Introduction to Proof-Writing

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Andre Kessler and Mitchell Lee

Why should you learn how to write a proof? There seems to be a general mentality which says "proofs are hard and I don't need them for most things. Why should I learn?" We assure you that nothing could be further from the truth. First, there are several contests we compete in on math team which require you to write out your solutions in the form of a proof. These include ARML Power, Mandelbrot Team, and HMMT Team. What's more, if you qualify for the USAMO, you will be facing a set of six proof-based problems. Knowing how to write out your ideas in a lucid manner is essential for doing well in these competitions. Second, learning how to write proofs helps elucidate the problem-solving process and helps you get used to organizing your thoughts well. Generally, solving more difficult proof problems will help improve your contest math skills, but the converse is not necessarily true. Finally, math team is by no means the be-all end-all of mathematics. In fact, math team barely scratches the surface of mathematics at all. Most math you will encounter later in life will be saturated with proofs, so it's best if you start now by learning how to deal with them. You may well have gotten through high school mathematics without really needing to prove what you learn. We guarantee that will not be the case in college.

1 What is a Proof?

The objective of a proof is to convince the reader that a particular assertion is true. Generally, you should assume that the reader is a subtle mix of equal parts dumb and brilliant – he or she knows all well known results that can be looked up in a small amount of time, makes no large jumps in logic, catches all mistakes, and wants to give you the lowest score possible while still adhering to his or her rubric. It's very easy to lose points due to lack of clarity in a proof if you don't know what you're doing. On the contrary, it's very, very difficult to lose points with a complete, correct proof if you know what you're doing. Learning the following general proof-writing tips and techniques will make your proofs much easier to write and understand.

1. If a problem asks for an answer, begin by stating your answer or claim, and then proceed to prove it.

2. If you have to prove multiple preliminary statements before tackling the main problem, separate these from the rest of the proof by labeling each as a "lemma" and putting them at the beginning or end of your main proof. Prove each lemma, and then use them later – don't intersperse your main proof with lots of minor ones.

3. Make sure it is clear when you are using a certain technique, such as induction, contradiction, or well-ordering, and note when you are done using this technique. For example, you might start an induction proof with "Let's induct on N" and end with "and thus the induction is complete."

4. Never force the grader to think. Prove all results which you use unless their proofs are trivial. It's better to include an unnecessary proof of a result than to leave out a necessary proof of a result.

5. Explain each large step you take – and explain smaller ones if you're not sure they're clear.

6. Cite well-known theorems if they have a name. Don't cite a result if it doesn't have a name, unless it is really well-known or trivial to prove.

7. Proofread your solutions and make absolutely sure that they have no mistakes. Your proofs are a form of formal writing and thus they should have correct grammar and style. Readers have a math mode and an English mode; you should try to avoid the number of times they switch back and forth. Don't start sentences with symbols[1] and beware of using symbols to convey too much information[2] all at once.

8. Organization, organization, organization! Following the previous tip will help with this, but make sure that your statements are in a clear, logical order. If the grader cannot follow the proof, you will not get many points.

9. Try to involve the grader in the proof by using first-person plural form (that is, "we" or "our"). This should help them follow your solution.

10. Writing an outline on scratch paper may help prevent you from leaving out a step and it will give you a sense of how to organize your claims.

11. Write clearly – scratch-work proofs or illegible handwriting do not translate into points. Graders are proof-readers, not mind-readers!

2 Specific Tips

The above items alone make most proofs pretty clear. However, there are some special instances in which separate rules should be followed, and here are some of them:

- Always include a diagram on geometry problems.
- Write out induction clearly. Make it clear that you are using induction, which variable you are inducting on, and what the base cases are.
- Write out casework clearly. Make it clear what the cases are and how each case is dealt with.
- It is unnecessary to write how you arrived at a proof; simply give the proof.
- Never write "it is easy to show;" instead write "it is not difficult to show."

Richard Rusczyk and Mathew Crawford have written an excellent article called "How to Write a Solution," available at `http://www.artofproblemsolving.com/Resources/articles.php?page=howtowrite`. It has examples of well-written solutions and badly-written solutions to clarify what the above tips mean. You should take a look at this article to see some specific examples of the techniques mentioned in this lecture.

3 Problems

Work on these problems individually, following the proof-writing techniques outlined above. Each page of your solutions **must** contain **both** your name and the problem number for it to be properly graded. Please write out the proofs to separate problems on separate pieces of paper. Feel free to use both sides of a sheet of paper for two pages of a single proof, but remember to move to a new sheet of paper on each new problem. Your solutions will be graded for both correctness and

[1] Example where this is bad: "$ax^2 + bx + c = 0$ has real roots if $b^2 - 4ac \geq 0$." Instead, say "The quadratic equation $ax^2 + bx + c$ has real roots if $b^2 - 4ac \geq 0$."

[2] Example where this is bad: "If $\Delta = b^2 - 4ac \geq 0$, then the roots are real." Instead, say "Set $\Delta = b^2 - 4ac$. If $\Delta \geq 0$, then the roots are real."

clarity. Each problem is worth the same number of points, so keep that in mind while working on the problems. They're 7 points each for a total of 70 points; there will also be a style multiplier on each problem which is a value between 0.0 and 1.0 that depends on the elegance of your solution, 1.0 being the best and 0.0 the worst. The problems are arranged in what we think to be a general order of increasing difficulty.

1. Prove that for $k \geq 6$, we have $3^k > 2^{k+3}$.
2. Prove that $\sqrt{2}$ is irrational.
3. Let n be a positive integer. Prove that $1 + 8 + 27 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.
4. Prove that the area of a triangle is equal to its inradius times its semiperimeter.
5. Prove *Bernoulli's Inequality*, which states that if $x \geq -1$ then $(1+x)^n \geq 1+nx$ for all natural numbers n. (Hint: induction)
6. Let $ABCD$ be a quadrilateral with an inscribed circle. Prove that $AB + CD = AD + BC$.
7. The numbers $1, 2, \ldots, 2010, 2011$ are written on the board. It is permitted to take two numbers a, b and replace them with $a - b$. Prove that if this operation is applied until there is only one number on the board, then this number is even.
8. Prove that among any six people, there are always three who know each other or three who are complete strangers.
9. Two rows of ten pegs are lined up and adjacent pegs are spaced 1 unit apart. Determine, with proof, the number of ways in which ten rubber-bands be looped around the pegs so that no peg does not contain a rubber band. (Rubber bands cannot stretch more than $\sqrt{2}$ units.)
10. Let F_n be the sequence defined by $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Prove that $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$ for all nonnegative integers m, n.