$X = (X_1, X_2, X_3) \qquad X \sim N_3(\mu, \Sigma)$

$$\mu = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix} \qquad |\rho| < \sqrt{2}/2$$

$$\Sigma = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix}$$

I need to calculate the portion of variance explained so I need the eigenvalues that are the zeros of this equation in λ

$$\det(\underbrace{\lambda I - \Sigma}_{=A}) = 0$$

$$A = \begin{bmatrix} \lambda-1 & -\rho & 0 \\ -\rho & \lambda-1 & -\rho \\ 0 & -\rho & \lambda-1 \end{bmatrix}$$

$$\det(A) = 0 + \rho \cdot \det\begin{pmatrix} \lambda-1 & -\rho \\ 0 & -\rho \end{pmatrix} + (\lambda-1)\cdot\det\begin{pmatrix} \lambda-1 & -\rho \\ -\rho & \lambda-1 \end{pmatrix}$$

$$= \rho \cdot (\rho(1-\lambda)) + (\lambda-1)\cdot((\lambda-1)^2 - \rho^2)$$

$$= \rho^2(1-\lambda) + (\lambda-1)^3 \boxed{-\rho^2(\lambda-1)} = \rho^2(1-\lambda)$$

$$= 2\rho^2(1-\lambda) + (\lambda-1)^3$$

$$= (\lambda-1)((\lambda-1)^2 - 2\rho^2)$$

eigenvalues

$$\lambda - 1 = 0 \rightarrow \lambda = 1$$

$$(\lambda-1)^2 - 2\rho^2 = 0 \rightarrow \underline{\lambda^2} \underline{-2\lambda} + \underline{1 - 2\rho^2} = 0$$

$$\lambda_1 = \frac{1 + \sqrt{1 - 1\cdot(1-2\rho^2)}}{1} = 1 + \sqrt{1 - 1 + 2\rho^2} = 1 + \sqrt{2}|\rho|$$

$$\lambda_2 = \frac{1 - \sqrt{1 - 1\cdot(1-2\rho^2)}}{1} = 1 - \sqrt{1 - 1 + 2\rho^2} = 1 - \sqrt{2}|\rho|$$

$\lambda_1 = 1+\sqrt{2}|\rho|, \quad \lambda_2 = 1, \quad \lambda_3 = 1-\sqrt{2}|\rho|$

$\text{trace}(\Sigma) = 3 = 1+1+1 \quad \text{or} \quad 1 + 1+\sqrt{2}|\rho| + 1-\sqrt{2}|\rho|$

1) Find ρ such that PC_1 and PC_2 account for more than 80 % of total variation of X

$$PVE = \frac{\lambda_1+\lambda_2}{\sum_{i=1}^{3}\lambda_i} = \frac{\lambda_1+\lambda_2}{\text{trace}(\Sigma)} = \frac{\lambda_1+\lambda_2}{3} = \frac{1+1+\sqrt{2}|\rho|}{3} = \frac{2+\sqrt{2}|\rho|}{3}$$

$$PVE > 0.8 \rightarrow \frac{2+\sqrt{2}|\rho|}{3} > 0.8$$

$$2+\sqrt{2}|\rho| > 2.4$$

$$\sqrt{2}|\rho| > 0.4 = \frac{2}{5}$$

$$|\rho| > \frac{2}{\sqrt{2}\cdot 5} \rightarrow \boxed{|\rho| > \frac{\sqrt{2}}{5}}$$

2) PC1 and PC2

$$A = \begin{bmatrix} \lambda-1 & -\rho & 0 \\ -\rho & \lambda-1 & -\rho \\ 0 & -\rho & \lambda-1 \end{bmatrix}$$

I calculate the eigenvectors solving this equation in X. substituting one eigenvalue at a time

$$(\underbrace{\lambda I - \Sigma}_{A})X = 0$$

$\lambda = 1$

$$\begin{bmatrix} 0 & -\rho & 0 \\ -\rho & 0 & -\rho \\ 0 & -\rho & 0 \end{bmatrix} \begin{bmatrix} e_2 \end{bmatrix} = 0$$

$-\rho y = 0 \rightarrow y = 0$

$-\rho x - \rho z = 0 \qquad x = -z$

$$e_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$\lambda = 1 + \sqrt{2}|\rho|$

$$\begin{bmatrix} \sqrt{2}|\rho| & -\rho & 0 \\ -\rho & \sqrt{2}|\rho| & -\rho \\ 0 & -\rho & \sqrt{2}|\rho| \end{bmatrix} \begin{bmatrix} e_1 \end{bmatrix} = 0$$

$\sqrt{2}|\rho|x - \rho y = 0$

$\sqrt{2}|\rho|z - \rho y = 0$

$\sqrt{2}|\rho|x - \sqrt{2}|\rho|z = 0$

$x = z$

$-2\rho x + \sqrt{2}|\rho| y = 0$

$y = \frac{2\rho}{\sqrt{2}|\rho|} x$

$$e_1 = \begin{bmatrix} 1 \\ \mathrm{sign}(\rho)\cdot\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \mathrm{sign}(\rho)/\sqrt{2} \\ 1/2 \end{bmatrix}$$

e_1 is similar to a mean of the three components of X, when $\rho < 0$ X_2 is multiplied by a negative value because it is the only one going on average in the "opposite direction".

e_2 only loads the first and the third components because they are the only ones that are not (directly) correlated so some difference between them is expected.

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$$\mu = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix} \qquad |\rho| < \sqrt{2}/2$$

3) $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} X_1 - X_2 \\ X_2 - X_3 \end{bmatrix}$ Z is a linear combination of the components of X

$$Z \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} X \sim N_2(\mu_Z, \Sigma_Z)$$

$$\mu_Z = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \mu = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\Sigma_Z = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \Sigma \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 1-\rho & \rho-1 & -\rho \\ \rho & 1-\rho & \rho-1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1-\rho-(\rho-1) & \rho-1+\rho \\ \rho-(1-\rho) & 1-\rho-(\rho-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2-2\rho & 2\rho-1 \\ 2\rho-1 & 2-2\rho \end{bmatrix}$$

$$Z \sim N\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2-2\rho & 2\rho-1 \\ 2\rho-1 & 2-2\rho \end{bmatrix} \right)$$

4) $\rho = -\frac{2}{3}$ plot on R

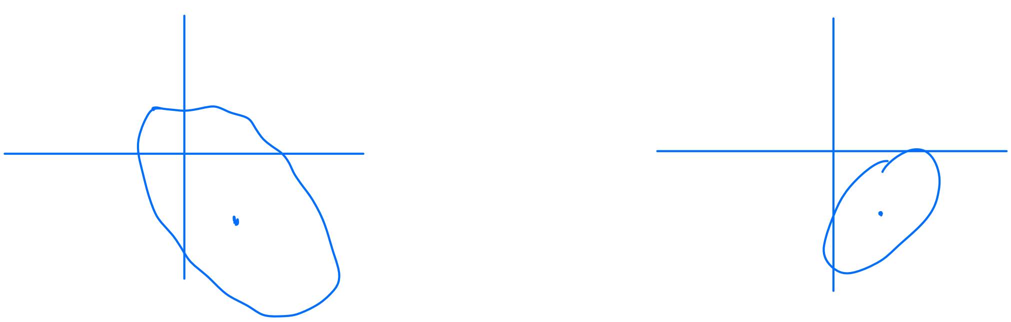
$$z \sim N\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2+\frac{4}{3} & -\frac{4}{3}-1 \\ -\frac{4}{3}-1 & 2+\frac{4}{3} \end{bmatrix}\right)$$

$$z \sim N\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} \frac{10}{3} & -\frac{7}{3} \\ -\frac{7}{3} & \frac{10}{3} \end{bmatrix}\right)$$

5) $\rho = \frac{2}{3}$

$$z \sim N\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2-\frac{4}{3} & +\frac{4}{3}-1 \\ \frac{4}{3}-1 & 2-\frac{4}{3} \end{bmatrix}\right)$$

$$z \sim N\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}\right)$$



expected plots