Solutions to Part I of Game Theory

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Solutions to Section I.1

1. To make your opponent take the last chip, you must leave a pile of size 1. So 1 is a P-position, and then 2, 3, and 4 are N-positions. Then 5 is a P-position, etc. The P-positions are 1, 5, 9, 13,..., i.e. the numbers equal to 1 mod 4.

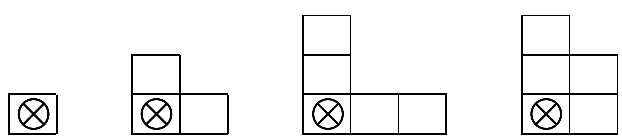
2.(a) The target positions are now 0, 7, 14, 21, etc.; i.e. anything divisible by 7.
(b) With 31 chips, you should remove 3, leaving 28.

3.(a) The target sums are 3, 10, 17, 24, and 31. If you start by choosing 3 and your opponent chooses 4, and this repeats four times, then the sum is 28, but there are no 3's left. You must choose 1 or 2 and he can then make the sum 31 and so win.
(b) Start with 5. If your opponent chooses 5 to get in the target series, you choose 2, and repeat 2 every time he chooses 5. When the sum is 26, it is his turn and there are no 5's left, so you will win. But if he ever departs from the target series, you can enter the series and win.

4.(a) The P-positions are the even numbers, $\{0, 2, 4, \ldots\}$.
(b) The P-positions are $\{0, 2, 4, 9, 11, 13, 18, 20, 22, \ldots\}$, the nonnegative integers equal to 0, 2, or 4 mod 9.
(c) The P-positions are $\{0, 3, 6, 9, \ldots\}$, the nonnegative integers divisible by 3.
(d) In (a), 100 is a P-position. In (b), 100 is an N-position since $100 = 1$ mod 9. It can be put into a P-position by subtracting 1 (or 6). In (c), 100 is an N-position. It can be put into a P-position by subtracting 1 (or 4 or 16 or 64).

5. The P-positions are those (m, n) with both m and n odd integers. If m and n are both odd, then any move will require putting an odd number of chips in two boxes; one of the two boxes would contain an even number of chips. If one of m and n is even, then we can empty the other box and put an odd number of chips in each box.

6. (a) In solving such problems, it is advisable to start the investigation with simpler positions and work up to the more difficult ones. Here are the simplest P-positions.



The last position shows that chomping at (3,1) is a winning move for the first player.

(b) The proof uses an argument, called "strategy stealing" that is useful in other problems as well. Consider removing the upper right corner. If this is a winning move, we are done. If not, then the second player has a winning reply. But whatever that reply is, the first player could have used it instead of the move he chose. (He could "steal" the second player's move.) Thus, in either case, the first player has a winning first move.

7. (a) Write the integer n in binary, $44 = (101100)_2$. A strategy that wins if it can be started out is to remove the smallest power of 2 in this expansion, in this case $4 = (100)_2$. Then the next player must leave a position for which this strategy can be continued. Another optimal first move for the first player is to remove $12 = (1100)_2$ chips. The only initial values of n for which the second player can win are the powers of 2: $n = 1, 2, 4, 8, 16, \ldots$.

(b) A strategy that wins is to remove the smallest Fibonacci number in the Zeckendorf expansion of n, (if possible). To see this, we note two things. First, if you do this, your opponent will be unable to take the smallest Fibnonacci number of the Zeckendorf expansion of the result, because it is greater than twice what you took. Second, if your opponent takes less than the smallest Fibonacci number in the Zeckendorf expension, you can again follow this strategy.

To prove this last sentence, suppose your opponent cannot take the smallest Fibonacci number in the Zeckendorf expansion of n. Let F_{n_0} represent this number, and suppose he takes $x < F_{n_0}$. The difference has a Zeckendorf expansion, $F_{n_0} - x = F_{n_1} + \cdots + F_{n_k}$, where F_{n_k} is the smallest. We must show $F_{n_k} \leq 2x$, i.e. that you can take F_{n_k}. We do this by contradiction. Suppose $2x < F_{n_k}$. Then x is less than the next lower Fibonacci number. This implies that when x is replaced by its Zeckendorf expansion, $x = F_{n_{k+1}} + \cdots + F_{n_\ell}$, we have

$$F_{n_0} = F_{n_1} + \cdots + F_{n_k} + x = F_{n_1} + \cdots + F_{n_k} + F_{n_{k+1}} + \cdots + F_{n_\ell}$$

which gives a second Zeckendorf expansions of F_{n_0}. This contradicts unicity.

For $n = 43 = 34 + 8 + 1$, the strategy requires that we take 1 chip. (Another optimal initial move is to remove 9 chips, leaving 34, since twice 9 is still smaller than 34.) The only initial values of n for which the second player can win are the Fibonacci numbers themselves: $n = 1, 2, 3, 5, 8, \ldots$.

8. (a) If the first player puts an S in the first square, the second player can win by putting an S in the last square. Then no matter what letter the first player puts in either empty square, the second player can complete an SOS.

(b) Player I can win by placing an S in the central square. Then if Player II plays on the left, say, without allowing I to win immediately, Player I plays an S in the last square. Now neither player can play on the right. But after Player II and then I play innoculously on the left, Player II must play on the right and lose.

(c) Call a square **x-rated** if no matter which letter a player places in the square, the other player can win immediately. It is not hard to show that the *only* way to make an x-rated square is to have it and another x-rated square between two S's as in (a). Thus, x-rated squares come in pairs. So, if n is even (like 2000) and if neither player makes an error allowing the opponent to win in one move, then after an even number of moves only x-rated squares will remain. It will then be Player I's turn and he must fill an x-rated square and so lose. However, Player II must make sure there is at least one x-rated pair. But this is easy to do if n is large say greater than 14. Just play an S in a square with at least three or four empty spots on either side. On your next move you will be able to make an x-rated pair on one side or the other. Generally, Player I wins if n is odd, and Player II wins if n is even.

(d) The case $n = 14$ is special. Player I begins by playing an O at position 7. Then if Player II plays an S at position 11, Player I plays an O at position 13, say, and then Player II cannot play an S at position 8 because Player I could win immediately with an S at position 6. The position is actually drawn. Player I can prevent Player II from making any x-rated squares.

Solutions to Section I.2

1.(a) $27 \oplus 17 = 10$.

(b) If $38 \oplus x = 25$, then $x = 38 \oplus 25 = 49$.

2.(a) The unique winning move is to remove 4 chips from the pile of 12 leaving 8.

(b) There are three winning moves; removing 8 chips from the pile of 17 or the pile of 19, or the pile of 23.

(c) Exactly the same answer as for (a) and (b).

3. We may identify a coin on a square labelled n with a nim pile of size n and a move of that coin to the left to a square labelled k as removing $n - k$ chips from the nim pile. Since the coins do not interact, this is exactly nim. The next player wins the displayed diagram by moving the coin on square 9 to square 0 (or moving the coin on square 10 to square 3, or by moving the coin on square 14 to square 7).

4.(a) Suppose there is an H in place n.
(1) Turning this H to T without turning over another coin corresponds to completely removing a pile of n chips.
(2) Turning this H to T and some T in place k to H, where $k < n$, corresponds to removing $n - k$ chips from a pile of n.
(3) Turning this H to T and some H in place k to T, where $k < n$, corresponds to removing two piles of sizes n and k. But this is equivalent to removing $n - k$ chips from the pile of size n, thus creating two piles of size k, which effectively cancel because $k \oplus k = 0$.

(b) Since $2 \oplus 5 \oplus 9 \oplus 10 \oplus 12 = 8$, we must reduce the 9, 10 or 12 by 8. One method is to turn the H in place 9 to T and the T in place 1 to H. Another would be to turn the H in place 10 to T and the H in place 2 to T.

5. The player who moves first wins. A row with n spaces between the checkers corresponds to a nim pile with n chips. So the given position corresponds to a nim position with piles of sizes 4, 2, 3, 5, 3, 6, 2, and 1. The nim sum of these numbers is 6. You can win, for example, by moving the checker in the sixth row six squares toward the other, making the nim sum 0. Now if the opponent moves away from you in some row, you can move in the same row to keep the nim sum the same. (Such a move is called reversible.) If he moves toward you in some row, the nim sum is no longer 0, so you can find some row such that moving toward him reduces the nim sum to 0. In this way, the game will eventually end and you will be the winner.

6. Any move from $(x_1, x_2, x_3, \ldots, x_n)$ in staircase nim changes exactly one of the numbers, $x_1, x_3, \ldots, x_k$. Moreover, any nim move from $(x_1, x_3, \ldots, x_k)$ can be achieved as a staircase nim move from $(x_1, x_2, x_3, \ldots, x_n)$ by reducing one of the numbers, $x_1, x_3, \ldots, x_k$. Therefore a winning strategy is to keep the odd numbered stairs as a P-position in nim.

7. (a) When expanded in base 2 and added without carry modulo 3, we find 2212. To change the first (most significant) column to a 0, we must reduce two numbers that are 8 or greater. We may change the 10 to a 5 and the 13 to a 5.

$$\begin{array}{rrcrr}
4 = & 100_2 & & 4 = & 100_2 \\
8 = & 1000_2 & & 8 = & 1000_2 \\
8 = & 1000_2 & & 8 = & 1000_2 \\
9 = & 1001_2 & & 9 = & 1001_2 \\
10 = & 1010_2 & & 5 = & 101_2 \\
13 = & \underline{1101_2} & & 5 = & \underline{101_2} \\
\text{sum mod } 3 = & 2212 & \longrightarrow & & 0000
\end{array}$$

(b) Let $x_i = \sum_{j=0}^{m} x_{ij}2^j$ be the base 2 expansion of x_i, where each x_{ij} is either 0 or 1 and m is sufficiently large. Let $\mathcal{P}$ be the set of all $(x_1, \ldots, x_n)$ such that for all j, $s_j \equiv \sum_{j=0}^{m} x_{ij} = 0 \pmod{k+1}$. (We refer to the vector s as the nim$_k$-sum of the x's. Note $0 \le s_j \le k$ for all j.) We show that $\mathcal{P}$ is the set of P-positions by following the proof of Theorem 1.

(1) *All terminal positions are in* $\mathcal{P}$. This is clear since $(0, \ldots, 0)$ is the only terminal position.

(2) *Every move from a position in* $\mathcal{P}$ *is to a position not in* $\mathcal{P}$. Suppose that $s_j = 0$ for all j, and that at most k of the x_i are reduced. Find the leftmost column j that is changed by one of these changes. If only one x_i had a 1 in position j, then s_j would be changed to k. If two x_i, then s_j would be changed to $k - 1$, etc. But at most k changes are made, so that s_j is changed into a number between 1 and k. Thus the move cannot be in $\mathcal{P}$.

(3) *From each position not in* $\mathcal{P}$*, there is a move to a position in* $\mathcal{P}$. The difficulty of finding a winning move is to select which piles of chips to reduce. The problem of finding how many chips to remove from each of the selected piles is easy and there are usually many solutions. The algorithm below finds which piles to select.

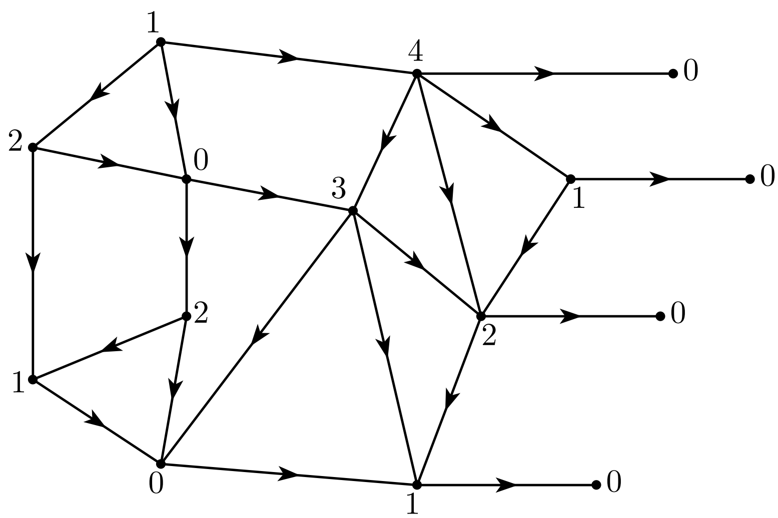
Find the leftmost column j with a nonzero s_j, and select any $t = s_j$ of the x_i with $x_{ij} = 1$. If $t = k$, you are done.

Let s' denote the nim$_k$-sum of the remaining x's, and find the leftmost column $j' < j$ such that $1 \le s'_{j'} < k - t$. If there is no such j', you are done and the t selected x's may be used. Otherwise, select any $t' = s'_m$ of the remaining x's with a 1 in position j' of their binary expansion. Then set $t = t + t'$, $j = j'$, and repeat this paragraph.

(c) Move as you would in normal Nim$_k$ until you would move to a position with all piles of size 1. Then move to leave 1 mod $k + 1$ piles instead of 0 mod $k + 1$ piles.

Solutions to Section I.3

1.



The Sprague-Grundy function.

2. The first few values of the SG function are as follows.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$g(x)$	0	1	0	1	2	3	2	0	1	0	1	2	3	...

Then pattern for the first 7 nonnegative integers repeats forever. We have

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } 2 \bmod 7 \\ 1 & \text{if } x = 1 \text{ or } 3 \bmod 7 \\ 2 & \text{if } x = 4 \text{ or } 6 \bmod 7 \\ 3 & \text{if } x = 5 \bmod 7. \end{cases}$$

3. The first few values of the SG function are as follows.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	...
$g(x)$	0	0	1	0	2	1	3	0	4	2	5	1	6	3	7	0	8	4	...

One may describe this function recursively as follows.

$$g(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ g((x-1)/2) & \text{if } x \text{ is odd.} \end{cases}$$

One may find $g(x)$ as follows. Take $x+1$ and factor out 2 as many times as possible (i.e. write $x+1 = 2^n y$ where y is an odd number). Then $g(x) = (y-1)/2$.

One may also write it as

$$
g(x) = \begin{cases} 0 & \text{if } x = 2^n - 1 \\ 1 & \text{if } x = 2^n 3 - 1 \\ 2 & \text{if } x = 2^n 5 - 1 \\ \vdots & \vdots \\ k & \text{if } x = 2^n(2k+1) - 1 \\ \vdots & \vdots \end{cases} \qquad \text{for } n = 0, 1, 2, \ldots
$$

4. (a) The first few values of the SG function are as follows.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\ldots$
$g(x)$	0	1	2	1	3	1	2	1	4	1	2	1	3	1	2	1	5	$\ldots$

It may be represented mathematically as

$$
g(x) = k + 1 \qquad \text{where } 2^k \text{ is the largest power of 2 dividing } x
$$

(b) The Sprague-Grundy finction for Aliquot is simply 1 less than the Sprague-Grundy function for Dim$^+$; so for $x \geq 1$, $g(x) = k$ where 2^k is the largest power of 2 dividing x.

5. The Sprague-Grundy function is

7	8	6	9	0	1	4	5
6	7	8	1	9	10	3	4
5	3	4	0	6	8	10	1
4	5	3	2	7	6	9	0
3	4	5	6	2	0	1	9
2	0	1	5	3	4	8	6
1	2	0	4	5	3	7	8
0	1	2	3	4	5	6	7

For larger boards, the entries seem to become chaotic, but Wythoff found that the zero entries have coordinates $(0,0), (1,2), (3,5), (4,7), (6,10), (8,13), (9,15), (11,18), \ldots$ with differences $0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots$, the first number in each pair being the smallest number that hasn't yet appeared. He also showed that the nth pair is $(\lfloor n\tau \rfloor, \lfloor n\tau^2 \rfloor)$, for $n = 0, 1, 2, \ldots$, where τ is the golden ratio $(1 + \sqrt{5})/2$.

6. (a) The Sprague-Grundy values are

5ω	$5\omega+1$	$5\omega+2$	$5\omega+3$	$5\omega+4$	$5\omega+5$	$5\omega+6$	
4ω	$4\omega+1$	$4\omega+2$	$4\omega+3$	$4\omega+4$	$4\omega+5$	$4\omega+6$	
3ω	$3\omega+1$	$3\omega+2$	$3\omega+3$	$3\omega+4$	$3\omega+5$	$3\omega+6$	
2ω	$2\omega+1$	$2\omega+2$	$2\omega+3$	$2\omega+4$	$2\omega+5$	$2\omega+6$	
ω	$\omega+1$	$\omega+2$	$\omega+3$	$\omega+4$	$\omega+5$	$\omega+6$	
0	1	2	3	4	5	6	

(b) The nim-sum of these transfinite Sprague-Grundy values follows the rule:

$$(x_1\omega + y_1) \oplus (x_2\omega + y_2) = (x_1 \oplus x_2)\omega + (y_1 \oplus y_2).$$

Therefore the Sprague-Grundy value of the given position is

$$(4\omega) \oplus (2\omega + 1) \oplus (\omega + 2) \oplus (5) = 7\omega + 6.$$

Since this is not zero, the position is an N-position. It can be moved to a P-position by moving the counter at 4ω down to $3\omega + 6$. There is no upper bound to how long the game can last, but every game ends in a finite number of moves.

(c) Yes.

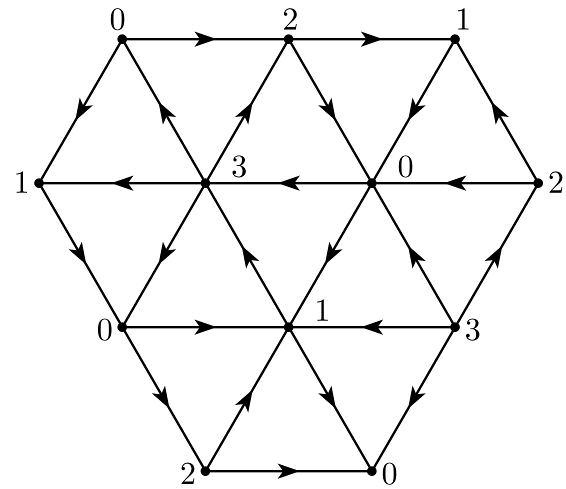
7. Suppose S consists of n numbers. Then no Sprague-Grundy value can be greater than n since the set $\{g(x-y) : y \in S\}$ contains at most n numbers. Let x_n be the largest of the numbers in S. There are exactly $(n+1)^{x_n}$ sequences of length x_n consisting of the integers from 0 to n. Therefore, when by time $(n+1)^{x_n} + 1$ there will have been two identical such sequences in the Sprague-Grundy sequence. From the second time on, the Sprague-Grundy sequence will proceed exactly the same as it did the first time.

8. We have $g(x) = \text{mex}\{g(x-y) : y \in S\}$, and $g^+(x) = \text{mex}\{0, \{g^+(x-y) : y \in S\}\}$. We will show $g^+(x) = g(x-1) + 1$ for $x \geq 1$ by induction on x. It is easily seen to be true for small values of x. Suppose it is true for all $x < z$. Then,

$$\begin{aligned} g^+(z) &= \text{mex}\{0, \{g^+(z-y) : y \in S\}\} = \text{mex}\{0, \{1 + g(x-y-1) : y \in S\}\} \\ &= 1 + \text{mex}\{g(x-y-1) : y \in S\} = 1 + g(x-1). \end{aligned}$$

9. (a) The Sprague-Grundy function does not exist for this graph. However, the terminal vertex is a P-position and the other vertex is an N-position.

(b) The Sprague-Grundy function exists here. However, backward induction does not succeed in finding it. The terminal vertex has SG-value 0, the vertex above it has SG-value 2, and of the two vertices above, the one on the left has SG-value 1 and the one on the right has SG-value 0. Those vertices of SG-value 0 are P-positions and the others are N-positions.



(c) The node at the bottom right, call it α, obviously has Sprague-Grundy value 0. But every other node can move to some node whose Sprague-Grundy value we don't know. Here is how we make progress. Consider the node, call it β, at the middle of the southwest edge. It can move to only two positions. But neither of these positions can have Sprague-Grundy value 0 since they can both move to α. So β must have Sprague-Grundy value 0. Continuing in a similar manner, we find:

Solutions to Section I.4

1. Remove any even number, or 1 chip if it is the whole pile. The SG-values of the first few numbers are

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$g(x)$	0	1	1	0	2	2	3	3	4	4	5	5	6	6	7	7	8	...

The general rule is $g(0) = 0$, g(1)=1, $g(2) = 1$, g(3)=0, and $g(x) = \lfloor x/2 \rfloor$ for $x \geq 4$, where $\lfloor x \rfloor$ represents the greatest integer less than or equal to x, sometimes called the floor of x.

2. Remove any multiple of 3 if it is not the whole pile, or the whole pile if it contains 2 (mod 3) chips. The SG-values of the first few numbers are

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$g(x)$	0	0	1	0	1	2	1	2	3	2	3	4	3	4	5	4	5	...

The general rule is $g(0) = 0$ and for $k \geq 0$,

$$\begin{aligned} g(3k+1) &= k \\ g(3k+2) &= k+1 \\ g(3k+3) &= k \end{aligned}$$

3. There are three piles of sizes 18, 17, and 7 chips. The first pile uses the rules of Exercise 1, the second pile uses the rules of Exercise 2, and the third pile uses the rules of nim. The respective SG-values are 9, 6, and 7, with nim-sum $(1001)_2 \hat{+} (0110)_2 \hat{+} (0111)_2 = (1000)_2 = 8$. The can be put into a position of nim-sum 0 by moving the first pile to a position of SG-value 1. This can be done by removing 16 chips from the pile of 18, leaving 2, which has SG-value 1.

4. (a) The given position represents 2 piles of sizes 1 and 11. From Table 4.1, the SG-values are 1 and 6, whose nim-sum is 7. Since the nim-sum is not zero, this is an N-position.

(b) We must change the SG-value 6 to SG-value 1. This may be done by knocking down pin number 6 (or pin number 10), leaving a position corresponding to 3 piles of sizes 1, 3, and 7, with SG-values 1, 3, and 2 respectively. This is a P-position since the nim-sum is 0.

5. Remove one chip and split if desired, or two chips without splitting. (a) The SG-values of the first few numbers are

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$g(x)$	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	...

We have $g(x) = x \pmod 4$, $0 \le g(x) \le 3$. This is periodic of period 4.

(b) Since 15 has SG-value 3, the moves to a P-position are those that remove 1 chip and split into two piles the nim-sum of whose SG-values is 0. For example, the move to two piles of sizes 1 and 13 is a winning move.

6. Remove two or more chips and split if desired, or one chip if it the whole pile. The SG-values of the first few numbers are

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$g(x)$	0	1	1	2	2	3	4	4	5	6	6	7	8	8	9	10	10	...

The general rule is

$$\begin{aligned} g(3k) &= 2k \\ g(3k+1) &= 2k \\ g(3k+2) &= 2k+1 \end{aligned}$$

for $k \ge 0$, except for $g(1) = 1$.

7. Remove any number of chips equal to 1 (mod 3) and split if desired. The SG-values of the first few numbers are

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
$g(x)$	0	1	0	1	2	3	2	3	4	5	4	5	6	7	6	7	8	...

The general rule is, for $k \ge 0$,

$$\begin{aligned} g(4k) &= 2k \\ g(4k+1) &= 2k+1 \\ g(4k+2) &= 2k \\ g(4k+3) &= 2k+1. \end{aligned}$$

8. (a) The loops divide the plane into regions. A move in a region with n dots divides that region into two regions with a and b dots, where $a+b$ is less than n but where a and b are otherwise arbitrary. We claim that a region with n dots has SG-value n, i.e. $g(n) = n$. (This may be seen by induction: Clearly $g(0) = 0$ since 0 is terminal. If $g(k) = k$ for all $k < n$, then $g(n) \ge n$ since all SG-values less than n can be reached in one move without splitting the region into two. But if a region of n dots is split into regions of size a and b with $a+b < k$, then since $a \oplus b \le a+b$, n cannot be obtained as the SG-value of a follower of n.) Thus a region of n dots corresponds to a nim pile of n chips.

(b) The given position corresponds to nim with three piles of sizes 3, 4 and 5. Since the nim-sum is 2, this is an N-position. An optimal move must reduce the 3 to a 1. This is achieved by drawing a loop through exactly two of the three free dots at the bottom of the figure.

9. (a) A loop in this game takes away 1 or 2 dots from a region and splits the region into two parts one of which may be empty of dots. This is exactly the same as the rules for Kayles.

(b) Using Table 4.1, $g(5) \oplus g(4) \oplus g(3) = 4 \oplus 1 \oplus 3 = 6$ so this is an N-position. An optimal move is to draw a closed loop through a dot from the innermost 5 dots such that exactly three dots stay inside the loop.

10. (a)

1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1	0	2	1	0	2	1	0	2	1	3

(b) The SG-values of 5, 8, and 13 are 2, 2, and 3 respectively. The winning first moves are (1) splitting 5 into 2 and 3, (2) splitting 8 into 2 and 6, and (3) splitting 13 into 5 and 8.

11. (a) $g(S_n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even, } n \geq 2. \end{cases}$

(b) When played on a line with n edges, the rules of the game are: (1) You may remove one chip if it is the whole pile, or (2) you may remove two chips from any pile and if desired split that pile into two parts. In the notation of Winning Ways, Chapter 4, this game is called .37 (or .6 if one counts vertices rather than edges). The Sprague-Grundy values up to $n = 10342$ have been computed without finding any periodic pattern. It is generally believed that none exists. Here are the first few values.

n	0	1	2	3	4	5	6	7	8	9	10	11	...
$g(L_n)$	0	1	2	0	1	2	3	1	2	3	4	0	...

(c) $g(C_n) = \begin{cases} 0 & \text{if } g(L_{n-2}) > 0 \\ 1 & \text{if } g(L_{n-2}) = 0 \end{cases}$. Because of (b), there seems to be no periodicity in the appearance of the 1's. But we can say that $g(C_n) = 0$ if n is even.

(d) Let $DS_{m,n}$ denote the stars S_m and S_n joined by an (additional) edge (so that $DS_{0,n} = S_{n+1}$, and $DS_{1,1} = L_3$). For $n \geq 0$, $g(DS_{0,n}) = g(DS_{n,0}) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$. For $m \geq 1$ and $n \geq 1$, $g(m,n) = \begin{cases} 0 & \text{if } m+n \text{ is even} \\ 3 & \text{if } m+n \text{ is odd} \end{cases}$.

(e) The first player wins the square lattice (i) by taking the central vertex and reducing the position to C_8 with Sprague-Grundy value 0 from (c). The second player wins the tic-tac-toe board by playing symmetrically about the center of the graph. To generalize to larger centrally symmetric graphs, we need to define the symmetry for an arbitrary graph, (V, E). Here is one way.

Suppose there exists a one-to-one map, g, of V onto V such that
(1) (graph preserving) $\{v_1, v_2\} \in E$ implies $\{g(v_1), g(v_2)\} \in E$
(2) (pairing) $u = g(v)$ implies $v = g(u)$
(3) (no fixed vertex) $v \neq g(v)$ for all $v \in V$
(4) (no fixed edge) $\{v_1, v_2\} \in E$ implies $\{v_1, v_2\} \neq \{g(v_2), g(v_1)\}$.

The second player wins such symmetrically paired graphs without fixed vertices or fixed edges, by playing symmetrically. Can the second player always win if we allow exactly one fixed edge in the mapping?

Solutions to Section I.5

1. (a) In Turning Turtles, the positions are labelled starting at 1, so the heads are in positions 3, 5, 6 and 9. The position has SG-value $3 \oplus 5 \oplus 6 \oplus 9 = 9$, so a winning move is to turn over the coin at position 9.
(b) In Twins, the labelling starts at 0, so the heads at in positions 2, 4, 5 and 8. The position has SG-value $2 \oplus 4 \oplus 5 \oplus 8 = 11$. . A winning move is to turn over the coins at positions 3 and 8.
(c) For the subtraction set $S = \{1, 3, 4\}$, the Sprague-Grundy sequence is

position x :	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14...
$g(x)$:	0	1	0	1	2	3	2	0	1	0	1	2	3	2	0...

The labelling starts at 0 so the heads are in positions 2, 4. 5 and 8, with a combined SG-value $0 \oplus 2 \oplus 3 \oplus 1 = 0$. This is a P-position.
(d) The labelling starts at 0 so the heads are in positions 2, 4, 5 and 8. In nim, this has SG-value $2 \oplus 4 \oplus 5 \oplus 8 = 11$. It can be moved to a position of SG-value 0 by turning over the coins at 3 and 8. Since this leaves an even number of heads, it is a P-position in Mock Turtles. The Mock Turtle did not need to be turned over.

2. (a) The maximum number of moves the game can last is n.
(b) Let T_n denote the maximum number of moves the game can last. This satisfies the recursion, $T_n = T_{n-1} + T_{n-2} + 1$ for $n > 2$ with initial values $T_1 = 1$ and $T_2 = 2$. We see that $T_n + 1$ is just the Fibonacci sequence, $2, 3, 5, 8, 13, 21 \ldots$. So T_n is the sequence $1, 2, 4, 7, 12, 20, \ldots$.
(c) This time T_n satisfies the recursion, $T_n = T_{n-1} + \cdots + T_1 + 1$ with initial condition $T_1 = 1$. So T_n is the sequence, $1, 2, 4, 8, 16, 32, \ldots$.

3. (a) Suppose we start the labelling from 0. Then a single heads in positions 0 or 1 is a terminal position and so receives SG-value 0. Continuing as in Mock Turtles, we find

position x :	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14...
$g(x)$:	0	0	1	2	4	7	8	11	13	14	16	19	21	22	25...

This is just the SG-sequence for Mock Turtles moved over two positions.
(b) To get nim out of this, we should have started labelling the positions of the coins from -2. The first two coins on the left are dummies. It doesn't matter whether they are heads or tails. The third coin on the left is the Mock Turtle. The P-positions in Triplets are exactly the P-positions in Mock Turtles when the first two coins on the left are ignored.

4. The SG-sequence for Rulerette is easily found to be

position x :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16...
$g(x)$:	0	1	0	2	0	1	0	4	0	1	0	2	0	1	0	8...

g(x) is half of the SG-value of x for Ruler except for x odd when $g(x) = 0$.

5. This becomes an impatient subtraction game mentioned in Exercise 3.8. The Sprague-Grundy function, $g^+(x)$ of this game is just $g(x-1)+1$, where $g(x)$ is the Sprague-Grundy function of the subtraction game.

6. (a) We have $6 \otimes 21 = 6 \otimes (16 \oplus 5) = (6 \otimes 16) + (6 \otimes 5) = 96 \oplus 8 = 104$.
(b) We have $25 \otimes 40 = (16 \oplus 9) \otimes (32 \oplus 8) = (16 \otimes 32) \oplus (16 \otimes 18) \oplus (9 \otimes 32) \oplus (9 \otimes 8)$.
Then using $16 \otimes 32 = 16 \otimes 16 \otimes 2 = 24 \otimes 2 = (16 \oplus 8) \otimes 2 = 32 \oplus 12 = 44$,
and $9 \otimes 32 = 9 \otimes 16 \otimes 2 = 13 \otimes 16 = 224$,
we have $25 \otimes 40 = 44 \oplus 128 \oplus 224 \oplus 5 = 79$.
(c) $1 \oslash 14 = 13$, so $15 \oslash 14 = 15 \otimes 13 = 12$.
(d) Since $14 \otimes 14 = 8$, we have $\sqrt{8} = 14$.
(e) $x^2 \oplus x \oplus 6$ is the same as $x \otimes (x \oplus 1) = 6$. Looking as Table 5.2, we see this occurs for $x = 14$ or $x = 15$.

7. (a) Suppose there exists a move in Turning Corners from (v_1, v_2) into a position of SG-value u. Then there is a $u_1 < v_1$ and a $u_2 < v_2$ such that $(u_1 \otimes u_2) \oplus (v_1 \otimes u_2) \oplus (u_1 \otimes v_2) = u$. Since $u_1 < g_1(x)$, there exists a move in G_1 to an SG-value u_1, turning over the coins, say, at positions $x_1, x_2, \ldots, x_m, x$, where all $x_i < x$. Similarly there exists a move in G_2 to an SG-value u_2 turning over coins, say, at positions $y_1, y_2, \ldots, y_n, y$, where all $y_j < y$. This implies

$$\begin{aligned} g_1(x_1) \oplus g_1(x_2) \oplus \cdots \oplus g_1(x_m) &= u_1 \qquad \text{and} \\ g_2(y_1) \oplus g_2(y_2) \oplus \cdots \oplus g_2(y_m) &= u_2. \end{aligned} \tag{1}$$

Then the move, $\{x_1, \ldots, x_m, x\} \times \{y_1, \ldots, y_n, y\}$ in $G_1 \times G_2$ results in SG-value

$$\begin{aligned} &\Big(\sum^{*}\sum^{*} g_1(x_i) \otimes g_2(y_j)\Big) \oplus \Big(\sum^{*} g_1(x_i) \otimes g_2(y)\Big) \oplus \Big(\sum^{*} g_1(x) \otimes g_2(y_j)\Big) \\ &\quad = ((g_1(x_1) \oplus \cdots \oplus g_1(x_m)) \otimes (g_2(y_1) \oplus \cdots \oplus g_2(y_n))) \\ &\qquad \oplus (g_1(x) \otimes (g_2(y_1) \oplus \cdots \oplus g_2(y_n)) \\ &\qquad \oplus (g_1(x_1) \oplus \cdots \oplus g_1(x_m)) \otimes g_2(y)) \\ &\quad = (u_1 \otimes u_2) \oplus (v_1 \otimes u_2) \oplus (u_1 \otimes v_2) = u \end{aligned} \tag{2}$$

where $\sum^{*}$ represents nim-sum. Conversely, for any move, $\{x_1, \ldots, x_m, x\} \times \{y_1, \ldots, y_n, y\}$, in $G_1 \times G_2$, we find u_1 and u_2 from (1). Then the same equation (2) shows that the corresponding move in Turning Corners has the same SG-value.
(b) We may conclude that the mex of the SG-values of the followers of (x, y) in $G_1 \times G_2$ is the same as the mex of the SG-values of the followers of (v_1, v_2) in Turning corners, implying $g_1(x) \otimes g_2(y) = v_1 \otimes v_2$.

8. (a) The table is

	1	2	1	4	1	2	1	8
1	1	2	1	4	1	2	1	8
2	2	3	2	8	2	3	2	12
4	4	8	4	6	4	8	4	11
7	7	9	7	4	10	9	7	15
8	8	12	8	11	8	12	8	13

(b) The given position has SG-value $2 \oplus 13 = 15$. A winning move must change the SG-value 13 to 2. In Turning corners, the move from (8,8) that that changes the SG-value 13 to 2 is the move with north west corner at (3,3). A move in Mock Turtles that changes $x = 5$ with $g_1(x) = 8$ into a position of SG-value 2, is the move that turns over 5, 2 and 1. A move in Ruler that change $y = 8$ with $g_2(8) = 9$ into a position with SG-value 2 is the move that turns over 8, 7, and 6. Therefore a winning move in the given position is $\{1, 2, 5\} \times \{6, 7, 8\}$. This gives

$$\begin{matrix} T & H & T & T & T & H & H & H \\ T & T & T & T & T & H & H & H \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & T & T & T \\ T & T & T & T & T & H & H & T \end{matrix}$$

which has SG-value 0.

9. (a) Since the game is symmetric, the SG-value of heads at (i, j) is the same as the SG-value of heads at (j, i). This implies that the SG-value of the initial position is 0. It is a P-position for all n. A simple winning strategy is to play symmetrically. If your opponent makes a move with (i, j) as the south east coin, you make the symmetric move at (j, i). Such a play keeps the game symmetric without heads along the diagonal. This holds true in any tartan game that is the square of some coin turning game.
(b) The SG-values of off-diagonal elements cancel, so the SG-value of the game is the sum of the SG-values on the diagonal. For $n = 1, 2, \ldots$, these are $1, 2, 3, 5, 4, 7, 6, 11, 10, 9, 8, 14, 15, 12, 13, \ldots$. One can show that this hits all positive integers without repeating, and is never 0. So this is a first player win. However there doesn't seem to be a simple winning strategy.

10. The SG-sequence for G_1 is

position x :	1	2	3	4	5	6	7	8	9	10	...	98	99	100
$g(x)$:	0	1	2	3	4	0	1	2	3	4	...	2	3	4.

For G_2, it is

position x :	1	2	3	4	5	6	7	8	...	97	98	99	100
$g(x)$:	1	2	1	4	1	2	1	8	...	1	2	1	4.

The coin at (100,100) has SG-value $4 \otimes 4 = 6$ and the coin at (4,1) has SG-value $3 \otimes 1 = 3$. You cam win by turning over the 8 coins at positions (x, y) with $x = 98, 100$ and $y = 97, 98, 99, 100$.

This works in any two-dimensional game which is the product of the two same one-domensional games.

Solutions to Section I.6

1. The SG-value of the three-leaf clover is 2. The SG-value of the girl is 3. The SG-value of the dog is 2. And the SG-value of the tree is 5. So there exists a winning move on the tree that reduces the SG-value to 3. The unique winning move is to hack the left branch of the rightmost branch completely away.

Solutions to Exercises of Section II.1.

1. The new payoff matrix is

$$\boxed{\begin{array}{cc} 1 & -2 \\ -2 & 4 \end{array}}$$

If player I uses the mixed strategy $(p, 1-p)$, the expected payoff is $-1p+2(1-p)$ if II uses column 1, and $2p-4(1-p)$ if II uses column 2. Equating these, we get $-p+2(1-p) = 2p-4(1-p)$ and solving for p gives $p = 2/3$. Use of this strategy guarantees that player I wins 0 on the average no matter what II does. Similarly, if II uses the same mixed strategy $(2/3, 1/3)$, II is guaranteed to win 0 on the average no matter what I does. Thus, 0 is the value of the game. Since the value of a game is zero, the game is fair by definition.

2. The payoff matrix is

$$\begin{array}{c|c} & \begin{array}{cc} \text{red } 2 & \text{black } 7 \end{array} \\ \begin{array}{l} \text{black Ace} \\ \text{red } 8 \end{array} & \begin{pmatrix} -2 & 1 \\ 8 & -7 \end{pmatrix} \end{array}$$

Solving $-2p+8(1-p) = p-7(1-p)$ gives $p = 5/6$ as the probability that Player I should use the black Ace. Similarly, $q = 5/9$ is the probability that Player II should use the red 2. The value is $-1/3$.

3. If Professor Moriarty stops at Canterbury with probability p and continues to Dover with probability $1-p$, then his average payoff is $100p$ if Holmes stops at Canterbury, and is $-50p+100(1-p)$ if Holmes continues to Dover. Equating these payoffs gives $250p = 100$, or $p = 2/5$. Use of this mixed strategy guarantees Moriarty an average payoff of $100p = 40$.

On the other hand, if Holmes stops at Canterbury with probability q and continues to Dover with probability $1-q$, then his average payoff is $100q - 50(1-q)$ if Moriarty stops at Canterbury, and is $100(1-q)$ if Holmes continues to Dover. Equating these gives $q = 3/5$. Use of this strategy holds Moriarty to an average payoff of 40.

The value of the game is 40, and so the game favors Moriarty. But, as related by Dr. Watson in *The Final Problem*, Holmes outwitted Moriarty once again and held the diabolical professor to a draw.

4. Without the side payment, the game matrix in cents is

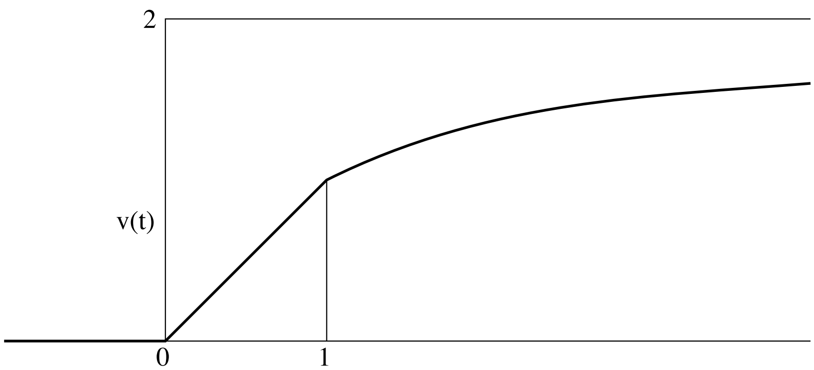
$$\begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 55 & 10 \\ 10 & 110 \end{pmatrix} \end{array}$$

Let p be the probability that Player I (Alex) uses row 1. Equating his payoffs if Player II uses cols 1 or 2 gives $55p+10(1-p) = 10p+110(1-p)$, or $145p = 100$ or $p = 20/29$. If Player I uses $(20/29, 9/29)$, his average payoff is $55(20/29)+10(1-9/29) = 1190/29$. Since this is $41\frac{1}{29}$, a side payment of 42 cents overcompensates slightly. With the side payment, the game is in Olaf's favor by 28/29 of one cent.

Solutions to Exercises of Section II.2.

1. The value is $-4/3$. The mixed strategy, (2/3,1/3), is optimal for I, and the mixed strategy (5/6,1/6) is optimal for II.

2. If $t \le 0$, the strategy pair $\langle 1,1\rangle$ is a saddle-point, and the value is $v(t)=0$. If $0 \le t \le 1$, the strategy pair $\langle 2,1\rangle$ is a saddle-point, and the value is $v(t)=t$. If $t>1$, there is no saddle-point; I's optimal strategy is $((t-1)/(t+1), 2/(t+1))$, II's optimal strategy is $(1/(t+1), t/(t+1))$, and the value is $v(t)=2t/(t+1)$.



3. Suppose that $\langle x,y\rangle$ and $\langle u,v\rangle$ are saddle-points. Look at the four numbers $a_{x,y}$, $a_{x,v}$, $a_{u,v}$, and $a_{u,y}$. We must have $a_{x,y} \le a_{x,v}$ since $a_{x,y}$ is the minimum in its row. Also, $a_{x,v} \le a_{u,v}$ since $a_{u,v}$ is the maximum of its column. Keep going: $a_{u,v} \le a_{u,y}$ since $a_{u,v}$ is the minimum of its row and $a_{u,y} \le a_{x,y}$ since $a_{x,y}$ is the maximum of its column. We have

$$a_{x,y} \le a_{x,v} \le a_{u,v} \le a_{u,y} \le a_{x,y}.$$

Since this begins and ends with the same number, we must have equality throughout: $a_{x,y} = a_{x,v} = a_{u,v} = a_{u,y} = a_{x,y}$. (This argument also works if $x=u$ or $y=v$.)

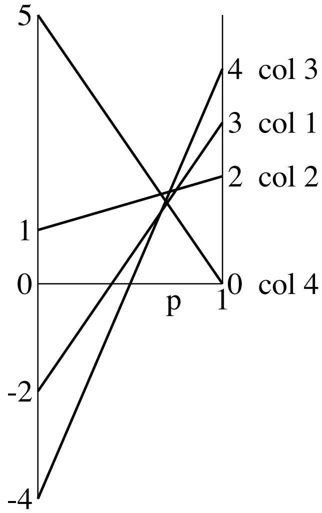
4. (a) Column 2 dominates column 1; then row 3 dominates row 4; then column 4 dominates column 3; then row 1 dominates row 2. The resulting submatrix consists of rows 1 and 3 vs. columns 2 and 4. Solving this 2 by 2 game and moving back to the original game we find that the value is 3/2, I's optimal strategy is $p=(1/2,0,1/2,0)$ and II's optimal strategy is $q=(0,3/8,0,5/8)$.

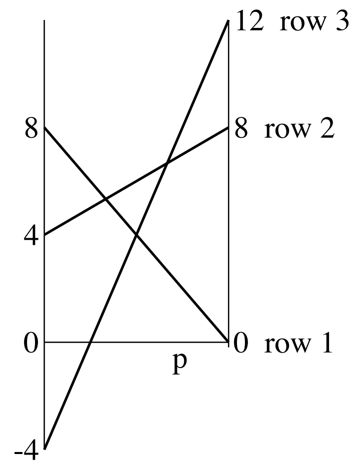
(b) Column 2 dominates column 4; then (1/2)row 1 + (1/2)row 2 dominates row 3; then (1/2) col 1 + (1/2)col 2 dominates col 3. The resulting 2 by 2 game is easily solved. Moving back to the original game we find that the value is 30/7, I's optimal strategy is (2/7,5/7,0) and II's optimal strategy is (3/7,4/7,0,0).

5. (a) From the graph on the left, we guess that Player II uses columns 1 and 4. Solving this 2 by 2 subgame gives

$$\begin{array}{c c} & \begin{array}{cc} 1/2 & 1/2 \end{array} \\ \begin{array}{c} 7/10 \\ 3/10 \end{array} & \begin{pmatrix} 3 & 0 \\ -2 & 5 \end{pmatrix} \end{array} \qquad \text{Value} = 1.5$$

We conjecture I's optimal strategy is (.7,.3), II's optimal strategy is (.5,0,0,.5), and the value is 1.5. Let us check how well I's strategy works on columns 2 and 3. For column 2, $2(.7)+1(.3)=1.7$ and for column 3, $4(.7)-4(.3)=1.6$, both greater than 1.5. This strategy guarantees I at least 1.5 so our conjecture is verified.





(b) (3/8)col 1 + (5/8)col 2 dominates col 3. Removing column 3 leaves a 3 by 2 game whose payoffs for a given q are displayed in the graph on the right. The upper envelope takes on its minimum value at the intersection of row 1 and row 2. Solving the 2 by 2 game in the upper left corner of the original matrix gives the solution. Player I's optimal strategy is (1/3,2/3,0), Player II's optimal strategy is (1/3,2/3,0), and the value is 16/3.

6. (a) The first row is dominated by the third; the seventh is dominated by the fifth. Then the third column is dominated by the first; the fourth is dominated by the second; the fifth column is dominated by the seventh. Then the middle row is dominated. When these three rows and columns are removed, the resulting matrix is the 4 by 4 identity matrix with value $v=1/4$ and optimal strategies giving equal weight 1/4 to each choice. This results in the optimal strategies $\boldsymbol{p}=(0,.25,.25,0,.25,.25,0)$ for I, and $\boldsymbol{q}=(.25,.25,0,0,0,.25,.25)$ for II.

(b) For all n, domination reduces the game matrix to the identity matrix. We find the value for arbitrary $n\geq 2$ to be $v_n=1/(2k)$ for $n=4k-2,4k-1,4k$, and $v_n=1/(2k+1)$ for $n=4k+1$. For n equal to 2 or 3, the optimal strategies are simple special cases. For $n\geq 4$, an optimal strategy for Player I is $\boldsymbol{p}=(p_1,p_2,\ldots,p_n)$, symmetric about its midpoint and such that for $i\leq(n+1)/2$,

$$p_i=\begin{cases} v_n & \text{if } i=2 \text{ or } 3 \bmod 4\\ 0 & \text{if } i=0 \text{ or } 1 \bmod 4.\end{cases}$$

Similarly, an optimal strategy for Player II is $\boldsymbol{q}=(q_1,\ldots,q_n)$, symmetric about its midpoint and such that for $j\leq(n+1)/2$,

$$q_j=\begin{cases} v_n & \text{if } j=1 \text{ or } 2 \bmod 4\\ 0 & \text{if } j=0 \text{ or } 3 \bmod 4.\end{cases}$$

7. If Player I uses $\boldsymbol{p}$ and Player II uses column 1, the average payoff is $(6/37)5+(20/37)4+(11/37)=121/37$. Similarly for columns 2, 3, 4 and 5, the average payoffs are

121/37, 160/37, 121/37, and 161/37. So Player I can guarantee an average payoff of at least 121/37 by using $\boldsymbol{p}$. Similarly, if Player II uses $\boldsymbol{q}$ and Player I uses rows 1, 2, 3, or 4, the average payoffs are 121/37, 121/37, 120/37, and 121/27 respectively. By using $\boldsymbol{q}$, Player II can keep the average payoff to at most 121/37. Thus, 121/37 is the value of the game and $\boldsymbol{p}$ and $\boldsymbol{q}$ are optimal strategies.

8. If $(52/143, 50/143.41/143)$ is optimal, then the value is the minimum of the inner product of this vector with the three columns of the matrix. This inner product with each of the three columns gives the same number, namely $96/143$, which is then the value.

9. The matrix is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

(1/2)row 1 + (1/2)row 3 dominates row 2; then (1/2)col 1 + (1/2)col 3 dominates row 2. Solving the resulting 2 by 2 game and moving back to the original game, we find the value is 1 and an optimal stratgey for I is (1/2,0,1/2), and an optimal strategy for II is (1/2,0,1/2). However, the pure strategy of choosing col 2 is also optimal. In fact it is better than the mixed strategy (1/2,0,1/2) whenever Player I makes the mistake of playing row 2.

10. In an $n \times n$ magic square, $\boldsymbol{A} = \left(a_{ij}\right)$, there is a number s such that $\sum_i a_{ij} = s$ for all j, and $\sum_j a_{ij} = s$ for all i. If Player I uses the mixed strategy $\boldsymbol{p} = (1/n, 1/n, \ldots, 1/n)$ his average payoff is $V = s/n$ no matter what Player II does. The same goes for player II, so the value is s/n and $\boldsymbol{p}$ is optimal for both players. In the example, $n = 4$ and $s = 34$, so the value of the game is $17/2$ and the optimal strategy is $(1/4, 1/4, 1/4, 1/4)$.

11. (a) First, 6 dominates 4 and 5. With 4 and 5 removed, C dominates D and F; A dominates E. Also, the mixture $(3/4)A + (1/4)C$ dominates B. Then with B, D, E and F removed, 3 dominates 2 and 1. The resulting 2 by 2 game

$$\begin{array}{c} \\ 3 \\ 6 \end{array} \begin{array}{c} \begin{array}{cc} A & C \end{array} \\ \begin{pmatrix} 18 & 31 \\ 23 & 19 \end{pmatrix} \end{array}$$

is easily solved. The value is $21 + \frac{14}{17}$. Optimal for Player I is $(0, 0, 4/17, 0, 0, 13/17)$ and optimal for Player II is $(12/17, 0, 5/17, 0, 0, 0)$.

(b) Neither player was using an optimal strategy. The German choice was very poor, and the Allies were lucky. (Or did they have inside information?)

Solutions to Exercises II.3.

1.(a) There is a saddle at row 2, column 3. The value is 1.

(b) The inverse is $\boldsymbol{A}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ -3 & 6 & -4 \end{pmatrix}$.

(c) The mixed strategy $(1/4, 1/2, 1/4)$, for example, is optimal for II.

(d) Equation (16) gives $\boldsymbol{q} = (2/5, 4/5, -1/5)$. Equations (16) are valid when $\boldsymbol{A}$ is nonsingular and Player I has an optimal strategy giving positive weight to each strategy. That is not the case here.

2.(a) If $d_i = 0$ for some i, then (row i, col i) is a saddlepoint of value zero. And row i and col i are optimal pure strategies for the players.

(b) If $d_i > 0$ and $d_j < 0$, then (row i, col j) is a saddlepoint of value zero. And row i and col j are optimal pure strategies for the players.

(c) If all $d_i < 0$, then the same analysis as in Section 3.3 holds. The value is $V = \sum_1^m 1/d_i$, and the players have the same optimal strategy, $(V/d_1, \ldots, V/d_m)$.

3. The matrix is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}.$$

This is a diagonal game of value $V = (1/2 + 1/4 + 1/8 + 1/16)^{-1} = 16/15$. The optimal strategy for both players is $(8/15, 4/15, 2/15, 1/15)$.

4. The matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 1/2 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}.$$

This is a triangular game. If V is the value and if (p_1, p_2, p_3, p_4) is Player I's optimal strategy, then equations (12) become $V = p_4 = p_3 + (1/2)p_4 = p_2 + (1/2)(p_3 + p_4) = p_1 + (1/2)(p_2 + p_3 + p_4)$. We may solve the equations one at a time to find $p_4 = V$, $p_3 = (1/2)V$, $p_2 = (1/4)V$ and $p_1 = (1/8)V$. Since the sum of the p's is one, we find $(\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1)V = 1$, so that $V = 8/15$. This is the value and $\boldsymbol{p} = (1/15, 2/15, 4/15, 8/15)$ is optimal for Player I and $\boldsymbol{q} = (8/15, 4/15, 2/15, 1/15)$ is optimal for II.

5. This is similar to Exercise 4. Equations (12) become:

$$\begin{aligned} p_n &= V \\ p_{n-1} - p_n &= V \\ p_{n-2} - p_{n-1} - p_n &= V \\ &\vdots \\ p_1 - p_2 - \cdots - p_{n-2} - p_{n-1} - p_n &= V \end{aligned}$$

The solution is $p_n = V$, $p_{n-1} = 2V, \ldots,$ $p_1 = 2^{n-1}V$. Since $1 = p_1 + p_2 + \cdots + p_n = [2^{n-1} + 2^{n-2} + \cdots + 1]V = [2^n - 1]V$, we find that the value is $V = 1/(2^n - 1)$. The optimal strategy for I (and for II also) is $(2^{n-1}, 2^{n-2}, \ldots, 2, 1)/(2^n - 1)$.

6. The matrix $\boldsymbol{A}$ has components $a_{ij} = 0$ for $i < j$ and $a_{ij} = b^{i-j}$ for $i \geq j$. It is easy to show that

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ b & 1 & 0 & & 0 \\ b^2 & b & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ b^{n-1} & b^{n-2} & \cdots & b & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -b & 1 & 0 & & 0 \\ 0 & -b & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & -b & 1 \end{pmatrix} = I$$

Therefore, we may use Theorem 3.2 to find the value V as the reciprocal of the sum of all the elements of $\boldsymbol{A}^{-1}$, $V = 1/(n - (n-1)b)$, and I's optimal strategy is proportional to the sums of the columns of $\boldsymbol{A}^{-1}$, $\boldsymbol{p} = (1-b, 1-b, \ldots, 1-b, 1)/(n - (n-1)b)$, and II's optimal strategy is $\boldsymbol{q} = (1, 1-b, 1-b, \ldots, 1-b)/(n-(n-1)b)$, proportional to the sums of the rows of $\boldsymbol{A}^{-1}$.

7. We may use Theorem 3.2 with $\boldsymbol{A}_n^{-1}$ replaced by $\boldsymbol{B}_n$. Since the sum of the ith row of $\boldsymbol{B}_n$ is 2^{i-1} (the binomial theorem), we have $\boldsymbol{B}_n\mathbf{1} = (1, 2, 4, \ldots, 2^{n-1})^T$, and so $\mathbf{1}^T\boldsymbol{B}_n\mathbf{1} = 2^n - 1$. Similarly, the sum of column j is $\sum_{k=j}^n \binom{k-1}{j-1} = \binom{n}{k}$ (easily proved by induction). So that $\mathbf{1}^T\boldsymbol{B} = (\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n})$. From this we may conclude that the value is $V = 1/(2^n - 1)$, the optimal strategy of Player I is $\boldsymbol{p} = (1, 2, \ldots, 2^{n-1})/(2^n - 1)$, and the optimal strategy of Player II is $\boldsymbol{q} = (\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n})/(2^n - 1)$.

8.(a) Assuming all strategies active, the optimal strategy for I satisfies, $p_1 = V$, $-p_1 + 2p_2 = V$ and $-p_1 + p_2 + 3p_3 = V$, from which we find $p_1 = V$, $p_2 = V$ and $p_3 = V/3$. Since $1 = p_1 + p_2 + p_3 = V + V + (1/3)V = (7/3)V$, we have $V = 3/7$ and $\boldsymbol{p} = (p_1, p_2, p_3) = (3/7, 3/7, 1/7)$. A similar analysis for Player II gives $\boldsymbol{q} = (q_1, q_2, q_3) = (5/7, 1/7, 1/7)$. Since $\boldsymbol{p}$ and $\boldsymbol{q}$ are nonnegative, these are the optimal strategies, and V is the value.

(b) If we subtract 1 from all entries of the matrix, we end up with a diagonal game with 1, 1/2, 1/3 and 1/4 along the diagonal. The value of that game is $1/10$, and the optimal strategies for both players is $(1/10, 2/10, 3/10, 4/10)$. The original game has value $11/10$ and the same optimal strategies.

(c) The last column is dominated by the first, and the bottom row is dominated by the mixture of row 1 and row 2 with probability 1/2 each. The resulting three by three matrix is a diagonal game with value $1/[(1/2)+(1/3)+(1/4)] = 12/13$. The optimal strategy for both players is $(6/13, 4/13, 3/13, 0)$.

9.(a) The matrix is

$$\begin{pmatrix} 0 & -2 & 1 & 1 & 1 & \dots \\ 2 & 0 & -2 & 1 & 1 & \dots \\ -1 & 2 & 0 & -2 & 1 & \\ -1 & -1 & 2 & 0 & -1 & \\ \vdots & \vdots & & & & \ddots \end{pmatrix}$$

(b) The game is symmetric and has value zero (if it exists). If the first five rows and columns are the active ones, the equations become

$$\begin{aligned} 2p_2 - \ p_3 - \ p_4 - \ p_5 &= 0 \\ -2p_1 \qquad + 2p_3 - \ p_4 - \ p_5 &= 0 \\ p_1 - 2p_2 \qquad + 2p_4 - \ p_5 &= 0 \\ p_1 + \ p_2 - 2p_3 \qquad + 2p_5 &= 0 \\ p_1 + \ p_2 + \ p_3 - 2p_4 \qquad &= 0 \end{aligned}$$

If we interchange (p_1, p_2) with (p_5, p_4) in these equations, we get the same set of equations. So in the solution, we must have $p_1 = p_5$ and $p_2 = p_4$. Using this, the top two equations become $p_2 = p_1 + p_3$ and $2p_3 = 3p_2 + p_1$, which together with $2p_1 + 2p_2 + p_3 = 1$ gives $p_1 = p_5 = 1/16$, $p_2 = p_4 = 5/16$ and $p_3 = 4/16$. If Player I uses $\boldsymbol{p} = (1/16, 5/16, 4/16, 5/16, 1/16, 0, 0, \dots)$ on the game with general n, then Player II will never use columns 6 or greater because the average payoff to Player I would be positive. Thus, the value is zero and $\boldsymbol{p}$ is optimal for both players.

10. (a) The matrix is

$$\begin{array}{c|c} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \dots \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \vdots \end{array} & \begin{pmatrix} 0 & -1 & 2 & 2 & 2 & 2 & \dots \\ 1 & 0 & -1 & -1 & -1 & 2 & \dots \\ -2 & 1 & 0 & -1 & -1 & -1 & \dots \\ -2 & 1 & 1 & 0 & -1 & -1 & \dots \\ -2 & 1 & 1 & 1 & 0 & -1 & \dots \\ -2 & -2 & 1 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \end{array}$$

One can see that columns $6, 7, \dots$ are all dominated by column 1. Similarly for rows. This reduces the game to a 5 by 5 matrix. Columns 3 and 4 are dominated by column 5. This reduces the game to 3 by 3.

(b) The game restricted to rows and columns 1, 2 and 5 has matrix $\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$, whose solution (see (25)) is $(1/4, 1/2, 1/4)$. It is easy to check that the mixed strategy

$(1/4, 1/2, 0, 0, 1/4, 0, \ldots)$ gives Player I an average payoff of at least 0 for every pure strategy of Player II. So this strategy is optimal for Player I, and by symmetry Player II as well.

11.(a) The matrix is skew-symmetric so this is a symmetric game. So the value is 0. To find an optimal strategy for I, we try (p_1, p_2, p_3) against the columns. The first column gives $-p_2 + 2p_3 = 0$ (since 0 is the value), and the second gives $p_1 - 3p_3 - 0$. We have $p_1 = 3p_3$ and $p_2 = 2p_3$. Then since the probabilities sum to 1, we have $3p_3 + 2p_3 + p_3 = 1$ or $p_3 = 1/6$. Then, $p_1 = 1/2$ and $p_2 = 1/3$. The optimal strategy for both players is $(1/2, 1/3, 1/6)$.

(b) This is a Latin square game, so $(1/3, 1/3, 1/3)$ is optimal for both players and the value is $v = (0 + 1 - 2)/3 = -1/3$.

(c) (1/4)row 1 + (1/4)row 2 + (1/4)row 3 + (1/4)row 4 dominates row 5. After removing row 5, the matrix is a Latin square. So $(1/4, 1/4, 1/4, 1/4)$ is optimal for II, and $(1/4, 1/4, 1/4, 1/4, 0)$ is optimal for I. The value is $v = (1 + 4 - 1 + 5)/4 = 9/4$.

12. The answer given by the Matrix Game Solver gives the same value and optimal strategy for Player I as in the text, but gives the optimal strategy for Player II as (7/90,32/90,48/90,3/90). This shows that although there may be a unique invariant optimal strategy, there may be other noninvariant optimal strategies as well. The simplex method only finds basic feasible solutions, and so will not find the invariant optimal solution (1/18,4/9,4/9,1/18), because it is not basic.

In (15), the middle row is strictly dominated by (3/4) the top row plus (1/4) the bottom row. Our solution and the one found by the Matrix Game Solver both give zero weight to (3,1) and (1,3).

13. (a) The reduced matrix has a saddle point.

$$\begin{array}{c c} & (1,0)^* \\ \begin{array}{c} (2,0)^* \\ (1,1) \end{array} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}$$

So the value is 1, $(1,1)$ (or $(2,0)^*$) is optimal for Player I and $(1,0)^*$ is optimal for Player II.

(b) The reduced matrix is

$$\begin{array}{c c} & \begin{array}{cc} (2,0)^* & (1,1) \end{array} \\ \begin{array}{c} (3,0)^* \\ (2,1)^* \end{array} & \begin{pmatrix} 3/2 & 1/2 \\ 0 & 2 \end{pmatrix} \end{array}$$

The value is 1. An optimal strategy for Player I is to use $(3,0)^*$ with probability 2/3, and $(2,1)^*$ with probability 1/3. This corresponds to playing $(3,0)$ and $(0,3)$ with probability 1/3 each, and $(2,1)$ and $(1,2)$ with probability 1/6 each. An optimal strategy for Player II is to use (1.1) with probability 1/2, and $(2,0)$ and $(0,2)$ with probability 1/4 each.

14. (a) In the matrix below, row 4 is dominated by (1/2)row 2 + (1/2)row 3. Then col 2 is dominated by (1/2)col 1 + (1/2)col 3. Then row 2 is dominated by (2/3)row 1 +

(1/3)row 3.

$$\begin{array}{c} \\ (4,0,0)^* \\ (3,1,0)^* \\ (2,2,0)^* \\ (2,1,1)^* \end{array}
\begin{array}{c} \begin{array}{ccc} (3,0,0)^* & (2,1,0)^* & (1,1,1) \end{array} \\
\begin{pmatrix} 4/3 & 2/3 & 0 \\ 1/3 & 1 & 1 \\ -1 & 4/3 & 3 \\ -1/3 & 1/3 & 2 \end{pmatrix} \end{array}$$

We find that (3/4,0,1/4,0) is optimal for Player I, (9/16,0,7/16) is optimal for Player II, and the value is 3/4.

(b) In the matrix below, row 4 is dominated by (1/2)row 3 + (1/2)row 5. But we might as well use the Matrix Game Solver directly.

$$\begin{array}{c} \\ (4,0,0,0)^* \\ (3,1,0,0)^* \\ (2,2,0,0)^* \\ (2,2,1,1)^* \\ (1,1,1,1) \end{array}
\begin{array}{c} \begin{array}{ccc} (3,0,0,0)^* & (2,1,0,0)^* & (1,1,1,0) \end{array} \\
\begin{pmatrix} 1 & 1/4 & -1/2 \\ 1/2 & 3/4 & 1/2 \\ -1/2 & 1 & 2 \\ 1/4 & 1/2 & 3/2 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

The value is 3/5, (0,4/5,0,0,1/5) is optimal for Player I, and (8/15,2/5,1/15) is optimal for Player II.

15. Consider the following strategies for Player II.

A: Start at the center square; if this is a hit continue with a 2, 4, 6, or 8 in random order each order equally likely; if this is a miss, shoot at the corners 1,3,7,9 in a random, equally likely order, and when a hit occurs, choose one of the two possible middle edge squares at random, then the other.

B: Start at the four middle edge squares, 2,4,6,8 in some random order; when a hit occurs, try the center next, the the possible corner squares.

C: Start at the four middle edge squares, 2,4,6,8 in some random order; when a hit occurs, try the possible corners next, then the center.

There are many other strategies for Player II, but they should be dominated by some mixture of these. In particular, starting at a corner square should be dominated by starting at a middle edge.

Using invariance, Player I has the two strategies, $[1,2]^*$ and $[2,5]^*$. Suppose Player I uses $[1,2]^*$ and Player II uses C. Then the first hit will occur on shot 1, 2, 3, or 4 with probability 1/4 each. After the first hit it takes on the average 1.5 more shots to get the other hit. The average number of shots then is

$$(1/4)(2.5) + (1/4)(3.5) + (1/4)(4.5) + (1/4)(5.5) = 4.$$

But if Player II starts off by shooting in the center before trying the corners, it will take one more shot on the average, namely 5. This gives the top row of the matrix below. The whole matrix turns out to be

$$\begin{array}{c} \\ [1,2]^* \\ [2,5]^* \end{array}
\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\
\begin{pmatrix} 5 & 5 & 4 \\ 3.5 & 3.5 & 5.5 \end{pmatrix} \end{array}$$

The first two columns are equivalent. Player I's optimal strategy is $(2/3, 1/3)$. This translates into choosing one of the 12 positions at random with probability 1/12 each. One optimal strategy for Player II is to randomize with equal probability between B and C. The value is 4.5.

16. Invariance reduces Player I to two strategies; choose 1 and 3 with probability 1/2 each, denoted by 1^*, and choose 3. Similarly, invariance and dominance reduces Player II to two strategies, we call A and B. For A, start with 2. For B, with probability 1/2, start with 1 and if it's not successful follow it with 3, and with probability 1/2 start with 3 if it's not successful and follow it with 1. This leads to a 2 by 2 game with matrix

$$\begin{array}{c} \\ 1^* \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 2 & 3/2 \\ 1 & 3 \end{pmatrix} \end{array}$$

The value is $9/5$. An optimal strategy for I is to choose 2 with probability $1/5$, and 1 or 3 equally likely with probability $2/5$ each. An optimal strategy for II is guess 2 first with probability $3/5$, and otherwise to guess 1 then 3, or 3 then 1 with probability $1/5$ each; that is, II never guesses 1 then 2 then 3 and never guesses 3 then 2 then 1.

17. To make Player I indifferent in choosing among rows 1 through k, Player II will choose $\boldsymbol{q} = (q_1, \ldots, q_k, 0, \ldots, 0)$ so that $u_i \sum_{i \neq j} q_j = V_k$ for $i = 1, \ldots, k$ for some constant V_k. Using $\sum_1^k q_j = 1$, this reduces to $(1 - q_i) = V_k/u_i$. Since $\sum_1^k (1 - q_i) = k - 1$, we have

$$V_k = \frac{k-1}{\sum_1^k 1/u_i} \qquad \text{and} \qquad q_i = \begin{cases} 1 - V_k/u_i & \text{for } i = 1, \ldots, k \\ 0 & \text{for } i = k+1, \ldots, m \end{cases} \tag{1}$$

The q_i are nondecreasing but we must have $q_i \geq 0$, which reduces to $V_k \leq u_k$. If $k < m$, we also require that Player I will not choose rows $k+1$ to m. This reduces to $u_{k+1} \leq V_k$. Therefore, if $u_{k+1} \leq V_k \leq u_k$, Player II can achieve V_k by using $\boldsymbol{q}$. (It is easy to show that $V_2 < u_2$ and that $V_k \leq u_k$ implies that $V_{k-1} \leq u_{k-1}$. This shows that such a k exists and is unique.)

To make Player II indifferent in choosing among columns 1 through k, Player I will choose $\boldsymbol{p} = (p_1, \ldots, p_k, 0, \ldots, 0)$ so that $\sum_1^k p_i u_i - p_j u_j = V_k$ for some constant V_k and $j = 1, \ldots, k$. This shows that p_j is equal to some constant over u_j for $j = 1, \ldots, k$. Using $\sum_1^k p_j = 1$, we find

$$p_j = \begin{cases} \dfrac{1/u_j}{\sum_1^k 1/u_i} & \text{for } j = 1, \ldots, k \\ \quad 0 & \text{for } j = k+1, \ldots, m \end{cases} \tag{2}$$

Solving for V_k shows it indeed has the same value as above. All the p_j are nonnegative, so we only have to show that Player II will not want to choose columns $k+1, \ldots, m$. The expected payoff is the same for each of these columns, namely, $\sum_1^k p_i u_i$ which is clearly greater than V_k, so Player I can achieve at least V_k.

In summary, find the largest k in $\{2, \ldots, m\}$ such that $(k-1)/u_k \leq \sum_1^k 1/u_i$. Then the value and the optimal strategies are given by (1) and (2).

18. The payoff matrix is $\boldsymbol{A}_n = (a_{ij})$, where

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that $\boldsymbol{A}_n \boldsymbol{B}_n = \boldsymbol{I}_n$, so that $\boldsymbol{A}_n^{-1} = \boldsymbol{B}_n$. The sum of the ith row of $\boldsymbol{A}_n^{-1}$ is $i(n+1-i)/2$. By symmetry these are also the column sums. Since they are all positive the game is completely mixed, and the optimal strategy, the same for both players, is proportional to $(n, 2(n-1), 3(n-2), \ldots, n)$, namely, $p(i) = 6i(n+1-i)/(n(n+1)(n+2))$. The sum of all numbers in $\boldsymbol{A}_n^{-1}$ is $n(n+1)(n+2)/12$, so the value is its reciprocal, $v_n = 12/(n(n+1)(n+2))$.

19. (a) n odd: Let $\boldsymbol{x} = (1,0,1,0,\ldots,1)^T$. Then $\boldsymbol{x}^T\boldsymbol{A} = (2,2,\ldots,2)$. There are $(n+1)/2$ 1's in $\boldsymbol{x}$, so $\boldsymbol{p} = 2\boldsymbol{x}/(n+1)$ is a mixed strategy for I that guarantees I will win $4/(n+1)$ no matter what column II chooses. The matrix is symmetric, so the same strategy guarantees II will lose $4/(n+1)$ no matter what I does. Thus, $\boldsymbol{p}$ is an optimal strategy for both I and II, and $4/(n+2)$ is the value.

(b) n even: Let $k = n/2$ and $\boldsymbol{x} = (k, 1, k-1, 2, \ldots, 1, k)^T$. Then $\boldsymbol{x}^t\boldsymbol{A} = (2k+1, 2k+1, \ldots, 2k+1)$. The sum of the elements of $\boldsymbol{x}$ is $k(k+1)$ so $\boldsymbol{p} = \boldsymbol{x}/(k(k+1))$ is a mixed strategy for I that guarantees I will win $(2k+1)/(k(k+1))$ no matter what column II chooses. The same strategy guarantees II will lose $(2k+1)/(k(k+1))$ no matter what I does. Thus, $\boldsymbol{p}$ is an optimal strategy for both I and II, and the value is $v = (2k+1)/(k(k+1)) = 4(n+1)/(n(n+2))$.

Solutions to Exercises of Section II.4.

1.(a) If Player II uses the mixed strategy, $(1/5, 1/5, 1/5, 2/5)$, I's expected payoff from rows 1, 2, and 3 are 17/5, 17/5, and 23/5 respectively. So I's Bayes strategy is row 3, giving expected payoff 23/5.

(b) If II guesses correctly that I will use the Bayes strategy against $(1/5, 1/5, 1/5, 2/5)$, she should choose column 3, giving Player I a payoff of -1.

2.(a) We have $b_{ij} = 5 + 2a_{ij}$ for all i and j. Hence, A and B have the same optimal strategies for the players, and the value of B is $\text{Val}(B) = 5 + 2\text{Val}(A) = 5$. The optimal strategy for I is $(6/11, 3/11, 2/11)$.

(b) Since we are given that $\text{Val}(A) = 0$, we may solve for the optimal $\boldsymbol{q}$ for II using the equations, $-q_2 + q_3 = 0$, and $2q_1 - 2q_3 = 0$. So $q_1 = q_3$ and $q_2 = q_3$. Since the probabilities sum to 1, all three must be equal to 1/3. So $(1/3, 1/3, 1/3)$ is optimal for Player II for both matrices A and B.

3.(a) Let $\epsilon > 0$ and let $\boldsymbol{q}$ be any element of Y^*. Then since $\sum_{j=1}^n q_j \to 1$ as $n \to \infty$, we have $\sum_{j=n}^\infty q_j \to 0$, so that there is an integer N such that $\sum_{j=N}^\infty q_j < \epsilon$. If Player I uses $i = N$, the expected payoff is $\sum_{j=1}^\infty L(N,j)q_j = \sum_{j=1}^{N-1} q_j - \sum_{j=N+1}^\infty q_j < 1 - 2\epsilon$. Thus for every $\boldsymbol{q} \in Y^*$, we have $\sup_{1\le i<\infty} \sum_{j=1}^\infty L(i,j)q_j \ge 1 - 2\epsilon$. Since this is true for all $\epsilon > 0$, it is also true for $\epsilon = 0$.

(b) Since (a) is true for all $\boldsymbol{q} \in Y^*$, we have $\overline{V} = \inf_{\boldsymbol{q}\in Y^*} \sup_{1\le i<\infty} \sum_{j=1}^\infty L(i,j)q_j \ge 1$. Since no payoff is greater than 1, we have $\overline{V} = 1$.

(c) The game is symmetric, so $\underline{V} = -\overline{V}$. Hence, $\underline{V} = -1$.

(d) Any strategy is minimax for Player I since any strategy guarantees an expected payoff of at least $\underline{V} = -1$.

4. The value of $\boldsymbol{A}$ is positive since the simple strategy (2/3)row 1 + (1/3)row 2 guarantees a positive return for Player I. But let's add 1 to $\boldsymbol{A}$ anyway to get $\boldsymbol{B}$:

$$\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 4 & -2 & 1 \end{pmatrix}.$$

The simplex tableau is displayed below on the left. We are asked to pivot in the second column. But there is only one positive number there, so we must pivot on the first row second column. We arrive at:

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & \\ \hline x_1 & 1 & 2 & 3 & 1 \\ x_2 & 3 & 0 & -1 & 1 \\ x_3 & 4 & -2 & 1 & 1 \\ \hline & -1 & -1 & -1 & 0 \end{array} \quad \longrightarrow \quad \begin{array}{c|ccc|c} & y_1 & x_1 & y_3 & \\ \hline y_2 & 1/2 & 1/2 & 3/2 & 1/2 \\ x_2 & 3 & 0 & -1 & 1 \\ x_3 & 5 & 1 & 4 & 2 \\ \hline & -1/2 & 1/2 & 1/2 & 1/2 \end{array}$$

There is still a negative element on the bottom edge so we continue. It is unique and in the first column, so we pivot in the first column. the smallest of the ratios is $1/3$ occuring in the second row. So we pivot on the second row first column to find:

$$\begin{array}{c|ccc|c} & y_1 & x_1 & y_3 & \\ \hline y_2 & 1/2 & 1/2 & 3/2 & 1/2 \\ x_2 & 3 & 0 & -1 & 1 \\ x_3 & 5 & 1 & 4 & 2 \\ \hline & -1/2 & 1/2 & 1/2 & 1/2 \end{array} \longrightarrow \begin{array}{c|ccc|c} & x_2 & x_1 & y_3 & \\ \hline y_2 & -1/6 & 1/2 & 5/3 & 1/3 \\ y_1 & 1/3 & 0 & -1/3 & 1/3 \\ x_3 & -5/3 & 1 & 17/3 & 1/3 \\ \hline & 1/6 & 1/2 & 1/3 & 2/3 \end{array}$$

From this we see that $\mathrm{Val}(\boldsymbol{B}) = 3/2$, so that $\mathrm{Val}(\boldsymbol{A}) = 1/2$. For either game $(p_1, p_2, p_3) = (3/4, 1/4, 0)$ is optimal for Player I and $(q_1, q_2, q_3) = (1/2, 1/2, 0)$ is optimal for Player II. You may use the sure-fire test to see that this is correct.

5. (a) For all $j = 0, 1, 2, \ldots,$

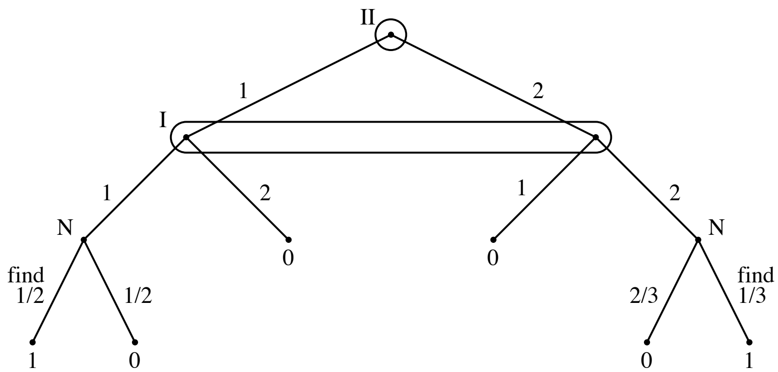
$$\begin{aligned} \sum_{i=0}^{\infty} p_i A(i,j) &= \sum_{i=0}^{j-1} \frac{1}{2^{(i+1)}}(-4^i) + \sum_{i=j+1}^{\infty} \frac{1}{2^{(i+1)}}(4^j) \\ &= -\frac{1}{2}(1 + 2 + \cdots + 2^{j-1}) + 4^j \frac{1}{2^{(j+2)}}(1 + \frac{1}{2} + \frac{1}{4} + \cdots) \\ &= -\frac{1}{2}(2^j - 1) + \frac{1}{2}2^j = \frac{1}{2}. \end{aligned}$$

(b) If both players use the mixed strategy, $\boldsymbol{p}$, the payoff is $\sum\sum p_i A(i,j) p_j$. The trouble is that the answer depends on the order of summation. If we sum over i first, we get $+1/2$, and if we sum over j first we get $-1/2$. In other words, Fubini's Theorem does not hold here. For Fubini's Theorem, we need $\sum\sum p_i |A(i,j)| p_j < \infty$, which is not the case here. The whole theory of using mixed strategies in games depends heavily on Utility Theory. In Utility Theory, at least as presented in Appendix 1, the utility functions are bounded. So it would seem most logical to restrict attention to games in which the payoff function, A, is bounded. That is certainly one way to avoid such examples. However, in many important problems the payoff function is unbounded, at least on one side, so one usually assumes that the payoff function is bounded below, say.

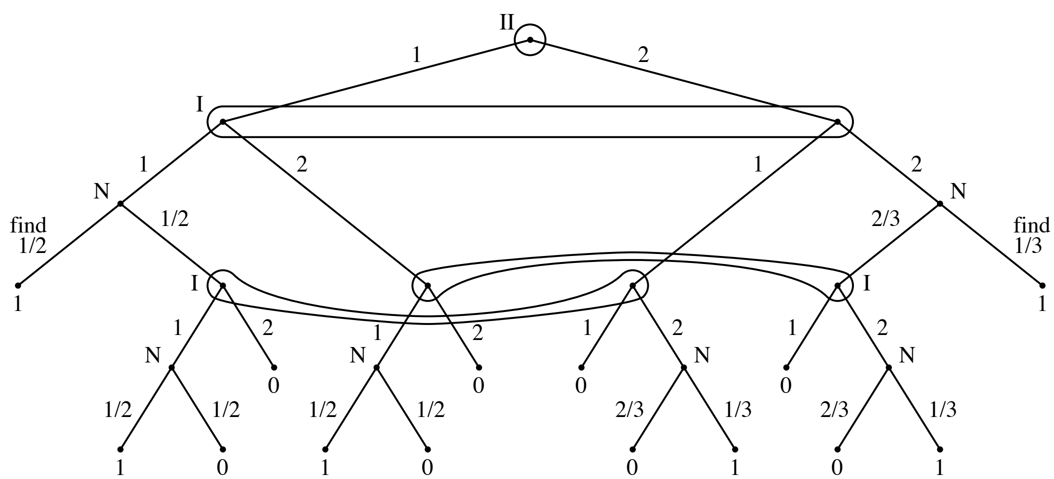
There is another way of dealing with the problem that is more germane to the example above, and that is by restricting the notion of a mixed strategy to be a probability distribution that gives weight to only a finite number of pure strategies. (Then Fubini's theorem holds because the summations are finite.) If this is done in the example, then one can easily see that the value of the game does not exist. This seems to be the "proper" solution of the game because it is just a blown-up version of the game, "the-player-that-chooses-the-larger-integer-wins".

Solutions to Exercises of Section II.5.

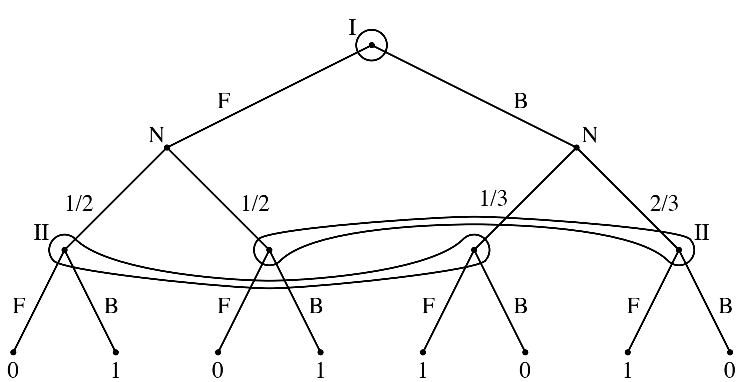
1. **The Silver Dollar.** I hides the dollar, II searches for it with random success.



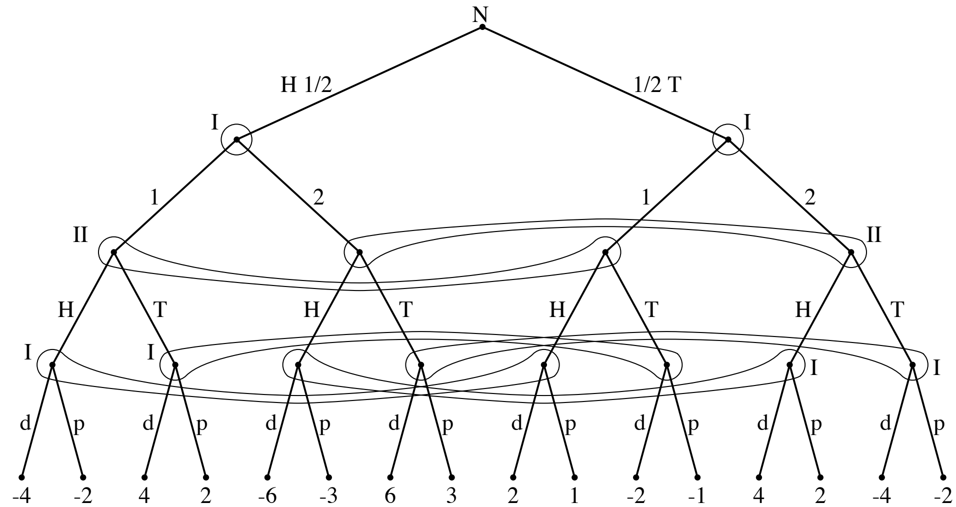
2. **Two Guesses for the Silver Dollar.** I hides the dollar, II searches for it twice with random success.



3. **Guessing the Probability of a Coin.** I chooses the fair (F) or biased (B) coin. II observes H or T on one toss of the coin and must guess which coin.



4. **A Forgetful Player.** A fair coin is tossed. I hears the outcome and bets 1 or 2. II guesses H or T. I forgets the toss and doubles or passes.



6. In Figure 2, replace 1/4 by p, 3/4 by $1-p$, and ±3 by $\pm(1+b)$. Again one can argue that Player I should bet with a winning card; he wins 1 if he checks, and wins at least 1 if he bets. In other words, as in the analysis in the text of the resulting 4×2 matrix, the first row dominates the third row and the second row dominates the fourth row. The top two rows of the matrix are

$$\begin{array}{c} \\ (b,b) \\ (b,c) \end{array} \begin{array}{c} \begin{array}{cc} \quad c \quad & \qquad f \end{array} \\ \begin{pmatrix} (1+b)(2p-1) & 1 \\ p(2+b)-1 & 2p-1 \end{pmatrix} \end{array}$$

If $(2p-1)(1+b)\ge 1$, (that is, if $p\ge(2+b)/(2+2b)$), there is a saddle-point in the upper right corner. The value of the game is 1, Player I should always bet, and Player II should always fold.

Otherwise, (if $p<(2+b)/(2+2b)$), the game does not have a saddle-point and we can use the straightforward method for solving two by two games. It is optimal for Player I to choose row 1 with probability $pb/((2+b)(1-p))$, and row 2 otherwise. It is optimal for Player II to choose column 1 with probability $2/(2+b)$, and column 2 otherwise. The value is $(4p(1+b)-(2+b))/(2+b)$.

7.(a) The strategic (normal) form is

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccc} (d,f) & (d,g) & (e,f) & (e,g) \end{array} \\ \begin{pmatrix} -1 & -1 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & 1 \end{pmatrix} \end{array}$$

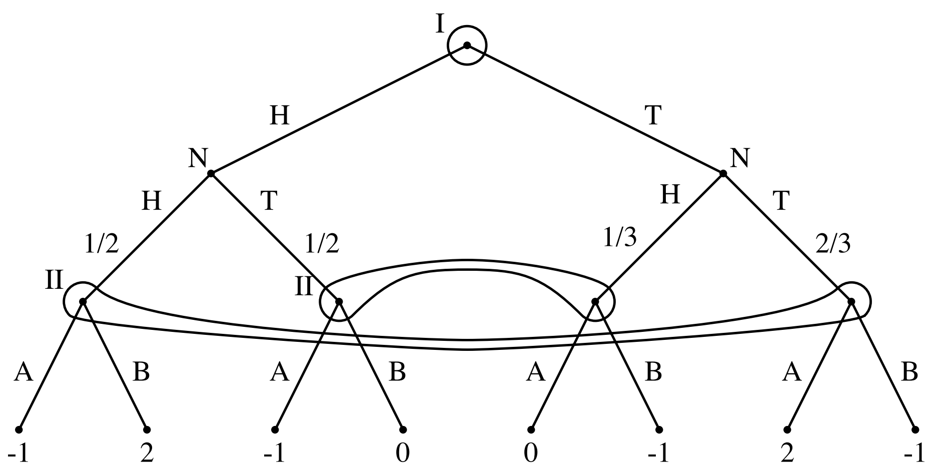
(b) Column 2 dominates columns 1 and 4. Then row 2 dominates row 1. The resulting two by two matrix is $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, with value 1/3. The optimal mixed strategy for Player I is $(0,2/3,1/3)$. The optimal mixed strategy for Player 2 is $(0,2/3,1/3,0)$.

8.(a) The strategic (normal) form is

$$\begin{array}{cc} & \begin{array}{cccc}(a,c) & (a,d) & (b,c) & (b,d)\end{array} \\ \begin{array}{c}(A,C)\\(A,D)\\(B,C)\\(B,D)\end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 3/2 & 1/2 & -1/2 & -3/2 \\ 1/2 & 3/2 & 1/2 & 3/2 \end{pmatrix}\end{array}$$

(b) Column 3 dominates column 2. The mixture, (2/3)row 2 + (1/3)row 3, dominates row 4. The mixture, (1/3)row 2 + (2/3)row 3, dominates row 1. The resulting two by three matrix is $\begin{pmatrix} 0 & 1 & 3 \\ 3/2 & -1/2 & -3/2 \end{pmatrix}$. The first two columns of this matrix are active. The value is $1/2$. An optimal mixed strategy for Player I in the original game is $(0, 2/3, 1/3, 0)$. An optimal mixed strategy for Player II is $(1/2, 0, 1/2, 0)$. It is interesting to note that Player I also has an optimal pure strategy, namely row 4.

9. (a)



(b) The matrix is $\begin{array}{cc} & \begin{array}{cccc}AA & AB & BA & BB\end{array} \\ \begin{array}{c}H\\T\end{array} & \begin{pmatrix} -1 & -1/2 & 1/2 & 1 \\ 4/3 & 1 & -2/3 & -1 \end{pmatrix}\end{array}$.

(c) Optimal for I $= (4/7, 3/7)$. Optimal for II $= (1/3, 0, 2/3, 0)$. Value $= 0$.

10. (a) The matrix is $\begin{array}{cc} & \begin{array}{cc}1 & 2\end{array} \\ \begin{array}{c}1\\2\end{array} & \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}\end{array}$. The mixed strategy $(2/5, 3/5)$ is optimal for both I and II. The value is 1/5.

(b) The matrix is $\begin{array}{cc} & \begin{array}{cc}1 & 2\end{array} \\ \begin{array}{c}11\\12\\12\\22\end{array} & \begin{pmatrix} 3/4 & 0 \\ 1/2 & 1/3 \\ 1/2 & 1/3 \\ 0 & 5/9 \end{pmatrix}\end{array}$. An optimal for I is $(0, 0, 10/13, 3/13)$. The optimal strategy for II is $(4/13, 9/13)$. The value is $5/13$.

(c) The matrix is $\begin{array}{c} \\ F \\ B \end{array}\begin{array}{c} \begin{array}{cccc} FF & FB & BF & BB \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 & 1 \\ 1 & 1/3 & 2/3 & 0 \end{pmatrix} \end{array}$. Optimal for I is $(4/7, 3/7)$. Optimal for II is $(1/7, 6/7, 0, 0)$. The value is $3/7$.

(d) The matrix is 64 by 4, much too large write out by hand. However, simple arguments show that most of Player I's pure strategies are dominated. First some notation. We denote Player I's pure strategies by a six-tuple, $ab; wxyz$, where a and b are 1 or 2 (the amount bet) for information sets I_1 and I_2 respectively, and and each of w, x, y and z are p or d (pass or double) for information sets I_3, I_4, I_5 and I_6 respectively. Thus, for example, $12; pdpp$ represents the strategy: Bet 1 with heads and 2 with tails; his partner passes unless 1 is bet and Player II guesses heads, in which case he doubles.

If Player I uses a strategy starting 12, then his partner upon hearing a bet of 1 and a guess of heads should pass rather than double since that means a loss of 2 rather than 4. Similarly, on hearing a bet of 1 and a guess of tails, his partner should double. Continuing in this way, we see that the strategy $12; pddp$ dominates all strategies beginning 12.

Similarly, we may see that the strategy $21; dppd$ dominates all strategies beginning 21, the strategy $11; pdxx$ (where x stands for "any") dominates all strategies beginning 11, and $22; xxpd$ dominates all strategies beginning 22. Thus, dominance reduces the game to the following 4 by 4 matrix.

$$\begin{array}{c} \\ 11; pdxx \\ 12; pddp \\ 21; dppd \\ 22; xxpd \end{array}\begin{array}{c} \begin{array}{cccc} HH & HT & TH & TT \end{array} \\ \begin{pmatrix} -1/2 & -1/2 & 1 & 1 \\ 1 & -2 & 4 & 1 \\ -1/2 & 4 & -2 & 5/2 \\ -1/2 & 1 & -1/2 & 1 \end{pmatrix} \end{array}$$

Now, we can see that col 1 dominates col 4. Moreover an equiprobable mixture of rows 2 and 3 dominate rows 1 and 4. This reduces the game to a 2 by 3 matrix which is easily solvable. For the above matrix, (0,3/5,2/5,0) is optimal for I, (4/5,1/5,0,0) is optimal for II and the value is 2/5.

We can describe Player I's strategy as follows. 60% of the time, Player I bets low on heads and high on tails, and his partner doubles when Player II is wrong. The other 40% of the time, Player I bets high on heads and low on tails, and his partner doubles when Player II is wrong.

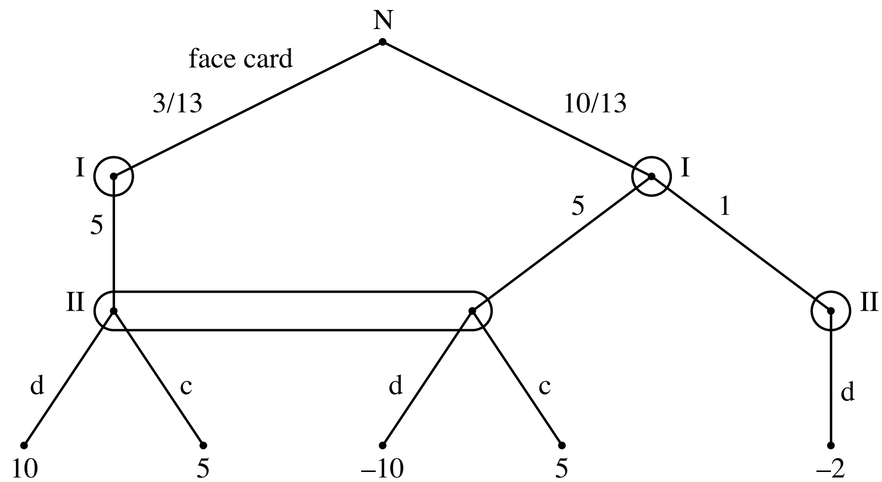
11. Suppose Player I uses f with probability p_1 and c with probability p_2 (and so g with probability $1-p_1$ and d with probability $1-p_2$). Suppose Player II uses a with probability q (and b with probability $1-q$). The average payoff is then

$$\begin{aligned} v &= p_1(q-(1-q)p_2) + (1-p_1(-(1-p_2)q + 2(1-q)) \\ &= q(4p_1+p_2-3) + 2 - p_1 - p_1 p_2. \end{aligned}$$

Therefore, Player II will choose $q = 0$ if $4p_1 + p_2 \geq 3$, and $q = 1$ if $4p_1 + p_2 < 3$. Against this, the best Player I can do is to use $p_1 = 1/2$ and $p_2 = 1$ (or $p_1 = 3/4$ and $p_2 = 0$) giving an average payoff of $1/2$. If q is announced, then Player I will use $p_1 = 1$ and

$p_2 = 0$ if $q \geq 2/3$, and $p_1 = 0$ and $p_2 = 1$ if $q < 2/3$. Against this, the best Player II can do is $q = 2/3$, which gives Player I an average payoff of $2/3$. Therefore, the value of the game does not exist if behavioral strategies must be used.

12. (a)



(b) The matrix is $\begin{array}{cc} & \begin{array}{cc} c & d \end{array} \\ \begin{array}{c} 5 \\ 1 \end{array} & \begin{pmatrix} 5 & -70/13 \\ -5/13 & 10/13 \end{pmatrix} \end{array}$.

(c) The strategy $(1/10, 9/10)$ is optimal for I, $(8/15, 7/15)$ is optimal for II and the value is $2/13$.

Solutions to Exercises of Section II.6.

1. G_1 has a saddle point. The value is 3, the pure strategy $(1,0)$ is optimal for I, and $(0,1)$ is optimal for II.
The value of G_2 is 3, the strategy $(.4,.6)$ is optimal for I, and $(.5,.5)$ is optimal for II.
The value of G_3 is -1, the strategy $(.5,.5)$ is optimal for I and for II.
The game G is thus equivalent to a game with matrix

$$\begin{pmatrix} 0 & 3 \\ 3 & -1 \end{pmatrix}.$$

Hence, the value of G is $9/7$, and the strategy $(4/7,3/7)$ is optimal for both players.

2. The games $G_{m,n}$ are defined by the induction

$$G_{m,n} = \begin{array}{c} \\ \text{inspect} \\ \text{wait} \end{array} \begin{array}{c} \begin{array}{cc} \text{act} & \text{wait} \end{array} \\ \begin{pmatrix} 1 & G_{m-1,n-1} \\ 0 & G_{m,n-1} \end{pmatrix} \end{array} \quad \text{for} \quad n = m+1, m+2, \ldots \quad \text{and} \quad m = 1, 2, \ldots$$

with boundary conditions $G_{0,n} = (0)$. Let $V_{m,n} = \text{Value}(G_{m,n})$. We have $V_{m,n} = 1$ for $m \geq n$ and $V_{0,n} = 0$ for $n \geq 1$. Also from the Example given in the text, we have $V_{1,n} = 1/n$. We compute the next few values,

$$V_{2,3} = \text{Value}\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} = \frac{2}{3} \qquad V_{2,4} = \text{Value}\begin{pmatrix} 1 & 1/3 \\ 0 & 2/3 \end{pmatrix} = \frac{2}{4}$$

and perhaps a few more, and then we conjecture $V_{m,n} = m/n$ for $0 \leq m \leq n$. Let us check this conjecture by induction. It is true for $m = 0$ and for $m = n$. Suppose that $0 < m < n$ and suppose the conjecture is true for all smaller values. Then,

$$\begin{aligned} V_{m,n} &= \text{Value}\begin{pmatrix} 1 & V_{m-1,n-1} \\ 0 & V_{m,n-1} \end{pmatrix} \\ &= \text{Value}\begin{pmatrix} 1 & (m-1)/(n-1) \\ 0 & m/(n-1) \end{pmatrix} \\ &= \frac{m}{n}. \end{aligned}$$

and the conjecture is verified. The optimal strategy for I in $G_{m,n}$ is the mixed strategy $(m/n, (n-m)/n)$, and the optimal strategy for II is $(1/n, (n-1/n)$. It is interesting to note that to play optimally player II does not need to keep track of how many times I has searched for him.

3. The induction is

$$G_n = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} G_{n-1} & 0 \\ 0 & G_{n-2} \end{pmatrix} \end{array} \qquad \text{for} \quad n = 2, 3, \ldots$$

and boundary conditions $G_0 = (1)$ and $G_1 = (1)$. Let $V_n = \text{Value}(G_n)$. The recursion for the V_n becomes

$$V_n = \text{Value}\begin{pmatrix} V_{n-1} & 0 \\ 0 & V_{n-2} \end{pmatrix} = \frac{V_{n-1}V_{n-2}}{V_{n-1}+V_{n-2}} \qquad \text{for} \quad n = 2,3,\ldots$$

with boundary conditions $V_0 = 1$ and $V_1 = 1$. Taking reciprocals of this equation, we find

$$\frac{1}{V_n} = \frac{1}{V_{n-1}} + \frac{1}{V_{n-2}} \qquad \text{for} \quad n = 2,3,\ldots$$

This is the recursion for the Fibonacci sequence. Since $1/V_0 = F_0$ and $1/V_1 = F_1$, we must have $1/V_n = F_n$. Hence we have $V_n = 1/F_n$ for all n and we may compute the optimal strategy for I and II to be $(F_{n-1}/F_n, F_{n-2}/F_n)$ for $n = 2,3,\ldots$.

4. Use of the first row implies $v_n \ge n+2$. Use of the first column implies $v_n \le n+3$. Since $n+2 \le v_n \le n+3$, none of the games have saddle points. So for $n = 0,1,\ldots,$

$$v_n = \text{Val}\begin{pmatrix} n+3 & n+2 \\ n+1 & v_{n+1} \end{pmatrix} = n+3 - \frac{2}{v_{n+1}-n}.$$

Let $w_n = v_n - n + 1$ for $n = 0,1,\ldots$. Then the w_n satisfy

$$w_n = 4 - \frac{2}{w_{n+1}}.$$

In fact, the w_n are the values of the games G'_n where

$$G'_n = 1 + \begin{pmatrix} 3 & 2 \\ 1 & G'_{n+1} \end{pmatrix} \qquad \text{for} \quad n = 0,1,\ldots$$

In game G'_n, I receives 1 from II and then the players choose row and column; if the players choose the second row second column, then I receives 1 from II and they next play G'_{n+1}. It may be seen that each of the games G'_n has the same structure. It is as if the players were playing the recursive game G' where

$$G' = 1 + \begin{pmatrix} 3 & 2 \\ 1 & G' \end{pmatrix}.$$

So all the games G'_n should have the same values. If so, denoting the common value by w, we would have $w = 4 - (2/w)$, or $w^2 - 4w + 2 = 0$. This has a unique solution in the interval $3 \le w \le 4$, namely $w = 2 + \sqrt{2}$. From this we have $v_n = n + 1 + \sqrt{2}$. The optimal strategies are the same for all games, namely,

$$\left(\frac{1+\sqrt{2}}{2+\sqrt{2}}, \frac{1}{2+\sqrt{2}}\right) = (.707\cdots, .293\cdots) \quad \text{is optimal for I for all games } G_n$$

$$\left(\frac{\sqrt{2}}{2+\sqrt{2}}, \frac{2}{2+\sqrt{2}}\right) = (.414\cdots, .586\cdots) \quad \text{is optimal for II for all games } G_n.$$

5. The game matrix of $G_{1,n}$ reduces to

$$\begin{pmatrix} 1 - \frac{n}{n+1}V_{n-1,1} & \frac{n}{n+1} - \frac{n}{n+1}V_{n-1,1} \\ 0 & 1 \end{pmatrix}.$$

(a) So Player I's optimal strategy uses odds $1 : 1/(n+1) = n+1 : 1$; i.e, he should bluff with probability $1/(n+2)$.
(b) Player II's optimal odds are $\frac{1}{n+1} + \frac{n}{n+1}V_{n-1,1} : 1 - \frac{n}{n+1}V_{n-1,1} = 1 + nV_{n-1,1} : n+1 - nV_{n-1,1}$; i.e., she should call with probability $(n+1-nV_{n-1,1})/(n+2) = V_{1,n}$.

6. (a) If $Q \geq 2$, the top row forever is optimal for I, the second column is optimal for II, and the value is $v = 2$. If $0 \leq Q \leq 2$, the top row forever is optimal for I, the first column forever is optimal for II, and the value is $v = Q$. If $Q \leq 0$, the bottom row is optimal for I, the first column forever is optimal for II, and the value is $v = 0$.
(b) If $Q \geq 1$, the value is $v = 1$, $(1,0,0)^\infty$ (i.e. the top row forever) is optimal for I, and $(0,1/2,1/2)^\infty$ is optimal for II. If $Q \leq 1$, the value is still $v = 1$, $(1,0,0)^\infty$ is optimal for II, and $(1-\epsilon, \epsilon/2, \epsilon/2)^\infty$ is ϵ-optimal for I (actually $\epsilon/(2-\epsilon)$-optimal)

7. Since in game G_3, I can choose the second row and II can choose the second column, we have $0 \leq v_3 \leq 1$. But since $v_3 \leq 1$, the most that I can hope to achieve in G_2 is $\mathrm{Val}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = 2/3$, so we have $0 \leq v_2 \leq 2/3$. Similarly, $1/2 \leq v_1 \leq 1$. So none of the game matrices have saddle points and we can write

$$v_1 = \frac{1}{2 - v_2} \qquad v_2 = \frac{2v_3}{2 + v_3} \qquad v_3 = \frac{1}{2 - v_1}.$$

Substitution of the first equation into the third and then the result into the second yields the quadratic equation, $v_2 = (4 - 2v_2)/(8 - 5v_2)$, or $5v_2^2 - 10v_2 + 4 = 0$. Solving this gives $v_2 = (5 - \sqrt{5})/5$ as the root less than 1. From this we can find $v_1 = (5 - \sqrt{5})/4$ and $v_3 = 3 - \sqrt{5}$. The complete solution is

G_1 : $\mathrm{Val}(G_1) = v_1 = (5 - \sqrt{5})/5 = .691\cdots$
Optimal for I = Optimal for II = $(v_1, 1 - v_1) = (.691\cdots, .309\cdots)$

G_2 : $\mathrm{Val}(G_2) = v_2 = (5 - \sqrt{5})/5 = .553\cdots$
Optimal for I = Optimal for II = $(\frac{5+\sqrt{5}}{10}, \frac{5-\sqrt{5}}{10}) = (.724\cdots, .276\cdots)$

G_3 : $\mathrm{Val}(G_3) = v_3 = 3 - \sqrt{5} = .764\cdots$
Optimal for I = Optimal for II = $(v_3, 1 - v_3) = (.764\cdots, .236\cdots)$

independent of Q.

8. Whatever the values of G_1, G_2 and G_3, the game G_1 is a latin square game and so has optimal strategies $(1/3, 1/3, 1/3)$ for both players. So we must have $v_1 = (v_1 + v_2 + v_3)/3$, or equivalently $2v_1 = v_2 + v_3$. From the form of G_2 and G_3, we see that $0 \leq v_2 \leq 2$ and $0 \leq v_3 \leq 1$. Hence, $0 \leq v_1 \leq 3/2$. We now see that G_2 does not have a saddle point, and that $0 \leq v_2 < 1$ so that G_3 does not have a saddle point either. We arrive at the three equations,

$$2v_1 = v_2 + v_3 \qquad v_2 = \frac{2v_1}{2 + v_1} \qquad v_3 = \frac{1}{2 - v_2}.$$

Eliminating v_1 and v_3 leads to a quadratic equation for v_2, namely $v_2^2 + 2v_2 - 1 = 0$. This has one positive root, namely $v_2 = \sqrt{2} - 1$. From this we can find $v_1 = (4\sqrt{2} - 2)/7$ and $v_3 = (3 + \sqrt{2})/7$. The complete solution is

G_1 : $\quad$ $\mathrm{Val}(G_1) = v_1 = (4\sqrt{2} - 2)/7 = .522\cdots$

Optimal for I = Optimal for II = $(1/3, 1/3, 1/3)$

G_2 : $\quad$ $\mathrm{Val}(G_2) = v_2 = \sqrt{2} - 1 = .414\cdots$

Optimal for I = Optimal for II = $(.793\cdots, .207\cdots)$

G_3 : $\quad$ $\mathrm{Val}(G_3) = v_3 = (3 + \sqrt{2})/7 = .631\cdots$

Optimal for I = Optimal for II = $(v_3, 1 - v_3) = (.631\cdots, .369\cdots)$

independent of Q.

9. This is a recursive game of the form

$$G = \begin{pmatrix} .8 + .2(-G^T) & .5 + .5(-G^T) \\ .6 + .4(-G^T) & .7 + .3(-G^T) \end{pmatrix}$$

and the value, v, of the game satisfies

$$v = \mathrm{Val}\begin{pmatrix} .8 - .2v & .5 - .5v \\ .6 - .4v & .7 - .3v \end{pmatrix}$$

The game is in favor of the server, so the value is between zero and one and the game does not have a saddle point. The optimal strategy for the server is to serve (high,low) with probabilities proportional to $(.1{+}.1v, .3{+}.3v)$, namely $(1/4, 3/4)$. The optimal strategy for the receiver is to receive (near,far) with probabilitites proportional to $(.2 + .2v, .2 + .2v)$, namely, $(1/2, 1/2)$. Using the second of these equalizing strategies, the value may be found to be $v = (1/2)(.8 - .2v) + (1/2)(.5 - .5v) = .65 - .35v$. Solving for v gives $v = 13/27 = .481\cdots$.

10. (a) We may think of the basic game, G, as the one in which player I chooses a number k to be the number of times he tosses the coin before challenging II. In this, player II has no choice and the matrix G is an $\infty \times 1$ matrix, which is to say an infinite dimensional column vector. The probability of tossing k heads in a row is p^k. Counting 1 for a win and -1 for a loss, the expected payoff given that I tosses k heads in a row is $p^k(-1) + (1 - p^k) = 1 - 2p^k$. Thus the kth component of G is $p^k(1 - 2p^k) + (1 - p^k)(-G^T)$. Whatever the value $v = \mathrm{Val}(G)$, Player I will choose k to maximize this. We have the equation

$$v = \max_k (p^k(1 - 2p^k) + (1 - p^k)(-v)).$$

Clearly $v > 0$, so there is a finite integer k at which the maximum is taken on, call it k_0. Then $v = p^{k_0}(1 - 2p^{k_0})/(2 - p^{k_0})$ and since v takes on its maximum value at k_0, we have

$$v = \max_k \left(\frac{p^k(1 - 2p^k)}{2 - p^k} \right).$$

When $p = .5$, evaluating $p^k(1-2p^k)/(2-p^k)$ at $k = 1, 2, 3, 4, \ldots$ gives 0, $1/14 = .0714\cdots$, $1/20 = .05$, $7/248 = .0282\cdots$, and so on, with a clear maximum at $k = 2$.

(b) For arbitrary p, there is a maximum value attainable by v. Replace p^k by y in the formula for v and write it as $f(y) = y(1-2y)/(2-y)$. Calculus gives $f'(y) = (2y^2 - 8y + 2)/(2-y)^2$, so the function $f(y)$ has a unique maximum on the interval (0,1) attained when $y^2 - 4y + 1 = 0$. The root of this equation in the interval (0,1) is $y = 2 - \sqrt{3}$, and the value attained there is $V^* = f(2-\sqrt{3}) = 7 - 4\sqrt{3} = .0718\cdots$. This is quite close to $.0714\cdots$ attainable when $p = .5$. If $p = 2 - \sqrt{3}$, then V^* is attainable with $k = 1$. As $p \to \infty$, it becomes easier and easier to choose k so that p^k is close to $2 - \sqrt{3}$ so the value converges to V^*.

11. The first row shows the value is at least 1, and the first column shows the value is at most 4. So $1 \le v \le 4$. Then we see by "down-up-down-up" that the game does not have a saddle-point, so

$$v = \mathrm{Val}\begin{pmatrix} 4 & 1 + (v/3) \\ 0 & 1 + (2v/3) \end{pmatrix} = \frac{4 + (8v/3)}{4 + (v/3)}.$$

This leads to the quadratic equation, $v^2 + 4v - 12 = 0$, which has solutions $v = -2 \pm 4$. Since v is positive, we have $v = 2$ as the value. The matrix becomes $\begin{pmatrix} 4 & 1 + (2/3) \\ 0 & 1 + (4/3) \end{pmatrix}$. Player I's stationary optimal strategy is $(1/2, 1/2)$, and Player II's stationary optimal strategy is $(1/7, 6/7)$.

12. We have

$$v(1) = \mathrm{Val}\begin{pmatrix} 2 & 2 + (v(2)/2) \\ 0 & 4 + (v(2)/2) \end{pmatrix} \qquad v(2) = \mathrm{Val}\begin{pmatrix} -4 & 0 \\ -2 + (v(1)/2) & -4 + (v(1)/2) \end{pmatrix}.$$

It may be difficult to guess that the matrices do not have saddle-points, so let us assume they do not and check later to see if this assumption is correct. If neither matrix has a saddle-point, then the equations become,

$$v(1) = \frac{8 + v(2)}{4} \qquad v(2) = \frac{16 - 2v(1)}{-6}.$$

Solving these equations simultaneously, we find $v(1) = 16/11$ and $v(2) = -24/11$. With these values the matrices above become

$$\begin{pmatrix} 2 & 2 - (12/11) \\ 0 & 4 - (12/11) \end{pmatrix} \qquad \begin{pmatrix} -4 & 0 \\ -2 + (8/11) & -4 + (8/11) \end{pmatrix}.$$

Since these do not have saddle-points, our assumption is valid and $v(1) = 16/11$ and $v(2) = -24/11$ are the values. In $G^{(1)}$, the optimal stationary strategies are $(8/11, 3/11)$ for I and $(1/2, 1/2)$ for II. In $G^{(2)}$, the optimal stationary strategies are $(1/3, 2/3)$ for I and $(6/11, 5/11)$ for II.

Solutions to Exercises of Section II.7.

1. (a)

(b) $d = (b+a)/2$ is the indifference equation for Player II at d. Indifference for II at c is the same as (3). Indifference for Player I at a gives $2c - \beta(c-1) = d$. Indifference of I at b gives $2c + (\beta+2)(b-c) - \beta(1-b) = 2b - d$.

(c) At $\beta = 2$, I get $a = 5/33$, $b = 23/33$, $c = 20/33$, and $d = 14/33$. The value is:
$v = (\beta+1)[(1-b)(b-c) - a(1-c)] + ac + (1-b)c - (1-b)(b-a) - (d-a)^2$

This turns out to be negative for all $\beta > 0$. In particular, I get $v(2) = -2/33$.

(d)

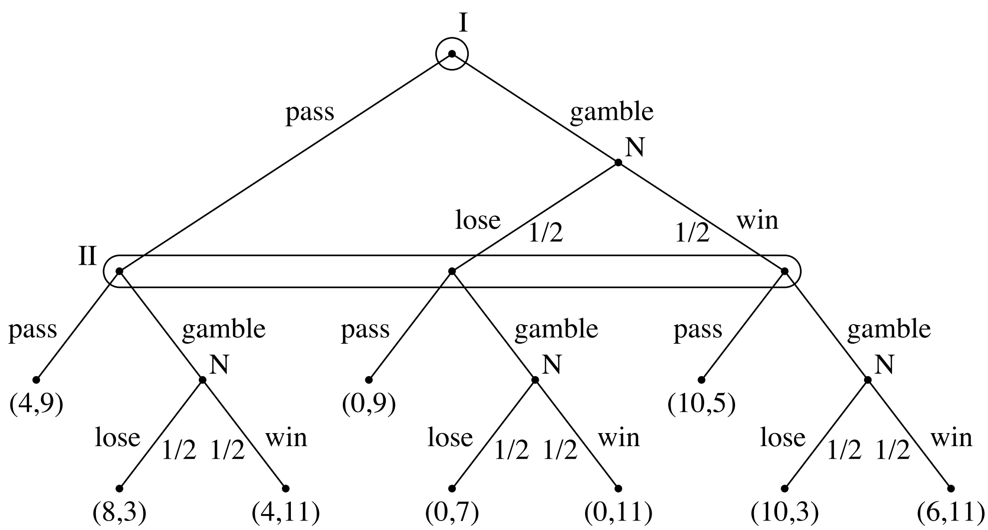
Solutions to Exercises of Section III.1.

1. The bimatrix is $\begin{array}{c c} & \begin{array}{cc} c & d \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} (13/4, 3) & (22/4, 3/4) \\ (4, 10/4) & (21/4, 2) \end{pmatrix} \end{array}.$

2. (a) Player I's maxmin strategy is $(1, 0)$ (i.e. row 1) guaranteeing him the safety level $v_I = 1$. Player II's maxmin strategy is $(1, 0)$ (i.e. column 1) guaranteeing her the safety level $v_{II} = 1$.

(b) Player I's maxmin strategy is $(1/2, 1/2)$ guaranteeing him the safety level $v_I = 5/2$. Player II's maxmin strategy is $(3/5, 2/5)$ guaranteeing her the safety level $v_{II} = 8$.

3. (a) There are many ways to draw the Kuhn tree. Here is one. The payoffs are in units of 100 dollars.



(b) The bimatrix is:

$$\begin{array}{c c} & \begin{array}{cc} \text{gamble} & \text{pass} \end{array} \\ \begin{array}{c} \text{gamble} \\ \text{pass} \end{array} & \begin{pmatrix} (4, 8) & (5, 7) \\ (6, 7) & (4, 9) \end{pmatrix} \end{array}$$

(c) Player I's safety level is $v_I = 14/3$. Player II's safety level is $v_{II} = 23/3$. Both maxmin strategies are $(2/3, 1/3)$.

4. Let Q denote the proportion of students in the class (excluding yourself) who choose row 2. If you choose row 2 you win $Q \cdot 6$ on the average. If you choose row 1, you win 4. So you should choose row 2 only if you predict that at least 2/3 of the rest of the class will choose row 2.

In my classes, only between 15% and 35% of the students chose row 2. If your classes are like mine, you should choose row 1.

Solutions to Exercises of Section III.2.

1. Let $\boldsymbol{p}_0$ denote the maxmin strategy of Player I, and let $(\boldsymbol{p},\boldsymbol{q})$ be any strategic equilibrium. Then, $v_I \le \boldsymbol{p}_0'\boldsymbol{A}\boldsymbol{q}$ since use of $\boldsymbol{p}_0$ guarantees Player I at least v_I no matter what Player II does. But also $\boldsymbol{p}_0'\boldsymbol{A}\boldsymbol{q} \le \boldsymbol{p}'\boldsymbol{A}\boldsymbol{q}$ since $\boldsymbol{p}$ is a best response to $\boldsymbol{q}$. This shows $v_I \le \boldsymbol{p}'\boldsymbol{A}\boldsymbol{q}$. Then $v_{II} \le \boldsymbol{p}'\boldsymbol{A}\boldsymbol{q}$ follows from symmetry.

2(a) The safety levels are $v_I = 2$ and $v_{II} = 16/5$. The corresponding MM strategies are $(0,1)$ (the second row) for Player I, and $(1/5,4/5)$ (the equalizing strategy on $\boldsymbol{B}$) for Player II. The unique SE is the PSE in the lower left corner with payoff $(2,4)$. It may be found by removing strictly dominated rows and columns.

(b) The safety levels are $v_I = 2$ and $v_{II} = 5/2$. The corresponding MM strategies are $(0,1)$ for Player I, and $(1/2,1/2)$ for Player II. There are no pure SE's, and the unique SE is the one using the equalizing strategies $(1/4,3/4)$ for Player I on Player II's payoff matrix, and $(1/2,1/2)$ for Player II on Player I's payoff matrix. The vector payoff is $(5/2,5/2)$. Note that Player II's equalizing strategy is not an optimal strategy on Player I's matrix.

(c) The safety levels are $v_I = 0$ and $v_{II} = 0$. The corresponding MM strategies are $(1,0)$ for Player I, and $(1,0)$ for Player II. There is no PSE. The unique SE is the one using equalizing strategies, $(3/4,1/4)$ for Player I and $(1/2,1/2)$ for Player II, with payoff vector $(0,0)$.

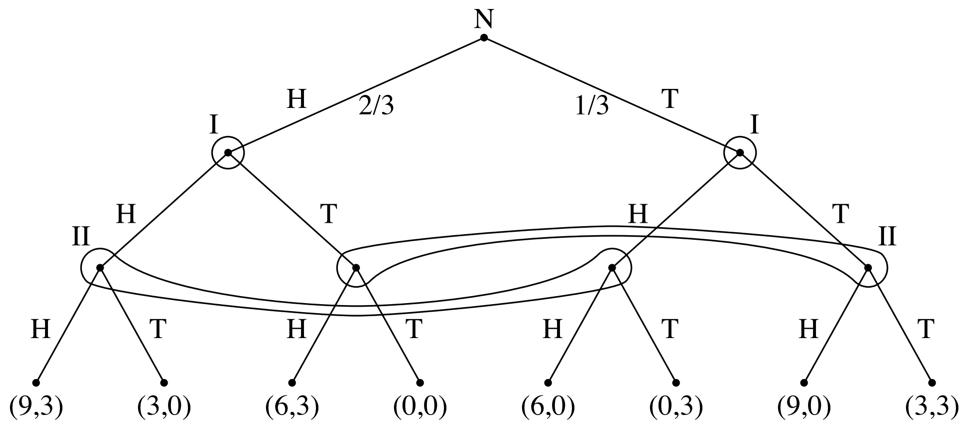
3. (a) The bimatrix is

$$\begin{array}{ll} & \begin{array}{cc} \text{chicken} & \text{iron nerves} \end{array} \\ \begin{array}{l} \text{chicken} \\ \text{iron nerves} \end{array} & \begin{pmatrix} (1,1) & (-1,2) \\ (2,-1) & (-2,-2) \end{pmatrix} \end{array}$$

(b) I's matrix has a saddle point with value $v_I = -1$, achievable if I uses the top row. Similarly, II's MM strategy is the first column, with value $v_{II} = -1$. Thus the safety levels are -1 for both players. Yet if both players play their MM strategies, the happy result is that they both receive $+1$.

(c) There are two PSE's: the lower left corner and the upper right corner. The third SE involves mixed strategies and may be found using equalization. The mixed strategy (1/2,1/2) for II is an equalizing strategy for I's matrix (even though it is not optimal there). Against this strategy, the average payoff to I is zero. Similarly, the strategy (1/2,1/2) for I is equalizing for II's matrix, giving an average payoff of zero. Thus, $((1/2,1/2),(1/2,1/2))$ is a mixed SE with payoff vector, (0,0).

4.(a)



(b)

$$\begin{array}{c} \\ \mathrm{HH} \\ \mathrm{HT} \\ \mathrm{TH} \\ \mathrm{TT} \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{HH} & \mathrm{HT} & \mathrm{TH} & \mathrm{TT} \end{array} \\ \begin{pmatrix} (8, 2^*) & (8^*, 2^*) & (2, 1) & (2, 1) \\ (9^*, 2) & (7, 3^*) & (5, 0) & (3^*, 1) \\ (6, 2) & (2, 0) & (4, 3^*) & (0, 1) \\ (7, 2^*) & (1, 1) & (7^*, 2^*) & (1, 1) \end{pmatrix} \end{array}$$

(c) There are two PSE's, those with double asterisks.

5.(a) We star the entries of the A matrix that are maxima of their column and entries of the B matrix that are maxima of their row.

$-3\ ,-4$	$2^*,-1$	$0\ ,\ \ 6^*$	$1^*,\ \ 1$
$2^*,\ \ 0$	$2^*,\ \ 2^*$	$-3\ ,\ \ 0$	$1^*,-2$
$2^*,-3$	$-5\ ,\ \ 1^*$	$-1\ ,-1$	$1^*,-3$
$-4\ ,\ \ 3^*$	$2^*,-5$	$1^*,\ \ 2$	$-3\ ,\ \ 1$

The only doubly starred entry occurs in the second row, second column, and hence the unique PSE is $\langle 2, 2\rangle$.

(b) Starring the entries in a similar manner leads to the matrix

$0\ ,\ \ 0$	$1^*,-1$	$1^*,\ \ 1^*$	$-1\ ,\ \ 0$
$-1\ ,\ \ 1^*$	$0\ ,\ \ 1^*$	$1^*,\ \ 0$	$0^*,\ \ 0$
$1^*,\ \ 0$	$-1\ ,-1$	$0\ ,\ \ 1^*$	$-1\ ,\ \ 1^*$
$1^*,-1$	$-1\ ,\ \ 0^*$	$1^*,-1$	$0^*,\ \ 0^*$
$1^*,\ \ 1^*$	$0\ ,\ \ 0$	$-1\ ,-1$	$0^*,\ \ 0$

We find there are three doubly starred squares, and hence three PSE's, namely, $\langle 5, 1\rangle$ and $\langle 1, 3\rangle$ and $\langle 4, 4\rangle$.

6.(a) $v_I = 0$ and $v_{II} = 2/3$.

(b) There is a unique PSE at row 2, column 1, with payoff vector (0,1).

(c) The mixed strategy $(1/3, 2/3)$ is the unique equalizing strategy for Player I. Column 1 is an equalizing strategy for II, but so is the mixture, $(0, 2/3, 1/3)$. More generally, any mixture of the form $(1-p, 2p/3, p/3)$ for $0 \le p \le 1$ is an equalizing strategy for II. Therefore, any of the strategy pairs, $(1/3, 2/3)$ for I and $(1-p, 2p/3, p/3)$ for $0 \le p \le 1$ for II, gives a strategic equilibrium. There are also some non-equalizing strategy pairs forming a strategic equilibrium, namely $(p, 1-p)$ for $0 \le p \le 1/3$ for I, and column 1 for II.

7. We are given $a_{1j} < \sum_{i=2}^m x_i a_{ij}$ for all j, where $x_i \ge 0$ and $\sum_{i=2}^m x_i = 1$. Suppose $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ is a strategic equilibrium. Then

$$\sum_j \sum_i p_i^* a_{ij} q_j^* \ge \sum_j \sum_i p_i a_{ij} q_j^* \quad \text{for all } \boldsymbol{p} = (p_1, \ldots, p_m). \tag{$*$}$$

We are to show $p_1^* = 0$. Suppose to the contrary that $p_1^* > 0$. Then

$$\begin{aligned}
\sum_j \sum_i p_i^* a_{ij} q_j^* &= \sum_j [p_1^* a_{1j} q_j^* + \sum_{i=2}^m p_i^* a_{ij} q_j^*] \\
&< \sum_j [p_1^* (\sum_{i=2}^m x_i a_{ij}) q_j^* + \sum_{i=2}^m p_i^* a_{ij} q_j^*] \quad \text{(strict inequality)} \\
&= \sum_j \sum_{i=2}^m (p_1^* x_i + p_i^*) a_{ij} q_j^* = \sum_j \sum_i p_i a_{ij} q_j^*
\end{aligned}$$

where $p_1 = 0$ and $p_i = p_1^* x_i + p_i^*$ for $i = 2, \ldots, m$. But The p's are nonnegative and add to one, so this contradicts (*).

8. (a) We have

$$A = \begin{pmatrix} 3 & 2 & 3 \\ 6 & 0 & 3 \\ 4 & 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 4 & 5 \end{pmatrix}$$

In A, the third row second col is a saddle point. So $v_I = 3$ and the third row is a maxmin strategy for Player I. In B, the row 3 is dominated by row 1, and col 2 is an equal probability mixture of col 1 and col 3. With these removed, the resulting 2 by 2 matrix has value $v_{II} = 2.5$. The maxmin strategy for Player II is $(1/4, 0, 3/4)$. (Another maxmin strategy for II is $(0, 1/2, 1/2.)$

(b) There are no PSE's. In A, row 1 is strictly dominated by row 3 and may be removed from consideration. Then in B, col 2 is strictly dominated by col 3 and may be removed. In the resulting 2 by 2 bimatrix game, there is a unique SE. It is given by equalizing strategies, $(1/3, 2/3)$ for I and $(1/3, 2/3)$ for II. In the original game, the unique SE is $(0, 1/3, 2/3)$ for Player I and $(1/3, 0, 2/3)$ for Player II. The equilibrium payoff is $(4, 4\frac{1}{3})$.

9. (a) At II's information set, a dominates b, so that vertex is worth $(1,0)$. Then at I's information set, B dominates A, so the PSE found by backward induction is (B,a), having payoff $(1,0)$. This is a subgame perfect PSE.

(b)

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} \quad a \quad & \quad b \quad \end{array} \\ \begin{pmatrix} (0,1) & (0,1) \\ (1,0) & (-10,-1) \end{pmatrix} \end{array}$$

(c) There are two PSE's, the lower left and the upper right. The lower left, (B,a), is the subgame perfect PSE. The upper right, (A,b), corresponds to the the PSE where Player I plays A because he believes Player II will play b. This is not subgame perfect because at Player II's vertex, it is not an equilibrium for Player II to play b.

10. There were 18 answers for Player I and 17 for Player II. The data is as follows.

I:	Stop at	No.	Score	II:	Stop at	No.	Score
	(1,1)	10	17		(0,3)	8	34
	(98,98)	2	882		(97,100)	2	806
	(99,99)	4	887		(98,101)	7	804
	never	2	880				

Scores for Player I ranged from 17 for those who selected $(1,1)$, to 887 for those who selected (99,99). Scores for Player II ranged from 34 for those who chose (0,3) to 806 for those who chose (97,100). Total scores ranged from 51 to 1693. Those who scored above 1000 received 5 points. Those who scored between 500 and 1000 received 3 points. Those who scored less than 100 recieved 1 point.

Solutions to Exercises of Section III.3.

1.(a) I's strategy space is $X = [0, \infty)$ and II's strategy space is $Y = [0, \infty)$. If I chooses $q_1 \in X$ and II chooses $q_2 \in Y$, the payoffs to I and II are

$$u_1(q_1, q_2) = q_1(a - q_1 - q_2)^+ - c_1 q_1, \qquad u_2(q_1, q_2) = q_2(a - q_1 - q_2)^+ - c_2 q_2$$

respectively. To find a PSE, we set derivatives to zero:

$$\frac{\partial}{\partial q_1} u_1(q_1, q_2) = a - 2q_1 - q_2 - c_1 = 0, \qquad \frac{\partial}{\partial q_2} u_2(q_1, q_2) = a - q_1 - 2q_2 - c_2 = 0.$$

The unique solution is (q_1^*, q_2^*), where

$$q_1^* = (a + c_2 - 2c_1)/3, \qquad q_2^* = (a + c_1 - 2c_2)/3.$$

Since we have assumed $c_1 < a/2$ and $c_2 < a/2$, both these production points are positive. Thus (q_1^*, q_2^*) is a PSE. Its payoff vector is $((a + c_2 - 2c_1)^2/9, (a + c_1 - 2c_2)^2/9)$.

(b) I's profit is $v_1(x, y) = x(17 - x - y) - x - 2 = x(16 - x - y) - 2$. II's profit is $v_2(x, y) = y(17 - x - y) - 3y - 1 = y(14 - x - y) - 1$. For fixed y, I should choose x so that $\partial v_1/\partial x = 16 - 2x - y = 0$. For fixed x, II should choose y so that $\partial v_2/\partial y = 14 - x - 2y = 0$. The equilibrium point is achieved if these two equations are satisfied simultaneously. This gives $x = 6$ and $y = 4$. The equilibrium payoff is $(36 - 2, 16 - 1) = (34, 15)$.

2. We assume $c < a$ — otherwise no company will produce anything. The payoff functions are

$$u_i(q_1, q_2, q_3) = q_i P(q_1 + q_2 + q_3) - c q_i = q_i[(a - q_1 - q_2 - q_3)^+ - c]$$

for $i = 1, 2, 3$. Assuming $q_1 + q_2 + q_3 < a$, there will be equilibrium production if the following three equations are satisfied:

$$\frac{\partial}{\partial q_i} u_i(q_1, q_2, q_3) = a - q_i - q_1 - q_2 - q_3 - c = 0$$

for $i = 1, 2, 3$. This solution is easily found to be $q_i = (a - c)/4$ for $i = 1, 2, 3$. This is the equilibrium production. The total production is $(3/4)(a - c)$, compared to $(2/3)(a - c)$ for the duopoly production, and $(1/2)(a - c)$ for the monopoly production.

3. The profit functions are

$$u_1(p_1, p_2) = (a - p_1 + b p_2)^+ (p_1 - c) \quad \text{and} \quad u_2(p_1, p_2) = (a - p_2 + b p_1)^+ (p_2 - c).$$

Knowing Player I's choice of p_1, Player II would choose p_2 to maximize $u_2(p_1, p_2)$. As in the Bertrand model with differentiated products, we find

$$\frac{\partial}{\partial p_2} u_2(p_1, p_2) = a - 2p_2 + bp_1 + c = 0 \quad \text{and hence} \quad p_2(p_1) = (a + bp_1 + c)/2.$$

Knowing Player II will use $p_2(p_1)$, Player I would choose p_1 to maximize $u_1(p_1, p_2(p_1))$. We have

$$\frac{\partial}{\partial p_1} u_1(p_1, p_2(p_1)) = a - p_1 + (b/2)(2 + bp_1 + c) - (p_1 - c)(2 - b^2)/2 = 0.$$

Hence, solving for p_1 and substituting into p_2 gives

$$p_1^* = \frac{a(2+b) + c(2+b-b^2)}{2(2-b^2)} \qquad \text{and} \qquad p_2^* = \frac{a+c}{2} + \frac{b}{2} \cdot \frac{a(2+b) + c(2+b-b^2)}{2(2-b^2)}$$

as the PSE.

Both p_1^* and p_2^* are greater than $(a+c)/(2-b)$, so both players charge more than in the Bertrand model. Surprisingly, both players receive more from the sequential PSE than they do from the PSE of the Bertrand model. However, Player I receives more than Player II. (This model is suspect. Do not assume these results hold in general.)

4. (a) $u(Q) = QP(Q) - Q$, so $u'(Q) = P(Q) + QP'(Q) - 1 = (3/4)Q^2 - 10Q + 25$. This quadratic function has roots $Q = 10/3$ and $Q = 10$. The maximum of $u(Q)$ on the interval $[0, 10]$ is at $Q = 10/3$, so this is the monopoly production. The monopoly price is $P(10/3) = 109/9$ and the return of this production is $u(10/3) = 1000/27 = 37+$.

(b) $u_1(q_1, q_2) = q_1 P(q_1 + q_2) - q_1$, so

$$\begin{aligned} \frac{\partial}{\partial q_1} u_1(q_1, \frac{5}{2}) &= P(q_1 + \frac{5}{2}) + q_1 p'(q_1 + \frac{5}{2}) - 1 \\ &= \frac{3}{4}(q_1^2 - 10q_1 + \frac{75}{4}) \end{aligned}$$

This has roots $q_1 = 5/2$ and $q_1 = 15/2$. The maximum occurs at $q_1 = 5/2$, and for $q_1 > 15/2$, $u_1(q_1, 5/2) = 0$. This shows that the optimal reply to $q_2 = 5/2$ is $q_1 = 5/2$. But the situation is symmetric, so the optimal reply of firm 2 to $q_1 = 5/2$ of firm 1, is $q_2 = 5/2$ also. This shows that $q_1 = q_2 = 5/2$ is a PSE.

5.

6.

7. (a) Setting the partial derivatives to zero,

$$\begin{aligned} \frac{\partial M_1}{\partial x} &= V \frac{y}{(x+y)^2} - C_1 = 0 \\ \frac{\partial M_2}{\partial y} &= V \frac{x}{(x+y)^2} - C_2 = 0, \end{aligned}$$

we see that $C_1x = C_2y$, from which it is easy to solve for x and y:

$$x = VC_2/(C_1+C_2)^2$$
$$y = VC_1/(C_1+C_2)^2$$

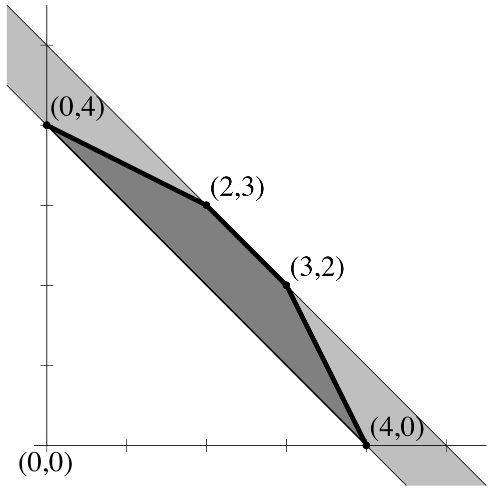
The profits are

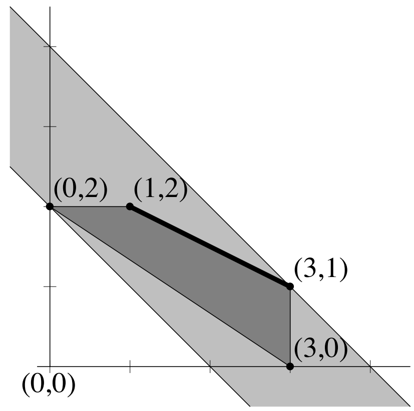
$$M_1 = C_2^2/(C_1+C_2)^2$$
$$M_2 = C_1^2/(C_1+C_2)^2$$

(b) If $V = 1$, $C_1 = 1$ and $C_2 = 2$, we find

$$x = 2/9 \qquad M_1 = 4/9$$
$$y = 1/9 \qquad M_2 = 1/9$$

Solutions to Exercises of Section III.4.

1. (a) 

(b) 

The light shaded region is the TU-feasible set. The dark shaded region is the NTU-feasible region. The NTU-Pareto optimal outcomes are the vectors along the heavy line. The TU-Pareto outcomes are the upper right lines of slope -1.

2. (a) The cooperative strategy is ((1,0),(1,0)) with sum $\sigma = 7$. The difference matrix $\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$ has a saddle point at the upper right with value $\delta = 0$. So $\boldsymbol{p}^* = (1, 0)$ and $\boldsymbol{q}^* = (0, 1)$ are the threat strategies and the disagreement point is $(0, 0)$. The TU-solution is $\boldsymbol{\varphi} = ((\sigma + \delta)/2, (\sigma - \delta)/2) = (7/2, 7/2)$. Since the cooperative strategy gives payoff $(4, 3)$, this requires a side payment of $1/2$ from I to II.

(b) The cooperative strategy is ((0,1),(0,1)) with sum $\sigma = 9$. The difference matrix $\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} 2 & 4 \\ 0 & -3 \end{pmatrix}$ has a saddle point at the upper left with value $\delta = 2$. So $\boldsymbol{p}^* = (1, 0)$ and $\boldsymbol{q}^* = (1, 0)$ are the threat strategies and the disagreement point is $(5, 3)$. The TU-solution is $\boldsymbol{\varphi} = ((\sigma + \delta)/2, (\sigma - \delta)/2) = (11/2, 7/2)$. Since the cooperative strategy gives payoff $(3, 6)$, this requires a side payment of $5/2$ from II to I.

3. (a) The cooperative strategy is ((1,0,0,0),(0,0,1,0)) with sum $\sigma = 6$. The difference matrix is

$$\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} 1 & 3 & -6 & 0 \\ 2 & 0 & -3 & 3 \\ 5 & -6 & 0 & 4 \\ -7 & 7 & -1 & -4 \end{pmatrix}$$

By the matrix game solver, the value is $\delta = -3/7$ and the threat strategies are $\boldsymbol{p}^* = (0, 0, 4/7, 3/7)$ and $\boldsymbol{q}^* = (0, 1/14, 13/14, 0)$. So the TU-solution is $\boldsymbol{\varphi} = ((\sigma + \delta)/2, (\sigma - \delta)/2) = (3 - \frac{3}{14}, 3 + \frac{3}{14})$. The disagreement point is $(-27/98, 15/98)$. The cooperative strategy gives payoff $(0, 6)$ so this requires a side payment of $2 + \frac{11}{14}$ from II to I.

(b) We have $\sigma = 2$. One cooperative strategy is $((0,0,0,0,1),(1,0,0,0))$. The difference matrix is

$$\begin{pmatrix} 0 & 2 & 0 & -1 \\ -2 & -1 & 1 & 0 \\ 1 & 0 & -1 & -2 \\ 2 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There is a saddle point at the lower right corner. The value is $\delta = 0$. The TU-solution is $(1,1)$. The threat strategies are $(0,0,0,0,1)$ and $(0,0,0,1)$. The disagreement point is $(0,0)$. There is no side payment.

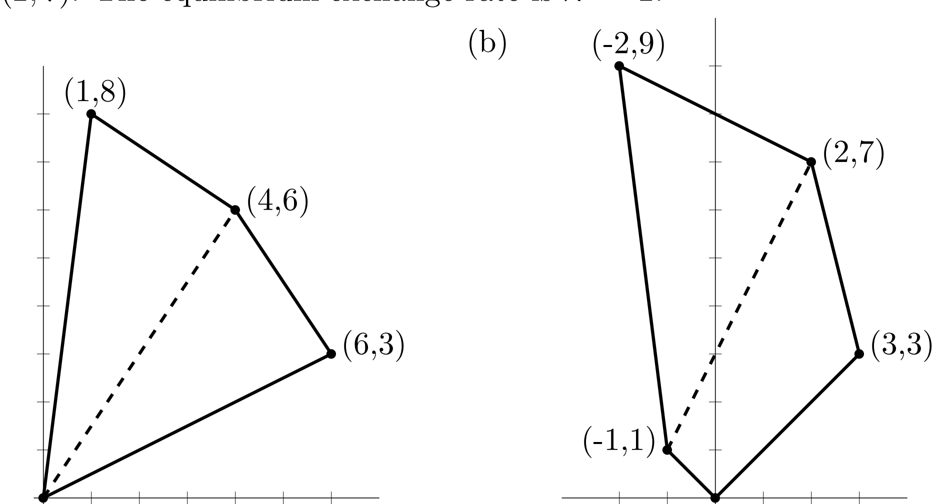
4. (a) The set of Pareto optimal points is the parabolic arc, $y = 4 - x^2$, from $x = 0$ to $x = 2$. We seek the point on this arc that maximizes the product $xy = x(4-x^2) = 4x - x^3$. Setting the derivative with respect to x to zero gives $4 - 3x^2 = 0$, or $x = 2/\sqrt{3}$. The corresponding value of y is $y = 4 - (4/3) = 8/3$. Hence the NTU solution is $(\bar{u}, \bar{v}) = (2/\sqrt{3}, 8/3)$.

(b) This time we seek the Pareto optimal point that maximizes the product $x(y-1) = x(3 - x^2) = 3x - x^3$. Setting the derivative with respect to x to zero gives $3 - 3x^2 = 0$, or $x = 1$. The corresponding value of y is $y = 4 - 1 = 3$. Hence the NTU solution is $(\bar{u}, \bar{v}) = (1, 3)$.

5. (a) The fixed threat point is $(u^*, v^*) = (0, 0)$. The set of Pareto optimal points consists of the two line segments, from (1,8) to (4,6) and from (4,6) to (6,3). The first has slope, $-2/3$, and the second has slope, $-3/2$. The slope of the line from (0,0) to (4,6) is $3/2$, exactly the negative of the slope of the second line. Thus, $(\bar{u}, \bar{v}) = (4, 6)$ is the NTU-solution. The equilibrium exchange rate is $\lambda^* = 3/2$.

(b) Both matrices, A and B, have saddle-points at the first row, second column. Therefore, the fixed threat point is $(u^*, v^*) = (-1, 1)$. The set of Pareto optimal points consists of the two line segments, from $(-2, 9)$ to (2,7) and from (2,7) to (3,3). The first has slope, $-1/2$, and the second has slope, -4. The slope of the line from $(-1, 1)$ to (2,7) is 2. Since this is between the negatives of the two neighboring slopes, the NTU-solution is $(\bar{u}, \bar{v}) = (2, 7)$. The equilibrium exchange rate is $\lambda^* = 2$.

(a) (b)



6. (a) Clearly, the NTU-solution must be on the line joining $(1,4)$ and $(5,2)$. For the TU-solution to be equal to the NTU-solution, we suspect that the slope of the λ-transformed line, from $(\lambda,4)$ to $(5\lambda,2)$, would be equal to -1. Since the slope of this line is $-2/(4\lambda)$, we have $\lambda^*=1/2$, in which case the game matrix becomes $\begin{pmatrix}(5/2,2) & (0,0)\\ (0,0) & (1/2,4)\end{pmatrix}$. This gives $\sigma=9/2$ and $\delta=0$, so that the TU-solution is $(9/4,9/4)$. The NTU-solution of the original matrix is obtained from this by dividing the first coordinate by λ^*, so that $\boldsymbol{\varphi}=(9/2,9/4)$.

(b) The lambda-transfer matrix is $\begin{pmatrix}(3\lambda,2) & (0,5)\\ (2\lambda,1) & (\lambda,0)\end{pmatrix}$. We see

$$\sigma(\lambda)=\begin{cases}5 & \text{if } \lambda\le 1\\ 3\lambda+2 & \text{if } \lambda\ge 1\end{cases}.$$

The difference matrix is

$$\lambda\boldsymbol{A}-\boldsymbol{B}=\begin{pmatrix}3\lambda-2 & -5\\ 2\lambda-1 & \lambda\end{pmatrix}.$$

This matrix has a saddle point no matter what be the value of $\lambda>0$. If $0<\lambda\le 1$, there is a saddle at $\langle 2,1\rangle$. If $\lambda\ge 1$, there is a saddle at $\langle 2,2\rangle$. Thus,

$$\delta(\lambda)=\begin{cases}2\lambda-1 & \text{if } 0<\lambda\le 1\\ \lambda & \text{if } \lambda\ge 1.\end{cases}$$

From (10),

$$\boldsymbol{\varphi}(\lambda)=(\frac{\sigma(\lambda)+\delta(\lambda)}{2\lambda},\frac{\sigma(\lambda)-\delta(\lambda)}{2}).$$

For $\lambda=1$, we find $\boldsymbol{\varphi}(\lambda)=(3,2)$. This is obviously feasible since it is the upper left entry of the originalbimatrix. So the NTU-solution is (3,2), and $\lambda^*=1$ is the equilibrium exchange rate.

7. (a) If Player II uses column 2, Player I is indifferent as to what he plays. If I uses $(1-p,p)$, Player II prefers column 2 to column 1 if $89(1-p)+98p\le 90(1-p)$, that is if $99p\le 1$. I's safety level is Val $\begin{pmatrix}11 & 10\\ 2 & 10\end{pmatrix}=10$, and II's safety level is Val $\begin{pmatrix}89 & 98\\ 90 & 0\end{pmatrix}=89+\frac{1}{11}$. At the equilibrium with $p=1/99$, the payoff vector is $(10,90(98/99))=(10,89+\frac{1}{11})$. Thus both players only get their safety levels.

(b) Working together, Player I and II can achieve $\sigma=100$. The difference matrix, $\boldsymbol{D}=\boldsymbol{A}-\boldsymbol{B}$, has value

$$\delta=\text{Val}\begin{pmatrix}-78 & -80\\ -96 & 10\end{pmatrix}=\frac{-78\cdot 10-80\cdot 98}{10-78+80+96}=-\frac{235}{3}=-78\frac{1}{3}.$$

Therefore, the TU solution is $\boldsymbol{\varphi}=((\sigma+\delta)/2,(\sigma-\delta)/2)=(10\frac{5}{6},89\frac{1}{6})$. This is on the line segment joining the top two payoff vectors of the game matrix, and so is in the NTU feasible set. Player I's threat strategy is $(106/108, 2/108) = (53/54, 1/54)$. Player II's threat strategy is $(90/108, 18/108) = (5/6, 1/6)$.

Part of the reason Player I is so strong in this game is that even if Player I carries out his threat strategy, $(53/54, 1/54)$, the best Player II can do against it is to play column 1, when the payoff to the players is $((10\frac{5}{6}, 89\frac{1}{6}, 0)$, the same as given by the NTU solution.

Solutions to Exercises of Section IV.1.

1. We find $v(\{1,2\}) = -v(\{3\}) = 4.4$, $v(\{1,3\}) = -v(\{2\}) = 4$, $v(\{2.3\}) = -v(\{1\}) = 1.5$, and $v(\emptyset) = v(N) = 0$.

$$
\text{(I,II)}: \begin{array}{c|cc|} & \multicolumn{2}{c}{\text{III}} \\ & 1 & 2 \\ \hline 1,1 & -1 & -3 \\ 1,2 & 4 & 5 \\ 2,1 & -3 & -2 \\ 2,2 & 6 & 2 \\ \hline \end{array}
\qquad
\text{(I,III)}: \begin{array}{c|cc|} & \multicolumn{2}{c}{\text{II}} \\ & 1 & 2 \\ \hline 1,1 & -1 & -3 \\ 1,2 & 4 & 5 \\ 2,1 & 2 & 6 \\ 2,2 & -2 & -3 \\ \hline \end{array}
\qquad
\text{(II,III)}: \begin{array}{c|cc|} & \multicolumn{2}{c}{\text{I}} \\ & 1 & 2 \\ \hline 1,1 & 2 & 1 \\ 1,2 & -1 & -12 \\ 2,1 & -1 & 4 \\ 2,2 & -10 & 1 \\ \hline \end{array}
$$

2. $v(\emptyset) = 0$, $v(1) = 6/10$, $v(2) = 2$, $v(3) = 1$, $v(12) = 5$, $v(13) = 4$, $v(23) = 3$ and $v(123) = 16$.

$$
\text{I:} \begin{array}{c|cccc|} & \multicolumn{4}{c}{\text{(II,III)}} \\ & 1,1 & 1,2 & 2,1 & 2,2 \\ \hline 1 & 1 & 3 & -1 & 3 \\ 2 & -1 & 1 & 7 & 3 \\ \hline \end{array}
$$

$$
\text{II:} \begin{array}{c|cccc|} & \multicolumn{4}{c}{\text{(I,III)}} \\ & 1,1 & 1,2 & 2,1 & 2,2 \\ \hline 1 & 2 & 0 & 2 & 0 \\ 2 & 6 & 2 & 5 & 2 \\ \hline \end{array}
$$

$$
\text{III:} \begin{array}{c|cccc|} & \multicolumn{4}{c}{\text{(I,II)}} \\ & 1,1 & 1,2 & 2,1 & 2,2 \\ \hline 1 & 1 & -3 & 4 & 4 \\ 2 & 1 & 1 & 3 & 1 \\ \hline \end{array}
$$

$$
\text{(I,II)}: \begin{array}{c|cc|} & \multicolumn{2}{c}{\text{III}} \\ & 1 & 2 \\ \hline 1,1 & 3 & 3 \\ 1,2 & 5 & 5 \\ 2,1 & 1 & 1 \\ 2,2 & 12 & 5 \\ \hline \end{array}
\qquad
\text{(I,III)}: \begin{array}{c|cc|} & \multicolumn{2}{c}{\text{II}} \\ & 1 & 2 \\ \hline 1,1 & 2 & -4 \\ 1,2 & 4 & 4 \\ 2,1 & 3 & 11 \\ 2,2 & 4 & 4 \\ \hline \end{array}
\qquad
\text{(II,III)}: \begin{array}{c|cc|} & \multicolumn{2}{c}{\text{I}} \\ & 1 & 2 \\ \hline 1,1 & 3 & 6 \\ 1,2 & 1 & 3 \\ 2,1 & 3 & 9 \\ 2,2 & 3 & 3 \\ \hline \end{array}
$$

3. (a) $v(\emptyset) = 0$, $v(1) = 2$, $v(2) = 2$ and $v(12) = 9$.

(b) Player 2's threat strategy and MM (safety level) strategy are the same, column 2. Player 1's threat strategy is row 1, while his MM strategy is row 2. His threat is not believable, whereas 2's threat is very believable. In addition, (row 2, col 1) is a PSE.

(c) $\sigma = 9$ and $\Delta = \begin{pmatrix} -2 & 3 \\ -2 & 1 \end{pmatrix}$, so $\delta = -2$. The TU-value is $\phi = ((9-2)/2, (7-2)/2) = (7/2, 11/2)$. Player 1 gets 5, but has to make a side payment of 3/2 to Player 2.

(d) The strategy spaces are taken to be $X_1 = \{\{1\},\{1,2\}\}$ and $X_2 = \{\{2\},\{1,2\}\}$. The bimatrix therefore is 2 by 2:

$$\begin{array}{cc} & \begin{array}{cc} \{2\} & \{1,2\} \end{array} \\ \begin{array}{c} \{1\} \\ \{1,2\} \end{array} & \begin{pmatrix} (2,2) & (2,2) \\ (2,2) & (9/2,9/2) \end{pmatrix} \end{array}$$

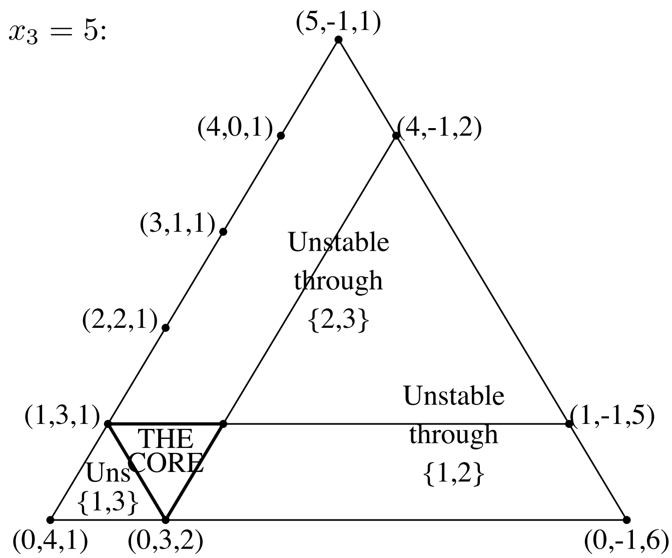
4. $N = \{1,2,3\}$. $v(\emptyset) = 0$. $v(\{1\}) = 1$, because Player 2 can always choose j within 1 of i. Similarly, $v(\{2\}) = 1$ because Player 1 can choose $i = 5$ say, and then Player 3 can choose k within 1 of j. $v(\{3\}) = 4$ is achieved by choosing $i = 5$ and $j = 10$, say. Similarly, $v(\{1,2\}) = 10$, $v(\{1,3\}) = 10$, $v(\{2,3\}) = 14$, and $v(N) = 18$.

Solutions to Exercises of Section IV.2.

1. A constant-sum game has $v(S) + v(N - S) = v(N)$ for all coalitions S. Therefore, a two-person constant-sum game has $v(\{1\}) + v(\{2\}) = v(\{1, 2\})$ and so is inessential.

2. We have $v(\{1\}) = \mathrm{Val}\begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = 8/5$, $v(\{2\}) = \mathrm{Val}\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix} = 8/5$, and $v(\{1, 2\}) = 6$. The set of imputations is $\{(x_1, x_2) : x_1 + x_2 = 6, x_1 \geq 8/5, x_2 \geq 8/5\}$, the line segment from $(8/5, 22/5)$ to $(22/5, 8/5)$. The core is the same set. In fact, for all 2-person games, the core is always the whole set of imputations.

3. On the plane $x_1 + x_2 + x_3 = 5$:



4. (a) The set of imputations is $\{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 3, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$. The core is

$$\{(x_1, x_2, x_3) : x_1+x_2+x_3 = 3,\ x_1+x_2 \geq a,\ x_1+x_3 \geq a,\ x_2+x_3 \geq a,\ \ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}.$$

For values of $a \leq 2$, the point $(1, 1, 1)$ is in the core. For values of $a > 2$, the core is empty, since summing the corresponding sides of the inequalities $x_1 + x_2 \geq a$, $x_1 + x_3 \geq a$, and $x_2 + x_3 \geq a$, gives $6 = 2(x_1 + x_2 + x_3) \geq 3a > 6$, a contradiction.

(b) The core consists of points (x_1, x_2, x_3, x_4) such that

$$\begin{array}{lllll}
x_1 \geq 0 & x_1 + x_2 \geq a & x_2 + x_3 \geq a & x_1 + x_2 + x_3 \geq b & x_1 + x_2 + x_3 + x_4 = 4. \\
x_2 \geq 0 & x_1 + x_3 \geq a & x_2 + x_4 \geq a & x_1 + x_2 + x_4 \geq b & \\
x_3 \geq 0 & x_1 + x_4 \geq a & x_3 + x_4 \geq a & x_1 + x_3 + x_4 \geq b & \\
x_4 \geq 0 & & & x_2 + x_3 + x_4 \geq b &
\end{array}$$

If $a \leq 2$ and $b \leq 3$, then $(1, 1, 1, 1)$ is in the core.
If $a > 2$, summing the inequalities involving a gives $12 = 3(x_1 + x_2 + x_3 + x_4) \geq 6a > 12$,

so the core is empty.
If $b > 3$, summing the inequalities involving b gives $12 = 3(x_1 + x_2 + x_3 + x_4) \geq 4b > 12$, so the core is empty.
Thus the core is non-empty if and only if $a \leq 2$ and $b \leq 3$. If v is superadditive, then a cannot be greater than two, so the condition reduces to $b \leq 3$.

(c) The core is nonempty if and only if $f(k)/k \leq f(n)/n$ for all $k = 1, \ldots, n$. To see this, note that if $f(k)/k \leq f(n)/n$ for all $k = 1, \ldots, n$, then $\boldsymbol{x} = (f(n)/n, \ldots, f(n)/n)$ is in the core since for any coalition S of size $|S| = k$, we have $\sum_{i \in S} x_i = kf(n)/n \leq f(k) = v(S)$. On the other hand, suppose that $f(k)/k > f(n)/n$ for some k. Then for any imputation $\boldsymbol{x}$, the coalition S consisting of those players with the k smallest x_i satisfies $(1/k)\sum_{i \in S} x_i \leq (1/n)\sum_{i \in N} x_i = f(n)/n < f(k)/k$. This means that $\boldsymbol{x}$ is unstable through S, and so cannot be in the core.

5. Suppose $\boldsymbol{x} = (x_1, \ldots, x_n)$ is in the core. For $\boldsymbol{x}$ to be stable, we must have $\sum_{j \neq i} x_j \geq v(N - \{i\})$ for all i. This is equivalent to $x_i \leq \delta_i$ for all i. However, summing over i gives $v(N) = \sum_1^n x_i \leq \sum_1^n \delta_i < v(N)$, a contradiction. Thus, the core must be empty.

6. Suppose Player 1 is a dummy. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ is in the core, it is an imputation, so $x_1 \geq v(\{1\}) = 0$. Suppose $x_1 > 0$. Then, $v(N - \{x_1\}) = v(N) = \sum_1^n x_i > \sum_2^n x_i$. So $\boldsymbol{x}$ is unstable through $N - \{x_1\}$ and cannot be in the core.

7.(a) If Players 1 and 2 are in P, and Players 3 and 4 are in Q, then $v(i) = 0$ for all i, $v(ij) = 1$ for all ij except $ij = 12$ and $ij = 34$, $v(ijk) = 1$ and $v(1234) = 2$. The core, where all coalitions are satisfied, is

$$\begin{aligned} C = \{(x_1, x_2, x_3, x_4) :& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_1 + x_2 + x_3 + x_4 = 2, \\ & x_1 + x_3 \geq 1, x_1 + x_4 \geq 1, x_2 + x_3 \geq 1, x_2 + x_4 \geq 1\} \end{aligned}$$

Since $x_1 + x_3 \geq 1$, $x_2 + x_4 \geq 1$ and $x_1 + x_2 + x_3 + x_4 = 2$, we must have $x_1 + x_3 = 1$, $x_2 + x_4 = 1$. Similarly, we have $x_1 + x_4 = 1$, $x_2 + x_3 = 1$. The core is therefore $C = \{(x_1, x_1, 1 - x_1, 1 - x_1) : 0 \leq x_1 \leq 1$. This is the line segment joining the points $(0, 0, 1, 1)$ and $(1, 1, 0, 0)$.

(b) If Players 1 and 2 are in P, and Players 3, 4 and 5 are in Q, then the core is

$$\begin{aligned} C = \{(x_1, x_2, x_3, x_4, x_5) :\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, \\ & x_1 + x_3 \geq 1, x_1 + x_4 \geq 1, x_1 + x_5 \geq 1, x_2 + x_3 \geq 1, x_2 + x_4 \geq 1, \\ & x_2 + x_5 \geq 1, x_1 + x_2 + x_3 + x_4 \geq 2, x_1 + x_2 + x_3 + x_5 \geq 2, \\ & x_1 + x_2 + x_4 + x_5 \geq 2, x_1 + x_2 + x_3 + x_4 + x_5 = 2.\} \end{aligned}$$

These inequalities imply that $x_3 = x_4 = x_5 = 0$, and therefore that $x_1 = x_2 = 1$. The core consists of the single point, $C = \{(1, 1, 0, 0, 0)\}$.

(c) If $|P| < |Q|$, the core is the imputation, $\boldsymbol{x}_P$, with $x_i = 1$ for $i \in P$ and $x_i = 0$ for $i \in Q$. If $|P| > |Q|$, the core is the imputation, $\boldsymbol{x}_Q$, with $x_i = 1$ for $i \in Q$ and $x_i = 0$ for $i \in P$. If $|P| = |Q|$, the core is the line segment joining $\boldsymbol{x}_P$ and $\boldsymbol{x}_Q$.

8. In the core, we have $x_i + x_k \geq 1$ for $i = 1, 2$ and $k = 3, 4, 5$. Also we have $x_i + x_j + x_k \geq 2$ for $i = 1, 2$ and $j, k = 3, 4, 5$. Finally, we have $x_1 + x_2 + x_3 + x_4 + x_5 = 3$.

Since, $x_1 + x_3 \geq 1$ and $x_2 + x_4 + x_5 \geq 2$, and $x_1 + x_2 + x_3 + x_4 + x_5 = 3$, we must have equality: $x_1 + x_3 = 1$ and $x_2 + x_4 + x_5 = 2$. Similarly, $x_1 + x_3 + x_4 = 2$, which with $x_1 + x_3 = 1$ implies $x_4 = 1$, etc. The core consists of the single point, $(0, 0, 1, 1, 1)$.

Solutions to Exercises of Section IV.3.

1. No player can get anything acting alone, so $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$. Players 2 and 3 can do nothing together, $v(\{2,3\}) = 0$, but $v(\{1,2\}) = 30$ and $v(\{1,3\}) = 40$. The object is also worth 40 to the grand coalition, $v(\{1,2,3\}) = 40$. (Player 3 will take the object and replace it by \$40, and the players must now decide how to split this money.) To find the Shapley value using Equation (4), we first find $c_\emptyset = c_{\{1\}} = c_{\{2\}} = c_{\{3\}} = c_{\{2,3\}} = 0$, and $c_{\{1,2\}} = 30$, $c_{\{1,3\}} = 40$ and $c_{\{1,2,3\}} = v(\{1,2,3\})-$ the sum of the above, so $c_{\{1,2,3\}} = 40 - 40 - 30 = -30$. Thus we have $v = 30w_{\{1,2\}} + 40w_{\{1,3\}} - 30w_{\{1,2,3\}}$, and consequently, $\phi(v) = 30\phi(w_{\{1,2\}}) + 40\phi(w_{\{1,3\}}) - 30\phi(w_{\{1,2,3\}})$.

From this we may compute

$$\begin{aligned}
\phi_1(v) &= 30/2 + 40/2 - 30/3 = 25\\
\phi_2(v) &= 30/2 + 0 - 30/3 = 5\\
\phi_3(v) &= 0 + 40/2 - 30/3 = 10.
\end{aligned}$$

So 3 gets the painting for \$30, of which \$25 goes to 1 and \$5 to 2. The core is

$$\begin{aligned}
C = \{(x_1,x_2,x_3) : x_1 \ge 30, x_2 \ge 0, x_3 \ge 0, x_1 + x_2 \ge 30,&\\
x_1 + x_3 \ge 40, x_2 + x_3 \ge 0, x_1 + x_2 + x_3 = 40\}&\\
= \{(x_1,x_2,x_3) : x_2 = 0, 30 \le x_1 \le 40, x_3 = 40 - x_1\}.&
\end{aligned}$$

The core gives player 2 nothing, while the Shapley value gives him 5. Without 2 present, 1 and 3 would probably agree on a price of 20. With 2 present, 1 is in a better bargaining position as he can play 3 off against 2.

2. We compute the Shapley value using Theorem 2. $\phi_1(v) = (1/3)\cdot 1 + (1/6)\cdot 2 + (1/6)\cdot 3 + (1/3)\cdot 3 = 13/6$. $\phi_2(v) = (1/3)\cdot 0 + (1/6)\cdot 1 + (1/6)\cdot 7 + (1/3)\cdot 7 = 22/6 = 11/3$. $\phi_3(v) = (1/3)\cdot(-4) + (1/6)\cdot(-2) + (1/6)\cdot 3 + (1/3)\cdot 4 = 1/6$.

3. $\sum_{j\in N}\phi_j(v) = v(N)$ from Axiom 1. We must show $\phi_i(v) \ge v(\{i\})$ for all $i \in N$. Since v is superadditive, $v(\{i\}) + v(S - \{i\}) \le v(S)$ for all S containing i. But since $\phi_i(v)$ is an average of numbers, $v(S) - v(S - \{i\})$, each of which is at least $v(\{i\})$, $\phi_i(v)$ itself must be at least $v(\{i\})$.

4.(a) By the symmetry axiom, $\phi_2(v) = \phi_3(v) = \cdots = \phi_n(v)$. If $1 \in S$, then $v(S) - v(S - \{i\}) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$. Therefore $\phi_2(v)$ is just the probability that 1 comes before 2 in the random ordering of the players. This probability is 1/2 by symmetry. So $\phi_2(v) = \phi_3(v) = \cdots = \phi_n(v) = 1/2$, and $\phi_1(v) = n - \phi_2(v) - \phi_3(v) - \cdots - \phi_n(v) = (n+1)/2$.

One may also use Equation (4) to compute the Shapley value. First show $c_{\{1\}} = c_{\{1,j\}} = 1$ for all $j = 2,\ldots,n$, and all other $c_S = 0$. Hence, $v = w_{\{1\}} + \sum_{j=2}^n w_{\{1,j\}}$, from which follows $\phi_1(v) = 1 + (n-1)/2 = (n+1)/2$ and $\phi_j(v) = 1/2$ for $j = 2,\ldots,n$.

(b) By the symmetry axiom, $\phi_1(v) = \phi_2(v)$ and $\phi_3(v) = \cdots = \phi_n(v)$. This time $\phi_3(v)$ is the probability that 1 or 2 is chosen before 3 in the random ordering. This is 1 minus the probability that 3 is chosen before 1 and 2, namely, $1 - (1/3) = 2/3$. So $\phi_3(v) = \cdots = \phi_n(v) = 2/3$ and $\phi_1(v) = \phi_2(v) = (1/2)(n - (n-2)(2/3)) = (n+4)/6$.

(c) By the symmetry axiom, $\phi_1 = \phi_2$ and $\phi_3 = \cdots = \phi_n$. If $3 \in S$, then $v(S) - v(S - \{3\})$ is 1 if 1 and 2 are in S and 0 otherwise. This implies that ϕ_3 is just the probability that 3 enters after both 1 and 2. Since each of 1, 2 and 3 have the same probability of entering after the other two, $\phi_3 = 1/3$. Then since $2\phi_1 + (n-2)\phi_3 = n$, we have $\phi_1 = \phi_2 = (n+1)/3$, and $\phi_3 = \cdots = \phi_n = 1/3$.

5 The answer is no. Here is a counterexample with $n = 4$ players. The minimal winning coalitions are $\{1,2\}$ and $\{3,4\}$. To be a weighted voting game, with weights w_i and quota q, we must have $w_1 + w_2 > q$ and $w_3 + w_4 > q$, so that $w_1 + w_2 + w_3 + w_4 > 2q$. On the other hand, $\{1,3\}$ and $\{2,4\}$ are losing coalitions so that $w_1 + w_3 \le q$ and $w_2 + w_4 \le q$. This gives $q_1 + q_2 + q_3 + q_4 \le 2q$, a contradiction.

6. The winning coalitions are $\{2,3\}$, $\{2,4\}$, $\{3,4\}$ and all supersets. It appears that 1 is a dummy, so $\phi_1(v) = 0$. Also, $\phi_2(v) = \phi_3(v) = \phi_4(v)$ from symmetry. Since the sum of the $\phi_i(v)$ must be 1, we have $\phi(v) = (0, 1/3, 1/3, 1/3)$.

7. Player 2 can be in only two winning coalitions that would be losing without him, namely, $S = \{1,2\}$ and $S = \{2,3,\ldots,n\}$. Hence,

$$\phi_2(v) = \frac{(2-1)!\,(n-2)!}{n!} + \frac{(n-2)!\,(n-(n-1))!}{n!} = \frac{2(n-2)!}{n!} = \frac{2}{n(n-1)}.$$

By symmetry,

$$\phi_3(v) = \cdots = \phi_n(v) = \frac{2}{n(n-1)}$$

and

$$\phi_1(v) = 1 - \phi_2(v) - \cdots - \phi_n(v) = 1 - (n-1)\frac{2}{n(n-1)} = \frac{n-2}{n}.$$

8. Let 1,2,3,4 denote the stockholders and let c denote the chairman of the board. The coalitions winning with 1 but losing without 1 are $\{1,4,c\}$ and $\{1,2,3\}$. So

$$\phi_1(v) = \frac{2!\,2!}{5!} + \frac{2!\,2!}{5!} = \frac{1}{30} + \frac{1}{30} = \frac{2}{30}.$$

The corresponding coalitions for 2 are $\{2,3,c\}$, $\{2,4\}$, $\{1,2,3\}$, $\{1,2,3,c\}$, $\{2,4,c\}$, and $\{1,2,4\}$. So $\phi_2(v) = 4(1/30) + 2(1/20) = 7/30$. Similarly, $\phi_3(v) = 7/30$, $\phi_4(v) = 12/30$, and $\phi_c(v) = 2/30$. The Shapley value is $(2/30, 7/30, 7/30, 12/30, 2/30)$, or, in terms of percentage power $(6.7\%, 23.3\%, 23.3\%, 40\%, 6.7\%)$.

9. (a) Let 1 denote the large party and 2, 3, 4 denote the smaller parties. Then the only winning coalitions that become losing without 2 are $S = \{1, 2\}$ and $S = \{2, 3, 4\}$. Hence,

$$\phi_2(v) = \frac{1!\,2!}{4!} + \frac{2!\,1!}{4!} = \frac{1}{6}.$$

By symmetry, $\phi_3(v) = \phi_4(v) = 1/6$ also, and hence $\phi_1(v) = 1/2$. The large party has half the power.

(b) Let 1, 2 denote the large parties and 3, 4, 5 denote the smaller ones. The only coalitions winning with 3 and losing without 3 are $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{2, 3, 4\}$, and $\{2, 3, 5\}$. Hence,

$$\phi_3(v) = 4\frac{2!\,2!}{5!} = \frac{2}{15}.$$

By symmetry, $\phi_4(v) = \phi_5(v) = 2/15$, and hence $\phi_1(v) = \phi_2(v) = 3/10$. The two large coalitions are less powerful than their size indicates.

10. There are three coalitions that are winning with 6 but losing without 6: $\{1, 2, 6\}$, $\{1, 3, 5, 6\}$, $\{2, 3, 5, 6\}$. Hence,

$$\phi_6(v) = \frac{2!\,3!}{6!} + 2\frac{3!\,2!}{6!} = \frac{3}{60}.$$

By symmetry, $\phi_5(v) = 3/60$. There are 7 coalitions winning with 4 but losing without 4: $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 3, 4, 5\}$, $\{1, 3, 4, 6\}$, $\{2, 3, 4, 5\}$, $\{2, 3, 4, 6\}$. Hence, $\phi_4(v) = 7/60$. There are 11 coalitions winning with 3 but losing without 3: $\{1, 2, 3\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 3, 4, 5\}$, $\{1, 3, 4, 6\}$, $\{1, 3, 5, 6\}$, $\{2, 3, 4, 5\}$, $\{2, 3, 4, 6\}$, $\{2, 3, 5, 6\}$, $\{1, 3, 4, 5, 6\}$, $\{2, 3, 4, 5, 6\}$. Hence,

$$\phi_3(v) = 3\frac{2!\,3!}{6!} + 6\frac{3!\,2!}{6!} + 2\frac{4!\,1!}{6!} = \frac{13}{60}.$$

Since $\phi_1(v) = \phi_2(v)$ by symmetry, we find that $\phi_1(v) = \phi_2(v) = 17/60$. The Shapley value is $(17/60, 17/60, 13/60, 7/60, 3/60, 3/60)$. In terms of percentage power, this is $(28.3\%, 28.3\%, 21.7\%, 11.7\%, 5\%, 5\%)$, which is much closer to the original intention found in Table 1.

11. Let $\phi_A(v)$ denote the Shapley value for A, of one of the big five, and let $\phi_a(v)$ denote the Shapley value of a, one of the smaller members. We must have $5\phi_A(v) + 10\phi_a(v) = 1$. Let us find $\phi_a(v)$. The only losing coalitions that become winning, when a is added to it, are the coalitions consisting of the big five and three of the other smaller nations. Thus in the random ordering, a must come in ninth and find all members of the big five there already. The number of such coalitions is the number of ways of choosing the three smaller nation members out of the remaining nine, namely $\binom{9}{3} = 9!/3!\,6!$. Thus

$$\phi_a(v) = \frac{8!\,6!}{15!} \cdot \frac{9!}{3!\,6!} = \frac{4}{15 \cdot 13 \cdot 11} = .001865\cdots.$$

Thus the 10 smaller nations have only 1.865% of the power, and each of the big five nations has 19.627% of the power.

12. The trip to A and return costs 14, so the value of A acting alone is $v(A) = 20 - 14 = 6$. Similarly, $v(B) = 20 - 16 = 4$, and $v(C) = 20 - 12 = 8$. If A and B combine forces, the travel cost is 17, the cost of the trip to A then to B and return. So, $v(AB) = 40 - 17 = 23$. Similarly, $v(AC) = 40 - 17 = 23$ and $v(BC) = 40 - 18 = 22$. If all three cities combine, the least travel cost is obtained using the route from H to A to B to C and return (or the reverse), for a total cost of 19. So $v(ABC) = 60 - 19 = 41$. From these, we may compute the Shapley value as

$$\begin{aligned}
\phi_A(v) &= \frac{1}{3}\cdot 6 + \frac{1}{6}\cdot 19 + \frac{1}{6}\cdot 15 + \frac{1}{3}\cdot 19 = 14 \\
\phi_B(v) &= \frac{1}{3}\cdot 4 + \frac{1}{6}\cdot 17 + \frac{1}{6}\cdot 14 + \frac{1}{3}\cdot 18 = 12.5 \\
\phi_C(v) &= \frac{1}{3}\cdot 8 + \frac{1}{6}\cdot 17 + \frac{1}{6}\cdot 18 + \frac{1}{3}\cdot 18 = 14.5
\end{aligned}$$

Thus, we require A to pay $20-14 = 6$, B to pay $20-12.5 = 7.5$, and C to pay $20-14.5 = 5.5$ for a total of 19.

13. Consider a permutation of the n players, $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, and let π' denote the reverse permutation, $\pi' = (\pi_n, \ldots, \pi_2, \pi_1)$. Consider player i and let z_i denote the sum of the contributions player i makes to the coalitions when he enters for these two permutations. Below, it is shown that $z_i = x_i$ if $i \in B$ and $z_i = y_i$ if $i \in C$. If so, then the Shapley value for player i in this game is $x_i/2$ if $i \in B$ and $y_i/2$ if $i \in C$.

Let S_0 denote the coalition player i finds upon entering, when the players enter in the order given by π, and let $S = S_0 \cup \{i\}$. Then if the players enter in the reverse order π', player i will find coalition $\bar{S}$ (the complement of S) there when he enters and he increases it to $\bar{S}_0$. The amount player i contributes to the grand coalition is $v(S) - v(S_0)$ when entering in the order given by π and $v(\bar{S}_0) - v(\bar{S})$ when entering in the order given by π. The sum is therefore

$$z_i = v(S) - v(S_0) + v(\bar{S}_0) - v(\bar{S}) = [v(S) - v(\bar{S})] - [v(\bar{S}_0) - v(S_0)]$$

Letting T denote the common value $T = \sum_B x_j = \sum_C y_j = v(N)$, we find

$$\begin{aligned}
v(S) - v(\bar{S}) &= \min\{\sum_{j\in S\cap B} x_j, \sum_{k\in S\cap C} y_k\} - \min\{\sum_{j\in \bar{S}\cap B} x_j, \sum_{k\in \bar{S}\cap C} y_k\} \\
&= \min\{\sum_{j\in S\cap B} x_j, \sum_{k\in S\cap C} y_k\} - \min\{T - \sum_{j\in S\cap B} x_j, T - \sum_{k\in S\cap C} y_k\} \\
&= \min\{\sum_{j\in S\cap B} x_j, \sum_{k\in S\cap C} y_k\} - T + \max\{\sum_{j\in S\cap B} x_j, \sum_{k\in S\cap C} y_k\} \\
&= \sum_{j\in S\cap B} x_j + \sum_{k\in S\cap C} y_k - T.
\end{aligned}$$

Similarly,

$$v(\bar{S}_0) - v(S_0) = \sum_{j\in\bar{S}_0\cap B} x_j + \sum_{k\in\bar{S}_0\cap C} y_k - T.$$

Since S plus $\bar{S}_0$ is the set of all players but including player i twice, the sum of the two previous displays is equal to $T + x_i + T - 2T = x_i$ if $i \in B$, and $T + T + y_i - 2T = y_i$ if $i \in C$.

14. (a) We compute the Shapley value, ϕ, by the method of Theorem 1. We have $c_\emptyset = 0$. For singleton coalitions, we have $c_{\{1\}} = 1$ and $c_{\{j\}} = 0$ for $j \neq 1$. For coalitions of two players, we have $c_{\{1,2\}} = 2 - 1 = 1$ and $c_{\{i,j\}} = 0$ for all other $i < j$. Continuing in the same way we find

$$c_{\{12\cdots k\}} = 1 \quad \text{for } k = 1, \ldots, n, \quad \text{and} \quad c_S = 0 \quad \text{for all other coalitions, } S.$$

We can check this by checking that $v(T) = \sum_S c_S w_S(T) = \sum_{k=1}^n w_{\{1,2,\ldots,k\}}(T)$. From this, we may conclude that $\phi_i(v) = \sum_{S \text{ containing } i} c_S/|S|$, or

$$\begin{aligned}
\phi_1(v) &= 1 + (1/2) + (1/3) + \cdots + (1/n) \\
\phi_2(v) &= \phantom{1 +{}} (1/2) + (1/3) + \cdots + (1/n) \\
\phi_3(v) &= \phantom{1 + (1/2) +{}} (1/3) + \cdots + (1/n) \\
&\cdots \\
\phi_n(v) &= \phantom{1 + (1/2) + (1/3) + \cdots +{}} (1/n)
\end{aligned}$$

(b) Similarly, $c_{\{12\cdots k\}} = a_k - a_{k-1}$ for $k = 1, \ldots, n$, and $c_S = 0$ for all other coalitions, S, where $a_0 = 0$. Then, $\phi_i(v) = \sum_i^n (a_j - a_{j-1})/j$ for $i = 1, \ldots, n$.

15.(a) Let S be an arbitrary coalition and let $m = \max\{i : i \in S\}$. Then,

$$v_k(S) = \begin{cases} -(c_k - c_{k-1}) & \text{if } k \leq m \\ 0 & \text{if } k > m \end{cases}$$

So $\sum_{k=1}^n v_k(S) = \sum_{k=1}^m -(c_k - c_{k-1}) = -c_m = v(S)$.

(b) Since the Shapley value is additive, $\phi_i(v) = \sum_{k=1}^n \phi_i(v_k)$. To compute $\phi_i(v_k)$, note that $\phi_i(v_k) = 0$ if $i < k$ and $\phi_i(v_k) = -(c_k - c_{k-1})P(i,k)$ for $i \geq k$, where $P(i,k)$ represents the probability that in a random ordering of the players into the grand coalition, player i is the first member of S_k to appear. $P(i,k)$ is just the probability that i is first in a random ordering of memberw of S_k, and so $P(i,k) = 1/(n-k+1)$, since there are $n - k + 1$ players in S_k. Therefore,

$$\phi_i(v) = \sum_{k=1}^{i} \frac{-(c_k - c_{k-1})}{n - k + 1}.$$

Thus, player 1 pays c_1/n, player 2 pays $c_1/n + (c_2 - c_1)/(n-1)$, etc. Since all n players use the first part of the airfield, each player pays c_1/n for this. Since players 2 through n use the second part of the airfield, they each pay $(c_2 - c_1)/(n-1)$, and so on.

16. The characteristic function is $v(S) = \begin{cases} 0 & \text{if } 0 \notin S \text{ or if } S = \{0\} \\ a_{k(S)} & \text{otherwise} \end{cases}$, where $k(S) = \min\{i : i \in S - \{0\}\}$. For $i \neq 0$,

$$\phi_i(v) = \sum_{S \in \mathcal{S}_i} \frac{|S|-1)!(m+1-|S|)!}{(m+1)!}(a_i - a_{k(S-\{i\})})$$

where $\mathcal{S}_i = \{S \subset N : 0 \in S, i \in S, k(S) = i\}$. This is because $v(S) - v(S - \{i\}) = 0$ unless $S \in \mathcal{S}$. Then,

$$\phi_i(v) = a_i \left[\sum_{S \in \mathcal{S}_i} \frac{|S|-1)!(m+1-|S|)!}{(m+1)!}\right] + \sum_{j=i+1}^{m} a_j \left[\sum_{S \in \mathcal{S}_{i,j}} \frac{|S|-1)!(m+1-|S|)!}{(m+1)!}\right]$$

where $\mathcal{S}_{i,j} = \{S \subset N : 0 \in S, i \in S, k(S - \{i\}) = j\}$.

The coefficient of a_i is just the probability that in a random ordering of all $m+1$ players, i enters after 0 but before $1, \ldots, i-1$. This is the same as the probability that in a random ordering of $0, 1, \ldots, i$, 0 enters first and i second, namely $1/((i+1)i)$.

The coefficient of a_j is just the probability that in a random ordering of all $m+1$ players, i enters after 0 and j but before $1, \ldots, j-1$. This is the same as the probability that in a random ordering of $0, 1, \ldots, j$, i enters third after 0 and j in some order, namely $2/((j+1)j(j-1))$. This gives

$$\phi_i(v) = \frac{a_i}{(i+1)i} - \sum_{j=i+1}^{m} \frac{2a_j}{(j+1)j(j-1)}.$$

Similarly,

$$\begin{aligned}\phi_0(v) &= \sum_{j=1}^{m} a_j P(0 \text{ enters after } j \text{ but before } 1, \ldots, j-1) \\ &= \sum_{j=1}^{m} \frac{a_j}{(j+1)j}.\end{aligned}$$

17.(a) If $v(N - \{i\}) = 1$ and if $\boldsymbol{x}$ is in the core, then $1 = v(N - \{i\}) \leq \sum_1^n x_j - x_i = 1 - x_i$, so $x_i = 0$. Thus, any player without veto power gets zero at every core point. If there are no veto players, then there can be no core points since we must have $\sum_1^n x_i = 1$.

(b) If i is a veto player, then $\boldsymbol{x} = \boldsymbol{e}_i$, the ith unit vector, is a core point, since if S is a winning coalition, then $i \in S$ and $\sum_{i \in S} x_i = 1 \geq v(S) = 1$, and if S is losing, then certainly $\sum_{i \in S} \geq v(T) = 0$.

(c) The core is the set of all vectors, $\boldsymbol{x}$, such that $\sum_1^n x_i = 1$, $x_i \geq 0$ for all $i \in N$, and $x_i = 0$ if i is not a veto player.

Solutions to Exercises of Section IV.4

1. The core is the set of imputations, $\boldsymbol{x}$, such that the excesses, $e(\boldsymbol{x}, S)$, are negative or zero for all coalitions, S. The nucleolus is an imputation that minimizes the largest of the excesses. If the core is not empty, there is an imputation, $\boldsymbol{x}$, with $e(\boldsymbol{x}, S) \le 0$ for all S. Therefore the nucleolus also satisfies $e(\boldsymbol{x}, S) \le 0$ for all S and so is in the core.

2. A constant-sum game satisfies $v(S) + v(S^c) = v(N)$ for all coalitions, S. The Shapley value for Player 1 in a three person game is

$$\begin{aligned}
\phi_1 &= \frac{1}{3}v(\{1\}) + \frac{1}{6}[v(\{1,2\}) - v(\{2\})] + \frac{1}{6}[v(\{1,3\}) - v(\{3\})] + \frac{1}{3}[v(N) - v(\{2,3\})] \\
&= \frac{1}{3}v(\{1\}) + \frac{1}{6}[v(\{N\}) - v(\{3\}) - v(\{2\})] + \frac{1}{6}[v(\{N\}) - v(\{2\}) - v(\{3\})] + \frac{1}{3}v(\{1\}) \\
&= \frac{1}{3}[v(N) + 2v(\{1\}) - v(\{2\}) - v(\{3\})]
\end{aligned}$$

and similarly for the other two players. The excess, $e(\boldsymbol{x}, \{1\}) = v(\{1\}) - x_1$, is the negative of the excess, $e(\boldsymbol{x}, \{2,3\}) = v(\{2,3\}) - x_2 - x_3 = v(N) - v(\{1\}) - x_1 - x_2 - x_3 + x_1 = -v(\{1\}) + x_1$, since $x_1 + x_2 + x_3 = v(N)$. Since $x_i \ge v(\{i\})$ for any imputation, the maximum excess is $\max\{x_1 - v(\{1\}), x_2 - v(\{2\}), x_3 - v(\{3\})\}$. This can be made a minimum by making all three terms equal: $x_1 - v(\{1\}) = x_2 - v(\{2\}) = x_3 - v(\{3\})$ which, together with $x_1 + x_2 + x_3 = v(N)$, determines the x_i to be the same as for the Shapley value.

3. (a) The core is the set of vectors (x_1, x_2, x_3) of non-negative numbers satisfying $x_1 + x_2 + x_3 = 1200$, $x_1 + x_2 \ge 1200$, $x_1 + x_3 \ge 1200$, and $x_2 + x_3 \ge 0$. If non-negative numbers satisfy $x_1 + x_2 + x_3 = 1200$ and $x_1 + x_2 \ge 1200$, we must have $x_3 = 0$. Similarly, we must have $x_2 = 0$. Therefore $x_1 = 1200$. The core consists of the single point $(1200, 0, 0)$. Since the nucleolus is in the core and the core consists of one point, that point must be the nucleolus.

(b) If the players enter the grand coalition in a random order, Player B can win only if Player A enters first and B second. This happens with probability 1/6. The amount won is 1200. So $\phi_B = (1/6)1200 = 200$. Similarly, $\phi_C = 200$, and then $\phi_A = 1200 - 200 - 200 = 800$.

(c) The Shapley value seems more reasonable to me. There is a danger that B and C will combine to demand say 1000, (500 each), so some payment to one or the other or both seems reasonable. It does not seem reasonable that A can play B and C against each other to be able to pay practically nothing.

4. The core consists of points $(x, 0, 40 - x)$ for $30 \le x \le 40$. We might try $(30, 0, 10)$ as a guess at the nucleolus. In the table below, we see the maximum excess is zero. The excess for either of the coalitions $\{2\}$ and $\{2,3\}$ cannot be made smaller without making

the other larger, so $x_2 = 0$. The excess for $\{1,3\}$ can be made smaller by increasing x_1. This increases the excess for $\{3\}$. These are equal for $x_1 = 35$. This gives $(35, 0, 5)$ as the nucleolus.

Coalition	Excess	$(30, 0, 10)$	$(35, 0, 5)$
$\{1\}$	$-x_1$	-30	-35
$\{2\}$	$-x_2$	0	0
$\{3\}$	$-x_3$	-10	-5
$\{1,2\}$	$30 - x_1 - x_2$	0	-5
$\{1,3\}$	$40 - x_1 - x_3$	0	0
$\{2,3\}$	$-x_2 - x_3$	-10	-5

The Shapley value is $(25, 5, 10)$. Player 2 receives 5 for just being there (to help Player 1). Since the Shapley value is not in the core and the core is not empty, we know that the nucleolus cannot be equal to the Shapley value. The nucleolus is always in the core when the core is not empty.

5. The Shapley value was found to be $(13/6, 22/6, 1/6)$ so we might try $(2, 3, 1)$ as an initial guess at the nucleolus. The largest excess occurs at either of the coalitions $\{1\}$ and $\{2,3\}$. One cannot be made larger without making the other smaller. So $x_1 = 2$ in the nucleolus. The next largest excess occurs at $\{2\}$ and $\{1,2\}$. These can be made smaller by making x_2 larger. This increases the excess of $\{1,3\}$. These are equal at $x_2 = 3.5$ and $x_3 = .5$. The nucleolus is $(2, 3.5, .5)$.

Coalition	Excess	$(2, 3, 1)$	$(2, 3.5, .5)$
$\{1\}$	$1 - x_1$	-1	-1
$\{2\}$	$-x_2$	-3	-3.5
$\{3\}$	$-4 - x_3$	-5	-4.5
$\{1,2\}$	$2 - x_1 - x_2$	-3	-3.5
$\{1,3\}$	$-1 - x_1 - x_3$	-4	-3.5
$\{2,3\}$	$2 - x_2 - x_3$	-1	-1

6. Since the characteristic function is symmetric in players 2 through n, we may assume the nucleolus is of the form $(x_1, x, x, \ldots, x)$ for some x_1 and x. To be an imputation we must have $x_1 + (n-1)x = v(N) = n$, so $x_1 = n - (n-1)x$. The excess for S not containing 1 is $e(\boldsymbol{x}, S) = -|S|x$. The excess for S containing 1 is $|S| - x_1 - (|S| - 1)x = -(n - |S|) + (n - |S|)x$. The smallest maximum excess is certainly less than 0 (since it is for $x = 0$) so we can see that $0 < x < 1$. The largest excess for S not containing 1 is $-x$ (when $|S| = 1$). The largest excess for S containing 1 is $-(1-x)$ (when $|S| = n-1$). The largest of these two is smallest when x is chosen to make then equal. This gives $x = 1/2$. Hence $x_1 = (n+1)/2$ and the nucleolus is the same as the Shapley value, $((n+1)/2, 1/2, \ldots, 1/2)$.

7. Player 1 is a dummy, so he gets zero. Players 2, 3 and 4 are symmetric, so they get the same amount, say x. Since the sum must be $v(N) = 1$, we have $3x = 1$ or $x = 1/3$. This gives $(0, 1/3, 1/3, 1/3)$ as the nucleolus.

8.(a) No player can profit without the others so $v(A) = v(B) = v(C) = 0$. Players A and B can build a road for 18 and receive 19 in return so $v(AB) = 19 - 18 = 1$. Similarly, $v(AC) = 0$, $v(BC) = 6$, and $v(ABC) = 8$, the latter requiring a road of cost 19.

(b) The Shapley values are: $\phi_A = (1/3)0 + (1/6)1 + (1/6)0 + (1/3)2 = 5/6$, $\phi_B = (1/3)0+(1/6)1+(1/6)6+(1/3)8 = 23/6$, and $\phi_C = (1/3)0+(1/6)0+(1/6)6+(1/3)7 = 20/6$. To build the road Player A pays $10-5/6 = 9+(1/6)$, Player B pays $9-(23/6) = 5+(1/6)$, and Player C pays $8 - (20/6) = 4 + (2/3)$, for a total of 19.

(c) Based on the Shapley value we might try a first guess of $(1, 4, 3)$. We must always have $x_1 + x_2 + x_3 = 8$. The maximum excess occurs for A and BC. One cannot be made smaller without making the other larger, so $x_1 = 1$. The next largest excess occurs for C, so we must make x_3 larger. But as C is made smaller, B and AC get larger. We choose $x_3 = 3.5$ because then all three will be equal. Thus $(1, 3.5, 3.5)$ is the nucleolus. Player A pays 9, Player B pays 5.5 and Player C pays 4.5 for a total of 19.

Coalition	Excess	$(1, 4, 3)$	$(1, 3.5, 3.5)$
$\{A\}$	$-x_1$	-1	-1
$\{B\}$	$-x_2$	-4	-3.5
$\{C\}$	$-x_3$	-3	-3.5
$\{A, B\}$	$1 - x_1 - x_2$	-4	-3.5
$\{A, C\}$	$-x_1 - x_3$	-4	-4.5
$\{B, C\}$	$6 - x_2 - x_3$	-1	-1

9. (a) $\phi_1 = 3/2$, $\phi_2 = \phi_3 = \phi_4 = 1/2$.

(b) $\nu_1 = 3/2$, $\nu_2 = \nu_3 = \nu_4 = 1/2$.

(c) If 1 enters the coalition and finds k peasants there, he wins $f(k)$. He is equally likely to enter at any of the positions 1 through $m + 1$, so his expected payoff is $\phi_1 = (f(1)+\cdots+f(m))/(m+1)$. The other players are symmetric and so receive equal amounts, $\phi_2 = \cdots = \phi_m = (f(m) - \phi_1)/m$.

(d) Players 2 through $m + 1$ are symmetric, so the nucleolus must be of the form $\nu = (f(m) - my, y, y, \ldots , y)$. The largest excess, $e(\nu, S)$ for S not containing 1 occurs at $|S| = 1$ with value $-y$, decreasing in y. The largest excess for S not containing 1 is $\max_{0\le k<m}[f(k) - f(m) + (m - k)y]$, increasing in y. The maximum excess is minimized when these are equal:

$$\max_{0\le k<m}[-(f(m) - f(k)) + (m - k - 1)y] = 0$$

These are lines with positive slope starting at a negative value. Therefore this equation is satisfied at the first root, $y = \min_{0\le k<m}[(f(m) - f(k))/(m - k + 1)]$. Thus, $\nu_2 = \cdots = \nu_{m+1} = \min\{(f(m) - f(k))/(m + 1 - k) : 0 \le k < m\}$, and $\nu_1 = f(m) - m * \nu_2$.

10.(a) Let us write the value in terms of profit. This normalizes the game so that $v(A) = v(B) = v(C) = v(D) = 0$. In addition, we have $v(AB) = v(CD) = 0$, $v(AC) = 4$,

$v(AD) = 8$, $v(BC) = 3$ and $v(BD) = 5$. Finally, $v(ABC) = 4$, $v(ABD) = 8$, $v(ACD) = 8$, $v(BCD) = 5$ and $v(ABCD) = 11$.

(b) $c_a = c_b = c_C = c_D = c_{AB} = c_{CD} = 0$, $c_{AC} = 4$, $c_{AD} = 8$, $c_{BC} = 3$, $c_{BD} = 5$, $c_{ABC} = -3$, $c_{ABD} = -5$, $c_{ACD} = -4$, $c_{BCD} = -3$, $c_{ABCD} = 6$. Therefore, the Shapley value is $\phi_A = 3.5$, $\phi_B = 1.833$, $\phi_C = 1.667$, $\phi_D = 4.0$.

(c) The nucleolus is $(3.5, 1.5, 1.5, 3.5)$. Under the nucleolus, A receives 13.5 for his house, B receives 21.5 for his house, C gets B's house for 21.5 and D gets A's house for 13.5. Under the Shapley value, A receives 13.5 for his house, B receives 21.833 for his house, C pays 21.333 and gets B's house and D pays 14 and gets A's house.

Coalition	Excess	$(3.5, 1.5, 1.5, 4.5)$
$\{A\}$	$-x_1$	-3.5
$\{B\}$	$-x_2$	-1.5
$\{C\}$	$-x_3$	-1.5
$\{D\}$	$-x_4$	-4.5
$\{A, B\}$	$-x_1 - x_2$	-5
$\{A, C\}$	$4 - x_1 - x_3$	-1
$\{A, D\}$	$8 - x_1 - x_4$	0
$\{B, C\}$	$3 - x_2 - x_3$	0
$\{B, D\}$	$5 - x_2 - x_4$	-1
$\{C, D\}$	$-x_3 - x_4$	-6
$\{A, B, C\}$	$x_4 - 7$	-2.5
$\{A, B, D\}$	$x_3 - 3$	-1.5
$\{A, C, D\}$	$x_2 - 3$	-1.5
$\{B, C, D\}$	$x_1 - 6$	-2.5

11. The nucleolus will certainly have (*) $x_1 \geq x_2 \geq \cdots \geq x_n$. Therefore, the maximum excess for a coalition S such that $v(S) = k$ occurs when $S = \{1, \ldots, k\}$, except for $k = 0$, when it occurs at $S = \{n\}$. The problem therefore reduces to minimizing

$$\max\{1 - x_1, 2 - x_1 - x_2, \ldots, (n-1) - x_1 - \cdots - x_{n-1}, -x_n\}$$

subject to (*) and $\sum_{i=1}^{n} x_i = n$. The last two are minimized when they are equal, giving $x_n = .5$. Then the previous one is minimized when $x_{n-1} = .5$, and so on down to $x_3 = .5$. Then the first two are minimized when they are equal, giving $x_2 = 1$, and therefore $x_1 = n/2$.

12. Since the game is symmetric in players 2 through n, we have $\nu_2 = \nu_3 = \cdots = \nu_n$. If $1 \in S$, the biggest excess, $e(\boldsymbol{\nu}, S)$, occurs when $S = \{1, 2\}$, in which case, $e(\boldsymbol{\nu}, S) = 1 - \nu_1 - \nu_2$. If $1 \notin S$, the biggest excess (and only positive excess) occurs at $S = \{2, \ldots, n\}$, in which case $e(\boldsymbol{\nu}, S) = 1 - \nu_2 - \cdots - \nu_n = 1 - (n-1)\nu_2$. The largest excess is minimized if these are equal: $1 - \nu_1 - \nu_2 = 1 - (n-1)\nu_2$. Since $\nu_1 + (n-1)\nu_2 = 1$, we may solve to find $\nu_2 = 1/(2n-1)$ and $\nu_1 = n/(2n-1)$.

Solutions to Exercises of Appendix 1.

1. Yes. This follows because the utility defined by (1) is linear on $\mathcal{P}^*$ in the sense that $u(\lambda p_1 + (1-\lambda)p_2) = \lambda u(p_1) + (1-\lambda)u(p_2)$. A1 is satisfied because $\lambda p_1 + (1-\lambda)q \preceq \lambda p_2 + (1-\lambda)q$ if and only if $u(\lambda p_1 + (1-\lambda)q) \le u(\lambda p_2 + (1-\lambda)q)$, if and only if $\lambda u(p_1) + (1-\lambda)u(q) \le \lambda u(p_2) + (1-\lambda)u(q)$, if and only if $\lambda u(p_1) \le \lambda u(p_2)$, if and only if $p_1 \preceq p_2$ (since $\lambda > 0$). Similarly for A2, if $u(p_1) < u(p_2)$ and $u(q)$ is any given number, we can find a $\lambda > 0$ sufficiently small so that $u(p_1) < \lambda u(q) + (1-\lambda)u(p_2)$.

2. For $\mathcal{P} = \{P_1, P_2\}$, define a preference on $\mathcal{P}^*$ to be

$$(1-\theta)P_1 + \theta P_2 \prec (1-\theta')P_1 + \theta' P_2 \qquad \text{if and only if} \qquad \theta < \theta' \text{ and } \theta' \ge 1/2.$$

This is the preference of a person who prefers P_2 to P_1 but has no preference between lotteries that give probability less than 1/2 to P_2. A1 is not satisfied since, taking $q = P_1$, $p_1 = P_1$ and $p_2 = P_2$, we have $p_1 \prec p_2$ but $\lambda p_1 + (1-\lambda)P_1 \simeq \lambda p_2 + (1-\lambda)P_1$ if $\lambda < 1/2$. A2 is still satisfied since if $p_1 \prec p_2$ and q is any other element of $\mathcal{P}^*$, we can take λ sufficiently small so that $p_1 \prec \lambda q + (1-\lambda)p_2$ and $\lambda q + (1-\lambda)p_1 \prec p_2$.

3. For $\mathcal{P} = \{P_1, P_2, P_3\}$, we may use $(\theta_1, \theta_2, \theta_3)$ to represent the element $\theta_1 P_1 + \theta_2 P_2 + \theta_3 P_3$, where $\theta_1 \ge 0$, $\theta_2 \ge 0$, $\theta_3 \ge 0$, and $\theta_1 + \theta_2 + \theta_3 = 1$. Define a preference on $\mathcal{P}^*$ to be

$$(\theta_1, \theta_2, \theta_3) \prec (\theta_1', \theta_2', \theta_3') \qquad \text{if and only if} \qquad \theta_1 > \theta_1' \text{ or } (\theta_1 = \theta_1' \text{ and } \theta_3 < \theta_3')$$

This is the preference of the person for whom it is the overriding consideration to avoid P_1 (death), but if two lotteries give the same probability to P_1, then the one that gives higher probability to P_3 is preferred. Then clearly A2 is not satisfied for $q = P_1$, $p_1 = P_2$ and $p_2 = P_3$. Checking A1 for $p_1 = (\theta_1, \theta_2, \theta_3)$ and $p_2 = (\theta_1', \theta_2', \theta_3')$ reduces to showing that both sides of (4) are equivalent to $\theta_1 > \theta_1'$ or ($\theta_1 = \theta_1'$ and $\theta_3 < \theta_3'$), for all q and $\lambda > 0$. This is checked by straightforward analysis.