Q1:

1. $\mathbf{x}+\mathbf{y} = \begin{pmatrix} 2 \\ 6 \\ 1 \\ 4 \\ 1+5i \end{pmatrix}$

2. $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\dagger * \mathbf{y} = \begin{pmatrix} 1 & 2 & 3 & 4 & -5i \end{pmatrix} * \begin{pmatrix} 1 \\ 4 \\ -2 \\ 0 \\ 1 \end{pmatrix}$
$= 1 \times 1 + 2 \times 4 - 3 \times 2 + 0 - 1 \times 5i = 3 - 5i$

3. $\mathbf{y}^\dagger \mathbf{x} = \begin{pmatrix} 1 & 4 & -2 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5i \end{pmatrix} = 1 \times 1 + 4 \times 2 - 2 \times 3 + 0 + 5i = 3 + 5i$

4. $\mathbf{x} \circ \mathbf{y} = \begin{pmatrix} 1 \times 1 \\ 2 \times 4 \\ 3 \times -2 \\ 4 \times 0 \\ 5i \times 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -6 \\ 0 \\ 5i \end{pmatrix}$

5. $\mathbf{x}\mathbf{y}^\dagger = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5i \end{pmatrix} * \begin{pmatrix} 1 & 4 & -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 * \begin{pmatrix} 1 & 4 & -2 & 0 & 1 \end{pmatrix} \\ 2 * \begin{pmatrix} 1 & 4 & -2 & 0 & 1 \end{pmatrix} \\ 3 * \begin{pmatrix} 1 & 4 & -2 & 0 & 1 \end{pmatrix} \\ 4 * \begin{pmatrix} 1 & 4 & -2 & 0 & 1 \end{pmatrix} \\ 5i * \begin{pmatrix} 1 & 4 & -2 & 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 4 & -2 & 0 & 1 \\ 2 & 8 & -4 & 0 & 2 \\ 3 & 12 & -6 & 0 & 3 \\ 4 & 16 & -8 & 0 & 4 \\ 5i & 20i & -10i & 0 & 5i \end{pmatrix}$

6. Because $\mathbf{x}\mathbf{y}^\dagger$ is the linear combination of $\mathbf{y}^\dagger$ and $\mathbf{y}^\dagger$ is a real matrix, so $\mathbf{y}^\dagger = \mathbf{y}^\mathbf{T}$ and $Rank(xy^\dagger) = Rank(y^\dagger) = Rank(y^T) = Rank(y) = 1$

Q2:

1. $\|\mathbf{x}\|_2 = (\sum_{i,j=1}^{n} |x_{i,j}|)^{\frac{1}{2}} = \sqrt{1^2 + 2^2 + 3^2 + 4^2 + |5i^2|} = \sqrt{55} = 7.4162$
2. $\|\mathbf{x}\mathbf{y}^\dagger\| = \sqrt{1210} = 34.7851$
   In MATLAB, use $norm(x, 2)$

Q3

1. For $\mathbf{A}$,

$$det(\mathbf{A}) = 1 * \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{vmatrix} - (-1) * \begin{vmatrix} -1 & 2 & 4 \\ 0 & 1 & 3 \\ i & 3 & 1 \end{vmatrix} + 0 * \begin{vmatrix} -1 & 1 & 4 \\ 0 & 2 & 3 \\ i & 4 & 1 \end{vmatrix} - (-i) * \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 1 \\ i & 4 & 3 \end{vmatrix}$$
$$= 20 + 8 + 2i + 3 - 2i = 31 > 0$$

Because $det(\mathbf{A}) > 0$, $\mathbf{A}$ is the full-rank matrix. $Rank(\mathbf{A}) = 4$

For any matrix $\mathbf{x}$, $Rank(\mathbf{x}) + nullity(\mathbf{x}) = n$, so the dimension of $\mathbf{x}$'s null space is:

$nullity(\mathbf{A}) = 4 - Rank(\mathbf{A}) = 0$

2. For $\mathbf{B}$,

$$\mathbf{B} = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & -1 \end{pmatrix} \overset{rref}{=} \begin{pmatrix} 1 & 4 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

$Rank(\mathbf{B}) = 2$

$nullity(\mathbf{B}) = 3 - 2 = 1$

3. For $\mathbf{C}$,

$$\mathbf{C} = \begin{pmatrix} 2 & 3 \\ -3 & 0.5 \end{pmatrix} \overset{rref}{=} \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$$

$Rank(\mathbf{C}) = 2$

$nullity(\mathbf{C}) = 2 - 2 = 0$

$$det(\mathbf{C}) = \begin{vmatrix} 2 & 3 \\ -3 & 0.5 \end{vmatrix} = 1 - (-9) = 10$$

$$\mathbf{C}^{-1} = \frac{1}{det(\mathbf{C})} \begin{pmatrix} 0.5 & -3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 0.05 & -0.3 \\ 0.3 & 0.2 \end{pmatrix}$$

$$\mathbf{C} + \mathbf{C}^T = \begin{pmatrix} 2 & 3 \\ -3 & 0.5 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 3 & 0.5 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

eigenvalues of $\mathbf{C} + \mathbf{C}^T$ is $\lambda_1 = 4, \lambda_2 = 1$

$$\mathbf{C} - \mathbf{C}^T = \begin{pmatrix} 2 & 3 \\ -3 & 0.5 \end{pmatrix} - \begin{pmatrix} 2 & -3 \\ 3 & 0.5 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ -6 & 0 \end{pmatrix}$$

eigenvalues of $\mathbf{C} - \mathbf{C}^T$ is $\lambda_1 = 6i, \lambda_2 = -6i$

Q4:

1. First, calculate the inverse of $\mathbf{A}\mathbf{A}^T$:

$$\mathbf{A}\mathbf{A}^T = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{pmatrix} * \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 10 & -4 \\ -4 & 9 \end{pmatrix} \Longrightarrow (\mathbf{A}\mathbf{A}^T)^{-1} = inv(\mathbf{A}\mathbf{A}) = \frac{1}{74}\begin{pmatrix} 9 & 4 \\ 4 & 10 \end{pmatrix} = \begin{pmatrix} 0.1216 & 0.0541 \\ 0.0541 & 0.1351 \end{pmatrix}$$

Then $\mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & -2 \end{pmatrix} * \begin{pmatrix} 9 & 4 \\ 4 & 10 \end{pmatrix} = \begin{pmatrix} 0.2297 & 0.3243 \\ -0.0541 & -0.1351 \\ 0.2568 & -0.1081 \end{pmatrix}$

2. $\mathbf{A}\mathbf{A}^+ = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{pmatrix} * \begin{pmatrix} 0.2297 & 0.3243 \\ -0.0541 & -0.1351 \\ 0.2568 & -0.1081 \end{pmatrix} = I_2$. So its dimension is 2.

3. $\mathbf{A}^+\mathbf{A} = \begin{pmatrix} 0.2297 & 0.3243 \\ -0.0541 & -0.1351 \\ 0.2568 & -0.1081 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 0.8784 & -0.3243 & 0.0405 \\ -0.3243 & 0.1351 & 0.1081 \\ 0.0405 & 0.1081 & 0.9865 \end{pmatrix}$ $\overset{rref}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Its dimension is 3×2.

Q5:

$\mathbf{ABC}(\mathbf{ABC})^{\mathbf{T}} = \mathbf{I} \Longrightarrow \mathbf{ABC} = (.)_{M\times Q}$ is semi-orthogonal if $M < Q$ or orthogonal if $M = Q$. So when $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are semi-orthogonal so that $\mathbf{AA^T} = \mathbf{I_M}; \mathbf{BB^T} = \mathbf{I_N}$; $\mathbf{CC^T} = \mathbf{I_P}$ if $M < N, N < P, P < Q$ or orthogonal so that $\mathbf{AA^T} = \mathbf{BB^T} = \mathbf{CC^T} = \mathbf{I_M}$ if $M = N = P = Q$, then:

$\mathbf{ABC}(\mathbf{ABC})^{\mathbf{T}} = \mathbf{AB}(\mathbf{CC^T})\mathbf{B^T A^T} = \mathbf{A}(\mathbf{BI_P B^T})\mathbf{A^T} = \mathbf{AI_N A^T} = \mathbf{I_M}$
So its $dim(.) = M$

Q6:

$$\mathbf{AA}^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix}1+i & 1-i & 1+i\\ 1-i & 1+i & 1-i\\ 1+i & 1-i & 1+i\end{pmatrix} * \frac{1}{\sqrt{2}}\begin{pmatrix}1-i & 1+i & 1-i\\ 1+i & 1-i & 1+i\\ 1-i & 1+i & 1-i\end{pmatrix} = \begin{pmatrix}3 & 0 & 3\\ 0 & 2 & 0\\ 3 & 0 & 3\end{pmatrix}$$

$$\mathbf{BB^T} = \mathbf{B} * \mathbf{B} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\mathbf{CC^T} = \frac{1}{3}\begin{pmatrix}1 & 2 & 2\\ -2 & -1 & 2\\ 2 & -2 & 1\end{pmatrix} * \frac{1}{3}\begin{pmatrix}1 & -2 & 2\\ 2 & -1 & -2\\ 2 & 2 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Because $\mathbf{B}$ is a symmetric matrix, $\mathbf{B} = \mathbf{B^T} \implies \mathbf{BB^T} = \mathbf{BB} = \mathbf{B^2}$
$\mathbf{A}$ is not a Hermitian matrix, not conjugate symmetric, therefore $\mathbf{AA}^\dagger \neq \mathbf{A^2}$.
$\mathbf{C}$ is not a symmetric matrix, therefore $\mathbf{CC^T} \neq \mathbf{C^2}$.

Q7:
$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = e^{-\frac{1}{2}\mathbf{x^T K^{-1} x}} * (-\frac{1}{2}\mathbf{x^T K^{-1} x})' = e^{-\frac{1}{2}\mathbf{x^T K^{-1} x}} * -\frac{1}{2}(2\mathbf{K^{-1} x}) = e^{-\frac{1}{2}\mathbf{x^T K^{-1} x}}\mathbf{K^{-1} x}$

As for its dimension, $\mathbf{K^{-1} x} = (.)_{N\times 1}$ and $\mathbf{x^T K^{-1} x} = (.)_{1\times 1}$
$\implies dim(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}) = dim(\mathbf{K^{-1} x}) = N \times 1$

Q8:

For $\mathbf{A}$, $det(\mathbf{A} - \lambda I) = 0 \Longrightarrow eig(A)$ in MATLAB: Then its eignevalues are:
$\lambda_1 = 11.0632, \lambda_2 = 2.922, \lambda_3 = -0.9791 + 0.3559i, \lambda_4 = -0.9791 - 0.3559i$
$\mathbf{A}$'s corresponding diagonal matrix $\mathbf{D}$ is:

$$\mathbf{D} = \begin{pmatrix} 11.0632 & & & \\ & 2.922 & & \\ & & -0.9791 + 0.3559i & \\ & & & -0.9791 - 0.3559i \end{pmatrix}$$

The corresponding nonsingular matrix $\mathbf{\Lambda}$ (via $[\mathbf{\Lambda}, \mathbf{D}] = \mathbf{eig}(\mathbf{A})$):

$$\mathbf{\Lambda} = \begin{pmatrix} -0.5140 & 0.5688 & 0.5378 + 0.0107i & 0.5378 - 0.0107i \\ -0.3665 & -0.7124 & 0.4333 - 0.1353i & 0.4333 + 0.1353i \\ -0.5291 & 0.2034 & -0.6356 & -0.6356 \\ -0.5671 & -0.3573 & -0.2862 + 0.1366i & -0.2862 - 0.1366i \end{pmatrix}$$

so that $\mathbf{\Lambda D \Lambda^{-1}} = \mathbf{A}$

For $\mathbf{B}$, same operation as $\mathbf{A}$, then its eigenvalues are:
$\lambda_1 = -2.1487, \lambda_2 = 0.0770, \lambda_3 = 4.3554, \lambda_4 = 9.7164$
$\mathbf{B}$'s diagonal matrix $\mathbf{D}$ is:

$$\mathbf{D} = \begin{pmatrix} -2.1487 & & & \\ & 0.0770 & & \\ & & 4.3554 & \\ & & & 9.7164 \end{pmatrix}$$

and corresponding nonsingular matrix $\mathbf{\Lambda} = \begin{pmatrix} 0.2041 & 0.6932 & -0.3318 & 0.6064 \\ 0.6640 & -0.2284 & 0.6081 & 0.3704 \\ -0.1256 & -0.6754 & -0.4595 & 0.5630 \\ -0.7083 & 0.1054 & 0.5559 & 0.4221 \end{pmatrix}$

Before calculating the eigenvalues, its easy to predict $\mathbf{B}$ has real eignevalues, since its a **real symmetric matrix**.

Proof: $\mathbf{B}$ is a real symmetric matrix, so $\mathbf{B} = \mathbf{B^T} = \bar{\mathbf{B}}$. Assume λ is a complex eigenvalue and $\mathbf{x}$ is the complex eigenvector of $\mathbf{B}$, and $\bar{\lambda}$ and $\bar{\mathbf{x}}$ are the corresponding conjugate complex and conjugate vector, then we have
$\mathbf{B\bar{x}} = \bar{\mathbf{B}}\bar{\mathbf{x}} = (\overline{\mathbf{Bx}}) = (\overline{\lambda\mathbf{x}}) = \bar{\lambda}\bar{\mathbf{x}}$

$$\mathbf{\bar{x}^T B x} = \mathbf{\bar{x}^T (Bx)} = \mathbf{\bar{x}^T}\lambda\mathbf{x} = \lambda\mathbf{\bar{x}^T x} \tag{1}$$

$$\mathbf{\bar{x}^T B x} = (\mathbf{x^T B^T})\mathbf{x} = (\mathbf{B\bar{x}})^{\mathbf{T}}\mathbf{x} = (\bar{\lambda}\mathbf{\bar{x}})^{\mathbf{T}}\mathbf{x} = \bar{\lambda}\mathbf{\bar{x}^T x} \tag{2}$$

Let (1)-(2), we get $(\lambda - \bar{\lambda})\mathbf{\bar{x}^T x} = 0$. Since $\mathbf{x}$ is not 0, $\lambda = \bar{\lambda}$. Therefore, λ is real.

As for A, the characteristic polynomials format of its eigenvalues is $det(\mathbf{A} - \lambda\mathbf{E}) = x^4 - 12x^3 + 6x^2 + 48x + 35 = 0$. According to **Descartes' rule of signs**, we could get the number of positive and negative real roots in a polynomial. Let $f(x) = x^4 - 12x^3 + 6x^2 + 48x + 35$ and $f(-x) = x^4 + 12x^3 + 6x^2 - 48x + 35$, there are both 2 sign changes, indicating 2 or 0 positive roots and 2 or 0 negative roots.

Therefore, $number of complex roots = n - p - q \geq 0$, where n is the degree of polynomials (here is 4), and p and q are the number of positive and negataive roots respectively. It's high likely that $\mathbf{A}$ has complex roots.

Q9:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 3 & 3 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 6 & 6 & 4 \end{pmatrix} \overset{rref}{=} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

After adding the noise $\mathbf{N} = \begin{pmatrix} -0.0065 & -0.0085 & -0.0020 & -0.0151 \\ 0.0118 & -0.0057 & 0.0059 & 0.0088 \\ -0.0076 & -0.0056 & -0.0085 & -0.0024 \\ -0.0111 & 0.0018 & 0.0080 & 0.0017 \end{pmatrix}$, the bottom row of $\mathbf{A}$ would be no longer 0. Since the noise row is less likely to be a linear combination of previous 3 rows, making the noised matrix $\mathbf{A} + N$ full rank, which means the inverse matrix exists.

However, because the noise only adds trivial value to original matrix, its eigenvalues changes little. The eigenvalues of $\mathbf{A}$ and eigenvalues of $\mathbf{A} + \mathbf{N}$ are:

$$\mathbf{vec}_{\lambda\mathbf{1}} = \begin{pmatrix} 10.5625 \\ 0.2188 + 0.5752i \\ 0.2188 - 0.5752i \\ 0.0000 \end{pmatrix}, \ \mathbf{vec}_{\lambda\mathbf{2}} = \begin{pmatrix} 10.5552 \\ 0.1969 + 0.5579i \\ 0.1969 - 0.5579i \\ 0.0320 \end{pmatrix}$$

$$\Delta\mathbf{vec}_{\lambda} = \mathbf{vec}_{\lambda\mathbf{1}} - \mathbf{vec}_{\lambda\mathbf{2}} = \begin{pmatrix} 0.0073 \\ 0.0219 + 0.0173i \\ 0.0219 - 0.0173i \\ -0.0320 \end{pmatrix} \text{ and } mean(\Delta\mathbf{vec}_{\lambda}) = 0.0048$$

We can see that the eigenvalues doesn't change a lot when $\alpha = 0.01$. If the α increase, the eigenvalues also change more and vice versa, as the following image shows.

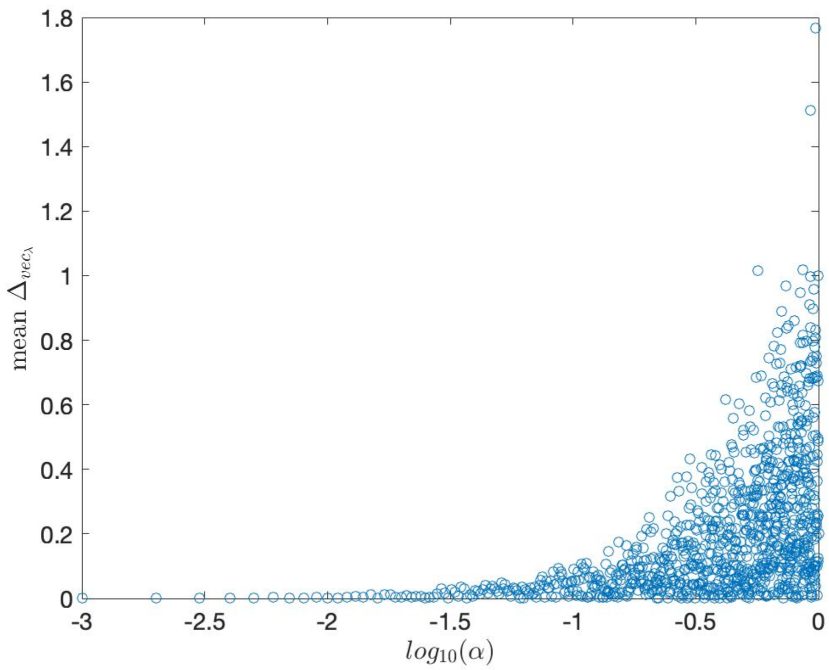

Figure 1: $\mathbf{A}$'s eigenvalues change under different noise amplitude α. Not axis y is the $mean(\Delta_{\mathbf{vec}_\lambda})$; axis x is the log_{10} value of α. We tested α from 0.001 to 1 with 0.001 as the stride.