Induction Solutions

Solutions to Review Problems

1. Show that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

Solution: We prove the result by induction on n. For the base case $n = 1$, we verify that $1^2 = 1 = \frac{(1)(2)(3)}{6}$.

For the inductive step, assume that the result holds for a given positive integer k, so that

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Then

$$\begin{aligned}
1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left(\frac{k(2k+1)}{6} + k + 1\right) \\
&= (k+1)\left(\frac{2k^2+k}{6} + \frac{6k+6}{6}\right) \\
&= (k+1) \cdot \frac{2k^2+7k+6}{6} \\
&= (k+1) \cdot \frac{(k+2)(2k+3)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}$$

Hence, the result holds for $k + 1$.

Thus, by induction, the result holds for all positive integers n.

2. Show that

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots$$

for all positive integers n.

Solution: It seems clear that we'll need to use the Fibonacci identity $F_n = F_{n-1} + F_{n-2}$. This makes it likely that the identity for n will depend on the identity for both $n-1$ and $n-2$, which means we'll need strong induction.

For $n = 1$, the given formula becomes $\binom{0}{0} = 1 = F_1$, and for $n = 2$, the given formula becomes $\binom{1}{0} = 1 = F_2$, so the formula holds for $n = 1$ and $n = 2$.

Assume that the formula holds for $n = 1, 2, \dots, k-1$ for a given positive integer $k \geq 3$. In particular,

$$\begin{aligned} F_{k-2} &= \binom{k-3}{0} + \binom{k-4}{1} + \binom{k-5}{2} + \cdots, \\ F_{k-1} &= \binom{k-2}{0} + \binom{k-3}{1} + \binom{k-4}{2} + \cdots. \end{aligned}$$

Adding these equations, we get

$$\begin{aligned} F_{k-2} + F_{k-1} &= \binom{k-3}{0} + \binom{k-4}{1} + \binom{k-5}{2} + \cdots \\ &= \quad + \binom{k-2}{0} + \binom{k-3}{1} + \binom{k-4}{2} + \cdots. \end{aligned}$$

But binomial coefficients with the same "top" and bottoms that differ by 1 can be added together, using Pascal's identity:

$$\binom{m}{r-1} + \binom{m}{r} = \binom{m+1}{r}.$$

Applying this to the above sum, we get

$$F_k = \binom{k-2}{0} + \binom{k-2}{1} + \binom{k-3}{2} + \cdots.$$

This is almost what we want. We just have to tweak the first term:

$$F_k = \binom{k-1}{0} + \binom{k-2}{1} + \binom{k-3}{2} + \cdots.$$

Hence, the formula holds for k.

Thus, by strong induction, the formula holds for all positive integers n.

3. In the plane, n lines are drawn so that no two lines are parallel and no three lines pass through the same point. How many regions do these n lines determine?

Solution: A little experimentation gives the following chart:

Lines	1	2	3	4	5	6
Regions	2	4	7	11	16	22

These are 1 more than the triangular numbers. So, we claim that the n lines determine $\frac{n(n+1)}{2} + 1$ regions, which we will prove by induction.

For the base case $n = 1$, we see that 1 line determines $\frac{1 \cdot 2}{2} + 1 = 2$ regions.

For the inductive step, assume that the result holds for a given positive integer k, so that k lines determine $\frac{k(k+1)}{2} + 1$ regions.

Consider $k + 1$ lines $\ell_1, \ell_2, \dots, \ell_{k+1}$. When we draw the k lines $\ell_1, \ell_2, \dots, \ell_k$, they will determine $\frac{k(k+1)}{2} + 1$ regions by the inductive hypothesis. When we draw line ℓ_{k+1}, it will intersect each of the other k lines at distinct points, giving us k intersection points that divide line ℓ_{k+1} into $k + 1$ segments. Each of these $k + 1$ segments divides a previous region into two new regions, so the number of regions is now

$$\frac{k(k+1)}{2} + 1 + k + 1 = (k+1)\left(\frac{k}{2} + 1\right) + 1 = \frac{(k+1)(k+2)}{2} + 1.$$

Hence, the result holds for $k + 1$, and by induction, for all positive integers n.

4. (Bernoulli's Inequality) Let n be a positive integer, and let $x \geq -1$. Show that

$$(1 + x)^n \geq 1 + nx.$$

Solution: We will prove the inequality using induction.

For the base case $n = 1$, the given inequality becomes $1 + x \geq 1 + x$, which is clearly true.

For the inductive step, assume that the inequality holds for a given positive integer k, so that

$$(1 + x)^k \geq 1 + kx$$

for all $x \geq -1$.

Multiplying both sides by $1 + x$ (which is nonnegative), we get

$$(1 + x)^{k+1} \geq (1 + kx)(1 + x) = 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x.$$

Hence, the inequality holds for $n = k + 1$, and by induction, for all positive integers n.

5. On a large, flat field, n people are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When n is odd, show that there is at least one person left dry. Is this always true when n is even? (Canada, 1987)

Solution: We will prove that when n is odd, there is at least one person left dry using induction. For the base case $n = 1$, the statement is clearly true.

For the inductive step, assume that the statement is true for some given odd positive integer k, and consider $k+2$ such people standing in a field. We need to eliminate 2 people somehow to use the inductive hypothesis, and the only two people that we know for sure will get wet are the two people closest to each other, since they'll shoot at each other. Specifically, since all the distances between two people are different, there is one distance d that is shorter than every other. Then the two people, say A and B, who are at distance d, will fire at each other.

If no one else fires at A or B, then we can apply the induction hypothesis to the k people other than A or B, and conclude that one of these k people must be left dry.

But if someone other than B also fires at A, say C, then A, B, and C have directed their 3 shots at 2 people, namely A and B. There are k people who are still dry, but only $(k + 2) - 3 = k - 1$ shots left, so (by the Pigeonhole principle) at least one person must be left dry. The argument is the same if someone other than A also fires at B.

Hence, the statement is true for $k + 2$ people, and by induction, for any odd positive number of people.

If n is even, then it is not necessarily true that at least one person is left dry. For example, you could arrange everyone into $n/2$ pairs, with the people in each pair standing close to each other, but with the pairs far apart. Then each person fires at the other person in the same pair.

6. (Fermat's Little Theorem) Let p be a prime number. Prove that $n^p - n$ is divisible by p for all positive integers n.

Solution: We prove the result by induction on n. For the base case $n = 1$, we have that $1^p - 1 = 0$ is divisible by p.

For the inductive step, assume that the result holds for a given positive integer k, so that $k^p - k$ is divisible by p.

We now examine $(k + 1)^p - (k + 1)$. We can expand the binomial:

$$\begin{aligned}(k+1)^p - (k+1) &= k^p + \binom{p}{1}k^{p-1} + \binom{p}{2}k^{p-2} + \cdots + \binom{p}{p-1}k + 1 - k - 1 \\ &= k^p + \binom{p}{1}k^{p-1} + \binom{p}{2}k^{p-2} + \cdots + \binom{p}{1}k - k \\ &= (k^p - k) + \left(\binom{p}{1}k^{p-1} + \binom{p}{2}k^{p-2} + \cdots + \binom{p}{1}k\right).\end{aligned}$$

By the inductive hypothesis, the first term above, $k^p - k$, is divisible by p. So it remains to show that the second term is also divisible by p. But each coefficient within the second term above is of the form

$$\binom{p}{i} = \frac{p!}{i!(p-i)!},$$

for some $1 \le i \le p - 1$, and the numerator $p!$ has a factor of p, but neither $i!$ nor $(p - i)!$ has a factor of p. So

$\binom{p}{i}$ is divisible by p, and hence the large second term in the equation above is also divisible by p.

Hence, $(k+1)^p - (k+1)$ is divisible by p. Thus, by induction, the result is true for all positive integers n.

7. Prove that for every positive integer n, the integers $1, 2, \dots, 2^{n+1}$ can be partitioned into two sets A and B, such that for all $0 \le i \le n$,

$$\sum_{x \in A} x^i = \sum_{x \in B} x^i.$$

For example, for $n = 2$, we can take $A = \{1, 4, 6, 7\}$ and $B = \{2, 3, 5, 8\}$, since

$$\begin{aligned} 1^0 + 4^0 + 6^0 + 7^0 &= 2^0 + 3^0 + 5^0 + 8^0, \\ 1^1 + 4^1 + 6^1 + 7^1 &= 2^1 + 3^1 + 5^1 + 8^1, \\ 1^2 + 4^2 + 6^2 + 7^2 &= 2^2 + 3^2 + 5^2 + 8^2. \end{aligned}$$

Solution: We prove the result using induction on n. For the base case $n = 1$, we can take $A = \{1, 4\}$ and $B = \{2, 3\}$, since

$$\begin{aligned} 1^0 + 4^0 &= 2^0 + 3^0, \\ 1^1 + 4^1 &= 2^1 + 3^1. \end{aligned}$$

The given example is our big clue how to go from a solution for 2^k and build a solution for 2^{k+1}. Notice that the numbers $1, 2, 3, 4$ are allocated to A and B in the $n = 2$ solution in exactly the same way as the $n = 1$ solution, and that the numbers $5, 6, 7, 8$ are allocated to A and B using the same pattern, but with the sets reversed.

1	2	3	4	Set A in red	Set B in blue
5	6	7	8		

So let's guess that pattern continues to hold, and verify it by induction.

Specifically, assume that the result holds for a given positive integer $n = k$, which means we can partition the integers $1, 2, \dots, 2^{k+1}$ into two sets $A_k = \{a_1, a_2, \dots, a_{2^k}\}$ and $B_k = \{b_1, b_2, \dots, b_{2^k}\}$ such that

$$\sum_{j=1}^{2^k} (a_j)^i = \sum_{j=1}^{2^k} (b_j)^i$$

for all $0 \le i \le k$. (Notice that we know that both A_k and B_k must each contain 2^k elements, because for $i = 0$, this equation becomes $|A_k| = |B_k|$.)

We are guessing that

$$\begin{aligned} A_{k+1} &= \{a_1, a_2, \dots, a_{2^k}\} \cup \{b_1 + 2^{k+1}, b_2 + 2^{k+1}, \dots, b_{2^k} + 2^{k+1}\}, \\ B_{k+1} &= \{b_1, b_2, \dots, b_{2^k}\} \cup \{a_1 + 2^{k+1}, a_2 + 2^{k+1}, \dots, a_{2^k} + 2^{k+1}\}. \end{aligned}$$

Note that the sets A_{k+1} and B_{k+1} form a partition of the integers $1, 2, \ldots, 2^{k+2}$. Now we need to check that each equation holds, so let i be given with $1 \le i \le k+1$. Then:

$$\sum_{x \in A_{k+1}} x^i = \sum_{j=1}^{2^k} (a_j)^i + \sum_{j=1}^{2^k} (b_j + 2^{k+1})^i.$$

We can use Binomial expansion to get

$$\sum_{x \in A_{k+1}} x^i = \sum_{j=1}^{2^k} (a_j)^i + \sum_{j=1}^{2^k} (b_j)^i + \binom{i}{1} 2^{k+1} \sum_{j=1}^{2^k} (b_j)^{i-1} + \binom{i}{2} (2^{k+1})^2 \sum_{j=1}^{2^k} (b_j)^{i-2} + \cdots + (2^{k+1})^i \sum_{j=1}^{2^k} 1. \tag{1}$$

Similarly,

$$\sum_{x \in B_{k+1}} x^i = \sum_{j=1}^{2^k} (b_j)^i + \sum_{j=1}^{2^k} (a_j)^i + \binom{i}{1} 2^{k+1} \sum_{j=1}^{2^k} (a_j)^{i-1} + \binom{i}{2} (2^{k+1})^2 \sum_{j=1}^{2^k} (a_j)^{i-2} + \cdots + (2^{k+1})^i \sum_{j=1}^{2^k} 1. \tag{2}$$

But the sum of the first two terms of (1) and (2) are clearly equal, and all of the corresponding remaining terms are equal by the inductive hypothesis.

Hence, the result holds for $k+1$, and by induction, for all positive integers n.