Intermediate International Trade

Jose Miguel Mora Casasola

Marcos Adamson

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For any material corrections, please write to:
casasola.economics@hotmail.com

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1 Introduction

The formalization of trade theory has very old origins. For example, Aristotle (ca. 350 B.C./2009, *Nicomachean Ethics*, V.5, 1133a) employed the concept of proportional modeling developed by the mathematician Euclid (ca. 300 B.C./1956, *Elements*, Book V), and formalized an exchange equation that already included assumptions commonly used by economists. If A is a builder, B a shoemaker, C a house, and D a pair of shoes, the relationship is established as $A : B = xD : C$. Two aspects are fundamental: determining x (the number of pairs of shoes equivalent to one house) and interpreting the ratio builder/shoemaker. The theory of international trade has examined this problem from different perspectives. Even the "new" theoretical approaches that have emerged in recent decades frequently, though not explicitly, rely on Schumpeterian concepts, in which entrepreneurs, "new markets," innovation, and market power play a central role, or draw on concepts from Newtonian physics, which itself was influenced by Aristotelian propositions.

It is therefore no coincidence that "Nothing exists in the world, except the blind forces of nature, that is not Greek in its origin" (Maine, cited in Livingstone, 1921). International trade theory is no exception. These are the conceptual foundations that later evolved into the propositions of the labor theory of value, which enabled the theories of absolute and comparative advantage. These theories can be extended to a larger number of goods and countries and explain trade patterns and the international organization of labor primarily through factor endowments, relative size, and technological differences.

The extension to two factors through the Heckscher–Ohlin (H–O) model, and its generalization in the Heckscher–Ohlin–Vanek (H–O–V) model, was enriched by a virtuous and intense cycle between empirical research and conceptual advances. This has been the evolution of trade theory—like many other fields of science—through empirical verification and the search for explanations of trade between relatively similar countries or of flows of goods in industries where close substitutes compete. As in other areas, increasing efforts with a stronger microeconomic foundation have emerged, seeking to explain how firms enter "new" markets and how global companies organize production processes across different locations and countries. These explanatory efforts are conventionally presented in textbooks and even in specialized journals as if isolated from entrepreneurial concepts, although in reality they are closely linked.

The relative availability of international trade data (values, volumes, identification of buyers and sellers, locations, among others) has also enabled the development of empirical explanations. These explanations do not necessarily follow models of welfare theory and are often ad hoc. Services pose an even greater challenge for the empirical verification of explanatory approaches. Similarly, Porter's concepts, product life-cycle theory, and the incorporation of more dynamic aspects—despite the absence of equations derived from microeconomic optimization models—have contributed detail and improved characterization of observed trade structures. Much of this description remains fundamentally linked to material transformation (production functions, technological relationships such as increasing returns, productivity, and technological change).

Trade relations have also acquired a strategic dimension, with the imposition of tariffs and trade barriers that modify the global equilibrium. This demands preparation to interpret and act upon regulatory and economic changes in this context. The concepts of "new markets" and the fundamental role of innovation, entrepreneurship, and market power—widely discussed since Schumpeter and influenced by Newtonian physics—are applied today in the analysis of international trade flows.

This document is the result of teaching international trade theory at the School of Economics of the University of Costa Rica. It addresses different topics, aiming to provide reference material with solid formal grounding, as well as exercises at an intermediate level of depth.

2 Microeconomic Overview

This chapter is a briefly review of: (i) Consumer Theory, (ii) Firm Theory and (iii) General Equilibrium in pure exchange.

2.1 Consumer Theory

Consumer theory analyzes how an individual chooses a consumption bundle to maximize utility (or satisfaction) given income and market prices. Dually, for a given utility level and given prices, the individual can choose the bundle that attains that utility at the minimum possible expenditure.
The individual's utility-maximization problem can be stated as follows:

$$\max_{x_1,\ldots,x_n} \quad U(x_1,\ldots,x_n) \tag{2.1.1}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} p_i x_i \leq m, \tag{2.1.2}$$

$$x_i \geq 0 \quad \forall i = 1,\ldots,n$$

Under non-satiation and an interior solution, the associated Lagrangian is

$$\mathcal{L}(x_1,\ldots,x_n,\lambda) = U(x_1,\ldots,x_n) + \lambda\left(m - \sum_{i=1}^{n} p_i x_i\right) \tag{2.1.3}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial U}{\partial x_i} - \lambda p_i = 0 \quad \forall i = 1,\ldots,n, \tag{2.1.4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - \sum_{i=1}^{n} p_i x_i = 0 \tag{2.1.5}$$

Combining (2.1.4) for any two goods i and j yields the optimality (Marginal Rate of Substitution = price ratio) condition:

$$\frac{\partial U/\partial x_i}{\partial U/\partial x_j} = \frac{p_i}{p_j} \tag{2.1.6}$$

Substituting the optimal condition into the equation (2.1.2) (budget constraint) gives the *Marshallian (ordinary) demand* for each good i:

$$x_i = x_i(m, p_1,\ldots,p_n) \tag{2.1.7}$$

Equation (2.1.7) states that the optimal quantity of good i depends on income and all prices. Plugging these demands into the utility function defines the *indirect utility function*:

$$v(m, p_1,\ldots,p_n) = U\big(x_1^M(m,\mathbf{p}),\ldots,x_n^M(m,\mathbf{p})\big) \tag{2.1.8}$$

The indirect utility $v(m, \mathbf{p})$ gives the maximum attainable utility at income m and price vector p and it is useful for comparing scenarios in which income and prices change simultaneously. By the envelope theorem, the Lagrange multiplier satisfies

$$\lambda^* = \frac{\partial v(m, \mathbf{p})}{\partial m}$$

so λ^* is the marginal utility of income.
Consider the following utility function

$$U(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}, \qquad \alpha_i > 0 \tag{2.1.9}$$

From (2.1.6),

$$\frac{x_i}{x_j} = \frac{\alpha_i p_j}{\alpha_j p_i} \tag{2.1.10}$$

Substituting into (2.1.2) yields the Marshallian demand

$$x_i(m, \mathbf{p}) = \frac{\alpha_i m}{p_i \sum_{k=1}^{n} \alpha_k} \tag{2.1.11}$$

Plugging (2.1.11) into (2.1.9) gives the indirect utility:

$$v(m, p_1, \dots, p_n) = \left(\frac{m}{\sum_{k=1}^{n} \alpha_k}\right)^{\sum_{k=1}^{n} \alpha_k} \prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} \tag{2.1.12}$$

For a target utility level $\bar{u}$, the dual problem is

$$\min_{x_1, \dots, x_n} \quad \sum_{i=1}^{n} p_i x_i \tag{2.1.13}$$

$$\text{s.t.} \quad U(x_1, \dots, x_n) \geq \bar{u}, \tag{2.1.14}$$

$$x_i \geq 0 \quad \forall i = 1, \dots, n$$

With an interior solution, the Lagrangian is

$$\mathcal{L}(x_1, \dots, x_n, \lambda) = \sum_{i=1}^{n} p_i x_i + \lambda\big(\bar{u} - U(x_1, \dots, x_n)\big) \tag{2.1.15}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_i} = p_i - \lambda \frac{\partial U}{\partial x_i} = 0 \quad \forall i = 1, \dots, n \tag{2.1.16}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{u} - U(x_1, \dots, x_n) = 0 \tag{2.1.17}$$

Combining the two-good first-order conditions in (2.1.16) once again produces the optimality condition found in (2.1.6). Substituting this condition into (2.1.14) yields the *Hicksian (compensated) demand*.

$$h_i = h_i(\bar{u}, p_1, \dots, p_n) \tag{2.1.18}$$

Substituting the Hicksian demand into (2.1.13) defines the resulting *expenditure function*:

$$e(\bar{u}, p_1, \dots, p_n) = \sum_{i=1}^{n} p_i h_i(\bar{u}, \mathbf{p}) \tag{2.1.19}$$

The Hicksian demand tells how much of each good is needed to achieve utility $\bar{u}$ at minimum cost, while the expenditure function gives that minimum cost. By the envelope theorem, the multiplier λ measures how much the minimum expenditure must increase to raise utility by one unit.

In the previous example, substituting (2.1.10) into the utility function yields to the Hicksian demand:

$$h_i(\bar{u}, \mathbf{p}) = \frac{\alpha_i \bar{u}^{\frac{1}{\sum_{i=1}^{n} \alpha_i}}}{p_i} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{\sum_{i=1}^{n} \alpha_i}}$$

Inserting this result into the expenditure function gives:

$$e(\bar{u}, \mathbf{p}) = \left[\bar{u}^{\frac{1}{\sum_{i=1}^{n} \alpha_i}} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\frac{\alpha_j}{\sum_{i=1}^{n} \alpha_i}}\right] \sum_{i=1}^{n} \alpha_i$$

2.1.1 Duality Properties

Let $v(m, \mathbf{p})$ denote the indirect utility function, $e(\bar{u}, \mathbf{p})$ the expenditure function, $x_i(m, \mathbf{p})$ the Marshallian (ordinary) demand, and $h_i(\bar{u}, \mathbf{p})$ the Hicksian (compensated) demand. Prices are $\mathbf{p} = (p_1, \dots, p_n)$, income is m, and $\bar{u}$ is a target utility level.

Roy's identity

$$x_i(m, \mathbf{p}) = -\frac{\frac{\partial v(m, \mathbf{p})}{\partial p_i}}{\frac{\partial v(m, \mathbf{p})}{\partial m}} \quad \forall i = 1, \dots, n \tag{2.1.20}$$

Shephard's lemma

$$h_i(\bar{u}, \mathbf{p}) = \frac{\partial e(\bar{u}, \mathbf{p})}{\partial p_i} \quad \forall i = 1, \dots, n \tag{2.1.21}$$

Indirect utility and expenditure functions are inverses (duality)

$$e\big(v(m, \mathbf{p}), \mathbf{p}\big) = m, \tag{2.1.22}$$

$$v\big(e(\bar{u}, \mathbf{p}), \mathbf{p}\big) = \bar{u} \tag{2.1.23}$$

Marshallian and Hicksian demands relationship

$$h_i(\bar{u}, \mathbf{p}) = x_i\big(e(\bar{u}, \mathbf{p}), \mathbf{p}\big), \tag{2.1.24}$$

$$x_i(m, \mathbf{p}) = h_i\big(v(m, \mathbf{p}), \mathbf{p}\big) \tag{2.1.25}$$

Given the Cobb–Douglas indirect utility function in equation (2.1.12), Roy's identity delivers the Marshallian demand:

$$x_i(m, \mathbf{p}) = \frac{\alpha_i\, m}{p_i \sum_{k=1}^{n} \alpha_k} \tag{2.1.26}$$

Equation (2.1.23) asserts that the indirect utility evaluated at the minimum expenditure equals the target utility. Substituting the form (2.1.12) and solving for $e(u, \mathbf{p})$ yields the minimum expenditure function

$$e(u, \mathbf{p}) = \Big[u \prod_{i=1}^{n} \Big(\tfrac{p_i}{\alpha_i}\Big)^{\alpha_i}\Big]^{\frac{1}{\sum_{k=1}^{n} \alpha_k}} \sum_{k=1}^{n} \alpha_k \tag{2.1.27}$$

Applying Shephard's lemma to the minimum expenditure function gives the Hicksian demand:

$$h_i(u, \mathbf{p}) = \frac{\alpha_i}{p_i} \Big[u \prod_{j=1}^{n} \Big(\tfrac{p_j}{\alpha_j}\Big)^{\alpha_j}\Big]^{\frac{1}{\sum_{k=1}^{n} \alpha_k}} \tag{2.1.28}$$

Solving the Marshallian demand for good i for p_i and substituting it into the indirect utility function restores the original utility function.

2.2 Firm Theory

In the theory of the firm, input choice is viewed either as profit maximization or, equivalently, as minimizing the cost of producing a given output level; the analysis first adopts the cost-minimization perspective.

$$\begin{aligned} \min_{z_1, z_2, \ldots, z_n} \quad & \sum_{i=1}^{n} w_i z_i \\ \text{s.t.} \quad & q(z_1, z_2, \ldots, z_n) \geq \bar{q} \\ & z_i \geq 0 \quad \forall i = 1, \ldots, n \end{aligned}$$

Assuming an interior solution, the Lagrangian for the cost-minimization problem is:

$$\mathcal{L}(z_1, \ldots, z_n, \lambda) \ = \ \sum_{i=1}^{n} w_i z_i \ + \ \lambda\big(\bar{q} - q(z_1, \ldots, z_n)\big)$$

By the envelope theorem, the Lagrange multiplier λ equals the marginal cost—the increase in minimum total cost required to produce one additional unit of output.

$$\frac{\partial \mathcal{L}}{\partial z_i} = w_i - \lambda \frac{\partial q(z_1, \ldots, z_n)}{\partial z_i} = 0 \quad \forall i = 1, \ldots, n,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{q} - q(z_1, \ldots, z_n) = 0$$

Combining the first-order conditions for any two inputs, i and j, implies that the marginal rate of technical substitution between them equals the ratio of their input prices.

$$\frac{\frac{\partial q(z_1,\ldots,z_n)}{\partial z_i}}{\frac{\partial q(z_1,\ldots,z_n)}{\partial z_j}} = \frac{w_i}{w_j} \quad \forall i, j = 1, \ldots, n \tag{2.2.1}$$

Substituting the optimality condition (2.2.1) back into the production constraint yields the *conditional input demand functions*, denoted by

$$z_i = z(\bar{q}, w_1, \ldots, w_n) \tag{2.2.2}$$

where $\bar{q}$ is the fixed output target and $\mathbf{w} = (w_1, \ldots, w_n)$ is the vector of input prices. Each function $z_i(\bar{q}, \mathbf{w})$ gives the amount of input that minimizes costs i required to produce units of output $\bar{q}$ at the prevailing prices, thus completing the solution to the firm's cost-minimization problem.
Substituting (2.2.2) into the cost objective yields the *minimum cost function*

$$C(\bar{q}, \mathbf{w}) = \sum_{i=1}^{n} w_i \, z_i(\bar{q}, \mathbf{w}), \tag{2.2.3}$$

which gives the least expenditure required to produce the target output $\bar{q}$ at input prices $\mathbf{w}$.
Define the *scale elasticity* as

$$\varepsilon_S = \sum_{i=1}^{n} \frac{\partial q(z_1, \ldots, z_n)}{\partial z_i} \frac{z_i}{q}$$

Classification follows immediately:

$$\varepsilon_S \begin{cases} > 1 & \text{Increasing Returns to Scale (IRS),} \\ = 1 & \text{Constant Returns to Scale (CRS),} \\ < 1 & \text{Decreasing Returns to Scale (DRS).} \end{cases}$$

Additionally, the *cost elasticity* with respect to output is

$$\varepsilon_C(q) = \frac{\partial C(\bar{q}, \mathbf{w})}{\partial q} \frac{q}{C(\bar{q}, \mathbf{w})} = \frac{MC(q)}{AC(q)},$$

Hence

$$\varepsilon_C(q) \begin{cases} < 1 & \text{Economies of scale } (AC \downarrow), \\ = 1 & \text{Constant returns to scale,} \\ > 1 & \text{Diseconomies of scale } (AC \uparrow). \end{cases}$$

To illustrate these concepts, consider the following production function:

$$q(z_1, \ldots, z_n) \;=\; \left(\sum_{i=1}^{n} a_i \, z_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \qquad \sigma > 1 \tag{2.2.4}$$

The marginal product of input z_i is defined as

$$\frac{\partial q(z_1, \ldots, z_n)}{\partial z_i} = \alpha_i z_i^{\frac{-1}{\sigma}} q \left(\sum_{i=1}^{n} a_i \, z_i^{\frac{\sigma-1}{\sigma}} \right)^{-1}$$

The (2.2.1) would yield to:

$$\left(\frac{z_k}{z_i} \right)^{\frac{1}{\sigma}} = \left(\frac{\alpha_k w_i}{\alpha_i w_k} \right)$$

Note that evaluating the *elasticity of substitution* between inputs i and j under the optimality condition yields

$$\sigma_{ij} \;=\; \frac{\partial \big(z_j/z_i\big)}{\partial \big(w_i/w_j\big)} \frac{\big(w_i/w_j\big)}{\big(z_j/z_i\big)} = \sigma$$

Because the elasticity of substitution, σ, remains constant for every input combination, the production function is called the *Constant Elasticity of Substitution (CES)* function.
Defining the Dixit–Stiglitz CES price index as

$$W(\mathbf{w}) \;=\; \left(\sum_{i=1}^{n} \alpha_i^{\sigma} p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{2.2.5}$$

The *conditional input demand* associated to this production function can be written as

$$z_i(\bar{q}, w) = \left(\frac{\alpha_i}{w_i} \right)^{\sigma} W^{\sigma} \bar{q}$$

Minimum cost function is

$$C(\bar{q}, \mathbf{w}) = q \sum_{i=1}^{n} w_i \left(\frac{\alpha_i}{w_i} \right)^{\sigma} W^{\sigma}$$

Note that $\varepsilon_S = 1$ and $\varepsilon_C(q) = 1$.
The supply curve for the firm if the good market is competitive is as follows

$$P = \sum_{i=1}^{n} w_i \left(\frac{\alpha_i}{w_i} \right)^{\sigma} W^{\sigma}$$

Now if we were to maximize the firms profit

$$\max_{z_1,\ldots,z_n} \pi \;=\; p\, q(z_1,\ldots,z_n) - \sum_{i=1}^{n} w_i z_i$$

Assuming an interior solution; the *First Order Conditions (FOC)*

$$p\,\frac{\partial q(z_1,\ldots,z_n)}{\partial z_i} \;=\; w_i\,, \qquad \forall i = 1,\ldots,n$$

Solving the system yields the *unconditional* (profit-maximizing) input demands

$$z_i^* \;=\; z_i(p,\mathbf{w}), \qquad i = 1,\ldots,n,$$

If the technology exhibits *decreasing returns to scale* (i.e. diseconomies of scale), marginal cost is increasing so supply curve is upward-sloping in the output price. Substituting the *unconditional* factor demands into the production function therefore yields the firm's supply function:

$$q = q(z_1(p,\mathbf{w}),\ldots,z_i(p,\mathbf{w}))$$

2.3 General Equilibrium in pure exchange

Suppose an economy with I consumers and n goods. Consumer j is endowed with $\boldsymbol{\omega}_j = (\omega_{1j},\ldots,\omega_{nj}) \in \mathbb{R}^n_+$. Through trade, every consumer tries to become *more satisfied* (i.e. reach a higher utility level) than at the initial endowment.
Considering a *Decentralized equilibrium*, consumer j solves

$$\max_{x_{1j},\ldots,x_{nj}} u_j(x_{1j},\ldots,x_{nj}) \tag{2.3.1}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} p_i x_{ij} \;=\; \sum_{i=1}^{n} p_i \omega_{ij} \tag{2.3.2}$$

The first-order (interior) optimality condition reads

$$\frac{\partial u_j/\partial x_{ij}}{\partial u_j/\partial x_{kj}} \;=\; \frac{p_i}{p_k}, \qquad \forall\, i \neq k.$$

The optimality condition, once substituted into equation (2.3.2) (the budget constraint) generates individual j's demand for each good. Repeating this procedure for every individual yields the complete system of demand functions—one for each of the n goods for each of the I individuals, i.e. $n \times I$ demand functions. Note that this is the same as getting the Marshallian demand for each good and replacing income m with the value of individual j's endowment, $\sum_{i=1}^{n} p_i \omega_{ij}$.

In equilibrium, by market clearing for every good i total demand equals total supply:

$$\sum_{j=1}^{I} x_{ij} \;=\; \sum_{j=1}^{I} \omega_{ij}, \qquad i = 1, \ldots, n.$$

The resulting system of n equations determines the equilibrium price vector $\big(p_1^*, \ldots, p_n^*\big)$ (defined up to a positive scalar normalization).

The equilibrium allocation is a *Pareto equilibrium*: no individual's utility can be increased without lowering someone else's. Varying individual endowments while maintaining the aggregate endowment fixed traces out the *Pareto set.* This set can be characterized by imposing, for each individual j, the optimality condition that equates the marginal substitution rate with the equilibrium price ratio, that is,

$$\frac{\partial u_j / \partial x_{ij}}{\partial u_j / \partial x_{kj}} = \frac{p_i^*}{p_k^*}$$

Note that some allocations make one or both individuals better off relative to their initial endowments while still allowing an increase in one person's utility without reducing the other's. This collection of allocations is called the *lens of trade.* Within this lens lies that segment of the *Pareto set* where no further utility gains are possible for anyone without harming someone else. The final allocation must therefore lie inside the lens of trade and on the Pareto set.

Take the following example with 2 individual and 2 goods; each individual's utility function is given by:

$$u_j(x_{1j}, x_{2j}) = \sqrt{x_{1j} x_{2j}} \qquad j = 1, 2$$

The resulting demand functions are:

$$x_{1j} = \frac{1}{2} \cdot \frac{p_1 \omega_{1j} + p_2 \omega_{2j}}{p_1} \qquad \wedge \qquad x_{2j} = \frac{1}{2} \cdot \frac{p_1 \omega_{1j} + p_2 \omega_{2j}}{p_2} \qquad j = 1, 2$$

Market clearing in equilibrium requires that

$$\begin{cases} \frac{1}{2} \cdot \frac{p_1 \omega_{11} + p_2 \omega_{21}}{p_1} + \frac{1}{2} \cdot \frac{p_1 \omega_{12} + p_2 \omega_{22}}{p_1} = \omega_{11} + \omega_{12} \\ \frac{1}{2} \cdot \frac{p_1 \omega_{11} + p_2 \omega_{21}}{p_2} + \frac{1}{2} \cdot \frac{p_1 \omega_{12} + p_2 \omega_{22}}{p_2} = \omega_{21} + \omega_{22} \end{cases}$$

Solving the first equation for the relative price yields:

$$\frac{p_2^*}{p_1^*} = \frac{(\omega_{11} + \omega_{12})}{(\omega_{21} + \omega_{22})} \tag{2.3.3}$$

The equilibrium *relative price* is unique, whereas the absolute price vector is determined only up to a positive scalar. For instance, in (2.3.3) one may set

$$p_2 = \lambda\big(\omega_{11} + \omega_{12}\big), \qquad p_1 = \lambda\big(\omega_{21} + \omega_{22}\big),$$

for any $\lambda > 0$, leaving the ratio p_1/p_2 unchanged.

Equation (2.3.3) states that the equilibrium *relative price* equals the ratio of total endowments. In words, the price of good 1 relative to good 2 equals the economy-wide endowment of good 2 relative to that of good 1, and vice-versa.

This illustrates *Walras's Law*: if a price vector clears total demand and supply in one market, it necessarily clears the remaining market. Concretely, choose any $\lambda > 0$ and set

$$p_2 = \lambda\big(\omega_{11} + \omega_{12}\big), \qquad p_1 = \lambda\big(\omega_{21} + \omega_{22}\big).$$

These prices equate aggregate demand and supply in the first market; by Walras's Law, they also clear the second market. More generally, if a price vector $(p_1^*, \ldots, p_n^*)$ clears $n-1$ markets, it clears all n markets.

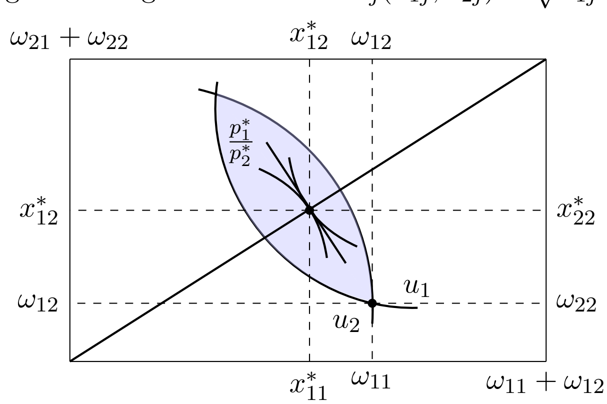
The Pareto-equilibrium allocation is as follows

$$x_{1j}^* = \frac{\omega_{1j}}{2} + \frac{\omega_{2j}}{2}\,\frac{\omega_{21} + \omega_{22}}{\omega_{11} + \omega_{12}}, \qquad x_{2j}^* = \frac{\omega_{1j}}{2}\,\frac{\omega_{21} + \omega_{22}}{\omega_{11} + \omega_{12}} + \frac{\omega_{2j}}{2}, \qquad j = 1, 2.$$

The Pareto-equilibrium allocation shown above depends on the *individual* endowments. If these endowments are redistributed—keeping the *aggregate* endowment constant—the Pareto allocation changes, yet the equilibrium relative price remains the one in (2.3.3). Hence a redistribution can raise one person's utility while lowering the other's.

The constancy of the relative price here is a special case: both agents share identical, symmetric preferences. In general, each Pareto equilibrium that arises from a given initial allocation is supported by its own relative-price vector, as stated by the Second Welfare Theorem. In this symmetric setting, however, every Pareto allocation is backed by the same price ratio in (2.3.3). Figure 1 illustrates the Edgeworth box for this example.

Figure 1: Edgeworth box for $u_j(x_{1j}, x_{2j}) = \sqrt{x_{1j}x_{2j}}$



Social–planner problem and the First Welfare Theorem Finally, note that a benevolent social planner who reallocates each good so that the aggregate assignment equals the aggregate endowment would implement exactly the same Pareto–efficient allocation that arises in the decentralized competitive equilibrium derived in (2.3.3). Formally, the planner solves

$$\begin{aligned} &\max_{x_{ij}\ \forall i,j} \sum_{j=1}^{I} \lambda_j\, u_j\big(x_{1j}, \ldots, x_{nj}\big) \\ &\text{s.t.} \qquad \sum_{j=1}^{I} x_{ij} \;=\; \sum_{j=1}^{I} \omega_{ij}, \qquad \forall\, i = 1, \ldots, n, \end{aligned} \tag{2.3.4}$$

where the Pareto weights λ_j in (2.3.4) are proportional to the marginal utility of the value of household j's original endowment. Because competitive markets already equate marginal rates of substitution across agents while respecting the resource constraints, the solution to (2.3.4) coincides with the decentralized equilibrium allocation—an illustration of the First Welfare Theorem.

2.4 Exercises

1. Consider a consumer whose preferences are represented by

$$U(x_1, x_2, \ldots, x_n) = x_k \prod_{i \neq k}^{n} (x_i - \theta_i), \qquad x_i > \theta_i \quad \forall i \neq k$$

   1. Derive the Marshallian demand functions.
   2. Obtain the indirect utility function.
   3. Derive the Hicksian (compensated) demand functions.
   4. Determine the expenditure function.

2. Let the expenditure function be

$$e(\bar{u}, \mathbf{p}) = \bar{u}p_1 - \frac{p_1^2}{4} \sum_{i=2}^{n} \frac{1}{p_i}, \qquad \bar{u} > \frac{p_1}{2} \sum_{i=2}^{n} \frac{1}{p_i}$$

   1. Derive the Marshallian demand functions.
   2. Obtain the indirect utility function.
   3. Derive the Hicksian (compensated) demand functions.
   4. Recover the underlying utility function.

3. Consider a firm with the production function

$$q(L, K) = [\max\{\min\{2L, K\}, \min\{L, 2K\}\}]^{\rho}, \qquad 0 < \rho < 1$$

   1. Derive the conditional factor demand functions.
   2. Determine the cost function.
   3. Derive the firm's supply function.
   4. Obtain the unconditional factor demands[1].

4. Consider a firm with the production function

$$q(L, K) = [\min\{\max\{2L, K\}, \max\{L, 2K\}\}]^{\rho}, \qquad 0 < \rho < 1$$

   1. Derive the conditional factor demand functions.
   2. Determine the cost function.
   3. Derive the firm's supply function.
   4. Obtain the unconditional factor demands.

[1] Consider $z_i(p, \mathbf{w}) = z_i(q(p, z_i(p, \mathbf{w})), \mathbf{w})$

5. Consider a pure-exchange economy with two consumers. Consumer A's utility is

$$u^A(x_{1A}, x_{2A}) = 2x_{1A} + x_{2A}$$

while consumer B's utility is

$$u^B(x_{1B}, x_{2B}) = \min\{x_{1B}, x_{2B}\}$$

Their endowments are $w^A = (0, \bar{w})$ and $w^B = (\bar{w}, 0)$

1. State the initial endowment point.
2. Determine the lens of trade.
3. Characterize the Pareto set.
4. Compute the equilibrium relative price.
5. Identify the set of equilibrium allocations.

6. A small town has n residents. Resident i is endowed with $\bar{w}$ units of good i and none of the other goods. Every resident's preferences are

$$u_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} \ln x_j$$

1. Derive the equilibrium relative prices.
2. Characterize the Pareto set.
3. Identify the equilibrium allocation set.
4. Show that equilibrium prices are independent of the endowments and explain the intuition.
5. Verify that the centralized allocation matches the competitive equilibrium.

3 The Export Condition and Ricardian Model

3.1 The Export Condition

Suppose there is Home (H) and (F) and n goods. Each good i requires a_i units of labor per unit of output, and a_i^* units abroad. The wage rate in Home is w, and in Foreign it is w^*. If e denotes the nominal exchange rate (units of Foreign currency per unit of Home currency), the unit cost of producing good i is

$$c_i^H = e \cdot w \cdot a_i, \qquad c_i^F = w^* \cdot a_i^*$$

Here, c_i^H is measured in Foreign currency by multiplying by e, while c_i^F is expressed in Foreign currency. This adjustment ensures cost to be measured in the same currency.

The fundamental export condition is that Home exports good i if it can supply the good at lower cost than Foreign:

$$c_i^H < c_i^F.$$

Substituting from above:

$$e \cdot w \cdot a_i < w^* \cdot a_i^* \tag{3.1.1}$$

Equation (3.1.1) can be written to state that Home exports good i if

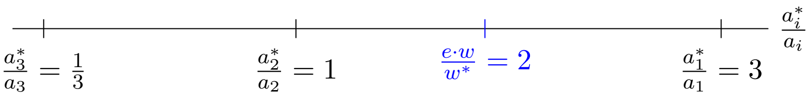
$$\frac{e \cdot w}{w^*} < \frac{a_i^*}{a_i} \tag{3.1.2}$$

Equation (3.1.2) can be interpreted as follows: Home exports good i whenever the relative wage—that is, the wage in Home expressed in terms of the Foreign wage—is lower than the relative cost of producing the good abroad, expressed in terms of Home's cost. The right-hand side of the inequality thus represents the relative unit labor requirements, capturing the notion of comparative efficiency.

To illustrate, consider three goods x_1, x_2, and x_3. The unit labor requirements in Home are $(1, 2, 3)$, while in Foreign they are $(3, 2, 1)$. Suppose the Home wage (in Foreign currency) is 2, and the Foreign wage (in Foreign currency) is 1. According to the export condition, Home will have a cost advantage in goods x_1, thereby producing and exporting them, while it will import x_2 and x_3 from Foreign.

Figure 2 illustrates this scenario. The goods are ordered from lowest to highest according to Home's relative unit requirements. Relative wage adjusted by exchange rate is in blue. All goods positioned to the right of relative wage are produced and exported by Home (and imported by Foreign), while those to the left are produced and exported by Foreign (and imported by Home).

Figure 2: Export condition in example given



3.1.1 Transport Costs

Consider the presence of transport costs, denoted by τ, which are assumed to be identical across countries and goods. The cost τ is measured in units of the good itself. Such costs are commonly referred to as *iceberg costs*, since τ represents the additional quantity of the good that must be shipped for one unit to arrive in the destination country.
The export condition for Home to supply good i under iceberg transport costs is given by

$$(1+\tau)e \cdot w \cdot a_i < w^* \cdot a_i^*, \tag{3.1.3}$$

which can be rewritten as

$$\frac{e \cdot w}{w^*} < \frac{a_i^*}{(1+\tau) \cdot a_i}, \tag{3.1.4}$$

where equation (3.1.4) expresses the condition in terms of relative wages.
Suppose that in the absence of transport costs, Home exports good i and Foreign exports good j. Once transport costs are introduced, the two goods may become nontraded if

$$(1+\tau)e \cdot w \cdot a_i > w^* \cdot a_i^*, \tag{3.1.5}$$

$$e \cdot w \cdot a_j < w^* \cdot a_j^*(1+\tau). \tag{3.1.6}$$

Although Home is assumed to have a comparative advantage in good i, a sufficiently high transport cost may lead condition (3.1.5) to hold, implying that Home no longer exports good i because the effective cost of exporting exceeds the cost of production in Foreign. Analogously, the same reasoning applies to Foreign with respect to good j in condition (3.1.6).
Equations (3.1.5) and (3.1.6) can be equivalently expressed as

$$\frac{(1+\tau)e \cdot w}{w^*} > \frac{a_i^*}{a_i}, \tag{3.1.7}$$

$$\frac{e \cdot w}{(1+\tau)w^*} < \frac{a_j^*}{a_j}. \tag{3.1.8}$$

Considering multiple goods, these inequalities define the range of nontraded goods. The relative requirements that satisfy equations (3.1.7) and (3.1.8) with equality determine the boundaries of the nontraded sector. Any good k whose relative requirement lies within these bounds will not be traded internationally.

3.2 The Ricardian Model of International Trade

Consider a world economy with two countries: Home (denoted by H) and Foreign (denoted by F). The economy produces two goods, indexed by q_1^i and q_2^i, where $i \in \{H, F\}$. Labor is the only factor of production, and each country is endowed with a fixed labor supply $L^i > 0$.
Technology is characterized by constant unit labor requirements: producing one unit of good q_1 in country i requires a_1^i units of labor, while producing one unit of good q_2 requires a_2^i units of labor. We assume $a_1^i, a_2^i > 0$ and constant returns to scale.

3.2.1 Production

The production function for good $i \in \{1, 2\}$ in country $j \in \{H, F\}$ is given by:

$$q_i^j = \frac{L_i^j}{a_i^j} \tag{3.2.1}$$

where L_i^j denotes the amount of labor allocated to sector i in country j.
This functional form reflects a fixed-coefficient technology: each unit of output q_i^j requires exactly a_i^j units of labor. Equivalently, $1/a_i^j$ is the marginal product of labor in sector i of country j.
Labor is perfectly mobile across sectors within a country but immobile across countries. The labor resource constraint is therefore:

$$L_1^j + L_2^j = L^j, \qquad j \in \{H, F\}, \tag{3.2.2}$$

which reflects the fact that total labor demand must equal the exogenously supply labor.

3.2.2 Production Possibility Frontier

To characterize the set of feasible output combinations, substitute $L_i^j = a_i^j q_i^j$ into the labor constraint:

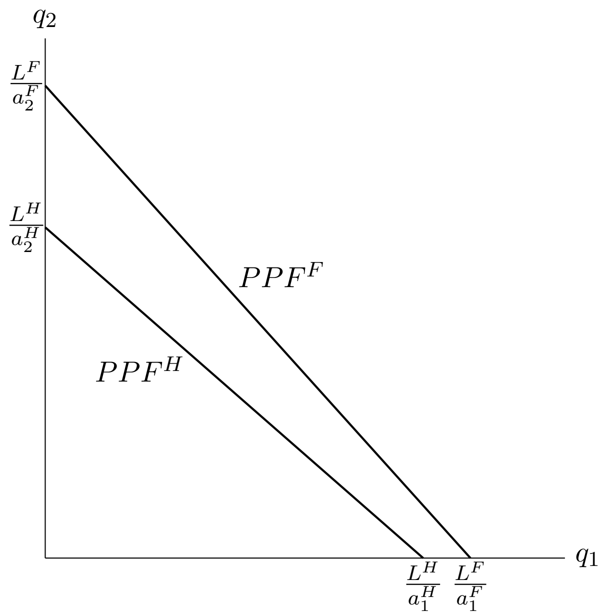
$$a_1^j q_1^j + a_2^j q_2^j = L^j, \qquad j \in \{H, F\}. \tag{3.2.3}$$

Equation (3.2.3) is a linear relationship between q_1^j and q_2^j with slope $-\frac{a_1^j}{a_2^j}$, which we refer to as the **relative requirements**. The intercepts are $\frac{L^j}{a_1^j}$ on the q_1^j axis and $\frac{L^j}{a_2^j}$ on the q_2^j axis. These represent the maximum quantity of each good that country j could produce if it devoted all its labor to that sector. Equation (3.2.3) thus defines the Production Possibility Frontier (PPF).
The linearity of the PPF follows directly from the assumption of constant unit labor requirements. Its slope, $-\frac{a_1^j}{a_2^j}$, measures the opportunity cost of producing one unit of q_1^j in terms of forgone units of q_2^j. Since this opportunity cost is constant, the model does not feature diminishing returns to specialization. As a result, **full specialization naturally emerges**

under trade.

Figure 3 illustrates the PPF in a case where Home has a comparative advantage in good q_1. The intercepts depend on each country's labor supply and labor requirements. Assuming $L^H = L^F$, the figure suggests that Foreign enjoys an absolute advantage in both goods, even though Home maintains a comparative advantage in good q_1.

Figure 3: Production Possibility Frontier



3.2.3 Autarky

In autarky, competitive equilibrium requires that the relative price of the two goods equals their opportunity cost in production. Let p_1^j and p_2^j denote the prices of goods q_1^j and q_2^j in country j, respectively. Under perfect competition and zero profits, the unit cost of producing good i must equal its price:

$$p_1^j = w^j a_1^j, \tag{3.2.4}$$
$$p_2^j = w^j a_2^j \tag{3.2.5}$$

where w^j is the wage in country j. Dividing equation (3.2.4) by equation (3.2.5) yields:

$$\frac{p_1^j}{p_2^j} = \frac{a_1^j}{a_2^j}. \tag{3.2.6}$$

Thus, in autarky, the relative price equals the constant marginal rate of transformation implied by the PPF.

3.2.4 Opening to Trade

When the economy opens to trade, the relevant relative price is the *world* relative price, $\frac{p_1^W}{p_2^W}$. Suppose that Home has a comparative advantage[2] in good q_1, meaning:

$$\frac{a_1^H}{a_2^H} < \frac{a_1^F}{a_2^F}, \tag{3.2.7}$$

where a_i^F are the unit labor requirements in Foreign.

Comparative advantage is therefore determined entirely by the ratio of unit labor requirements across goods and countries. If Home has a comparative advantage in q_1, then by construction, Foreign must have a comparative advantage in q_2.

It is also possible for one country to have an *absolute advantage*[3] in both goods. Absolute advantage is defined by direct productivity levels, while comparative advantage arises from relative productivity differences and ultimately governs trade patterns.

Figure 4 illustrates the relative offer curve of the model once the economy opens to trade. The derivation, assuming that Home has a comparative advantage in good q_1, proceeds as follows:

- If the world relative price $\frac{p_1^W}{p_2^W}$ is lower than both Home's autarky price ratio $\frac{a_1^H}{a_2^H}$ and Foreign's autarky price ratio $\frac{a_1^F}{a_2^F}$, then both Home and Foreign fully specialize in the production of q_2. In this case:

  $$q_2^W = \frac{L^H}{a_2^H} + \frac{L^F}{a_2^F}, \qquad q_1^W = 0$$

  Thus, $\frac{q_1^W}{q_2^W} = 0$, since both countries specialize in q_2 (whose relative price exceeds its relative cost in both economies). This corresponds to the segment A–B in Figure 4.

- If the world relative price $\frac{p_1^W}{p_2^W}$ lies between Home's autarky price ratio $\frac{a_1^H}{a_2^H}$ and Foreign's autarky price ratio $\frac{a_1^F}{a_2^F}$, then Home fully specializes in q_1 and Foreign in q_2. In this case:

  $$q_1^W = \frac{L^H}{a_1^H}, \qquad q_2^W = \frac{L^F}{a_2^F}$$

  Therefore,

  $$\frac{q_1^W}{q_2^W} = \frac{L^H/a_1^H}{L^F/a_2^F}$$

  This scenario is the most economically relevant: each country specializes in the good for which it has comparative advantage. It corresponds to the segment C–D in Figure 4.

[2] A country has a comparative advantage in good i if its relative cost of producing i is lower than that of the other country. In other words, Home sacrifices less of good j to produce one unit of i compared to Foreign.

[3] A country has an absolute advantage in good i if, with the same resources, it can produce more of i than the other country. Formally, country j has absolute advantage in good i if $a_i^j < a_i^{-j}$, where $-j$ denotes the other country.

- If the world relative price $\frac{p_1^W}{p_2^W}$ is higher than both Home's and Foreign's autarky price ratios, then both countries fully specialize in q_1. In this case:

$$q_1^W = \frac{L^H}{a_1^H} + \frac{L^F}{a_1^F}, \qquad q_2^W = 0$$

  Hence, $\frac{q_1^W}{q_2^W} = \infty$, as both countries allocate all resources to q_1. This corresponds to the segment D–∞ on the horizontal axis in Figure 4.

- If the world relative price equals Home's autarky price ratio, $\frac{p_1^W}{p_2^W} = \frac{a_1^H}{a_2^H}$, then Home is indifferent between producing q_1, q_2, or any combination of both. If it participates in trade, it may choose any production in its PPF, while Foreign specializes in q_2. In this case:

$$\frac{q_1^W}{q_2^W} \in \left[0,\ \frac{L^H/a_1^H}{L^F/a_2^F}\right]$$

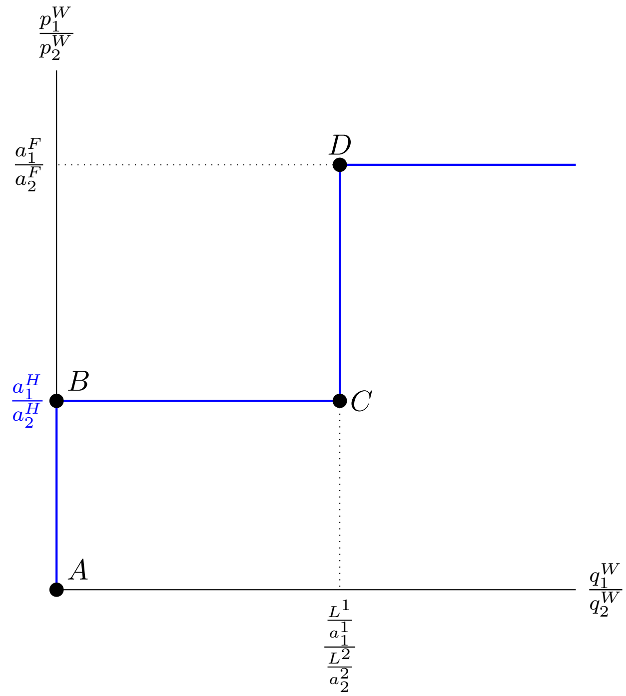
  This corresponds to the segment B–C in Figure 4.

- If the world relative price equals Foreign's autarky price ratio, $\frac{p_1^W}{p_2^W} = \frac{a_1^F}{a_2^F}$, then Foreign is indifferent between producing q_1, q_2, or any combination of both. If it participates in trade, it may choose any production in its PPF, while Home specializes in q_1. In this case:

$$\frac{q_1^W}{q_2^W} \in \left[\frac{L^H/a_1^H}{L^F/a_2^F},\ \infty\right]$$

  This corresponds to the segment D–∞ in Figure 4.

Figure 4: Relative Market in the Ricardian model



As discussed earlier, the equilibrium arises when each country specializes in a different good. Figure 5 illustrates this equilibrium outcome.

Assume that both countries share a homothetic utility function. For example, let preferences be represented by

$$U_j(q_1, q_2) = q_1 q_2, \qquad j \in H, F$$

The corresponding optimality condition is:

$$\frac{q_2}{q_1} = \frac{p_1}{p_2} \tag{3.2.8}$$

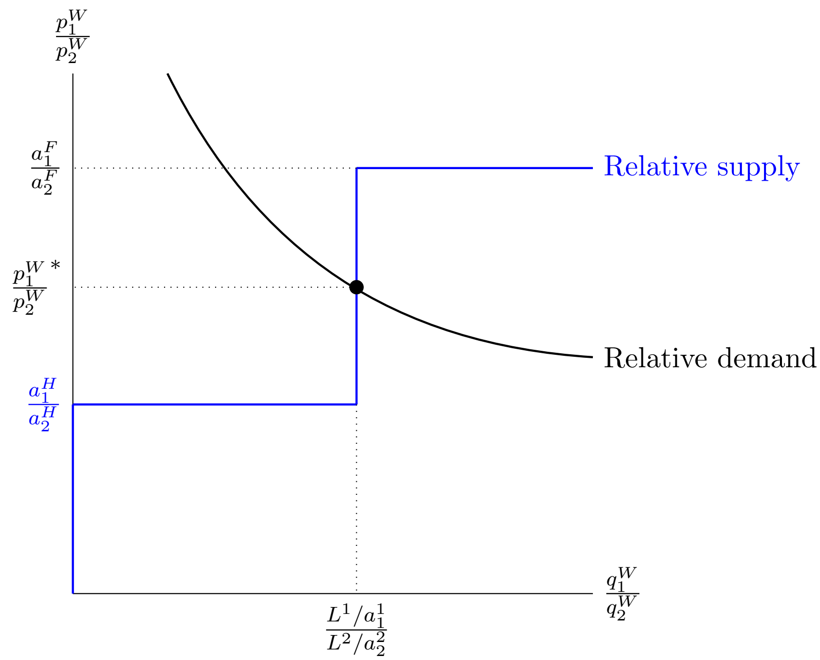
Equation (3.2.8) can be expressed as

$$\frac{p_1}{p_2} = \frac{1}{q_1/q_2} \tag{3.2.9}$$

Equation (3.2.9) represents the relative demand. This curve has a negative slope and is convex in relative quantities. To determine the equilibrium world relative price, the ratio $\frac{q_1^W}{q_2^W}$ can be substituted for the case in which each country specializes in the good for which it holds a comparative advantage. The resulting expression yields the equilibrium world relative price:

$$\frac{p_1^W}{p_2^W} = \frac{L^2/a_2^2}{L^1/a_1^1} \tag{3.2.10}$$

Figure 5: Relative Market in the Ricardian model



For the relative demand curve to intersect the vertical segment of the relative supply curve at the relative quantity $\frac{L^H/a_1^H}{L^F/a_2^F}$ in Figure 5, the equilibrium world relative price must satisfy

$$\left(\frac{p_1^W}{p_2^W}\right)^* \in \left]\frac{a_1^H}{a_2^H},\ \frac{a_1^F}{a_2^F}\right[$$

3.3 Exercises

1. Consider the Export Condition studied here (Section 3.1) where there are three goods and three countries. Table 1. reports the unit labor requirements for each good in each country.

Requirements	**A**	**B**	**C**
a_1	2	3	4
a_2	4	3	1
a_3	1	2	4

   1. Suppose it is known that country A produces and exports good x_3, country B produces and exports good x_1, and country C produces and exports good x_2. What must be true about the relative wages, expressed in the currency of country C?

2. Based on the export condition (Section 3.1), consider an economy with n goods, where wages are identical across countries, expressed in a common currency. The labor requirements are specified as

$$(a_1, a_2, \ldots, a_n) = (1, 2, \ldots, n), \quad (a_1^*, a_2^*, \ldots, a_n^*) = (n, n-1, \ldots, 1).$$

   1. Determine which goods are produced and exported by Home, and which goods are produced and exported by Foreign.
   2. Assume a transport cost of $\frac{1}{4}$ and $n = 10$. Identify the set of goods produced and exported by Home and those produced and exported by Foreign. Additionally, determine whether any goods become nontraded.
   3. Consider a transport cost equal to τ and n goods. Characterize the goods produced and exported by Home and Foreign. Furthermore, assess the existence of nontraded goods as a function of τ. Demonstrate that the range of nontraded goods expands as the transport cost increases.

3. Consider the Ricardian model (Section 3.2) where country A requires 2 units of labor to produce one unit of q_1 and 1 unit of labor to produce one unit of q_2. Country B requires 4 units of labor to produce one unit of q_1 and 3 units of labor to produce one unit of q_2. The labor endowment in each country is not specified.

   1. Identify which country has absolute advantage and which has comparative advantage.
   2. Derive the Production Possibility Frontier (PPF) for each country.
   3. Obtain the world relative supply.
   4. Assuming preferences are represented by $U(q_1, q_2) = q_1^2 q_2$ in both countries, determine equilibrium production, relative prices, and wages under autarky.

5. Using the same utility function, determine equilibrium production, relative prices, and wages under international trade.
6. Propose a method to evaluate whether each country is better off under trade compared to autarky.

4. Consider the Ricardian model (Section 3.2) where Home has a comparative advantage in q_2. Suppose the economy is open to trade and equilibrium occurs at a point where each country fully specializes in the good for which it has comparative advantage. Answer the following questions and explain the underlying intuition:

   1. Why can Home still benefit from trade even if it has an absolute advantage in both goods?
   2. What happens if the population in Home increases? Is Home better off? Is Foreign better off?
   3. What happens if the population in Foreign decreases? Is Home better off? Is Foreign better off?
   4. What happens if technology improves in Home at the same proportional rate for both goods? Is Home better off? Is Foreign worse off?
   5. What happens if technology improves in Foreign for good q_1 only? Is Home better off? Is Foreign worse off?

4 Comparative Advantage, Trade, and Payments in a Ricardian Model with a Continuum of Goods (Dornbusch et al., 1977)

Dornbusch et al. (1977) extend the basic Ricardian framework to a continuum of goods. Consider two countries, Home (H) and Foreign (F), and a single factor of production, labor, with endowments L^H and L^F, respectively. Labor is assumed to be perfectly mobile across sectors within each country but immobile across countries. Markets operate under perfect competition and exhibit constant returns to scale. Wages are denoted by w^H in Home and w^F in Foreign. The exchange rate, defined as the price of one unit of Foreign's currency in terms of Home's currency, is represented by e^H.

4.1 Supply

Relative efficiency between the two countries for any good z is defined as

$$A(z) = \frac{a^F(z)}{a^H(z)}, \qquad A'(z) < 0 \tag{4.1.1}$$

where $a^H(z)$ denotes the unit labor requirement (units of labor necessary to produce one unit of good z) in Home, and $a^F(z)$ represents the corresponding requirement in Foreign.

Goods are represented by a continuum indexed by $z \in [0, 1]$, ordered such that Home's comparative advantage decreases with z. This implies that goods located near $z = 0$ are those in which the relative productive efficiency of Home with respect to Foreign is most evident, whereas goods located near $z = 1$ are those in which the relative productive efficiency of Foreign is most pronounced. This ordering justifies the assumption $A'(z) < 0$ in equation (4.1.1).

Under free trade, each good z is produced in the country with the lower unit cost. The export condition requires that Home produces z if

$$a^H(z)\, e^H w^H \leq a^F(z)\, w^F \quad \Leftrightarrow \quad \frac{e^H w^H}{w^F} \leq \frac{a^F(z)}{a^H(z)}$$

The cutoff good $\tilde{z}$ is defined at the point where the unit costs are exactly equal across countries. This condition determines the supply schedule for the model:

$$\frac{e^H w^H}{w^F} = \frac{a^F(z)}{a^H(z)} \tag{4.1.2}$$

All goods with $z < \tilde{z}$ are produced by Home (the range in which it holds comparative advantage), while all goods with $z > \tilde{z}$ are produced by Foreign.

Given constant returns to scale and perfect competition, the price of good z equals its marginal cost. Hence, the relative price of two goods produced in Home is

$$\frac{p^H(z)}{p^H(z')} = \frac{w^H \cdot a^H(z)}{w^H \cdot a^H(z')} = \frac{a^H(z)}{a^H(z')}$$

which corresponds to the relative labor requirements between the two goods.
By contrast, the relative price of a Home-produced good z in terms of a Foreign-produced good z'' is given by

$$\frac{p^H(z)}{p^F(z'')} = \frac{e^H w^H \cdot a^H(z)}{w^F \cdot a^F(z'')}$$

4.2 Demand

Preferences are assumed to be identical and homothetic across countries. Let $b^H(z)$ denote the *expenditure-share density function* for good z in Home, which must satisfy

$$\int_0^1 b^H(z)\, dz = 1, \qquad b^H(z) = \frac{p^H(z)\, x^H(z)}{y^H} > 0 \tag{4.2.1}$$

where $p^H(z)$ is the price of good z in Home, $x^H(z)$ is the quantity consumed in Home, and y^H denotes Home's income. Hence, $b^H(z)$ represents the fraction of home's income spent on good z. Because preferences are identical across countries, it follows that $b^H(z) = b^F(z)$.
Define

$$B(\tilde{z}) = \int_0^{\tilde{z}} b^H(z)\, dz \tag{4.2.2}$$

where $B(\tilde{z})$ is the share of global spending on Home-produced goods.
World income is given by

$$Y^W = e^H w^H L^H + w^F L^F \tag{4.2.3}$$

In equilibrium, the value of Home's exports must equal the value of its production:

$$B(\tilde{z})\, Y^W = e^H w^H L^H \tag{4.2.4}$$

Substituting (4.2.2) and (4.2.3) into (4.2.4) and solving for relative factor prices yields the demand side of the model:

$$\frac{e^H w^H}{w^F} = \left(\frac{\int_0^{\tilde{z}} b^H(z)\, dz}{1 - \int_0^{\tilde{z}} b^H(z)\, dz} \right) \frac{L^F}{L^H} \tag{4.2.5}$$

Note that $1 - \int_0^{\tilde{z}} b^H(z)\, dz$ represents the share of global expenditure on Foreign-produced goods.
Equation (4.2.5) can be interpreted as follows: if the range of domestically produced goods were to expand at constant relative wages, demand for domestic labor (and goods) would rise while demand for Foreign labor would decline. To restore equilibrium, an increase in the

domestic relative wage is required. Consequently, equation (4.2.5) is upward-sloping in relative wages.

Another way to understand why equation (4.2.5) is upward-sloping is that, as when z increases, the share of income allocated to home goods rises, while the share allocated to foreign goods declines. Consequently, $\left(\frac{\int_0^{\tilde{z}} b^H(z)\,dz}{1-\int_0^{\tilde{z}} b^H(z)\,dz}\right)$ increases.

An alternative formulation of the trade balance condition is

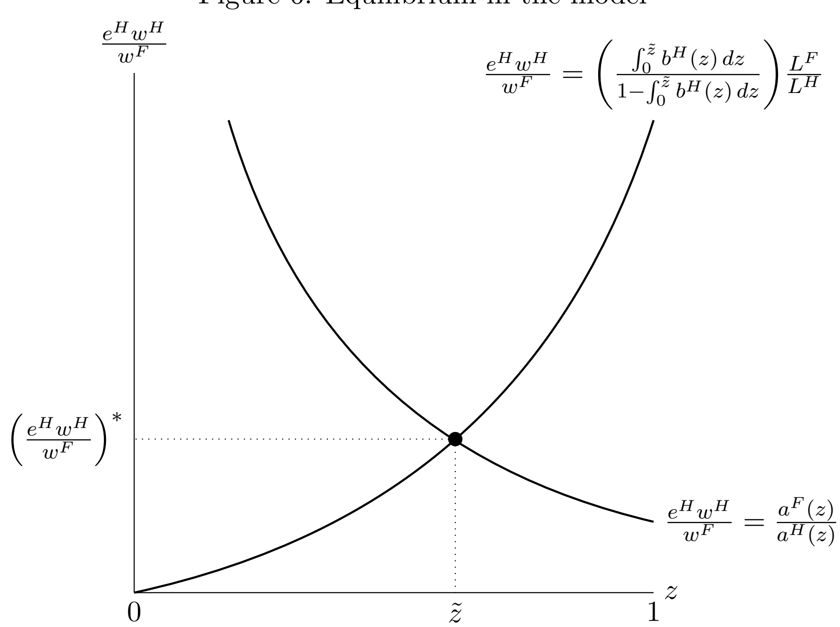
$$e^H w^H L^H \left(1 - \int_0^{\tilde{z}} b^H(z)\,dz\right) = w^F L^F \int_0^{\tilde{z}} b^H(z)\,dz \tag{4.2.6}$$

Equation (4.2.6) states that equilibrium in trade requires imports (the left-hand side) to equal exports (the right-hand side). The schedule implied by (4.2.6) is upward-sloping because an expansion in the range of goods produced at Home, at constant relative wages, reduces imports and increases exports. The resulting imbalance must be corrected through an increase in Home's relative wage.

4.3 Equilibrium

Equilibrium is attained at the relative wage that equalizes the right-hand sides of equations (4.1.2) and (4.2.5). This condition determines $\tilde{z}$, the equilibrium cutoff that delineates the range of goods produced and exported by Home from those produced and exported by Foreign. Figure 6 illustrates the equilibrium in this framework.

Figure 6: Equilibrium in the model



Note that, given a home wage w^H, all goods produced and exported by Home (i.e., $z \in [0, \tilde{z}]$) have a price equal to the Home autarky price for that z, expressed in foreign currency.

Meanwhile, all goods produced and exported by Foreign are priced at the foreign autarky price. Formally, this can be written as

$$p(z) = \begin{cases} e^H w^H a^H(z), & z \in [0, \tilde{z}], \\ w^F a^F(z), & z \in [\tilde{z}, 1]. \end{cases} \tag{4.3.1}$$

If welfare in Home is measured by its real wage, observe that for those goods exported by Home, the real wage in equilibrium remains the same as under autarky. However, for goods imported by Home, the real wage is higher, since the foreign price of these goods is lower than the autarky price, given the export condition. This implies that Home is better off when trade is opened. The same intuition applies symmetrically to the foreign country.
To illustrate these mechanisms more clearly, consider a discrete setting with n goods, where the two countries are characterized by the following production functions:

$$x_i^H = \frac{1}{i} L_i^H, \qquad x_i^F = \frac{1}{(n-i+1)} L_i^F$$

The utility function is specified as

$$U^j(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^i, \qquad j = H, F$$

The supply schedule is given by

$$\frac{e^H w^H}{w^F} = \frac{(n-i+1)}{i} \tag{4.3.2}$$

The fraction of national income spent on good i is

$$b_i = \frac{2i}{n(n+1)}$$

Accordingly, the fraction of world income devoted to goods produced by Home is

$$\sum_{i=1}^{i^*} \frac{2i}{n(n+1)} = \frac{i^*(i^*+1)}{n(n+1)}$$

where i^* denotes the cutoff good separating production between Home and Foreign. This condition implies the following demand schedule:

$$\frac{e^H w^H}{w^F} = \left(\frac{\frac{i^*(i^*+1)}{n(n+1)}}{1 - \frac{i^*(i^*+1)}{n(n+1)}} \right) \frac{L^F}{L^H} \tag{4.3.3}$$

The cutoff i^* is determined by combining (4.3.2) and (4.3.3) and solving for i^*:

$$\frac{(n - i^* + 1)}{i^*} = \left(\frac{\frac{i^*(i^*+1)}{n(n+1)}}{1 - \frac{i^*(i^*+1)}{n(n+1)}} \right) \frac{L^F}{L^H} \tag{4.3.4}$$

Expression (4.3.4) implies that i^* increases with either n or L^H, whereas i^* decreases with an increase in L^F.

Note expression (4.3.4) is increasing in i^* if n or L^H increases and i^* decreases if L^F increases.

4.4 The Price-Specie Flow Mechanism

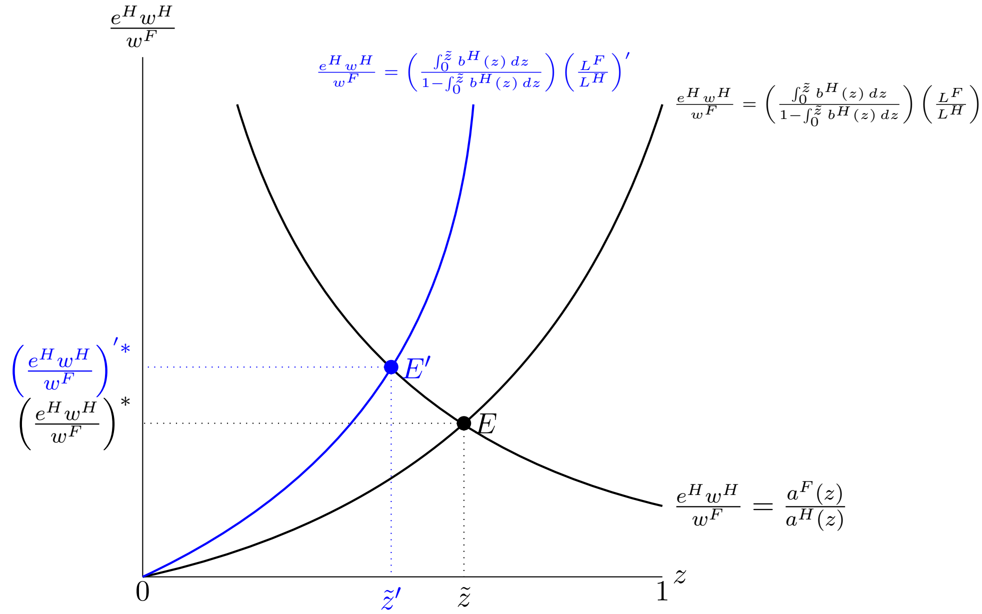
This mechanism was first articulated by Hume (1752) in which gold or silver served as the means of international settlement. Suppose that Home initially runs a trade surplus. The excess demand for Home goods implies that foreign buyers must pay in specie, leading to an inflow of gold or silver into Home. As specie enters the economy, the domestic money supply expands, generating an increase in the price level of Home goods. This raises Home's wage measured in specie, thereby eroding the set of goods for which Home has a comparative cost advantage. Goods that were previously competitive for Home shift toward the Foreign production margin. Consequently, Home's exports contract and imports expand until there is a balance in trade.

4.5 Comparative statics in the model

Equilibrium in Figure 6 is determined by preferences, technology in both Home and Foreign, and the labor endowments of each country. Suppose there is an increase in the relative labor endowment $\frac{L^F}{L^H}$. This shift raises the right-hand side of equation (4.2.5), thereby shifting the demand schedule upward.

At the initial equilibrium, this adjustment generates an imbalance in equation (4.2.5) (or equivalently in equation (4.2.6)), as the value of Home's exports exceeds the value of its imports. According to the Price–Specie Flow Mechanism, this imbalance induces an increase in wages at Home. The resulting rise in relative wages reduces Home's competitiveness, causing the cutoff $\bar{z}$ to decline until a new equilibrium is reached at point E', as illustrated in Figure 7.

Figure 7: Effect of an increase in relative labor endowment $\frac{L^F}{L^H}$



Suppose that w^H remains constant while w^F decreases, thereby raising Home's relative wage in the new equilibrium shown in Figure 7. The welfare analysis can then be summarized as follows:

1. For $z \in [0, \tilde{z}']$, the price of z remains unchanged after the increase in the foreign relative labor endowment. This is because neither w^H nor $a^H(z)$ is affected.

2. For $z \in (\tilde{z}, 1]$, the price of z is lower after the increase in foreign labor endowment because w^F falls while $a^F(z)$ remains unchanged.

3. For $z \in (\tilde{z}, \tilde{z}')$, the price of z becomes

$$p'(z) = w'^F a^F(z),$$

which is strictly lower than its equilibrium price before the change in the foreign relative labor endowment, that is,

$$p(z) = e^H w^H a^H(z),$$

due to the export condition.

From this analysis, and measuring welfare by the relative wage, it follows that Home is better off after the increase in the foreign relative labor endowment, while Foreign is worse off. Home benefits because the price of z is lower for $z > \tilde{z}'$ while its wage remains constant. In contrast, the foreign relative wage does not change for $z \in [\tilde{z}, 1]$, but it decreases for $z < \tilde{z}$.

4.6 Exercises

1. Examine the behavior of the exchange rate in recent years for Costa Rica, measured in colones per U.S. dollar.

Table 1: Average exchange rate: colones per U.S. dollar

Year	Bid	Ask
2022	₡644	₡651
2023	₡541	₡547
2024	₡512	₡518
2025[4]	₡502	₡508
Average 2022 to 2025	**571**	**577**

   1. Based on the model developed in this section, analyze the implications of this evolution for Costa Rica. Illustrate your explanation with a graph.

2. Using the model studied in this section, analyze the effects of the following changes (support your answer with graphs):

   1. An increase in the population (and therefore the labor force) in Home.
   2. A decrease in the population (and therefore the labor force) in Foreign.
   3. An improvement in technology in Home.
   4. An improvement in technology in Foreign.
   5. A stronger preference for goods produced in Foreign.

3. Consider a continuum of goods indexed by $z \in [0, 1]$, with the following production functions for Home and Foreign:

$$x^H(z) = \frac{L^H(z)}{(\alpha\sqrt{z} - b\sqrt[3]{z^2})}, \qquad x^F(z) = \frac{L^F(z)}{\sqrt{z}}$$

Preferences in both countries are represented by the utility function

$$U = \int_0^1 \ln(x)\, dx$$

Initially, the labor endowment is the same in both countries.

   1. Derive the supply curve implied by the model.
   2. Derive the demand curve implied by the model.

3. Determine the cutoff (borderline) between goods produced by Home and those produced by Foreign. Also, compute relative wages in equilibrium.
4. Compute the production of each good.
5. Suppose Foreign's labor endowment increases by 5%. Analyze how this change affects the equilibrium and provide the economic intuition behind the result.

5 General Equilibrium in production

Two equilibrium concepts are distinguished: (i) Equilibrium with only production, and (ii) Equilibrium with consumption and production. Together, these concepts provide the theoretical foundation for the *Heckscher–Ohlin–Vanek* model.

5.1 Equilibrium with only production

Consider an economy that produces two goods, q_1 and q_2, using two factors of production—labour (L) and capital (K). The aggregate endowments of these factors are $\bar{L}$ and $\bar{K}$, respectively. Each sector is described by a production function exhibiting constant returns to scale (i.e. a function homogeneous of degree 1).

Assume that the two sectors differ in factor intensities: good q_i is *capital-intensive* relative to good q_j, meaning

$$\frac{K_i^*(q_i,w,r)}{L_i^*(q_i,w,r)} > \frac{K_j^*(q_j,w,r)}{L_j^*(q_j,w,r)}$$

This ordering is invariant for all admissible factor-price pairs (w,r); that is, there is *no factor-intensity reversal*.

Solving the problem:

$$\begin{aligned} \min_{L_i,K_i} \quad & wL_i + rK_i \\ \text{s.t.} \quad & q_i = q_i(L_i,K_i)\,, \qquad \forall i = 1,2 \end{aligned}$$

The optimization delivers the conditional factor demands $L_i^*(q_i,w,r)$ and $K_i^*(q_i,w,r)$. Substituting these expressions into the total cost function generates the minimum total cost (same as unit cost as per constant returns to scale) for the good i, denoted $C_i(q_i,w,r)$. Differentiating C_i with respect to the output yields the marginal cost and because the production function exhibits constant returns to scale, profits are zero, and under perfect competition the good's price equals its (constant) marginal cost

$$p_i \;=\; \frac{\partial C_i(q_i,w,r)}{\partial q_i} = c_i(w,r)$$

Let a_{Li} and a_{Ki} denote the labour and capital required to produce one unit of good i ($q_i = 1$):

$$a_{Li} \equiv L_i^*(1,w,r), \qquad a_{Ki} \equiv K_i^*(1,w,r)$$

Consequently, the zero-profit conditions can be expressed as

$$\begin{cases} p_1 = w\,a_{L1} + r\,a_{K1}, \\ p_2 = w\,a_{L2} + r\,a_{K2} \end{cases} \tag{5.1.1}$$

where prices equal unit costs for each sector.

Factor–market clearing (full utilization of labor and capital) requires

$$\begin{cases} \bar{L} = L_i^*(q_i, w, r) + L_j^*(q_j, w, r), \\ \bar{K} = K_i^*(q_i, w, r) + K_j^*(q_j, w, r) \end{cases} \tag{5.1.2}$$

which is equivalent to

$$\begin{cases} \bar{L} = a_{Li}q_i + a_{Lj}q_j, \\ \bar{K} = a_{Ki}q_i + a_{Kj}q_j \end{cases}$$

Solving this linear system yields the equilibrium outputs (q_i^*, q_j^*).

Sector i is *labor-intensive* relative to sector j if

$$\frac{a_{Li}}{a_{Ki}} > \frac{a_{Lj}}{a_{Kj}} \quad \Longleftrightarrow \quad \frac{L_i^*}{K_i^*} > \frac{L_j^*}{K_j^*}$$

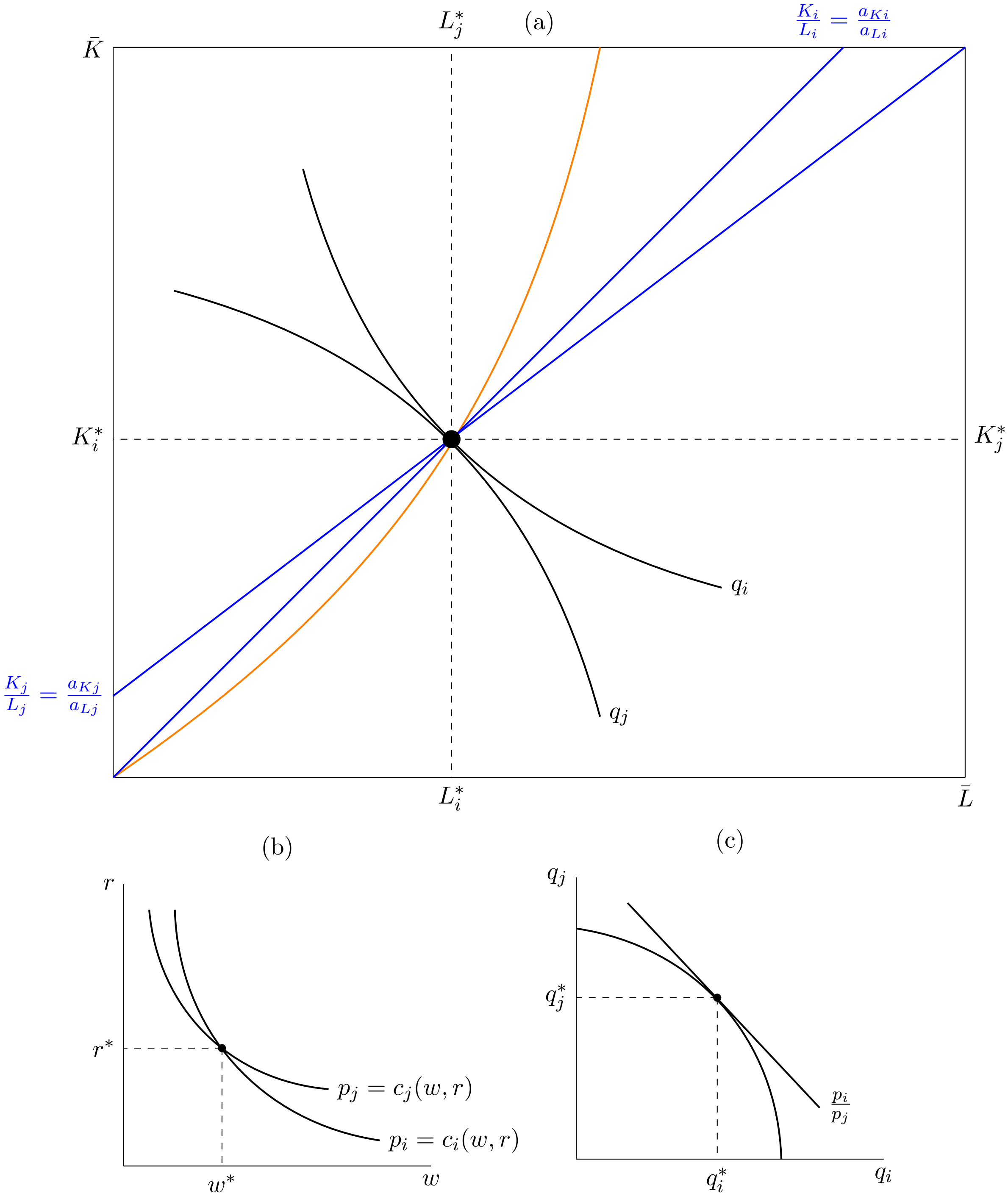
This ranking is invariant to all admissible factor-price pairs (w, r), reflecting the assumption of *no factor-intensity reversal*.

The equilibrium allocation of labor and capital in each sector must satisfy the optimal factor ratio:

$$\frac{L_i}{K_i} = \frac{a_{Li}}{a_{Ki}}, \qquad i = 1, 2 \tag{5.1.3}$$

Figure 8 summarizes the equilibrium. Panel (a) presents the Production Edgeworth box: the equilibrium allocation is the point where the two marginal rates of technical substitution coincide, lying on the production Pareto set (orange curve). The blue rays correspond to equation (5.1.3). Panel (b) plots equations (5.1.1); the intersection of the two lines determines the equilibrium factor–price. Panel (c) depicts the production–possibility frontier, whose slope—the marginal rate of transformation—equals the relative price of the goods.

Figure 8: Equilibrium in production



Stolper–Samuelson Theorem: Suppose good i is labour-intensive and its relative price rises ($dp_i > dp_j$). Then the real wage of labor, w, increases, whereas the real return to capital, r, decreases. Moreover, the proportional change in w exceeds that in p_1, while the change in r is smaller.

As established above, under competitive markets the price of each good equals its unit cost:

$$\begin{cases} p_1 = c_1(w, r), \\ p_2 = c_2(w, r) \end{cases} \tag{5.1.4}$$

Taking total differentials of system (5.1.4) and applying Shepard's lemma yields

$$\begin{cases} dp_1 = a_{L1}\,dw + a_{K1}\,dr, \\ dp_2 = a_{L2}\,dw + a_{K2}\,dr, \end{cases} \tag{5.1.5}$$

Assume that $dp_2 = 0$, so that the price of good 2 remains constant, while p_1 increases. The system (5.1.5) then implies

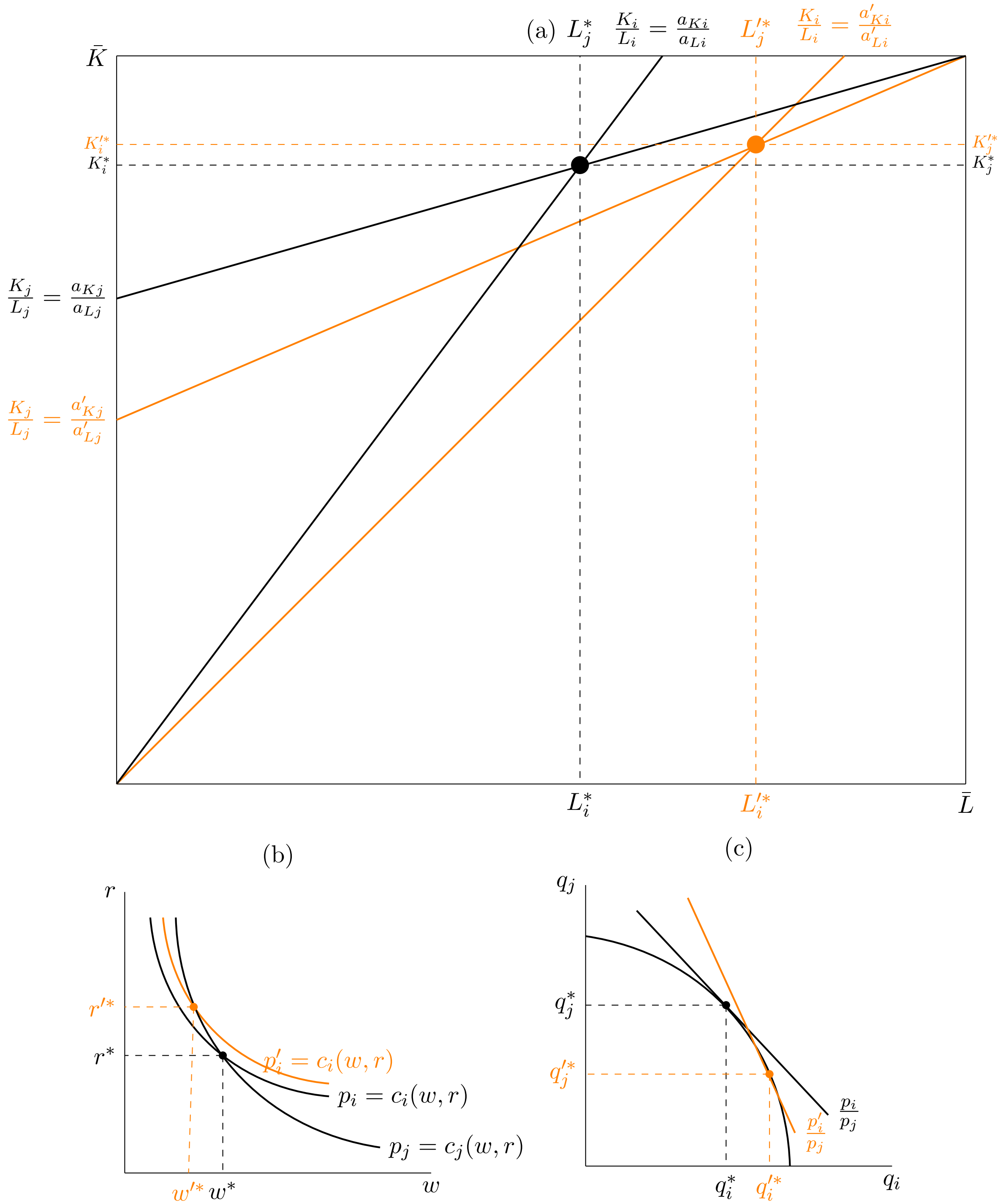
$$dr \;=\; \frac{a_{L2}\,dp_1}{a_{K1}a_{L2} - a_{L1}a_{K2}} \tag{5.1.6}$$

The sign of (5.1.6) depends on factor intensities. If q_1 is labor-intensive, the denominator is negative and thus $dr < 0$; if good 1 is capital-intensive, then $dr > 0$.
Similarly, the change in the wage rate is given by

$$dw \;=\; \frac{a_{K2}\,dp_1}{a_{K1}a_{L2} - a_{L1}a_{K2}} \tag{5.1.7}$$

Equation (5.1.7) is positive when q_1 is labor-intensive and negative when it is capital-intensive. Equations (5.1.6) and (5.1.7) formalize the Stolper–Samuelson theorem: an increase in the relative price of the labor-intensive good raises the real wage and lowers the return to capital, whereas an increase in the relative price of the capital-intensive good raises the return to capital and lowers the real wage.
Figure 9 illustrates the Stolper–Samuelson theorem in the case where sector i is *capital-intensive* and its output price p_i increases.

Figure 9: Stolper–Samuelson Theorem



Rybczynski Theorem: With goods prices fixed, an increase in the labor endowment expands the output of the labor-intensive good more than proportionally and reduces the output of the other good; analogously, a rise in the capital endowment enlarges the capital-intensive sector and contracts the labour-intensive one.

When factor endowments change, the market-clearing conditions in equation (5.1.2) become

$$\begin{cases} a_{Li}\, dq_i + a_{Lj}\, dq_j = d\bar{L}, \\ a_{Ki}\, dq_i + a_{Kj}\, dq_j = d\bar{K} \end{cases}$$

Suppose an increase in $\bar{L}$, therefore

$$(a_{Kj}a_{Li} - a_{Lj}a_{Ki})\, dq_i = a_{Kj} d\bar{L}$$

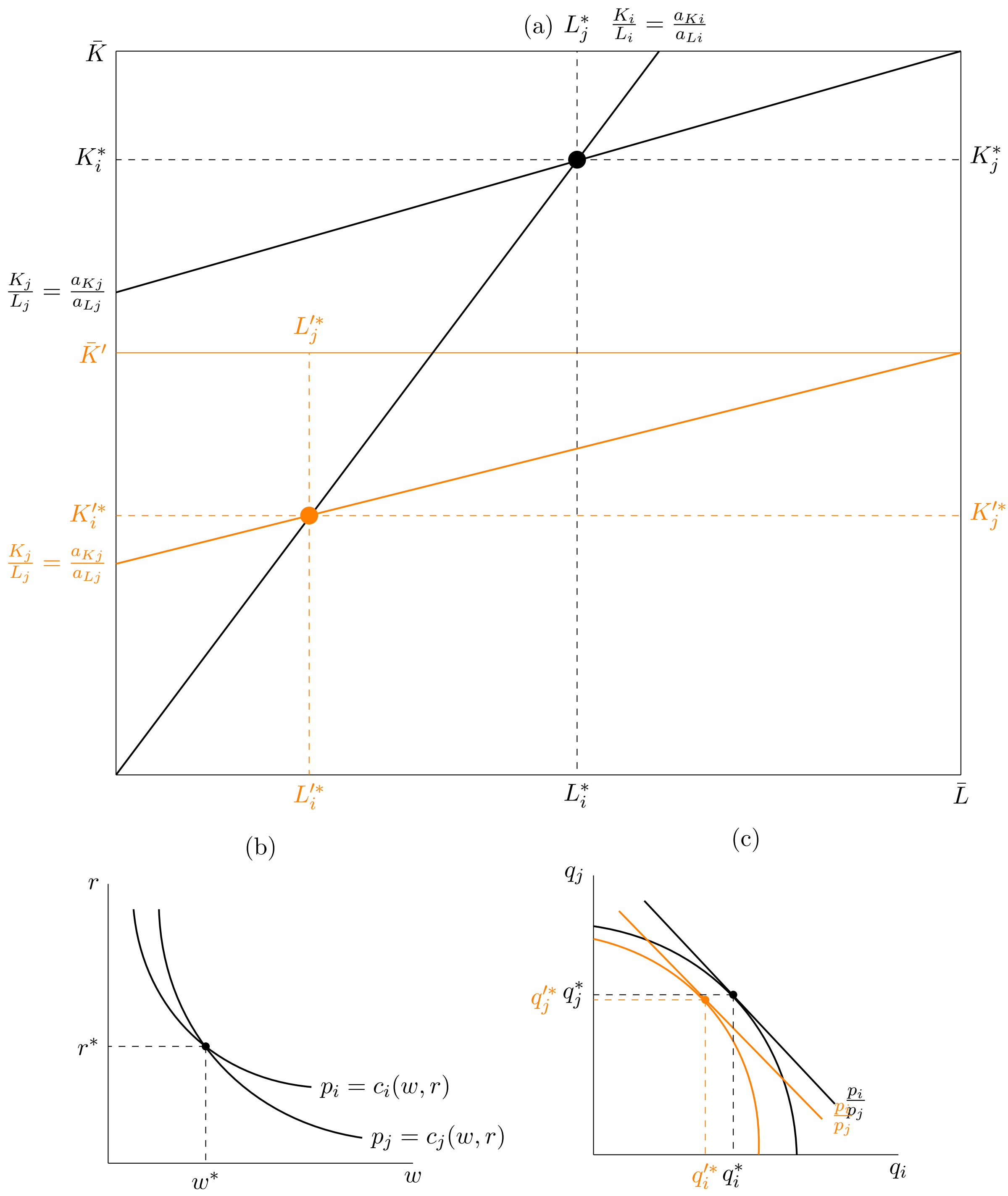
If good i is labour-intensive, then $\frac{a_{Li}}{a_{Ki}} > \frac{a_{Lj}}{a_{Kj}}$, implying $a_{Kj}a_{Li} - a_{Lj}a_{Ki} > 0$; hence $dq_i > 0$. Substituting back gives

$$dq_j = \frac{d\bar{L} - a_{Li} dq_i}{a_{Lj}} < 0$$

Thus, an increase in labour endowment expands the labour-intensive sector and contracts the capital-intensive one.

Figure 10 illustrates the Rybczynski theorem when sector i is *capital-intensive* and the aggregate capital endowment $\bar{K}$ falls, while goods prices and factor prices remain unchanged. In panel (c) the production–possibility frontier shifts inward; the maximum feasible output of good i (q_i-intercept) contracts by a larger amount than that of good j, reflecting the disproportionate impact on the capital-intensive sector.

Figure 10: Rybczynski Theorem



To illustrate, consider an economy endowed with $\bar{L}$ units of labor and $\bar{K}$ units of capital. The two sectors exhibit production functions given by

$$q_1 = L^{\alpha} K^{1-\alpha}, \qquad q_2 = L^{1-\alpha} K^{\alpha}, \qquad \frac{1}{2} < \alpha < 1.$$

Conditional factor demands are:

$$\begin{cases} L_1^*(q_1, w, r) = q_1 \left(\frac{\alpha}{1-\alpha} \cdot \frac{r}{w}\right)^{1-\alpha} \\ K_1^*(q_1, w, r) = q_1 \left(\frac{1-\alpha}{\alpha} \cdot \frac{w}{r}\right)^{\alpha} \\ L_2^*(q_2, w, r) = q_2 \left(\frac{1-\alpha}{\alpha} \cdot \frac{r}{w}\right)^{\alpha} \\ K_2^*(q_2, w, r) = q_2 \left(\frac{\alpha}{1-\alpha} \cdot \frac{w}{r}\right)^{1-\alpha} \end{cases}$$

Equivalently:

$$\begin{cases} a_{L1}(w, r) = \left(\frac{\alpha}{1-\alpha} \cdot \frac{r}{w}\right)^{1-\alpha} \\ a_{K1(w,r)} = \left(\frac{1-\alpha}{\alpha} \cdot \frac{w}{r}\right)^{\alpha} \\ a_{L2}(w, r) = \left(\frac{1-\alpha}{\alpha} \cdot \frac{r}{w}\right)^{\alpha} \\ a_{K2}(w, r) = \left(\frac{\alpha}{1-\alpha} \cdot \frac{w}{r}\right)^{1-\alpha} \end{cases}$$

The marginal costs for each sector are as follows:

$$\begin{cases} p_1 = c_1(w, r) = w \cdot \left(\frac{\alpha}{1-\alpha} \cdot \frac{r}{w}\right)^{1-\alpha} + r \cdot \left(\frac{1-\alpha}{\alpha} \cdot \frac{w}{r}\right)^{\alpha} \\ p_2 = c_2(w, r) = w \cdot \left(\frac{1-\alpha}{\alpha} \cdot \frac{r}{w}\right)^{\alpha} + r \cdot \left(\frac{\alpha}{1-\alpha} \cdot \frac{w}{r}\right)^{1-\alpha} \end{cases}$$

Solving the above system for the factor–price ratio yields:

$$\frac{w}{r} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{2\alpha-1}}$$

Because $\alpha > \frac{1}{2}$, sector 1 is labour-intensive:

$$\frac{a_{L1}}{a_{K1}} = \frac{\alpha}{1-\alpha} \ > \ \frac{a_{L2}}{a_{K2}} = \frac{1-\alpha}{\alpha}$$

Factor–market clearing implies:

$$\begin{cases} \bar{L} = q_1 \left(\frac{\alpha}{1-\alpha} \cdot \frac{r}{w}\right)^{1-\alpha} + q_2 \left(\frac{1-\alpha}{\alpha} \cdot \frac{r}{w}\right)^{\alpha} \\ \bar{K} = q_1 \left(\frac{1-\alpha}{\alpha} \cdot \frac{w}{r}\right)^{\alpha} + q_2 \left(\frac{\alpha}{1-\alpha} \cdot \frac{w}{r}\right)^{1-\alpha} \end{cases}$$

Solving the system yields

$$q_1 = \frac{\bar{L}\left(\dfrac{\alpha}{1-\alpha}\dfrac{w}{r}\right)^{1-\alpha} - \bar{K}\left(\dfrac{1-\alpha}{\alpha}\dfrac{r}{w}\right)^{\alpha}}{\left(\dfrac{\alpha}{1-\alpha}\right)^{2(1-\alpha)} - \left(\dfrac{1-\alpha}{\alpha}\right)^{2\alpha}},$$

$$q_2 = \frac{\bar{K}\left(\dfrac{\alpha}{1-\alpha}\dfrac{r}{w}\right)^{1-\alpha} - \bar{L}\left(\dfrac{1-\alpha}{\alpha}\dfrac{w}{r}\right)^{\alpha}}{\left(\dfrac{\alpha}{1-\alpha}\right)^{2(1-\alpha)} - \left(\dfrac{1-\alpha}{\alpha}\right)^{2\alpha}}.$$

The contract curve is obtained by equating the marginal rates of technical substitution (MRTS) in both sectors:

$$\begin{aligned}\frac{\partial q_1/\partial L_1}{\partial q_1/\partial K_1} = \frac{\partial q_2/\partial L_2}{\partial q_2/\partial K_2} &\implies \frac{K_1}{L_1} = \frac{K_2}{L_2} \\ &\implies \frac{K_1}{L_1} = \frac{\bar{K}-K_1}{\bar{L}-L_1} \implies K_1 = \frac{\bar{K}}{\bar{L}}\,L_1.\end{aligned}$$

5.2 Equilibrium with consumption and production

Consider a closed economy with two firms. Firm 1 produces output q_1 and Firm 2 produces output q_2. Both firms employ labor L and capital K, and each technology exhibits constant returns to scale. The aggregate factor endowment is $(\bar{L}, \bar{K})$. Social welfare is represented by the utility function $u(x_1, x_2)$, defined over the two consumption goods x_1 and x_2.

In a competitive, decentralized equilibrium, each agent maximizes their own objective function. The representative consumer solves

$$\max_{x_1,x_2} u(x_1,x_2) \quad \text{s.t.} \quad w\bar{L} + r\bar{K} + \pi_1(p_1,w,r) + \pi_2(p_2,w,r) = p_1x_1 + p_2x_2,$$

where w and r are the prices of labor and capital, respectively, and $\pi_j(p_j, w, r)$ denotes the profit of firm $j \in \{1, 2\}$. Since both production technologies display *constant returns to scale* and markets are perfectly competitive, equilibrium profits satisfy $\pi_1 = \pi_2 = 0$. The representative consumer's budget constraint therefore simplifies to

$$p_1x_1 + p_2x_2 = w\bar{L} + r\bar{K}, \tag{5.2.1}$$

where the right-hand side represents the total income from the factor endowment $(\bar{L}, \bar{K})$.

The (interior) first–order condition equates the marginal rate of substitution to the relative price ratio:

$$\frac{\partial u(x_1,x_2)/\partial x_1}{\partial u(x_1,x_2)/\partial x_2} = \frac{p_1}{p_2}$$

Combining the optimality condition with the budget constraint (5.2.1) yields the demand functions

$$x_i = x_i\big(w, r, \bar{L}, \bar{K}, p_1, p_2\big), \qquad i \in \{1, 2\}$$

For firm j, the cost-minimization problem is

$$\min_{K_j, L_j} \; C_j(q_j, w, r) = rK_j + wL_j \quad \text{s.t.} \quad q_j = q_j(K_j, L_j)$$

where $q_j(K_j, L_j)$ is the production function for sector j. The first-order condition equates the *marginal rate of technical substitution* to the ratio of input prices:

$$\frac{\partial q_j(K_j, L_j)/\partial L_j}{\partial q_j(K_j, L_j)/\partial K_j} \;=\; \frac{w}{r}$$

Solving the programme yields the conditional (input demand) functions

$$K_j = K_j(q_j, w, r), \qquad L_j = L_j(q_j, w, r)$$

Under perfect competition and constant returns to scale, price equals marginal cost

$$p_j \;=\; c_j(q_j, w, r)$$

Considering all output is consumed, and households hold no initial endowments of the goods, equilibrium in the goods markets requires

$$x_i = q_i, \qquad i = 1, 2$$

Equilibrium in the factor markets equates total factor demand with the aggregate endowment:

$$\bar{L} = L_1 + L_2, \qquad \bar{K} = K_1 + K_2$$

A *competitive equilibrium* is a collection of prices

$$\big(p_1^*, p_2^*, w^*, r^*\big)$$

and allocations

$$\big(q_1^*, q_2^*, x_1^*, x_2^*, K_1^*, K_2^*, L_1^*, L_2^*\big)$$

that jointly satisfy

$$
\begin{cases}
x_i^* &= q_i^*, & i = 1,2, & \text{(goods clearing)} \\
x_i^* &= x_i\big(w^*, r^*, \bar{L}, \bar{K}, p_1^*, p_2^*\big), & i = 1,2, & \text{(household demand)} \\
L_j^* &= L_j\big(q_j^*, w^*, r^*\big), & j = 1,2, & \text{(labour demand)} \\
K_j^* &= K_j\big(q_j^*, w^*, r^*\big), & j = 1,2, & \text{(capital demand)} \\
p_j^* &= c_j\big(w^*, r^*\big), & j = 1,2, & \text{(zero-profit)} \\
\bar{L} &= L_1^* + L_2^*, & & \text{(labour market)} \\
\bar{K} &= K_1^* + K_2^*. & & \text{(capital market)}
\end{cases}
$$

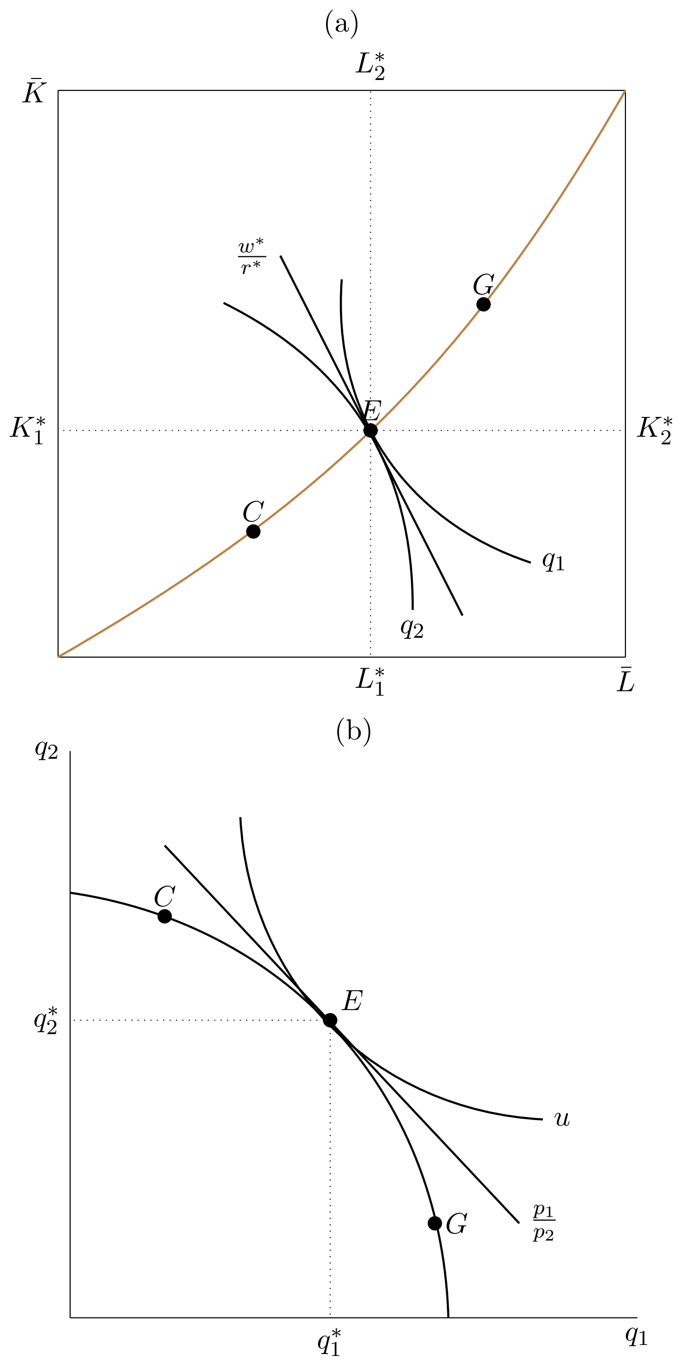
Equations (5.2.2) constitute a system of twelve equations in the twelve unknowns

$$
(p_1^*, p_2^*, w^*, r^*, q_1^*, q_2^*, x_1^*, x_2^*, K_1^*, K_2^*, L_1^*, L_2^*) \tag{5.2.3}
$$

whose solution yields the competitive-equilibrium prices and allocation.

Figure 11 summarizes the general-equilibrium. Panel (a) displays the production Edgeworth box, with the contract curve—the locus of Pareto-efficient factor allocations—highlighted. Panel (b) maps those efficient allocations into output space, tracing the *Production-Possibility Frontier* (PPF). Equilibrium is found at the point where the PPF is tangent to the indifference curve, so that the marginal rate of transformation equals the marginal rate of substitution.

Figure 11: General Equilibrium in $2 \times 2 \times 1$



Let the representative household have Cobb–Douglas preferences

$$u(x_1, x_2) = x_1 x_2$$

and suppose each firm $j \in \{1, 2\}$ produces with the Cobb–Douglas technology

$$q_j(K_j, L_j) = K_j^{1/2} L_j^{1/2}$$

Optimal consumption satisfies

$$x_1 = \frac{w\bar{L} + r\bar{K}}{2p_1}, \qquad x_2 = \frac{w\bar{L} + r\bar{K}}{2p_2}$$

Cost minimization yields the conditional factor–demand functions

$$K_j = q_j\sqrt{\frac{w}{r}}, \qquad L_j = q_j\sqrt{\frac{r}{w}}$$

and the (zero–profit) pricing condition

$$p_j = 2\sqrt{wr}$$

Adding up across firms,

$$\bar{K} = (q_1 + q_2)\sqrt{\frac{w}{r}}, \qquad \bar{L} = (q_1 + q_2)\sqrt{\frac{r}{w}}$$

Taking the ratio of the two equations gives the equilibrium

$$\frac{w}{r} = \frac{\bar{K}}{\bar{L}} \quad \Longrightarrow \quad w^* = r^*\frac{\bar{K}}{\bar{L}}$$

Since all output is consumed, goods-market clearing implies $x_i = q_i$ for $i = 1, 2$. Substituting $p_1 = p_2 = 2\sqrt{wr}$ into the household demand functions then yields to

$$q_1 = q_2 = \frac{1}{2}\sqrt{\bar{K}\bar{L}}$$

Factor demands at the optimum.

$$K_1 = K_2 = \frac{\bar{K}}{2}, \qquad L_1 = L_2 = \frac{\bar{L}}{2}$$

Choosing $r^* = 1$ (any positive normalization works), the equilibrium is

$$\begin{aligned} &\left(p_1^*, p_2^*, w^*, r^*, q_1^*, q_2^*, x_1^*, x_2^*, K_1^*, K_2^*, L_1^*, L_2^*\right) \\ = &\left(2\sqrt{\tfrac{\bar{K}}{\bar{L}}},\ 2\sqrt{\tfrac{\bar{K}}{\bar{L}}},\ \tfrac{\bar{K}}{\bar{L}},\ 1,\ \tfrac{1}{2}\sqrt{\bar{K}\bar{L}},\ \tfrac{1}{2}\sqrt{\bar{K}\bar{L}},\ \tfrac{1}{2}\sqrt{\bar{K}\bar{L}},\ \tfrac{1}{2}\sqrt{\bar{K}\bar{L}},\ \tfrac{\bar{K}}{2},\ \tfrac{\bar{K}}{2},\ \tfrac{\bar{L}}{2},\ \tfrac{\bar{L}}{2}\right) \end{aligned}$$

5.3 Exercises

1. Consider the general-equilibrium production model in Section 5.1. Sector i is capital-intensive.

    1. The economy opens to international trade and the relative price of good i rises above its autarky level. Using a graphical approach, describe
        1. the impact on equilibrium factor prices,
        2. the resulting changes in the output of each good and distribution of factor between sectors, and
        3. the economic intuition behind these effects.
    2. Keeping the economy open, suppose a natural disaster cuts the aggregate capital stock in half. Reassess your answers: how are factor prices and output levels now affected?

2. Consider the production economy of Section 5.1. Output is produced according to

$$q_1 = L_1 + 2K_1, \qquad q_2 = 2L_2 + K_2, \qquad \frac{1}{2} < \frac{w}{r} < 2$$

    1. Express the factor–price ratio $\frac{w}{r}$ as a function of the goods–price ratio $\frac{p_1}{p_2}$. Determine the output levels q_1^* and q_2^*, and illustrate the results.
    2. Explain why $\frac{1}{2} < \frac{w}{r} < 2$ must hold. Derive the corresponding admissible range for $\frac{p_1}{p_2}$.
    3. A storm halves the aggregate capital stock. Re-evaluate part 1 using the graphical approach and relate your findings to the relevant theorem.
    4. Suppose the economy remains open and p_1 doubles. Re-do part 1 graphically and link the outcome to the theorem discussed.
    5. Now both prices double. How does the equilibrium adjust, and what is the intuition?
    6. If $\frac{w}{r} = 1$ and the endowments are $(\bar{L}, \bar{K}) = (1, 1)$, what is the optimal allocation?

3. In the general-equilibrium model with consumption and production (Section 5.2), the representative consumer has Cobb–Douglas preferences.

    1. Assume the consumer's preferences shift, making good 2 relatively more preferred than before. How does this shift affect the relative price $\frac{p_2}{p_1}$, the output mix (q_1, q_2), and the allocation of labor and capital across sectors? Illustrate the adjustments a graphical approach.
    2. Total factor productivity rises in sector 2. Analyze the impact on $\frac{p_2}{p_1}$, (q_1, q_2), and factor allocation, and depict the outcome graphically.
    3. Repeat part 2 assuming Leontief preferences. How do the results change? Explain the roles of income and substitution effects.

4. Consider an economy described as in Equilibrium with consumption and production. The representative consumer has Cobb–Douglas preferences.

    1. Suppose the preferences of the indivual changes and now good 2 is more prefered that what it was. How do this change the relative prices of the goods and the optimal produced in each sector? How do capital and labor distribution between sector change? Use graphical approach
    2. Suppose an increase in technology for sector q_2. How do this change the relative prices of the goods and the optimal produced in each sector? How do capital and labor distribution between sector change? Use graphical approach (5.2.3)? Use graphical approach
    3. Suppose an increase in technology for sector q_2 but with preferences by Leontief utility function, how do your previous asnwers change? What does income and substitution effect has to be related with this?

5. In the general-equilibrium model with consumption and production (Section 5.2), the representative consumer has preferences given by

$$u(x_1, x_2) = \min\{x_1, x_2\}$$

    Firms produce according to

$$q_1 = \left(L_1^{1/2} + 4K_1^{1/2}\right)^2, \qquad q_2 = \left(4L_2^2 + K_2^2\right)^{1/2}$$

    1. Find the competitive-equilibrium vector (5.2.3) and illustrate it with the chapter's graphical approach.

6. In the general-equilibrium model with consumption and production (Section 5.2), the representative consumer has preferences given by

$$u(x_1, x_2) = x_1 x_2$$

    Additionally, the consumer is endowed with ϕ_1 for good 1 and ϕ_2 for good 2. Firms produce according to

$$q_1 = \min\{\frac{L_1}{2}, K_1\}, \qquad q_2 = \min\{L_2, \frac{K_2}{2}\}$$

    1. Find the competitive-equilibrium vector (5.2.3) and illustrate it with the chapter's graphical approach.

6 The Armington Model

6.1 Introduction

The Armington model, introduced by Armington (1969), provides a framework to analyze international trade when goods are differentiated by their country of origin. Unlike the Ricardian, which assume goods are homogeneous across producers, the Armington model postulates that even within the same sector, goods from different countries are imperfect substitutes. This assumption introduces product differentiation and plays a central role in modern empirical trade analysis.

6.2 The model

Consider a world economy consisting of n countries, where each country produces a differentiated variety of a common type of good. The total population of country j is denoted by L_j, and each individual is endowed with one unit of the domestically variety and none of the other varieties. Let w_j denote the endowment value of variety j for a representative consumer j.[5] The aggregate value of country j's endowment across all individuals defines the value of country j's gross domestic product (GDP).

$$X_j = L_j w_j \tag{6.2.1}$$

Consumers exhibit preferences over the full set of available varieties. Let x_{kj} denote the quantity of the variety associated with country k that is consumed by a representative individual in country j. The preferences of the representative consumer are assumed to be described by a Constant Elasticity of Substitution (CES) utility function of the form

$$U_j = \left(\sum_{k=1}^{n} \alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{6.2.2}$$

where $\sigma > 1$ denotes the elasticity of substitution across varieties, and α_{kj} represents the taste parameter associated with variety k in country j.

The budget constraint of a representative consumer j is expressed as

$$w_j = \sum_{k=1}^{n} p_{kj}\, x_{kj} \tag{6.2.3}$$

where p_{kj} denotes the price of variety k in country j. The right-hand side of equation (6.2.3) corresponds to the total expenditure of the representative consumer in country j on all available varieties.

Furthermore, assume the presence of iceberg trade costs $\tau_{kj} \geq 1$, such that τ_{kj} units must be

[5] Since the representative consumer in country j is endowed with one unit of variety j, it follows that $w_j = p_{jj}$, where p_{jj} denotes the price of variety j in country j.

shipped from country k to country j for one unit of variety k to be delivered in country j. Under this assumption, the consumer price of variety k in country j is given by

$$p_{kj} = w_k \tau_{kj} \tag{6.2.4}$$

The corresponding optimality condition when maximizing the utility function in equation (6.2.2) subject to the budget constraint in equation (6.2.3) is

$$\frac{\alpha_{ij}^{\frac{1}{\sigma}} x_{kj}^{\frac{1}{\sigma}}}{\alpha_{kj}^{\frac{1}{\sigma}} x_{ij}^{\frac{1}{\sigma}}} = \frac{p_{ij}}{p_{kj}} \tag{6.2.5}$$

The CES price index in country j is defined as

$$P_j = \left(\sum_{k=1}^{n} \alpha_{kj} p_{kj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{6.2.6}$$

The Marshallian demand of the representative consumer in country j for variety i, expressed in terms of the CES price index in equation (6.2.6), is given by

$$x_{ij} = \alpha_{ij} p_{ij}^{-\sigma} \frac{w_j}{P_j^{1-\sigma}} \tag{6.2.7}$$

Equation (6.2.7) can equivalently be written to express the Marshallian demand in terms of trade costs as

$$x_{ij} = \alpha_{ij} w_i^{-\sigma} \tau_{ij}^{-\sigma} \frac{w_j}{P_j^{1-\sigma}} \tag{6.2.8}$$

Denoting by $p_{ij}\, x_{ij}$ the expenditure of the representative consumer in country j on variety i, equation (6.2.8) can be rewritten as

$$p_{ij} x_{ij} = \alpha_{ij} w_i^{1-\sigma} \tau_{ij}^{1-\sigma} \frac{w_j}{P_j^{1-\sigma}} \tag{6.2.9}$$

6.3 From the Representative Consumer to the Aggregate Economy

Let $\tilde{x}_{ij}$ denote the total consumption of variety i by the population in country j. This equals the demand of the representative consumer in country j for variety i, multiplied by the population size L_j. The total expenditure on variety i by the population in country j is obtained by multiplying both sides of equation (6.2.9) by L_j.

$$p_{ij} \tilde{x}_{ij} = \alpha_{ij} w_i^{1-\sigma} \tau_{ij}^{-\sigma} \frac{L_j w_j}{P_j^{1-\sigma}} \tag{6.3.1}$$

Using equation (6.2.1), the total expenditure on variety i by the population in country j is defined in terms of country j's GDP as follows

$$p_{ij}\tilde{x}_{ij} = \alpha_{ij} w_i^{1-\sigma} \tau_{ij}^{-\sigma} \frac{X_j}{P_j^{1-\sigma}} \tag{6.3.2}$$

Additionally, country i's GDP is defined via the income approach as the total revenue earned from selling its variety to all countries, that is

$$X_i = \sum_{j=1}^{n} L_j p_{ij} x_{ij} \tag{6.3.3}$$

The right-hand side of equation (6.3.3) represents the sum of the total expenditures by the populations of all countries on variety i.
Substituting equation (6.2.9) into equation (6.3.3) yields

$$X_i = \sum_{j=1}^{n} L_j w_i^{1-\sigma} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \alpha_{ij}\, w_j \tag{6.3.4}$$

Using equation (6.2.1) for country j, equation (6.3.4) can be rewritten as

$$X_i = \sum_{j=1}^{n} w_i^{1-\sigma} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \alpha_{ij}\, X_j \tag{6.3.5}$$

Equation (6.3.5) can be equivalently expressed by isolating the domestic price term $w_i^{1-\sigma}$ as

$$w_i^{1-\sigma} = \frac{X_i}{\sum_{j=1}^{n} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \alpha_{ij}\, X_j} \tag{6.3.6}$$

Finally, substituting equation (6.3.6) into (6.3.2) leads to the gravity equation of the model:

$$p_{ij}\tilde{x}_{ij} = \frac{\alpha_{ij} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} X_i X_j}{\sum_{j=1}^{n} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \alpha_{ij}\, X_j} \tag{6.3.7}$$

Equation (6.3.7) expresses bilateral expenditure $\tilde{x}_{ij}$ as a function of the exporter's income X_i, the importer's income X_j, on the taste parameter α_{ij}, trade costs τ_{ij}, the elasticity of substitution σ, and the multilateral resistance terms captured in the denominator.

6.4 Exercises

1. Indicate whether each of the following propositions is true or false. Justify your answer.

    1. Countries z, x, and y share identical preferences and trade costs. However, country x is larger than country y. Country z will always trade more with x than with y.
    2. If preferences for variety x increase in country z (*ceteris paribus*), trade between z and x will increase, while trade between z and y will decrease.
    3. If the price of good x in country z increases (*ceteris paribus*), trade between x and z will decrease, while trade between y and z will remain unchanged.
    4. If the trade cost of exporting good x to country z increases (*ceteris paribus*), trade between x and z will decrease, while trade between y and z will remain unchanged.
    5. If GDP increases in all countries except z and y (*ceteris paribus*), trade between z and y will remain constant.

2. Suppose a pandemic reduces the population by half in all countries $h+1, \ldots, n$, within a set of countries indexed by $k = 1, \ldots, h, \ldots, n$. Analyze how this shock affects trade flows between countries. Consider all possible bilateral cases (four in total).

3. Suppose there is migration from country i to country j. Analyze the effects of this migration on the following:

    1. Country j's imports of good i.
    2. Country j's imports of good k.
    3. Country i's imports of good k.

7 The Specific Factors Model

The Specific Factors Model constitutes a natural extension of the Ricardian framework by introducing multiple factors of production and relaxing the assumption of full mobility. It captures the idea that some factors are mobile across sectors, while others remain sector-specific. This asymmetry in factor mobility generates sectoral distributional effects from trade that are absent in the Ricardian model.

7.1 The Model

Consider a small economy that produces two goods, denoted q_1 and q_2. The production of good q_1 employs capital K, which is specific to sector q_1, together with labor L_1, whereas the production of good q_2 uses land T, which is specific to sector q_2, together with labor L_2. Labor is perfectly mobile across sectors, and the total labor endowment is denoted by $\bar{L}$. Under the assumption of full employment, it follows that $\bar{L} = L_1 + L_2$. The economy is thus endowed with three factors of production: one mobile factor (labor) and two sector-specific factors (capital and land). Note that sector q_1 employs the entire endowment of capital, while sector q_2 employs the entire endowment of land, since q_1 does not use land and q_2 does not use capital in production.

The production functions are assumed to be strictly concave and to exhibit constant returns to scale with respect to labor and the specific factor within each sector:

$$q_1 = q_1(K, L_1), \qquad q_2 = q_2(T, L_2) \tag{7.1.1}$$

Marginal products of labor are positive and diminishing, implying the existence of downward-sloping labor demand schedules in each sector.

Under perfect competition, factors are remunerated according to the value of their marginal products. Since labor is perfectly mobile across sectors, the wage rate w is equalized across sectors and satisfies

$$w \;=\; p_1 \cdot \frac{\partial q_1}{\partial L_1} \;=\; p_2 \cdot \frac{\partial q_2}{\partial L_2} \tag{7.1.2}$$

where p_1 and p_2 denote the world prices of goods q_1 and q_2, respectively. The allocation of labor is determined at the point where the value of the marginal product of labor is equal across sectors.

Owners of the specific factors receive returns given by the residual product once labor is compensated. In sector q_1, the return to capital is

$$r_K = p_1 \cdot \frac{\partial q_1}{\partial \bar{K}} \tag{7.1.3}$$

while in sector q_2, the return to land is

$$r_T = p_2 \cdot \frac{\partial q_2}{\partial \bar{T}} \tag{7.1.4}$$

Under the assumption of constant returns to scale, it follows that

$$p_1 q_1 = wL_1 + r_k \bar{K} \tag{7.1.5}$$

$$p_2 q_2 = wL_2 + r_t \bar{T} \tag{7.1.6}$$

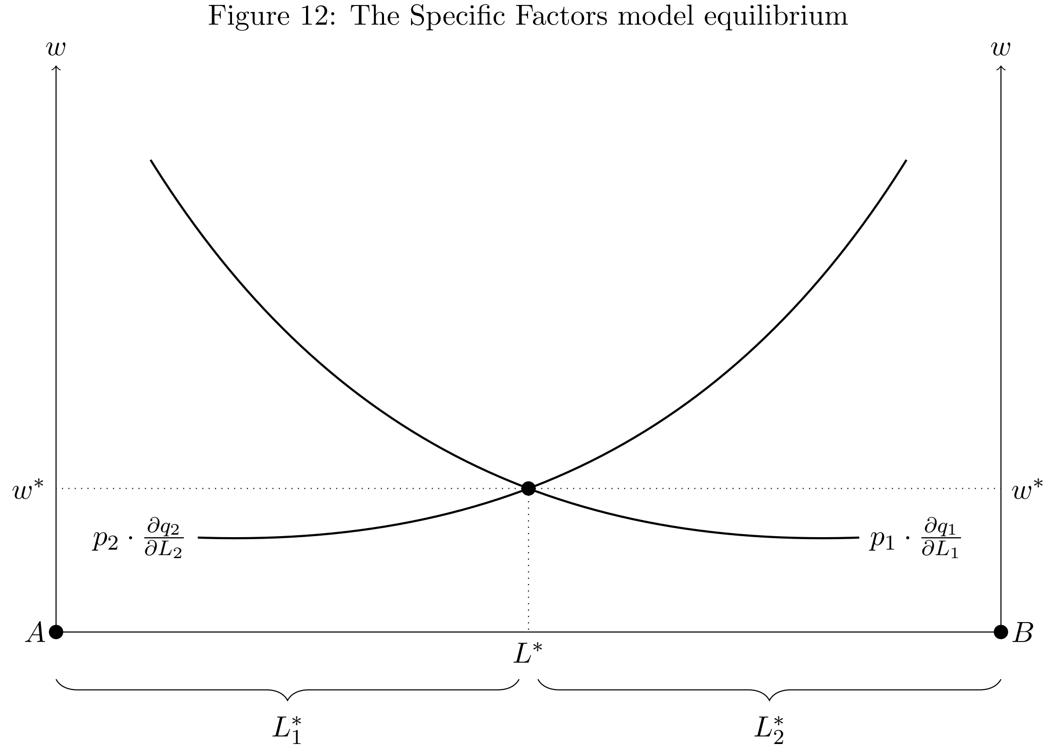
Equations (7.1.3) and (7.1.4) can equivalently be written as

$$r_K = \frac{p_1 \cdot q_1 - wL_1}{\bar{K}}, \tag{7.1.7}$$

$$r_T = \frac{p_2 \cdot q_2 - wL_2}{\bar{T}} \tag{7.1.8}$$

Figure 12 illustrates the equilibrium in the model. The wage is represented on the vertical axis and labor on the horizontal axis. The marginal product value curves of both sectors are depicted, with point A serving as the origin for sector q_1 and point B as the origin for sector q_2. The intersection of the marginal product value curves determines the equilibrium wage and the allocation of labor between sectors.

Figure 12: The Specific Factors model equilibrium



These specific-factor returns are not equalized across sectors. Consequently, changes in goods prices translate into changes in the distribution of income among capital owners, landowners, and workers.

7.2 Welfare in the Specific Factors Model

Welfare is measured in terms of the purchasing power of the return to each factor. For workers, the real wage in terms of good q_1 and in terms of good q_2 can be expressed as

$$\frac{w}{p_1} = \frac{\partial q_1}{\partial L_1}, \tag{7.2.1}$$

$$\frac{w}{p_2} = \frac{\partial q_2}{\partial L_2} \tag{7.2.2}$$

For capital owners, the real return to capital can be written as

$$\frac{r_K}{p_1} = \frac{\partial q_1}{\partial \bar{K}}, \tag{7.2.3}$$

$$\frac{r_K}{p_2} \tag{7.2.4}$$

For landowners, the real return to land is

$$\frac{r_T}{p_1}, \tag{7.2.5}$$

$$\frac{r_T}{p_2} = \frac{\partial q_2}{\partial \bar{T}} \tag{7.2.6}$$

7.3 Impact of Trade in Specific Factors Model

The introduction of trade alters the relative prices faced by the economy as the country integrates into world markets. Suppose that the world price of good q_1 increases while the price of good q_2 remains constant. This raises the value of the marginal product of labor in sector q_1, thereby attracting labor from sector q_2 into sector q_1. As a result, wages increase due to the higher value of marginal products of labor in both sectors (the value of the marginal product in q_2 rises as labor employed in that sector declines). This adjustment is illustrated in Figure 13.

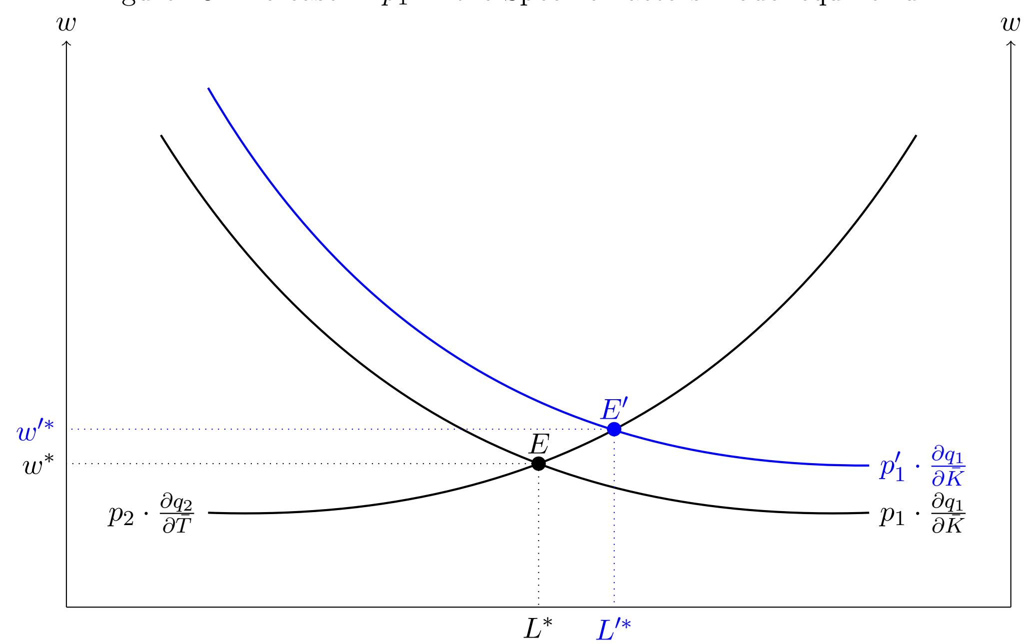
The return to capital in sector q_1 increases as the use of additional labor raises the productivity of capital, whereas the return to land in sector q_2 declines due to the withdrawal of labor from that sector.

Workers, being mobile, experience an ambiguous effect: the real wage may either increase or decrease. In equation (7.2.1), both w and p_1 increase; however, p_1 rises proportionally more than w since the right-hand side decreases, leading to a decline in the real wage expressed in terms of good q_1. In contrast, according to equation (7.2.2), the purchasing power in terms of good q_2 increases because the wage rises while p_2 remains constant.

Capital owners experience a welfare improvement. In equation (7.2.3), the purchasing power of the return to capital in terms of good q_1 increases, since the right-hand side rises and therefore the left-hand side also increases (as noted previously, r_K increases while p_1 also rises) making real return to capital increases. In equation (7.2.4), the purchasing power in terms of good q_2 also increases because the return to capital rises while p_2 remains constant.

Landowners experience a welfare loss. In equation (7.2.5), the return to land decreases while p_1 increases, causing the real return to land measured in units of good q_1 to decline. In equation (7.2.6), the return to land also decreases while p_2 remains constant, leading to a decline in the real return to land measured in units of good q_2.

Figure 13: Increase in p_1 in the Specific Factors model equilibrium



The central insight of the Specific Factors Model is that trade generates both aggregate gains and redistribution across factors. At the national level, the economy is unambiguously better off, as trade enables specialization and provides access to a broader set of consumption possibilities. Within the economy, however, specific factors are affected asymmetrically: owners of the factor tied to the expanding sector gain, whereas owners of the factor tied to the contracting sector lose. The welfare of the mobile factor is ambiguous, as its outcome depends on the composition of its consumption bundle.

This distributional conflict underscores the political economy dimension of trade. Although the economy as a whole benefits, opposition to trade may arise from groups whose incomes decline. The model therefore provides a theoretical foundation for understanding sector-based lobbying and political resistance to trade, as observed in practice.

7.4 Exercises

1. Within the framework of the Specific Factors Model, analyze the effects of the following scenarios on sectoral production, factor allocation, and the welfare of factor owners:

    1. A technological improvement in sector q_1.
    2. An increase in the endowment of the factor specific to sector q_2.
    3. A decrease in the relative price $\frac{p_1}{p_2}$.
    4. A reduction in the endowment of the mobile factor.

2. Consider the following production functions for sectors q_1 and q_2:

$$q_1 = L_1^{\frac{1}{4}} K^{\frac{3}{4}}, \qquad q_2 = L_2^{\frac{3}{4}} T^{\frac{1}{4}}$$

    where the economy is endowed with $\bar{L}$ units of labor, $\bar{K}$ units of capital, and $\bar{T}$ units of land.

    1. Determine the allocation of factors across sectors.
    2. Derive the level of output in each sector.
    3. Compute the returns to each factor of production.
    4. Suppose that all factor endowments are equal to one (normalized to 100%) and that prices are equal to one. Determine the allocation of factors, the output in each sector, and the returns to each factor. Illustrate the results using the diagram developed in this section.

8 The Neoclassical Model of International Trade and Offer Curve

8.1 The Neoclassical Model of International Trade

The neoclassical model of international trade extends the Ricardian framework by introducing multiple factors of production and incorporating the principle of diminishing marginal returns. In contrast to the Ricardian model, which explains trade solely through cross-country differences in labor productivity, the neoclassical approach highlights how variations in factor endowments and production technologies shape comparative advantage.

8.2 Introduction to the model

Consider a world economy composed of two countries, i and j, each producing two goods, denoted by q_{1z} and q_{2z}, where $z = i, j$. Production in each country requires two factors of production—labor (L) and capital (K)—which are perfectly mobile across sectors within a country but immobile across countries. The aggregate factor endowment of country z is represented by $(\bar{L}_z, \bar{K}_z)$. Each good is produced according to a neoclassical production function that exhibits constant returns to scale and diminishing marginal returns with respect to each input. Moreover, sectors differ in their factor intensities, meaning that one good is relatively more labor-intensive while the other is relatively more capital-intensive. As a result, the Production Possibility Frontier (PPF) of each country is strictly concave to the origin, reflecting increasing opportunity costs.

Additionally, the utility function of the representative consumer in each country is assumed to be homothetic and strictly convex in consumption, resulting to concave indifference curves.

8.3 Tracing the PPF for an Economy with Two Factors and Two Sectors

In section 5.2, it was established that in a model with two goods and two factors, production efficiency requires the equality of the *marginal rate of technical substitution (MRTS)* across sectors. This condition defines the *contract curve* in factor space. Each allocation of labor and capital located on the contract curve represents an efficient distribution of resources, implying that an increase in the output of one good cannot occur without a reduction in the output of the other.

This concept serves as the basis for tracing the Production Possibility Frontier (PPF). For every feasible allocation of labor to one sector, there exists a corresponding level of capital that satisfies the efficiency condition along the contract curve. The remaining quantities of labor and capital are allocated to the other sector. Each efficient allocation determines the production levels of both goods and corresponds to a specific point on the PPF. Varying the allocation of labor across all feasible values traces the complete PPF of the country.

8.4 Autarky Equilibrium in the model

Additionally, in section 5.2, the autarky equilibrium is determined simultaneously in goods and factor markets. The demand side in the goods markets is driven by consumer preferences, expressed as

$$x_{1z} = x_{1z}(\bar{L}_z, \bar{K}_z, w, r, p_1, p_2) \tag{8.4.1}$$

$$x_{2z} = x_{2z}(\bar{L}_z, \bar{K}_z, w, r, p_1, p_2) \tag{8.4.2}$$

Equilibrium in the goods markets requires that

$$q_{1z} = x_{1z}(\bar{L}_z, \bar{K}_z, w, r, p_1, p_2) \tag{8.4.3}$$

$$q_{2z} = x_{2z}(\bar{L}_z, \bar{K}_z, w, r, p_1, p_2) \tag{8.4.4}$$

The conditional factor demands in each sector are given by

$$L_{1z} = L_{1z}(q_{1z}, w, r) \tag{8.4.5}$$

$$K_{1z} = K_{1z}(q_{1z}, w, r) \tag{8.4.6}$$

$$L_{2z} = L_{2z}(q_{2z}, w, r) \tag{8.4.7}$$

$$K_{2z} = K_{2z}(q_{2z}, w, r) \tag{8.4.8}$$

Equilibrium in the factor markets implies that

$$\bar{L}_z = L_{1z}(q_{1z}, w, r) + L_{2z}(q_{2z}, w, r) \tag{8.4.9}$$

$$\bar{K}_z = K_{1z}(q_{1z}, w, r) + K_{2z}(q_{2z}, w, r) \tag{8.4.10}$$

Since production functions exhibit constant returns to scale, the unit cost functions determine prices as follows:

$$p_{1z} = c_{1z}(w, r) \tag{8.4.11}$$

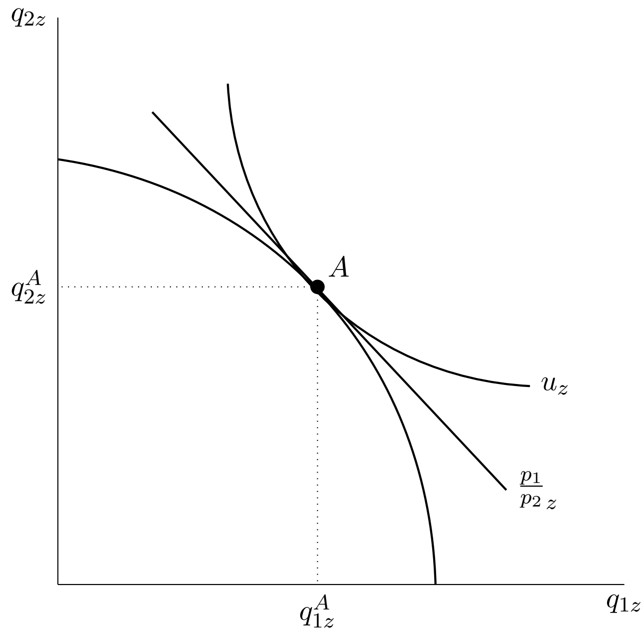
$$p_{2z} = c_{2z}(w, r) \tag{8.4.12}$$

indicating that prices are equal to marginal costs.

Solving the system composed of equations (8.4.1), (8.4.2), (8.4.9), (8.4.10), (8.4.11), and (8.4.12) yields the autarky equilibrium for country z.

The autarky equilibrium is illustrated in Figure 14. Equilibrium occurs at the tangency point between the Production Possibility Frontier (PPF) and the highest attainable indifference curve. This point determines the equilibrium levels of production and consumption for both goods, as well as the relative autarky prices. The relative goods prices, in turn, determine the relative factor prices, and together with the production levels, they define the allocation of factors between sectors. Point A represents the autarky equilibrium since, given the PPF, it yields the maximum attainable utility; any other point along the PPF would correspond to a lower indifference curve.

Figure 14: Autarky Equilibrium in the Neoclassical model



8.5 Trade Equilibrium in the Neoclassical model

The world relative price, $\left(\frac{p_1}{p_2}\right)^W$, will lie between the autarky relative prices of both countries. If $\left(\frac{p_1}{p_2}\right)^W$ is below the lower autarky relative price, both countries will import q_1 and export q_2, generating an excess demand for q_1 and an excess of supply for q_2. Conversely, if $\left(\frac{p_1}{p_2}\right)^W$ is above the higher autarky relative price, both countries will attempt to export q_1 and import q_2, creating an excess supply of q_1 and an excess demand for q_2.

When the world relative price $\left(\frac{p_1}{p_2}\right)^W$ lies between the autarky relative prices, one country will export one good while the other will export the alternative good. If the world relative price $\left(\frac{p_1}{p_2}\right)^W$ is higher than the autarky relative price, good q_1 will be exported and good q_2 imported. Conversely, if the world relative price is lower than the autarky relative price, good q_1 will be imported and good q_2 exported. A specific world relative price will equalize the desired export quantity of one country with the desired import quantity of the other. By *Walras' Law*, this price will also ensure equilibrium in the market for the remaining good.

Production under trade conditions is determined at the point where the slope of the Production Possibility Frontier (PPF) equals the world relative price, that is, where the *marginal rate of transformation (MRT)* corresponds to $\left(\frac{p_1}{p_2}\right)^W$. From equations (8.4.9), (8.4.10), (8.4.11), and

(8.4.12), the production levels under trade can be obtained for a given world relative price. Consumption under trade is determined by the condition that the *marginal rate of substitution (MRS)* equals the world relative price, expressed as

$$\frac{\frac{\partial U}{\partial q_{1z}}}{\frac{\partial U}{\partial q_{2z}}} = \left(\frac{p_1}{p_2}\right)^W \tag{8.5.1}$$

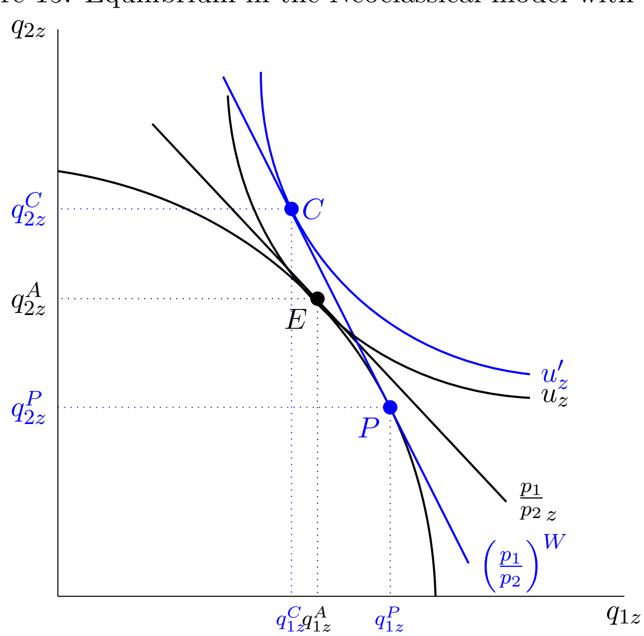
The consumer's budget constraint under trade is given by

$$\left(\frac{p_1}{p_2}\right)^W q_{1z}^P + q_{2z}^P = \left(\frac{p_1}{p_2}\right)^W q_{1z}^C + q_{2z}^C \tag{8.5.2}$$

where q_{1z}^P and q_{2z}^P represent the production of goods under trade, and q_{1z}^C and q_{2z}^C denote the corresponding consumption quantities. Equations (8.5.1) and (8.5.2) yield the consumption bundle under trade.

Figure 15 illustrates the trading equilibrium for the case in which the world relative price of good 1 is higher than its autarky level. The production point is denoted by P, while the consumption point is represented by C. Welfare increases relative to autarky, as the economy reaches a higher indifference curve. International trade enables countries to consume beyond their autarky production possibility frontier by exploiting differences in relative costs and factor endowments.

Figure 15: Equilibrium in the Neoclassical model with trade



The export and import quantities for each good can be derived once the economy engages in trade. Figure 16 illustrates exports and imports under trade. The difference between q_{1z}^P and q_{1z}^C represents the exports of q_1, whereas the difference between q_{2z}^C and q_{2z}^P corresponds to the imports of q_2. In the figure, the height of the blue triangle represents the volume of imports, while the base corresponds to the volume of exports.

Figure 16: Exports and Imports in the Neoclassical model with trade

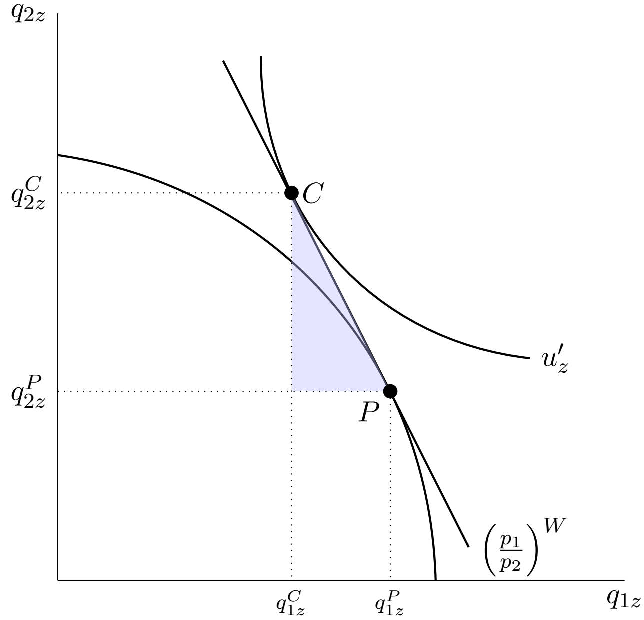


Figure 17 illustrates exports and imports under different world relative prices. It can be observed that larger deviations between the world relative price and the autarky relative price are associated with greater volumes of exports and imports.

Figure 17: Equilibrium in the Neoclassical model with trade at different terms of trade

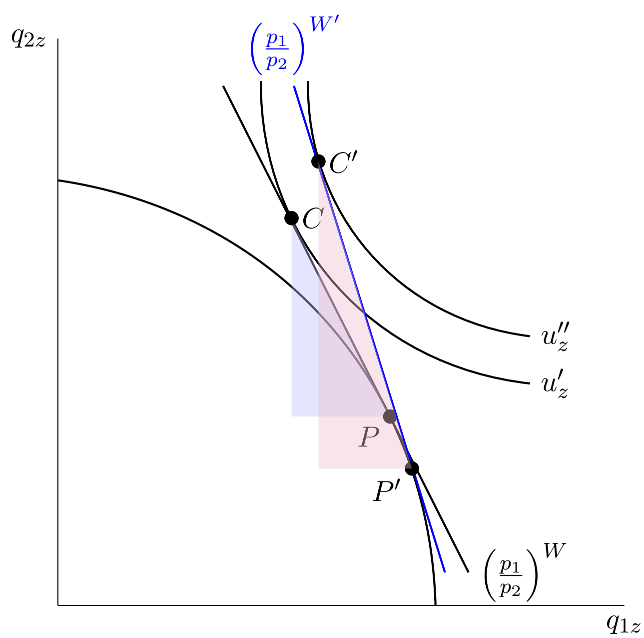


Figure 18 presents the offer curve (OC) under the assumption that the world relative price of good 1 is higher than its autarky level. The horizontal axis represents the quantity of exports of q_1, while the vertical axis represents the corresponding imports of q_1. The offer curve illustrates the combinations of exports and imports associated with different relative prices. The origin, where exports and imports are both equal to zero, corresponds to the autarky relative price.

Figure 18: Offer curve for country z

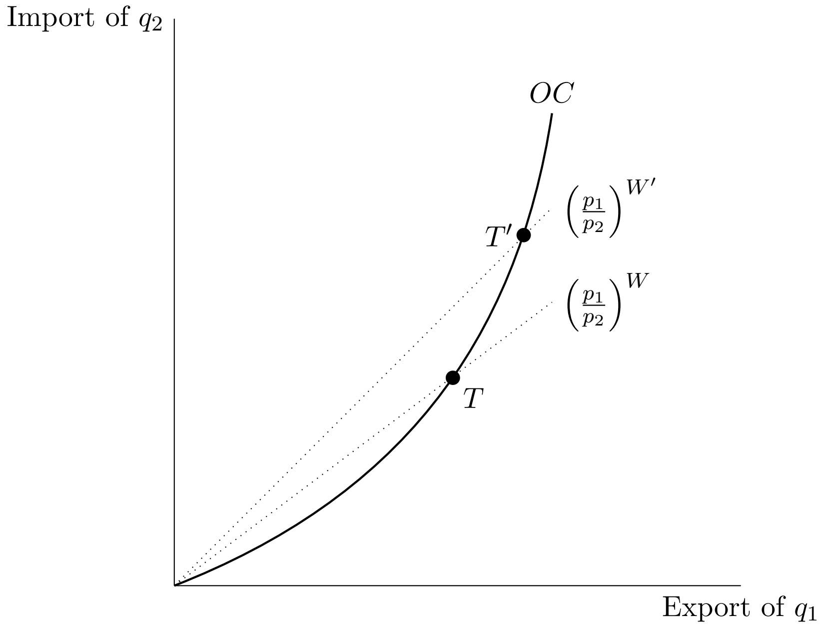
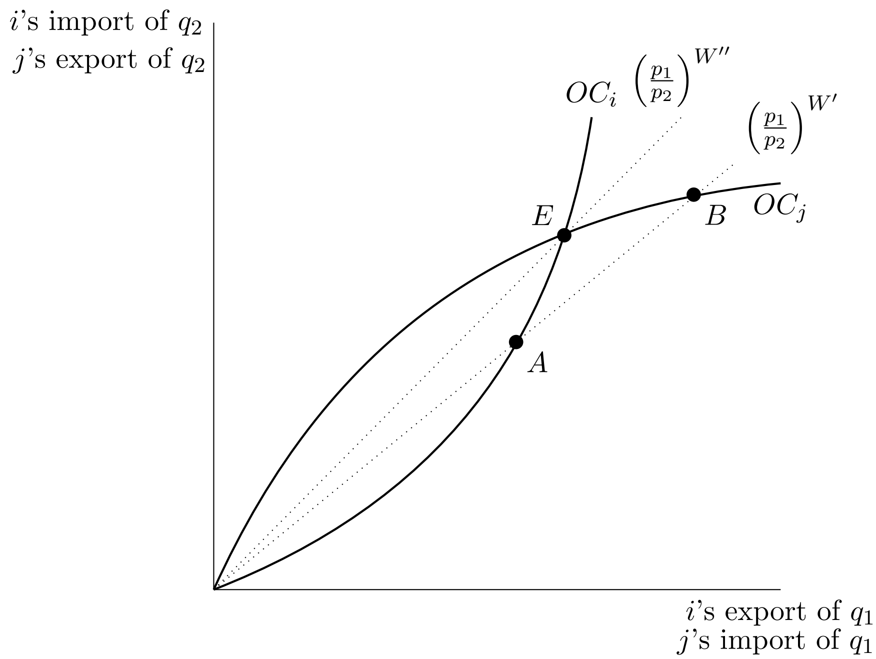


Figure 19 depicts the offer curves for countries i and j, assuming that the world relative price is higher than country i's autarky relative price but lower than that of country j. As previously discussed, the trading equilibrium occurs at a world relative price ($\left(\frac{p_1}{p_2}\right)^{W''}$), where country i's imports of good q_2 are exactly equal to country j's exports of q_2 and the analogous condition holds for q_1.

Figure 19: Trade Equilibrium



8.6 Minimum conditions to Trade

The minimum condition for trade to occur is that countries possess different autarky relative prices. When this condition is met, both countries benefit from trade through specialization following the opening of international markets.

In section 5.2, it was established that the autarky relative price is determined simultaneously in two markets: (i) the goods market and (ii) the factor markets. In the goods market, demand is determined by consumer preferences, represented by the utility function, while supply depends on the production technology, represented by the production functions. In the factor markets, the supply of factors is defined by the endowments, whereas factor demand depends on the production functions and the level of production, which in turn depends on consumption.

These three components—(i) the utility function, (ii) the production functions, and (iii) the factor endowments—jointly determine the autarky relative price. If at least one of these elements differs across countries, their autarky relative prices will diverge, thereby creating the conditions for mutually beneficial trade

Figure 20 illustrates the case in which both countries share identical production functions and identical factor endowments. Consequently, they exhibit the same Production Possibility Frontier (PPF). However, the countries differ in consumer preferences, represented by distinct utility functions, which lead to different autarky relative prices.

In figure 20, points A and B represent the autarky equilibria for countries i and j, respectively. A world relative price lying between these autarky relative prices (shown in red) results in production under trade at point E for both countries. Under these conditions, consumption occurs at points C and D, each located on a higher indifference curve, indicating welfare gains from trade.

Figure 20: Trade between countries with identical PPF and endowments

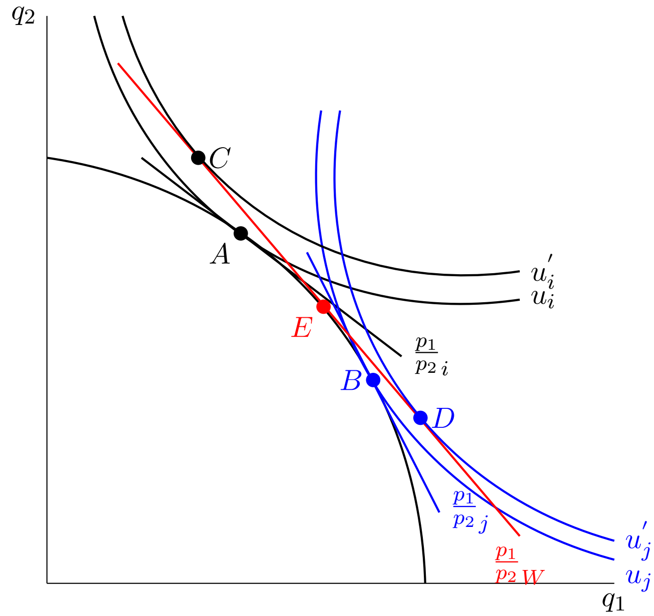
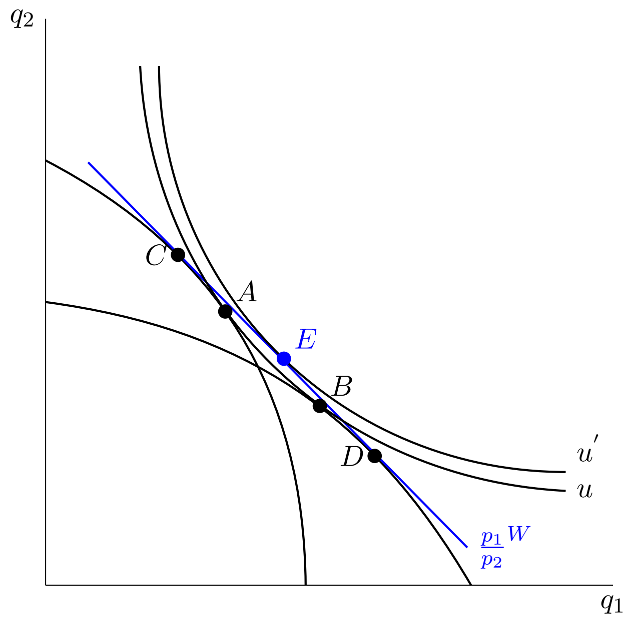


Figure 20 illustrates the case in which both countries possess identical factor endowments and identical preferences, represented by the same utility function. However, the countries differ in production technology, as reflected in distinct production functions. Because production technologies vary across countries, their Production Possibility Frontiers (PPFs) differ, leading to distinct autarky relative prices.

In figure 20 , one country's maximum attainable output of good q_2 exceeds that of the other, while the other country can produce a higher maximum quantity of good q_1. This representation is merely illustrative; the specific shape and position of each PPF depend on the functional form of the underlying production technologies.

Figure 21: Trade between countries with identical preferences and endowments



If both countries share identical technologies (production functions) and identical preferences (utility functions) but differ in their relative factor endowments, they will still exhibit different autarky relative prices. Consequently, their Production Possibility Frontiers (PPFs) will also differ.[6]

The specific configuration of the PPFs across countries depends on the respective factor endowments. For instance, if one country is relatively abundant in one factor while the other is relatively abundant in the alternative factor, the scenario resembles that depicted in Figure 20. Conversely, if one country possesses larger quantities of both factors, its PPF will completely encompass that of the other country.

This last case corresponds precisely to the **Heckscher–Ohlin Model**. In this model, two countries produce two goods using two factors of production. Both countries share identical production technologies and identical preferences. However, even under these conditions, differences in relative factor endowments lead to distinct autarky relative prices. As a result, countries engage in trade and achieve higher levels of welfare through specialization according to their comparative advantage.

8.7 Equilibrium in the Neoclassical $2 \times 2 \times 1$ model

In an economy with two countries and two goods but only one factor of production, the autarky equilibrium has the same graphical representation as in Figure 14. However, solving for the autarky equilibrium in this case is simpler, as the Production Possibility Frontier (PPF) can

[6] If endowments differ in absolute terms but relative endowments are identical across countries, PPFs will differ while autarky relative prices will remain the same. In this case, there would be no incentive to engage in trade. In particular, one country's PPF will encompass that of the other.

be obtained explicitly. In section 8.3, the Production Possibility Frontier (PPF) was derived implicitly by considering that each point on the contract curve represents an efficient allocation of factors and, therefore, corresponds to a point on the PPF. Different allocations of factors can thus be used to determine the implicit form of the PPF.

The optimal consumption and production in autarky occur where the *marginal rate of transformation* (MRT) equals the *marginal rate of substitution* (MRS). Given that the PPF is expressed explicitly, the equilibrium condition can be solved as

$$\frac{\partial q_{2z}}{\partial q_{1z}} = \frac{\frac{\partial U}{\partial q_{1z}}}{\frac{\partial U}{\partial q_{2z}}} \tag{8.7.1}$$

where the left-hand side of equation (8.7.1) represents the MRT, and the right-hand side represents the MRS.

Using equation (8.7.1) and the explicit form of the PPF, the autarky equilibrium can be readily determined.

When the economy opens to trade, production is determined at the point where the MRT equals the world relative price, that is,

$$\frac{\partial q_{2z}}{\partial q_{1z}} = \left(\frac{p_1}{p_2}\right)^W \tag{8.7.2}$$

The optimal consumption under trade is characterized by the same condition described in equation (8.5.1)

8.8 Case study

Consider an economy composed of two countries, denoted by i and j, and two production factors: labor (L) and capital (K). Both countries share identical production technologies, defined as

$$q_{1z} = L_{1z}^{\frac{1}{4}} K_{1z}^{\frac{3}{4}}, \quad q_{2z} = L_{2z}^{\frac{3}{4}} K_{2z}^{\frac{1}{4}}, \quad z = i, j \tag{8.8.1}$$

The representative consumer in each country has preferences with the following utility function:

$$U_z = \sqrt{x_{1z} x_{2z}} \tag{8.8.2}$$

The factor endowments of each country are given by $(\bar{L}_i, \bar{K}_i) = (100, 200)$ for country i, and $(\bar{L}_j, \bar{K}_j) = (200, 100)$ for country j

8.8.1 Production Possibility Frontier (PPF)

The contract curve is given by

$$K_{1z} = \frac{9\bar{K}_z L_{1z}}{\bar{L}_z + 8L_{1z}}, \qquad z = i, j \tag{8.8.3}$$

Each point on the contract curve corresponds to an efficient allocation of factors and, therefore, to a point on the Production Possibility Frontier (PPF). The PPF can thus be expressed as

$$\begin{aligned}\mathrm{PPF}_z = \Bigg\{ (q_{1z}, q_{2z}) \ : \exists\, L_{1z} \in [0, \bar{L}_z], \quad & q_{1z} = L_{1z}^{\frac{1}{4}} \left(\tfrac{9\bar{K}_z L_{1z}}{\bar{L}_z + 8L_{1z}} \right)^{\frac{3}{4}}, \\ & q_{2z} = \left(\bar{L}_z - L_{1z}\right)^{\frac{3}{4}} \left(\bar{K}_z - \tfrac{9\bar{K}_z L_{1z}}{\bar{L}_z + 8L_{1z}} \right)^{\frac{1}{4}} \Bigg\}\end{aligned} \tag{8.8.4}$$

Equation (8.8.4) characterizes the PPF. For each feasible allocation of L_{1z}, the contract curve determines the corresponding optimal amount of K_{1z}. These two values simultaneously define the remaining factor allocations, L_{2z} and K_{2z}, which together determine the output levels of both sectors

8.8.2 Autarky

From equations (8.4.3) and (8.4.4), the equilibrium in the goods markets is given by

$$q_{1z} = \frac{(w\bar{L}_z + r\bar{K}_z)}{2p_1}, \qquad q_{2z} = \frac{(w\bar{L}_z + r\bar{K}_z)}{2p_2}, \qquad z = i, j \tag{8.8.5}$$

From equations (8.4.9) and (8.4.10), the equilibrium in the factor markets can be expressed as

$$\bar{L}_z = 3^{-\frac{3}{4}} q_{1z} \left(\frac{r}{w}\right)^{\frac{3}{4}} + 3^{\frac{1}{4}} q_{2z} \left(\frac{r}{w}\right)^{\frac{1}{4}} \tag{8.8.6}$$

$$\bar{K}_z = 3^{\frac{1}{4}} q_{1z} \left(\frac{w}{r}\right)^{\frac{1}{4}} + 3^{-\frac{3}{4}} q_{2z} \left(\frac{w}{r}\right)^{\frac{3}{4}} \tag{8.8.7}$$

In addition, from equations (8.4.11) and (8.4.12), the unit cost functions imply that

$$p_1 = \left(3^{-\frac{3}{4}} + 3^{\frac{1}{4}}\right) w^{\frac{1}{4}} r^{\frac{3}{4}} \tag{8.8.8}$$

$$p_2 = \left(3^{-\frac{3}{4}} + 3^{\frac{1}{4}}\right) w^{\frac{3}{4}} r^{\frac{1}{4}} \tag{8.8.9}$$

Combining equations (8.8.8) and (8.8.9) yields the relative factor price relationship:

$$\frac{r}{w} = \left(\frac{p_1}{p_2}\right)^2 \tag{8.8.10}$$

Equation (8.8.10), together with the goods and factor market equilibrium conditions, allows the determination of the autarky relative price:

$$\frac{p_1}{p_2} = \sqrt{\frac{\bar{L}_z}{\bar{K}_z}} \tag{8.8.11}$$

Once the autarky relative price equilibrium has been determined, the corresponding equilibrium factor prices, levels of consumption and production, and the allocation of factors across sectors can be derived by substitution into the preceding equations.

8.8.3 Trade Equilibrium

Substituting equation (8.8.10) into equations (8.8.6) and (8.8.7) defines the production levels of each sector when the economy is open to trade and faces relative prices equal to $\left(\frac{p_1}{p_2}\right)^W$:

$$\bar{L}_z = 3^{-\frac{3}{4}} q_{1z} \left(\frac{p_1}{p_2}^W\right)^{\frac{3}{2}} + 3^{\frac{1}{4}} q_{2z} \left(\frac{p_1}{p_2}^W\right)^{\frac{1}{2}} \tag{8.8.12}$$

$$\bar{K}_z = 3^{\frac{1}{4}} q_{1z} \left(\frac{p_1}{p_2}^W\right)^{-\frac{1}{2}} + 3^{-\frac{3}{4}} q_{2z} \left(\frac{p_1}{p_2}^W\right)^{-\frac{3}{2}} \tag{8.8.13}$$

Given the representative consumer preferences, equation (8.5.1) can be expressed as

$$\frac{x_{2z}}{x_{1z}} = \frac{p_1}{p_2}^W \tag{8.8.14}$$

Substituting equation (8.8.14) into the budget constraint (8.5.2) and solving yields

$$x_{1z} = \frac{1}{2}\left(q_{1z} + \frac{p_2}{p_1}^W q_{2z}\right) \tag{8.8.15}$$

$$x_{2z} = \frac{1}{2}\left(\frac{p_1}{p_2}^W q_{1z} + q_{2z}\right) \tag{8.8.16}$$

Solving for q_{1z} and q_{2z} in equations (8.8.12) and (8.8.13), and substituting the results into equations (8.8.15) and (8.8.16), yields the consumption levels under trade. The relative offer curve can then be derived by considering different values of the world relative price.

8.9 Exercises

1. Consider an economy in which labor is the only factor of production. There are two countries, i and j, each endowed with L_i and L_j units of labor, respectively. Both countries share the same production technology given by

$$q_{1k} = \sqrt{L_{1k}}, \quad q_{2k} = \sqrt{L_{2k}}, \quad k = i, j$$

However, preferences differ across countries and are represented by the following utility functions:

$$U_i = x_{1i}^2 x_{2i}, \quad U_j = x_{1j} x_{2j}^2$$

   1. Derive the Production Possibility Frontier (PPF) for each country and determine their respective Marginal Rates of Transformation (MRT).
   2. Determine the autarky equilibrium for each country.
   3. Compute production and consumption levels for each country when trade is allowed, given a world relative price $\frac{p_1}{p_2}^W$.
   4. Derive an expression for the world relative price in equilibrium under free trade.
   5. Assume both countries have identical labor endowments, each equal to 100. Demonstrate that both countries gain from trade and explain the intuition behind this result, given that they share the same technology and labor endowments. Determine the equilibrium world relative price and demonstrate that it lies between the autarky relative prices of both countries.

2. Considering the example proposed in Section 8.8 and using Excel (all computations must be performed in Excel), for each country:

   1. Plot the Production Possibility Frontier (PPF).
   2. Estimate an approximate Marginal Rate of Transformation (MRT).
   3. Determine explicitly the autarky equilibrium, including relative goods prices, relative factor prices, consumption, production, and the allocation of factors across sectors.
   4. Demonstrate that in equilibrium, the autarky relative price approximates the MRT estimated previously.
   5. Show that, in equilibrium, no higher level of utility can be attained.
   6. Approximate and plot the offer curve when both countries are opened to trade.
   7. Determine the world relative price[7] and the equilibrium for each country. Verify that both world markets are in equilibrium at that relative price.
   8. Demonstrate that both countries achieve higher welfare when engaging in trade.

[7] This can be computed using the *Solver* tool in Excel.

3. Based on the neoclassical model, consider an increase in country i's preference for good q_2. Assume that under trade, country i imports q_1 both before and after the change in preferences.

    1. Using Figures 14 and 15, illustrate the new production and consumption points under autarky and trade. Provide the economic intuition for the resulting changes.
    2. Using Figure 18, illustrate how country i's offer curve shifts. Explain how this shift reflects a change in the willingness to trade and interpret the underlying intuition.
    3. Using Figure 19, analyze how the world relative price equilibrium may change. Provide the economic intuition for this result.
    4. Assess whether it is possible for country i to become an exporter of q_1 following the preference change. Specify the conditions under which this could occur and justify your reasoning.
    5. Discuss how your previous answers would change if the preference shift were instead toward good q_1.

9 The Heckscher–Ohlin–Vanek Model

9.1 Introduction

This chapter builds on the fundamental concepts developed in Chapter 5 and Chapter 8. As discussed in Section 8.6, the Heckscher–Ohlin model constitutes a particular case of the neo-classical framework in which countries share identical technologies (production functions) and preferences (utility functions) but differ in their relative factor endowments—and consequently in their absolute endowments. This model highlights the role of factor endowments, rather than technological differences, as the fundamental source of comparative advantage. Vanek later extended and generalized this framework to encompass n factors and n goods

In this chapter, an additional assumption is introduced: **relative prices are exogenous**. This implies that the country under analysis takes the relative prices of goods as given and exerts no influence over their determination. This assumption can be interpreted as the country being sufficiently small relative to the rest of the world so that its actions have no effect on world relative prices.

It should be noted that in Chapter 8, where there were two countries, each country influenced the determination of the world relative prices, making them endogenous to the model. In fact, as illustrated in the offer curve equilibrium in Figure 19, the equilibrium world relative price is determined by the intersection of both countries' offer curves.

The Heckscher–Ohlin–Vanek (HOV) model represents the natural multi-good, multi-factor extension of the canonical Heckscher–Ohlin framework. Its central prediction is that the pattern of trade can be described in terms of the factor content of net exports, thereby linking observed trade flows directly to relative factor endowments across countries. This formulation generalizes the Heckscher–Ohlin theorem and provides a tractable empirical framework for testing the relationship between factor abundance and trade patterns.

9.2 The model

Consider a world economy composed of m countries, each endowed with a given quantity of the n factors of production. The relative factor endowments of each country differ from the world relative endowments. All countries produce n goods using identical, concave production functions that exhibit constant returns to scale. Sectors differ in their factor intensities. Preferences are identical and homothetic across countries, ensuring that demand patterns depend solely on relative prices.

9.3 Prices Equal its Marginal Cost

As discussed in Chapter 5—and implicitly applicable to Chapter 8—the condition that prices equal marginal costs for the case of two factors and two goods can be expressed as

$$\begin{cases} p_1 = w\, a_{L1} + r\, a_{K1}, \\ p_2 = w\, a_{L2} + r\, a_{K2} \end{cases} \tag{9.3.1}$$

In matrix form, this relationship generalizes to

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \tag{9.3.2}$$

where a_{ij} denotes the quantity of factor i required to produce one unit of good j—that is, the conditional factor demand per unit of output.
The factor requirement can also be derived from Shephard's Lemma:

$$a_{ij}(w_1, \ldots, w_n) = \frac{\partial c_j(w_1, \ldots, w_n)}{\partial w_i} \tag{9.3.3}$$

Hence, equation (9.3.2) can be written more compactly as

$$\mathbf{P} = \mathbf{AW} \tag{9.3.4}$$

where $\mathbf{P}$ is the $n \times 1$ vector of goods prices, $\mathbf{A}$ is the $n \times n$ matrix of factor requirements, and $\mathbf{W}$ is the $n \times 1$ vector of factor prices.

9.4 Equilibrium in the Factor Markets

As discussed in Chapter 5—and equally applicable to Chapter 8—the equilibrium conditions in the factor markets for the case of two factors and two goods can be expressed as

$$\begin{cases} \bar{L} = a_{Li} q_i + a_{Lj} q_j, \\ \bar{K} = a_{Ki} q_i + a_{Kj} q_j \end{cases} \tag{9.4.1}$$

In matrix form, this relationship generalizes to

$$\begin{bmatrix} \bar{e}_{1i} \\ \bar{e}_{2i} \\ \vdots \\ \bar{e}_{ni} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} q_{1i} \\ q_{2i} \\ \vdots \\ q_{ni} \end{bmatrix} \tag{9.4.2}$$

Equation (9.4.2) can be written more compactly as

$$\mathbf{E}_i = \mathbf{A}\mathbf{Q}_i \tag{9.4.3}$$

where $\mathbf{E}_i$ denotes the $n \times 1$ vector of factor endowments for country i, and $\mathbf{Q}_i$ represents the $n \times 1$ vector of outputs for the same country. As these quantities vary across countries, the subscript i is used to identify the respective country.
This relationship also holds at the world level:

$$\mathbf{E}_W = \mathbf{A}\mathbf{Q}_W \tag{9.4.4}$$

If the matrix of factor requirements, $\mathbf{A}$, is invertible, equation (9.4.3) and (9.4.4) can be rewritten as

$$\mathbf{Q}_k = \mathbf{A}^{-1}\mathbf{E}_k, \quad k = i, W \tag{9.4.5}$$

9.5 Factor Price Equalization

If countries share identical technologies (production functions), are open to trade, and face the same world relative prices, then factor prices will be equalized across countries. The intuition behind this result is that, as factor prices are determined by equation (9.3.2), identical production technologies imply identical marginal costs across sectors. Because all countries face the same world relative prices, they will consequently have identical factor prices.
It should be noted that equation (9.3.4) applies uniformly to all countries. No country-specific subscript is required, as the vector of goods prices $\mathbf{P}$ is identical across countries, the factor requirement matrix $\mathbf{A}$ is the same due to identical production technologies, and the vector of factor prices $\mathbf{W}$ is equalized across countries by the *Factor Price Equalization* theorem.

9.6 Consumption Allocation

Under identical and homothetic preferences, each country's share of world income determines its share in world consumption. Let α_i denote the share of country i in world income, with $\sum_{i=1}^{m} \alpha_i = 1$. This share can be measured as

$$\alpha_i = \frac{\mathbf{P}^\top \mathbf{Q}_i}{\mathbf{P}^\top \mathbf{Q}_W} \tag{9.6.1}$$

Consumption in country i can therefore be expressed as

$$\mathbf{X}_i = \alpha_i \mathbf{X}_W \tag{9.6.2}$$

where $\mathbf{X}_i$ is the vector of country i's consumption and $\mathbf{X}_W$ represents world consumption. In equilibrium, world consumption equals world production, $\mathbf{X}_W = \mathbf{Q}_W$. Hence, equation (9.6.2) can be rewritten as

$$\mathbf{X}_i = \alpha_i \mathbf{Q}_W \tag{9.6.3}$$

9.7 Factor Content of Trade

The factor content of trade is defined is the factor embodied in a country's net exports. The factor content vector of trade for country i is

$$A\mathbf{T}_i = A(\mathbf{Q}_i - \mathbf{C}_i) \tag{9.7.1}$$

Note that $\mathbf{T}_i$ denotes the vector of net exports for country i, expressed in units of goods. When premultiplied by matrix $\mathbf{A}$, that is, $\mathbf{AT}_i$, the expression is measured in terms of factors. Equation (9.7.2) can be expressed As

$$A\mathbf{T}_i = \mathbf{E}_i - \alpha_i \mathbf{E}_W \tag{9.7.2}$$

Thus the factor content of trade equals the difference between a country's factor endowment and its consumption share of world factor endowments.

9.8 The Heckscher–Ohlin–Vanek Theorem

The Heckscher–Ohlin Theorem states that, in the $2 \times 2 \times 2$ model, if a country is relatively abundant in factor i, it will export the good that is intensive in the use of that factor and import the other good. In this context, a country is considered labor-abundant if its relative labor endowment (with respect to capital) exceeds that of the other country—or, equivalently, if it exceeds the world's relative labor endowment; otherwise, it is capital-abundant. It should be noted that for this theorem to hold, countries must differ in their relative factor endowments, and sectors must differ in their factor intensities.

Extending this logic to the case of n factors and n goods, the Heckscher–Ohlin–Vanek Theorem states that country i will be a net exporter of the services of factors in which it is relatively abundant and a net importer of the services of factors in which it is relatively scarce. Formally,

$$\mathbf{AT}_i = \mathbf{E}_i - \alpha_i \mathbf{E}_W \tag{9.8.1}$$

Each element of the resulting vector $\mathbf{AT}_i$ represents the quantity of factor services embodied in net exports for country i, where a positive entry indicates a net export and a negative entry indicates a net import of the corresponding factor.

Suppose factors are ordered from the most to the least relatively abundant (compared to the world). All factors whose relative endowment ratio exceeds the country's share of world income are net exported, while those below that threshold are net imported. Formally,

$$\frac{e_{ji}}{e_{jW}} \geq \cdots \geq \alpha_i \geq \cdots \geq \frac{e_{ki}}{e_{kW}} \tag{9.8.2}$$

Hence, factor j will be a net export (its corresponding entry in $\mathbf{AT}_i$ is positive) if country i's relative endowment exceeds α_i, whereas factor k will be a net import (its corresponding entry in $\mathbf{AT}_i$ is negative) if it falls below this value

9.9 Stolper–Samuelson Theorem

Since countries take prices as exogenous, in an economy with two goods and two factors, the *Stolper–Samuelson Theorem* applies exactly as described in Chapter 5.
For the general case of n factors and n goods, the relationship can be expressed in matrix form as

$$\Delta \mathbf{P} = \mathbf{A} \Delta \mathbf{W} \tag{9.9.1}$$

The generalized result for n factors is that a change in goods prices benefits the factors used intensively in the production of goods whose prices increase, and harms the factors used intensively in the production of goods whose prices decline. Consequently, trade-induced changes in relative prices generate redistributive effects across factors, consistent with the HOV prediction that countries export the goods of their relatively abundant factors.

9.10 Rybczynski Theorem

In an economy with two goods and two factors, the *Rybczynski Theorem* applies exactly as described in Chapter 5. As previously discussed, the Rybczynski Theorem analyzes the effect of changes in factor endowments on output levels under fixed goods prices. In this setting, factor prices $\mathbf{W}$ remain constant, ensuring that the input coefficients a_{fi} are fixed.
In matrix form, the *Rybczynski Theorem* can be expressed as

$$\Delta \mathbf{E}_i = \mathbf{A} \Delta \mathbf{Q}_i \tag{9.10.1}$$

If the matrix of factor requirements $\mathbf{A}$ is invertible, equation (9.10.1) can be rewritten as

$$\Delta \mathbf{Q}_i = \mathbf{A}^{-1} \Delta \mathbf{E}_i \tag{9.10.2}$$

The Rybczynski Theorem then states that an increase in the endowment of factor j expands the output of the good that uses factor j intensively and contracts the output of other goods. This result highlights how changes in factor endowments influence the composition of production, even when goods prices remain constant.

9.11 Exercises

1. Using the matrix approach developed in this chapter, suppose country i is endowed with $(\bar{L}_i, \bar{K}_i) = (0.4, 0.6)$, while the world is endowed with $(\bar{L}_W, \bar{K}_W) = (1, 1)$. World prices are given by $(p_1, p_2) = (1, 1)$. The total cost functions for each sector are

$$CT_1 = 2q_1 \left(\frac{w}{\frac{1}{3}}\right)^{\frac{1}{3}} \left(\frac{r}{\frac{2}{3}}\right)^{\frac{2}{3}}, \quad CT_2 = 2q_2 \left(\frac{w}{\frac{2}{3}}\right)^{\frac{2}{3}} \left(\frac{r}{\frac{1}{3}}\right)^{\frac{1}{3}}$$

   1. Determine the vector of factor prices, $\mathbf{W}$.
   2. Compute the vector of production for country i, $\mathbf{Q}_i$.
   3. Compute the world production vector, $\mathbf{Q}_W$.
   4. Determine the consumption vector for country i, $\mathbf{X}_i$.
   5. Derive the vector of net exports for country i, $\mathbf{T}_i$.
   6. Verify that the Heckscher–Ohlin theorem holds.
   7. Verify that the *Factor Price Equalization Theorem* holds.

2. Considering the example developed in Section 8.8:

   1. Demonstrate that equation (9.8.1) holds.
   2. Verify that the Heckscher–Ohlin–Vanek (HOV) theorem is satisfied.

3. Suppose an economy with three factors (labor, capital, and land) and three sectors. World prices are equal to 1 for all goods. Country i is endowed with 10, 20, and 30 units of labor, capital, and land, respectively, while the world endowment for each factor is 100. The production functions are given by

$$\begin{aligned} q_1 &= \min\{3L_1,\ 2K_1,\ T_1\} \\ q_2 &= \min\{L_2,\ K_2,\ T_2\} \\ q_3 &= \min\{L_3,\ 2K_3,\ 3T_3\} \end{aligned}$$

   1. Determine $\mathbf{E}_i$ and $\mathbf{E}_W$.
   2. Compute $\mathbf{AT}_i$.
   3. Verify that the HOV theorem holds.
   4. Suppose the land endowment of country i increases by 10%. Determine $\mathbf{Q}_i'$ and demonstrate that the *Rybczynski Theorem* holds.
   5. Suppose p_3 increases to 2, while all other prices remain unchanged. Determine the new vector of factor prices, $\mathbf{W}'$, and demonstrate that the *Stolper–Samuelson Theorem* holds.

10 The Input–Output Model

The Input–Output (I–O) model, originally developed by Leontief, constitutes a fundamental analytical framework in international economics for examining the interdependence among industries and the transmission of trade shocks across sectors. It provides a systematic approach to understanding how final demand and production structures jointly determine output levels within an economy. The model is particularly valuable for identifying and quantifying both the direct and indirect effects of changes in demand.

10.1 The Model

Consider an economy composed of n productive sectors. Each sector produces a good that can be used either as an intermediate input by other sectors or as a final good consumed domestically or exported. Let x_i denote the total output of sector i, z_{ij} the quantity of good i used as an intermediate input by sector j, and b_i the final demand for good i. The final demand comprises household consumption, government expenditure, inventory accumulation or capital formation, and foreign demand. Market clearing for each good requires

$$x_i = \sum_{j=1}^{n} z_{ij} + b_i, \quad j = 1, \ldots, n \tag{10.1.1}$$

Additionally, it is assumed that the total output of sector i equals the total value of its inputs, which include all intermediate inputs used by sector i and the value added generated within the sector. This relationship can be expressed as

$$x_i = \sum_{j=1}^{n} z_{ji} + v_i, \quad j = 1, \ldots, n \tag{10.1.2}$$

where v_i in equation (10.1.2) represents the value added by sector i, which may include the returns to labor, capital, and land, as well as taxes and profits.

Table 2 presents the Input–Output table for an economy with three sectors. In practice, each entry in Table 2 is measured in monetary units, ensuring that all values in the table are expressed in the same currency

Table 2: The Input-Output table for three sectors

		Sectors 1	2	3	b	Total input
Sectors	**1**	z_{11}	z_{12}	z_{13}	b_1	x_1
	2	z_{21}	z_{22}	z_{23}	b_2	x_2
	3	z_{31}	z_{32}	z_{33}	b_3	x_3
Value added		v_1	v_2	v_3		
Total input		x_1	x_2	x_3		

Defining the input coefficients as $a_{ij} = \frac{z_{ij}}{x_j}$, which represent the quantity of good i required per unit of output in sector j[8], the system can be expressed compactly in matrix form. Let $\mathbf{x}$ denote the $n \times 1$ vector of total outputs, $\mathbf{b}$ the $n \times 1$ vector of final demands, and $A = [a_{ij}]$ the $n \times n$ matrix of technical coefficients. Then

$$\mathbf{x} = A\mathbf{x} + \mathbf{b} \tag{10.1.3}$$

Rearranging equation (10.1.3) yields to

$$(I - A)\mathbf{x} = \mathbf{b} \tag{10.1.4}$$

If the inverse of the matrix $(I - A)$ exists, total output can be expressed from equation (10.1.4) as

$$\mathbf{x} = (I - A)^{-1}\mathbf{b} \tag{10.1.5}$$

The matrix $(I - A)^{-1}$, known as the *Leontief inverse*, captures both the direct and indirect production requirements within the economy. Each element (i, j) of this matrix represents the total output from sector i required to satisfy one additional unit of final demand in sector j, taking into account the entire network of intermediate linkages across sectors.

10.2 Comparative Statics in the I–O Model

The input–output framework is particularly well suited for comparative static analysis of demand shocks. Consider a marginal change $\Delta\mathbf{b}$ in final demand. The corresponding change in total output is given by

$$\Delta\mathbf{x} = (I - A)^{-1}\Delta\mathbf{b}. \tag{10.2.1}$$

Equation (10.2.1) implies that any disturbance to final demand is proportionally transmitted throughout the production network. Specifically, an increase in the final demand for sector i

[8] If the entries of the Input–Output table are measured in monetary units, a_{ij} represents the amount of monetary units spent in sector i by sector j to generate one monetary unit of output.

leads to a more than proportional rise in its total output, as sector i requires intermediate inputs from other sectors, which in turn demand additional inputs from sector i, and so on. The same mechanism applies in the case of a decrease in final demand, generating a proportional contraction that propagates through the different sectors.
Furthermore, the new output vector can be expressed using equation (10.1.5) as

$$\mathbf{x}' = (I - A)^{-1}\mathbf{b}' \tag{10.2.2}$$

10.3 Case Study

Consider the following Input–Output table for an economy composed of two sectors: goods and services

Table 3: Input–Output example for an economy with goods and services sectors

	Goods	**Services**	**Final demand**
Goods	1	3	6
Services	2	4	5
Labor	7	4	

It follows that the production vector is

$$\mathbf{x} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

The corresponding matrix of technical coefficients, A, is given by

$$A = \begin{bmatrix} \frac{1}{10} & \frac{3}{11} \\ \frac{2}{10} & \frac{5}{11} \end{bmatrix}$$

The Leontief inverse matrix, $(I - A)^{-1}$, is

$$(I - A)^{-1} = \begin{bmatrix} 1.23 & 0.53 \\ 0.39 & 1.74 \end{bmatrix}$$

Using equation (10.1.5), it can be verified that

$$(I - A)^{-1}\mathbf{b} = \begin{bmatrix} 1.23 & 0.53 \\ 0.39 & 1.74 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix} = \mathbf{x}$$

Suppose now that final demand in the goods sector decreases by two units. The corresponding change in final demand is

$$\Delta\mathbf{b} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Using equation (10.2.2), the resulting change in the production vector is obtained as

$$(I - A)^{-1}\Delta\mathbf{b} = \begin{bmatrix} 1.23 & 0.53 \\ 0.39 & 1.74 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2.46 \\ -0.77 \end{bmatrix} = \Delta\mathbf{x}$$

It is noteworthy that, even though the final demand for the services sector remains unchanged, its output decreases because the goods sector reduces its use of intermediate inputs from services as a result of the decline in goods output.

10.4 Exercises

1. Consider a country that categorizes its production into three sectors. The technical coefficients matrix is given by

Table 4: Technical coefficients matrix A

	Sector 1	**Sector 2**	**Sector 3**
Sector 1	0.15	0.09	0.12
Sector 2	0.03	0.09	0.08
Sector 3	0.03	0.03	0.12

The vector of final demand (in local currency) is

Table 5: Final demand vector **b** (local currency)

	Household demand	**Government demand**	**Foreign demand**
Sector 1	9	6	7
Sector 2	8	10	9
Sector 3	6	7	7

1. Determine the total output vector **x** for the country.
2. Determine the value added generated by each sector.
3. Suppose the government seeks to raise total output by increasing public expenditure. The targeted increases in sectoral output are 30%, 15%, and 15% for sectors 1, 2, and 3, respectively. Determine the required increase in government spending by sector to achieve these targets.

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