Q5.1

$$\mathcal{F}\{exp(-a^2x^2)\} = \int_{-\infty}^{\infty} dx\ exp(-a^2x^2)exp(-i2\pi ux)$$

$$= \int_{-\infty}^{\infty} dx\ exp(-a^2(x^2 + \frac{i2\pi u}{a^2}x - \frac{\pi^2u^2}{a^4}) - \frac{\pi^2u^2}{a^2})$$

$$= exp(-\frac{\pi^2u^2}{a^2}) \int_{-\infty}^{\infty} dx\ exp(-a^2(x + \frac{i\pi u}{a^2})^2)$$

Let $x' = x + \frac{i\pi u}{a}$, then

$$\mathcal{F} = exp(-\frac{\pi^2u^2}{a^2}) \int_{-\infty}^{\infty} dx' exp(-a^2x'^2)$$

The later term is Gaussian Integral, $\int_{-\infty}^{\infty} dx' exp(-a^2x'^2) = \sqrt{\frac{\pi}{a^2}}$, then

$$\mathcal{F}\{exp(-a^2x^2)\} = \frac{\sqrt{\pi}}{a} exp(-\frac{\pi^2u^2}{a^2})$$

Q5.2

Let $x' = x - x_0$, and according to shift theorem

$\mathcal{F}\{g(t-t_0)\} = e^{-i2\pi u t_0}\mathcal{F}\{g(t')\}$, then

$\mathcal{F}\{h(x')\} = exp(-i2\pi ut_0)\mathcal{F}\{h(x')\} = exp(-i2\pi ut_0)\mathcal{F}\{\frac{1}{\sqrt{2\pi}\sigma}exp(-(\frac{1}{\sqrt{2}\sigma})^2 x'^2)\}$

$= exp(-i2\pi ut_0)\frac{1}{\sqrt{2\pi}\sigma}\mathcal{F}\{exp(-(\frac{1}{\sqrt{2}\sigma})^2 x'^2)\}$

Use Problem 5.1 which means $a = \frac{1}{\sqrt{2}\sigma}$, then

$\mathcal{F}\{h(x)\} = exp(-i2\pi ut_0) * \frac{1}{\sqrt{2\pi}\sigma} * \frac{\sqrt{\pi}}{\frac{1}{\sqrt{2}\sigma}}exp(-\frac{\pi^2 u^2}{(\frac{1}{\sqrt{2}\sigma})^2})$

$= exp(-2\pi^2 u^2\sigma^2 - i2\pi ut_0)$

Q5.3

Use convolution theorem, $\mathcal{F}\{h(x)cos(2\pi u_0 x)\} = \mathcal{F}\{h(x)\}\mathcal{F}\{cos(2\pi u_0 x)\}$
$= H(x)\mathcal{F}\{cos(2\pi u_0 x)\}$
Based on Euler's equation, $cos(x) = \frac{1}{2}[e^{ix} + e^{-ix}]$, then
$\mathcal{F}\{cos(2\pi u_0 x)\} = \displaystyle\int_{-\infty}^{\infty} dx\ \tfrac{1}{2}[exp(i2\pi u_0 x) + exp(-i2\pi u_0 x)]exp(-i2\pi ux)$
$= \frac{1}{2}\displaystyle\int_{-\infty}^{\infty} dx\ exp(-i2\pi(u-u_0)x) + \frac{1}{2}\displaystyle\int_{-\infty}^{\infty} dx\ exp(-i2\pi(u+u_0)x)$
$= \frac{1}{2}[\delta(u-u_0) + \delta(u+u_0)]$
So $\mathcal{F} = H(x) * \frac{1}{2}[\delta(u-u_0) + \delta(u+u_0)] = \frac{1}{2}[H(u-u_0) + H(u+u_0)]$

Q5.4

a.

$$\frac{d}{dt}rect(\frac{t}{2T_0}) = \begin{cases} \infty & t = -T_0 \\ -\infty & t = T_0 \\ 0 & t \neq -T_0 \ or \ t \neq T_0 \end{cases} \longrightarrow$$

$\frac{d}{dt}rect(\frac{t}{2T_0}) = \delta(t+T_0) - \delta(t+T_0)$

Use forward CT-FT, then

$\mathcal{F}\{\frac{d}{dt}rect(\frac{t}{2T_0})\} = \mathcal{F}\{\delta(t+T_0)-\delta(t-T_0)\} = \mathcal{F}\{\delta(t+T_0)\}-\mathcal{F}\{\delta(t-T_0)\}$

$= exp(i2\pi uT_0) - exp(-i2\pi uT_0)$

$= cos(2\pi uT_0) + i\ sin(2\pi uT_0) - (cos(-2\pi uT_0) + i\ sin(-2\pi uT_0))$

$= i2sin(2\pi uT_0)$

b.

According to derivative theorem, $\mathcal{F}\{\frac{d}{dt}f'(t)\} = i2\pi uF(u)$. Then use CT-FT theorm as equation 5.21,

$$F(u) = \int_{-\infty}^{\infty} dt\ f(t)exp(-i2\pi ut)$$

$$= \int_{-\infty}^{\infty} dt\ [rect(\tfrac{t}{2T_0})]exp(-i2\pi ut) = 2T_0sinc(2T_0u),$$

Therefore, $\mathcal{F}\{\frac{d}{dt}rect(\frac{t}{2T_0})\} = i2\pi u * 2T_0sinc(2T_0u)$

$= i2\pi u * 2T_0\frac{sin(2\pi uT_0)}{2\pi uT_0} = i2sin(2\pi uT_0)$

Q5.5

Let $circ(r/a) = g(x, y)$, then

$$\mathcal{F}_{2D}g(x,y) = G(u,v) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\ g(x,y)exp(-i2\pi(ux+vy))$$

Since $r^2 = x^2 + y^2$ and $\rho^2 = u^2 + v^2$, transform to polar coordinates where $x = rcos\theta, y = rsin\theta, u = \rho cos\varphi, v = \rho sin\varphi$, then

$$\mathcal{F}_{2D}g(x,y) = \int_0^a dr \int_0^{2\pi} d\theta r\ exp(-i2\pi(r\rho cos\theta cos\varphi + r\rho sin\theta sin\varphi))$$

$$= \int_0^a dr\ r \int_0^{2\pi} d\theta\ exp(-i2\pi r\rho cos(\theta - \varphi))$$

Because $J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\ exp(-iacos(\theta - \varphi))$, then $exp(-i2\pi r\rho cos(\theta - \varphi)) = 2\pi J_0(2\pi r\rho)$.

Put it into previous equation,

$$\mathcal{F}_{2D}g(x,y) = \int_0^a dr\ r * 2\pi J_0(2\pi r\rho) = \frac{1}{\rho}\int_0^a dr\ 2\pi r\rho J_0(2\pi r\rho)$$

Consider $\int_0^\alpha d\beta\ \beta J_0(\beta) = \alpha J_1(\alpha)$ and let $w = 2\pi r\rho$, then:

$$\int_0^a dr\ 2\pi r\rho J_0(2\pi r\rho) = \frac{1}{2\pi\rho}\int_0^{2\pi a\rho} dw\ wJ_0(w) = aJ_1(2\pi a\rho)$$

Also, $jinc(\rho) = 2J_1(2\pi\rho)/2\pi\rho$,Therefore,

$$\mathcal{F}_{2D}circ(r/a) = \frac{1}{\rho}aJ_1(2\pi a\rho) = \frac{1}{\rho}a\pi * a\rho jinc(a\rho) = \pi a^2\ jinc(a\rho)$$

Q5.6

a.

Use Euler theorem, $cos(\Omega_n t) = \frac{1}{2}(exp(i\Omega_n t) + exp(-i\Omega_n t))$
Replace it into FID function. Let $\Omega = 2\pi u$ and apply FT

$$\mathcal{F}\{FID(t)\} = \int_{-\infty}^{\infty} dt\,[M_0+\sum_{n=1}^{3} M_n exp(-t/T_n)cos(\Omega_n t)]step(t)exp(-i\Omega t)$$

$$= M_0 \int_0^{\infty} dt\ \ exp(-i\Omega t) + \sum_{n=1}^{3} M_n \int_0^{\infty} dt\ \ \tfrac{1}{2}exp(-t/T_n)(exp(i(\Omega_n - \Omega)t) + exp(-i(\Omega_n + \Omega)t))$$

where

$$\int_0^{\infty} dt\ exp(-t/T_n)exp(i(\Omega_n - \Omega)t)$$
$$= \int_0^{\infty} dt\ exp(-(1/T_n - i(\Omega_n - \Omega))t) = \frac{1}{1/T_n - i(\Omega_n - \Omega)}$$

and

$$\int_0^{\infty} dt\ exp(-t/T_n)exp(-i(\Omega_n + \Omega)t)$$
$$= \int_0^{\infty} dt\ exp(-(1/T_n + i(\Omega_n + \Omega))t) = \frac{1}{1/T_n + i(\Omega_n + \Omega)}$$

Therefore, the later term is:
$$\sum_{n=1}^{3} M_n \frac{i\Omega + 1/T_n}{(1/Tn)^2 + \Omega_n^2 - \Omega^2 + 2i\Omega/T_n}$$

Finally,
$$\mathcal{F}\{FID(t)\} = \frac{M_0}{i\Omega} + \Sigma_{n=1}^{3} M_n \frac{i\Omega + 1/T_n}{(1/Tn)^2 + \Omega_n^2 - \Omega^2 + 2i\Omega/T_n}, \text{ where } \Omega = 2\pi u$$

b.

In equation a, easy to find the first M_0 term belongs to the imaginary part. For the rest term, multiply both numerator and denominator by $(1/Tn)^2 + \Omega_n^2 - \Omega^2 - 2i\Omega/T_n$, then the real part should be :

$$\sum_{n=1}^{3} M_n \frac{\frac{1}{T_n}(\frac{1}{T_n} + \Omega_n^2 - \Omega^2) + 2\Omega^2 T_n}{((1/T_n)^2 + \Omega_n^2 - \Omega^2)^2 + (2\Omega T_n)^2}$$

c.

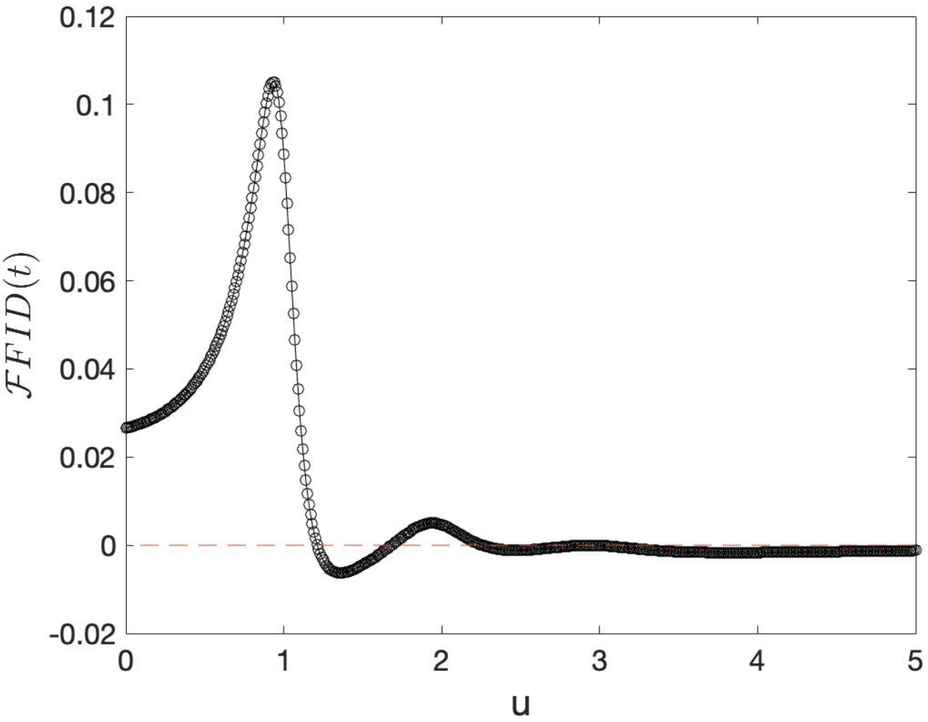

Figure 1: Real part of $\mathcal{F}\{FID(t)\}$ vs frequency u

Using following function to compute each real part of M_n:

```
function res = real(t0, m0, u0, u)
    o=2*pi*u;
    o0=2*pi*u0;
    a=1/t0;
    res=m0*(a*(a+o0^2-o.^2) + 2*o*t0)./((a^2+o0^2-o.^2).^2+(2*o*t0).^2);
end
```

Q5.7

a.

because f_2 is the time shift copy of f_1 with lower magnitude, so $f_2(t) = af_1(t - t_0)$, then the net output $g(t)$ would be the sum of f_1 and f_2.
$g(t) = f_1(t) + af_1(t - t_0)$

b.

the power spectrum of g is $G(u) = \mathcal{F}g$, so

$$\mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} dt\ (f_1(t) + af_1(t - t_0))exp(-i2\pi ut)$$

$$= \int_{-\infty}^{\infty} dt\ f_1(t)exp(-i2\pi ut) + a\int_{-\infty}^{\infty} dt\ f_1(t-t_0)exp(-i2\pi u(t-t_0))exp(-i2\pi ut_0)$$

$$= F(u) + a * exp(-i2\pi ut_0)F(u)$$

use euler's equation, then

$$\mathcal{F}\{g(t)\} = F(u) + a * (cos(2\pi ut_0) - isin(2\pi ut_0))F(u)$$

$$= (1 + a * cos(2\pi ut_0))F(u) - ia * sin(2\pi ut_0)F(u)$$

Since the predicted $F(u)$ is Gaussian-shaped spectrum, then $G(u)$ woule be like this shape with the following code.

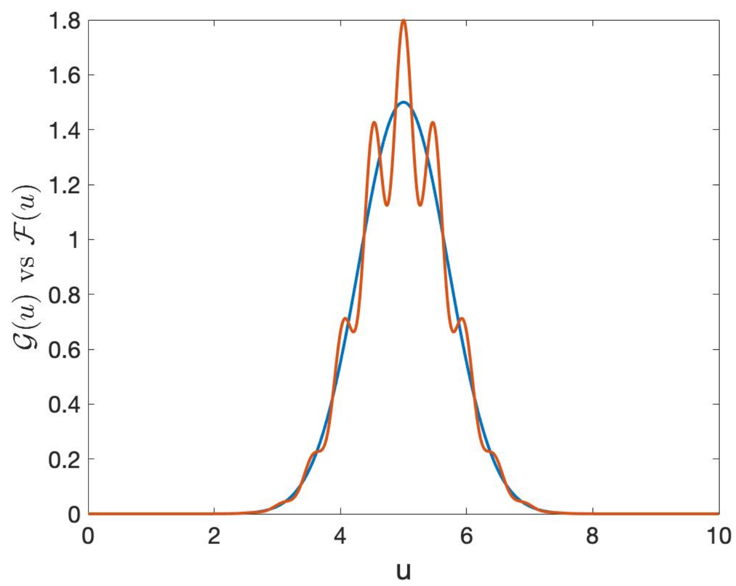

Figure 2: $\mathcal{G}\{u\}$ and $\mathcal{G}$ vs frequency u

```
u=-0:0.01:10;sig=1;t0=2;a=0.2;u0=5;amp=1.5;
fu = amp*exp(-(u-u0).^2/sig^2);
gu = fu.*(1+a*cos(2*pi*u*t0));
```

Q5.8

From problem 5.2, we could get $\mathcal{F}\{c(t)\} = exp(-2\pi^2 u^2 \sigma^2)$

From problem 5.3, $\mathcal{F}\{0.5cos(2\pi u_0 t)\} = \frac{1}{4}[\delta(u - u_0) + \delta(u + u_0)]$

So $\mathcal{F}\{g(t)\} = exp(-18\pi^2 u^2/u_0^2) + 0.25[\delta(u - u_0) + \delta(u + u_0)]$

The cosine signal would become a intensive pulse after CT-FT which locates at $u = 60$Hz, where most of $c(t)$ output at this frequency would be zero. Therefore, the data could be used after filtering the cosine signal. **c.**

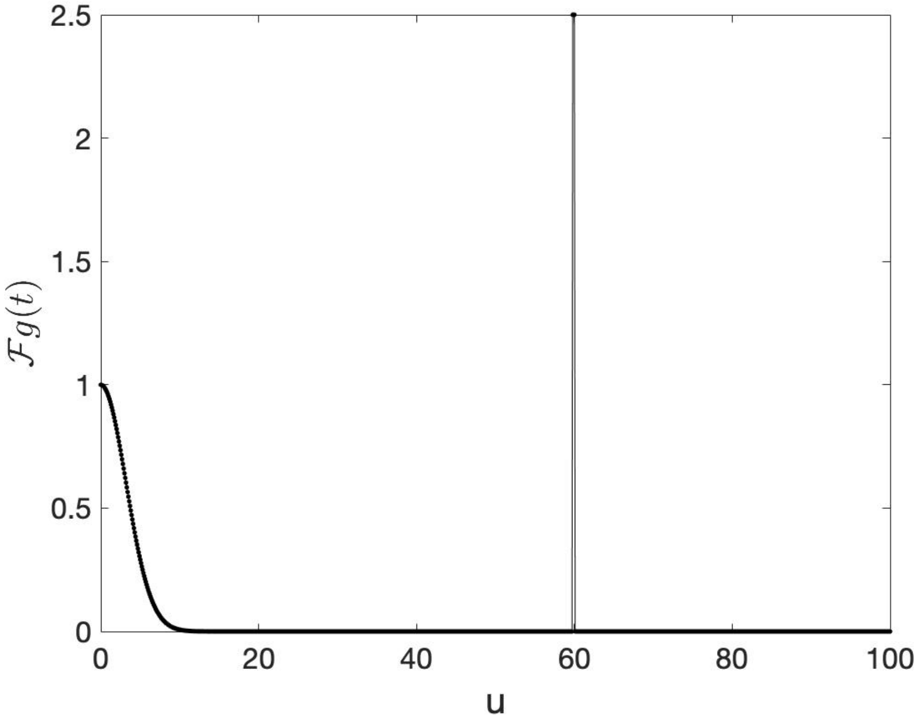

Figure 3: Real part of $\mathcal{F}\{g(t)\}$ vs frequency u

Q5.9

$g(t) = \int_{-\infty}^{\infty} dt'\ h(t-t')f(at'-b)$

$= \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d(u)H(u)exp(i2\pi u(t-t')) \int_{-\infty}^{\infty} du'F(u')exp(i2\pi u'(at'-$
$b))$

$= \int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} duH(u)F(u')exp(i2\pi ut)exp(-i2\pi u'b) \int_{-\infty}^{\infty} dt'exp(-i2\pi(u-$
$au')t')$

$= \int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} duH(u)F(u')exp(i2\pi(ut-u'b))\delta(u-au')$

$= \int_{-\infty}^{\infty} du'\ H(au')F(u')exp(i2\pi(at-b)u')$

Apply FT, $\mathcal{F}g(t) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du'\ H(au')F(u')exp(i2\pi(at-b)u')exp(-i2\pi ut)$

$= \int_{-\infty}^{\infty} du'H(au')F(u')exp(-i2\pi u'b) \int_{-\infty}^{\infty} dtexp(-i2\pi(u-au')t)$

$= \int_{-\infty}^{\infty} du'H(au')F(u')exp(-i2\pi u'b)\delta(u-au')$

$= H(u)F(\frac{u}{a})exp(-i2\pi u\frac{b}{a})$

Q5.10 (Thanks **Charles Marchini** & **Joseph Tibbs**)
See equation **3.31**, for continuous function, if $Z = X + Y$ where X and Y are independent with each other. Then we have:
$p_Z(z) = \int_{-\infty}^{\infty} dx \; p_X(x)p_Y(z-x)$

But here because X and Y are poisson distribution, which only have positive values, so we need to get the discrete version of previous equation: $p_Z(z) = \sum_{x=1}^{z} p_X(x)p_Y(z-x)$.
Put $p_X(x) = \lambda_x^x exp(-\lambda_x)/x!$ and $p_Y(z-x) = \lambda_y^{z-x} exp(-\lambda_y)/(z-x)!$ into previous equation, then we have:
$p_Z(z) = \sum_{x=1}^{z} \frac{\lambda_x^x exp(-\lambda_x)}{x!} \frac{\lambda_y^{z-x} exp(-\lambda_y)}{(z-x)!}$
$= \sum_{x=1}^{z} \frac{z!}{x!(z-x)!} \frac{\lambda_x^x \lambda_y^{z-x} exp(-\lambda_x) exp(-\lambda_y)}{z!}$
$= \frac{exp(-(\lambda_x+\lambda_y))}{z!} \sum_{x=1}^{z} \binom{z}{x} \lambda_x^x \lambda_y^{z-x}$
$= \frac{exp(-(\lambda_x+\lambda_y))}{z!} (\lambda_x + \lambda_y)^z$
$= \frac{(\lambda_x+\lambda_y)^z}{z!} exp(-(\lambda_x + \lambda_y))$
Let $\lambda_z = \lambda_x + \lambda_y$, $p_Z(z) = \lambda_z^z exp(-\lambda_z)/z!$. The result shows that the sum of two independent poi