New Zealand Mathematical Olympiad Committee

2011 Squad Assignment Three

Number Theory

Due: Monday 14th March 2011

1. The two pairs of consecutive natural numbers $(8, 9)$ and $(288, 289)$ have the following property: in each pair, each number contains each of its prime factors to a power not less than 2. Prove that there are infinitely many such pairs of consecutive natural numbers.

2. Suppose that N is a positive integer such that there are exactly 2005 ordered pairs (x, y) of positive integers satisfying
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$
Prove that N is a perfect square.

3. Find all quadruples (a, b, p, n) of positive integers such that p is prime and
$$a^3 + b^3 = p^n.$$

4. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n^2$ for all values of n?

5. Let
$$f(n) = 1 + n + n^2 + \cdots + n^{2010}.$$
Prove that for every integer m with $2 \leq m \leq 2010$, there is no non-negative integer n such that $f(n)$ is divisible by m.

6. An integer m is a *perfect power* if there exist positive integers a and n with $n > 1$ such that $m = a^n$.
   (a) Prove that there exist 2011 distinct positive integers such that no subset of them sums to a perfect power.
   (b) Prove that there exist 2011 distinct positive integers such that every subset of them sums to a perfect power.

7. Let p be a prime, and let $q(x)$ be a polynomial with integer co-efficients such that $q(0) = 0$, $q(1) = 1$, and $q(n)$ is congruent to 0 or 1 mod p for all $n \in \mathbb{N}$. Show that the degree of q is at least $p - 1$.

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