Q5.12

Since $\epsilon(t') = \epsilon_0 sin\Omega t$, place it into $\sigma(t)$:

$$\sigma(t) = \int_{-\infty}^{t} dt' \ G(t-t') \frac{d\epsilon_0 sin\Omega t'}{dt'} = \int_{-\infty}^{t} dt' \ G(t-t')\epsilon_0 \Omega cos\Omega t$$

Let $\tau = t - t'$, so

$\sigma(t) = \epsilon_0 \Omega \int_{\infty}^{0} d\tau \ (-G(\tau))cos(\Omega(t-\tau))$

$= \epsilon_0 \Omega \int_{0}^{\infty} d\tau \ G(\tau)cos(\Omega(t-\tau))$

$= \epsilon_0 \Omega \int_{0}^{\infty} d\tau \ G(\tau)(cos\Omega t cos\Omega\tau + sin\Omega t sin\Omega\tau)$

$= \epsilon_0[(\Omega \int_{0}^{\infty} d\tau \ G(\tau)sin\Omega\tau)sin\Omega t + (\Omega \int_{0}^{\infty} d\tau \ G(\tau)cos\Omega\tau)cos\Omega t]$

$= \epsilon_0(G'(\Omega)sin\Omega t + G''(\Omega)cos\Omega t)$

Q6.1

$\mathbf{A}, \mathbf{B}$ is unitary matrix, so $\mathbf{A}^\dagger\mathbf{A} = I = \mathbf{A^{-1}}\mathbf{A}$ and $\mathbf{B}^\dagger\mathbf{B} = I = \mathbf{B^{-1}}\mathbf{B}$

(a). $\mathbf{A}^\dagger\mathbf{A} = I$

(b). $\mathbf{A}^\dagger\mathbf{A^{-1}} = \mathbf{A}^\dagger\mathbf{A}^\dagger = \mathbf{A}^{\dagger 2}$

(c). $(c\mathbf{A}\mathbf{A}^\dagger)^\dagger = (c\mathbf{I})^\dagger = \bar{c}\mathbf{I}$

(d). $\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^\dagger \end{pmatrix}$

Q6.3

$\langle \mathbf{x}, \mathbf{y} \rangle \mathbf{z} = \mathbf{x}^\dagger \mathbf{y} \mathbf{z}$, where $\mathbf{x}^\dagger \mathbf{y}$ is multiplication of $1 \times N$ and $N \times 1$ matrix and the result is scalar m. Therefore,
$\langle \mathbf{x}, \mathbf{y} \rangle \mathbf{z} = m\mathbf{z} = \mathbf{z}m = \mathbf{z}\mathbf{x}^\dagger \mathbf{y}$

Need to expand into elements details
$\langle \mathbf{x}, \mathbf{y} \rangle \mathbf{z} = \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{z}[n]$

$$= \sum_{m=0}^{M-1} x^*[m] y[m] z[n]$$

$$= \sum_{m=0}^{M-1} x_m^* y_m z_n$$

$$= \sum_{m=0}^{M-1} z_n x_m^* y_m$$

$$= \sum_{m=0}^{M-1} [zx^*]_{nm} y_m$$

$= [\mathbf{z}\mathbf{x}^\dagger \mathbf{y}]_n$

Q6.4

(a). $\mathbf{y} = \mathbf{A}\mathbf{x} = \sum_{n=1}^{N} A[m,n]x[n]$

(b). $\mathbf{x} = [\mathbf{A}^{\dagger}\mathbf{y}] = \sum_{m=1}^{M} [A^{\dagger}]_{nm} y_m = \sum_{m=1}^{M} A^*[n,m]y[m]$

(c). $\mathbf{y} = \mathcal{A}(x(t')) = \int_{-\infty}^{\infty} dt' \ a_m(t')x(t')$

(d). $x(t') = \mathbf{A}^{\dagger}\mathbf{y}(t') = \sum_{m=1}^{M} a_m^*(t')y_m$

(e). $y(t) = \mathcal{A}\{x(t')\} = \int_{-\infty}^{\infty} dt' \ a(t,t')x(t')$

(f). $x(t') = \mathcal{A}^{\dagger}\{y(t)\} = \int_{-\infty}^{\infty} dt \ a^*(t,t')y(t)$

Assume $\mathbf{y}$ is $m \times 1$ matrix, and $\mathbf{x}$ is $n \times 1$ matrix,

(a). It is a discrete-to-discrete transfromation in transformation matrix form. $\mathbf{A}$ is $m \times n$ matrix.

$$\mathbf{y} = \mathbf{A}\mathbf{x} = (\sum_{i=1}^{n} \mathbf{A}_{ji}\mathbf{x}_i), j = (1, ...m)$$

If $\mathbf{A}$ is fourier basis vectors, according to §6.8 and forward DFT,

$$\mathbf{y} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \mathbf{x}[n]exp(-i2\pi nk/N) = \mathbf{Y}[k]$$

(b). Similarly, $\mathbf{A}^{\dagger}$ is $n \times m$ transformation matrix.

$$\mathbf{x} = [\mathbf{A}^{\dagger}\mathbf{y}] = (\sum_{i=1}^{m} \mathbf{A}_{ji}^{\dagger}\mathbf{y}_{\mathbf{i}}), j = (1, ..., n)$$

Use §5.8 inverse DFT, since the unitary transformation constant is $1/\sqrt{N}$, then

$$\mathbf{x} = [\mathbf{A}^{\dagger}\mathbf{y}] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{y}[k]exp(i2\pi kn/N) = \mathbf{X}[n]$$

(c). It's a continuous-to-discrete transformation. $\mathcal{A}$ is a transformation operator. According to §5.4 forward FS, so

$$\mathbf{y} = \mathcal{A}\{x(t')\} = \int_{-\infty}^{\infty} dt'\, \mathcal{A}(x(t')) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} dt'\, x(t') exp(-i2\pi kt'/T_0) = \mathbf{Y}[k]$$

(d). It's a dicrete-to-continuous transformation. According to §5.4 inverse FS, so

$$x(t') = [\mathbf{A}^{\dagger}\mathbf{y}](t') = \sum_{k=-\infty}^{\infty} \mathbf{A}^{\dagger}[k]\mathbf{y}[k] = \sum_{k=-\infty}^{\infty} \mathbf{y}[k] exp(i2\pi kt'/T_0)$$

(f). It's a forward CT-FT as §5.5. So

$$y(t) = \mathcal{A}\{x(t')\} = \int_{-\infty}^{\infty} dt'\, \mathcal{A}(x(t')) = \int_{-\infty}^{\infty} dt'\, x(t') exp(-i2\pi tt')$$

(g). It's a inverse CT-FT as §5.5. So

$$x(t') = \mathcal{A}^{\dagger}\{y(t)\} = \int_{-\infty}^{\infty} dt\, \mathcal{A}^{\dagger}(y(t)) = \int_{-\infty}^{\infty} dt\, y(t) exp(i2\pi t't)$$

Q6.5

Way 1: (Thanks Joseph Tibbs)

Move e to the left of equation so that $\mathbf{g} - \mathbf{e} = \mathbf{Hf}$. Then apply $\mathbf{Q}^\dagger$ to both side,

$$\mathbf{Q}^\dagger(\mathbf{g} - \mathbf{e}) = \mathbf{Q}^\dagger\mathbf{Hf}$$

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}(g[n] - e[n])exp(-i2\pi nk/N) = \mathbf{Q}^\dagger\mathbf{Hf} = \mathbf{\Lambda Q}^\dagger\mathbf{f}$$

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}g'[n]exp(-i2\pi nk/N) = \mathbf{\Lambda Q}^\dagger\mathbf{f}, Let\ g' = g - e$$

$$G'(k) = H(k)F(k),\ use\ equation\ 6.8$$

Way 2:

Apply forward fourier operater to $\mathbf{g} = \mathbf{HF} + \mathbf{e}$, then

$\mathbf{Q}^\dagger\mathbf{g} = \mathbf{Q}^\dagger\mathbf{Hf} + \mathbf{Q}^\dagger\mathbf{e}$.

Apply equation 6.7 and 6.8,

$\mathbf{Q}^\dagger\mathbf{g} = \mathbf{\Lambda Q}^\dagger\mathbf{f} + \mathbf{Q}^\dagger\mathbf{e}$

$$\implies G(k) = H(k)F[k] + \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}e[n]exp(-i2\pi nk/N)$$

$= H(k)F(k) + E(k)$

Because $\mathbf{e}$ exists constantly in the time domain, then as example 5.4.1 shows, the narrow functions in one domain imply broad functions in the other and vice versa. In this case, the $E(k)$ would have non-zero values in a very narrow width in frequency domain while the left values are all zeros. Therefore, $G(k) \approx H(k)F(k)$ which is the Fourier convolution theorem.

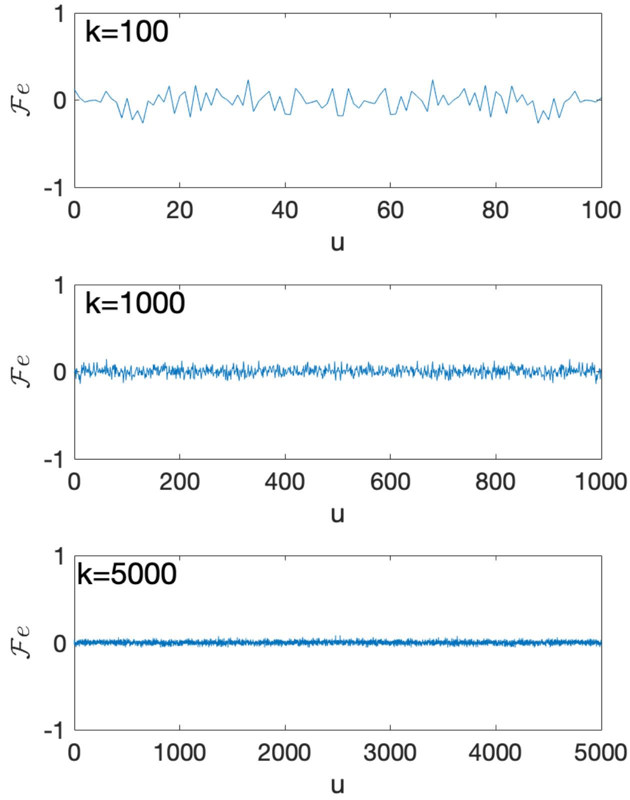

Figure 1: $\mathcal{F}e$ in frequency domain with different k

Code:

```
hz=100;t=0:1/hz:1;uu=0:hz;a=2;
noi=randn(length(t),1)*a;
gu = fft(noi)/hz;
```

Frequency k has impact on the transformed value. In theory, $dt \to 0$, which means $k = 1/dt \to \infty$. Then $E(k) = \mathcal{F}e \to 0$

Q6.6

As equation 6.3, fourier operator matrix

$$\mathbf{Q}_{4\times 4} = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & exp(i2\pi/4) & exp(i2\pi * 2/4) & exp(i2\pi * 3/4) \\ 1 & exp(i2\pi 2 * 1/4) & exp(i2\pi 2 * 2/4) & exp(i2\pi 2 * 3/4) \\ 1 & exp(i2\pi 3 * 1/4) & exp(i2\pi 3 * 2/4) & exp(i2\pi 3 * 3/4) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & exp(i0.5\pi) & exp(i\pi) & exp(i1.5\pi) \\ 1 & exp(i\pi) & exp(i2\pi) & exp(i3\pi) \\ 1 & exp(i1.5\pi) & exp(i3\pi) & exp(i4.5\pi) \end{pmatrix}$$

$\mathbf{Q_{4\times 4}}$ is symmetric. Use euler's equation, then

$$\mathbf{Q_{4\times 4}} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & cos(0.5\pi) + isin(0.5\pi) & cos(\pi) + isin(\pi) & cos(1.5\pi) + isin(1.5\pi) \\ 1 & cos(\pi) + isin(\pi) & cos(2\pi) + isin(2\pi) & cos(3\pi) + isin(3\pi) \\ 1 & cos(1.5\pi) + isin(1.5\pi) & cos(3\pi) + isin(3\pi) & cos(4.5\pi) + isin(4.5\pi) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

$$\mathbf{Q}^\dagger \mathbf{Q} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} * \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = I_{4\times 4} = \mathbf{Q}\mathbf{Q}^\dagger$$

So $\mathbf{Q_{4\times 4}}$ is unitary.

Q6.7

According to equation 5.30 in §5.9:

$$F(u,v) = \mathcal{F}f(x,y) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\ f(x,y)exp(-i2\pi(ux+vy))$$

$$= \int_{-\infty}^{\infty} dx\ Arect\frac{x-x_0}{X_0}exp(-i2\pi ux)\int_{-\infty}^{\infty} dy\ Brect\frac{y-y_0}{Y_0}exp(-i2\pi vy)$$

Let $x' = x - x_0$ and use fourier shift theorem, then

$$\int_{-\infty}^{\infty} dx\ Arect\frac{x-x_0}{X_0}exp(-i2\pi ux)$$

$$= Aexp(-i2\pi ux_0)\int_{-\infty}^{\infty} dx'\ rect\frac{x'}{X_0}exp(-i2\pi ux')$$

$$= Aexp(-i2\pi ux_0)\int_{-X_0/2}^{X_0/2} dx'\ exp(-i2\pi ux')$$

$$= Aexp(-i2\pi ux_0)\frac{(exp(-i2\pi uX_0/2) - exp(-i2\pi u(-X_0/2)))}{-i2\pi u}$$

$$= AX_0exp(-i2\pi ux_0)\frac{sin(\pi uX_0)}{\pi uX_0}$$

$$= AX_0exp(-i2\pi uX_0)sinc(uX_0) \qquad when\ sinc(t) = sin(\pi t)/(\pi t)$$

Do the same thing for the y part, then

$$F(u,v) = AX_0exp(-i2\pi ux_0)sinc(uX_0)BY_0exp(-i2\pi vy_0)sinc(vY_0)$$