Appendix A

FOURIER TRANSFORM

This appendix gives the definition of the one-, two-, and three-dimensional Fourier transforms as well as their properties.

A.1 One-Dimensional Fourier Transform

If we have a one-dimensional (1-D) function $f(x)$, its Fourier transform $F(m)$ is defined as

$$F(m)=\int_{-\infty}^{\infty} f(x)\exp(-2\pi imx)dx\,, \tag{A.1.1}$$

where m is a variable in the Fourier space. Usually it is termed the frequency in the Fourier domain. If x is a variable in the spatial domain, m is called the spatial frequency. If x represents time, then m is the temporal frequency which denotes, for example, the colour of light in optics or the tone of sound in acoustics.

In this book, we call Eq. (A.1.1) the inverse Fourier transform because a minus sign appears in the exponent. Eq. (A.1.1) can be symbolically expressed as

$$F(m) = \mathcal{F}^{-1}\{f(x)\}. \tag{A.1.2}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform in Eq. (A.1.1).

Therefore, the direct Fourier transform in this case is given by

$$f(x)=\int_{-\infty}^{\infty} F(m)\exp(2\pi imx)dm, \tag{A.1.3}$$

which can be symbolically rewritten as

$$f(x) = \mathcal{F}\{F(m)\}. \tag{A.1.4}$$

Substituting Eq. (A.1.2) into Eq. (A.1.4) results in

$$f(x) = \mathcal{F}\mathcal{F}^{-1}\{f(x)\}. \tag{A.1.5}$$

Therefore we have the following unity relation:

$$\mathcal{F}\mathcal{F}^{-1} = 1. \tag{A.1.6}$$

It means that performing a direct Fourier transform and an inverse Fourier transform of a function $f(x)$ successively leads to no change.

Using the identity $\exp(ix) = \cos x + i\sin x$ and Eq. (A.1.1), one has

$$F(m) = A(m) - iB(m), \tag{A.1.7}$$

where

$$\begin{cases} A(m) = \int_{-\infty}^{\infty} f(x)\cos(2\pi mx)dx, \\ B(m) = \int_{-\infty}^{\infty} f(x)\sin(2\pi mx)dx. \end{cases} \tag{A.1.8}$$

A.2 Two-Dimensional Fourier Transform

In a similar way, the direct and inverse Fourier transforms of a two-dimensional (2-D) function $f(x, y)$ can be expressed as

$$f(x, y) = \int_{-\infty}^{\infty} F(m, n)\exp[2\pi i(mx + ny)]dmdn, \tag{A.2.1}$$

and

$$F(m, n) = \int_{-\infty}^{\infty} f(x, y)\exp[-2\pi i(mx + ny)]dxdy, \tag{A.2.2}$$

respectively.

If x and y are spatial coordinates, the exponent in Eq. (A.2.2), $\exp[-2\pi i(mx + ny)]$, represents a plane wave if we recognize that $k_x = 2\pi m$ and $k_y = 2\pi n$. Here k_x and k_y are the components of the wave vector $\boldsymbol{k}$ in the x and y directions, i.e.

$$k_x^2 + k_y^2 = k = \frac{2\pi}{\lambda}, \tag{A.2.3}$$

where λ is the illumination wavelength. In other words, the inverse Fourier transform of a spatial function is equivalent to resolving the function into a series of plane waves propagating in different directions and the direct Fourier transform means that the original function is represented by a superposition of these plane waves each of which has a particular weighting factor.

A.3 Three-Dimensional Fourier Transform

For a three-dimensional (3-D) function $f(x, y, z)$, we have the direct and inverse 3-D Fourier transforms as follows:

$$f(x,y,z)=\int_{-\infty}^{\infty} F(m,n,s)\exp[2\pi i(mx+ny+sz)]dmdnds\,, \tag{A.3.1}$$

and

$$F(m,n,s)=\int_{-\infty}^{\infty} f(x,y,z)\exp[-2\pi i(mx+ny+sz)]dxdydz\,. \tag{A.3.2}$$

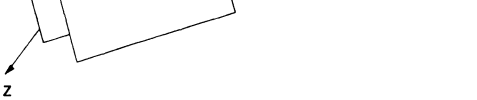

Fig. A.3.1 A plane wave corresponds to an exponent in Eq. (A.3.1).

According to the discussion in Section A.2, the exponent in Eqs. (A.3.2) denotes a plane wave with the wave-vector components, k_x, k_y and k_z which are given by

$$\begin{cases} k_x = 2\pi m, \\ k_y = 2\pi n, \\ k_z = 2\pi s. \end{cases} \tag{A.3.3}$$

If $2\pi(mx + ny + sz) = \text{constant} = A$, this equation gives a series of parallel planes. The phase on these planes is constant. If $A = 2\pi j$ $(j = 0, \pm1, \pm2\ldots)$, the phase difference between two adjacent planes is 2π. The spatial periods along the x, y and z axes are

$$\begin{cases} \lambda_x = \dfrac{2\pi}{k_x} = \dfrac{1}{m}, \\ \lambda_y = \dfrac{2\pi}{k_y} = \dfrac{1}{n}, \\ \lambda_z = \dfrac{2\pi}{k_z} = \dfrac{1}{s}. \end{cases} \tag{A.3.4}$$

Here Eq. (A.3.3) has been used. As we expect, m, n, and s correspond to the spatial frequencies in the x, y and z directions, respectively. A spatial frequency vector $\boldsymbol{m}$ can be introduced with three components m, n, s. Thus the direct and inverse 3-D Fourier transforms can be expressed in a compact form:

$$f(\boldsymbol{x}) = \int_{-\infty}^{\infty} F(\boldsymbol{m}) \exp(2\pi i \boldsymbol{m} \bullet \boldsymbol{r}) d\boldsymbol{m}, \tag{A.3.5}$$

and

$$F(\boldsymbol{m}) = \int_{-\infty}^{\infty} f(\boldsymbol{r}) \exp(-2\pi i \boldsymbol{m} \bullet \boldsymbol{r}) d\boldsymbol{r}, \tag{A.3.6}$$

where the vector $\boldsymbol{r}$ has three components x, y and z.

A.4 Fourier Transform Theorems

In this section, we give the Fourier transform theorems without giving the derivation procedure. The theorems are given for a 1-D case but the 2-D and 3-D forms of the theorems can be easily written down.

a) Similarity theorem

If $F(m) = \mathcal{F}\{f(x)\}$, then

$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|} F\left(\frac{m}{a}\right). \tag{A.4.1}$$

It means that a "stretching" of the coordinate in the x space leads to a contraction of the coordinates in the Fourier space and a change of the Fourier transform by a factor of $1/|a|$.

b) Shift theorem

If $F(m) = \mathcal{F}\{f(x)\}$, then

$$\mathcal{F}\{f(x - a)\} = F(m)\exp(-2\pi ima). \tag{A.4.2}$$

It means that a translation of a function in the x space leads to a linear phase shift in the Fourier space.

c) Parsval's theorem

If $F(m) = \mathcal{F}\{f(x)\}$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(m)|^2 dm, \tag{A.4.3}$$

which is a statement of the conservation of energy in physics.

d) Convolution theorem

If $F(m) = \mathcal{F}\{f(x)\}$ and $G(m) = \mathcal{F}\{g(x)\}$, then

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi\right\} = F(m)G(m), \tag{A.4.4}$$

or

$$\mathcal{F}\{f(x) \otimes g(x)\} = F(m)G(m). \tag{A.4.5}$$

This theorem implies that the Fourier transform of a convolution operation of two functions in the x space is simply equivalent to the product of their corresponding Fourier transforms.

e) Autocorrelation theorem

If $F(m) = \mathcal{F}\{f(x)\}$, then

$$\mathcal{F}\left\{\int_{-\infty}^{\infty} f(\xi)f^*(\xi - x)d\xi\right\} = |F(m)|^2, \tag{A.4.6}$$

or

$$\mathcal{F}\left\{|f(x)|^2\right\} = \int_{-\infty}^{\infty} F(\xi)F^*(\xi - m)d\xi. \tag{A.4.7}$$

It is noted that the autocorrelation theorem is a special case of the convolution theorem if we let $g(x) = f^*(-x)$.

Appendix B

HANKEL TRANSFORM

In this appendix, a special form of the two-dimensional (2-D) Fourier transform, called the Hankel transform, is discussed.

Let us start with the 2-D Fourier transform in Cartesian coordinates, which is given by

$$F(m,n) = \int\int_{-\infty}^{\infty} f(x, y) \exp[2\pi i(xm + yn)]dxdy. \tag{B.1.1}$$

The Hankel transform is the 2-D Fourier transform in a polar coordinate. The function $f(x, y)$ can be represented by a function $f(r, \phi)$ if the following coordinate transformation is adopted:

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi, \end{cases} \tag{B.1.2}$$

and

$$\begin{cases} m = l\cos\psi, \\ n = l\sin\psi, \end{cases} \tag{B.1.3}$$

where r and φ are the polar coordinates in the x - y plane, whereas l and ψ are the polar coordinates in the m - n plane. Therefore Eq. (B.1.1) can be rewritten as

$$F(l,\theta) = \int_0^{2\pi}\int_0^{\infty} f(r,\phi) \exp[2\pi irl\cos(\varphi - \psi)]rdrd\varphi. \tag{B.1.4}$$

In the case of a circularly symmetric system, $f(r, \phi) = f(r)$. Thus the Fourier transform of $f(r)$ is also circularly symmetric, which may be denoted by $F(l)$. Finally, Eq. (B.1.4) can reduce to

$$F(l) = \int_0^\infty f(r) J_0(2\pi r l) 2\pi r dr, \tag{B.1.5}$$

which is called the Hankel transform. In this expression,

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(\pm ix \cos\varphi) d\varphi \tag{B.1.6}$$

is a Bessel function of the first kind of order zero.

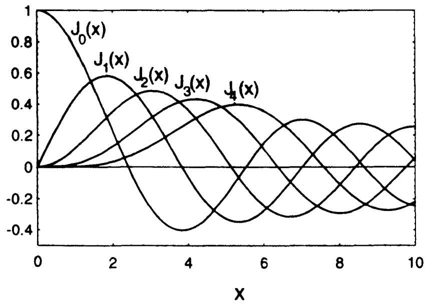

Fig. B.1.1 Bessel functions of the first kind of the first five orders, $J_0(x)$, $J_1(x)$, $J_2(x)$, $J_3(x)$ and $J_4(x)$.

If $f(r)$ is a uniform function within a radius a:

$$f(r) = \begin{cases} 1 \quad , \quad r \le a, \\ 0 \quad , \quad r > a, \end{cases} \tag{B.1.7}$$

thus using the identity

$$\int_o^x x_0 J_0(x_0) dx_0 = x J_1(x), \tag{B.1.8}$$

one can derive an analytical expression for Eq. (B.1.5):

$$F(l) = \pi a^2 \left[\frac{2J_1(2\pi a l)}{2\pi a l} \right], \tag{B.1.9}$$

where $J_1(x)$ is a Bessel function of the first kind of order one. Fig. B.1.1 gives the Bessel functions of the first kind of the first four orders. The function $2J_1(x)/x$ is also termed the Airy function and is shown in Fig. B.1.2.

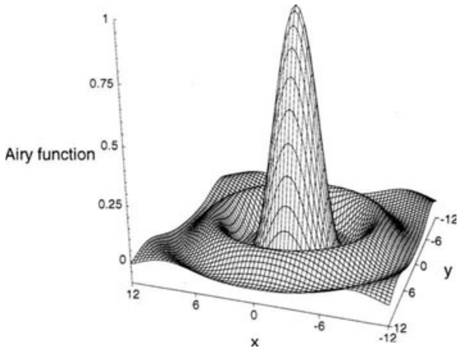

Fig. B.1.2 Airy function $2J_1(r)/r$: 2-D behaviour.

Appendix C

DELTA FUNCTIONS

This appendix summarizes the main properties of delta functions. For a one-dimensional (1-D) problem, the definition of a delta function $\delta(x)$ is given by

$$\delta(x)=\begin{cases}\infty \quad , \quad x=0, \\ \\ 0 \quad , \quad x\neq 0,\end{cases} \tag{C.1.1}$$

and

$$\int_{-\infty}^{\infty}\delta(x)dx=1. \tag{C.1.2}$$

Eqs. (C.1.1) and (C.1.2) give a complete definition of a delta function. Mathematically, a delta function represents an infinity at the origin ($x = 0$). Physically, it represents an impulse response or action. For example, in optical imaging, a point illumination source or a point detector can be represented by a delta function.

In general, a delta function can be defined with respect to an arbitrary position x_0. In this case, Eqs. (C. 1.1) and (C.1.2) should be rewritten as

$$\delta(x-x_0)=\begin{cases}\infty \quad , \quad x=x_0, \\ \\ 0 \quad , \quad x\neq x_0,\end{cases} \tag{C.1.3}$$

and

$$\int_{-\infty}^{\infty}\delta(x-x_0)dx=1. \tag{C.1.4}$$

One of the important properties of delta functions is mathematically expressed as

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0), \tag{C.1.5}$$

or

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0). \tag{C.1.6}$$

This property means that the integration of the product of a delta function and an arbitrary function $f(x)$ is equivalent to taking the value of the function $f(x)$ at $x = x_0$. If $f(x)$ denotes a physical quantity, Eq. (C.1.5) represents the selection of the value of the physical quantity at x_0.

Now, let us assume $f(x)$ to be exp($-2\pi imx$). The inverse Fourier transform of a delta function can be easily derived, according to Eq. (A.1.5), as

$$F(m) = \int_{-\infty}^{\infty} \delta(x - x_0) \exp(-2\pi imx) dx = \exp(-2\pi ix_0). \tag{C.1.7}$$

According to the definition of the direct Fourier transform in Eq. (A.1.3), a delta function can be expressed as a integration form given by

$$\delta(x - x_0) = \int_{-\infty}^{\infty} \exp[2\pi im(x - x_0)] dm. \tag{C.1.8}$$

A delta function can be represented by a series of normal functions in the limiting forms. Two examples of these limiting forms are

$$\delta_a(x) = \sqrt{\frac{a}{\pi}} \exp(-ax^2) \quad , \quad a \to 0, \tag{C.1.9}$$

and

$$\delta_a(x) = \frac{a}{\pi} \sin c(ax) \quad , \quad a \to 0, \tag{C.1.10}$$

where

$$\sin c(x) = \frac{\sin x}{x}. \tag{C.1.11}$$

The above definitions and discussions also apply for the two- and three-dimensional cases.

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