Sequences

In this chapter, we are concerned with infinite sequences of either integers or, more generally, real numbers. Although it is no longer one of the main phyla of questions in the IMO (combinatorics, algebra, geometry and number theory), sequences do feature very prominently.

Generating functions

To manipulate sequences, it is useful to be able to represent them algebraically as a power series known as a *generating function*.

- A sequence $\{A_0, A_1, A_2, A_3, \ldots\}$ has the *ordinary generating function* $a_o(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots$. **[Definition of OGF]**

We can add and multiply ordinary generating functions, which correspond to addition and convolution of their respective sequences.

1. Suppose we have two sequences, $\{A_n\}$ and $\{B_n\}$, with ordinary generating functions $a_o(x)$ and $b_o(x)$, respectively. Let $\{C_n\}$ and $\{D_n\}$ have ordinary generating functions $a_o(x) + b_o(x)$ and $a_o(x)\, b_o(x)$, respectively. Show that $C_n = A_n + B_n$ and $D_n = A_0 B_n + A_1 B_{n-1} + \ldots + A_n B_0$. **[Addition and convolution]**

2. Find a closed form for $1 + x + x^2 + \ldots$, the ordinary generating function of $\{1, 1, 1, \ldots\}$. Hence find an ordinary generating function for the sequence of natural numbers, $\{1, 2, 3, \ldots\}$, and the triangular numbers, $\{1, 3, 6, 10, \ldots\}$.

3. Suppose $\{A_0, A_1, A_2, A_3, \ldots\}$ has ordinary generating function $a_o(x)$. What sequence has ordinary generating function $\frac{d}{dx} a_o(x)$?

4. Hence find the sequence with ordinary generating function $\ln(1 - x)$.

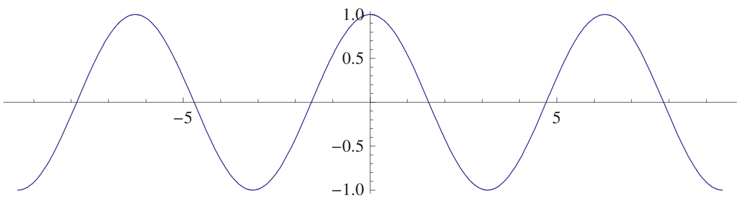
When differentiating $A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots$, we obtain the sequence $A_1 + 2 A_2 x + 3 A_3 x^2 + 4 A_4 x^3 + \ldots$. As an operation on sequences, this is a left shift followed by a (somewhat annoying) multiplication of each term by a different scalar. We can remove this inelegance by defining a more complicated *exponential generating function*, or *EGF*.

- A sequence $\{B_0, B_1, B_2, B_3, \ldots\}$ has the *exponential generating function* $b_e(x) = \frac{B_0}{0!} + \frac{B_1}{1!} x + \frac{B_2}{2!} x^2 + \frac{B_3}{3!} x^3 + \ldots$. **[Definition of EGF]**

If we differentiate it, we obtain the exponential generating function of the sequence $\{B_1, B_2, B_3, \ldots\}$, which is simply the original sequence shifted to the left. The sequence $\{1, 1, 1, \ldots\}$ is invariant when shifted to the left, so its exponential generating function (namely $f(x) = e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \ldots$) is invariant under differentiation, and is thus a solution to the differential equation $f(x) = f'(x)$.

5. Show that the exponential generating function of $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$ is a solution of the differential equation $f''(x) + f(x) = 0$.

This type of differential equation is encountered in the mechanics of mass-spring systems. This particular solution is the function $f(x) = \cos x$; the general solution is $f(x) = A \cos x + B \sin x$, corresponding to the sequence $\{A, B, -A, -B, A, B, -A, -B, \ldots\}$. For this reason, a stretched spring with a suspended mass will oscillate periodically in simple harmonic motion.



Some basic sequences and their exponential generating functions are given below.

Sequence	EGF
$\{1, 1, 1, 1, \ldots\}$	e^x
$\{1, -1, 1, -1, 1, \ldots\}$	e^{-x}
$\{1, 0, 1, 0, 1, \ldots\}$	$\cosh(x)$
$\{0, 1, 0, 1, 0, \ldots\}$	$\sinh(x)$
$\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$	$\cos(x)$
$\{0, 1, 0, -1, 0, 1, 0, -1, \ldots\}$	$\sin(x)$
$\{1, 2, 4, 8, 16, \ldots\}$	e^{2x}
$\{1, 2, 3, 4, 5, \ldots\}$	$e^x x$

6. Find a sequence $\{F_0, F_1, F_2, \ldots\}$ whose exponential generating function satisfies the differential equation $f''(x) = f'(x) + f(x)$.

In general, the solution to any homogeneous linear differential equation is the exponential generating function of a sequence defined by a *linear recurrence relation*.

Linear recurrence relations

Suppose we have a sequence defined by the *linear recurrence relation* $A_{n+k} = \alpha_0 A_n + \alpha_1 A_{n+1} + \alpha_2 A_{n+2} + \cdots + \alpha_{k-1} A_{n+k-1}$. This is linear and homogeneous, which means that for any two sequences $\{A_n\}$ and $\{B_n\}$ satisfying the equation, so does the sequence $\{\lambda A_n + \mu B_n\}$. It is also determined entirely by the values of $\{A_0, A_1, \ldots, A_{k-1}\}$, so there are k degrees of freedom in the solution set.

If a sequence $\{B_n = x^n\}$ satisfies the recurrence relation, then we have $x^k = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{k-1} x^{k-1}$. By the fundamental theorem of algebra, we can rearrange and factorise this into k linear terms. Assuming that this polynomial has l distinct roots, $\beta_1, \beta_2, \ldots, \beta_l$, we obtain the general solution $\{A_n = P_1(n)\,\beta_1{}^n + P_2(n)\,\beta_2{}^n + \ldots + P_l(n)\,\beta_l{}^n\}$. P_i is a polynomial of degree $m_i - 1$, where m_i is the multiplicity of the root β_i. Specifically, when all the roots are distinct, all values of $P_i(n)$ are constants. It is possible to verify that each term satisfies the linear recurrence relation, so the general solution is valid. Also, it has k degrees of freedom, so there can be no other solutions.

Probably the simplest non-trivial linear recurrence relation is the *Fibonacci sequence* $\{F_n\}$ with $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. It was discovered by Leonardo of Pisa when contemplating a problem about the exponential growth of a rabbit population. It has the closed-form expression $F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$, where ϕ and ψ are the positive and negative roots, respectively, of the equation $x^2 = x + 1$.

- $F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. **[Binet's formula]**

This enables one to compute a closed form for the exponential generating function of $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$, and thus find a closed-form solution to the differential equation $f''(x) = f'(x) + f(x)$.

7. Find a closed form for the exponential generating function of $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$.

8. Show that, for all $a, b \in \mathbb{Z}^+$, we have $F_a \mid F_{ab}$.

9. Prove that $F_n{}^2 = F_{n+1} F_{n-1} - (-1)^n$. **[Cassini's identity]**

10. Show that the Fibonacci sequence has the ordinary generating function $f_o(x) = \frac{x}{1-x-x^2}$, and thus find a rational approximation to 0.01010203050813213455.

Every sequence generated by a linear recurrence relation has an ordinary generating function expressible as a ratio between two polynomials.

Consider the sequence $\{A_i \pmod p\}$. As the sequence is generated by the previous k terms, each of which can be one of n values, the sequence must eventually cycle with period $P \le n^k$. The recurrence relation is deterministic in both directions, so it must be completely periodic. Indeed, this bound can be slightly improved upon, as k consecutive zeros would result in a sequence that is identically zero. Hence, $P \le n^k - 1$. Sequences where equality holds are known as *maximal*. For example, the non-zero sequence with recurrence relation $X_{n+3} \equiv X_{n+1} + X_n \pmod 2$ has a period of $2^3 - 1$ and is displayed below (where red and blue discs indicate 0 and 1, respectively).

…  …

In a maximal sequence, every non-empty string of k digits appears precisely once in each cycle. Together with the property that the sequence has a long period, linear recurrence sequences are used as simple pseudo-random number generators. A refined algorithm, known as the *Mersenne twister*, has a period of $2^{19937} - 1$ and is almost indistinguishable from random data.

11. Let n be an integer greater than 1. In a circular arrangement of n lamps $L_0, L_1, \ldots, L_{n-1}$, each one of which can be either ON or OFF, we begin with the configuration where all lamps are ON. We carry out a sequence of steps, $\text{Step}_0, \text{Step}_1, \ldots$. If L_{j-1} is ON, then Step_j changes the status of L_j (from ON to OFF or *vice-versa*), but does not change the status of any other lamps; if L_{j-1} is OFF, then Step_j does not change anything at all. Show that:

- There is a positive integer $M(n)$ such that all lamps are ON again after $M(n)$ steps.
- If n has the form 2^k, then all lamps are on after $n^2 - 1$ steps.
- If n has the form $2^k + 1$, then all lamps are on after $n^2 - n + 1$ steps. **[IMO 1993, Question 6]**

Positional systems

When we write numbers in decimal notation, we let '$d_k\, d_{k-1} \ldots d_2\, d_1\, d_0$' represent the non-negative integer $d_k\, 10^k + d_{k-1}\, 10^{k-1} + \ldots + 10\, d_1 + d_0$, where each digit $d_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Note that this is unique; an integer cannot have two valid representations. This can be generalised to other bases (or *radices*) such as binary and ternary. For example, we can uniquely express any non-negative integer in binary as a sum of distinct powers of two.

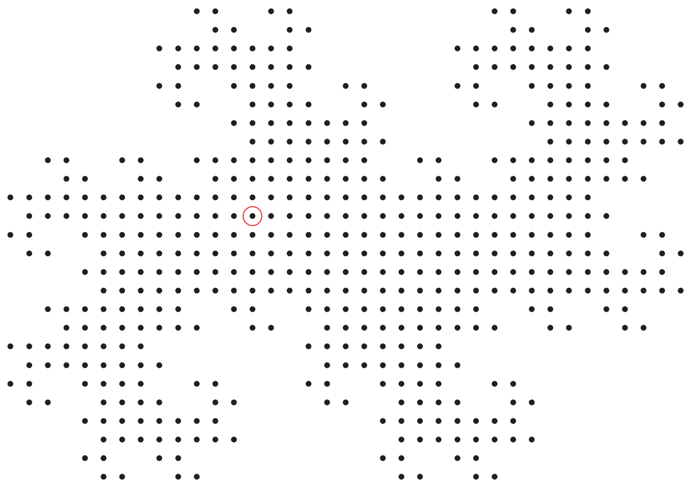
- The n-ary representation '$d_k\, d_{k-1} \ldots d_2\, d_1\, d_0$', where each digit $d_i \in \{0, 1, \ldots, n-1\}$, uniquely represents the integer $\sum_{i=0}^{k} n^i\, d_i$.

A certain type of functional equation, known as a *binary functional equation* or *Monk equation*, is best approached by expressing the numbers in binary.

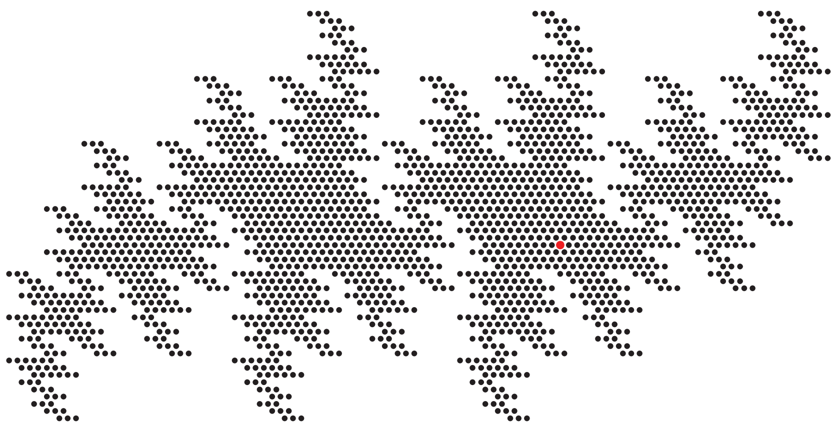
12. A function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is defined using the initial term $f(0) = 0$ and the recurrence relations $f(2n) = \frac{1}{2} f(n)$ and $f(2n+1) = 1 + f(2n)$. How many integers x exist such that $x = 2^{20} f(x)$?

There are some more interesting variants of positional systems. *Balanced ternary* has $n = 3$ and $d_i \in \{-1, 0, 1\}$, as opposed to the $\{0, 1, 2\}$ of ordinary base-3. There exist unique expressions in balanced ternary for every integer, as opposed to merely non-negative integers. By contrast with the modern binary computers we use today, there was an early computer (Setun), which operated in balanced ternary.

The system with $n = i - 1$ and $d_i \in \{0, 1\}$ is even better, as it can represent any Gaussian integer. The Gaussian integers with representations using at most $k + 1$ digits are the 2^{k+1} points on a space-filling fractal known as the *twindragon curve*. The example for $k = 8$ is shown below, with the origin encircled red.



13. Prove that every Eisenstein integer has a unique representation in the positional system with base $n = \omega - 1$ and digits $\{0, 1, 2\}$.



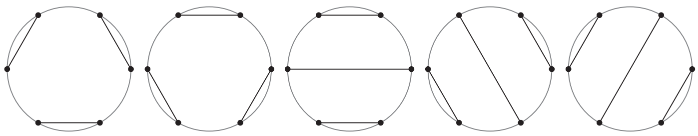
A particularly interesting positional notation is the *Zeckendorf representation*. There is a unique way to express any non-negative integer as the sum of Fibonacci numbers, no two of which are consecutive. For example, $100 = 89 + 8 + 3$.

14. Determine whether there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = f(n) + n$ for all $n \in \mathbb{N}$.

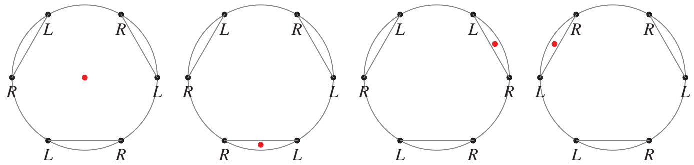
15. A set A of integers is called *sum-full* if $A \subseteq A + A$, *i.e.* each element $a \in A$ is the sum of some pair of (not necessarily distinct) elements $b, c \in A$. A set A of integers is said to be *zero-sum-free* if 0 is the only integer that cannot be expressed as the sum of the elements of a finite non-empty subset of A. Does there exist a set of integers which is both sum-full and zero-sum-free? **[EGMO 2012, Question 4, Dan Schwarz]**

Catalan sequence

We define the nth *Catalan number*, C_n, to be the number of ways of pairing $2\,n$ points on the circumference of a circle with n non-intersecting chords. For example, we have $C_3 = 5$:



Each of these pairings divides the interior of the circle into $n + 1$ regions. We can place an 'observer' in any region. Now label each point with a L or R depending on whether it is connected to a point to the left or right of itself when viewed by the observer. For example, the first pairing can lead to any of the following labellings, depending on where the observer is positioned:



As every labelling of $2\,n$ points with n Ls and n Rs can result in a valid pairing and position of observer, and every pairing and position of observer results in a unique labelling, we have a bijection between the two. As there are C_n pairings and $n + 1$ regions in which to place the observer, there are $(n + 1)\,C_n$ different labellings. However, there are clearly $\binom{2\,n}{n}$ different labellings, so we have $C_n = \frac{1}{n+1}\binom{2\,n}{n}$.

16. Show that the number of valid balanced strings S of $2\,n$ parentheses is given by C_n. (There must be n left parentheses, n right parentheses, and for all $k \le 2\,n$, the number of right parentheses in the first k symbols of S cannot exceed the number of left parentheses.)

17. An insect walks on the integer lattice $\mathbb{Z}^2$, beginning at $(0, 0)$. After $2\,n$ steps, it reaches (n, n).
- How many different paths could the insect have taken?
- Assuming that, for all points $(x,\ y)$ on the path, $x \ge y$, how many different paths are possible?

18. Prove that $C_{n+1} = C_0\,C_n + C_1\,C_{n-1} + C_2\,C_{n-2} + \ldots + C_n\,C_0$. **[Segner's recurrence relation]**

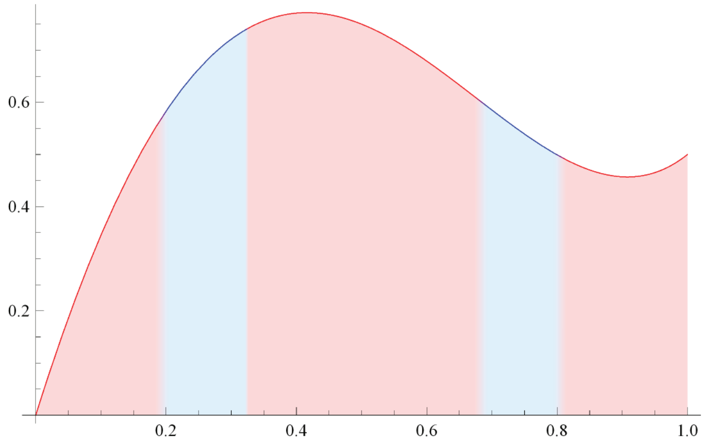
19. Let $c_o(x) = C_0 + C_1\,x + C_2\,x^2 + \ldots$ be the ordinary generating function for the Catalan numbers. Show that $x\,c_o(x)^2 - c_o(x) + 1 = 0$, and thus that $c_o(x) = \frac{1-\sqrt{1-4\,x}}{2\,x}$.

L-systems

Suppose we have a string S consisting entirely of the symbols A and B. We define $f(S)$ by simultaneously replacing every A with $A\,B$ and every B with $B\,A$. For example, starting with a single A, we have $f(A) = A\,B$, $f(f(A)) = A\,B\,B\,A$, $f^3(A) = A\,B\,B\,A\,B\,A\,A\,B$, and so on. The limit of this is an infinite sequence $A\,B\,B\,A\,B\,A\,A\,B\,B\,A\,A\,B\,A\,B\,B\,A\ldots$, known as the *Thue-Morse sequence*. This process of repeatedly applying *substitution rules* to every symbol in a string is known as an *L-system*.

20. A sequence $\{x_i\}$ is defined by $x_1 = 1$ and the recurrences $x_{2\,k} = -x_k$ and $x_{2\,k-1} = (-1)^{k+1}\,x_k$ (for all $k \in \mathbb{Z}^+$). Prove that, for all $n \ge 1$, we have $x_1 + x_2 + \ldots + x_n \ge 0$. **[IMO 2010 shortlist, Question A4]**

21. Prove that we can divide the interval [0, 1] into finitely many intervals, alternately coloured red and blue, such that $\int_{\text{red}} P(x)\, dx = \int_{\text{blue}} P(x)\, dx$ for all polynomials P of degree 2013.



One of the more interesting L-systems is the *golden string*, generated by iterating the system $S \to g(S)$ with substitution rules $X \to X\,Y$ and $Y \to X$ to the initial string X. The first few iterations are $X \to X\,Y \to X\,Y\,X \to X\,Y\,X\,X\,Y \to X\,Y\,X\,X\,Y\,X\,Y\,X \to \ldots$.

22. Show that the number of symbols in $g^n(X)$ is given by F_{n+2}, i.e. the $(n+2)$th term of the Fibonacci sequence.

23. Prove that $g^n(X)$ can be expressed as the concatenation of two palindromic substrings for all $n \in \mathbb{Z}^+$. Moreover, find the lengths of the palindromic substrings for all $n \geq 3$. (A string is described as *palindromic* if it reads the same in both directions. For example, $L\,E\,V\,E\,L$ and $R\,A\,C\,E\,C\,A\,R$ are palindromic strings. The empty string is also considered to be palindromic.)

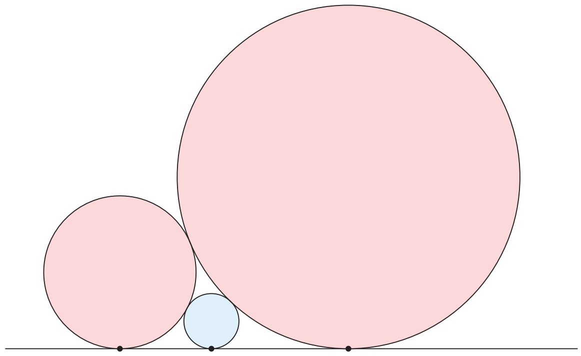
24. Hence prove that, for all $n \in \mathbb{Z}^+$, removing the last two symbols of $g^n(X)$ results in a palindromic string.

The golden string may be regarded as a one-dimensional analogue of the Penrose tiling. Indeed, if you know where to look, you will be able to find the golden string recurring throughout any Penrose tiling. Additionally, the last digit of the Zeckendorf representation of n (for all non-negative integers n) forms the golden string.

Farey sequences

Suppose we have two tangent circles resting on the real line. Circle Γ_1 is positioned at $\frac{p_1}{q_1}$ and has a diameter of $\frac{1}{{q_1}^2}$. Similarly, circle Γ_2 is positioned at $\frac{p_2}{q_2}$ and has a diameter of $\frac{1}{{q_2}^2}$.

25. Prove that $p_1\,q_2 + 1 = p_2\,q_1$.



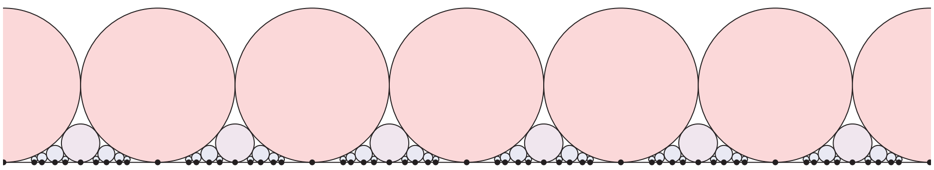
We now position a smaller circle, Γ_3, externally tangent to the two larger circles and the real line.

26. Show that Γ_3 is tangent to the line at $\frac{p_3}{q_3} = \frac{p_1+p_2}{q_1+q_2}$.

This gives a geometrical relationship between the radii of the circles.

- Suppose circles Γ_1 and Γ_2 are tangent to each other, and one of the outer common tangents is l. Let Γ_3 be a third circle tangent internally to Γ_1, Γ_2 and l. Then $\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$, where r_i is the radius of Γ_i. **[Sangaku problem]**

If we begin with a circle of unit diameter for each positive integer, and iterate this process infinitely, we create a pattern known as the *Ford circles*. This has an elegant symmetry associated with modular forms and certain tilings of the hyperbolic plane.



Taking only the circles where $q \le n$ and confining ourselves to the interval [0, 1], we generate a *Farey sequence*, F_n. For example, $F_5 = \left\{0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1\right\}$. Each term is the *mediant* of the two neighbouring terms. For instance, $\frac{1}{3}$ is situated between $\frac{1}{4}$ and $\frac{2}{5}$, and $\frac{1}{3} = \frac{1+2}{4+5}$. This enables a Farey sequence to be extrapolated in both directions from two adjacent terms. It turns out that Rademacher's proof of the formula for the partition numbers involves Ford circles and Farey sequences.

Returning to the Sangaku problem, there is a generalisation known as *Descartes' theorem*, named after the philosopher who said *'cogito ergo sum'* and invented Cartesian coordinates. If we have four circles, which are pairwise externally tangent, then there is a quadratic relationship between the reciprocals of the radii.

- $\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right)^2 = 2\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}\right)$. **[Descartes' theorem]**

$\frac{1}{r_4}$ is thus the solution to a quadratic equation. If we choose the other root (and multiply by -1 to make it positive), we obtain the reciprocal of the radius of the *circumscribed* circle, rather than the inscribed circle.

Egyptian fractions

The Fibonacci sequence, Catalan sequence and powers of two grow reasonably quickly, namely *exponentially*. By comparison, *Sylvester's sequence* grows even more quickly (*doubly-exponentially*), with the first few terms being $\{2, 3, 7, 43, 1807, 3\,263\,443, \ldots\}$. This is defined with the initial term $s_0 = 2$ together with the recurrence relation $s_n = 1 + s_0\, s_1\, s_2 \ldots s_{n-1}$.

27. Prove that the terms in Sylvester's sequence are pairwise coprime.

This is a direct proof that there are infinitely many primes, as no two terms in Sylvester's sequence share a prime factor. Euclid's proof of the infinitude of primes is similar, but with a proof by contradiction instead.

28. Show that $\frac{1}{s_0} + \frac{1}{s_1} + \frac{1}{s_2} + \ldots = 1$.

As decimal expansions, continued fractions and ratios had not been invented, the ancient Egyptians expressed fractions as the sum of reciprocals of distinct positive integers. It is a remarkable fact that *Egyptian fractions* can represent any positive rational number. One algorithm which proves the possibility of this is the following:

- Initially express $\frac{a}{b}$ as $\frac{1}{b} + \frac{1}{b} + \frac{1}{b} + \ldots + \frac{1}{b}$.
- If there are multiple copies of $\frac{1}{n}$, replace one of them with $\frac{1}{n+1} + \frac{1}{n(n+1)}$.
- Repeat the previous step until all unit fractions are distinct.

As arbitrarily large fractions can be generated in this manner, we have a proof that the series $\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges to infinity. It does so rather slowly, with the first n terms tending to $\ln(n) + \gamma$, where γ is the *Euler-Mascheroni constant*. Also, we have yet another proof of the infinitude of primes, because we can factorise $\zeta(1)$ as $\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \ldots\right)\left(1 + \frac{1}{5} + \frac{1}{25} + \ldots\right)\ldots$. Each term is the sum of a geometric series, resulting in the product expansion $\zeta(1) = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \ldots \cdot \frac{p}{p-1} \cdot \ldots$. As each of the terms is finite, but the product is infinite, there must be infinitely many primes.

Another algorithm for generating Egyptian fraction expansions is the *greedy algorithm*, where we choose the largest unused unit fraction less than or equal to the remainder. For example, if we wanted to express $\frac{4}{5}$ as an Egyptian fraction, we would first subtract $\frac{1}{2}$, resulting in $\frac{3}{10}$, followed by $\frac{1}{4}$, resulting in $\frac{1}{20}$, and finally $\frac{1}{20}$, resulting in the expansion $\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}$.

If, instead, we restrict ourselves to choosing the largest *odd* unit fraction at each point, the process may continue forever. For example, applying this algorithm to $\frac{1}{2}$ generates the remainder of Sylvester's sequence. This is similarly the case for all fractions with even denominators. It is an open problem as to whether the process necessarily terminates for all fractions with odd denominators.

Fermat numbers

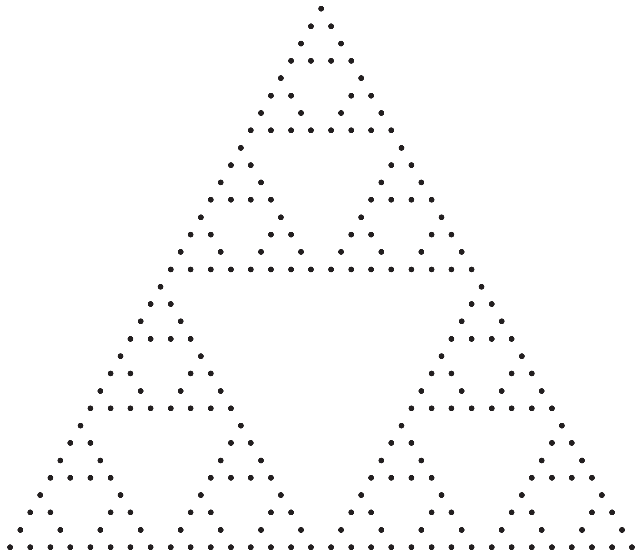
Another doubly-exponential sequence is the sequence of *Fermat numbers* of the form $2^{2^n} + 1$, namely $\{3, 5, 17, 257, 65\,537, \ldots\}$. You may notice that the first five Fermat numbers are prime, known as *Fermat primes*. Fermat conjectured that all Fermat numbers are prime, with the first counter-example found by Euler.

- $2^{2^5} + 1 = 4\,294\,967\,297 = 641 \cdot 6\,700\,417$. **[Euler's factorisation]**

It is unknown whether there are infinitely many Fermat primes. Only the first five are known to be prime; the next 28 have been proved to be composite!

29. Prove that if $2^k + 1$ is prime, then it is a Fermat prime.

A regular n-gon of unit side length is *constructible* if and only if it can be constructed using a compass and straightedge. Equivalently, this means that the Cartesian coordinates of each of the vertices can be expressed as a finite combination of integers together with the operations $\{+, -, \times, \div, \sqrt{\ }\}$. Gauss proved that a regular n-gon is constructible if and only if $n = 2^a p_1 p_2 \ldots p_k$, where each p_i is a distinct Fermat prime. The only known odd values for n are thus products of distinct Fermat primes. Expressed in binary, these form the first 32 rows of Pascal's triangle modulo 2, namely {1, 11, 101, 1111, 10 001, …}. When each 1 is replaced with a dot and 0 is replaced with an empty space, this is the fifth-order approximation to the Sierpinski triangle.



Certain geometrical constructions, such as trisecting the angle, are possible with paper folding but not with Euclidean constructions. Using origami, we can apply the operations $\left\{+, -, \times, \div, \sqrt{}, \sqrt[3]{}\right\}$ to Cartesian coordinates, and thus reach points unattainable with compass and straightedge alone. In this new system, a regular n-gon is constructible if and only if $n = 2^a 3^b q_1 q_2 \ldots q_k$, where each q_i is a distinct *Pierpont prime* (prime expressible in the form $2^n 3^m + 1$). Fermat primes are, by definition, a subset of Pierpont primes.

Solutions

1. We have $a_o(x) + b_o(x) = A_0 + B_0 + A_1\,x + B_1\,x + \cdots$. Similarly, $a_o(x)\,b_o(x) = \left(A_0 + A_1\,x + A_2\,x^2 + \ldots\right)\left(B_0 + B_1\,x + B_2\,x^2 + \ldots\right)$. Expanding the brackets results in $A_0\,B_0 + (A_0\,B_1 + A_1\,B_0)\,x + (A_0\,B_2 + A_1\,B_1 + A_2\,B_0)\,x^2 + \ldots$.

2. The geometric series $1 + x + x^2 + \ldots$ is given by $\frac{1}{1-x}$. As $\{1, 2, 3, \ldots\}$ is the convolution of $\{1, 1, 1, \ldots\}$ with itself, its ordinary generating function is $\frac{1}{(1-x)^2}$. The convolution of this with $\{1, 1, 1, \ldots\}$ gives the triangular numbers with ordinary generating function $\frac{1}{(1-x)^3}$.

3. Differentiating each term, we obtain $\frac{d}{dx}\,a_o(x) = A_1 + 2\,A_2\,x + 3\,A_3\,x^2 + 4\,A_4\,x^3 + \ldots$. This is the ordinary generating function of $\{A_1, 2\,A_2, 3\,A_3, \ldots\}$.

4. We already know that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$. This can be integrated to yield $c + x + \frac{1}{2}\,x^2 + \frac{1}{3}\,x^3 + \ldots$. As $\ln(1-x)$ evaluates to zero when x is zero, the constant term is zero. Hence, $\ln(1-x)$ is the ordinary generating function of $\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$.

5. Shift the sequence two places to the left and add it to the original sequence. This results in the zero sequence, the EGF of which is the zero function.

6. This has the recurrence relation $F_{n+2} = F_{n+1} + F_n$. A possible solution is the Fibonacci sequence, $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$.

7. $f_e(x) = \frac{1}{\sqrt{5}}\left(1 + \phi\,x + \frac{(\phi\,x)^2}{2!} + \frac{(\phi\,x)^3}{3!} + \ldots\right) - \frac{1}{\sqrt{5}}\left(1 + \psi\,x + \frac{(\psi\,x)^2}{2!} + \frac{(\psi\,x)^3}{3!} + \ldots\right)$. This is equal to $\frac{1}{\sqrt{5}}\left(e^{\phi\,x} - e^{\psi\,x}\right)$.

8. The ratio $\frac{F_{a\,b}}{F_a} = \frac{\phi^{a\,b}-\psi^{a\,b}}{\phi^a-\psi^a} = \phi^{a(b-1)} + \phi^{a(b-2)}\,\psi^a + \ldots + \phi^a\,\psi^{a(b-2)} + \psi^{a(b-1)}$ is a symmetric polynomial in ϕ and ψ, so is expressible as a polynomial with integer coefficients in $\phi\,\psi$ and $\phi + \psi$. As the elementary symmetric polynomials are themselves integers, so is the ratio $\frac{F_{a\,b}}{F_a}$.

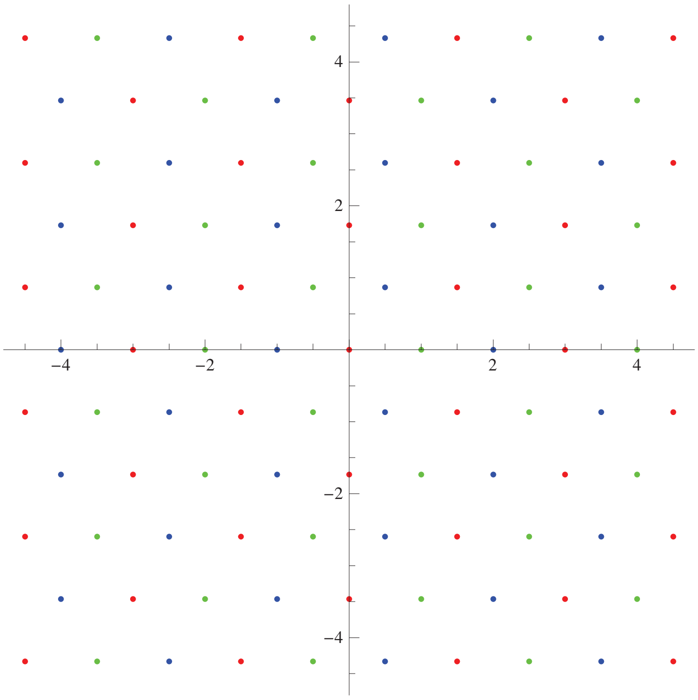
9. $\frac{(\phi^n-\psi^n)^2}{(\phi-\psi)^2} - \frac{\left(\phi^{n-1}-\psi^{n-1}\right)\left(\phi^{n+1}-\psi^{n+1}\right)}{(\phi-\psi)^2} = \frac{\phi^{2\,n}+\psi^{2\,n}-2\,(\phi\,\psi)^n}{5} - \frac{\phi^{2\,n}+\psi^{2\,n}+3\,(\phi\,\psi)^n}{5} = -(\phi\,\psi)^n = -(-1)^n$.

10. $f_o(x) = x + x^2 + 2\,x^3 + 3\,x^4 + 5\,x^5 + \cdots$. It is straightforward to verify from the recurrence relation that $f_0(x) = x + x\,f_o(x) + x^2\,f_o(x)$. Rearranging, we obtain the closed form $f_o(x) = \frac{x}{1-x-x^2}$. Setting $x = 0.01$ gives us the rational approximation $\frac{0.01}{1-0.01-0.0001} = \frac{0.01}{0.9899} = \frac{100}{9899}$.

11. This system is deterministic and finite, so must necessarily eventually become cyclic. As it is reversible as well, it must be completely cyclic; this solves the first part of the question. We can represent the state of the system using the ordinary generating function $L_{n-1} + L_{n-2}\,x + L_{n-3}\,x^2 + L_{n-4}\,x^3 + \ldots + L_0\,x^{n-1}$, where 0 and 1 correspond to OFF and ON, respectively. This is a polynomial $\mathbb{Z}_2[x]$, as each coefficient can be either 0 or 1 and we consider addition modulo 2. If we follow each step with a rotation to the left (such that L_i moves into the position of L_{i-1}), then we only need to alter the state of L_0 depending on L_{n-1}. Refer to this composite operation as Step*. The lamp alteration is equivalent to the operation $P(x) \to P(x) - 1 + x^{n-1}$ (if applicable, or the identity function otherwise), and the rotation is equivalent to $P(x) \to x\,P(x)$. So, Step* performs the polynomial operation $P(x) \to x\,P(x)\left(\text{mod } x^n + x^{n-1} + 1\right)$; this modulus is the characteristic

polynomial associated with this linear feedback shift register. If we apply m steps in succession, then we obtain the polynomial $x^m P(x) \left(\text{mod } x^n + x^{n-1} + 1\right)$. If m is the period of the system, then $x^m P(x) \equiv P(x)$ and thus $x^m \equiv 1$. For the second part of the question, we get $x^{n^2} \equiv (x^n)^n \equiv \left(x^{n-1} + 1\right)^n$. As $n = 2^k$, all binomial coefficients apart from $\binom{2^k}{0} = 1$ and $\binom{2^k}{2^k} = 1$ are even (consider Pascal's triangle modulo 2, which approximates the Sierpinski triangle), thus equal to zero in the field $\mathbb{Z}_2$. So, $\left(x^{n-1} + 1\right)^n$ expands to give $x^{n^2-n} + 1$. With the identities $x^{n^2} \equiv x^{n^2-n} + 1$ and $x^{n^2} \equiv x^{n^2-n} + x^{n^2-1}$, we obtain $x^{n^2-1} \equiv 1$, and thus $(\text{Step}^*)^{n^2-1}$ is the identity function. For the third part of the question, we get
$x^{n^2-1} \equiv \left(x^{n+1}\right)^{n-1} \equiv \left(x^{n+1}\right)^{2^k} \equiv (x^n + x)^{2^k} \equiv x^{n(n-1)} + x^{n-1} \equiv x^{n^2-n} + x^{n-1}$. Hence,
$x^{n^2} \equiv x^{n^2-1} + x^{n^2-n} \equiv x^{n-1} + 2\,x^{n^2-n} \equiv x^{n-1}$. Dividing throughout by x^{n-1} results in $x^{n^2-n+1} \equiv 1$, so $(\text{Step}^*)^{n^2-n+1}$ is the identity function.

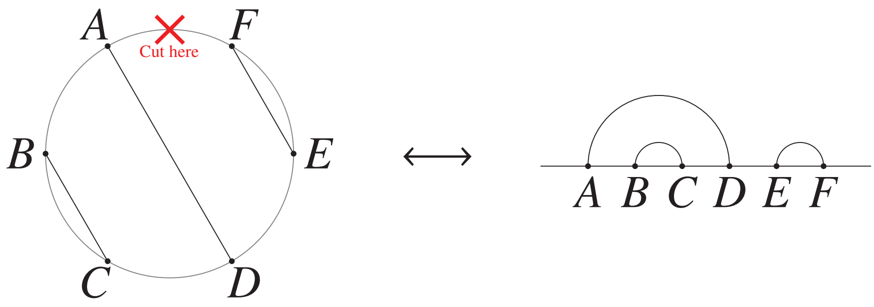
12. The function 'reflects' the binary representation. For example, it maps 10 011 101 to 1.0111001. Multiplying by 2^{20} means that it is subsequently shifted 20 places to the left. The fixed points of this function are the palindromes of length 21, of which there are 2^{11}.



13. Colour the Eisenstein integers red, green and blue, as in the diagram above. If we multiply all Eisenstein integers by $\omega - 1$, we obtain the red Eisenstein integers (a copy of the Eisenstein integers rotated by $\frac{5}{6}\pi$ and scaled by $\sqrt{3}$). The green and blue Eisenstein integers can be obtained by translating the red Eisenstein integers by 1 and 2, respectively. Hence, if no Eisenstein integer has two representations using k digits, then no Eisenstein integer has two representations using $k + 1$ digits. By induction, all representations are unique. However, we have yet to show that all Eisenstein integers can be obtained in this manner. If all Eisenstein integers within the hexagon with vertices $\left\{1, -\omega^2, \omega, -1, \omega^2, -\omega\right\}$ can be represented (which can be shown using trial and error), then we can represent all Eisenstein integers within the hexagon with vertices $\left\{1\,(\omega - 1), -\omega^2(\omega - 1), \omega(\omega - 1), -1\,(\omega - 1), \omega^2(\omega - 1), -\omega(\omega - 1)\right\}$ by multiplying those Eisenstein integers by $\omega - 1$ and translating by 1 and 2 to fill in the gaps. By induction, we can represent all Eisenstein integers within an arbitrarily large hexagon centred about the origin, and therefore all Eisenstein integers.

14. Yes, consider the function which shifts the Zeckendorf representation of a positive integer one position to the left. Due to the Fibonacci recurrence relation, $f(f(n)) = f(n) + n$.

15. Yes, for example $\{1, -2, 3, -5, 8, -13, 21, \dots\}$. This is sum-full as each term is the sum of the next two terms. To prove that it is zero-sum-free, consider all numbers expressible as the sum of the first k terms of the sequence. We can prove from a trivial base case and simple inductive argument that this is $\{1 - F_{k+1}, 2 - F_{k+1}, \dots, -2, -1, 1, 2, \dots, F_{k+2} - 2, F_{k+2} - 1\}$ for odd k, and the negation thereof for even k. The limiting set is the set of nonzero integers.



16. We can biject between these strings and non-intersecting pairings of points on a circle by cutting the circle at a given point and 'unfolding' it, as demonstrated above.

17. The insect can only move right or up at each step, as otherwise it would take too long to reach (n, n). There must be n moves to the right and n moves up, so there are $\binom{2n}{n}$ possible paths in the first part of the problem. For the second part of the problem, we represent a horizontal move with a left parenthesis and a vertical move with a right parenthesis. This reduces the problem to the previous question, so there are $C_n = \frac{1}{n+1}\binom{2n}{n}$ paths with this constraint.

18. Consider $2n + 2$ points on the circumference of a circle, and label one vertex A. Choose another vertex B such that the number of vertices on each arc AB is even, and join A and B with a chord. Let the number of points right of the chord be $2k$; the number of points left of the chord must be $2(n-k)$. There are C_k non-intersecting pairings of the vertices to the left of the chord, and C_{n-k} non-intersecting pairings of vertices to the right of the chord, giving a total of $C_k C_{n-k}$ pairings. Repeating this for each location of B gives $C_0 C_n + C_1 C_{n-1} + \dots + C_n C_0$ possible pairings. This must be equal to C_{n+1}, by definition.

19. $x\, c_o(x)^2 = x\left(C_0 + C_1 x + C_2 x^2 + \dots\right)^2 = C_0{}^2 x + (C_0 C_1 + C_1 C_0)\, x^2 + (C_0 C_2 + C_1 C_1 + C_2 C_0)\, x^3 + \dots$, which simplifies to $C_1 x + C_2 x^2 + \dots = c_o(x) - C_0 = c_o(x) - 1$ by Segner's recurrence relation. So, $x\, c_o(x)^2 - c_o(x) + 1 = 0$, and we can obtain $c_o(x)$ by the Babylonian formula for the roots of a quadratic equation. Specifically, we have $c_o(x) = \frac{1 \pm \sqrt{(-1)^2 - 4x}}{2x}$. As $c_o(x) \to 1$ when $x \to 0$, the correct root is $c_o(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$.

20. We define a sequence $\{y_i\}$ such that:

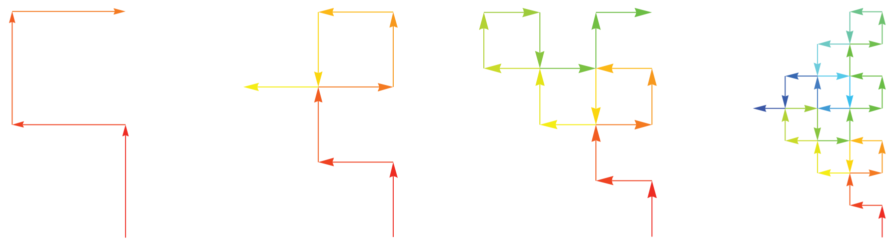
- $y_i = A$ if k is even and $x_k = 1$;
- $y_i = B$ if k is odd and $x_k = 1$;
- $y_i = C$ if k is even and $x_k = -1$;
- $y_i = D$ if k is odd and $x_k = -1$.

Consider the string Y_l obtained by concatenating the first 2^l elements of $\{y_i\}$. It is straightforward to verify that $Y_0 = B$, and that Y_{l+1} can be obtained from Y_l by applying the L-system with substitution rules: $A \to D\,C$, $B \to B\,C$, $C \to B\,A$, and $D \to D\,A$. The first few terms of $\{Y_i\}$ are $B \to B\,C \to B\,C\,B\,A \to B\,C\,B\,A\,B\,C\,D\,C \to \dots$. Interpret these strings as sequences of instructions for moving an insect on the integer lattice $\mathbb{Z}^2$:

- A: move east;
- B: move north;
- C: move west;

- D: move south;

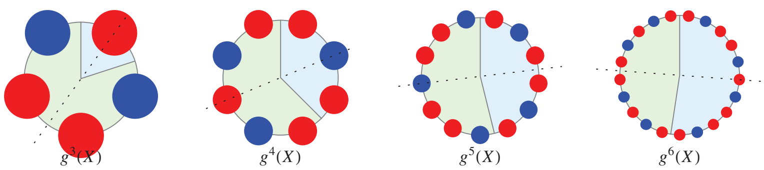
The pattern generated by this is a space-filling curve bounded by the lines $y = -x$ and $x = 0$ (with the first few iterations displayed below). The substitution rules effectively reflect the path in a NNW line through the origin, combined with a dilation of scale factor $\sqrt{2}$. Hence, the insect always remains in the octant bounded by the lines $y = -x$ and $x = 0$. Remaining to the right of $y = -x$ is equivalent to the condition $a_n + b_n \geq c_n + d_n$ (for all $n \in \mathbb{Z}^+$), where $a_n = |\{k : y_k = A\}|$, *et cetera*. This is in turn equivalent to the desired inequality.



21. We can scale the set of intervals without affecting anything, so let's colour $[-1, 1]$ instead. Note that if we have a set of monic polynomials of degrees $0, 1, \ldots, n$, inclusive, which satisfy the condition ($\int_{\text{red}} P(x)\, dx = \int_{\text{blue}} P(x)\, dx$), then the condition holds for all linear combinations of them (namely all polynomials of degree $\leq n$). If we have a colouring that is symmetric (i.e. x is coloured identically to $-x$ for all $x \in [-1, 1]$), then all odd functions satisfy the condition. Similarly, if we have a colouring that is antisymmetric (i.e. x is coloured oppositely to $-x$ for all $x \in [-1, 1]$), then all even functions satisfy the condition. Begin with the colouring C_0, where the interval $[-1, 0]$ is red and $[0, 1]$ is blue. Clearly, all constant functions (polynomials of degree 0) satisfy the condition. Let C^* denote the complement of C, where an interval is red in C^* if and only if it is blue in C. We can scale C_0 to the interval $[-1, 0]$, and scale $C_0{}^*$ to the interval $[0, 1]$, and append them to form the colouring C_1. As C_0 works for all degree-0 polynomials, so must the scaled copies of C_0 and $C_0{}^*$, and thus also C_1. As C_1 is symmetric, it must also work for the function $y = x$, and therefore all linear polynomials. We define C_2 by performing this operation on C_1, resulting in an antisymmetric colouring which must also satisfy $y = x^2$, and therefore all quadratics. Continuing in this manner, we obtain C_{2013} which works for all degree-2013 polynomials. Note that this colouring is related to the Thue-Morse sequence.

22. Observe that $g(g(X)) = X\,Y\,X$ is the concatenation of $g(X) = X\,Y$ followed by X. Hence, $g^{n+2}(X)$ is the concatenation of $g^{n+1}(X)$ followed by $g^n(X)$, and thus $|g^{n+2}(X)| = |g^{n+1}(X)| + |g^n(X)|$. This is the recurrence relation for the Fibonacci sequence.

23. As opposed to considering strings of symbols, consider instead beads on a necklace (positioned at the kth roots of unity, where k is the length of the string). The substitution rules $X \to X\,Y$ and $Y \to X$ are then equivalent to the alternative rules $X \to \langle\, Y \,\rangle$ and $Y \to \langle\,\rangle$, where we consider $\langle$ and $\rangle$ to be 'half-beads', where $\rangle\langle = X$, followed by a rotation of the entire necklace by one half-bead to restore the correct orientation. As the substitution rules map each symbol to a palindromic string, and the initial necklace has bilateral symmetry, then all subsequent necklaces have bilateral symmetry. We then reflect the 'start' of the necklace in this axis of symmetry to produce another position. Cutting at these two positions will clearly result in two palindromic strings. Using this idea of the half-bead substitution (which preserves the axis of symmetry) followed by a rotation, it is straightforward to show that $g^n(X)$ has palindromic substrings of length $F_{n+1} - 2$ and $F_n + 2$, where F_n is the nth Fibonacci number.



24. We already proved, in the previous exercise, that the first $F_{n+2} - 2$ symbols of $g^{n+1}(X)$ is a palindrome. As we obtain $g^n(X)$ by taking the first F_{n+2} symbols of the infinite golden string, $g^\omega(X)$, the first $F_{n+2} - 2$ symbols of $g^n(X)$ also form a palindrome.

25. Consider the trapezium formed by the centres of the circles and the points of tangency with the real line. The hypotenuse has length $\frac{1}{2}\left(\frac{1}{{q_1}^2} + \frac{1}{{q_2}^2}\right)$, the base has length $\frac{p_2}{q_2} - \frac{p_1}{q_1}$ and the difference between the left and right heights is $\pm\frac{1}{2}\left(\frac{1}{{q_1}^2} - \frac{1}{{q_2}^2}\right)$. Applying Pythagoras' theorem, we obtain $\frac{1}{4}\left(\frac{1}{{q_1}^2} + \frac{1}{{q_2}^2}\right)^2 = \left(\frac{p_2}{q_2} - \frac{p_1}{q_1}\right)^2 + \frac{1}{4}\left(\frac{1}{{q_1}^2} - \frac{1}{{q_2}^2}\right)^2$. Expanding, this results in $\left(\frac{p_2}{q_2} - \frac{p_1}{q_1}\right)^2 = \frac{1}{{q_1}^2\,{q_2}^2}$, so $\frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{1}{q_1\,q_2}$. Multiplying by $q_1\,q_2$ gives the identity $p_2\,q_1 - p_1\,q_2 = 1$.

26. We have three equations, namely $p_2\,q_1 - p_1\,q_2 = 1$, $p_3\,q_1 - p_1\,q_3 = 1$, and $p_2\,q_3 - p_3\,q_2 = 1$. Subtracting the third equation from the second gives us $(p_1 + p_2)\,q_3 = (q_1 + q_2)\,p_3$, which rearranges to give $\frac{p_3}{q_3} = \frac{p_1+p_2}{q_1+q_2}$.

27. Let $j > i$. Then $s_j = k\,s_i + 1$, so $s_j \equiv 1 \pmod{s_i}$. By applying Euclid's algorithm, s_i and s_j have a greatest common divisor of 1.

28. By reverse-engineering the definition, we get the recurrence relation $s_{n+1} - 1 = s_n(s_n - 1)$. Assume that $\frac{1}{s_0} + \frac{1}{s_1} + \ldots + \frac{1}{s_k} = \frac{s_{k+1-2}}{s_{k+1}-1}$. Adding the next reciprocal would give us $\frac{s_{k+1}-2}{s_{k+1}-1} + \frac{1}{s_{k+1}} = \frac{{s_{k+1}}^2-s_{k+1}-1}{{s_{k+1}}^2-s_{k+1}} = \frac{s_{k+2}-2}{s_{k+2}-1}$. So, by induction, this tends towards 1.

29. Assume that k is not a power of two, so $k = l\,p$ for some odd prime p and integer l. Then, $\left(2^l + 1\right)\left(1 - 2^l + 2^{2\,l} - \ldots + 2^{(p-1)\,l}\right) = 2^k + 1$, thus proving that $2^k + 1$ is composite.