Art of Problem Solving
WOOT 2010-11

Double-Good Catalan

Richard Rusczyk

Back when I first started looking at Olympiad problems as a student, I remember that most of the solutions fell into one of two categories. The first category consisted of completely incomprehensible solutions. These used notation or invoked theorems that I had never seen before. While they at least gave me something to try to research, they didn't help much. (This was back before the internet—researching notation or theorems wasn't simply a matter of going to the AoPS site and asking on the forum.) The second category was even more maddening—each solution was clear and easy to understand, but there was absolutely no indication as to how I might have come up with the solution on my own unless I had seen essentially the same problem before.

Through WOOT, we hope to help you on both fronts. Many of the classes will be designed to expand your mathematical toolboxes. However, these tools are less important than our second purpose, which is to teach how to attack problems that are not just like problems you already know how to do. This latter skill is the heart of problem solving. It's much harder to teach and much harder to learn, but it's far more important than the individual tools we use to solve specific problems.

Hopefully, you will start to see more than just the final solution when you look at the solution to an Olympiad problem. This final solution is often much like the tip of an iceberg. It's the visible part that sits on top of a mountain of invisible effort that led to it. The problem solving is not in the tip—it's in all the mass below.

Most classes and texts focus primarily on the tip of the iceberg, with an occasional glimpse below the waterline. However, in this article, I'll instead dissect a single problem I solved for inclusion in our *Intermediate Counting & Probability* text. The problem is very difficult (I think), and it took me a long time to solve it. I'll start this article with the background material I knew prior to starting on the problem. Then, I'll explain how I attacked the problem, including all the blind alleys, false leads, harebrained ideas, and, finally, my successful approach.

1 Background: The Catalan Numbers

This section is excerpted from Art of Problem Solving's textbook Intermediate Counting & Probability by David Patrick. This book is an excellent resource for mastering advanced counting techniques, including those that are very valuable for the AIME, for national Olympiads, and for various areas of higher mathematics.

These first two sections are structured just like the Intermediate Counting & Probability text. They start with a series of problems. You should try to solve these problems before continuing. Then, we present the solutions to the problems, along with important observations about the results.

Double-Good Catalan

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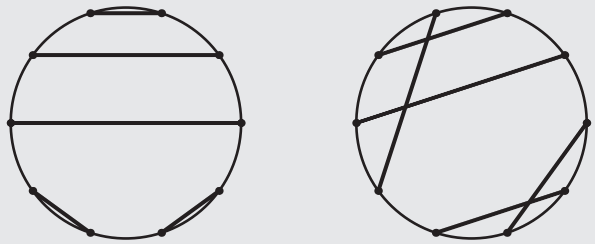
Problem 1.1: In how many ways can 10 people sitting around a circular table simultaneously shake hands (so that there are 5 handshakes going on), such that no two people cross arms? For example, the handshake arrangement on the left side below is valid, but the arrangement on the right side is invalid.



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(a) Compute by hand the number of handshake arrangements for 2, 4, or 6 people sitting around a table.

(b) It's a bit hard to do it by hand for 8 people (you can try if you like), so we'll look for a more clever approach. Pick one person (out of the 8); how many people can he shake hands with?

(c) For each possible handshake for the first person in (b), in how many ways can the rest of the table shake hands?

(d) Use your answers from (b) and (c) to count the number of 8-person handshake arrangements.

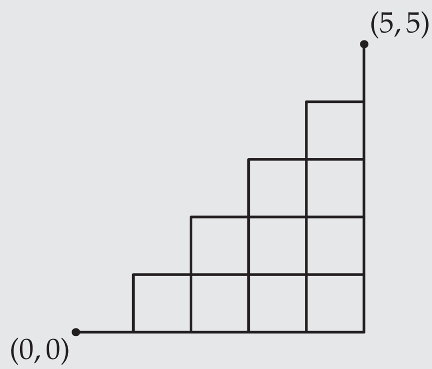
(e) Can you extend your reasoning from (b)-(d) above to solve the 10-person problem?

Problem 1.2: How many ways are there to arrange 5 open parentheses "(" and 5 closed parentheses ")" such that the parentheses "balance," meaning that, as we read left-to-right, there are never more ")"s than "("s? For example, the arrangement ((()())()) is valid, but the arrangement (()()))(() is invalid.

(a) Compute by hand the number of arrangements for 1, 2, and 3 pairs of parentheses. Do your answers look familiar?

(b) Try to find a 1-1 correspondence between arrangements of n pairs of parentheses and handshake arrangements of $2n$ people (from Problem 1.1).

Problem 1.3: How many 10-step paths are there from $(0,0)$ to $(5,5)$ on the grid below?



(a) It may be tempting to answer $\frac{1}{2}\binom{10}{5} = 126$. Explain why this is incorrect.

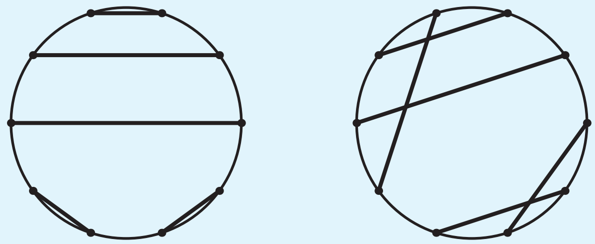
(b) Compute by hand the number of paths on the half-grid to $(1,1)$, $(2,2)$, and $(3,3)$. Notice anything familiar?

(c) Try to find a 1-1 correspondence between solutions to this problem and solutions to one of the two previous problems.

We'll explore several problems that look very different on the surface, but that actually all have the same underlying structure. As we work through these problems, try to keep them all in the back of your mind, with an eye towards the features in the various problems that are similar.

Sidenote: **Eugène Catalan 1814–1894**
The Catalan numbers (which we will be exploring in this section) are named after the 19th-century mathematician Eugène Catalan. He is also known for his conjecture (made in 1844) that 8 and 9 are the only consecutive positive integers that are perfect powers ($8 = 2^3$ and $9 = 3^2$). This conjecture remained unproven until 2002, when it was proved by Preda Mihăilescu.

Problem 1.1: In how many ways can 10 people sitting around a circular table simultaneously shake hands (so that there are 5 handshakes going on), such that no two people cross arms? For example, the handshake arrangement on the left side below is valid, but the arrangement on the right side is invalid.



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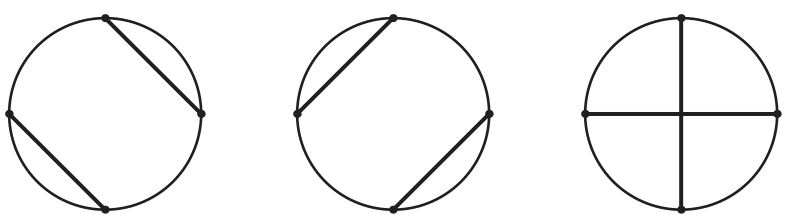
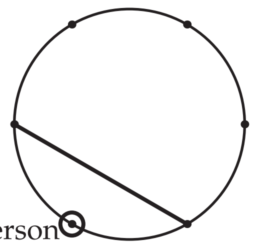
Solution for Problem 1.1: As we often do, we can experiment on smaller versions of the same problem, in order to get some idea for what's going on in general.

If there are 2 people, then there is obviously only one way for them to shake hands.

If there are 3 people, then there's no way that they can all shake hands, because there will always be an odd person left out. In general, we must always have an even number of people.

If there are 4 people, then there are 3 ways for them to shake hands (pick one of the people, and choose one of the other 3 people to shake hands with him; the other two people are then forced to shake with each other). But one of these ways is illegal: the pairs of people sitting across from each other cannot shake hands, since their arms would cross. So there are only 2 legal handshake configurations. In the figure below, we see the two legal handshake configurations on the left, and the 3$^{\text{rd}}$ (illegal) configuration on the right.

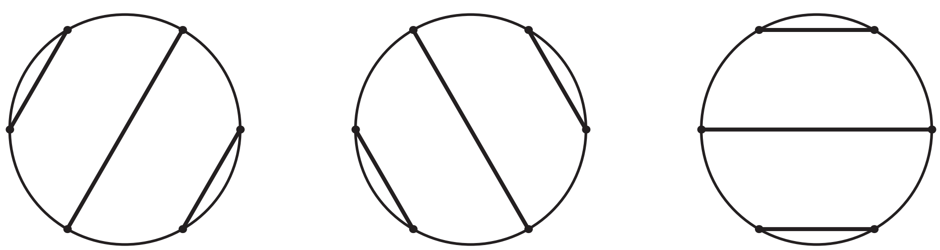




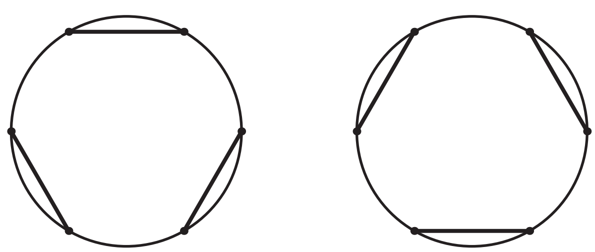
If there are 6 people, then things get a bit more complicated. The first thing to note is that no one can shake hands with the person sitting 2 positions away from them on the left or on the right, because if they did, they'd "cut off" a person who would not be able to shake hands with anyone, as in the figure on the left.

This leaves us with two cases.

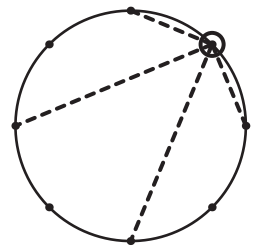
Case 1: Some pair of people who are directly across from each other shake hands. There can only be one such pair, since two or more such pairs would cross each other at the center of the table. There are 3 choices for a pair of opposite people, and once we have chosen such a pair, the rest of the handshakes are fixed (the two people on each side of the central handshake must shake with each other). These three cases are shown in the figure below:



Case 2: Everybody shakes hands with one of his/her neighbors. There are two possibilities, depending on whether a specific person shakes hands to the left or to the right, as shown in the figure below:

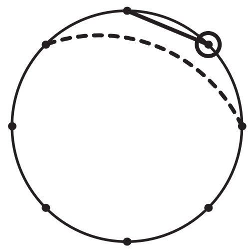


So there are a total of $3 + 2 = 5$ ways for 6 people to legally shake hands.



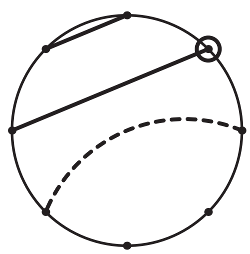
When we get up to 8 people, it's starting to get too complicated to list all the configurations. So let's look at it from a particular person's point-of-view. As in the 6-person case above, we cannot leave an odd number of people on either side of this person's handshake. So our initial person cannot shake hands with anybody that is an even number of people away. In the figure to the right, we show a circled initial person, and his allowed handshakes are shown by dashed lines. Note that each of these handshakes ends at a person who is an odd number of people away from our initial (circled) person.

If the initial person shakes hands with a neighbor, we can think of the remaining 6 people as being on a smaller circle, as in the figure below:



These 6 remaining people have 5 ways to shake, just as in the 6-person problem. Since the initial person has 2 neighbors with whom to shake, this means that there are 2(5) = 10 handshake arrangements that start with our initial person shaking hands with a neighbor.

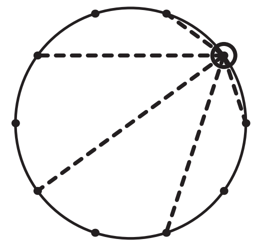
Otherwise, our initial person has to shake with a person who is 3 positions to his left or right. Once this is done, the two people who are "cut off" from the rest must shake with each other, and the other 4 people form a 4-person mini-table that can shake in 2 ways:



This gives another 2(2) = 4 handshake arrangements, since there are 2 choices of the person that is 3 away from the original person, and then 2 choices to finish the handshaking at the 4-person mini-table.

Therefore, there are 10 + 4 = 14 ways for 8 people to shake hands.

Finally, we can use this same strategy for 10 people. Choose an initial person. This person has 5 choices for whom to shake hands with, as shown in the picture to the right. If he shakes with one of his neighbors (2 choices), then the remaining 8 people form a mini-table that can shake in 14 ways. If he shakes with a person 3 positions away (2 choices), then 2 people are cut off (and must shake), and the other 6 people form a mini-table that can shake in 5 ways. If he shakes with the person directly opposite (1 choice), then each side of the table has a group of 4 people, each of which can shake in 2 ways.



Therefore, the number of handshake arrangements for 10 people is

$$2(14) + 2(5) + 1(2)(2) = 28 + 10 + 4 = 42.$$

□

Before we go on, let's list the numbers that we found while working through the previous problem:

Number of people	2	4	6	8	10
Number of handshake configurations	1	2	5	14	42

Keep these numbers in mind as we continue through this section.

Problem 1.2: How many ways are there to arrange 5 open parentheses "(" and 5 closed parentheses ")" such that the parentheses "balance," meaning that, as we read left-to-right, there are never more ")"s than "("s? For example, the arrangement ((()())()) is valid, but the arrangement (()()))(() is invalid.

Solution for Problem 1.2: As we often do, let's experiment with small values.

If we have 1 set of parentheses, then we only have one possibility: ().

If we have 2 sets of parentheses, we can either nest them as (()), or we can list both pairs one after the other as ()(). So there are 2 possibilities.

If we have 3 sets of parentheses, then a little experimentation will show that there are 5 possibilities:

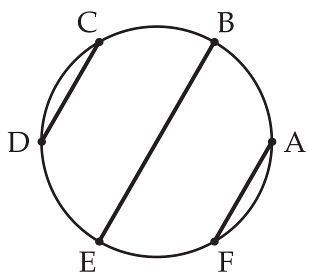
()()(), (())(), ()(()), (()()), ((())).

Hmmm. . . , 1, 2, 5, ... Do you recognize these numbers? They are the same numbers that we got for the number of non-crossing handshakes of people sitting at a round table in Problem 1.1. Perhaps there is a connection between the two problems.

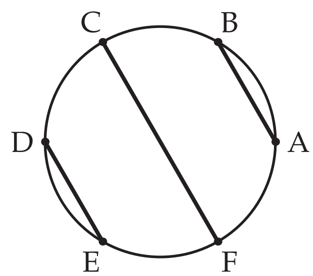
Concept: When you see the same answer for two different problems, look for a connection, or better yet, for a 1-1 correspondence between them.

Since the parentheses come in pairs, it's natural to think that in any 1-1 correspondence between parenthesis-arrangements and valid handshakes around a table, each set of parentheses will represent two people shaking hands. The fact that the parentheses must be properly nested will somehow correspond to the condition that handshakes cannot cross.

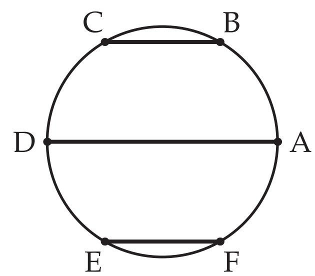
For instance, we can list all of the arrangements of 3 pairs of parentheses, and their corresponding handshake arrangements. We'll label both the parentheses and the people with the letters A through F, and note how each pair of parentheses corresponds to a pair of people that are shaking hands.



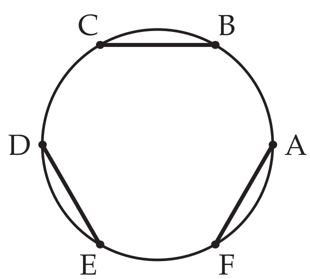
((( )))
A B C D E F



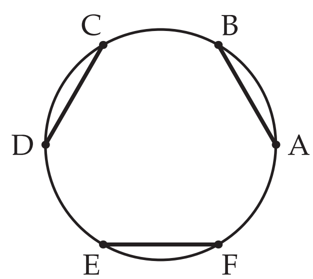
( ) (( ))
A B C D E F



(( )) ( )
A B C D E F



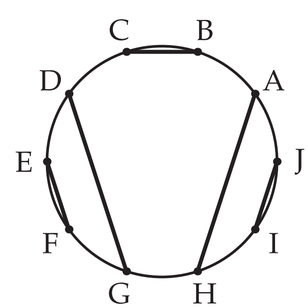
(( ) ( ))
A B C D E F



( ) ( ) ( )
A B C D E F

Let's see this further in an example with 5 sets of parentheses and 10 people around a table. We'll label the people around the table A through J, and the parentheses will also be labeled with A through J as we read from left to right. Each matching pair of parentheses corresponds to a handshake. A sample correspondence is shown below.

(( ) (( ))) ( )
A B C D E F G H I J $\leftrightarrow$



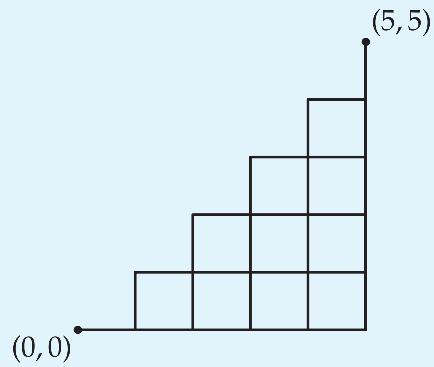
This leads to a 1-1 correspondence

$$\left\{\begin{array}{l}\text{parenthesis arrangements of } n\\ \text{pairs of parentheses}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{handshake arrangements of } 2n\\ \text{people around a table}\end{array}\right\}.$$

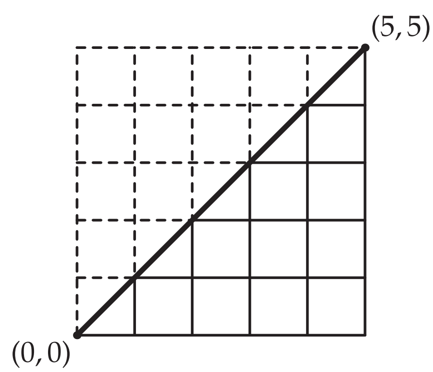
Thus the answer to our problem is the same as the number of handshake arrangements of 10 people, which is 42. □

That's two problems so far involving the sequence 1,2,5,12,42,. . . . You should therefore not be surprised by what you will find in the next problem.

Problem 1.3: How many 10-step paths are there from $(0, 0)$ to $(5, 5)$ on the grid below?



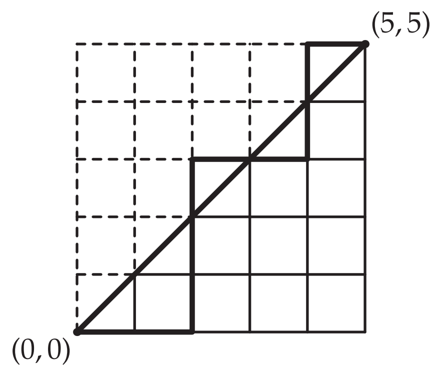
Solution for Problem 1.3: The first thing that we notice is that the grid shown is exactly the part of the full 5×5 grid that is below the main diagonal, as shown below:



This might suggest the following quick "solution":

Bogus Solution: We know that there are $\binom{10}{5} = 252$ paths on the full grid. Since we only have the lower-half of the grid to work with, that means that we have $\frac{252}{2} = 126$ paths on the lower-half of the grid.

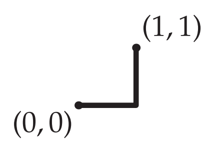
This is of course absurd, as there are many paths that pass through both halves of the grid, like the one shown below:



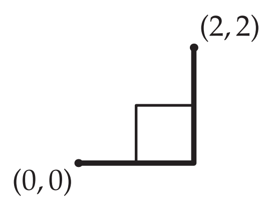
So how can we count the paths that only go below the main diagonal?

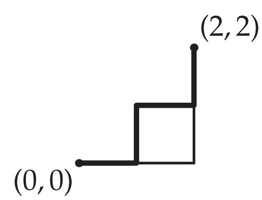
Once again, let's count the paths in some smaller cases.

If the half-grid is 1×1, then there's only one path:

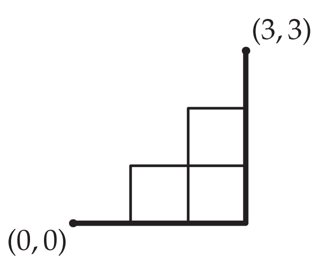


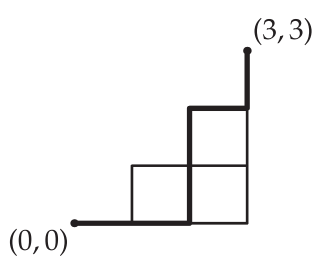
If the half-grid is 2×2, then there are 2 paths:

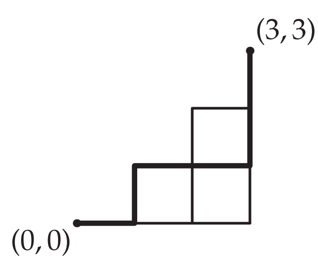


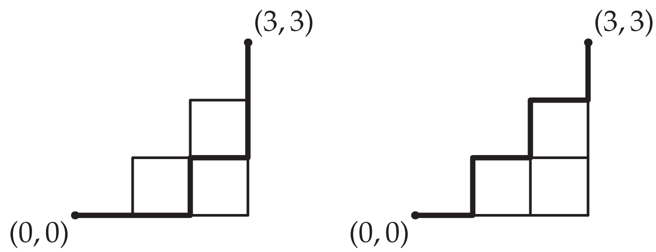


If the half-grid is 3×3, then there are 5 paths:









There are those numbers again: 1,2,5,.... So we'll once again look for a 1-1 correspondence between this problem and one of the previous problems. Since each path from $(0,0)$ to (n,n) consists of n moves up and n moves to the right, we think to try to find a correspondence between these paths and lists of n "("s and n ")"s.

Indeed, we can make a 1-1 correspondence

$$\{\text{balanced expressions with } n \text{ pairs of parentheses}\} \quad \leftrightarrow \quad \{\text{paths on an } n\times n \text{ grid below the diagonal}\},$$

by letting each "(" represent a move to the right and each ")" represent a move up. As long as there are more "("s than ")"s, there will be more rights than ups, and the path will never cross above the main diagonal of the $n \times n$ grid.

Therefore, there are 42 paths on the 5×5 half-grid, since there are 42 possible nested expressions with 5 pairs of parentheses. □

2 Formulas for the Catalan Numbers

This section is excerpted from Art of Problem Solving's new textbook Intermediate Counting & Probability by David Patrick.

Problem 2.1: Can you write a recurrence relation for the Catalan numbers?

Problem 2.2: Compare the n^{th} Catalan number with the binomial coefficient $\binom{2n}{n}$. Do you notice any pattern?

Problem 2.3: Find a 1-1 correspondence between:

$$\left\{\begin{matrix}\text{paths from } (0,0) \text{ to } (n,n) \text{ that go above the main} \\ \text{diagonal}\end{matrix}\right\} \quad \leftrightarrow \quad \{\text{paths from } (0,0) \text{ to } (n-1,n+1)\}.$$

Problem 2.4: Find a formula for the n^{th} Catalan number.

As we've seen in the problems in the previous section, the n^{th} Catalan number can be defined as:

- the number of ways that $2n$ people sitting around a table can shake hands, so that no two handshakes cross arms;
- the number of ways to write n "("s and n ")"s such that the parentheses are balanced;
- the number of $2n$-step paths on a rectangular grid from $(0,0)$ to (n,n) that do not cross above the main diagonal.

It would be nice if we could easily compute the Catalan numbers. For now, let's focus on the recursive definition.

Problem 2.1: What is the recurrence relation for the Catalan numbers?

Solution for Problem 2.1: We've actually already seen it in the problems in the previous section. For each of the problems in the previous section, we can break down the problem of size n into cases, where each case is composed of two smaller problems whose sizes add to $n-1$.

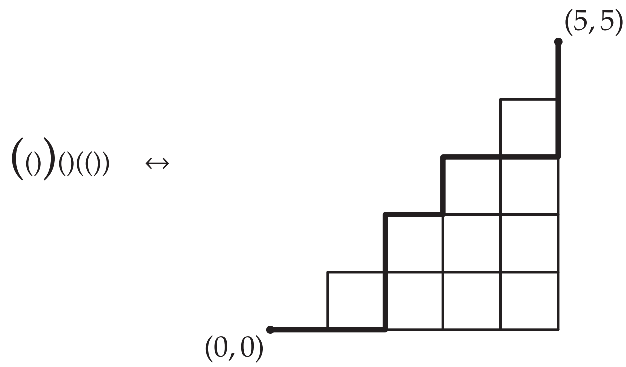
For instance, in Problem 1.1, we start with a table with $2n$ people. Once we place the initial handshake, we are left with two smaller tables with $2(n-1)$ people combined.

In Problem 1.2, we can look at the first parenthesis on the left and its corresponding closing parenthesis. This splits the rest of the parentheses into two groups: those that are inside this first pair, and those that are to the right of this pair. For example, in the following 10-pair nesting, the first set of parentheses (in bold) splits the rest of the parentheses into a 6-pair group (inside the bold parentheses) and a 3-pair group (to the right of the bold parentheses):

$$\Big(\,(()())(())()\,\Big)(())()$$

The first set of parentheses will always split the remaining $n-1$ pairs into two groups of balanced parentheses, although one of the groups may be empty.

We can use the 1-1 correspondence between Problem 1.2 and Problem 1.3 to see how to set up the recursion for the paths on the half-grid from $(0,0)$ to (n,n). The idea is that the end of the *first* complete set of parentheses corresponds to the place where the path *first* touches the diagonal after leaving $(0,0)$. For example, we show a 5-parentheses nesting and its corresponding path in the figure below. The first set of parentheses is shown in bold, and it corresponds to the path's first touching of the main diagonal at $(2,2)$.



The path is now broken into 2 paths on 2 smaller half-grids.

In all of these problems, the solution is the n^{th} Catalan number C_n, and we arrive at the solution by breaking up the problem into a sum of two smaller problems. Specifically, we see that C_n is the sum of all possible products of the form C_kC_l where $k + l = n - 1$. That is,

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0 = \sum_{k=0}^{n-1} C_kC_{n-1-k}.$$

The sequence starts at $C_0 = 1$. □

We can once again verify this recursion for the numbers that we've already computed:

$$\begin{aligned}
C_0 &= 1,\\
C_1 &= C_0C_0 = 1,\\
C_2 &= C_0C_1 + C_1C_0 = 1 + 1 = 2,\\
C_3 &= C_0C_2 + C_1C_1 + C_2C_0 = 2 + 1 + 1 = 5,\\
C_4 &= C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 = 5 + 2 + 2 + 5 = 14,\\
C_5 &= C_0C_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_0 = 14 + 5 + 4 + 5 + 14 = 42.
\end{aligned}$$

Let's continue and compute the next couple of Catalan numbers:

$$\begin{aligned}
C_6 &= C_0C_5 + C_1C_4 + C_2C_3 + C_3C_2 + C_4C_1 + C_5C_0 = 42 + 14 + 10 + 10 + 14 + 42 = 132,\\
C_7 &= C_0C_6 + C_1C_5 + C_2C_4 + C_3C_3 + C_4C_2 + C_5C_1 + C_6C_0 = 132 + 42 + 28 + 25 + 28 + 42 + 132 = 429.
\end{aligned}$$

So now we have a recursive formula for the Catalan numbers. However, it is somewhat unsatisfying. Not only it is recursive, but each Catalan number depends on *all* of the preceding Catalan numbers, not just the one or two immediately prior. It would be much nicer to have a closed-form formula into which we could plug some value of n and have C_n just pop out. But where can we begin to find such a formula?

Problem 1.3 looks most promising, as it's most related to a problem that we feel like we understand well and know how to find a formula for, namely paths on a grid from $(0, 0)$ to (n, n). We know that, without any restrictions, there are $\binom{2n}{n}$ such paths. So that's a good place to start.

Problem 2.2: Compare the n^{th} Catalan number with the binomial coefficient $\binom{2n}{n}$. Do you notice any pattern?

Solution for Problem 2.2: Let's list the first 7 Catalan numbers and the first 7 values of $\binom{2n}{n}$ and see if we notice anything.

n	1	2	3	4	5	6	7
C_n	1	2	5	14	42	132	429
$\binom{2n}{n}$	2	6	20	70	252	924	3432

It's not too clear how to find a pattern between these two rows of numbers, but the one column that might jump out at you is the $n = 4$ column with the numbers 14 and 70, since $70 = 5(14)$. This might cause you to notice that $\binom{2n}{n}$ always appears to be a multiple of C_n. Let's expand our chart:

n	1	2	3	4	5	6	7
C_n	1	2	5	14	42	132	429
$\binom{2n}{n}$	2	6	20	70	252	924	3432
$\frac{\binom{2n}{n}}{C_n}$	2	3	4	5	6	7	8

Now we have strong experimental evidence that $C_n = \frac{1}{n+1}\binom{2n}{n}$. □

Of course, observing a pattern is not a proof. Let's further examine the "paths on a grid" problem and see what else we might determine. We know that $\binom{2n}{n}$ counts the number of paths from $(0, 0)$ to (n, n) on a rectangular grid. We also know that C_n counts the number of these paths that don't go above the diagonal. So $\binom{2n}{n} - C_n$ counts the number of paths that *do* go above the diagonal. Since we suspect that $C_n = \frac{1}{n+1}\binom{2n}{n}$, we suspect that the number of paths that go above the diagonal should be:

$$\binom{2n}{n} - \frac{1}{n+1}\binom{2n}{n} = \frac{n}{n+1}\binom{2n}{n}.$$

We do a bit of algebraic manipulation with this quantity:

$$\frac{n}{n+1}\binom{2n}{n} = \frac{n(2n)!}{(n+1)n!n!} = \frac{(2n)!}{(n+1)!(n-1)!} = \binom{2n}{n-1}.$$

This last quantity gives us an idea for a 1-1 correspondence:

Problem 2.3: Find a 1-1 correspondence between:

$$\left\{\begin{array}{l}\text{paths from } (0,0) \text{ to } (n,n) \text{ that go above the main} \\ \text{diagonal}\end{array}\right\} \leftrightarrow \{\text{paths from } (0,0) \text{ to } (n-1,n+1)\}.$$

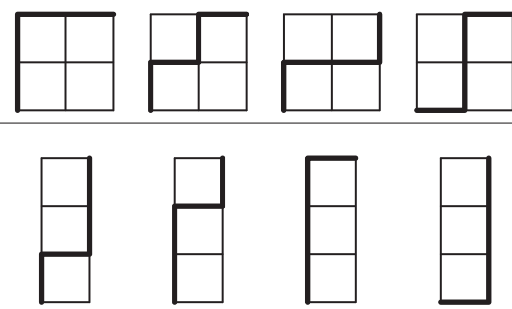
Solution for Problem 2.3: This can be a bit tricky to see, so let's play with the $n = 2$ case.

There are $\binom{4}{2} = 6$ paths from $(0,0)$ to $(2,2)$, and we know that $C_2 = 2$ of them stay on or below the main diagonal, so the other 4 go above the diagonal. We also know that there are $\binom{4}{1} = 4$ paths from $(0,0)$ to $(1,3)$. (Good—there are the same number of paths in each category, which is a necessity for there to be a 1-1 correspondence.)

Let's draw the 4 paths in each category, and see if we can match them up. (I'm going to help you out and list them in the order that we will match them—see if you can find the correspondence.)

Paths from $(0,0)$ to $(2,2)$ that go above the main diagonal

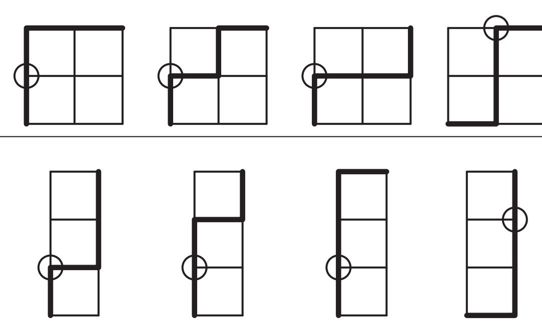
Paths from $(0,0)$ to $(1,3)$



In each column, let's start at $(0,0)$, and let's mark (with a circle) the point on each path where the two paths differ. In other words, the path from $(0,0)$ to the circled point is the same in both paths, but after the circled point, one path goes up whereas the other goes right.

Paths from $(0,0)$ to $(2,2)$ that go above the main diagonal

Paths from $(0,0)$ to $(1,3)$



We see that in each column, the path from $(0,0)$ to the circled point in both pictures is the same. However, what's more interesting is what happens after the circled point. Compare the paths after the circled point in both pictures of a column. They're mirror images of each other!

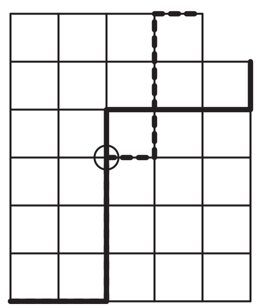
To be more precise, let's list the paths using "r" for a step to the right and "u" for a step up. We'll place in bold all of the steps after the circled point.

Paths from $(0,0)$ to $(2,2)$ that go above the main diagonal	u**urr**	ur**ur**	urr**u**	r**uur**
Paths from $(0,0)$ to $(1,3)$	u**ruu**	uu**ru**	uuu**r**	r**uuu**

Note that the unbolded parts of the paths—the parts between $(0,0)$ and the circled point—are identical, and the bolded parts of the paths—the parts between the circled point and the end—are exactly reversed.

This suggests a general strategy for finding a 1-1 correspondence. Given a path from $(0,0)$ to (n,n) that goes above the diagonal, circle the *first* point at which the path crosses above the diagonal. Then, reverse all steps past the circled point: change ups to rights and rights to ups.

Here's an example where $n = 5$. The original path is shown as solid, and the new path (after the transformation described above) is shown as dashed.



Note that the solid path, before the circled point, has one more up step than right step. After the circled point, the solid path has one more right step than up step (since the circled point lies one "up" step above the diagonal). After the reversal transformation, the dashed path has, after the circled point, one more up step than right step. Hence, starting at $(0,0)$, the combined new path has 2 more up steps than right steps. Since it still has $2n$ steps in total, it must have $n+1$ up steps and $n-1$ right steps, and thus the path ends at $(n-1, n+1)$.

This process is clearly reversible, and hence we have a 1-1 correspondence:

$$\left\{\begin{matrix}\text{paths from } (0,0) \text{ to } (n,n) \text{ that go above the main} \\ \text{diagonal}\end{matrix}\right\} \quad \leftrightarrow \quad \{\text{paths from } (0,0) \text{ to } (n-1,n+1)\}.$$

□

Problem 2.4: Find a formula for the n^{th} Catalan number.

Solution for Problem 2.4: We know that the n^{th} Catalan number is the number of paths from $(0,0)$ to (n,n) that don't go above the diagonal. However, we know from Problem 2.3 that the paths that do go above the diagonal are in 1-1 correspondence with paths to $(n-1,n+1)$. Since there are $\binom{2n}{n}$ paths from $(0,0)$ to (n,n) and $\binom{2n}{n-1}$ paths from $(0,0)$ to $(n-1,n+1)$, we have that

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}.$$

This does not exactly look like what we conjectured in Problem 2.2, so let's try to simplify it a bit. We start by writing out the expressions for the binomial coefficients:

$$C_n = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!}.$$

We can factor out like terms and simplify:

$$C_n = \frac{(2n)!}{(n-1)!n!}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{(2n)!}{(n-1)!n!} \cdot \frac{1}{n(n+1)} = \frac{(2n)!}{(n+1)!n!}.$$

Removing an $(n+1)$ term from the denominator gives us our result:

$$C_n = \frac{1}{n+1}\left(\frac{(2n)!}{n!n!}\right) = \frac{1}{n+1}\binom{2n}{n}.$$

□

Important: The formula for the n^{th} Catalan number is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

3 The Problem

A *double-good* nesting of order n is an arrangement of $2n$ ")"s and n "("s such that as we read left-to-right, the number of ")"s that have appeared at any point is no more than 2 times the number of "("s that have appeared to that point. For example, the complete list of the double-good nestings of order 2 is

())())
()()))
(())))

Prove that the number of double-good nestings of order n is $\dfrac{1}{2n+1}\dbinom{3n}{n}$.

This is one of the hardest problems in the *Intermediate Counting & Probability* text, so don't feel bad if you don't make much progress. It took me several hours to find the solution. In the second half of this WOOT article, which we'll release after you've had a week to work on the problem, I'll describe my exploration of the problem, together with the solution I found.