Pigeonhole principle

Problem 1. *Let $n \geq 3$ be odd and let $a_1, a_2, \ldots, a_n$ be a rearrangement of the integers $1, 2, \ldots, n$. Prove that $(a_1 - 1)(a_2 - 2) \ldots (a_n - n)$ is even.*

Problem 2. *Let A be any set of 19 distinct integers chosen from the arithmetic progression 1,4,7,10,...,100. Prove that there must be two distinct integers in A whose sum is 104.*

Problem 3. *Let $A = \{500, 501, 502, ..., 550\}$. Prove that we cannot partition A in 5 subsets with the sum of elements of each subset strictly less than 5555.*

Problem 4. *Let S be a square of side 2, and choose 9 points inside S. Show that 3 of these points may be chosen which are the vertices of a triangle of area $\leq 1/2$.*

Problem 5. *Let S be a square of side 1 and 5 points in the interior of S. Show that there are 2 points among these 5 points, such that the distance between them is less than $\sqrt{2}/2$.*

Problem 6. *Nineteen points are chosen inside a regular hexagon whose sides have length 1. Prove that two of these points may be chosen whose distance apart is at most $1/\sqrt{3}$.*

Problem 7. *For a class with two or more students, show that at least two students have the same number of friends. Assume that you cannot be your own friend. Also assume that if I am your friend, then you are my friend (and vice versa).*

Problem 8. *Mr. and Mrs. Smith went to a party attended by 15 other couples. Various handshakes took place during the party. In the end Mrs, Smith asked each person at the party how many handshakes did they have. To her surprise, each person gave a different answer. How many hand shakes did Mr. Smith have? (Here we assume that no person shakes hand with his/her spouse and of course with itself).*

Problem 9. *Given a positive integer n, show that there exists a positive integer containing only the digits 0 and 1 in its decimal notation, and which is divisible by n. If n is not divisible by 2 or 5, then there is a multiple of n containing only the digit 1 in its decimal notation.*

Problem 10. *Given a set of $n+1$ positive integers, none of which exceeds $2n$, show that at least one member of the set must divide another member of the set.*

Problem 11. *Let S be a set of seven distinct positive integers which are less than or equal to 24. Prove that S has two subsets whose sums are equal.*

Problem 12. *Let n be a positive integer and x a positive real number. Show there exist integers a, b such that $1 \leq a \leq n-1$ and $|ax - b| \leq \dfrac{1}{n}$.*

Problem 13. *Prove there exist integers a, b, c, not all 0 and each of absolute value less than 10^6 such that $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$.*

Problem 14. *Each square of a 3×7 chessboard is colored either black or white. Show that there is a rectangle with sides parallel to the edges of the board all of whose corners are squares of the same color. Does the statement hold for a 3×6 chessboard?*