Chapter1
Preliminaries to Complex Analysis

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Abstract

This chapter is devoted to the exposition of basic preliminary material which we use extensively throughout of this book. I will help you solve serval exercises in this chapter, there may be some spelling mistakes or even wrong methods.

The sweeping development of mathematics during the last two centuries is due in large part to the introduction of complex numbers; paradoxically, this is based on the seemingly absurd notion that there are numbers whose squares are negative.

E. Borel, 1952

Exercise 1 is very easy, so we skip it and begin with Exercise 2.

Exercise 2. Let $<\cdot,\cdot>$ denote the usual inner product in $\mathbb{R}^2$. In other words, if $Z = (x_1, y_1)$ and $W = (x_2, y_2)$, then

$$<Z, W> = x_1x_2 + y_1y_2.$$

Similarly, we may define a Hermitian inner product $(\cdot,\cdot)$ in $\mathbb{C}$ by

$$(z, w) = z\overline{w}.$$

The term Hermitian is used to describe the fact that $(\cdot,\cdot)$ is not symmetric, but rather satisfies the relation

$$(z, w) = z\overline{w} \qquad for \quad all \quad z, w \in \mathbb{C}.$$

Show that

$$<Z, W> = \frac{1}{2}[(z, w) + (w, z)] = Re(z, w),$$

Where we use the usual identification $z = x + iy \in \mathbb{C}$ with$(x, y) \in \mathbb{R}^2$.
Solution 2. Though this is a straightforward calculation, but we have two ways to solve it.
We suppose that $z = z_1 + iz_2, w = w_1 + iw_2$, so

$$\frac{1}{2}[(z,w)+(w,z)] = \frac{1}{2}(z\overline{w}+w\overline{z}) = Re(z\overline{w}) = Re\left((z_1 + iz_2)\left(w_1 - iw_2\right)\right) = z_1w_1+z_2w_2 \qquad Way1$$

$$\frac{1}{2}[(z,w)+(w,z)] = \frac{1}{2}(z\overline{w}+w\overline{z}) = \frac{1}{2}\left[(z_1+iz_2)(w_1-iw_2)+(z_1-iz_2)(w_1+iw_2)\right] = z_1w_1+z_2w_2 \qquad Way2$$

Exercise 3. With $\omega = se^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where n is a natural number. How many solutions are there?
Solution 3. We suppose that $z = te^{i\beta}$, where $t \geq 0$ as $s \geq 0$ and n is a natural number. Then we have

$$t^n e^{ni\beta} = se^{i\varphi}$$

We easily find that $z = s^{\frac{1}{n}} e^{\frac{\varphi}{n} + \frac{2\pi ik}{n}}$, where $k = 0, 1, \cdots, n-1$ and $s^{\frac{1}{n}}$ is the real nth root of s.
In summary, there are n solutions as k has n values.

Exercise 4. Show that it is impossible to define a total ordering on $\mathbb{C}$. In other words, one cannot find a relation $\succ$ between complex numbers so that:

(i) For any two complex numbers z,w, one and only one of the following is true: $z \succ \omega, \omega \succ z$ or $z = \omega$.

(ii) For all $z_1, z_2, z_3 \in \mathbb{C}$ the relation $z_1 \succ z_2$ implies $z_1 + z_3 \succ z_2 + z_3$.

(iii) Moreover, for all $z_1, z_2, z_3 \in \mathbb{C}$ with $z_3 \succ 0$, then $z_1 \succ z_2$ implies $z_1 z_3 \succ z_2 z_3$.

Solution 4. We can conclude that $i = 0$. If not, just for a contradiction, that $i \succ 0$, then $-1 = i \cdot i \succ 0 \cdot i = 0$. Now we may suppose that $0 \succ i$, but also similarly, $0 = 0 \cdot i \succ i \cdot i = -1$. So we must have $i = 0$. But then for all $z \in \mathbb{C}$ we have $z \cdot i = z \cdot 0 = 0$. So this relation would not give a trivial total ordering.

Exercise 5 and Exercise 6 is not so easy to write down, so I skip them and we turn to Exercise 7.

Exercise 7. The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

(a) Let z, w be two complex numbers such that $\overline{z}w \neq 1$.Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right|<1 \quad if\,|z|<1and\,|w|<1,$$

and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right|=1 \quad if\,|z|=1or\,|w|=1.$$

(b) Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$F: z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

(i) F maps the unit disc to itself(that is, $F: \mathbb{D} \to \mathbb{D}$),and is holomorphic.

(ii) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ of $|z| = 1$.

(iv) $F: \mathbb{D} \to \mathbb{D}$ is bijective.

Solution 7.

(a) We have two ways to solve it.

$[Way1]$It is a straightforward calculation.

We suppose that $z = z_1 + iz_2$,$w = w_1 + w_2$, then to calculate $\left|\frac{w-z}{1-\overline{w}z}\right|$ we just need to calculate $\frac{|w-z|}{|1-\overline{w}z|}$, that is,

$$\frac{w_1 - z_1 + i(w_2 - z_2)}{1 - w_1 z_1 - w_2 z_2 + i(w_1 z_2 - w_2 z_1)}$$

We need to compare it with 1,

that is, to compare $w_1 - z_1 + i(w_2 - z_2)$ with $1 - w_1 z_1 - w_2 z_2 + i(w_1 z_2 - w_2 z_1)$,

that is, to compare $(w_1 - z_1)^2$ with $(1 - w_1 z_1 - w_2 z_2)^2 + (w_1 z_2 - w_2 z_1)^2$,

that is, to compare $|w|^2 + |z|^2 - 2(w_1 z_1 + w_2 z_2)$ with $|w|^2 |z|^2 + 1 - 2(w_1 z_1 + w_2 z_2)$, that is, to compare $|w|^2 + |z|^2$ with $|w|^2 |z|^2 + 1$.
When $|w| < 1, |z| < 1$, $(1 - |w|^2)(1 - |z|^2) > 0$.
So we know that $\left|\frac{w-z}{1-\overline{w}z}\right|$<1
When $|w| = 1 or\, |z| = 1$, $|w|^2 + |z|^2 = |w|^2 |z|^2 + 1$, so$\left|\frac{w-z}{1-\overline{w}z}\right| = 1$
$[Way2]$We use the Maximum modulus principle, page 92, Theorem 4.5 to solve it. Maybe I should retell it.

Theorem 1. ***(Maximum modulus principle)*** *If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.*

Suppose that $[w] < 1$ and $[z] = 1$, then we have

$$\left|\frac{w-z}{1-\overline{w}z}\right| = \left|\frac{w-z}{\overline{z}-\overline{w}}\right| = 1$$

Since $\left|\frac{1}{\overline{z}}\right| = 1$. And since that $[w] < 1$, we see that the function $f(z) := \frac{w-z}{1-\overline{w}z}$ is holomorphic in $\mathbb{D}$. It is easy to find out that it is not a constant, so it satisfies $|f(z)| < 1$ as we use the Maximum modulus principle.

(b)

(i) We already show that $F(\mathbb{D}) \subset \mathbb{D}$, what we need to do is just to prove that it is holomorphic.

$$\forall z \in \mathbb{D}, \lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = \lim_{h\to 0} \frac{\frac{w-(z+h)}{1-\overline{w}(z+h)} - \frac{w-z}{1-\overline{w}z}}{h}$$
$$= \lim_{h\to 0} \frac{w\overline{w}h - h}{[1-\overline{w}(h+z)][1-\overline{w}z]h} = \frac{w\overline{w} - 1}{(1-\overline{w}z)^2}$$

So $F(z)$ is continuous at each point in $\mathbb{D}$, that means F is holomorphic in the unit disc.

(ii) $F(0) = \frac{w-0}{1-\overline{w}\cdot 0} = w$, $F(w) = \frac{w-w}{1-w\overline{w}} = 0$

(iii) We already explain it in (a) that $F(\partial\mathbb{D}) \subseteq \mathbb{D}$.

(iv) $F \circ F = \frac{w - \frac{w-z}{1-\overline{w}z}}{1 - \frac{w-z}{1-\overline{w}z}} = z$, that means $F : \mathbb{D} \to \mathbb{D}$ is a identity mapping, so it is also a bijection.

Exercise 8 is also very easy, so we skip it and turn to Exercise 9.

Exercise 9. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad and \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

Use these equations to show that the logarithm function defined by

$$logz = logr + i\theta \qquad where \quad z = re^{i\theta} with - \pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.
Solution 9. A straight calculation can solve it.
We know that $z = x + iy$,$x = \rho\cos\varphi$,$y = \rho\sin\varphi$
then

$$f(z) = u(x,y) + iv(x,y), \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

so we have

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \rho} = \frac{\partial u}{\partial x}\cdot\cos\varphi + \frac{\partial u}{\partial y}\cdot\sin\varphi$$

Similarly, we also have

$$\frac{\partial v}{\partial \varphi} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \varphi} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \varphi} = \frac{\partial v}{\partial x}\cdot(-\rho\sin\varphi) + \frac{\partial v}{\partial y}\cdot\rho\cos\varphi$$

so we have

$$\frac{\partial u}{\partial \rho} = \frac{1}{\rho}\frac{\partial v}{\partial \varphi}$$

just continue to calculate,

$$\frac{\partial v}{\partial \rho} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \rho} = \frac{\partial v}{\partial x}\cdot\cos\varphi + \frac{\partial v}{\partial y}\cdot\sin\varphi$$

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \varphi} = \frac{\partial u}{\partial x}\cdot(-\rho\sin\varphi) + \frac{\partial u}{\partial y}\cdot\rho\sin\varphi$$

so we have

$$\frac{\partial v}{\partial \rho} = -\frac{1}{\rho}\frac{\partial u}{\partial \varphi}$$

which means

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

And we check it as followed,

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \frac{\partial v}{\partial \theta} = 1$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = 0, \frac{\partial v}{\partial r} = 0$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

so $f(z) = logz$ is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$

Exercise 10. Show that

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \triangle,$$

where $\triangle$ is the **Laplacian**

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Solution 10. To prove it, we shall begin with

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} \cdot \frac{\partial z}{\partial x} + \frac{\partial}{\partial \overline{z}} \cdot \frac{\partial \overline{z}}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z} \cdot \frac{\partial z}{\partial y} + \frac{\partial}{\partial \overline{z}} \cdot \frac{\partial}{\partial \overline{z}} \cdot \frac{\partial \overline{z}}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \overline{z}}$$

$$\frac{\partial}{\partial x^2} = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z \partial \overline{z}} + \frac{\partial^2}{\partial \overline{z}^2} + \frac{\partial^2}{\partial \overline{z} \partial z}$$

$$\frac{\partial}{\partial y^2} = -\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z \partial \overline{z}} - \frac{\partial^2}{\partial \overline{z}^2} + \frac{\partial^2}{\partial \overline{z} \partial z}$$

$$\triangle = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} = 2\frac{\partial^2}{\partial z \partial \overline{z}} + 2\frac{\partial^2}{\partial \overline{z} \partial z}$$

so lastly we get

$$\triangle = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}$$

Exercise 11. Use Exercise 10 to prove that if f is holomorphic in the open set Ω, then the real and imaginary parts of f are **harmonic**; that is, their Laplacian is zero.

Solution 11.

$$\triangle u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$$\triangle v = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial y \partial x}$$

$f = u + iv$ is a holomorphic function and u, v is continuous in the defined area. Thus,

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad , \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

So the real and imaginary parts of f is harmonic; that is, their Laplacian: $\triangle u, \triangle v$ is zero.

Exercise 12. Consider the function defined by

$$f\left(x + iy\right) = \sqrt{|x|\,|y|}, where \quad x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

Solution 12. We can easily get that $u = \sqrt{|x|\,|y|}$ and $v = 0$
v is continuously differentiable while u is differentiable but not continuously differentiable.
At the origin, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$, and as $v = 0$, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$, which means that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

Exercise 13. Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

(i) $Re\,(f)$ is constant
(ii) $Im\,(f)$ is constant
(iii) $|f|$ is constant

one can conclude that f is constant.
Solution 13. Let $f\,(z) = f\,(x, y) = u\,(x, y) + iv\,(x, y)$, where $z = x + iy$.

(i) Since $Re\,(f)$ is constant,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

as we know that f is holomorphic in an open set Ω, by the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0$$

then in Ω,

$$f'\,(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0$$

thus $f(z)$ is constant.

(ii) Since $Im\,(f)$ is constant,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

by the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = 0$$

Thus, in Ω,

$$f'\,(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0$$

thus $f(z)$ is constant.

(iii) We first give a mostly correct argument; the reader should pay attention to find the difficulty. Since $|f| = \sqrt{x^2 + y^2}$ is constant, we have

$$0 = \frac{\partial(u^2 + v^2)}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$
$$0 = \frac{\partial(u^2 + v^2)}{\partial x} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$$

Plug in the Cauchy-Riemann equations and we get

$$u\frac{\partial v}{\partial y}+v\frac{\partial v}{\partial x}=0$$
$$-u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=0$$

thus we have

$$\frac{\partial v}{\partial x}=\frac{v}{u}\frac{\partial v}{\partial y}$$
$$\frac{u^2+v^2}{u}\frac{\partial v}{\partial y}=0$$

which means $u^2+v^2=0$ or $\frac{\partial v}{\partial y}=0$. If $u^2+v^2=0$, then, since u,v are real, $u=v=0$, and thus $f=0$ which is constant. Thus we may assume u^2+v^2 equal a non-zero constant, and we may divide by it. We multiply both sides by u and find $\frac{\partial v}{\partial y}=0$, plug back in the early equation, we get $\frac{\partial v}{\partial x}=0$, by Cauchy-Riemann equations, $\frac{\partial u}{\partial x}=0$

$$f'(z)=\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=0$$

Thus f is constant.

Why is the above only mostly a proof? The problem is we have a division by u, and need to make sure everything is well-defined. Specifically, we need to know that u is never zero. We do have $f'=0$ except at points where $u=0$, but we would need to investigate that a bit more.

Let's turn back to

$$0=\frac{\partial(u^2+v^2)}{\partial x}=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}$$
$$0=\frac{\partial(u^2+v^2)}{\partial x}=2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}$$

Plug in the Cauchy-Riemann equations and we get

$$u\frac{\partial v}{\partial y}+v\frac{\partial v}{\partial x}=0$$
$$-u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=0$$

We multiply the first equation u and the second by v, and obtain

$$u^2\frac{\partial v}{\partial y}+uv\frac{\partial v}{\partial x}=0$$
$$-uv\frac{\partial v}{\partial x}+v^2\frac{\partial v}{\partial y}=0$$

Adding the two yields

$$u^2\frac{\partial v}{\partial y}+v^2\frac{\partial v}{\partial y}=0$$

or equivalently

$$(u^2 + v^2)\frac{\partial v}{\partial y} = 0$$

We now argue in a similar manner as before, except now we don't have the annoying u in the denominator. If $u^2 + v^2 = 0$ then $u = v = 0$, else we can divide by $u^2 + v^2 = 0$ and find $\frac{\partial v}{\partial y} = 0$. Arguing along these lines finishes the proof.

One additional remark by $\mathscr{Jeff\ \ Meng}$: we can trivially pass from results on patrials with respect to v to those with respect to u by noting that if $f = u + iv$ has constant magnitude, so too does $g = if = -v + iu$, which essentially switches the roles of u and v, Though this isn't needed for this problem, arguments such as this can be very useful.

The following is from $\mathscr{Steven\ \ Miller}$. Let's consider the proof. If $|f| = 0$ the problem is trivial as then $f = 0$, so we assume $|f|$ equals a non-zero constant. As $|f|$ is constant, $|f|^2 = f\overline{f}$ is constant. By the quotient rule, the ratio of two holomorphic functions is holomorphic, assuming the denominator is non-zero. We thus find $\overline{f} = \frac{|f|^2}{f}$ is holomorphic. Thus f and $\overline{f}$ are holomorphic, and satisfy the Cauchy-Riemann equations. Applying these to $f = ui + iv$ yields

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

while applying to $\overline{f} = u + i(-v)$ gives

$$\frac{\partial u}{\partial x} = \frac{\partial(-v)}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial(-v)}{\partial x}$$

Adding these equations together yields

$$2\frac{\partial u}{\partial x} = 0, 2\frac{\partial u}{\partial y} = 0$$

Thus u is constant, and by (i) implies that f is constant. If we didn't want to use that we could subtract rather than add, and similarly find that v is constant.

Exercise 14 and Exercise 15 is the proof of **summations by parts** and **Abel's theorem**, the solution is similar to Real Analysis, so we skip and just retell them.

Theorem 2. ***(summations by parts)*** *Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0 = 0$, then*

$$\sum_{n=M}^{N} a_n b_n = a_N b_N - a_M b_M - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

Theorem 3. ***(Abel's theorem)*** *Suppose $\sum_{n=1}^{\infty} a_n$ converges, then*

$$\lim_{r \to 1, r<1} \sum_{n=1}^{\infty} r^n a_n = \sum_{n=1}^{\infty} a_n$$

As to Exercise 16,17, we use Hadamard's formula and directly solve them. Actually, we can use Real Analysis to solve them, so I skip them.

Exercise 18. Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.
Solution 18. We suppose

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

and for any $z_0 \in \Omega$, then write $z = z_0 + (z - z_0)$, as we know that $f(z_0)$ converges, we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} a_n [z_0 + (z - z_0)]^n \\ &= \sum_{n=0}^{+\infty} a_n [\sum_{i=0}^{n} C_n^i z_0^{n-i} (z - z_0)^i]^n \\ &= \sum_{n=0}^{+\infty} b_n (z - z_0)^n \end{aligned}$$

where

$$b_n = \sum_{i=0}^{+\infty} C_{n+i}^n a_{n+i} z_0^i$$

and b_n actually also converges.

Exercise 19. Prove the following:

(i) The power series $\sum nz^n$ does not converge on any point of the unit circle.
(ii) The power series $\sum z^n/n^2$ converges at any point of the unit circle.
(iii) The power series $\sum z^n/n$ converges at every point of the unit circle except $z = 1$.
Solution 19.
(i) On the unit circle, we suppose $z = \cos\theta + i\sin\theta, \quad \theta \in [0, 2\pi)$, then $a_n = nz^n = n\cos\theta + in\sin\theta$

We know that $\lim_{n\to\infty} n\cos n\theta$ doesn't exist $\forall\theta \in (0, 2\pi)$. But when $\theta = 0$, we have $\lim_{n\to\infty} n\cos n\theta = +\infty$, so $\lim_{n\to+\infty} a_n$ doesn't exist too.

Now we can deduce that $\sum_{n=1}^{+\infty} a_n$ diverges, that is, The power series $\sum nz^n$ does not converge on any point of the unit circle.

(ii) On the unit circle, we suppose $z = \cos\theta + i\sin\theta, \quad \theta \in [0, 2\pi)$, then $\alpha_n = z^n/n^2 = \cos n\theta/n^2 + i\sin\theta/n^2 = a_n + ib_n$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty} a_n$ converges.

Similarly,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty} b_n$ converges.

In summary, $\sum_{n=1}^{\infty} \alpha$ also converges as to be proved above.

(iii) On the unit circle, we suppose $z = \cos\theta + i\sin\theta, \quad \theta \in [0, 2\pi)$, then $\alpha_n = z^n/n = \frac{\cos n\theta}{n} + \frac{i\sin n\theta}{n} = a_n + ib_n$

When $\theta = 0$, that is, when $z = 1$, $\sum_{n=1}^{+\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ obviously diverges.

When $\theta \in (0, 2\pi)$, $\sum \frac{\cos n\theta}{n}$ and $\sum \frac{\sin n\theta}{n}$ both converges by **Dirichlet discriminance**.

For the integrity of the proof, I will explain it briefly.

As can be easily proved, $\sum \cos n\theta$'s partial sum sequence is $\frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} - \frac{1}{2}$, which is bounded when $\theta \in (0, 2\pi)$, $\left\{\frac{1}{n}\right\}$ is monotonically decreasing and tending to zero, so $\sum \frac{\cos n\theta}{n}$ converges. Similarly, $\sum \frac{\sin n\theta}{n}$ also converges.

In summary, the power series $\sum z^n/n$ converges at every point of the unit circle except $z = 1$.

Exercise 20. Expand $(1-z)^{-m}$ in powers of z. Here m is a fixed positive integer. Also, show that if

$$(1-z)^{-m} = \sum_{n=0}^{\infty} a_n z^n$$

then one obtains the following asymptotic relation for the coefficients:

$$a_n \sim \frac{1}{(m-1)!} n^{m-1} \quad as \quad n \to \infty$$

Solution 20. Directly use **Taylor's formula** at zero, we get

$$(1-z)^{-m} = \sum_{n=0}^{\infty} a_n z^n$$

where

$$a_n = \frac{(m+n-1)!}{(m-1)!n!}$$

To prove

$$a_n \sim \frac{1}{(m-1)!} n^{m-1} \quad as \quad n \to \infty$$

we just need to prove

$$\frac{(m+n-1)!}{(m-1)!n!} \sim \frac{1}{(m-1)!} n^{m-1} \quad as \quad n \to \infty$$

that is, to prove

$$(n+1)\cdots(m+n-1) \sim n^{m-1}$$

The number of items in the left formula is actually $m-1$, thus we just need to prove

$$(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{m-1}{n})\sim 1$$

We already learn that m is a fixed positive integer and $n\to\infty$, thus the number of items in the left formula is limited, we can directly prove it.

Exercise 21. Show that for $|z|<1$, one has

$$\frac{z}{1-z^2}+\frac{z^2}{1-z^4}+\cdots+\frac{z^{2^n}}{1-z^{2^{n+1}}}+\cdots=\frac{z}{1-z}$$

and

$$\frac{z}{1+z}+\frac{2z^2}{1+z^2}+\cdots+\frac{2^k z^{2^k}}{1+z^{2^k}}+\cdots=\frac{z}{1-z}$$

Justify any change in the order of summation.

Solution 21. For the first formula, a calculation can solve it.

$$\begin{aligned}
&-\frac{z}{1-z}+\frac{z}{1-z^2}+\frac{z^2}{1-z^4}+\cdots+\frac{z^{2^n}}{1-z^{2^{n+1}}}+\cdots\\
&=\frac{-z^2}{1-z^2}+\frac{z^2}{1-z^4}+\cdots+\frac{z^{2^n}}{1-z^{2^{n+1}}}+\cdots\\
&=\frac{-z^4}{1-z^4}++\cdots+\frac{z^{2^n}}{1-z^{2^{n+1}}}+\cdots\\
&\cdots\\
&=\lim_{n\to\infty}\frac{z^{2^n}}{1-z^{2^n}}
\end{aligned}$$

for $|z|<1$, the formula equals 0, which means

$$\frac{z}{1-z^2}+\frac{z^2}{1-z^4}+\cdots+\frac{z^{2^n}}{1-z^{2^{n+1}}}+\cdots=\frac{z}{1-z}$$

As for the second formula, similarly,

$$\begin{aligned}
&-\frac{z}{1-z}+\frac{z}{1+z}+\frac{2z^2}{1+z^2}+\cdots+\frac{2^k z^{2^k}}{1+z^{2^k}}+\cdots\\
&=\frac{-2z^2}{1-z^2}+\frac{2z^2}{1+z^2}+\cdots+\frac{2^k z^{2^k}}{1+z^{2^k}}+\cdots\\
&=\frac{-2^2 z^4}{1-z^4}+\cdots+\frac{2^k z^{2^k}}{1+z^{2^k}}+\cdots\\
&\cdots\\
&=\lim_{n\to\infty}\frac{-2^n z^{2^n}}{1+z^{2^n}}
\end{aligned}$$

for $|z| < 1$, the formula equals 0, which means

$$\frac{z}{1+z} + \frac{2z^2}{1+z^2} + \cdots + \frac{2^k z^{2^k}}{1+z^{2^k}} + \cdots = \frac{z}{1-z}$$

There must be some concise solutions, but the solution here is very easy to come out.

Exercise 22. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of positive integers. A subset $S \subset \mathbb{N}$ is said to be in arithmetic progression if

$$S = \{a, a+d, a+2d, a+3d, \cdots\}$$

where $a, d \in \mathbb{N}$. Here d is called the step of S.

Show that $\mathbb{N}$ cannot be partitioned into a finite number of subsets that are in arithmetic progression with distinct steps (except for the trivial case $a = d = 1$).
Solution 22. Suppose $S_{a,d} = \{a, a+d, a+2d, a+3d, \cdots\}$

$$\sum_{n \in S_{a,d}} z^n = z^a + z^{a+d} + z^{a+2d} + \cdots + z^{a+nd} + \cdots = \lim_{n\to\infty} \frac{z^a}{1-z^d}\left(1 - z^{(n+1)d}\right)$$

let $|z| < 1$, we have

$$\sum_{n \in S_{a,d}} z^n = \frac{z^a}{1-z^d}$$

at the same time, we have

$$\sum_{n \in \mathbb{N}^*} z^n = \frac{z}{1-z}$$

To find a finite number of subsets with different elements and steps d,we can just pick by

$$1, 3, 5, 7, 9 \cdots ; 2, 6, 10, \cdots ; \cdots$$

so we can get

$$\begin{aligned} S_n &= \sum_{n \in S_{a_1,d_1}} z^n + \sum_{n \in S_{a_2,d_2}} z^n + \cdots + \sum_{n \in S_{a_n,d_n}} z^n \\ &= \frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^n}}{1-z^{2^{n+1}}} \end{aligned}$$

then

$$S_n + \frac{1}{1-z^{P2^{n+1}}} = \frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^{n-1}}}{1-z^{2^n}} + \frac{1+z^{2^n}}{1-z^{2^{n+1}}}$$

$$\therefore S_n = \frac{z}{1-z} - \frac{1}{1-z^{2^{n+1}}}$$

$$\therefore \lim_{n\to\infty} S_n = \frac{z}{1-z}$$

To any subsets with finite numbers, such as $S_{a_1,d_1}, S_{a_2,d_2}, \cdots, S_{a_n,d_n}$

$$\sum S_{a_i,d_i} < \lim_{n\to\infty} S_n = \frac{z}{1-z}$$

So to subsets with finite numbers, if a and d cannot be 1 at the same time, it's impossible that $\sum S_{a_i,d_i} = \frac{z}{1-z}$.Q.E.D.

Exercise 23. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & if \quad x \leq 0 \\ e^{-1/x^2} & if \quad x > 0 \end{cases}$$

Prove that f is indefinitely differentiable on $\mathbb{R}$, and that $f^{(n)}(0) = 0$ for all $n \geq 1$.

Conclude that f does not have a converging power series expansion $\sum_{n=0}^{\infty} a_n x^n$ for x near the origin.

Solution 23. When $x \in (-\infty, 0]$, $f^{(0)}(x) = 0$ is obvious. So it's derivable of any order and

$$\lim_{x\to 0^-} f^{(n)}(x) = 0$$

When $x \in (0, +\infty)$, $n = 1$, $f'(x) = \frac{-2}{x^3} e^{-\frac{1}{x^2}}$ Use **mathematical methods of induction**, then suppose when $n = k$,

$$f^{(k)}(x) = e^{-\frac{1}{x^2}} \cdot P(x), \quad P(x) = \sum C_i \frac{1}{x^i}$$

then when $n = k + 1$, we can deduce that

$$\begin{aligned} f^{(k+1)}(x) &= e^{-\frac{1}{x^2}} \cdot P'(x) - \frac{2}{x^3} \\ &= e^{-\frac{1}{x^2}} \cdot \left[P'(x) - \frac{2}{x^3} P(x) \right] \\ &= e^{-\frac{1}{x^2}} \cdot Q(x) \end{aligned}$$

where

$$Q(x) = \sum C_i' \frac{1}{x^i}$$

and

$$\lim_{n\to\infty} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0$$

Obviously, $f(x)$'s derivable of any order of k can be expressed as $e^{-\frac{1}{x^2}} \cdot P(x)$, where $P(x)$ is a polynomial function of x. And when $x \in (0, +\infty)$, $P(x)$ is derivable everywhere, so $f(x)$ can be derived at any order.

Then in a small neighborhood $(0, \delta)$, $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot (-\frac{1}{x^2})^n$.

As to

$$\lim_{n\to\infty, x\to 0} \frac{1}{n!} \cdot (-\frac{1}{x^2})^n$$

$\forall n, M > 0$, we can find x, when x satisfies

$$x < min\left\{ \sqrt[2n]{\frac{1}{n!M}}, \delta \right\}$$

we have

$$\left| \frac{1}{n!} \cdot (-\frac{1}{x^2})^n \right| > M$$

so the series does not converge near the origin.

Exercise 25. The next three calculations provide some insight into Cauchys theorem, which we treat in the next chapter.

(i) Evaluate the integrals

$$\int_\gamma z^n dz$$

for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

(ii) Same question as before, but with γ any circle not containing the origin.

(iii) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

Solution 25.

(i) Suppose $z = re^{i\theta}$, $\theta \in [0, 2\pi)$, then we have

$$\begin{aligned}
\int_\gamma z^n dz &= \int_\gamma r^n e^{i\theta n} dre^{i\theta} \\
&= r^{n+1} \int_0^{2\pi} ie^{i\theta(n+1)} d\theta \\
&= \frac{r^{n+1}}{n+1} \cdot \int_0^{2\pi} de^{i(n+1)\theta} \\
&= \frac{r^{n+1}}{n+1} \cdot \left[e^{i(n+1)2\pi} - 1 \right] \\
&= 0
\end{aligned}$$

(ii) Suppose $z = re^{i\theta} + ke^{i\varphi}$, $\theta, \varphi \in [0, 2\pi)$. φ, r, k here are all constant.

Then

$$\int_\gamma z^n dz = \frac{z^{n+1}}{n+1}\Big|_\gamma = \frac{(re^{i\theta}+ke^{i\varphi})^{n+1}}{n+1}\Big|_0^{2\pi}$$
$$= \frac{(re^{i2\pi}+ke^{i\varphi})^{n+1}}{n+1} - \frac{(r+ke^{i\varphi})^{n+1}}{n+1}$$
$$= \frac{(r+ke^{i\varphi})^{n+1}}{n+1} - \frac{(r+ke^{i\varphi})^{n+1}}{n+1}$$
$$= 0$$

(iii)

$$\int_\gamma \frac{1}{(z-a)(z-b)}dz = \frac{1}{a-b}\int_\gamma \left(\frac{1}{z-a} - frac1z-b\right)dz$$

equals

$$\int_\gamma \frac{1}{z-a}dz - \int_\gamma \frac{1}{z-b}dz = 2\pi i$$

let

$$z' = z-a = re^{i\theta} - a = r'e^{i\theta'}$$

$$\int_\gamma \frac{1}{z-a}d(z-a) = \int_\gamma \frac{1}{z'}dz'$$
$$= \int_{\gamma \frac{ir'e^{i\theta'}}{r'e^{i\theta'}}} d\theta'$$
$$= i\theta\Big|_{\arcsin\frac{r-a}{r'}}^{2\pi+\arcsin\frac{r-a}{r'}}$$
$$= 2\pi i$$

here θ from 0 to 2π, θ' from $\arcsin\frac{r-a}{r'}$ to $2\pi + \arcsin\frac{r-a}{r'}$

We know that $\int_\gamma \frac{1}{z-b}dz$ is analytic,

$$\therefore \int_\gamma \frac{1}{z-b}dz = 0$$

In summary,

$$\int_\gamma \frac{1}{(z-a)(z-b)}dz = \frac{1}{a-b}\int_\gamma \left(\frac{1}{z-a} - frac1z-b\right)dz = \frac{2\pi i}{a-b}$$

Q.E.D. Or we have another concrete proof, I'll show you behind.

Firstly we calculate $\int_\gamma \frac{1}{z-a}dz$.

$$\int_\gamma \frac{1}{z-a}dz = \int_\gamma \frac{d(z-a)}{z-a} = \int_C \frac{r_m d(e^{i\theta'}) + d(r_m)\cdot^{i\theta'}}{r_m e^{i\theta'}}$$

$$z - a = r_m e^{i\theta'}$$
$$r_m^2 = r_a^2 + r^2 - 2rr_a \cos(\theta - \theta_a)$$

thus we have

$$\begin{aligned}
&= \int_C \frac{ir_m e^{i\theta'} d\theta' + \frac{rr_a \sin(\theta-\theta_a)+r_m\cdot e^{i\theta'}}{r_m} d\theta}{r_m e^{i\theta'}} \\
&= \int_0^{2\pi} id\theta + rr_a \int_0^{2\pi} \frac{\sin(\theta-\theta_a)}{r_m^2} d\theta \\
&= 2\pi i + rr_a \int_0^{2\pi} \frac{\sin(\theta-\theta_a)}{r_a^2 + r^2 - 2rr_a\cos(\theta-\theta_a)} d\theta \\
&= 2\pi i + 2 \int_0^{2\pi} \frac{d[r_a^2 + r^2 - 2rr_a\cos(\theta-\theta_a)]}{r_a^2 + r^2 - 2rr_a\cos(\theta-\theta_a)} \\
&= 2\pi i + 2ln \left| r_a^2 + r^2 - 2rr_a\cos(\theta-\theta_a) \right| \Big|_0^{2\pi} \\
&= 2\pi i
\end{aligned}$$

Similarly, we calculate $\int_\gamma \frac{1}{z-b} dz$ then, and we get

$$\int_\gamma \frac{1}{z-b} dz = 0$$

Exercise 26. Suppose f is continuous in a region Ω. Prove that any two primitives of f (if they exist) differ by a constant.
Solution 26. We suppose F, G are any two primitive functions of f. Suppose $F = F - G$, then

$$[H]' = [F - G]' = F' - G' = f - f = 0$$

we may suppose $H = u + iv$, because f is continuous, so F, G is analytic, therefore H is also analytic.

$$\therefore \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$$

by Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$$

$$\therefore u = c_1, v = c_2, \quad c_1, c_2 \quad are \quad constants.$$

$$\therefore F - G = c_1 + ic_2$$

Q.E.D.

$$\omega = \frac{i}{z} = \frac{i}{x+iy} = \frac{x}{x^2+y^2} \cdot i + \frac{y}{x^2+y^2}$$

$$\omega = x' + iy'$$

by comparison,

$$x' = \frac{y}{x^2+y^2}$$

$$y' = \frac{x}{x^2+y^2}$$

And

$$x^2 + y^2 - 2y = 0$$

we get

$$x' = \frac{1}{2}$$