Generating Functions

1 Definitions

In this handout, we introduce generating functions and discuss their uses and applications. Generating functions are powerful tools, as they provide a compact way of working with sequences, and can reduce combinatorial problems to mere algebra.

Given a sequence a_0, a_1, a_2, ... of real numbers, the *generating function* of this sequence is defined as

$$a_0 + a_1 x + a_2 x^2 + \cdots .$$

We treat this "function" as a formal power series in x, meaning that we can add, subtract, and multiply (and even divide) such power series, but we do not worry if the series converges for any particular value of x.

Given a generating function $A(x)$, let $[x^n]A(x)$ denote the coefficient of x^n in $A(x)$. Hence, if

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots ,$$

then $[x^n]A(x) = a_n$.

2 Sums, Products, and Quotients of Generating Functions

Given two sequences (a_n) and (b_n), let $A(x)$ and $B(x)$ denote their generating functions, respectively, so

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + \cdots , \\ B(x) &= b_0 + b_1 x + b_2 x^2 + \cdots . \end{aligned}$$

If we take the sum of $A(x)$ and $B(x)$, then we get

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots ,$$

so $A(x) + B(x)$ is the generating function of the sequence $(a_n + b_n)$. More generally, $c_1 A(x) + c_2 B(x)$ is the generating function of the sequence $(c_1 a_n + c_2 b_n)$, for any constants c_1 and c_2.

However, if we take the product of $A(x)$ and $B(x)$, then we get

$$\begin{aligned} A(x)B(x) &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n. \end{aligned}$$

The sequence of coefficients in $A(x)B(x)$ is known as the *convolution* of the sequences (a_n) and (b_n), which we will discuss in more detail in Section 5. We can also divide $A(x)$ by $B(x)$ to obtain another generating function, but only if a certain condition is met. Since we can think of $\frac{A(x)}{B(x)}$ as the product of $A(x)$ and $\frac{1}{B(x)}$, we can first ask when $\frac{1}{B(x)}$ is a well-defined generating function.

If $\frac{1}{B(x)}$ is a well-defined generating function, say

$$\frac{1}{B(x)} = C(x) = c_0 + c_1 x + c_2 x^2 + \cdots ,$$

then $C(x)$ must satisfy $B(x)C(x) = 1$. The left-hand side expands as

$$B(x)C(x) = b_0c_0 + (b_0c_1 + b_1c_0)x + (b_0c_2 + b_1c_1 + b_2c_0)x^2 + \cdots ,$$

and the right-hand side is

$$1 = 1 + 0x + 0x^2 + \cdots .$$

Comparing the coefficients on both sides, we obtain the infinite system of equations

$$\begin{aligned} b_0c_0 &= 1, \\ b_0c_1 + b_1c_0 &= 0, \\ b_0c_2 + b_1c_1 + b_2c_0 &= 0, \\ b_0c_3 + b_1c_2 + b_2c_1 + b_3c_0 &= 0, \\ &\vdots \end{aligned}$$

In particular, from the first equation $b_0c_0 = 1$, we see that the coefficient b_0 must satisfy $b_0 \neq 0$. Conversely, if $b_0 \neq 0$, then we can construct a sequence (c_n) that satisfies the system of equations above as follows: First, we set $c_0 = 1/b_0$. Then all subsequent terms are given recursively by

$$c_n = -\frac{1}{b_0} \sum_{k=1}^{n} b_k c_{n-k}.$$

Since $B(0) = b_0$, we conclude that $\frac{1}{B(x)}$ is a well-defined generating function if and only if $B(0) \neq 0$.

The quotient $\frac{A(x)}{B(x)}$ can be a well-defined generating function even if $B(0) = 0$, but there must be enough factors of x in the numerator $A(x)$ to cancel all the factors of x in the denominator $B(x)$. More precisely, if s and t are the smallest indices such that $a_s \neq 0$ and $b_t \neq 0$, then the quotient $\frac{A(x)}{B(x)}$ is a well-defined generating function if and only if $s \geq t$.

3 Linearly Recurrent Sequences

If a sequence satisfies a linear recurrence, then we can write down its generating function, and in certain cases, use the generating function to solve the sequence.

Problem. Find the generating function of the geometric sequence $a_n = ar^n$.

Solution. In this geometric sequence, each term is obtained by multiplying the previous term by r. We use this property to find a closed form of the generating function as follows. (The same technique is used to find the formula for a geometric series.)

Let

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + \cdots \\ &= a + arx + ar^2 x^2 + \cdots \\ &= a(1 + rx + r^2 x^2 + \cdots). \end{aligned}$$

Then

$$\begin{aligned} rxA(x) &= rx \cdot a(1 + rx + r^2 x^2 + \cdots) \\ &= a(rx + r^2 x^2 + r^3 x^3 + \cdots). \end{aligned}$$

Subtracting these equations, we get

$$A(x) - rxA(x) = a,$$

so

$$A(x) = \frac{a}{1 - rx}.$$

In other words, the generating function of the geometric sequence (ar^n) is given by

$$a + arx + ar^2 x^2 + \cdots = \frac{a}{1 - rx}.$$

In particular, taking $a = 1$ and $r = 1$, we get

$$1 + x + x^2 + \cdots = \frac{1}{1 - x}.$$

Problem. Find the generating function of the Fibonacci sequence (F_n).

Solution. Recall that the Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. We use the recurrence to find a closed form of the generating function as follows. Let

$$F(x) = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots .$$

Then

$$\begin{aligned} xF(x) &= F_0 x + F_1 x^2 + F_2 x^3 + F_3 x^4 + \cdots , \\ x^2 F(x) &= F_0 x^2 + F_1 x^3 + F_2 x^4 + F_3 x^5 + \cdots , \end{aligned}$$

so

$$\begin{aligned} F(x) - xF(x) - x^2F(x) &= F_0 + (F_1 - F_0)x + (F_2 - F_1 - F_0)x^2 + (F_3 - F_2 - F_1)x^3 + \cdots \\ &= x. \end{aligned}$$

Hence,

$$F(x) = \frac{x}{1 - x - x^2}.$$

We can use this generating function to find an explicit formula for F_n. First, we factor the denominator as

$$1 - x - x^2 = (1 - \tau x)(1 - \sigma x),$$

which expands as

$$1 - x - x^2 = 1 - (\tau + \sigma)x + \tau\sigma x^2,$$

so $\tau + \sigma = 1$ and $\tau\sigma = -1$. By Vieta's formulas, τ and σ are the roots of the quadratic equation $t^2 - t - 1 = 0$, which are $\frac{1 \pm \sqrt{5}}{2}$. Hence, let $\tau = \frac{1+\sqrt{5}}{2}$ and $\sigma = \frac{1-\sqrt{5}}{2}$.

By the method of partial fractions, there exist constants A and B such that

$$\frac{x}{(1 - \tau x)(1 - \sigma x)} = \frac{A}{1 - \tau x} + \frac{B}{1 - \sigma x}.$$

Multiplying both sides by $(1 - \tau x)(1 - \sigma x)$, we get

$$x = A(1 - \sigma x) + B(1 - \tau x) = (A + B) - (\sigma A + \tau B)x.$$

Hence, we obtain the system of equations

$$\begin{aligned} A + B &= 0, \\ \sigma A + \tau B &= -1. \end{aligned}$$

From the first equation, $B = -A$. Substituting into the second equation, we get $\sigma A - \tau A = (\sigma - \tau)A = -1$, so $A = \frac{1}{\tau - \sigma} = \frac{1}{\sqrt{5}}$. Then $B = -\frac{1}{\sqrt{5}}$. Therefore,

$$F(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \tau x} - \frac{1}{1 - \sigma x} \right).$$

By the generating function of a geometric sequence,

$$\begin{aligned} \frac{1}{1 - \tau x} &= 1 + \tau x + \tau^2 x^2 + \cdots, \\ \frac{1}{1 - \sigma x} &= 1 + \sigma x + \sigma^2 x^2 + \cdots, \end{aligned}$$

so

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \tau x} - \frac{1}{1 - \sigma x} \right) \\ &= \frac{1}{\sqrt{5}} [(1 + \tau x + \tau^2 x^2 + \cdots) - (1 + \sigma x + \sigma^2 x^2 + \cdots)] \\ &= \frac{\tau - \sigma}{\sqrt{5}} x + \frac{\tau^2 - \sigma^2}{\sqrt{5}} x^2 + \frac{\tau^3 - \sigma^3}{\sqrt{5}} x^3 + \cdots . \end{aligned}$$

Thus, the coefficient of x^n in $F(x)$ is equal to

$$F_n = \frac{\tau^n - \sigma^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right].$$

Problem. Let

$$\frac{3 - x^2}{1 - x^2 - x^3} = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Find a_{10}.

Solution. From the given expression,

$$\begin{aligned} 3 - x^2 &= (1 - x^2 - x^3)(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= a_0 + a_1 x + (a_2 - a_0) x^2 + (a_3 - a_1 - a_0) x^3 + (a_4 - a_2 - a_1) x^4 + \cdots . \end{aligned}$$

Hence, $a_0 = 3$, $a_1 = 0$, $a_2 - a_0 = -1$, and $a_n - a_{n-2} - a_{n-3} = 0$ for all $n \ge 3$. We see that $a_2 = a_0 - 1 = 2$. Using the recurrence, we find $a_{10} = 17$.

Another way to find the first few coefficients, which serve as the initial terms of the recurrence, is as follows:

$$\begin{aligned} \frac{3 - x^2}{1 - x^2 - x^3} &= \frac{3 - x^2}{1 - (x^2 + x^3)} \\ &= (3 - x^2)[1 + (x^2 + x^3) + (x^2 + x^3)^2 + \cdots] \\ &= (3 - x^2)(1 + x^2 + x^3 + 2x^4 + \cdots) \\ &= 3 + 2x^2 + 3x^3 + 2x^4 + \cdots . \end{aligned}$$

More generally, suppose a sequence (a_n) is defined by the linear recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-k} + \cdots + c_k a_{n-k}.$$

Then the polynomial

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k$$

is the *characteristic polynomial* of the sequence (a_n), and the generating function of the sequence (a_n) is of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots = \frac{P(x)}{1 - c_1 x - c_2 x^2 - \cdots - c_k x^k},$$

where $P(x)$ is a polynomial. This result tells us that in terms of generating functions, rational functions correspond to linearly recurrent sequences.

Exercises

1. The sequence (a_n) is defined by $a_0 = 0$, $a_1 = 1$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for all $n \ge 2$. Find the generating function of this sequence, and use it to solve for a_n.

2. The sequences (a_n) and (b_n) are defined by $a_0 = 1$, $b_0 = 0$, and
$$\begin{aligned} a_n &= 3a_{n-1} + 4b_{n-1}, \\ b_n &= 2a_{n-1} + 3b_{n-1} \end{aligned}$$
for all $n \ge 1$. Find the generating functions of these sequences, and use them to solve for a_n and b_n.

3. Let
$$\frac{1}{1 + x + x^2 + x^3 + x^4 + x^5 + x^6} = a_0 + a_1 x + a_2 x^2 + \cdots .$$
Find a_n.

4. For a positive integer n, let a_n denote the the number of permutations π of the numbers 1, 2, …, n, such that $|\pi(i) - i| \le 2$ for all $1 \le i \le n$. The generating function of the sequence (a_n) is given by
$$\frac{1 - x}{1 - 2x - 2x^3 + x^5}.$$
Using this generating function or otherwise, find a_8.

4 Partial Sums and Finite Differences

In Section 2, we saw that

$$\begin{aligned} &(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots . \end{aligned}$$

If we set each b_i equal to 1, then we get

$$\begin{aligned} &(a_0 + a_1 x + a_2 x^2 + \cdots)(1 + x + x^2 + \cdots) \\ &= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots . \end{aligned}$$

The sequence a_0, $a_0 + a_1$, $a_0 + a_1 + a_2$, $\dots$ is known as the sequence of *partial sums* of the sequence a_0, a_1, a_2, $\dots$. Thus, if the generating function of the sequence (a_n) is

$$a_0 + a_1 x + a_2 x^2 + \cdots = A(x),$$

then the generating function of the sequence of partial sums is given by

$$a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots = A(x)(1 + x + x^2 + \cdots) = \frac{A(x)}{1 - x}.$$

On the other hand, suppose we multiply the function $A(x)$ by $1 - x$ instead of divide by $1 - x$:

$$\begin{aligned}(1 - x)A(x) &= (1 - x)(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + \cdots .\end{aligned}$$

The sequence $a_1 - a_0$, $a_2 - a_1$, $a_3 - a_2$, $\dots$ is known as the sequence of *finite differences* of the sequence a_0, a_1, a_2, $\dots$. Hence, the generating function of the sequence of finite differences is given by

$$(a_1 - a_0)x + (a_2 - a_1)x^2 + (a_3 - a_2)x^3 + \cdots = (1 - x)A(x) - A(0).$$

This is a particularly useful formula when a_n is a polynomial in n, because in such a case, the sequence of finite differences is another polynomial in n, whose degree is one less.

Problem. Find the generating function of the sequence $a_n = n^2$.

Solution. Let

$$A(x) = \sum_{n=0}^{\infty} n^2 x^n = 0^2 + 1^2 x + 2^2 x^2 + 3^2 x^3 + \cdots .$$

Then

$$\begin{aligned}(1 - x)A(x) &= (1 - x)(0^2 + 1^2 x + 2^2 x^2 + 3^2 x^3 + \cdots + n^2 x^n + \cdots) \\ &= 0^2 + (1^2 - 0^2)x + (2^2 - 1^2)x^2 + (3^2 - 2^2)x^3 + \cdots + [n^2 - (n - 1)^2]x^n + \cdots \\ &= x + 3x^2 + 5x^3 + \cdots + (2n - 1)x^n + \cdots ,\end{aligned}$$

and

$$\begin{aligned}(1 - x)^2 A(x) &= (1 - x)[x + 3x^2 + 5x^3 + 7x^4 + \cdots + (2n - 1)x^n + \cdots] \\ &= x + (3 - 1)x^2 + (5 - 3)x^3 + (7 - 5)x^4 + \cdots + [(2n - 1) - (2n - 3)]x^n + \cdots \\ &= x + 2x^2 + 2x^3 + 2x^4 + \cdots \\ &= x + 2x^2(1 + x + x^2 + \cdots) \\ &= x + \frac{2x^2}{1 - x} \\ &= \frac{x + x^2}{1 - x}.\end{aligned}$$

Hence,

$$A(x) = \sum_{n=0}^{\infty} n^2 x^n = \frac{x + x^2}{(1 - x)^3}.$$

Exercises

5. Find the generating function of the sequence $a_n = n^3$.
6. Show that $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$ for all nonnegative integers n, where F_n denotes the n^{th} Fibonacci number.

5 Generating Functions and Combinatorics

To describe how generating functions relate to problems in combinatorics, we start with a simple example.

Problem. Matt has a basket containing 4 apples and 5 bananas. Matt wants an even number of apples and at least two bananas. In how many ways can Matt choose 6 pieces of fruit? (The apples are distinguishable, as are the bananas.)

Solution. Matt can choose 2 apples and 4 bananas, or 4 apples and 2 bananas. Hence, Matt can choose 6 pieces of fruit in

$$\binom{4}{2}\binom{5}{4} + \binom{4}{4}\binom{5}{2} = 40$$

ways.

What if we wanted to find the number of ways Matt could choose n pieces of fruit? To make things more definite, let a_i be the number of ways Matt can choose i apples, so $a_i = \binom{4}{i}$ if $0 \le i \le 4$ and i is even, and $a_i = 0$ otherwise. Let b_i be the number of ways Matt can choose i bananas, so $b_i = \binom{5}{i}$ if $2 \le i \le 5$, and $b_i = 0$ otherwise. Let c_n be the number of ways Matt can choose n pieces of fruit. If Matt chooses k apples, then he also chooses $n - k$ bananas, which he can do in a total of $a_k b_{n-k}$ ways. Summing over $0 \le k \le n$, we find

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_0 b_n.$$

We recognize the sum in the right-hand side as the n^{th} term in the convolution of the two sequences (a_n) and (b_n). Hence, if we let $A(x)$, $B(x)$, and $C(x)$ be the generating functions of the sequences (a_n), (b_n), and (c_n), respectively, then $C(x) = A(x)B(x)$. These generating functions are

$$\begin{aligned} A(x) &= \binom{4}{0} + \binom{4}{2}x^2 + \binom{4}{4}x^4 \\ &= 1 + 6x^2 + x^4, \\ B(x) &= \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5 \\ &= 10x^2 + 10x^3 + 5x^4 + x^5, \end{aligned}$$

and so

$$\begin{aligned} C(x) &= A(x)B(x) \\ &= (1 + 6x^2 + x^4)(10x^2 + 10x^3 + 5x^4 + x^5) \\ &= 10x^2 + 10x^3 + 65x^4 + 61x^5 + 40x^6 + 16x^7 + 5x^8 + x^9. \end{aligned}$$

This generating function tells us exactly how many ways Matt can choose n pieces of fruit. For example, the number of ways that Matt can choose 6 pieces of fruit is 40, the coefficient of x^6, as computed above.

More generally, we have the following result.

Theorem. (The Product Formula) We are given two disjoint sets $\mathcal{A}$ and $\mathcal{B}$. We are also given that the number of ways to choose n elements from $\mathcal{A}$ is a_n, and the number of ways to choose n elements from $\mathcal{B}$ is b_n. Let c_n be the number of ways to choose n elements (as a subset) from $\mathcal{A} \cup \mathcal{B}$.

Let $A(x)$, $B(x)$, and $C(x)$ be the generating functions of the sequences (a_n), (b_n), and (c_n), respectively. Then $C(x) = A(x)B(x)$.

Note. The number of ways to choose elements from a set can be the traditional $\binom{n}{k}$ formula, but it can be completely arbitrary. For example, in the problem above, Matt may decide that if he chooses 4 bananas, then there are only 2 sets of 4 bananas that he likes. In this case, $b_4 = 2$ instead of $b_4 = \binom{5}{4} = 5$.

Example. Paige has a collection of two algebra books (which are identical) and three geometry books (which are identical). Let c_n be the number of ways Paige can choose n of her books. Find the generating function of (c_n).

Solution. Since both algebra books are identical, there is only one way to choose i algebra books, for $0 \le i \le 2$. Hence, the generating function corresponding to the algebra books is $1 + x + x^2$. Similarly, the generating function corresponding to the geometry books is $1 + x + x^2 + x^3$. Therefore, the generating function of (c_n) is

$$(1 + x + x^2)(1 + x + x^2 + x^3) = 1 + 2x + 3x^2 + 3x^3 + 2x^4 + x^5.$$

To verify the coefficients in the generating function, we list the possible combinations of algebra books and geometry books in the following table:

n	Combinations of n books	Number of Combinations
0	No books	1
1	One algebra book, one geometry book	2
2	Two algebra books, one algebra book and one geometry book, two geometry books	3
3	Two algebra books and one geometry book, one algebra book and two geometry books, three geometry books	3
4	Two algebra books and two geometry books, one algebra book and three geometry books,	2
5	Two algebra books and three geometry books	1

Example. Let s_n denote the number of ways to obtain a sum of n by rolling two standard six-sided dice. What is the generating function of (s_n)?

Solution. The generating function of a single die is $x + x^2 + x^3 + x^4 + x^5 + x^6$, so the generating function of two dice is

$$\begin{aligned}&(x + x^2 + x^3 + x^4 + x^5 + x^6)^2\\ &= x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}.\end{aligned}$$

For example, the number of ways to obtain the sum 10 is 3, the coefficient of x^{10}. (These three ways are rolling a 4 and a 6, a 5 and a 5, and a 6 and a 4.)

Problem. We have three pennies, four nickels, and two quarters. Find the generating function of the number of ways we can make change for n cents.

Solution. We assume that the coins of each denomination are indistinguishable, so for example, when considering the combinations of pennies, only the number of pennies is relevant. Hence, there is only one way to form i cents from the three pennies, for $0 \le i \le 3$, and there are no ways otherwise, so the generating function corresponding to the pennies is $1 + x + x^2 + x^3$.

Similarly, there is only one way to form i cents from the four nickels if $0 \le i \le 20$ and i is divisible by 5, and there are no ways otherwise, so the generating function corresponding to the four nickels is $1 + x^5 + x^{10} + x^{15} + x^{20}$. Finally, the generating function corresponding to the two quarters is $1 + x^{25} + x^{50}$. Therefore, the generating function of the number of ways we can make change for n cents is

$$(1 + x + x^2 + x^3)(1 + x^5 + x^{10} + x^{15} + x^{20})(1 + x^{25} + x^{50}).$$

What if the coins are distinguishable? If the coins are distinguishable, then for the pennies, for example, the question is not how many pennies are present, but whether each penny is present or not (since we can tell them apart). The generating function for each penny is $1 + x$, and there are three pennies, so the generating function corresponding to the pennies is now $(1 + x)^3$. Similarly, the generating function corresponding to the nickels is $(1+x^5)^4$, and the generating function corresponding to the quarters is $(1+x^{25})^2$. Therefore, if the coins are distinguishable, then the generating function of the number of ways we can make change for n cents is

$$(1+x)^3(1+x^5)^4(1+x^{25})^2.$$

Exercises

7. Find the number of ways to collect \$15 from 20 people if each of the first 19 people can give a dollar or nothing, and the twentieth person can give either \$1, \$5, or nothing.

8. Show that the generating function of the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = n,$$

where $0 \le x_1 \le x_2 \le x_3 \le x_4$, is

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}.$$

6 The Binomial Theorem

Consider the product

$$(1+x)^n = \underbrace{(1+x)(1+x)\cdots(1+x)}_{n \text{ factors}},$$

where n is a positive integer. The coefficient of x^k in this product is the number of ways we can choose a term from each of the n factors, such that their product is equal to x^k. We see that k of these terms must be equal to x and the remaining $n-k$ of these terms must be equal to 1. The number of ways to choose k xs among n xs is simply $\binom{n}{k}$, so

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

This is, of course, the **Binomial Theorem**. Hence, we can view $(1+x)^n$ as the generating function of $\binom{n}{k}$, where n is fixed integer.

We can generalize this result: The **Generalized Binomial Theorem** states that

$$(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots,$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!},$$

and α can be any real number.

Example. The generating function of $\sqrt{1+x} = (1+x)^{1/2}$ is

$$\begin{aligned}
&\binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{1}x + \binom{\frac{1}{2}}{2}x^2 + \binom{\frac{1}{2}}{3}x^3 + \cdots \\
&= 1 + \frac{\frac{1}{2}}{1!}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \cdots \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^3 + \frac{1}{16}x^4 - \cdots .
\end{aligned}$$

Can we derive the generating function of $\binom{n}{k}$, where k is a fixed integer? The next problem shows that we can.

Problem. Let n be a nonnegative integer. Show that

$$\frac{1}{(1-x)^{n+1}} = \binom{n}{n} + \binom{n+1}{n}x + \binom{n+2}{n}x^2 + \cdots .$$

Solution. Consider the product

$$\frac{1}{(1-x)^{n+1}} = \underbrace{(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots)}_{n+1 \text{ factors}}.$$

The coefficient of x^k is the number of ways we can choose a term from each of the $n+1$ factors, such that their product is equal to x^k. If the $n+1$ terms we choose are x^{a_1}, x^{a_2}, $\dots$, $x^{a_{n+1}}$, then

$$a_1 + a_2 + \cdots + a_{n+1} = k.$$

Hence, the coefficient of x^k is the number of solutions to this equation in nonnegative integers.

To count the number of solutions, we represent each solution as a string of $*$ symbols (stars) and $|$ symbols (bars). (This is known as a *stars-and-bars* argument.) For example, for $n = 4$ and $k = 9$, the solution $2+0+3+1+3 = 9$ becomes the string

$$** \,||\, *** \,|\, * \,|\, ***.$$

Hence, for each solution, we obtain a string consisting of k stars and n bars. Furthermore, every arrangement of k stars and n bars corresponds to a different solution, so the number of solutions is $\binom{n+k}{n}$, which is also the coefficient of x^k.

Problem. Derive a formula for the sum $1^2 + 2^2 + \cdots + n^2$.

Solution. In a previous problem, we derived that

$$1^2x + 2^2x^2 + 3^2x^3 + \cdots = \frac{x+x^2}{(1-x)^3}.$$

We can obtain the sequence of partial sums by dividing by $1-x$:

$$1^2x + (1^2+2^2)x^2 + (1^2+2^2+3^2)x^3 + \cdots = \frac{x+x^2}{(1-x)^4}.$$

We want to express this fraction as the sum of fractions of the form $\frac{c}{(1-x)^k}$. We can use the method of partial fractions, or we can use a substitution: Let $y = 1-x$. Then $x = 1-y$, and

$$\begin{aligned}\frac{x+x^2}{(1-x)^4} &= \frac{(1-y)+(1-y)^2}{y^4}\\ &= \frac{y^2-3y+2}{y^4}\\ &= \frac{1}{y^2} - \frac{3}{y^3} + \frac{2}{y^4}\\ &= \frac{1}{(1-x)^2} - \frac{3}{(1-x)^3} + \frac{2}{(1-x)^4}.\end{aligned}$$

By the previous result,

$$\begin{aligned}\frac{1}{(1-x)^2} - \frac{3}{(1-x)^3} + \frac{2}{(1-x)^4} &= \left[\binom{1}{1} + \binom{2}{1}x + \binom{3}{1}x^2 + \cdots\right]\\ &\quad - 3\left[\binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 + \cdots\right]\\ &\quad + 2\left[\binom{3}{3} + \binom{4}{3}x + \binom{5}{3}x^2 + \cdots\right].\end{aligned}$$

The coefficient of x^n in this expression is

$$\begin{aligned}\binom{n+1}{1} - 3\binom{n+2}{2} + 2\binom{n+3}{3} &= n+1-3\cdot\frac{(n+2)(n+1)}{2} + 2\cdot\frac{(n+3)(n+2)(n+1)}{6}\\ &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\\ &= \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

Exercises

9. The sequence (a_n) is defined by $a_0 = 0$, $a_1 = 2$, and $a_n = 4a_{n-1} - 4a_{n-2}$ for all $n \ge 2$. Find the generating function of this sequence, and use it to solve for a_n.

10. How many solutions in positive integers are there to the equation $y_1+y_2+y_3+y_4 = 30$ such that no y_i is greater than 12?

11. Let n be a positive integer. Show that

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots ,$$

where F_n denotes the n^{th} Fibonacci number.

Hint:

$$\frac{1}{1-x-x^2} = 1 + (x+x^2) + (x+x^2)^2 + \cdots .$$

12. Expand

$$\frac{1}{(1-x)^{n+1}} = (1-x)^{-n-1}$$

using the generalized Binomial theorem.

7 Roots of Unity Filter

Suppose we are given a generating function

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots ,$$

and we want to extract the series

$$a_0 + a_3 x^3 + a_6 x^6 + \cdots .$$

We can do so using a *roots of unity filter.*

Let $\omega = e^{2\pi i/3}$, so $\omega^3 = 1$ and $\omega \neq 1$. Then

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots , \\ A(\omega x) &= \sum_{n=0}^{\infty} \omega^n a_n x^n = a_0 + \omega a_1 x + \omega^2 a_2 x^2 + \omega^3 a_3 x^3 + \cdots , \\ A(\omega^2 x) &= \sum_{n=0}^{\infty} \omega^{2n} a_n x^n = a_0 + \omega^2 a_1 x + \omega^4 a_2 x^2 + \omega^6 a_3 x^3 + \cdots . \end{aligned}$$

Hence,

$$A(x) + A(\omega x) + A(\omega^2 x) = \sum_{n=0}^{\infty} (1 + \omega^n + \omega^{2n}) a_n x^n.$$

If n is divisible by 3, then $1 + \omega^n + \omega^{2n} = 1 + 1 + 1 = 3$. Otherwise,

$$1 + \omega^n + \omega^{2n} = \frac{\omega^{3n} - 1}{\omega^n - 1} = 0.$$

(It is not hard to check that $\omega^n = 1$ if and only if n is divisible by 3, so the denominator is nonzero.)

Therefore,

$$A(x) + A(\omega x) + A(\omega^2 x) = 3a_0 + 3a_3x^3 + 3a_6x^6 + \cdots,$$

which means

$$a_0 + a_3x^3 + a_6x^6 + \cdots = \frac{1}{3}[A(x) + A(\omega x) + A(\omega^2 x)].$$

More generally, we can use the n^{th} roots of unity to extract every n^{th} term of a generating function.

Problem. An "unfair" coin has a 2/3 probability of turning up heads. If this coin is tossed 50 times, what is the probability that the total number of heads is even? (AHSME, 1992)

Solution. Let $p = 2/3$ and $q = 1/3$. We wish to compute

$$p^{50} + \binom{50}{2}p^{48}q^2 + \binom{50}{4}p^{47}q^4 + \cdots + q^{50}.$$

By the Binomial theorem,

$$(p + qx)^{50} = p^{50} + \binom{50}{1}p^{49}qx + \binom{50}{2}p^{48}q^2x^2 + \cdots + q^{50}x^{50}.$$

Taking $x = 1$, we get

$$(p + q)^{50} = p^{50} + \binom{50}{1}p^{49}q + \binom{50}{2}p^{48}q^2 + \cdots + q^{50}.$$

Taking $x = -1$, we get

$$(p - q)^{50} = p^{50} - \binom{50}{1}p^{49}q + \binom{50}{2}p^{48}q^2 + \cdots + q^{50}.$$

Adding these equations and dividing by 2, we find

$$p^{50} + \binom{50}{2}p^{48}q^2 + \binom{50}{4}p^{47}q^4 + \cdots + q^{50} = \frac{1}{2}[(p + q)^{50} + (p - q)^{50}] = \frac{1}{2}\left(1 + \frac{1}{3^{50}}\right).$$

Problem. For a nonnegative integer n, let

$$S_n = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots.$$

(a) Show that

$$S_n = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$$

for all $n \geq 0$.

(b) Find the generating function of the sequence (S_n).

Solution. (a) Let $\omega = e^{2\pi i/3}$, so $\omega^3 = 1$ and $\omega \neq 1$. Then $\omega^3 - 1 = 0$, which factors as

$$(\omega - 1)(\omega^2 + \omega + 1) = 0.$$

Since $\omega - 1 \neq 0$, ω satisfies the equation $\omega^2 + \omega + 1 = 0$.

By the Binomial theorem,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots .$$

Then from our work above,

$$S_n = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots = \frac{1}{3}[(1+1)^n + (1+\omega)^n + (1+\omega^2)^n].$$

Since $\omega^2 + \omega + 1 = 0$, we have that

$$1 + \omega = -\omega^2 = -e^{4\pi i/3} = e^{\pi i} \cdot e^{4\pi i/3} = e^{7\pi i/3} = e^{\pi i/3},$$

and

$$1 + \omega^2 = -\omega = -e^{2\pi i/3} = e^{\pi i} \cdot e^{2\pi i/3} = e^{5\pi i/3} = e^{-\pi i/3},$$

so

$$\begin{aligned} S_n &= \frac{1}{3}(2^n + e^{n\pi i/3} + e^{-n\pi i/3}) \\ &= \frac{1}{3}\left(2^n + \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right) \\ &= \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right). \end{aligned}$$

(b) We can write

$$S_n = \frac{1}{3}[2^n + (-\omega)^n + (-\omega^2)^n].$$

Then the generating function of the sequence (S_n) is

$$\begin{aligned}
&\frac{1}{3}\sum_{n=0}^{\infty}[2^n + (-\omega)^n + (-\omega^2)^n]x^n \\
&= \frac{1}{3}\sum_{n=0}^{\infty} 2^n x^n + \frac{1}{3}\sum_{n=0}^{\infty}(-\omega)^n x^n + \frac{1}{3}\sum_{n=0}^{\infty}(-\omega^2)^n x^n \\
&= \frac{1}{3(1-2x)} + \frac{1}{3(1+\omega x)} + \frac{1}{3(1+\omega^2 x)} \\
&= \frac{1}{3(1-2x)} + \frac{1+\omega x + 1 + \omega^2 x}{3(1+\omega x)(1+\omega^2 x)} \\
&= \frac{1}{3(1-2x)} + \frac{2+(\omega+\omega^2)x}{3[1+(\omega+\omega^2)x+\omega^3 x^2]} \\
&= \frac{1}{3(1-2x)} + \frac{2-x}{3(1-x+x^2)} \\
&= \frac{(1-x)^2}{(1-2x)(1-x+x^2)}.
\end{aligned}$$

We can also derive the generating function by writing

$$\begin{aligned}
&S_0 + S_1 x + S_2 x^2 + S_3 x^3 + S_4 x^4 + \cdots \\
&= \binom{0}{0} + \binom{1}{0}x + \binom{2}{0}x^2 + \left[\binom{3}{0} + \binom{3}{3}\right]x^3 + \left[\binom{4}{0} + \binom{4}{3}\right]x^4 + \cdots.
\end{aligned}$$

This sum is equal to

$$\sum_{k=0}^{\infty}\left[\binom{3k}{3k}x^{3k} + \binom{3k+1}{3k}x^{3k+1} + \binom{3k+2}{3k}x^{3k+2} + \cdots\right].$$

We know that

$$\binom{3k}{3k} + \binom{3k+1}{3k}x + \binom{3k+2}{3k}x^2 + \cdots = \frac{1}{(1-x)^{3k+1}},$$

so the generating function of the sequence (S_n) is

$$\begin{aligned}
&S_0 + S_1 x + S_2 x^2 + S_3 x^3 + S_4 x^4 + \cdots \\
&= \sum_{k=0}^{\infty} \left[\binom{3k}{3k} x^{3k} + \binom{3k+1}{3k} x^{3k+1} + \binom{3k+2}{3k} x^{3k+2} + \cdots \right] \\
&= \sum_{k=0}^{\infty} \frac{x^{3k}}{(1-x)^{3k+1}} \\
&= \frac{1}{1-x} + \frac{x^3}{(1-x)^4} + \frac{x^6}{(1-x)^7} + \cdots \\
&= \frac{\frac{1}{1-x}}{1 - \frac{x^3}{(1-x)^3}} \\
&= \frac{(1-x)^2}{(1-x)^3 - x^3} \\
&= \frac{(1-x)^2}{(1-2x)(1-x+x^2)}.
\end{aligned}$$

Exercises

13. Given $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, express

$$a_1 x + a_4 x^4 + a_7 x^7 + \cdots$$

in terms of $A(x)$.

14. Show that

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = 2^{n/2} \cos \frac{n\pi}{4},$$

and that

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \cdots = 2^{n/2} \sin \frac{n\pi}{4}.$$

8 Partitions

Given a positive integer n, a *partition* of n is an unordered sum of positive integers that sum to n. For example, there are five partitions of the number 4, namely 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$. Since the order does not matter, the sums $3+1$ and $1+3$ represent the same partition. Let $p(n)$ denote the number of partitions of n. The generating function of the partition function $p(n)$ is

$$\begin{aligned} &p(0) + p(1)x + p(2)x^2 + \cdots \\ &= (1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)\cdots \\ &= \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}. \end{aligned}$$

There is no simple closed formula for $p(n)$, but we can still derive many interesting partition identities using generating functions.

Problem. Prove that the number of partitions of n with distinct parts is equal to the number of partitions of n with only odd parts.

Solution. If we have a partition with distinct parts, then each part appears at most once, so the generating function of the number of partitions of n with distinct parts is

$$(1+x)(1+x^2)(1+x^3)\cdots.$$

The generating function of the number of partitions of n with only odd parts is

$$\begin{aligned} &(1 + x + x^2 + x^3 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots) \\ &= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}. \end{aligned}$$

To make this generating function look more like the first generating function, we multiply the numerator and denominator by $(1-x^2)(1-x^4)(1-x^6)\cdots$:

$$\begin{aligned} &\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} \\ &= \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} \\ &= \frac{(1+x)(1-x)(1+x^2)(1-x^2)(1+x^3)(1-x^3)\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} \\ &= (1+x)(1+x^2)(1+x^3)\cdots. \end{aligned}$$

Thus, the two generating functions coincide, which means that the number of such partitions is equal.

Exercises

15. Find a simple expression for

$$\prod_{k=0}^{\infty}(1+x^{2^k}) = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots.$$

16. Prove that the number of partitions of n into odd parts greater than 1 equals the number of partitions of n into distinct parts that are not powers of 2.

9 Compositions

Given a positive integer n, a *composition* of n is an ordered sum of positive integers that sum to n. For example, there are eight compositions of the number 4, namely 4, $3+1$, $2+2$, $1+3$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$.

Problem. Find the number of compositions of n.

Solution. To make the problem simpler, we first consider the number of compositions of n with k parts, where $1 \le k \le n$. In other words, we want the number of solutions to the equation

$$a_1 + a_2 + \cdots + a_k = n$$

in positive integers. This reminds us of our work in Section 6, where we found that the number of solutions to the equation

$$a_1 + a_2 + \cdots + a_{n+1} = k$$

was the coefficient of x^k in

$$\underbrace{(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)\cdots(1 + x + x^2 + \cdots)}_{n+1 \text{ factors}}.$$

Analogously, we see that the number of solutions to the equation

$$a_1 + a_2 + \cdots + a_k = n$$

in positive integers is the coefficient of x^n in

$$(x + x^2 + x^3 + \cdots)^k = \underbrace{(x + x^2 + x^3 + \cdots)(x + x^2 + x^3 + \cdots)\cdots(x + x^2 + x^3 + \cdots)}_{k \text{ factors}}.$$

(Note how each factor is $x + x^2 + x^3 + \cdots$ here, because the a_i can only be positive integers; in particular, no a_i can be equal to 0.)

Hence, summing from 1 to n, we find that the number of compositions of n is the coefficient of x^n in

$$1 + (x + x^2 + x^3 + \cdots) + (x + x^2 + x^3 + \cdots)^2 + \cdots + (x + x^2 + x^3 + \cdots)^n.$$

We include an initial term of 1 to stand for the null composition. However, there are no terms of degree n in $(x + x^2 + x^3 + \cdots)^k$ for $k > n$, so the number of compositions of n is also the coefficient of x^n in

$$\begin{aligned} &1 + (x + x^2 + x^3 + \cdots) + (x + x^2 + x^3 + \cdots)^2 + (x + x^2 + x^3 + \cdots)^3 + \cdots \\ &= 1 + \frac{x}{1-x} + \left(\frac{x}{1-x}\right)^2 + \left(\frac{x}{1-x}\right)^3 + \cdots. \end{aligned}$$

This is an infinite geometric series with first term 1 and common ratio $\frac{x}{1-x}$, so it is equal to

$$\begin{aligned}
\frac{1}{1-\frac{x}{1-x}} &= \frac{1-x}{1-x-x} \\
&= \frac{1-x}{1-2x} \\
&= \frac{1-2x+x}{1-2x} \\
&= 1 + \frac{x}{1-2x} \\
&= 1 + x(1 + 2x + 4x^2 + 8x^3 + \cdots) \\
&= 1 + x + 2x^2 + 4x^3 + 8x^4 + \cdots .
\end{aligned}$$

Hence, the number of compositions of n is 2^{n-1}.

More generally, let $f(n)$ be a function taking the positive integers to the real numbers, and let $F(x)$ be the generating function of the sequence $(f(n))$, so

$$F(x) = f(1)x + f(2)x^2 + f(3)x^3 + \cdots .$$

Then the coefficient of x^n in

$$[F(x)]^k = \underbrace{[f(1)x + f(2)x^2 + f(3)x^3 + \cdots][f(1)x + f(2)x^2 + f(3)x^3 + \cdots]\cdots[f(1)x + f(2)x^2 + f(3)x^3 + \cdots]}_{k \text{ factors}}$$

is the sum of all products of the form

$$f(a_1)f(a_2)\cdots f(a_k),$$

where $a_1 + a_2 + \cdots + a_k$ is a composition of n. Hence, by the same argument as above, if we let

$$s(n) = \sum_{a_1+a_2+\cdots+a_k=n} f(a_1)f(a_2)\cdots f(a_k), \qquad (*)$$

where the sum is taken over all compositions $a_1 + a_2 + \cdots + a_k$ of n, then the generating function of $s(n)$ is given by

$$1 + F(x) + F(x)^2 + \cdots = \frac{1}{1-F(x)}.$$

For example, if we take $f(n) = 1$ for all positive integers n, then equation $(*)$ simply becomes

$$s(n) = \sum_{a_1+a_2+\cdots+a_k=n} 1,$$

so $s(n)$ is the number of compositions of n. Also,

$$F(x) = x + x^2 + x^3 + \cdots = \frac{x}{1-x},$$

and the generating function of $s(n)$ is

$$\frac{1}{1-F(x)} = \frac{1}{1-\frac{x}{1-x}} = \frac{1-x}{1-2x},$$

as derived before.

Problem. Find the number of compositions of n, where each part is equal to 1 or 2. For example, for $n = 4$, there are five such compositions, namely $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$.

Solution. We set $f(n) = 1$ if n is equal to 1 or 2, and $f(n) = 0$ otherwise. Then

$$f(a_1)f(a_2)\cdots f(a_k)$$

is equal to 1 if each part a_i is equal to 1 or 2, and 0 otherwise. Hence, the sum

$$s(n) = \sum_{a_1+a_2+\cdots+a_k=n} f(a_1)f(a_2)\cdots f(a_k)$$

counts the number of compositions we seek.

We have that $F(x) = x + x^2$, so the generating function of $s(n)$ is

$$\frac{1}{1-F(x)} = \frac{1}{1-x-x^2}.$$

We know that

$$\frac{x}{1-x-x^2}$$

is the generating function of the Fibonacci sequence, so $s(n) = F_{n+1}$.

Problem. Find

$$s(n) = \sum_{a_1+a_2+\cdots+a_k} a_1a_2\cdots a_k,$$

where the sum is taken over all compositions $a_1+a_2+\cdots+a_k$ of n. For example, $s(3) = 3+2\cdot1+1\cdot2+1\cdot1\cdot1 = 8$.

Solution. We take $f(n) = n$. Then

$$F(x) = x + 2x^2 + 3x^3 + \cdots.$$

Multiplying by $1-x$, we get

$$(1-x)F(x) = (1-x)(x+2x^2+3x^3+\cdots) = x + x^2 + x^3 + \cdots = \frac{x}{1-x},$$

so

$$F(x) = \frac{x}{(1-x)^2}.$$

Then the generating function of $s(n)$ is

$$\begin{aligned}
\frac{1}{1-F(x)} &= \frac{1}{1-\frac{x}{(1-x)^2}} \\
&= \frac{(1-x)^2}{(1-x)^2-x} \\
&= \frac{1-2x+x^2}{1-3x+x^2} \\
&= \frac{1-3x+x^2+x}{1-3x+x^2} \\
&= 1+\frac{x}{1-3x+x^2}.
\end{aligned}$$

Expressing the fraction as a generating function, we get

$$\frac{x}{1-3x+x^2} = x + 3x^2 + 8x^3 + 21x^4 + 55x^5 + \cdots.$$

These coefficients look like alternate Fibonacci terms: $F_2 = 1$, $F_4 = 3$, $F_6 = 8$, and so on. To obtain these terms, we apply a roots of unity filter to the generating function

$$F_0 + F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \cdots = \frac{x}{1-x-x^2}.$$

Substituting $-x$ for x, we get

$$F_0 - F_1 x + F_2 x^2 - F_3 x^3 + F_4 x^4 + \cdots = \frac{-x}{1+x-x^2}.$$

Adding these equations and dividing by 2, we get

$$\begin{aligned}
F_0 + F_2 x^2 + F_4 x^4 + \cdots &= \frac{1}{2}\left(\frac{x}{1-x-x^2} - \frac{x}{1+x-x^2}\right) \\
&= \frac{x(1+x-x^2) - x(1-x-x^2)}{2(1-x-x^2)(1+x-x^2)} \\
&= \frac{x^2}{(1-x^2)(1-x^2)^2 - x^2} \\
&= \frac{x^2}{1-3x^2+x^4}.
\end{aligned}$$

Hence,

$$F_0 + F_2 x + F_4 x^2 + \cdots = \frac{x}{1-3x+x^2},$$

as desired. We conclude that $s(n) = F_{2n}$.

Exercises

17. Find the number of compositions of n, where each part is at least 2.
18. Find the number of compositions of n, where each part is odd.
19. Show that
$$s(n) = f(1)s(n-1) + f(2)s(n-2) + \cdots + f(n-1)s(1) + f(n)$$
for all positive integers n, where $s(n)$ is as defined in $(*)$.

10 Words

A *word* is a string of letters, taken from some alphabet. For example, $ABAABA$ is a word consisting of the letters A and B. There are many problems we can ask about words; for example, how many n-letter words are there, consisting of the letters A and B, that do not contain any consecutive As? We can use generating functions to solve some of these word problems.

Our first step is to define an arithmetic on words. If we have two words, such as ab and baa, we can add them, just as we add algebraic expressions, to get $ab + baa$. Addition of words is commutative, so
$$ab + baa = baa + ab.$$
We can also multiply two words, by concatenating them. For example, $ab \times baa = abbaa$. However, multiplication of words is not commutative, because $baa \times ab = baaab$, and this is not the same word as $abbaa$. We also have the empty word, which we represent by 1. The product of 1 any word is the word itself, so 1 is the multiplicative identity; for example, $1 \times ab = ab \times 1 = ab$.

Thus, we have defined a kind of arithmetic on words, with addition and multiplication. Here is a typical calculation in this arithmetic:
$$(1 + a + ab)(1 + bb) = 1(1 + bb) + a(1 + bb) + ab(1 + bb) = 1 + bb + a + abb + ab + abbb.$$
In our first problem, we derive a basic generating function for words made from the letters a and b.

Problem. A word is composed of the letters a and b, so that no two consecutive letters are the same, such as $ababa$. How many such words of length n are there?

Solution. First, we list all words composed of the letters a and b, where no two consecutive letters are the same, including the empty word:
$$1,\ a,\ b,\ ab,\ ba,\ aba,\ bab,\ \ldots.$$
We then let $W(a, b)$ be the sum of all these words, which is our generating function on words:
$$W(a, b) = 1 + a + b + ab + ba + aba + bab + \cdots.$$
We want to find the number of such words of length n, for each positive integer n.

If we replace the letters a and b by a variable x, then we get

$$W(x,x) = 1 + x + x + xx + xx + xxx + xxx + \cdots .$$

Treating x as an ordinary variable, this expression simplifies to

$$W(x,x) = 1 + 2x + 2x^2 + 2x^3 + \cdots .$$

This step is notable for two reasons. First, by substituting a variable x for the letters a and b, we have turned an expression of words into an expression of variables, which we can simplify algebraically. Second, every word of length n collapses to the term x^n, so the number of such words is simply the coefficient of x^n. We conclude that the number of such of words of length n for any positive integer n is 2.

This result may seem completely evident, but the real significance of this problem is finding a closed algebraic expression for the generating function W. Substituting the ordinary variables u and v for the letters a and b, respectively, we get

$$W(u,v) = 1 + u + v + uv + vu + uvu + vuv + \cdots ,$$

which simplifies as

$$\begin{aligned}
W(u,v) &= 1 + u + v + 2uv + u^2v + uv^2 + 2u^2v^2 + u^3v^2 + u^2v^3 + \cdots \\
&= 1 + (2uv + 2u^2v^2 + 2u^3v^3 + \cdots) + (u + v + u^2v + uv^2 + u^3v^2 + u^2v^3 + \cdots) \\
&= 1 + 2uv(1 + uv + u^2v^2 + \cdots) + (u+v)(1 + uv + u^2v^2 + \cdots) \\
&= 1 + 2uv \cdot \frac{1}{1-uv} + (u+v) \cdot \frac{1}{1-uv} \\
&= \frac{(1-uv) + (2uv) + (u+v)}{1-uv} \\
&= \frac{1 + u + v + uv}{1-uv} \\
&= \frac{(1+u)(1+v)}{1-uv}.
\end{aligned}$$

This generating function W is important because it serves as a template for more general word problems. As another example, we use W to derive the number of words with no restrictions.

Problem. A word is composed of the letters a and b, with no restrictions on the letters. How many such words of length n are there?

Solution. Suppose we take the generating function

$$W(a,b) = 1 + a + b + ab + ba + aba + bab + \cdots ,$$

and replace a and b with $a + aa + aaa + \cdots$ and $b + bb + bbb \cdots$, respectively. Then

$$\begin{aligned}
&W(a + aa + aaa + \cdots, b + bb + bbb + \cdots)\\
&= 1 + (a + aa + aaa + \cdots) + (b + bb + bbb + \cdots)\\
&\quad + (a + aa + aaa + \cdots)(b + bb + bbb + \cdots)\\
&\quad + (b + bb + bbb + \cdots)(a + aa + aaa + \cdots)\\
&\quad + (a + aa + aaa + \cdots)(b + bb + bbb + \cdots)(a + aa + aaa + \cdots)\\
&\quad + (b + bb + bbb + \cdots)(a + aa + aaa + \cdots)(b + bb + bbb + \cdots) + \cdots .
\end{aligned}$$

If we expand this expression, in words, we will find that every word composed of the letters a and b appears exactly once. For example, the word $abbbaa$ appears exactly once, in the expansion of the product

$$(a + aa + aaa + \cdots)(b + bb + bbb + \cdots)(a + aa + aaa + \cdots).$$

Hence, applying the same technique of replacing the letters a and b by the same variable x, the generating function of the number of such words is given by

$$W(x + xx + xxx + \cdots, x + xx + xxx + \cdots).$$

Since

$$x + xx + xxx + \cdots = x + x^2 + x^3 + \cdots = \frac{x}{1-x},$$

the generating function is

$$\begin{aligned}
W\left(\frac{x}{1-x}, \frac{x}{1-x}\right) &= \frac{(\frac{x}{1-x} + 1)(\frac{x}{1-x} + 1)}{1 - (\frac{x}{1-x})^2}\\
&= \frac{(\frac{1}{1-x})^2}{1 - (\frac{x}{1-x})^2}\\
&= \frac{1}{(1-x)^2 - x^2}\\
&= \frac{1}{1-2x}\\
&= 1 + 2x + 4x^2 + 8x^3 + \cdots .
\end{aligned}$$

Therefore, the number of such words of length n is 2^n, as expected.

Problem. Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14? (AIME, 2008)

Solution. We take the generating function

$$W(a, b) = 1 + a + b + ab + ba + aba + bab + \cdots ,$$

and replace a and b with $AA + AAAA + \cdots$ and $B + BBB + \cdots$, respectively. Thus,

$$\begin{aligned}&W(AA + AAAA + \cdots, B + BBB + \cdots)\\&= 1 + (AA + AAAA + \cdots) + (B + BBB + \cdots)\\&\quad + (AA + AAAA + \cdots)(B + BBB + \cdots)\\&\quad + (B + BBB + \cdots)(AA + AAAA + \cdots) + \cdots\end{aligned}$$

is the generating function of such words. Then the generating function of the number of such words is

$$\begin{aligned}&W(xx + xxxx + \cdots, x + xxx + \cdots)\\&= W(x^2 + x^4 + \cdots, x + x^3 + \cdots)\\&= W\left(\frac{x^2}{1-x^2}, \frac{x}{1-x^2}\right)\\&= \frac{(\frac{x^2}{1-x^2} + 1)(\frac{x}{1-x^2} + 1)}{1 - \frac{x^2}{1-x^2} \cdot \frac{x}{1-x^2}}\\&= \frac{(x^2 + 1 - x^2)(x + 1 - x^2)}{(1-x^2)(1-x^2) - x^3}\\&= \frac{1 + x - x^2}{1 - 2x^2 - x^3 + x^4}.\end{aligned}$$

Let

$$\frac{1 + x - x^2}{1 - 2x^2 - x^3 + x^4} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

From the denominator of the generating function, we see that $a_n = 2a_{n-2} + a_{n-3} - a_{n-4}$ for all $n \geq 4$. Also, the initial terms are $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, and $a_3 = 3$, based on the words B, AA, AAB, BAA, and BBB. From here, we easily compute that $a_{14} = 172$.

Exercises

20. For an arbitrary positive integer n, consider all possible words of n letters A and B and denote by p_n the number of words containing neither $AAAA$ nor BBB. Calculate the value of

$$\frac{p_{2004} - p_{2002} - p_{1999}}{p_{2001} + p_{2000}}.$$

(Czech-Slovak, 2004)

21. Let S denote the set of strings where each letter is A or B, such that every A is next to an A and every B is next to a B. For example, the string $AAABBAA$ is allowed, but the string $AAAB$ is not.

    (a) Let $f(n, m)$ denote the number of strings in S with n As and m Bs. Find a recursive formula for $f(n, m)$.

(b) Let $g(n)$ denote the number of strings in S with n letters. Find $g(n)$.

22. Show that the generating function of the words composed of the letters a, b, and c, such that no two consecutive letters are the same, is

$$W(a, b, c) = \frac{(1 + a)(1 + b)(1 + c)}{1 - ab - ac - bc - 2abc}.$$

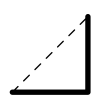
Generalize to n letters.

23. The U.S. Social Security numbers consist of 9 digits (with initial zeros permitted). How many such numbers are there that do not contain any digit three or more times consecutively? (Crux Mathematicorum, Problem 879)

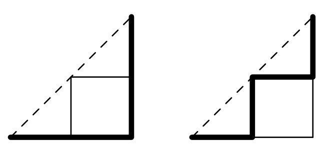
11 The Catalan Numbers

For a positive integer n, let C_n denote the number of paths in the coordinate plane that start at $(0, 0)$ and end at (n, n), consisting of unit steps moving up and to the right, such that the path stays below the line $y = x$ (but the path can touch this line). These numbers C_n are known as the *Catalan numbers*. (This is but one of many, many possible definitions of these numbers.) To get a feel for these numbers, we compute the first few values.

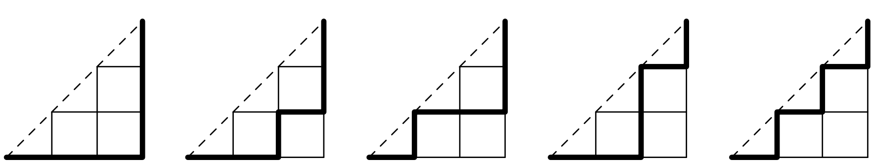
For $n = 1$, there is only path, so $C_1 = 1$.



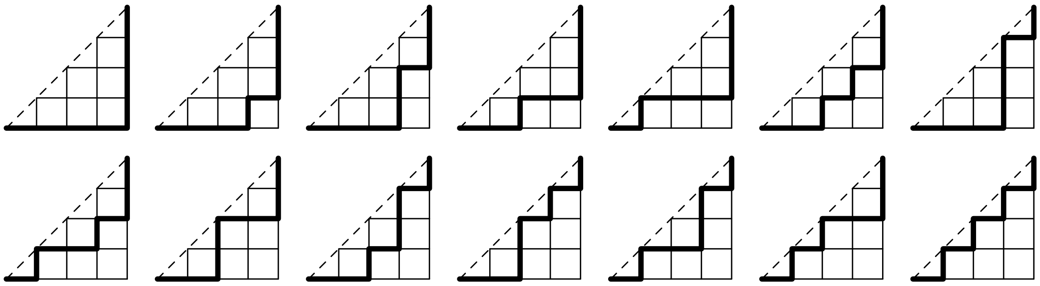
For $n = 2$, there are two paths, so $C_2 = 2$.



For $n = 3$, there are five paths, so $C_3 = 5$.

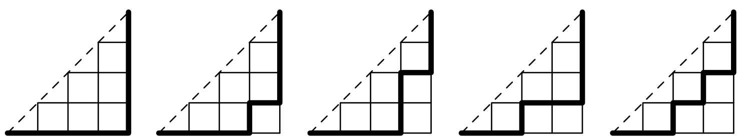


For $n = 4$, there are 14 paths, so $C_4 = 14$.



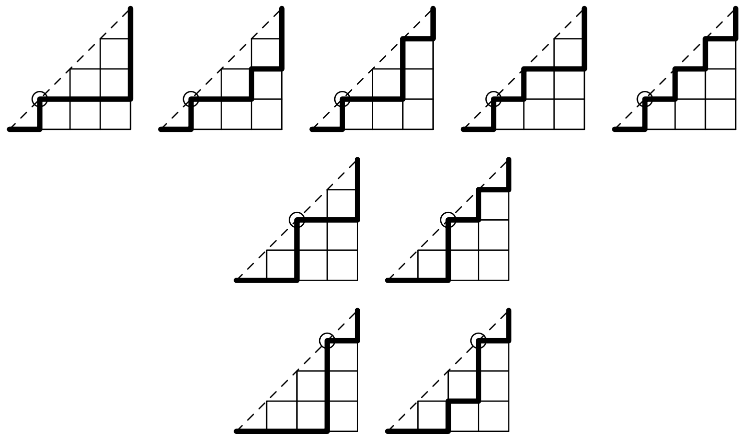
Can we find a way to compute C_n without drawing all the possible paths?

We take a closer look at the 14 paths from $(0, 0)$ to $(4, 4)$. How many of these paths do not touch the line $y = x$, other than at $(0, 0)$ and $(4, 4)$? Counting, we find that there are five such paths. Note that this number is equal to $C_3 = 5$.



This makes sense, because if a path from $(0, 0)$ to $(4, 4)$ does not touch the line $y = x$ (other than at $(0, 0)$ and $(4, 4)$), then the portion of the path that goes from $(1, 0)$ to $(4, 3)$ stays below the line $y = x - 1$. The number of such paths is clearly equal to $C_3 = 5$. More generally, the number of paths from $(0, 0)$ to (n, n) that do not touch the line $y = x$, other than at $(0, 0)$ and (n, n), is C_{n-1}.

The remaining paths touch the line $y = x$ at some point, other than at $(0, 0)$ and $(4, 4)$, and we can classify them as to when they first touch this line. Of the 9 remaining paths, 5 touch the line first at $(1, 1)$, 2 touch the line first at $(2, 2)$, and 2 touch the line first at $(3, 3)$.



How many paths from $(0,0)$ to (n,n) first touch the line $y=x$ at (k,k), where $1 \le k \le n-1$?

For $k=1$, there is only one path from $(0,0)$ to $(1,1)$. Then there are C_{n-1} possible paths from $(1,1)$ to (n,n).

For $k \ge 2$, the portion of the path from $(0,0)$ to (k,k) does not touch the line $y=x$ (other than at $(0,0)$ and (k,k)), so there are C_{k-1} possible paths from $(0,0)$ to (k,k). Then there are C_{n-k} possible paths from (k,k) to (n,n), resulting in $C_{k-1}C_{n-k}$ possible paths from $(0,0)$ to (n,n).

Summing over $1 \le k \le n-1$, and including the C_{n-1} paths that do not touch the line $y=x$ (other than at $(0,0)$ and (n,n)), we find

$$C_n = C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \cdots + C_{n-2}C_1 + C_{n-1}.$$

If we set $C_0 = 1$, then we can re-write the equation above as

$$C_n = C_0C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \cdots + C_{n-2}C_1 + C_{n-1}C_0.$$

In addition to providing a recursion for the Catalan numbers, we recognize the sum in the right-hand side as the $(n-1)^{\text{st}}$ term in the convolution of the sequence (C_n) with itself. So, let

$$C(x) = C_0 + C_1x + C_2x^2 + \cdots.$$

Then

$$\begin{aligned} C(x)^2 &= C_0^2 + (C_0C_1 + C_1C_0)x + (C_0C_2 + C_1C_1 + C_2C_0)x^2 + \cdots \\ &= C_1 + C_2x + C_3x^2 + \cdots, \end{aligned}$$

so

$$\begin{aligned} xC(x)^2 &= C_1x + C_2x^2 + C_3x^3 + \cdots \\ &= C(x) - 1. \end{aligned}$$

Hence,

$$xC(x)^2 - C(x) + 1 = 0.$$

By the quadratic formula,

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

By the generalized Binomial theorem,

$$\begin{aligned}\sqrt{1-4x} &= (1-4x)^{1/2} \\ &= 1 + \binom{\frac{1}{2}}{1}(-4x) + \binom{\frac{1}{2}}{2}(-4x)^2 + \cdots .\end{aligned}$$

For $n \ge 1$, the coefficient of x^n in $\sqrt{1-4x}$ is given by

$$\begin{aligned}\binom{\frac{1}{2}}{n}(-4)^n &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots[\frac{1}{2}-(n-1)]}{n!}\cdot(-4)^n \\ &= \frac{\frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})\cdots(\frac{3-2n}{2})}{n!}\cdot(-4)^n \\ &= (-1)^{n-1}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}\cdot(-4)^n \\ &= -\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\cdot 2^n.\end{aligned}$$

Multiplying the numerator and denominator by $2\cdot 4\cdot 6\cdots(2n-2)$, we get

$$\begin{aligned}-\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\cdot 2^n &= -\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdots(2n-3)(2n-2)}{2\cdot 4\cdot 6\cdots(2n-2)n!}\cdot 2^n \\ &= -\frac{(2n-2)!}{2^{n-1}1\cdot 2\cdot 3\cdots(n-1)n!}\cdot 2^n \\ &= -\frac{2(2n-2)!}{n!(n-1)!}.\end{aligned}$$

Hence,

$$\sqrt{1-4x} = 1 - \frac{2\cdot 0!}{1!0!}x - \frac{2\cdot 2!}{2!1!}x^2 - \frac{2\cdot 4!}{3!2!}x^3 - \cdots .$$

We have derived that

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

For this expression to represent a well-defined generating function, we must be able to cancel the factor of x in the denominator with a factor of x in the numerator. This can occur only if we take the negative square root, so

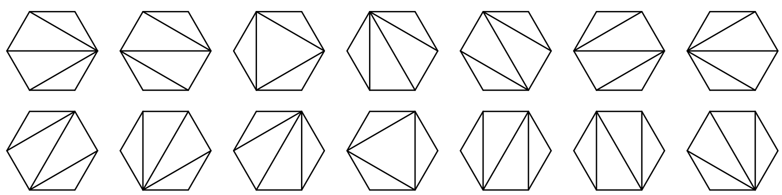
$$\begin{aligned} C(x) &= \frac{1-\sqrt{1-4x}}{2x} \\ &= \frac{1-(1-\frac{2\cdot 0!}{1!0!}x-\frac{2\cdot 2!}{2!1!}x^2-\frac{2\cdot 4!}{3!2!}x^3-\cdots)}{2x} \\ &= \frac{0!}{1!0!}+\frac{2!}{2!1!}x+\frac{4!}{3!2!}x^2+\cdots. \end{aligned}$$

Therefore, the coefficient of x^n in $C(x)$ is

$$C_n = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\cdot\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}.$$

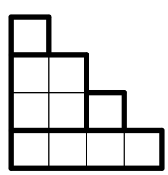
Exercises

24. Given a polygon, we can partition the polygon into triangles using the diagonals of the polygon, such that every vertex of a triangle is also a vertex of the polygon. Such a partition is called a *triangulation*. Show that the number of triangulations of a regular n-gon is C_{n-2}. The 14 triangulations of a hexagon are shown below.

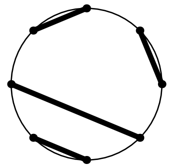


25. For a positive integer n, an n-*staircase* is a figure consisting of unit squares, with one square in the first row, two squares in the second row, and so on, up to n squares in the n^{th} row, such that all the left-most squares in each row are aligned vertically.

    The n-staircase is tiled with n rectangles, such that each rectangle has integer side lengths. Show that the number of such tilings is C_n. A tiled 4-staircase is shown below.



26. Given a positive integer n, take $2n$ points evenly spaced on a circle. We divide the $2n$ points into n pairs, and draw the line segment connecting the two points in the same pair. Show that the number of ways to divide the $2n$ points into n pairs, so that no two line segments intersect, is C_n. An example for $n = 4$ is shown below.



27. Find all n such that C_n is odd.

12 The Principle of Inclusion-Exclusion

For any two sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The left-hand side $|A \cup B|$ counts all the elements that are in at least one of the sets A and B. We can also count this number by adding all the elements in A and B to get $|A| + |B|$, but we have double-counted all the elements that are in both A and B, so we must subtract $|A \cap B|$.

For three sets A, B, and C, the number of elements that are in at least one of the sets A, B, and C is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

In general, for n sets A_1, A_2, $\dots$, A_n, the number of elements that are in at least one of the sets A_1, A_2, $\dots$, A_n is

$$\begin{aligned} |A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| &+ |A_2| + \dots + |A_n| \\ &- (|A_1 \cap A_2| + |A_1 \cap A_3| + \dots + |A_{n-1} \cap A_n|) \\ &+ (|A_1 \cap A_2 \cap A_3| + \dots + |A_{n-2} \cap A_{n-1} \cap A_n|) \\ &- (|A_1 \cap A_2 \cap A_3 \cap A_4| + \cdots) \\ &+ \cdots \\ &+ (-1)^{n-1} |A_1 \cap A_2 \cap \dots \cap A_n|. \end{aligned}$$

Instead, what if we want to count the number of elements that are in **exactly** one set? For example, for two sets, the number of elements that are in exactly one of the sets A and B is

$$|A| + |B| - 2|A \cap B|.$$

For three sets, the number of elements that are in exactly one of the sets A, B, and C is

$$|A| + |B| + |C| - 2|A \cap B| - 2|A \cap C| - 2|B \cap C| + 3|A \cap B \cap C|.$$

Can we generalize this to n sets? We can also generalize another way, asking how many elements are in exactly k sets.

Let A_1, A_2, $\dots$, A_n be subsets of a set S, and for $0 \le t \le n$, let e_t be the number of elements in S that appear in exactly t of the sets A_1, A_2, $\dots$, A_n.

For $1 \le r \le n$, let

$$N_r = \sum_{\substack{I \subseteq \{1,2,\dots,n\} \\ |I|=r}} \left| \bigcap_{i \in I} A_i \right|,$$

and set $N_0 = |S|$. In other words, N_r is the sum of all terms of the form

$$|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_r}|,$$

where $\{i_1, i_2, \dots, i_r\}$ is taken over all r-subsets of $\{1, 2, \dots, n\}$. For example, for $n = 3$,

$$\begin{aligned} N_1 &= |A_1| + |A_2| + |A_3|, \\ N_2 &= |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|, \\ N_3 &= |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

Thus, we have already encountered these expressions in the Inclusion-Exclusion formula. Our goal is to express e_t in terms of N_0, N_1, N_2, $\dots$, N_n.

Let x be an element in S. Assume that x appears in exactly t of the sets A_1, A_2, $\dots$, A_n, say A_{i_1}, A_{i_2}, $\dots$, A_{i_t}. Then x is counted $\binom{t}{r}$ times in the sum N_r, once for each of the $\binom{t}{r}$ subsets of $\{i_1, i_2, \dots, i_t\}$ containing r elements. Therefore, summing over all $0 \le t \le n$, we find

$$N_r = \sum_{t=0}^{n} \binom{t}{r} e_t.$$

Now, let

$$E(x) = \sum_{t=0}^{n} e_t x^t, \quad N(x) = \sum_{r=0}^{n} N_r x^r.$$

Then

$$\begin{aligned} N(x) &= \sum_{r=0}^{n} N_r x^r \\ &= \sum_{r=0}^{n} \left(\sum_{t=0}^{n} \binom{t}{r} e_t \right) x^r \\ &= \sum_{t=0}^{n} e_t \left(\sum_{r=0}^{n} \binom{t}{r} x^r \right) \\ &= \sum_{t=0}^{n} e_t (x+1)^t \\ &= E(x+1). \end{aligned}$$

Hence,

$$\begin{aligned} E(x) &= N(x-1) \\ &= \sum_{r=0}^{n} N_r (x-1)^r \\ &= \sum_{r=0}^{n} N_r \left(\sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} x^i \right). \end{aligned}$$

Therefore, the coefficient of x^t is

$$e_t = \sum_{r=t}^{n} (-1)^{r-t} \binom{r}{t} N_r.$$

Example. For $t = 1$, the number of elements in S that appear in exactly one of the sets A_1, A_2, $\dots$, A_n is

$$e_1 = \sum_{r=1}^{n} (-1)^{r-1} \binom{r}{1} N_r = N_1 - 2N_2 + 3N_3 + \cdots + (-1)^{n-1} n N_n.$$

Exercises

28. For $1 \le t \le n$, let f_t be the number of elements in S that appear in **at least** t of the sets A_1, A_2, $\dots$, A_n, and let

$$F(x) = \sum_{t=0}^{n} f_t x^t.$$

   (a) Show that

$$F(x) = \frac{xE(x) - E(1)}{x-1}.$$

   (b) Show that for $1 \le t \le n$,

$$f_t = \sum_{r=t}^{n} (-1)^{r-t} \binom{r-1}{t-1} N_r.$$

13 Miscellaneous Exercises

1. Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

   Prove that, for each integer $n \ge 0$, there is an integer m such that $a_n^2 + a_{n+1}^2 = a_m$. (Putnam, 1999)

2. If the expansion in powers of x of the function
$$\frac{1}{(1-ax)(1-bx)}$$
is given by $c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$, prove that the expansion in powers of x of the function
$$\frac{1+abx}{(1-abx)(1-a^2x)(1-b^2x)}$$
is given by $c_0^2 + c_1^2x + c_2^2x^2 + c_3^2x^3 + \cdots$. (Putnam, 1939)

3. (Vandermonde's Identity) Let m, n, and k be nonnegative integers such that $k \le m$ and $k \le n$. Prove that
$$\binom{m+n}{k} = \binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \binom{m}{2}\binom{n}{k-2} + \cdots + \binom{m}{k}\binom{n}{0}.$$

4. Let
$$\prod_{n=1}^{1996}(1+nx^{3^n}) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m},$$
where a_1, a_2, ..., a_m are nonzero and $k_1 < k_2 < \cdots < k_m$. Find a_{1996}. (Turkey, 1996)

5. (Hockey Stick Identity) Let n and k be nonnegative integers, with $n \ge k$. Prove that
$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

6. Let
$$\frac{1}{1-x-xy} = \sum_{n,m=0}^{\infty} a_{n,m}x^ny^m.$$
Find $a_{n,m}$.

7. Find the sequence (a_n) if $a_0 = 1$ and
$$\sum_{k=0}^{n} a_ka_{n-k} = 1$$
for all $n \ge 0$.

8. Let
$$(1+x+x^2+x^3+x^4)^{496} = a_0 + a_1x + a_2x^2 + \cdots + a_{1984}x^{1984}.$$
Determine $\gcd(a_3, a_8, a_{13}, \ldots, a_{1983})$. (IMO Proposal, 1983)