## Problems for Final Mock

Friday July 10, 1998

**C1** For $n > 1$, let $A(n)$ denote the number of partitions of $n$ in which every term appears at least twice, and let $B(n)$ denote the number of partitions of $n$ in which no terms are 1 and no two terms are consecutive integers. Show that $A(n) = B(n)$ for all $n > 1$.

**C2** 65 beetles are placed on the squares of a 9x9 chess board. At each time step, every beetle moves one square either vertically or horizontally. No beetle moves horizontally for two consecutive time steps or vertically for two consecutive time steps. Prove that after some number of steps, one of the squares will contain at least two beetles.

**C3** What is the maximum number of knights which can be placed on an 8x8 chess board so that no knight is attacked by two other knights?

**C4** Each square in an infinite array of squares is coloured either red or white in such a way that every 2x3 or 3x2 rectangle of squares contains exactly 2 red squares. How many red squares are there in a 9x11 rectangle of squares?

**C5** An $nxn$ matrix is filled with the integers 1, 2, ..., $n$ such that every integer appears exactly $n$ times. Show that it is possible to choose one entry from each column such that the set of chosen integers is exactly $\{1, 2, \ldots, n\}$.

**S1** Let $\{a_k\}$ be a sequence of non-negative real numbers such that $a_k - 2a_{k+1} + a_{k+2} \geq 0$ and $\sum_{j=1}^{k} a_j \leq 1 \quad \forall k = 1,2,\ldots$ Prove that

$$0 \leq (a_k - a_{k+1}) < \frac{2}{k^2} \qquad \forall k = 1,2,\ldots$$

**S2** $a_1, a_2, \ldots$ are non-negative real numbers such that $a_{n+m} \leq a_n + a_m$ for all positive integers $m$, $n$. Prove that

$$a_n \leq ma_1 + \left(\frac{n}{m} - 1\right)a_m \qquad \forall n \geq m$$

**G** In acute angled triangle ABC, let AD, BE be altitudes and let AP, BQ be internal bisectors. Denote by I, O the incenter and the circumcenter of the triangle respectively. Prove that the points D, E and I are collinear if and only if P, Q and O are collinear.

**I** Let ABC be an equilateral triangle and let P be a point in its interior. Let the lines AP, BP and CP meet the sides BC, CA, AB at the points $A_1$, $B_1$, $C_1$ respectively. Prove that

$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A$$

**N** Let $n$ be a positive integer. Show that $\left(\sqrt{1998} + \sqrt{1999}\right)^n = \sqrt{m} + \sqrt{m-1}$ for some positive integer $m$.

Cmon