2005 MOSP Homework

Congratulations on your excellent performance on the AMC, AIME, and USAMO tests, which has earned you an invitation to attend the Math Olympiad Summer Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

To better prepare yourself for MOSP, you need to work on the following homework problems, which come from last year's National Olympiads, from countries all around the world. Even if some may seem difficult, you should dedicate a significant amount of effort to think about them—don't give up right away. All of you are highly talented, but you may have a disappointing start if you do not put in enough energy here. At the beginning of the program, written solutions will be submitted for review by MOSP graders, and you will present your solutions and ideas during the first few study sessions.

You are encouraged to use the email list to discuss these and other interesting math problems. Also, if you have any questions about these homework problems, please feel free to contact us.

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1
Problems for the Red Group

Algebra

1.1. Let a and b be nonnegative real numbers. Prove that

$$\sqrt{2}\left(\sqrt{a(a+b)^3}+b\sqrt{a^2+b^2}\right)\leq 3(a^2+b^2).$$

1.2. Determine if there exist four polynomials such that the sum of any three of them has a real root while the sum of any two of them does not.

1.3. Let a, b, and c be real numbers. Prove that

$$\begin{aligned}&\sqrt{2(a^2+b^2)}+\sqrt{2(b^2+c^2)}+\sqrt{2(c^2+a^2)}\\ \geq\ &\sqrt{3[(a+b)^2+(b+c)^2+(c+a)^2]}.\end{aligned}$$

1.4. Let $x_1, x_2, \ldots, x_5$ be nonnegative real numbers such that $x_1 + x_2 + x_3 + x_4 + x_5 = 5$. Determine the maximum value of $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5$.

1.5. Let S be a finite set of positive integers such that none of them has a prime factor greater than three. Show that the sum of the reciprocals of the elements in S is smaller than three.

1.6. Find all function $f : \mathbb{Z} \to \mathbb{R}$ such that $f(1) = 5/2$ and that

$$f(x)f(y) = f(x+y) + f(x-y)$$

for all integers x and y.

1.7. Let $x_{1,1}, x_{2,1}, \ldots, x_{n,1}$, $n \geq 2$, be a sequence of integers and assume that not all $x_{i,1}$ are equal. For $k \geq 2$, if sequence $\{x_{i,k}\}_{i=1}^n$ is defined, we define sequence $\{x_{i,k+1}\}_{i=1}^n$ as

$$x_{i,k+1} = \frac{1}{2}\left(x_{i,k} + x_{i+1,k}\right),$$

for $i = 1, 2, \ldots, n$, (where $x_{n+1,k} = x_{1,k}$). Show that if n is odd then there exist indices j and k such that $x_{j,k}$ is not an integer.

Combinatorics

1.8. Two rooks on a chessboard are said to be attacking each other if they are placed in the same row or column of the board.

(a) There are eight rooks on a chessboard, none of them attacks any other. Prove that there is an even number of rooks on black fields.

(b) How many ways can eight mutually non-attacking rooks be placed on the 9×9 chessboard so that all eight rooks are on squares of the same color.

1.9. Set $S = \{1, 2, \ldots, 2004\}$. We denote by d_1 the number of subset of S such that the sum of elements of the subset has remainder 7 when divided by 32. We denote by d_2 the number of subset of S such that the sum of elements of the subset has remainder 14 when divided by 16. Compute d_1/d_2.

1.10. In a television series about incidents in a conspicuous town there are n citizens staging in it, where n is an integer greater than 3. Each two citizens plan together a conspiracy against one of the other citizens. Prove that there exists a citizen, against whom at least $\sqrt{n}$ other citizens are involved in the conspiracy.

1.11. Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there are three players A, B, and C such that A beats B, B beats C, and C beats A. Such a triple of player is called a *cycle*. Determine the number of maximum cycles such a tournament can have.

1.12. Determine if it is possible to choose nine points in the plane such that there are $n = 10$ lines in the plane each of which passes through exactly three of the chosen points. What if $n = 11$?

1.13. Let n be a positive integer. Show that

$$\frac{1}{\binom{n}{1}} + \frac{1}{2\binom{n}{2}} + \frac{1}{3\binom{n}{3}} + \cdots + \frac{1}{n\binom{n}{n}}$$
$$= \frac{1}{2^{n-1}} + \frac{1}{2 \cdot 2^{n-2}} + \frac{1}{3 \cdot 2^{n-3}} + \cdots + \frac{1}{n \cdot 2^0}.$$

1.14. A segment of length 2 is divided into n, $n \geq 2$, subintervals. A square is then constructed on each subinterval. Assume that the sum of the areas of all such squares is greater than 1. Show that under this assumption one can always choose two subintervals with total length greater than 1.

Geometry

1.15. Isosceles triangle ABC, with $AB = AC$, is inscribed in circle ω. Point D lies on arc $\widehat{BC}$ not containing A. Let E be the foot of perpendicular from A to line CD. Prove that $BC + DC = 2DE$.

1.16. Let ABC be a triangle, and let D be a point on side AB. Circle ω_1 passes through A and D and is tangent to line AC at A. Circle ω_2 passes through B and D and is tangent to line BC at B. Circles ω_1 and ω_2 meet at D and E. Point F is the reflection of C across the perpendicular bisector of AB. Prove that points D, E, and F are collinear.

1.17. Let M be the midpoint of side BC of triangle ABC ($AB > AC$), and let AL be the bisector of the angle A. The line passing through M perpendicular to AL intersects the side AB at the point D. Prove that $AD + MC$ is equal to half the perimeter of triangle ABC.

1.18. Let ABC be an obtuse triangle with $\angle A > 90^\circ$, and let r and R denote its inradius and circumradius. Prove that

$$\frac{r}{R} \le \frac{a \sin A}{a+b+c}.$$

1.19. Let ABC be a triangle. Points D and E lie on sides BC and CA, respectively, such that $BD = AE$. Segments AD and BE meet at P. The bisector of angle BCA meet segments AD and BE at Q and R, respectively. Prove that

$$\frac{PQ}{AD} = \frac{PR}{BE}.$$

1.20. Consider the three disjoint arcs of a circle determined by three points of the circle. We construct a circle around each of the midpoint of every arc which goes the end points of the arc. Prove that the three circles pass through a common point.

1.21. Let ABC be a triangle. Prove that

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \ge 4\left(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}\right).$$

Number Theory

1.22. Find all triples (x, y, z) in integers such that $x^2 + y^2 + z^2 = 2^{2004}$.

1.23. Suppose that n is s positive integer. Determine all the possible values of the first digit after the decimal point in the decimal expression of the number $\sqrt{n^3 + 2n^2 + n}$.

1.24. Suppose that p and q are distinct primes and S is a subset of $\{1, 2, \ldots, p-1\}$. Let $N(S)$ denote the number of ordered q-tuples $(x_1, x_2, \ldots, x_q)$ with $x_i \in S$, $1 \le i \le q$, such that

$$x_1 + x_2 + \cdots + x_q \equiv 0 \pmod{p}.$$

1.25. Let p be an odd prime. Prove that

$$\sum_{k=1}^{p-1} k^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}.$$

1.26. Find all ordered triple (a, b, c) of positive integers such that the value of the expression

$$\left(b - \frac{1}{a}\right)\left(c - \frac{1}{b}\right)\left(a - \frac{1}{c}\right)$$

is an integer.

1.27. Let $a_1 = 0$, $a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for all positive integers n. Show that there exists an increasing infinite arithmetic progression of integers, which has no number in common in the sequence $\{a_n\}_{n\ge 0}$.

1.28. Let a, b, and c be pairwise distinct positive integers, which are side lengths of a triangle. There is a line which cuts both the area and the perimeter of the triangle into two equal parts. This line cuts the longest side of the triangle into two parts with ratio $2 : 1$. Determine a, b, and c for which the product abc is minimal.

2
Problems for the Blue Group

Algebra

2.1. Let $a_0, a_1, \dots a_n$ be integers, not all zero, and all at least -1. Given $a_0+2a_1+2^2a_2+\cdots+2^na_n = 0$, prove that $a_0+a_1+\cdots+a_n > 0$.

2.2. The sequence of real numbers $\{a_n\}$, $n \in \mathbb{N}$ satisfies the following condition: $a_{n+1} = a_n(a_n + 2)$ for any $n \in \mathbb{N}$. Find all possible values for a_{2004}.

2.3. Determine all polynomials $P(x)$ with real coefficients such that

$$(x^3 + 3x^2 + 3x + 2)P(x - 1) = (x^3 - 3x^2 + 3x - 2)P(x).$$

2.4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^3) - f(y^3) = (x^2 + xy + y^2)(f(x) - f(y)).$$

2.5. Let $a_1, a_2, \dots, a_{2004}$ be non-negative real numbers such that $a_1 + \cdots + a_{2004} \le 25$. Prove that among them there exist at least two numbers a_i and a_j $(i \ne j)$ such that

$$\left|\sqrt{a_i} - \sqrt{a_j}\right| \le \frac{5}{2003}.$$

2.6. Let c be a fixed positive integer, and $\{x_k\}_{k=1}^{\infty}$ be a sequence such that $x_1 = c$ and

$$x_n = x_{n-1} + \left\lfloor \frac{2x_{n-1} - 2}{n} \right\rfloor$$

for $n \ge 2$. Determine the explicit formula of x_n in terms of n and c.

(Here $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.)

2.7. Let n be a positive integer with n greater than one, and let $a_1, a_2, \dots, a_n$ be positive integers such that $a_1 < a_2 < \cdots < a_n$ and

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \le 1.$$

Prove that

$$\left(\frac{1}{a_1^2 + x^2} + \frac{1}{a_2^2 + x^2} + \cdots + \frac{1}{a_n^2 + x^2}\right)^2 \le \frac{1}{2} \cdot \frac{1}{a_1(a_1 - 1) + x^2}$$

for all real numbers x.

Combinatorics

2.8. Consider all binary sequences (sequences consisting of 0's and 1's). In such a sequence the following four types of operation are allowed: (a) $010 \to 1$, (b) $1 \to 010$, (c) $110 \to 0$, and (d) $0 \to 110$. Determine if it is possible to obtain sequence

$$1\underbrace{00\ldots0}_{2003}$$

from the sequence

$$\underbrace{0\ldots0}_{2003}1.$$

2.9. Exactly one integer is written in each square of an n by n grid, $n \geq 3$. The sum of all of the numbers in any 2×2 square is even and the sum of all the numbers in any 3×3 square is even. Find all n for which the sum of all the numbers in the grid is necessarily even.

2.10. Let T be the set of all positive integer divisors of 2004^{100}. What is the largest possible number of elements that a subset S of T can have if no element of S is an integer multiple of any other element of S.

2.11. Consider an infinite array of integer. Assume that each integer is equal to the sum of the integers immediately above and immediately to the left. Assume that there exists a row R_0 such that all the number in the row are positive. Denote by R_1 the row below row R_0, by R_2 the row below row R_1, and so on. Show that for each positive integer n, row R_n cannot contain more than n zeros.

2.12. Show that for nonnegative integers m and n,

$$\begin{aligned}&\frac{\binom{m}{0}}{n+1} - \frac{\binom{m}{1}}{n+2} + \cdots + (-1)^m \frac{\binom{m}{m}}{n+m+1} \\ &= \frac{\binom{n}{0}}{m+1} - \frac{\binom{n}{1}}{m+2} + \cdots + (-1)^n \frac{\binom{n}{n}}{m+n+1}.\end{aligned}$$

2.13. A $10 \times 10 \times 10$ cube is made up up from 500 white unit cubes and 500 black unit cubes, arranged in such a way that every two unit cubes that shares a face are in different colors. A *line* is a $1 \times 1 \times 10$ portion of the cube that is parallel to one of cube's edges. From the initial cube have been removed 100 unit cubes such that 300 lines of the cube has exactly one missing cube.

Determine if it is possible that the number of removed black unit cubes is divisible by 4.

2.14. Let $\mathcal{S}$ be a set of points in the plane satisfying the following conditions:

(a) there are seven points in $\mathcal{S}$ that form a convex heptagon; and

(b) for any five points in $\mathcal{S}$, if they form a convex pentagon, then there is point in $\mathcal{S}$ lies in the interior of the pentagon.

Determine the minimum value of the number of elements in $\mathcal{S}$.

Geometry

2.15. Let $ABCDEF$ be a equilateral convex hexagon with $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F$. Prove that lines AD, BE, and CF are concurrent.

2.16. Let $ABCD$ be a convex quadrilateral. Let P, Q be points on sides BC and DC respectively such that $\angle BAP = \angle DAQ$. Show that the area of triangles ABP and ADQ is equal if and only if the line through the orthocenters of these triangles is orthogonal to AC.

2.17. Circles S_1 and S_2 meet at points A and B. A line through A is parallel to the line through the centers of S_1 and S_2 and meets S_1 and S_2 again C and D respectively. Circle S_3 having CD as its diameter meets S_1 and S_2 again at P and Q respectively. Prove that lines CP, DQ, and AB are concurent.

2.18. The incircle O of an isosoceles triangle ABC with $AB = AC$ meets BC, CA, AB at K, L, M respectively. Let N be the intersection of lines OL and KM and let Q be the intersection of lines BN and CA. Let P be the foot of the perpendicular from A to BQ. If we assume that $BP = AP + 2PQ$, what are the possible values of AB/BC?

2.19. Let $ABCD$ be a cyclic quadrilateral such that $AB \cdot BC = 2 \cdot AD \cdot DC$. Prove that its diagonals AC and BD satisfy the inequality $8BD^2 \le 9AC^2$.

2.20. A circle which is tangent to sides AB and BC of triangle ABC is also tangent to its circumcircle at point T. If I in the incenter of triangle ABC, show that $\angle ATI = \angle CTI$.

2.21. Points E, F, G, and H lie on sides AB, BC, CD, and DA of a convex quadrilateral $ABCD$ such that

$$\frac{AE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CG}{GD} \cdot \frac{DH}{HA} = 1.$$

Points A, B, C, and D lie on sides H_1E_1, E_1F_1, F_1G_1, and G_1H_1 of a convex quadrilateral $E_1F_1G_1H_1$ such that $E_1F_1 \parallel EF$, $F_1G_1 \parallel FG$, $G_1H_1 \parallel GH$, and $H_1E_1 \parallel HE$.

Given that $\frac{E_1A}{AH_1} = a$, express $\frac{F_1C}{CG_1}$ in terms of a.

Number Theory

2.22. We call a natural number **3-partite** if the set of its divisors can be partitioned into 3 subsets each with the same sum. Show that there exist infinitely many 3-partite numbers.

2.23. Find all real numbers x such that

$$\lfloor x^2 - 2x \rfloor + 2\lfloor x \rfloor = \lfloor x \rfloor^2.$$

(For real number x, $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.)

2.24. Prove that the equation $a^3 - b^3 = 2004$ does not have any solutions in positive integers.

2.25. Find all prime numbers p and q such that $3p^4 + 5p^4 + 15 = 13p^2q^2$.

2.26. Does there exist an infinite subset S of the natural numbers such that for every $a, b \in S$, the number $(ab)^2$ is divisible by $a^2 - ab + b^2$?

2.27. A positive integer n is *good* if n can be written as the sum of 2004 positive integers $a_1, a_2, \ldots, a_{2004}$ such that $1 \le a_1 < a_2 < \cdots < a_{2004}$ and a_i divides a_{i+1} for $i = 1, 2, \ldots, 2003$. Show that there are only finitely many positive integers that are not good.

2.28. Let n be a natural number and $f_1, f_2, \ldots, f_n$ be polynomials with integers coefficients. Show that there exists a polynomial $g(x)$ which can be factored (with at least two terms of degree at least 1) over the integers such that $f_i(x) + g(x)$ cannot be factored (with at least two terms of degree at least 1) over the integers for every i.

3
Problems for the Black Group

Algebra

3.1. Given real numbers x, y, z such that $xyz = -1$, show that

$$x^4 + y^4 + z^4 + 3(x + y + z) \geq \frac{x^2}{y} + \frac{x^2}{z} + \frac{y^2}{x} + \frac{y^2}{z} + \frac{z^2}{y} + \frac{z^2}{x}.$$

3.2. Let x, y, z be positive real numbers and $x + y + z = 1$. Prove that

$$\sqrt{xy + z} + \sqrt{yz + x} + \sqrt{zx + y} \geq 1 + \sqrt{xy} + \sqrt{yz} + \sqrt{zx}.$$

3.3. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that

(a) $f(1) = 1$

(b) $f(n+2) + (n^2 + 4n + 3)f(n) = (2n + 5)f(n + 1)$ for all $n \in \mathbb{N}$

(c) $f(n)$ divides $f(m)$ if $m > n$.

3.4. Does there exist a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x^2y + f(x + y^2)) = x^3 + y^3 + f(xy)?$$

3.5. Find the smallest real number p such that the inequality

$$\sqrt{1^2 + 1} + \sqrt{2^2 + 1} + \cdots + \sqrt{n^2 + 1} \leq \frac{1}{2}n(n + p)$$

holds for all natural numbers n.

3.6. Solve the system of equations

$$\begin{cases} x^2 = \frac{1}{y} + \frac{1}{z}, \\ y^2 = \frac{1}{z} + \frac{1}{x}, \\ z^2 = \frac{1}{x} + \frac{1}{y}. \end{cases}$$

in the real numbers.

3.7. Find all positive integers n for which there are distinct integers $a_1, \ldots, a_n$ such that

$$\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{n}{a_n} = \frac{a_1 + \cdots + a_n}{2}.$$

Combinatorics

3.8. Let X be a set with n elements and $0 \leq k \leq n$. Let $a_{n,k}$ be the maximum number of permutations of the set X such that every two of them have at least k common components (where a common component of f and g is an $x \in X$ such that $f(x) = g(x)$). Let $b_{n,k}$ be the maximum number of permutations of the set X such that every two of them have at most k common components.

(a) Show that $a_{n,k} \cdot b_{n,k-1} \leq n!$.

(b) Let p be prime, and find the exact value of $a_{p,2}$.

3.9. A regular 2004-sided polygon is given, with all of its diagonals drawn. After some sides and diagonals are removed, every vertex has at most five segments coming out of it. Prove that one can color the vertices with two colors such that at least $3/5$ of the remaining segments have ends with different colors.

3.10. Squares of an $n \times n$ table ($n \geq 3$) are painted black and white as in a chessboard. A move allows one to choose any 2×2 square and change all of its squares to the opposite color. Find all such n that there is a finite number of the moves described after which all squares are the same color.

3.11. A convex 2004-sided polygon P is given such that no four vertices are cyclic. We call a triangle whose vertices are vertices of P *thick* if all other 2001 vertices of P lie inside the circumcircle of the triangle, and *thin* if they all lie outside its circumcircle. Prove that the number of thick triangles is equal to the number of thin triangles.

3.12. A group consists of n tourists. Among every three of them there are at least two that are not familiar. For any partition of the group into two busses, there are at least two familiar tourists in one bus. Prove that there is a tourist who is familiar with at most two fifth of all the tourists.

3.13. A computer network is formed by connecting 2004 computers by cables. A set S of these computers is said to be *independent* if no pair of computers of S is connected by a cable. Suppose that the number of cables used is the minimum number possible such that the size of any independent set is at most 50. Let $c(L)$ be the number of cables connected to computer L. Show that for any distinct computers A and B, $c(A) = c(B)$ if they are connected by a cable and $|c(A) - c(B)| \leq 1$ otherwise. Also, find the number of cables used in the network.

3.14. Eight problems were given to each of 30 students. After the test was given, point values of the problems were determined as follows: a problem is worth n points if it is not solved by exactly

n contestants (no partial credit is given, only zero marks or full marks).

(a) Is it possible that the contestant having got more points that any other contestant had also solved less problems than any other contestant?

(b) Is it possible that the contestant having got less points than any other contestant has solved more problems than any other contestant?

Geometry

3.15. A circle with center O is tangent to the sides of the angle with the vertex A at the points B and C. Let M be a point on the larger of the two arcs BC of this circle (different from B and C) such that M does not lie on the line AO. Lines BM and CM intersect the line AO at the points P and Q respectively. Let K be the foot of the perpendicular drawn from P to AC and L be the foot of the perpendicular drawn from Q to AB. Prove that the lines OM and KL are perpendicular.

3.16. Let I be the incenter of triangle ABC, and let A_1, B_1, and C_1 be arbitrary points lying on segments AI, BI, and CI, respectively. The perpendicular bisectors of segments AA_1, BB_1, and CC_1 form triangles $A_2B_2C_2$. Prove that the circumcenter of triangle $A_2B_2C_2$ coincides with the circumcenter of triangle ABC if and only if I is the orthocenter of triangle $A_1B_1C_1$.

3.17. Points M and M' are isogonal conjugates in the triangle ABC (i.e. $\angle BAM = \angle M'AC$, $\angle ACM = \angle M'CB$, $\angle CBM = \angle M'BA$). We draw perpendiculars MP, MQ, MR and $M'P'$, $M'Q'$, $M'R'$ to the sides BC, AC, AB respectively. Let $QR, Q'R'$ and $RP, R'P'$ and $PQ, P'Q'$ intersect at E, F, G respectively. Show that the lines EA, FB, and GC are parallel.

3.18. Let $ABCD$ be a convex quadrilateral and let K, L, M, N be the midpoints of sides AB, BC, CD, DA respectively. Let NL and KM meet at point T. Show that

$$\frac{8}{3}[DNTM] < [ABCD] < 8[DNTM],$$

where $[P]$ denotes area of P.

3.19. Let $ABCD$ be a cyclic quadrilateral such that $AB \cdot BC = 2 \cdot AD \cdot DC$. Prove that its diagonals AC and BD satisfy the inequality $8BD^2 \leq 9AC^2$.

3.20. Given a convex quadrilateral $ABCD$. The points P and Q are the midpoints of the diagonals AC and BD respectively. The line PQ intersects the lines AB and CD at N and M respectively. Prove that the circumcircles of triangles NAP, NBQ, MQD, and MPC have a common point.

3.21. Let $ABCD$ be a cyclic quadrilateral who interior angle at B is 60 degrees. Show that if $BC = CD$, then $CD + DA = AB$. Does the converse hold?

Number Theory

3.22. Let n be a natural number and $f_1, f_2, \ldots, f_n$ be polynomials with integers coefficients. Show that there exists a polynomial $g(x)$ which can be factored (with at least two terms of degree at least 1) over the integers such that $f_i(x) + g(x)$ cannot be factored (with at least two terms of degree at least 1) over the integers for every i.

3.23. Let a, b, c, and d be positive integers satisfy the following properties:

(a) there are exactly 2004 pairs of real numbers (x, y) with $0 \leq x, y \leq 1$ such that both $ax + by$ and $cx + dy$ are integers

(b) $\gcd(a, c) = 6$.

Find $\gcd(b, d)$.

3.24. For any positive integer n, the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is written in the lowest form $\frac{p_n}{q_n}$; that is p_n and q_n are relative prime positive integers. Find all n such that p_n is divisible by 3.

3.25. Prove that there does not exist an integer $n > 1$ such that n divides $3^n - 2^n$.

3.26. Find all integer solutions to

$$y^2(x^2 + y^2 - 2xy - x - y) = (x + y)^2(x - y).$$

3.27. Let p be a prime number, and let $0 \leq a_1 < a_2 < \cdots < a_m < p$ and $0 \leq b_1 < b_2 < \cdots < b_n < p$ be arbitrary integers. Denote by k the number of different remainders of $a_i + b_j$, $1 \leq i \leq m$ and $1 \leq j \leq n$, modulo p. Prove that

(i) if $m + n > p$, then $k = p$;

(ii) if $m + n \leq p$, then $k \geq m + n - 1$,

3.28. Let A be a finite subset of prime numbers and a be a positive integer. Show that the number of positive integers m for which all prime divisors of $a^m - 1$ are in A is finite.

4
Selected Problems from MOSP 2003 and 2004 Tests

Algebra

4.1.

1. Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be real numbers such that

$$(a_1^2+a_2^2+\cdots+a_n^2-1)(b_1^2+b_2^2+\cdots+b_n^2-1) > (a_1b_1+a_2b_2+\cdots+a_nb_n-1)^2.$$

Show that $a_1^2 + a_2^2 + \cdots + a_n^2 > 1$ and $b_1^2 + b_2^2 + \cdots + b_n^2 > 1$.

4.2. Let a, b and c be positive real numbers. Prove that

$$\frac{a^2+2bc}{(a+2b)^2+(a+2c)^2}+\frac{b^2+2ca}{(b+2c)^2+(b+2a)^2}+\frac{c^2+2ab}{(c+2a)^2+(c+2b)^2} \le \frac{1}{2}.$$

4.3. Let a, b, and c be positive real numbers. Prove that

$$\left(\frac{a+2b}{a+2c}\right)^3 + \left(\frac{b+2c}{b+2a}\right)^3 + \left(\frac{c+2a}{c+2b}\right)^3 \ge 3.$$

4.4. Prove that for any nonempty finite set S, there exists a function $f : S \times S \to S$ satisfying the following conditions:

(a) for all $a, b \in S$, $f(a, b) = f(b, a)$;

(b) for all $a, b \in S$, $f(a, f(a, b)) = b$;

(c) for all $a, b, c, d \in S$, $f(f(a, b), f(c, d)) = f(f(a, c), f(b, d))$.

4.5. Find all pairs (x, y) of real numbers with $0 < x < \frac{\pi}{2}$ such that

$$\frac{(\sin x)^{2y}}{(\cos x)^{y^2/2}} + \frac{(\cos x)^{2y}}{(\sin x)^{y^2/2}} = \sin(2x).$$

4.6. Prove that in any acute triangle ABC,

$$\cot^3 A+\cot^3 B+\cot^3 C+6\cot A\cot B\cot C \ge \cot A+\cot B+\cot C.$$

4.7. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \ge \frac{3}{\sqrt[3]{abc}\left(1+\sqrt[3]{abc}\right)}.$$

4.8. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(m+n)f(m-n) = f(m^2)$$

for $m, n \in \mathbb{N}$.

4.9. Let A, B, C be real numbers in the interval $(0, \frac{\pi}{2})$. Prove that

$$\frac{\sin A \sin(A-B)\sin(A-C)}{\sin(B+C)} + \frac{\sin B \sin(B-C)\sin(B-A)}{\sin(C+A)} + \frac{\sin C \sin(C-A)\sin(C-B)}{\sin(A+B)} \geq 0.$$

4.10. For a pair of integers a and b, with $0 < a < b < 1000$, set $S \subseteq \{1, 2, \ldots, 2003\}$ is called a *skipping set* for (a, b) if for any pair of elements $s_1, s_2 \in S$, $|s_1 - s_2| \notin \{a, b\}$. Let $f(a, b)$ be the maximum size of a skipping set for (a, b). Determine the maximum and minimum values of f.

4.11. Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(yf(x)-1) = x^2 f(y) - f(x)$$

for all real numbers x and y.

4.12. Show that there is an infinite sequence of positive integers

$$a_1, a_2, a_3, \ldots$$

such that

(i) each positive integer occurs exactly once in the sequence, and

(ii) each positive integer occurs exactly once in the sequence $|a_1 - a_2|, |a_2 - a_3|, \ldots$.

4.13. Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be real numbers in the interval $[1, 2]$ with $a_1^2 + a_2^2 + \cdots + a_n^2 = b_1^2 + b_2^2 + \cdots + b_n^2$. Determine the minimum value of constant c such that

$$\frac{a_1^3}{b_1} + \frac{a_2^3}{b_2} + \cdots + \frac{a_n^3}{b_n} \leq c(a_1^2 + a_2^2 + \cdots + a_n^2).$$

4.14. Let x, y, z be nonnegative real numbers with $x^2 + y^2 + z^2 = 1$. Prove that

$$1 \leq \frac{z}{1+xy} + \frac{x}{1+yz} + \frac{y}{1+zx} \leq \sqrt{2}.$$

4.15. Let n be a positive number, and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that

$$x_1 + x_2 + \cdots + x_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}.$$

Prove that

$$\frac{1}{n-1+x_1}+\frac{1}{n-1+x_2}+\cdots+\frac{1}{n-1+x_n}\le 1.$$

4.16. Let x, y, and z be real numbers. Prove that

$$xyz(2x+2y-z)(2y+2z-x)(2z+2x-y)+[x^2+y^2+z^2-2(xy+yz+zx)](xy+yz+zx)^2\ge 0.$$

4.17. Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)f(yf(x)-1)=x^2f(y)-f(x)$$

for all real numbers x and y.

Combinatorics

4.18. Let $\mathbb{N}$ denote the set of positive integers, and let S be a set. There exists a function $f:\mathbb{N}\to S$ such that if x and y are a pair of positive integers with their difference being a prime number, then $f(x)\neq f(y)$. Determine the minimum number of elements in S.

4.19. Let n be a integer with $n\ge 2$. Determine the number of non-congruent triangles with positive integer side lengths two of which sum to n.

4.20. Jess has 3 pegs and disks of different sizes. Jess is supposed to transfer the disks from one peg to another, and the disks have to be sorted so that for any peg the disk at the bottom is the largest on that peg. (Discs above the bottom one may be in any order.) There are n disks sorted from largest on bottom to smallest on top at the start. Determine the minimum number of moves (moving one disk at a time) needed to move the disks to another peg sorted in the same order.

4.21. Let set $S=\{1,2,\ldots,n\}$ and set T be the set of all subsets of S (including S and the empty set). One tries to choose three (not necessarily distinct) sets from the set T such that either two of the chosen sets are subsets of the third set or one of the chosen set is a subset of both of the other two sets. In how many ways can this be done?

4.22. Let $\mathbb{N}$ denote the set of positive integers, and let $f:\mathbb{N}\to\mathbb{N}$ be a function such that $f(m)>f(n)$ for all $m>n$. For positive integer m, let $g(m)$ denote the number of positive integers n such that $f(n) < m$. Express

$$\sum_{i=1}^{m} g(i) + \sum_{k=1}^{n} f(k)$$

in closed form.

4.23. Given that it is possible to use 501 discs of radius two to cover a rectangular sheet of paper, determine if 2004 discs of radius one can always cover the same piece of paper.

4.24. [7pts] The usual procedure for shuffling a deck of n cards involves first splitting the deck into two blocks and then merging the two blocks to form a single deck in such a way that the order of the cards within each block is not changed. A trivial cut in which one block is empty is allowed.

(a) How many different permutations of a deck of n cards can be produced by a single shuffle?

(b) How many of these single shuffle permutations can be inverted by another such single shuffle, putting the deck back in order?

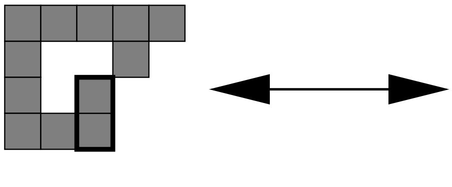
4.25. A robot is placed on an infinite square grid; it is composed of a (finite) connected block of units occupying one square each. A *valid subdivision* of the robot is a partition of its units into two connected pieces which meet along a single unbroken line segment. The robot moves as follows: it may divide into a valid subdivision, then one piece may slide one square sideways so that the result is again a valid subdivision, at which point the pieces rejoin. (See diagram for examples.)

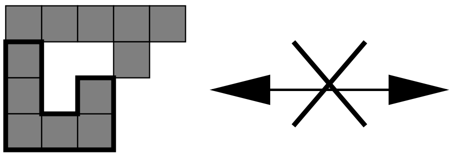
We say a position of the robot (i.e., a connected block of squares in the plane) is *row-convex* if

(a) the robot does not occupy only a single row or only a single column, and

(b) no row meets the robot in two or more separate connected blocks.

Prove that from any row-convex position in the plane, the robot can move to any other row-convex position in the plane.





4.26. Let n be a positive integer, and let S_n be the set of all positive integer divisors of n (including 1 and n). Prove that at most half of the elements of S_n end in the digit 3.

4.27. Let $\mathcal{S}$ denote the set of points inside and on the boundary of a regular hexagon with side length 1 unit. Find the smallest value of r such that the points of $\mathcal{S}$ can be colored in three colors in such a way that any two points with the same color are less than r units apart.

4.28. Let m be a positive integer. In the coordinate plane, we call a polygon $A_1A_2\cdots A_n$, *admissible* if

(i) each side is parallel to one of the coordinate axes;

(ii) its perimeter $A_1A_2 + A_2A_3 + \cdots + A_nA_1$ is equal to m.

(iii) for $1 \le i \le n$, $A_i = (x_i, y_i)$ is a lattice point;

(iv) $x_1 \le x_2 \le \cdots \le x_n$; and

(v) $y_1 = y_n = 0$, and there is k with $1 < k < m$ such that $y_1 \le y_2 \cdots \le y_k$ and $y_k \ge y_{k+1} \ge \cdots \ge y_n$.

Determine the number of admissible polygons as a function in m. (Two admissible polygons are consider distinct if one can be obtained by the other via a composition of reflections and rotations.)

4.29. In the coordinate plane, color the lattice points which have both coordinates even black and all other lattice points white. Let P be a polygon with black points as vertices. Prove that any white point on or inside P lies halfway between two black points, both of which lie on or inside P.

4.30. Let n be a positive integer greater than or equal to 4. Let S be a sphere with center O and radius 1, and let $H_1, H_2, \dots H_n$ be n hemispheres with center O and radius 1. Sphere S is covered by hemispheres $H_1, H_2, \dots, H_n$. Prove that one can find positive integers i_1, i_2, i_3, i_4 such that sphere S is covered by $H_{i_1}, H_{i_2}, H_{i_3}, H_{i_4}$.

4.31. Let n be a positive integer. Consider sequences $a_0, a_1, \dots, a_n$ such that $a_i \in \{1, 2, \dots, n\}$ for each i and $a_n = a_0$.

(a) Call such a sequence *good* if for all $i = 1, 2, \dots, n$, $a_i - a_{i-1} \not\equiv i \pmod{n}$. Suppose that n is odd. Find the number of good sequences.

(b) Call such a sequence *great* if for all $i = 1, 2, \dots, n$, $a_i - a_{i-1} \not\equiv i, 2i \pmod{n}$. Suppose that n is an odd prime. Find the number of great sequences.

4.32. For each positive integer n, let D_n be the set of all positive divisors of $2^n 3^n 5^n$. Find the maximum size of a subset S of D_n in which no element of S is a proper divisor of any other.

4.33. Let $A = (0, 0, 0)$ be the origin in the three dimensional coordinate space. The *weight* of a point is the sum of the absolute values of its coordinates. A point is a *primitive lattice point* if all its coordinates are integers with their greatest common divisor equal to 1. A square $ABCD$ is called a *unbalanced primitive integer square* if it has integer side length and the points B and D are primitive lattice points with different weights.

Show that there are infinitely many unbalanced primitive integer squares $AB_iC_iD_i$ such that the plane containing the squares are not parallel to each other.

4.34. A 2004×2004 array of points is drawn. Find the largest integer n such that it is possible to draw a convex n-sided polygon whose vertices lie on the points of the array.

Geometry

4.35. Let ABC be an acute triangle. Let A_1 be the foot of the perpendicular from A to side BC, and let A_B and A_C be the feet of the perpendiculars from A_1 to sides AB and AC, respectively.

Line ℓ_A passes through A and is perpendicular to line A_BA_C. Lines ℓ_B and ℓ_C are defined analogously. Prove that lines ℓ_A, ℓ_B, and ℓ_C are concurrent.

4.36. Let n be a positive integer. Given n non-overlapping circular discs on a rectangular piece of paper, prove that one can cut the piece of paper into convex polygonal pieces each of which contains exactly one disc.

4.37. Let ABC be an acute-angled scalene triangle, and let H, I, and O be its orthocenter, incenter, and circumcenter, respectively. Circle ω passes through points H, I, and O. Prove that if one of the vertices of triangle ABC lies on circle ω, then there is one more vertex lies on ω.

4.38. A circle is inscribed in trapezoid $ABCD$, with $AD \parallel BC$ and $AD > BC$. Diagonals AC and BD meet at P. Prove that $\angle APD$ is obtuse.

4.39. Let ABC be a triangle, and let M be the midpoint of side BC. The circumcircle of triangle ACM meets side AB again at D (other than A). Points E and F lie on segments CB and CA, respectively, such that $CE = EM$ and $CF = 3FA$. Suppose that $EF \perp BC$. Prove that $\angle ABC = \angle DEF$.

4.40. Let $ABCD$ be a tetrahedron such that triangles ABC, BCD, CDA, and DAB all have the same inradius. Is it necessary that all four triangles be congruent?

4.41. A convex polygon is called *balanced* if for any interior point P, the sum of distance from P to the lines containing the sides of the polygon is a constant. Two convex polygons $A_1A_2\dots A_n$ and $B_1B_2\dots B_n$ have mutually parallel sides. Prove that $A_1A_2\dots A_n$ is balanced if and only if $B_1B_2\dots B_n$ is balanced.

4.42. Let ABC be a triangle with circumcircle ω, and let P be a point inside triangle ABC. Line ℓ is tangent to circle ω at C. Points D, E, F are the feet of perpendiculars from P to lines ℓ, AC, BC, respectively. Prove that

$$PD \cdot AB = PE \cdot BC + PF \cdot CA.$$

4.43. Let ABC be a triangle and let D be a point in its interior. Construct a circle ω_1 passing through B and D and a circle ω_2 passing through C and D such that the point of intersection of

ω_1 and ω_2 other than D lies on line AD. Denote by E and F the points where ω_1 and ω_2 intersect side BC, respectively, and by X and Y the intersections of lines DF, AB and DE, AC, respectively. Prove that $XY \parallel BC$.

4.44. Let $ABCD$ be a convex cyclic quadrilateral, and let bisectors of $\angle A$ and $\angle B$ meet at point E. Points P and Q lie on sides AD and BC, respectively, such that PQ passes through E and $PQ \parallel CD$. Prove that $AP + BQ = PQ$.

4.45. Let ABC be an acute triangle with O and I as its circumcenter and incenter, respectively. The incircle of triangle ABC touches its sides at D, E, and F respectively. Let G be the centroid of triangle DEF. Prove that $OG \geq 7IG$.

4.46. Let ABC be an acute triangle with $AB \neq AC$, and let D be the foot of perpendicular from A to line BC. Point P is on altitude AD. Rays BP and CP meet sides AC and AB at E and F, respectively. If $BFEC$ is cyclic, prove that P is the orthocenter of triangle ABC.

4.47. Let $ABCD$ be a cyclic quadrilateral. Diagonals AC and BD meet at P. Points E, F, G, and H are the feet of perpendiculars from P to sides AB, BC, CD, and DA, respectively. Prove that lines BD, EH, and FG are either concurrent or parallel to each other.

4.48. Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, PC intersect sides BC, CA, AB at D, E, F, respectively. Prove that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$

if and only if P lies on at least one of the medians of triangle ABC. (Here $[XYZ]$ denotes the area of triangle XYZ.)

4.49. In triangle ABC three distinct triangles are inscribed, similar to each other, but not necessarily similar to triangle ABC, with corresponding points on corresponding sides of triangle ABC. Prove that if two of these triangles share a vertex, than the third one does as well.

4.50. Let ABC be an isosceles right triangle with $\angle A = 90^\circ$ and $AB = 1$, D the midpoint of $\overline{BC}$ and E and F two other points on the side BC. Let M be the second point of intersection of the circumcircles of triangles ADE and ABF. Denote by N the second point of intersection of the line AF with the circumcircle of triangle ACE and by P the second point of intersection of the line AD with the circumcircle of triangle AMN. Find with proof the distance from A to P.

4.51. Let ABC be a triangle. Circle ω passes through B and C and meet sides AB and AC again at C' and B', respectively. Let H and H' be the orthocenters of triangles ABC and $AB'C'$, respectively. Prove that lines BB', CC', and HH' are concurrent.

4.52. Let $\mathbb{T}$ be the set of triangles ABC for which there is a point D on BC such that segments AB, BD, AD, DC and AC have integral lengths and $\angle ACD = \frac{1}{2}\angle ABC = \frac{1}{3}\angle ADB$.

(a) Characterize all triples $\{a, b, c\}$ that are sets of side lengths of triangles in $\mathbb{T}$.

(b) Find the triangle of minimum area in $\mathbb{T}$.

5.53. Let ABC be a triangle. Points D, E, F are on sides BC, CA, AB, respectively, such that $DC+CE = EA+AF = FB+BD$. Prove that

$$DE + EF + FD \geq \frac{1}{2}(AB + BC + CA).$$

4.54. Let ABC be a triangle with ω and I with incircle and incenter, respectively. Circle ω touches the sides AB, BC, and CA at points C_1, A_1, and B_1, respectively. Segments AA_1 and BB_1 meet at point G. Circle ω_A is centered at A with radius AB_1. Circles ω_B and ω_C are defined analogously. Circles ω_A, ω_B, and ω_C are externally tangent to circle ω_1. Circles ω_A, ω_B, and ω_C are internally tangent to circle ω_2. Let O_1 and O_2 be the centers of ω_1 and ω_2, respectively. Lines A_1B_1 and AB meet at C_2, and lines A_1C_1 and AC meet at B_2. Prove that points I, G, O_1, and O_2 lie on a line ℓ that is perpendicular to line B_2C_2.

4.55. Convex quadrilateral $ABCD$ is inscribed in circle ω. The bisector of $\angle ADC$ of passes through the incenter of triangle ABC. Let M be an arbitrary point on arc ADC of ω. Denote by P and Q the incenters of triangles ABM and BCM.

(a) Prove that all triangles DPQ are similar, regardless of the position of point M.

(b) Suppose that $\angle BAC = \alpha$ and $\angle BCA = \beta$. Express $\angle PDQ$ and the ratio DP/DQ in terms of α and β.

4.56. Let $ABCD$ be a convex quadrilateral inscribed in circle ω, which has center at O. Lines BA and DC meet at point P. Line PO intersects segments AC and BD at E and F, respectively. Construct point Q in a such way that $QE \perp AC$ and $QF \perp BD$. Prove that triangles ACQ and BDQ have the same area.

4.57. Let $\overline{AH_1}, \overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute scalene triangle ABC. The incircle of triangle ABC is tangent to $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at T_1, T_2, and T_3, respectively. For $k = 1, 2, 3$, let P_i be the point on line H_iH_{i+1} (where $H_4 = H_1$) such that $H_iT_iP_i$ is an acute isosceles triangle with $H_iT_i = H_iP_i$. Prove that the circumcircles of triangles $T_1P_1T_2, T_2P_2T_3, T_3P_3T_1$ pass through a common point.

Number Theory

4.58. Let r be an integer with $1 < r < 2003$. Prove that the arithmetic progression

$$\{2003n + r \mid n = 1, 2, 3, \ldots\}$$

contains infinitely many perfect power integers.

4.59. For positive integers m and n define

$$\phi_m(n) = \begin{cases} \phi(n) & \text{if } n \text{ divides } m; \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi(n)$ counts the number of positive integers between 1 and n which are relatively prime to n. Show that $\sum_{d|n} \phi_m(d) = \gcd(m, n)$.

4.60. Let $\{a_1, a_2, \ldots, a_{\phi(2005)}\}$ be the set of positive integers less than 2005 and relatively prime to 2005. Compute

$$\left| \prod_{k=1}^{\phi(2005)} \cos\left(\frac{a_k \pi}{2005}\right) \right|.$$

4.61. Find all triples of nonnegative integers (x, y, z) for which $4^x + 4^y + 4^z$ is the square of an integer.

4.62. Given eight distinct positive integers not exceeding 2004, prove that there are four of them, say a, b, c, and d, such that

$$4 + d \leq a + b + c \leq 4d.$$

4.63. Find all polynomials $p(x)$ with integer coefficients such that for each positive integer n, the number $2^n - 1$ is divisible by $p(n)$.

4.64. Determine if there exists a polynomial $Q(x)$ of degree at least 2 with nonnegative integer coefficients such that for each prime p, $Q(p)$ is also a prime.

4.65. Let a, b, c be nonzero integers such that both

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \quad \text{and} \quad \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$$

are integers. Prove that $|a| = |b| = |c|$.

4.66. Let $a_1, a_2, \ldots, a_n$ be a finite integer geometrical sequence and let k a positive integer, relatively prime to n. Prove that $a_1^k + a_2^k + \cdots + a_n^k$ is divisible by $a_1 + a_2 + \cdots + a_n$.

4.67. Let m and n be positive integers such that 2^m divides the number $n(n+1)$. Prove that 2^{2m-2} divides the number $1^k + 2^k + \cdots + n^k$, for all positive odd numbers k with $k > 1$.

4.68. Find all ordered triples of primes (p, q, r) such that

$$p \mid q^r + 1, \quad q \mid r^p + 1, \quad r \mid p^q + 1.$$

4.69. Let $\mathbb{N}_0^+$ and $\mathbb{Q}$ be the set of nonnegative integers and rational numbers, respectively. Define the function $f : \mathbb{N}_0^+ \to \mathbb{Q}$ by $f(0) = 0$ and

$$f(3n + k) = -\frac{3f(n)}{2} + k, \quad \text{for } k = 0, 1, 2.$$

Prove that f is one-to-one, and determine its range.

4.70. Let $a_1, a_2, \ldots, a_n$ be integers such that all the subset sums $a_{i_1} + a_{i_2} + \cdots + a_{i_k}$, for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, are nonzero. Prove that it is possible to partition the set of positive integers into finitely many subsets $S_1, S_2, \ldots, S_m$ in a such way that if S_i $(1 \leq i \leq m)$ has at least n elements, then $a_1x_1 + a_2x_2 + \cdots + a_nx_n \neq 0$, where $x_1, x_2, \ldots, x_n$ are arbitrary distinct elements in S_i.

5
Final Round of 2004 Russia Mathematics Olympiad

9.1 Each lattice point in the plane is colored with one of three colors, and each of the three colors is used at least once. Show that there exists a right-angled triangle whose vertices have three different colors.

9.2 Let $ABCD$ be a quadrilateral with an inscribed circle. The external angle bisectors at A and B intersect at K, the external angle bisectors at B and C intersect at L, the external angle bisectors at C and D intersect at M, and the external angle bisectors at D and A intersect at N. Let K_1, L_1, M_1 and N_1 be the orthocenters of triangles ABK, BCL, CDM and DAN respectively. Show that quadrilateral $K_1L_1M_1N_1$ is a parallelogram.

9.3 There are 2004 boxes on a table, each containing a single ball. I know that some of the balls are white and that the number of white balls is even. I am allowed to indicate any two boxes and ask whether at least one of them contains a white ball. What is the smallest number of questions needed to determine two boxes, each of which is guaranteed to contain a white ball?

9.4 Let $n > 3$ and let $x_1, x_2, \ldots, x_n$ be positive real numbers whose product is 1. Prove that

$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \cdots + \frac{1}{1+x_n+x_nx_1} > 1.$$

9.5 Are there pairwise distinct positive integers m, n, p, q satisfying $m+n = p+q$ and $\sqrt{m} + \sqrt[3]{n} = \sqrt{p} + \sqrt[3]{q} > 2004$?

9.6 There are 2004 telephones in a cabinet. Each pair of telephones is connected by a cable, which is colored in one of four colors. Each of the four colors appears on at least one cable. Can one always select some of the telephones so that among their pairwise cable connections exactly three different colors occur?

9.7 The natural numbers from 1 to 100 are arranged on a circle in such a way that each number is either larger than both of its neighbors or smaller than both of its neighbors. A pair of adjacent numbers is called "good" if when it is removed the circle of remaining numbers still has the above property. What is the smallest possible number of good pairs?

9.8 Let ABC be an acute triangle with circumcenter O. Let T be the circumcenter of triangle AOC and let M be the midpoint of segment AC. Points D and E are selected on lines AB and BC respectively such that $\angle BDM = \angle BEM = \angle ABC$. Show that lines BT and DE are perpendicular.

10.1 $= 9.1$.

10.2 $= 9.3$.

10.3 Let $ABCD$ be a quadrilateral with both an inscribed circle and a circumscribed circle. The incircle of quadrilateral $ABCD$ touches the sides AB, BC, CD and DA at points K, L, M and N respectively. The external angle bisectors at A and B intersect at K', the external angle bisectors at B and C intersect at L', the external angle bisectors at C and D intersect at M', and the external angle bisectors at D and A intersect at N'. Prove that the lines KK', LL', MM' and NN' pass through a common point.

10.4 $= 9.4$.

10.5 A sequence of nonnegative rational numbers $a_1, a_2, \ldots$ satisfies $a_m + a_n = a_{mn}$ for all m, n. Show that not all elements of the sequence can be distinct.

10.6 A country has 1001 cities, each pair of which is connected by a one-way street. Exactly 500 roads begin in each city and exactly 500 roads end in each city. Now an independent republic containing 668 of the 1001 cities breaks off from the country. Prove that it is possible to travel between any two cities in the republic without leaving the republic.

10.7 A triangle T is contained inside a polygon M which has a point of symmetry. Let T' be the reflection of T through some point P inside T. Prove that at least one vertex of T' lies in or on the boundary of M.

10.8 Does there exist a natural number $n > 10^{1000}$ such that $10 \nmid n$ and it is possible to exchange two distinct nonzero digits in the decimal representation of n, leaving the set of prime divisors the same?

11.1 $= 10.1 = 9.1$.

11.2 Let I_a and I_b be the centers of the excircles of traingle ABC opposite A and B, respectively. Let P be a point on the circumcircle ω of ABC. Show that the midpoint of the segment connecting the circumcenters of triangles I_aCP and I_bCP is the center of ω.

11.3 The polynomials $P(x)$ and $Q(x)$ satisfy the property that for a certain polynomial $R(x, y)$, the identity $P(x)-P(y) = R(x, y)(Q(x)-Q(y))$ holds. Prove that there exists a polynomial $S(x)$ such that $P(x) = S(Q(x))$.

11.4 The cells of a 9×2004 rectangular array contain the numbers 1 to 2004, each 9 times. Furthermore, any two numbers in the same column differ by at most 3. Find the smallest possible value for the sum of the numbers in the first row.

11.5 Let $M = \{x_1, \ldots, x_{30}\}$ be a set containing 30 distinct positive real numbers, and let A_n denote the sum of the products of elements of M taken n at a time, $1 \le n \le 30$. Prove that if $A_{15} > A_{10}$, then $A_1 > 1$.

11.6 Prove that for $N > 3$, there does not exist a finite set S containing more than $2N$ pairwise non-collinear vectors in the plane satisfying:

(i) for any N vectors in S, there exist $N-1$ more vectors in S such that the sum of the $2N-1$ vectors is the zero vector;

(ii) for any N vectors in S, there exist N more vectors in S such that the sum of the $2N$ vectors is the zero vector.

11.7 A country contains several cities, some pairs of which are connected by airline flight service (in both directions). Each such pair of cities is serviced by one of k airlines, such that for each airline, all flights that the airline offers contain a common endpoint. Show that it is possible to partition the cities into $k+2$ groups such that no two cities from the same group are connected by a flight path.

11.8 A parallelepiped is cut by a plane, giving a hexagon. Suppose there exists a rectangle R such that the hexagon fits in R; that is, the rectangle R can be put in the plane of the hexagon so that the hexagon is completely covered by the rectangle. Show that one of the faces of the parallelepiped also fits in R.

6
Selected Problems from Chinese IMO Team Training in 2003 and 2004

Algebra

5.1. Let x and y be positive integers with $x < y$. Find all possible integer values of

$$P = \frac{x^3 - y}{1 + xy}.$$

5.2. Let ABC be a triangle, and let x be a nonnegative number. Prove that

$$a^x \cos A + b^x \cos B + c^x \cos C \le \frac{1}{2}(a^x + b^x + c^x).$$

5.3. Given integer n with $n \ge 2$, determine the number of ordered n-tuples of integers $(a_1, a_2, \ldots, a_n)$ such that

(i) $a_1 + a_2 + \cdots + a_n \ge n^2$; and

(ii) $a_1^2 + a_2^2 + \cdots + a_n^2 \le n^3 + 1$.

5.4. Let a, b, and c denote the side lengths of a triangle with perimeter no greater than 2π. Prove that there is a triangle with side lengths $\sin a, \sin b$, and $\sin c$.

5.5. Let n be an integer greater than 1. Determine the largest real number λ, in terms of n, such that

$$a_n^2 \ge \lambda(a_1 + a_2 + \cdots + a_{n-1}) + 2a_n.$$

for all positive integers $a_1, a_2, \ldots, a_n$ with $a_1 < a_2 < \cdots < a_n$.

5.6. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$, for $n = 1, 2, \ldots$. Prove that

$$1 - \frac{1}{2003^{2003}} < \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2003}} < 1.$$

5.7. Let $a_1, a_2, \ldots, a_{2n}$ be real numbers such that $\sum_{i=1}^{2n-1}(a_{i+1} - a_i)^2 = 1$. Determine the maximum value of

$$(a_{n+1} + a_{n+2} + \cdots + a_{2n}) - (a_1 + a_2 + \cdots + a_n).$$

5.8. Let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that there is a k, $1 \le k \le n$, such that

$$\left|\sum_{i=1}^{k} a_i - \sum_{i=k+1}^{n} a_i\right| \le \max\{|a_1|, |a_2|, \ldots, |a_n|\}.$$

5.9. Let n be a fixed positive integer. Determine the smallest positive real number λ such that for any $\theta_1, \theta_2, \ldots, \theta_n$ in the interval

$(0, \frac{\pi}{2})$, if the product

$$\tan\theta_1 \tan\theta_2 \cdots \tan\theta_n = 2^{\frac{n}{2}},$$

then the sum

$$\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n \le \lambda.$$

5.10. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $a_1 = 3$, $a_2 = 7$, and $a_n^2 + 5 = a_{n-1}a_{n+1}$ for $n \ge 2$. Prove that if $a_n + (-1)^n$ is a prime, then $n = 3^m$ for some nonnegative integer m.

5.11. Let k be a positive integer. Prove that $\sqrt{k+1} - \sqrt{k}$ is not the real part of the complex number z with $z^n = 1$ for some positive integer n.

5.12. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that the system of equations

$$\begin{array}{ccccccc}
 & - & 2x_1 & + & x_2 & = & -a_1x_1, \\
x_1 & - & 2x_2 & + & x_3 & = & -a_2x_2, \\
x_2 & - & 2x_3 & + & x_4 & = & -a_3x_3, \\
 & \cdots & & & & \cdots & \\
x_{n-2} & - & 2x_{n-1} & + & x_n & = & -a_{n-1}x_{n-1}, \\
x_{n-1} & - & 2x_n & & & = & -a_nx_n,
\end{array}$$

has a solution $(x_1, x_2, \ldots, x_n)$ in real numbers with $(x_1, x_2, \ldots, x_n) \ne (0, 0, \ldots, 0)$. Prove that

$$a_1 + a_2 + \cdots + a_n \ge \frac{4}{n+1}.$$

5.13. Let a, b, c, d be positive real numbers with $ab + cd = 1$. For $i = 1, 2, 3, 4$, points $P_i = (x_i, y_i)$ are on the unit circle. Prove that

$$(ay_1+by_2+cy_3+dy_4)^2+(ax_4+bx_3+cx_2+dx_1)^2 \le 2\left(\frac{a^2+b^2}{ab} + \frac{c^2+d^2}{cd}\right).$$

5.14. Determine all functions $f, g\ \mathbb{R} \to \mathbb{R}$ such that

$$f(x + yg(x)) = g(x) + xf(y)$$

for all real numbers x and y.

5.15. Let n be a positive integer, and let $a_0, a_1, \ldots, a_{n-1}$ be complex numbers with

$$|a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2 \le 1.$$

Let $z_1, z_2, \dots z_n$ be the (complex) roots of the polynomial

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Prove that

$$|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 \le n.$$

Combinatorics

5.16. Let T be a real number satisfying the property: For any non-negative real numbers a, b, c, d, e with their sum equal to 1, it is possible to arrange them around a circle such that the products of any two neighboring numbers are no greater than T. Determine the minimum value of T.

5.17. Let $A = \{1, 2, \dots, 2002\}$ and $M = \{1001, 2003, 3005\}$. A subset B of A is called *M-free* if the sum of any pairs of elements b_1 and b_2 in B, $b_1 + b_2$ is not in M. An ordered pair of subset (A_1, A_2) is called a *M-partition* of A if A_1 and A_2 is a partition of A and both A_1 and A_2 are M-free. Determine the number of M-partitions of A.

5.18. Let $S = (a_1, a_2, \dots, a_n)$ be the longest binary sequence such that for $1 \le i < j \le n-4$, $(a_i, a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4}) \neq (a_j, a_{j+1}, a_{j+2}, a_{j+3}, a_{j+4})$. Prove that

$$(a_1, a_2, a_3, a_4) = (a_{n-3}, a_{n-2}, a_{n-1}, a_n).$$

5.19. For integers r, let $S_r = \sum_{j=1}^{n} b_j z_j^r$ where b_j are complex numbers and z_j are nonzero complex numbers. Prove that

$$|S_0| \le n \max_{0<|r|\le n} |S_r|.$$

5.20. Let n be a positive integer. Three letters (x_1, x_2, x_3), with each x_i being an element in $\{R, G, B\}$, is called a *set* if $x_1 = x_2 = x_3$ or $\{x_1, x_2, x_3\} = \{R, G, B\}$. Claudia is playing the following game: She is given a row of n of letters $a_{1,1}, a_{1,2}, \dots, a_{1,n}$ with each letter $a_{1,i}$ being either A, or B, or C. Claudia then builds a triangular array of letter with the given row as the initial row on the top. She writes down $n-1$ letters $a_{2,1}, a_{2,2}, \dots, a_{2,n-1}$ in the second row such that $(a_{1,i}, a_{1,i+1}, a_{2,i}$ form a set for each $1 \le i \le n-1$. She then writes down $n-2$ letters in the third row by applying the similar rules to letter in the second row, and so on. The games ends when she obtain the single letter $a_{n,1}$ in the n^{th} row. For example, Claudia will obtain the following triangular array with $(a_1, a_2, \ldots, a_6) = (A, C, B, C, B, A))$ as the given initial row:

$$\begin{array}{ccccccccccc}
R & & G & & B & & G & & B & & R \\
& B & & R & & R & & R & & C & \\
& & G & & R & & R & & B & & \\
& & & B & & R & & G & & & \\
& & & & G & & B & & & & \\
& & & & & R & & & & &
\end{array}$$

An initial row is *balanced* if there are equal numbers of A, B, C in the row. An initial row is *good* if $a_{1,1}, a_{1,n}, a_{n,1}$ form a set.

(a) Find all the integers n such that all possible initial sequence are good.

(b) Find all the integers n such that all possible balanced initial sequences are good.

5.21. Let S be a set such that

(a) Each element of S is a positive integer no greater than 100;

(b) For any two distinct elements a and b in S, there exists an element c in S such that a and c, b and c are relatively prime; and

(c) For any two distinct elements a and b in S, there exists a third element d in S such that a and d, b and d are not relatively prime to each other.

Determine the maximum number of elements S can have.

5.22. Ten people are applying for a job. The job selection committee decides to interview the candidates one by one. The order of candidates being interviewed is random. Assume that all the candidates have distinct abilities. For $1 \le k \le 10$. The following policies are set up within the committee:

(i) The first three candidates interviewed will not be fired;

(ii) For $4 \le i \le 9$, if the i^{th} candidate interviewed is more capable than all the previously interviewed candidates, then this candidate is hired and the interview process is terminated;

(iii) The 10^{th} candidate interviewed will be hired.

Let P_k denote the probability that the k^{th} most able person is hired under the selection policies. Show that,

(a) $P_1 > P_2 > \cdots > P_8 = P_9 = P_{10}$; and

(b) that there are more than 70% chance that one of the three most able candidates is hired and there are no more than 10% chance that one of the three least able candidates is hired.

5.23. Let A be a subset of the set $\{1, 2, \ldots, 29\}$ such that for any integer k and any elements a and b in A (a and b are not necessarily distinct), $a+b+30k$ is not the product of two consecutive integers. Find the maximum number of elements A can have.

5.24. Let set $S = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R},\ 1 \le i \le n\}$, and let A be a finite subset of S. For any pair of elements $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in A, define $d(a, b) = (|a_1 - b_1|, |a_2 - b_2|, \ldots, |a_n - b_n|)$ and $D(A) = \{d(a, b) \mid a, b \in A\}$. Prove that the set $D(A)$ contains more elements than the set A does.

5.25. Let n be a positive integer, and let $A_1, A_2, \ldots A_{n+1}$ be nonempty subsets of the set $\{1, 2, \ldots, n\}$. Prove that there exists nonempty and nonintersecting index sets $I_1 = \{i_1, i_2, \ldots, i_k\}$ and $I_2 = \{j_1, j_2, \ldots, j_m\}$ such that

$$A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} = A_{j_1} \cup A_{j_2} \cup \cdots \cup A_{j_m}.$$

5.26. Integers $1, 2, \ldots, 225$ are arranged in a 15×15 array. In each row, the five largest numbers are colored red. In each column, the five largest numbers are colored blue. Prove that there are at least 25 numbers are colored purple (that is, colored in both red and blue).

5.27. Let $p(x)$ be a polynomial with real coefficients such that $p(0) = p(n)$. Prove that there are n distinct pairs of real numbers (x, y) such that $y - x$ is a positive integer and $p(x) = p(y)$.

5.28. Determine if there is an $n \times n$ array with entries $-1, 0, 1$ such that the $2n$ sums of all the entries in each row and column are all different, where (1) $n = 2003$; and (2) $n = 2004$.

5.29. Let P be a 1000-sided regular polygon. Some of its diagonals were drawn to obtain a *triangulation* the polygon P. (The region inside P is cut into triangular regions, and the diagonals drawn only intersect at the vertices of P.) Let n be the number of different lengths of the drawn diagonals. Determine the minimum value of

n.

5.30. Let m and n be positive integers with $m \geq n$, and let $ABCD$ be a rectangle with $A = (0,0)$, $B = (m,0)$, $C = (m,n)$, and $D = (0,n)$. Rectangle $ABCD$ is tiled by mn unit squares. There is a bug in each unit square. At a certain moment, Ben shouts at the bugs: "Move!!!" Each bug can choose independently whether or not to follow Ben's order and bugs do not necessarily move to an adjacent square. After the moves are finished, Ben noticed that if bug a and bug b were neighbors before the move, then either they are still neighbors or they are in the same square after the move. (Two bugs are called *neighbors* if the square they are staying share a common edge.) Prove that, after the move, there are n bugs such that the centers of the squares they are staying are on a line of slope 1.

Geometry

5.31. Quadrilateral $ABCD$ is inscribed in circle with AC a diameter of the circle and $BD \perp AC$. Diagonals AC and BD intersect at E. Extend segment DA through A to F. Extend segment BA through A to G such that $DG \parallel BF$. Extend segment GF through F to H such that $CH \perp GH$. Prove that points B, E, F, H lie on a circle.

5.32. Let ABC be a triangle. Points D, E, and F are on segments AB, AC, and DE, respectively. Prove that

$$\sqrt[3]{[BDF]} + \sqrt[3]{[CEF]} \leq \sqrt[3]{[ABC]}$$

and determine the conditions when the equality holds.

5.33. Circle ω is inscribed in convex quadrilateral $ABCD$, and it touches sides AB, BC, CD, and DA at A_1, B_1, C_1, and D_1, respectively. Let E, F, G, and H be the midpoints of A_1B_1, B_1C_1, C_1D_1, and D_1A_1, respectively. Prove that quadrilateral $EFGH$ is a rectangle if and only if $ABCD$ is cyclic.

5.34. Let $ABCD$ be a cyclic convex quadrilateral with $\angle A = 60^\circ$, $BC = CD = 1$. Rays AB and DC meet E, and rays AD and BC meet at F. Suppose that the the perimeters of triangles BCE and CDF are integers. Compute the perimeter of quadrilateral $ABCD$.

5.35. Let $ABCD$ be a convex quadrilateral. Diagonal AC bisects $\angle BAD$. Let E be a point on side CD. Segments BE and AC intersect at F. Extend segment DF through F to intersect segment BC at G. Prove that $\angle GAC = \angle EAC$.

5.36. Four lines are given in the plane such that each three form a non-degenerate, non-equilateral triangle. Prove that, if it is true that one line is parallel to the Euler line of the triangle formed by the other three lines, then this is true for each of the lines.

5.37. Let ABC be an acute triangle with I and H be its incenter and orthocenter, respectively. Let B_1 and C_1 be the midpoints of $\overline{AC}$ and $\overline{AB}$ respectively. Ray B_1I intersects $\overline{AB}$ at $B_2 \neq B$. Ray C_1I intersects ray AC at C_2 with $C_2A > CA$. Let K be the intersection of $\overline{BC}$ and $\overline{B_2C_2}$. Prove that triangles BKB_2 and CKC_2 have the same area if and only if A, I, A_1 are collinear, where A_1 is the circumcenter of triangle BHC.

5.38. In triangle ABC, $AB \neq AC$. Let D be the midpoint of side BC, and let E be a point on median AD. Let F be the foot of perpendicular from E to side BC, and let P be a point on segment EF. Let M and N be the feet of perpendiculars from P to sides AB and AC, respectively. Prove that M, E, and N are collinear if and only if $\angle BAP = \angle PAC$.

5.39. [7pts] Let ABC be a triangle with $AB = AC$. Let D be the foot of perpendicular from C to side AB, and let M be the midpoint of segment CD. Let E be the foot of perpendicular from A to line BM, and let F be the foot of perpendicular from A to line CE. Prove that

$$AF \leq \frac{AB}{3}.$$

5.40. Let ABC be an acute triangle, and let D be a point on side BC such that $\angle BAD = \angle CAD$. Points E and F are the foot of perpendiculars from D to sides AC and AB, respectively. Let H be the intersection of segments BE and CF. The circumcircle of triangle AFH meets line BE again at G. Prove that segments BG, GE, BF can be the sides of a right triangle.

5.41. Let $A_1A_2A_3A_4$ be a cyclic quadrilateral that also has an inscribed circle. Let B_1, B_2, B_3, B_4, respectively, be the points on sides $A_1A_2, A_2A_3, A_3A_4, A_4A_1$ at which the inscribed circle is tangent to the quadrilateral. Prove that

$$\left(\frac{A_1A_2}{B_1B_2}\right)^2 + \left(\frac{A_2A_3}{B_2B_3}\right)^2 + \left(\frac{A_3A_4}{B_3B_4}\right)^2 + \left(\frac{A_4A_1}{B_4B_1}\right)^2 \geq 8.$$

5.42. Triangle ABC is inscribed in circle ω. Line ℓ passes through A and is tangent to ω. Point D lies on ray BC and point P is in the plane such that D, A, P lie on ℓ in that order. Point U is on segment CD. Line PU meets segments AB and AC at R and S, respectively. Circle ω and line PU intersect at Q and T. Prove that if $QR = ST$, then $PQ = UT$.

5.43. Two circles ω_1 and ω_2 (in the plane) meet at A and B. Points P and Q are on ω_1 and ω_2, respectively, such that line PQ is tangent to both ω_1 and ω_2, and B is closer to line PQ than A. Triangle APQ is inscribed in circle ω_3. Point S is such that lines PS and QS are tangent to ω_3 at P and Q, respectively. Point H is the image of B reflecting across line PQ. Prove that A, H, S are collinear.

5.44. Convex quadrilateral $ABCD$ is inscribed in circle ω. Let P be the intersection of diagonals AC and BD. Lines AB and CD meet at Q. Let H be the orthocenter of triangle ADQ. Let M and N be the midpoints of diagonals AC and BD, respectively. Prove that $MN \perp PH$.

5.45. Let ABC be a triangle with I as its incenter. Circle ω is centered at I and lies inside triangle ABC. Point A_1 lies on ω such that $IA_1 \perp BC$. Points B_1 and C_1 are defined analogously. Prove that lines AA_1, BB_1, and CC_1 are concurrent.

Number Theory

5.46. Given integer a with $a > 1$, an integer m is *good* if $m = 200a^k + 4$ for some integer k. Prove that, for any integer n, there is a degree n polynomial with integer coefficients such that $p(0), p(1), \ldots, p(n)$ are distinct good integers.

5.47. Find all the ordered triples (a, m, n) of positive integers such that $a \geq 2$, $m \geq 2$, and $a^m + 1$ divides $a^n + 203$.

5.48. Determine if there exists a positive integer n such that n has exactly 2000 prime divisors, n is not divisible by a square of a prime number, and $2^n + 1$ is divisible by n.

5.49. [7pts] Determine if there exists a positive integer n such that n has exactly 2000 prime divisors, n is not divisible by a square of a prime number, and $2^n + 1$ is divisible by n.

5.50. A positive integer u if called *boring* if there are only finitely many triples of positive integers (n, a, b) such that $n! = u^a - u^b$. Determine all the boring integers.

5.51. Find all positive integers n, $n > 1$, such that all the divisors of n, not including 1, can be written in the form of $a^r + 1$, where a and r are positive integers with $r > 1$.

5.52. Determine all positive integers m satisfying the following property: There exists prime p_m such that $n^m - m$ is not divisible by p_m for all integers n.

5.53. Let m and n be positive integers. Find all pairs of positive integers (x, y) such that

$$(x + y)^m = x^n + y^n.$$

5.54. Let p be a prime, and let $a_1, a_2, \dots, a_{p+1}$ be distinct positive integers. Prove that there are indices i and j, $1 \le i < j \le p + 1$, such that

$$\frac{\max\{a_i, a_j\}}{\gcd(a_i, a_j)} \ge p + 1.$$

5.55. For positive integer $n = p_1^{a_1} p_2^{a_2} \cdot p_m^{a_m}$, where $p_1, p_2, \dots, p_n$ are distinct primes and $a_1, a_2, \dots, a_m$ are positive integers, $d(n) = \max\{p_1^{a_1}, p_2^{a_2}, \dots, p_m^{a_m}\}$ is called the *greatest prime power divisor* of n. Let $n_1, n_2, \dots,$ and n_{2004} be distinct positive integers with $d(n_1) = d(n_2) = \cdots = d(n_{2004})$. Prove that there exists integers $a_1, a_2, \dots,$ and a_{2004} such that infinite arithmetic progressions $\{a_i, a_i + n_i, a_i + 2n_i, \dots\}$, $i = 1, 2, \dots, 2004$, are pairwise disjoint.

5.56. Let N be a positive integer such that for all integers $n > N$, the set $S_n = \{n, n + 1, \dots, n + 9\}$ contains at least one number that has at least three distinct prime divisors. Determine the minimum value of N.

5.57. Determine all the functions from the set of positive integers to the set of real numbers such that

(a) $f(n + 1) \ge f(n)$ for all positive integers n; and

(b) $f(mn) = f(m)f(n)$ for all relatively prime positive integers m and n.

5.58. Let n be a positive integer. Prove that there is a positive integer k such that digit 7 appears for at least $\lfloor 2n/3 \rfloor$ times among the last n digits of the decimal representation of 2^k.

5.59. An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, 2, \ldots, n$, the i^{th} row and i^{th} column together contain all elements of S. Determine all the values n for which silver matrices exist.

5.60. Let n be a positive integer. Determine the largest integer $f(n)$ such that

$$\binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}}$$

is divisible by $2^{f(n)}$.

7
Selected Problems from Romania Contests in 2004