Problems for Materials in Appendix A

M.F. Insana, 7.16.2020

1. For $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5i \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ 4 \\ -2 \\ 0 \\ 1 \end{pmatrix}$,
find $\mathbf{x}+\mathbf{y}$, $\langle \mathbf{x},\mathbf{y}\rangle$, $\mathbf{y}^\dagger\mathbf{x}$, $\mathbf{x}\circ\mathbf{y}$, $\mathbf{x}\mathbf{y}^\dagger$. What is the rank of $\mathbf{x}\mathbf{y}^\dagger$. Explain your answer.

2. Find vector and matrix 2-norms, $\|\mathbf{x}\|_2$, $\|\mathbf{x}\mathbf{y}^\dagger\|_2$.

3. Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & -i \\ -1 & 1 & 2 & 4 \\ 0 & 2 & 1 & 3 \\ i & 4 & 3 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & -1 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 2 & 3 \\ -3 & 0.5 \end{pmatrix}$. Find the rank of each matrix. What are the dimensions of the null spaces? Find the determinants and the one inverse $\mathbf{C}^{-1}$. Finally, before computing eigenvalues in MATLAB determine them for $\mathbf{C}+\mathbf{C}^\top$ and $\mathbf{C}-\mathbf{C}^\top$.

4. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{pmatrix}$. Find $\mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}$ where all the elements in your answer are integers. This is a pseudoinverse of $\mathbf{A}$ that is often expresses as $\mathbf{A}^+$. It is not a true matrix inverse, which doesn't exist because $\mathbf{A}$ is not square. What are the dimensions of $\mathbf{A}\mathbf{A}^+$ and $\mathbf{A}^+\mathbf{A}$?

5. Let $\mathbf{A} \in \mathbb{R}^{M\times N}$, $\mathbf{B} \in \mathbb{R}^{N\times P}$, and $\mathbf{C} \in \mathbb{R}^{P\times Q}$. What property must hold for each matrix so that $\mathbf{ABC}(\mathbf{ABC})^\top = \mathbf{I}$. What is the dimension of $\mathbf{I}$?

6. Which of the following matrices are orthonormal?
$$\mathbf{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 1-i & 1+i \\ 1-i & 1+i & 1-i \\ 1+i & 1-i & 1+i \end{pmatrix},$$
$$\mathbf{B} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$
$$\mathbf{C} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}.$$
Why is $\mathbf{B}\mathbf{B}^\top = \mathbf{B}^2$ while $\mathbf{A}\mathbf{A}^\dagger \neq \mathbf{A}^2$ and $\mathbf{C}\mathbf{C}^\top \neq \mathbf{C}^2$?

7. Consider the scalar function of a vector variable,
$$f(\mathbf{x}) = \exp\left(-\frac{1}{2}\mathbf{x}^\top\mathbf{K}^{-1}\mathbf{x}\right),$$
where data vector $\mathbf{x} \in \mathbb{R}^{N\times 1}$. $\mathbf{K}$ is the covariance matrix of the data. It is computed from the expected value of the outer product of $\mathbf{x}$ with itself, i.e., $\mathcal{E}\{\mathbf{x}\mathbf{x}^\top\}$. Find $df(\mathbf{x})/d\mathbf{x}$, which is no longer a scalar function. What is its dimension?

8. Diagonalize the following full-rank matrices:
$$\mathbf{A} = \begin{pmatrix} 4 & 1 & 4 & 2 \\ 1 & 2 & 1 & 4 \\ 4 & 3 & 4 & 1 \\ 3 & 4 & 4 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 4 & 1 & 4 & 2 \\ 1 & 2 & 1 & 4 \\ 4 & 1 & 4 & 1 \\ 2 & 4 & 1 & 2 \end{pmatrix}$$
Before doing this, I could predict that $\mathbf{A}$ has complex eigenvalues whereas $\mathbf{B}$ has real eigenvalues. What are the properties of the matrices that enabled this prediction?

9. If
$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 3 & 3 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 6 & 6 & 4 \end{pmatrix};,$$
we find $\mathbf{A}^{-1}$ doesn't exist and the reason is simple. The fourth row is twice the second row so $\mathbf{A}$ is singular, only rank 3. Computing its inverse in MATLAB gives nonsense. However, if I add a little noise to $\mathbf{A}$ using `rng(1);a=0.01;B=A+a*randn(4)`, I can find an inverse. This seems nuts! I start with a singular matrix and add noise and my ability to estimate an inverse improves! Explain why this is true and what happens as you modify `a`.