CS271: Project 1

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1. Prove Theorem 3.1 on page 48:

For any two functions $f(n)$ and $g(n)$, $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Proof. Assume that $f(n) = \Theta(g(n))$. We will prove that $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. By the definition of Θ-notation, since $f(n) = \Theta(g(n))$, there exists positive constants c_1, c_2, and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$. Since $0 \leq c_1 g(n) \leq f(n)$ for all $n \geq n_0$, where c_1 and n_0 are positive constants, $f(n) = \Omega(g(n))$. As $0 \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$ with positive constants c_2 and n_0, $f(n) = O(g(n))$.
Assume that $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. We will prove that $f(n) = \Theta(g(n))$. Because $f(n) = \Omega(g(n))$, there exists positive constants c_3, n_3 such that $0 \leq c_3 g(n) \leq f(n)$ for all $n \geq n_3$. Since $f(n) = O(g(n))$, there exists positive constants c_4, n_4 such that $0 \leq f(n) \leq c_4 g(n)$ for all $n \geq n_4$. If we pick $n_5 = max\{n_3, n_4\}$, then $0 \leq c_3 g(n) \leq f(n) \leq c_4 g(n)$ for all $n \geq n_5$. Thus, $f(n) = \Theta(g(n))$.
Therefore, for any two functions $f(n)$ and $g(n)$, $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. □

2. Prove the following using the definitions of O, Ω, and Θ.

(a) $n^2 + 3n - 20 = O(n^2)$

Proof. We need to find positive constants c and n_0 such that $n^2 + 3n - 20 \leq cn^2$ for all $n \geq n_0$. Dividing both sides by n^2 gives $1 + 3/n - 20/n^2 \leq c$. This inequality holds if we choose $c = 2$ and $n_0 = 4$. Since there exists $c = 2$ and $n_0 = 4$ such that $0 \leq n^2 + 3n - 20 \leq cn^2$ for all $n \geq n_0$, we have that $n^2 + 3n - 20 = O(n^2)$. □

(b) $n - 2 = \Omega(n)$

Proof. We need to find positive constants c and n_0 such that $n - 2 \geq cn$ for all $n \geq n_0$. Dividing both sides by n gives $1 - 2/n \geq c$. We have that

$$
\begin{aligned}
1 - \frac{2}{n} &\geq 1 - \frac{2}{4} \quad \text{(when } n \geq 4\text{)} \\
&\geq \frac{1}{2}
\end{aligned}
$$

Thus, we can choose $n_0 = 4$ and $c = 1/2$ so that $n - 2 \geq cn \geq 0$ for all $n \geq n_0$. Therefore, $n - 2 = \Omega(n)$. $\square$

(c) $\log_{10} n + 4 = \Theta(\log_2 n)$

Proof. First, we will prove that $\log_{10} n + 4 = \Omega(\log_2 n)$. We need to find positive constants c_1 and n_1 such that $\log_{10} n + 4 \geq c_1 \log_2 n \geq 0$ for all $n \geq n_1$. We have that $\log_{10} n + 4 = \log_{10} 2. \log_2 n + 4 \geq c_1 \log_2 n$. Dividing both sides by $\log_2 n$ gives $\log_{10} 2 + 4/\log_2 n \geq c_1$. This inequality holds when $n \geq 2$ and $c_1 = 0.3$. Since there exists positive constants $c_1 = 0.3$ and $n_1 = 2$ such that $\log_{10} n + 4 \geq c_1 \log_2 n \geq 0$ for all $n \geq n_1$, $\log_{10} n + 4 = \Omega(\log_2 n)$.
Next, we will prove that $\log_{10} n + 4 = O(\log_2 n)$, meaning that we have to find c_2 and n_2 such that $\log_{10} n + 4 \leq c_2 \log_2 n$ for all $n \geq n_2$. Dividing both sides by $\log_2 n$ gives $\log_{10} 2 + 4/\log_2 n \leq c_2$. When $n \geq 16$, we have

$$
\begin{aligned}
\log_{10} 2 + \frac{4}{\log_2 n} &\leq \log_{10} 2 + \frac{4}{\log_2 16} \\
&= \log_{10} 2 + 1 \\
&< 2
\end{aligned}
$$

Hence, by choosing $c_2 = 2$ and $n_2 = 16$, we have $0 \leq \log_{10} n + 4 \leq c_2 \log_2 n$ for all $n \geq n_2$. Therefore, $\log_{10} n + 4 = O(\log_2 n)$.
Since $\log_{10} n + 4 = \Omega(\log_2 n)$ and $\log_{10} n + 4 = O(\log_2 n)$, it must be the case that $\log_{10} n + 4 = \Theta(\log_2 n)$ (by the theorem we proved in question 1). $\square$

(d) $2^{n+1} = O(2^n)$

Proof. To prove that $2^{n+1} = O(2^n)$, we will find positive constants c and n_0 such that $0 \leq 2^{n+1} \leq c2^n$ for all $n \geq n_0$. Since $2^{n+1} = 2 \times 2^n \leq c2^n$ when $c \geq 2$, we can choose $c = 2$ and $n_0 = 1$ so that $0 \leq 2^{n+1} \leq c2^n$ for all $n \geq n_0$. Hence, $2^{n+1} = O(2^n)$. $\square$

(e) $\ln n = \Theta(\log_2 n)$

Proof. We will first prove that $\ln n = \Omega(\log_2 n)$. We need to find positive constants c_1 and n_1 such that $\ln n \geq c_1 \log_2 n$ for all $n \geq n_1$. Dividing both sides by $\log_2 n$ gives $\ln 2 \geq c_1$. This inequality holds if we choose $c_1 = 0.5$ and $n_1 = 1$. As there exists $c_1 = 0.5$ and $n_1 = 1$ such that $\ln n \geq c_1 \log_2 n \geq 0$ for all $n \geq n_1$, $\ln n = \Omega(\log_2 n)$.
Next, we will prove that $\ln n = O(\log_2 n)$. We need to find positive constants

c_2 and n_2 such that $0 \leq \ln n \leq c_2 \log_2 n$ for all $n \geq n_2$. We have that $\ln n = \ln 2. \log_2 n \leq c_2 \log_2 n$ when $c_2 \geq \ln 2$ and $n \geq 1$. Thus, we can choose $c_2 = 1$ and $n_2 = 1$ so that $0 \leq \ln n \leq c_2 \log_2 n$ for all $n \geq n_2$. Hence, $\ln n = O(\log_2 n)$. By the theorem we proved in question 1, $\ln n = \Omega(\log_2 n)$ and $\ln n = O(\log_2 n)$ imply that $\ln n = \Theta(\log_2 n)$. □

(f) $n^\epsilon = \Omega(\log_2 n)$ for any $\epsilon > 0$.

Proof. We will prove that there exists positive constants c and n_0 such that $n^\epsilon \geq c \log_2 n \geq 0$ for all $n \geq n_0$. For any $\epsilon > 0$, we can choose $0 < c \leq \epsilon$ so that

$$\begin{aligned} c \log_2 n &\leq \epsilon \log_2 n \\ &= \log_2 n^\epsilon \\ &\leq n^\epsilon - 1 && \text{(when } n \geq 1) \\ &\leq n^\epsilon \end{aligned}$$

Thus, we can choose $c = \epsilon$ and $n_0 = 1$ so that $n^\epsilon \geq c \log_2 n \geq 0$ for all $n \geq n_0$. Hence, $n^\epsilon = \Omega(\log_2 n)$ for any $\epsilon > 0$. □

3. For each of the following recurrences, find a tight upper bound for $T(n)$. Prove that each is correct using induction. In each case, assume that $T(n)$ is constant for $n \leq 2$ and that floor division applies to all recurrences.

(a) $T(n) = 2T(n/2) + n^3$

Proof. We will prove that $T(n) = O(n^3)$. We will adopt the inductive hypothesis that $T(n) \leq cn^3$ for all $n \geq n_0$, where c and n_0 are positive constants. Assume by induction that $T(k) \leq ck^3$ for all $n_0 \leq k < n$. If $n \geq 2n_0$, we have that

$$\begin{aligned} T(n) &= 2T(n/2) + n^3 \\ &\leq 2c\frac{n^3}{8} + n^3 \\ &= \frac{cn^3}{4} + n^3 \\ &= cn^3 - \frac{3cn^3}{4} + n^3 \\ &\leq cn^3 \quad (\text{when } c \geq \frac{4}{3}) \end{aligned}$$

Hence, the inductive case holds. Next, we will prove that the inductive hypothesis holds for the base case of the induction, that is, $T(n) \leq cn^3$ when $n_0 \leq n < 2n_0$. We can choose $n_0 = 1$. We have $T(1) \leq c \leq c1^3$ when $c \geq T(1)$, where $T(1)$ is constant by assumption. Thus, the base case holds. Therefore, we have $T(n) \leq cn^3$ for all $n \geq 1$, which implies that $T(n) = O(n^3)$. □

(b) $T(n) = T(9n/10) + n$

Proof. We will prove that $T(n) = O(n)$. We want to find positive constants c and n_0 such that $T(n) \leq cn$ for all $n \geq n_0$. Assume by induction that $T(k) \leq ck$ for all $n_0 \leq k < n$. If $n \geq (10/9)n_0$, we have that

$$\begin{aligned}
T(n) &= T(9n/10) + n \\
&\leq c.\frac{9n}{10} + n \\
&= cn - \frac{cn}{10} + n \\
&\leq cn \quad \text{(when } c \geq 10\text{)}
\end{aligned}$$

Thus, the inductive case is true. We will prove that the inductive hypothesis holds for the base case of the induction, that is, $T(n) \leq cn$ for all $n_0 \leq n < (10/9)n_0$. If we pick $n_0 = 1$, the base case becomes $T(n) \leq cn$ for all $1 \leq n < 10/9$. Since by assumption $T(1)$ is constant, we have $T(1) \leq c.1$ when $c \geq T(1)$. Therefore, the base case holds. Since $T(n) \leq n$ for all $n \geq 1$, it must be the case that $T(n) = O(n)$. $\square$

(c) $T(n) = 7T(n/3) + n^2$

Proof. The solution to this recurrence is that $T(n) = O(n^2)$. We will adopt the inductive hypothesis that $T(n) \leq cn^2$ for all $n \geq n_0$, where c and n_0 are positive constants. Assume by induction that $T(k) \leq ck^2$ for all $n_0 \leq k < n$. If $n \geq 3n_0$, we have that

$$\begin{aligned}
T(n) &= 7T(n/3) + n^2 \\
&\leq 7\frac{cn^2}{9} + n^2 \\
&= cn^2 - \frac{2cn^2}{9} + n^2 \\
&\leq cn^2 \quad \text{(when } c \geq \tfrac{9}{2}\text{)}
\end{aligned}$$

Therefore, the inductive case holds. We will prove that the inductive hypothesis holds for the base case of the induction, that is, $T(n) \leq cn^2$ when $n_0 \leq n < 3n_0$. We can pick $n_0 = 1$. Since $T(1)$ and $T(2)$ are constants, we have $T(1) \leq c1^2$ and $T(2) \leq c1^2$ when $c \geq \max\{T(1), T(2)\}$. Thus, the base case holds. Therefore, we have $T(n) \leq cn^2$ for all $n_0 \geq 1$, which implies that $T(n) = O(n^2)$ $\square$

(d) $T(n) = T(\sqrt{n}) + 1$

Proof. The solution to this recurrence is that $T(n) = O(\log_2 \log_2 n)$. We will adopt the inductive hypothesis that $T(n) \leq c \log_2 \log_2 n$ for all $n \geq n_0$, where

c and n_0 are positive constants. Assume by induction that $T(k) \le c \log_2 \log_2 k$ for all $n_0 \le k < n$. If $n \ge n_0^2$, we have

$$
\begin{aligned}
T(n) &= T(\sqrt{n}) + 1 \\
&\le c \log_2 \log_2 \sqrt{n} + 1 \\
&= c \log_2 \log_2 n^{\frac{1}{2}} + 1 \\
&= c \log_2 (\frac{1}{2} \log_2 n) + 1 \\
&= c \log_2 \log_2 n + c \log_2 (\frac{1}{2}) + 1 \\
&= c \log_2 \log_2 n - c + 1 \\
&\le c \log_2 \log_2 n \quad \text{(when } c \ge 1\text{)}
\end{aligned}
$$

Hence, the inductive hypothesis holds for the inductive case. We will prove that the inductive hypothesis holds for the base case of the induction, that is, $T(n) \le c \log_2 \log_2 n$ when $n_0 \le n < n_0^2$. If we pick $n_0 = \sqrt{5}$, the base case becomes $T(n) \le c \log_2 \log_2 n$ when $\sqrt{5} \le n < 5$, so we will check $T(3)$ and $T(4)$. Since $T(n)$ is constant for all $n \le 2$, $T(3) = T(\sqrt{3}) + 1 = T(1) + 1$ and $T(4) = T(2) + 1$ are also constant. When $c \ge \max\{T(3), T(4)\}$, we have $T(3) \le c < c \log_2 \log_2 3$ and $T(4) \le c < c \log_2 \log_2 4$. Therefore, the base case holds. Since $T(n) \le c \log_2 \log_2 n$ for all $n \ge \sqrt{5}$, it must be the case that $T(n) = O(\log_2 \log_2 n)$. □

(e) $T(n) = T(n-1) + \log_2 n$

Proof. We will prove that $T(n) = O(n \log_2 n)$. We want to find positive constants c and n_0 such that $T(n) \le cn \log_2 n$ for all $n \ge n_0$. Assume by induction that $T(k) \le ck \log_2 k$ for all $n_0 \le k < n$. If $n \ge n_0 + 1$, we have that

$$
\begin{aligned}
T(n) &= T(n-1) + \log_2 n \\
&\le c(n-1) \log_2 (n-1) + \log_2 n \\
&= cn \log_2 (n-1) - c \log_2 (n-1) + \log_2 n \\
&\le cn \log_2 (n-1) - 3 \log_2 (n-1) + \log_2 n \quad \text{(when } c \ge 3\text{)} \\
&\le cn \log_2 (n-1) \quad \text{(when } n \ge 3\text{)} \\
&\le cn \log_2 n
\end{aligned}
$$

Thus, the inductive case is true. We will prove that the inductive hypothesis holds for the base case of the induction, that is, $T(n) \le cn \log_2 n$ for all $n_0 \le n < n_0 + 1$. If we pick $n_0 = 3$, the base case becomes $T(n) \le cn \log_2 n$ for all $3 \le n < 4$. We have $T(3) = T(2) + \log_2 3 \le c < c.3 \log_2 3$ (when $c \ge T(2) + \log_2 3$, where $T(2)$ is a constant). Therefore, the base case holds. Hence, we have $T(n) \le cn \log_2 n$ for all $n \ge 3$, which implies that $T(n) = O(cn \log_2 n)$. □

4. *Carefully* prove by induction that the i^{th} Fibonacci number satisfies the equality

$$F_i = \frac{\phi^i - \widehat{\phi}^i}{\sqrt{5}},$$

where $\phi = (1+\sqrt{5})/2$ is the golden ratio and $\widehat{\phi} = (1-\sqrt{5})/2$ is its conjugate.

Proof. We will prove this with strong induction.

Hypothesis: Let $P(i)$: $F_i = \dfrac{\phi^i - \widehat{\phi}^i}{\sqrt{5}}$ for every non-negative integer i.

Base case: When $i = 0$, the right-hand side of $P(0)$ is $\dfrac{\phi^0 - \widehat{\phi}^0}{\sqrt{5}} = \dfrac{1-1}{\sqrt{5}} = 0 = F_0$ (based on the definition of Fibonacci numbers). Hence, $P(0)$ is true.

When $i = 1$, the right-hand side of $P(1)$ is $\dfrac{\phi^1 - \widehat{\phi}^1}{\sqrt{5}} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}} = \dfrac{\sqrt{5}}{\sqrt{5}} = 1 = F_1$ (based on the definition of Fibonacci numbers). Hence, $P(1)$ is true.

Inductive step: We fix some $k \geq 1$ and assume that the hypothesis $P(i)$ is true for all $0 \leq i \leq k$.

We want to show that $P(k+1)$ is true, that is

$$F_{k+1} = \frac{\phi^{k+1} - \widehat{\phi}^{k+1}}{\sqrt{5}}$$

Since F_{k+1} is the $(k+1)$-th Fibonacci number, using the recursive definition of Fibonacci numbers and the inductive hypothesis, we have

$$
\begin{aligned}
F_{k+1} &= F_k + F_{k-1} \\
&= \frac{\phi^k - \widehat{\phi}^k}{\sqrt{5}} + \frac{\phi^{k-1} - \widehat{\phi}^{k-1}}{\sqrt{5}} \quad \text{(by inductive hypothesis)} \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1} \right] \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{1+\sqrt{5}}{2} + 1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{1-\sqrt{5}}{2} + 1\right) \right] \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{3-\sqrt{5}}{2}\right) \right] \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{6+2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{6-2\sqrt{5}}{4}\right) \right] \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^2 \right] \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}} \\
&= \frac{\phi^{k+1} - \widehat{\phi}^{k+1}}{\sqrt{5}}
\end{aligned}
$$

Hence, $P(k+1)$ is true.
Wrap-up: Since the base cases hold for $i = 0$ and $i = 1$, by the inductive case, we see that $P(2)$ holds. Since $P(0)$, $P(1)$, and $P(2)$ hold, it must be the case that $P(3)$ holds. Continuing in this manner, we can prove that $P(i)$ is true for all $i \geq 0$.

$\square$