Classical Inequalities

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1 Introduction

This section will start with some basic facts and exercises. Frequent users of this discipline can just skim over the notation and take a look at formulas that talk about generalities in which the theorems will be shown.

The reason for starting with basic principles is the intention to show that the theory is simple enough to be completely derived on 20 pages without using any high-level mathematics. If you take a look at the first theorem and compare it with some scary inequality already mentioned in the table of contents, you will see how huge is the path that we will bridge in so few pages. And that will happen on a level accessible to a beginning high-school student. Well, maybe I exaggerated in the previous sentence, but the beginning high-school student should read the previous sentence again and forget about this one.

Theorem 1. *If x is a real number, then $x^2 \geq 0$. The equality holds if and only if $x = 0$.*

No proofs will be omitted in this text. Except for this one. We have to acknowledge that this is very important inequality, everything relies on it, ..., but the proof is so easy that it makes more sense wasting the space and time talking about its triviality than actually proving it. Do you know how to prove it? Hint: "A friend of my friend is my friend"; "An enemy of my enemy is my friend". It might be useful to notice that "An enemy of my friend is my enemy" and "A friend of my enemy is my enemy", but the last two facts are not that useful for proving theorem 1.

I should also write about the difference between "$\geq$" and "$>$"; that something weird happens when both sides of an inequality are multiplied by a negative number, but I can't imagine myself doing that. People would hate me for real.

Theorem 2. *If $a,b \in \mathbb{R}$ then:*

$$a^2+b^2 \geq 2ab. \tag{1}$$

The equality holds if and only if $a=b$.

Proof. After subtracting $2ab$ from both sides the inequality becomes equivalent to $(a-b)^2 \geq 0$, which is true according to theorem 1. $\square$

Problem 1. *Prove the inequality $a^2+b^2+c^2 \geq ab+bc+ca$, if a,b,c are real numbers.*

Solution. If we add the inequalities $a^2+b^2 \geq 2ab$, $b^2+c^2 \geq 2bc$, and $c^2+a^2 \geq 2ca$ we get $2a^2+2b^2+2c^2 \geq 2ab+2bc+2ca$, which is equivalent to what we are asked to prove. $\triangle$

Problem 2. *Find all real numbers a,b,c, and d such that*

$$a^2+b^2+c^2+d^2 = a(b+c+d).$$

Solution. Recall that $x^2+y^2 \geq 2xy$, where the equality holds if and only if $x=y$. Applying this inequality to the pairs of numbers $(a/2,b)$, $(a/2,c)$, and $(a/2,d)$ yields:

$$\frac{a^2}{4}+b^2 \geq ab, \quad \frac{a^2}{4}+c^2 \geq ac, \quad \frac{a^2}{4}+d^2 \geq ad.$$

Note also that $a^2/4 > 0$. Adding these four inequalities gives us $a^2+b^2+c^2+d^2 \geq a(b+c+d)$. Equality can hold only if all the inequalities were equalities, i.e. $a^2=0$, $a/2=b$, $a/2=c$, $a/2=d$. Hence $a=b=c=d=0$ is the only solution of the given equation. $\triangle$

Problem 3. *If a,b,c are positive real numbers that satisfy $a^2+b^2+c^2=1$, find the minimal value of*

$$S = \frac{a^2b^2}{c^2}+\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}.$$

Solution. If we apply the inequality $x^2+y^2 \geq 2xy$ to the numbers $x=\frac{ab}{c}$ and $y=\frac{bc}{a}$ we get

$$\frac{a^2b^2}{c^2}+\frac{b^2c^2}{a^2} \geq 2b^2. \tag{2}$$

Similarly we get

$$\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2} \geq 2c^2, \text{ and} \tag{3}$$

$$\frac{c^2a^2}{b^2}+\frac{a^2b^2}{c^2} \geq 2a^2. \tag{4}$$

Summing up (2), (3), and (4) gives $2\left(\frac{a^2b^2}{c^2}+\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}\right) \geq 2(a^2+b^2+c^2)=2$, hence $S \geq 1$. The equality holds if and only if $\frac{ab}{c}=\frac{bc}{a}=\frac{ca}{b}$, i.e. $a=b=c=\frac{1}{\sqrt{3}}$. $\triangle$

Problem 4. *If x and y are two positive numbers less than* 1*, prove that*

$$\frac{1}{1-x^2}+\frac{1}{1-y^2}\geq\frac{2}{1-xy}.$$

Solution. Using the inequality $a+b\geq 2\sqrt{ab}$ we get $\frac{1}{1-x^2}+\frac{1}{1-y^2}\geq\frac{2}{\sqrt{(1-x^2)(1-y^2)}}$. Now we notice that $(1-x^2)(1-y^2)=1+x^2y^2-x^2-y^2\leq 1+x^2y^2-2xy=(1-xy)^2$ which implies $\frac{2}{\sqrt{(1-x^2)(1-y^2)}}\geq\frac{2}{1-xy}$ and this completes the proof. △

Since the main focus of this text is to present some more advanced material, the remaining problems will be harder then the ones already solved. For those who want more of the introductory-type problems, there is a real hope that this website will soon get some text of that sort. However, nobody should give up from reading the rest, things are getting very interesting.

Let us return to the inequality (1) and study some of its generalizations. For $a,b\geq 0$, the consequence $\frac{a+b}{2}\geq\sqrt{ab}$ of (1) is called the Arithmetic-Geometric mean inequality. Its left-hand side is called the arithmetic mean of the numbers a and b, and its right-hand side is called the geometric mean of a and b. This inequality has its analogue:

$$\frac{a+b+c}{3}\geq\sqrt[3]{abc},\ a,b,c\geq 0.$$

More generally, for a sequence $x_1,\dots,x_n$ of positive real numbers, the Arithmetic-Geometric mean inequality holds:

$$\frac{x_1+x_2+\cdots+x_n}{n}\geq\sqrt[n]{x_1\cdot x_2\cdots x_n}. \tag{5}$$

These two inequalities are highly non-trivial, and there are variety of proofs to them. We did (5) for $n=2$. If you try to prove it for $n=3$, you would see the real trouble. What a person tortured with the case $n=3$ would never suspect is that $n=4$ is much easier to handle. It has to do something with 4 being equal $2\cdot 2$ and $3\neq 2\cdot 2$. I believe you are not satisfied by the previous explanation but you have to accept that the case $n=3$ comes after the case $n=4$. The induction argument follows these lines, but (un)fortunately we won't do it here because that method doesn't allow generalizations that we need.

Besides (5) we have the inequality between quadratic and arithmetic mean, namely

$$\sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}}\geq\frac{x_1+x_2+\cdots+x_n}{n}. \tag{6}$$

The case of equality in (5) and (6) occurs if and only if all the numbers $x_1,\dots,x_n$ are equal.

Arithmetic, geometric, and quadratic means are not the only means that we will consider. There are infinitely many of them, and there are infinitely many inequalities that generalize (5) and (6). The beautiful thing is that we will consider all of them at once. For appropriately defined means, a very general inequality will hold, and the above two inequalities will ended up just being consequences.

Definition 1. *Given a sequence $x_1,x_2,\dots,x_n$ of positive real numbers, the mean of order r, denoted by $M_r(x)$ is defined as*

$$M_r(x)=\left(\frac{x_1^r+x_2^r+\cdots+x_n^r}{n}\right)^{\frac{1}{r}}. \tag{7}$$

Example 1. *$M_1(x_1,\dots,x_n)$ is the arithmetic mean, while $M_2(x_1,\dots,x_n)$ is the geometric mean of the numbers $x_1,\dots,x_n$.*

M_0 can't be defined using the expression (7) but we will show later that as r approaches 0, M_r will approach the geometric mean. The famous mean inequality can be now stated as

$$M_r(x_1,\dots,x_n) \leq M_s(x_1,\dots,x_n), \quad \text{for } 0 \leq r \leq s.$$

However we will treat this in slightly greater generality.

Definition 2. *Let $m = (m_1,\dots,m_n)$ be a fixed sequence of non-negative real numbers such that $m_1+m_2+\cdots+m_n = 1$. Then the weighted mean of order r of the sequence of positive reals $x = (x_1,\dots,x_n)$ is defined as:*

$$M_r^m(x) = \left(x_1^r m_1 + x_2^r m_2 + \cdots + x_n^r m_n\right)^{\frac{1}{r}}. \tag{8}$$

Remark. Sequence m is sometimes called a sequence of masses, but more often it is called a measure, and $M_r^m(x)$ is the L^r norm with repsect to the Lebesgue integral defined by m. I didn't want to scare anybody. I just wanted to emphasize that this hard-core math and not something coming from physics.

We will prove later that as r tends to 0, the weighted mean $M_r^m(x)$ will tend to the weighted geometric mean of the sequence x defined by $G^m(x) = x_1^{m_1}\cdot x_2^{m_2}\cdots x_n^{m_n}$.

Example 2. *If $m_1 = m_2 = \cdots = \frac{1}{n}$ then $M_r^m(x) = M_r(x)$ where $M_r(x)$ is previously defined by the equation (7).*

Theorem 3 (General Mean Inequality)**.** *If $x = (x_1,\dots,x_n)$ is a sequence of positive real numbers and $m = (m_1,\dots,m_n)$ another sequence of positive real numbers satisfying $m_1+\cdots+m_n = 1$, then for $0 \leq r \leq s$ we have $M_r^m(x) \leq M_s^m(x)$.*

The proof will follow from the Hölders inequality.

2 Convex Funtions

To prove some of the fundamental results we will need to use convexity of certain functions. Proofs of the theorems of Young, Minkowski, and Hölder will require us to use very basic facts – you should be fine if you just read the definition 3 and example 3. However, the section on Karamata's inequality will require some deeper knowledge which you can find here.

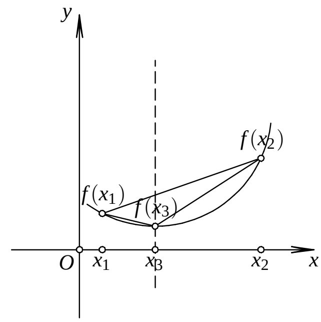
Definition 3. *The function $f:[a,b]\to\mathbb{R}$ is convex if for any $x_1,x_2\in[a,b]$ and any $\lambda\in(0,1)$ the following inequality holds:*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2). \tag{9}$$

Function is called concave if $-f$ is convex. If the inequality in (9) is strict then the function is called strictly convex.

Now we will give a geometrical interpretation of convexity. Take any $x_3 \in (x_1, x_2)$. There is $\lambda \in (0,1)$ such that $x_2 = \lambda x_1 + (1-\lambda)x_3$. Let's paint in green the line passing through x_3 and parallel to the y axis. Let's paint in red the chord connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Assume that the green line and the red chord intersect at the yellow point. The y coordinate (also called the height) of the yellow point is:

$$\lambda f(x_1) + (1-\lambda) f(x_2).$$



The inequality (9) means exactly that the the green line will intersect the graph of a function below the red chord. If f is strictly convex then the equality can hold in (9) if and only if $x_1 = x_2$.

Example 3. *The following functions are convex:* e^x, x^p *(for* $p \geq 1$, $x > 0$*),* $\frac{1}{x}$ *(*$x \neq 0$*), while the functions* $\log x$ *(*$x > 0$*),* $\sin x$ *(*$0 \leq x \leq \pi$*) ,* $\cos x$ *(*$-\pi/2 \leq x \leq \pi/2$*) are concave.*

All functions mentioned in the previous example are elementary functions, and proving the convexity/concavity for them would require us to go to the very basics of their foundation, and we will not do that. In many of the examples and problems respective functions are slight modifications of elementary functions. Their convexity (or concavity) is something we don't have to verify. However, we will develop some criteria for verifying the convexity of more complex combinations of functions.

Let us take another look at our picture above and compare the slopes of the three drawn lines. The line connecting $(x_1, f(x_1))$ with $(x_3, f(x_3))$ has the smallest slope, while the line connecting $(x_3, f(x_3))$ with $(x_2, f(x_2))$ has the largest slope. In the following theorem we will state and prove that the convex function has always an "increasing slope".

Theorem 4. *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function and* $a \leq x_1 < x_3 < x_2 \leq b$. *Then*

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x_3)}{x_2 - x_3}. \tag{10}$$

Proof. We can write $x_3 = \lambda x_1 + (1-\lambda)x_2$ for some $\lambda \in (0,1)$. More precisely $\lambda = \frac{x_2 - x_3}{x_2 - x_1}$, and $1 - \lambda = \frac{x_3 - x_1}{x_2 - x_1}$. From (9) we get

$$f(x_3) \leq \frac{x_2 - x_3}{x_2 - x_1} f(x_1) + \frac{x_3 - x_1}{x_2 - x_1} f(x_2).$$

Subtracting $f(x_1)$ from both sides of the last inequality yields $f(x_3) - f(x_1) = -\frac{x_3 - x_1}{x_2 - x_1} f(x_1) + \frac{x_3 - x_1}{x_2 - x_1} f(x_2)$ giving immediately the first inequality of (10). The second inequality of (10) is obtained in an analogous way. $\square$

The rest of this chapter is using some of the properties of limits, continuity and differentiability. If you are not familiar with basic calculus, you may skip that part, and you will be able to understand most of what follows. The theorem 6 is the tool for verifying the convexity for differentiable functions that we mentioned before. The theorem 5 will be used it in the proof of Karamata's inequality.

Theorem 5. *If* $f : (a,b) \to \mathbb{R}$ *is a convex function, then* f *is continuous and at every point* $x \in (a,b)$ *it has both left and right derivative* $f'_-(x)$ *and* $f'_+(x)$. *Both* f'_- *and* f'_+ *are increasing functions on* (a,b) *and* $f'_-(x) \leq f'_+(x)$.

Solution. The theorem 10 implies that for fixed x the function $\varphi(t)=\frac{f(t)-f(x)}{t-x}$, $t\neq x$ is an increasing function bounded both by below and above. More precisely, if t_0 and t_1 are any two numbers from (a,b) such that $t_0<x<t_1$ we have:

$$\frac{f(x)-f(t_0)}{x-t_0}\leq\varphi(t)\leq\frac{f(t_1)-f(x)}{t_1-x}.$$

This specially means that there are $\lim_{t\to x-}\varphi(t)$ and $\lim_{t\to x+}\varphi(t)$. The first one is precisely the left, and the second one – the right derivative of φ at x. Since the existence of both left and right derivatives implies the continuity, the statement is proved. □

Theorem 6. *If $f:(a,b)\to\mathbb{R}$ is a twice differentiable function. Then f is convex on (a,b) if and only if $f''(x)\geq 0$ for every $x\in(a,b)$. Moreover, if $f''(x)>0$ then f is strictly convex.*

Proof. This theorem is the immediate consequence of the previous one. □

3 Inequalities of Minkowski and Hölder

Inequalities presented here are sometimes called weighted inequalities of Minkowski, Hölder, and Cauchy-Schwartz. The standard inequalities are easily obtained by placing $m_i=1$ whenever some m appears in the text below. Assuming that the sum $m_1+\cdots+m_n=1$ one easily get the generalized (weighted) mean inequalities, and additional assumption $m_i=1/n$ gives the standard mean inequalities.

Lemma 1. *If $x,y>0$, $p>1$ and $\alpha\in(0,1)$ are real numbers, then*

$$(x+y)^p\leq\alpha^{1-p}x^p+(1-\alpha)^{1-p}y^p. \tag{11}$$

The equality holds if and only if $\frac{x}{\alpha}=\frac{y}{1-\alpha}$.

Proof. For $p>1$, the function $\varphi(x)=x^p$ is strictly convex hence $(\alpha a+(1-\alpha)b)^p\leq\alpha a^p+(1-\alpha)b^p$. The equality holds if and only if $a=b$. Setting $x=\alpha a$ and $y=(1-\alpha)b$ we get (11) immediately. □

Lemma 2. *If $x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n$ and $m_1,m_2,\dots,m_n$ are three sequences of positive real numbers and $p>1$, $\alpha\in(0,1)$, then*

$$\sum_{i=1}^n(x_i+y_i)^pm_i\leq\alpha^{1-p}\sum_{i=1}^n x_i^pm_i+(1-\alpha)^{1-p}\sum_{i=1}^n y_i^pm_i. \tag{12}$$

The equality holds if and only if $\frac{x_i}{y_i}=\frac{\alpha}{1-\alpha}$ for every i, $1\leq i\leq n$.

Proof. From (11) we get $(x_i+y_i)^p\leq\alpha^{1-p}x_i^p+(1-\alpha)^{1-p}y_i^p$. Multiplying by m_i and adding as $1\leq i\leq n$ we get (12). The equality holds if and only if $\frac{x_i}{y_i}=\frac{\alpha}{1-\alpha}$. □

Theorem 7 (Minkowski)**.** *If x_1, x_2, …, x_n, y_1, y_2, …, y_n, and m_1, m_2, …, m_n are three sequences of positive real numbers and $p>1$, then*

$$\left(\sum_{i=1}^n(x_i+y_i)^pm_i\right)^{1/p}\leq\left(\sum_{i=1}^n x_i^pm_i\right)^{1/p}+\left(\sum_{i=1}^n y_i^pm_i\right)^{1/p}. \tag{13}$$

The equality holds if and only if the sequences (x_i) and (y_i) are proportional, i.e. if and only if there is a constant λ such that $x_i=\lambda y_i$ for $1\leq i\leq n$.

Proof. For any $\alpha \in (0,1)$ we have inequality (12). Let us write

$$A = \left(\sum_{i=1}^{n} x_i^p m_i\right)^{1/p}, \quad B = \left(\sum_{i=1}^{n} y_i^p m_i\right)^{1/p}.$$

In new terminology (12) reads as

$$\sum_{i=1}^{n} (x_i + y_i)^p m_i \le \alpha^{1-p} A^p + (1-\alpha)^{1-p} B^p. \tag{14}$$

If we choose α such that $\frac{A}{\alpha} = \frac{B}{1-\alpha}$, then (11) implies $\alpha^{1-p}A^p + (1-\alpha)^{1-p}B^p = (A+B)^p$ and (14) now becomes

$$\sum_{i=1}^{n} (x_i + y_i)^p m_i = \left[\left(\sum_{i=1}^{n} x_i^p m_i\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p m_i\right)^{1/p}\right]^p$$

which is equivalent to (13). $\square$

Problem 5 (SL70)**.** *If $u_1, \dots, u_n, v_1, \dots, v_n$ are real numbers, prove that*

$$1 + \sum_{i=1}^{n} (u_i + v_i)^2 \le \frac{4}{3}\left(1 + \sum_{i=1}^{n} u_i^2\right)\left(1 + \sum_{i=1}^{n} v_i^2\right).$$

When does equality hold?

Solution. Let us set $a = \sqrt{\sum_{i=1}^{n} u_i^2}$ and $b = \sqrt{\sum_{i=1}^{n} v_i^2}$. By Minkowski's inequality (for $p = 2$) we have $\sum_{i=1}^{n} (u_i + v_i)^2 \le (a+b)^2$. Hence the LHS of the desired inequality is not greater than $1 + (a+b)^2$, while the RHS is equal to $4(1+a^2)(1+b^2)/3$. Now it is sufficient to prove that

$$3 + 3(a+b)^2 \le 4(1+a^2)(1+b^2).$$

The last inequality can be reduced to the trivial $0 \le (a-b)^2 + (2ab-1)^2$. The equality in the initial inequality holds if and only if $u_i/v_i = c$ for some $c \in \mathbb{R}$ and $a = b = 1/\sqrt{2}$. $\triangle$

Theorem 8 (Young)**.** *If $a, b > 0$ and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{15}$$

Equality holds if and only if $a^p = b^q$.

Proof. Since $\varphi(x) = e^x$ is a convex function we have that $e^{\frac{1}{p}x + \frac{1}{q}y} \le \frac{1}{p}e^x + \frac{1}{q}e^y$. The equality holds if and only if $x = y$, and the inequality (15) is immediately obtained by placing $a = e^{x/p}$ and $b = e^{y/q}$. The equality holds if and only if $a^p = b^q$. $\square$

Lemma 3. *If $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, m_1, m_2, \dots, m_n$ are three sequences of positive real numbers and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and $\alpha > 0$, then*

$$\sum_{i=1}^{n} x_i y_i m_i \le \frac{1}{p} \cdot \alpha^p \cdot \sum_{i=1}^{n} x_i^p m_i + \frac{1}{q} \cdot \frac{1}{\alpha^q} \cdot \sum_{i=1}^{n} y_i^q m_i. \tag{16}$$

The equality holds if and only if $\frac{\alpha^p x_i^p}{p} = \frac{y_i^q}{q\alpha^q}$ for $1 \le i \le n$.

Proof. From (15) we immediately get $x_iy_i=(\alpha x_i)\frac{y_i}{\alpha}\leq\frac{1}{p}\cdot\alpha^px_i^p+\frac{1}{q}\cdot\frac{1}{\alpha^q}y_i^q$. Multiplying by m_i and adding as $i=1,2,\dots,n$ we get (16). The inequality holds if and only if $\frac{\alpha^px_i^p}{p}=\frac{y_i^q}{q\alpha^q}$ for $1\leq i\leq n$. $\square$

Theorem 9 (Hölder)**.** *If $x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n,m_1,m_2,\dots,m_n$ are three sequences of positive real numbers and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\sum_{i=1}^n x_iy_im_i\leq\left(\sum_{i=1}^n x_i^pm_i\right)^{1/p}\cdot\left(\sum_{i=1}^n y_i^qm_i\right)^{1/q}. \tag{17}$$

The equality holds if and only if the sequences (x_i^p) and (y_i^q) are proportional.

Proof. The idea is very similar to the one used in the proof of Minkowski's inequality. The inequality (16) holds for any positive constant α. Let

$$A=\left(\alpha^p\sum_{i=1}^n x_i^pm_i\right)^{1/p},\ B=\left(\frac{1}{\alpha^q}\sum_{i=1}^n y_i^qm_i\right)^{1/q}.$$

By Young's inequality we have that $\frac{1}{p}A^p+\frac{1}{q}B^q=AB$ if $A^p=B^q$. Equivalently $\alpha^p\sum_{i=1}^n x_i^pm_i=\frac{1}{\alpha^q}\sum_{i=1}^n y_i^qm_i$. Choosing such an α we get

$$\sum_{i=1}^n x_iy_im_i\leq\frac{1}{p}A^p+\frac{1}{q}B^q=AB=\left(\sum_{i=1}^n x_i^pm_i\right)^{1/p}\cdot\left(\sum_{i=1}^n y_i^qm_i\right)^{1/q}.\ \square$$

Problem 6. *If $a_1,\dots,a_n$ and $m_1,\dots,m_n$ are two sequences of positive numbers such that $a_1m_1+\cdots+a_nm_n=\alpha$ and $a_1^2m_1+\cdots+a_n^2m_n=\beta^2$, prove that $\sqrt{a_1}m_1+\cdots+\sqrt{a_n}m_n\geq\frac{\alpha^{3/2}}{\beta}$.*

Solution. We will apply Hölder's inequality on $x_i=a_i^{1/3}$, $y_i=a_i^{2/3}$, $p=\frac{3}{2}$, $q=3$:

$$\alpha=\sum_{i=1}^n a_im_i\leq\left(\sum_{i=1}^n a_i^{1/2}m_i\right)^{2/3}\cdot\left(\sum_{i=1}^n a_i^2m_i\right)^{1/3}=\left(\sum_{i=1}^n\sqrt{a_i}m_i\right)^{2/3}\cdot\beta^{2/3}.$$

Hence $\sum_{i=1}^n\sqrt{a_i}m_i\geq\frac{\alpha^{3/2}}{\beta}$. $\triangle$

Proof of the theorem 3. $M_r^m=\left(\sum_{i=1}^n x_i^r\cdot m_i\right)^{1/r}$. We will use the Hölders inequality for $y_i=1$, $p=\frac{s}{r}$, and $q=\frac{p}{1-p}$. Then we get

$$M_r^m\leq\left(\sum_{i=1}^n x_i^{rp}\cdot m_i\right)^{\frac{1}{pr}}\cdot\left(\sum_{i=1}^n 1^q\cdot m_i\right)^{p/(1-p)}=M_s.\ \square$$

Problem 7. *(SL98) Let x, y, and z be positive real numbers such that $xyz=1$. Prove that*

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)}\geq\frac{3}{4}.$$

Solution. The given inequality is equivalent to

$$x^3(x+1)+y^3(y+1)+z^3(z+1)\geq \frac{3}{4}(1+x+y+z+xy+yz+zx+xyz).$$

The left-hand side can be written as $x^4+y^4+z^4+x^3+y^3+z^3=3M_4^4+3M_3^3$. Using $xy+yz+zx\leq x^2+y^2+z^2=3M_2^2$ we see that the right-hand side is less than or equal to $\frac{3}{4}(2+3M_1+3M_2^2)$. Since $M_1\geq 3\sqrt[3]{xyz}=1$, we can further say that the right-hand side of the required inequality is less than or equal to $\frac{3}{4}(5M_1+3M_2^2)$. Since $M_4\geq M_3$, and $M_1\leq M_2\leq M_3$, the following inequality would imply the required statement:

$$3M_3^4+3M_3^3\geq \frac{3}{4}(5M_3+3M_3^2).$$

However the last inequality is equivalent to $(M_3-1)(4M_3^2+8M_3+5)\geq 0$ which is true because $M_3\geq 1$. The equality holds if and only if $x=y=z=1$. △

Theorem 10 (Weighted Cauchy-Schwartz). *If x_i, y_i are real numbers, and m_i positive real numbers, then*

$$\sum_{i=1}^{n}x_iy_im_i \quad\leq\quad \sqrt{\sum_{i=1}^{n}x_i^2m_i}\cdot\sqrt{\sum_{i=1}^{n}y_i^2m_i}. \tag{18}$$

Proof. After noticing that $\sum_{i=1}^{n}x_iy_im_i\leq\sum_{i=1}^{n}|x_i|\cdot|y_i|m_i$, the rest is just a special case ($p=q=2$) of the Hölder's inequality. □

Problem 8. *If a, b, and c are positive numbers, prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{(a+b+c)^2}{ab+bc+ca}.$$

Solution. We will apply the Cauchy-Schwartz inequality with $x_1=\sqrt{\frac{a}{b}}$, $x_2=\sqrt{\frac{b}{c}}$, $x_3=\sqrt{\frac{c}{a}}$, $y_1=\sqrt{ab}$, $y_2=\sqrt{bc}$, and $y_3=\sqrt{ca}$. Then

$$\begin{aligned}a+b+c &= x_1y_1+x_2y_2+x_3y_3\leq\sqrt{x_1^2+x_2^2+x_3^2}\cdot\sqrt{y_1^2+y_2^2+y_3^2}\\ &= \sqrt{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}\cdot\sqrt{ab+bc+ca}.\end{aligned}$$

Theorem 11. *If $a_1,\dots,a_n$ are positive real numbers, then*

$$\lim_{r\to 0}M_r(a_1,\dots,a_n)=a_1^{m_1}\cdot a_2^{m_2}\cdots a_n^{m_n}.$$

Proof. This theorem is given here for completeness. It states that as $r\to 0$ the mean of order r approaches the geometric mean of the sequence. Its proof involves some elementary calculus, and the reader can omit the proof.

$$M_r(a_1,\dots,a_n)=e^{\frac{1}{r}\log(a_1^rm_1+\cdots+a_n^rm_n)}.$$

Using the L'Hospitale's theorem we get

$$\begin{aligned}\lim_{r\to 0}\frac{1}{r}\log(a_1^rm_1+\cdots+a_n^rm_n) &= \lim_{r\to 0}\frac{m_1a_1^r\log a_1+\cdots+m_na_n^r\log a_n}{a_1^rm_1+\cdots+a_n^rm_n}\\ &= m_1\log a_1+\cdots+m_n\log a_n\\ &= \log\left(a_1^{m_1}\cdots a_n^{m_n}\right).\end{aligned}$$

The result immediately follows. □

4 Inequalities of Schur and Muirhead

Definition 4. *Let $\sum! F(a_1,\dots,a_n)$ be the sum of $n!$ summands which are obtained from the function $F(a_1,\dots,a_n)$ making all permutations of the array (a).*

We will consider the special cases of the functio F, i.e. when $F(a_1,\dots,a_n)=a_1^{\alpha_1}\cdot\dots\cdot a_n^{\alpha_n}$, $\alpha_i\geq 0$.

If (α) is an array of exponents and $F(a_1,\dots,a_n)=a_1^{\alpha_1}\cdot\dots\cdot a_n^{\alpha_n}$ we will use $T[\alpha_1,\dots,\alpha_n]$ instead of $\sum! F(a_1,\dots,a_n)$, if it is clear what is the sequence (a).

Example 4. *$T[1,0,\dots,0]=(n-1)!\cdot(a_1+a_2+\cdots+a_n)$, and $T[\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}]=n!\cdot\sqrt[n]{a_1\cdot\dots\cdot a_n}$. The AM-GM inequality is now expressed as:*

$$T[1,0,\dots,0]\geq T\left[\frac{1}{n},\dots,\frac{1}{n}\right].$$

Theorem 12 (Schur)**.** *For $\alpha\in\mathbb{R}$ and $\beta>0$ the following inequality holds:*

$$T[\alpha+2\beta,0,0]+T[\alpha,\beta,\beta]\geq 2T[\alpha+\beta,\beta,0]. \tag{19}$$

Proof. Let (x,y,z) be the sequence of positive reals for which we are proving (19). Using some elementary algebra we get

$$\begin{aligned}&\frac{1}{2}T[\alpha+2\beta,0,0]+\frac{1}{2}T[\alpha,\beta,\beta]-T[\alpha+\beta,\beta,0]\\ =\ & x^\alpha(x^\beta-y^\beta)(x^\beta-z^\beta)+y^\alpha(y^\beta-x^\beta)(y^\beta-z^\beta)+z^\alpha(z^\beta-x^\beta)(z^\beta-y^\beta).\end{aligned}$$

Without loss of generality we may assume that $x\geq y\geq z$. Then in the last expression only the second summand may be negative. If $\alpha\geq 0$ then the sum of the first two summands is ≥ 0 because $x^\alpha(x^\beta-y^\beta)(x^\beta-z^\beta)\geq x^\alpha(x^\beta-y^\beta)(y^\beta-z^\beta)\geq y^\alpha(x^\beta-y^\beta)(y^\beta-z^\beta)=-y^\alpha(x^\beta-y^\beta)(y^\beta-z^\beta)$. Similarly for $\alpha<0$ the sum of the last two terms is ≥ 0. □

Example 5. *If we set $\alpha=\beta=1$, we get*

$$x^3+y^3+z^3+3xyz\geq x^2y+xy^2+y^2z+yz^2+z^2x+zx^2.$$

Definition 5. *We say that the array (α) majorizes array (α'), and we write that in the following way $(\alpha')\prec(\alpha)$, if we can arrange the elements of arrays (α) and (α') in such a way that the following three conditions are satisfied:*

1. *$\alpha_1'+\alpha_2'+\cdots+\alpha_n'=\alpha_1+\alpha_2+\cdots+\alpha_n$;*
2. *$\alpha_1'\geq\alpha_2'\geq\cdots\geq\alpha_n'$ i $\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_n$.*
3. *$\alpha_1'+\alpha_2'+\cdots+\alpha_\nu'\leq\alpha_1+\alpha_2+\cdots+\alpha_\nu$, for all $1\leq\nu<n$.*

Clearly, $(\alpha)\prec(\alpha)$.

Theorem 13 (Muirhead)**.** *The necessairy and sufficient condition for comparability of $T[\alpha]$ and $T[\alpha']$, for all positive arrays (a), is that one of the arrays (α) and (α') majorizes the other. If $(\alpha')\prec(\alpha)$ then*

$$T[\alpha']\leq T[\alpha].$$

Equality holds if and only if (α) and (α') are identical, or when all a_is are equal.

Proof. First, we prove the necessity of the condition. Setting that all elements of the array a are equal to x, we get that

$$x^{\sum \alpha_i'} \leq x^{\sum \alpha_i}.$$

This can be satisfied for both large and small xs only if the condition 1 from the definition is satisfied. Now we put $a_1 = \cdots, a_\nu = x$ and $a_{\nu+1} = \cdots = a_n = 1$. Comparing the highest powers of x in expressions $T[\alpha]$ and $T[\alpha']$, knowing that for sufficiently large x we must have $T[\alpha'] \leq T[\alpha]$, we conclude that $\alpha_1' + \cdots + \alpha_\nu' \leq \alpha_1 + \cdots + \alpha_\nu$.

Now we will proof the sufficiency of the condition. The statement will follow from the following two lemmas. We will define one linear operation L on the set of the exponents (α). Suppose that α_k and α_l are two different exponents of (α) such that $\alpha_k > \alpha_l$. We can write

$$\alpha_k = \rho + \tau, \quad \alpha_l = \rho - \tau \quad (0 < \tau \leq \rho).$$

If $0 \leq \sigma < \tau \leq \rho$, define the array $(\alpha') = L(\alpha)$ in the following way:

$$\begin{cases} \alpha_k' = \rho + \sigma = \frac{\tau+\sigma}{2\tau}\alpha_k + \frac{\tau-\sigma}{2\tau}\alpha_l, \\ \alpha_l' = \rho - \sigma = \frac{\tau-\sigma}{2\tau}\alpha_k + \frac{\tau+\sigma}{2\tau}\alpha_l, \\ \alpha_\nu' = \alpha_\nu, \quad (\nu \neq k, \nu \neq l). \end{cases}$$

The definition of this mapping doesn't require that some of the arrays (α) and (α') is in non-decreasing order.

Lemma 4. *If $(\alpha') = L(\alpha)$, then $T[\alpha'] \leq T[\alpha]$, and equality holds if and only if all the elements of (a) are equal.*

Proof. We may rearrange the elements of the sequence such that $k = 1$ i $l = 2$. Then we have

$$\begin{aligned} & T[\alpha] - T[\alpha'] \\ = \quad & \sum ! a_3^{\alpha_3} \cdots a_n^{\alpha_n} \cdot (a_1^{\rho+\tau} a_2^{\rho-\tau} + a_1^{\rho-\tau} a_2^{\rho+\tau} - a_1^{\rho+\sigma} a_2^{\rho-\sigma} - a_1^{\rho-\sigma} a_2^{\rho+\sigma}) \\ = \quad & \sum ! (a_1 a_2)^{\rho-\tau} a_3^{\alpha_3} \cdots a_n^{\alpha_n} (a_1^{\tau+\sigma} - a_2^{\tau+\sigma})(a_1^{\tau-\sigma} - a_2^{\tau-\sigma}) \geq 0. \end{aligned}$$

Eaquality holds if and only if a_is are equal. □

Lemma 5. *If $(\alpha') \prec (\alpha)$, but (α') and (α) are different, then (α') can be obtained from (α) by succesive application of the transformation L.*

Proof. Denote by m the number of differences $\alpha_\nu - \alpha_\nu'$ that are $\neq 0$. m is a positive integer and we will prove that we can apply operation L in such a way that after each of applications, number m decreases (this would imply that the procedure will end up after finite number of steps). Since $\sum(\alpha_\nu - \alpha_\nu') = 0$, and not all of differences are 0, there are positive and negative differences, but the first one is positive. We can find such k and l for which:

$$\alpha_k' < \alpha_k, \quad \alpha_{k+1}' = \alpha_{k+1}, \ldots, \alpha_{l-1}' = \alpha_{l-1}, \quad \alpha_l' > \alpha_l.$$

($\alpha_l - \alpha_l'$ is the first negative difference, and $\alpha_k - \alpha_k'$ is the last positive difference before this negative one). Let $\alpha_k = \rho + \tau$ and $\alpha_l = \rho - \tau$, define σ by

$$\sigma = \max\{|\alpha_k' - \rho|, |\alpha_l' - \rho|\}.$$

At least one of the following two equalities is satisfied:

$$\alpha_l' - \rho = -\sigma, \quad \alpha_k' - \rho = \sigma,$$

because $\alpha'_k > \alpha'_l$. We also have $\sigma < \tau$, because $\alpha'_k < \alpha_k$ i $\alpha'_l > \alpha_l$. Let

$$\alpha''_k = \rho + \sigma, \quad \alpha''_l = \rho - \sigma, \quad \alpha''_\nu = \alpha_\nu \quad (\nu \neq k, \nu \neq l).$$

Now instead of the sequence (α) we will consider the sequence (α''). Number m has decreased by at least 1. It is easy to prove that the sequence (α'') is increasing and majorizes (α'). Repeating this procedure, we will get the sequence (α') which completes the proof of the second lemma, and hence the Muirhead's theorem. □ □

Example 6. *AM-GM is now the consequence of the Muirhead's inequality.*

Problem 9. *Prove that for positive numbers a, b and c the following equality holds:*

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$

Solution. After multiplying both left and right-hand side of the required inequality with $abc(a^3+b^3+abc)(b^3+c^3+abc)(c^3+a^3+abc)$ we get that the original inequality is equivalent to

$$\begin{gathered}\tfrac{3}{2}T[4,4,1] + 2T[5,2,2] + \tfrac{1}{2}T[7,1,1] + \tfrac{1}{2}T[3,3,3] \leq \\ \leq \tfrac{1}{2}T[3,3,3] + T[6,3,0] + \tfrac{3}{2}T[4,4,1] + \tfrac{1}{2}T[7,1,1] + T[5,2,2]\end{gathered}$$

which is true because Muirhead's theorem imply that $T[5,2,2] \leq T[6,3,0]$. △

More problems with solutions using Muirhead's inequality can be found in the section "Problems".

5 Inequalities of Jensen and Karamata

Theorem 14 (Jensen's Inequality). *If f is convex function and $\alpha_1, \dots, \alpha_n$ sequence of real numbers such that $\alpha_1 + \cdots + \alpha_n = 1$, than for any sequence $x_1, \dots, x_n$ of real numbers, the following inequality holds:*

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n).$$

Remark. If f is concave, then $f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \geq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$.

Example 7. *Using Jensen's inequality prove the generalized mean inequality, i.e. that for every two sequences of positive real numbers $x_1, \dots, x_n$ and $m_1, \dots, m_n$ such that $m_1 + \cdots + m_n = 1$ the following inequality holds:*

$$m_1 x_1 + m_2 x_2 + \cdots + m_n x_n \geq x_1^{m_1} \cdot x_2^{m_2} \cdots x_n^{m_n}.$$

Theorem 15 (Karamata's inequalities). *Let f be a convex function and $x_1, \dots, x_n$, $y_1, y_2, \dots, y_n$ two non-increasing sequences of real numbers. If one of the following two conditions is satisfied:*

(a) $(y) \prec (x)$;

(b) $x_1 \geq y_1$, $x_1 + x_2 \geq y_1 + y_2$, $x_1 + x_2 + x_3 \geq y_1 + y_2 + y_3$, $\dots$, $x_1 + \cdots + x_{n-1} \geq y_1 + \cdots + y_{n-1}$, $x_1 + \cdots + x_n \geq y_1 + \cdots + y_n$ *and f is increasing;*

then

$$\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(y_i). \tag{20}$$

Proof. Let $c_i = \frac{f(y_i)-f(x_i)}{y_i-x_i}$, for $y_i \neq x_i$, and $c_i = f'_+(x_i)$, for $x_i = y_i$. Since f is convex, and x_i, y_i are decreasing sequences, c_i is non-increasing (because is represents the "slope" of f on the interval between x_i and y_i). We now have

$$\begin{aligned}\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(y_i) &= \sum_{i=1}^{n} c_i(x_i - y_i) = \sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{n} c_i y_i \\ &= \sum_{i=1}^{n}(c_i - c_{i+1})(x_1 + \cdots + x_i) \\ &\quad - \sum_{i=1}^{n}(c_i - c_{i+1})(y_1 + \cdots + y_i), \qquad (21)\end{aligned}$$

here we define c_{n+1} to be 0. Now, denoting $A_i = x_1 + \cdots + x_i$ and $B_i = y_1 + \cdots + y_i$ (21) can be rearranged to

$$\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(y_i) = \sum_{i=1}^{n-1}(c_i - c_{i+1})(A_i - B_i) + c_n \cdot (A_n - B_n).$$

The sum on the right-hand side of the last inequality is non-negative because c_i is decreasing and $A_i \geq B_i$. The last term $c_n(A_n - B_n)$ is zero under the assumption (a). Under the assumption (b) we have that $c_n \geq 0$ (f is increasing) and $A_n \geq B_n$ and this implies (20). □

Problem 10. *If $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ are two sequences of positive real numbers which satisfy the following conditions:*

$$a_1 \geq b_2,\ a_1a_2 \geq b_1b_2,\ a_1a_2a_3 \geq b_1b_2b_3,\ \cdots \geq a_1a_2\cdots a_n \geq b_1b_2\cdots b_n,$$

prove that

$$a_1 + a_2 + \cdots + a_n \geq b_1 + b_2 + \cdots + b_n.$$

Solution. Let $a_i = e^{x_i}$ and $b_i = e^{y_i}$. We easily verify that the conditions (b) of the Karamata's theorem are satisfied. Thus $\sum_{i=1}^{n} e^{y_i} \geq \sum_{i=1}^{n} e^{x_i}$ and the result immediately follows. △

Problem 11. *If $x_1, \ldots, x_n \in [-\pi/6, \pi/6]$, prove that*

$$\cos(2x_1 - x_2) + \cos(2x_2 - x_3) + \cdots + \cos(2x_n - x_1) \leq \cos x_1 + \cdots + \cos x_n.$$

Solution. Rearrange $(2x_1 - x_2, 2x_2 - x_3, \ldots, 2x_n - x_1)$ and $(x_1, \ldots, x_n)$ in two non-increasing sequences $(2x_{m_1} - x_{m_1+1}, 2x_{m_2} - x_{m_2+1}, \ldots, 2x_{m_n} - x_{m_n+1})$ and $(x_{k_1}, x_{k_2}, \ldots, x_{k_n})$ (here we assume that $x_{n+1} = x_1$. We will verify that condition (a) of the Karamata's inequality is satisfied. This follows from

$$\begin{aligned}&(2x_{m_1} - x_{m_1+1} + \cdots + 2x_{m_l} - x_{m_l+1}) - (x_{k_1} + \cdots + x_{k_l}) \\ \geq\ &(2x_{k_1} - x_{k_1+1} + \cdots + 2x_{k_l} - x_{k_l+1}) - (x_{k_1} + \cdots + x_{k_l}) \\ =\ &(x_{k_1} + \cdots x_{k_l}) - (x_{k_1+1} + \cdots + x_{k_l+1}) \geq 0.\end{aligned}$$

The function $f(x) = -\cos x$ is convex on $[-\pi/2, \pi/2]$ hence Karamata's inequality holds and we get

$$-\cos(2x_1 - x_2) - \cdots - \cos(2x_n - x_1) \geq -\cos x_1 - \cdots - \cos x_n,$$

which is obviously equivalent to the required inequality. △

6 Chebyshev's inequalities

Theorem 16 (Chebyshev's inequalities)**.** *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be real numbers. Then*

$$n\sum_{i=1}^{n} a_ib_i \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq n\sum_{i=1}^{n} a_ib_{n+1-i}. \tag{22}$$

The two inequalities become equalities at the same time when $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

The Chebyshev's inequality will follow from the following generalization (placing $m_i = \frac{1}{n}$ for the left part, and the right inequality follows by applying the left on a_i and $c_i = -b_{n+1-i}$).

Theorem 17 (Generalized Chebyshev's Inequality)**.** *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be any real numbers, and $m_1, \dots, m_n$ non-negative real numbers whose sum is 1. Then*

$$\sum_{i=1}^{n} a_ib_im_i \geq \left(\sum_{i=1}^{n} a_im_i\right)\left(\sum_{i=1}^{n} b_im_i\right). \tag{23}$$

The inequality become an equality if and only if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

Proof. From $(a_i - a_j)(b_i - b_j) \geq 0$ we get:

$$\sum_{i,j}(a_i - a_j)(b_i - b_j)m_im_j \geq 0. \tag{24}$$

Since $\left(\sum_{i=1}^{n} a_im_i\right)\cdot\left(\sum_{i=1}^{n} b_im_i\right) = \sum_{i,j} a_ib_jm_im_j$, (24) implies that

$$\begin{aligned} 0 &\leq \sum_{i,j} a_ib_im_im_j - \sum_{i,j} a_ib_jm_im_j - \sum_{i,j} a_jb_im_jm_i + \sum_{i,j} a_jb_jm_im_j \\ &= 2\left[\sum_i a_ib_im_i - \left(\sum_i a_im_i\right)\left(\sum_i b_im_i\right)\right]. \ \square \end{aligned}$$

Problem 12. *Prove that the sum of distances of the orthocenter from the sides of an acute triangle is less than or equal to $3r$, where the r is the inradius.*

Solution. Denote $a = BC$, $b = CA$, $c = AB$ and let S_{ABC} denote the area of the triangle ABC. Let d_A, d_B, d_C be the distances from H to BC, CA, AB, and A', B', C' the feet of perpendiculars from A, B, C. Then we have $ad_a + bd_b + cd_c = 2(S_{BCH} + S_{ACH} + S_{ABH}) = 2P$. On the other hand if we assume that $a \geq b \geq c$, it is easy to prove that $d_A \geq d_B \geq d_C$. Indeed, $a \geq b$ implies $\angle A \geq \angle B$ hence $\angle HCB' \leq \angle HCA'$ and $HB' \leq HA'$. The Chebyshev's inequality implies

$$(a+b+c)r = 2P = ad_a + bd_b + cd_c \geqslant \frac{1}{3}(a+b+c)(d_a+d_b+d_c). \ \triangle$$

7 Problems

1. If $a, b, c, d > 0$, prove that

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2.$$

2. Prove that
$$\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\geq\frac{a+b+c}{3},$$
for $a,b,c>0$.

3. If $a,b,c,d,e,f>0$, prove that
$$\frac{ab}{a+b}+\frac{cd}{c+d}+\frac{ef}{e+f}\leq\frac{(a+c+e)(b+d+f)}{a+b+c+d+e+f}.$$

4. If $a,b,c\geq 1$, prove that
$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}\leq\sqrt{c(ab+1)}.$$

5. Let $a_1,a_2,\dots,a_n,b_1,b_2,\dots,b_n$ be positive real numbers. Prove that
$$\left(\sum_{i\neq j}a_ib_j\right)^2\geq\left(\sum_{i\neq j}a_ia_j\right)\left(\sum_{i\neq j}b_ib_j\right).$$

6. If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ for $x,y,z>0$, prove that
$$(x-1)(y-1)(z-1)\geq 8.$$

7. Let $a,b,c>0$ satisfy $abc=1$. Prove that
$$\frac{1}{\sqrt{b+\frac{1}{a}+\frac{1}{2}}}+\frac{1}{\sqrt{c+\frac{1}{b}+\frac{1}{2}}}+\frac{1}{\sqrt{a+\frac{1}{c}+\frac{1}{2}}}\geq\sqrt{2}.$$

8. Given positive numbers a,b,c,x,y,z such that $a+x=b+y=c+z=S$, prove that $ay+bz+cx<S^2$.

9. Let a,b,c be positive real numbers. Prove the inequality
$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c+\frac{4(a-b)^2}{a+b+c}.$$

10. Determine the maximal real number a for which the inequality
$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2\geq a(x_1x_2+x_2x_3+x_3x_4+x_4x_5)$$
holds for any five real numbers x_1,x_2,x_3,x_4,x_5.

11. If $x,y,z\geq 0$ and $x+y+z=1$, prove that
$$0\leq xy+yz+zx-2xyz\leq\frac{7}{27}.$$

12. Let a,b and c be positive real numbers such that $abc=1$. Prove that
$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

13. If a, b and c are positive real numbers, prove that:

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2}\geq 3\cdot\frac{ab+bc+ca}{a+b+c}.$$

14. (IMO05) Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\geq 0.$$

15. Let $a_1, \dots, a_n$ be positive real numbers. Prove that

$$\frac{a_1^3}{a_2}+\frac{a_2^3}{a_3}+\cdots+\frac{a_n^3}{a_1}\geq a_1^2+a_2^2+\cdots+a_n^2.$$

16. Let $a_1, \dots, a_n$ be positive real numbers. Prove that

$$(1+a_1)(1+a_2)\cdots(1+a_n)\leq\left(1+\frac{a_1^2}{a_2}\right)\cdot\left(1+\frac{a_2^2}{a_3}\right)\cdot\cdots\cdot\left(1+\frac{a_n^2}{a_1}\right).$$

17. If a, b, and c are the lengths of the sides of a triangle, s its semiperimeter, and $n \geq 1$ an integer, prove that

$$\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^n}{a+b}\geq\left(\frac{2}{3}\right)^{n-2}\cdot s^{n-1}.$$

18. Let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ $(n \geq 2)$ and

$$\frac{1}{1+x_1}+\frac{1}{1+x_2}+\cdots+\frac{1}{1+x_n}=1.$$

Prove that

$$\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}\geq(n-1)\left(\frac{1}{\sqrt{x_1}}+\frac{1}{\sqrt{x_2}}+\cdots+\frac{1}{\sqrt{x_n}}\right).$$

19. Suppose that any two members of certain society are either *friends* or *enemies*. Suppose that there is total of n members, that there is total of q pairs of friends, and that in any set of three persons there are two who are enemies to each other. Prove that there exists at least one member among whose enemies we can find at most $q\cdot\left(1-\frac{4q}{n^2}\right)$ pairs of friends.

20. Given a set of unit circles in the plane whose total area is S. Prove that among those circles there exist certain number of non-intersecting circles whose total area is $\geq \frac{2}{9}S$.

8 Solutions

1. Denote by L the left-hand side of the required inequality. If we add the first and the third summand of L we get

$$\frac{a}{b+c}+\frac{c}{d+a}=\frac{a^2+c^2+ad+bc}{(b+c)(a+d)}.$$

We will bound the denominator of the last fraction using the inequality $xy \le (x+y)^2/4$ for appropriate x and y. For $x = b+c$ and $y = a+d$ we get $(b+c)(a+d) \le (a+b+c+d)^2/4$. The equality holds if and only if $a+d = b+c$. Therefore

$$\frac{a}{b+c} + \frac{c}{d+a} \ge 4\frac{a^2+c^2+ad+bc}{(a+b+c+d)^2}.$$

Similarly $\frac{b}{c+d} + \frac{d}{a+b} \ge 4\frac{b^2+d^2+ab+cd}{(a+b+c+d)^2}$ (with the equality if and only if $a+b = c+d$) implying

$$\begin{aligned} & \frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \\ \ge \quad & 4\frac{a^2+b^2+c^2+d^2+ad+bc+ab+cd}{(a+b+c+d)^2} \\ = \quad & 4\frac{a^2+b^2+c^2+d^2+(a+c)(b+d)}{[(a+c)+(b+d)]^2}. \end{aligned}$$

In order to solve the problem it is now enough to prove that

$$2\frac{a^2+b^2+c^2+d^2+(a+c)(b+d)}{[(a+c)+(b+d)]^2} \ge 1. \tag{25}$$

After multiplying both sides of (25) by $[(a+c)+(b+d)]^2 = (a+c)^2+(b+d)^2$ it becomes equivalent to $2(a^2+b^2+c^2+d^2) \ge (a+c)^2+(b+d)^2 = a^2+b^2+c^2+d^2+2ac+2bd$. It is easy to see that the last inequality holds because many terms will cancel and the remaining inequality is the consequence of $a^2+c^2 \ge 2ac$ and $b^2+d^2 \ge 2bc$. The equality holds if and only if $a = c$ and $b = d$.

2. We first notice that

$$\frac{a^3-b^3}{a^2+ab+b^2} + \frac{b^3-c^3}{b^2+bc+c^2} + \frac{c^3-a^3}{c^2+ca+a^2} = 0.$$

Hence it is enough to prove that

$$\frac{a^3+b^3}{a^2+ab+b^2} + \frac{b^3+c^3}{b^2+bc+c^2} + \frac{c^3+a^3}{c^2+ca+a^2} \ge \frac{2(a+b+c)}{3}.$$

However since $3(a^2-ab+b^2) \ge a^2+ab+b^2$,

$$\frac{a^3+b^3}{a^2+ab+b^2} = (a+b)\frac{a^2-ab+b^2}{a^2+ab+b^2} \ge \frac{a+b}{3}.$$

The equality holds if and only if $a = b = c$.

Second solution. First we prove that

$$\frac{a^3}{a^2+ab+b^2} \ge \frac{2a-b}{3}. \tag{26}$$

Indeed after multiplying we get that the inequality is equivalent to $a^3+b^3 \ge ab(a+b)$, or $(a+b)(a-b)^2 \ge 0$ which is true. After adding (26) with two similar inequalities we get the result.

3. We will first prove that

$$\frac{ab}{a+b}+\frac{cd}{c+d}\leq\frac{(a+c)(b+d)}{a+b+c+d}. \tag{27}$$

As is the case with many similar inequalities, a first look at (27) suggests to multiply out both sides by $(a+b)(c+d)(a+b+c+d)$. That looks scary. But we will do that now. In fact you will do, I will not. I will just encourage you and give moral support (try to imagine me doing that). After you multiply out everything (do it twice, to make sure you don't make a mistake in calculation), the result will be rewarding. Many things cancel out and what remains is to verify the inequality $4abcd\leq a^2d^2+b^2c^2$ which is true because it is equivalent to $0\leq(ad-bc)^2$. The equality holds if and only if $ad=bc$, or $\frac{a}{b}=\frac{c}{d}$.

Applying (27) with the numbers $A=a+c$, $B=b+d$, $C=e$, and $D=f$ yields:

$$\frac{(a+c)(b+d)}{a+b+c+d}+\frac{ef}{e+f}\leq\frac{(A+C)(B+D)}{A+B+C+D}=\frac{(a+c+e)(b+d+f)}{a+b+c+d+e+f},$$

and the required inequality is proved because (27) can be applied to the first term of the left-hand side. The equality holds if and only if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$.

4. To prove the required inequality we will use the similar approach as in the previous problem. First we prove that

$$\sqrt{a-1}+\sqrt{b-1}\leq\sqrt{ab}. \tag{28}$$

Squaring both sides gives us that the original inequality is equivalent to

$$\begin{aligned} & a+b-2+2\sqrt{(a-1)(b-1)}\leq ab \\ \Leftrightarrow\quad & 2\sqrt{(a-1)(b-1)}\leq ab-a-b+2=(a-1)(b-1)+1. \end{aligned} \tag{29}$$

The inequality (29) is true because it is of the form $x+1\geq 2\sqrt{x}$ for $x=(a-1)(b-1)$.

Now we will apply (28) on numbers $A=ab+1$ and $B=c$ to get

$$\sqrt{ab}+\sqrt{c-1}=\sqrt{A-1}+\sqrt{B-1}\leq\sqrt{AB}=\sqrt{(ab+1)c}.$$

The first term of the left-hand side is greater than or equal to $\sqrt{a-1}+\sqrt{b-1}$ which proves the statement. The equality holds if and only if $(a-1)(b-1)=1$ and $ab(c-1)=1$.

5. Let us denote $p=\sum_{i=1}^n a_i, q=\sum_{i=1}^n b_i, k=\sum_{i=1}^n a_i^2, l=\sum_{i=1}^n b_i^2$, and $m=\sum_{i=1}^n a_ib_i$. The following equalities are easy to verify:

$$\sum_{i\neq j}a_ib_j=pq-m,\ \sum_{i\neq j}a_ia_j=p^2-k,\ \text{and}\ \sum_{i\neq j}b_ib_j=q^2-l,$$

so the required inequality is equivalent to

$$(pq-m)^2\geq(p^2-k)(q^2-l)\Leftrightarrow lp^2-2qm\cdot p+m^2+q^2k-kl\geq 0.$$

Consider the last expression as a quadratic equation in p, i.e. $\varphi(p)=lp^2-2qm\cdot p+q^2k-kl$. If we prove that its discriminant is less than or equal to 0, we are done. That condition can be written as:

$$q^2m^2-l(m^2+q^2k-kl)\leq 0\Leftrightarrow(lk-m^2)(q^2-l)\geq 0.$$

The last inequality is true because $q^2 - l = \sum_{i \neq j} b_i b_j > 0$ (b_i are positive), and $lk - m^2 \geq 0$ (Cauchy-Schwartz inequality). The equality holds if and only if $lk - m^2 = 0$, i.e. if the sequences (a) and (b) are proportional.

6. This is an example of a problem where we have some conditions on x, y, and z. Since there are many reciprocals in those conditions it is natural to divide both sides of the original inequality by xyz. Then it becomes

$$\left(1 - \frac{1}{x}\right) \cdot \left(1 - \frac{1}{y}\right) \cdot \left(1 - \frac{1}{z}\right) \geq \frac{8}{xyz}. \tag{30}$$

However $1 - \frac{1}{x} = \frac{1}{y} + \frac{1}{z}$ and similar relations hold for the other two terms of the left-hand side of (30). Hence the original inequality is now equivalent to

$$\left(\frac{1}{y} + \frac{1}{z}\right) \cdot \left(\frac{1}{z} + \frac{1}{x}\right) \cdot \left(\frac{1}{x} + \frac{1}{y}\right) \geq \frac{8}{xyz},$$

and this follows from $\frac{1}{x} + \frac{1}{y} \geq 2\frac{1}{\sqrt{xy}}$, $\frac{1}{y} + \frac{1}{z} \geq 2\frac{1}{\sqrt{yz}}$, and $\frac{1}{z} + \frac{1}{x} \geq 2\frac{1}{\sqrt{zx}}$. The equality holds if and only if $x = y = z = 3$.

7. Notice that

$$\frac{1}{2} + b + \frac{1}{a} + \frac{1}{2} > 2\sqrt{\frac{1}{2} \cdot \left(b + \frac{1}{a} + \frac{1}{2}\right)}.$$

This inequality is strict for any two positive numbers a and b. Using the similar inequalities for the other two denominators on the left-hand side of the required inequality we get:

$$\begin{aligned} & \frac{1}{\sqrt{b + \frac{1}{a} + \frac{1}{2}}} + \frac{1}{\sqrt{c + \frac{1}{b} + \frac{1}{2}}} + \frac{1}{\sqrt{a + \frac{1}{c} + \frac{1}{2}}} \\ > \quad & \sqrt{2}\left(\frac{1}{1 + \frac{1}{a} + b} + \frac{1}{1 + \frac{1}{b} + c} + \frac{1}{1 + \frac{1}{c} + a}\right). \end{aligned} \tag{31}$$

The last expression in (31) can be transformed using $\frac{1}{1+\frac{1}{a}+b} = \frac{a}{1+a+ab} = \frac{a}{1+\frac{1}{c}+a}$ and $\frac{1}{1+\frac{1}{b}+c} = \frac{1}{c(ab+a+1)} = \frac{\frac{1}{c}}{1+\frac{1}{c}+a}$. Thus

$$\begin{aligned} & \sqrt{2}\left(\frac{1}{1 + \frac{1}{a} + b} + \frac{1}{1 + \frac{1}{b} + c} + \frac{1}{1 + \frac{1}{c} + a}\right) \\ = \quad & \sqrt{2} \cdot \frac{1 + \frac{1}{c} + a}{1 + \frac{1}{c} + a} = \sqrt{2}. \end{aligned}$$

The equality can never hold.

8. Denote $T = S/2$. One of the triples (a, b, c) and (x, y, z) has the property that at least two of its members are greater than or equal to T. Assume that (a, b, c) is the one, and choose $\alpha = a - T$, $\beta = b - T$, and $\gamma = c - T$. We then have $x = T - \alpha$, $y = T - \beta$, and $z = T - \gamma$. Now the required inequality is equivalent to

$$(T + \alpha)(T - \beta) + (T + \beta)(T - \gamma) + (T + \gamma)(T - \alpha) < 4T^2.$$

After simplifying we get that what we need to prove is

$$-(\alpha\beta+\beta\gamma+\gamma\alpha)<T^2. \tag{32}$$

We also know that at most one of the numbers α, β, γ is negative. If all are positive, there is nothing to prove. Assume that $\gamma<0$. Now (32) can be rewritten as $-\alpha\beta-\gamma(\alpha+\beta)<T^2$. Since $-\gamma<T$ we have that $-\alpha\beta-\gamma(\alpha+\beta)<-\alpha\beta+T(\alpha+\beta)$ and the last term is less than T since $(T-\alpha)(T-\beta)>0$.

9. Starting from $\frac{(a-b)^2}{b}=\frac{a^2}{b}-2a+b$ and similar equalitites for $(b-c)^2/c$ and $(c-a)^2/a$ we get the required inequality is equivalent to

$$(a+b+c)\left(\frac{(a-b)^2}{b}+\frac{(b-c)^2}{a}+\frac{(c-a)^2}{b}\right)\geq 4(a-b)^2. \tag{33}$$

By the Cauchy-Schwartz inequality we have that the left-hand side of (33) is greater than or equal to $(|a-b|+|b-c|+|c-a|)^2$. (33) now follows from $|b-c|+|c-a|\geq|a-b|$.

10. Note that

$$\begin{aligned} & x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 \\ = & \left(x_1^2+\frac{x_2^2}{3}\right)+\left(\frac{2x_2^2}{3}+\frac{x_3^2}{2}\right)+\left(\frac{x_3^2}{2}+\frac{2x_4^2}{3}\right)+\left(\frac{x_4^2}{3}+x_5^2\right). \end{aligned}$$

Now applying the inequality $a^2+b^2\geq 2ab$ we get

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2\geq\frac{2}{\sqrt{3}}(x_1x_2+x_2x_3+x_3x_4+x_4x_5).$$

This proves that $a\geq\frac{2}{\sqrt{3}}$. In order to prove the other inequality it is sufficient to notice that for $(x_1,x_2,x_3,x_4,x_5)=(1,\sqrt{3},2,\sqrt{3},1)$ we have

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=\frac{2}{\sqrt{3}}(x_1x_2+x_2x_3+x_3x_4+x_4x_5).$$

11. Since $xy+yz+zx-2xyz=(x+y+z)(xy+yz+zx)-2xyz=T[2,1,0]+\frac{1}{6}T[1,1,1]$ the left part of the inequality follows immediately. In order to prove the other part notice that

$$\frac{7}{27}=\frac{7}{27}(x+y+z)^3=\frac{7}{27}\left(\frac{1}{2}T[3,0,0]+3T[2,1,0]+T[1,1,1]\right).$$

After multiplying both sides by 54 and cancel as many things as possible we get that the required inequality is equivalent to:

$$12T[2,1,0]\leq 7T[3,0,0]+5T[1,1,1].$$

This inequality is true because it follows by adding up the inequalities $2T[2,1,0]\leq 2T[3,0,0]$ and $10T[2,1,0]\leq 5T[3,0,0]+5T[1,1,1]$ (the first one is a consequence of the Muirhead's and the second one of the Schur's theorem for $\alpha=\beta=1$).

12. The expressions have to be homogenous in order to apply the Muirhead's theorem. First we divide both left and right-hand side by $(abc)^{\frac{4}{3}} = 1$ and after that we multiply both sides by $a^3b^3c^3(a+b)(b+c)(c+a)(abc)^{\frac{4}{3}}$. The inequality becomes equivalent to
$$2T\left[\frac{16}{3},\frac{13}{3},\frac{7}{3}\right]+T\left[\frac{16}{3},\frac{16}{3},\frac{4}{3}\right]+T\left[\frac{13}{3},\frac{13}{3},\frac{10}{3}\right]\geq 3T[5,4,3]+T[4,4,4].$$
The last inequality follows by adding the following three which are immediate consequences of the Muirhead's theorem:
$$\begin{aligned}&1.\quad 2T\left[\tfrac{16}{3},\tfrac{13}{3},\tfrac{7}{3}\right]\geq 2T[5,4,3],\\ &2.\quad T\left[\tfrac{16}{3},\tfrac{16}{3},\tfrac{4}{3}\right]\geq T[5,4,3],\\ &3.\quad T\left[\tfrac{13}{3},\tfrac{13}{3},\tfrac{10}{3}\right]\geq T[4,4,4].\end{aligned}$$
The equality holds if and only if $a=b=c=1$.

13. The left-hand side can be easily transformed into $\frac{a^3(b+c)}{b^3+c^3}+\frac{b^3(c+a)}{c^3+a^3}+\frac{c^3(a+b)}{a^3+b^3}$. We now multiply both sides by $(a+b+c)(a^3+b^3)(b^3+c^3)(c^3+a^3)$. After some algebra the left-hand side becomes
$$\begin{aligned}L=\quad & T[9,2,0]+T[10,1,0]+T[9,1,1]+T[5,3,3]+2T[4,4,3]\\ &+T[6,5,0]+2T[6,4,1]+T[6,3,2]+T[7,4,0]+T[7,3,1],\end{aligned}$$
while the right-hand side transforms into
$$D=3(T[4,4,3]+T[7,4,0]+T[6,4,1]+T[7,3,1]).$$
According to Muirhead's theorem we have:
$$\begin{aligned}&1.\quad T[9,2,0]\geq T[7,4,0],\\ &2.\quad T[10,1,0]\geq T[7,4,0],\\ &3.\quad T[6,5,0]\geq T[6,4,1],\\ &4.\quad T[6,3,2]\geq T[4,4,3].\end{aligned}$$
The Schur's inequality gives us $T[4,2,2]+T[8,0,0]\geq 2T[6,2,0]$. After multiplying by abc, we get:
$$5.\quad T[5,3,3]+T[9,1,1]\geq T[7,3,1].$$
Adding up $1,2,3,4$, 5, and adding $2T[4,4,3]+T[7,4,0]+2T[6,4,1]+T[7,3,1]$ to both sides we get $L\geq D$. The equality holds if and only if $a=b=c$.

14. Multiplying the both sides with the common denominator we get
$$T_{5,5,5}+4T_{7,5,0}+T_{5,2,2}+T_{9,0,0}\geq T_{5,5,2}+T_{6,0,0}+2T_{5,4,0}+2T_{4,2,0}+T_{2,2,2}.$$
By Schur's and Muirhead's inequalities we have that $T_{9,0,0}+T_{5,2,2}\geq 2T_{7,2,0}\geq 2T_{7,1,1}$. Since $xyz\geq 1$ we have that $T_{7,1,1}\geq T_{6,0,0}$. Therefore
$$T_{9,0,0}+T_{5,2,2}\geq 2T_{6,0,0}\geq T_{6,0,0}+T_{4,2,0}.$$
Moreover, Muirhead's inequality combined with $xyz\geq 1$ gives us $T_{7,5,0}\geq T_{5,5,2}$, $2T_{7,5,0}\geq 2T_{6,5,1}\geq 2T_{5,4,0}$, $T_{7,5,0}\geq T_{6,4,2}\geq T_{4,2,0}$, and $T_{5,5,5}\geq T_{2,2,2}$. Adding these four inequalities to (1) yields the desired result.

15. Let $a_i = e^{x_i}$ and let $(m_1, \dots, m_n)$, $(k_1, \dots, k_n)$ be two permutations of $(1, \dots, n)$ for which the sequences $(3x_{m_1} - x_{m_1+1}, \dots, 3x_{m_n} - x_{m_n+1})$ and $(2x_{k_1}, \dots, 2x_{k_n})$ are non-increasing. As above we assume that $x_{n+1} = x_n$. Similarly as in the problem 11 from the section 5 we prove that $(2x_{k_i}) \prec (3x_{m_i} - x_{m_i+1})$. The function $f(x) = e^x$ is convex so the Karamata's implies the required result.

16. Hint: Choose x_i such that $a_i = e^{x_i}$. Sort the sequences $(2x_1 - x_2, \dots, 2x_n - x_1)$ and $(x_1, \dots, x_n)$ in non-increasing order, prove that the first majorizes the second, and apply Karamata's inequality with the convex function $f(x) = 1 + e^x$.

17. Applying the Chebyshev's inequality first we get

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geq \frac{a^n + b^n + c^n}{3} \cdot \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right).$$

The Cauchy-Schwartz inequality gives:

$$2(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right) \geq 9,$$

and the inequality $M_n \geq M_2$ gives

$$\frac{a^n + b^n + c^n}{3} \geq \left(\frac{a+b+c}{3} \right)^n.$$

In summary

$$\begin{aligned} \frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} &\geq \left(\frac{a+b+c}{3} \right)^n \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right) \\ &\geq \frac{1}{3} \cdot \frac{1}{2} \cdot \left(\frac{2}{3} s \right)^{n-1} \cdot 9 = \left(\frac{2}{3} \right)^{n-2} s^{n-1}. \end{aligned}$$

18. It is enough to prove that

$$\begin{aligned} & \left(\sqrt{x_1} + \frac{1}{\sqrt{x_1}} \right) + \left(\sqrt{x_2} + \frac{1}{\sqrt{x_2}} \right) + \cdots + \left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}} \right) \\ \geq \quad & n \left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}} \right), \end{aligned}$$

or equivalently

$$\begin{aligned} & \left(\frac{1+x_1}{\sqrt{x_1}} + \cdots + \frac{1+x_n}{\sqrt{x_n}} \right) \left(\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} \right) \\ \geq \quad & n \cdot \left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}} \right). \end{aligned}$$

Consider the function $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} = \frac{x+1}{\sqrt{x}}, x \in (0, +\infty)$. It is easy to verify that f is non-decreasing on$(1, +\infty)$ and that $f(x) = f\left(\frac{1}{x}\right)$ for every $x > 0$. Furthermore from the given conditions it follows that only x_1 can be less than 1 and that $\frac{1}{1+x_2} \leq 1 - \frac{1}{1+x_1} = \frac{x_1}{1+x_1}$. Hence $x_2 \geq \frac{1}{x_1}$. Now it is clear that (in both of the cases $x_1 \geq 1$ and $x_1 < 1$):

$$f(x_1) = f\left(\frac{1}{x_1}\right) \leq f(x_1) \leq \cdots \leq f(x_n).$$

This means that the sequence $\left(\frac{1+x_k}{x_k}\right)_{k=1}^{n}$ is non-decreasing. Thus according to the Chebyshev's inequality we have:

$$\begin{aligned} & \left(\frac{1+x_1}{\sqrt{x_1}} + \cdots + \frac{1+x_n}{\sqrt{x_n}}\right)\left(\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n}\right) \\ \geq \quad & n \cdot \left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right). \end{aligned}$$

The equality holds if and only if $\frac{1}{1+x_1} = \cdots = \frac{1}{1+x_n}$, or $\frac{1+x_1}{\sqrt{x_1}} = \cdots = \frac{1+x_n}{\sqrt{x_n}}$, which implies that $x_1 = x_2 = \cdots = x_n$. Thus the equality holds if and only if $x_1 = \cdots = x_n = n-1$.

19. Denote by S the set of all members of the society, by A the set of all pairs of friends, and by N the set of all pairs of enemies. For every $x \in S$, denote by $f(x)$ number of friends of x and by $F(x)$ number of pairs of friends among enemies of x. It is easy to prove:

$$q = |A| = \frac{1}{2}\sum_{x \in S} f(x);$$

$$\sum_{\{a,b\} \in A} (f(a) + f(b)) = \sum_{x \in S} f^2(x).$$

If a and b are friends, then the number of their common enemies is equal to $(n-2) - (f(a) - 1) - (f(b) - 1) = n - f(a) - f(b)$. Thus

$$\frac{1}{n}\sum_{x \in S} F(x) = \frac{1}{n}\sum_{\{a,b\} \in A} (n - f(a) - f(b)) = q - \frac{1}{n}\sum_{x \in S} f^2(x).$$

Using the inequality between arithmetic and quadratic mean on the last expression, we get

$$\frac{1}{n}\sum_{x \in S} F(x) \leq q - \frac{4q^2}{n^2}$$

and the statement of the problem follows immediately.

20. Consider the partition of plane π into regular hexagons, each having inradius 2. Fix one of these hexagons, denoted by γ. For any other hexagon x in the partition, there exists a unique translation τ_x taking it onto γ. Define the mapping $\varphi : \pi \to \gamma$ as follows: If A belongs to the interior of a hexagon x, then $\varphi(A) = \tau_x(A)$ (if A is on the border of some hexagon, it does not actually matter where its image is).

The total area of the images of the union of the given circles equals S, while the area of the hexagon γ is $8\sqrt{3}$. Thus there exists a point B of γ that is covered at least $\dfrac{S}{8\sqrt{3}}$ times, i.e., such that $\varphi^{-1}(B)$ consists of at least $\frac{S}{8\sqrt{3}}$ distinct points of the plane that belong to some of the circles. For any of these points, take a circle that contains it. All these circles are disjoint, with total area not less than $\frac{\pi}{8\sqrt{3}}S \geq 2S/9$.