WOOT 2010–11

Practice AIME 3 Solutions

Answer Key

1. 095	6. 096	11. 289
2. 143	7. 189	12. 337
3. 512	8. 315	13. 023
4. 324	9. 075	14. 013
5. 209	10. 077	15. 008

1. Find the smallest positive integer n such that the fractions

$$\frac{19}{n+21}, \quad \frac{20}{n+22}, \quad \frac{21}{n+23}, \quad \dots, \quad \frac{91}{n+93}$$

are all irreducible.

Solution. The fraction $\frac{k}{n+k+2}$ is irreducible if and only if k and $n+k+2$ are relatively prime, or equivalently, $n+2$ and k are relatively prime. The positive integer k ranges over all positive integers from 19 to 91, which means $n+2$ must be relatively prime to all the primes from 2 to 89. Therefore, $n+2$ must be at least 97, so n must be at least 95.

2. In a convex polygon with 18 sides, the angles are all positive integers (when measured in degrees) and form a nonconstant arithmetic sequence. Find the measure of the smallest angle, in degrees.

Solution. Let the angles be a, $a+2d$, $\dots$, $a+17d$, in degrees. Then the sum of the angles is

$$18a + \frac{17 \cdot 18}{2} \cdot d = 180 \cdot 16,$$

which becomes $18a + 153d = 2880$. Dividing both sides by 9, we get $2a + 17d = 320$. Taking both sides modulo 17, we get $2a \equiv 14 \pmod{17}$. Since 2 is relatively prime to 17, we may divide both sides by 2, to get $a \equiv 7 \pmod{17}$.

Since the polygon is convex, each of the angles is less than 180 degrees. In particular, $a + 17d < 180$. Therefore, $a = (2a+17d) - (a+17d) > 320 - 180 = 140$. Since $2a + 17d = 320$ and d is positive, $a < 320/2 = 160$. The only positive integer in the range $140 < a < 160$ that satisfies $a \equiv 7 \pmod{17}$ is $a = 143$. (The corresponding value of d is 2.)

3. For two sets A and B, let $f(A, B)$ denote the number of elements that are either in A or B, but not both. For example, $f(\{1,2,3,4,6\}, \{2,3,6,7\}) = 3$, because of the elements 1, 4, and 7.

For every ordered pair of sets (A, B), where A and B are subsets of $\{1,2,3,4\}$, we compute $f(A, B)$. Find the sum of $f(A, B)$ taken over all such ordered pairs (A, B).

Solution. We find the sum of $f(A, B)$ by counting how many times each element in the set $\{1,2,3,4\}$ is counted in this sum.

Consider an element x in the set $\{1,2,3,4\}$. The element x is counted in $f(A, B)$ if either x is in A but not in B, or x is in B but not in A. Then any subset of the three remaining elements (for a total of 8) can be added to A, and any subset (for another total of 8) can be added to B.

There are four elements in the set $\{1, 2, 3, 4\}$, so the sum of $f(A, B)$ over all such ordered pairs (A, B) is $2 \cdot 8 \cdot 8 \cdot 4 = 512$.

4. A floor can be covered with n identical square tiles of a certain size. If a certain smaller square tile is used, then $n + 76$ tiles are required. Find n.

   Solution. Let the side lengths of the square tiles be x and y, where $x > y$, and let a be one of the side lengths of the floor. Then $a = qx = py$ for some positive integers p and q, and the area of the floor is $nx^2 = (n + 76)y^2$. Therefore,
$$\frac{n+76}{n} = \frac{x^2}{y^2} = \frac{p^2}{q^2}.$$
   Let $d = \gcd(p, q)$, and let $p = du$ and $q = dv$, so u and v are relatively prime. Then
$$\frac{n+76}{n} = \frac{p^2}{q^2} = \frac{d^2u^2}{d^2v^2} = \frac{u^2}{v^2}.$$
   Hence, $n + 76 = ku^2$ and $n = kv^2$ for some positive integer k, so $ku^2 - kv^2 = 76$, which factors as $k(u + v)(u - v) = 2^2 \cdot 19$.

   The difference between $u + v$ and $u - v$ is $2v$, so they must have the same parity, i.e. they are both even or both odd. If both $u + v$ and $u - v$ are even, then we must have $u + v = 2 \cdot 19$, $u - v = 2$, and $k = 1$, which means $u = (2 \cdot 9 + 2)/2 = 20$ and $v = (2 \cdot 19 - 2)/2 = 18$, which are not relatively prime.

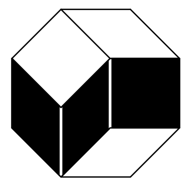
   Hence, both $u + v$ and $u - v$ are odd. Then we must have $u + v = 19$, $u - v = 1$, and $k = 4$, so $v = (19 - 1)/2 = 9$, and $n = 4 \cdot 9^2 = 324$.

5. There exist unique integers x and y such that
$$(2 - \sqrt{3})^4 x + (2 - \sqrt{3})^5 y = 1.$$
   Find x.

   Solution 1. Expanding, we get $(2 - \sqrt{3})^4 = 97 - 56\sqrt{3}$ and $(2 - \sqrt{3})^5 = 362 - 209\sqrt{3}$, so
$$\begin{aligned}(2 - \sqrt{3})^4 x + (2 - \sqrt{3})^5 y &= (97 - 56\sqrt{3})x + (362 - 209\sqrt{3})y \\ &= 97x + 362y - (56x + 209y)\sqrt{3} \\ &= 1.\end{aligned}$$
   Hence, $97x + 362y = 1$ and $56x + 209y = 0$. Solving this system of equations, we get $x = 209$ and $y = -56$.

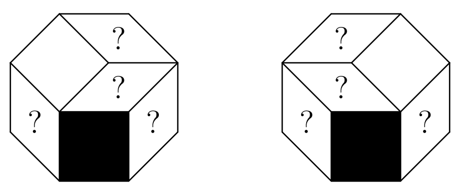
   Solution 2. Multiplying both sides by $(2 + \sqrt{3})^4$, we get
$$x + (2 - \sqrt{3})y = (2 + \sqrt{3})^4,$$
   which becomes $x + 2y - y\sqrt{3} = 97 + 56\sqrt{3}$. Hence, $x + 2y = 97$ and $y = -56$, so $x = 97 - 2 \cdot (-56) = 209$.

6. Mr. X has six tiles that are all rhombi with side length 1: one black square, one white square, two identical black rhombi with angles 45° and 135°, and two identical white rhombi with angles 45° and 135°. In how many different ways can Mr. X tile a regular octagon with side length 1 using these six tiles? An example of such a tiling is shown below.

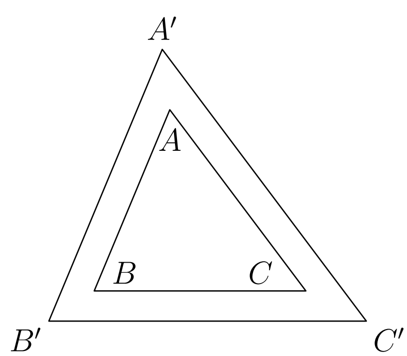


(Even if one tiling can be obtained from another tiling by rotating or reflecting the octagon, they are still considered different.)

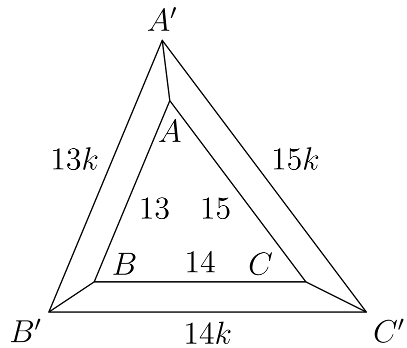
Solution. First, we choose the side of the octagon that the black square is adjacent to, which can be done in 8 different ways. After the position of the black square has been chosen, the white square can be placed in two possible positions, as shown below. Then the remaining four non-square rhombi can be colored in $\binom{4}{2} = 6$ ways, each way producing a different tiling. Therefore, there are a total of $8 \cdot 2 \cdot 6 = 96$ different tilings.



7. Let ABC and $A'B'C'$ be two triangles, such that triangle ABC lies inside $A'B'C'$ and the corresponding sides are parallel. The distance between each pair of corresponding sides is 2. Find the area of triangle $A'B'C'$ if the sides of triangle ABC are 13, 14, and 15.



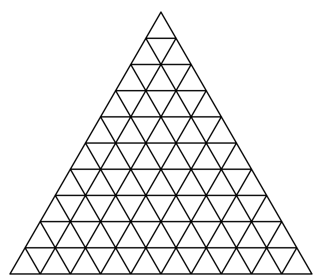
Solution. Without loss of generality, let $BC = 14$, $AC = 15$, and $AB = 13$. Triangles ABC and $A'B'C'$ are similar, so $B'C' = 14k$, $A'C' = 15k$, and $A'B' = 13k$ for some real number $k > 1$.



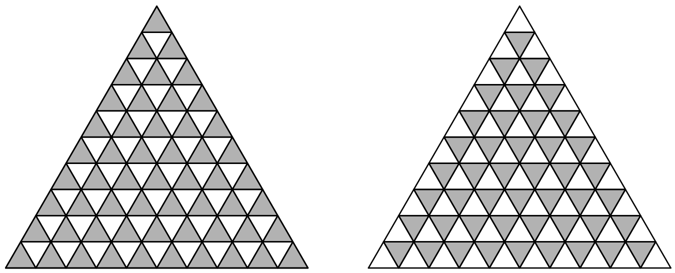
The area of trapezoid $BCC'B'$ is $\frac{1}{2} \cdot 2 \cdot (14k + 14) = 14k + 14$. Similarly, the areas of trapezoids $ACC'A'$ and $ABB'A'$ are $15k + 15$ and $13k + 13$, respectively. By Heron's formula, the area of triangle ABC is 84, so the area of triangle $A'B'C'$ is $84 + 13k + 13 + 14k + 14 + 15k + 15 = 42k + 126$.

But the area of triangle $A'B'C'$ is also $84k^2$, so $84k^2 - 42k - 126 = 42(2k - 3)(k + 1) = 0$. Since $k > 1$, $k = 3/2$. Therefore, the area of triangle $A'B'C'$ is $84 \cdot (3/2)^2 = 189$.

8. An equilateral triangle is divided into 100 smaller equilateral triangles, as shown below. Find the number of equilateral triangles of any size in the diagram.



Solution. Let the side length of the large equilateral triangle be 10, so each small equilateral triangle has side length 1. There are two types of equilateral triangles, namely those that have the same orientation as the large equilateral triangle, and those that have the opposite orientation. Let T_n denote $n(n + 1)/2$, the n^{th} triangular number.



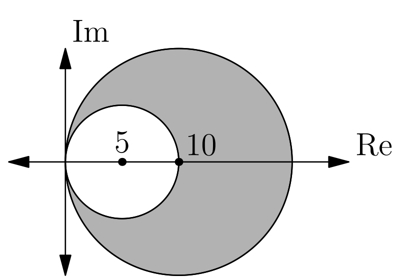
First, we count the equilateral triangles that have the same orientation. There are T_{10} such equilateral triangles of side length 1, as shown above and to the left. Similarly, there are T_9 such equilateral triangles of side length 2, T_8 such equilateral triangles of side length 3, and so on, until $T_1 = 1$ such equilateral triangle of side length 10, for a total of $T_{10} + T_9 + T_8 + \cdots + T_1 = 55 + 45 + 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1 = 220$.

Next, we count the equilateral triangles that have the opposite orientation. There are T_9 such equilateral triangles of side length 1, as shown above and to the right. Similarly, there are T_7 such equilateral triangles of side length 2, T_5 such equilateral triangles of side length 3, T_3 such equilateral triangles of side length 2, and $T_1 = 1$ such equilateral triangle of side length 5, for a total of $T_9 + T_7 + T_5 + T_3 + T_1 = 45 + 28 + 15 + 6 + 1 = 95$.

Therefore, the total number of equilateral triangles is $220 + 95 = 315$.

9. Let S be the set of complex numbers z such that the real part of $1/\overline{z}$ lies between $1/20$ and $1/10$. As a subset of the complex plane, the area of S can be expressed in the form $k\pi$, where k is a positive integer. Determine k.

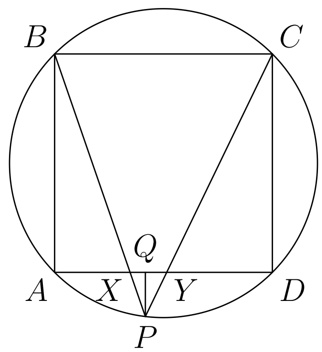
   Solution. Let $z = x + yi$, where x and y are real numbers, so the real part of
$$\frac{1}{\overline{z}} = \frac{1}{x - yi} = \frac{x + yi}{x^2 + y^2}$$
   is $x/(x^2 + y^2)$. Then
$$\frac{x}{x^2 + y^2} \geq \frac{1}{20} \quad \Leftrightarrow \quad x^2 + y^2 \leq 20x \quad \Leftrightarrow \quad (x - 10)^2 + y^2 \leq 100$$
   and
$$\frac{x}{x^2 + y^2} \leq \frac{1}{10} \quad \Leftrightarrow \quad x^2 + y^2 \geq 10x \quad \Leftrightarrow \quad (x - 5)^2 + y^2 \geq 25.$$



Hence, S is the set of complex numbers that lie inside the circle with center 10 and radius 10, and outside the circle with center 5 and radius 5. The latter circle lies inside the former circle, so the area of S is $100\pi - 25\pi = 75\pi$, which means $k = 75$.

10. Let ω be the circumcircle of square $ABCD$, and let P be a point on arc AD of ω. Let X and Y be the intersections of $\overline{PB}$ and $\overline{PC}$ with $\overline{AD}$, respectively. If $AX = 5$ and $DY = 7$, then XY can be expressed in the form $\sqrt{m} - n$, where m and n are positive integers. Find $m + n$.

Solution 1. More generally, let $AX = a$ and $DY = b$, and let s be the side length of square $ABCD$. Let Q be the projection of P onto AD, and let $h = PQ$.



Then triangles BAX and PQX are similar, so $QX = ah/s$. Similarly, $QY = bh/s$. Hence,

$$a + \frac{ah}{s} + \frac{bh}{s} + b = s,$$

which implies $(a + b)(s + h) = s^2$.

Also, by Pythagoras, $BX = \sqrt{s^2 + a^2}$, and $PX = h/s \cdot \sqrt{s^2 + a^2}$. By power of a point on X, $BX \cdot PX = AX \cdot DX$, which becomes

$$\frac{h}{s}(s^2 + a^2) = a\left(\frac{ah}{s} + \frac{bh}{s} + b\right).$$

This simplifies as $ab(s + h) = hs^2$. Dividing these equations, we get $h = ab/(a + b)$. Hence,

$$(a + b)\left(s + \frac{ab}{a + b}\right) = s^2 \quad \Rightarrow \quad s^2 - (a + b)s - ab = 0.$$

By the quadratic formula,

$$s = \frac{a + b \pm \sqrt{a^2 + 6ab + b^2}}{2}.$$

Since $\sqrt{a^2 + 6ab + b^2} > \sqrt{a^2 + 2ab + b^2} = a + b$, we must take the root with the plus sign. Therefore,

$$XY = s - (a + b) = \frac{\sqrt{a^2 + 6ab + b^2} - a - b}{2}.$$

In particular, when $a = 5$ and $b = 7$, $XY = \sqrt{71} - 6$. The final answer is $71 + 6 = 77$.

Solution 2. From right triangles BAX and CDY, $\tan\angle BXA = s/a$ and $\tan\angle CYD = s/b$, so

$$\tan(\angle BXA + \angle CYD) = \frac{\tan\angle BXA + \tan\angle CYD}{1 - \tan\angle BXA \cdot \tan\angle CYD} = \frac{s/a + s/b}{1 - s/a \cdot s/b} = \frac{(a+b)s}{ab - s^2}.$$

But $\angle BXA + \angle CYD = \angle PXY + \angle PYX = 180^\circ - \angle XPY = 180^\circ - 45^\circ = 135^\circ$, so $\tan(\angle BXA + \angle CYD) = \tan 135^\circ = -1$. Therefore, $s(a+b) = s^2 - ab$, or $s^2 - (a+b)s - ab = 0$. We can then proceed as in Solution 1.

11. Let

$$S = \sum_{i,j,k} \frac{1}{3^{i+j+k}},$$

where the sum is taken over all ordered triples of nonnegative integers (i, j, k), where i, j, and k are distinct, i.e. $i \neq j$, $i \neq k$, and $j \neq k$. Then S can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

Solution 1. Let S be the set of ordered triples of nonnegative integers (i, j, k), and let A, B, and C be the sets of ordered triples of nonnegative integers (i, j, k), for which $i = j$, $i = k$, and $j = k$, respectively. Then by the Principle of Inclusion-Exclusion,

$$\begin{aligned}\sum_{A\cup B\cup C} \frac{1}{3^{i+j+k}} &= \sum_{A} \frac{1}{3^{i+j+k}} + \sum_{B} \frac{1}{3^{i+j+k}} + \sum_{C} \frac{1}{3^{i+j+k}} \\ &\quad - \sum_{A\cap B} \frac{1}{3^{i+j+k}} - \sum_{A\cap C} \frac{1}{3^{i+j+k}} - \sum_{B\cap C} \frac{1}{3^{i+j+k}} \\ &\quad + \sum_{A\cap B\cap C} \frac{1}{3^{i+j+k}}.\end{aligned}$$

First, we compute

$$\sum_{A} \frac{1}{3^{i+j+k}}.$$

If $(i, j, k) \in A$, then $i = j$, so we can re-write the sum as

$$\sum_{A} \frac{1}{3^{i+j+k}} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{3^{2i+k}} = \sum_{i=0}^{\infty} \frac{1}{9^i} \sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{1}{1 - 1/9} \cdot \frac{1}{1 - 1/3} = \frac{9}{8} \cdot \frac{3}{2} = \frac{27}{16}.$$

Similarly,

$$\sum_{B} \frac{1}{3^{i+j+k}} = \sum_{C} \frac{1}{3^{i+j+k}} = \frac{27}{16}.$$

Now, note that $A \cap B = A \cap B = B \cap C = A \cap B \cap C$, because each of these sets is the set of ordered triples (i, j, k) where $i = j = k$. Hence,

$$\sum_{A\cap B} \frac{1}{3^{i+j+k}} = \sum_{A\cap C} \frac{1}{3^{i+j+k}} = \sum_{B\cap C} \frac{1}{3^{i+j+k}} = \sum_{A\cap B\cap C} \frac{1}{3^{i+j+k}}$$
$$= \sum_{i=0}^{\infty} \frac{1}{3^{3i}} = \sum_{i=0}^{\infty} \frac{1}{27^i} = \frac{1}{1-1/27} = \frac{27}{26}.$$

Therefore,

$$\sum_{A\cup B\cup C} \frac{1}{3^{i+j+k}} = 3 \cdot \frac{27}{16} - 2 \cdot \frac{27}{26} = \frac{621}{208}.$$

The set $A \cup B \cup C$ is the set of ordered triples (i, j, k) where at least two of i, j, and k are equal. The sum of $1/3^{i+j+k}$ taken over all ordered triples is

$$\sum_{i,j,k=0}^{\infty} \frac{1}{3^{i+j+k}} = \sum_{i=0}^{\infty} \frac{1}{3^i} \sum_{j=0}^{\infty} \frac{1}{3^j} \sum_{k=0}^{\infty} \frac{1}{3^k} = \left(\frac{1}{1-1/3}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8},$$

so the sum of $1/3^{i+j+k}$ taken over all ordered triples (i, j, k) where i, j, and k are distinct is equal to

$$\frac{27}{8} - \frac{621}{208} = \frac{81}{208}.$$

The final answer is $81 + 208 = 289$.

Solution 2. Since i, j, and k are distinct, we can compute the given sum by taking the sum of $1/3^{i+j+k}$ over all ordered triples (i, j, k) such that $i < j < k$, and then multiplying by $3!$ to account for all permutations. Hence,

$$S = 3! \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \sum_{k=j+1}^{\infty} \frac{1}{3^{i+j+k}}.$$

We can write this sum as

$$S = 6 \sum_{i=0}^{\infty} \frac{1}{3^i} \sum_{j=i+1}^{\infty} \frac{1}{3^j} \sum_{k=j+1}^{\infty} \frac{1}{3^k}.$$

We have that

$$\sum_{k=j+1}^{\infty} \frac{1}{3^k} = \frac{1/3^{j+1}}{1-1/3} = \frac{3}{2} \cdot \frac{1}{3^{j+1}} = \frac{1}{2 \cdot 3^j},$$

so

$$S = 6 \sum_{i=0}^{\infty} \frac{1}{3^i} \sum_{j=i+1}^{\infty} \frac{1}{3^j} \cdot \frac{1}{2 \cdot 3^j} = 3 \sum_{i=0}^{\infty} \frac{1}{3^i} \sum_{j=i+1}^{\infty} \frac{1}{9^j}.$$

We have that

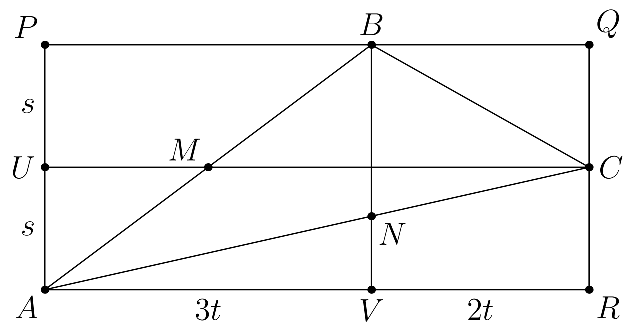
$$\sum_{j=i+1}^{\infty} \frac{1}{9^j} = \frac{1/9^{i+1}}{1-1/9} = \frac{9}{8} \cdot \frac{1}{9^{i+1}} = \frac{1}{8 \cdot 9^i},$$

so

$$S = 3\sum_{i=0}^{\infty} \frac{1}{3^i} \cdot \frac{1}{8 \cdot 9^i} = \frac{3}{8}\sum_{i=0}^{\infty} \frac{1}{27^i} = \frac{3}{8} \cdot \frac{1}{1-1/27} = \frac{81}{208}.$$

12. In triangle ABC, $AB = 30$ and $AC = 41$. Let M be the midpoint of $\overline{AB}$, and let N be the point on side $\overline{AC}$ such that $AN : NC = 3 : 2$. Find BC^2 if $\overline{BN}$ is perpendicular to $\overline{CM}$.

Solution. Construct lines through A and B parallel to CM, and lines through A and C parallel to BN. Since BN is perpendicular to CM, these four lines determine a rectangle $APQR$, as shown. Let U be the intersection of CM and AP, and let V be the intersection of BN and AR.



Since triangles AMU and ABP are similar, U is the midpoint of AP. Let $s = AU = UP$. Similarly, triangles ANV and ACR are similar, so $AV : VR = AN : NC = 3 : 2$. Let $AV = 3t$ and $VR = 2t$.

Then by Pythagoras, $AB^2 = AP^2+PB^2$, which becomes $900 = 4s^2+9t^2$, and $AC^2 = AR^2+RC^2$, which becomes $1681 = 25t^2+s^2$. Solving for s and t, we find $s = 9$ and $t = 8$. Therefore, $BC^2 = s^2+4t^2 = 337$.

13. Let x and y be angles such that $\sin x + \sin y = \frac{27}{25}$ and $\cos x + \cos y = \frac{39}{25}$. Then

$$\tan\frac{x}{2} + \tan\frac{y}{2}$$

can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

Solution 1. Let $\alpha = x/2$ and $\beta = y/2$, so $x = 2\alpha$ and $y = 2\beta$. Then

$$\begin{aligned} e^{2i\alpha} + e^{2i\beta} &= e^{ix} + e^{iy} \\ &= \cos x + i\sin x + \cos y + i\sin y \\ &= (\cos x + \cos y) + i(\sin x + \sin y) \\ &= \frac{39+27i}{25}. \end{aligned}$$

We can also write

$$\begin{aligned} e^{2i\alpha} + e^{2i\beta} &= e^{i(\alpha+\beta)}[e^{i(\alpha-\beta)} + e^{i(\beta-\alpha)}] \\ &= e^{i(\alpha+\beta)}[\cos(\alpha-\beta) + i\sin(\alpha-\beta) + \cos(\beta-\alpha) + i\sin(\beta-\alpha)] \\ &= 2\cos(\alpha-\beta)e^{i(\alpha+\beta)}, \end{aligned}$$

so

$$2\cos(\alpha-\beta)e^{i(\alpha+\beta)} = \frac{39+27i}{25},$$

or

$$\cos(\alpha-\beta)[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] = \frac{39+27i}{50}.$$

The absolute value of this complex number is

$$\frac{\sqrt{39^2+27^2}}{50} = \frac{\sqrt{2250}}{50} = \frac{15\sqrt{10}}{50} = \frac{3}{\sqrt{10}},$$

and

$$\frac{39+27i}{50} = \frac{3}{\sqrt{10}} \cdot \frac{13+9i}{5\sqrt{10}}.$$

Hence,

$$\begin{aligned} \cos(\alpha-\beta) &= \frac{3}{\sqrt{10}}, \\ \cos(\alpha+\beta) &= \frac{13}{5\sqrt{10}}, \\ \sin(\alpha+\beta) &= \frac{9}{5\sqrt{10}}. \end{aligned}$$

We want the value of

$$\begin{aligned} \tan\frac{x}{2} + \tan\frac{y}{2} &= \tan\alpha + \tan\beta \\ &= \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}. \end{aligned}$$

The numerator is

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta) = \frac{9}{5\sqrt{10}}.$$

The denominator is

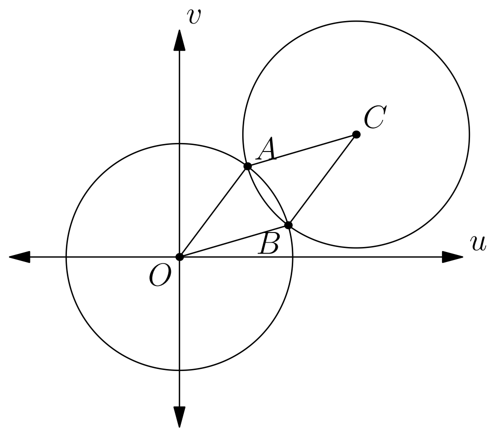
$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)] = \frac{1}{2}\left(\frac{3}{\sqrt{10}} + \frac{13}{5\sqrt{10}}\right) = \frac{14}{5\sqrt{10}}.$$

Therefore,

$$\tan\frac{x}{2}+\tan\frac{y}{2}=\frac{9/(5\sqrt{10})}{14/(5\sqrt{10})}=\frac{9}{14}.$$

The final answer is $9+14=23$.

Solution. In the coordinate plane, let $O=(0,0)$, $A=(\cos x,\sin x)$, $B=(\cos y,\sin y)$, and $C=(\cos x+\cos y,\sin x+\sin y)=(\frac{39}{25},\frac{27}{25})$. Then the points O, A, B, and C form the vertices of a rhombus with side length 1. Hence, A and B are the intersections of the unit circles centered at O and C.



The equations of the unit circles centered at O and C are

$$u^2+v^2=1,$$
$$\left(u-\frac{39}{25}\right)^2+\left(v-\frac{27}{25}\right)^2=1.$$

Subtracting these equations, we get

$$\frac{2\cdot 39}{25}\cdot u+\frac{2\cdot 27}{25}\cdot v-\frac{39^2}{25^2}-\frac{27^2}{25^2}=0,$$

which simplifies to $13u+9v=15$.

Then $v=(15-13u)/9$. Substituting into $u^2+v^2=1$, we get

$$u^2+\frac{(15-13u)^2}{9^2}=1,$$

which simplifies to

$$250u^2-390u+144=2(5u-3)(25u-24)=0,$$

so $u=3/5$ or $u=24/25$. If $u=3/5$, then $v=4/5$. If $u=24/25$, then $v=7/25$. Hence, the points A and B are $(\frac{3}{5},\frac{4}{5})$ and $(\frac{24}{25},\frac{7}{25})$ in some order.

Without loss of generality, assume that

$$A = \left(\frac{3}{5}, \frac{4}{5}\right) \quad \text{and} \quad B = \left(\frac{24}{25}, \frac{7}{25}\right).$$

Then $\cos x = \frac{3}{5}$ and $\sin x = \frac{4}{5}$, so

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{4/5}{1+3/5} = \frac{1}{2}.$$

Also, $\cos y = \frac{24}{25}$ and $\sin y = \frac{7}{25}$, so

$$\tan\frac{y}{2} = \frac{\sin y}{1+\cos y} = \frac{7/25}{1+24/25} = \frac{1}{7}.$$

Therefore,

$$\tan\frac{x}{2} + \tan\frac{y}{2} = \frac{1}{2} + \frac{1}{7} = \frac{9}{14}.$$

14. Find the number of permutations a_1, a_2, a_3, ..., a_{101} of the numbers 2, 3, 4, ..., 102, such that a_k is divisible by k for all $1 \le k \le 101$.

Solution. There exists an index m_1, $1 \le m_1 \le 101$, such that $a_{m_1} = 102$, so m_1 divides 102. If $m_1 \neq 1$, then there exists an index m_2 such that $a_{m_2} = m_1$, so m_2 divides m_1. If $m_2 \neq 1$, then there exists an index m_3 such that $a_{m_3} = m_2$, so m_3 divides m_2, and so on. Eventually, we will find that some term in this sequence, say m_n, must be equal to 1.

Now, among the values of a_1, a_2, ..., a_{101} (which are 2, 3, 4, ..., 102 in some order), the values 102, m_1, m_2, ..., m_{n-1} have already been taken. Similarly, among the indices 1, 2, ..., 101, the indices m_1, m_2, ..., m_{n-1}, $m_n = 1$ have already been taken. Hence, the set of remaining values coincides with the set of remaining indices, which means that the permutation fixes each of these indices. (In other words, if $1 \le k \le 101$ and k does not appear among the taken indices m_1, m_2, ..., $m_n = 1$, then $a_k = k$.)

Thus, the set of possible permutations is in 1–1 correspondence with the set of sequences $(m_0, m_1, m_2, \dots, m_n)$, such that $m_0 = 102$, $m_n = 1$, and m_i divides m_{i-1} for all $1 \le i \le n$. There are precisely 13 such sequences:

$$\begin{gathered}(102, 51, 17, 1),\\ (102, 51, 3, 1),\\ (102, 51, 1),\\ (102, 34, 17, 1),\\ (102, 34, 2, 1),\\ (102, 34, 1),\\ (102, 6, 3, 1),\\ (102, 6, 2, 1),\\ (102, 6, 1),\\ (102, 17, 1),\\ (102, 3, 1),\\ (102, 2, 1),\\ (102, 1).\end{gathered}$$

15. Let
$$p(x) = -\frac{7}{6}x^2 + \frac{11}{2}x - \frac{7}{3}.$$

The set of real numbers $\{1, 2, 4\}$ has the property that $p(1) = 2$, $p(2) = 4$, and $p(4) = 1$.

There exists another set of three distinct real numbers $\{a, b, c\}$ such that $p(a) = b$, $p(b) = c$, and $p(c) = a$. Find $a + b + c$.

Solution. Let $f(x) = p(p(p(x))) - x$, a polynomial of degree 8. Then

$$f(1) = p(p(p(1))) - 1 = p(p(2)) - 1 = p(4) - 1 = 0,$$

so 1 is a root of $f(x) = 0$. Similarly, 2, 4, a, b, and c are also roots of $f(x) = 0$. This accounts for six of the roots of $f(x) = 0$.

Let r and s be the roots of $p(x) = x$. Then

$$f(r) = p(p(p(r))) - r = p(p(r)) - r = p(r) - r = 0,$$

so r is a root of $f(x) = 0$. Similarly, s is a root of $f(x) = 0$. Thus, we have accounted for all eight roots of $f(x) = 0$.

By definition, r and s are the roots of

$$p(x) - x = -\frac{7}{6}x^2 + \frac{9}{2}x - \frac{7}{3} = 0.$$

By Vieta's Formulas, $r + s = (9/2)/(7/6) = 27/7$.

To simplify the expansion of $f(x)$, let $A = -7/6$, $B = 11/2$, and $C = -7/3$. Then $p(x) = Ax^2+Bx+C$, so

$$\begin{aligned} p(p(x)) &= A(Ax^2 + Bx + C)^2 + B(Ax^2 + Bx + C) + C \\ &= A^3x^4 + 2A^2Bx^3 + \cdots, \\ p(p(p(x))) &= A(A^3x^4 + 2A^2Bx^3 + \cdots)^2 + B(A^3x^4 + 2A^2Bx^3 + \cdots) + C \\ &= A^7x^8 + 4A^6Bx^7 + \cdots. \end{aligned}$$

Hence, again by Vieta's Formulas, the sum of the roots of $f(x) = p(p(p(x))) - x$ is

$$1 + 2 + 4 + a + b + c + r + s = -\frac{4A^6B}{A^7} = -\frac{4B}{A} = \frac{132}{7}.$$

Therefore,
$$a + b + c = \frac{132}{7} - 1 - 2 - 4 - \frac{27}{7} = 8.$$