Chapter2
Cauchy's Theorem and Its Applications

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Abstract

This chapter focuses on Cauchy's Theorem and some related integral calculation. I will help you solve serval exercises in this chapter, there may be some spelling mistakes or even wrong methods.

The solution of a large number of problems can be reduced, in the last analysis, to the evaluation of definite integrals; thus mathematicians have been much occupied with this task... However, among many results obtained, a number were initially discovered by the aid of a type of induction based on the passage from real to imaginary. Often passage of this kind led directly to remarkable results. Nevertheless this part of the theory, as has been observed by Laplace, is subject to various difficulties... After having reflected on this subject and brought together various results mentioned above, I hope to establish the passage from the real to the imaginary based on a direct and rigorous analysis; my researches have thus led me to the method which is the object of this memoir...

A. L. Cauchy, 1827

Exercise 1. Prove that

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

These are the **Fresnel integrals**. Here, $\int_0^\infty$ is interpreted as $\lim_{R\to\infty}\int_0^R$.

Solution 1. Let $f(z) = e^{iz^2}$. We integrate $f(z)$ around a circular sector of radius R running from $\theta = 0$ to $\frac{\pi}{4}$ just as the figure showed.

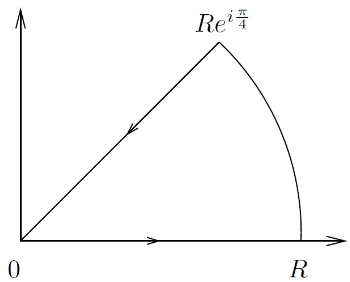

Figure 1: The contour in Exercise 1

Along the x axis the integral is

$$\int_0^R e^{ix^2}dx$$

Along the curved part we have $z = Re^{i\theta}$ and the integral is

$$\int_0^{\frac{\pi}{4}} e^{iR^2e^{2i\theta}} iRe^{i\theta}d\theta = iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}\cdot e^{i[\theta+R^2\cos 2\theta]}d\theta$$

Finally, along the segment at angle $\frac{\pi}{4}$ we have $z = re^{i\pi/4}$ and the integral is

$$\int_R^0 e^{ir^2e^{i\pi/2}}e^{i\pi/4}dr$$

The total integral is zero since f is analytic everywhere. As $R\to\infty$, the integral over the third piece approaches

$$-e^{i\pi/4}\int_0^\infty e^{-x^2}dx = -e^{i\pi/4}\cdot\frac{\sqrt{\pi}}{2} = -\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i$$

To estimate the integral over the curved piece, we shall take its absolute value, and we use that when $0 \leq \varphi \leq \pi/2$, we have $\sin\varphi \geq \frac{\varphi}{\pi/2}$

$$\begin{aligned}
&\left| iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}\cdot e^{i[\theta+R^2\cos 2\theta]}d\theta \right| \\
&\leq R\int_0^{\frac{\pi}{4}} \left| e^{-R^2\sin 2\theta}\cdot e^{i[\theta+R^2\cos 2\theta]} \right| d\theta \\
&= R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}d\theta \\
&\leq R\int_0^{\frac{\pi}{4}} e^{-4R^2\theta/\pi}d\theta \\
&= -\frac{\pi}{4R}e^{-4R^2\theta/\pi}\Big|_0^{\frac{\pi}{4}} \\
&= \frac{\pi(1-e^{-R^2})}{4R}
\end{aligned}$$

As $R \to \infty$, this result approaches zero and we have

$$\int_0^\infty e^{ix^2}dx - \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i = 0$$

Take real and imaginary parts, we have

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

Exercise 2. Show that

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$$

Solution 2. Here we consider the function $f(z) = \frac{e^{iz}}{z}$, around an indented semi-circular contour bounded by circles of radius ϵ and R in the upper half plane.

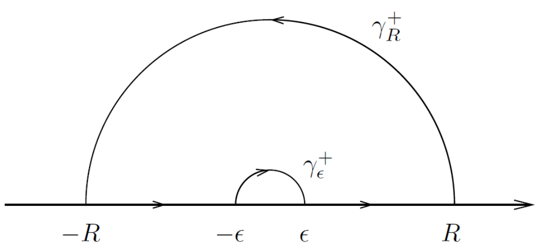

Figure 2: The contour in Exercise 2

The integrals along the two portions of the real axis add up to

$$\int_{-R}^{-\epsilon} \frac{\cos x + i\sin x}{x} dx + \int_{\epsilon}^{R} \frac{\cos x + i\sin x}{x} dx = 2i \int_{\epsilon}^{R} \frac{\sin x}{x} dx$$

because cosine is even and sine is odd. The integral around the arc of radius R tends to zero as $R \to \infty$, by the Jordan lemma; since this lemma isn't mentioned in the book, here's a proof for this specific case:

One the arc, $z = Re^{i\theta}$ so the integral is

$$\int_0^{\pi} \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta} d\theta = i \int_0^{\pi} e^{-R\sin\theta} e^{iR\cos\theta} d\theta$$

we take its absolute value

$$\begin{aligned} \left| i \int_0^{\pi} e^{-R\sin\theta} e^{iR\cos\theta} d\theta \right| &\leq \int_0^{\pi} \left| e^{-R\sin\theta} e^{iR\cos\theta} \right| d\theta \\ &= \int_0^{\pi} e^{-R\sin\theta} d\theta \\ &= 2\int_0^{\pi/2} e^{-R\sin\theta} d\theta \end{aligned}$$

Now as we have shown before in Exercise 1, we have $\sin\theta \geq \frac{\theta}{\pi/2}$ when $0 \leq \theta \leq \pi/2$, thus we continue to estimate its value.

$$2\int_0^{\pi/2} e^{-R\sin\theta} d\theta \leq 2\int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = -\frac{\pi e^{-2R\theta/\pi}}{R}\Big|_0^{\pi/2} = \frac{\pi(1-e^{-R})}{R}$$

which tends to 0 as $R \to \infty$

Finally, the integral over the inner semicircle tends to $-\pi i$, which can be easily proved and I'll show you behind.

As we know, $\frac{e^{iz}}{z} = \frac{1}{z} + O(1)$ as $z \to 0$, and since the length of the semicircle is tending to zero, the integral over it approaches the integral of $\frac{1}{z}$ over it, which is

$$\int_{\pi}^{0} \frac{1}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta = -\int_0^{\pi} i d\theta = -\pi i$$

Putting the pieces together and letting $R \to \infty$ and $\epsilon \to 0$, we have

$$2i \int_0^{\infty} \frac{\sin x}{x} dx - \pi i = 0 \Rightarrow \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Exercise 3. Evaluate the integrals

$$\int_0^{\infty} e^{-ax}\cos bx dx \quad and \quad \int_0^{\infty} e^{-ax}\sin bx dx, \quad a > 0$$

by integrating e^{-Az}, $A = \sqrt{a^2+b^2}$, over an appropriate sector with angle ω, with $\cos\omega = a/A$.

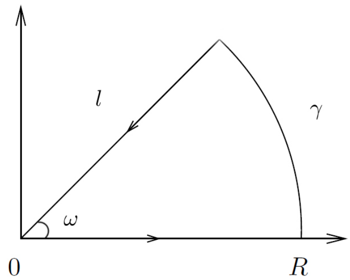

Figure 3: The contour in Exercise 3

Solution 3. We integrate $f(z) = e^{-Az}$, where $A = \sqrt{a^2+b^2}$ around a circular sector of radius R running from $\theta = 0$ to ω, with $\omega = a/A$ just as the figure showed.

$$\int_0^R e^{-Ax} dx + \int_\gamma e^{-Az} dz + \int_{l^-} e^{-Az} dz = 0$$

For the first piece, we can directly calculate it

$$\int_0^R e^{-Ax} dx = -\frac{1}{A} e^{-Ax} \Big|_0^R$$

when $R \to \infty$, we know it approaches $\frac{1}{A}$.
For the second piece,

$$|e^{-Az}| = \frac{1}{e^{AR|e^{i\theta}|}} = \frac{1}{e^{AR}}$$

$$\therefore |\int_\gamma f(z)dz| \leq \sup_{z\in\gamma} |f(z)| \cdot length\gamma = \frac{1}{e^{AR}} \cdot R\omega$$

when $r \to \infty$, it approaches 0.

Lastly, for the third piece, we plug $z = re^{i\omega}$ in it and we have

$$\begin{aligned} -\int_{l^-} e^{-Az} dz &= \int_0^R e^{-Are^{i\omega}} \cdot e^{i\omega} dr \\ &= \int_0^R e^{-Ar(\cos\omega + i\sin\omega)} e^{i\omega} dr \\ &= \int_0^R e^{-Ar\cos\omega} e^{-iAr\sin\omega} e^{i\omega} dr \end{aligned}$$

let $A = \sqrt{a^2+b^2}, \cos\omega = \frac{a}{\sqrt{a^2+b^2}}, \sin\omega = \frac{b}{\sqrt{a^2+b^2}}$, thus we know

$$\begin{aligned}
&= \int_0^R e^{i\omega} \cdot e^{-ar} \cdot e^{-ibr} dr \\
&= e^{i\omega} \int_0^R e^{-ar} \cdot [\cos br - i \sin br] dr \\
&= (\cos\omega + i\sin\omega) \left[\int_0^R e^{-ar} \cos br dr - i \int_0^R e^{-ar} \sin br dr \right]
\end{aligned}$$

Letting $R \to \infty$, thus we have

$$\lim_{R\to\infty} (\cos\omega + i\sin\omega) \left[\int_0^R e^{-ar} \cos br dr - i \int_0^R e^{-ar} \sin br dr \right] = \frac{1}{A}(a+bi)[I_1 - iI_2]$$

while $I_1 = \int_0^R e^{-ar} \cos br dr$ and $I_2 = \int_0^R e^{-ar} \sin br dr$, putting them together, we have

$$\frac{1}{A}(a+bi)[I_1 - iI_2] = \frac{1}{A}$$

thus we have

$$\begin{cases} aI_1 + bI_2 = 1 \\ aI_2 = bI_1 \end{cases}$$

We calculate it and we get

$$\begin{cases} \int_0^\infty e^{-ax} \cos bx dx = \frac{a}{a^2+b^2} \\ \int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2+b^2} \end{cases}$$

Exercise 4. Prove that for all $\xi \in \mathbb{C}$ we have

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx$$

Solution 4. Let $\xi \in a+bi$ with $a, b \in \mathbb{R}$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx &= \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x(a+bi)} dx \\
&= \int_{-\infty}^{\infty} e^{-\pi(x^2-2bx)} e^{-2\pi i a x} dx \\
&= e^{\pi b^2} \int_{-\infty}^{\infty} e^{-\pi(x-b)^2} e^{-2\pi i a x} dx \\
&= e^{\pi b^2} e^{-2\pi i a b} \int_{-\infty}^{\infty} e^{-\pi(x-b)^2} e^{-2\pi i a(x-b)} dx \\
&= e^{\pi b^2} e^{-2\pi i a b} \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi i a u} du \\
&= e^{\pi b^2} e^{-2\pi i a b} e^{-\pi a^2} \\
&= e^{-\pi(a+bi)^2} \\
&= e^{-\pi \xi^2}.
\end{aligned}$$

we use the fact that if $\xi \in \mathbb{R}$, then

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi} dx$$

which you can check it in **Complex Analysis** on page 42.

Exercise 5. Suppose f is continuously complex differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to show that

$$\int_T f(z) dz = 0$$

This provides a proof of Goursat's theorem under the additional assumption that f' is continuous.

Solution 5. Write $f(z)$ as $f(x, y) = u(x, y) + iv(x, y)$ where u, v are real-valued and $z = x + iy$. Then $dz = dx + idy$ so

$$\begin{aligned}
\oint_T f(z) dz &= \oint_T [u(x,y) + iv(x,y)] (dx + idy) \\
&= \oint_T udx - vdy + i \oint_T vdx + udy \\
&= \iint \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + i \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \\
&= 0
\end{aligned}$$

by the Cauchy-Riemann equations.

Exercise 6. Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point ω inside T. Prove that if f is bounded near ω, then

$$\int_T f(z)dz = 0$$

Solution 6. Let γ_ϵ be a circle of radius ϵ centered at ω, where ϵ is sufficiently small that γ_ϵ lies within the interior of T. Since f is holomorphic in the region R between T and γ_ϵ,

$$\int_{\partial R} f(z)dz = \int_T f(z)dz - \int_{\gamma_\epsilon} f(z)dz = 0$$

which means that

$$\int_T f(z)dz = \int_{\gamma_\epsilon} f(z)dz$$

But f is bounded near ω and the length of γ_ϵ goes to 0 as $\epsilon \to 0$, so $\int_{\gamma_\epsilon} f(z)dz \to 0$ and therefore $\int_T f(z)dz = 0$.
Actually, if we're not allowed to use Cauchy's theorem for a region bounded by two curves, one can use a "keyhole contour" instead; the result is the same.

Exercise 7. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = sup_{z,\omega\in\mathbb{D}}|f(z) - f(\omega)|$ of the image of f satisfies

$$2|f'(0)| \leq d$$

Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.
Solution 7. By the Cauchy derivative formula, we have

$$f'(0) = \frac{1}{2\pi i}\oint_{C_r} \frac{f(\zeta)}{(\zeta)^2}d\zeta$$

where C_r is the circle of radius r centered at zero with $0 < r < 1$. Substituting $-\zeta$ for ζ and adding the two equations yields, we have Then we take its absolute value and get

$$\begin{aligned}
|2f'(0)| &= \left|\frac{1}{2\pi i}\oint_{C_r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2}d\zeta\right| \\
&\leq \frac{1}{2\pi}\oint_{C_r} \left|\frac{f(\zeta) - f(-\zeta)}{\zeta^2}\right| d\zeta \\
&= \frac{1}{2\pi}\oint \frac{|f(\zeta) - f(-\zeta)|}{r^2}d\zeta \\
&\leq \frac{1}{2\pi}\oint \frac{M_r}{r^2}d\zeta \\
&= \frac{M_r}{r} \\
&\leq \frac{d}{r}
\end{aligned}$$

where

$$M_r = \sup_{|\zeta|=1} |f(\zeta) - f(-\zeta)|$$

Letting $r \to 1$ yields the desired result.

Exercise 8. If f is a holomorphic function on the strip $-1 < y < 1$, $x \in \mathbb{R}$ with

$$|f(z)| \leq A\,(1 + |z|)^{\eta}\,, \quad \eta \; is\; a\; fixed\; real\; number$$

for all z in that strip, show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that

$$\left|f^{(n)}(x)\right| \leq A_n\,(1 + |x|)^{\eta}\,, \quad for\; all\; x \in \mathbb{R}$$

Solution 8. Using the Cauchy inequalities to solve it.
For any x, consider a circle C centered at x of radius $\frac{1}{2}$, for $z \in C$,

$$1 + |z| \leq 1 + |x| + |z - x| \leq \frac{3}{2} + |x| < 2(1 + |x|)$$

So

$$|f(z)| \leq A(1 + |z|)^{\eta} \leq A2^{\eta}(1 + |x|)^{\eta}$$

When we suppose $\|f\|_C = \sup_{z\in C} |f(z)|$, we have $\|f\|_C \leq A2^{\eta}(1 + |x|^{\eta})$, thus by Cauchy inequalities,

$$\begin{aligned} f^{(n)}(x) &\leq \frac{n!\|f\|_C}{(\frac{1}{2})^n} \\ &\leq n!2^n A2^{\eta}(1 + |x|)^{\eta} \\ &= A_n(1 + |x|)^{\eta} \end{aligned}$$

with $A_n = n!2^n A2^{\eta}$

Exercise 9. Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\varphi(z_0) = z_0 \quad and \quad \varphi'(z_0) = 1$$

then φ is linear.
Solution 9. If we let $f(z) = \varphi(z + z_0) - z_0$ for $z \in \Omega - z_0$, we have $f(z) \in \Omega - z_0$ and f is linear if φ is. Thus, we may assume that $z_0 = 0$. Expanding in a power series around 0, we have $\varphi(z) = z + a_2 z^2 + \cdots$
Suppose a_n is the first nonzero coefficient with $n > 1$. Then $\varphi(z) = z + a_n z^n + O(z^{n+1})$, if $\varphi_k(z) = \varphi \circ \cdots \circ \varphi$, then as we have proved the case that $k = 1$, we use the mathematical induction to prove that $\varphi_k(z) = z + ka_n z^n + O(z^{n+1})$. If it's true for k, it follows that

$$\begin{aligned} \varphi_{k+1}(z) &= \left(z + ka_n z^n + O(z^{n+1})\right) + a_n \left(z + ka_n z^n + O(z^{n+1})\right)^n \\ &+ O\left(\left(z + ka_n z^n + O(z^{n+1})\right)^{n+1}\right) \\ &= z + (k+1)a_n z^n + O(z^{n+1}) \end{aligned}$$

Now we shall let $r > 0$ and $z \in \Omega$ for $|z| \le r$. By the Cauchy inequalities,

$$\left|\varphi_k^{(n)}(0)\right| \le \frac{n!\|\varphi_k\|_r}{r^n}$$

where

$$\|\varphi_k\|_r = \sup_{|z|=r} |\varphi_k(z)|$$

But $\varphi_k(z) \in \Omega$ is bounded, so there must have a constant independent of n and k, which we call it M. And we simply calculate $\varphi_k^{(n)}$, $\varphi_k(z) = z + ka_nz^n + O(z^{n+1})$, we have $\varphi_k^{(n)} = kn!a_n$. Then we plug them in the pervious formula, we have

$$kn!a_n \le \frac{n!M}{r^n} \Rightarrow a_n \le \frac{M}{kr^n}$$

for all k. Letting $k \to \infty$, we have $a_n = 0$. Thus, there cannot be no nonzero terms of order $n > 1$ in the power series expansion of φ, which means that φ is linear.

Exercise 11. Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

(i) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi$$

(ii) Show that

$$Re\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}$$

Solution 11.

(i) We may start with RHS,

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi \\
&= \frac{1}{4\pi}\int_0^{2\pi} f(Re^{i\varphi})\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z} + \frac{Re^{-i\varphi}+\overline{z}}{Re^{-i\varphi}-\overline{z}}\right)d\varphi \\
&= \frac{1}{4\pi}\int_0^{2\pi} f(Re^{i\varphi})\left(2\frac{Re^{i\varphi}}{Re^{i\varphi}-z} - 1 + 1 - 2\frac{\overline{z}}{\overline{z}-Re^{-i\varphi}}\right)d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{Re^{i\varphi}d\varphi}{Re^{i\varphi}-z} - \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{\overline{z}d\varphi}{\overline{z}-Re^{-i\varphi}} \\
&= \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\varphi})\frac{d(Re^{i\varphi})}{Re^{i\varphi}-z} - \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{Re^{i\varphi}d\varphi}{Re^{i\varphi}-R^2/\overline{z}} \\
&= \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\varphi})\frac{d(Re^{i\varphi})}{Re^{i\varphi}-z} - \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\varphi})\frac{d(Re^{i\varphi})}{Re^{i\varphi}-R^2/\overline{z}}
\end{aligned}$$

By the Cauchy integral formula, the first part is equal to $f(z)$ and the latter is equal to zero as we replace $Re^{i\varphi}$ by ζ, we have

$$\frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\varphi})\frac{d(Re^{i\varphi})}{Re^{i\varphi}-R^2/\overline{z}} = \frac{1}{2\pi i}\int_0^{2\pi} f(\zeta)\frac{d\zeta}{\zeta - R^2/\overline{z}}$$

since it is analytic on and inside the circle of radius R.

(ii) A straightforward calculation can solve it.

$$\begin{aligned}
Re\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) &= Re\left(\frac{R\cos\gamma + r + iR\sin\gamma}{R\cos\gamma - r + iR\sin\gamma}\right)\\
&= Re\left(\frac{[R\cos\gamma + r + iR\sin\gamma][R\cos\gamma - r - iR\sin\gamma]}{[R\cos\gamma - r]^2 + R^2\sin^2\gamma}\right)\\
&= Re\left(\frac{R^2\cos^2\gamma - r^2 + R^2\sin^2\gamma}{R^2 - 2Rr\cos\gamma + r^2}\right)\\
&= \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}
\end{aligned}$$

Exercise 12. Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\Delta u(x,y) = 0$$

for all $(x,y)\in\mathbb{D}$.

(i) Prove that there exists a holomorphic function f on the unit disc such that

$$Re(f) = u$$

Also show that the imaginary part of f is uniquely defined up to an additive (real) constant.

(ii) Deduce from this result, and from Exercise 11, the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in the unit disc and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\varphi)u(\varphi)$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma) = \frac{1-r^2}{1-2r\cos\gamma + r^2}$$

Solution 12.

(i) For quadratic differentiable real valued function u, use the proof of path independence based on Green's formula, we suppose

$$P(x,y) = -\frac{\partial u}{\partial y}, Q(x,y) = \frac{\partial u}{\partial x}$$

Thus

$$\frac{\partial P}{\partial y} = -\frac{\partial^2 u}{\partial y^2}, \frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

as

$$\triangle u(x, y) = 0, \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

that means

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Thus in the unit disc $\mathbb{D}$, letting $v(x, y)$ satisfies

$$dv = Pdx + Qdy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

thus

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

which satisfy the Cauchy-Riemann equations, thus $f(z) = u(x, y) + iv(x, y)$, $f(z)$ is holomorphic and $Re(f) = u(x, y)$, v is determined by u. If c is a constant, we have $d(v + c) = dv$.

(ii) By Exercise 11, we have

$$u(z) + iv(z) = \frac{1}{2\pi}\int_0^{2\pi} \left[u(e^{i\varphi}) + iv(e^{i\varphi})\right] Re\left(\frac{e^{i\varphi} + z}{e^{i\varphi} - z}\right) d\varphi$$

so

$$\begin{aligned}
u(re^{i\theta}) &= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})Re\left(\frac{e^{i\varphi} + re^{i\theta}}{e^{i\varphi} - re^{i\theta}}\right) d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})Re\left(\frac{e^{i(\varphi-\theta)} + r}{e^{i(\varphi-\theta)} - r}\right) d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi)u(e^{i\varphi})d\varphi
\end{aligned}$$

Exercise 13. Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n \left(z - z_0\right)^n$$

is equal to 0. Prove that f is a polynomial.

Solution 13. There is a lemma remain to be proved to solve this Exercise.

Lemma 1. *Let $S \subset \mathbb{C}$ be a subset of the plane with no accumulation points. Then S is at most countable.*

Proof. For each $x \in S$, since x is not an accumulation point of S, $\exists r_x > 0$ such that $B_{r_x} \cap S = \{x\}$. Then $\left\{B_{r_x/2}(x) : x \in S\right\}$ is a disjoint family of open sets; since each contains a distinct rational point, it is at most countable. But this set is bijective with S, so S is at most countable.

Now suppose that f is not a polynomial. Then none of its derivatives can be identically zero, because if $f^{(n)}$ were identically zero, then $f^{(k)}$ would be zero for $k \geq n$ and f would be a polynomial of degree $\geq n-1$. Since the derivatives of f are entire functions that are not everywhere zero, the set of zeros of $f^{(n)}$ has no accumulation points, so it is at most countable by the lemma. The set of zeros of any derivative of f must then be countable since it is a countable union of countable sets. But by hypothesis, every point $z \in \mathbb{C}$ is a zero of some derivative of f, since if $f(z) = \sum c_n(z-z_0)^n$ and $c_k = 0$, then $\frac{d^k}{dz^k} f(z)\Big|_{z_0} = 0$. Since $\mathbb{C}$ is uncountable, this is a contradiction, so f must be a polynomial.

Exercise 14. Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$$

Solution 14. By replacing z with z/z_0, we may assume that $z_0 = 1$. Now let Ω be an open set containing $\overline{\mathbb{D}}$ such that f is holomorphic on Ω except for a pole at 1. Then

$$g(z) = f(z) - \sum_{i=1}^{N} \frac{a_{-j}}{(z-1)^j}$$

is holomorphic on Ω for some N and $a_{-1}, \cdots, a_{-N}$, where N is the order of the pole at 1. Next, we note that Ω must contain some disk of radius $1+\delta$ with $\delta > 0$: the set $\{z : |z| \leq 2\} \setminus \Omega$ is compact, so its image under the map $z \mapsto |z|$ is also compact and hence attains a lower bound, which must be strictly greater than 1 since the unit circle is contained in Ω. Now since g converges on the disc $|z| < 1+\delta$, we can expand it in a power series $\sum_{n=0}^{\infty} b_n z^n$ on this disk, and we must have $b_n \to 0$.(This follows from the fact that $\lim_{n\to\infty} \sup \frac{b_{n+1}}{b_n} < 1$ when the radius of convergence is greater than 1.) Now for $|z| < 1$, we have

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{j=1}^{N} \frac{a_{-j}}{(z-1)^j} + \sum_{n=0}^{\infty} b_n z^n$$

Using the fact that

$$\begin{aligned}\frac{1}{(z-1)^j} &= \frac{(-1)^{j-1}}{(j-1)!}\frac{d^{j-1}}{dz^{j-1}}\frac{1}{z-1}\\ &= \frac{(-1)^j}{(j-1)!}\sum_{s=0}^{\infty}(s+j-1)!z^s \quad for\ |z|<1\end{aligned}$$

we can write

$$\begin{aligned}&\sum_{n=0}^{\infty} a_n z^n = \sum_{s=0}^{\infty}\left[\sum_{j=1}^{N}\frac{(-1)^j a_{-j}}{(j-1)!}(s+j-1)!\right] z^s + \sum_{n=0}^{\infty} b_n z^n\\ &\Rightarrow a_n = P(n) + b_n\end{aligned}$$

where $P(n)$ is a polynomial in n of degree at most $N-1$. Here the rearrangements of the series are justified by the fact that all these series converge uniformly on compact subsets of $\mathbb{D}$. Since $b_n \to 0$, $\frac{a_n}{a_{n+1}} \to \lim \frac{P(n)}{P(n+1)} = 1$. (Every polynomial P has the property that $\frac{P(n)}{P(n+1)} \to 1$ since if the leading coefficient is $c_k n^k$, $\frac{P(n)}{P(n+1)} \approx \frac{cn^k}{c(n+1)^k} = \left(1-\frac{1}{n+1}\right)^k \to 1$)

Or if we want to explain it more clearly, because that $\sum_{n=0}^{\infty} a_n z^n$ is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle, we suppose that $z_0 = e^{i\theta}$ and $|z_0| = 1$.
Thus, without loss of generality, we can suppose that there exists a pole at $z_0 = 1$ with the order of N, which means that

$$f(z) = \frac{b_N}{(1-z)^N} + \frac{b_{N-1}}{(1-z)^{N-1}} + \cdots + \frac{b_1}{1-z} + G(z)$$

and $G(z)$ is holomorphic in the neighborhood of z_0, thus $G(z) = \sum_{n=0}^{\infty} c_n z^n$ holds for any z in the unit disc.

$$f(z) = \sum_{n=0}^{\infty}\left(\sum_{m=1}^{N} b_m \binom{n+m-1}{m-1} + c_n\right) z^n, \ \lim_{n\to+\infty} c_n = 0$$

$$\Rightarrow a_n = \sum_{m=1}^{N} b_m \binom{n+m-1}{m-1} + c_n$$

thus

$$\frac{a_n}{a_{n+1}} = \frac{b_n\binom{n+N-1}{N-1} + \cdots + b_3\binom{n+2}{2} + b_2\binom{n+1}{1} + b_1 + c_n}{b_{n+1}\binom{n+N}{N-1} + \cdots + b_3\binom{n+3}{2} + b_2\binom{n+2}{1} + b_1 + c_{n+1}}$$

when $N=1$,

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{b_1+c_n}{b_1+c_{n+1}} = 1$$

if $m \geq 2$, for $k \leq N-1$,

$$\frac{\binom{n+k-1}{k-1}}{\binom{n+N}{N-1}} \leq \frac{\binom{n+k}{k-1}}{\binom{n+k-1}{k-1}} = \frac{\frac{(n+k)\cdots(n+2)}{(k-1)!}}{\frac{(n+N)\cdots(n+2)}{(N-1)!}} = \frac{(N-1)\cdots k}{(n+N)\cdots(n+k+1)} \to 0$$

when $n \to \infty$. And

$$\frac{\binom{n+N-1}{N-1}}{\binom{n+N}{N-1}} = \frac{n+1}{n+N} \to 1$$

when $n \to \infty$. thus

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{b_n}{b_{n+1}} = 1$$

Exercise 15. Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Prove that if

$$|f(z)| = 1 \quad \textit{whenever } |z| = 1$$

then f is constant.

Solution 15. Define

$$F(z) = \begin{cases} f(z) & |z| \leq 1 \\ \frac{1}{\overline{f(\frac{1}{\bar{z}})}} & |z| > 1 \end{cases}$$

Then F is obviously continuous for both $|z| < 1$ and $|z| > 1$; for $|z| = 1$, we clearly have continuity from the inside, and if $\omega \to z$ with $|\omega| > 1$, then $\frac{1}{\bar{\omega}} \to \frac{1}{\bar{z}} = z$ and $F(\omega) = \frac{1}{\overline{f(\frac{1}{\bar{\omega}})}} \to \frac{1}{\overline{f(z)}} = f(z) = F(z)$. Hence F is continuous everywhere. It is known to be holomorphic for $|z| < 1$. For $|z| > 1$ we can compute $\frac{\partial f}{\partial \bar{z}} = 0$; alternatively, if Γ is any contour lying in the region $|z| > 1$, let Γ' be the image of Γ under the map $\omega = \frac{1}{z}$. Then Γ' is a contour lying in the region $|\omega| < 1$ and excluding the region from its interior (since the point at infinity does not lie within Γ), so

$$\oint_\Gamma F(z)dz = \oint_{\Gamma'} \frac{1}{\overline{f(\overline{\omega})}} \frac{-d\omega}{\omega^2} = 0$$

since $\frac{1}{\overline{f(\overline{\omega})}\omega^2}$ is analytic on and inside Γ'. To show F is analytic at points on the unit circle we follow the same procedure as with the Schwarz reflection principle, by subdividing a triangle which crosses the circle into triangles which either have a vertex on the circle or an edge lying "along" the circle (i.e. a chord of the circle). In the former case we may move the vertex by ϵ to conclude that the integral around the triangle is zero. In the case where a side of the triangle is a chord of the circle, we subdivide into smaller triangles (take the midpoint of the circular arc spanned by the chord) until the chord lies within ϵ of the circle and apply the same argument. The result is that F is entire. But F is bounded since $f(\overline{\mathbb{D}})$ is a compact set which excludes 0 and hence exclude a neighborhood of zero, so $\frac{1}{f}$ is bounded on $\mathbb{D}$. Since F is a bounded entire function, it is constant, so f is constant. (Liouville's theorem)