IMO Winter Camp 2006

Mock Olympiad 1 – January 5, 2006

1. Find all pairs of positive integers such that the last digit of their sum is 3, their difference is a prime, and their product is the square of a positive integer.

2. In an acute-angled triangle ABC, circles k_1 with diameter AC and k_2 with diameter BC are drawn. Let E be the foot of B on AC and F be the foot of A on BC. Furthermore, let L and N be the points in which the line BE intersects k_1 (with L on the segment BE), and K and M be the points in which the line AF intersects k_2 (with K on the segment AF).

   Prove that $KLMN$ is a cyclic quadrilateral.

3. Let x, y, and z be positive real numbers such that $xyz = 1$. Prove that
$$\frac{1}{1+x+y} + \frac{1}{1+x+z} + \frac{1}{1+y+z} \leq 1.$$

4. Determine whether the following assertion is true: For any ordering of the numbers 1, 2, 3, ..., 15, these numbers can be coloured using at most four colours, in such a way that all numbers of any given colour form, in the given ordering, a monotone (i.e. increasing or decreasing) sequence. (A sequence consisting of a single member is monotone.)

5. Find all functions f, taking the positive integers to the positive integers, such that
$$f(m - n + f(n)) = f(m) + f(n)$$
for all positive integers m and n.