Homework 2: Generalized Functions & Idealized Imaging

1. A simple example of idealized imaging. An object function is given

$$f(x) = \cos(2x)\sin(\frac{x}{4}). \tag{1}$$

Let the one-dimensional C→D imaging kernel be: $\mathrm{rect}(\frac{2x}{\pi})$. "Image" the object via convolution and plot both the image and the object on the same axes for $x \in [0, 8\pi]$. Then, construct the adjoint of the forward operator (H) and plot H^*g on the same axes as before. Thoroughly label your plot. *Hints*: use a `for` loop and think carefully about integration limits. Also, pyplot.step() might be helpful.

2. Building generalized functions. One of the motivations behind generalized functions is that they "behave" like functions under mathematical operations such as integration. Compute the convolution of $\mathrm{rect}(\frac{x}{a}) * \mathrm{rect}(\frac{x}{a})$ and thus derive the triangle function. *Hint*: first compute the integral of a Heaviside step function and then expand one of the rect() functions in terms of Heaviside step functions.

3. Effect of views and filtering on object estimates. Suppose a discrete object $f(r, c)$ perfectly fits into the RxC field of view of a discrete detector. Let the object be zero everywhere except at two points:

$$f(r,c) = \begin{cases} 64 & r = c = R/2 \\ 128 & r = c = R/4 \\ 0 & \text{else} \end{cases} \tag{2}$$

a. Let $R = C = 64$ and the initial numbers of views be $N = 256$. Use `skimage.transform.radon()` and plot a sinogram for this object. You might also vary the number of views and initial point values in order to convince yourself that you understand the appearance of the sinograms.

b. Apply `skimage.transform.iradon()` with `filter_name=None` to the sinogram and thus compute an estimate of the object, $\widehat{f(r,c)}$. Plot $\widehat{f(r,c)}$ and estimate both the width of the known, nonzero points and the value at those points. Subjectively, is $\widehat{f(r,c)}$ a "good" estimate of $f(r, c)$?

c. Repeat parts a and b for the $N \in \{256, 128, 64, 32, 16, 8\}$. Summarize your results in a plot and comment on how the number of views impacts the width of and value estimates. *Hint*: a bar chart or boxplot will suffice.

d. Let `filter_name='ramp'` and repeat part c.

4. The Ram-Lak filter. The Ram-Lak filter, RL(r), is one popular filter convolved with sinograms as part of filtered back-projection. RL(r) may be thought of as the difference between a rectangle function and a triangle function, each centered at zero and extending from $-k_{\max}$ to $k_{\max}$. Derive the Ram-Lak filter that can be convolved with a one-dimensional sinogram like those generated in the previous exercise. *Hint*: RL(r) can be derived straightforwardly from the inverse Fourier transform.

5. Fourier transform of a "frequency comb." Consider the object from the last problem on the last homework as well-sampled object $f(x, y)$ that we'll call "real." You are also given a two-dimensional comb filter:

$$c(x, y) = \sum_{m=-\infty}^{\infty} \delta(x - Tm) \sum_{n=-\infty}^{\infty} \delta(y - Tn) \tag{3}$$

where T is such that every fourth pixel is selected.

a. Apply $c(x, y)$ to $f(x, y)$ such that the only information that you have about the original object is a discretized point profile through $f(x, y)$. Call this diminished object $f_d(x_o, y_o)$.

b. Image the original object with the Fourier transform to obtain $\mathcal{F}[f_d(x_o, y_o)] = g(u, v)$.

c. Derive the Fourier transform of $c(x, y)$. Call this $\widetilde{c}(u, v)$. *Hint*: this derivation can be easy, if you recall the rules about deltas and Fourier transforms.

d. Apply $\widetilde{c}(u, v)$ to $g(u, v)$. Call this result $g_d(u, v)$.

e. Invert $g_d(u, v)$ with the inverse Fourier transform. Does $\mathcal{F}^{-1}[g_d(u, v)] = f_d(x, y)$? Why does your answer make sense?

6. Diffraction through a circular aperture. Suppose that a coherent beam of light illuminates a large, two-dimensional screen with a small circular aperture of radius a given by:

$$\mathrm{circ}(r/a) = \begin{cases} 0 & r > a \\ 1 & r \leq a \end{cases} \tag{4}$$

a. Apply the Fourier transform analytically to obtain a continuous image of the aperture. *Hint*: this is a famous problem and the answer involves a named integral function that you don't have to actually evaluate.

b. Vary the size of the aperture and make some plots of the image. Describe the patterns that you see and thus learn the effect of aperture size on the image.

c. Let the aperture radius be 1 mm and assume that a beam of wavelength 100 pm much larger than the aperture is aimed perfectly along the center axis of the aperture. Assume a perfectly continuous detector and image the aperture at a propagation distance of 10 cm and then again for a distance of 100 cm. Make some plots, describe how your images differ, and why this difference makes sense.

d. Now assume that each element in a 1024x1024-element detector is a 32 x 32 $\mu\mathrm{m}^2$ square. Recompute both images from part c and comment on the effect of discretization.