FUNCTIONAL EQUATIONS

2001 Winter Camp + 2003

Functional equations appear frequently on IMO contests. Here are two problems that highlight some of the common techniques that you can use to solve hard functional equations.

this represents a function that takes rational numbers to rational numbers.

1. Find all functions $f: \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2001$ and $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.

It is not difficult to see that $f(x) = 2001x$ is a solution that satisfies the given conditions. Let us prove that this is the only solution.

Let $y = 1$. Then $f(x+1) = f(x) + f(1) = f(x) + 2001$ for all x. Since $f(1) = 2001$, we have $f(2) = f(1+1) = f(1) + f(1) = 2001 \cdot 2$, and by induction we can easily prove that $f(n) = 2001n$ for all integers $n \geq 1$.

Let $x = y = 0$. Then $f(0) = 2f(0) \Rightarrow \underline{f(0) = 0}$, and letting $y = -x$, we get $0 = f(0) = f(x + (-x)) = f(x) + f(-x) \Rightarrow f(-x) = -f(x)$ for all $x \in \mathbb{Q}$. Thus, we have $f(-1) = -2001$, $f(-2) = -4002$, etc. Hence, we have shown that $f(n) = 2001n$ for <u>all integers n</u>. But we want to prove that $f(x) = 2001x$ for all <u>rational</u> x.

Let's prove that $f(x) = 2001x$ for all rational numbers of the form $\frac{1}{b}$ (where $b \in \mathbb{N}$), and then we will extend it to all rational numbers.

We have $f(\frac{2}{b}) = f(\frac{1}{b}) + f(\frac{1}{b})$, $f(\frac{3}{b}) = f(\frac{2}{b}) + f(\frac{1}{b}) = f(\frac{1}{b}) + f(\frac{1}{b}) + f(\frac{1}{b})$, and continuing this, we see that $2001 = f(1) = f(\frac{b}{b}) = \underbrace{f(\frac{1}{b}) + f(\frac{1}{b}) + \cdots + f(\frac{1}{b})}_{b \text{ times}} = b \cdot f(\frac{1}{b})$, so $f(\frac{1}{b}) = 2001 \cdot (\frac{1}{b})$, as required.

Let $\frac{a}{b}$ be a positive rational number (so $a, b > 0$). From above, $f(\frac{1}{b}) = \frac{2001}{b}$, and so $f(\frac{a}{b}) = \underbrace{f(\frac{1}{b}) + f(\frac{1}{b}) + \cdots + f(\frac{1}{b})}_{a \text{ times}} = a \cdot f(\frac{1}{b}) = a \cdot \frac{2001}{b} = 2001 \cdot \frac{a}{b}$. So the claim is true for all positive rational numbers. Finally, using the fact that $f(-x) = -f(x)$, we conclude that $f(x) = 2001x$ for all negative rational numbers as well.

Therefore we have proven that the only function satisfying the given conditions is $\boxed{f(x) = 2001x}$, $x \in \mathbb{Q}$.

As an aside: say we changed $f: \mathbb{Q} \to \mathbb{Q}$ to $f: \mathbb{R} \to \mathbb{R}$. Would the unique solution still be $f(x) = 2001x$? How about if we changed $f: \mathbb{Q} \to \mathbb{Q}$ to $f: \mathbb{R} \to \mathbb{R}$ but specified that the function must be <u>continuous</u>? What would happen then?

Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(x^2+f(y)) = y + (f(x))^2$, for all $x, y \in \mathbb{R}$. (1992 IMO, Question #2).

Let $f(0)=t$. Then letting $x=0$, we have $f(f(y)) = y+(f(0))^2 = y+t^2$, for all $y \in \mathbb{R}$. —①

Let $f(p)=q$. Then $f(q) = f(f(p)) = p+t^2$.

From ①, we have $f(f(p^2+f(q))) = [p^2+f(q)]+t^2$. —②

Let's evaluate $f(f(p^2+f(q)))$ another way. By substituting $x=p$ and $y=q$ into our original functional equation, we get $f(p^2+f(q)) = q+(f(p))^2 = q+q^2$.

So $f(f(p^2+f(q))) = f(q+q^2) = f(q^2+f(p)) = p+(f(q))^2 = p+(p+t^2)^2$.

Hence, from ②, we get $p+(p+t^2)^2 = [p^2+f(q)]+t^2 \Rightarrow$ $p+p^2+2pt^2+t^4 = p^2+p+t^2+t^2 \Rightarrow t^2[t^2+2p-1]=0$, and this holds for all $p \in \mathbb{R}$. Thus, we must have $t^2=0$, i.e. $t=0$. Therefore, $\underline{f(0)=0}$.

So now we have established that $f(0)=0$. So $f(p)=q$ implies that $f(q) = p+0^2 = p$. Here are two different proofs that $f(x)=x$:

<u>Solution 1:</u> From our functional equation, we have $f(x^2+f(y)) = y+(f(x))^2 \geq y$, for all $x, y \in \mathbb{R}$ —③. We shall show that $f(x)=x$, for all $x \in \mathbb{R}$.

Suppose $f(p)=q$ and $q>p$. Let $r=q-p>0$. Since $f(p)=q$, we have $f(q)=p$.

Then $p = f(q) = f((q-p)+p) = f((\sqrt{r})^2+p) = f((\sqrt{r})^2+f(q)) \geq q$ by ③.

So we have $p \geq q$, contradicting $q>p$. By symmetry we can show that the case $q<p$ leads to a contradiction too. So we must have $\boxed{f(x)=x}$ for all $x \in \mathbb{R}$.

<u>Solution 2:</u> Let $x, y \in \mathbb{R}$. Let $x=z^2$ for some z, and $f(y)=w$ for some w. Then $f(w)=y$, and $f(x)=f(z^2)=(f(z))^2$ which we get from substituting $y=0$ and $x=z$ into our original functional equation. Then $f(x+y) = f(z^2+f(w)) = w+(f(z))^2 = f(y)+f(x)$. Letting $y=-x$, we get $f(0)=f(-x)+f(x) \Rightarrow f(-x)=-f(x)$ for all $x \in \mathbb{R}$. Thus, we get $\underline{f(x-y)=f(x)-f(y)}$, for all real x and y. Using this, we show that $f(x)$ must equal x for all $x \in \mathbb{R}$.

Suppose $f(p)=q$ and $q>p$. Let $r=q-p>0$. Then $f(r)=f(q-p)=f(q)-f(p)=p-q<0$. So $f(r)<0$ with $r>0$. Let $r=t^2$ for some t. Then $f(r)=f(t^2)=(f(t))^2 \geq 0$, and this contradicts $f(r)<0$. Similarly, if $q<p$, we let $r=p-q$ and establish the same contradiction. Therefore, we must have $\boxed{f(x)=x}$ for all $x \in \mathbb{R}$.