11.1

According to equation 11.21, $\mathcal{L}\{\ddot{g}(t)\} = s^2G(s) - sg(0) - \dot{g}(0)$, then we apply Laplace transform to both sides of equation:

$$\begin{aligned}
\mathcal{L}\{\ddot{g}(t) - 4\dot{g}(t) + 3g(t)\} &= \mathcal{L}f(t) \\
s^2G(s) - sg(0) - \dot{g}(0) - 4(sG(s) - g(0)) + 3G(s) &= F(s) \\
let\ g(0) = \dot{g}(0) = 0;\ (s^2 - 4s + 3)G(s) &= F(s) \\
H(s) = G(s)/F(s) &= \frac{1}{(s-1)(s-3)}
\end{aligned}$$

Here we can see that the nontrivial poles of this impulse response system are 1 and 3, which is larger than 0. Therefore, it's an unstable system. Then according to appendix F.11, we have

$h(t) = \mathcal{L}^{-1}\{H(s)\} = \dfrac{1}{a-b}(exp(-bt) - exp(-at))$ and $a = -1, b = -3$. So

$h(t) = \dfrac{1}{2}(e^t + e^{3t})$. Also, the exponentially increasing function indicates it's unstable.

Since $f(t) = f_p step(t)$, for $g(t)$, $g(t) = [h*f](t) = \displaystyle\int_{-\infty}^{\infty} dt' h(t-t')f(t') \rightarrow$

$$g(t) = \int_{-\infty}^{\infty} dt' \frac{1}{2}(exp(t-t') + exp(3(t-t')))f_p step(t')$$

$$= \frac{f_p}{2}(exp(t)\int_0^{\infty} dt' exp(-t') + exp(3t)\int_0^{\infty} dt' exp(-3t'))$$

$$= \frac{f_p}{2}(exp(t) + exp(3t)/3)$$

11.2

Like 11.1,

$$\mathcal{L}(\ddot{g}(t) + a^2 g(t)) = \mathcal{L}(f(t))$$

$$s^2 G(s) - sg(0) - \dot{g}(0) + a^2 G(s) = F(s)$$

$$(s^2 + a^2)G(s) = F(s); let\ g(0) = \dot{g}(0) = 0$$

$$H(s) = \frac{G(s)}{F(s)} = \frac{1}{s^2 + a^2}$$

According to Appendix F.17 and equation 11.18,

$h(t) = 1/2\pi \int_{\sigma - i\infty}^{\sigma + i\infty} ds \frac{1}{a} \frac{a}{s^2 + a^2} exp(st)$

$= (1/a) * 1/2\pi \int_{\sigma - i\infty}^{\sigma + i\infty} ds \frac{a}{s^2 + a^2} exp(st)$

$= sin(at)/a$

Because $a > 0$, the response function's poles are on the imaginary axis, where $\sigma = 0$. So it's a nondamping system, which is marginally stable.

Since $f(t) = cos(at)$, for $g(t)$, $g(t) = \int_{-\infty}^{\infty} dt' h(t - t') f(t')$

$= \int_{-\infty}^{\infty} dt' cos(at') sin(a(t - t'))/a$

$= \frac{1}{a} \int_{-\infty}^{\infty} dt' cos(at')(sin(at)cos(at') - cos(at)sin(at'))$

$= \frac{1}{a} \int_{-\infty}^{\infty} dt' sin(at)cos(at')^2 - cos(at)sin(at')cos(at')$

$= \frac{1}{a}(sin(at) \int_{-\infty}^{\infty} dt' \frac{1}{2}(1 - cos(2at')) - cos(at) \int_{-\infty}^{\infty} dt' \frac{1}{2} sin(2at'))$

$= \frac{1}{a} sin(at) * 1/2 + 0 + 0$

$= \frac{sin(at)}{2a}$

11.3

Like 11.1 and 11.2, apply Laplace transform to both sides:

$$\begin{aligned}
\mathcal{L}\{m\ddot{x}(t) + b\dot{x}(t) + kx(t)\} &= \mathcal{L}\{f(t)\} \\
m(s^2X(s) - sx(0) - \dot{x}(0)) + b(sX(s) - x(0)) + kX(s) &= F(s) \\
(ms^2 + bs + k)X(s) &= F(s) \\
H(s) = \frac{X(s)}{F(s)} &= \frac{1}{ms^2 + bs + k}
\end{aligned}$$

So $H(s) = \dfrac{1}{(s - \dfrac{-b+\sqrt{b^2-4mk}}{2m})(s - \dfrac{-b-\sqrt{b^2-4mk}}{2m})}$

According to Appendix F.11,

let $x_1 = \dfrac{b+\sqrt{b^2-4mk}}{2m}$; $x_2 = \dfrac{b-\sqrt{b^2-4mk}}{2m}$

Then $h(t) = \mathcal{L}^{-1}H(s) = \dfrac{1}{x_1 - x_2}(exp(-x_2 t) - exp(-x_1 t))$

$= \dfrac{m}{\sqrt{b^2-4mk}}(exp(-\dfrac{b-\sqrt{b^2-4mk}}{2m}t) - (exp(-\dfrac{b+\sqrt{b^2-4mk}}{2m}t))$

We know that object would always have mass, meaning $m > 0$.

(1) When $\sqrt{b^2-4mk} \geq 0$, the poles are on the real axis. If $b - \sqrt{b^2-4mk} < 0$ which means at least one pole in on the positive real axis, this systems would be unstable. The response would increase exponentially. If $b + \sqrt{b^2-4mk} > b - \sqrt{b^2-4mk} > 0$ which means both nontrivial poles are on the positive real axis, this could be a stable system. The response would decay exponentially.

(2) When $\sqrt{b^2-4mk} < 0$, the poles woule be on the imaginary plane, indicating the system would have oscilating component. If $b < 0$ which means the real part is on the positive real plane, it's an unstable system, wiht oscilatingly increasing component. If $b > 0$, which means the poles are on the negative imaginary plane, the system is stable, with oscilatingly decreasing component. If $b = 0$ which means the poles are on the imaginary axis, it's marginally stable, oscilating without decreasing.

11.4

Apply Laplace transform to both sides:

$$\mathcal{L}\{\ddot{g} + 3\dot{g} + 2g\} = 0$$

$$s^2G(s) - sg(0) - \dot{g}(0) + 3(sG(s) - g(0)) + 2G(s) = 0$$

$$let\ g(0) = 1; \dot{g}(0) = 0;\ (s^2 + 3s + 2)G(s) = s + 3;$$

$$G(s) = \frac{s+3}{(s+1)(s+2)}$$

Therefore, the system has nontrivial poles at $s = -1, -2$ and zeros at $s = -3$.

According to Appendix F.11 and F.16,

$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\frac{s}{(s+1)(s+2)} + \frac{3}{(s+1)(s+2)}\}$

$= \frac{e^{-t} - 2e^{-2t}}{1-2} + 3\frac{1}{1-2}(e^{-2t} - e^{-t})$

$= 2e^{-2t} - e^{-t} + 3(e^{-t} - e^{-2t})$

$= 2e^{-t} - e^{-2t}$

Then we compute the left part of equation by using $g(t)$,

$\ddot{g} + 3\dot{g} + 2g = 2e^{-t} - 4e^{-2t} + 3(-2e^{-t} + 2e^{-2t}) + 2(2e^{-t} - e^{-2t})$

$= (2 - 6 + 4)e^{-t} + (-4 + 6 - 2)e^{-2t}$

$= 0$ So our computed $g(t)$ matches the equation.