Self-made Problems In Number Theory

1. Find all positive integers n there are positive integers $a_1, a_2, \ldots, a_k$ strictly less than n and pairwisely distinct satisfying

$$n \left| \left(\sum_{i=1}^{k} a_i + \prod_{i=1}^{k} \right)^2 - 1 \right.$$

for some positive integer $1 < k < n - 1$.

Solution. Let S_n be the set of positive integers less than or equal to n. Then S_n has $\varphi(n)$ elements, where $\varphi(n)$ is the Euler function. We prove that these elements satisfy the given property for $k = \varphi(n)$. Say,

$$S_n = \{a_1, a_2, ..., a_{k-1}, a_k\}$$

$$N = \left(\sum_{i=1}^{k} a_i + \prod_{i=1}^{k} \right)^2 - 1$$

with $a_1 < a_2 < ... < a_{k-1} < a_k$. For $n > 1$, every $a_i \in S_n$ is strictly less than n. Since otherwise $\gcd(n, n) = n > 1$ would hold. F

Lemma 1.

$$n \left| \sum_{i=1}^{k} a_i \right.$$

Proof. From Euclidean algorithm, if $\gcd(a, n) = 1$, then

$$\gcd(n, n - a) = \gcd(a, n) = 1$$

Therefore, $a_i + a_{k-i} = n$, and $\varphi(n)$ is even. So, $n | a_i + a_{k-i}$ for $0 < i < \frac{k}{2}$. This gives

$$n \left| \sum_{i=1}^{k} a_i \right.$$

□

Lemma 2. *Let P_A be the product of elements of set A. Then,*

$$P_{S_n}^2 \equiv 1 \pmod{n}$$

Proof. Let $a \in S_n$. Then, all ai are distinct modulo n, otherwise we would have

$$ai \equiv aj \pmod{n}$$

implying

$$n | a(i - j)$$

with $\gcd(n, a) = 1$ and $|i - j| < n$. Take any $a \in S_n$. Then, for any $a_i \in S_n$, there is a unique j such that

$$a_i a_j \equiv a \pmod p$$

i.e. two of them pair up for a. Running them over S_n, we have

$$a_1 a_2 \cdots a_{k-1} a_k \equiv a^{\frac{k}{2}} \pmod n$$

Squaring, we have

$$P_{S_n}^2 \equiv \left(a^{\frac{k}{2}}\right)^2 \equiv a^k \equiv 1 \pmod n$$

□

Finally,

$$\begin{aligned} N &= \left(\sum_{i=1}^{k} a_i + \prod_{i=1}^{k}\right)^2 - 1 \\ &\equiv \left(\prod_{i=1}^{k}\right)^2 - 1 \pmod n \\ &\equiv 0 \pmod n \end{aligned}$$

So, every $n > 1$ satisfies the property.

We assume the following notations.

- *s.t.* is the short form of *such that*
- qr is the short form of *quadratic residue*.
- $a|b$ means b is divisible by a.
- $(a, b) = \gcd(a, b)$ is the greatest common divisor of a and b.
- $lcm(a, b) = [a, b]$ is the least common multiple of a and b.
- $a \perp b$ denotes $(a, b) = 1$ or a and b are co-prime.
- $\tau(n)$ is the number of divisors of n.
- $\sigma(n)$ is the sum of divisors of n.
- $\mathbb{P}$ denotes the set of primes.
- $p^\alpha || n$ or $\nu_p(n) = \alpha$ means α is the greatest positive integer such that $p^\alpha | n$. In other words, $p^\alpha | n$ and $p^{\alpha+1} \not| n$.
- $\omega(n)$ is the number of distinct prime factors of n.
- $\varphi(n)$ is the number of positive integers less than or equal to n and co-prime to n.
- $ord_m(a) = x$ denotes x is the order of $a \pmod m$ i.e. x is the smallest positive integer s.t. $a^x \equiv 1 \pmod m$.
- $\pi(n)$ is the number of primes less or equal to n.

1. Problems

2. Find all $(m, n) \in \mathbb{N}^2$ s.t. $n^2 + 3m^2$ and $n + 3m$ both are perfect cubes.

3. Let c_n be the smallest positive integer s.t. $1 < c_n < n$ and $c_n \perp n$. Prove that c_n exists except for some finite n.

4. Find all n such that $\varphi(n)|n$.

5. Prove that for a prime $p > 2$, the set of complete residue class $\pmod p$ can be divided into two subsets of equal number of elements with sum of each group divisible by p.

6. A number having only one prime factor can't be a perfect number.

7. $\sigma(64n^2)$ is odd for any n.

8. Find all odd n s.t. $2013|F_n$.

9. Take any $2n$ integers[1] where $n > 2$. Consider all pair-wise differences we can possibly have from them. Let's denote the product of these $\binom{2n}{2}$ differences by S. Prove that S is divisible by

$$2^{n^2-n} \cdot (2n-1)(2n-3)(2n-5)$$

10. p is a prime of the form $3k+2$. Prove that, there exists a set of $p-1$ elements which forms a complete set of residue class $\pmod p$ and sum of elements is divisible by p^2.

11. If n is odd then $\tau(F_n) \geq \tau(n)$.

12. For all even n, $\tau(F_{2n}) \geq \tau(n)$.

13. If p is a prime and x is a positive integer s.t. $p > x^2 - x + 1$ then

$$\omega\left((x+1)^p - (x^p+1)\right) \geq 4$$

14. For any prime $p > 2$, there are two positive integers u, v s.t. uv^{-1} is a qr of p with $u, v < \dfrac{p}{2}$.

15. The number of numbers less than or equal to n having odd sum of divisors is

$$\lfloor \sqrt{n} \rfloor + \left\lfloor \sqrt{\frac{n}{2}} \right\rfloor$$

16. Find all sequence of positive integers $\{a_i\}_{i=0}^{\infty}$ s.t.

$$[a_i, a_{i+1}] = (a_{i+1}, a_{i+2})$$

17. Find all primes p and $(a, b) \in \mathbb{N}^2$ s.t. $a^p + b^p$ is a perfect power of a prime.

18. Let $a \in \mathbb{N}$. Then, $a^{a-1} - 1$ is never square-free[2].

19. Show that $\forall n, 81|10^{n+1} - 9n - 10$.

[1] positive or negative whatever, including 0

[2] a is square-free if it has no square factor i.e. there is no x s.t. $x^2|a$.

20. Let p be a prime. Find all perfect numbers having p factors exactly.

21. Find all $(a, b) \in \mathbb{N}^2$ s.t. $7^a + 11^b$ is a perfect square.

22. Let $m, n, a_1, a_2, \ldots, a_n$ be positive integers s.t. $\forall i, a_i + m$ is a prime. Let

$$N = \prod_{i=1}^{n} p_i^{a_i}$$

and S be the number of ways to write N as a product of m positive integers. Calculate the remainder of S upon division by m^n.

23. Solve in positive integers:

$$\sum_{i=1}^{8} n_i^{10} = 19488391$$

24. Find all $n \in \mathbb{N}$ s.t. $n|2^{n!} - 1$.

25. Show that there exists an infinite pairs $(a, b) \in \mathbb{N}^2$ s.t. $\dfrac{a^k + b^k}{a^k b^k + 1}$ is a perfect k^{th} power.

26. Solve in positive integers

$$a^n + b^n = (kac)^{mn}$$

27. Let a, b are positive integers s.t. $a \perp b$ and $p \in \mathbb{P}$ s.t. $p|x^6 + 64$. Find all pairs (a, b) s.t.

$$2013|\frac{a^2 + b^2}{p}$$

28. Find all integers $n > 2$ there are positive integers $a_1, a_2, \ldots, a_k$ less than or equal to n and pairwisely distinct s.t.

$$n|\left(\sum_{i=1}^{k} a_i + \prod_{i=1}^{k}\right)^2 - 1$$

29. Find all n such that the sum of number of divisors of divisors of n is n.

30. Say a, n, d are positive integers where $a + id \in \mathbb{P}$ for $i = 0$ to $n - 1$. Define $f(n) = 1$ if $n = a$, $f(n) = 1$ otherwise. Let $\pi(n)$ be the number of primes strictly less than n. Show that,

$$T = 2^{\frac{d}{2}} + 1$$

has at least $2^{2^{\pi(n)-1}-1-f(n)}$.

31. Let's define General Fibonacci Number[3] as

$$G_n = \begin{cases} a \text{ if } n = 0 \\ b \text{ if } n = 1 \\ G_{n-1} + G_{n-2} \text{ if } n > 1 \end{cases}$$

Prove that $\left|G_{n+1}G_{n-1} - G_n^2\right|$ is independent of n.

[3] We shall maintain this notation through this whole note.

32. Determine true or false: G_{2n+1} has no prime factor of the form $4n+3$ for an infinite a, b. Alternatively, $\exists x, y \in \mathbb{N} : G_n = x^2 + y^2$.

33. Define $k(n)$ as:

$$k(n) = \sum_{d|n, d+1 \in \mathbb{P}} 1$$

and $C(n)$ is the number of positive integers x so that $x|a^n - a$ for all a. Prove that $C(n) \geq 2^{k(n)}$.

34. Let G be a group with $ord(G) = n$. Find all G with n elements where n is a Carmichael number[4].

35. $p \in \mathbb{P}, p > 5$.

$$X_n = \sum_{x_1+\ldots+x_n=p} \prod_{i=1}^{n} \binom{p}{x_i}$$

Find all k s.t.

$$p^3 \mid \sum_{i=1}^{k} X_i$$

36. Find all $n \in \mathbb{N}$ s.t.

$$\tau(n) = \varphi(n)$$

37. Find all n s.t. $\tau(n) = \pi(n)$.

38. Find all n s.t. $\varphi(n) = \pi(n)$.

39. Find all n that satisfies $\varphi(n) - 1|n - 2$.

40. Define c_n as the smallest positive integer s.t. $1 < c_n < n - 1$ and $\gcd(c_n, n) = 1$.

(a) Find all n s.t. c_n does not exist

(b) Prove that there are an infinite n s.t. $ord_n(c_n) = \varphi(n)$.

41. *Prove that, if $p(n)$ is the greatest odd divisor of n then $\pi(n) \geq \dfrac{n}{p(n)}$.

42. *Define

$$n!! = n(n-2)\cdots$$

Using the notation defined above, prove that,

$$\prod_{p \leq n} p > (2\left(\frac{n}{p(n)}\right) - 1)!!$$

[4] i.e. n is composite and $n|a^n - a$ for all a.