Appendix: Dynamic Model of a Self-Pressurized N_2O Tank with Closed-Loop Throttle and Helium Supercharge

A. Purpose & Scope

This appendix documents the dynamic simulation used to design and evaluate a liquid N_2O feed system that:

- Tracks commanded mass-flow setpoints with a metering element (valve/orifice) subject to a first-order actuator.
- Avoids pre-valve flashing upstream of the injector manifold by commanding helium supercharge (via a regulator + injector orifice) so that $P_{\text{up}} \geq P_{\text{sat}}(T) + P_{\text{margin}}$.
- Accounts for upstream line losses (Darcy–Weisbach + minor K's), tank self-refrigeration, and a two-node wall thermal model (liquid-wet and vapor-wet wall).
- Provides startup geometry and hydraulics checks and logs time histories for valve state, line ΔP drops, Reynolds numbers, P_{sat}, P_{He}, and tank temperature/masses.
- Schedules helium partial pressure so that $P_{\text{up}} \geq P_c + \Delta P_{\text{inj,req}} + P_{\text{margin}}$ with $P_c = \dot{m}\, c^*/A_t$.

Theory—non-condensable ullage (N_2O + He) and single-phase delivery. In a vessel containing a condensable (N_2O) and an inert gas (He), which when modeled together is treated as a non-condensable ullage, the N_2O vapor partial pressure is pinned to $P_{\text{sat}}(T)$ assuming equilibrium for the entire process. Since Helium adds only *partial* pressure to the ullage; it does not affect $P_{\text{sat}}(T)$. The total tank pressure is $P_{\text{tank}} = P_{\text{sat}}(T) + P_{\text{He}}$. Since the contents of the tank are two phase if the system flashes tremendous efficiency losses will be observed, and bottleneck the operation time of the system. Flashing (and cavitation inception): is a local hydrodynamic event which occurs wherever the *local static* liquid pressure falls *below* the saturation pressure at the *local* temperature, $P_{\text{s}}(x) < P_{\text{sat}}(T_{\text{local}}(x))$, resulting in vapor forming and collapsing thus degrading control within the system. On a P–T chart this is crossing under the saturation curve at T_{local}, i.e., the liquid becomes superheated. If pressure later recovers above P_{sat}, the vapor collapses (cavitation); if it remains below, the flow flashes and stays two-phase. The control objective is to maintain a positive *subcooling margin* along the entire pre-valve path, across self-refrigeration and peak flow. Practically, this explains the reasoning behind the addition of He overpressure to cover hydrostatics and line/valve losses with margin, prefer larger/shorter/smoother plumbing, draw from a bottom outlet, and use valve trims that resist cavitation.

Why we do not hold tank pressure constant. In a fixed-throat engine, chamber pressure is not a commanded input but a consequence of mass flow, $P_c = \dot{m}\, c^*/A_t$. The injector sees $\Delta P_{\text{inj}} = P_{\text{up}} - P_c$ with $P_{\text{up}} = P_{\text{tank}} - \Delta P_{\text{line}}$ and $\Delta P_{\text{line}} \sim k\, \dot{m}^2$. If we held P_{tank} constant while throttling, then $\Delta P_{\text{inj}} \approx (P_{\text{tank}} - \Delta P_{\text{line}}) - P_c$ would *collapse* at low $\dot{m}$, degrading atomization, upsetting element balance (O/F), and weakening hydraulic decoupling that protects stability. For self-pressurizing oxidizers (e.g. N_2O), P_{tank} naturally follows $P_{\text{sat}}(T)$ as the tank auto-refrigerates; enforcing a constant P_{tank} would require heavy heating or large helium usage (inert mass, residuals) without directly controlling thrust. What matters for predictable performance is maintaining the *required* injector drop and single-phase margin, i.e. scheduling the supply so

$$P_{\text{tank}} \;\gtrsim\; P_c \;+\; \Delta P_{\text{line}} \;+\; \Delta P_{\text{inj,req}} \;+\; \text{margin},$$

while commanding thrust via $\dot{m}$ (valves) with fixed injector areas. This keeps P_c tracking $\dot{m}$ as physics dictates, yet preserves injector behavior across the throttle band.

B. Inputs and Outputs

Key inputs

- **Tank & initial state:** volume V_{tank}, initial ullage fraction, T_0, ambient T_{amb}.
- **Wall model:** thickness t_{wall}, ρ_{wall}, $c_{p,\text{wall}}$, internal heat transfer coefficient's (HTC's) $h_{\text{in,liq}}$, $h_{\text{in,vap}}$, external HTC h_{out}.
- **Feedline:** tube spec $\Rightarrow$ $(D_{\text{pipe}}, \varepsilon)$, length L_{pipe}, component counts $\{N_i\}$ with K_i for frictional losses per fitting.
- **Valve:** discharge coefficient C_d, area bounds $[A_{\min}, A_{\max}]$ (internally estimated), actuator time constant τ_{valve} for the mass flow rate control valve.
- **Setpoints:** $\dot{m}_{\text{sp}}(t)$; optional base He overpressure set $\Delta P_{\text{He,base}}$ (MPa) to maintain the margin.
- **He system:** bottle $(P_0, T_{\text{b}}, V_{\text{b}})$, regulator time constant τ_{reg}, injector orifice diameter d_{He} and $C_{d,\text{He}}$.
- **No-flash controller:** enable flag and margin P_{margin} (MPa).
- **Integration:** step Δt, horizon t_{end}.
- **Engine (fixed):** characteristic velocity c^*, throat area A_t (expansion ratio ε and L^* for context).
- **Injector (lumped):** total geometric area A_{inj} and discharge coefficient $C_{d,\text{inj}}$; $\Delta P_{\text{inj,req}}$ is computed, not prescribed.

Key outputs

- Time series: $T(t)$, $P_{\text{sat}}(t)$, $P_{\text{He}}(t)$, $P_{\text{tank}}(t)$, $P_{\text{up}}(t)$, $\dot{m}(t)$, $\dot{m}_{\text{sp}}(t)$, $A(t)$, line drops $(\Delta P_{\text{maj}}, \Delta P_{\min}, \Delta P_{\text{line}})$, $\text{Re}(t)$, $v_{\text{pipe}}(t)$, $f(t)$, $m_\ell(t)$, $m_v(t)$, $m_{\text{He}}(t)$.
- Helium usage (integrated), regulator pressure history $P_{\text{reg}}(t)$.
- Geometry and wetted areas $A_\ell(t)$, $A_v(t)$; wall node temperatures $T_{w,\ell}(t)$, $T_{w,v}(t)$.
- $P_c(t) = \dot{m}(t)\, c^*/A_t$.
- $\Delta P_{\text{inj,req}}(t) = \dfrac{1}{2\,\rho_\ell(T(t))} \left(\dfrac{\dot{m}(t)}{C_{d,\text{inj}}\, A_{\text{inj}}} \right)^2$.
- Subcooling margin at the valve inlet: $P_{\text{up}}(t) - \left(P_c(t) + \Delta P_{\text{inj,req}}(t)\right)$.

C. Core Assumptions

1. **Thermo:** Bulk liquid/vapor in saturation: $P_{\text{sat}} = P_{\text{sat}}(T)$, with property calls from CoolProp/REFPROP or a validated table. Liquid properties (ρ_ℓ, μ_ℓ) evaluated at T.
2. **Helium:** Treated as ideal gas in ullage at tank T; high pressure source bottle ideal gas at fixed T_{b}. Injector orifice uses perfect-gas compressible flow (choked/subsonic) with discharge coefficient.
3. **Hydraulics:** Assumption of single-phase liquid line to the valve; Darcy–Weisbach with Churchill friction; standard minor K's; horizontal run with the addition of hydrostatic pressure $\rho g \Delta z$ if needed.

4. **Valve:** Quasi-steady orifice equation with first-order actuator dynamics.

5. **Wall heat transfer:** Two well-mixed wall nodes (liquid-wet and vapor-wet) exchanging with fluid and ambient via lumped HTC's, which will be used to represent the ullage contact points and the liquid contact points individually.

6. **Control:** No-flash controller imposes a *required* He partial pressure so that a predicted P_{up} at $\dot{m}_{\text{sp}}$ meets the margin. Regulator dynamics low-pass this command and are limited by bottle capability.

7. **Injector model:** the injector is a lumped hydraulic orifice with total area A_{inj} and coefficient $C_{d,\text{inj}}$; the required drop $\Delta P_{\text{inj,req}}$ is enforced for atomization, not a detailed spray model.

D. Thermodynamics and Phase Constraints

D.1 Saturation and ullage composition

The ullage past $t = 0$ contains N_2O vapor and He gas. By Dalton's law:

$$P_{\text{tank}} = P_{\text{sat}}(T) + P_{\text{He}}, \qquad P_{\text{He}} = \frac{m_{\text{He}} R_{\text{He}} T}{V_v}, \tag{1}$$

where V_v is the ullage volume and R_{He} is the specific gas constant of He.

D.2 Isochoric phase constraint (mass–volume closure)

At each step the tank is maintained at saturation. With total tank volume V_{tank} being a fixed value leading to the isochoric assumption which can be modeled by,

$$V_{\text{tank}} = \frac{m_\ell}{\rho_\ell(T)} + \frac{m_v}{\rho_v(T)}. \tag{2}$$

From the creation of a mass balance around the tank; Let $\Delta m_{\text{out}} = \dot{m}\,\Delta t$ be the liquid outflow mass (assumed to leave as a single phase liquid). Define the *evaporated mass* in the step as $\Delta m_{\text{evap}} = m_v^{n+1} - m_v^n$. Then from the liquid phasing two sources of outflow and the ullage/vapor experiencing only an in-flow of mass it can be said that:

$$m_\ell^{n+1} = m_\ell^n - \Delta m_{\text{out}} - \Delta m_{\text{evap}}, \tag{3}$$

$$m_v^{n+1} = m_v^n + \Delta m_{\text{evap}}. \tag{4}$$

Substitute into (2) with properties at T^{n+1} and solve algebraically for m_v^{n+1}:

$$m_v^{n+1}\left(1 - \frac{\rho_v}{\rho_\ell}\right) = \rho_v\left[V_{\text{tank}} - \frac{m_\ell^n - \Delta m_{\text{out}} + m_v^n}{\rho_\ell}\right], \qquad \Rightarrow \quad m_v^{n+1} = \frac{\rho_v\left[V_{\text{tank}} - (m_\ell^n - \Delta m_{\text{out}} + m_v^n)/\rho_\ell\right]}{1 - \rho_v/\rho_\ell}. \tag{5}$$

All ρ's are evaluated at T^{n+1}.

D.3 Energy balance with wall heat

We derive the working energy equation from the first law of thermodynamics for a rigid, well-mixed tank control volume (CV) that contains liquid N_2O (m_ℓ), N_2O vapor (m_v), and helium (m_{He}). Kinetic/potential energies and shaft work are negligible; the tank exchanges heat with the wall and ejects liquid through the metering element.

D.3.1 First law for the tank CV. Let U_f be the internal energy of the *fluid* in the tank. The first law reads

$$\frac{dU_f}{dt} \;=\; \underbrace{Q_{w\to f}}_{\substack{\text{heat from wall}\\ \text{into fluid}}} \;+\; \sum_{\text{in}} \dot{m}\, h_{\text{in}} \;-\; \sum_{\text{out}} \dot{m}\, h_{\text{out}}. \tag{6}$$

Here h denotes specific enthalpy carried with mass crossing the CV boundary. In our system the only appreciable mass outflow is liquid propellant at the tank face $\Rightarrow h_{\text{out}} \approx h_\ell(T)$. Helium inflow enthalpy is small versus N_2O latent effects and is neglected in the baseline model (see the conservative variant below).

The fluid internal energy is decomposed as

$$U_f \;\approx\; m_\ell\, u_\ell(T) \;+\; m_v\, u_v(T), \qquad u_\ell(T) \approx c_{p,\ell} T, \tag{7}$$

i.e. liquid sensible energy plus a vapor term whose *net* effect is retained through the latent heat $h_{fg}(T) \equiv h_v(T) - h_\ell(T)$.

D.3.2 Species balances and phase change. With liquid outflow $\dot{m}$ and interfacial evaporation rate $\dot{m}_{\text{evap}}$ (liquid $\to$ vapor),

$$\frac{dm_\ell}{dt} = -\,\dot{m} - \dot{m}_{\text{evap}}, \qquad\qquad \frac{dm_v}{dt} = +\,\dot{m}_{\text{evap}}. \tag{8}$$

D.3.3 Reduction to an operational energy equation. Differentiate (7), substitute (8), and collect terms:

$$\frac{dU_f}{dt} \;=\; \underbrace{m_\ell c_{p,\ell}\,\dot{T}}_{\text{sensible (liquid)}} \;-\; \underbrace{\dot{m}\, u_\ell(T)}_{\text{removed with outflow}} \;+\; \underbrace{\big(u_v(T) - u_\ell(T)\big)\,\dot{m}_{\text{evap}}}_{\text{phase change}}. \tag{9}$$

Insert (9) into (6) and place the outflow term on the right:

$$m_\ell c_{p,\ell}\,\dot{T} \;+\; \big(u_v - u_\ell\big)\,\dot{m}_{\text{evap}} \;-\; \dot{m}\, u_\ell \;=\; Q_{w\to f} \;-\; \dot{m}\, h_\ell. \tag{10}$$

Using $h_\ell - u_\ell = P_{\text{tank}}/\rho_\ell$, we see that the two outflow terms differ only by the small flow-work contribution $\dot{m}\, P_{\text{tank}}/\rho_\ell$. Dropping this small term (it is $\ll$ latent + wall heat for N_2O shots) and replacing $u_v - u_\ell$ with h_{fg} yields the continuous *operational* balance

$$m_\ell c_{p,\ell}\,\dot{T} \;+\; \dot{m}_{\text{evap}}\, h_{fg}(T) \;=\; Q_{w\to f}. \tag{11}$$

The wall-to-fluid heat is the sum of liquid-wet and vapor-wet contributions,

$$Q_{w\to f} \;=\; h_{\text{in,liq}}\, A_\ell\, (T_{w,\ell} - T) \;+\; h_{\text{in,vap}}\, A_v\, (T_{w,v} - T), \tag{12}$$

which is positive when walls are hotter than the bulk fluid.

D.3.4 Semi-implicit time discretization and residual. Over one step $t^n \to t^{n+1} = t^n + \Delta t$, we enforce (11) in an energetically consistent, numerically robust way by evaluating the strongest T-dependencies at the *new* time. Specifically,

$$\underbrace{m_\ell^n\, c_{p,\ell}\,\big(T^{n+1} - T^n\big)}_{\text{sensible (liquid)}} \;+\; \underbrace{\big(m_v^{n+1} - m_v^n\big)\, h_{fg}\Big(T^{n+1}\Big)}_{\text{latent (evap./cond.)}} \;-\; \underbrace{\Big[h_{\text{in,liq}} A_\ell (T_{w,\ell}^n - T^{n+1}) + h_{\text{in,vap}} A_v (T_{w,v}^n - T^{n+1})\Big]\Delta t}_{Q_{w\to f}\Delta t} \;=\; 0 \tag{13}$$

- m_v^{n+1} is obtained from the isochoric closure (2) using (5) with properties at T^{n+1}; this couples latent uptake to phase equilibrium at the new state.

- m_ℓ^n is used in the sensible term (the difference to m_ℓ^{n+1} is $O(\Delta t)$ and avoids extra nonlinearity).
- Wall temperatures $T_{w,\bullet}^n$ appear explicitly; the wall ODEs (Sec.) are then advanced to $n{+}1$ after the fluid solve.

We define the scalar residual $\mathcal{R}(T^{n+1})$ as the left-hand side of (13) and solve $\mathcal{R}(T^{n+1}) = 0$ by bisection within a safe bracket $T^{n+1} \in \big[T^n - \Delta T_b,\ \min\{T^n, T_{\max} - \varepsilon\}\big]$, with $\Delta T_b \sim 25K$. Over this bracket $\mathcal{R}$ is continuous and (for physical states) monotone increasing in T^{n+1} because $c_{p,\ell} > 0$, $h_{fg} > 0$, and the $m_v^{n+1}(T)$ map from (5) is smooth, ensuring robust convergence without Jacobians.

D.3.5 Conservative variant (including small flow-work and He sensible terms). If desired, one can retain the small liquid flow-work and He sensible enthalpy to make the step fully conservative:

$$m_\ell^n c_{p,\ell}(T^{n+1} - T^n) + \left(m_v^{n+1} - m_v^n\right) h_{fg}(T^{n+1}) - \Big[h_{\text{in,liq}} A_\ell (T_{w,\ell}^n - T^{n+1}) + h_{\text{in,vap}} A_v (T_{w,v}^n - T^{n+1})\Big]\Delta t - \underbrace{\dot{m}^n \frac{P_1}{\rho_\ell(}}_{\text{liquid f}} \tag{14}$$

In typical N_2O operating envelopes these extra terms are small compared to the latent and wall-heat terms, so (13) is preferred for simplicity and numerical conditioning.

D.3.6 Assumptions and sign conventions.

1. **Well-mixed bulk:** a single bulk T represents both phases (local subcooling is handled in the line model and no-flash constraint).
2. **Saturation constraint:** phases remain at equilibrium at T^{n+1}; m_v^{n+1} follows from mass–volume closure.
3. **Wall heat sign:** $Q_{w\to f} > 0$ when walls are hotter than the fluid (heat into fluid).
4. **Neglected terms:** liquid flow-work and He sensible enthalpy are negligible to first order; include via (14) if margins are tight.

D.3.7 Physical interpretation. Equation (13) encodes "*heat in = sensible + latent uptake*" over Δt. If the walls are colder than the bulk ($Q_{w\to f} < 0$), the residual drives condensation ($m_v^{n+1} < m_v^n$) or cooling ($T^{n+1} < T^n$) to satisfy energy conservation; if walls are hotter, it drives evaporation and/or warming.

E. Wall Thermal Submodel (Two Nodes)

We model the tank wall as two lumped thermal-capacitance nodes: a *liquid-wet* node with area A_ℓ and a *vapor-wet* node with area A_v. Each node exchanges heat with the bulk fluid (at temperature T) and with the ambient (at T_{amb}). The two nodes are not directly coupled in this baseline model (axial conduction across the phase-contact line is addressed in the extensions below), and thus variables/equations that have no dependency on the phase can be written with a bullet attached as in $A_\bullet$. The areal wall heat capacity is

$$C_{w,\bullet} \;=\; \rho_{\text{wall}}\, c_{p,\text{wall}}\, t_{\text{wall}}\, A_\bullet, \qquad \bullet \in \{\ell, v\},$$

which assumes the wall is isothermal through its thickness (a standard *lumped* assumption, justified by a small Biot number; see E.5).

E.1 Energy balances for the wall nodes. Applying the first law to each wall control mass (no phase change, negligible storage of mechanical energy):

$$C_{w,\ell}\,\frac{dT_{w,\ell}}{dt} = \underbrace{h_{\text{out}}A_\ell(T_{\text{amb}}-T_{w,\ell})}_{\text{ambient}\to\text{wall}} \;+\; \underbrace{h_{\text{in,liq}}A_\ell(T-T_{w,\ell})}_{\text{fluid}\to\text{wall}}, \tag{15}$$

$$C_{w,v}\,\frac{dT_{w,v}}{dt} = \underbrace{h_{\text{out}}A_v(T_{\text{amb}}-T_{w,v})}_{\text{ambient}\to\text{wall}} \;+\; \underbrace{h_{\text{in,vap}}A_v(T-T_{w,v})}_{\text{fluid}\to\text{wall}}. \tag{16}$$

Equations (15)–(16) are linear, stable first-order ODEs. Note the *sign convention*: a positive right-hand side warms the wall node.

E.2 Heat passed to the fluid (consistency with Sec. D.3). The heat *into the fluid* that appears in the fluid energy residual (Eq. (13)) is the negative of the fluid→wall terms above:

$$Q_{w\to f} \;=\; h_{\text{in,liq}}A_\ell(T_{w,\ell}-T) \;+\; h_{\text{in,vap}}A_v(T_{w,v}-T), \tag{17}$$

so $Q_{w\to f}>0$ when the wall is hotter than the fluid. Energy conservation over a step Δt implies

$$\Delta U_{w,\ell}+\Delta U_{w,v} \;=\; -\,(Q_{w\to f}+Q_{\text{amb}\to w})\,\Delta t, \qquad Q_{\text{amb}\to w}=h_{\text{out}}\big[A_\ell(T_{\text{amb}}-T_{w,\ell})+A_v(T_{\text{amb}}-T_{w,v})\big],$$

which is satisfied by integrating (15)–(16) with the same $\{A_\ell, A_v\}$ used in (17).

E.3 Closed-form solution and time constants. With $A_\bullet$, h's, and T held constant over one step, each node has the exact solution

$$T_{w,\bullet}(t) = T_{\infty,\bullet} \;+\; (T_{w,\bullet}(0)-T_{\infty,\bullet})\,\exp\!\Big(-t/\tau_{w,\bullet}\Big), \tag{18}$$

$$\tau_{w,\bullet} = \frac{C_{w,\bullet}}{h_{\text{out}}A_\bullet + h_{\text{in},\bullet}A_\bullet} = \frac{\rho_{\text{wall}}\,c_{p,\text{wall}}\,t_{\text{wall}}}{h_{\text{out}}+h_{\text{in},\bullet}}, \tag{19}$$

$$T_{\infty,\bullet} = \frac{h_{\text{out}}A_\bullet\,T_{\text{amb}} + h_{\text{in},\bullet}A_\bullet\,T}{h_{\text{out}}A_\bullet + h_{\text{in},\bullet}A_\bullet}, \qquad \bullet\in\{\ell,v\}. \tag{20}$$

Thus, the wall node exponentially approaches a weighted average of ambient and fluid temperatures with time constant $\tau_{w,\bullet}$.

E.4 Discretization choices. The code currently advances (15)–(16) by *explicit Euler*:

$$T_{w,\ell}^{n+1} = T_{w,\ell}^{n} + \frac{\Delta t}{C_{w,\ell}}\Big[h_{\text{out}}A_\ell(T_{\text{amb}}-T_{w,\ell}^{n}) - h_{\text{in,liq}}A_\ell(T_{w,\ell}^{n}-T^{n+1})\Big], \tag{21}$$

$$T_{w,v}^{n+1} = T_{w,v}^{n} + \frac{\Delta t}{C_{w,v}}\Big[h_{\text{out}}A_v(T_{\text{amb}}-T_{w,v}^{n}) - h_{\text{in,vap}}A_v(T_{w,v}^{n}-T^{n+1})\Big]. \tag{22}$$

This is unconditionally *consistent* and conditionally *stable*. A convenient stability criterion for forward Euler on a linear first-order decay is

$$\Delta t \;\lesssim\; 2\,\tau_{w,\bullet} \qquad \text{(practical choice: } \Delta t \le 0.3\text{–}0.5\,\tau_{w,\bullet}\text{)}. \tag{23}$$

Because $\tau_{w,\bullet}$ does not depend on area (areas cancel in (20)), explicit stability is insensitive to fill fraction. If one desires a larger time step or stricter conservation, use either:

- the *exact exponential* update from (20) with $\{T, T_{\text{amb}}, h\}$ frozen over Δt, or
- a semi-implicit (backward Euler) step:

$$T_{w,\bullet}^{n+1} = \frac{T_{w,\bullet}^{n} + \frac{\Delta t}{C_{w,\bullet}}\,(h_{\text{out}}A_\bullet T_{\text{amb}} + h_{\text{in},\bullet}A_\bullet T^{n+1})}{1+\frac{\Delta t}{C_{w,\bullet}}\,(h_{\text{out}}A_\bullet + h_{\text{in},\bullet}A_\bullet)}.$$

E.5 Validity of the lumped wall (Biot number). The lumped-capacitance assumption requires the *through-thickness* Biot number to be small:

$$\mathrm{Bi}_\bullet \;\equiv\; \frac{(h_{\mathrm{out}} + h_{\mathrm{in},\bullet})\, t_{\mathrm{wall}}}{k_{\mathrm{wall}}} \;\ll\; 1.$$

For stainless steel ($k_{\mathrm{wall}} \sim 15 W\, m^{-1}\, K^{-1}$), $t_{\mathrm{wall}} \sim 5mm$, $h_{\mathrm{out}} \sim 5$to 15, and $h_{\mathrm{in,liq}} \sim 500$to $1500\, W\, m^{-2}\, K^{-1}$, one finds $\mathrm{Bi}_\ell \lesssim 0.1$–$0.6$ and $\mathrm{Bi}_v \ll 0.1$; the lumped model is adequate for screening and control design. If $\mathrm{Bi} \gtrsim 0.3$ persistently, a through-thickness wall layer or a 1D conduction model is recommended.

E.6 Handling changing wetted areas. The areas A_ℓ and A_v depend on fill level and are recomputed each step from geometry (upright cylinder). Two practical details:

1. *Degenerate areas:* when $A_\bullet \to 0$ we set $dT_{w,\bullet}/dt \to 0$ to avoid division by zero and simply *carry* $T_{w,\bullet}$ forward (the code's guard does this). Physically, a vanishing patch has vanishing thermal mass.
2. *Contact switching:* when a patch transitions from vapor-wet to liquid-wet (or vice versa), the governing $h_{\mathrm{in},\bullet}$ changes abruptly. Smoothing $h_{\mathrm{in},\bullet}$ over a narrow fill interval or using the exact/exponential update reduces small numerical spikes in $Q_{w\to f}$.

E.7 Parameter guidance (orders of magnitude). Typical ranges used in cryogenic/oxidizer plumbing for screening:

$h_{\mathrm{in,liq}} \sim 500$to $1500\, W\, m^{-2}\, K^{-1}$, $\quad h_{\mathrm{in,vap}} \sim 5$to $30\, W\, m^{-2}\, K^{-1}$, $\quad h_{\mathrm{out}} \sim 5$to $15\, W\, m^{-2}\, K^{-1}$ (natural convecti

Film boiling or vigorous nucleation at the liquid interface can push $h_{\mathrm{in,liq}}$ higher (into the few $\mathrm{kW\,m^{-2}\,K^{-1}}$ range); if suspected, treat $h_{\mathrm{in,liq}}$ as a function of $(T_{w,\ell} - T)$ or cap with vendor correlations.

E.8 Extensions (when needed).

- *Radiation:* add $q_{\mathrm{rad}} = \varepsilon\sigma\left(T_{\mathrm{sur}}^4 - T_{w,\bullet}^4\right)$; for small excursions, linearize into an effective $h_{\mathrm{rad}} = 4\varepsilon\sigma T_{\mathrm{ref}}^3$ and add to h_{out}.
- *Axial wall conduction:* couple the nodes by $k_{\mathrm{wall}} \frac{A_{\mathrm{bridge}}}{L_{\mathrm{bridge}}}(T_{w,\ell} - T_{w,v})$ if the phase-contact line is thermally short.
- *Through-thickness model:* if Bi is not small, split each node into inner/outer layers or solve 1D conduction with Robin boundary conditions to T and T_{amb}.

F. Feedline Hydraulics and Pre-Valve Pressure

We treat the pre-valve line as single-phase, incompressible liquid with constant ID D, length L, and roughness ε. Between the tank outlet (node 0) and the valve inlet (node 1), the steady mechanical-energy equation (Bernoulli with head losses) gives

$$P_0 + \frac{1}{2}\rho_\ell v_0^2 + \rho_\ell g z_0 \;=\; P_1 + \frac{1}{2}\rho_\ell v_1^2 + \rho_\ell g z_1 \;+\; \rho_\ell g\, h_L, \tag{24}$$

where the total head loss h_L is the sum of major (wall friction) and minor (fittings) contributions. For a constant-ID run ($v_0 \simeq v_1 \equiv v$) this reduces to the standard pressure-drop form

$$\Delta P_{\mathrm{line}} \;\equiv\; P_0 - P_1 \;=\; \left(f\,\frac{L}{D} + K_{\mathrm{tot}}\right)\left(\tfrac{1}{2}\rho_\ell v^2\right) \;+\; \rho_\ell g\,\Delta z, \qquad \Delta z \equiv z_1 - z_0. \tag{25}$$

Here $K_{\mathrm{tot}} = \sum_i N_i K_i$ is the sum of minor-loss coefficients referenced to the *pipe* velocity v, and f is the Darcy friction factor.

F.1 Kinematics and Reynolds number. With area $A = \pi D^2/4$ and mass-flow rate $\dot{m}$,

$$v \;=\; \frac{\dot{m}}{\rho_\ell A} \;=\; \frac{4\dot{m}}{\rho_\ell \pi D^2}, \qquad \mathrm{Re} \;=\; \frac{\rho_\ell v D}{\mu_\ell}. \tag{26}$$

F.2 Friction factor correlation (Darcy). We use laminar $f = 64/\mathrm{Re}$ for $\mathrm{Re} < 2300$ and Churchill's explicit turbulent correlation otherwise:

$$f = 8\left[\left(\frac{8}{\mathrm{Re}}\right)^{12} + \frac{1}{(A+B)^{1.5}}\right]^{1/12}, \tag{27}$$

$$A = \left[-2.457\ln\!\left((7/\mathrm{Re})^{0.9} + 0.27(\varepsilon/D)\right)\right]^{16}, \qquad B = \left(\frac{37530}{\mathrm{Re}}\right)^{16}. \tag{28}$$

(We use the *Darcy* f; the Fanning factor is $f_F = f/4$.)

F.3 Minor losses and equivalent length. Minor components (entrances, elbows, tees, valves, contractions/expansions) are modeled by

$$\Delta P_{\mathrm{min}} = K_{\mathrm{tot}}\left(\tfrac{1}{2}\rho_\ell v^2\right), \qquad K_{\mathrm{tot}} = \sum_i N_i K_i. \tag{29}$$

Equivalently, one may fold minors into an "equivalent length" $L_{\mathrm{eq}} = (K_{\mathrm{tot}} D)/f$ and write $\Delta P_{\mathrm{line}} = f\frac{L+L_{\mathrm{eq}}}{D}\left(\frac{1}{2}\rho_\ell v^2\right) + \rho_\ell g\,\Delta z$. (We keep the split form in the code for transparency.)

F.4 Hydrostatics and optional acceleration head. The elevation term in (25) is *added* when the valve sits above the tank outlet ($\Delta z > 0$). For fast command ramps, a transient "acceleration head"

$$\Delta P_{\mathrm{acc}} \approx \rho_\ell L\,\frac{dv}{dt}$$

can appear; it is neglected in the base model but may reach $\mathcal{O}(0.1)$ MPa for $L \sim 1.5m$ and $dv/dt \sim 50m\,s^{-2}$.

F.5 Pre-valve pressure and Euler-number view. With the tank total pressure $P_{\mathrm{tank}} = P_{\mathrm{sat}}(T) + P_{\mathrm{He}}$, the pressure just upstream of the metering element is

$$P_{\mathrm{up}} \;=\; P_{\mathrm{tank}} - \Delta P_{\mathrm{line}}. \tag{30}$$

It is convenient to interpret ΔP_{line} via the Euler number:

$$\mathrm{Eu} \;\equiv\; \frac{\Delta P_{\mathrm{line}} - \rho_\ell g\,\Delta z}{\frac{1}{2}\rho_\ell v^2} \;=\; f\,\frac{L}{D} + K_{\mathrm{tot}}. \tag{31}$$

For a given geometry and fluid state, Eu is set by Re (through f) and roughness ε/D.

F.6 Closed-form capacity and design inversion (with iteration). Substituting $v = \dot{m}/(\rho_\ell A)$ into (25),

$$\Delta P_{\mathrm{line}} = \left(f\,\frac{L}{D} + K_{\mathrm{tot}}\right)\frac{\dot{m}^2}{2\,\rho_\ell\,A^2} + \rho_\ell g\,\Delta z, \qquad A = \frac{\pi D^2}{4}. \tag{32}$$

For *design* one often seeks $\dot{m}$ for a permissible drop $\Delta P^\star$. Because f depends on $\mathrm{Re} \propto \dot{m}$, (32) is solved by a 1–3 step fixed-point:

1. Guess $\dot{m}^{(0)}$ (or $f^{(0)}$); compute $\mathrm{Re}^{(0)}$, $f^{(0)}$.

2. Update $\dot{m}^{(1)}$ from (32) holding $f^{(0)}$: $\dot{m}^{(1)} = \sqrt{\frac{2\,\rho_\ell\,A^2(\Delta P^\star - \rho_\ell g\,\Delta z)}{f^{(0)}\,L/D + K_{\rm tot}}}$.

3. Recompute $\mathrm{Re}^{(1)}$, $f^{(1)}$; repeat until convergence (typically rapid with Churchill).

In turbulent, hydraulically smooth flow, the scaling is roughly $\Delta P_{\rm line} \propto D^{-5}$ (because $v \sim D^{-2}$ and f is weakly D-dependent), so small increases in D yield large drops in ΔP.

F.7 Practical checks and ranges.

- **Regime:** Ensure $\mathrm{Re} \gtrsim 4 \times 10^3$ if you assume turbulent correlations. If Re dips near transition, expect higher uncertainty.
- **Roughness:** Drawn 316 SS has $\varepsilon \sim 1.5_m$, so $\varepsilon/D \sim 2{\times}10^{-4}$ for $D \approx 7.8mm$ (3/8 OD×0.035 in tube).
- **Velocity:** For oxidizer feeds, $v \sim 2-10m\,s^{-1}$ is typical; very high v raises K-losses, noise, and cavitation risk at fittings.
- **Hydrostatics:** Include $\rho_\ell g\,\Delta z$ whenever the outlet is significantly above/below the tank outlet.
- **Single-phase validity:** This section is valid provided $P_{\rm up} - P_{\rm sat}(T) \geq 0$ (with margin). If not, the line is two-phase and (25) underpredicts the true drop.

G. Metering Element and Actuation

The metering element is modeled as a single hydraulic restriction with effective area A and discharge coefficient C_d. Let the available differential across the trim be

$$\Delta P_{\rm valve} \;\equiv\; \max(P_{\rm up} - P_{\rm back},\, 0), \qquad \rho_\ell \;\equiv\; \rho_\ell(T). \tag{33}$$

Under single-phase, noncavitating conditions, the mass-flow rate follows the classical orifice law

$$\dot{m} \;=\; C_d\,A\,\sqrt{2\,\rho_\ell\,\Delta P_{\rm valve}}\,, \qquad A_{\rm ff} \;=\; \frac{\dot{m}_{\rm sp}}{C_d\,\sqrt{2\,\rho_\ell\,\Delta P_{\rm valve}}} \tag{34}$$

where $A_{\rm ff}$ is the feedforward area required to meet the instantaneous mass-flow setpoint $\dot{m}_{\rm sp}$ given the currently available $\Delta P_{\rm valve}$.

G.1 Derivation and meaning of C_d. Between the pre-valve plane (1) and the vena contracta (vc), Bernoulli with a contraction and internal loss gives $P_1 - P_{\rm vc} \approx \frac{1}{2}\rho_\ell v_{\rm vc}^2$, with $v_{\rm vc} = \dot{m}/(\rho_\ell A_{\rm vc})$ and $A_{\rm vc} = C_c A_{\rm geom}$ the contracted jet area. Accounting for velocity profile and additional internal loss via a velocity coefficient C_v, one obtains $\dot{m} = C_c C_v A_{\rm geom}\sqrt{2\rho_\ell(P_1 - P_2)}$, so $C_d \equiv C_c C_v$ is an empirical aggregate (typically 0.6–0.85 for sharp to well-rounded trims). Equation (34) simply writes this with $A \equiv A_{\rm geom}$ and $\Delta P_{\rm valve} \equiv P_1 - P_2$.

G.2 Relation to vendor C_v/K_v data. Vendors often specify a capacity coefficient:

$$Q = N_1\,C_v\,\sqrt{\frac{\Delta P_{\rm valve}}{\mathrm{SG}}} \quad \text{(US units: } Q \text{ in gpm, } \Delta P \text{ in psi, SG} = \rho_\ell/\rho_{\rm water}),$$

or, in metric, $Q\,[\mathrm{m}^3/\mathrm{h}] = K_v\sqrt{\Delta P\,[\mathrm{bar}]/\mathrm{SG}}$. Given $Q = \dot{m}/\rho_\ell$, one can back out an *effective area*

$$A_{\rm eff} = \frac{Q}{\sqrt{2\Delta P_{\rm valve}/\rho_\ell}} \quad \Rightarrow \quad C_d A \approx A_{\rm eff},$$

and thus map a specified C_v or K_v to $C_d A$ using the appropriate unit conversions. In our code we manipulate A directly (with fixed C_d); a C_v-based actuator can be supported by the same equations after a static conversion.

G.3 Cavitation/flash limits (scope note). Equation (34) holds provided the flow remains single-phase through the trim. If the vena contracta static pressure drops below saturation, $P_{\mathrm{vc}} < P_{\mathrm{sat}}(T_{\mathrm{vc}})$, incipient cavitation occurs; if the downstream recovery leaves $P_2 > P_{\mathrm{sat}}$, bubbles collapse (cavitation), otherwise persistent flashing occurs. Industrial valve standards introduce cavitation coefficients (e.g. F_L, F_F) and a choked (liquid) limit

$$Q_{\max} \sim N_1 C_v \, F_L \sqrt{\frac{P_1 - F_F P_{\mathrm{sat}}}{\mathrm{SG}}},$$

beyond which increasing $\Delta P_{\mathrm{valve}}$ does not increase Q. Our design avoids this regime by enforcing a pre-valve *subcooling margin* (Sec.) so that the trim operates noncavitating.

G.4 Sensitivities and control authority. Linearizing (34) at an operating point $(A^\star, \Delta P^\star)$:

$$\frac{\partial \dot{m}}{\partial A}\Big|_\star = C_d \sqrt{2\rho_\ell \Delta P^\star}, \qquad \frac{\partial \dot{m}}{\partial \Delta P}\Big|_\star = \frac{C_d A^\star}{2}\sqrt{\frac{2\rho_\ell}{\Delta P^\star}}. \tag{35}$$

These slopes guide actuator sizing and loop tuning: as ΔP falls, $\partial \dot{m}/\partial \Delta P$ grows, and the feedforward A_{ff} rises nonlinearly.

G.5 Feedforward area and numerical guards. Because $A_{\mathrm{ff}} \propto 1/\sqrt{\Delta P_{\mathrm{valve}}}$, a small $\Delta P_{\mathrm{valve}}$ can produce impractically large A_{ff}. We therefore clip

$$\Delta P_{\mathrm{valve,eff}} = \max\left(\Delta P_{\mathrm{valve}},\, \Delta P_{\min}\right), \qquad A_{\mathrm{ff}} = \frac{\dot{m}_{\mathrm{sp}}}{C_d\sqrt{2\rho_\ell \Delta P_{\mathrm{valve,eff}}}},$$

with a small floor $\Delta P_{\min}$ to avoid numerical blow-up when $P_{\mathrm{up}} \approx P_{\mathrm{back}}$.

G.6 Actuator dynamics, saturation, and discrete update. We model the actuator as a first-order lag to a saturated command:

$$A_{\mathrm{cmd}} = \mathrm{sat}\big(A_{\mathrm{ff}},\, A_{\min}, A_{\max}\big), \qquad \frac{dA}{dt} = \frac{A_{\mathrm{cmd}} - A}{\tau_{\mathrm{valve}}}. \tag{36}$$

In discrete time ($t^n \to t^{n+1} = t^n + \Delta t$), forward Euler gives

$$A^{n+1} = A^n + \frac{\Delta t}{\tau_{\mathrm{valve}}}\Big(A^n_{\mathrm{cmd}} - A^n\Big), \tag{37}$$

which is stable for $\Delta t \lesssim 2\tau_{\mathrm{valve}}$ (practically, choose $\Delta t \le 0.2$–$0.5\,\tau_{\mathrm{valve}}$). Because A itself is the plant input (not an integral of control error), saturation does not induce classical integral windup.

G.7 Characterization (trim law). Real trims are not strictly "area-linear" with stem position; many are *equal-percentage* or *quick-opening*. If a stem position $\xi \in [0,1]$ is the actuator variable, include a static map $A(\xi)$; the control law is unchanged after replacing A by $A(\xi)$ in (34) and (36). The present model uses A directly, i.e. an idealized linear trim.

G.8 Compressibility and liquid bulk modulus (optional refinement). For very large drops relative to the liquid bulk modulus K_ℓ, a compressibility correction slightly reduces the ideal $\sqrt{\Delta P}$ gain (so-called *liquid choking* due to density change). A simple correction writes

$$\dot{m} \;\approx\; C_d A \sqrt{\frac{2\rho_\ell \Delta P_{\mathrm{valve}}}{1 + \beta\, \Delta P_{\mathrm{valve}}/K_\ell}}, \quad \beta \sim \mathcal{O}(1).$$

In our operating envelope, $\Delta P_{\mathrm{valve}} \ll K_\ell$ and this effect is negligible compared to uncertainties in C_d and upstream hydraulics.

G.9 Practical guidance.

- Ensure ΔP_{valve} stays comfortably positive at all setpoints; otherwise A_{ff} will saturate and tracking will fail.
- Choose $A_{\max}$ from the *largest* required $\dot{m}_{\text{sp}}$ at the *smallest* expected ΔP_{valve} (coldest tank, largest line losses).
- Keep C_d conservative (lower bound) during early design; revisit with vendor C_v/K_v data.
- If noise is present on P_{up} or P_{back}, low-pass ΔP_{valve} before computing A_{ff} to avoid chattering.

G.10 Injector pressure-drop requirement The lumped injector (total area A_{inj}, discharge $C_{d,\text{inj}}$) requires

$$\Delta P_{\text{inj,req}} = \frac{1}{2\,\rho_\ell(T)} \left(\frac{\dot{m}}{C_{d,\text{inj}}\, A_{\text{inj}}} \right)^2. \tag{38}$$

At setpoint we use $\dot{m}_{\text{sp}}$ to form $\Delta P^{\text{sp}}_{\text{inj,req}}$.

H. Helium System: Regulator and Injector

The helium (He) subsystem provides a controllable *partial pressure* in the ullage to maintain a positive pre-valve subcooling margin while tracking the demanded propellant flow. We treat the bottle as an ideal-gas reservoir, the regulator as a first-order, rate-limited source that cannot exceed supply, and the injector as a compressible orifice discharging into the tank ullage.

State and algebraic relations. With ullage volume V_v, bulk tank temperature T, and ullage helium mass m_{He},

$$P_{\text{He}} = \frac{m_{\text{He}} R_{\text{He}} T}{V_v}, \qquad P_{\text{tank}} = P_{\text{sat}}(T) + P_{\text{He}}, \tag{39}$$

where $R_{\text{He}} = 2077.1 J\, kg^{-1}\, K^{-1}$. Equation (39) couples the He mass balance to the flash-avoidance objective.

H.1 No-flash controller (pressure target generation)

The controller enforces a *pre-valve* margin at the *requested* flow, using the single-phase line model from Sec. . Using properties at the current tank temperature T, predict the line drop at the setpoint:

$$\Delta P_{\text{line,pred}}(\dot{m}_{\text{sp}}) = \Delta P_{\text{maj}}(\dot{m}_{\text{sp}}) + \Delta P_{\text{min}}(\dot{m}_{\text{sp}}). \tag{40}$$

$$P_c^{\text{pred}} = \frac{\dot{m}_{\text{sp}}\, c^*}{A_t}, \tag{41}$$

$$\Delta P^{\text{sp}}_{\text{inj,req}} = \frac{1}{2\,\rho_\ell(T)} \left(\frac{\dot{m}_{\text{sp}}}{C_{d,\text{inj}}\, A_{\text{inj}}} \right)^2. \tag{42}$$

To guarantee single-phase at the valve inlet ($P_{\text{up}} \geq P_{\text{sat}} + P_{\text{margin}}$) *at the setpoint* we require

$$P_{\text{He,req}} \;\geq\; \Delta P_{\text{line,pred}}(\dot{m}_{\text{sp}}) + P_{\text{margin}}. \tag{43}$$

We also allow a user base overpressure $\Delta P_{\mathrm{He,base}}$ (e.g., to desensitize to modeling error). The controller selects

$$\boxed{P_{\mathrm{He,target}} = \max\Big(P_c^{\mathrm{pred}} + \Delta P_{\mathrm{inj,req}}^{\mathrm{sp}} + \Delta P_{\mathrm{line,pred}} + P_{\mathrm{margin}} - P_{\mathrm{sat}}(T),\ \Delta P_{\mathrm{He,base}}\Big).} \tag{44}$$

Comment. Because $\Delta P_{\mathrm{line,pred}} \propto \dot{m}_{\mathrm{sp}}^2$ (Sec.), $P_{\mathrm{He,target}}$ naturally rises with demanded throughput; smoothing $\dot{m}_{\mathrm{sp}}$ reduces He demand transients.

H.2 Regulator dynamics and supply ceiling

The He supply is a bottle of volume V_{b} at temperature T_{b} with mass $m_{\mathrm{He,b}}$. In the nominal model the bottle is isothermal (a conservative simplification for short shots), so

$$P_{\mathrm{bottle}} = \frac{m_{\mathrm{He,b}} R_{\mathrm{He}} T_{\mathrm{b}}}{V_{\mathrm{b}}}. \tag{45}$$

A single-stage regulator attempts to realize $P_{\mathrm{reg,target}}$ but cannot exceed a safety margin below supply, $P_{\mathrm{reg,max}} = \max(P_{\mathrm{bottle}} - \Delta P_{\mathrm{margin,reg}}, 0)$. Its outlet pressure tracks first-order:

$$\frac{dP_{\mathrm{reg}}}{dt} = \frac{\min\big(P_{\mathrm{reg,target}},\ P_{\mathrm{reg,max}}\big) - P_{\mathrm{reg}}}{\tau_{\mathrm{reg}}}, \tag{46}$$

with time constant τ_{reg}. In discrete time ($t^n \to t^{n+1} = t^n + \Delta t$),

$$P_{\mathrm{reg}}^{n+1} = P_{\mathrm{reg}}^{n} + \frac{\Delta t}{\tau_{\mathrm{reg}}}\left[\min\big(P_{\mathrm{reg,target}}^{n},\ P_{\mathrm{reg,max}}^{n}\big) - P_{\mathrm{reg}}^{n}\right].$$

Notes. (i) A real regulator exhibits droop (flow-dependent offset). For higher fidelity one can add a static term $-S\,\dot{m}_{\mathrm{He}}$ on the right of (46) with slope S. (ii) If the command exceeds $P_{\mathrm{reg,max}}$, the loop becomes supply-limited; P_{He} will then fall short of (43) unless demand is reduced.

H.3 Injector orifice (compressible flow)

Helium passes from regulator pressure $P_u = P_{\mathrm{reg}}$ at temperature T through an injector of area $A_{\mathrm{He}} = \pi d_{\mathrm{He}}^2/4$ and discharge coefficient $C_{d,\mathrm{He}}$ into the tank ullage at $P_d = P_{\mathrm{tank}}$ (Eq. (39)). Treating He as a perfect gas with ratio of specific heats $\gamma = 1.66$, the isentropic nozzle relations give the critical pressure ratio

$$r_\star \equiv \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} \approx 0.487, \qquad r \equiv \frac{P_d}{P_u}. \tag{47}$$

The mass flow is then

$$\dot{m}_{\mathrm{He}} = \begin{cases} C_{d,\mathrm{He}} A_{\mathrm{He}}\, P_u \sqrt{\dfrac{\gamma}{R_{\mathrm{He}} T}} \left(\dfrac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}, & r \le r_\star \quad \text{(choked)}, \\[2ex] C_{d,\mathrm{He}} A_{\mathrm{He}}\, P_u \sqrt{\dfrac{2\gamma}{(\gamma-1) R_{\mathrm{He}} T}\left(r^{2/\gamma} - r^{(\gamma+1)/\gamma}\right)}, & r > r_\star \quad \text{(subsonic)}. \end{cases} \tag{48}$$

Bottle and ullage masses evolve as

$$\frac{dm_{\mathrm{He,b}}}{dt} = -\,\dot{m}_{\mathrm{He}}, \qquad \frac{dm_{\mathrm{He}}}{dt} = \dot{m}_{\mathrm{He}}, \tag{49}$$

and P_{He} follows from (39). Because P_{tank} appears in r, (48) is coupled to the fluid thermodynamics and line hydraulics (through P_{up}) in a physically consistent way.

Throughput and sizing heuristics. At choked conditions, the injector delivers a nearly constant *mass flux*

$$G_\star = \frac{\dot{m}_{\mathrm{He}}}{A_{\mathrm{He}}} = C_{d,\mathrm{He}}\, P_u \sqrt{\frac{\gamma}{R_{\mathrm{He}} T}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}},$$

so the integrated He added over a burst Δt is roughly $m_{\mathrm{He,add}} \approx G_\star A_{\mathrm{He}} \Delta t$ while P_u and T are quasi-constant. This provides a quick check that the chosen d_{He} and supply can meet transient demands before running the full simulation.

H.4 Coupling to tank pressure and energy

Given $m_{\mathrm{He}}^{n+1} = m_{\mathrm{He}}^{n} + \dot{m}_{\mathrm{He}}^{n} \Delta t$ and the new bulk T^{n+1} from the fluid energy solve (Sec. D.3), the updated He partial pressure is

$$P_{\mathrm{He}}^{n+1} = \frac{m_{\mathrm{He}}^{n+1} R_{\mathrm{He}} T^{n+1}}{V_v^{n+1}}, \qquad P_{\mathrm{tank}}^{n+1} = P_{\mathrm{sat}}(T^{n+1}) + P_{\mathrm{He}}^{n+1}.$$

Energy note. The baseline model neglects the sensible enthalpy of the injected He in the fluid energy residual (it is small relative to N_2O latent/sensible terms for our envelopes). If margins are tight, include a term $-\dot{m}_{\mathrm{He}} c_{p,\mathrm{He}} (T_{\mathrm{in}} - T)$ in (14) (cf. Sec. D.3.5).

H.5 Implementation details and safeguards

- **Smoothing and limits.** Low-pass $P_{\mathrm{He,target}}$ with a short time constant to avoid chattering of P_{reg} when $\dot{m}_{\mathrm{sp}}$ steps. Enforce $P_{\mathrm{reg}} \geq P_{\mathrm{tank}}$ for forward flow; otherwise set $\dot{m}_{\mathrm{He}} = 0$.
- **Supply-limited regime.** When $P_{\mathrm{reg,target}} > P_{\mathrm{reg,max}}$, the bottle cannot sustain the requested overpressure; the no-flash condition (43) may be violated at high setpoints. The controller should either lower $\dot{m}_{\mathrm{sp}}$ or accept reduced margin.
- **Geometry.** Place the injector above the liquid level or with a dip tube that vents in the ullage to avoid direct He sparging into the liquid, which would degrade metering.
- **Real-gas effects.** For very high pressures ($\gtrsim 20 MPa$) or deep bottle blowdown, real-gas Z-factors can be applied in (45) and (48). For our ranges, the ideal-gas assumption is adequate within other design uncertainties.

I. Simulation Algorithm (One Step)

This section formalizes the one-step march from t^n to $t^{n+1} = t^n + \Delta t$ and clarifies the causal ordering, couplings, numerical choices, and diagnostic checks. The solver is *partitioned* but *consistent*: hydraulics and controls are advanced explicitly on the current thermophysical state; the tank thermodynamics are then updated semi-implicitly (Sec. D.3) with wall nodes advanced in a stable, explicit manner (Sec. E).

I.1 State vector and auxiliaries. At time t^n we hold

$$\mathbf{x}^n = (T^n,\ m_\ell^n,\ m_v^n,\ m_{\mathrm{He}}^n,\ m_{\mathrm{He,b}}^n,\ A^n,\ P_{\mathrm{reg}}^n,\ T_{w,\ell}^n,\ T_{w,v}^n),$$

and compute properties at T^n:

$$(P_{\mathrm{sat}}(T^n),\ \rho_\ell(T^n),\ \rho_v(T^n),\ \mu_\ell(T^n),\ h_{fg}(T^n)).$$

Geometric auxiliaries follow from m_ℓ^n (Sec. C, Sec. E):

$$V_\ell^n = \frac{m_\ell^n}{\rho_\ell(T^n)}, \qquad V_v^n = V_{\mathrm{tank}} - V_\ell^n, \qquad A_\ell^n,\ A_v^n = \mathcal{A}(m_\ell^n; \text{tank geom.}).$$

I.2 Single-pass update (baseline). For transparency and speed the code uses a single, ordered sweep per step:

1. **Read state & properties.** Evaluate $P_{\mathrm{sat}}(T^n)$, $\rho_\ell(T^n)$, $\mu_\ell(T^n)$; form V_ℓ^n, V_v^n and wetted areas A_ℓ^n, A_v^n.

2. **No-flash target (feedforward helium).** With the commanded setpoint $\dot{m}_{\mathrm{sp}}(t^n)$, predict single-phase line losses using Sec. F at *current* properties:

$$\Delta P_{\mathrm{line,pred}} = \Delta P_{\mathrm{maj}}(\dot{m}_{\mathrm{sp}}; T^n) + \Delta P_{\mathrm{min}}(\dot{m}_{\mathrm{sp}}; T^n).$$

   Compute P_c^{pred} by (41) and $\Delta P_{\mathrm{inj,req}}^{\mathrm{sp}}$ by (42); then form $P_{\mathrm{He,target}}$ via (44).
   Demand a valve-inlet margin (Sec. H.1):

$$P_{\mathrm{He,target}} = \max\big(\Delta P_{\mathrm{line,pred}} + P_{\mathrm{margin}},\ \Delta P_{\mathrm{He,base}}\big), \quad P_{\mathrm{reg,target}} = P_{\mathrm{sat}}(T^n) + P_{\mathrm{He,target}}.$$

3. **Regulator tracking & bottle ceiling.** Update the regulator outlet (first order, Sec. H.2) with supply cap

$$P_{\mathrm{reg,max}}^n = \max\left(\frac{m_{\mathrm{He,b}}^n R_{\mathrm{He}} T_{\mathrm{b}}}{V_{\mathrm{b}}} - \Delta P_{\mathrm{margin,reg}},\ 0\right), \quad P_{\mathrm{reg}}^{n+1/2} = P_{\mathrm{reg}}^n + \frac{\Delta t}{\tau_{\mathrm{reg}}}\Big(\min(P_{\mathrm{reg,target}}, P_{\mathrm{reg,max}}^n) - P_{\mathrm{reg}}^n\Big).$$

4. **He injector flow & partial pressure.** Using $P_u = P_{\mathrm{reg}}^{n+1/2}$, $P_d = P_{\mathrm{tank}}^n = P_{\mathrm{sat}}(T^n) + P_{\mathrm{He}}^n$, compute $\dot{m}_{\mathrm{He}}^n$ via (48). Update masses explicitly:

$$m_{\mathrm{He,b}}^{n+1} = m_{\mathrm{He,b}}^n - \dot{m}_{\mathrm{He}}^n \Delta t, \qquad m_{\mathrm{He}}^{n+1/2} = m_{\mathrm{He}}^n + \dot{m}_{\mathrm{He}}^n \Delta t,$$

   and form a provisional He partial pressure using T^n, V_v^n:

$$P_{\mathrm{He}}^{n+1/2} = \frac{m_{\mathrm{He}}^{n+1/2} R_{\mathrm{He}} T^n}{V_v^n}, \quad P_{\mathrm{tank}}^{n+1/2} = P_{\mathrm{sat}}(T^n) + P_{\mathrm{He}}^{n+1/2}.$$

5. **Line hydraulics and pre-valve pressure.** With $P_{\mathrm{tank}}^{n+1/2}$, compute a *trial* valve drop $\Delta P_{\mathrm{valve}}^{\mathrm{trial}} = \max(P_{\mathrm{tank}}^{n+1/2} - P_{\mathrm{back}}, 0)$. Evaluate the line model (Sec. F) at the trial flow (or last flow) to obtain $\Delta P_{\mathrm{line}}^n$ and

$$P_{\mathrm{up}}^n = P_{\mathrm{tank}}^{n+1/2} - \Delta P_{\mathrm{line}}^n, \qquad \Delta P_{\mathrm{valve}}^n = \max(P_{\mathrm{up}}^n - P_{\mathrm{back}}, 0).$$

6. **Feedforward metering & actuator step.** Compute the feedforward area

$$A_{\mathrm{ff}}^n = \frac{\dot{m}_{\mathrm{sp}}(t^n)}{C_d \sqrt{2\rho_\ell(T^n)\ \max(\Delta P_{\mathrm{valve}}^n, \Delta P_{\mathrm{min}})}},$$

   clip to $[A_{\mathrm{min}}, A_{\mathrm{max}}]$, and advance the actuator (Sec. G)

$$A^{n+1} = A^n + \frac{\Delta t}{\tau_{\mathrm{valve}}}(\mathrm{sat}(A_{\mathrm{ff}}^n) - A^n).$$

   The actual mass flow then follows

$$\dot{m}^{n+1} = C_d\, A^{n+1} \sqrt{2\rho_\ell(T^n)\, \Delta P_{\mathrm{valve}}^n}.$$

7. **Recompute line metrics for logging.** With $\dot{m}^{n+1}$ and the current properties, recompute $\Delta P_{\mathrm{maj}}^n$, $\Delta P_{\mathrm{min}}^n$, $\Delta P_{\mathrm{line}}^n$, P_{up}^n for output and checks.

8. **Tank thermodynamics (semi-implicit).** Solve the energy residual (13) for T^{n+1} by bisection on a safe bracket (e.g., $T^n - 25K$ to T^n). Using (5) evaluate m_v^{n+1}, then set

$$m_\ell^{n+1} = m_\ell^n - \dot{m}^{n+1}\Delta t - (m_v^{n+1} - m_v^n).$$

9. **He pressure closure with new T, V_v.** Update wetted areas and volumes using T^{n+1}: $V_\ell^{n+1} = m_\ell^{n+1}/\rho_\ell(T^{n+1})$, $V_v^{n+1} = V_{\text{tank}} - V_\ell^{n+1}$. Close the He partial pressure with the *new* state:

$$P_{\text{He}}^{n+1} = \frac{m_{\text{He}}^{n+1/2} R_{\text{He}} T^{n+1}}{V_v^{n+1}}, \qquad P_{\text{tank}}^{n+1} = P_{\text{sat}}(T^{n+1}) + P_{\text{He}}^{n+1}.$$

10. **Wall nodes.** Advance $T_{w,\ell}$, $T_{w,v}$ with explicit Euler or the exact exponential (Sec. E.4), using A_ℓ^{n+1}, A_v^{n+1} and T^{n+1}.

11. **Guards & termination.** Stop the run if (i) $m_\ell^{n+1} \le \epsilon_m$, (ii) T^{n+1} leaves the valid property range, or (iii) any geometry/volume becomes non-physical (negative). Clamp $A_\bullet \to 0$ limits when a wetted patch vanishes.

I.3 Coupling options (inner Picard loop). The baseline uses a single explicit pass. If tighter coupling is needed (e.g., large $\dot{m}_{\text{sp}}$ steps, strong He transients), wrap Steps 2–8 in a 1–3 iteration Picard loop at fixed Δt:

$$\text{iterate } \{\Delta P_{\text{line}},\ P_{\text{up}},\ A,\ \dot{m}\} \quad \text{until} \quad \max\Big(\frac{|\dot{m}^{(k)} - \dot{m}^{(k-1)}|}{\dot{m}^{(k)}},\ \frac{|P_{\text{up}}^{(k)} - P_{\text{up}}^{(k-1)}|}{P_{\text{up}}^{(k)}}\Big) < \varepsilon.$$

Convergence is usually monotone because (i) the line model is quadratic in $\dot{m}$ and (ii) the feedforward mapping $A(\Delta P)$ is Lipschitz on $[\Delta P_{\min}, \infty)$.

I.4 Time-step selection and stability. Choose Δt to satisfy the most restrictive first-order lags:

$$\Delta t \lesssim 0.2 \min\left(\tau_{\text{valve}},\ \tau_{\text{reg}},\ \tau_{w,\ell},\ \tau_{w,v}\right),$$

with wall time constants from Sec. E.3. For the semi-implicit thermodynamics, the bisection is unconditionally convergent provided the bracket covers the expected ΔT_b (typically $\sim 10K$–$30K$ per short step under strong evaporation).

I.5 Conservation diagnostics (recommended). Track stepwise residuals to verify consistency:

$$\mathcal{M}_{\text{N2O}}^n = \Big[m_\ell^{n+1} + m_v^{n+1}\Big] - \Big[m_\ell^n + m_v^n - \dot{m}^{n+1}\Delta t\Big], \tag{50}$$

$$\mathcal{M}_{\text{He}}^n = \Big[m_{\text{He}}^{n+1/2} + m_{\text{He,b}}^{n+1}\Big] - \Big[m_{\text{He}}^n + m_{\text{He,b}}^n\Big], \tag{51}$$

$$\mathcal{E}_{\text{fluid}}^n = m_\ell^n c_{p,\ell}(T^{n+1} - T^n) + (m_v^{n+1} - m_v^n) h_{fg}(T^{n+1}) - Q_{w\to f}^{n+1}\Delta t \ \stackrel{!}{=}\ 0, \tag{52}$$

with $Q_{w\to f}$ from (17). In practice $|\mathcal{M}|/\text{inventory} \lesssim 10^{-6}$ and $|\mathcal{E}|/\text{latent} \lesssim 10^{-5}$ indicate healthy steps.

I.6 Event handling and safeguards.

- **Degenerate areas:** If $A_\ell \to 0$ or $A_v \to 0$, freeze the corresponding wall node (Sec. E.6).
- **Pressure floors:** Use $\Delta P_{\min} \sim 1kPa$ when inverting $\sqrt{\Delta P}$ for A_{ff} to avoid blow-up as $P_{\text{up}} \to P_{\text{back}}$.

- **Forward flow only:** Enforce $P_{\text{reg}} \geq P_{\text{tank}}$; otherwise set $\dot{m}_{\text{He}} = 0$.
- **Property bounds:** Clip any trial T^{n+1} to $[T_{\min} + \epsilon,\ T_{\max} - \epsilon]$ during bracketing to keep property calls defined.

I.7 Computational notes. Cache property evaluations that repeat within the step; the most expensive call is $P_{\text{sat}}(T)$ and density pair (ρ_ℓ, ρ_v). Bracketing for the energy solve can reuse $h_{fg}(T)$ evaluations. With the baseline single-pass scheme, the algorithm scales $\mathcal{O}(1)$ per step and is robust for Δt in the $5ms$–$50ms$ range typical of hardware-in-the-loop studies.

Summary. The ordered sweep keeps the hydraulics/control causal and uses an unconditionally convergent semi-implicit closure for the two-phase tank thermodynamics. Optional Picard iterations further reduce lag at negligible cost if aggressive setpoint profiles are exercised.

J. Numerical Methods and Stability Notes

This section records the numerical choices that make the simulator robust, fast, and auditable. We cover the root solve for the tank thermal state, explicit/implicit time-stepping constraints, actuator/regulator lags, supply ceilings, and model-consistency checks.

J.1 Energy residual solve for T^{n+1} (root finding). Let the scalar residual be (Sec. D.3)

$$\mathcal{R}(T^{n+1}) \equiv m_\ell^n c_{p,\ell}\,(T^{n+1}-T^n)+\Big(m_v^{n+1}(T^{n+1})-m_v^n\Big)\,h_{fg}(T^{n+1})-\Big[h_{\text{in,liq}}A_\ell^n\,(T_{w,\ell}^n-T^{n+1})+h_{\text{in,vap}}A_v^n\,(T_{w,v}^n-T^{n+1} \tag{53}$$

where $m_v^{n+1}(T)$ is given by the isochoric closure (5) evaluated with properties at T. The step temperature T^{n+1} satisfies $\mathcal{R}(T^{n+1}) = 0$.

Bracketing. Define a conservative bracket

$$T_{\min}^{\text{br}} = \max(T_{\text{prop,min}} + \varepsilon,\ T^n - \Delta T_b), \qquad T_{\max}^{\text{br}} = \min(T^n,\ T_{\text{prop,max}} - \varepsilon),$$

with $\Delta T_b \sim 25K$ and a small guard ε. On this interval, $\mathcal{R}$ is continuous and, for physical states, monotone increasing because $c_{p,\ell} > 0$, $h_{fg} > 0$, and $m_v^{n+1}(T)$ is a smooth, increasing function of T from (5). Bisection is therefore *globally convergent*.

Hybrid accelerate. For speed, a safeguarded Newton–secant can be used once the bracket is tight:

$$\text{(i) Bisection until } |\mathcal{R}| \text{ decreases by an order of magnitude;} \tag{54}$$

$$\text{(ii) } T \leftarrow T - \frac{\mathcal{R}(T)}{\mathcal{R}'(T)} \quad \text{if the Newton step stays inside the bracket, else bisect.} \tag{55}$$

A practical Jacobian proxy $\mathcal{R}'(T)$ is $\mathcal{R}' \approx m_\ell^n c_{p,\ell} + (m_v^{n+1} - m_v^n)\,h'_{fg} + \left(\frac{\partial m_v^{n+1}}{\partial T}\right) h_{fg} + (h_{\text{in,liq}}A_\ell^n + h_{\text{in,vap}}A_v^n)\Delta t$, or simply use a secant update.

Stopping. Terminate when both criteria hold:

$$|T^{k+1} - T^k| \leq \tau_T^{\text{abs}} + \tau_T^{\text{rel}}\,|T^{k+1}|, \qquad |\mathcal{R}(T^{k+1})| \leq \tau_{\mathcal{R}},$$

with $\tau_T^{\text{abs}} \sim 1 * 10^{-4}K$, $\tau_T^{\text{rel}} \sim 10^{-8}$, and $\tau_{\mathcal{R}}$ chosen so that the implied heat error is $\ll 10^{-5}$ of the latent term for the step.

Pathological states. If m_ℓ^n is vanishing or $1 - \rho_v/\rho_\ell$ is tiny, (5) can be ill-conditioned. Clamp the denominator with a small floor and (optionally) switch to a fully explicit m_v update for the last few grams to exit gracefully.

J.2 Time step selection and stability. Choose Δt against the fastest first-order dynamics and explicit thermal nodes:

$$\Delta t \;\lesssim\; 0.2\min\Big(\tau_{\text{valve}},\, \tau_{\text{reg}},\, \tau_{w,\ell},\, \tau_{w,v}\Big), \qquad \tau_{w,i} \equiv \frac{C_{w,i}}{(h_{\text{out}} + h_{\text{in},i})A_i}, \quad i \in \{\ell, v\}. \tag{56}$$

This keeps the explicit Euler wall updates well inside their stability regions and preserves control causality. If using the *exact exponential* update for the wall nodes (Sec. E.4), the wall dynamics become unconditionally stable; then (56) reduces to the control lags. As the liquid film shrinks ($A_\ell \downarrow$), $C_{w,\ell} \propto A_\ell$ falls and $\tau_{w,\ell}$ can become very small; either (i) switch to the exponential update or (ii) reduce Δt adaptively near drain-out.

J.3 Actuator saturation and windup. Because the plant input is the *area* A obeying $\dot{A} = (A_{\text{cmd}} - A)/\tau_{\text{valve}}$, there is no integral state on control error; saturation at $A_{\min}/A_{\max}$ therefore does not accumulate windup. If a position controller on stem travel is later added, apply standard anti-windup clamping.

J.4 Regulator ceiling and supply-limited behavior. The cap $P_{\text{reg,max}} = P_{\text{bottle}} - \Delta P_{\text{margin,reg}}$ (see (45)) prevents demanding an outlet above supply. As $m_{\text{He,b}}$ decays, $P_{\text{reg,max}}$ falls; when $P_{\text{reg,target}} > P_{\text{reg,max}}$ the system becomes *supply-limited*. In that regime, P_{He} may not meet the no-flash requirement (43) at high $\dot{m}_{\text{sp}}$. Operational mitigations: (i) reduce $\dot{m}_{\text{sp}}$, (ii) raise $\Delta P_{\text{He,base}}$ earlier in the shot, or (iii) upsize V_b/P_0.

J.5 Hydraulics nonlinearity and fixed-point convergence. The line drop scales quadratically with flow, $\Delta P_{\text{line}} \approx (fL/D + K_{\text{tot}})\,\dot{m}^2/(2\rho_\ell A^2)$, while $f = f(\text{Re})$ depends on $\dot{m}$ only weakly in turbulent flow. A one–three step Picard iteration (Sec. I.3) is typically sufficient to converge the coupled set $\{\Delta P_{\text{line}}, P_{\text{up}}, A, \dot{m}\}$. A contraction estimate follows from Churchill's correlation: for $\text{Re} \gtrsim 10^4$ and small roughness, $|\partial f/\partial \dot{m}|$ is $\mathcal{O}(10^{-5})$–$\mathcal{O}(10^{-4})$ per % change, ensuring rapid monotone convergence.

J.6 Model validity and consistency monitor. The flashing controller is derived from a *single-phase* prediction at the setpoint. Once the commanded margin is achieved, the model is self-consistent because

$$P_{\text{up}} - P_{\text{sat}}(T) \;\geq\; P_{\text{margin}} \;>\; 0$$

at the actual flow. As a runtime check, flag a warning if the logged margin falls below a threshold (e.g., $0.5\,P_{\text{margin}}$) for more than a few time steps; this typically indicates supply-limited He, underestimated K's, or a larger-than-anticipated temperature drop.

J.7 Error controls, conservation, and step adaptivity. At each step, compute the conservation diagnostics (Sec. I.5). Recommended thresholds:

$$\frac{|\mathcal{M}_{\text{N2O}}|}{m_\ell + m_v} \leq 10^{-8}, \quad \frac{|\mathcal{M}_{\text{He}}|}{m_{\text{He}} + m_{\text{He,b}}} \leq 10^{-10}, \quad \frac{|\mathcal{E}_{\text{fluid}}|}{|(m_v^{n+1} - m_v^n)h_{fg}|} \leq 10^{-6}.$$

If any bound is exceeded, *reject* the step, halve Δt, and recompute; if well below, consider increasing Δt by up to 25% to speed long holds.

J.8 Floating-point conditioning and guards. Use `float64`. Protect divisions with small floors:

$$A \geq A_{\min}^{\text{num}}, \quad \mu_\ell \geq \mu_{\min}^{\text{num}}, \quad 1 - \rho_v/\rho_\ell \geq \delta_\rho,$$

with $A_{\min}^{\text{num}} \sim 1 * 10^{-12} m^2$, $\mu_{\min}^{\text{num}} \sim 1 * 10^{-6} Pa\,s$, $\delta_\rho \sim 10^{-6}$. When inverting ΔP inside a square root, use a floor $\Delta P_{\min} \sim 1kPa$ to avoid division blow-up near zero differential.

J.9 Property evaluations and caching. The expensive calls are $P_{\mathrm{sat}}(T)$ and $\{\rho_\ell(T), \rho_v(T), h_{fg}(T)\}$. Inside the root solve, cache property values at trial T to avoid repeated evaluations at the same points. If CoolProp is not available, pre-interpolate the tabulated arrays onto a fine T grid and use linear or monotone cubic interpolation.

J.10 Computational cost and scaling. Per step cost is $\mathcal{O}(1)$: one hydraulics evaluation, one or two orifice evaluations, and $\mathcal{O}(10)$–$\mathcal{O}(40)$ residual evaluations during bisection (often fewer with hybrid Newton). At $\Delta t \in [5ms, 20ms]$, a $40s$ profile runs in $\mathcal{O}(10^3)$ steps and completes in milliseconds–seconds on a laptop.

J.11 Reproducibility and regression tests. Fix all inputs and property sources to ensure deterministic runs. Include simple regression cases:

1. *No He, no line losses:* $K_{\mathrm{tot}} = 0$, $L = 0$, confirming $\dot{m}$ tracks with $\Delta P_{\mathrm{valve}} = P_{\mathrm{sat}} - P_{\mathrm{back}}$.
2. *He-only step:* $\dot{m}_{\mathrm{sp}} = 0$ and a jump in $P_{\mathrm{reg,target}}$; check m_{He} gain = bottle loss.
3. *Isothermal check:* $h_{\mathrm{in},\bullet} = h_{\mathrm{out}} = 0$; energy residual reduces to latent + sensible terms.

Summary. A bracketed (safeguarded) root solve for T^{n+1}, small-step explicit updates for controls and hydraulics (with optional Picard refinement), and conservative floors/ceilings on ill-conditioned expressions together yield a numerically stable and physically faithful simulator throughout the operating envelope.

L. Limitations and Extensions

The present model is intentionally compact to enable rapid trade studies and control design. This section documents what is *not* captured and outlines credible extensions for higher-fidelity analysis when moving toward qualification or certification.

L.1 Valve internals and cavitation metrics (out of scope). The metering element is represented by a lumped C_dA (Sec. G) with no explicit vena–contracta model. True cavitation and flashing limits depend on proprietary trim geometry through vendor indices (e.g., liquid pressure recovery factor F_L, critical pressure ratio factor F_F). In practice one should:

- Map operating points (P_1, P_2, T, ρ_ℓ) into vendor curves and enforce the vendor choked-liquid limit
$$Q_{\max} \approx N_1 C_v F_L \sqrt{\frac{P_1 - F_F P_{\mathrm{sat}}(T)}{\mathrm{SG}}}$$
with the appropriate unit constants N_1.
- Include trim-specific *incipient cavitation* and *constant cavitation* indices if noise/erosion limits are relevant.
- Replace (34) by a C_v-based dynamic map $Q(\Delta P, \mathrm{stroke})$ when vendor characterization is available.

L.2 Two-phase flow in the feedline (not modeled). The line model (Sec. F) assumes single-phase liquid. If the design permits or intentionally uses two-phase upstream of the valve, the momentum balance must include acceleration and slip:
$$\Delta P = \underbrace{\int f \frac{L}{D} \frac{\rho_m v_m^2}{2}\, dx}_{\text{wall}} + \underbrace{\int K(x) \frac{\rho_m v_m^2}{2}\, dx}_{\text{minor}} + \underbrace{\int \dot{m}\, dv_m}_{\text{acceleration}},$$

with mixture density ρ_m, velocity v_m, and a closure for vapor quality x and slip ratio $S = v_v/v_\ell$. Candidate closures:

- **HEM (homogeneous equilibrium):** $S = 1$, thermodynamic equilibrium; simplest, optimistic for pressure drop.
- **HRM (homogeneous relaxation):** adds a finite equilibration time toward saturation; captures overshoots during fast transients.
- **Separated-flow (Lockhart–Martinelli/Chisholm):** friction multipliers $\phi_\ell^2(x, S)$ with empirical C-factors.

The no-flash controller in Sec. H.1 should then be re-derived to target a *two-phase* P_{up} criterion or to regulate vapor quality at the valve inlet.

L.3 Wall model fidelity (lumped two-node approximation). The wall is represented by two lumped nodes with constant film coefficients (Sec. E). Not captured:

- Axial/radial conduction and thermal stratification along the shell, penetrations, and supports.
- State-dependent HTC variation (natural/forced convection, film condensation/boiling, frosting).
- Radiative exchange to environment and within insulation.

Extensions:

1. **Grey-box CHT:** partition the shell axially into N segments, each with separate (A_ℓ, A_v) and film coefficients; integrate N ODEs for $T_{w,\ell}^{(i)}, T_{w,v}^{(i)}$.
2. **1D conduction:** add a radial node in the insulation with $k(T)$ and an external radiation term $\sigma\varepsilon(T_{\text{amb}}^4 - T_{\text{surf}}^4)$.
3. **Film correlations:** switch $h_{\text{in},\bullet}$ based on regime maps (laminar/turbulent natural convection, condensation Nusselt law, etc.).

L.4 Thermophysical properties and phase model. We assume saturation thermodynamics (one temperature T and $P_{\text{sat}}(T)$) and ideal-gas He (Sec. D). Limitations:

- **Near-critical behavior:** As $T \to T_c$, $\rho_\ell \downarrow$, $\rho_v \uparrow$, and $h_{fg} \to 0$; tabular fallbacks may lose accuracy. Prefer CoolProp/REFPROP in this regime.
- **Real-gas He:** At very high supply pressures ($\gtrsim 20\,\text{MPa}$) or low temperatures, include compressibility factor Z in bottle state and injector flow.
- **Multicomponent ullage:** Trace N_2O dissolved in He and vice versa are neglected; Dalton's law is used with $P_{\text{tank}} = P_{\text{sat}} + P_{\text{He}}$.

L.5 Helium supply and regulator physics. The bottle is modeled isothermal and the regulator has a simple first-order lag (Secs. H.2–H.3). Missing effects:

- **Bottle blowdown cooling:** Real bottles cool during discharge; add an energy balance

$$m_b c_{p,b} \frac{dT_b}{dt} = -\dot{m}_{\text{He}} h_{\text{out}} - h_{\text{wall}} A_b (T_b - T_{\text{amb}})$$

  and use real-gas enthalpy $h_{\text{out}}(P, T)$.

- **Regulator droop/hysteresis:** Add a static flow-dependent droop $P_{\mathrm{reg}} = P_{\mathrm{set}} - S\,\dot{m}_{\mathrm{He}}$ and rate limits on opening/closing.
- **Downstream plumbing:** Include line volume/compliance between regulator and injector for transient studies (pressure wave travel, choked length).

L.6 Tank fluid dynamics (uniform bulk assumption). The tank liquid and vapor are well-mixed in the baseline model. Not represented:

- **Thermal stratification:** vertical gradients in T and density; can bias P_{sat} and wall heat fluxes.
- **Slosh and acceleration fields:** non-horizontal interfaces under vehicle motion; add hydrostatic head $\rho g \Delta z$ and dynamic pressure terms.
- **Free-surface mass transfer limits:** finite interfacial area and transport resistances; HRM can emulate finite-rate evaporation/condensation.

L.7 Feedline dynamics and transients. The line is quasi-steady; acoustics and waterhammer are neglected. For fast valve moves or long, small-ID lines:

- Add a **liquid compressibility** state and wave speed $a = \sqrt{K_\ell/\rho_\ell}$; solve 1D waterhammer (Method of Characteristics) to bound dP/dt and peak loads.
- Model **entrained microvapor** (microcavitation) for fast ramps; insert a small compliance element at the valve inlet.

L.8 Control architecture extensions. Current logic is feedforward (area from setpoint) with a no-flash He scheduler (Sec. H.1). Enhancements:

- Add **feedback** on $\dot{m}$ (flowmeter) with PI + feedforward for setpoint tracking under disturbances.
- Schedule C_d or $A(\xi)$ versus Reynolds number and trim position for improved linearity.
- Supervisory logic to **de-rate** $\dot{m}_{\mathrm{sp}}$ when He becomes supply-limited or when the flashing margin falls below a threshold.

L.9 Uncertainty quantification and design margins. Major contributors: K-coefficients ($\pm$20–50% typical without vendor data), roughness ε, C_d, property data, and unmodeled dynamics. Recommended practice:

1. **Sensitivity sweep:** vary each parameter over its plausible range and track $\min(P_{\mathrm{up}} - P_{\mathrm{sat}})$ and peak He usage.
2. **Monte Carlo:** apply independent priors and design P_{margin} to achieve a target probability of no-flash (e.g., $> 99\%$).
3. **Test correlation:** back-fit C_d, K_{tot}, and regulator droop to hot-fire or flow-stand data; update the model.

L.10 Verification & validation (V&V) roadmap.

- **Code verification:** unit tests for conservation identities (Sec. I.5), regression baselines, and dimensional checks.
- **Model validation:** compare predicted $\dot{m}(t)$, $P_{\text{tank}}(t)$, $P_{\text{up}}(t)$, and He consumption to instrumented runs; refine C_d, K's, and HTC's.
- **Acceptance envelope:** document the range of T, P, $\dot{m}$, and ullage fractions over which residuals and conservation errors remain below specified tolerances.

L.11 Safety and operability notes.

- The model does not address **$\mathbf{N_2O}$ decomposition hazards**, ignition sources, or relief sizing; these require dedicated safety analyses.
- Ensure adequate **venting and relief** for He over-pressurization scenarios (stuck regulator, blocked injector).
- Instrumentation accuracy (pressure, temperature, flow) and sampling rates should be specified to support the controller's assumptions and to detect off-nominal states.

Summary. The baseline simulator is appropriate for sizing, control concept development, and He consumption budgeting. For certification-level predictions or operation at the edge of the envelope (near-critical T, aggressive transients, or intentional two-phase), the extensions above should be implemented and correlated to test.

M. Default Parameters (Representative)

Parameter	Nominal value	Notes
V_{tank}	$0.034\,\text{m}^3$	34 L cylinder, upright
Ullage (init.)	20%	V_v/V_{tank}
T_0	$293.15\,\text{K}$	$20\,^\circ\text{C}$ initial
C_d	0.80	Metering discharge coefficient
τ_{valve}	$0.10\,\text{s}$	First-order actuator
$h_{\text{in,liq}}$	$800\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$	Internal HTC, liquid-wet
$h_{\text{in,vap}}$	$15\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$	Internal HTC, vapor-wet
h_{out}	$8\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$	External HTC
t_{wall}	5 mm	Uniform wall
Tube spec	1/2 OD $\times$ 0.035 in	Drawn 316 SS (ID ≈ 10.9 mm)
L_{pipe}	1.5 m	Horizontal
K_{tot}	From counts	Inlet + two 45° elbows + ball valve
He injector d_{He}	1.2 mm	Circular orifice
$C_{d,\text{He}}$	0.85	He orifice discharge coefficient
$P_{0,\text{bottle}}$	20 MPa	200 bar
T_{b}	300 K	Bottle temperature
V_{b}	6 L	Bottle free volume
τ_{reg}	0.15 s	Regulator lag
P_{margin}	0.25 MPa	No-flash margin
Engine / Injector		
c^*	1382.22 m/s	Verified at $P_c \approx 2.413$ MPa (350 psi)
A_t	$4.474\,36 \times 10^{-4}\,\text{m}^2$	Nozzle throat area
Expansion ratio ε	3.6631	Context
L^*	0.74836	Context
A_{inj}	*[fill in]*	Total injector geometric area
$C_{d,\text{inj}}$	0.80	Assumed

N. Glossary of Symbols

Symbol	Definition (SI units)
V_{tank}	Tank internal volume (m^3)
D, H	Tank diameter, height (m)
A_ℓ, A_v	Liquid-wet and vapor-wet wall areas (m^2)
T	Tank bulk temperature (K)
$P_{\text{sat}}(T)$	N_2O saturation pressure at T (Pa)
m_ℓ, m_v	Liquid and vapor N_2O masses (kg)
ρ_ℓ, ρ_v	Liquid and vapor densities ($\text{kg}\,\text{m}^{-3}$)
μ_ℓ	Liquid viscosity ($\text{Pa}\,\text{s}$)
$h_{fg}(T)$	Latent heat of vaporization ($\text{J}\,\text{kg}^{-1}$)
$c_{p,\ell}$	Liquid specific heat ($\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$)
$T_{w,\ell}, T_{w,v}$	Liquid-wet and vapor-wet wall temperatures (K)
$h_{\text{in,liq}}, h_{\text{in,vap}}$	Internal HTC's ($\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$)
h_{out}	External HTC ($\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$)
t_{wall}	Wall thickness (m)
$C_{w,\bullet}$	Wall thermal capacitance for node $\bullet$ ($\text{J}\,\text{K}^{-1}$)
P_{He}	He partial pressure in ullage (Pa)
m_{He}	He mass in ullage (kg)
R_{He}	He gas constant ($2077.1\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$)
P_{tank}	Total tank pressure $= P_{\text{sat}} + P_{\text{He}}$ (Pa)
D_{pipe}	Pipe inside diameter (m)
L_{pipe}	Pipe length (m)
ε	Pipe roughness (m)
K_{tot}	Sum of minor-loss coefficients (-)
Re	Reynolds number (-)
f	Darcy friction factor (-)
$\Delta P_{\text{maj}}, \Delta P_{\text{min}}$	Major/minor line pressure drops (Pa)
ΔP_{line}	Total line pressure drop (Pa)
P_{up}	Pre-valve pressure (Pa)
P_{back}	Downstream/back pressure (Pa)
A	Metering area (m^2)
C_d	Metering discharge coefficient (-)
τ_{valve}	Valve actuator time constant (s)
$\dot{m}, \dot{m}_{\text{sp}}$	Actual and setpoint mass flow ($\text{kg}\,\text{s}^{-1}$)
P_{reg}	Regulator outlet pressure (Pa)
τ_{reg}	Regulator time constant (s)
A_{He}	He injector area (m^2)
$C_{d,\text{He}}$	He injector discharge coefficient (-)
$m_{\text{He,b}}, V_{\text{b}}, T_{\text{b}}$	He bottle mass, volume, temperature
P_{bottle}	He bottle pressure (Pa)
P_{margin}	Required pre-valve margin above P_{sat} (Pa)
A_{inj}	Total geometric injector area (m^2)
$C_{d,\text{inj}}$	Injector discharge coefficient (-)
c^*	Characteristic velocity ($\text{m}\,\text{s}^{-1}$)
A_t	Nozzle throat area (m^2)
P_c	Chamber pressure $= \dot{m}\,c^*/A_t$ (Pa)
$\Delta P_{\text{inj,req}}$	Injector required pressure drop (Pa)

O. Practical Guidance

- Use *conservative* K values until vendor data are available; re-run with final components.
- For long runs or smaller ID, increase P_{margin} and/or smooth $\dot{m}_{\text{sp}}$ ramps.
- Verify numerical robustness by halving Δt; results should be insensitive.
- For vertical separations, add $\rho_\ell g \Delta z$ to ΔP_{line}.
- Size A_{inj} so $\Delta P_{\text{inj,req}}$ meets spec at $\dot{m}_{\max}$; otherwise $\Delta P_{\text{inj,req}} \propto \dot{m}^2$ will drive helium demand and margin.