Homework #3

1. **(20 points)** The joint probability mass function of discrete random variables X and Y taking values x = 1, 2, 3 and y = 1, 2, 3, respectively, is given by a formula $f_{XY}(x, y) = c*(x + y)$. Determine the following:

a) **(2 points)** Find c

Answer: $\sum_R f(x,y) = c*(2+3+4+3+4+5+4+5+6) = 1$, c*36 = 1. Thus, c = 1/36

b) **(2 points)** Find probability of the event where X = 1 and Y < 3

Answer: $P(X=1, Y<3) = f_{XY}(1,1) + f_{XY}(1,2) = \frac{1}{36}(2+3) = 5/36$

c) **(2 points)** Find marginal probability $P_Y(Y = 2)$

Answers: $P(Y=2) = f_{XY}(1,2) + f_{XY}(2,2) + f_{XY}(3,2) = \frac{1}{36}(3+4+5) = 1/3$

d) **(2 points)** Marginal probability distribution of the random variable X

Answers: marginal distribution of *X*

x	$f_X(x) = f_{XY}(x,1) + f_{XY}(x,2) + f_{XY}(x,3)$
1	1/4
2	1/3
3	5/12

e) **(2 points)** E(X), E(Y), V(X), and V(Y)

Answers:

$E(X) = (1\times\frac{1}{4}) + (2\times\frac{1}{3}) + (3\times\frac{5}{12}) = 13/6 = 2.167$

$V(X) = E(X=1)*(1-2.167)^2 + E(X=2)*(2-2.167)^2 + E(X=3)*(3-2.167)^2 = 0.6389$

$E(Y) = 2.167$
$V(Y) = 0.6389$

f) **(2 points)** Find conditional probability distribution of Y given that X = 1

Answers: $f_{Y|X}(y) = \dfrac{f_{XY}(1,y)}{f_X(1)}$

y	$f_{Y\|X}(y)$
1	(2/36)/(1/4)=2/9
2	(3/36)/(1/4)=1/3
3	(4/36)/(1/4)=4/9

g) **(2 points)** Conditional probability distribution of X given that Y = 2

Answers: $f_{X|Y}(x) = \dfrac{f_{XY}(x,2)}{f_Y(2)}$ and $f_Y(2) = f_{XY}(1,2) + f_{XY}(2,2) + f_{XY}(3,2) = \frac{12}{36} = 1/3$

x	$f_{X\|Y}(x)$
1	(3/36)/(1/3)=1/4
2	(4/36)/(1/3)=1/3
3	(5/36)/(1/3)=5/12

h) **(2 points)** Are X and Y independent?
Answers: Since $f_{XY}(1,1)=2/36 \neq 9/36*9/36=2f_X(x)f_Y(y)$, X and Y are not independent.

i) **(2 points)** What is the covariance for X and Y?
Answers: cov(X,Y) = <XY>-<X><Y>=(1/36)*(2*1+3*2+4*3+3*2+4*4+5*6+4*3+5*6+6*9)-2.167*2.167 = -0.0292

j) **(2 points)** What is the correlation for X and Y?
Answers: corr(X,Y) = -0.0292/0.6389 = -0.0457

2. **(8 points)** A random variable X has density function $f(X=x)=c(x+x^3)$ for $x \in [0,1]$ and $f(X=x)=0$ otherwise.

a) **(2 points)** Determine c.
Answer: c = 4/3.

b) **(2 points)** Compute E(1/X)
Answer: E(1/X) = 16/9

c) **(4 points)** Determine the probability density function of $Y = X^2$

Answer: $P(Y=y)=\frac{dP(X \le \sqrt{y})}{dy}=\frac{d\int_0^{\sqrt{y}} c(x+x^3)dx}{dy}=c(y^{1/2}+y^{3/2})\frac{1}{2\sqrt{y}}=\frac{2}{3}(1+y)$.

3. **(10 points)** Toss an unfair coin 4 times. The probability of head is 0.6 and that of tail is 0.4. Let X be the total number of heads among the first two tosses and Y the total number of heads among the last two tosses.

a) **(4 points)** Write down the joint probability mass fraction of X and Y.
Answers:

x/y	0	1	2	Margin
0	$0.4^4 = 0.0256$	$0.4^2 \times 2 \times 0.6 \times 0.4 = 0.0768$	$0.4^2 \times 0.6^2 = 0.0576$	0.16
1	$2 \times 0.6 \times 0.4 \times 0.4^2 = 0.0768$	$2 * 0.4 * 0.6 \times 2 \times 0.6 \times 0.4 = 0.2304$	$2 \times 0.6 \times 0.4 \times 0.6^2 = 0.1728$	0.48
2	$0.6^2 \times 0.4^2 = 0.0576$	$0.6^2 \times 2 \times 0.6 \times 0.4 = 0.1728$	$0.6^4 = 0.1296$	0.36
Margin	0.16	0.48	0.36	1.00

b) **(2 points)** Are X and Y independent? Please explain.
Answers: Independent.

c) **(4 points)** Compute the conditional probability $P(X \geq Y | X \geq 1)$
Answers:
$$P(X \geq Y \mid X \geq 1) = \frac{P(X \geq Y, X \geq 1)}{P(X \geq 1)} = \frac{0.0768 + 0.2304 + 0.1296 + 0.1728 + 0.0576}{0.48 + 0.36} = 0.7943$$

4. **(6 points)** A random variable X is the average of p independent random variables X_k, i.e., $X = \frac{1}{p}\sum_{k=0}^{p} X_k$, Calculate the expectation and the variance of X for three different cases:

a) **(2 points)** When all X_k are independent uniform continuous random variables in the interval (0,1)

Answers: $E(X) = \frac{1}{2}, V(X) = \frac{1}{12p}$

b) **(2 points)** When all X_k are independent exponential random variables with PDF $P(X_k = x) = \lambda_k e^{-\lambda_k x}$

Answers: $E(X) = \frac{1}{p}\sum_k \lambda_k^{-1}, V(X) = \frac{1}{p^2}\sum_k \lambda_k^{-2}$

c) **(2 points)** When all X_k are Independent normal random variables but each one has its own mean μ_k and its own standard deviation σ_k

Answers: $E(X) = \frac{1}{p}\sum_k \mu_k, V(X) = \frac{1}{p^2}\sum_k \sigma_k^2$

5. **(4 points)** Suppose random variables X, Y have standard derivations, $\sigma_X = 2$ and $\sigma_Y = 6$, respectively, and correlation coefficient corr(X, Y) = −1/3.

(a) **(2 points)** Find cov(X, Y).

Answer: $\operatorname{cov}(X,Y) = Corr(X,Y) * \sigma_X * \sigma_Y = -4$

(b) **(2 points)** Find Var(4X − 2Y).

Answers:
$Var(4X - 2Y) = 16 * Var(X) + 4 * \text{Var(Y)} - 16 * \text{Cov(X,Y)} = 16 * 4 + 4 * 36 - 16 * (-4) = 272$

6. **(12 points)** Time interval separating subsequent bus arrivals at a stop is an exponential random variable with mean 20 minutes. Steve and Andrew work at the same place and each will be late to work unless they board a bus on or before 8:40am. Steve comes to the bus stop exactly at 8am. Andrew also comes to the same bus stop but at a random time, uniformly distributed between 8am and 8:30am. Both of them take the first bus that arrives.

(a) **(4 points)** What is the probability that Steve will be late for work tomorrow?

Answers: $P(Steve \text{ late}) = 1 - P(T < 40) = 1 - \frac{1}{20}\int_0^{40} e^{-t/20} dt = e^{-2} = 0.1353$

(b) **(4 points)** What is the probability that Andrew will be late for work tomorrow?

Answers:

$$P(\text{Andrew late}) = \int_0^{30} \frac{dx}{30} P(T >= 40 \,|\, \mathrm{T} > \mathrm{x}) = \int_0^{30} \frac{dx}{30} e^{-(40-x)/20} = \frac{e^{-2}}{30}\int_0^{30} e^{x/20} dx = \frac{20e^{-2}}{30}\left(e^{30/20} - 1\right) = 0.3141$$

(c) **(4 points)** What is the probability that Steve and Andrew will ride the same bus?

Probability that Steve will not leave by the time x when Andrew comes is exp(-x/20).

It needs to be integrated over Int_0^30 dx/30 exp(-x/20)=

Answers: $P(\text{Steve and Andrew meet}) = \int_0^{30} \frac{dx}{30} e^{-x/20} = \frac{20}{30}(1 - \mathrm{e}^{-30/20}) = 0.5179$