2001/2003

Hard Functional Equation Problems

Notation: $\mathbb{N}$ denotes the set of positive integers, $\mathbb{W}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers, and $\mathbb{Q}$ denotes the set of rational numbers.

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)f(y) - f(xy) = x + y$ for all $x, y \in \mathbb{R}$.

2. Find all functions $f : \mathbb{Z} \to \mathbb{R}$ such that $f(0) \neq 0$, $f(1) = \frac{5}{2}$, and $f(x)f(y) = f(x+y) + f(x-y)$ for all $x, y \in \mathbb{Z}$.

3. Find all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(xy) = 2f(x)f(y)$, for all $x, y \in \mathbb{R}^+$.

4. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + f(y)) = f(x) + y$ for all $x, y \in \mathbb{R}$.

5. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $xf(y) - yf(x) = (x - y)f(xy)$ for all $x, y \in \mathbb{R}$.

6. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x - f(y)) = 1 - x - y$ for all $x, y \in \mathbb{R}$.

7. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x+y) + 1$ for all $x, y \in \mathbb{Q}$.

8. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(n+k) + f(n-k) = 2f(k)f(n)$ for all integers n and k, and there exists an integer N for which $|f(n)| \leq N$ for all n.

9. $f : \mathbb{N} \to \mathbb{W}$ is a function such that $f(m+n) - f(m) - f(n)$ equals 0 or 1 for all $m, n \in \mathbb{N}$. Also, $f(2) = 0, f(3) > 0$, and $f(9999) = 3333$. Determine $f(1982)$.

   (1982 IMO, Question 1)

10. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(xf(y)) = yf(x)$ for all $x, y \in \mathbb{R}^+$ and $f(x) \to 0$ as $x \to \infty$.

    (1983 IMO)

11. $f : \mathbb{N} \to \mathbb{W}$ is a function such that

    (a) $f(mn) = f(m) + f(n)$ for all $m, n \in \mathbb{N}$.

    (b) $f(n) = 0$ whenever the units digit of n is 3.

    (c) $f(10) = 0$.

    Determine the value of $f(1985)$.

    (Proposed for the 1985 IMO)

12. Find all functions $f : \mathbb{W} \to \mathbb{W}$ such that $f(m + f(n)) = f(f(m)) + f(n)$ for all $m, n \in \mathbb{W}$.

    (1996 IMO, Question 3)

13. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions satisfying the equation $f(x+y)+f(x-y) = 2f(x)g(y)$ for all $x, y \in \mathbb{R}$. Prove that if $f(x)$ is not identically zero, and if $|f(x)| \leq 1$ for all x, then $|g(y)| \leq 1$ for all y.

(1972 IMO, Question 5)

14. Find all polynomials P in two variables, with the following properties:

 (a) For a positive integer n and all real t, x, y, $P(tx, ty) = t^n P(x, y)$.

 (b) $P(b+c, a) + P(c+a, b) + P(a+b, c) = 0$ for all real a, b, c.

 (c) $P(1, 0) = 1$.

(1975 IMO, Question 6)

15. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(n+1) > f(f(n))$ for each $n \in \mathbb{N}$. Prove that $f(n) = n$ for each n.

(1977 IMO, Question 6)

16. The set of all positive integers is the union of two disjoint subsets $\{f(1), f(2), f(3), \ldots\}$, and $\{g(1), g(2), g(3), \ldots\}$ where $f(1) < f(2) < f(3) < \ldots$, $g(1) < g(2) < g(3) < \ldots$, and $g(n) = f(f(n)) + 1$ for all $n \geq 1$. Determine $f(240)$.

(1978 IMO, Question 6)

17. The function $f(x, y)$ satisfies

 (a) $f(0, y) = y + 1$.

 (b) $f(x+1, 0) = f(x, 1)$.

 (c) $f(x+1, y+1) = f(x, f(x+1, y))$.

Determine the value of $f(4, 1981)$.

(1981 IMO, Question 6)

18. $f : \mathbb{N} \to \mathbb{N}$ is a function such that $f(n^2 f(m)) = m(f(n))^2$, for all $m, n \in \mathbb{N}$. Determine the least possible value of $f(1998)$.

(1998 IMO, Question 6).