Homework 3: Decompositions, Projections and Pseudo-inverses

1. Manipulation of Pseudo-inverses. The second Penrose equation is $\mathcal{H}^{\#}\mathcal{H}\mathcal{H}^{\#} = \mathcal{H}^{\#}$. Does the singular value decomposition of the Moore-Penrose pseudo-inverse:

$$\mathcal{H}^{+} = \sum_{k=1}^{R} \frac{1}{\sqrt{\mu_k}} u_k v_k^* \tag{1}$$

satisfy the given Penrose equation?

2. Derivation of the right-inverse form of the pseudo-inverse. In class, we used singular value decomposition and some properties of eigenvectors to derive a limiting representation of the Moore-Penrose pseudo-inverse:

$$\mathcal{H}^{+} = \lim_{\eta \to 0^+} (\mathcal{H}^*\mathcal{H} + \eta\mathcal{I}_{\mathbb{U}})^{-1}\mathcal{H}^* \tag{2}$$

Follow a similar procedure to show that

$$\mathcal{H}^{+} = \lim_{\eta \to 0^+} \mathcal{H}^*(\mathcal{H}\mathcal{H}^* + \eta\mathcal{I}_{\mathbb{V}})^{-1} \tag{3}$$

and explain the effective difference in these two operators.

3. Derivation of a Useful Projection Operator. In class, we derived the projection operator, $\mathcal{P}_{\text{null}}$, in terms of a pseudo-inverse by operating on a generic object. We then also showed that the expected behavior of this operator was consistent with a Penrose equation. Follow a similar procedure to derive $\mathcal{P}_{\text{cons}}$ and show that it behaves as it should.

4. Functions of Operators. For this exercise, assume that an operator A has a spectral decomposition and therefore $A^n = \sum_k \lambda_k^n P_k$ where P_k is a projection operator.

a. Show that if $f(x)$ can be expressed as a power series then $f(A) = \sum_k f(\lambda_k)P_k$. Can you think of a good use of this relation?

b. Suppose that $\mathcal{H}$ is Hermitian. Use a spectral decomposition to show that $\exp(i\mathcal{H})$ is unitary. Call this operator $\mathcal{U}$. *Hint*: to prove something is unitary, aim for a clever rewrite of an identity operator and exploit the completeness of a projection operator.

c. Suppose that $\{\mu_k\}$ the eigenvalues of $\mathcal{U}$ and $\{\lambda_k\}$ are the eigenvalues of $\mathcal{H}$. Use spectral decomposition to show the relationship between $\{\mu_k\}$ and $\{\lambda_k\}$.

BONUS: Use mathematical induction to prove the assumption given at the beginning of the exercise.