Art of Problem Solving

WOOT

Class Transcript 09/23 - Pigeonhole Principle

MellowMelon 7:30:17 pm
WOOT 2013-14: Pigeonhole Principle

MellowMelon 7:30:22 pm
Today, we will be discussing the Pigeonhole Principle.

MellowMelon 7:30:35 pm
The basic version of the Pigeonhole Principle states that if $n+1$ objects (say, pigeons) are distributed among n holes, then one of the holes must contain at least 2 objects.

MellowMelon 7:30:43 pm
A more general version of the Pigeonhole Principle states that if $kn+1$ objects are distributed among n holes, then one of the holes must contain at least $k+1$ objects.

MellowMelon 7:30:55 pm
Often, the tricky part is figuring out exactly how to apply the Pigeonhole Principle, and to what objects. So one important step is determining exactly what are the pigeons and what are the holes.

MellowMelon 7:31:05 pm
Let A be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, ..., 100. Prove that there must be two distinct integers in A whose sum is 104.

MellowMelon 7:31:33 pm
As with last week, the class title makes this problem a bit easier... So let's start with the question: What tells us that the Pigeonhole Principle may work here?

MSTang 7:32:13 pm
"there must be..."

zhuangzhuang 7:32:13 pm
the "exists" word

mathlover3737 7:32:13 pm
Prove that there *must* be

Tuxianeer 7:32:13 pm
must be; existence

minimario 7:32:13 pm
"There must be"

soy_un_chemisto 7:32:13 pm
"there must be"

ssk9208 7:32:13 pm
We have to prove an existence

MellowMelon 7:32:15 pm
The words "Prove that there must be...".

MellowMelon 7:32:21 pm
The Pigeonhole Principle is often well-suited for existential problems; that is, problems that ask us to prove that something exists. We don't necessarily have to find it, we just have to prove that there is something with a given property.

MellowMelon 7:32:27 pm
What can we do here?

MellowMelon 7:33:20 pm
We should think about the pigeons and holes.

MellowMelon 7:33:23 pm
What are the pigeons?

chenjamin 7:34:09 pm
20 distinct integers

ssk9208 7:34:09 pm
the 20 integers

zhuangzhuang 7:34:09 pm
The 20 numbers chosen

brian22 7:34:09 pm
the numbers in A

MellowMelon 7:34:12 pm
We're trying to find "two distinct integers" that share a property, so it makes sense that the pigeons are the 20 distinct integers.

MellowMelon 7:34:23 pm
(In contrast, using the whole arithmetic progression as the pigeons won't do much.)

MellowMelon 7:34:33 pm
This means that in order to use the Pigeonhole Principle naively, we'll need at most 19 holes in order to force 2 of the pigeons to be in the same hole. Otherwise we'll have to find more pigeons or get rid of some holes.

MellowMelon 7:34:44 pm
With that in mind, any idea about the pigeonholes?

mathlover3737 7:35:46 pm
make holes {1}, {4, 100}, {7, 97}, ... {52}

NT2048 7:35:46 pm
(4, 100), (7, 97), ect

Tuxianeer 7:35:46 pm
make holes pairs that add to 104

MSTang 7:35:46 pm
Pairs of numbers that sum to 104

djmathman 7:35:46 pm
Partition the set of integers into the sets {4,100},{7,97}, and so on

brian22 7:35:46 pm
pairs of integers that sum to 104?

chenjamin 7:35:46 pm
We can only put one integer from each pair that adds up to 104

CyclicRain 7:35:46 pm
pairs that add up to 104

MellowMelon 7:36:01 pm
We want "two pigeons in the same hole" to be a solution to our problem, so we can form pigeonholes by pairing the numbers so that the numbers in each pair add up to 104. We're basically working backwards here.

MellowMelon 7:36:16 pm
The pairs are that sum to 104 are then {4, 100}, {7, 97}, ..., {49, 55}. Are those all of the holes?

MSTang 7:37:00 pm
Also, {1} and {52}

Tuxianeer 7:37:00 pm
no, also {1} and {52}, numbers without a pair

joshxiong 7:37:00 pm
We still have {1} and {52}

djmathman 7:37:00 pm
Well, no, since we have the singletons {1} and {52}

sujaykazi 7:37:00 pm
1 and 52 get left out, so those make 2 more holes.

chenjamin 7:37:00 pm
{1} and {52} are also holes

MellowMelon 7:37:03 pm
Note that the numbers 1 and 52 are leftover. Those are two holes as well.

MellowMelon 7:37:11 pm
How many holes do we have?

mathcool2009 7:37:41 pm
so 18 holes in total

vincenthuang75025 7:37:41 pm
18

SuperSnivy 7:37:41 pm
18

Tuxianeer 7:37:41 pm
18

Michelangelo 7:37:41 pm
18

MSTang 7:37:41 pm
18

mathcool2009 7:37:41 pm
18

Rogman 7:37:41 pm
18

mathlover3737 7:37:41 pm
18?

MellowMelon 7:37:42 pm
We have 16 pairs, and two singletons. There are 18 holes. By the Pigeonhole principle, ...

MSTang 7:38:46 pm
Two pigeons must be in the same hole

Tuxianeer 7:38:46 pm
we must have two pairs of numbers adding to 104

mathlover3737 7:38:46 pm
there must be at least 2 numbers in one hole, and since there can only be one number in the sets {1} and {52}, there must be a pair.

RocketSingh 7:38:46 pm
We must have 2 things in a hole

CyclicRain 7:38:46 pm
two holes have two pigeons

zhuangzhuang 7:38:46 pm
2 elements must go into a 2 element set. Thus, they sum to 104. QED.

mathcool2009 7:38:46 pm
at least 1 set must contain at least 2 elements of A

MellowMelon 7:38:51 pm
By the Pigeonhole Principle, two of our 20 elements in A appear in the same pair, since there are only 18 holes. These two numbers add up to 104, as desired.

MellowMelon 7:39:05 pm
Note that only 19 numbers need to be chosen to guarantee the result.

MSTang 7:39:28 pm
Then could we show that there has to be 2 such pairs?

MellowMelon 7:39:33 pm
Yes, our solution shows that as well.

MellowMelon 7:39:39 pm
Prove that, given a set of ten distinct two-digit numbers, it is always possible to find two disjoint subsets whose elements have the same sum.

MellowMelon 7:40:08 pm
Again, "it is possible to find". Pigeonhole principle!

MellowMelon 7:40:14 pm
Any idea what the pigeons should be?

MellowMelon 7:41:18 pm
We want to find two subsets with the same sum. Think: when we say "two pigeons are in the same hole", how should we define pigeons and holes to get what we want?

RocketSingh 7:42:43 pm
make the sums hole and the subsets piegons

Tuxianeer 7:42:43 pm
subsets and sums

MSTang 7:42:43 pm
pigeons = subsets and holes = sums

brian22 7:42:43 pm
well the "two pigeons" might be the two disjoint subsets

tareyza 7:42:43 pm
the subsets would be the pigeons

SuperSnivy 7:42:43 pm
subsets are the pigeons, sum is the holes

mathcool2009 7:42:43 pm
the pigeons are all the possible nonempty subsets of the 10 numbers and the holes are all the possible sums of those subsets

zhuangzhuang 7:42:43 pm
The subsets are the pigeons

sujaykazi 7:42:43 pm
The pidgeons should be all of the possible subsets created from the 10 given numbers.

MellowMelon 7:42:44 pm
We make the subsets the pigeons, and the sums of those subsets the holes. Then "two pigeons are in the same hole" means "two subsets have the same sum". That sounds like something we'd like to show!

MellowMelon 7:42:59 pm
So let's try things that way first. How many pigeons are there?

RocketSingh 7:44:19 pm
1023

vincenthuang75025 7:44:19 pm
1023 (or `1024 including empty set)

zhuangzhuang 7:44:19 pm
1023 Pidgeons

mathcool2009 7:44:19 pm
1023

joshxiong 7:44:19 pm
There are $2^1 0 - 1 = 1023$ (we do not need the empty set)

MellowMelon 7:44:20 pm
The pigeons are the nonempty subsets of the set of ten numbers. There are $2^{10} - 1 = 1023$ such subsets.

MellowMelon 7:44:39 pm
Some of you are pointing out that we may not want to include the whole set, and that there are 1022 pigeons. That's viable, but to make the possible sums easier to compute later, we'll go ahead and throw it in. Consider this an optimization to keep in mind if we come up a bit short later.

MellowMelon 7:44:54 pm
So we'll use 1023 pigeons for now. How many holes are there?

MSTang 7:46:21 pm
936

Rogman 7:46:21 pm
99+98+97....+90?

Tuxianeer 7:46:21 pm
maximum 990 (actually less) mimimum 10 so less than 980

chenjamin 7:46:21 pm
Possible sums from 10-945 so 936

MellowMelon 7:46:23 pm
The pigeonholes are the sums of the numbers in the subsets. The smallest possible sum is 10. The largest possible sum is 90 + 91 + 92 + ... + 99 = 945. Any number in between is possible.

MellowMelon 7:46:42 pm
I also like Tuxianeer's answer. We're not on the AIME here; we can just get a rough bound less than 1023 and we'll be fine.

MellowMelon 7:46:55 pm
How do we finish off the problem?

david_sun 7:47:58 pm
so there are 936 holes, and 936<1023

mathcool2009 7:47:58 pm
there are more pigeons than holes!!

vincenthuang75025 7:47:58 pm
we no have enough space, 936<1022

RocketSingh 7:47:58 pm
1023 > 936

Rogman 7:47:58 pm
Since there are more pigeons than holes, there must be at least one pair of subsets with matching sums!

MellowMelon 7:48:01 pm
Now, the answer seems evident: We have 1023 subsets, and 936 possible sums, so by the Pigeonhole Principle, there must be two subsets with the same sum.

MellowMelon 7:48:04 pm
QED?

zhuangzhuang 7:48:42 pm
What if the sets are not disjoint?

Tuxianeer 7:48:42 pm
no they might not be disjoint

brian22 7:48:42 pm
NO

djmathman 7:48:42 pm
Well kind of; we still need to prove that the sets are disjoint.

mathcool2009 7:48:42 pm
no, the two subsets could overlap

mathlover3737 7:48:42 pm
disjoint subsets..?

minimario 7:48:42 pm
No

chenjamin 7:48:42 pm
subsets must be disjoint

ssk9208 7:48:42 pm
No. "disjoint"

MellowMelon 7:48:43 pm
No! These two subsets may not be disjoint. We need a pair of disjoint subsets! What can we do if the two subsets are not disjoint?

Tuxianeer 7:49:13 pm
if the are not disjoint remove the intersection from both to get disjoint subsets with the same sum

sujaykazi 7:49:13 pm
Remove the common terms.

RocketSingh 7:49:13 pm
Remove common terms

mathcool2009 7:49:13 pm
subtract their overlap!

zhuangzhuang 7:49:13 pm
If the sets have any common elements, take out the common elements from both.

Arithmophobia 7:49:13 pm
subtract the common elements

djmathman 7:49:13 pm
Subtract their common elements!

chenjamin 7:49:13 pm
delete the joint terms from both subsets

blippy1998 7:49:13 pm
just remove the intersection?

brian22 7:49:13 pm
remove the common element from both sets?

MellowMelon 7:49:15 pm
Fortunately, this is easy to deal with. We remove any common elements from both subsets. Removing common elements from both subsets clearly decreases the sums of the elements by the same amount, and we can't end up with empty subsets (because the subsets we started with were different), so we end up with two disjoint subsets with the same sum.

MellowMelon 7:49:30 pm
Now we can write QED!

RocketSingh 7:49:37 pm
Yay we're done

mathcool2009 7:49:37 pm
yayyy QED

MellowMelon 7:49:43 pm
Eight players participate in a tennis tournament. Every pair of players plays exactly one game, and there are no ties. Prove that there are four players A_1 , A_2 , A_3 , and A_4 such that A_i won against A_j for $1 \le i < j \le 4$.

MellowMelon 7:50:07 pm
"Prove that there are..." Pigeonhole principle!

MellowMelon 7:50:27 pm
... except I'm not sure what the pigeons or holes are. Any ideas about that? Or ideas in general?

MSTang 7:51:19 pm
There are 28 games 😛

mathlover3737 7:51:19 pm
(8 c 2) = 28 total games

MellowMelon 7:51:20 pm
A total of $\binom{8}{2} = 28$ games were played in total. This might be useful. What else?

MellowMelon 7:51:35 pm
Let's try a constructive approach -- we'll try to pick the four players, and we'll see what obstacles we encounter as we go.

MellowMelon 7:51:46 pm
How should we get started with that?

MSTang 7:52:36 pm
label the players

AopsKevin 7:52:36 pm
start with A1

MellowMelon 7:52:40 pm
We have to start somewhere, so let's look for a candidate for A_1, the player who wins against A_2, A_3, and A_4 in our chain of four players.

MellowMelon 7:52:46 pm
If we wanted to pick one player as A_1, what might we look for?

Michelangelo 7:53:41 pm
Assume A1 is the player with the most wins?

mathlover3737 7:53:41 pm
the player who won the most games

Michelangelo 7:53:41 pm
the person with the most wins

soy_un_chemisto 7:53:41 pm
a player that wins against at least 3 other players

Rogman 7:53:41 pm
the one with the most wins

joshxiong 7:53:41 pm
Person with the maximum amount of wins

RocketSingh 7:53:41 pm
The Biggest Winner

MellowMelon 7:53:44 pm
We might look for someone who won a lot of games. So let's pick A_1 to be the player who won the most games.

MellowMelon 7:54:17 pm
(There were many suggestions about picking players who won at least three times. I went with the above suggestion because, for one thing, we know there is a player that won the most games. We are not yet sure whether there are players who won 3 times...)

MellowMelon 7:54:26 pm
What do we know about A_1?

MellowMelon 7:55:02 pm
Can we determine how many games she must have won?

mathlover3737 7:55:57 pm
won at least 4 games

sujaykazi 7:55:57 pm
A1 won at least 4 games, because there are 28 wins and only 8 players.

RocketSingh 7:55:57 pm
We wins against at least 4 people

Tuxianeer 7:55:57 pm
won at least 4 games by pigeonhole

zhuangzhuang 7:55:57 pm
At least 4

brian22 7:55:57 pm
at least 4

MellowMelon 7:56:08 pm
Ah, we use pigeonhole and our earlier result about how many games there were.

MellowMelon 7:56:17 pm
A total of $\binom{8}{2} = 28$ games were played in total. Since $28 = 8 \cdot 3 + 4$, then by the Pigeonhole Principle, one of the eight players won at least four games.

MellowMelon 7:56:31 pm
An alternate way to derive this result:

tareyza 7:56:33 pm
each player plays 7 games, and for every win there is a loss, so somehow had to have won 4+ games

MellowMelon 7:57:01 pm
Each player plays seven games. In all, there were an equal number of wins and losses, so someone won at least half their games.

MellowMelon 7:57:15 pm
Regardless of how we find it, we know that A_1 won at least four games. Suppose A_1 won against players $B, C, D,$ and $E.$ These will be our candidates for the other A 's.

MellowMelon 7:57:24 pm
What can we do with players B, C, D, and E?

Michelangelo 7:58:34 pm
consider them a separate set. that is, make another "tournament" with just them

SuperSnivy 7:58:34 pm
find the one with the most wins out of B,C,D,E

sujaykazi 7:58:34 pm
Look at the 6 matches among those 4 players.

brian22 7:58:34 pm
pit them against each other and see who within that subset won the most games

RocketSingh 7:58:34 pm
A_2 is the biggest winner out of these guys.

MellowMelon 7:58:42 pm
As with the set of all eight players, we can look at who among B, C, D, and E won the most games. But only considering the games against each other.

MellowMelon 7:59:03 pm
Remember, A_1 needs to beat A_2, A_3, A_4 , so we need our other three players from this set only.

MellowMelon 7:59:27 pm
If we take the one with the most wins in our mini-tournament between B, C, D and E , can we get a bound on how many games they won?

tareyza 7:59:59 pm
we know that among the four of them, one beat at least 2 others

Tuxianeer 7:59:59 pm
show one won 2 games by pigeonhole, with 6 games and 4 people

MSTang 7:59:59 pm
Someone in the set $\{B, C, D, E\}$ won at least twice

mathlover3737 7:59:59 pm
at least 2

Rogman 7:59:59 pm
2-3

brian22 7:59:59 pm
at least 2

sujaykazi 7:59:59 pm
at least 2

SuperSnivy 7:59:59 pm
at least 2

chenjamin 7:59:59 pm
at least 2

Michelangelo 7:59:59 pm
6 total games and 4 people so 2 wins at least

RocketSingh 7:59:59 pm
Someone must have won 2 games by PHP

MellowMelon 8:00:00 pm
The four players B, C, D, and E played a total of $\binom{4}{2} = 6$ games against each other. Since $6 = 1 \cdot 4 + 2$, then by the Pigeonhole Principle, one of B, C, D, and E won at least two games against each other.

MellowMelon 8:00:19 pm
Or, alternately, each player played 3 games, and someone won at least half their games. That person has at least two wins.

MellowMelon 8:00:30 pm
Okay, now that we've shown this, what do we do with it?

mathlover3737 8:01:03 pm
let them be A_2

MSTang 8:01:03 pm
Let that person be A_2 and continue on

Michelangelo 8:01:03 pm
take the remaining people and repeat the process

RocketSingh 8:01:03 pm
Look at the 2 losers

brian22 8:01:03 pm
same thing we did last time let's look at the two people he beat

ssk9208 8:01:03 pm
Take the player who won two games as A_2

MellowMelon 8:01:05 pm
We can set A_2 to be the player who won at least two games, because A_2 only needs to win against players A_3 and A_4.

MellowMelon 8:01:11 pm
Next?

sujaykazi 8:01:44 pm
Just take the two players that the second guy beat.

zhuangzhuang 8:01:44 pm
Then, take the people A2 beat, choose the one that won the match with the other as A3.

mathlover3737 8:01:44 pm
The two players that A_2 won against should be A_3 and A_4, where A_3 beat A_4

brian22 8:01:44 pm
put A3 against A4

Tuxianeer 8:01:44 pm
now, of the two people A_2 beat, one beat the other, and we are done

Michelangelo 8:01:44 pm
1 game so someone won a game and we are done

tareyza 8:01:44 pm
among A_3 and A_4, one beat the other

zhuangzhuang 8:01:44 pm
Look at the winner of those remaing 2 people.

chenjamin 8:01:44 pm
Look at the match between the two losers

Rogman 8:01:44 pm
make a "tournament" between A_3 and A_4

RocketSingh 8:01:44 pm
We know between A_3 and A_4 someone won

mathcool2009 8:01:44 pm
let A_3 and A_4 be the losers to A_2, and A_3 beat A_4

djmathman 8:01:44 pm
Consider the result of the match between the two people A_2 beat to determine which one should be A_3 and which one A_4 .

MellowMelon 8:01:47 pm
Take the two players that A_2 won against. They played against each other, so let A_3 be the player who won and let A_4 be the player who lost.

MellowMelon 8:01:56 pm
This gives us our chain of four players: A_1, A_2, A_3, and A_4.

Rogman 8:02:04 pm
QED!

MellowMelon 8:02:08 pm
And the problem is solved. :)

MellowMelon 8:02:24 pm
We had to apply the pigeonhole principle several times here. Sometimes it's not just a matter of "here's the pigeons, here's the holes, and QED".

MellowMelon 8:02:35 pm
Next up:

MellowMelon 8:02:38 pm
Suppose that 1985 points are given inside a unit cube. Show that one can always choose 32 of them in such a way that every (possibly degenerate) closed polygon with these points as vertices has total length less than $8\sqrt{3}$.

MellowMelon 8:03:16 pm
I'm getting some reactions of fright here. Don't be intimidated now...

MellowMelon 8:03:20 pm
Let's look at the big picture. What's going to be the overall strategy for using the Pigeonhole Principle here?

brian22 8:03:58 pm
total length= perimeter?

MellowMelon 8:04:08 pm
Ah, yes, total length means the perimeter.

MellowMelon 8:05:17 pm
We want to be able to ensure that 32 of the points will be in some "hole", and that the hole is small enough to ensure that the condition about the closed polygon will hold.

MellowMelon 8:05:28 pm
It's still not clear what the holes are, so let's look at the condition and try to figure out what the hole should be.

MellowMelon 8:05:43 pm
If we have up to 32 points forming a polygon, what's a sufficient condition for the perimeter to be less than $8\sqrt{3}$?

mathlover3737 8:06:34 pm
8sqrt3 / 32 = (sqrt 3)/4

mathlover3737 8:06:34 pm
The distance between any two points is less than (sqrt 3) / 4

Michelangelo 8:06:34 pm
each side is less than sqrt{3}/4

MellowMelon 8:06:41 pm
If each side is less that $\frac{1}{4}\sqrt{3}$, we'll be set, because then the perimeter is at most $32\left(\frac{1}{4}\sqrt{3}\right) = 8\sqrt{3}$. (This is a version of the Pigeonhole Principle "in reverse".)

MellowMelon 8:07:05 pm
Getting a lot of questions about this, so let me address it:

MSTang 8:07:13 pm
How can we know the polygon is 2D?

Rogman 8:07:13 pm
but how do we ensure 32 of the 1985 are even coplanar?

MellowMelon 8:07:39 pm
For now, let's just pretend we can have polygons that aren't coplanar. This condition is too complicated to worry about to start with. ☹

MellowMelon 8:08:01 pm
If we find we're struggling to get our bounds to work, then we can try to work it in.

MellowMelon 8:08:18 pm
Back to above: the holes need to be small enough so that no two points are more than $\frac{1}{4}\sqrt{3}$ **apart. Any ideas on how to guarantee that?**

tareyza 8:09:21 pm
(1/4)sqrt(3) is one fourth the length of the longest diagonal of the cube, suggesting to separate the cube into 4^3 regions?

Tuxianeer 8:09:21 pm
make the holes cubes with side length 1/4

mathlover3737 8:09:21 pm
space diagonal of a unit cube has length sqrt 3

sujaykazi 8:09:21 pm
Divide the cube into 64 0.25*0.25*0.25 cubes.

SuperSnivy 8:09:21 pm
cut the cube into 64 smaller cubes

mathcool2009 8:09:21 pm
cut the unit cube into 64 tiny cubes!

Rogman 8:09:21 pm
divide the cube into cubes of side length 1/4

MellowMelon 8:09:30 pm
If a cube has side length $\frac{1}{4}$ **, then any two points are at least as close together as the two opposite vertices at either end of a space diagonal, which are** $\frac{1}{4}\sqrt{3}$ **apart.**

MellowMelon 8:09:46 pm
The $\sqrt{3}$ **should remind us of the distance between two opposite vertices of a unit cube, particularly since a unit cube is given in the problem. That's the motivation here.**

MellowMelon 8:10:06 pm
So we'd like our holes to be small cubes with side length $\frac{1}{4}$ **, and we can divide the unit cube into** $4^3 = 64$ **cubes of side length** $\frac{1}{4}$ **. Then what?**

mathlover3737 8:11:14 pm
By the Pigeonhole Principle, there must be at least 32 points in one of the small cubes.

djmathman 8:11:14 pm
Use the points as pigeons.

MellowMelon 8:11:24 pm
Right, it's time for the pigeonhole principle. We just need to see how to apply it...

Tuxianeer 8:12:05 pm
make the holes cubes with side length 1/4: works since 31*64=1984<1985 (woah they cut this one close)

MSTang 8:12:05 pm
1985 = 31 * 64 + 1, so close!

david_sun 8:12:05 pm
64*31<1985, so there must be one cube with at least 32 points

chenjamin 8:12:05 pm
1985/64 > 31

sujaykazi 8:12:05 pm
There are 64 holes, and 1985 pidgeons. 1985/64=31 with remainder 1.

ssk9208 8:12:05 pm
$64 \cdot 31 = 1984$

joshxiong 8:12:05 pm
Since $1985 = 64 * 31 + 1$, we can conclude that there are at least 32 points in one of the smaller cubes by PHP.

bengals 8:12:05 pm
1985 divided by 64

Michelangelo 8:12:05 pm
that's about 31.02 i think...

MellowMelon 8:12:16 pm
Note that $1985 = 31 \cdot 64 + 1$. Hence, by the Pigeonhole Principle, one of these small cubes must contain at least 32 of the 1985 chosen points.

MellowMelon 8:12:30 pm
QED?

tareyza 8:13:26 pm

Michelangelo 8:13:26 pm
yes...?

MellowMelon 8:13:30 pm
Yes, basically.

MSTang 8:13:33 pm
equality case...?

MellowMelon 8:13:54 pm
I guess we do need that strict inequality, but if we are careful to specify that none of the smaller unit cubes contain their whole boundary, we will have it.

RocketSingh 8:14:01 pm
Coplanar

Raninf 8:14:01 pm
what about that coplanar problem?

MellowMelon 8:14:17 pm
We dealt with it by proving a stronger problem. Convex polygons in three dimensions? Sure, why not?

MellowMelon 8:14:30 pm
We solved this problem by working backwards. We started with the clues in the problem, then found what the pigeonholes should be, to set up the Pigeonhole Principle argument. Whether we start with the pigeonholes or end with them, we should choose whichever direction makes the problem easier.

MellowMelon 8:14:51 pm
Let $u_1, u_2, u_3, \ldots$ be a sequence of integers satisfying the recurrence relation

$$u_{n+2} = u_{n+1}^2 - u_n.$$

Suppose $u_1 = 39$ and $u_2 = 45$. Prove that 1986 divides infinitely many terms of the sequence.

MellowMelon 8:15:19 pm
What can we try here?

RocketSingh 8:15:57 pm
lets experiment a bit

Tuxianeer 8:15:57 pm
trying out terms

Rogman 8:15:57 pm
plugging in some numbers

soy_un_chemisto 8:15:57 pm
see if there is a pattern mod 1986

MellowMelon 8:15:59 pm
First, we can try computing the first few terms of the sequence.

MellowMelon 8:16:14 pm
We get $u_3 = u_2^2 - u_1 = 45^2 - 39 = 2025 - 39 = 1986.$

MellowMelon 8:16:22 pm
... Whoa, bullseye!

MellowMelon 8:16:31 pm
How about u_4 ?

Tuxianeer 8:17:15 pm
-45 mod 1986

MSTang 8:17:15 pm
1986^2 - 45

Tuxianeer 8:17:15 pm
$-45 \pmod 1)986$

mathlover3737 8:17:15 pm
-45 (mod 1986)

zhuangzhuang 8:17:15 pm
big number

soy_un_chemisto 8:17:15 pm
$-45 \pmod{1986}$

PikachuKiller9001 8:17:15 pm
1986^2-45

MellowMelon 8:17:16 pm
Then $u_4 = (1986)^2 - 45 =$ a big number. The numbers grow very quickly from here. We can find that it's -45 mod 1986, but the residues only get worse as we go on....

MellowMelon 8:17:26 pm
If we felt like computing, we could get the sequence

MellowMelon 8:17:29 pm
39, 45, 0, 1941, 39, 1566, 1593, 1947, 1914, 1251, 105, 1830, 399, ...

Rogman 8:17:46 pm
oh gosh

MellowMelon 8:17:48 pm
at which point we start to realize we are probably wasting our time and should start thinking more generally.

MellowMelon 8:18:10 pm
But $u_3 = 1986$ ... that wasn't an accident, right? Can we take advantage of that?

Tuxianeer 8:18:40 pm
but what if the pattern shows up after one more?

MellowMelon 8:18:48 pm
Advice for you: never go to a casino or play the lottery. 🙂

MellowMelon 8:19:52 pm
Any ideas about the problem? We have a term divisible by 1986. What about the sequence would allow us to find more such terms?

RocketSingh 8:21:19 pm
try to prove it is periodical

soy_un_chemisto 8:21:19 pm
it somehow becoming cyclic mod 1986

MellowMelon 8:21:22 pm
If we could show that the sequence was periodic after reducing modulo 1986 (and that the period includes the third term $u_3 = 1986$), then it would follow that infinitely many terms are divisible by 1986.

MellowMelon 8:21:33 pm
How can we show that the sequence of residues is periodic?

chenjamin 8:22:29 pm
Time to use Pidgeonhole

Rogman 8:22:29 pm
pigeonhole?

MellowMelon 8:22:44 pm
This sounds like something we can pigeonhole...

MellowMelon 8:23:34 pm
We might try making the pigeons the terms and the holes residues mod 1986, but that doesn't quite work. Notice we computed that both a_1 and a_5 were 39 mod 1986, but it wasn't periodic after that!

MellowMelon 8:23:36 pm
How can we save this?

MellowMelon 8:24:43 pm
Each term of the sequence depends only on the previous two terms -- how does that help?

soy_un_chemisto 8:25:01 pm
given a sufficiently large consecutive set of terms of the sequence, there are 2 pairs of consecutive terms that are equal

Tuxianeer 8:25:01 pm
pairs of terms, since the recurrence uses the previous 2 terms

RocketSingh 8:25:01 pm
Look at sets of 2 consec terms

ssk9208 8:25:01 pm
Take consecutive pairs mod 1986?

zhuangzhuang 8:25:01 pm
look at ordered pairs of 2 numbers modulos

brian22 8:25:01 pm
pairs

MellowMelon 8:25:11 pm
If any pair of consecutive residues mod 1986 appears more than once in the sequence, then all terms after those two terms must be equal too.

MellowMelon 8:25:28 pm
That is, if $u_a \equiv u_b$ and $u_{a+1} \equiv u_{b+1}$ (both modulo 1986), then $u_{a+2} \equiv u_{b+2}$ because of the recurrence relation, and then $u_{a+3} \equiv u_{b+3}$, and so on. So the sequence becomes periodic (and the period divides $b-a$).

MellowMelon 8:25:57 pm
Okay, so that gives us a nice criterion to try to nail with an application of pigeonhole. How do we do it?

Tuxianeer 8:27:20 pm
there are only 1986^2 possible pairs of terms

RocketSingh 8:27:20 pm
Consider 1986^2 +1 pairs

soy_un_chemisto 8:27:20 pm
we have 1986^2 possible consecutive sequences mod 1986 , and an infinite amount of things to choose from

Bg1 8:27:20 pm
we have infinite pigeons and only limited holes of mod pairs

ssk9208 8:27:20 pm
There are 1986^2 possible pairs, so we take the first 1986^2+1 numbers as the pigeons and the possible pairs as the holes.

brian22 8:27:20 pm
INFINITELY MANY PIGEONS!!! PIGEONS THAT MULTIPLY LIKE FIBBONACCI'S RABBITS!!

MellowMelon 8:27:23 pm
It's the "infinite" Pigeonhole Principle. There are only 1986^2 possible pairs modulo 1986, but we have infinitely many pairs in our sequence!

MellowMelon 8:27:50 pm
Whenever we have to place infinitely many pigeons into finitely many holes, we must have some hole with more than one pigeon. Or rather, we must have one hole with infinitely many pigeons! (My mind just exploded...)

MellowMelon 8:28:07 pm
So the sequence is periodic mod 1986, by applying pigeonhole like this.

MellowMelon 8:28:10 pm
QED?

brian22 8:28:28 pm
once again, you asked, so NO

MellowMelon 8:28:30 pm
But it was yes last time!

MellowMelon 8:29:04 pm
You should actually think about it. 😛 You have to do that on a real contest you know...

brian22 8:29:10 pm
NO

MSTang 8:29:10 pm
No!

mathlover3737 8:29:10 pm
it might not include a term == 0 mod 1986

Tuxianeer 8:29:10 pm
no 1986 might not be in the periodic part

RocketSingh 8:29:10 pm
no, the square does not work backwards right

sujaykazi 8:29:10 pm
But what if there are no other instances of the mod equaling zero after u(3)?

joshxiong 8:29:10 pm
No; we must show that a term that is divisible by 1986 is included within the sequence.

MellowMelon 8:29:38 pm
No, because showing that the sequence (u_i) of residues modulo 1986 is *eventually* periodic is not enough. We also need to show that the term $u_3 = 1986$ also lies in the periodic bit. The sequence (u_i) may not become periodic until long after this third term.

MellowMelon 8:29:51 pm
How can we show that the period includes the term u_3?

ssk9208 8:31:26 pm
We take the equal pairs and work backwards!

MellowMelon 8:31:36 pm
We can try to trace the period backwards.

MellowMelon 8:31:50 pm
That is, if we have a pair of consecutive elements that are equal, maybe we can see if we can deduce what the term before was. How could we do that?

soy_un_chemisto 8:33:08 pm
just use the definition of the sequence

brian22 8:33:08 pm
going backwards is the same as going forwards right?

Tuxianeer 8:33:08 pm
$u_n = u_{n+1}^2 - u_{n+2}$: same in reverse

MellowMelon 8:33:11 pm
We can write down

$$u_n = u_{n+1}^2 - u_{n+2}.$$

That's actually the same recurrence we had before! So we could apply the recurrence in either direction, actually.

MellowMelon 8:33:26 pm
Does that show u_3 is in the period?

Tuxianeer 8:34:07 pm
yes

cerberus88 8:34:07 pm
yes

mathlover3737 8:34:07 pm
yes - we can keep going backwards until u_3

soy_un_chemisto 8:34:07 pm
yes

zhuangzhuang 8:34:07 pm
Yes. It works in both directions, now.

MellowMelon 8:34:11 pm
Indeed, all the terms of the sequence are part of the period. We conclude that infinitely many terms of the sequence are divisible by 1986.

ssk9208 8:34:32 pm
$YAY!!!$

brian22 8:34:32 pm
QED

mathlover3737 8:34:35 pm
Part b: how long is the period? 🙂

MellowMelon 8:34:43 pm
I have this one ready. The next appearance of the residues 39 and 45 is $u_{1339} \equiv 39$, $u_{1340} \equiv 45 \pmod{1986}$.

MellowMelon 8:34:54 pm
So that's why I stopped computing out terms early on.

MellowMelon 8:40:11 pm
And now back to problems!

MellowMelon 8:40:30 pm
Let x be a real number, and let $n \geq 2$ be a positive integer. Prove that one of the numbers $x, 2x, \ldots, (n-1)x$ differs from an integer by at most $\frac{1}{n}$.

MellowMelon 8:40:40 pm
What tells us to consider the Pigeonhole Principle?

cerberus88 8:41:12 pm
one of the numbers

chenjamin 8:41:12 pm
Prove that one...

sujaykazi 8:41:12 pm
"one of the numbers"

RocketSingh 8:41:12 pm
prove that one of ...

MSTang 8:41:12 pm
one of the numbers

Tuxianeer 8:41:13 pm
existence

MellowMelon 8:41:14 pm
Again, those words "Prove that one of the numbers...". Pigeonhole is a good way to show existence.

MellowMelon 8:41:22 pm
To apply the Pigeonhole Principle, we should think about what the pigeons and holes are. What might the pigeons be?

MSTang 8:42:22 pm
the numbers?

Rogman 8:42:22 pm
the terms of the sequence

joshxiong 8:42:22 pm
The set of numbers

Tuxianeer 8:42:22 pm
the numbers x,2x,..

cerberus88 8:42:25 pm
x, 2x, ..., (n-1)x

MellowMelon 8:42:27 pm
The pigeons are easy: the $n-1$ numbers $x, 2x, \ldots, (n-1)x$.

MellowMelon 8:42:30 pm
What are the holes?

MellowMelon 8:43:37 pm
So far, all the suggestions for holes I've gotten have somewhere between an infinite number of holes and n holes... can't really work with that when we have $n-1$ pigeons.

MellowMelon 8:43:52 pm
Moreover, it's not yet clear what having two numbers in the same "hole" does for us. So much for "the pigeons are easy".

MellowMelon 8:44:15 pm
But I do like some of the suggestions for the holes anyway:

sujaykazi 8:44:27 pm
The spaces between the integers that are more than 1/n away from an integer.

PikachuKiller9001 8:44:27 pm
1/n difference

mathlover3737 8:44:27 pm
(0 through 1/n), (1/n through 2/n), ... (n-1/n through 1)

AopsKevin 8:44:27 pm
holes are differences to 1/n

MellowMelon 8:44:48 pm
It feels like we should be looking at the n pieces of an interval, each with block $1/n$.

MellowMelon 8:45:15 pm
How can we get an idea like this to work? In the first place, our numbers might be scattered all over the real line...

Tuxianeer 8:46:28 pm
fractional part: 1/n,2/n,...

RocketSingh 8:46:28 pm
fractional part

joshxiong 8:46:28 pm
If we consider fractional part of each of the numbers, the holes are 0 to 1/n, 1/n to 2/n, etc.

tRIG 8:46:28 pm
non integer part and which a/n to (a+1)/n it falls in

blippy1998 8:46:28 pm
could we try this such that all the numbers are between 0 and 1?

Raninf 8:46:28 pm
take just the fractional part?

MellowMelon 8:46:38 pm
The only relevant parts of the numbers are their fractional parts. Recall that the fractional part of a number $\{x\}$ is defined as

$$\{x\} = x - \lfloor x \rfloor,$$

where $\lfloor x \rfloor$ is the floor function. (For example, the fractional part of 23.57 is 0.57, and the fractional part of -1.8 is 0.2.)

MellowMelon 8:46:57 pm
When does a number differ from an integer by at most $\frac{1}{n}$, in terms of the fractional part?

joshxiong 8:48:15 pm
If it is in the first or last hole.

mikechen 8:48:15 pm
less than 1/n or more than (n-1)/n

RocketSingh 8:48:15 pm
frac part<1/n or > 1 - 1/n

sjewu 8:48:15 pm
{x} ≤ 1/n or {x} ≥ n-1/n

Raninf 8:48:15 pm
when the fractional part is between 0 and 1/n or (n-1)/n and 1

MellowMelon 8:48:18 pm
A number differs from an integer by at most $\frac{1}{n}$ when the fractional part is less than or equal to $\frac{1}{n}$, or it is greater than or equal to $\frac{n-1}{n}$.

MellowMelon 8:48:32 pm
So if we're going to proceed by contradiction, and we assume that none of our numbers work, what do we know about their fractional parts?

Tuxianeer 8:49:19 pm
(1/n,(n-1)/n) - interval

joshxiong 8:49:19 pm
There are (n-2) holes left.

Rogman 8:49:19 pm
between those two fractions

ssk9208 8:49:19 pm
none of them lies in the first or the last hole

nilaisarda 8:49:19 pm
they are between 1/n and n-1/n

cerberus88 8:49:19 pm
between 1/n and (n-1)/n

MellowMelon 8:49:23 pm
The fractional parts of our $n-1$ numbers all lie in the interval $\left(\frac{1}{n}, \frac{n-1}{n}\right)$.

MellowMelon 8:49:42 pm
Now if we combine this with our earlier splitting of the interval, it looks like the holes can be the $n-2$ intervals:

$$\left(\frac{1}{n}, \frac{2}{n}\right), \left[\frac{2}{n}, \frac{3}{n}\right), \ldots, \left[\frac{n-2}{n}, \frac{n-1}{n}\right).$$

MellowMelon 8:49:56 pm
(Being careful about open/closed here will save us some pain later.)

MellowMelon 8:50:03 pm
What's the natural way to make use of these holes?

cerberus88 8:51:00 pm
put birds in there

ssk9208 8:51:00 pm
And one interval contains at least two fractional parts

brian22 8:51:00 pm
BUT there are n-1 terms! so pigeonhole

bengals 8:51:00 pm
Compare the number of holes to the pigeons

MSTang 8:51:00 pm
Pigeonhole!

joshxiong 8:51:05 pm
There exists a hole with at least two pigeons.

MellowMelon 8:51:06 pm
We make the pigeons the the fractional parts of the $n-1$ numbers. By the Pigeonhole Principle, two of the fractional parts, say $\{ix\}$ and $\{jx\}$ (for $i < j$) lie in the same interval of length $\frac{1}{n}$.

MellowMelon 8:51:21 pm
That's not quite what the problem asked us for though. How do we proceed?

zhuangzhuang 8:52:41 pm
Take the difference between the two numbers in the same interval, if there exist any.

Michelangelo 8:52:41 pm
find what are the pigeons and apply pigeonhole to see there must be 2 in one of them and subtract those two

RocketSingh 8:52:41 pm
Subtract them

MSTang 8:52:41 pm
Take their differences

Michelangelo 8:52:41 pm
subtract the two

ssk9208 8:52:41 pm
And the diferrence of them is also one of those multiples of x!

MellowMelon 8:52:44 pm
We can look at $\{jx\} - \{ix\}$. That's going to have a fractional part that's in the region we want. But it's not a term in our sequence...

zhuangzhuang 8:53:53 pm
Take {|j-i|x}.

Tuxianeer 8:53:53 pm
then take the j-ith term.

nilaisarda 8:53:53 pm
Take jx - ix, this is less than 1/n

sujaykazi 8:53:53 pm
Since {ix} and {jx} lie in the same interval, that means {jx - ix} will be within 1/n of an integer.

MSTang 8:53:53 pm
Well, $\{jx - ix\}$

joshxiong 8:53:53 pm
Yes it is! (j-i) is positive and less than n, so (j-i)x is one of the number in our sequence.

MellowMelon 8:54:04 pm
Ah, so we might think that $\{(j-i)x\}$ is related to $\{jx\}$ and $\{ix\}$.

MellowMelon 8:54:16 pm
At first, we might think that they are equal and we are done, but this is not necessarily the case. For example, if $x = 0.3$, $i = 2$, and $j = 5$, then

$$\{jx\} - \{ix\} = \{1.5\} - \{0.6\} = 0.5 - 0.6 = -0.1$$

and

$$\{(j-i)x\} = \{0.9\} = 0.9.$$

MellowMelon 8:54:27 pm
What is true instead?

Tuxianeer 8:55:17 pm
their fractional parts are equal

MSTang 8:55:17 pm
their difference is an integer

mathlover3737 8:55:17 pm
fractional part of that?

soy_un_chemisto 8:55:17 pm
if you can think mod 1, it works

zhuangzhuang 8:55:17 pm
Take fract parts

MellowMelon 8:55:26 pm
It is certainly true that $\{(j-i)x\}$ and $\{jx\} - \{ix\}$ have the same fractional part.

MellowMelon 8:55:56 pm
One of the easiest ways to prove this would just be to recall that $jx, ix, (j-i)x$ were all originally terms in our sequence. (Which many of you have been hollering at me for awhile...)

MellowMelon 8:56:05 pm
I'll leave the details for your own time.

MellowMelon 8:56:33 pm
The end result of this fact is that the fractional part of $\{(j-i)x\}$ is also within $\frac{1}{n}$ of an integer. And therefore?

ssk9208 8:57:07 pm
$YAY?$

Michelangelo 8:57:07 pm
QED

nilaisarda 8:57:07 pm
QED! Hooray!

mathlover3737 8:57:07 pm
at least one of the numbers in the sequence is within 1/n of an integer

MSTang 8:57:07 pm
QED?

zhuangzhuang 8:57:07 pm
We have a term that has the desired property. QED.

MellowMelon 8:57:08 pm
But $(j-i)x$ is one of our numbers, so that's our number! One of the numbers must be within $\frac{1}{n}$ of an integer, and we're done.

MellowMelon 8:57:21 pm
(This was a little weird: we assumed that what we wanted to show wasn't true, trying to get a contradiction, but the contradiction that we got was that what we wanted to show was true anyway. Contradiction arguments don't often work out that way.)

MellowMelon 8:58:01 pm
(Usually when this happens, a way to write it up is to just use cases. Here, we might have said "If there are no numbers with fractional parts in [0,1/n) or (n-1/n, 1), then we..." and then the complicated stuff.)

MellowMelon 8:58:08 pm
Here's a question: Can we do better than $\frac{1}{n}$? In other words, can we replace $\frac{1}{n}$ by a smaller number, so that the problem is still true?

zhuangzhuang 8:58:42 pm
No, take n=2 and x=0.5.

mathlover3737 8:58:42 pm
Probably not - if n=2, 1/2 is the best you can do

MellowMelon 8:58:46 pm
Can't do better for $n = 2$ ...

djmathman 8:59:00 pm
No, just take $x = \frac{1}{n}$.

zhuangzhuang 8:59:00 pm
Take {x}=1/n.

sjewu 8:59:00 pm
1/n, 2/n, ..., n-1/n

MellowMelon 8:59:07 pm
In general, this is the best we can do. If we set $x = \frac{1}{n}$, then the $n-1$ numbers $x, 2x,$ ..., $(n-1)x$ become

$$\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n},$$

so we can't do better than $\frac{1}{n}$.

MellowMelon 8:59:26 pm
On to the next one...

MellowMelon 8:59:28 pm
Prove that there exist integers a , b , c , not all zero and each of absolute value less than one million, such that

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$

Tuxianeer 9:00:19 pm
"exist"

brian22 9:00:19 pm
start with this: Prove that there exsist = pigeonhole

MellowMelon 9:00:21 pm
There's that word "exists" though... so the Pigeonhole Principle might come to mind. Though it seems like a bizarre problem to use it on.

MellowMelon 9:00:31 pm
Any guess what the pigeons are?

brian22 9:01:35 pm
all possible numbers for a b and c

MSTang 9:01:35 pm
possible values of $a + b\sqrt{2} + c\sqrt{3}$?

MellowMelon 9:01:41 pm
It seems like the pigeons are the numbers of the form

$$a + b\sqrt{2} + c\sqrt{3},$$

where $|a|, |b|, |c| < 10^6$.

MellowMelon 9:02:11 pm
But if we proved two of these pigeons were in the same hole... that might not do so much for us. I can't think of a set of holes where that would help.

Tuxianeer 9:02:48 pm
you could subtract the two

MellowMelon 9:02:55 pm
Right, based on our last problem, we might do subtraction.

MellowMelon 9:03:02 pm
Will we get a valid number if we do that?

MSTang 9:04:00 pm
maybe not

djmathman 9:04:00 pm
Oh, right, not necessarily, since it might violate the absolute value less than one million condition.

MellowMelon 9:04:02 pm
Not necessarily: for example $600000\sqrt{2}-(-700000\sqrt{2})=1300000\sqrt{2}$ is a difference of two of our numbers that is not one of our valid numbers.

MellowMelon 9:04:25 pm
So if we got two of the above pigeons in the same hole, we might have nothing to work with.

MellowMelon 9:04:39 pm
Any ideas for how to modify the pigeons to fix this?

nilaisarda 9:05:52 pm
Well, what if we took a,b,c to have absolute value ¡ 500000?

Rogman 9:05:52 pm
restrict the original ones to less than 500000

MSTang 9:05:52 pm
Prove it for $|a|,|b|,|c|<\dfrac{10^6}{2}$?

zhuangzhuang 9:05:52 pm
For an initial argument only use half the numbers and if there is a fail fall back on the other half of the numbers

mikechen 9:05:52 pm
a,b,c<=500000

Raninf 9:05:52 pm
halve the range so that |a|, |b|, |c| < (10^6)/2 ?

MellowMelon 9:06:35 pm
We can try to only consider the numbers of the form

$$a+b\sqrt{2}+c\sqrt{3},$$

where $|a|,|b|,|c|<10^6/2$. The difference of any two of these numbers will be a valid number.

MellowMelon 9:07:18 pm
Alternately, we need only consider the numbers of the form

$$a+b\sqrt{2}+c\sqrt{3},$$

where $0\le a,b,c<10^6$. (That is, numbers with nonnegative coefficients.) This is the same set as above, just translated a bit. I'm going to work with this set to make the computations a bit easier.

MellowMelon 9:07:32 pm
Does everyone understand why we can use the above numbers as the pigeons?

MellowMelon 9:08:00 pm
How many such numbers are there?

djmathman 9:09:37 pm
10^{18} since no two of the $a+b\sqrt{2}+c\sqrt{3}$ can be the same

nilaisarda 9:09:37 pm
10^18

Rogman 9:09:37 pm
1000000 for each

Tuxianeer 9:09:37 pm
10^18

MSTang 9:09:37 pm
10^{18}

Rogman 9:09:37 pm
10^18

MellowMelon 9:09:39 pm
There are $(10^6)^3=10^{18}$ such numbers, and these will form the pigeons. (I am including 0, since taking a difference will ensure we don't get all zeroes in the end. But the logic probably works for $10^{18}-1$ too.)

MellowMelon 9:10:17 pm
When we're taking differences between these numbers, we need to figure out what kind of range they lie in.

MellowMelon 9:10:29 pm
Can we get some bounds on all 10^{18} of our numbers?

djmathman 9:11:46 pm
Well, they all do have to be less than or equal to $(10^6-1)(1+\sqrt{2}+sqrt3)$.

MSTang 9:11:46 pm
All numbers x satisfy $|x| \le 10^6 \cdot (1+\sqrt{2}+\sqrt{3})$

mathlover3737 9:11:46 pm
0 <= every number <=10^6(1+sqrt2+sqrt3)

RocketSingh 9:11:46 pm
10^6(1 + sqrt2 + sqrt3) is the biggest 0 is the smallest

MellowMelon 9:11:52 pm
The minimum is clearly 0. The maximum is

$$\begin{aligned} d &= (10^6-1)+(10^6-1)\sqrt{2}+(10^6-1)\sqrt{3} \\ &= (10^6-1)(1+\sqrt{2}+\sqrt{3}), \end{aligned}$$

so all 10^{18} numbers of this form lie in the interval $[0,d]$.

MellowMelon 9:12:04 pm
What's next?

brian22 9:12:28 pm
holes

vincenthuang75025 9:12:28 pm
holes

MellowMelon 9:12:29 pm
Let's define some holes. What would be good holes?

djmathman 9:13:20 pm
Intervals of length 10^{-11} ?

Tuxianeer 9:13:20 pm
fractional part /10^11

brian22 9:13:20 pm
width 10^-11

RocketSingh 9:13:20 pm
10^-11 slots

zhuangzhuang 9:13:20 pm
The succesive increments of 10^-11 in length up to d, with d<9*10^6

vincenthuang75025 9:13:20 pm
Intervals of 10^{-11} ?

MellowMelon 9:13:30 pm
We can use intervals of width 10^{-11} as our holes.

MellowMelon 9:13:42 pm
But how many holes do we need to make sure all of our pigeons will have to fit in a hole we've made for it?

RocketSingh 9:14:55 pm
10^18 - 1

Michelangelo 9:14:55 pm
less than 10^18

djmathman 9:14:55 pm
at most $10^{18}-1$

Tuxianeer 9:14:55 pm
10^18-1

MellowMelon 9:14:59 pm
If we can show there are strictly less than 10^{18} holes, then our pigeonhole principle application will work smoothly. What inequality do we need to prove to show this is the case?

djmathman 9:16:42 pm
Uhh... $10^{11}(10^6-1)(1+\sqrt{2}+\sqrt{3}) \le 10^{18}-1$?

Michelangelo 9:16:42 pm
(10^6-1)(1+sqrt(2)+sqrt(3)) / (10^-11)<10^18

MellowMelon 9:16:46 pm
We need to prove that $\frac{d}{10^{18}-1} < 10^{-11}$.

MellowMelon 9:16:53 pm
Let's substitute in d **, so that what we want to show is that**

$$\frac{(10^6-1)(1+\sqrt{2}+\sqrt{3})}{10^{18}-1} < 10^{-11}.$$

Tuxianeer 9:17:12 pm
the factors cancel

MSTang 9:17:12 pm
Factor difference of cubes

MellowMelon 9:17:18 pm
We can certainly notice factorizations...

Michelangelo 9:17:43 pm
can't we just make approximations since the numbers are so large?

MellowMelon 9:17:48 pm
but really, the inequality is so weak that we may as well just make gross approximations and still have room to spare. Remember that $1+\sqrt{2}+\sqrt{3}$ **is way less than 10.**

MellowMelon 9:18:04 pm
We can get the bound formally in a lot of ways (since it is so loose). Here is one way:

$$\frac{(10^6-1)(1+\sqrt{2}+\sqrt{3})}{10^{18}-1} < \frac{5(10^6-1)}{10^{18}-1} = \frac{5}{10^{12}+10^6+1} < \frac{5}{10^{12}} < \frac{10}{10^{12}} = 10^{-11}.$$

MellowMelon 9:18:53 pm
So the inequality is true. How does that finish the problem?

MSTang 9:19:50 pm
Take two in the same hole and subtract, QED

RocketSingh 9:19:50 pm
We can use pigeon hole

Tuxianeer 9:19:50 pm
pigeonhole works

mathlover3737 9:19:50 pm
just subtract the two in the same hole and we're done

Rogman 9:19:50 pm
by pigeonhole

tRIG 9:19:50 pm
there must be two number that have fractional part when subtracted < 10^-11 and their subtraction is in the set as well

djmathman 9:19:50 pm
We can find two sets of positive integers (a,b,c) such that their values of $a+b\sqrt{2}+c\sqrt{3}$ differ by no more than 10^{-11} , so then just use the fact that the a,b,c can be negative to QED.

MellowMelon 9:19:52 pm
By the Pigeonhole Principle, two of the 10^{18} **numbers of the form**

$$a+b\sqrt{2}+c\sqrt{3},$$

where $0 \le a,b,c < 10^6$ **lie in the same subinterval, and each subinterval is less than** 10^{-11} **across. Thus their difference, in absolute value, is less than** 10^{-11} **. This difference gives us the desired number.**

MellowMelon 9:20:11 pm
(Also, I have no idea what the number is.)

MellowMelon 9:20:50 pm
I have 10 minutes left. I have no chance of finishing this next problem in 10 minutes, but we'll see where we get.

MellowMelon 9:20:58 pm
I won't go too far over because we might be here for a very long time if I finish it...

MellowMelon 9:21:06 pm
We are given a sequence of n^2+1 **distinct real numbers. Show that there exists a subsequence containing** $n+1$ **terms that is increasing or decreasing.**

MellowMelon 9:21:21 pm
By the way, such a subsequence is called monotonic.

MellowMelon 9:21:23 pm
Does everyone understand what the problem is asking?

Tuxianeer 9:22:28 pm
is a subsequence consecutive terms or not?

MellowMelon 9:22:32 pm
No, not necessarily consecutive.

MellowMelon 9:22:47 pm
To get a feel for the problem and clarify things, let's start with an example, say for n = 3.

MellowMelon 9:22:56 pm
Consider the sequence
8, 2, 5, 3, 9, 7, 10, 4, 6, 1
of 10 terms. Can you find a subsequence (preserving order) containing 4 terms that is increasing or decreasing?

mathlover3737 9:23:23 pm
2, 3, 7, 10

djmathman 9:23:23 pm
2,3,7,10

MSTang 9:23:23 pm
8, 5, 3, 1

SuperSnivy 9:23:23 pm
8, 5, 3, 1

Tuxianeer 9:23:23 pm
2,3,4,6

tRIG 9:23:23 pm
2 5 7 10

MellowMelon 9:23:25 pm
There are many examples, like 2, 3, 4, 6 (which is increasing) or 8, 5, 3, 1 (which is decreasing).

CyclicRain 9:23:35 pm
so you have to preserve order?

MellowMelon 9:23:41 pm
Yes. 1, 2, 3, 4 is not valid.

MellowMelon 9:24:24 pm
Also, we might explore why n^2+1 terms are necessary. Can anyone come up with a counterexample for n^2 terms?

mathlover3737 9:24:45 pm
n=1 🙂

MellowMelon 9:24:50 pm
True. :) How about for some higher n ?

Arithmophobia 9:25:41 pm
2143

Tuxianeer 9:25:41 pm
2,4,1,3

MSTang 9:25:41 pm
n = 2, the sequence is 3 - 1 - 4 - 2

Tuxianeer 9:25:41 pm
n=2: 2,4,1,3

sujaykazi 9:25:41 pm
n=2 3,4,1,2

MellowMelon 9:25:48 pm
Here's several counterexamples for $n = 2$.

MellowMelon 9:25:57 pm
For $n = 3$, we could take 7, 8, 9, 4, 5, 6, 1, 2, 3.

MellowMelon 9:26:18 pm
In general, you can follow some patterns here to come up with counterexamples for any n .

MellowMelon 9:26:34 pm
(Exploring counterexamples is a very important part of mathematical reasoning, and it can sometimes guide us when we're finding proofs of the actual problem.)

tRIG 9:26:40 pm
is it a coincidence that if we have n holes, we gurantee n+1 pigeons in a hole?

NT2048 9:26:40 pm
it seems like we should use n holes for n^2+1 pigeons...

MellowMelon 9:26:53 pm
So as for actually solving the problem, pigeonhole seems obvious. n holes, n^2+1 pigeons...

MellowMelon 9:27:02 pm
I think I know what you want to make the pigeons. Can you tell me what the holes are?

brian22 9:27:21 pm
the n+1 terms

MellowMelon 9:27:28 pm
... which we don't know exist yet. ☹

Rogman 9:28:24 pm
all subsequences of length n+1

NT2048 9:28:24 pm
perhaps something like (1, 2, ... n), (n+1, n+2 ... 2n), ect?

Michelangelo 9:28:24 pm
groups of n consecutive terms?

MellowMelon 9:28:33 pm
These are all viable ideas, and they get the n holes bound we want.

MellowMelon 9:28:49 pm
The real issue I'm seeing here... we need a subsequence that is increasing or decreasing.

MellowMelon 9:28:58 pm
We don't even know which one we want yet!

Rogman 9:29:18 pm
it seems that either always works...

MellowMelon 9:29:37 pm
Can anyone find an example where we get an increasing subsequence of $n+1$ terms but neither a decreasing one of $n+1$ terms nor an increasing one of $n+2$ terms?

MellowMelon 9:30:44 pm
In the interest of time, I'll throw one out, a modification of one given above: 7, 8, 9, 4, 5, 6, 1, 2, 3, 10.

Tuxianeer 9:30:53 pm
4,5,1,2,3

MellowMelon 9:30:57 pm
Here's one for $n=2$ as well.

MellowMelon 9:31:06 pm
So as you can see, we need to pick carefully. Do we want increasing, or decreasing?

djmathman 9:31:34 pm
Let's WLOG and say increasing

MellowMelon 9:31:40 pm
How can you WLOG this though?

MSTang 9:32:21 pm
You can flip the sequence around to get from increasing to decreasing

zhuangzhuang 9:32:21 pm
We can consider 2 at once, it and its reflection

djmathman 9:32:21 pm
Well, we can take the set of numbers and reverse the order to WLOG

nilaisarda 9:32:21 pm
take the inverse of all the number, that's a new sequence with increasing/decreasing flipped?

NT2048 9:32:21 pm
you can invert the sequence to change increasing to decreasing

RocketSingh 9:32:21 pm
if you flip the signs

tRIG 9:32:21 pm
if its decreasing, flip the sequence around

Davidcao 9:32:21 pm
You could reverse the same sequences to get decreasing

MellowMelon 9:32:23 pm
Ah, that's true. Though considering both simultaneously could get complicated...

MellowMelon 9:33:42 pm
I will cut the class time here. Feel free to discuss the problem further in this room, although don't post it on the message board until the end of the week...

MellowMelon 9:33:52 pm
SUMMARY

MellowMelon 9:33:55 pm
The Pigeonhole Principle is one of the important techniques for proving that certain objects must exist. Almost every existential problem involving finite sets (and some infinite sets) involves the Pigeonhole Principle. It is simple to state, but it can be used to prove some very deep results.

MellowMelon 9:34:06 pm
When applying the Pigeonhole Principle, be clear what the pigeons and what the holes are. (And sometimes, you may have to apply it more than once!).

MellowMelon 9:34:23 pm
See you in two weeks, and good luck if you keep working on the last problem!