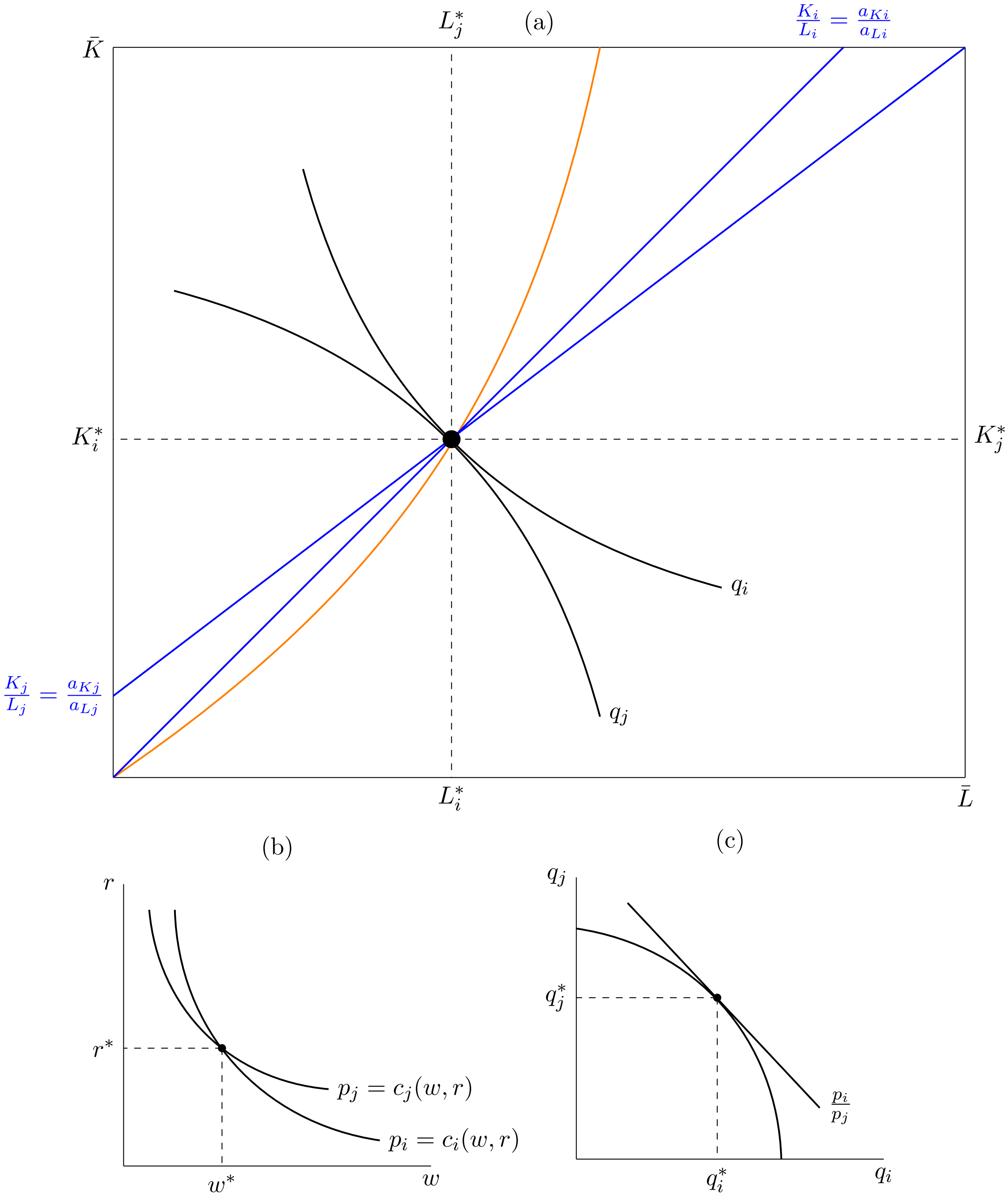
This ranking is invariant to all admissible factor-price pairs (w, r), reflecting the assumption of *no factor-intensity reversal.*

The equilibrium allocation of labor and capital in each sector must satisfy the optimal factor ratio:

$$\frac{L_i}{K_i} = \frac{a_{Li}}{a_{Ki}}, \qquad i = 1, 2 \tag{5.1.3}$$

Figure 8 summarizes the equilibrium. Panel (a) presents the Production Edgeworth box: the equilibrium allocation is the point where the two marginal rates of technical substitution coincide, lying on the production Pareto set (orange curve). The blue rays correspond to equation (5.1.3). Panel (b) plots equations (5.1.1); the intersection of the two lines determines the equilibrium factor–price. Panel (c) depicts the production–possibility frontier, whose slope—the marginal rate of transformation—equals the relative price of the goods.

Figure 8: Equilibrium in production



Stolper–Samuelson Theorem: Suppose good i is labour-intensive and its relative price rises ($dp_i > dp_j$). Then the real wage of labor, w, increases, whereas the real return to capital, r, decreases. Moreover, the proportional change in w exceeds that in p_1, while the change in r is smaller.

As established above, under competitive markets the price of each good equals its unit cost:

$$\begin{cases} p_1 = c_1(w, r), \\ p_2 = c_2(w, r) \end{cases} \tag{5.1.4}$$

Taking total differentials of system (5.1.4) and applying Shepard's lemma yields

$$\begin{cases} dp_1 = a_{L1}\, dw + a_{K1}\, dr, \\ dp_2 = a_{L2}\, dw + a_{K2}\, dr, \end{cases} \tag{5.1.5}$$

Assume that $dp_2 = 0$, so that the price of good 2 remains constant, while p_1 increases. The system (5.1.5) then implies

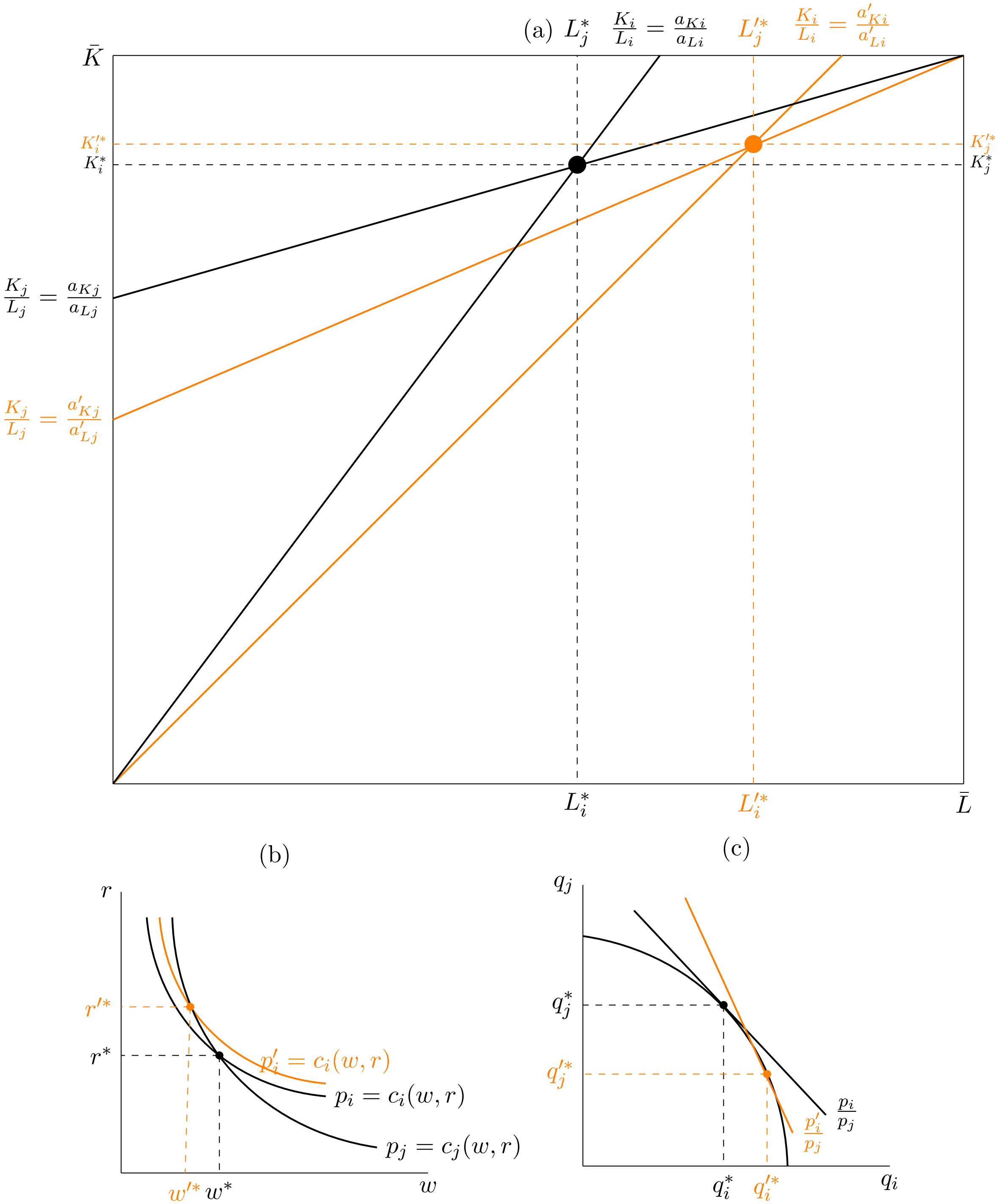
$$dr \;=\; \frac{a_{L2}\, dp_1}{a_{K1}a_{L2} - a_{L1}a_{K2}} \tag{5.1.6}$$

The sign of (5.1.6) depends on factor intensities. If q_1 is labor-intensive, the denominator is negative and thus $dr < 0$; if good 1 is capital-intensive, then $dr > 0$.
Similarly, the change in the wage rate is given by

$$dw \;=\; \frac{a_{K2}\, dp_1}{a_{K1}a_{L2} - a_{L1}a_{K2}} \tag{5.1.7}$$

Equation (5.1.7) is positive when q_1 is labor-intensive and negative when it is capital-intensive. Equations (5.1.6) and (5.1.7) formalize the Stolper–Samuelson theorem: an increase in the relative price of the labor-intensive good raises the real wage and lowers the return to capital, whereas an increase in the relative price of the capital-intensive good raises the return to capital and lowers the real wage.
Figure 9 illustrates the Stolper–Samuelson theorem in the case where sector i is *capital-intensive* and its output price p_i increases.

Figure 9: Stolper–Samuelson Theorem



Rybczynski Theorem: With goods prices fixed, an increase in the labor endowment expands the output of the labor-intensive good more than proportionally and reduces the output of the other good; analogously, a rise in the capital endowment enlarges the capital-intensive sector and contracts the labour-intensive one.

When factor endowments change, the market-clearing conditions in equation (5.1.2) become

$$\begin{cases} a_{Li}\,dq_i + a_{Lj}\,dq_j = d\bar{L}, \\ a_{Ki}\,dq_i + a_{Kj}\,dq_j = d\bar{K} \end{cases}$$

Suppose an increase in $\bar{L}$, therefore

$$(a_{Kj}a_{Li} - a_{Lj}a_{Ki})\,dq_i = a_{Kj}d\bar{L}$$

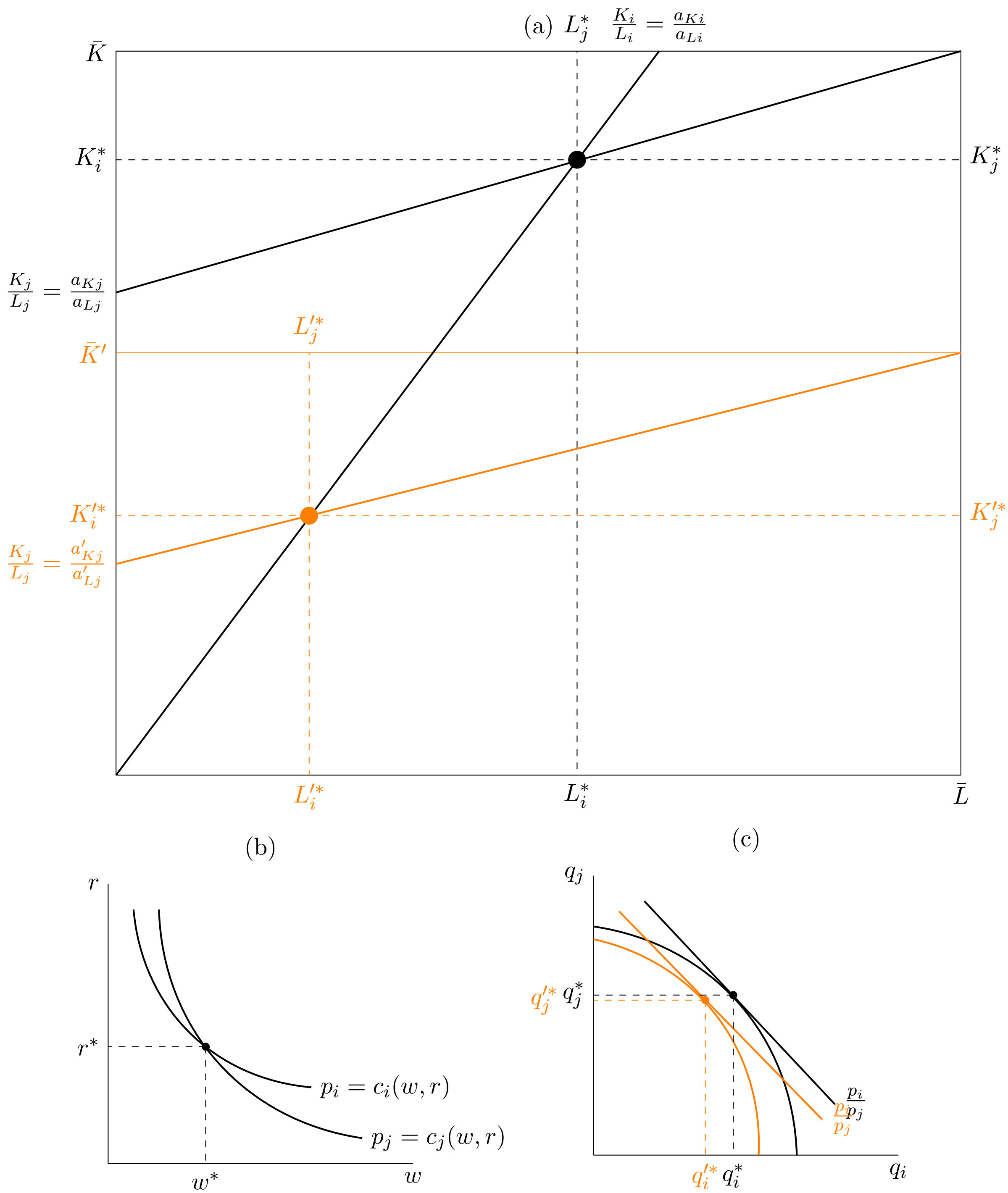
If good i is labour-intensive, then $\frac{a_{Li}}{a_{Ki}} > \frac{a_{Lj}}{a_{Kj}}$, implying $a_{Kj}a_{Li} - a_{Lj}a_{Ki} > 0$; hence $dq_i > 0$. Substituting back gives

$$dq_j = \frac{d\bar{L} - a_{Li}dq_i}{a_{Lj}} < 0$$

Thus, an increase in labour endowment expands the labour-intensive sector and contracts the capital-intensive one.

Figure 10 illustrates the Rybczynski theorem when sector i is *capital-intensive* and the aggregate capital endowment $\bar{K}$ falls, while goods prices and factor prices remain unchanged. In panel (c) the production–possibility frontier shifts inward; the maximum feasible output of good i (q_i-intercept) contracts by a larger amount than that of good j, reflecting the disproportionate impact on the capital-intensive sector.

Figure 10: Rybczynski Theorem



To illustrate, consider an economy endowed with $\bar{L}$ units of labor and $\bar{K}$ units of capital. The two sectors exhibit production functions given by

$$q_1 = L^\alpha K^{1-\alpha}, \qquad q_2 = L^{1-\alpha} K^\alpha, \qquad \frac{1}{2} < \alpha < 1.$$

Conditional factor demands are:

$$\begin{cases} L_1^*(q_1,w,r) = q_1\left(\frac{\alpha}{1-\alpha}\cdot\frac{r}{w}\right)^{1-\alpha} \\ K_1^*(q_1,w,r) = q_1\left(\frac{1-\alpha}{\alpha}\cdot\frac{w}{r}\right)^{\alpha} \\ L_2^*(q_2,w,r) = q_2\left(\frac{1-\alpha}{\alpha}\cdot\frac{r}{w}\right)^{\alpha} \\ K_2^*(q_2,w,r) = q_2\left(\frac{\alpha}{1-\alpha}\cdot\frac{w}{r}\right)^{1-\alpha} \end{cases}$$

Equivalently:

$$\begin{cases} a_{L1}(w,r) = \left(\frac{\alpha}{1-\alpha}\cdot\frac{r}{w}\right)^{1-\alpha} \\ a_{K1(w,r)} = \left(\frac{1-\alpha}{\alpha}\cdot\frac{w}{r}\right)^{\alpha} \\ a_{L2}(w,r) = \left(\frac{1-\alpha}{\alpha}\cdot\frac{r}{w}\right)^{\alpha} \\ a_{K2}(w,r) = \left(\frac{\alpha}{1-\alpha}\cdot\frac{w}{r}\right)^{1-\alpha} \end{cases}$$

The marginal costs for each sector are as follows:

$$\begin{cases} p_1 = c_1(w,r) = w\cdot\left(\frac{\alpha}{1-\alpha}\cdot\frac{r}{w}\right)^{1-\alpha} + r\cdot\left(\frac{1-\alpha}{\alpha}\cdot\frac{w}{r}\right)^{\alpha} \\ p_2 = c_2(w,r) = w\cdot\left(\frac{1-\alpha}{\alpha}\cdot\frac{r}{w}\right)^{\alpha} + r\cdot\left(\frac{\alpha}{1-\alpha}\cdot\frac{w}{r}\right)^{1-\alpha} \end{cases}$$

Solving the above system for the factor–price ratio yields:

$$\frac{w}{r} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{2\alpha-1}}$$

Because $\alpha > \frac{1}{2}$, sector 1 is labour-intensive:

$$\frac{a_{L1}}{a_{K1}} = \frac{\alpha}{1-\alpha} \; > \; \frac{a_{L2}}{a_{K2}} = \frac{1-\alpha}{\alpha}$$

Factor–market clearing implies:

$$\begin{cases} \bar{L} = q_1\left(\frac{\alpha}{1-\alpha}\cdot\frac{r}{w}\right)^{1-\alpha} + q_2\left(\frac{1-\alpha}{\alpha}\cdot\frac{r}{w}\right)^{\alpha} \\ \bar{K} = q_1\left(\frac{1-\alpha}{\alpha}\cdot\frac{w}{r}\right)^{\alpha} + q_2\left(\frac{\alpha}{1-\alpha}\cdot\frac{w}{r}\right)^{1-\alpha} \end{cases}$$

Solving the system yields

$$q_1 = \frac{\bar{L}\left(\dfrac{\alpha}{1-\alpha}\,\dfrac{w}{r}\right)^{1-\alpha} - \bar{K}\left(\dfrac{1-\alpha}{\alpha}\,\dfrac{r}{w}\right)^{\alpha}}{\left(\dfrac{\alpha}{1-\alpha}\right)^{2(1-\alpha)} - \left(\dfrac{1-\alpha}{\alpha}\right)^{2\alpha}},$$

$$q_2 = \frac{\bar{K}\left(\dfrac{\alpha}{1-\alpha}\,\dfrac{r}{w}\right)^{1-\alpha} - \bar{L}\left(\dfrac{1-\alpha}{\alpha}\,\dfrac{w}{r}\right)^{\alpha}}{\left(\dfrac{\alpha}{1-\alpha}\right)^{2(1-\alpha)} - \left(\dfrac{1-\alpha}{\alpha}\right)^{2\alpha}}.$$

The contract curve is obtained by equating the marginal rates of technical substitution (MRTS) in both sectors:

$$\begin{aligned} \frac{\partial q_1/\partial L_1}{\partial q_1/\partial K_1} = \frac{\partial q_2/\partial L_2}{\partial q_2/\partial K_2} &\implies \frac{K_1}{L_1} = \frac{K_2}{L_2} \\ &\implies \frac{K_1}{L_1} = \frac{\bar{K}-K_1}{\bar{L}-L_1} \implies K_1 = \frac{\bar{K}}{\bar{L}}\,L_1. \end{aligned}$$

5.2 Equilibrium with consumption and production

Consider a closed economy with two firms. Firm 1 produces output q_1 and Firm 2 produces output q_2. Both firms employ labor L and capital K, and each technology exhibits constant returns to scale. The aggregate factor endowment is $(\bar{L}, \bar{K})$. Social welfare is represented by the utility function $u(x_1, x_2)$, defined over the two consumption goods x_1 and x_2.
In a competitive, decentralized equilibrium, each agent maximizes their own objective function. The representative consumer solves

$$\max_{x_1,x_2}\; u(x_1,x_2) \quad \text{s.t.} \quad w\bar{L} + r\bar{K} + \pi_1(p_1,w,r) + \pi_2(p_2,w,r) = p_1x_1 + p_2x_2,$$

where w and r are the prices of labor and capital, respectively, and $\pi_j(p_j, w, r)$ denotes the profit of firm $j \in \{1, 2\}$. Since both production technologies display *constant returns to scale* and markets are perfectly competitive, equilibrium profits satisfy $\pi_1 = \pi_2 = 0$. The representative consumer's budget constraint therefore simplifies to

$$p_1x_1 + p_2x_2 = w\bar{L} + r\bar{K}, \tag{5.2.1}$$

where the right-hand side represents the total income from the factor endowment $(\bar{L}, \bar{K})$.
The (interior) first–order condition equates the marginal rate of substitution to the relative price ratio:

$$\frac{\partial u(x_1,x_2)/\partial x_1}{\partial u(x_1,x_2)/\partial x_2} = \frac{p_1}{p_2}$$

Combining the optimality condition with the budget constraint (5.2.1) yields the demand functions

$$x_i = x_i\big(w, r, \bar{L}, \bar{K}, p_1, p_2\big), \qquad i \in \{1, 2\}$$

For firm j, the cost-minimization problem is

$$\min_{K_j,\, L_j} \; C_j(q_j, w, r) = rK_j + wL_j \quad \text{s.t.} \quad q_j = q_j(K_j, L_j)$$

where $q_j(K_j, L_j)$ is the production function for sector j. The first-order condition equates the *marginal rate of technical substitution* to the ratio of input prices:

$$\frac{\partial q_j(K_j, L_j)/\partial L_j}{\partial q_j(K_j, L_j)/\partial K_j} \; = \; \frac{w}{r}$$

Solving the programme yields the conditional (input demand) functions

$$K_j = K_j(q_j, w, r), \qquad L_j = L_j(q_j, w, r)$$

Under perfect competition and constant returns to scale, price equals marginal cost

$$p_j \; = \; c_j(q_j, w, r)$$

Considering all output is consumed, and households hold no initial endowments of the goods, equilibrium in the goods markets requires

$$x_i = q_i, \qquad i = 1, 2$$

Equilibrium in the factor markets equates total factor demand with the aggregate endowment:

$$\bar{L} = L_1 + L_2, \qquad \bar{K} = K_1 + K_2$$

A *competitive equilibrium* is a collection of prices

$$\big(p_1^*, p_2^*, w^*, r^*\big)$$

and allocations

$$\big(q_1^*, q_2^*, x_1^*, x_2^*, K_1^*, K_2^*, L_1^*, L_2^*\big)$$

that jointly satisfy

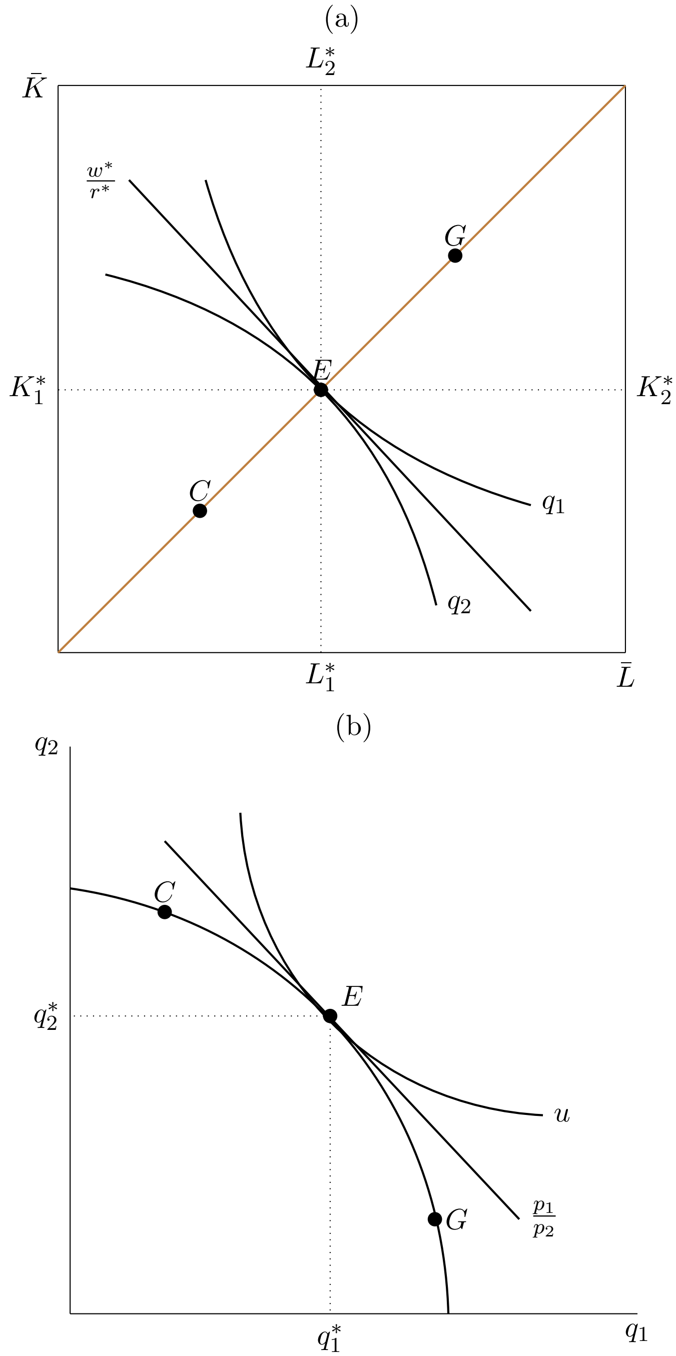
$$
\begin{cases}
x_i^* = q_i^*, & i = 1,2, \qquad \text{(goods clearing)} \\
x_i^* = x_i\big(w^*, r^*, \bar{L}, \bar{K}, p_1^*, p_2^*\big), & i = 1,2, \text{(household demand)} \\
L_j^* = L_j\big(q_j^*, w^*, r^*\big), & j = 1,2, \qquad \text{(labour demand)} \\
K_j^* = K_j\big(q_j^*, w^*, r^*\big), & j = 1,2, \qquad \text{(capital demand)} \\
p_j^* = c_j\big(w^*, r^*\big), & j = 1,2, \qquad \text{(zero-profit)} \\
\bar{L} = L_1^* + L_2^*, & \text{(labour market)} \\
\bar{K} = K_1^* + K_2^*. & \text{(capital market)}
\end{cases}
$$

Equations (5.2.2) constitute a system of twelve equations in the twelve unknowns

$$(p_1^*, p_2^*, w^*, r^*, q_1^*, q_2^*, x_1^*, x_2^*, K_1^*, K_2^*, L_1^*, L_2^*) \tag{5.2.3}$$

whose solution yields the competitive-equilibrium prices and allocation.
Figure 11 summarizes the general-equilibrium. Panel (a) displays the production Edgeworth box, with the contract curve—the locus of Pareto-efficient factor allocations—highlighted. Panel (b) maps those efficient allocations into output space, tracing the *Production-Possibility Frontier* (PPF). Equilibrium is found at the point where the PPF is tangent to the indifference curve, so that the marginal rate of transformation equals the marginal rate of substitution.

Figure 11: General Equilibrium in $2 \times 2 \times 1$



Let the representative household have Cobb–Douglas preferences

$$u(x_1, x_2) = x_1 x_2$$

and suppose each firm $j \in \{1, 2\}$ produces with the Cobb–Douglas technology

$$q_j(K_j, L_j) = K_j^{1/2} L_j^{1/2}$$

Optimal consumption satisfies

$$x_1 = \frac{w\bar{L} + r\bar{K}}{2p_1}, \qquad x_2 = \frac{w\bar{L} + r\bar{K}}{2p_2}$$

Cost minimization yields the conditional factor–demand functions

$$K_j = q_j\sqrt{\frac{w}{r}}, \qquad L_j = q_j\sqrt{\frac{r}{w}}$$

and the (zero–profit) pricing condition

$$p_j = 2\sqrt{wr}$$

Adding up across firms,

$$\bar{K} = (q_1 + q_2)\sqrt{\frac{w}{r}}, \qquad \bar{L} = (q_1 + q_2)\sqrt{\frac{r}{w}}$$

Taking the ratio of the two equations gives the equilibrium

$$\frac{w}{r} = \frac{\bar{K}}{\bar{L}} \quad \Longrightarrow \quad w^* = r^*\frac{\bar{K}}{\bar{L}}$$

Since all output is consumed, goods-market clearing implies $x_i = q_i$ for $i = 1, 2$. Substituting $p_1 = p_2 = 2\sqrt{wr}$ into the household demand functions then yields to

$$q_1 = q_2 = \frac{1}{2}\sqrt{\bar{K}\bar{L}}$$

Factor demands at the optimum.

$$K_1 = K_2 = \frac{\bar{K}}{2}, \qquad L_1 = L_2 = \frac{\bar{L}}{2}$$

Choosing $r^* = 1$ (any positive normalization works), the equilibrium is

$$\begin{aligned} &\Big(p_1^*, p_2^*, w^*, r^*, q_1^*, q_2^*, x_1^*, x_2^*, K_1^*, K_2^*, L_1^*, L_2^*\Big) \\ = &\Big(2\sqrt{\tfrac{\bar{K}}{\bar{L}}},\ 2\sqrt{\tfrac{\bar{K}}{\bar{L}}},\ \tfrac{\bar{K}}{\bar{L}},\ 1,\ \tfrac{1}{2}\sqrt{\bar{K}\bar{L}},\ \tfrac{1}{2}\sqrt{\bar{K}\bar{L}},\ \tfrac{1}{2}\sqrt{\bar{K}\bar{L}},\ \tfrac{1}{2}\sqrt{\bar{K}\bar{L}},\ \tfrac{\bar{K}}{2},\ \tfrac{\bar{K}}{2},\ \tfrac{\bar{L}}{2},\ \tfrac{\bar{L}}{2}\Big) \end{aligned}$$

5.3 Exercises

1. Consider the general-equilibrium production model in Section 5.1. Sector i is capital-intensive.

    1. The economy opens to international trade and the relative price of good i rises above its autarky level. Using a graphical approach, describe
        1. the impact on equilibrium factor prices,
        2. the resulting changes in the output of each good and distribution of factor between sectors, and
        3. the economic intuition behind these effects.
    2. Keeping the economy open, suppose a natural disaster cuts the aggregate capital stock in half. Reassess your answers: how are factor prices and output levels now affected?

2. Consider the production economy of Section 5.1. Output is produced according to

$$q_1 = L_1 + 2K_1, \qquad q_2 = 2L_2 + K_2, \qquad \frac{1}{2} < \frac{w}{r} < 2$$

    1. Express the factor–price ratio $\frac{w}{r}$ as a function of the goods–price ratio $\frac{p_1}{p_2}$. Determine the output levels q_1^* and q_2^*, and illustrate the results.
    2. Explain why $\frac{1}{2} < \frac{w}{r} < 2$ must hold. Derive the corresponding admissible range for $\frac{p_1}{p_2}$.
    3. A storm halves the aggregate capital stock. Re-evaluate part 1 using the graphical approach and relate your findings to the relevant theorem.
    4. Suppose the economy remains open and p_1 doubles. Re-do part 1 graphically and link the outcome to the theorem discussed.
    5. Now both prices double. How does the equilibrium adjust, and what is the intuition?
    6. If $\frac{w}{r} = 1$ and the endowments are $(\bar{L}, \bar{K}) = (1, 1)$, what is the optimal allocation?

3. In the general-equilibrium model with consumption and production (Section 5.2), the representative consumer has Cobb–Douglas preferences.

    1. Assume the consumer's preferences shift, making good 2 relatively more preferred than before. How does this shift affect the relative price $\frac{p_2}{p_1}$, the output mix (q_1, q_2), and the allocation of labor and capital across sectors? Illustrate the adjustments a graphical approach.
    2. Total factor productivity rises in sector 2. Analyze the impact on $\frac{p_2}{p_1}$, (q_1, q_2), and factor allocation, and depict the outcome graphically.
    3. Repeat part 2 assuming Leontief preferences. How do the results change? Explain the roles of income and substitution effects.

4. Consider an economy described as in Equilibrium with consumption and production. The representative consumer has Cobb–Douglas preferences.

    1. Suppose the preferences of the indivual changes and now good 2 is more prefered that what it was. How do this change the relative prices of the goods and the optimal produced in each sector? How do capital and labor distribution between sector change? Use graphical approach
    2. Suppose an increase in technology for sector q_2. How do this change the relative prices of the goods and the optimal produced in each sector? How do capital and labor distribution between sector change? Use graphical approach (5.2.3)? Use graphical approach
    3. Suppose an increase in technology for sector q_2 but with preferences by Leontief utility function, how do your previous asnwers change? What does income and substitution effect has to be related with this?

5. In the general-equilibrium model with consumption and production (Section 5.2), the representative consumer has preferences given by

$$u(x_1, x_2) = \min\{x_1, x_2\}$$

    Firms produce according to

$$q_1 = \left(L_1^{1/2} + 4K_1^{1/2}\right)^2, \qquad q_2 = \left(4L_2^2 + K_2^2\right)^{1/2}$$

    1. Find the competitive-equilibrium vector (5.2.3) and illustrate it with the chapter's graphical approach.

6. In the general-equilibrium model with consumption and production (Section 5.2), the representative consumer has preferences given by

$$u(x_1, x_2) = x_1 x_2$$

    Additionally, the consumer is endowed with ϕ_1 for good 1 and ϕ_2 for good 2. Firms produce according to

$$q_1 = \min\{\frac{L_1}{2}, K_1\}, \qquad q_2 = \min\{L_2, \frac{K_2}{2}\}$$

    1. Find the competitive-equilibrium vector (5.2.3) and illustrate it with the chapter's graphical approach.

6 The Armington Model

6.1 Introduction

The Armington model, introduced by Armington (1969), provides a framework to analyze international trade when goods are differentiated by their country of origin. Unlike the Ricardian, which assume goods are homogeneous across producers, the Armington model postulates that even within the same sector, goods from different countries are imperfect substitutes. This assumption introduces product differentiation and plays a central role in modern empirical trade analysis.

6.2 The model

Consider a world economy consisting of n countries, where each country produces a differentiated variety of a common type of good. The total population of country j is denoted by L_j, and each individual is endowed with one unit of the domestically variety and none of the other varieties. Let w_j denote the endowment value of variety j for a representative consumer j.[5] The aggregate value of country j's endowment across all individuals defines the value of country j's gross domestic product (GDP).

$$X_j = L_j w_j \tag{6.2.1}$$

Consumers exhibit preferences over the full set of available varieties. Let x_{kj} denote the quantity of the variety associated with country k that is consumed by a representative individual in country j. The preferences of the representative consumer are assumed to be described by a Constant Elasticity of Substitution (CES) utility function of the form

$$U_j = \left(\sum_{k=1}^{n} \alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{6.2.2}$$

where $\sigma > 1$ denotes the elasticity of substitution across varieties, and α_{kj} represents the taste parameter associated with variety k in country j.

The budget constraint of a representative consumer j is expressed as

$$w_j = \sum_{k=1}^{n} p_{kj}\, x_{kj} \tag{6.2.3}$$

where p_{kj} denotes the price of variety k in country j. The right-hand side of equation (6.2.3) corresponds to the total expenditure of the representative consumer in country j on all available varieties.

Furthermore, assume the presence of iceberg trade costs $\tau_{kj} \geq 1$, such that τ_{kj} units must be

[5] Since the representative consumer in country j is endowed with one unit of variety j, it follows that $w_j = p_{jj}$, where p_{jj} denotes the price of variety j in country j.

shipped from country k to country j for one unit of variety k to be delivered in country j. Under this assumption, the consumer price of variety k in country j is given by

$$p_{kj} = w_k \tau_{kj} \tag{6.2.4}$$

The corresponding optimality condition when maximizing the utility function in equation (6.2.2) subject to the budget constraint in equation (6.2.3) is

$$\frac{\alpha_{ij}^{\frac{1}{\sigma}} x_{kj}^{\frac{1}{\sigma}}}{\alpha_{kj}^{\frac{1}{\sigma}} x_{ij}^{\frac{1}{\sigma}}} = \frac{p_{ij}}{p_{kj}} \tag{6.2.5}$$

The CES price index in country j is defined as

$$P_j = \left(\sum_{k=1}^{n} \alpha_{kj} p_{kj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{6.2.6}$$

The Marshallian demand of the representative consumer in country j for variety i, expressed in terms of the CES price index in equation (6.2.6), is given by

$$x_{ij} = \alpha_{ij} p_{ij}^{-\sigma} \frac{w_j}{P_j^{1-\sigma}} \tag{6.2.7}$$

Equation (6.2.7) can equivalently be written to express the Marshallian demand in terms of trade costs as

$$x_{ij} = \alpha_{ij} w_i^{-\sigma} \tau_{ij}^{-\sigma} \frac{w_j}{P_j^{1-\sigma}} \tag{6.2.8}$$

Denoting by $p_{ij}\, x_{ij}$ the expenditure of the representative consumer in country j on variety i, equation (6.2.8) can be rewritten as

$$p_{ij} x_{ij} = \alpha_{ij} w_i^{1-\sigma} \tau_{ij}^{1-\sigma} \frac{w_j}{P_j^{1-\sigma}} \tag{6.2.9}$$

6.3 From the Representative Consumer to the Aggregate Economy

Let $\tilde{x}_{ij}$ denote the total consumption of variety i by the population in country j. This equals the demand of the representative consumer in country j for variety i, multiplied by the population size L_j. The total expenditure on variety i by the population in country j is obtained by multiplying both sides of equation (6.2.9) by L_j.

$$p_{ij} \tilde{x}_{ij} = \alpha_{ij} w_i^{1-\sigma} \tau_{ij}^{-\sigma} \frac{L_j w_j}{P_j^{1-\sigma}} \tag{6.3.1}$$

Using equation (6.2.1), the total expenditure on variety i by the population in country j is defined in terms of country j's GDP as follows

$$p_{ij}\tilde{x}_{ij} = \alpha_{ij} w_i^{1-\sigma} \tau_{ij}^{-\sigma} \frac{X_j}{P_j^{1-\sigma}} \tag{6.3.2}$$

Additionally, country i's GDP is defined via the income approach as the total revenue earned from selling its variety to all countries, that is

$$X_i = \sum_{j=1}^{n} L_j p_{ij} x_{ij} \tag{6.3.3}$$

The right-hand side of equation (6.3.3) represents the sum of the total expenditures by the populations of all countries on variety i.
Substituting equation (6.2.9) into equation (6.3.3) yields

$$X_i = \sum_{j=1}^{n} L_j w_i^{1-\sigma} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \alpha_{ij}\, w_j \tag{6.3.4}$$

Using equation (6.2.1) for country j, equation (6.3.4) can be rewritten as

$$X_i = \sum_{j=1}^{n} w_i^{1-\sigma} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \alpha_{ij}\, X_j \tag{6.3.5}$$

Equation (6.3.5) can be equivalently expressed by isolating the domestic price term $w_i^{1-\sigma}$ as

$$w_i^{1-\sigma} = \frac{X_i}{\sum_{j=1}^{n} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \alpha_{ij}\, X_j} \tag{6.3.6}$$

Finally, substituting equation (6.3.6) into (6.3.2) leads to the gravity equation of the model:

$$p_{ij}\tilde{x}_{ij} = \frac{\alpha_{ij} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} X_i X_j}{\sum_{j=1}^{n} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \alpha_{ij}\, X_j} \tag{6.3.7}$$

Equation (6.3.7) expresses bilateral expenditure $\tilde{x}_{ij}$ as a function of the exporter's income X_i, the importer's income X_j, on the taste parameter α_{ij}, trade costs τ_{ij}, the elasticity of substitution σ, and the multilateral resistance terms captured in the denominator.

6.4 Exercises

1. Indicate whether each of the following propositions is true or false. Justify your answer.

    1. Countries z, x, and y share identical preferences and trade costs. However, country x is larger than country y. Country z will always trade more with x than with y.
    2. If preferences for variety x increase in country z (*ceteris paribus*), trade between z and x will increase, while trade between z and y will decrease.
    3. If the price of good x in country z increases (*ceteris paribus*), trade between x and z will decrease, while trade between y and z will remain unchanged.
    4. If the trade cost of exporting good x to country z increases (*ceteris paribus*), trade between x and z will decrease, while trade between y and z will remain unchanged.
    5. If GDP increases in all countries except z and y (*ceteris paribus*), trade between z and y will remain constant.

2. Suppose a pandemic reduces the population by half in all countries $h+1,\ldots,n$, within a set of countries indexed by $k=1,\ldots,h,\ldots,n$. Analyze how this shock affects trade flows between countries. Consider all possible bilateral cases (four in total).

3. Suppose there is migration from country i to country j. Analyze the effects of this migration on the following:

    1. Country j's imports of good i.
    2. Country j's imports of good k.
    3. Country i's imports of good k.

7 The Specific Factors Model

The Specific Factors Model constitutes a natural extension of the Ricardian framework by introducing multiple factors of production and relaxing the assumption of full mobility. It captures the idea that some factors are mobile across sectors, while others remain sector-specific. This asymmetry in factor mobility generates sectoral distributional effects from trade that are absent in the Ricardian model.

7.1 The Model

Consider a small economy that produces two goods, denoted q_1 and q_2. The production of good q_1 employs capital K, which is specific to sector q_1, together with labor L_1, whereas the production of good q_2 uses land T, which is specific to sector q_2, together with labor L_2. Labor is perfectly mobile across sectors, and the total labor endowment is denoted by $\bar{L}$. Under the assumption of full employment, it follows that $\bar{L} = L_1 + L_2$. The economy is thus endowed with three factors of production: one mobile factor (labor) and two sector-specific factors (capital and land). Note that sector q_1 employs the entire endowment of capital, while sector q_2 employs the entire endowment of land, since q_1 does not use land and q_2 does not use capital in production.
The production functions are assumed to be strictly concave and to exhibit constant returns to scale with respect to labor and the specific factor within each sector:

$$q_1 = q_1(K, L_1), \qquad q_2 = q_2(T, L_2) \tag{7.1.1}$$

Marginal products of labor are positive and diminishing, implying the existence of downward-sloping labor demand schedules in each sector.
Under perfect competition, factors are remunerated according to the value of their marginal products. Since labor is perfectly mobile across sectors, the wage rate w is equalized across sectors and satisfies

$$w \;=\; p_1 \cdot \frac{\partial q_1}{\partial L_1} \;=\; p_2 \cdot \frac{\partial q_2}{\partial L_2} \tag{7.1.2}$$

where p_1 and p_2 denote the world prices of goods q_1 and q_2, respectively. The allocation of labor is determined at the point where the value of the marginal product of labor is equal across sectors.
Owners of the specific factors receive returns given by the residual product once labor is compensated. In sector q_1, the return to capital is

$$r_K = p_1 \cdot \frac{\partial q_1}{\partial \bar{K}} \tag{7.1.3}$$

while in sector q_2, the return to land is

$$r_T = p_2 \cdot \frac{\partial q_2}{\partial \bar{T}} \tag{7.1.4}$$

Under the assumption of constant returns to scale, it follows that

$$p_1 q_1 = wL_1 + r_k \bar{K} \tag{7.1.5}$$

$$p_2 q_2 = wL_2 + r_t \bar{T} \tag{7.1.6}$$

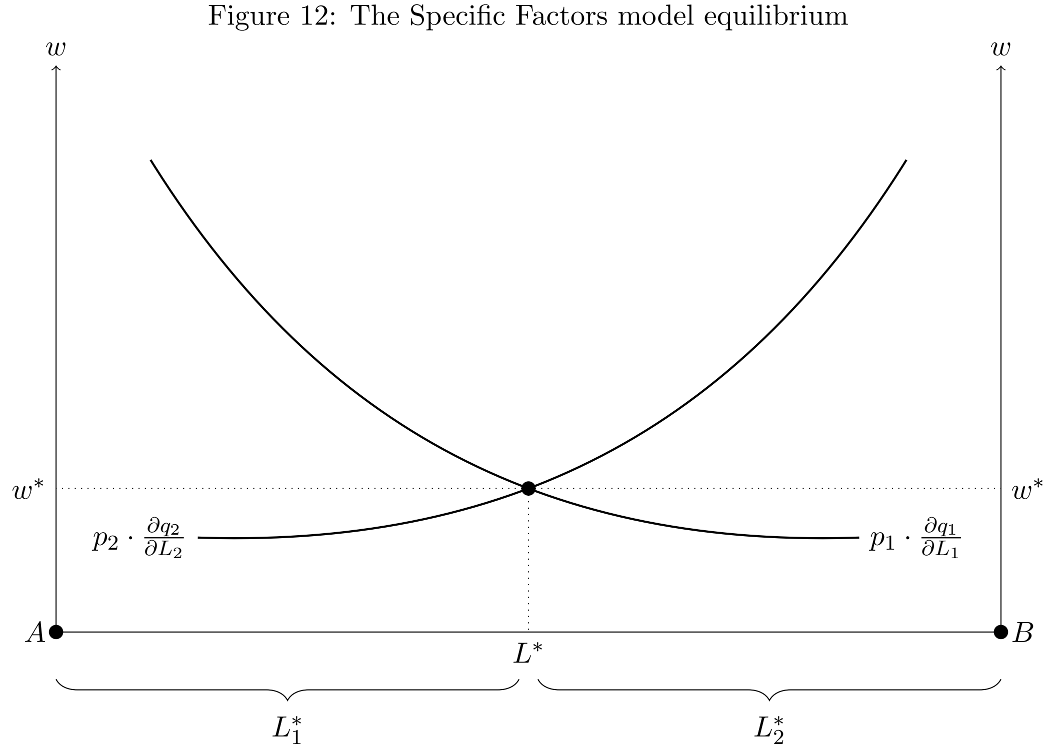
Equations (7.1.3) and (7.1.4) can equivalently be written as

$$r_K = \frac{p_1 \cdot q_1 - wL_1}{\bar{K}}, \tag{7.1.7}$$

$$r_T = \frac{p_2 \cdot q_2 - wL_2}{\bar{T}} \tag{7.1.8}$$

Figure 12 illustrates the equilibrium in the model. The wage is represented on the vertical axis and labor on the horizontal axis. The marginal product value curves of both sectors are depicted, with point A serving as the origin for sector q_1 and point B as the origin for sector q_2. The intersection of the marginal product value curves determines the equilibrium wage and the allocation of labor between sectors.

Figure 12: The Specific Factors model equilibrium



These specific-factor returns are not equalized across sectors. Consequently, changes in goods prices translate into changes in the distribution of income among capital owners, landowners, and workers.

7.2 Welfare in the Specific Factors Model

Welfare is measured in terms of the purchasing power of the return to each factor. For workers, the real wage in terms of good q_1 and in terms of good q_2 can be expressed as

$$\frac{w}{p_1} = \frac{\partial q_1}{\partial L_1}, \tag{7.2.1}$$

$$\frac{w}{p_2} = \frac{\partial q_2}{\partial L_2} \tag{7.2.2}$$

For capital owners, the real return to capital can be written as

$$\frac{r_K}{p_1} = \frac{\partial q_1}{\partial \bar{K}}, \tag{7.2.3}$$

$$\frac{r_K}{p_2} \tag{7.2.4}$$

For landowners, the real return to land is

$$\frac{r_T}{p_1} \tag{7.2.5}$$

$$\frac{r_T}{p_2} = \frac{\partial q_2}{\partial \bar{T}} \tag{7.2.6}$$

7.3 Impact of Trade in Specific Factors Model

The introduction of trade alters the relative prices faced by the economy as the country integrates into world markets. Suppose that the world price of good q_1 increases while the price of good q_2 remains constant. This raises the value of the marginal product of labor in sector q_1, thereby attracting labor from sector q_2 into sector q_1. As a result, wages increase due to the higher value of marginal products of labor in both sectors (the value of the marginal product in q_2 rises as labor employed in that sector declines). This adjustment is illustrated in Figure 13.

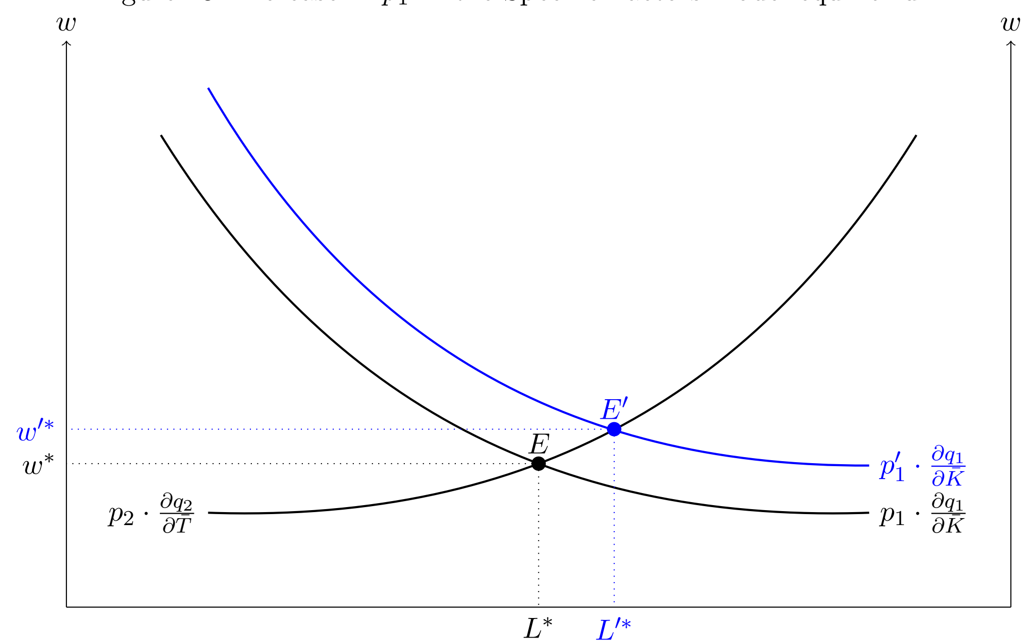
The return to capital in sector q_1 increases as the use of additional labor raises the productivity of capital, whereas the return to land in sector q_2 declines due to the withdrawal of labor from that sector.

Workers, being mobile, experience an ambiguous effect: the real wage may either increase or decrease. In equation (7.2.1), both w and p_1 increase; however, p_1 rises proportionally more than w since the right-hand side decreases, leading to a decline in the real wage expressed in terms of good q_1. In contrast, according to equation (7.2.2), the purchasing power in terms of good q_2 increases because the wage rises while p_2 remains constant.

Capital owners experience a welfare improvement. In equation (7.2.3), the purchasing power of the return to capital in terms of good q_1 increases, since the right-hand side rises and therefore the left-hand side also increases (as noted previously, r_K increases while p_1 also rises) making real return to capital increases. In equation (7.2.4), the purchasing power in terms of good q_2 also increases because the return to capital rises while p_2 remains constant.

Landowners experience a welfare loss. In equation (7.2.5), the return to land decreases while p_1 increases, causing the real return to land measured in units of good q_1 to decline. In equation (7.2.6), the return to land also decreases while p_2 remains constant, leading to a decline in the real return to land measured in units of good q_2.

Figure 13: Increase in p_1 in the Specific Factors model equilibrium



The central insight of the Specific Factors Model is that trade generates both aggregate gains and redistribution across factors. At the national level, the economy is unambiguously better off, as trade enables specialization and provides access to a broader set of consumption possibilities. Within the economy, however, specific factors are affected asymmetrically: owners of the factor tied to the expanding sector gain, whereas owners of the factor tied to the contracting sector lose. The welfare of the mobile factor is ambiguous, as its outcome depends on the composition of its consumption bundle.

This distributional conflict underscores the political economy dimension of trade. Although the economy as a whole benefits, opposition to trade may arise from groups whose incomes decline. The model therefore provides a theoretical foundation for understanding sector-based lobbying and political resistance to trade, as observed in practice.

7.4 Exercises

1. Within the framework of the Specific Factors Model, analyze the effects of the following scenarios on sectoral production, factor allocation, and the welfare of factor owners:

    1. A technological improvement in sector q_1.
    2. An increase in the endowment of the factor specific to sector q_2.
    3. A decrease in the relative price $\frac{p_1}{p_2}$.
    4. A reduction in the endowment of the mobile factor.

2. Consider the following production functions for sectors q_1 and q_2:

$$q_1 = L_1^{\frac{1}{4}} K^{\frac{3}{4}}, \qquad q_2 = L_2^{\frac{3}{4}} T^{\frac{1}{4}}$$

    where the economy is endowed with $\bar{L}$ units of labor, $\bar{K}$ units of capital, and $\bar{T}$ units of land.

    1. Determine the allocation of factors across sectors.
    2. Derive the level of output in each sector.
    3. Compute the returns to each factor of production.
    4. Suppose that all factor endowments are equal to one (normalized to 100%) and that prices are equal to one. Determine the allocation of factors, the output in each sector, and the returns to each factor. Illustrate the results using the diagram developed in this section.

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