Ex1

$$\mathcal{H}^+ = \sum_{k=1}^{R} \frac{1}{\sqrt{\mu_k}} u_k v_k^* \rightarrow \mathcal{H} = \sum_{n=1}^{R} \sqrt{\mu_n} v_n u_n^*$$

$$\mathcal{H}^+ \mathcal{H} \mathcal{H}^+ = \sum_{k=1}^{R} \frac{1}{\sqrt{\mu_k}} u_k v_k^* \sum_{n=1}^{R} \sqrt{\mu_n} v_n u_n^* \sum_{k=1}^{R} \frac{1}{\sqrt{\mu_k}} u_k v_k^*$$

$$Since \ \sum_{k=1}^{R} v_k^* \sum_{n=1}^{R} v_k = \delta_{kn}$$

$$Then \ = \sum_{k=1}^{R} u_k u_k^* \sum_{k=1}^{R} \frac{1}{\sqrt{\mu_k}} u_k v_k^*$$

$$Similarly, \ \sum_{k=1}^{R} u_k^* \sum_{k=1}^{R} u_k = \delta_{kk}$$

$$Then, \ = \sum_{k=1}^{R} u_k \frac{1}{\sqrt{\mu_k}} v_k^*$$

$$= \sum_{k=1}^{R} \frac{1}{\sqrt{\mu_k}} u_k v_k^* = \mathcal{H}^+$$

So $\mathcal{H}^+ \mathcal{H} \mathcal{H}^+ = \mathcal{H}^+$ is satisfied.

Ex2

Let $\mathcal{H} = \sum_{k=1}^{R} \sqrt{\mu_k} v_k u_k^*; \mathcal{H}^* = \sum_{k=1}^{R} \sqrt{\mu_k} u_k v_k^*; \mathcal{I}_{\mathbb{V}} = \sum_{k=1}^{R} v_k v_k^*,$

$$\mathcal{H} * \mathcal{H}^* + \eta \mathcal{I}_{\mathbb{V}} = \sum_{k=1}^{R} \sqrt{\mu_k} v_k u_k^* \sum_{n=1}^{R} \sqrt{\mu_n} u_n v_n^* + \eta \sum_{k=1}^{N} v_k v_k^*$$

$$= \sum_{k=1}^{R} \mu_k v_k v_k^* + \sum_{k=1}^{N} \eta v_k v_k^*$$

$$= \sum_{k=1}^{N} (\mu_k + \eta) v_k v_k^*, \ since\ v_k = 0\ if\ k \in [R+1, N]$$

Then $(\mathcal{H}\mathcal{H}^* + \eta \mathcal{I}_{\mathbb{V}})^{-1} = \sum_{k=1}^{N} \frac{1}{\mu_k + \eta} v_k v_k^*$

Apply $\mathcal{H}^*$ to it, $\mathcal{H}^*(\mathcal{H}\mathcal{H}^* + \eta \mathcal{I}_{\mathbb{V}}) = \sum_{k=1}^{R} \sqrt{\mu_k} u_k v_k^* \sum_{k=1}^{N} \frac{1}{\mu_k + \eta} v_k v_k^*$

$$= \sum_{k=1}^{N} \frac{\sqrt{\mu_k}}{\mu_k + \eta} u_k v_k^*$$

So $\lim_{\eta \to 0} \mathcal{H}^*(\mathcal{H}\mathcal{H}^* + \eta \mathcal{I}_{\mathbb{V}}) = \sum_{k=1}^{R} \frac{1}{\sqrt{\mu_k}} u_k v_k^* = \mathcal{H}^+$

The previous $\mathcal{H}^+$ is the left inverse of $\mathcal{H}$, when the dimension of $\mathcal{U}$ is higher than $\mathcal{V}$. The later $\mathcal{H}^+$ is the right inverse of $\mathcal{H}$, when the dimension of $\mathcal{U}$ is lower than $\mathcal{V}$. The previous one faces solution problem and the later one faces uniqueness problem.

Ex3

Suppose $g = \sum_n \beta_n v_n$.

Consider $H^+ g = \sum_k \dfrac{1}{\sqrt{\mu_k}} u_k u_k^* \sum_n \beta_n v_n = \sum_k \dfrac{1}{\sqrt{\mu_k}} \beta_k u_k$

$HH^+ g = \sum_k \sqrt{\mu_k} v_k u_k^* \sum_k \dfrac{1}{\sqrt{\mu_k}} \beta_k u_k = \sum_k \beta_k v_k = g$

So $P_{cons} = HH^+$.

$P_{incons} = I_{\mathbb{V}} - P_{cons} = I_{\mathbb{V}} - HH^+$

$P_{incons} g = g_{null} \rightarrow H^+ P_{incons} g = 0$

$H^+(I_{\mathbb{V}} - HH^+)g = 0$

$(H^+ - H^+ HH^+)g = 0$

$H^+ = H^+ HH^+$, which satisfies the second Penrose equation.

Ex4

a. Let $f(x) = \sum_{n=1}^{R} a_n x^n$, so $f(\lambda_k) = \sum_{n=1}^{R} a_n \lambda_k^n$ and $f(A) = \sum_{n=1}^{R} a_n A^n$

Plug in $A^n = \sum_k \lambda_k^n P_k$, then

$$f(A) = \sum_{n=1}^{R} a_n \sum_k \lambda_k^n P_k = \sum_k P_k \sum_{n=1}^{R} a_n \lambda_k^n = \sum_k f(\lambda_k) P_k$$

b. Since $\mathcal{H}$ is hermitian, $\mathcal{H} = \mathcal{H}^*$, then

$$\mathcal{U}^* = exp(i\mathcal{H})^* = exp(-i\mathcal{H}^*) = exp(-i\mathcal{H}) = \frac{1}{exp(i\mathcal{H})} = \frac{1}{\mathcal{U}} = \mathcal{U}^{-1}$$

So $\mathcal{U}^* = \mathcal{U}^{-1}$, $\mathcal{U}$ is unitary.

c. Since $\mathcal{U}$ is unitary, let $\mathcal{U} = QUQ^*$, where Q is unitary and $U = diag(\mu_1, \mu_2, , , , , \mu_k)$ is a diagonal matrix.

$log\mathcal{U} = Qlog(U)Q^* = i\mathcal{H} \longrightarrow \mathcal{H} = Q(-ilog(U))Q^*$

Let $\mathcal{H} = Q\Lambda Q^* = Qdiag(\lambda_1, \lambda_2, .., \lambda_k)Q$

So $diag(\lambda_1, \lambda_2, ..., \lambda_k) = diag(-ilog\mu_1, -ilog\mu_2, ..., -ilog\mu_k)$

$\lambda_k = -ilog\mu \rightarrow \mu_k = exp(i\lambda_k)$

Extra: Assume $A = Q\Lambda Q^*$, where Q is unitary.

So $A^n = (Q\Lambda Q^*)(Q\Lambda Q^*)...(Q\Lambda Q^*)$

$= Q\Lambda(Q^*Q)\Lambda(Q^*...Q)\Lambda Q^* = Q\Lambda^n Q^*$. The eigenvalue of A^n is $\{\lambda_1^n, \lambda_2^n, ..., \lambda_k^n\}$.

Suppose $A = \sum_k \lambda_k P_k$, then $A^n = \sum_k \lambda_k^n P_k$