IMO Winter Camp 2006

*Inequalities – January 6, 2006*

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1. The sum of two positive real numbers is less than their product. Prove that their sum is greater than 4.

2. The real-valued function $f$ satisfies $f(\tan 2x) = \tan^4 x + \cot^4 x$ for all real $x$. Prove that $f(\sin x) + f(\cos x) \geq 196$ for all real $x$.

3. Let $a$, $b$, $c$, $d$, and $e$ be positive real numbers such that $abcde = 1$. Prove that $a^4 + b^4 + c^4 + d^4 + e^4 \geq a + b + c + d + e$.

4. Let $x$ and $y$ be real numbers. Prove that $x^2 + y^2 + 1 \geq xy + x + y$, and find when equality occurs.

5. Let $0 \leq u, v, w \leq 1$, and $u + v + w = 1$. Show that

$$\sqrt{12uvw} + u^2 + v^2 + w^2 \leq 1.$$

6. Let $a_1, a_2, a_3, \ldots$ be a sequence of non-negative real numbers such that $a_k - 2a_{k+1} + a_{k+2} \geq 0$, and $\sum_{j=1}^{k} a_j \leq 1$ for all $k \geq 1$. Prove that

$$0 \leq a_{k+1} - a_k < \frac{2}{k^2}$$

for all $k \geq 1$.

7. Let $f : [0,1] \to [0,1]$ be an increasing function (i.e. if $x < y$, then $f(x) < f(y)$), with $f(0) = 0$ and $f(1) = 1$. Show that

$$f\left(\frac{1}{10}\right)+f\left(\frac{2}{10}\right)+\cdots+f\left(\frac{9}{10}\right)+f^{-1}\left(\frac{1}{10}\right)+f^{-1}\left(\frac{2}{10}\right)+\cdots+f^{-1}\left(\frac{9}{10}\right) \leq \frac{99}{10}.$$

8. Let $x$, $y$, and $z$ be non-negative real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+x)(1+z)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

9. Given $n$ points on the sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$, show that the sum of the squares of the distances between them does not exceed $n^2$.

10. Let $a$, $b$, and $c$ be non-negative real numbers such that $a + b + c \geq abc$. Prove that $a^2 + b^2 + c^2 \geq \sqrt{3}abc$.

11. For positive real numbers $a$, $b$, $c$, $d$, $e$, and $f$, prove that

$$\frac{ab}{a+b} + \frac{cd}{c+d} + \frac{ef}{e+f} \leq \frac{(a+c+e)(b+d+f)}{a+b+c+d+e+f}.$$

12. Prove that

$$27(a+b)^2(a+c)^2(b+c)^2 \geq 64(a+b+c)^3 abc$$

for all positive real numbers $a$, $b$, and $c$.

13. Let $a_1, a_2, \ldots, a_n > 0$. Prove that

$$\frac{a_1^2}{a_2} + \frac{a_2^2}{a_3} + \cdots + \frac{a_n^2}{a_1} \geq a_1 + a_2 + \cdots + a_n.$$

14. Let $a$, $b$, $c$ be real numbers satisfying $a < b < c$, $a + b + c = 6$, and $ab + bc + ca = 9$. Prove that $0 < a < 1 < b < 3 < c < 4$.

15. Let $x$, $y$, $z$ be positive real numbers such that $1/x + 1/y + 1/z = 1$ and $x + y + z = 10$. Find the minimum and maximum values of $A = x^2 + y^2 + z^2$.

16. Let $a, b, c > 0$. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

17. Let $ABC$ be an acute triangle with circumradius $R$. Let $m_a$, $m_b$, and $m_c$ denote the lengths of the medians. Show that $m_a + m_b + m_c > 4R$.