Intermediate International Trade

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1 Introduction

The formalization of trade theory has very old origins. For example, Aristotle (ca. 350 B.C./2009, *Nicomachean Ethics*, V.5, 1133a) employed the concept of proportional modeling developed by the mathematician Euclid (ca. 300 B.C./1956, *Elements*, Book V), and formalized an exchange equation that already included assumptions commonly used by economists. If A is a builder, B a shoemaker, C a house, and D a pair of shoes, the relationship is established as $A : B = xD : C$. Two aspects are fundamental: determining x (the number of pairs of shoes equivalent to one house) and interpreting the ratio builder/shoemaker. The theory of international trade has examined this problem from different perspectives. Even the "new" theoretical approaches that have emerged in recent decades frequently, though not explicitly, rely on Schumpeterian concepts, in which entrepreneurs, "new markets," innovation, and market power play a central role, or draw on concepts from Newtonian physics, which itself was influenced by Aristotelian propositions.

It is therefore no coincidence that "Nothing exists in the world, except the blind forces of nature, that is not Greek in its origin" (Maine, cited in Livingstone, 1921). International trade theory is no exception. These are the conceptual foundations that later evolved into the propositions of the labor theory of value, which enabled the theories of absolute and comparative advantage. These theories can be extended to a larger number of goods and countries and explain trade patterns and the international organization of labor primarily through factor endowments, relative size, and technological differences.

The extension to two factors through the Heckscher–Ohlin (H–O) model, and its generalization in the Heckscher–Ohlin–Vanek (H–O–V) model, was enriched by a virtuous and intense cycle between empirical research and conceptual advances. This has been the evolution of trade theory—like many other fields of science—through empirical verification and the search for explanations of trade between relatively similar countries or of flows of goods in industries where close substitutes compete. As in other areas, increasing efforts with a stronger microeconomic foundation have emerged, seeking to explain how firms enter "new" markets and how global companies organize production processes across different locations and countries. These explanatory efforts are conventionally presented in textbooks and even in specialized journals as if isolated from entrepreneurial concepts, although in reality they are closely linked.

The relative availability of international trade data (values, volumes, identification of buyers and sellers, locations, among others) has also enabled the development of empirical explanations. These explanations do not necessarily follow models of welfare theory and are often ad hoc. Services pose an even greater challenge for the empirical verification of explanatory approaches. Similarly, Porter's concepts, product life-cycle theory, and the incorporation of more dynamic aspects—despite the absence of equations derived from microeconomic optimization models—have contributed detail and improved characterization of observed trade structures. Much of this description remains fundamentally linked to material transformation (production functions, technological relationships such as increasing returns, productivity, and technological change).

Trade relations have also acquired a strategic dimension, with the imposition of tariffs and trade barriers that modify the global equilibrium. This demands preparation to interpret and act upon regulatory and economic changes in this context. The concepts of "new markets" and the fundamental role of innovation, entrepreneurship, and market power—widely discussed since Schumpeter and influenced by Newtonian physics—are applied today in the analysis of international trade flows.

This document is the result of teaching international trade theory at the School of Economics of the University of Costa Rica. It addresses different topics, aiming to provide reference material with solid formal grounding, as well as exercises at an intermediate level of depth.

2 Microeconomic Overview

This chapter is a briefly review of: (i) Consumer Theory, (ii) Firm Theory and (iii) General Equilibrium in pure exchange.

2.1 Consumer Theory

Consumer theory analyzes how an individual chooses a consumption bundle to maximize utility (or satisfaction) given income and market prices. Dually, for a given utility level and given prices, the individual can choose the bundle that attains that utility at the minimum possible expenditure.

The individual's utility-maximization problem can be stated as follows:

$$\max_{x_1,\dots,x_n} \quad U(x_1,\dots,x_n) \tag{2.1.1}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} p_i x_i \le m, \tag{2.1.2}$$

$$x_i \ge 0 \quad \forall i = 1,\dots,n$$

Under non-satiation and an interior solution, the associated Lagrangian is

$$\mathcal{L}(x_1,\dots,x_n,\lambda) = U(x_1,\dots,x_n) + \lambda\left(m - \sum_{i=1}^{n} p_i x_i\right) \tag{2.1.3}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial U}{\partial x_i} - \lambda p_i = 0 \quad \forall i = 1,\dots,n, \tag{2.1.4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - \sum_{i=1}^{n} p_i x_i = 0 \tag{2.1.5}$$

Combining (2.1.4) for any two goods i and j yields the optimality (Marginal Rate of Substitution = price ratio) condition:

$$\frac{\partial U/\partial x_i}{\partial U/\partial x_j} = \frac{p_i}{p_j} \tag{2.1.6}$$

Substituting the optimal condition into the equation (2.1.2) (budget constraint) gives the *Marshallian (ordinary) demand* for each good i:

$$x_i = x_i(m, p_1,\dots,p_n) \tag{2.1.7}$$

Equation (2.1.7) states that the optimal quantity of good i depends on income and all prices. Plugging these demands into the utility function defines the *indirect utility function*:

$$v(m, p_1,\dots,p_n) = U\big(x_1^M(m,\mathbf{p}),\dots,x_n^M(m,\mathbf{p})\big) \tag{2.1.8}$$

The indirect utility $v(m, \mathbf{p})$ gives the maximum attainable utility at income m and price vector p and it is useful for comparing scenarios in which income and prices change simultaneously. By the envelope theorem, the Lagrange multiplier satisfies

$$\lambda^* = \frac{\partial v(m, \mathbf{p})}{\partial m}$$

so λ^* is the marginal utility of income.
Consider the following utility function

$$U(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}, \qquad \alpha_i > 0 \tag{2.1.9}$$

From (2.1.6),

$$\frac{x_i}{x_j} = \frac{\alpha_i p_j}{\alpha_j p_i} \tag{2.1.10}$$

Substituting into (2.1.2) yields the Marshallian demand

$$x_i(m, \mathbf{p}) = \frac{\alpha_i m}{p_i \sum_{k=1}^{n} \alpha_k} \tag{2.1.11}$$

Plugging (2.1.11) into (2.1.9) gives the indirect utility:

$$v(m, p_1, \ldots, p_n) = \left(\frac{m}{\sum_{k=1}^{n} \alpha_k}\right)^{\sum_{k=1}^{n} \alpha_k} \prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} \tag{2.1.12}$$

For a target utility level $\bar{u}$, the dual problem is

$$\min_{x_1, \ldots, x_n} \quad \sum_{i=1}^{n} p_i x_i \tag{2.1.13}$$

$$\text{s.t.} \quad U(x_1, \ldots, x_n) \geq \bar{u}, \tag{2.1.14}$$

$$x_i \geq 0 \quad \forall i = 1, \ldots, n$$

With an interior solution, the Lagrangian is

$$\mathcal{L}(x_1, \ldots, x_n, \lambda) = \sum_{i=1}^{n} p_i x_i + \lambda\big(\bar{u} - U(x_1, \ldots, x_n)\big) \tag{2.1.15}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_i} = p_i - \lambda \frac{\partial U}{\partial x_i} = 0 \quad \forall i = 1, \ldots, n \tag{2.1.16}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{u} - U(x_1, \ldots, x_n) = 0 \tag{2.1.17}$$

Combining the two-good first-order conditions in (2.1.16) once again produces the optimality condition found in (2.1.6). Substituting this condition into (2.1.14) yields the *Hicksian (compensated) demand*.

$$h_i = h_i(\bar{u}, p_1, \dots, p_n) \tag{2.1.18}$$

Substituting the Hicksian demand into (2.1.13) defines the resulting *expenditure function*:

$$e(\bar{u}, p_1, \dots, p_n) = \sum_{i=1}^{n} p_i h_i(\bar{u}, \mathbf{p}) \tag{2.1.19}$$

The Hicksian demand tells how much of each good is needed to achieve utility $\bar{u}$ at minimum cost, while the expenditure function gives that minimum cost. By the envelope theorem, the multiplier λ measures how much the minimum expenditure must increase to raise utility by one unit.

In the previous example, substituting (2.1.10) into the utility function yields to the Hicksian demand:

$$h_i(\bar{u}, \mathbf{p}) = \frac{\alpha_i \bar{u}^{\frac{1}{\sum_{i=1}^{n} \alpha_i}}}{p_i} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j} \right)^{\frac{\alpha_j}{\sum_{i=1}^{n} \alpha_i}}$$

Inserting this result into the expenditure function gives:

$$e(\bar{u}, \mathbf{p}) = \left[\bar{u}^{\frac{1}{\sum_{i=1}^{n} \alpha_i}} \prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j} \right)^{\frac{\alpha_j}{\sum_{i=1}^{n} \alpha_i}} \right] \sum_{i=1}^{n} \alpha_i$$

2.1.1 Duality Properties

Let $v(m, \mathbf{p})$ denote the indirect utility function, $e(\bar{u}, \mathbf{p})$ the expenditure function, $x_i(m, \mathbf{p})$ the Marshallian (ordinary) demand, and $h_i(\bar{u}, \mathbf{p})$ the Hicksian (compensated) demand. Prices are $\mathbf{p} = (p_1, \dots, p_n)$, income is m, and $\bar{u}$ is a target utility level.

Roy's identity

$$x_i(m, \mathbf{p}) = -\frac{\dfrac{\partial v(m, \mathbf{p})}{\partial p_i}}{\dfrac{\partial v(m, \mathbf{p})}{\partial m}} \quad \forall i = 1, \dots, n \tag{2.1.20}$$

Shephard's lemma

$$h_i(\bar{u}, \mathbf{p}) = \frac{\partial e(\bar{u}, \mathbf{p})}{\partial p_i} \quad \forall i = 1, \dots, n \tag{2.1.21}$$

Indirect utility and expenditure functions are inverses (duality)

$$e\big(v(m, \mathbf{p}), \mathbf{p}\big) = m, \tag{2.1.22}$$

$$v\big(e(\bar{u}, \mathbf{p}), \mathbf{p}\big) = \bar{u} \tag{2.1.23}$$

Marshallian and Hicksian demands relationship

$$h_i(\bar{u},\mathbf{p}) = x_i\big(e(\bar{u},\mathbf{p}),\mathbf{p}\big), \tag{2.1.24}$$
$$x_i(m,\mathbf{p}) = h_i\big(v(m,\mathbf{p}),\mathbf{p}\big) \tag{2.1.25}$$

Given the Cobb–Douglas indirect utility function in equation (2.1.12), Roy's identity delivers the Marshallian demand:

$$x_i(m,\mathbf{p}) = \frac{\alpha_i\, m}{p_i \sum_{k=1}^{n} \alpha_k} \tag{2.1.26}$$

Equation (2.1.23) asserts that the indirect utility evaluated at the minimum expenditure equals the target utility. Substituting the form (2.1.12) and solving for $e(u,\mathbf{p})$ yields the minimum expenditure function

$$e(u,\mathbf{p}) = \left[u \prod_{i=1}^{n} \left(\tfrac{p_i}{\alpha_i}\right)^{\alpha_i}\right]^{\frac{1}{\sum_{k=1}^{n}\alpha_k}} \sum_{k=1}^{n} \alpha_k \tag{2.1.27}$$

Applying Shephard's lemma to the minimum expenditure function gives the Hicksian demand:

$$h_i(u,\mathbf{p}) = \frac{\alpha_i}{p_i}\left[u \prod_{j=1}^{n} \left(\tfrac{p_j}{\alpha_j}\right)^{\alpha_j}\right]^{\frac{1}{\sum_{k=1}^{n}\alpha_k}} \tag{2.1.28}$$

Solving the Marshallian demand for good i for p_i and substituting it into the indirect utility function restores the original utility function.

2.2 Firm Theory

In the theory of the firm, input choice is viewed either as profit maximization or, equivalently, as minimizing the cost of producing a given output level; the analysis first adopts the cost-minimization perspective.

$$\begin{aligned} \min_{z_1,z_2,\ldots,z_n} \quad & \sum_{i=1}^{n} w_i z_i \\ \text{s.t.} \quad & q(z_1,z_2,\ldots,z_n) \geq \bar{q} \\ & z_i \geq 0 \quad \forall i = 1,\ldots,n \end{aligned}$$

Assuming an interior solution, the Lagrangian for the cost-minimization problem is:

$$\mathcal{L}(z_1,\ldots,z_n,\lambda) \;=\; \sum_{i=1}^{n} w_i z_i \;+\; \lambda\big(\bar{q} - q(z_1,\ldots,z_n)\big)$$

By the envelope theorem, the Lagrange multiplier λ equals the marginal cost—the increase in minimum total cost required to produce one additional unit of output.

$$\frac{\partial \mathcal{L}}{\partial z_i} = w_i - \lambda \frac{\partial q(z_1, \ldots, z_n)}{\partial z_i} = 0 \quad \forall i = 1, \ldots, n,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{q} - q(z_1, \ldots, z_n) = 0$$

Combining the first-order conditions for any two inputs, i and j, implies that the marginal rate of technical substitution between them equals the ratio of their input prices.

$$\frac{\frac{\partial q(z_1,\ldots,z_n)}{\partial z_i}}{\frac{\partial q(z_1,\ldots,z_n)}{\partial z_j}} = \frac{w_i}{w_j} \quad \forall i, j = 1, \ldots, n \tag{2.2.1}$$

Substituting the optimality condition (2.2.1) back into the production constraint yields the *conditional input demand functions*, denoted by

$$z_i = z(\bar{q}, w_1, \ldots, w_n) \tag{2.2.2}$$

where $\bar{q}$ is the fixed output target and $\mathbf{w} = (w_1, \ldots, w_n)$ is the vector of input prices. Each function $z_i(\bar{q}, \mathbf{w})$ gives the amount of input that minimizes costs i required to produce units of output $\bar{q}$ at the prevailing prices, thus completing the solution to the firm's cost-minimization problem.

Substituting (2.2.2) into the cost objective yields the *minimum cost function*

$$C(\bar{q}, \mathbf{w}) = \sum_{i=1}^{n} w_i \, z_i(\bar{q}, \mathbf{w}), \tag{2.2.3}$$

which gives the least expenditure required to produce the target output $\bar{q}$ at input prices $\mathbf{w}$.
Define the *scale elasticity* as

$$\varepsilon_S = \sum_{i=1}^{n} \frac{\partial q(z_1, \ldots, z_n)}{\partial z_i} \frac{z_i}{q}$$

Classification follows immediately:

$$\varepsilon_S \begin{cases} > 1 & \text{Increasing Returns to Scale (IRS),} \\ = 1 & \text{Constant Returns to Scale (CRS),} \\ < 1 & \text{Decreasing Returns to Scale (DRS).} \end{cases}$$

Additionally, the *cost elasticity* with respect to output is

$$\varepsilon_C(q) = \frac{\partial C(\bar{q}, \mathbf{w})}{\partial q} \frac{q}{C(\bar{q}, \mathbf{w})} = \frac{MC(q)}{AC(q)},$$

Hence

$$\varepsilon_C(q)\begin{cases} < 1 & \text{Economies of scale } (AC\downarrow), \\ = 1 & \text{Constant returns to scale}, \\ > 1 & \text{Diseconomies of scale } (AC\uparrow). \end{cases}$$

To illustrate these concepts, consider the following production function:

$$q(z_1,\ldots,z_n) \;=\; \left(\sum_{i=1}^{n} a_i\, z_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \qquad \sigma > 1 \tag{2.2.4}$$

The marginal product of input z_i is defined as

$$\frac{\partial q(z_1,\ldots,z_n)}{\partial z_i} = \alpha_i z_i^{\frac{-1}{\sigma}}\, q\left(\sum_{i=1}^{n} a_i\, z_i^{\frac{\sigma-1}{\sigma}}\right)^{-1}$$

The (2.2.1) would yield to:

$$\left(\frac{z_k}{z_i}\right)^{\frac{1}{\sigma}} = \left(\frac{\alpha_k w_i}{\alpha_i w_k}\right)$$

Note that evaluating the *elasticity of substitution* between inputs i and j under the optimality condition yields

$$\sigma_{ij} \;=\; \frac{\partial\big(z_j/z_i\big)}{\partial\big(w_i/w_j\big)}\,\frac{\big(w_i/w_j\big)}{\big(z_j/z_i\big)} = \sigma$$

Because the elasticity of substitution, σ, remains constant for every input combination, the production function is called the *Constant Elasticity of Substitution (CES)* function.
Defining the Dixit–Stiglitz CES price index as

$$W(\mathbf{w}) \;=\; \left(\sum_{i=1}^{n} \alpha_i^{\sigma} p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \tag{2.2.5}$$

The *conditional input demand* associated to this production function can be written as

$$z_i(\bar{q}, w) = \left(\frac{\alpha_i}{w_i}\right)^{\sigma} W^{\sigma}\bar{q}$$

Minimum cost function is

$$C(\bar{q},\mathbf{w}) = q\sum_{i=1}^{n} w_i\left(\frac{\alpha_i}{w_i}\right)^{\sigma} W^{\sigma}$$

Note that $\varepsilon_S = 1$ and $\varepsilon_C(q) = 1$.
The supply curve for the firm if the good market is competitive is as follows

$$P = \sum_{i=1}^{n} w_i\left(\frac{\alpha_i}{w_i}\right)^{\sigma} W^{\sigma}$$

Now if we were to maximize the firms profit

$$\max_{z_1,\ldots,z_n} \pi \;=\; p\,q(z_1,\ldots,z_n) - \sum_{i=1}^{n} w_i z_i$$

Assuming an interior solution; the *First Order Conditions (FOC)*

$$p\,\frac{\partial q(z_1,\ldots,z_n)}{\partial z_i} \;=\; w_i\,, \qquad \forall i = 1,\ldots,n$$

Solving the system yields the *unconditional* (profit-maximizing) input demands

$$z_i^* \;=\; z_i(p,\mathbf{w}), \qquad i = 1,\ldots,n,$$

If the technology exhibits *decreasing returns to scale* (i.e. diseconomies of scale), marginal cost is increasing so supply curve is upward-sloping in the output price. Substituting the *unconditional* factor demands into the production function therefore yields the firm's supply function:

$$q = q(z_1(p,\mathbf{w}),\ldots,z_i(p,\mathbf{w}))$$

2.3 General Equilibrium in pure exchange

Suppose an economy with I consumers and n goods. Consumer j is endowed with $\boldsymbol{\omega}_j = (\omega_{1j},\ldots,\omega_{nj}) \in \mathbb{R}^n_+$. Through trade, every consumer tries to become *more satisfied* (i.e. reach a higher utility level) than at the initial endowment.
Considering a *Decentralized equilibrium*, consumer j solves

$$\max_{x_{1j},\ldots,x_{nj}} u_j(x_{1j},\ldots,x_{nj}) \tag{2.3.1}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} p_i x_{ij} \;=\; \sum_{i=1}^{n} p_i \omega_{ij} \tag{2.3.2}$$

The first-order (interior) optimality condition reads

$$\frac{\partial u_j/\partial x_{ij}}{\partial u_j/\partial x_{kj}} \;=\; \frac{p_i}{p_k}, \qquad \forall\, i \neq k.$$

The optimality condition, once substituted into equation (2.3.2) (the budget constraint) generates individual j's demand for each good. Repeating this procedure for every individual yields the complete system of demand functions—one for each of the n goods for each of the I individuals, i.e. $n \times I$ demand functions. Note that this is the same as getting the Marshallian demand for each good and replacing income m with the value of individual j's endowment, $\sum_{i=1}^{n} p_i \omega_{ij}$.

In equilibrium, by market clearing for every good i total demand equals total supply:

$$\sum_{j=1}^{I} x_{ij} = \sum_{j=1}^{I} \omega_{ij}, \qquad i = 1, \dots, n.$$

The resulting system of n equations determines the equilibrium price vector $\left(p_1^*, \dots, p_n^*\right)$ (defined up to a positive scalar normalization).
The equilibrium allocation is a *Pareto equilibrium*: no individual's utility can be increased without lowering someone else's. Varying individual endowments while maintaining the aggregate endowment fixed traces out the *Pareto set.* This set can be characterized by imposing, for each individual j, the optimality condition that equates the marginal substitution rate with the equilibrium price ratio, that is,

$$\frac{\partial u_j / \partial x_{ij}}{\partial u_j / \partial x_{kj}} = \frac{p_i^*}{p_k^*}$$

Note that some allocations make one or both individuals better off relative to their initial endowments while still allowing an increase in one person's utility without reducing the other's. This collection of allocations is called the *lens of trade.* Within this lens lies that segment of the *Pareto set* where no further utility gains are possible for anyone without harming someone else. The final allocation must therefore lie inside the lens of trade and on the Pareto set.
Take the following example with 2 individual and 2 goods; each individual's utility function is given by:

$$u_j(x_{1j}, x_{2j}) = \sqrt{x_{1j} x_{2j}} \qquad j = 1, 2$$

The resulting demand functions are:

$$x_{1j} = \frac{1}{2} \cdot \frac{p_1 \omega_{1j} + p_2 \omega_{2j}}{p_1} \qquad \wedge \qquad x_{2j} = \frac{1}{2} \cdot \frac{p_1 \omega_{1j} + p_2 \omega_{2j}}{p_2} \qquad j = 1, 2$$

Market clearing in equilibrium requires that

$$\begin{cases} \frac{1}{2} \cdot \frac{p_1 \omega_{11} + p_2 \omega_{21}}{p_1} + \frac{1}{2} \cdot \frac{p_1 \omega_{12} + p_2 \omega_{22}}{p_1} = \omega_{11} + \omega_{12} \\ \frac{1}{2} \cdot \frac{p_1 \omega_{11} + p_2 \omega_{21}}{p_2} + \frac{1}{2} \cdot \frac{p_1 \omega_{12} + p_2 \omega_{22}}{p_2} = \omega_{21} + \omega_{22} \end{cases}$$

Solving the first equation for the relative price yields:

$$\frac{p_2^*}{p_1^*} = \frac{(\omega_{11} + \omega_{12})}{(\omega_{21} + \omega_{22})} \tag{2.3.3}$$

The equilibrium *relative price* is unique, whereas the absolute price vector is determined only up to a positive scalar. For instance, in (2.3.3) one may set

$$p_2 = \lambda\big(\omega_{11} + \omega_{12}\big), \qquad p_1 = \lambda\big(\omega_{21} + \omega_{22}\big),$$

for any $\lambda > 0$, leaving the ratio p_1/p_2 unchanged.

Equation (2.3.3) states that the equilibrium *relative price* equals the ratio of total endowments. In words, the price of good 1 relative to good 2 equals the economy-wide endowment of good 2 relative to that of good 1, and vice-versa.

This illustrates *Walras's Law*: if a price vector clears total demand and supply in one market, it necessarily clears the remaining market. Concretely, choose any $\lambda > 0$ and set

$$p_2 = \lambda\big(\omega_{11} + \omega_{12}\big), \qquad p_1 = \lambda\big(\omega_{21} + \omega_{22}\big).$$

These prices equate aggregate demand and supply in the first market; by Walras's Law, they also clear the second market. More generally, if a price vector $(p_1^*, \ldots, p_n^*)$ clears $n-1$ markets, it clears all n markets.

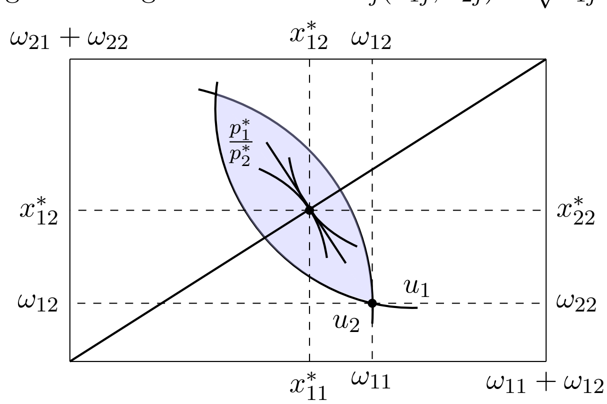
The Pareto-equilibrium allocation is as follows

$$x_{1j}^* = \frac{\omega_{1j}}{2} + \frac{\omega_{2j}}{2}\,\frac{\omega_{21} + \omega_{22}}{\omega_{11} + \omega_{12}}, \qquad x_{2j}^* = \frac{\omega_{1j}}{2}\,\frac{\omega_{21} + \omega_{22}}{\omega_{11} + \omega_{12}} + \frac{\omega_{2j}}{2}, \qquad j = 1, 2.$$

The Pareto-equilibrium allocation shown above depends on the *individual* endowments. If these endowments are redistributed—keeping the *aggregate* endowment constant—the Pareto allocation changes, yet the equilibrium relative price remains the one in (2.3.3). Hence a redistribution can raise one person's utility while lowering the other's.

The constancy of the relative price here is a special case: both agents share identical, symmetric preferences. In general, each Pareto equilibrium that arises from a given initial allocation is supported by its own relative-price vector, as stated by the Second Welfare Theorem. In this symmetric setting, however, every Pareto allocation is backed by the same price ratio in (2.3.3). Figure 1 illustrates the Edgeworth box for this example.

Figure 1: Edgeworth box for $u_j(x_{1j}, x_{2j}) = \sqrt{x_{1j}x_{2j}}$



Social–planner problem and the First Welfare Theorem Finally, note that a benevolent social planner who reallocates each good so that the aggregate assignment equals the aggregate endowment would implement exactly the same Pareto–efficient allocation that arises in the decentralized competitive equilibrium derived in (2.3.3). Formally, the planner solves

$$\begin{aligned} &\max_{x_{ij}\ \forall i,j} \sum_{j=1}^{I} \lambda_j\, u_j\big(x_{1j}, \ldots, x_{nj}\big) \\ &\text{s. t.} \qquad \sum_{j=1}^{I} x_{ij} \;=\; \sum_{j=1}^{I} \omega_{ij}, \qquad \forall\, i = 1, \ldots, n, \end{aligned} \tag{2.3.4}$$

where the Pareto weights λ_j in (2.3.4) are proportional to the marginal utility of the value of household j's original endowment. Because competitive markets already equate marginal rates of substitution across agents while respecting the resource constraints, the solution to (2.3.4) coincides with the decentralized equilibrium allocation—an illustration of the First Welfare Theorem.

2.4 Exercises

1. Consider a consumer whose preferences are represented by

$$U(x_1, x_2, \ldots, x_n) = x_k \prod_{i \neq k}^{n} (x_i - \theta_i), \qquad x_i > \theta_i \quad \forall i \neq k$$

    1. Derive the Marshallian demand functions.
    2. Obtain the indirect utility function.
    3. Derive the Hicksian (compensated) demand functions.
    4. Determine the expenditure function.

2. Let the expenditure function be

$$e(\bar{u}, \mathbf{p}) = \bar{u} p_1 - \frac{p_1^2}{4} \sum_{i=2}^{n} \frac{1}{p_i}, \qquad \bar{u} > \frac{p_1}{2} \sum_{i=2}^{n} \frac{1}{p_i}$$

    1. Derive the Marshallian demand functions.
    2. Obtain the indirect utility function.
    3. Derive the Hicksian (compensated) demand functions.
    4. Recover the underlying utility function.

3. Consider a firm with the production function

$$q(L, K) = [\max\{\min\{2L, K\}, \min\{L, 2K\}\}]^{\rho}, \qquad 0 < \rho < 1$$

    1. Derive the conditional factor demand functions.
    2. Determine the cost function.
    3. Derive the firm's supply function.
    4. Obtain the unconditional factor demands[1].

4. Consider a firm with the production function

$$q(L, K) = [\min\{\max\{2L, K\}, \max\{L, 2K\}\}]^{\rho}, \qquad 0 < \rho < 1$$

    1. Derive the conditional factor demand functions.
    2. Determine the cost function.
    3. Derive the firm's supply function.
    4. Obtain the unconditional factor demands.

[1] Consider $z_i(p, \mathbf{w}) = z_i(q(p, z_i(p, \mathbf{w})), \mathbf{w})$

5. Consider a pure-exchange economy with two consumers. Consumer A's utility is

$$u^A(x_{1A}, x_{2A}) = 2x_{1A} + x_{2A}$$

while consumer B's utility is

$$u^B(x_{1B}, x_{2B}) = \min\{x_{1B}, x_{2B}\}$$

Their endowments are $w^A = (0, \bar{w})$ and $w^B = (\bar{w}, 0)$

   1. State the initial endowment point.
   2. Determine the lens of trade.
   3. Characterize the Pareto set.
   4. Compute the equilibrium relative price.
   5. Identify the set of equilibrium allocations.

6. A small town has n residents. Resident i is endowed with $\bar{w}$ units of good i and none of the other goods. Every resident's preferences are

$$u_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} \ln x_j$$

   1. Derive the equilibrium relative prices.
   2. Characterize the Pareto set.
   3. Identify the equilibrium allocation set.
   4. Show that equilibrium prices are independent of the endowments and explain the intuition.
   5. Verify that the centralized allocation matches the competitive equilibrium.

3 The Export Condition and Ricardian Model

3.1 The Export Condition

Suppose there is Home (H) and (F) and n goods. Each good i requires a_i units of labor per unit of output, and a_i^* units abroad. The wage rate in Home is w, and in Foreign it is w^*. If e denotes the nominal exchange rate (units of Foreign currency per unit of Home currency), the unit cost of producing good i is

$$c_i^H = e \cdot w \cdot a_i, \qquad c_i^F = w^* \cdot a_i^*$$

Here, c_i^H is measured in Foreign currency by multiplying by e, while c_i^F is expressed in Foreign currency. This adjustment ensures cost to be measured in the same currency.

The fundamental export condition is that Home exports good i if it can supply the good at lower cost than Foreign:

$$c_i^H < c_i^F.$$

Substituting from above:

$$e \cdot w \cdot a_i < w^* \cdot a_i^* \tag{3.1.1}$$

Equation (3.1.1) can be written to state that Home exports good i if

$$\frac{e \cdot w}{w^*} < \frac{a_i^*}{a_i} \tag{3.1.2}$$

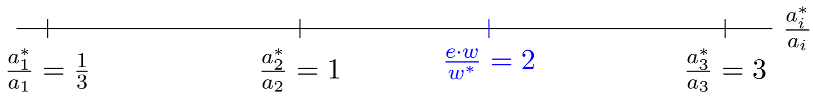
Equation (3.1.2) can be interpreted as follows: Home exports good i whenever the relative wage—that is, the wage in Home expressed in terms of the Foreign wage—is lower than the relative cost of producing the good abroad, expressed in terms of Home's cost. The right-hand side of the inequality thus represents the relative unit labor requirements, capturing the notion of comparative efficiency.

To illustrate, consider three goods x_1, x_2, and x_3. The unit labor requirements in Home are $(1, 2, 3)$, while in Foreign they are $(3, 2, 1)$. Suppose the Home wage (in Foreign currency) is 2, and the Foreign wage (in Foreign currency) is 1. According to the export condition, Home will have a cost advantage in goods x_1 and x_2, thereby producing and exporting them, while it will import x_3 from Foreign.

Figure 2 illustrates this scenario. The goods are ordered from lowest to highest according to Home's relative unit requirements. Relative wage adjusted by exchange rate is in blue. All goods positioned to the left of relative wage are produced and exported by Home (and imported by Foreign), while those to the right are produced and exported by Foreign (and imported by Home).

Figure 2 represent this scenario, all goods are ordenated from less to higher in relative requirements for Home and we mark the relative wages adjusted to the same currency and note that all goods to the left of relative wages are produced and export by Home (imported by Foreign) and all goods to the right are produced and export by Foreign (imported by Home).

Figure 2: Export condition in example given



3.2 The Ricardian Model of International Trade

Consider a world economy with two countries: Home (denoted by H) and Foreign (denoted by F). The economy produces two goods, indexed by q_1^i and q_2^i, where $i \in \{H, F\}$. Labor is the only factor of production, and each country is endowed with a fixed labor supply $L^i > 0$.
Technology is characterized by constant unit labor requirements: producing one unit of good q_1 in country i requires a_1^i units of labor, while producing one unit of good q_2 requires a_2^i units of labor. We assume $a_1^i, a_2^i > 0$ and constant returns to scale.

3.2.1 Production

The production function for good $i \in \{1, 2\}$ in country $j \in \{H, F\}$ is given by:

$$q_i^j = \frac{L_i^j}{a_i^j} \tag{3.2.1}$$

where L_i^j denotes the amount of labor allocated to sector i in country j.
This functional form reflects a fixed-coefficient technology: each unit of output q_i^j requires exactly a_i^j units of labor. Equivalently, $1/a_i^j$ is the marginal product of labor in sector i of country j.
Labor is perfectly mobile across sectors within a country but immobile across countries. The labor resource constraint is therefore:

$$L_1^j + L_2^j = L^j, \qquad j \in \{H, F\}, \tag{3.2.2}$$

which reflects the fact that total labor demand must equal the exogenously supply labor.

3.2.2 Production Possibility Frontier

To characterize the set of feasible output combinations, substitute $L_i^j = a_i^j q_i^j$ into the labor constraint:

$$a_1^j q_1^j + a_2^j q_2^j = L^j, \qquad j \in \{H, F\}. \tag{3.2.3}$$

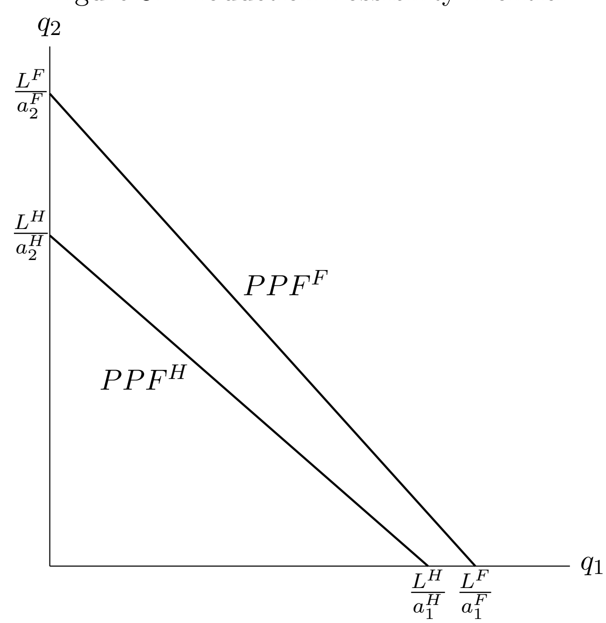
Equation (3.2.3) is a linear relationship between q_1^j and q_2^j with slope $-\frac{a_1^j}{a_2^j}$, which we refer to as the **relative requirements**. The intercepts are $\frac{L^j}{a_1^j}$ on the q_1^j axis and $\frac{L^j}{a_2^j}$ on the q_2^j

axis. These represent the maximum quantity of each good that country j could produce if it devoted all its labor to that sector. Equation (3.2.3) thus defines the Production Possibility Frontier (PPF).

The linearity of the PPF follows directly from the assumption of constant unit labor requirements. Its slope, $-\frac{a_1^j}{a_2^j}$, measures the opportunity cost of producing one unit of q_1^j in terms of forgone units of q_2^j. Since this opportunity cost is constant, the model does not feature diminishing returns to specialization. As a result, **full specialization naturally emerges under trade**.

Figure 3 illustrates the PPF in a case where Home has a comparative advantage in good q_1. The intercepts depend on each country's labor supply and labor requirements. Assuming $L^H = L^F$, the figure suggests that Foreign enjoys an absolute advantage in both goods, even though Home maintains a comparative advantage in good q_1.

Figure 3: Production Possibility Frontier



3.2.3 Autarky

In autarky, competitive equilibrium requires that the relative price of the two goods equals their opportunity cost in production. Let p_1^j and p_2^j denote the prices of goods q_1^j and q_2^j in country j, respectively. Under perfect competition and zero profits, the unit cost of producing good i must equal its price:

$$p_1^j = w^j a_1^j, \tag{3.2.4}$$

$$p_2^j = w^j a_2^j \tag{3.2.5}$$

where w^j is the wage in country j. Dividing equation (3.2.4) by equation (3.2.5) yields:

$$\frac{p_1^j}{p_2^j} = \frac{a_1^j}{a_2^j}. \tag{3.2.6}$$

Thus, in autarky, the relative price equals the constant marginal rate of transformation implied by the PPF.

3.2.4 Opening to Trade

When the economy opens to trade, the relevant relative price is the *world* relative price, $\frac{p_1^W}{p_2^W}$. Suppose that Home has a comparative advantage[2] in good q_1, meaning:

$$\frac{a_1^H}{a_2^H} < \frac{a_1^F}{a_2^F}, \tag{3.2.7}$$

where a_i^F are the unit labor requirements in Foreign.

Comparative advantage is therefore determined entirely by the ratio of unit labor requirements across goods and countries. If Home has a comparative advantage in q_1, then by construction, Foreign must have a comparative advantage in q_2.

It is also possible for one country to have an *absolute advantage*[3] in both goods. Absolute advantage is defined by direct productivity levels, while comparative advantage arises from relative productivity differences and ultimately governs trade patterns.

Figure 4 illustrates the relative offer curve of the model once the economy opens to trade. The derivation, assuming that Home has a comparative advantage in good q_1, proceeds as follows:

- If the world relative price $\frac{p_1^W}{p_2^W}$ is lower than both Home's autarky price ratio $\frac{a_1^H}{a_2^H}$ and Foreign's autarky price ratio $\frac{a_1^F}{a_2^F}$, then both Home and Foreign fully specialize in the production of q_2. In this case:

  $$q_2^W = \frac{L^H}{a_2^H} + \frac{L^F}{a_2^F}, \qquad q_1^W = 0$$

  Thus, $\frac{q_1^W}{q_2^W} = 0$, since both countries specialize in q_2 (whose relative price exceeds its relative cost in both economies). This corresponds to the segment A–B in Figure 4.

- If the world relative price $\frac{p_1^W}{p_2^W}$ lies between Home's autarky price ratio $\frac{a_1^H}{a_2^H}$ and Foreign's

[2] A country has a comparative advantage in good i if its relative cost of producing i is lower than that of the other country. In other words, Home sacrifices less of good j to produce one unit of i compared to Foreign.

[3] A country has an absolute advantage in good i if, with the same resources, it can produce more of i than the other country. Formally, country j has absolute advantage in good i if $a_i^j < a_i^{-j}$, where $-j$ denotes the other country.

autarky price ratio $\frac{a_1^F}{a_2^F}$, then Home fully specializes in q_1 and Foreign in q_2. In this case:

$$q_1^W = \frac{L^H}{a_1^H}, \qquad q_2^W = \frac{L^F}{a_2^F}$$

Therefore,

$$\frac{q_1^W}{q_2^W} = \frac{L^H/a_1^H}{L^F/a_2^F}$$

This scenario is the most economically relevant: each country specializes in the good for which it has comparative advantage. It corresponds to the segment C–D in Figure 4.

- If the world relative price $\frac{p_1^W}{p_2^W}$ is higher than both Home's and Foreign's autarky price ratios, then both countries fully specialize in q_1. In this case:

$$q_1^W = \frac{L^H}{a_1^H} + \frac{L^F}{a_1^F}, \qquad q_2^W = 0$$

Hence, $\frac{q_1^W}{q_2^W} = \infty$, as both countries allocate all resources to q_1. This corresponds to the segment D–∞ on the horizontal axis in Figure 4.

- If the world relative price equals Home's autarky price ratio, $\frac{p_1^W}{p_2^W} = \frac{a_1^H}{a_2^H}$, then Home is indifferent between producing q_1, q_2, or any combination of both. If it participates in trade, it may choose any production in its PPF, while Foreign specializes in q_2. In this case:

$$\frac{q_1^W}{q_2^W} \in \left[0,\ \frac{L^H/a_1^H}{L^F/a_2^F}\right]$$

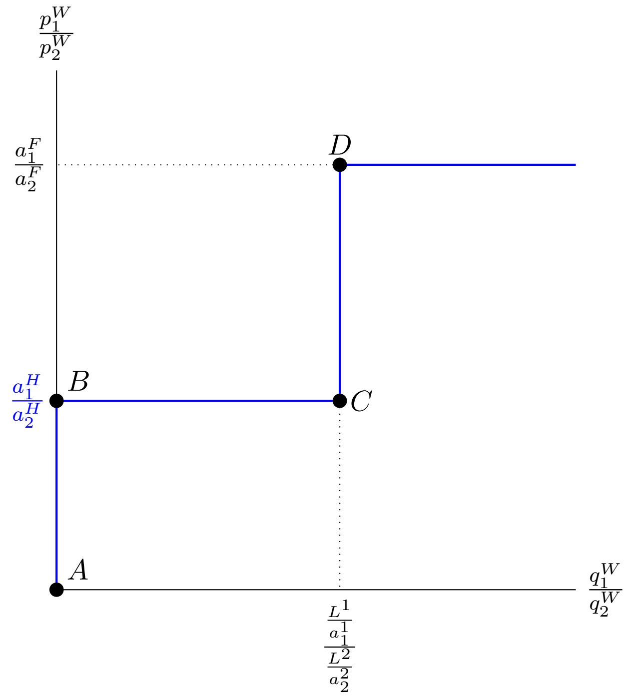
This corresponds to the segment B–C in Figure 4.

- If the world relative price equals Foreign's autarky price ratio, $\frac{p_1^W}{p_2^W} = \frac{a_1^F}{a_2^F}$, then Foreign is indifferent between producing q_1, q_2, or any combination of both. If it participates in trade, it may choose any production in its PPF, while Home specializes in q_1. In this case:

$$\frac{q_1^W}{q_2^W} \in \left[\frac{L^H/a_1^H}{L^F/a_2^F},\ \infty\right]$$

This corresponds to the segment D–∞ in Figure 4.

Figure 4: Relative Market in the Ricardian model



As discussed earlier, the equilibrium arises when each country specializes in a different good. Figure 5 illustrates this equilibrium outcome.
Assume that both countries share a homothetic utility function. For example, let preferences be represented by

$$U_j(q_1, q_2) = q_1 q_2, \qquad j \in H, F$$

The corresponding optimality condition is:

$$\frac{q_2}{q_1} = \frac{p_1}{p_2} \tag{3.2.8}$$

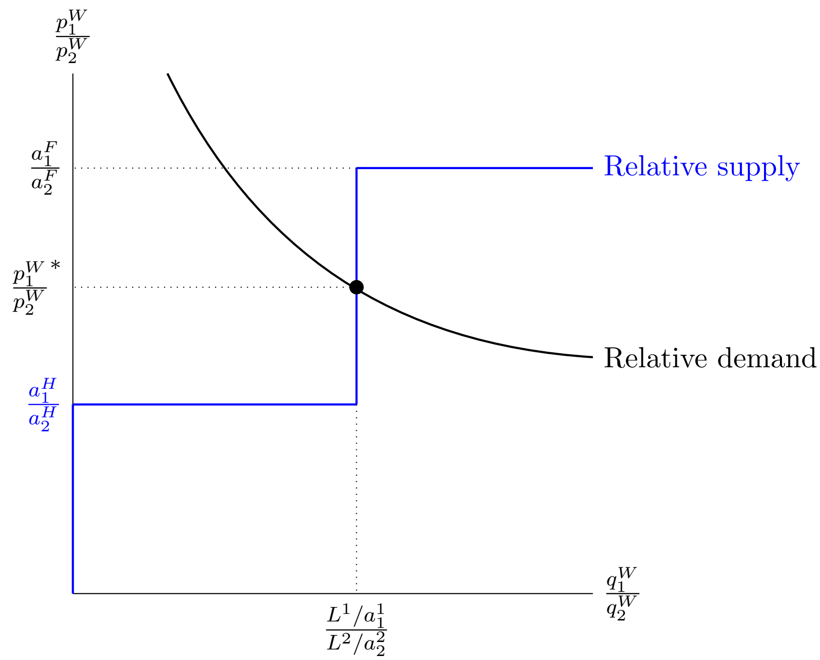
Equation (3.2.8) can be expressed as

$$\frac{p_1}{p_2} = \frac{1}{q_1/q_2} \tag{3.2.9}$$

Equation (3.2.9) represents the relative demand. This curve has a negative slope and is convex in relative quantities. To determine the equilibrium world relative price, the ratio $\frac{q_1^W}{q_2^W}$ can be substituted for the case in which each country specializes in the good for which it holds a comparative advantage. The resulting expression yields the equilibrium world relative price:

$$\frac{p_1^W}{p_2^W} = \frac{L^2/a_2^2}{L^1/a_1^1} \tag{3.2.10}$$

Figure 5: Relative Market in the Ricardian model



For the relative demand curve to intersect the vertical segment of the relative supply curve at the relative quantity $\frac{L^H/a_1^H}{L^F/a_2^F}$ in Figure 5, the equilibrium world relative price must satisfy

$$\left(\frac{p_1^W}{p_2^W}\right)^* \in \left]\frac{a_1^H}{a_2^H}, \frac{a_1^F}{a_2^F}\right[$$

3.3 Exercises

1. Consider the Export Condition studied here (Section 3.1) where there are three goods and three countries. Table 1. reports the unit labor requirements for each good in each country.

Requirements	**A**	**B**	**C**
a_1	2	3	4
a_2	4	3	1
a_3	1	2	4

    1. Suppose it is known that country A produces and exports good x_3, country B produces and exports good x_1, and country C produces and exports good x_2. What must be true about the relative wages, expressed in the currency of country C?

2. Consider the Ricardian model (Section 3.2) where country A requires 2 units of labor to produce one unit of q_1 and 1 unit of labor to produce one unit of q_2. Country B requires 4 units of labor to produce one unit of q_1 and 3 units of labor to produce one unit of q_2. The labor endowment in each country is not specified.

    1. Identify which country has absolute advantage and which has comparative advantage.
    2. Derive the Production Possibility Frontier (PPF) for each country.
    3. Obtain the world relative supply.
    4. Assuming preferences are represented by $U(q_1, q_2) = q_1^2 q_2$ in both countries, determine equilibrium production, relative prices, and wages under autarky.
    5. Using the same utility function, determine equilibrium production, relative prices, and wages under international trade.
    6. Propose a method to evaluate whether each country is better off under trade compared to autarky.

3. Consider the Ricardian model (Section 3.2) where Home has a comparative advantage in q_2. Suppose the economy is open to trade and equilibrium occurs at a point where each country fully specializes in the good for which it has comparative advantage. Answer the following questions and explain the underlying intuition:

    1. Why can Home still benefit from trade even if it has an absolute advantage in both goods?
    2. What happens if the population in Home increases? Is Home better off? Is Foreign better off?

3. What happens if the population in Foreign decreases? Is Home better off? Is Foreign better off?
4. What happens if technology improves in Home at the same proportional rate for both goods? Is Home better off? Is Foreign worse off?
5. What happens if technology improves in Foreign for good q_1 only? Is Home better off? Is Foreign worse off?