Inequalities

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Most analysts spend half their time hunting through the literature for inequalities they want to use, but cannot prove.

- H. Bohr

Overture

Inequalities are useful in all fields of Mathematics. The aim of this *problem-oriented* book is to present elementary techniques in the theory of inequalities. Our target readers are challenging high schools students and undergraduate students. We will meet fundamental theorems including Schur's Inequality, Muirhead's Theorem, Hölder's Theorem, Jensen's Inequality, The Cauchy-Schwarz Inequality, The AM-GM-HM Inequality, The Power Mean Inequality, and The Hardy-Littlewood-Pólya Inequality. The given techniques or heuristics in this book are just the tip of the inequalities iceberg. It simply means that young students should find creative methods to attack the problems and build up their own heuristics. We would greatly appreciate hearing about comments and corrections from our readers. **Have fun!**

Acknowledgement

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IMO CODE (from http://www.imo-official.org)

AFG	Afghanistan	ALB	Albania	ALG	Algeria
ARG	Argentina	ARM	Armenia	AUS	Australia
AUT	Austria	AZE	Azerbaijan	BAH	Bahrain
BGD	Bangladesh	BLR	Belarus	BEL	Belgium
BEN	Benin	BOL	Bolivia	BIH	BIH
BRA	Brazil	BRU	Brunei	BGR	Bulgaria
KHM	Cambodia	CMR	Cameroon	CAN	Canada
CHI	Chile	CHN	CHN	COL	Colombia
CIS	CIS	CRI	Costa Rica	HRV	Croatia
CUB	Cuba	CYP	Cyprus	CZE	Czech Republic
CZS	Czechoslovakia	DEN	Denmark	DOM	Dominican Republic
ECU	Ecuador	EST	Estonia	FIN	Finland
FRA	France	GEO	Georgia	GDR	GDR
GER	Germany	HEL	Greece	GTM	Guatemala
HND	Honduras	HKG	Hong Kong	HUN	Hungary
ISL	Iceland	IND	India	IDN	Indonesia
IRN	Islamic Republic of Iran	IRL	Ireland	ISR	Israel
ITA	Italy	JPN	Japan	KAZ	Kazakhstan
PRK	PRK	KOR	Republic of Korea	KWT	Kuwait
KGZ	Kyrgyzstan	LVA	Latvia	LIE	Liechtenstein
LTU	Lithuania	LUX	Luxembourg	MAC	Macau
MKD	MKD	MAS	Malaysia	MLT	Malta
MRT	Mauritania	MEX	Mexico	MDA	Republic of Moldova
MNG	Mongolia	MNE	Montenegro	MAR	Morocco
MOZ	Mozambique	NLD	Netherlands	NZL	New Zealand
NIC	Nicaragua	NGA	Nigeria	NOR	Norway
PAK	Pakistan	PAN	Panama	PAR	Paraguay
PER	Peru	PHI	Philippines	POL	Poland
POR	Portugal	PRI	Puerto Rico	ROU	Romania
RUS	Russian Federation	SLV	El Salvador	SAU	Saudi Arabia
SEN	Senegal	SRB	Serbia	SCG	Serbia and Montenegro
SGP	Singapore	SVK	Slovakia	SVN	Slovenia
SAF	South Africa	ESP	Spain	LKA	Sri Lanka
SWE	Sweden	SUI	Switzerland	SYR	Syria
TWN	Taiwan	TJK	Tajikistan	THA	Thailand
TTO	Trinidad and Tobago	TUN	Tunisia	TUR	Turkey
NCY	NCY	TKM	Turkmenistan	UKR	Ukraine
UAE	United Arab Emirates	UNK	United Kingdom	USA	United States of America
URY	Uruguay	USS	USS	UZB	Uzbekistan
VEN	Venezuela	VNM	Vietnam	YUG	Yugoslavia

IMO CODE (continued)

BIH	Bosnia and Herzegovina
CHN	People's Republic of China
CIS	Commonwealth of Independent States
FRG	Federal Republic of Germany
GDR	German Democratic Republic
MKD	The Former Yugoslav Republic of Macedonia
NCY	Turkish Republic of Northern Cyprus
PRK	Democratic People's Republic of Korea
USS	Union of the Soviet socialist republics

Contents

Chapter 1

Symmetry

Each problem that I solved became a rule, which served afterwards to solve other problems.

- R. Descartes

1.1 Exploiting Symmetry

We begin with the following example.

Example 1. *Let a, b, c be positive real numbers. Prove the inequality*

$$\frac{a^4+b^4}{a+b}+\frac{b^4+c^4}{b+c}+\frac{c^4+a^4}{c+a} \geq a^3+b^3+c^3.$$

First Solution. After brute-force computation, i.e, clearing denominators, we reach

$$a^5b+a^5c+b^5c+b^5a+c^5a+c^5b \geq a^3b^2c+a^3bc^2+b^3c^2a+b^3ca^2+c^3a^2b+c^3ab^2.$$

Now, we deduce

$$\begin{aligned}
& a^5b+a^5c+b^5c+b^5a+c^5a+c^5b \\
= \ & a\left(b^5+c^5\right)+b\left(c^5+a^5\right)+c\left(a^5+b^5\right) \\
\geq \ & a\left(b^3c^2+b^2c^3\right)+b\left(c^3a^2+c^2b^3\right)+c\left(c^3a^2+c^2b^3\right) \\
= \ & a^3b^2c+a^3bc^2+b^3c^2a+b^3ca^2+c^3a^2b+c^3ab^2.
\end{aligned}$$

Here, we used the the auxiliary inequality

$$x^5+y^5 \geq x^3y^2+x^2y^3,$$

where $x, y \geq 0$. Indeed, we obtain the equality

$$x^5+y^5-x^3y^2-x^2y^3=\left(x^3-y^3\right)\left(x^2-y^2\right).$$

It is clear that the final term $\left(x^3-y^3\right)\left(x^2-y^2\right)$ is always non-negative. □

Here goes a more economical solution without the brute-force computation.

Second Solution. The trick is to observe that the right hand side admits a nice decomposition:

$$a^3 + b^3 + c^3 = \frac{a^3+b^3}{2} + \frac{b^3+c^3}{2} + \frac{c^3+a^3}{2}.$$

We then see that the inequality has the *symmetric* face:

$$\frac{a^4+b^4}{a+b} + \frac{b^4+c^4}{b+c} + \frac{c^4+a^4}{c+a} \geq \frac{a^3+b^3}{2} + \frac{b^3+c^3}{2} + \frac{c^3+a^3}{2}.$$

Now, the symmetry of this expression gives the *right* approach. We check that, for $x, y > 0$,

$$\frac{x^4+y^4}{x+y} \geq \frac{x^3+y^3}{2}.$$

However, we obtain the identity

$$2\left(x^4+y^4\right) - \left(x^3+y^3\right)(x+y) = x^4 + y^4 - x^3y - xy^3 = \left(x^3-y^3\right)(x-y).$$

It is clear that the final term $\left(x^3-y^3\right)(x-y)$ is always non-negative. □

Delta 1. [LL 1967 POL] *Prove that, for all $a, b, c > 0$,*

$$\frac{a^8+b^8+c^8}{a^3b^3c^3} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Delta 2. [LL 1970 AUT] *Prove that, for all $a, b, c > 0$,*

$$\frac{a+b+c}{2} \geq \frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b}$$

Delta 3. [SL 1995 UKR] *Let n be an integer, $n \geq 3$. Let $a_1, \cdots, a_n$ be real numbers such that $2 \leq a_i \leq 3$ for $i = 1, \cdots, n$. If $s = a_1 + \cdots + a_n$, prove that*

$$\frac{{a_1}^2+{a_2}^2-{a_3}^2}{{a_1}^2+{a_2}^2+{a_3}^2} + \frac{{a_2}^2+{a_3}^2-{a_4}^2}{{a_2}^2+{a_3}^2+{a_4}^2} + \cdots + \frac{{a_n}^2+{a_1}^2-{a_2}^2}{{a_n}^2+{a_1}^2+{a_2}^2} \leq 2s - 2n.$$

Delta 4. [SL 2006] *Let $a_1, \cdots, a_n$ be positive real numbers. Prove the inequality*

$$\frac{n}{2(a_1+a_2+\cdots+a_n)} \sum_{1\leq i<j\leq n} a_i a_j \geq \sum_{1\leq i<j\leq n} \frac{a_i a_j}{a_i+a_j}$$

Epsilon 1. *Let a, b, c be positive real numbers. Prove the inequality*

$$\left(1+a^2\right)\left(1+b^2\right)\left(1+c^2\right) \geq (a+b)(b+c)(c+a).$$

Show that the equality holds if and only if $(a, b, c) = (1, 1, 1)$.

Epsilon 2. (Poland 2006) Let a, b, c be positive real numbers with $ab + bc + ca = abc$. Prove that

$$\frac{a^4+b^4}{ab(a^3+b^3)} + \frac{b^4+c^4}{bc(b^3+c^3)} + \frac{c^4+a^4}{ca(c^3+a^3)} \geq 1.$$

Epsilon 3. (APMO 1996) Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

1.2 Breaking Symmetry

We now learn how to break the symmetry. Let's attack the following problem.

Example 2. *Let a, b, c be non-negative real numbers. Show the inequality*

$$a^4 + b^4 + c^4 + 3\,(abc)^{\frac{4}{3}} \geq 2\left(a^2b^2 + b^2c^2 + c^2a^2\right).$$

There are many ways to prove this inequality. In fact, it can be proved either with Shur's Inequality or with Popoviciu's Inequality. Here, we try to give another proof. One natural starting point is to apply The AM-GM Inequality to obtain the estimations

$$c^4 + 3\,(abc)^{\frac{4}{3}} \geq 4\left(c^4 \cdot abc \cdot abc \cdot abc\right)^{\frac{1}{4}} = 4abc^2$$

and

$$a^4 + b^4 \geq 2a^2b^2.$$

Adding these two inequalities, we obtain

$$a^4 + b^4 + c^4 + 3\,(abc)^{\frac{4}{3}} \geq 2a^2b^2 + 4abc^2.$$

Hence, it now remains to show that

$$2a^2b^2 + 4abc^2 \geq 2\left(a^2b^2 + b^2c^2 + c^2a^2\right)$$

or equivalently

$$0 \geq 2c^2(a-b)^2,$$

which is clearly untrue in general. It is *reversed*! However, we can exploit the above idea to finsh the proof.

Proof. Using the symmetry of the inequality, we break the symmetry. Since the inequality is symmetric, we may consider the case $a, b \geq c$ only. Since The AM-GM Inequality implies the inequality $c^4 + 3\,(abc)^{\frac{4}{3}} \geq 4abc^2$, we obtain the estimation

$$\begin{aligned}
& a^4 + b^4 + c^4 + 3\,(abc)^{\frac{4}{3}} - 2\left(a^2b^2 + b^2c^2 + c^2a^2\right) \\
\geq\; & \left(a^4 + b^4 - 2a^2b^2\right) + 4abc^2 - 2\left(b^2c^2 + c^2a^2\right) \\
=\; & \left(a^2 - b^2\right)^2 - 2c^2\,(a-b)^2 \\
=\; & (a-b)^2\left((a+b)^2 - 2c^2\right).
\end{aligned}$$

Since we have $a, b \geq c$, the last term is clearly non-negative. $\square$

Epsilon 4. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

Epsilon 5. (USA 1980) Prove that, for all positive real numbers a, b, c,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

Epsilon 6. [AE, p. 186] Show that, for all $a, b, c \in [0, 1]$,

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \leq 2.$$

Epsilon 7. **[SL 2006 KOR]** *Let a, b, c be the lengths of the sides of a triangle. Prove the inequality*

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

Epsilon 8. *Let $f(x, y) = xy\left(x^3 + y^3\right)$ for $x, y \geq 0$ with $x + y = 2$. Prove the inequality*

$$f(x, y) \leq f\left(1 + \frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{3}}\right) = f\left(1 - \frac{1}{\sqrt{3}}, 1 + \frac{1}{\sqrt{3}}\right).$$

Epsilon 9. *Let $a, b \geq 0$ with $a + b = 1$. Prove that*

$$\sqrt{a^2+b} + \sqrt{a+b^2} + \sqrt{1+ab} \leq 3.$$

Show that the equality holds if and only if $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$.

Epsilon 10. (USA 1981) Let ABC be a triangle. Prove that

$$\sin 3A + \sin 3B + \sin 3C \leq \frac{3\sqrt{3}}{2}.$$

The above examples say that, in general, *symmetric problems does not have symmetric solutions.* We now introduce an extremely useful inequality when we make the ordering assmption.

Epsilon 11. (Chebyshev's Inequality) Let $x_1, \cdots, x_n$ and $y_1, \cdots y_n$ be two monotone increasing sequences of real numbers:

$$x_1 \leq \cdots \leq x_n,\ y_1 \leq \cdots \leq y_n.$$

Then, we have the estimation

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right).$$

Corollary 1.2.1. (The AM-HM Inequality) Let $x_1, \cdots, x_n > 0$. Then, we have

$$\frac{x_1 + \cdots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \cdots \frac{1}{x_n}}$$

or

$$\frac{1}{x_1} + \cdots \frac{1}{x_n} \geq \frac{n^2}{x_1 + \cdots + x_n}.$$

The equality holds if and only if $x_1 = \cdots = x_n$.

Proof. Since the inequality is symmetric, we may assume that $x_1 \leq \cdots \leq x_n$. Since we have

$$-\frac{1}{x_1} \leq \cdots \leq -\frac{1}{x_n},$$

Chebyshev's Inequality shows that

$$-n = \left(x_1 \cdot \frac{-1}{x_1} + \cdots + x_1 \cdot \frac{-1}{x_1}\right) \geq \frac{1}{n}\left(x_1 + \cdots + x_n\right)\left(\frac{-1}{x_1} + \cdots \frac{-1}{x_n}\right).$$

$\square$

Remark 1.2.1. *In Chebyshev's Inequality, we do not require that the variables are positive. It also implies that if $x_1 \le \cdots \le x_n$ and $y_1 \ge \cdots \ge y_n$, then we have the reverse estimation*

$$\sum_{i=1}^{n} x_i y_i \le \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right).$$

Epsilon 12. (United Kingdom 2002) For all $a, b, c \in (0,1)$, show that

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \ge \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}.$$

Epsilon 13. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}.$$

We now present three different proofs of Nesbitt's Inequality:

Proposition 1.2.1. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}.$$

Proof 1. *We denote $\mathcal{L}$ the left hand side. Since the inequality is symmetric in the three variables, we may assume that $a \ge b \ge c$. Since $\frac{1}{b+c} \ge \frac{1}{c+a} \ge \frac{1}{a+b}$, Chebyshev's Inequality yields that*

$$\begin{aligned}
\mathcal{L} &\ge \frac{1}{3}(a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \\
&= \frac{1}{3}\left(\frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b}\right) \\
&= 3\left(1 + \frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 + \frac{c}{a+b}\right) \\
&= \frac{1}{3}(3+\mathcal{L}),
\end{aligned}$$

so that $\mathcal{L} \ge \frac{3}{2}$, as desired.

Proof 2. *We now break the symmetry by a suitable normalization. Since the inequality is symmetric in the three variables, we may assume that $a \ge b \ge c$. After the substitution $x = \frac{a}{c}, y = \frac{b}{c}$, we have $x \ge y \ge 1$. It becomes*

$$\frac{\frac{a}{c}}{\frac{b}{c}+1} + \frac{\frac{b}{c}}{\frac{a}{c}+1} + \frac{1}{\frac{a}{c}+\frac{b}{c}} \ge \frac{3}{2}$$

or

$$\frac{x}{y+1} + \frac{y}{x+1} \ge \frac{3}{2} - \frac{1}{x+y}.$$

We first apply The AM-GM Inequality to deduce

$$\frac{x+1}{y+1} + \frac{y+1}{x+1} \ge 2 \quad or \quad \frac{x}{y+1} + \frac{y}{x+1} \ge 2 - \frac{1}{y+1} + \frac{1}{x+1}.$$

It is now enough to show that

$$2 - \frac{1}{y+1} + \frac{1}{x+1} \ge \frac{3}{2} - \frac{1}{x+y} \quad \Leftrightarrow \quad \frac{1}{2} - \frac{1}{y+1} \ge \frac{1}{x+1} - \frac{1}{x+y} \quad \Leftrightarrow \quad \frac{y-1}{2(1+y)} \ge \frac{y-1}{(x+1)(x+y)}.$$

However, the last inequality clearly holds for $x \ge y \ge 1$.

Proof 3. *As in the previous proof, we may assume $a \geq b \geq 1 = c$. We present a proof of*

$$\frac{a}{b+1} + \frac{b}{a+1} + \frac{1}{a+b} \geq \frac{3}{2}.$$

Let $A = a + b$ and $B = ab$. What we want to prove is

$$\frac{a^2 + b^2 + a + b}{(a+1)(b+1)} + \frac{1}{a+b} \geq \frac{3}{2}$$

or

$$\frac{A^2 - 2B + A}{A + B + 1} + \frac{1}{A} \geq \frac{3}{2}$$

or

$$2A^3 - A^2 - A + 2 \geq B(7A - 2).$$

Since $7A - 2 > 2(a + b - 1) > 0$ and $A^2 = (a + b)^2 \geq 4ab = 4B$, it's enough to show that

$$4(2A^3 - A^2 - A + 2) \geq A^2(7A - 2) \quad \Leftrightarrow \quad A^3 - 2A^2 - 4A + 8 \geq 0.$$

However, it's easy to check that $A^3 - 2A^2 - 4A + 8 = (A - 2)^2(A + 2) \geq 0$.

1.3 Symmetrizations

We now attack non-symmetrical inequalities by transforming them into symmetric ones.

Example 3. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}.$$

First Solution. We break the homogeneity. After the substitution $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$, it becomes

$$a^2 + b^2 + c^2 \geq a + b + c.$$

We now obtain

$$a^2 + b^2 + c^2 \geq \frac{1}{3}(a+b+c)^2 \geq (a+b+c)(abc)^{\frac{1}{3}} = a + b + c.$$

□

Epsilon 14. (APMO 1991) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers such that $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. Show that

$$\frac{{a_1}^2}{a_1 + b_1} + \cdots + \frac{{a_n}^2}{a_n + b_n} \geq \frac{a_1 + \cdots + a_n}{2}.$$

Epsilon 15. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x}{2x+y} + \frac{y}{2y+z} + \frac{z}{2z+x} \leq 1.$$

Epsilon 16. *Let x, y, z be positive real numbers with $x + y + z = 3$. Show the cyclic inequality*

$$\frac{x^2}{x^2+xy+y^2} + \frac{y^2}{y^2+yz+z^2} + \frac{z^2}{z^2+zx+x^2} \geq 1.$$

Epsilon 17. [SL 1985 CAN] *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{x^2+yz} + \frac{y^2}{y^2+zx} + \frac{z^2}{z^2+xy} \leq 2.$$

Epsilon 18. [SL 1990 THA] *Let $a, b, c, d \geq 0$ with $ab + bc + cd + da = 1$. show that*

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+d} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

Delta 5. [SL 1998 MNG] *Let $a_1, \cdots, a_n$ be positive real numbers such that $a_1 + \cdots + a_n < 1$. Prove that*

$$\frac{a_1 \cdots a_n (1 - a_1 - \cdots - a_n)}{(a_1 + \cdots + a_n)(1-a_1)\cdots(1-a_n)} \leq \frac{1}{n^{n+1}}.$$

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?

- P. Halmos, *I Want to be a Mathematician*, MAA Spectrum, 1985

Chapter 2

Geometric Inequalities

Think geometrically, prove algebraically.

- J. Tate

2.1 Ravi Substitution

Many inequalities are simplified by some suitable substitutions. We begin with a classical inequality in triangle geometry. What is the first[1] *nontrivial* geometric inequality?

Theorem 2.1.1. (Chapple 1746, Euler 1765) Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Then, we have $R \geq 2r$ and the equality holds if and only if ABC is equilateral.

Proof. Let $BC = a$, $CA = b$, $AB = c$, $s = \frac{a+b+c}{2}$ and $S = [ABC]$.[2] Recall the well-known identities:

$$S = \frac{abc}{4R},\ S = rs,\ S^2 = s(s-a)(s-b)(s-c).$$

Hence, the inequality $R \geq 2r$ is equivalent to $\frac{abc}{4S} \geq 2\frac{S}{s}$ or $abc \geq 8\frac{S^2}{s}$ or $abc \geq 8(s-a)(s-b)(s-c)$. We need to prove the following. $\square$

Theorem 2.1.2. (A. Padoa) Let a, b, c be the lengths of a triangle. Then, we have

$$abc \geq 8(s-a)(s-b)(s-c)$$

or

$$abc \geq (b+c-a)(c+a-b)(a+b-c)$$

Here, the equality holds if and only if $a = b = c$.

Proof. We exploit The Ravi Substitution. Since a, b, c are the lengths of a triangle, there are positive reals x, y, z such that $a = y+z$, $b = z+x$, $c = x+y$. (Why?) Then, the inequality is $(y+z)(z+x)(x+y) \geq 8xyz$ for x, y, $z > 0$. However, we get

$$(y+z)(z+x)(x+y) - 8xyz = x(y-z)^2 + y(z-x)^2 + z(x-y)^2 \geq 0.$$

$\square$

[1] The first geometric inequality is the Triangle Inequality: $AB + BC \geq AC$

[2] In this book, $[P]$ stands for the area of the polygon P.

Does the above inequality hold for arbitrary positive reals a, b, c? Yes ! It's possible to prove the inequality without the additional condition that a, b, c are the lengths of a triangle :

Theorem 2.1.3. *Whenever x, y, $z > 0$, we have*

$$xyz \geq (y+z-x)(z+x-y)(x+y-z).$$

Here, the equality holds if and only if $x = y = z$.

Proof. Since the inequality is symmetric in the variables, without loss of generality, we may assume that $x \geq y \geq z$. Then, we have $x+y > z$ and $z+x > y$. If $y+z > x$, then x, y, z are the lengths of the sides of a triangle. In this case, by the previous theorem, we get the result. Now, we may assume that $y+z \leq x$. Then, it is clear that $xyz > 0 \geq (y+z-x)(z+x-y)(x+y-z)$. □

The above inequality holds when some of x, y, z are zeros:

Theorem 2.1.4. *Let x, y, $z \geq 0$. Then, we have $xyz \geq (y+z-x)(z+x-y)(x+y-z)$.*

Proof. Since $x, y, z \geq 0$, we can find *strictly positive* sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ for which

$$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ \lim_{n\to\infty} z_n = z.$$

The above theorem says that

$$x_n y_n z_n \geq (y_n + z_n - x_n)(z_n + x_n - y_n)(x_n + y_n - z_n).$$

Now, taking the limits to both sides, we get the result.

□

We now notice that, when x, y, $z \geq 0$, the equality $xyz = (y+z-x)(z+x-y)(x+y-z)$ does not guarantee that $x = y = z$. In fact, for $x, y, z \geq 0$, the equality $xyz = (y+z-x)(z+x-y)(x+y-z)$ implies that

$$x = y = z \ \text{ or } \ x = y, z = 0 \ \text{ or } \ y = z, x = 0 \ \text{ or } \ z = x, y = 0.$$

(Verify this!) It's straightforward to verify the equality

$$xyz - (y+z-x)(z+x-y)(x+y-z) = x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y).$$

Hence, it is a particular case of Schur's Inequality.

Delta 6. *Let R and r denote the radii of the circumcircle and incircle of the right triangle ABC. Show that*

$$R \geq (1+\sqrt{2})r.$$

When does the equality hold ?

Delta 7. [LL 1988 ESP] *Let ABC be a triangle with inradius r and circumradius R. Show that*

$$\sin\frac{A}{2}\sin\frac{B}{2} + \sin\frac{B}{2}\sin\frac{C}{2} + \sin\frac{C}{2}\sin\frac{A}{2} \leq \frac{5}{8} + \frac{r}{4R}.$$

Epsilon 19. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

The Ravi Substitution is useful for inequalities for the lengths a, b, c of a triangle. After The Ravi Substitution, we can remove the condition that they are the lengths of the sides of a triangle.

Epsilon 20. [IMO 1983/6 USA] *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Delta 8. (Darij Grinberg) Let a, b, c be the lengths of a triangle. Show the inequalities

$$a^3+b^3+c^3+3abc-2b^2a-2c^2b-2a^2c\geq 0,$$

and

$$3a^2b+3b^2c+3c^2a-3abc-2b^2a-2c^2b-2a^2c\geq 0.$$

Delta 9. [LL 1983 UNK] *Show that if the sides a,b,c of a triangle satisfy the equation*

$$2\left(ab^2+bc^2+ca^2\right)=a^2b+b^2c+c^2a+3abc$$

then the triangle is equilateral. Show also that the equation can be satisfied by positive real numbers that are not the sides of a triangle.

Delta 10. [IMO 1991/1 USS] *Prove for each triangle ABC the inequality*

$$\frac{1}{4}<\frac{IA\cdot IB\cdot IC}{l_A\cdot l_B\cdot l_C},$$

where I is the incenter and l_A,l_B,l_C are the lengths of the angle bisectors of ABC.

We now discuss Weitzenböck's Inequality and related theorems.

Epsilon 21. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2+b^2+c^2\geq 4\sqrt{3}S.$$

Epsilon 22. (Hadwiger-Finsler Inequality) For any triangle ABC with sides a, b, c and area F, the following inequality holds.

$$2ab+2bc+2ca-(a^2+b^2+c^2)\geq 4\sqrt{3}F.$$

Here is a simultaneous generalization of Weitzenböck's Inequality and Nesbitt's Inequality.

Epsilon 23. (Tsintsifas) Let p,q,r be positive real numbers and let a,b,c denote the sides of a triangle with area F. Then, we have

$$\frac{p}{q+r}a^2+\frac{q}{r+p}b^2+\frac{r}{p+q}c^2\geq 2\sqrt{3}F.$$

Epsilon 24. (The Neuberg-Pedoe Inequality) Let a_1,b_1,c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2,b_2,c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$${a_1}^2({b_2}^2+{c_2}^2-{a_2}^2)+{b_1}^2({c_2}^2+{a_2}^2-{b_2}^2)+{c_1}^2({a_2}^2+{b_2}^2-{c_2}^2)\geq 16F_1F_2.$$

Notice that it's a generalization of Weitzenböck's Inequality. Carlitz observed that The Neuberg-Pedoe Inequality can be deduced from Aczél's Inequality.

Epsilon 25. (Aczél's Inequality) If $a_1,\cdots,a_n,b_1,\cdots,b_n>0$ satisfies the inequality

$${a_1}^2\geq{a_2}^2+\cdots+{a_n}^2\ \text{ and }\ {b_1}^2\geq{b_2}^2+\cdots+{b_n}^2,$$

then the following inequality holds.

$$a_1b_1-(a_2b_2+\cdots+a_nb_n)\geq\sqrt{\left({a_1}^2-({a_2}^2+\cdots+{a_n}^2)\right)\left({b_1}^2-\left({b_2}^2+\cdots+{b_n}^2\right)\right)}$$

2.2 Trigonometric Methods

In this section, we employ trigonometric methods to attack geometric inequalities.

Theorem 2.2.1. (The Erdős-Mordell Theorem) If from a point P inside a given triangle ABC perpendiculars PH_1, PH_2, PH_3 are drawn to its sides, then

$$PA + PB + PC \geq 2(PH_1 + PH_2 + PH_3).$$

This was conjectured by Paul Erdős in 1935, and first proved by Mordell in the same year. Several proofs of this inequality have been given, using Ptolemy's Theorem by André Avez, angular computations with similar triangles by Leon Bankoff, area inequality by V. Komornik, or using trigonometry by Mordell and Barrow.

Proof. [MB] We transform it to a trigonometric inequality. Let $h_1 = PH_1$, $h_2 = PH_2$ and $h_3 = PH_3$. Apply the Since Law and the Cosine Law to obtain

$$\begin{aligned}
PA\sin A = \overline{H_2H_3} &= \sqrt{{h_2}^2 + {h_3}^2 - 2h_2h_3\cos(\pi - A)},\\
PB\sin B = \overline{H_3H_1} &= \sqrt{{h_3}^2 + {h_1}^2 - 2h_3h_1\cos(\pi - B)},\\
PC\sin C = \overline{H_1H_2} &= \sqrt{{h_1}^2 + {h_2}^2 - 2h_1h_2\cos(\pi - C)}.
\end{aligned}$$

So, we need to prove that

$$\sum_{\text{cyclic}} \frac{1}{\sin A}\sqrt{{h_2}^2 + {h_3}^2 - 2h_2h_3\cos(\pi - A)} \geq 2(h_1 + h_2 + h_3).$$

The main trouble is that the left hand side has too heavy terms with square root expressions. Our strategy is to find a lower bound without square roots. To this end, we express the terms inside the square root as the sum of two squares.

$$\begin{aligned}
\overline{H_2H_3}^2 &= {h_2}^2 + {h_3}^2 - 2h_2h_3\cos(\pi - A)\\
&= {h_2}^2 + {h_3}^2 - 2h_2h_3\cos(B + C)\\
&= {h_2}^2 + {h_3}^2 - 2h_2h_3(\cos B\cos C - \sin B\sin C).
\end{aligned}$$

Using $\cos^2 B + \sin^2 B = 1$ and $\cos^2 C + \sin^2 C = 1$, one finds that

$$\overline{H_2H_3}^2 = (h_2\sin C + h_3\sin B)^2 + (h_2\cos C - h_3\cos B)^2.$$

Since $(h_2\cos C - h_3\cos B)^2$ is clearly nonnegative, we get $\overline{H_2H_3} \geq h_2\sin C + h_3\sin B$. Hence,

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{\sqrt{{h_2}^2 + {h_3}^2 - 2h_2h_3\cos(\pi - A)}}{\sin A} &\geq \sum_{\text{cyclic}} \frac{h_2\sin C + h_3\sin B}{\sin A}\\
&= \sum_{\text{cyclic}} \left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B}\right)h_1\\
&\geq \sum_{\text{cyclic}} 2\sqrt{\frac{\sin B}{\sin C}\cdot\frac{\sin C}{\sin B}}h_1\\
&= 2h_1 + 2h_2 + 2h_3.
\end{aligned}$$

□

Epsilon 26. **[SL 2005 KOR]** *In an acute triangle ABC, let D, E, F, P, Q, R be the feet of perpendiculars from A, B, C, A, B, C to BC, CA, AB, EF, FD, DE, respectively. Prove that*

$$p(ABC)p(PQR) \geq p(DEF)^2,$$

where $p(T)$ denotes the perimeter of triangle T .

Epsilon 27. **[IMO 2001/1 KOR]** *Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A. Assume that $\angle BCA \geq \angle ABC + 30°$. Prove that $\angle CAB + \angle COP < 90°$.*

Epsilon 28. **[IMO 1961/2 POL]** (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Epsilon 29. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$${a_1}^2({b_2}^2 + {c_2}^2 - {a_2}^2) + {b_1}^2({c_2}^2 + {a_2}^2 - {b_2}^2) + {c_1}^2({a_2}^2 + {b_2}^2 - {c_2}^2) \geq 16F_1F_2.$$

We close this section with Barrows' Inequality stronger than The Erdös-Mordell Theorem. We need the following trigonometric inequality:

Proposition 2.2.1. *Let $x, y, z, \theta_1, \theta_2, \theta_3$ be real numbers with $\theta_1 + \theta_2 + \theta_3 = \pi$. Then,*

$$x^2 + y^2 + z^2 \geq 2(yz\cos\theta_1 + zx\cos\theta_2 + xy\cos\theta_3).$$

Proof. Using $\theta_3 = \pi - (\theta_1 + \theta_2)$, it's an easy job to check the following identity

$$x^2 + y^2 + z^2 - 2(yz\cos\theta_1 + zx\cos\theta_2 + xy\cos\theta_3) = (z - (x\cos\theta_2 + y\cos\theta_1))^2 + (x\sin\theta_2 - y\sin\theta_1)^2.$$

□

Corollary 2.2.1. *Let p, q, and r be positive real numbers. Let θ_1, θ_2, and θ_3 be real numbers satisfying $\theta_1 + \theta_2 + \theta_3 = \pi$. Then, the following inequality holds.*

$$p\cos\theta_1 + q\cos\theta_2 + r\cos\theta_3 \leq \frac{1}{2}\left(\frac{qr}{p} + \frac{rp}{q} + \frac{pq}{r}\right).$$

Proof. Take $(x, y, z) = \left(\sqrt{\frac{qr}{p}}, \sqrt{\frac{rp}{q}}, \sqrt{\frac{pq}{r}}\right)$ and apply the above proposition. □

Epsilon 30. (Barrow's Inequality) Let P be an interior point of a triangle ABC and let U, V, W be the points where the bisectors of angles BPC, CPA, APB cut the sides BC,CA,AB respectively. Then, we have

$$PA + PB + PC \geq 2(PU + PV + PW).$$

Epsilon 31. [AK] Let $x_1, \cdots, x_4$ be positive real numbers. Let $\theta_1, \cdots, \theta_4$ be real numbers such that $\theta_1 + \cdots + \theta_4 = \pi$. Then, we have

$$x_1\cos\theta_1 + x_2\cos\theta_2 + x_3\cos\theta_3 + x_4\cos\theta_4 \leq \sqrt{\frac{(x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4)(x_1x_4 + x_2x_3)}{x_1x_2x_3x_4}}.$$

2.3 Tossing onto Real Plane

Example 4. *Let I be the incenter of the triangle ABC with $BC = a$, $CA = b$ and $AB = c$. Prove that, for all points X,*

$$aXA^2 + bXB^2 + cXC^2 \geq abc.$$

Solution. This geometric inequality follows from the following geometric identity:

$$aXA^2 + bXB^2 + cXC^2 = (a+b+c)XI^2 + abc.$$ [3]

There are many ways to establish this identity. To euler[4] it, we toss the picture on the real plane $\mathbb{R}^2$ so that $A(c\cos B, c\sin B)$, $B(0,0)$ and $C(a,0)$. Letting r be the inradius of $\triangle ABC$ and $s = \frac{a+b+c}{2}$, we get $I(s-b, r)$. It is well-known that

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}.$$

Set $X(p,q)$. On the one hand, we obtain

$$\begin{aligned}
aXA^2 + bXB^2 + cXC^2 &= a\left[(p-c\cos B)^2 + (q-c\sin B)^2\right] + b\left(p^2+q^2\right) + c\left[(p-a)^2+q^2\right] \\
&= (a+b+c)p^2 - 2acp(1+\cos B) + (a+b+c)q^2 - 2acq\sin B + ac^2 + a^2c \\
&= 2sp^2 - 2acp\left(1 + \frac{a^2+c^2-b^2}{2ac}\right) + 2sq^2 - 2acq\frac{[\triangle ABC]}{\frac{1}{2}ac} + ac^2 + a^2c \\
&= 2sp^2 - p(a+c+b)(a+c-b) + 2sq^2 - 4q[\triangle ABC] + ac^2 + a^2c \\
&= 2sp^2 - p(2s)(2s-2b) + 2sq^2 - 4qsr + ac^2 + a^2c \\
&= 2sp^2 - 4s(s-b)p + 2sq^2 - 4rsq + ac^2 + a^2c.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
(a+b+c)XI^2 + abc &= 2s\left[(p-(s-b))^2 + (q-r)^2\right] \\
&= 2s\left[p^2 - 2(s-b)p + (s-b)^2 + q^2 - 2qr + r^2\right] \\
&= 2sp^2 - 4s(s-b)p + 2s(s-b)^2 + 2sq^2 - 4rsq + 2sr^2 + abc.
\end{aligned}$$

It thus follows that

$$\begin{aligned}
aXA^2 + bXB^2 + cXC^2 - (a+b+c)XI^2 - abc &= ac^2 + a^2c - 2s(s-b)^2 - 2sr^2 - abc \\
&= ac(a+c) - 2s(s-b)^2 - 2(s-a)(s-b)(s-c) - abc \\
&= ac(a+c-b) - 2s(s-b)^2 - 2(s-a)(s-b)(s-c) \\
&= 2ac(s-b) - 2s(s-b)^2 - 2(s-a)(s-b)(s-c) \\
&= 2(s-b)\left[ac - s(s-b) - 2(s-a)(s-c)\right].
\end{aligned}$$

However, we compute $ac - s(s-b) - 2(s-a)(s-c) = -2s^2 + (a+b+c)s = 0$. □

Delta 11. [SL 1988 UNK] *The triangle ABC is acute-angled. Let L be any line in the plane of the triangle and let u, v, w be lengths of the perpendiculars from A, B, C respectively to L. Prove that*

$$u^2\tan A + v^2\tan B + w^2\tan C \geq 2\triangle,$$

where $\triangle$ is the area of the triangle, and determine the lines L for which equality holds.

[3] **[SL 1988 SGP]**

[4] euler v. (in Mathematics) transform the geometric identity in triangle geometry to trigonometric or algebraic identity.

Epsilon 32. [IMO 1961/2 POL] (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Epsilon 33. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$${a_1}^2({b_2}^2 + {c_2}^2 - {a_2}^2) + {b_1}^2({c_2}^2 + {a_2}^2 - {b_2}^2) + {c_1}^2({a_2}^2 + {b_2}^2 - {c_2}^2) \geq 16F_1F_2.$$

2.4 Tossing onto Complex Plane

In this section, we discuss some applications of complex numbers to geometric inequality. Every complex number corresponds to a unique point in the complex plane. The standard symbol for the set of all complex numbers is $\mathbb{C}$, and we also refer to the complex plane as $\mathbb{C}$. The key idea we use here is the fact that we can identify the *points* in the real plane $\mathbb{R}^2$ as *numbers* in $\mathbb{C}$. The main tool is the following fundamental inequality.

Theorem 2.4.1. (Triangle Inequality) If $z_1, \cdots, z_n \in \mathbb{C}$, then $|z_1| + \cdots + |z_n| \geq |z_1 + \cdots + z_n|$.

Proof. Induction on n. □

Theorem 2.4.2. (Ptolemy's Inequality) For any points A, B, C, D in the plane, we have

$$\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{DA} \geq \overline{AC} \cdot \overline{BD}.$$

Proof. Let a, b, c and 0 be complex numbers that correspond to A, B, C, D in the complex plane $\mathbb{C}$. It then becomes

$$|a-b| \cdot |c| + |b-c| \cdot |a| \geq |a-c| \cdot |b|.$$

Applying the Triangle Inequality to the identity $(a-b)c + (b-c)a = (a-c)b$, we get the result. □

Remark 2.4.1. *Investigate the equality case in Ptolemy's Inequality.*

Delta 12. [SL 1997 RUS] *Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$, $EF = FA$. Prove that*

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

When does the equality occur?

Epsilon 34. [TD] Let P be an arbitrary point in the plane of a triangle ABC with the centroid G. Show the following inequalities

(1) $\overline{BC} \cdot \overline{PB} \cdot \overline{PC} + \overline{AB} \cdot \overline{PA} \cdot \overline{PB} + \overline{CA} \cdot \overline{PC} \cdot \overline{PA} \geq \overline{BC} \cdot \overline{CA} \cdot \overline{AB}$,

(2) $\overline{PA}^3 \cdot \overline{BC} + \overline{PB}^3 \cdot \overline{CA} + \overline{PC}^3 \cdot \overline{AB} \geq 3\overline{PG} \cdot \overline{BC} \cdot \overline{CA} \cdot \overline{AB}$.

Epsilon 35. (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$${a_1}^2({b_2}^2 + {c_2}^2 - {a_2}^2) + {b_1}^2({c_2}^2 + {a_2}^2 - {b_2}^2) + {c_1}^2({a_2}^2 + {b_2}^2 - {c_2}^2) \geq 16F_1F_2.$$

Epsilon 36. [SL 2002 KOR] *Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E, respectively. Prove that $\overline{AB} + \overline{AC} \geq 4\overline{DE}$.*

Inspiration is needed in geometry, just as much as in poetry. - A. Pushkin

Chapter 3

Three Terrific Techniques

> A long time ago an older and well-known number theorist made some disparaging remarks about Paul Erdős's work. You admire Erdős's contributions to mathematics as much as I do, and I felt annoyed when the older mathematician flatly and definitively stated that all of Erdős's work could be "reduced" to a few tricks which Erdős repeatedly relied on in his proofs. What the number theorist did not realize is that other mathematicians, even the very best, also rely on a few tricks which they use over and over. **Take Hilbert.** The second volume of Hilbert's collected papers contains Hilbert's papers in invariant theory. I have made a point of reading some of these papers with care. It is sad to note that some of Hilbert′s beautiful results have been completely forgotten. But on reading the proofs of Hilbert's striking and deep theorems in invariant theory, it was surprising to verify that Hilbert's proofs relied on the same few tricks. **Even Hilbert had only a few tricks!**
>
> - G-C Rota, *Ten Lessons I Wish I Had Been Taught*, Notices of the AMS, Jan. 1997

3.1 Trigonometric Substitutions

If you are faced with an integral that contains square root expressions such as

$$\int \sqrt{1-x^2}\,dx,\ \int \sqrt{1+y^2}\,dy,\ \int \sqrt{z^2-1}\,dz$$

then trigonometric substitutions such as $x = \sin t$, $y = \tan t$, $z = \sec t$ are very useful. We will learn that making a suitable *trigonometric* substitution simplifies the given inequality.

Epsilon 37. (APMO 2004/5) Prove that, for all positive real numbers a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

Epsilon 38. (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1.$$

Prove that $abcd \geq 3$.

Epsilon 39. (Korea 1998) Let x, y, z be the positive reals with $x+y+z=xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\leq\frac{3}{2}.$$

Since the function $f(t)=\frac{1}{\sqrt{1+t^2}}$ is not concave on $\mathbb{R}^+$, we cannot apply Jensen's Inequality directly. However, the function $f(\tan\theta)$ is concave on $\left(0,\frac{\pi}{2}\right)$!

Proposition 3.1.1. *In any acute triangle ABC, we have* $\cos A+\cos B+\cos C\leq\frac{3}{2}$.

Proof. Since $\cos x$ is concave on $\left(0,\frac{\pi}{2}\right)$, it's a direct consequence of Jensen's Inequality. □

We note that the function $\cos x$ is not concave on $(0,\pi)$. In fact, it's convex on $\left(\frac{\pi}{2},\pi\right)$. One may think that the inequality $\cos A+\cos B+\cos C\leq\frac{3}{2}$ doesn't hold for any triangles. However, it's known that it holds for all triangles.

Proposition 3.1.2. *In any triangle ABC, we have*

$$\cos A+\cos B+\cos C\leq\frac{3}{2}.$$

First Proof. It follows from $\pi-C=A+B$ that

$$\cos C=-\cos(A+B)=-\cos A\cos B+\sin A\sin B$$

or

$$3-2(\cos A+\cos B+\cos C)=(\sin A-\sin B)^2+(\cos A+\cos B-1)^2\geq 0.$$

□

Second Proof. Let $BC=a$, $CA=b$, $AB=c$. Use The Cosine Law to rewrite the given inequality in the terms of a, b, c :

$$\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}\leq\frac{3}{2}.$$

Clearing denominators, this becomes

$$3abc\geq a(b^2+c^2-a^2)+b(c^2+a^2-b^2)+c(a^2+b^2-c^2),$$

which is equivalent to $abc\geq(b+c-a)(c+a-b)(a+b-c)$. □

We remind that the *geometric* inequality $R\geq 2r$ is equivalent to the *algebraic* inequality $abc\geq(b+c-a)(c+a-b)(a+b-c)$. We now find that, in the proof of the above theorem, $abc\geq(b+c-a)(c+a-b)(a+b-c)$ is equivalent to the *trigonometric* inequality $\cos A+\cos B+\cos C\leq\frac{3}{2}$. One may ask that

> in any triangles ABC, is there a *natural* relation between $\cos A+\cos B+\cos C$ and $\frac{R}{r}$, where R and r are the radii of the circumcircle and incircle of ABC?

Theorem 3.1.1. *Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Then, we have*

$$\cos A+\cos B+\cos C=1+\frac{r}{R}.$$

Proof. Use the algebraic identity

$$a(b^2+c^2-a^2)+b(c^2+a^2-b^2)+c(a^2+b^2-c^2)=2abc+(b+c-a)(c+a-b)(a+b-c).$$

We leave the details for the readers. □

Delta 13. *(a) Let p, q, r be the positive real numbers such that $p^2 + q^2 + r^2 + 2pqr = 1$. Show that there exists an acute triangle ABC such that $p = \cos A$, $q = \cos B$, $r = \cos C$.*
(b) Let $p, q, r \geq 0$ with $p^2 + q^2 + r^2 + 2pqr = 1$. Show that there are $A, B, C \in \left[0, \frac{\pi}{2}\right]$ with $p = \cos A$, $q = \cos B$, $r = \cos C$, and $A + B + C = \pi$.

Epsilon 40. (USA 2001) Let a, b, and c be nonnegative real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that $0 \leq ab + bc + ca - abc \leq 2$.

3.2 Algebraic Substitutions

We know that some inequalities in triangle geometry can be treated by the *Ravi* substitution and *trigonometric* substitutions. We can also transform the given inequalities into easier ones through some clever *algebraic* substitutions.

Epsilon 41. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Epsilon 42. [IMO 1995/2 RUS] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

Epsilon 43. (Korea 1998) Let x, y, z be the positive reals with $x+y+z=xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\leq\frac{3}{2}.$$

We now prove a classical theorem in various ways.

Proposition 3.2.1. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 4. *After the substitution $x=b+c,\ y=c+a,\ z=a+b$, it becomes*

$$\sum_{\text{cyclic}}\frac{y+z-x}{2x}\geq\frac{3}{2}\quad or\quad\sum_{\text{cyclic}}\frac{y+z}{x}\geq 6,$$

which follows from The AM-GM Inequality as following:

$$\sum_{\text{cyclic}}\frac{y+z}{x}=\frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}+\frac{x}{z}+\frac{y}{z}\geq 6\left(\frac{y}{x}\cdot\frac{z}{x}\cdot\frac{z}{y}\cdot\frac{x}{y}\cdot\frac{x}{z}\cdot\frac{y}{z}\right)^{\frac{1}{6}}=6.$$

Proof 5. *We make the substitution*

$$x=\frac{a}{b+c},\ y=\frac{b}{c+a},\ z=\frac{c}{a+b}.$$

It follows that

$$\sum_{\text{cyclic}}f(x)=\sum_{\text{cyclic}}\frac{a}{a+b+c}=1,\quad where\ \ f(t)=\frac{t}{1+t}.$$

Since f is concave on $(0,\infty)$, Jensen's Inequality shows that

$$f\left(\frac{1}{2}\right)=\frac{1}{3}=\frac{1}{3}\sum_{\text{cyclic}}f(x)\leq f\left(\frac{x+y+z}{3}\right)\quad or\quad f\left(\frac{1}{2}\right)\leq f\left(\frac{x+y+z}{3}\right).$$

Since f is monotone increasing, this implies that

$$\frac{1}{2}\leq\frac{x+y+z}{3}\quad or\quad\sum_{\text{cyclic}}\frac{a}{b+c}=x+y+z\geq\frac{3}{2}.$$

Proof 6. *As in the previous proof, it suffices to show that*

$$T\geq\frac{1}{2},\ \ where\ \ T=\frac{x+y+z}{3}\ \ and\ \ \sum_{\text{cyclic}}\frac{x}{1+x}=1.$$

One can easily check that the condition

$$\sum_{\text{cyclic}}\frac{x}{1+x}=1$$

becomes $1=2xyz+xy+yz+zx$. By The AM-GM Inequality, we have

$$1=2xyz+xy+yz+zx\leq 2T^3+3T^2\ \Rightarrow\ 2T^3+3T^2-1\geq 0\ \Rightarrow\ (2T-1)(T+1)^2\geq 0\ \Rightarrow\ T\geq\frac{1}{2}.$$

Epsilon 44. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1.$$

Epsilon 45. *Let a, b, c be positive real numbers satisfying $a + b + c = 1$. Show that*

$$\frac{a}{a + bc} + \frac{b}{b + ca} + \frac{\sqrt{abc}}{c + ab} \le 1 + \frac{3\sqrt{3}}{4}.$$

Epsilon 46. (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1 + a^4} + \frac{1}{1 + b^4} + \frac{1}{1 + c^4} + \frac{1}{1 + d^4} = 1.$$

Prove that $abcd \ge 3$.

Delta 14. [SL 1993 USA] *Prove that*

$$\frac{a}{b + 2c + 3d} + \frac{b}{c + 2d + 3a} + \frac{c}{d + 2a + 3b} + \frac{d}{a + 2b + 3c} \ge \frac{2}{3}$$

for all positive real numbers a, b, c, d.

Epsilon 47. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sqrt{x + y + z} \ge \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

Epsilon 48. (Belarus 1998) Prove that, for all $a, b, c > 0$,

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{a + b}{b + c} + \frac{b + c}{c + a} + 1.$$

Delta 15. [IMO 1969 USS] *Under the conditions $x_1, x_2 > 0$, $x_1 y_1 > {z_1}^2$, and $x_2 y_2 > {z_2}^2$, prove the inequality*

$$\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \le \frac{1}{x_1 y_1 - {z_1}^2} + \frac{1}{x_2 y_2 - {z_2}^2}.$$

Epsilon 49. [SL 2001] *Let $x_1, \cdots, x_n$ be arbitrary real numbers. Prove the inequality.*

$$\frac{x_1}{1 + {x_1}^2} + \frac{x_2}{1 + {x_1}^2 + {x_2}^2} + \cdots + \frac{x_n}{1 + {x_1}^2 + \cdots + {x_n}^2} < \sqrt{n}.$$

Delta 16. [LL 1987 FRA] *Given n real numbers $0 \le t_1 \le t_2 \le \cdots \le t_n < 1$, prove that*

$$\left(1 - {t_n}^2\right)\left(\frac{t_1}{(1 - {t_1}^2)^2} + \frac{{t_2}^2}{(1 - {t_2}^3)^2} + \cdots + \frac{{t_n}^n}{(1 - {t_n}^{n+1})^2}\right) < 1.$$

3.3 Establishing New Bounds

The following examples give a nice description of the title of this section.

Example 5. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\geq\frac{x}{y}+\frac{y}{z}+\frac{z}{x}.$$

Second Solution. We first use the auxiliary inequality $t^2 \geq 2t-1$ to deduce

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\geq 2\frac{x}{y}-1+2\frac{y}{z}-1+2\frac{z}{x}-1.$$

It now remains to check that

$$2\frac{x}{y}-1+2\frac{y}{z}-1+2\frac{z}{x}-1\geq\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

or equivalently

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq 3.$$

However, The AM-GM Inequality shows that

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq 3\left(\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x}\right)^{\frac{1}{3}}=3.$$

□

Proposition 3.3.1. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 7. *From $\left(\frac{a}{b+c}-\frac{1}{2}\right)^2\geq 0$, we deduce that*

$$\frac{a}{b+c}\geq\frac{1}{4}\cdot\frac{\frac{8a}{b+c}-1}{\frac{a}{b+c}+1}=\frac{8a-b-c}{4(a+b+c)}.$$

It follows that

$$\sum_{\text{cyclic}}\frac{a}{b+c}\geq\sum_{\text{cyclic}}\frac{8a-b-c}{4(a+b+c)}=\frac{3}{2}.$$

Proof 8. *We claim that*

$$\frac{a}{b+c}\geq\frac{3a^{\frac{3}{2}}}{2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right)}\quad \text{or}\quad 2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right)\geq 3a^{\frac{1}{2}}(b+c).$$

The AM-GM inequality gives $a^{\frac{3}{2}}+b^{\frac{3}{2}}+b^{\frac{3}{2}}\geq 3a^{\frac{1}{2}}b$ and $a^{\frac{3}{2}}+c^{\frac{3}{2}}+c^{\frac{3}{2}}\geq 3a^{\frac{1}{2}}c$. Adding these two inequalities yields $2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right)\geq 3a^{\frac{1}{2}}(b+c)$, as desired. Therefore, we have

$$\sum_{\text{cyclic}}\frac{a}{b+c}\geq\frac{3}{2}\sum_{\text{cyclic}}\frac{a^{\frac{3}{2}}}{a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}}=\frac{3}{2}.$$

Epsilon 50. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2.$$

Some cyclic inequalities can be established by finding some clever bounds. Suppose that we want to establish that

$$\sum_{\text{cyclic}} F(x,y,z) \geq C$$

for some given constant $C \in \mathbb{R}$. Whenever we have a function G such that, for all $x, y, z > 0$,

$$F(x,y,z) \geq G(x,y,z)$$

and

$$\sum_{\text{cyclic}} G(x,y,z) = C,$$

we then deduce that

$$\sum_{\text{cyclic}} F(x,y,z) \geq \sum_{\text{cyclic}} G(x,y,z) = C.$$

For instance, if a function F satisfies the inequality

$$F(x,y,z) \geq \frac{x}{x+y+z}$$

for all $x, y, z > 0$, then F obeys the inequality

$$\sum_{\text{cyclic}} F(x,y,z) \geq 1.$$

As we saw in the above two proofs of Nesbitt's Inequality, there are various lower bounds. One day, I tried finding a new lower bound of $(x+y+z)^2$ where $x, y, z > 0$. There are well-known lower bounds such as $3(xy+yz+zx)$ and $9(xyz)^{\frac{2}{3}}$. But I wanted to find quite different one. So, I tried breaking the symmetry. Notice that

$$(x+y+z)^2 = x^2 + y^2 + z^2 + xy + xy + yz + yz + zx + zx.$$

I then applied The AM-GM Inequality to the right hand side except the term x^2 :

$$y^2 + z^2 + xy + xy + yz + yz + zx + zx \geq 8x^{\frac{1}{2}} y^{\frac{3}{4}} z^{\frac{3}{4}}\ .$$

It follows that

$$(x+y+z)^2 \geq x^2 + 8x^{\frac{1}{2}} y^{\frac{3}{4}} z^{\frac{3}{4}} = x^{\frac{1}{2}} \left(x^{\frac{3}{2}} + 8y^{\frac{3}{4}} z^{\frac{3}{4}} \right).$$

This gives a proof of the following inequality:

Epsilon 51. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

Epsilon 52. [IMO 2005/3 KOR] *Let x, y, and z be positive numbers such that $xyz \geq 1$. Prove that*

$$\frac{x^5-x^2}{x^5+y^2+z^2} + \frac{y^5-y^2}{y^5+z^2+x^2} + \frac{z^5-z^2}{z^5+x^2+y^2} \geq 0.$$

Epsilon 53. (KMO Weekend Program 2007) Prove that, for all $a, b, c, x, y, z > 0$,

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z}.$$

Epsilon 54. (USAMO Summer Program 2002) Let a, b, c be positive real numbers. Prove that

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} + \left(\frac{2b}{c+a}\right)^{\frac{2}{3}} + \left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3.$$

Epsilon 55. (APMO 2005) Let a, b, c be positive real numbers with $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

Delta 17. **[SL 1996 SVN]** *Let a, b, and c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \leq 1.$$

Delta 18. **[SL 1971 YUG]** *Prove the inequality*

$$\frac{a_1+a_3}{a_1+a_2} + \frac{a_2+a_4}{a_2+a_3} + \frac{a_3+a_1}{a_3+a_4} + \frac{a_4+a_2}{a_4+a_1} \geq 4$$

where $a_1, a_2, a_3, a_4 > 0$.

There is a simple way to find new bounds for given differentiable functions. We begin to show that every supporting lines are tangent lines in the following sense.

Proposition 3.3.2. (The Characterization of Supporting Lines) Let f be a real valued function. Let $m, n \in \mathbb{R}$. Suppose that

(1) $f(\alpha) = m\alpha + n$ for some $\alpha \in \mathbb{R}$,
(2) $f(x) \geq mx + n$ for all x in some interval (ϵ_1, ϵ_2) including α, and
(3) f is differentiable at α.

Then, the supporting line $y = mx + n$ of f is the tangent line of f at $x = \alpha$.

Proof. Let us define a function $F : (\epsilon_1, \epsilon_2) \longrightarrow \mathbb{R}$ by $F(x) = f(x) - mx - n$ for all $x \in (\epsilon_1, \epsilon_2)$. Then, F is differentiable at α and we obtain $F'(\alpha) = f'(\alpha) - m$. By the assumption (1) and (2), we see that F has a local minimum at α. So, the first derivative theorem for local extreme values implies that $0 = F'(\alpha) = f'(\alpha) - m$ so that $m = f'(\alpha)$ and that $n = f(\alpha) - m\alpha = f(\alpha) - f'(\alpha)\alpha$. It follows that $y = mx + n = f'(\alpha)(x - \alpha) + f(\alpha)$. □

Proposition 3.3.3. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 9. *We may normalize to $a + b + c = 1$. Note that $0 < a, b, c < 1$. The problem is now to prove*

$$\sum_{\text{cyclic}} f(a) \geq \frac{3}{2} \Leftrightarrow \frac{f(a)+f(b)+f(c)}{3} \geq f\left(\frac{1}{3}\right), \text{ where } f(x) = \frac{x}{1-x}.$$

The equation of the tangent line of f at $x = \frac{1}{3}$ is given by $y = \frac{9x-1}{4}$. We claim that

$$f(x) \geq \frac{9x-1}{4}$$

for all $x \in (0, 1)$. It immediately follows from the equality

$$f(x) - \frac{9x-1}{4} = \frac{(3x-1)^2}{4(1-x)}.$$

Now, we conclude that

$$\sum_{\text{cyclic}} \frac{a}{1-a} \geq \sum_{\text{cyclic}} \frac{9a-1}{4} = \frac{9}{4}\sum_{\text{cyclic}} a - \frac{3}{4} = \frac{3}{2}.$$

The above argument can be generalized. If a function f has a supporting line at some point on the graph of f, then f satisfies Jensen's Inequality in the following sense.

Theorem 3.3.1. (Supporting Line Inequality) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose that $\alpha \in [a, b]$ and $m \in \mathbb{R}$ satisfy

$$f(x) \geq m(x - \alpha) + f(\alpha)$$

for all $x \in [a, b]$. Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. Then, the following inequality holds

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\alpha)$$

for all $x_1, \cdots, x_n \in [a, b]$ such that $\alpha = \omega_1 x_1 + \cdots + \omega_n x_n$. In particular, we obtain

$$\frac{f(x_1) + \cdots + f(x_n)}{n} \geq f\left(\frac{s}{n}\right),$$

where $x_1, \cdots, x_n \in [a, b]$ with $x_1 + \cdots + x_n = s$ for some $s \in [na, nb]$.

Proof. It follows that

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq \omega_1[m(x_1 - \alpha) + f(\alpha)] + \cdots + \omega_1[m(x_n - \alpha) + f(\alpha)] = f(\alpha).$$

□

We can apply the supporting line inequality to deduce Jensen's inequality for differentiable functions.

Lemma 3.3.1. *Let $f : (a, b) \longrightarrow \mathbb{R}$ be a convex function which is differentiable twice on (a, b). Let $y = l_\alpha(x)$ be the tangent line at $\alpha \in (a, b)$. Then, $f(x) \geq l_\alpha(x)$ for all $x \in (a, b)$. So, the convex function f admits the supporting lines.*

Proof. Let $\alpha \in (a, b)$. We want to show that the tangent line $y = l_\alpha(x) = f'(\alpha)(x - \alpha) + f(\alpha)$ is the supporting line of f at $x = \alpha$ such that $f(x) \geq l_\alpha(x)$ for all $x \in (a, b)$. However, by Taylor's Theorem, we can find a real number θ_x between α and x such that

$$f(x) = f(\alpha) + f'(\alpha)(x - \alpha) + \frac{f''(\theta_x)}{2}(x - \alpha)^2 \geq f(\alpha) + f'(\alpha)(x - \alpha).$$

□

Theorem 3.3.2. (The Weighted Jensen's Inequality) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous convex function which is differentiable twice on (a, b). Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n \in [a, b]$,

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

First Proof. By the continuity of f, we may assume that $x_1, \cdots, x_n \in (a, b)$. Now, let $\mu = \omega_1\, x_1 + \cdots + \omega_n\, x_n$. Then, $\mu \in (a, b)$. By the above lemma, f has the tangent line $y = l_\mu(x) = f'(\mu)(x - \mu) + f(\mu)$ at $x = \mu$ satisfying $f(x) \geq l_\mu(x)$ for all $x \in (a, b)$. Hence, the supporting line inequality shows that

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq \omega_1 f(\mu) + \cdots + \omega_n f(\mu) = f(\mu) = f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

□

We note that the cosine function is concave on $\left[0, \frac{\pi}{2}\right]$ and convex on $\left[\frac{\pi}{2}, \pi\right]$. Non-convex functions can be locally convex and have supporting lines at some points. This means that the supporting line inequality is a powerful tool because we can also produce Jensen-type inequalities for non-convex functions. We now remind again that the cosine function is *not* convex on $[0, \pi]$.

Proposition 3.3.4. *In any triangle ABC, we have $\cos A + \cos B + \cos C \leq \frac{3}{2}$.*

Third Proof. Let $f(x) = -\cos x$. Our goal is to establish a three-variables inequality

$$\frac{f(A)+f(B)+f(C)}{3} \geq f\left(\frac{\pi}{3}\right),$$

where $A, B, C \in (0, \pi)$ with $A + B + C = \pi$. We compute $f'(x) = \sin x$. The equation of the tangent line of f at $x = \frac{\pi}{3}$ is given by $y = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}$. To apply the supporting line inequality, we need to show that

$$-\cos x \geq \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}$$

for all $x \in (0, \pi)$. It is a one-variable inequality! We omit the proof.[1] $\square$

Epsilon 56. (Titu Andreescu, Gabriel Dospinescu) Let x, y, and z be real numbers such that $x, y, z \leq 1$ and $x + y + z = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \leq \frac{27}{10}.$$

Epsilon 57. (Japan 1997) Let a, b, and c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}.$$

Sleep with problem. - R. Bott

[1] In other words... differentiate!

Chapter 4

Homogenizations and Normalizations

Mathematicians do not study objects, but relations between objects.

- H. Poincaré

4.1 Homogenizations

Many inequality problems come with constraints such as $ab = 1$, $xyz = 1$, $x + y + z = 1$. A non-homogeneous *symmetric* inequality can be transformed into a homogeneous one. Then we apply two powerful theorems: Shur's Inequality and Muirhead's theorem. We begin with a simple example.

Example 6. (Hungary, 1996) Let a and b be positive real numbers with $a + b = 1$. Prove that

$$\frac{a^2}{a+1} + \frac{b^2}{b+1} \geq \frac{1}{3}.$$

Solution. Using the condition $a + b = 1$, we can reduce the given inequality to homogeneous one:

$$\frac{1}{3} \leq \frac{a^2}{(a+b)(a+(a+b))} + \frac{b^2}{(a+b)(b+(a+b))}$$

or

$$a^2b + ab^2 \leq a^3 + b^3,$$

which follows from

$$(a^3 + b^3) - (a^2b + ab^2) = (a-b)^2(a+b) \geq 0.$$

The equality holds if and only if $a = b = \frac{1}{2}$. □

Theorem 4.1.1. *Let a_1, a_2, b_1, b_2 be positive real numbers such that $a_1 + a_2 = b_1 + b_2$ and $max(a_1, a_2) \geq max(b_1, b_2)$. Let x and y be nonnegative real numbers. Then, we have*

$$x^{a_1}y^{a_2} + x^{a_2}y^{a_1} \geq x^{b_1}y^{b_2} + x^{b_2}y^{b_1}.$$

Proof. Without loss of generality, we can assume that $a_1 \geq a_2, b_1 \geq b_2, a_1 \geq b_1$. If x or y is zero, then it clearly holds. So, we assume that both x and y are nonzero. It follows from $a_1 + a_2 = b_1 + b_2$ that $a_1 - a_2 = (b_1 - a_2) + (b_2 - a_2)$. It's easy to check

$$\begin{aligned} x^{a_1}y^{a_2} + x^{a_2}y^{a_1} - x^{b_1}y^{b_2} - x^{b_2}y^{b_1} &= x^{a_2}y^{a_2}\left(x^{a_1-a_2} + y^{a_1-a_2} - x^{b_1-a_2}y^{b_2-a_2} - x^{b_2-a_2}y^{b_1-a_2}\right) \\ &= x^{a_2}y^{a_2}\left(x^{b_1-a_2} - y^{b_1-a_2}\right)\left(x^{b_2-a_2} - y^{b_2-a_2}\right) \\ &= \frac{1}{x^{a_2}y^{a_2}}\left(x^{b_1} - y^{b_1}\right)\left(x^{b_2} - y^{b_2}\right) \geq 0. \end{aligned}$$

□

Remark 4.1.1. *When does the equality hold in the above theorem?*

We now introduce two summation notations. Let $\mathcal{P}(x, y, z)$ be a three variables function of x, y, z. Let us define

$$\sum_{\text{cyclic}} \mathcal{P}(x, y, z) = \mathcal{P}(x, y, z) + \mathcal{P}(y, z, x) + \mathcal{P}(z, x, y)$$

and

$$\sum_{\text{sym}} \mathcal{P}(x, y, z) = \mathcal{P}(x, y, z) + \mathcal{P}(x, z, y) + \mathcal{P}(y, x, z) + \mathcal{P}(y, z, x) + \mathcal{P}(z, x, y) + \mathcal{P}(z, y, x).$$

Here, we have some examples:

$$\sum_{\text{cyclic}} x^3y = x^3y + y^3z + z^3x, \quad \sum_{\text{sym}} x^3 = 2(x^3 + y^3 + z^3),$$

$$\sum_{\text{sym}} x^2y = x^2y + x^2z + y^2z + y^2x + z^2x + z^2y, \quad \sum_{\text{sym}} xyz = 6xyz$$

Example 7. *Let x, y, z be positive real numbers. Show the cyclic inequality*

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}.$$

Third Solution. We break the homogeneity. After the substitution $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$, it becomes

$$a^2 + b^2 + c^2 \geq a + b + c.$$

Using the constraint $abc = 1$, we now impose the homogeneity to this as follows:

$$a^2 + b^2 + c^2 \geq (abc)^{\frac{1}{3}}(a + b + c).$$

After setting $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$, it then becomes

$$x^6 + y^6 + z^6 \geq x^4yz + xy^4z + xyz^4.$$

We now deduce

$$\sum_{\text{cyclic}} x^6 = \sum_{\text{cyclic}} \frac{x^6 + y^6}{2} \geq \sum_{\text{cyclic}} \frac{x^4y^2 + x^2y^4}{2} = \sum_{\text{cyclic}} x^4\left(\frac{y^2 + z^2}{2}\right) \geq \sum_{\text{cyclic}} x^4yz.$$

□

Epsilon 58. [IMO 1984/1 FRG] *Let x, y, z be nonnegative real numbers such that $x + y + z = 1$. Prove that*

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

Epsilon 59. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sqrt{x + y + z} \geq \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

4.2 Schur's Inequality and Muirhead's Theorem

Theorem 4.2.1. (Schur's Inequality) Let x, y, z be nonnegative real numbers. For any $r > 0$, we have

$$\sum_{\text{cyclic}} x^r(x-y)(x-z) \geq 0.$$

Proof. Since the inequality is symmetric in the three variables, we may assume without loss of generality that $x \geq y \geq z$. Then the given inequality may be rewritten as

$$(x-y)[x^r(x-z) - y^r(y-z)] + z^r(x-z)(y-z) \geq 0,$$

and every term on the left-hand side is clearly nonnegative. $\square$

Remark 4.2.1. *When does the equality hold in Schur's Inequality?*

Delta 19. *Disprove the following proposition: for all $a, b, c, d \geq 0$ and $r > 0$, we have*

$$a^r(a-b)(a-c)(a-d) + b^r(b-c)(b-d)(b-a) + c^r(c-a)(c-c)(a-d) + d^r(d-a)(d-b)(d-c) \geq 0.$$

Delta 20. **[LL 1971 HUN]** *Let a, b, c, d, e be real numbers. Prove the expression*

$$(a-b)(a-c)(a-d)(a-e)+(b-a)(b-c)(b-d)(b-e)+(c-a)(c-b)(c-d)(c-e)+(d-a)(d-b)(d-c)(a-e)+(e-a$$

is nonnegative.

The following special case of Schur's Inequality is useful :

$$\sum_{\text{cyclic}} x(x-y)(x-z) \geq 0 \Leftrightarrow 3xyz + \sum_{\text{cyclic}} x^3 \geq \sum_{\text{sym}} x^2y \Leftrightarrow \sum_{\text{sym}} xyz + \sum_{\text{sym}} x^3 \geq 2\sum_{\text{sym}} x^2y.$$

Epsilon 60. *Let x, y, z be nonnegative real numbers. Then, we have*

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

Epsilon 61. *Let $t \in (0, 3]$. For all $a, b, c \geq 0$, we have*

$$(3-t) + t(abc)^{\frac{2}{t}} + \sum_{\text{cyclic}} a^2 \geq 2\sum_{\text{cyclic}} ab.$$

Epsilon 62. (APMO 2004/5) Prove that, for all positive real numbers a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

Epsilon 63. **[IMO 2000/2 USA]** *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Epsilon 64. (Tournament of Towns 1997) Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1.$$

Delta 21. [TZ, p.142] Prove that for any acute triangle ABC,

$$\cot^3 A + \cot^3 B + \cot^3 C + 6\cot A \cot B \cot C \geq \cot A + \cot B + \cot C.$$

Delta 22. (Korea 1998) Let I be the incenter of a triangle ABC. Prove that

$$IA^2 + IB^2 + IC^2 \geq \frac{BC^2 + CA^2 + AB^2}{3}.$$

Delta 23. [IN, p.103] Let a, b, c be the lengths of a triangle. Prove that

$$a^2b + a^2c + b^2c + b^2a + c^2a + c^2b > a^3 + b^3 + c^3 + 2abc.$$

Delta 24. (Surányi's Inequality) Show that, for all $x_1, \cdots, x_n \geq 0$,

$$(n-1)\left({x_1}^n + \cdots {x_n}^n\right) + nx_1 \cdots x_n \geq (x_1 + \cdots + x_n)\left({x_1}^{n-1} + \cdots {x_n}^{n-1}\right).$$

Epsilon 65. (Muirhead's Theorem) Let $a_1, a_2, a_3, b_1, b_2, b_3$ be non-negative real numbers such that

$$a_1 \geq a_2 \geq a_3,\ b_1 \geq b_2 \geq b_3,\ a_1 \geq b_1,\ a_1 + a_2 \geq b_1 + b_2,\ a_1 + a_2 + a_3 = b_1 + b_2 + b_3.$$

(In this case, we say that the vector $\mathrm{a} = (a_1, a_2, a_3)$ majorizes the vector $\mathrm{b} = (b_1, b_2, b_3)$ and write $\mathrm{a} \succ \mathrm{b}$.) For all positive real numbers x, y, z, we have

$$\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} \geq \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.$$

Remark 4.2.2. *The equality holds if and only if $x = y = z$. However, if we allow $x = 0$ or $y = 0$ or $z = 0$, then one may easily check that the equality holds (after assuming $a_1, a_2, a_3 > 0$ and $b_1, b_2, b_3 > 0$) if and only if*

$$x = y = z \ \textit{ or } \ x = y,\ z = 0 \ \textit{ or } \ y = z,\ x = 0 \ \textit{ or } \ z = x,\ y = 0.$$

We can apply Muirhead's Theorem to establish Nesbitt's Inequality.

Proposition 4.2.1. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 10. *Clearing the denominators of the inequality, it becomes*

$$2\sum_{\text{cyclic}} a(a+b)(a+c) \geq 3(a+b)(b+c)(c+a)$$

or

$$\sum_{\text{sym}} a^3 \geq \sum_{\text{sym}} a^2b.$$

Epsilon 66. **[IMO 1995/2 RUS]** *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Epsilon 67. (Iran 1996) Let x, y, z be positive real numbers. Prove that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

Epsilon 68. *Let x, y, z be nonnegative real numbers with $xy + yz + zx = 1$. Prove that*

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{5}{2}.$$

4.3 Normalizations

In the previous sections, we transformed non-homogeneous inequalities into homogeneous ones. On the other hand, homogeneous inequalities also can be normalized in *various* ways. We offer two alternative solutions of the problem 8 by normalizations :

Epsilon 69. [IMO 2001/2 KOR] *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Epsilon 70. [IMO 1983/6 USA] *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Epsilon 71. (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{(a^2b+b^2c+c^2a)\left(ab^2+bc^2+ca^2\right)}\geq abc+\sqrt[3]{(a^3+abc)\left(b^3+abc\right)\left(c^3+abc\right)}$$

Epsilon 72. [IMO 1999/2 POL] *Let n be an integer with $n \geq 2$.*

(a) Determine the least constant C such that the inequality

$$\sum_{1\leq i<j\leq n} x_ix_j(x_i^2+x_j^2)\leq C\left(\sum_{1\leq i\leq n} x_i\right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

Delta 25. [SL 1991 POL] *Let n be a given integer with $n \geq 2$. Find the maximum value of*

$$\sum_{1\leq i<j\leq n} x_ix_j(x_i+x_j),$$

where $x_1, \cdots, x_n \geq 0$ and $x_1+\cdots+x_n = 1$.

We close this section with another proofs of Nesbitt's Inequality.

Proposition 4.3.1. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 11. *We may normalize to $a+b+c = 1$. Note that $0 < a, b, c < 1$. The problem is now to prove*

$$\sum_{\text{cyclic}}\frac{a}{b+c}=\sum_{\text{cyclic}} f(a)\geq\frac{3}{2},\ \ \textit{where}\ \ f(x)=\frac{x}{1-x}.$$

Since f is convex on $(0, 1)$, Jensen's Inequality shows that

$$\frac{1}{3}\sum_{\text{cyclic}} f(a)\geq f\left(\frac{a+b+c}{3}\right)=f\left(\frac{1}{3}\right)=\frac{1}{2}\ \ \textit{or}\ \ \sum_{\text{cyclic}} f(a)\geq\frac{3}{2}.$$

Proof 12. (Cao Minh Quang) Assume that $a+b+c = 1$. Note that $ab+bc+ca \leq \frac{1}{3}(a+b+c)^2 = \frac{1}{3}$. More strongly, we establish that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq 3-\frac{9}{2}(ab+bc+ca)$$

or

$$\left(\frac{a}{b+c}+\frac{9a(b+c)}{4}\right)+\left(\frac{b}{c+a}+\frac{9b(c+a)}{4}\right)+\left(\frac{c}{a+b}+\frac{9c(a+b)}{4}\right)\geq 3.$$

The AM-GM inequality shows that

$$\sum_{\text{cyclic}}\frac{a}{b+c}+\frac{9a(b+c)}{4}\geq\sum_{\text{cyclic}} 2\sqrt{\frac{a}{b+c}\cdot\frac{9a(b+c)}{4}}=\sum_{\text{cyclic}} 3a=3.$$

4.4 Cauchy-Schwarz Inequality and Hölder's Inequality

We begin with the following famous theorem:

Theorem 4.4.1. (The Cauchy-Schwarz Inequality) Whenever $a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$,

$$({a_1}^2 + \cdots + {a_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (a_1b_1 + \cdots + a_nb_n)^2.$$

First Proof. Let $A = \sqrt{{a_1}^2 + \cdots + {a_n}^2}$ and $B = \sqrt{{b_1}^2 + \cdots + {b_n}^2}$. In the case when $A = 0$, we get $a_1 = \cdots = a_n = 0$. Thus, the given inequality clearly holds. So, we may assume that $A, B > 0$. We may normalize to

$$1 = {a_1}^2 + \cdots + {a_n}^2 = {b_1}^2 + \cdots + {b_n}^2.$$

Hence, we need to to show that

$$|a_1b_1 + \cdots + a_nb_n| \leq 1.$$

We now apply the AM-GM inequality to deduce

$$|x_1y_1 + \cdots + x_ny_n| \leq |x_1y_1| + \cdots + |x_ny_n| \leq \frac{{x_1}^2 + {y_1}^2}{2} + \cdots + \frac{{x_n}^2 + {y_n}^2}{2} = 1.$$

□

Second Proof. It immediately follows from The Lagrange Identity:

$$\left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right) - \left(\sum_{i=1}^{n} a_ib_i\right)^2 = \sum_{1\leq i<j\leq n} (a_ib_j - a_jb_i)^2.$$

□

Delta 26. [IMO 2003/5 IRL] *Let n be a positive integer and let $x_1 \leq \cdots \leq x_n$ be real numbers. Prove that*

$$\left(\sum_{1\leq i,j\leq n} |x_i - x_j|\right)^2 \leq \frac{2\left(n^2-1\right)}{3} \sum_{1\leq i,j\leq n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \cdots, x_n$ is an arithmetic progression.

Delta 27. (Darij Grinberg) Suppose that $0 < a_1 \leq \cdots \leq a_n$ and $0 < b_1 \leq \cdots \leq b_n$ be real numbers. Show that

$$\frac{1}{4}\left(\sum_{k=1}^{n} a_k\right)^2\left(\sum_{k=1}^{n} b_k\right)^2 > \left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} {b_k}^2\right) - \left(\sum_{k=1}^{n} a_kb_k\right)^2$$

Delta 28. [LL 1971 AUT] *Let a, b, c be positive real numbers, $0 < a \leq b \leq c$. Prove that for any $x, y, z > 0$ the following inequality holds:*

$$\left(\frac{(a+c)^2}{4ac}\right)(x+y+z)^2 \geq (ax+by+cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right).$$

Delta 29. [LL 1987 AUS] *Let $a_1, a_2, a_3, b_1, b_2, b_3$ be positive real numbers. Prove that*

$$(a_1b_2 + a_1b_3 + a_2b_1 + a_2b_3 + a_3b_1 + a_3b_2)^2 \geq 4\left(a_1a_2 + a_2a_3 + a_3a_1\right)\left(b_1b_2 + b_2b_3 + b_3b_1\right)$$

and show that the two sides of the inequality are equal if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

Delta 30. [PF] Let $a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$. Suppose that $x \in [0, 1]$. Show that

$$\left(\sum_{i=1}^{n} {a_i}^2 + 2x\sum_{i<j} a_ia_j\right)\left(\sum_{i=1}^{n} {b_i}^2 + 2x\sum_{i<j} b_ib_j\right) \geq \left(\sum_{i=1}^{n} a_ib_i + x\sum_{i\leq j} a_ib_j\right)^2.$$

Delta 31. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Show that*

$$\begin{cases} (1)\ \sqrt{(a_1+\cdots+a_n)(b_1+\cdots+b_n)} \geq \sqrt{a_1b_1}+\cdots+\sqrt{a_nb_n}, \\ (2)\ \frac{{a_1}^2}{b_1}+\cdots+\frac{{a_n}^2}{b_n} \geq \frac{(a_1+\cdots+a_n)^2}{b_1+\cdots+b_n}, \\ (3)\ \frac{a_1}{{b_1}^2}+\cdots+\frac{a_n}{{b_n}^2} \geq \frac{1}{a_1+\cdots+a_n}\left(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}\right)^2, \\ (4)\ \frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n} \geq \frac{(a_1+\cdots+a_n)^2}{a_1b_1+\cdots+a_nb_n}. \end{cases}$$

Delta 32. [SL 1993 USA] *Prove that*

$$\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c} \geq \frac{2}{3}$$

for all positive real numbers a, b, c, d.

Epsilon 73. (APMO 1991) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers such that $a_1+\cdots+a_n = b_1+\cdots+b_n$. Show that

$$\frac{{a_1}^2}{a_1+b_1}+\cdots+\frac{{a_n}^2}{a_n+b_n} \geq \frac{a_1+\cdots+a_n}{2}.$$

Epsilon 74. *Let $a, b \geq 0$ with $a+b=1$. Prove that*

$$\sqrt{a^2+b}+\sqrt{a+b^2}+\sqrt{1+ab} \leq 3.$$

Show that the equality holds if and only if $(a,b)=(1,0)$ or $(a,b)=(0,1)$.

Epsilon 75. [LL 1992 UNK] (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

We now apply The Cauchy-Schwarz Inequality to prove Nesbitt's Inequality.

Proposition 4.4.1. (Nesbitt) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 13. *Applying The Cauchy-Schwarz Inequality, we have*

$$((b+c)+(c+a)+(a+b))\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq 3^2.$$

It follows that

$$\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b} \geq \frac{9}{2}$$

or

$$3+\sum_{\text{cyclic}}\frac{a}{b+c} \geq \frac{9}{2}.$$

Proof 14. *The Cauchy-Schwarz Inequality yields*

$$\sum_{\text{cyclic}}\frac{a}{b+c}\sum_{\text{cyclic}}a(b+c) \geq \left(\sum_{\text{cyclic}}a\right)^2$$

or

$$\sum_{\text{cyclic}}\frac{a}{b+c} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3}{2}.$$

Epsilon 76. (Gazeta Matematicã) Prove that, for all $a, b, c > 0$,

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4} \geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.$$

Epsilon 77. (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)} \geq abc + \sqrt[3]{(a^3 + abc)(b^3 + abc)(c^3 + abc)}$$

Epsilon 78. (Andrei Ciupan) Let a, b, c be positive real numbers such that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \geq 1.$$

Show that $a + b + c \geq ab + bc + ca$.

We now illustrate normalization techniques to establish classical theorems. Using the same idea in the proof of The Cauchy-Schwarz Inequality, we find a natural generalization :

Theorem 4.4.2. *Let $a_{ij}(i, j = 1, \cdots, n)$ be positive real numbers. Then, we have*

$$({a_{11}}^n + \cdots + {a_{1n}}^n) \cdots ({a_{n1}}^n + \cdots + {a_{nn}}^n) \geq (a_{11}a_{21} \cdots a_{n1} + \cdots + a_{1n}a_{2n} \cdots a_{nn})^n.$$

Proof. The inequality is homogeneous. We make the normalizations:

$$({a_{i1}}^n + \cdots + {a_{in}}^n)^{\frac{1}{n}} = 1$$

or

$${a_{i1}}^n + \cdots + {a_{in}}^n = 1,$$

for all $i = 1, \cdots, n$. Then, the inequality takes the form

$$a_{11}a_{21} \cdots a_{n1} + \cdots + a_{1n}a_{2n} \cdots a_{nn} \leq 1$$

or

$$\sum_{i=1}^{n} a_{i1} \cdots a_{in} \leq 1.$$

Hence, it suffices to show that, for all $i = 1, \cdots, n$,

$$a_{i1} \cdots a_{in} \leq \frac{1}{n}$$

where ${a_{i1}}^n + \cdots + {a_{in}}^n = 1$. To finish the proof, it remains to show the following *homogeneous* inequality. □

Theorem 4.4.3. (The AM-GM Inequality) Let $a_1, \cdots, a_n$ be positive real numbers. Then, we have

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Proof. Since it's homogeneous, we may rescale $a_1, \cdots, a_n$ so that $a_1 \cdots a_n = 1$. [1] We want to show that

$$a_1 \cdots a_n = 1 \implies a_1 + \cdots + a_n \geq n.$$

The proof is by induction on n. If $n = 1$, it's trivial. If $n = 2$, then we get $a_1 + a_2 - 2 = a_1 + a_2 - 2\sqrt{a_1a_2} = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$. Now, we assume that it holds for some positive integer $n \geq 2$. And let $a_1, \cdots, a_{n+1}$ be positive numbers such that $a_1 \cdots a_n a_{n+1}$=1. We may assume that $a_1 \geq 1 \geq a_2$. (Why?) It follows that $a_1a_2 + 1 - a_1 - a_2 = (a_1 - 1)(a_2 - 1) \leq 0$ so that $a_1a_2 + 1 \leq a_1 + a_2$. Since $(a_1a_2)a_3 \cdots a_n = 1$, by the induction hypothesis, we have

$$a_1a_2 + a_3 + \cdots + a_{n+1} \geq n.$$

It follows that $a_1 + a_2 - 1 + a_3 + \cdots + a_{n+1} \geq n$. □

[1] Set $x_i = \frac{a_i}{(a_1 \cdots a_n)^{\frac{1}{n}}}$ $(i = 1, \cdots, n)$. Then, we get $x_1 \cdots x_n = 1$ and it becomes $x_1 + \cdots + x_n \geq n$.

We now make simple observation. Let $a, b > 0$ and $m, n \in \mathbf{N}$. Take $x_1 = \cdots = x_m = a$ and $x_{m+1} = \cdots = x_{x_{m+n}} = b$. Applying the AM-GM inequality to $x_1, \cdots, x_{m+n} > 0$, we obtain

$$\frac{ma + nb}{m+n} \geq (a^m b^n)^{\frac{1}{m+n}} \quad \text{or} \quad \frac{m}{m+n}a + \frac{n}{m+n}b \geq a^{\frac{m}{m+n}} b^{\frac{n}{m+n}}.$$

Hence, for all positive *rational* numbers ω_1 and ω_2 with $\omega_1 + \omega_2 = 1$, we get

$$\omega_1\, a + \omega_2\, b \geq a^{\,\omega_1} b^{\,\omega_2}.$$

We now immediately have

Theorem 4.4.4. *Let ω_1, $\omega_2 > 0$ with $\omega_1 + \omega_2 = 1$. For all x, $y > 0$, we have*

$$\omega_1\, x + \omega_2\, y \geq x^{\,\omega_1} y^{\,\omega_2}.$$

Proof. We can choose a sequence $a_1, a_2, a_3, \cdots \in (0, 1)$ of rational numbers such that

$$\lim_{n\to\infty} a_n = \omega_1.$$

Set $b_i = 1 - a_i$, where $i \in \mathbb{N}$. Then, $b_1, b_2, b_3, \cdots \in (0, 1)$ is a sequence of rational numbers with

$$\lim_{n\to\infty} b_n = \omega_2.$$

From the previous observation, we have $a_n\, x + b_n\, y \geq x^{a_n} y^{b_n}$. By taking the limits to both sides, we get the result. $\square$

We may extend the above arguments to the n-variables. We see that the AM-GM inequality implies that

Theorem 4.4.5. (The Weighted AM-GM Inequality) Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n > 0$, we have

$$\omega_1\, x_1 + \cdots + \omega_n\, x_n \geq {x_1}^{\omega_1} \cdots {x_n}^{\omega_n}.$$

Also, it is a straightforward consequence of the concavity of $\ln x$. Indeed, The Weighted Jensen's Inequality shows that

$$\ln(\omega_1\, x_1 + \cdots + \omega_n\, x_n) \geq \omega_1 \ln(x_1) + \cdots + \omega_n \ln(x_n) = \ln({x_1}^{\omega_1} \cdots {x_n}^{\omega_n}).$$

Recall that The AM-GM Inequality is used to deduce the theorem 18, which is a generalization of The Cauchy-Schwarz Inequality. Since we now get the *weighted* version of The AM-GM Inequality, we establish *weighted* version of The Cauchy-Schwarz Inequality.

Epsilon 79. (Hölder's Inequality) Let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^{m} \left(\prod_{j=1}^{n} {x_{ij}}^{\omega_j} \right).$$

Life is good for only two things, discovering mathematics and teaching mathematics. - S. Poisson

Chapter 5

Minima and Maxima

Differentiate!

- S-S Chern

5.1 Extreme Value Problems

We first remind standard stuffs in mulitivariable calculus.

Definition 5.1.1. (Global Extremum vs. Local Extremum) Let $\mathcal{F} : D \subset \mathbb{R}^n \to \mathbb{R}$ be a function. We say that $\mathcal{F}$ has a global minimum value at $p \in D$ if we have $\mathcal{F}(\mathrm{x}) \geq \mathcal{F}(p)$ for all $\mathrm{x} \in D$. We say that $\mathcal{F}$ has a local minimum at $p \in D$ if there exists an open ball $\mathrm{B}_r(p)$ centered at p with radius $r > 0$ satisfying that $\mathcal{F}(\mathrm{x}) \geq \mathcal{F}(p)$ for all $\mathrm{x} \in D \cap \mathrm{B}_r(p)$. Analogusly, we say that $\mathcal{F}$ has a global maximum value at $p \in D$ if the inequality $\mathcal{F}(\mathrm{x}) \leq \mathcal{F}(p)$ holds for all $\mathrm{x} \in D$. We say that $\mathcal{F}$ has a local maximum at $p \in D$ if there exists an open ball $\mathrm{B}_r(p)$ such that $\mathcal{F}(\mathrm{x}) \leq \mathcal{F}(p)$ for all $\mathrm{x} \in D \cap \mathrm{B}_r(p)$.

Theorem 5.1.1. (The Extreume Value Theorem) Let X be a compact (or equivalently, closed and bounded) set in $\mathbb{R}^n$ and let $\mathcal{F} : X \to \mathbb{R}$ be a continuous function. Then, the function $\mathcal{F}$ takes on its global minimum and global maximum.

Theorem 5.1.2. (The First Derivative Test) Let U be an open set in $\mathbb{R}^n$ and let $\mathcal{F} : U \to \mathbb{R}$ be a differentiable function. If the function $\mathcal{F}$ has a local extremum at $p \in U$, then p is a critical point of $\mathcal{F}$, that is, $\nabla\mathcal{F}(p) = \mathbf{0}_{\mathbb{R}^n}$, where $\nabla\mathcal{F}(\mathrm{x}) = \left(\frac{\partial\mathcal{F}}{\partial x_1}, \cdots, \frac{\partial\mathcal{F}}{\partial x_n}\right)$ denotes the gradient vector of $\mathcal{F}$ at $\mathrm{x} = (x_1, \cdots, x_n)$.

Theorem 5.1.3. (The Second Derivative Test) Let $\mathcal{F} : U \to \mathbb{R}$ be a function of class $\mathcal{C}^2$, where U is an open set in $\mathbb{R}^n$. Assume that F has a critical point at $p \in U$

a. If the Hessian $H\mathcal{F}(p)$ is positive definite, then $\mathcal{F}(p)$ is a local minimum value.
b. If the Hessian $H\mathcal{F}(p)$ is negative definite, then $\mathcal{F}(p)$ is a local maximum value.

Theorem 5.1.4. (Sylvester's Criterion) A symmetric matrix of real numbers is positive definite if and only if the determinant of its upper-left squre submatrices is always positive.

Example 8. *Let x, y, z be positive real numbers with $xyz = 2$. Find the minimum value of $xy + \frac{1}{2}yz + \frac{1}{2}zx$.*

This problem admits a one-line-proof by using The AM-GM Inequality. However, here we review how to use multivariable calculus to prove the multivariable inequalities.

Solution. Our job is to find the global minimum of the two-variables function $\mathcal{F}: U \to \mathbb{R}$ defined by

$$\mathcal{F}(x, y) = xy + \frac{1}{x} + \frac{1}{y}$$

where $U = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0\}$, which is open in in $\mathbb{R}^2$.

Step 1. We first determine all criticial points of the function $\mathcal{F}$. We find that

$$\nabla \mathcal{F} = (\mathcal{F}_x, \mathcal{F}_y) = \left(y - \frac{2}{x^2}, x - \frac{1}{y^2} \right)$$

vanishes on U only at the point $(x, y) = (1, 1) \in U$. Since U is open in $\mathbb{R}^2$, by The First Derivative Test, we see that $(1, 1)$ is a unique candidate of local extremum point of $\mathcal{F}$. We now find that, by Sylvester's Criterion, the Hessian of $\mathcal{F}$ at $(1, 1)$

$$H\mathcal{F}(1,1) = \begin{pmatrix} \frac{2}{x^3} & 1 \\ 1 & \frac{2}{y^3} \end{pmatrix}_{(1,1)} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

is positive definite. The Second Derivative Test guarantees that $\mathcal{F}(1, 1) = 2$ is a *local* minimum of $\mathcal{F}$. We also observe that $\mathcal{F} \to \infty$ as $x \to 0^+$, $x \to \infty$, $y \to 0^+$ or $y \to \infty$. (In other words, the value of $\mathcal{F}$ is very big near the boundary of the domain.) We now show that $\mathcal{F}(1, 1) = 3$ is the *global* minimum of $\mathcal{F}$.

Step 2. We construct a barriar $\mathcal{W}$ around the point $(1, 1)$ as follows:

$$\mathcal{W} = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{1}{10} < x < 100,\ \frac{1}{10} < y < 100,\ xy < 10 \right\}.$$

We claim that the strict inequality $\mathcal{F}(\mathrm{x}) > 3$ holds for all $\mathrm{x} \in U - \overline{\mathcal{W}}$:

Case 1. $0 < x \leq \frac{1}{10}$: We obtain $\mathcal{F}(x, y) = xy + \frac{1}{x} + \frac{1}{y} > x \geq 10 > 3$.

Case 2. $0 < y \leq \frac{1}{10}$: We obtain $\mathcal{F}(x, y) = xy + \frac{1}{x} + \frac{1}{y} > y \geq 10 > 3$.

Case 3. $xy \geq 10$: We obtain $\mathcal{F}(x, y) = xy + \frac{1}{x} + \frac{1}{y} > xy \geq 10 > 3$.

In other words, the inequality $\mathcal{F}(\mathrm{x}) > 3$ holds for all points outside (including boundary) the barriar $\mathcal{W}$. Since $\mathcal{F}(1, 1) = 3$, in particular, the function $\mathcal{F}|_{\overline{\mathcal{W}}}$ *cannot* attain its global minimum on the boundary $\partial\mathcal{W}$.

Step 3. Since $\overline{\mathcal{W}}$ is compact, by The Extreume Value Theorem, the function $\mathcal{F}|_{\overline{\mathcal{W}}}$ attains its global minimum. Let $\mathcal{F}(p)$ be the global minimum value of $\mathcal{F}|_{\overline{\mathcal{W}}}$, where $p \in \overline{\mathcal{W}}$. By Step 2, wee see that $p \in \partial\mathcal{W}$ is impossible. In other words, $p \in \mathcal{W}$. Hence, $\mathcal{F}(p)$ gives a global minimum (and so a local minimum) of the function $\mathcal{F}|_{\mathcal{W}}$ over the open set $\mathcal{W}$. By the argument in Step 1 again, we conclude that $p = (1, 1)$. Since $\mathcal{F}(p) = \mathcal{F}(1, 1) = 3$ is the global minimum value of $\mathcal{F}|_{\overline{\mathcal{W}}}$, we know that, for all $x \in \overline{\mathcal{W}}$,

$$\mathcal{F}(\mathrm{x}) \geq \mathcal{F}(1, 1) = 3.$$

Combining results, we see that the inequality holds $\mathcal{F}(\mathrm{x}) \geq 3$ for all $\mathrm{x} \in U$. We therefore conclue that $\mathcal{F}(1, 1) = 3$ is the *global* minimum of $\mathcal{F}$. $\square$

5.2 Increasing Function Theorem

Even for multivariable inequalitie, *in practice*, techniques in one-variable calculus are more powerful (and easy-to-use) than ones from multivariate calculus.

Theorem 5.2.1. (The Increasing Function Theorem) Let $f : (a,b) \longrightarrow \mathbb{R}$ be a differentiable function. If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotone increasing on (a,b). If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on (a,b).

Proof. We first consider the case when $f'(x) > 0$ for all $x \in (a,b)$. Let $a < x_1 < x_2 < b$. We want to show that $f(x_1) < f(x_2)$. Applying The Mean Value Theorem, we find some $c \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $f'(c) > 0$, this equation means that $f(x_2) - f(x_1) > 0$. In case when $f'(x) \geq 0$ for all $x \in (a,b)$, we can also apply the Mean Value Theorem to get the result. □

Delta 33. [LL 1979 HEL] *Show that*

$$\frac{20}{60} < \sin 20^\circ < \frac{21}{60}.$$

Epsilon 80. (Ireland 2000) Let $x, y \geq 0$ with $x + y = 2$. Prove that $x^2y^2(x^2 + y^2) \leq 2$.

Epsilon 81. [IMO 1984/1 FRG] *Let x, y, z be nonnegative real numbers such that $x + y + z = 1$. Prove that*

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

Delta 34. *Let $a, b, c \in [0,1]$ be real numbers such that $a + b + c = 2$. Prove that*

$$2 \leq a^3 + b^3 + c^3 + 4abc \leq \frac{9}{4}.$$

Delta 35. [SL 1993 VNM] *Let a, b, c, d be four non-negative numbers satisfying $a + b + c + d = 1$. Prove the inequality*

$$abc + bcd + cda + dab \leq \frac{1}{27} + \frac{176}{27}abcd.$$

Epsilon 82. [IMO 2000/2 USA] *Let a, b, c be positive numbers such that $abc = 1$. Prove that*

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Any good idea can be stated in fifty words or less. - S. Ulam

Chapter 6

Convexity and Its Applications

> It gives me the same pleasure when someone else proves a good theorem as when I do it myself.
>
> - E. Landau

6.1 Jensen's Inequality

In the previous chapter, we deduced the weighted AM-GM inequality from The AM-GM Inequality. We use the same idea to study the following functional inequalities.

Epsilon 83. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then, the followings are equivalent.*

(1) For all $n \in \mathbb{N}$, the following inequality holds.

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a, b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$.

(2) For all $n \in \mathbb{N}$, the following inequality holds.

$$r_1 f(x_1) + \cdots + r_n f(x_n) \geq f(r_1\, x_1 + \cdots + r_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a, b]$ and $r_1, \cdots, r_n \in \mathbb{Q}^+$ with $r_1 + \cdots + r_n = 1$.

(3) For all $N \in \mathbb{N}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_N)}{N} \geq f\left(\frac{y_1 + \cdots + y_N}{N}\right)$$

for all $y_1, \cdots, y_N \in [a, b]$.

(4) For all $k \in \{0, 1, 2, \cdots\}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_{2^k})}{2^k} \geq f\left(\frac{y_1 + \cdots + y_{2^k}}{2^k}\right)$$

for all $y_1, \cdots, y_{2^k} \in [a, b]$.

(5) We have $\frac{1}{2} f(x) + \frac{1}{2} f(y) \geq f\left(\frac{x+y}{2}\right)$ for all $x, y \in [a, b]$.

(6) We have $\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$ for all $x, y \in [a, b]$ and $\lambda \in (0, 1)$.

Definition 6.1.1. *A real valued function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be convex if the inequality*

$$\lambda f(x) + (1 - \lambda)f(y) \geq f(\lambda x + (1 - \lambda)y)$$

holds for all $x, y \in [a, b]$ and $\lambda \in (0, 1)$.

The above proposition says that

Corollary 6.1.1. (Jensen's Inequality) If $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous convex function, then for all $x_1, \cdots, x_n \in [a, b]$, we have

$$\frac{f(x_1) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 + \cdots + x_n}{n}\right).$$

Delta 36. [SL 1998 AUS] *Let $r_1, \cdots, r_n$ be real numbers greater than or equal to 1. Prove that*

$$\frac{1}{r_1 + 1} + \cdots + \frac{1}{r_n + 1} \geq \frac{n}{\sqrt[n]{r_1 \cdots r_n} + 1}.$$

Corollary 6.1.2. (The Weighted Jensen's Inequality) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous convex function. Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n \in [a, b]$, we have

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

In fact, we can almost drop the continuity of f. As an exercise, show that every convex function on $[a, b]$ is continuous on (a, b). Hence, every convex function on $\mathbb{R}$ is continuous on $\mathbb{R}$. By the above result again, we get

Corollary 6.1.3. (The Convexity Criterion I) If a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ satisfies the midpoint convexity

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x + y}{2}\right)$$

for all $x, y \in [a, b]$, then the function f is convex on $[a, b]$.

Delta 37. (The Convexity Criterion II) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function which are differentiable twice in (a, b). Show that (1) $f''(x) \geq 0$ for all $x \in (a, b)$ if and only if (2) f is convex on (a, b).

We now present an inductive proof of The Weighted Jensen's Inequality. It turns out that we can completely drop the continuity of f.

Third Proof. It clearly holds for $n = 1, 2$. We now assume that it holds for some $n \in \mathbb{N}$. Let $x_1, \cdots, x_n, x_{n+1} \in [a, b]$ and $\omega_1, \cdots, \omega_{n+1} > 0$ with $\omega_1 + \cdots + \omega_{n+1} = 1$. Since we have the equality

$$\frac{\omega_1}{1 - \omega_{n+1}} + \cdots + \frac{\omega_n}{1 - \omega_{n+1}} = 1,$$

by the induction hypothesis, we obtain

$$\begin{aligned}
& \omega_1 f(x_1) + \cdots + \omega_{n+1} f(x_{n+1}) \\
= \;& (1 - \omega_{n+1}) \left(\frac{\omega_1}{1 - \omega_{n+1}} f(x_1) + \cdots + \frac{\omega_n}{1 - \omega_{n+1}} f(x_n)\right) + \omega_{n+1} f(x_{n+1}) \\
\geq \;& (1 - \omega_{n+1}) f\left(\frac{\omega_1}{1 - \omega_{n+1}} x_1 + \cdots + \frac{\omega_n}{1 - \omega_{n+1}} x_n\right) + \omega_{n+1} f(x_{n+1}) \\
\geq \;& f\left((1 - \omega_{n+1})\left[\frac{\omega_1}{1 - \omega_{n+1}} x_1 + \cdots + \frac{\omega_n}{1 - \omega_{n+1}} x_n\right] + \omega_{n+1} x_{n+1}\right) \\
= \;& f(\omega_1 x_1 + \cdots + \omega_{n+1} x_{n+1}).
\end{aligned}$$

□

6.2 Power Mean Inequality

The notion of convexity is one of the most important concepts in analysis. Jensen's Inequality is the most powerful tool in theory of inequalities. The Power Mean Inequality can be proved by applying Jensen's inequality in two ways. We begin with two simple lemmas.

Lemma 6.2.1. *Let a, b, and c be positive real numbers. Let us define a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by*

$$f(x) = \ln\left(\frac{a^x + b^x + c^x}{3}\right),$$

where $x \in \mathbb{R}$. Then, we obtain $f'(0) = \ln{(abc)}^{\frac{1}{3}}$.

Proof. We compute $f'(x) = \frac{a^x \ln a + b^x \ln b + c^x \ln c}{a^x + b^x + c^x}$. Then, $f'(0) = \frac{\ln a + \ln b + \ln c}{3} = \ln{(abc)}^{\frac{1}{3}}$. □

Epsilon 84. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Suppose that f is monotone increasing on $(0, \infty)$ and monotone increasing on $(-\infty, 0)$. Then, f is monotone increasing on $\mathbb{R}$.*

Epsilon 85. (Power Mean inequality for Three Variables) Let a, b, and c be positive real numbers. We define a function $M_{(a,b,c)} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$M_{(a,b,c)}(0) = \sqrt[3]{abc}\,,\quad M_{(a,b,c)}(r) = \left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}} \quad (r \neq 0).$$

Then, $M_{(a,b,c)}$ is a monotone increasing continuous function.

In particular, we deduce The RMS-AM-GM-HM Inequality for three variables.

Corollary 6.2.1. *For all positive real numbers a, b, and c, we have*

$$\sqrt{\frac{a^2 + b^2 + c^2}{3}} \geq \frac{a + b + c}{3} \geq \sqrt[3]{abc} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}.$$

Proof. The Power Mean Inequality implies that

$$M_{(a,b,c)}(2) \geq M_{(a,b,c)}(1) \geq M_{(a,b,c)}(0) \geq M_{(a,b,c)}(-1).$$

□

Delta 38. [SL 2004 THA] *Let $a, b, c > 0$ and $ab + bc + ca = 1$. Prove the inequality*

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

Delta 39. [SL 1998 RUS] *Let x, y, and z be positive real numbers such that $xyz = 1$. Prove that*

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

Delta 40. [LL 1992 POL] *For positive real numbers a, b, c, define*

$$A = \frac{a + b + c}{3},\ G = (abc)^{\frac{1}{3}},\ H = \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}.$$

Prove that

$$\left(\frac{A}{G}\right)^3 \geq \frac{1}{4} + \frac{3}{4} \cdot \frac{A}{H}.$$

Using the convexity of $x \ln x$ or the convexity of x^λ ($\lambda \geq 1$), we can also establish the monotonicity of the power means for n positive real numbers.

Theorem 6.2.1. (The Power Mean Inequality) Let $x_1, \cdots, x_n$ be positive real numbers. The power mean of order r is defined by

$$M_{(x_1,\cdots,x_n)}(0) = \sqrt[n]{x_1 \cdots x_n}\,, \quad M_{(x_1,\cdots,x_n)}(r) = \left(\frac{x_1^r + \cdots + {x_n}^r}{n}\right)^{\frac{1}{r}} \quad (r \neq 0).$$

Then, the function $M_{(x_1,\cdots,x_n)} : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and monotone increasing.

Corollary 6.2.2. (The Geometric Mean as a Limit) Let $x_1, \cdots, x_n > 0$. Then,

$$\sqrt[n]{x_1 \cdots x_n} = \lim_{r \to 0} \left(\frac{{x_1}^r + \cdots + {x_n}^r}{n}\right)^{\frac{1}{r}}.$$

Theorem 6.2.2. (The RMS-AM-GM-HM Inequality) For all $x_1, \cdots, x_n > 0$, we have

$$\sqrt{\frac{{x_1}^2 + \cdots + {x_n}^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}.$$

Delta 41. [SL 2004 IRL] *Let $a_1, \cdots, a_n$ be positive real numbers, $n > 1$. Denote by g_n their geometric mean, and by $A_1, \cdots, A_n$ the sequence of arithmetic means defined by*

$$A_k = \frac{a_1 + \cdots + a_k}{k},\ k = 1, \cdots, n.$$

Let G_n be the geometric mean of $A_1, \cdots, A_n$. Prove the inequality

$$n + 1 \geq \sqrt[n]{\frac{G_n}{A_n}} + \frac{g_n}{G_n}$$

and establish the cases of equality.

6.3 Hardy-Littlewood-Pólya Inequality

We first meet a famous inequality proved by the Romanian mathematician T. Popoviciu.

Theorem 6.3.1. (Popoviciu's Inequality) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a convex function. For all $x, y, z \in [a, b]$, we have

$$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \geq 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{z+x}{2}\right).$$

Proof. We break the symmetry. Since the inequality is symmetric, we may assume that $x \leq y \leq z$.

Case 1. $y \geq \frac{x+y+z}{3}$: The key idea is to make the following *geometric* observation:

$$\frac{z+x}{2},\ \frac{x+y}{2} \in \left[x,\ \frac{x+y+z}{3}\right].$$

It guarantees the existence of two positive weights $\lambda_1, \lambda_2 \in [0, 1]$ satisfying that

$$\begin{cases} \frac{z+x}{2} = (1-\lambda_1)\,x + \lambda_1 \frac{x+y+z}{3}, \\ \frac{x+y}{2} = (1-\lambda_2)\,x + \lambda_2 \frac{x+y+z}{3}, \\ \lambda_1 + \lambda_2 = \frac{3}{2}. \end{cases}$$

Now, Jensen's inequality shows that

$$\begin{aligned} & f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \\ \leq\ & (1-\lambda_2)\,f(x) + \lambda_2 f\left(\frac{x+y+z}{3}\right) + \frac{f(y)+f(z)}{2} + (1-\lambda_1)\,f(x) + \lambda_1 f\left(\frac{x+y+z}{3}\right) \\ \leq\ & \frac{1}{2}\left(f(x)+f(y)+f(z)\right) + \frac{3}{2} f\left(\frac{x+y+z}{3}\right). \end{aligned}$$

The proof of the second case uses the same idea.

Case 2. $y \leq \frac{x+y+z}{3}$: We make the following *geometric* observation:

$$\frac{z+x}{2},\ \frac{y+z}{2} \in \left[\frac{x+y+z}{3}, z\right].$$

It guarantees the existence of two positive weights $\mu_1, \mu_2 \in [0, 1]$ satisfying that

$$\begin{cases} \frac{z+x}{2} = (1-\mu_1)\,z + \mu_1 \frac{x+y+z}{3}, \\ \frac{y+z}{2} = (1-\mu_2)\,z + \mu_2 \frac{x+y+z}{3}, \\ \mu_1 + \mu_2 = \frac{3}{2}. \end{cases}$$

Jensen's inequality implies that

$$\begin{aligned} & f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \\ \leq\ & \frac{f(x)+f(y)}{2} + (1-\mu_2)\,f(z) + \mu_2 f\left(\frac{x+y+z}{3}\right) + (1-\mu_1)\,f(z) + \mu_1 f\left(\frac{x+y+z}{3}\right) \\ \leq\ & \frac{1}{2}\left(f(x)+f(y)+f(z)\right) + \frac{3}{2} f\left(\frac{x+y+z}{3}\right). \end{aligned}$$

□

Epsilon 86. *Let x, y, z be nonnegative real numbers. Then, we have*

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

Extending the proof of Popoviciu's Inequality, we can establish a majorization inequality.

Definition 6.3.1. *We say that a vector* $\mathrm{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ *majorizes a vector* $\mathrm{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$ *if we have*

(1) $x_1 \geq \cdots \geq x_n$, $y_1 \geq \cdots \geq y_n$,
(2) $x_1 + \cdots + x_k \geq y_1 + \cdots + y_k$ *for all* $1 \leq k \leq n-1$,
(3) $x_1 + \cdots + x_n = y_1 + \cdots + y_n$.

In this case, we write $\mathrm{x} \succ \mathrm{y}$.

Theorem 6.3.2. (The Hardy-Littlewood-Pólya Inequality) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a convex function. Suppose that $(x_1, \cdots, x_n)$ majorizes $(y_1, \cdots, y_n)$, where $x_1, \cdots, x_n, y_1, \cdots, y_n \in [a, b]$. Then, we obtain

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

Epsilon 87. *Let ABC be an acute triangle. Show that*

$$\cos A + \cos B + \cos C \geq 1.$$

Epsilon 88. *Let ABC be a triangle. Show that*

$$\tan^2\left(\frac{A}{4}\right) + \tan^2\left(\frac{B}{4}\right) + \tan^2\left(\frac{C}{4}\right) \leq 1.$$

Epsilon 89. *Use The Hardy-Littlewood-Pólya Inequality to deduce Popoviciu's Inequality.*

Epsilon 90. **[IMO 1999/2 POL]** *Let n be an integer with $n \geq 2$.*

Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.

It's not that I'm so smart, it's just that I stay with problems longer. - A. Einstein

Chapter 7

Epsilons

God has a transfinite book with all the theorems and their best proofs

- P. Erdős

1 Let a, b, c be positive real numbers. Prove the inequality

$$\left(1+a^2\right)\left(1+b^2\right)\left(1+c^2\right) \geq (a+b)(b+c)(c+a).$$

Show that the equality holds if and only if $(a, b, c) = (1, 1, 1)$.

Solution. The inequality has the *symmetric* face:

$$\left(1+a^2\right)\left(1+b^2\right) \cdot \left(1+b^2\right)\left(1+c^2\right) \cdot \left(1+c^2\right)\left(1+a^2\right) \geq (a+b)^2(b+c)^2(c+a)^2.$$

Now, the symmetry of this expression gives the *right* approach. We check that, for $x, y > 0$,

$$\left(1+x^2\right)\left(1+y^2\right) \geq (x+y)^2$$

with the equality $xy = 1$. However, it immediately follows from the identity

$$\left(1+x^2\right)\left(1+y^2\right) - (x+y)^2 = (1-xy)^2.$$

It is easy to check that the equality in the original inequalty occurs only when $a = b = c = 1$. $\square$

2 (Poland 2006) Let a, b, c be positive real numbers with $ab + bc + ca = abc$. Prove that

$$\frac{a^4 + b^4}{ab(a^3 + b^3)} + \frac{b^4 + c^4}{bc(b^3 + c^3)} + \frac{c^4 + a^4}{ca(c^3 + a^3)} \geq 1.$$

Solution. We first notice that the constraint can be written as

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

It is now enough to establish the auxiliary inequality

$$\frac{x^4 + y^4}{xy(x^3 + y^3)} \geq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)$$

or

$$2\left(x^4 + y^4\right) \geq \left(x^3 + y^3\right)(x + y),$$

where $x, y > 0$. However, we obtain

$$2\left(x^4 + y^4\right) - \left(x^3 + y^3\right)(x + y) = x^4 + y^4 - x^3 y - x y^3 = \left(x^3 - y^3\right)(x - y) \geq 0.$$

□

3 (APMO 1996) Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}.$$

Proof. The left hand side admits the following decomposition

$$\frac{\sqrt{c+a-b}+\sqrt{a+b-c}}{2}+\frac{\sqrt{a+b-c}+\sqrt{b+c-a}}{2}+\frac{\sqrt{b+c-a}+\sqrt{c+a-b}}{2}.$$

We now use the inequality $\frac{\sqrt{x}+\sqrt{y}}{2} \leq \sqrt{\frac{x+y}{2}}$ to deduce

$$\frac{\sqrt{c+a-b}+\sqrt{a+b-c}}{2} \leq \sqrt{a},$$

$$\frac{\sqrt{a+b-c}+\sqrt{b+c-a}}{2} \leq \sqrt{b},$$

$$\frac{\sqrt{b+c-a}+\sqrt{c+a-b}}{2} \leq \sqrt{c}.$$

Adding these three inequalities, we get the result. □

4 Let a, b, c be the lengths of a triangle. Show that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2.$$

Proof. Since the inequality is symmetric in the three variables, we may assume that $a \leq b \leq c$. We obtain

$$\frac{a}{b+c} \leq \frac{a}{a+b}, \quad \frac{b}{c+a} \leq \frac{b}{a+b}, \quad \frac{c}{a+b}<1.$$

Adding these three inequalities, we get the result. □

5 (USA 1980) Prove that, for all positive real numbers a, b, c,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

Solution. Since the inequality is symmetric in the three variables, we may assume that $a \leq b \leq c$. Our first step is to bring the estimation

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} \leq \frac{a}{a+b+1} + \frac{b}{a+b+1} + \frac{c}{a+b+1} \leq \frac{a+b+c}{a+b+1}.$$

It now remains to check that

$$\frac{a+b+c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

or

$$(1-a)(1-b)(1-c) \leq \frac{1-c}{a+b+1}$$

or

$$(1-a)(1-b)(a+b+1) \leq 1.$$

We indeed obtain the estimation

$$(1-a)(1-b)(a+b+1) \leq (1-a)(1-b)(1+a)(1+b) = \left(1-a^2\right)\left(1-b^2\right) \leq 1.$$

$\square$

6 ([AE], p. 186) Show that, for all $a, b, c \in [0, 1]$,

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \leq 2.$$

Proof. Since the inequality is symmetric in the three variables, we may begin with the assumption $0 \leq a \geq b \geq c \leq 1$. We first give term-by-term estimation:

$$\frac{a}{1+bc} \leq \frac{a}{1+ab}, \quad \frac{b}{1+ca} \leq \frac{b}{1+ab}, \quad \frac{c}{1+ab} \leq \frac{1}{1+ab}.$$

Summing up these three, we reach

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \leq \frac{a+b+1}{1+ab}.$$

We now want to show the inequality

$$\frac{a+b+1}{1+ab} \leq 2$$

or

$$a + b + 1 \leq 2 + 2ab$$

or

$$a + b \leq 1 + 2ab.$$

However, it is immediate that $1 + 2ab - a - b = ab + (1-a)(1-b)$ is clearly non-negative. $\square$

7 **[SL 2006 KOR]** Let a, b, c be the lengths of the sides of a triangle. Prove the inequality

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

Solution. Since the inequality is symmetric in the three variables, we may assume that $a \geq b \geq c$. We claim that

$$\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 1$$

and

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \leq 2.$$

It is clear that the denominators are positive. So, the first inequality is equivalent to

$$\sqrt{a}+\sqrt{b} \geq \sqrt{a+b-c}+\sqrt{c}.$$

or

$$\left(\sqrt{a}+\sqrt{b}\right)^2 \geq \left(\sqrt{a+b-c}+\sqrt{c}\right)^2$$

or

$$\sqrt{ab} \geq \sqrt{c(a+b-c)}$$

or

$$ab \geq c(a+b-c),$$

which immediately follows from $(a-c)(b-c) \geq 0$. Now, we prove the second inequality. Setting $p=\sqrt{a}+\sqrt{b}$ and $q=\sqrt{a}-\sqrt{b}$, we obtain $a-b=pq$ and $p \geq 2\sqrt{c}$. It now becomes

$$\frac{\sqrt{c-pq}}{\sqrt{c}-q}+\frac{\sqrt{c+pq}}{\sqrt{c}+q} \leq 2.$$

We now apply The Cauchy-Schwartz Inequality to deduce

$$\begin{aligned}
\left(\frac{\sqrt{c-pq}}{\sqrt{c}-q}+\frac{\sqrt{c+pq}}{\sqrt{c}+q}\right)^2 &\leq \left(\frac{c-pq}{\sqrt{c}-q}+\frac{c+pq}{\sqrt{c}+q}\right)\left(\frac{1}{\sqrt{c}-q}+\frac{1}{\sqrt{c}+q}\right) \\
&= \frac{2\left(c\sqrt{c}-pq^2\right)}{c-q^2}\cdot\frac{2\sqrt{c}}{c-q^2} \\
&= 4\,\frac{c^2-\sqrt{c}pq^2}{\left(c-q^2\right)^2} \\
&\leq 4\,\frac{c^2-2cq^2}{\left(c-q^2\right)^2} \\
&\leq 4\,\frac{c^2-2cq^2+q^4}{\left(c-q^2\right)^2} \\
&\leq 4.
\end{aligned}$$

We find that the equality holds if and only if $a=b=c$. □

8 Let $f(x,y) = xy\left(x^3+y^3\right)$ for $x, y \geq 0$ with $x+y=2$. Prove the inequality

$$f(x,y) \leq f\left(1+\frac{1}{\sqrt{3}}, 1-\frac{1}{\sqrt{3}}\right) = f\left(1-\frac{1}{\sqrt{3}}, 1+\frac{1}{\sqrt{3}}\right).$$

First Solution. We write $(x,y) = (1+\epsilon, 1-\epsilon)$ for some $\epsilon \in (-1,1)$. It follows that

$$\begin{aligned} f(x,y) &= (1+\epsilon)(1-\epsilon)\left((1+\epsilon)^3+(1-\epsilon)^3\right) \\ &= \left(1-\epsilon^2\right)\left(6\epsilon^2+2\right) \\ &= -6\left(\epsilon^2-\frac{1}{3}\right)^2+\frac{8}{3} \\ &\leq \frac{8}{3} \\ &= f\left(1\pm\frac{1}{\sqrt{3}}, 1\mp\frac{1}{\sqrt{3}}\right). \end{aligned}$$

□

Second Solution. The AM-GM Inequality gives

$$f(x,y) = xy(x+y)\left((x+y)^2-3xy\right) = 2xy(4-4xy) \leq \frac{2}{3}\left(\frac{3xy+(4-3xy)}{2}\right)^2 = \frac{8}{3}.$$

□

9 Let $a, b \geq 0$ with $a+b=1$. Prove that

$$\sqrt{a^2+b}+\sqrt{a+b^2}+\sqrt{1+ab} \leq 3.$$

Show that the equality holds if and only if $(a, b)=(1,0)$ or $(a, b)=(0,1)$.

First Solution. We may begin with the assumption $a \geq \frac{1}{2} \geq b$. The AM-GM Inequality yields

$$2+b \geq 1+(1+ab) \geq 2\sqrt{1+ab}$$

with the equality $b=0$. We next show that

$$3+a \geq 4\sqrt{a^2-a+1}$$

or

$$(3+a)^2 \geq 16\left(a^2-a+1\right)$$

or

$$(15a-7)(1-a) \geq 0.$$

Since we have $a \in \left[\frac{1}{2}, 1\right]$, the inequality clearly holds with the equality $a=1$. Since we have

$$a^2+b=a^2-a+1=a+(1-a)^2=a+b^2$$

we conclude that

$$2\sqrt{a^2+b}+2\sqrt{a+b^2}+2\sqrt{1+ab} \leq 3+a+(2+b)=6.$$

□

10 (USA 1981) Let ABC be a triangle. Prove that

$$\sin 3A + \sin 3B + \sin 3C \leq \frac{3\sqrt{3}}{2}.$$

Solution. We observe that the sine function is *not* cocave on $[0, 3\pi]$ and that it is negative on $(\pi, 2\pi)$. Since the inequality is symmetric in the three variables, we may assume that $A \leq B \leq C$. Observe that $A + B + C = \pi$ and that $3A, 3B, 3C \in [0, 3\pi]$. It is clear that $A \leq \frac{\pi}{3} \leq C$.

We see that either $3B \in [2\pi, 3\pi)$ or $3C \in (0, \pi)$ is impossible. In the case when $3B \in [\pi, 2\pi)$, we obtain the estimation

$$\sin 3A + \sin 3B + \sin 3C \leq 1 + 0 + 1 = 2 < \frac{3\sqrt{3}}{2}.$$

So, we may assume that $3B \in (0, \pi)$. Similarly, in the case when $3C \in [\pi, 2\pi]$, we obtain

$$\sin 3A + \sin 3B + \sin 3C \leq 1 + 1 + 0 = 2 < \frac{3\sqrt{3}}{2}.$$

Hence, we also assume $3C \in (2\pi, 3\pi)$. Now, our assmptions become $A \leq B < \frac{1}{3}\pi$ and $\frac{2}{3}\pi < C$. After the substitution $\theta = C - \frac{2}{3}\pi$, the trigonometric inequality becomes

$$\sin 3A + \sin 3B + \sin 3\theta \leq \frac{3\sqrt{3}}{2}.$$

Since $3A, 3B, 3\theta \in (0, \pi)$ and since the sine function is concave on $[0, \pi]$, Jensen's Inequality gives

$$\sin 3A + \sin 3B + \sin 3\theta \leq 3 \sin\left(\frac{3A + 3B + 3\theta}{3}\right) = 3 \sin\left(\frac{3A + 3B + 3C - 2\pi}{3}\right) = 3 \sin\left(\frac{\pi}{3}\right).$$

Under the assumption $A \leq B \leq C$, the equality occurs only when $(A, B, C) = \left(\frac{1}{9}\pi, \frac{1}{9}\pi, \frac{7}{9}\pi\right)$. □

11 (Chebyshev's Inequality) Let $x_1, \cdots, x_n$ and $y_1, \cdots y_n$ be two monotone increasing sequences of real numbers:

$$x_1 \leq \cdots \leq x_n,\ y_1 \leq \cdots \leq y_n.$$

Then, we have the estimation

$$\sum_{i=1}^{n} x_i y_i \geq \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} y_i \right).$$

Proof. We observe that two sequences are similarly ordered in the sense that

$$(x_i - x_j)(y_i - y_j) \geq 0$$

for all $1 \leq i, j \leq n$. Now, the given inequality is an immediate consequence of the identity

$$\frac{1}{n} \sum_{i=1}^{n} x_i y_i - \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right) \frac{1}{n} \left(\sum_{i=1}^{n} y_i \right) = \frac{1}{n^2} \sum_{1 \leq i,j \leq n} (x_i - x_j)(y_i - y_j).$$

□

12 (United Kingdom 2002) For all $a, b, c \in (0,1)$, show that

$$\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}.$$

First Solution. Since the inequality is symmetric in the three variables, we may assume that $a \geq b \geq c$. Then, we have $\frac{1}{1-a} \geq \frac{1}{1-b} \geq \frac{1}{1-c}$. By Chebyshev's Inequality, The AM-HM Inequality and The AM-GM Inequality, we obtain

$$\begin{aligned}
\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c} &\geq \frac{1}{3}(a+b+c)\left(\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\right)\\
&\geq \frac{1}{3}(a+b+c)\left(\frac{9}{(1-a)+(1-b)+(1-c)}\right)\\
&= \frac{1}{3}\left(\frac{a+b+c}{3-(a+b+c)}\right)\\
&\geq \frac{1}{3}\cdot\frac{3\sqrt[3]{abc}}{3-3\sqrt[3]{abc}}
\end{aligned}$$

□

13 **[IMO 1995/2 RUS]** Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

First Solution. After the substitution $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$, we get $xyz = 1$. The inequality takes the form

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

Since the inequality is symmetric in the three variables, we may assume that $x \geq y \geq z$. Observe that $x^2 \geq y^2 \geq z^2$ and $\frac{1}{y+z} \geq \frac{1}{z+x} \geq \frac{1}{x+y}$. Chebyshev's Inequality and The AM-HM Inequality offer the estimation

$$\begin{aligned}
\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} &\geq \frac{1}{3}\left(x^2+y^2+z^2\right)\left(\frac{1}{y+z} + \frac{1}{z+x} + \frac{1}{x+y}\right) \\
&\geq \frac{1}{3}\left(x^2+y^2+z^2\right)\left(\frac{9}{(y+z)+(z+x)+(x+y)}\right) \\
&= \frac{3}{2} \cdot \frac{x^2+y^2+z^2}{x+y+z}.
\end{aligned}$$

Finally, we have $x^2+y^2+z^2 \geq \frac{1}{3}(x+y+z)^2 \geq (x+y+z)\sqrt[3]{xyz} = x+y+z$. □

14 (APMO 1991) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers such that $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. Show that

$$\frac{{a_1}^2}{a_1 + b_1} + \cdots + \frac{{a_n}^2}{a_n + b_n} \geq \frac{a_1 + \cdots + a_n}{2}.$$

First Solution. The key observation is the following identity:

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i} = \frac{1}{2} \sum_{i=1}^{n} \frac{{a_i}^2 + {b_i}^2}{a_i + b_i},$$

which is equivalent to

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i} = \sum_{i=1}^{n} \frac{{b_i}^2}{a_i + b_i},$$

which immediately follows from

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i} - \sum_{i=1}^{n} \frac{{b_i}^2}{a_i + b_i} = \sum_{i=1}^{n} \frac{{a_i}^2 - {b_i}^2}{a_i + b_i} = \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i = 0.$$

Our strategy is to establish the following *symmetric* inequality

$$\frac{1}{2} \sum_{i=1}^{n} \frac{{a_i}^2 + {b_i}^2}{a_i + b_i} \geq \frac{a_1 + \cdots + a_n + b_1 + \cdots + b_n}{4}.$$

It now remains to check the the auxiliary inequality

$$\frac{a^2 + b^2}{a + b} \geq \frac{a + b}{2},$$

where $a, b > 0$. Indeed, we have $2\left(a^2 + b^2\right) - (a + b)^2 = (a - b)^2 \geq 0$. $\square$

15 Let x, y, z be positive real numbers. Show the cyclic inequality

$$\frac{x}{2x+y} + \frac{y}{2y+z} + \frac{z}{2z+x} \leq 1.$$

Solution. We first break the homogeneity. The original inequality can be rewritten as

$$\frac{1}{2+\frac{y}{x}} + \frac{1}{2+\frac{z}{y}} + \frac{1}{2+\frac{x}{z}} \leq 1$$

The key idea is to employ the substitution

$$a = \frac{y}{x},\ b = \frac{z}{y},\ c = \frac{x}{z}.$$

It follows that $abc = 1$. It now admits the symmetry in the variables:

$$\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} \leq 1$$

Clearing denominators, it becomes

$$(2+a)(2+b) + (2+b)(2+c) + (2+c)(2+a) \leq (2+a)(2+b)(2+c)$$

or

$$12 + 4(a+b+c) + ab + bc + ca \leq 8 + 4(a+b+c) + 2(ab+bc+ca) + 1$$

or

$$3 \leq ab + bc + ca.$$

Applying The AM-GM Inequality, we obtain $ab + bc + ca \geq 3\,(abc)^{\frac{1}{3}} = 3.$ □

16 Let x, y, z be positive real numbers with $x+y+z=3$. Show the cyclic inequality

$$\frac{x^3}{x^2+xy+y^2}+\frac{y^3}{y^2+yz+z^2}+\frac{z^3}{z^2+zx+x^2} \geq 1.$$

Proof. Let $f(x, y, z)$ denote the left hand side of the inequality. The key observation is to employ the identity $f(x, y, z)=f(y, z, x)$. Indeed, we find that

$$\begin{aligned}
f(x, y, z)-f(y, z, x) & = \frac{x^3-y^3}{x^2+xy+y^2}+\frac{y^3-z^3}{y^2+yz+z^2}+\frac{z^3-x^3}{z^2+zx+x^2} \\
& = (x-y)+(y-z)+(z-x) \\
& = 0.
\end{aligned}$$

Our strategy is to establish the following *symmetric* inequality

$$\frac{f(x, y, z)+f(y, z, x)}{2} \geq 1.$$

or

$$\frac{x^3+y^3}{x^2+xy+y^2}+\frac{y^3+z^3}{y^2+yz+z^2}+\frac{z^3+x^3}{z^2+zx+x^2} \geq 2.$$

It now remains to check the the auxiliary inequality

$$\frac{a^3+b^3}{a^2+ab+b^2} \geq \frac{a+b}{3},$$

where $a, b>0$. Indeed, we obtain the equality

$$3\left(a^3+b^3\right)-(a+b)\left(a^2+ab+b^2\right)=2(a+b)(a-b)^2.$$

We now conclude that

$$\frac{x^3+y^3}{x^2+xy+y^2}+\frac{y^3+z^3}{y^2+yz+z^2}+\frac{z^3+x^3}{z^2+zx+x^2} \geq \frac{x+y}{3}+\frac{y+z}{3}+\frac{z+x}{3}=2.$$

□

17 **[SL 1985 CAN]** Let x, y, z be positive real numbers. Show the cyclic inequality

$$\frac{x^2}{x^2+yz} + \frac{y^2}{y^2+zx} + \frac{z^2}{z^2+xy} \le 2.$$

First Solution. We first break the homogeneity. The original inequality can be rewritten as

$$\frac{1}{1+\frac{yz}{x^2}} + \frac{1}{1+\frac{zx}{y^2}} + \frac{1}{1+\frac{xy}{z^2}} \le 2$$

The key idea is to employ the substitution

$$a = \frac{yz}{x^2},\ b = \frac{zx}{y^2},\ c = \frac{z^2}{xy}.$$

It then follows that $abc = 1$. It now admits the symmetry in the variables:

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \le 2$$

Since it is symmetric in the three variables, we may break the symmetry. Let's assume $a \le b, c$. Since it is obvious that $\frac{1}{1+a} < 1$, it is enough to check the estimation

$$\frac{1}{1+b} + \frac{1}{1+c} \le 1$$

or equivalently

$$\frac{2+b+c}{1+b+c+bc} \le 1$$

or equivalently

$$bc \ge 1.$$

However, it follows from $abc = 1$ and from $a \le b, c$ that $a \le 1$ and so that $bc \ge 1$. $\square$

18 **[SL 1990 THA]** Let $a, b, c, d \geq 0$ with $ab + bc + cd + da = 1$. show that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

Solution. Since the constraint $ab+bc+cd+da = 1$ is *not* symmetric in the variables, we *cannot* consider the case when $a \geq b \geq c \geq d$ only. We first make the observation that

$$a^2 + b^2 + c^2 + d^2 = \frac{a^2+b^2}{2} + \frac{b^2+c^2}{2} + \frac{c^2+d^2}{2} + \frac{d^2+a^2}{2} \geq ab + bc + cd + da = 1.$$

Our strategy is to establish the following result. It is *symmetric* and more *stronger*.

Let $a, b, c, d \geq 0$ with $a^2 + b^2 + c^2 + d^2 \geq 1$. Then, we obtain

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

We now exploit the symmetry! Since everything is symmetric in the variables, we may assume that $a \geq b \geq c \geq d$. Two applications of Chebyshev's Inequality and one application of The AM-GM Inequality yield

$$\begin{aligned}
& \frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \\
\geq\ & \frac{1}{4}\left(a^3+b^3+c^3+d^3\right)\left(\frac{1}{b+c+d} + \frac{1}{c+d+a} + \frac{1}{d+a+b} + \frac{1}{a+b+c}\right) \\
\geq\ & \frac{1}{4}\left(a^3+b^3+c^3+d^3\right)\frac{4^2}{(b+c+d)+(c+d+a)+(d+a+b)+(a+b+c)} \\
\geq\ & \frac{1}{4^2}\left(a^2+b^2+c^2+d^2\right)(a+b+c+d)\frac{4^2}{3(a+b+c+d)} \\
=\ & \frac{1}{3}.
\end{aligned}$$

$\square$

19 **[IMO 2000/2 USA]** Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1.$$

First Solution. Since $abc = 1$, we can make the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for some positive real numbers x, y, z.[1] Then, it becomes a well-known symmetric inequality:

$$\left(\frac{x}{y} - 1 + \frac{z}{y}\right)\left(\frac{y}{z} - 1 + \frac{x}{z}\right)\left(\frac{z}{x} - 1 + \frac{y}{x}\right) \le 1$$

or

$$xyz \ge (y + z - x)(z + x - y)(x + y - z).$$

$\square$

[1] For example, take $x = 1$, $y = \frac{1}{a}$, $z = \frac{1}{ab}$.

20 **[IMO 1983/6 USA]** Let a, b, c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

First Solution. After setting $a=y+z$, $b=z+x$, $c=x+y$ for $x,y,z>0$, it becomes

$$x^3z+y^3x+z^3y\geq x^2yz+xy^2z+xyz^2$$

or

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq x+y+z.$$

However, an application of The Cauchy-Schwarz Inequality gives

$$(y+z+x)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)\geq(x+y+z)^2.$$

$\square$

21 **[IMO 1961/2 POL]** (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

First Solution. Write $a = y + z$, $b = z + x$, $c = x + y$ for $x, y, z > 0$. It's equivalent to

$$((y + z)^2 + (z + x)^2 + (x + y)^2)^2 \geq 48(x + y + z)xyz,$$

which can be obtained as following :

$$((y + z)^2 + (z + x)^2 + (x + y)^2)^2 \geq 16(yz + zx + xy)^2 \geq 16 \cdot 3(xy \cdot yz + yz \cdot zx + xy \cdot yz).$$

Here, we used the well-known inequalities $p^2 + q^2 \geq 2pq$ and $(p + q + r)^2 \geq 3(pq + qr + rp)$. $\square$

22 (Hadwiger-Finsler Inequality) For any triangle ABC with sides a, b, c and area F, the following inequality holds.

$$2ab + 2bc + 2ca - (a^2 + b^2 + c^2) \geq 4\sqrt{3}F.$$

First Proof. After the substitution $a = y + z$, $b = z + x$, $c = x + y$, where $x, y, z > 0$, it becomes

$$xy + yz + zx \geq \sqrt{3xyz(x + y + z)},$$

which follows from the identity

$$(xy + yz + zx)^2 - 3xyz(x + y + z) = \frac{(xy - yz)^2 + (yz - zx)^2 + (zx - xy)^2}{2}.$$

□

Second Proof. We now present a convexity proof. It is easy to deduce

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{2ab + 2bc + 2ca - (a^2 + b^2 + c^{2)}}{4F}.$$

Since the function $\tan x$ is convex on $\left(0, \frac{\pi}{2}\right)$, Jensen's Inequality implies that

$$\frac{2ab + 2bc + 2ca - (a^2 + b^2 + c^{2)}}{4F} \geq 3\tan\left(\frac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3}\right) = \sqrt{3}.$$

□

23 (Tsintsifas) Let p, q, r be positive real numbers and let a, b, c denote the sides of a triangle with area F. Then, we have

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq 2\sqrt{3}F.$$

Proof. (V. Pambuccian) By Hadwiger-Finsler Inequality, it suffices to show that

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q}c^2 \geq \frac{1}{2}(a+b+c)^2 - (a^2+b^2+c^2)$$

or

$$\left(\frac{p+q+r}{q+r}\right)a^2 + \left(\frac{p+q+r}{r+p}\right)b^2 + \left(\frac{p+q+r}{p+q}\right)c^2 \geq \frac{1}{2}(a+b+c)^2$$

or

$$((q+r)+(r+p)+(p+q))\left(\frac{1}{q+r}a^2 + \frac{1}{r+p}b^2 + \frac{1}{p+q}c^2\right) \geq (a+b+c)^2.$$

However, this is a straightforward consequence of The Cauchy-Schwarz Inequality. □

24 (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

First Proof. ([LC1], Carlitz) We begin with the following lemma.

Lemma 7.0.1. *We have*

$$a_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) + b_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + c_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) > 0.$$

Proof. Observe that it's equivalent to

$$(a_1{}^2 + b_1{}^2 + c_1{}^2)(a_2{}^2 + b_2{}^2 + c_2{}^2) > 2(a_1{}^2a_2{}^2 + b_1{}^2b_2{}^2 + c_1{}^2c_2{}^2).$$

From Heron's Formula, we find that, for $i = 1, 2$,

$$16F_i{}^2 = (a_i{}^2 + b_i{}^2 + c_i{}^2)^2 - 2(a_i{}^4 + b_i{}^4 + c_i{}^4) > 0 \quad \text{or} \quad a_i{}^2 + b_i{}^2 + c_i{}^2 > \sqrt{2(a_i{}^4 + b_i{}^4 + c_i{}^4)}\ .$$

The Cauchy-Schwarz Inequality implies that

$$(a_1{}^2 + b_1{}^2 + c_1{}^2)(a_2{}^2 + b_2{}^2 + c_2{}^2) > 2\sqrt{(a_1{}^4 + b_1{}^4 + c_1{}^4)(a_2{}^4 + b_2{}^4 + c_2{}^4)} \geq 2(a_1{}^2a_2{}^2 + b_1{}^2b_2{}^2 + c_1{}^2c_2{}^2).$$

□

By the lemma, we obtain

$$L = a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) > 0,$$

Hence, we need to show that

$$L^2 - (16F_1{}^2)(16F_2{}^2) \geq 0.$$

One may easily check the following identity

$$L^2 - (16F_1{}^2)(16F_2{}^2) = -4(UV + VW + WU),$$

where

$$U = b_1{}^2c_2{}^2 - b_2{}^2c_1{}^2,\ V = c_1{}^2a_2{}^2 - c_2{}^2a_1{}^2 \ \text{ and } \ W = a_1{}^2b_2{}^2 - a_2{}^2b_1{}^2.$$

Using the identity

$$a_1{}^2U + b_1{}^2V + c_1{}^2W = 0 \ \text{ or } \ W = -\frac{a_1{}^2}{c_1{}^2}U - \frac{b_1{}^2}{c_1{}^2}V,$$

one may also deduce that

$$UV + VW + WU = -\frac{a_1{}^2}{c_1{}^2}\left(U - \frac{c_1{}^2 - a_1{}^2 - b_1{}^2}{2a_1{}^2}V\right)^2 - \frac{4a_1{}^2b_1{}^2 - (c_1{}^2 - a_1{}^2 - b_1{}^2)^2}{4a_1{}^2c_1{}^2}V^2.$$

It follows that

$$UV + VW + WU = -\frac{a_1{}^2}{c_1{}^2}\left(U - \frac{c_1{}^2 - a_1{}^2 - b_1{}^2}{2a_1{}^2}V\right)^2 - \frac{16F_1{}^2}{4a_1{}^2c_1{}^2}V^2 \leq 0.$$

□

Second Proof. ([LC2], Carlitz) We rewrite it in terms of $a_1, b_1, c_1, a_2, b_2, c_2$:

$$({a_1}^2+{b_1}^2+{c_1}^2)({a_2}^2+{b_2}^2+{c_2}^2)-2({a_1}^2{a_2}^2+{b_1}^2{b_2}^2+{c_1}^2{c_2}^2)$$
$$\geq \sqrt{\left(({a_1}^2+{b_1}^2+{c_1}^2)^2-2({a_1}^4+{b_1}^4+{c_1}^4)\right)\left(({a_2}^2+{b_2}^2+{c_2}^2)^2-2({a_2}^4+{b_2}^4+{c_2}^4)\right)}.$$

We employ the following substitutions

$$x_1={a_1}^2+{b_1}^2+{c_1}^2, x_2=\sqrt{2}\,{a_1}^2, x_3=\sqrt{2}\,{b_1}^2, x_4=\sqrt{2}\,{c_1}^2,$$
$$y_1={a_2}^2+{b_2}^2+{c_2}^2, y_2=\sqrt{2}\,{a_2}^2, y_3=\sqrt{2}\,{b_2}^2, y_4=\sqrt{2}\,{c_2}^2.$$

We now observe

$${x_1}^2>{x_2}^2+{y_3}^2+{x_4}^2 \text{ and } {y_1}^2>{y_2}^2+{y_3}^2+{y_4}^2.$$

We now apply Aczél's inequality to get the inequality

$$x_1y_1-x_2y_2-x_3y_3-x_4y_4 \geq \sqrt{\left({x_1}^2-({x_2}^2+{y_3}^2+{x_4}^2)\right)\left({y_1}^2-({y_2}^2+{y_3}^2+{y_4}^2)\right)}.$$

□

25 (Aczél's Inequality) If $a_1, \cdots, a_n, b_1, \cdots, b_n > 0$ satisfies the inequality

$$a_1^2 \geq a_2^2 + \cdots + a_n^2 \text{ and } b_1^2 \geq b_2^2 + \cdots + b_n^2,$$

then the following inequality holds.

$$a_1 b_1 - (a_2 b_2 + \cdots + a_n b_n) \geq \sqrt{(a_1^2 - (a_2^2 + \cdots + a_n^2)) \left(b_1^2 - \left(b_2^2 + \cdots + b_n^2\right)\right)}$$

Proof. [AI] The Cauchy-Schwarz Inequality shows that

$$a_1 b_1 \geq \sqrt{(a_2^2 + \cdots + a_n^2)(b_2^2 + \cdots + b_n^2)} \geq a_2 b_2 + \cdots + a_n b_n.$$

Then, the above inequality is equivalent to

$$(a_1 b_1 - (a_2 b_2 + \cdots + a_n b_n))^2 \geq \left(a_1^2 - \left(a_2^2 + \cdots + a_n^2\right)\right) \left(b_1^2 - \left(b_2^2 + \cdots + b_n^2\right)\right).$$

In case $a_1^2 - (a_2^2 + \cdots + a_n^2) = 0$, it's trivial. Hence, we now assume that $a_1^2 - (a_2^2 + \cdots + a_n^2) > 0$. The main trick is to think of the following quadratic polynomial

$$\mathcal{P}(x) = (a_1 x - b_1)^2 - \sum_{i=2}^{n} (a_i x - b_i)^2 = \left(a_1^2 - \sum_{i=2}^{n} a_i^2\right) x^2 + 2\left(a_1 b_1 - \sum_{i=2}^{n} a_i b_i\right) x + \left(b_1^2 - \sum_{i=2}^{n} b_i^2\right).$$

We now observe that

$$\mathcal{P}\left(\frac{b_1}{a_1}\right) = -\sum_{i=2}^{n} \left(a_i \left(\frac{b_1}{a_1}\right) - b_i\right)^2.$$

Since $\mathcal{P}\left(\frac{b_1}{a_1}\right) \leq 0$ and since the coefficient of x^2 in the quadratic polynomial P is positive, $\mathcal{P}$ should have at least one real root. Therefore, $\mathcal{P}$ has nonnegative discriminant. It follows that

$$\left(2\left(a_1 b_1 - \sum_{i=2}^{n} a_i b_i\right)\right)^2 - 4\left(a_1^2 - \sum_{i=2}^{n} a_i^2\right)\left(b_1^2 - \sum_{i=2}^{n} b_i^2\right) \geq 0.$$

□

26 **[SL 2005 KOR]** In an acute triangle ABC, let D, E, F, P, Q, R be the feet of perpendiculars from A, B, C, A, B, C to BC, CA, AB, EF, FD, DE, respectively. Prove that

$$p(ABC)p(PQR) \geq p(DEF)^2,$$

where $p(T)$ denotes the perimeter of triangle T .

Solution. Let's euler this problem. Let ρ be the circumradius of the triangle ABC. It's easy to show that $BC = 2\rho \sin A$ and $EF = 2\rho \sin A \cos A$. Since $DQ = 2\rho \sin C \cos B \cos A$, $DR = 2\rho \sin B \cos C \cos A$, and $\angle FDE = \pi - 2A$, the Cosine Law gives us

$$\begin{aligned} QR^2 &= DQ^2 + DR^2 - 2DQ \cdot DR \cos(\pi - 2A) \\ &= 4\rho^2 \cos^2 A \left[(\sin C \cos B)^2 + (\sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C \cos(2A)\right] \end{aligned}$$

or

$$QR = 2\rho \cos A \sqrt{f(A,B,C)},$$

where

$$f(A,B,C) = (\sin C \cos B)^2 + (\sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C \cos(2A).$$

So, what we need to attack is the following inequality:

$$\left(\sum_{\text{cyclic}} 2\rho \sin A\right)\left(\sum_{\text{cyclic}} 2\rho \cos A \sqrt{f(A,B,C)}\right) \geq \left(\sum_{\text{cyclic}} 2\rho \sin A \cos A\right)^2$$

or

$$\left(\sum_{\text{cyclic}} \sin A\right)\left(\sum_{\text{cyclic}} \cos A \sqrt{f(A,B,C)}\right) \geq \left(\sum_{\text{cyclic}} \sin A \cos A\right)^2.$$

Our job is now to find a reasonable lower bound of $\sqrt{f(A,B,C)}$. Once again, we express $f(A,B,C)$ as the sum of two squares. We observe that

$$\begin{aligned} f(A,B,C) &= (\sin C \cos B)^2 + (\sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C \cos(2A) \\ &= (\sin C \cos B + \sin B \cos C)^2 + 2 \sin C \cos B \sin B \cos C \left[-1 + \cos(2A)\right] \\ &= \sin^2(C+B) - 2 \sin C \cos B \sin B \cos C \cdot 2 \sin^2 A \\ &= \sin^2 A \left[1 - 4 \sin B \sin C \cos B \cos C\right]. \end{aligned}$$

So, we shall express $1 - 4 \sin B \sin C \cos B \cos C$ as the sum of two squares. The trick is to replace 1 with $\left(\sin^2 B + \cos^2 B\right)\left(\sin^2 C + \cos^2 C\right)$. Indeed, we get

$$\begin{aligned} 1 - 4 \sin B \sin C \cos B \cos C &= \left(\sin^2 B + \cos^2 B\right)\left(\sin^2 C + \cos^2 C\right) - 4 \sin B \sin C \cos B \cos C \\ &= (\sin B \cos C - \sin C \cos B)^2 + (\cos B \cos C - \sin B \sin C)^2 \\ &= \sin^2(B-C) + \cos^2(B+C) \\ &= \sin^2(B-C) + \cos^2 A. \end{aligned}$$

It therefore follows that

$$f(A,B,C) = \sin^2 A \left[\sin^2(B-C) + \cos^2 A\right] \geq \sin^2 A \cos^2 A$$

so that

$$\sum_{\text{cyclic}} \cos A \sqrt{f(A,B,C)} \geq \sum_{\text{cyclic}} \sin A \cos^2 A.$$

So, we can complete the proof if we establish that

$$\left(\sum_{\text{cyclic}} \sin A\right)\left(\sum_{\text{cyclic}} \sin A\cos^2 A\right) \geq \left(\sum_{\text{cyclic}} \sin A\cos A\right)^2.$$

Indeed, one sees that it's a direct consequence of The Cauchy-Schwarz Inequality

$$(p+q+r)(x+y+z) \geq (\sqrt{px}+\sqrt{qy}+\sqrt{rz})^2,$$

where p, q, r, x, y and z are positive real numbers.

$\square$

Remark 7.0.1. *Alternatively, one may obtain another lower bound of $f(A,B,C)$:*

$$\begin{aligned} f(A,B,C) &= (\sin C\cos B)^2 + (\sin B\cos C)^2 + 2\sin C\cos B\sin B\cos C\cos(2A) \\ &= (\sin C\cos B - \sin B\cos C)^2 + 2\sin C\cos B\sin B\cos C\,[1+\cos(2A)] \\ &= \sin^2(B-C) + 2\frac{\sin(2B)}{2}\cdot\frac{\sin(2C)}{2}\cdot 2\cos^2 A \\ &\geq \cos^2 A\sin(2B)\sin(2C). \end{aligned}$$

Then, we can use this to offer a lower bound of the perimeter of triangle PQR:

$$p(PQR) = \sum_{\text{cyclic}} 2\rho\cos A\sqrt{f(A,B,C)} \geq \sum_{\text{cyclic}} 2\rho\cos^2 A\sqrt{\sin 2B\sin 2C}$$

So, one may consider the following inequality:

$$p(ABC)\sum_{\text{cyclic}} 2\rho\cos^2 A\sqrt{\sin 2B\sin 2C} \geq p(DEF)^2$$

or

$$\left(2\rho\sum_{\text{cyclic}} \sin A\right)\left(\sum_{\text{cyclic}} 2\rho\cos^2 A\sqrt{\sin 2B\sin 2C}\right) \geq \left(2\rho\sum_{\text{cyclic}} \sin A\cos A\right)^2.$$

or

$$\left(\sum_{\text{cyclic}} \sin A\right)\left(\sum_{\text{cyclic}} \cos^2 A\sqrt{\sin 2B\sin 2C}\right) \geq \left(\sum_{\text{cyclic}} \sin A\cos A\right)^2.$$

However, it turned out that this doesn't hold. Disprove this!

27 **[IMO 2001/1 KOR]** Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A. Assume that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.

Solution. The angle inequality $\angle CAB + \angle COP < 90^\circ$ can be written as $\angle COP < \angle PCO$. This can be shown if we establish the length inequality $OP > PC$. Since the power of P with respect to the circumcircle of ABC is $OP^2 = R^2 - BP \cdot PC$, where R is the circumradius of the triangle ABC, it becomes $R^2 - BP \cdot PC > PC^2$ or $R^2 > BC \cdot PC$. We euler this. It's an easy job to get $BC = 2R \sin A$ and $PC = 2R \sin B \cos C$. Hence, we show the inequality $R^2 > 2R \sin A \cdot 2R \sin B \cos C$ or $\sin A \sin B \cos C < \frac{1}{4}$. Since $\sin A < 1$, it suffices to show that $\sin A \sin B \cos C < \frac{1}{4}$. Finally, we use the angle condition $\angle C \geq \angle B + 30^\circ$ to obtain the trigonometric inequality

$$\sin B \cos C = \frac{\sin(B+C) - \sin(C-B)}{2} \leq \frac{1 - \sin(C-B)}{2} \leq \frac{1 - \sin 30^\circ}{2} = \frac{1}{4}.$$

□

28 **[IMO 1961/2 POL]** (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Second Proof. [AE, p.171] Let ABC be a triangle with sides $BC = a$, $CA = b$ and $AB = c$. After taking the point P on the same side of BC as the vertex A so that $\triangle PBC$ is equilateral, we use The Cosine Law to deduce the geometric identity

$$\begin{aligned} AP^2 &= b^2 + c^2 - 2bc\cos\left|C - \frac{\pi}{6}\right| \\ &= b^2 + c^2 - 2bc\cos\left(C - \frac{\pi}{6}\right) \\ &= b^2 + c^2 - bc\cos C - \sqrt{3}bc\sin C \\ &= b^2 + c^2 - \frac{b^2 + c^2 - a^2}{2} - 2\sqrt{3}K \end{aligned}$$

which implies the geometric inequality

$$b^2 + c^2 - \frac{b^2 + c^2 - a^2}{2} \geq 2\sqrt{3}K$$

or equivalently

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

□

29 (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

Third Proof. [DP2] We take the point P on the same side of B_1C_1 as the vertex A_1 so that $\triangle PB_1C_1 \sim \triangle A_2B_2C_2$. Now, we use The Cosine Law to deduce the geometric identity

$$\begin{aligned}
& a_2{}^2A_1P^2 \\
= \; & a_2{}^2b_1{}^2 + b_2{}^2a_1{}^2 - 2a_1a_2b_1b_2\cos|C_1 - C_2| \\
= \; & a_2{}^2b_1{}^2 + b_2{}^2a_1{}^2 - 2a_1a_2b_1b_2\cos(C_1 - C_2) \\
= \; & a_2{}^2b_1{}^2 + b_2{}^2a_1{}^2 - \frac{1}{2}(2a_1b_1\cos C_1)(2a_2b_2\cos C_2) - 8\left(\frac{1}{2}a_1b_1\sin C_1\right)\left(\frac{1}{2}a_2b_2\sin C_2\right) \\
= \; & a_2{}^2b_1{}^2 + b_2{}^2a_1{}^2 - \frac{1}{2}\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right)\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right) - 8F_1F_2,
\end{aligned}$$

which implies the geometric inequality

$$a_2{}^2b_1{}^2 + b_2{}^2a_1{}^2 - \frac{1}{2}\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right)\left(a_1{}^2 + b_1{}^2 - c_1{}^2\right) \geq 8F_1F_2$$

or equivalently

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

□

30 (Barrow's Inequality) Let P be an interior point of a triangle ABC and let U, V, W be the points where the bisectors of angles BPC, CPA, APB cut the sides BC,CA,AB respectively. Then, we have

$$PA + PB + PC \geq 2(PU + PV + PW).$$

Proof. ([MB] and [AK]) Let $d_1 = PA$, $d_2 = PB$, $d_3 = PC$, $l_1 = PU$, $l_2 = PV$, $l_3 = PW$, $2\theta_1 = \angle BPC$, $2\theta_2 = \angle CPA$, and $2\theta_3 = \angle APB$. We need to show that $d_1 + d_2 + d_3 \geq 2(l_1 + l_2 + l_3)$. It's easy to deduce the following identities

$$l_1 = \frac{2d_2d_3}{d_2 + d_3}\cos\theta_1,\ l_2 = \frac{2d_3d_1}{d_3 + d_1}\cos\theta_2,\ \text{ and }\ l_3 = \frac{2d_1d_2}{d_1 + d_2}\cos\theta_3,$$

It now follows that

$$l_1 + l_2 + l_3 \leq \sqrt{d_2d_3}\cos\theta_1 + \sqrt{d_3d_1}\cos\theta_2 + \sqrt{d_1d_2}\cos\theta_3 \leq \frac{1}{2}\left(d_1 + d_2 + d_3\right).$$

$\square$

31 ([AK], Abi-Khuzam) Let $x_1, \cdots, x_4$ be positive real numbers. Let $\theta_1, \cdots, \theta_4$ be real numbers such that $\theta_1 + \cdots + \theta_4 = \pi$. Then, we have

$$x_1 \cos\theta_1 + x_2 \cos\theta_2 + x_3 \cos\theta_3 + x_4 \cos\theta_4 \le \sqrt{\frac{(x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4)(x_1x_4 + x_2x_3)}{x_1x_2x_3x_4}}.$$

Proof. Let $p = \frac{{x_1}^2+{x_2}^2}{2x_1x_2} + \frac{{x_3}^2+{x_4}^2}{2x_3x_4}$ $q = \frac{x_1x_2+x_3x_4}{2}$ and $\lambda = \sqrt{\frac{p}{q}}$. In the view of $\theta_1 + \theta_2 + (\theta_3 + \theta_4) = \pi$ and $\theta_3 + \theta_4 + (\theta_1 + \theta_2) = \pi$, we have

$$x_1 \cos\theta_1 + x_2 \cos\theta_2 + \lambda \cos(\theta_3 + \theta_4) \le p\lambda = \sqrt{pq},$$

and

$$x_3 \cos\theta_3 + x_4 \cos\theta_4 + \lambda \cos(\theta_1 + \theta_2) \le \frac{q}{\lambda} = \sqrt{pq}.$$

Since $\cos(\theta_3 + \theta_4) + \cos(\theta_1 + \theta_2) = 0$, adding these two above inequalities yields

$$x_1 \cos\theta_1 + x_2 \cos\theta_2 + x_3 \cos\theta_3 + x_4 \cos\theta_4 \le 2\sqrt{pq} = \sqrt{\frac{(x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4)(x_1x_4 + x_2x_3)}{x_1x_2x_3x_4}}.$$

$\square$

32 **[IMO 1961/2 POL]** (Weitzenböck's Inequality) Let a, b, c be the lengths of a triangle with area S. Show that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

Third Proof. ([RW], R. Weitzenböck) Let ABC be a triangle with sides a, b, and c. To euler it, we toss the picture on the real plane $\mathbb{R}^2$ with the coordinates $A(\alpha, \beta)$, $B\left(-\frac{a}{2}, 0\right)$ and $C\left(\frac{a}{2}, 0\right)$. Now, we obtain

$$\left(a^2 + b^2 + c^2\right)^2 - \left(4\sqrt{3}S\right)^2 = \left(\frac{3}{2}a^2 + \left(\alpha^2 - \beta^2\right)\right)^2 + 16\alpha^2\beta^2 \geq 0.$$

$\square$

33 (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \geq 16F_1F_2.$$

Fourth Proof. (By a participant from KMO[2] summer program.) We toss $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ onto the real plane $\mathbb{R}^2$:

$$A_1(0, p_1),\ B_1(p_2, 0),\ C_1(p_3, 0),\ A_2(0, q_1),\ B_2(q_2, 0),\ \text{and } C_2(q_3, 0).$$

It therefore follows from the inequality $x^2 + y^2 \geq 2|xy|$ that

$$\begin{aligned}
& a_1{}^2(b_2{}^2 + c_2{}^2 - a_2{}^2) + b_1{}^2(c_2{}^2 + a_2{}^2 - b_2{}^2) + c_1{}^2(a_2{}^2 + b_2{}^2 - c_2{}^2) \\
= \ & (p_3 - p_2)^2(2q_1{}^2 + 2q_1q_2) + (p_1{}^2 + p_3{}^2)(2q_2{}^2 - 2q_2q_3) + (p_1{}^2 + p_2{}^2)(2q_3{}^2 - 2q_2q_3) \\
= \ & 2(p_3 - p_2)^2q_1{}^2 + 2(q_3 - q_2)^2p_1{}^2 + 2(p_3q_2 - p_2q_3)^2 \\
\geq \ & 2((p_3 - p_2)q_1)^2 + 2((q_3 - q_2)p_1)^2 \\
\geq \ & 4|(p_3 - p_2)q_1| \cdot |(q_3 - q_2)p_1| \\
= \ & 16F_1F_2\ .
\end{aligned}$$

□

[2]Korean Mathematical Olympiads

34 [TD] Let P be an arbitrary point in the plane of a triangle ABC with the centroid G. Show the following inequalities

(1) $\overline{BC}\cdot\overline{PB}\cdot\overline{PC}+\overline{AB}\cdot\overline{PA}\cdot\overline{PB}+\overline{CA}\cdot\overline{PC}\cdot\overline{PA}\geq\overline{BC}\cdot\overline{CA}\cdot\overline{AB}$ and
(2) $\overline{PA}^3\cdot\overline{BC}+\overline{PB}^3\cdot\overline{CA}+\overline{PC}^3\cdot\overline{AB}\geq 3\overline{PG}\cdot\overline{BC}\cdot\overline{CA}\cdot\overline{AB}$.

Solution. We only check the first inequality. We regard A, B, C, P as complex numbers and assume that P corresponds to 0. We're required to prove that

$$|(B-C)BC|+|(A-B)AB|+|(C-A)CA|\geq|(B-C)(C-A)(A-B)|.$$

It remains to apply The Triangle Inequality to the algebraic identity

$$(B-C)BC+(A-B)AB+(C-A)CA=-(B-C)(C-A)(A-B).$$

□

35 (The Neuberg-Pedoe Inequality) Let a_1, b_1, c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2, b_2, c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have

$$a_1{}^2(b_2{}^2+c_2{}^2-a_2{}^2)+b_1{}^2(c_2{}^2+a_2{}^2-b_2{}^2)+c_1{}^2(a_2{}^2+b_2{}^2-c_2{}^2)\geq 16F_1F_2.$$

Fifth Proof. ([GC], G. Chang) We regard A, B, C, A', B', C' as complex numbers and assume that C corresponds to 0. Rewriting the both sides in the inequality in terms of complex numbers, we get

$$\begin{aligned} & a_1{}^2(b_2{}^2+c_2{}^2-a_2{}^2)+b_1{}^2(c_2{}^2+a_2{}^2-b_2{}^2)+c_1{}^2(a_2{}^2+b_2{}^2-c_2{}^2) \\ = \;& 2\left(|A'|^2|B|^2+|A|^2|B'|^2\right)-\left(A\overline{B}+\overline{A}B\right)\left(A'\overline{B'}+\overline{A'}B\right) \end{aligned}$$

and

$$16F_1F_2=\pm\left(\overline{A}B-A\overline{B}\right)\left(A'\overline{B'}+\overline{A'}B'\right),$$

where the sign begin chose to make the right hand positive. According to whether the triangle ABC and the triangle $A'B'C'$ have the same orientation or not, we obtain either

$$a_1{}^2(b_2{}^2+c_2{}^2-a_2{}^2)+b_1{}^2(c_2{}^2+a_2{}^2-b_2{}^2)+c_1{}^2(a_2{}^2+b_2{}^2-c_2{}^2)-16F_1F_2=2\left|AB'-A'B\right|^2$$

or

$$a_1{}^2(b_2{}^2+c_2{}^2-a_2{}^2)+b_1{}^2(c_2{}^2+a_2{}^2-b_2{}^2)+c_1{}^2(a_2{}^2+b_2{}^2-c_2{}^2)-16F_1F_2=2\left|A\overline{B'}-\overline{A'}B\right|^2.$$

This completes the proof. □

36 **[SL 2002 KOR]** Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E, respectively. Prove that $\overline{AB} + \overline{AC} \geq 4\overline{DE}$.

Solution. Let $\overline{AF} = x, \overline{BF} = y, \overline{CF} = z$ and let $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. We can toss the pictures on $\mathbb{C}$ so that the points F, A, B, C, D, and E are represented by the complex numbers 0, x, $y\omega$, $z\omega^2$, d, and e. It's an easy exercise to establish that $\overline{DF} = \frac{xz}{x+z}$ and $\overline{EF} = \frac{xy}{x+y}$. This means that $d = -\frac{xz}{x+z}\omega$ and $e = -\frac{xy}{x+y}\omega$. We're now required to prove that

$$|x - y\omega| + |z\omega^2 - x| \geq 4\left|\frac{-zx}{z+x}\omega + \frac{xy}{x+y}\omega^2\right|.$$

Since $|\omega| = 1$ and $\omega^3 = 1$, we have $|z\omega^2 - x| = |\omega(z\omega^2 - x)| = |z - x\omega|$. Therefore, we need to prove

$$|x - y\omega| + |z - x\omega| \geq \left|\frac{4zx}{z+x} - \frac{4xy}{x+y}\omega\right|.$$

More strongly, we establish that $|(x - y\omega) + (z - x\omega)| \geq \left|\frac{4zx}{z+x} - \frac{4xy}{x+y}\omega\right|$ or $|p - q\omega| \geq |r - s\omega|$, where $p = z + x$, $q = y + x$, $r = \frac{4zx}{z+x}$ and $s = \frac{4xy}{x+y}$. It's clear that $p \geq r > 0$ and $q \geq s > 0$. It follows that

$$|p - q\omega|^2 - |r - s\omega|^2 = (p - q\omega)\overline{(p - q\omega)} - (r - s\omega)\overline{(r - s\omega)} = (p^2 - r^2) + (pq - rs) + (q^2 - s^2) \geq 0.$$

It's easy to check that the equality holds if and only if $\triangle ABC$ is equilateral. $\square$

37 (APMO 2004/5) Prove that, for all positive real numbers a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

First Solution. Choose $A, B, C \in \left(0, \frac{\pi}{2}\right)$ with $a = \sqrt{2}\tan A$, $b = \sqrt{2}\tan B$, and $c = \sqrt{2}\tan C$. Using the well-known trigonometric identity $1 + \tan^2\theta = \frac{1}{cos^2\theta}$, one may rewrite it as

$$\frac{4}{9} \geq \cos A\cos B\cos C\left(\cos A\sin B\sin C + \sin A\cos B\sin C + \sin A\sin B\cos C\right).$$

One may easily check the following trigonometric identity

$$\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C - \sin A\sin B\cos C.$$

Then, the above trigonometric inequality takes the form

$$\frac{4}{9} \geq \cos A\cos B\cos C\left(\cos A\cos B\cos C - \cos(A+B+C)\right).$$

Let $\theta = \frac{A+B+C}{3}$. Applying The AM-GM Inequality and Jesen's Inequality, we have

$$\cos A\cos B\cos C \leq \left(\frac{\cos A + \cos B + \cos C}{3}\right)^3 \leq \cos^3\theta.$$

We now need to show that

$$\frac{4}{9} \geq \cos^3\theta(\cos^3\theta - \cos 3\theta).$$

Using the trigonometric identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \text{ or } \cos^3\theta - \cos 3\theta = 3\cos\theta - 3\cos^3\theta,$$

it becomes

$$\frac{4}{27} \geq \cos^4\theta\left(1 - \cos^2\theta\right),$$

which follows from The AM-GM Inequality

$$\left(\frac{\cos^2\theta}{2}\cdot\frac{\cos^2\theta}{2}\cdot\left(1-\cos^2\theta\right)\right)^{\frac{1}{3}} \leq \frac{1}{3}\left(\frac{\cos^2\theta}{2} + \frac{\cos^2\theta}{2} + \left(1-\cos^2\theta\right)\right) = \frac{1}{3}.$$

One find that the equality holds if and only if $\tan A = \tan B = \tan C = \frac{1}{\sqrt{2}}$ if and only if $a = b = c = 1$. □

38 (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1.$$

Prove that $abcd \geq 3$.

First Solution. We can write $a^2=\tan A$, $b^2=\tan B$, $c^2=\tan C$, $d^2=\tan D$, where $A, B, C, D \in \left(0, \frac{\pi}{2}\right)$. Then, the algebraic identity becomes the following trigonometric identity :

$$\cos^2 A+\cos^2 B+\cos^2 C+\cos^2 D=1.$$

Applying The AM-GM Inequality, we obtain

$$\sin^2 A=1-\cos^2 A=\cos^2 B+\cos^2 C+\cos^2 D \geq 3\left(\cos B\cos C\cos D\right)^{\frac{2}{3}}.$$

Similarly, we obtain

$$\sin^2 B \geq 3\left(\cos C\cos D\cos A\right)^{\frac{2}{3}}, \sin^2 C \geq 3\left(\cos D\cos A\cos B\right)^{\frac{2}{3}}, \text{ and } \sin^2 D \geq 3\left(\cos A\cos B\cos C\right)^{\frac{2}{3}}.$$

Multiplying these four inequalities, we get the result! $\square$

39 (Korea 1998) Let x, y, z be the positive reals with $x+y+z=xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\leq\frac{3}{2}.$$

First Solution. We give a convexity proof. We can write $x=\tan A$, $y=\tan B$, $z=\tan C$, where $A,B,C\in\left(0,\frac{\pi}{2}\right)$. Using the fact that $1+\tan^2\theta=\left(\frac{1}{\cos\theta}\right)^2$, we rewrite it in the terms of A, B, C :

$$\cos A+\cos B+\cos C\leq\frac{3}{2}.$$

It follows from $\tan(\pi-C)=-z=\frac{x+y}{1-xy}=\tan(A+B)$ and from $\pi-C, A+B\in(0,\pi)$ that $\pi-C=A+B$ or $A+B+C=\pi$. Hence, it suffices to show the following. □

40 (USA 2001) Let a, b, and c be nonnegative real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that $0 \leq ab + bc + ca - abc \leq 2$.

Solution. Notice that $a, b, c > 1$ implies that $a^2+b^2+c^2+abc > 4$. If $a \leq 1$, then we have $ab+bc+ca-abc \geq (1-a)bc \geq 0$. We now prove that $ab + bc + ca - abc \leq 2$. Letting $a = 2p$, $b = 2q$, $c = 2r$, we get $p^2 + q^2 + r^2 + 2pqr = 1$. By the above exercise, we can write

$$a = 2\cos A,\ b = 2\cos B,\ c = 2\cos C \ \text{ for some } A, B, C \in \left[0, \frac{\pi}{2}\right] \text{ with } A + B + C = \pi.$$

We are required to prove

$$\cos A\cos B + \cos B\cos C + \cos C\cos A - 2\cos A\cos B\cos C \leq \frac{1}{2}.$$

One may assume that $A \geq \frac{\pi}{3}$ or $1 - 2\cos A \geq 0$. Note that

$$\cos A\cos B+\cos B\cos C+\cos C\cos A-2\cos A\cos B\cos C = \cos A(\cos B+\cos C)+\cos B\cos C(1-2\cos A).$$

We apply Jensen's Inequality to deduce $\cos B + \cos C \leq \frac{3}{2} - \cos A$. Note that $2\cos B\cos C = \cos(B - C) + \cos(B + C) \leq 1 - \cos A$. These imply that

$$\cos A(\cos B + \cos C) + \cos B\cos C(1 - 2\cos A) \leq \cos A\left(\frac{3}{2} - \cos A\right) + \left(\frac{1-\cos A}{2}\right)(1 - 2\cos A).$$

However, it's easy to verify that $\cos A\left(\frac{3}{2} - \cos A\right) + \left(\frac{1-\cos A}{2}\right)(1 - 2\cos A) = \frac{1}{2}$. □

41 **[IMO 2001/2 KOR]** Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

First Solution. To remove the square roots, we make the following substitution :

$$x=\frac{a}{\sqrt{a^2+8bc}},\ \ y=\frac{b}{\sqrt{b^2+8ca}},\ \ z=\frac{c}{\sqrt{c^2+8ab}}.$$

Clearly, $x, y, z \in (0, 1)$. Our aim is to show that $x + y + z \geq 1$. We notice that

$$\frac{a^2}{8bc}=\frac{x^2}{1-x^2},\ \ \frac{b^2}{8ac}=\frac{y^2}{1-y^2},\ \ \frac{c^2}{8ab}=\frac{z^2}{1-z^2}\ \ \Longrightarrow\ \ \frac{1}{512}=\left(\frac{x^2}{1-x^2}\right)\left(\frac{y^2}{1-y^2}\right)\left(\frac{z^2}{1-z^2}\right).$$

Hence, we need to show that

$$x+y+z\geq 1,\ \text{where } 0<x,y,z<1 \text{ and } (1-x^2)(1-y^2)(1-z^2)=512(xyz)^2.$$

However, $1 > x + y + z$ implies that, by The AM-GM Inequality,

$(1-x^2)(1-y^2)(1-z^2)>((x+y+z)^2-x^2)((x+y+z)^2-y^2)((x+y+z)^2-z^2)=(x+x+y+z)(y+z)$

$(x+y+y+z)(z+x)(x+y+z+z)(x+y)\geq 4(x^2yz)^{\frac{1}{4}}\cdot 2(yz)^{\frac{1}{2}}\ \cdot 4(y^2zx)^{\frac{1}{4}}\cdot 2(zx)^{\frac{1}{2}}\ \cdot 4(z^2xy)^{\frac{1}{4}}\cdot 2(xy)^{\frac{1}{2}}$

$= 512(xyz)^2$. This is a contradiction ! $\square$

42 **[IMO 1995/2 RUS]** Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Second Solution. After the substitution $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$, we get $xyz = 1$. The inequality takes the form

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

It follows from The Cauchy-Schwarz Inequality that

$$[(y+z) + (z+x) + (x+y)]\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) \geq (x+y+z)^2$$

so that, by The AM-GM Inequality,

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{x+y+z}{2} \geq \frac{3(xyz)^{\frac{1}{3}}}{2} = \frac{3}{2}.$$

$\square$

43 (Korea 1998) Let x, y, z be the positive reals with $x+y+z=xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\leq\frac{3}{2}.$$

Second Solution. The starting point is letting $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$. We find that $a+b+c=abc$ is equivalent to $1=xy+yz+zx$. The inequality becomes

$$\frac{x}{\sqrt{x^2+1}}+\frac{y}{\sqrt{y^2+1}}+\frac{z}{\sqrt{z^2+1}}\leq\frac{3}{2}$$

or

$$\frac{x}{\sqrt{x^2+xy+yz+zx}}+\frac{y}{\sqrt{y^2+xy+yz+zx}}+\frac{z}{\sqrt{z^2+xy+yz+zx}}\leq\frac{3}{2}$$

or

$$\frac{x}{\sqrt{(x+y)(x+z)}}+\frac{y}{\sqrt{(y+z)(y+x)}}+\frac{z}{\sqrt{(z+x)(z+y)}}\leq\frac{3}{2}.$$

By the AM-GM inequality, we have

$$\frac{x}{\sqrt{(x+y)(x+z)}}=\frac{x\sqrt{(x+y)(x+z)}}{(x+y)(x+z)}\leq\frac{1}{2}\frac{x[(x+y)+(x+z)]}{(x+y)(x+z)}=\frac{1}{2}\left(\frac{x}{x+z}+\frac{x}{x+z}\right).$$

In a like manner, we obtain

$$\frac{y}{\sqrt{(y+z)(y+x)}}\leq\frac{1}{2}\left(\frac{y}{y+z}+\frac{y}{y+x}\right)\quad\text{and}\quad\frac{z}{\sqrt{(z+x)(z+y)}}\leq\frac{1}{2}\left(\frac{z}{z+x}+\frac{z}{z+y}\right).$$

Adding these three yields the required result. □

44 **[IMO 2000/2 USA]** Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \le 1.$$

Second Solution. ([IV], Ilan Vardi) Since $abc = 1$, we may assume that $a \ge 1 \ge b$. [3] It follows that

$$1-\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) = \left(c+\frac{1}{c}-2\right)\left(a+\frac{1}{b}-1\right)+\frac{(a-1)(1-b)}{a}.$$ [4]

□

Third Solution. As in the first solution, after the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for $x, y, z > 0$, we can rewrite it as $xyz \ge (y+z-x)(z+x-y)(x+y-z)$. Without loss of generality, we can assume that $z \ge y \ge x$. Set $y - x = p$ and $z - x = q$ with $p, q \ge 0$. It's straightforward to verify that

$$xyz-(y+z-x)(z+x-y)(x+y-z) = (p^2-pq+q^2)x+(p^3+q^3-p^2q-pq^2).$$

Since $p^2 - pq + q^2 \ge (p-q)^2 \ge 0$ and $p^3 + q^3 - p^2q - pq^2 = (p-q)^2(p+q) \ge 0$, we get the result. □

Fourth Solution. (From the IMO 2000 Short List) Using the condition $abc = 1$, it's straightforward to verify the equalities

$$2 = \frac{1}{a}\left(a-1+\frac{1}{b}\right)+c\left(b-1+\frac{1}{c}\right),$$
$$2 = \frac{1}{b}\left(b-1+\frac{1}{c}\right)+a\left(c-1+\frac{1}{a}\right),$$
$$2 = \frac{1}{c}\left(c-1+\frac{1}{a}\right)+b\left(a-1+\frac{1}{c}\right).$$

In particular, they show that at most one of the numbers $u = a-1+\frac{1}{b}$, $v = b-1+\frac{1}{c}$, $w = c-1+\frac{1}{a}$ is negative. If there is such a number, we have

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) = uvw < 0 < 1.$$

And if $u, v, w \ge 0$, The AM-GM Inequality yields

$$2 = \frac{1}{a}u+cv \ge 2\sqrt{\frac{c}{a}uv},\ \ 2 = \frac{1}{b}v+aw \ge 2\sqrt{\frac{a}{b}vw},\ \ 2 = \frac{1}{c}w+aw \ge 2\sqrt{\frac{b}{c}wu}.$$

Thus, $uv \le \frac{a}{c}$, $vw \le \frac{b}{a}$, $wu \le \frac{c}{b}$, so $(uvw)^2 \le \frac{a}{c}\cdot\frac{b}{a}\cdot\frac{c}{b} = 1$. Since $u, v, w \ge 0$, this completes the proof. □

[3] Why? Note that the inequality is not symmetric in the three variables. Check it!
[4] For a verification of the identity, see [IV].

45 Let a, b, c be positive real numbers satisfying $a+b+c=1$. Show that

$$\frac{a}{a+bc}+\frac{b}{b+ca}+\frac{\sqrt{abc}}{c+ab}\leq 1+\frac{3\sqrt{3}}{4}.$$

Solution. We want to establish that

$$\frac{1}{1+\frac{bc}{a}}+\frac{1}{1+\frac{ca}{b}}+\frac{\sqrt{\frac{ab}{c}}}{1+\frac{ab}{c}}\leq 1+\frac{3\sqrt{3}}{4}.$$

Set $x=\sqrt{\frac{bc}{a}}$, $y=\sqrt{\frac{ca}{b}}$, $z=\sqrt{\frac{ab}{c}}$. We need to prove that

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{z}{1+z^2}\leq 1+\frac{3\sqrt{3}}{4},$$

where $x,y,z>0$ and $xy+yz+zx=1$. It's not hard to show that there exists $A,B,C\in(0,\pi)$ with

$$x=\tan\frac{A}{2}, y=\tan\frac{B}{2}, z=\tan\frac{C}{2}, \text{ and } A+B+C=\pi.$$

The inequality becomes

$$\frac{1}{1+\left(\tan\frac{A}{2}\right)^2}+\frac{1}{1+\left(\tan\frac{B}{2}\right)^2}+\frac{\tan\frac{C}{2}}{1+\left(\tan\frac{C}{2}\right)^2}\leq 1+\frac{3\sqrt{3}}{4}$$

or

$$1+\frac{1}{2}(\cos A+\cos B+\sin C)\leq 1+\frac{3\sqrt{3}}{4}$$

or

$$\cos A+\cos B+\sin C\leq\frac{3\sqrt{3}}{2}.$$

□

Note that $\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$. Since $\left|\frac{A-B}{2}\right|<\frac{\pi}{2}$, this means that

$$\cos A+\cos B\leq 2\cos\left(\frac{A+B}{2}\right)=2\cos\left(\frac{\pi-C}{2}\right).$$

It will be enough to show that

$$2\cos\left(\frac{\pi-C}{2}\right)+\sin C\leq\frac{3\sqrt{3}}{2},$$

where $C\in(0,\pi)$. This is a one-variable inequality.[5] It's left as an exercise for the reader.

[5] *Differentiate!* Shiing-Shen Chern

46 (Latvia 2002) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1.$$

Prove that $abcd \geq 3$.

Second Solution. (given by Jeong Soo Sim at the KMO Weekend Program 2007) We need to prove the inequality $a^4b^4c^4d^4 \geq 81$. After making the substitution

$$A=\frac{1}{1+a^4},\ B=\frac{1}{1+b^4},\ C=\frac{1}{1+c^4},\ D=\frac{1}{1+d^4},$$

we obtain

$$a^4=\frac{1-A}{A},\ b^4=\frac{1-B}{B},\ c^4=\frac{1-C}{C},\ d^4=\frac{1-D}{D}.$$

The constraint becomes $A+B+C+D=1$ and the inequality can be written as

$$\frac{1-A}{A}\cdot\frac{1-B}{B}\cdot\frac{1-C}{C}\cdot\frac{1-D}{D}\geq 81.$$

or

$$\frac{B+C+D}{A}\cdot\frac{C+D+A}{B}\cdot\frac{D+A+B}{C}\cdot\frac{A+B+C}{D}\geq 81.$$

or

$$(B+C+D)(C+D+A)(D+A+B)(A+B+C)\geq 81ABCD.$$

However, this is an immediate consequence of The AM-GM Inequality:

$$(B+C+D)(C+D+A)(D+A+B)(A+B+C)\geq 3\,(BCD)^{\frac{1}{3}}\cdot 3\,(CDA)^{\frac{1}{3}}\cdot 3\,(DAB)^{\frac{1}{3}}\cdot 3\,(ABC)^{\frac{1}{3}}.$$

$\square$

47 **[LL 1992 UNK]** (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

First Solution. We begin with the algebraic substitution $a = \sqrt{x-1}$, $b = \sqrt{y-1}$, $c = \sqrt{z-1}$. Then, the condition becomes

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{1}{1+c^2} = 2 \quad \Leftrightarrow \quad a^2b^2 + b^2c^2 + c^2a^2 + 2a^2b^2c^2 = 1$$

and the inequality is equivalent to

$$\sqrt{a^2+b^2+c^2+3} \geq a+b+c \quad \Leftrightarrow \quad ab+bc+ca \leq \frac{3}{2}.$$

Let $p = bc$, $q = ca$, $r = ab$. Our job is to prove that $p + q + r \leq \frac{3}{2}$ where $p^2 + q^2 + r^2 + 2pqr = 1$. Now, we can make the trigonometric substitution

$$p = \cos A,\ q = \cos B,\ r = \cos C \ \text{ for some } A, B, C \in \left(0, \frac{\pi}{2}\right) \text{ with } A + B + C = \pi.$$

What we need to show is now that $\cos A + \cos B + \cos C \leq \frac{3}{2}$. It follows from Jensen's Inequality. □

48 (Belarus 1998) Prove that, for all $a, b, c > 0$,

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b}{b+c} + \frac{b+c}{c+a} + 1.$$

Solution. After writing $x = \frac{a}{b}$ and $y = \frac{c}{b}$, we get

$$\frac{c}{a} = \frac{y}{x}, \quad \frac{a+b}{b+c} = \frac{x+1}{1+y}, \quad \frac{b+c}{c+a} = \frac{1+y}{y+x}.$$

One may rewrite the inequality as

$$x^3y^2 + x^2 + x + y^3 + y^2 \geq x^2y + 2xy + 2xy^2.$$

Apply The AM-GM Inequality to obtain

$$\frac{x^3y^2 + x}{2} \geq x^2y, \quad \frac{x^3y^2 + x + y^3 + y^3}{2} \geq 2xy^2, \quad x^2 + y^2 \geq 2xy.$$

Adding these three inequalities, we get the result. The equality holds if and only if $x = y = 1$ or $a = b = c$. $\square$

49 **[SL 2001]** Let $x_1, \cdots, x_n$ be arbitrary real numbers. Prove the inequality.

$$\frac{x_1}{1+{x_1}^2} + \frac{x_2}{1+{x_1}^2+{x_2}^2} + \cdots + \frac{x_n}{1+{x_1}^2+\cdots+{x_n}^2} < \sqrt{n}.$$

First Solution. We only consider the case when $x_1, \cdots, x_n$ are all nonnegative real numbers.(Why?)[6] Let $x_0 = 1$. After the substitution $y_i = {x_0}^2 + \cdots + {x_i}^2$ for all $i = 0, \cdots, n$, we obtain $x_i = \sqrt{y_i - y_{i-1}}$. We need to prove the following inequality

$$\sum_{i=0}^{n} \frac{\sqrt{y_i - y_{i-1}}}{y_i} < \sqrt{n}.$$

Since $y_i \geq y_{i-1}$ for all $i = 1, \cdots, n$, we have an upper bound of the left hand side:

$$\sum_{i=0}^{n} \frac{\sqrt{y_i - y_{i-1}}}{y_i} \leq \sum_{i=0}^{n} \frac{\sqrt{y_i - y_{i-1}}}{\sqrt{y_i y_{i-1}}} = \sum_{i=0}^{n} \sqrt{\frac{1}{y_{i-1}} - \frac{1}{y_i}}$$

We now apply the Cauchy-Schwarz inequality to give an upper bound of the last term:

$$\sum_{i=0}^{n} \sqrt{\frac{1}{y_{i-1}} - \frac{1}{y_i}} \leq \sqrt{n \sum_{i=0}^{n} \left(\frac{1}{y_{i-1}} - \frac{1}{y_i}\right)} = \sqrt{n \left(\frac{1}{y_0} - \frac{1}{y_n}\right)}.$$

Since $y_0 = 1$ and $y_n > 0$, this yields the desired upper bound $\sqrt{n}$.

$\square$

Second Solution. We may assume that $x_1, \cdots, x_n$ are all nonnegative real numbers. Let $x_0 = 0$. We make the following *algebraic* substitution

$$t_i = \frac{x_i}{\sqrt{{x_0}^2 + \cdots + {x_i}^2}},\ c_i = \frac{1}{\sqrt{1+{t_i}^2}} \text{ and } s_i = \frac{t_i}{\sqrt{1+{t_i}^2}}$$

for all $i = 0, \cdots, n$. It's an easy exercise to show that $\frac{x_i}{{x_0}^2+\cdots+{x_i}^2} = c_0 \cdots c_i s_i$. Since $s_i = \sqrt{1-{c_i}^2}$, the desired inequality becomes

$$c_0 c_1 \sqrt{1-{c_1}^2} + c_0 c_1 c_2 \sqrt{1-{c_2}^2} + \cdots + c_0 c_1 \cdots c_n \sqrt{1-{c_n}^2} < \sqrt{n}.$$

Since $0 < c_i \leq 1$ for all $i = 1, \cdots, n$, we have

$$\sum_{i=1}^{n} c_0 \cdots c_i \sqrt{1-{c_i}^2} \leq \sum_{i=1}^{n} c_0 \cdots c_{i-1} \sqrt{1-{c_i}^2} = \sum_{i=1}^{n} \sqrt{(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2}.$$

Since $c_0 = 1$, by The Cauchy-Schwarz Inequality, we obtain

$$\sum_{i=1}^{n} \sqrt{(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2} \leq \sqrt{n \sum_{i=1}^{n} [(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2]} = \sqrt{n\,[1 - (c_0 \cdots c_n)^2]}.$$

$\square$

[6] $\frac{x_1}{1+{x_1}^2} + \frac{x_2}{1+{x_1}^2+{x_2}^2} + \cdots + \frac{x_n}{1+{x_1}^2+\cdots+{x_n}^2} \leq \frac{|x_1|}{1+{x_1}^2} + \frac{|x_2|}{1+{x_1}^2+{x_2}^2} + \cdots + \frac{|x_n|}{1+{x_1}^2+\cdots+{x_n}^2}$.

50 Let a, b, c be the lengths of a triangle. Show that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2.$$

Solution. We don't employ The Ravi Substitution! It follows from the triangle inequality that

$$\sum_{\text{cyclic}}\frac{a}{b+c}<\sum_{\text{cyclic}}\frac{a}{\frac{1}{2}(a+b+c)}=2.$$

$\square$

51 **[IMO 2001/2 KOR]** Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Second Solution. We proved the estimation, for $x, y, z > 0$,

$$x+y+z \geq \sqrt{x^{\frac{1}{2}}\left(x^{\frac{3}{2}}+8y^{\frac{3}{4}}z^{\frac{3}{4}}\right)}.$$

It follows that

$$\sum_{\text{cyclic}}\frac{x^{\frac{3}{4}}}{\sqrt{x^{\frac{3}{2}}+8y^{\frac{3}{4}}z^{\frac{3}{4}}}}\geq\sum_{\text{cyclic}}\frac{x}{x+y+z}=1.$$

After the substitution $x = a^{\frac{4}{3}}, y = b^{\frac{4}{3}}$, and $z = c^{\frac{4}{3}}$, it now becomes

$$\sum_{\text{cyclic}}\frac{a}{\sqrt{a^2+8bc}}\geq 1.$$

□

52 **[IMO 2005/3 KOR]** Let x, y, and z be positive numbers such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

First Solution. It's equivalent to the following inequality

$$\left(\frac{x^2 - x^5}{x^5 + y^2 + z^2} + 1\right) + \left(\frac{y^2 - y^5}{y^5 + z^2 + x^2} + 1\right) + \left(\frac{z^2 - z^5}{z^5 + x^2 + y^2} + 1\right) \leq 3$$

or

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \leq 3.$$

With The Cauchy-Schwarz Inequality and the fact that $xyz \geq 1$, we have

$$(x^5 + y^2 + z^2)(yz + y^2 + z^2) \geq (x^2 + y^2 + z^2)^2 \text{ or } \frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} \leq \frac{yz + y^2 + z^2}{x^2 + y^2 + z^2}.$$

Taking the cyclic sum and $x^2 + y^2 + z^2 \geq xy + yz + zx$ give us

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \leq 2 + \frac{xy + yz + zx}{x^2 + y^2 + z^2} \leq 3.$$

□

Second Solution. The main idea is to think of 1 as follows :

$$\frac{x^5}{x^5 + y^2 + z^2} + \frac{y^5}{y^5 + z^2 + x^2} + \frac{z^5}{z^5 + x^2 + y^2} \geq 1 \geq \frac{x^2}{x^5 + y^2 + z^2} + \frac{y^2}{y^5 + z^2 + x^2} + \frac{z^2}{z^5 + x^2 + y^2}.$$

We first show the left-hand. It follows from $y^4 + z^4 \geq y^3 z + yz^3 = yz(y^2 + z^2)$ that

$$x(y^4 + z^4) \geq xyz(y^2 + z^2) \geq y^2 + z^2 \text{ or } \frac{x^5}{x^5 + y^2 + z^2} \geq \frac{x^5}{x^5 + xy^4 + xz^4} = \frac{x^4}{x^4 + y^4 + z^4}.$$

Taking the cyclic sum, we have the required inequality. It remains to show the right-hand. As in the first solution, The Cauchy-Schwarz Inequality and $xyz \geq 1$ imply that

$$(x^5 + y^2 + z^2)(yz + y^2 + z^2) \geq (x^2 + y^2 + z^2)^2 \text{ or } \frac{x^2(yz + y^2 + z^2)}{(x^2 + y^2 + z^2)^2} \geq \frac{x^2}{x^5 + y^2 + z^2}.$$

Taking the cyclic sum, we have

$$\sum_{\text{cyclic}} \frac{x^2(yz + y^2 + z^2)}{(x^2 + y^2 + z^2)^2} \geq \sum_{\text{cyclic}} \frac{x^2}{x^5 + y^2 + z^2}.$$

Our job is now to establish the following homogeneous inequality

$$1 \geq \sum_{\text{cyclic}} \frac{x^2(yz + y^2 + z^2)}{(x^2 + y^2 + z^2)^2} \Leftrightarrow (x^2 + y^2 + z^2)^2 \geq 2\sum_{\text{cyclic}} x^2y^2 + \sum_{\text{cyclic}} x^2yz \Leftrightarrow \sum_{\text{cyclic}} x^4 \geq \sum_{\text{cyclic}} x^2yz.$$

However, by The AM-GM Inequality, we obtain

$$\sum_{\text{cyclic}} x^4 = \sum_{\text{cyclic}} \frac{x^4 + y^4}{2} \geq \sum_{\text{cyclic}} x^2y^2 = \sum_{\text{cyclic}} x^2\left(\frac{y^2 + z^2}{2}\right) \geq \sum_{\text{cyclic}} x^2yz.$$

□

Remark 7.0.2. *Here is an alternative way to reach the right hand side inequality. We claim that*

$$\frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \frac{x^2}{x^5+y^2+z^2}.$$

We do this by proving

$$\frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \frac{x^2yz}{x^4+y^3z+yz^3}$$

because $xyz \geq 1$ *implies that*

$$\frac{x^2yz}{x^4+y^3z+yz^3} = \frac{x^2}{\frac{x^5}{xyz}+y^2+z^2} \geq \frac{x^2}{x^5+y^2+z^2}.$$

Hence, we need to show the homogeneous inequality

$$(2x^4+y^4+z^4+4x^2y^2+4x^2z^2)(x^4+y^3z+yz^3) \geq 4x^2yz(x^2+y^2+z^2)^2.$$

However, this is a straightforward consequence of The AM-GM Inequality.

$$\begin{aligned}
& (2x^4+y^4+z^4+4x^2y^2+4x^2z^2)(x^4+y^3z+yz^3) - 4x^2yz(x^2+y^2+z^2)^2 \\
= \;& (x^8+x^4y^4+x^6y^2+x^6y^2+y^7z+y^3z^5)+(x^8+x^4z^4+x^6z^2+x^6z^2+yz^7+y^5z^3) \\
& +2(x^6y^2+x^6z^2)-6x^4y^3z-6x^4yz^3-2x^6yz \\
\geq \;& 6\sqrt[6]{x^8\cdot x^4y^4\cdot x^6y^2\cdot x^6y^2\cdot y^7z\cdot y^3z^5}+6\sqrt[6]{x^8\cdot x^4z^4\cdot x^6z^2\cdot x^6z^2\cdot yz^7\cdot y^5z^3} \\
& +2\sqrt{x^6y^2\cdot x^6z^2}-6x^4y^3z-6x^4yz^3-2x^6yz \\
= \;& 0.
\end{aligned}$$

Taking the cyclic sum, we obtain

$$1 = \sum_{\text{cyclic}} \frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \sum_{\text{cyclic}} \frac{x^2}{x^5+y^2+z^2}.$$

Third Solution. (by an IMO 2005 contestant Iurie Boreico[7] from Moldova) We establish that

$$\frac{x^5-x^2}{x^5+y^2+z^2} \geq \frac{x^5-x^2}{x^3(x^2+y^2+z^2)}.$$

It follows immediately from the identity

$$\frac{x^5-x^2}{x^5+y^2+z^2} - \frac{x^5-x^2}{x^3(x^2+y^2+z^2)} = \frac{(x^3-1)^2x^2(y^2+z^2)}{x^3(x^2+y^2+z^2)(x^5+y^2+z^2)}.$$

Taking the cyclic sum and using $xyz \geq 1$, we have

$$\sum_{\text{cyclic}} \frac{x^5-x^2}{x^5+y^2+z^2} \geq \frac{1}{x^5+y^2+z^2}\sum_{\text{cyclic}}\left(x^2-\frac{1}{x}\right) \geq \frac{1}{x^5+y^2+z^2}\sum_{\text{cyclic}}\left(x^2-yz\right) \geq 0.$$

$\square$

[7] He received the special prize for this solution.

53 (KMO Weekend Program 2007) Prove that, for all $a, b, c, x, y, z > 0$,

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z}.$$

Solution. (by Sanghoon at the KMO Weekend Program 2007) We need the following lemma:

Lemma 7.0.2. *For all* $p, q, \omega_1, \omega_2 > 0$, *we have*

$$\frac{pq}{p+q} \leq \frac{{\omega_1}^2 p + {\omega_2}^2 q}{(\omega_1 + \omega_2)^2}.$$

Proof. After expanding, it becomes

$$(p+q)\left({\omega_1}^2 p + {\omega_2}^2 q\right) - (\omega_1 + \omega_2)^2 pq \geq 0.$$

However, it can be written as

$$(\omega_1 p - \omega_2 q)^2 \geq 0.$$

□

Now, taking $(p, q, \omega_1, \omega_2) = (a, x, x+y+z, a+b+c)$ in the lemma, we get

$$\frac{ax}{a+x} \leq \frac{(x+y+z)^2 a + (a+b+c)^2 x}{(x+y+z+a+b+c)^2}.$$

Similarly, we obtain

$$\frac{by}{b+y} \leq \frac{(x+y+z)^2 b + (a+b+c)^2 y}{(x+y+z+a+b+c)^2}$$

and

$$\frac{cz}{c+z} \leq \frac{(x+y+z)^2 c + (a+b+c)^2 z}{(x+y+z+a+b+c)^2}.$$

Adding the above three inequalities, we get

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(x+y+z)^2(a+b+c) + (a+b+c)^2(x+y+z)}{(x+y+z+a+b+c)^2}.$$

or

$$\frac{ax}{a+x} + \frac{by}{b+y} + \frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z},$$

as desired.

□

54 (USAMO Summer Program 2002) Let a, b, c be positive real numbers. Prove that

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} + \left(\frac{2b}{c+a}\right)^{\frac{2}{3}} + \left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3.$$

Proof. Establish the inequality

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} \geq 3\left(\frac{a}{a+b+c}\right).$$

□

55 (APMO 2005) Let a, b, c be positive real numbers with $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

Proof. Use the auxiliary inequality

$$\frac{1}{\sqrt{1+x^3}} \geq \frac{2}{2+x^2}.$$

□

56 (Titu Andreescu, Gabriel Dospinescu) Let x, y, and z be real numbers such that $x, y, z \leq 1$ and $x + y + z = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \leq \frac{27}{10}.$$

Solution. Employ the following inequality

$$\frac{1}{1+t^2} \leq -\frac{27}{50}\left(t-2\right),$$

where $t \leq 1$. □

57 (Japan 1997) Let a, b, and c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5}.$$

Solution. Because of the homogeneity of the inequality, we may normalize to $a+b+c=1$. It takes the form

$$\frac{(1-2a)^2}{(1-a)^2+a^2}+\frac{(1-2b)^2}{(1-b)^2+b^2}+\frac{(1-2c)^2}{(1-c)^2+c^2}\geq\frac{3}{5}$$

or

$$\frac{1}{2a^2-2a+1}+\frac{1}{2b^2-2b+1}+\frac{1}{2c^2-2c+1}\leq\frac{27}{5}.$$

We find that the equation of the tangent line of $f(x)=\frac{1}{2x^2-2x+1}$ at $x=\frac{1}{3}$ is given by

$$y=\frac{54}{25}x+\frac{27}{25}$$

and that

$$f(x)-\left(\frac{54}{25}x+\frac{27}{25}\right)=-\frac{2(3x-1)^2(6x+1)}{25(2x^2-2x+1)}\leq 0.$$

for all $x>0$. It follows that

$$\sum_{\text{cyclic}} f(a)\leq\sum_{\text{cyclic}}\frac{54}{25}a+\frac{27}{25}=\frac{27}{5}.$$

□

58 **[IMO 1984/1 FRG]** Let x, y, z be nonnegative real numbers such that $x + y + z = 1$. Prove that $0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$.

First Solution. Using the constraint $x + y + z = 1$, we reduce the inequality to homogeneous one:

$$0 \leq (xy + yz + zx)(x + y + z) - 2xyz \leq \frac{7}{27}(x + y + z)^3.$$

The left hand side inequality is trivial because it's equivalent to

$$0 \leq xyz + \sum_{\text{sym}} x^2 y.$$

The right hand side inequality simplifies to

$$7 \sum_{\text{cyclic}} x^3 + 15xyz - 6 \sum_{\text{sym}} x^2 y \geq 0.$$

In the view of

$$7 \sum_{\text{cyclic}} x^3 + 15xyz - 6 \sum_{\text{sym}} x^2 y = \left(2 \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2 y\right) + 5\left(3xyz + \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2 y\right),$$

it's enough to show that

$$2 \sum_{\text{cyclic}} x^3 \geq \sum_{\text{sym}} x^2 y$$

and

$$3xyz + \sum_{\text{cyclic}} x^3 \geq \sum_{\text{sym}} x^2 y.$$

The first inequality follows from

$$2 \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2 y = \sum_{\text{cyclic}} (x^3 + y^3) - \sum_{\text{cyclic}} (x^2 y + xy^2) = \sum_{\text{cyclic}} (x^3 + y^3 - x^2 y - xy^2) \geq 0.$$

The second inequality can be rewritten as

$$\sum_{\text{cyclic}} x(x - y)(x - z) \geq 0,$$

which is a particular case of Schur's Theorem. □

59 **[LL 1992 UNK]** (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

Second Solution. After the algebraic substitution $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$, we are required to prove that

$$\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \geq \sqrt{\frac{1-a}{a}}+\sqrt{\frac{1-b}{b}}+\sqrt{\frac{1-c}{c}},$$

where $a, b, c \in (0,1)$ and $a+b+c=2$. Using the constraint $a+b+c=2$, we obtain a homogeneous inequality

$$\sqrt{\frac{1}{2}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \sqrt{\frac{\frac{a+b+c}{2}-a}{a}}+\sqrt{\frac{\frac{a+b+c}{2}-b}{b}}+\sqrt{\frac{\frac{a+b+c}{2}-c}{c}}$$

or

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}},$$

which immediately follows from The Cauchy-Schwarz Inequality

$$\sqrt{[(b+c-a)+(c+a-b)+(a+b-c)]\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}}.$$

$\square$

60 Let x, y, z be nonnegative real numbers. Then, we have

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

First Solution. By Schur's Inequality and The AM-GM Inequality, we have

$$3xyz + \sum_{\text{cyclic}} x^3 \geq \sum_{\text{cyclic}} x^2y + xy^2 \geq \sum_{\text{cyclic}} 2(xy)^{\frac{3}{2}}.$$

$\square$

61 Let $t \in (0,3]$. For all $a,b,c \geq 0$, we have

$$(3-t)+t(abc)^{\frac{2}{t}}+\sum_{\text{cyclic}} a^2 \geq 2\sum_{\text{cyclic}} ab.$$

Proof. After setting $x=a^{\frac{2}{3}}$, $y=b^{\frac{2}{3}}$, $z=c^{\frac{2}{3}}$, it becomes

$$3-t+t(xyz)^{\frac{3}{t}}+\sum_{\text{cyclic}} x^3 \geq 2\sum_{\text{cyclic}} (xy)^{\frac{3}{2}}.$$

By the previous epsilon, it will be enough to show that

$$3-t+t(xyz)^{\frac{3}{t}} \geq 3xyz,$$

which is a straightforward consequence of the weighted AM-GM inequality :

$$\frac{3-t}{3}\cdot 1+\frac{t}{3}(xyz)^{\frac{3}{t}} \geq 1^{\frac{3-t}{3}}\left((xyz)^{\frac{3}{t}}\right)^{\frac{t}{3}}=3xyz.$$

One may check that the equality holds if and only if $a=b=c=1$. $\square$

Remark 7.0.3. *In particular, we obtain non-homogeneous inequalities*

$$\frac{5}{2}+\frac{1}{2}(abc)^4+a^2+b^2+c^2 \geq 2(ab+bc+ca),$$
$$2+(abc)^2+a^2+b^2+c^2 \geq 2(ab+bc+ca),$$
$$1+2abc+a^2+b^2+c^2 \geq 2(ab+bc+ca).$$

62 **(APMO 2004/5)** Prove that, for all positive real numbers a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

Second Solution. After expanding, it becomes

$$8+(abc)^2+2\sum_{\text{cyclic}} a^2b^2+4\sum_{\text{cyclic}} a^2 \geq 9\sum_{\text{cyclic}} ab.$$

From the inequality $(ab-1)^2+(bc-1)^2+(ca-1)^2 \geq 0$, we obtain

$$6+2\sum_{\text{cyclic}} a^2b^2 \geq 4\sum_{\text{cyclic}} ab.$$

Hence, it will be enough to show that

$$2+(abc)^2+4\sum_{\text{cyclic}} a^2 \geq 5\sum_{\text{cyclic}} ab.$$

Since $3(a^2+b^2+c^2) \geq 3(ab+bc+ca)$, it will be enough to show that

$$2+(abc)^2+\sum_{\text{cyclic}} a^2 \geq 2\sum_{\text{cyclic}} ab,$$

which is a particular case of the previous epsilon. □

63 **[IMO 2000/2 USA]** Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Second Solution. It is equivalent to the following homogeneous inequality:

$$\left(a - (abc)^{1/3} + \frac{(abc)^{2/3}}{b}\right)\left(b - (abc)^{1/3} + \frac{(abc)^{2/3}}{c}\right)\left(c - (abc)^{1/3} + \frac{(abc)^{2/3}}{a}\right) \leq abc.$$

After the substitution $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$, it becomes

$$\left(x^3 - xyz + \frac{(xyz)^2}{y^3}\right)\left(y^3 - xyz + \frac{(xyz)^2}{z^3}\right)\left(z^3 - xyz + \frac{(xyz)^2}{x^3}\right) \leq x^3y^3z^3,$$

which simplifies to

$$\left(x^2y - y^2z + z^2x\right)\left(y^2z - z^2x + x^2y\right)\left(z^2x - x^2y + y^2z\right) \leq x^3y^3z^3$$

or

$$3x^3y^3z^3 + \sum_{\text{cyclic}} x^6y^3 \geq \sum_{\text{cyclic}} x^4y^4z + \sum_{\text{cyclic}} x^5y^2z^2$$

or

$$3(x^2y)(y^2z)(z^2x) + \sum_{\text{cyclic}} (x^2y)^3 \geq \sum_{\text{sym}} (x^2y)^2(y^2z)$$

which is a special case of Schur's Inequality. □

64 (Tournament of Towns 1997) Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \le 1.$$

Solution. We can rewrite the given inequality as following :

$$\frac{1}{a+b+(abc)^{1/3}} + \frac{1}{b+c+(abc)^{1/3}} + \frac{1}{c+a+(abc)^{1/3}} \le \frac{1}{(abc)^{1/3}}.$$

We make the substitution $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$. Then, it becomes

$$\frac{1}{x^3+y^3+xyz} + \frac{1}{y^3+z^3+xyz} + \frac{1}{z^3+x^3+xyz} \le \frac{1}{xyz}$$

which is equivalent to

$$xyz \sum_{\text{cyclic}} (x^3+y^3+xyz)(y^3+z^3+xyz) \le (x^3+y^3+xyz)(y^3+z^3+xyz)(z^3+x^3+xyz)$$

or

$$\sum_{\text{sym}} x^6y^3 \ge \sum_{\text{sym}} x^5y^2z^2 \quad !$$

We now obtain

$$\begin{aligned}
\sum_{\text{sym}} x^6y^3 &= \sum_{\text{cyclic}} x^6y^3 + y^6x^3 \\
&\ge \sum_{\text{cyclic}} x^5y^4 + y^5x^4 \\
&= \sum_{\text{cyclic}} x^5(y^4+z^4) \\
&\ge \sum_{\text{cyclic}} x^5(y^2z^2+y^2z^2) \\
&= \sum_{\text{sym}} x^5y^2z^2.
\end{aligned}$$

□

65 (Muirhead's Theorem) Let $a_1, a_2, a_3, b_1, b_2, b_3$ be real numbers such that

$$a_1 \geq a_2 \geq a_3 \geq 0, b_1 \geq b_2 \geq b_3 \geq 0, a_1 \geq b_1, a_1 + a_2 \geq b_1 + b_2, a_1 + a_2 + a_3 = b_1 + b_2 + b_3.$$

Let x, y, z be positive real numbers. Then, we have

$$\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} \geq \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.$$

Solution. We distinguish two cases.

Case 1. $b_1 \geq a_2$: It follows from $a_1 \geq a_1 + a_2 - b_1$ and from $a_1 \geq b_1$ that $a_1 \geq max(a_1 + a_2 - b_1, b_1)$ so that $max(a_1, a_2) = a_1 \geq max(a_1 + a_2 - b_1, b_1)$. From $a_1 + a_2 - b_1 \geq b_1 + a_3 - b_1 = a_3$ and $a_1 + a_2 - b_1 \geq b_2 \geq b_3$, we have $max(a_1 + a_2 - b_1, a_3) \geq max(b_2, b_3)$. It follows that

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} &= \sum_{\text{cyclic}} z^{a_3} (x^{a_1} y^{a_2} + x^{a_2} y^{a_1}) \\
&\geq \sum_{\text{cyclic}} z^{a_3} (x^{a_1+a_2-b_1} y^{b_1} + x^{b_1} y^{a_1+a_2-b_1}) \\
&= \sum_{\text{cyclic}} x^{b_1} (y^{a_1+a_2-b_1} z^{a_3} + y^{a_3} z^{a_1+a_2-b_1}) \\
&\geq \sum_{\text{cyclic}} x^{b_1} (y^{b_2} z^{b_3} + y^{b_3} z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.
\end{aligned}$$

Case 2. $b_1 \leq a_2$: It follows from $3b_1 \geq b_1 + b_2 + b_3 = a_1 + a_2 + a_3 \geq b_1 + a_2 + a_3$ that $b_1 \geq a_2 + a_3 - b_1$ and that $a_1 \geq a_2 \geq b_1 \geq a_2 + a_3 - b_1$. Therefore, we have $max(a_2, a_3) \geq max(b_1, a_2 + a_3 - b_1)$ and $max(a_1, a_2 + a_3 - b_1) \geq max(b_2, b_3)$. It follows that

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1} y^{a_2} z^{a_3} &= \sum_{\text{cyclic}} x^{a_1} (y^{a_2} z^{a_3} + y^{a_3} z^{a_2}) \\
&\geq \sum_{\text{cyclic}} x^{a_1} (y^{b_1} z^{a_2+a_3-b_1} + y^{a_2+a_3-b_1} z^{b_1}) \\
&= \sum_{\text{cyclic}} y^{b_1} (x^{a_1} z^{a_2+a_3-b_1} + x^{a_2+a_3-b_1} z^{a_1}) \\
&\geq \sum_{\text{cyclic}} y^{b_1} (x^{b_2} z^{b_3} + x^{b_3} z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1} y^{b_2} z^{b_3}.
\end{aligned}$$

□

66 **[IMO 1995/2 RUS]** Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Third Solution. It's equivalent to

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2(abc)^{4/3}}.$$

Set $a = x^3, b = y^3, c = z^3$ with $x, y, z > 0$. Then, it becomes

$$\sum_{\text{cyclic}} \frac{1}{x^9(y^3+z^3)} \geq \frac{3}{2x^4y^4z^4}.$$

Clearing denominators, this can be rewritten as

$$\sum_{\text{sym}} x^{12}y^{12} + 2\sum_{\text{sym}} x^{12}y^9z^3 + \sum_{\text{sym}} x^9y^9z^6 \geq 3\sum_{\text{sym}} x^{11}y^8z^5 + 6x^8y^8z^8$$

or

$$\left(\sum_{\text{sym}} x^{12}y^{12} - \sum_{\text{sym}} x^{11}y^8z^5\right) + 2\left(\sum_{\text{sym}} x^{12}y^9z^3 - \sum_{\text{sym}} x^{11}y^8z^5\right) + \left(\sum_{\text{sym}} x^9y^9z^6 - \sum_{\text{sym}} x^8y^8z^8\right) \geq 0,$$

By Muirhead's Theorem, every term on the left hand side is nonnegative. □

67 (Iran 1996) Let x, y, z be positive real numbers. Prove that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

Solution. It's equivalent to

$$4\sum_{\text{sym}} x^5y + 2\sum_{\text{cyclic}} x^4yz + 6x^2y^2z^2 - \sum_{\text{sym}} x^4y^2 - 6\sum_{\text{cyclic}} x^3y^3 - 2\sum_{\text{sym}} x^3y^2z \geq 0.$$

We rewrite this as following

$$\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^4y^2\right) + 3\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^3y^3\right) + 2xyz\left(3xyz + \sum_{\text{cyclic}} x^3 - \sum_{\text{sym}} x^2y\right) \geq 0.$$

By Muirhead's Theorem and Schur's Inequality, it's a sum of three nonnegative terms. $\square$

68 Let x, y, z be nonnegative real numbers with $xy + yz + zx = 1$. Prove that

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{5}{2}.$$

Solution. Using $xy + yz + zx = 1$, we homogenize the given inequality as following :

$$(xy + yz + zx)\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right)^2 \geq \left(\frac{5}{2}\right)^2$$

or

$$4\sum_{\text{sym}} x^5y + \sum_{\text{sym}} x^4yz + 14\sum_{\text{sym}} x^3y^2z + 38x^2y^2z^2 \geq \sum_{\text{sym}} x^4y^2 + 3\sum_{\text{sym}} x^3y^3$$

or

$$\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^4y^2\right) + 3\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^3y^3\right) + xyz\left(\sum_{\text{sym}} x^3 + 14\sum_{\text{sym}} x^2y + 38xyz\right) \geq 0.$$

By Muirhead's Theorem, we get the result. In the above inequality, without the condition $xy+yz+zx = 1$, the equality holds if and only if $x = y, z = 0$ *or* $y = z, x = 0$ *or* $z = x, y = 0$. Since $xy + yz + zx = 1$, the equality occurs when $(x, y, z) = (1, 1, 0), (1, 0, 1), (0, 1, 1)$. □

69 **[IMO 2001/2 KOR]** Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Third Solution. We offer a convexity proof. We make the substitution

$$x=\frac{a}{a+b+c},\ y=\frac{b}{a+b+c},\ z=\frac{c}{a+b+c}.$$

The inequality becomes

$$xf(x^2+8yz)+yf(y^2+8zx)+zf(z^2+8xy)\geq 1,$$

where $f(t)=\frac{1}{\sqrt{t}}$.[8] Since f is convex on $\mathbb{R}^+$ and $x+y+z=1$, we apply (the weighted) Jensen's Inequality to obtain

$$xf(x^2+8yz)+yf(y^2+8zx)+zf(z^2+8xy)\geq f(x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy)).$$

Note that $f(1)=1$. Since the function f is strictly decreasing, it suffices to show that

$$1\geq x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy).$$

Using $x+y+z=1$, we homogenize it as

$$(x+y+z)^3\geq x(x^2+8yz)+y(y^2+8zx)+z(z^2+8xy).$$

However, it is easily seen from

$$(x+y+z)^3-x(x^2+8yz)-y(y^2+8zx)-z(z^2+8xy)=3[x(y-z)^2+y(z-x)^2+z(x-y)^2]\geq 0.$$

□

Fourth Solution. We begin with the substitution

$$x=\frac{bc}{a^2},y=\frac{ca}{b^2},z=\frac{ab}{c^2}.$$

Then, we get $xyz=1$ and the inequality becomes

$$\frac{1}{\sqrt{1+8x}}+\frac{1}{\sqrt{1+8y}}+\frac{1}{\sqrt{1+8z}}\geq 1$$

which is equivalent to

$$\sum_{\text{cyclic}}\sqrt{(1+8x)(1+8y)}\geq\sqrt{(1+8x)(1+8y)(1+8z)}.$$

After squaring both sides, it's equivalent to

$$8(x+y+z)+2\sqrt{(1+8x)(1+8y)(1+8z)}\sum_{\text{cyclic}}\sqrt{1+8x}\geq 510.$$

Recall that $xyz=1$. The AM-GM Inequality gives us $x+y+z\geq 3$,

$$(1+8x)(1+8y)(1+8z)\geq 9x^{\frac{8}{9}}\cdot 9y^{\frac{8}{9}}\cdot 9z^{\frac{8}{9}}=729\ \text{ and }\ \sum_{\text{cyclic}}\sqrt{1+8x}\geq\sum_{\text{cyclic}}\sqrt{9x^{\frac{8}{9}}}\geq 9(xyz)^{\frac{4}{27}}=9.$$

Using these three inequalities, we get the result. □

[8] Dividing by $a+b+c$ gives the equivalent inequality $\sum_{\text{cyclic}}\frac{\frac{a}{a+b+c}}{\sqrt{\frac{a^2}{(a+b+c)^2}+\frac{8bc}{(a+b+c)^2}}}\geq 1$.

70 **[IMO 1983/6 USA]** Let a, b, c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Second Solution. We present a convexity proof. After setting $a=y+z$, $b=z+x$, $c=x+y$ for $x,y,z>0$, it becomes

$$x^3z+y^3x+z^3y\geq x^2yz+xy^2z+xyz^2$$

or

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq x+y+z.$$

Since it's homogeneous, we can restrict our attention to the case $x+y+z=1$. Then, it becomes

$$yf\left(\frac{x}{y}\right)+zf\left(\frac{y}{z}\right)+xf\left(\frac{z}{x}\right)\geq 1,$$

where $f(t)=t^2$. Since f is convex on $\mathbb{R}$, we apply (the weighted) Jensen's Inequality to obtain

$$yf\left(\frac{x}{y}\right)+zf\left(\frac{y}{z}\right)+xf\left(\frac{z}{x}\right)\geq f\left(y\cdot\frac{x}{y}+z\cdot\frac{y}{z}+x\cdot\frac{z}{x}\right)=f(1)=1.$$

□

71 (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{\left(a^2b + b^2c + c^2a\right)\left(ab^2 + bc^2 + ca^2\right)} \geq abc + \sqrt[3]{\left(a^3 + abc\right)\left(b^3 + abc\right)\left(c^3 + abc\right)}$$

First Solution. Dividing by abc, it becomes

$$\sqrt{\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right)\left(\frac{c}{a} + \frac{a}{b} + \frac{b}{c}\right)} \geq abc + \sqrt[3]{\left(\frac{a^2}{bc} + 1\right)\left(\frac{b^2}{ca} + 1\right)\left(\frac{c^2}{ab} + 1\right)}.$$

After the substitution $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$, we obtain the constraint $xyz = 1$. It takes the form

$$\sqrt{(x + y + z)(xy + yz + zx)} \geq 1 + \sqrt[3]{\left(\frac{x}{z} + 1\right)\left(\frac{y}{x} + 1\right)\left(\frac{z}{y} + 1\right)}.$$

From the constraint $xyz = 1$, we find two identities

$$\left(\frac{x}{z} + 1\right)\left(\frac{y}{x} + 1\right)\left(\frac{z}{y} + 1\right) = \left(\frac{x + z}{z}\right)\left(\frac{y + x}{x}\right)\left(\frac{z + y}{y}\right) = (z + x)(x + y)(y + z),$$

$$(x + y + z)(xy + yz + zx) = (x + y)(y + z)(z + x) + xyz = (x + y)(y + z)(z + x) + 1.$$

Letting $p = \sqrt[3]{(x + y)(y + z)(z + x)}$, the inequality we want to prove now becomes

$$\sqrt{p^3 + 1} \geq 1 + p.$$

Applying The AM-GM Inequality yields

$$p \geq \sqrt[3]{2\sqrt{xy} \cdot 2\sqrt{yz} \cdot 2\sqrt{zx}} = 2.$$

or

$$\left(p^3 + 1\right) - (1 + p)^2 = p(p + 1)(p - 2) \geq 0.$$

$\square$

72 **[IMO 1999/2 POL]** Let n be an integer with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

First Solution. (Marcin E. Kuczma[9]) For $x_1 = \cdots = x_n = 0$, it holds for any $C \geq 0$. Hence, we consider the case when $x_1 + \cdots + x_n > 0$. Since the inequality is homogeneous, we may normalize to $x_1 + \cdots + x_n = 1$. We denote

$$F(x_1, \cdots, x_n) = \sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2).$$

From the assumption $x_1 + \cdots + x_n = 1$, we have

$$F(x_1, \cdots, x_n) = \sum_{1 \leq i < j \leq n} {x_i}^3 x_j + \sum_{1 \leq i < j \leq n} x_i {x_j}^3 = \sum_{1 \leq i \leq n} {x_i}^3 \sum_{j \neq i} x_i = \sum_{1 \leq i \leq n} {x_i}^3 (1 - x_i)$$

$$= \sum_{i=1}^{n} x_i ({x_i}^2 - {x_i}^3).$$

We claim that $C = \frac{1}{8}$. It suffices to show that $F(x_1, \cdots, x_n) \leq \frac{1}{8} = F\left(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0\right)$.

Lemma 7.0.3. $0 \leq x \leq y \leq \frac{1}{2}$ *implies* $x^2 - x^3 \leq y^2 - y^3$.

Proof. Since $x + y \leq 1$, we get $x + y \geq (x+y)^2 \geq x^2 + xy + y^2$. Since $y - x \geq 0$, this implies that $y^2 - x^2 \geq y^3 - x^3$ or $y^2 - y^3 \geq x^2 - x^3$, as desired. □

Case 1. $\frac{1}{2} \geq x_1 \geq x_2 \geq \cdots \geq x_n$*:*

$$\sum_{i=1}^{n} x_i ({x_i}^2 - {x_i}^3) \leq \sum_{i=1}^{n} x_i \left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 \right) = \frac{1}{8} \sum_{i=1}^{n} x_i = \frac{1}{8}.$$

Case 2. $x_1 \geq \frac{1}{2} \geq x_2 \geq \cdots \geq x_n$*: Let* $x_1 = x$ *and* $y = 1 - x = x_2 + \cdots + x_n$*. Since* $y \geq x_2, \cdots, x_n$*,*

$$F(x_1, \cdots, x_n) = x^3 y + \sum_{i=2}^{n} x_i ({x_i}^2 - {x_i}^3) \leq x^3 y + \sum_{i=2}^{n} x_i (y^2 - y^3) = x^3 y + y(y^2 - y^3).$$

Since $x^3 y + y(y^2 - y^3) = x^3 y + y^3(1 - y) = xy(x^2 + y^2)$*, it remains to show that*

$$xy(x^2 + y^2) \leq \frac{1}{8}.$$

Using $x + y = 1$*, we homogenize the above inequality as following.*

$$xy(x^2 + y^2) \leq \frac{1}{8}(x+y)^4.$$

However, we immediately find that $(x+y)^4 - 8xy(x^2 + y^2) = (x-y)^4 \geq 0$*.*

□

[9] I slightly modified his solution in [Au99].

73 (APMO 1991) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers such that $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. Show that

$$\frac{{a_1}^2}{a_1 + b_1} + \cdots + \frac{{a_n}^2}{a_n + b_n} \geq \frac{a_1 + \cdots + a_n}{2}.$$

Second Solution. By The Cauchy-Schwarz Inequality, we have

$$\sum_{i=1}^{n} a_i + b_i \sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i} \geq \left(\sum_{i=1}^{n}\right)^2$$

or

$$\sum_{i=1}^{n} \frac{{a_i}^2}{a_i + b_i} \geq \frac{\left(\sum_{i=1}^{n}\right)^2}{\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i} = \frac{1}{2}\sum_{i=1}^{n} a_i$$

□

74 Let $a, b \geq 0$ with $a + b = 1$. Prove that

$$\sqrt{a^2 + b} + \sqrt{a + b^2} + \sqrt{1 + ab} \leq 3.$$

Show that the equality holds if and only if $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$.

Second Solution. The Cauchy-Schwarz Inequality shows that

$$\begin{aligned}
\sqrt{a^2 + b} + \sqrt{a + b^2} + \sqrt{1 + ab} &\leq \sqrt{3\left(a^2 + b + a + b^2 + 1 + ab\right)} \\
&= \sqrt{3\left(a^2 + ab + b^2 + a + b + 1\right)} \\
&\leq \sqrt{3\left((a + b)^2 + a + b + 1\right)} \\
&= 3.
\end{aligned}$$

□

75 **[LL 1992 UNK]** (Iran 1998) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Third Solution. We first note that

$$\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z} = 1.$$

Apply The Cauchy-Schwarz Inequality to deduce

$$\sqrt{x+y+z} = \sqrt{(x+y+z)\left(\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z}\right)} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

$\square$

76 (Gazeta Matematică) Prove that, for all $a, b, c > 0$,

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4} \ge a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.$$

Solution. We obtain the chain of equalities and inequalities

$$\begin{aligned}
\sum_{\text{cyclic}} \sqrt{a^4+a^2b^2+b^4} &= \sum_{\text{cyclic}} \sqrt{\left(a^4+\frac{a^2b^2}{2}\right)+\left(b^4+\frac{a^2b^2}{2}\right)} \\
&\ge \frac{1}{\sqrt{2}} \sum_{\text{cyclic}} \left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{b^4+\frac{a^2b^2}{2}}\right) \qquad \text{(Cauchy – Schwarz)} \\
&= \frac{1}{\sqrt{2}} \sum_{\text{cyclic}} \left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{a^4+\frac{a^2c^2}{2}}\right) \\
&\ge \sqrt{2} \sum_{\text{cyclic}} \sqrt[4]{\left(a^4+\frac{a^2b^2}{2}\right)\left(a^4+\frac{a^2c^2}{2}\right)} \qquad \text{(AM – GM)} \\
&\ge \sqrt{2} \sum_{\text{cyclic}} \sqrt{a^4+\frac{a^2bc}{2}} \qquad \text{(Cauchy – Schwarz)} \\
&= \sum_{\text{cyclic}} \sqrt{2a^4+a^2bc}\,.
\end{aligned}$$

□

77 (KMO Winter Program Test 2001) Prove that, for all $a, b, c > 0$,

$$\sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)} \geq abc + \sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}$$

Second Solution. (based on work by an winter program participant) We obtain

$$\begin{aligned}
& \sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)} \\
= \ & \frac{1}{2}\sqrt{[b(a^2+bc)+c(b^2+ca)+a(c^2+ab)][c(a^2+bc)+a(b^2+ca)+b(c^2+ab)]} \\
\geq \ & \frac{1}{2}\left(\sqrt{bc}(a^2+bc)+\sqrt{ca}(b^2+ca)+\sqrt{ab}(c^2+ab)\right) & \text{(Cauchy} - \text{Schwarz)} \\
\geq \ & \frac{3}{2}\sqrt[3]{\sqrt{bc}(a^2+bc)\cdot\sqrt{ca}(b^2+ca)\cdot\sqrt{ab}(c^2+ab)} & \text{(AM} - \text{GM)} \\
= \ & \frac{1}{2}\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)} + \sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)} \\
\geq \ & \frac{1}{2}\sqrt[3]{2\sqrt{a^3\cdot abc}\cdot 2\sqrt{b^3\cdot abc}\cdot 2\sqrt{c^3\cdot abc}} + \sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)} & \text{(AM} - \text{GM)} \\
= \ & abc + \sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}.
\end{aligned}$$

□

78 (Andrei Ciupan) Let a, b, c be positive real numbers such that

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\geq 1.$$

Show that $a+b+c\geq ab+bc+ca$.

First Solution. (Andrei Ciupan) By applying The Cauchy-Schwarz Inequality, we obtain

$$(a+b+1)(a+b+c^2)\geq (a+b+c)^2$$

or

$$\frac{1}{a+b+1}\leq \frac{c^2+a+b}{(a+b+c)^2}.$$

Now by summing cyclically, we obtain

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\leq \frac{a^2+b^2+c^2+2(a+b+c)}{(a+b+c)^2}$$

But from the condition, we can see that

$$a^2+b^2+c^2+2(a+b+c)\geq (a+b+c)^2,$$

and therefore

$$a+b+c\geq ab+bc+ca.$$

We see that the equality occurs if and only if $a=b=c=1$. □

Second Solution. (Cezar Lupu) We first observe that

$$2\geq \sum_{\text{cyclic}}\left(1-\frac{1}{a+b+1}\right)=\sum_{\text{cyclic}}\frac{a+b}{a+b+1}=\sum_{\text{cyclic}}\frac{(a+b)^2}{(a+b)^2+a+b}.$$

Apply The Cauchy-Schwarz Inequality to get

$$2\geq \sum_{\text{cyclic}}\frac{(a+b)^2}{(a+b)^2+a+b}\geq \frac{(\sum a+b)^2}{\sum (a+b)^2+a+b}=\frac{4\sum a^2+8\sum ab}{2\sum a^2+2\sum ab+2\sum a}.$$

or

$$a+b+c\geq ab+bc+ca.$$

□

79 (Hölder's Inequality) Let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^{m} \left(\prod_{j=1}^{n} {x_{ij}}^{\omega_j} \right).$$

Proof. Because of the homogeneity of the inequality, we may rescale $x_{1j}, \cdots, x_{mj}$ so that $x_{1j} + \cdots + x_{mj} = 1$ for each $j \in \{1, \cdots, n\}$. Then, we need to show that

$$\prod_{j=1}^{n} 1^{\omega_j} \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j} \quad \text{or} \quad 1 \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j}.$$

The Weighted AM-GM Inequality provides that

$$\sum_{j=1}^{n} \omega_j x_{ij} \geq \prod_{j=1}^{n} {x_{ij}}^{\omega_j} \quad (i \in \{1, \cdots, m\}) \implies \sum_{i=1}^{m} \sum_{j=1}^{n} \omega_j x_{ij} \geq \sum_{i=1}^{m} \prod_{j=1}^{n} {x_{ij}}^{\omega_j}.$$

However, we immediately have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \omega_j x_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} \omega_j x_{ij} = \sum_{j=1}^{n} \omega_j \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{j=1}^{n} \omega_j = 1.$$

□

80 (Ireland 2000) Let $x, y \geq 0$ with $x + y = 2$. Prove that $x^2y^2(x^2 + y^2) \leq 2$.

First Solution. After homogenizing it, we need to prove

$$2\left(\frac{x+y}{2}\right)^6 \geq x^2y^2(x^2+y^2) \text{ or } (x+y)^6 \geq 32x^2y^2(x^2+y^2).$$

(Now, forget the constraint $x + y = 2$!) In case $xy = 0$, it clearly holds. We now assume that $xy \neq 0$. Because of the homogeneity of the inequality, this means that we may normalize to $xy = 1$. Then, it becomes

$$\left(x+\frac{1}{x}\right)^6 \geq 32\left(x^2+\frac{1}{x^2}\right) \text{ or } p^3 \geq 32(p-2).$$

where $p = \left(x + \frac{1}{x}\right)^2 \geq 4$. Our job is now to minimize $F(p) = p^3 - 32(p-2)$ on $[4, \infty)$. Since $F'(p) = 3p^2 - 32 \geq 0$, where $p \geq \sqrt{\frac{32}{3}}$, F is (monotone) increasing on $[4, \infty)$. So, $F(p) \geq F(4) = 0$ for all $p \geq 4$. □

Second Solution. As in the first solution, we prove that $(x + y)^6 \geq 32(x^2 + y^2)(xy)^2$ for all $x, y \geq 0$. In case $x = y = 0$, it's clear. Now, if $x^2 + y^2 > 0$, then we may normalize to $x^2 + y^2 = 2$. Setting $p = xy$, we have $0 \leq p \leq \frac{x^2+y^2}{2} = 1$ and $(x + y)^2 = x^2 + y^2 + 2xy = 2 + 2p$. It now becomes

$$(2+2p)^3 \geq 64p^2 \text{ or } p^3 - 5p^2 + 3p + 1 \geq 0.$$

We want to minimize $F(p) = p^3 - 5p^2 + 3p + 1$ on $[0, 1]$. We compute $F'(p) = 3\left(p - \frac{1}{3}\right)(p - 3)$. We find that F is monotone increasing on $[0, \frac{1}{3}]$ and monotone decreasing on $[\frac{1}{3}, 1]$. Since $F(0) = 1$ and $F(1) = 0$, we conclude that $F(p) \geq F(1) = 0$ for all $p \in [0, 1]$. □

Third Solution. We show that $(x + y)^6 \geq 32(x^2 + y^2)(xy)^2$ where $x \geq y \geq 0$. We make the substitution $u = x + y$ and $v = x - y$. Then, we have $u \geq v \geq 0$. It becomes

$$u^6 \geq 32\left(\frac{u^2+v^2}{2}\right)\left(\frac{u^2-v^2}{4}\right)^2$$

or

$$u^6 \geq (u^2+v^2)(u^2-v^2)^2.$$

Notice that $u^4 \geq u^4 - v^4 \geq 0$ and that $u^2 \geq u^2 - v^2 \geq 0$. So, we have

$$u^6 \geq (u^4-v^4)(u^2-v^2) = (u^2+v^2)(u^2-v^2)^2.$$

□

Remark 7.0.4. *This is a particular case of the following proposition:*

Proposition 7.0.1. *Let x, y, z be non-negative real numbers. Then, we have*

$$\left(x^2+y^2\right)\left(y^2+z^2\right)\left(z^2+x^2\right) \leq \frac{1}{32}(x+y+z)^6.$$

Indeed, taking $z = 0$ and $x + y = 2$ in the proposition yields the above inequality.

81 **[IMO 1984/1 FRG]** Let x, y, z be nonnegative real numbers such that $x + y + z = 1$. Prove that $0 \le xy + yz + zx - 2xyz \le \frac{7}{27}$.

Second Solution. Let $f(x, y, z) = xy + yz + zx - 2xyz$. We may assume that $0 \le x \le y \le z \le 1$. Since $x+y+z = 1$, this implies that $x \le \frac{1}{3}$. It follows that $f(x, y, z) = (1-3x)yz+xyz+zx+xy \ge 0$. Applying The AM-GM Inequality, we obtain $yz \le \left(\frac{y+z}{2}\right)^2 = \left(\frac{1-x}{2}\right)^2$. Since $1 - 2x \ge 0$, this implies that

$$f(x, y, z) = x(y + z) + yz(1 - 2x) \le x(1 - x) + \left(\frac{1 - x}{2}\right)^2 (1 - 2x) = \frac{-2x^3 + x^2 + 1}{4}.$$

Our job is now to maximize a one-variable function $F(x) = \frac{1}{4}(-2x^3 + x^2 + 1)$, where $x \in \left[0, \frac{1}{3}\right]$. Since $F'(x) = \frac{3}{2}x\left(\frac{1}{3} - x\right) \ge 0$ on $\left[0, \frac{1}{3}\right]$, we conclude that $F(x) \le F(\frac{1}{3}) = \frac{7}{27}$ for all $x \in \left[0, \frac{1}{3}\right]$. $\square$

82 **[IMO 2000/2 USA]** Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Fifth Solution. (based on work by an IMO 2000 contestant from Japan) Since $abc = 1$, at least one of a, b, c is greater than or equal to 1. Say $b \geq 1$. Putting $c = \frac{1}{ab}$, it becomes

$$\left(a - 1 + \frac{1}{b}\right)(b - 1 + ab)\left(\frac{1}{ab} - 1 + \frac{1}{a}\right) \leq 1$$

or

$$a^3b^3 - a^2b^3 - ab^3 - a^2b^2 + 3ab^2 - ab + b^3 - b^2 - b + 1 \geq 0.$$

Setting $x = ab$, it becomes $f_b(x) \geq 0$, where

$$f_b(t) = t^3 + b^3 - b^2t - bt^2 + 3bt - t^2 - b^2 - t - b + 1.$$

Fix a positive number $b \geq 1$. We need to show that $F(t) := f_b(t) \geq 0$ for all $t \geq 0$. It follows from $b \geq 1$ that the cubic polynomial $F'(t) = 3t^2 - 2(b+1)t - (b^2 - 3b + 1)$ has two real roots

$$\frac{b + 1 - \sqrt{4b^2 - 7b + 4}}{3} \quad \text{and} \quad \lambda = \frac{b + 1 + \sqrt{4b^2 - 7b + 4}}{3}.$$

Since F has a local minimum at $t = \lambda$, we find that $F(t) \geq Min\,\{F(0),\ F(\lambda)\}$ for all $t \geq 0$. We have to prove that $F(0) \geq 0$ and $F(\lambda) \geq 0$. We have $F(0) = b^3 - b^2 - b + 1 = (b-1)^2(b+1) \geq 0$. It remains to show that $F(\lambda) \geq 0$. Notice that λ is a root of $F^{/}(t)$. After long division, we get

$$F(t) = F'(t)\left(\frac{1}{3}t - \frac{b+1}{9}\right) + \frac{1}{9}\left((-8b^2 + 14b - 8)t + 8b^3 - 7b^2 - 7b + 8\right).$$

Putting $t = \lambda$, we have

$$F(\lambda) = \frac{1}{9}\left((-8b^2 + 14b - 8)\lambda + 8b^3 - 7b^2 - 7b + 8\right).$$

Thus, our job is now to establish that, for all $b \geq 0$,

$$(-8b^2 + 14b - 8)\left(\frac{b + 1 + \sqrt{4b^2 - 7b + 4}}{3}\right) + 8b^3 - 7b^2 - 7b + 8 \geq 0,$$

which is equivalent to

$$16b^3 - 15b^2 - 15b + 16 \geq (8b^2 - 14b + 8)\sqrt{4b^2 - 7b + 4}\,.$$

Since both $16b^3 - 15b^2 - 15b + 16$ and $8b^2 - 14b + 8$ are positive,[10] it's equivalent to

$$(16b^3 - 15b^2 - 15b + 16)^2 \geq (8b^2 - 14b + 8)^2(4b^2 - 7b + 4)$$

or

$$864b^5 - 3375b^4 + 5022b^3 - 3375b^2 + 864b \geq 0 \quad \text{or} \quad 864b^4 - 3375b^3 + 5022b^2 - 3375b + 864 \geq 0.$$

Let $G(x) = 864x^4 - 3375x^3 + 5022x^2 - 3375x + 864$. We prove that $G(x) \geq 0$ for all $x \in \mathbb{R}$. We find that

$$G'(x) = 3456x^3 - 10125x^2 + 10044x - 3375 = (x-1)(3456x^2 - 6669x + 3375).$$

Since $3456x^2 - 6669x + 3375 > 0$ for all $x \in \mathbb{R}$, we find that $G(x)$ and $x - 1$ have the same sign. It follows that G is monotone decreasing on $(-\infty, 1]$ and monotone increasing on $[1, \infty)$. We conclude that G has the global minimum at $x = 1$. Hence, $G(x) \geq G(1) = 0$ for all $x \in \mathbb{R}$. $\square$

[10] It's easy to check that $16b^3 - 15b^2 - 15b + 16 = 16(b^3 - b^2 - b + 1) + b^2 + b > 16(b^2 - 1)(b - 1) \geq 0$ and $8b^2 - 14b + 8 = 8(b-1)^2 + 2b > 0$.

83 Let $f : [a, b] \longrightarrow \mathbb{R}$ be a **continuous** function. Then, the followings are equivalent.

(1) For all $n \in \mathbb{N}$, the following inequality holds.

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a, b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$.

(2) For all $n \in \mathbb{N}$, the following inequality holds.

$$r_1 f(x_1) + \cdots + r_n f(x_n) \geq f(r_1\, x_1 + \cdots + r_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a, b]$ and $r_1, \cdots, r_n \in \mathbb{Q}^+$ with $r_1 + \cdots + r_n = 1$.

(3) For all $N \in \mathbb{N}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_N)}{N} \geq f\left(\frac{y_1 + \cdots + y_N}{N}\right)$$

for all $y_1, \cdots, y_N \in [a, b]$.

(4) For all $k \in \{0, 1, 2, \cdots\}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_{2^k})}{2^k} \geq f\left(\frac{y_1 + \cdots + y_{2^k}}{2^k}\right)$$

for all $y_1, \cdots, y_{2^k} \in [a, b]$.

(5) We have $\frac{1}{2} f(x) + \frac{1}{2} f(y) \geq f\left(\frac{x+y}{2}\right)$ for all $x, y \in [a, b]$.

(6) We have $\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y)$ for all $x, y \in [a, b]$ and $\lambda \in (0, 1)$.

Solution. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) is obvious.

(2) $\Rightarrow$ (1) **:** Let $x_1, \cdots, x_n \in [a, b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. One may see that there exist positive rational sequences $\{r_k(1)\}_{k\in\mathbb{N}}$, $\cdots$, $\{r_k(n)\}_{k\in\mathbb{N}}$ satisfying

$$\lim_{k\to\infty} r_k(j) = w_j \ \ (1 \leq j \leq n) \ \text{ and } \ r_k(1) + \cdots + r_k(n) = 1 \ \text{ for all } \ k \in \mathbb{N}.$$

By the hypothesis in (2), we obtain $r_k(1) f(x_1) + \cdots + r_k(n) f(x_n) \geq f(r_k(1)\, x_1 + \cdots + r_k(n)\, x_n)$. Since f is continuous, taking $k \to \infty$ to both sides yields the inequality

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

(3) $\Rightarrow$ (2) **:** Let $x_1, \cdots, x_n \in [a, b]$ and $r_1, \cdots, r_n \in \mathbb{Q}^+$ with $r_1 + \cdots + r_n = 1$. We can find a positive integer $N \in \mathbb{N}$ so that Nr_1, $\cdots$, $Nr_n \in \mathbb{N}$. For each $i \in \{1, \cdots, n\}$, we can write $r_i = \frac{p_i}{N}$, where $p_i \in \mathbb{N}$. It follows from $r_1 + \cdots + r_n = 1$ that $N = p_1 + \cdots + p_n$. Then, (3) implies that

$$\begin{aligned} & r_1 f(x_1) + \cdots + r_n f(x_n) \\ = \ & \frac{\overbrace{f(x_1) + \cdots + f(x_1)}^{p_1 \text{ terms}} + \cdots + \overbrace{f(x_n) + \cdots + f(x_n)}^{p_n \text{ terms}}}{N} \\ \geq \ & f\left(\frac{\overbrace{x_1 + \cdots + x_1}^{p_1 \text{ terms}} + \cdots + \overbrace{x_n + \cdots + x_n}^{p_n \text{ terms}}}{N}\right) \\ = \ & f(r_1\, x_1 + \cdots + r_n\, x_n). \end{aligned}$$

(4) $\Rightarrow$ (3) **:** Let $y_1, \cdots, y_N \in [a,b]$. Take a large $k \in \mathbb{N}$ so that $2^k > N$. Let $a = \frac{y_1+\cdots+y_N}{N}$. Then, (4) implies that

$$\begin{aligned} & \frac{f(y_1)+\cdots+f(y_N)+(2^k-n)f(a)}{2^k} \\ = \ & \frac{f(y_1)+\cdots+f(y_N)+\overbrace{f(a)+\cdots+f(a)}^{(2^k-N)\text{ terms}}}{2^k} \\ \geq \ & f\left(\frac{y_1+\cdots+y_N+\overbrace{a+\cdots+a}^{(2^k-N)\text{ terms}}}{2^k}\right) \\ = \ & f(a) \end{aligned}$$

so that

$$f(y_1)+\cdots+f(y_N) \geq Nf(a) = Nf\left(\frac{y_1+\cdots+y_N}{N}\right).$$

(5) $\Rightarrow$ (4) **:** We use induction on k. In case $k = 0, 1, 2$, it clearly holds. Suppose that (4) holds for some $k \geq 2$. Let $y_1, \cdots, y_{2^{k+1}} \in [a,b]$. By the induction hypothesis, we obtain

$$\begin{aligned} & f(y_1)+\cdots+f(y_{2^k})+f(y_{2^k+1})+\cdots+f(y_{2^{k+1}}) \\ \geq \ & 2^k f\left(\frac{y_1+\cdots+\ y_{2^k}}{2^k}\right)+2^k f\left(\frac{y_{2^k+1}+\cdots+\ y_{2^{k+1}}}{2^k}\right) \\ = \ & 2^{k+1}\frac{f\left(\frac{y_1+\cdots+\ y_{2^k}}{2^k}\right)+f\left(\frac{y_{2^k+1}+\cdots+\ y_{2^{k+1}}}{2^k}\right)}{2} \\ \geq \ & 2^{k+1} f\left(\frac{\frac{y_1+\cdots+\ y_{2^k}}{2^k}+\frac{y_{2^k+1}+\cdots+\ y_{2^{k+1}}}{2^k}}{2}\right) \\ = \ & 2^{k+1} f\left(\frac{y_1+\cdots+\ y_{2^{k+1}}}{2^{k+1}}\right). \end{aligned}$$

Hence, (4) holds for $k+1$. This completes the induction.

So far, we've established that (1), (2), (3), (4), (5) are all equivalent. Since (1) $\Rightarrow$ (6) $\Rightarrow$ (5) is obvious, this completes the proof. $\square$

84 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Suppose that f is monotone increasing on $(0, \infty)$ and monotone increasing on $(-\infty, 0)$. Then, f is monotone increasing on $\mathbb{R}$.

Proof. We first show that f is monotone increasing on $[0, \infty)$. By the hypothesis, it remains to show that $f(x) \geq f(0)$ for all $x > 0$. For all $\epsilon \in (0, x)$, we have $f(x) \geq f(\epsilon)$. Since f is continuous at 0, we obtain

$$f(x) \geq \lim_{\epsilon \to 0+} f(\epsilon) = f(0).$$

Similarly, we find that f is monotone increasing on $(-\infty, 0]$. We now show that f is monotone increasing on $\mathbb{R}$. Let x and y be real numbers with $x > y$. We want to show that $f(x) \geq f(y)$. In case $0 \notin (x, y)$, we get the result by the hypothesis. In case $x \geq 0 \geq y$, it follows that $f(x) \geq f(0) \geq f(y)$. $\square$

85 (Power Mean Inequality for Three Variables) Let a, b, and c be positive real numbers. We define a function $M_{(a,b,c)} : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$M_{(a,b,c)}(0) = \sqrt[3]{abc}\,, \quad M_{(a,b,c)}(r) = \left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}} \quad (r \neq 0).$$

Then, $M_{(a,b,c)}$ is a monotone increasing continuous function.

First Proof. Write $M(r) = M_{(a,b,c)}(r)$. We first establish that M is continuous. Since M is continuous at r for all $r \neq 0$, it's enough to show that

$$\lim_{r\longrightarrow 0} M(r) = \sqrt[3]{abc}.$$

Let $f(x) = \ln\left(\frac{a^x+b^x+c^x}{3}\right)$, where $x \in \mathbb{R}$. Since $f(0) = 0$, the lemma 2 implies that

$$\lim_{r\longrightarrow 0} \frac{f(r)}{r} = \lim_{r\longrightarrow 0} \frac{f(r) - f(0)}{r - 0} = f'(0) = \ln \sqrt[3]{abc}\,.$$

Since e^x is a continuous function, this means that

$$\lim_{r\longrightarrow 0} M(r) = \lim_{r\longrightarrow 0} e^{\frac{f(r)}{r}} = e^{\ln \sqrt[3]{abc}} = \sqrt[3]{abc}.$$

Now, we show that M is monotone increasing. It will be enough to establish that M is monotone increasing on $(0, \infty)$ and monotone increasing on $(-\infty, 0)$. We first show that M is monotone increasing on $(0, \infty)$. Let $x \geq y > 0$. We want to show that

$$\left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}} \geq \left(\frac{a^y + b^y + c^y}{3}\right)^{\frac{1}{y}}.$$

After the substitution $u = a^y$, $v = a^y$, $w = a^z$, it becomes

$$\left(\frac{u^{\frac{x}{y}} + v^{\frac{x}{y}} + w^{\frac{x}{y}}}{3}\right)^{\frac{1}{x}} \geq \left(\frac{u + v + w}{3}\right)^{\frac{1}{y}}.$$

Since it is homogeneous, we may normalize to $u + v + w = 3$. We are now required to show that

$$\frac{G(u) + G(v) + G(w)}{3} \geq 1,$$

where $G(t) = t^{\frac{x}{y}}$, where $t > 0$. Since $\frac{x}{y} \geq 1$, we find that G is convex. Jensen's inequality shows that

$$\frac{G(u) + G(v) + G(w)}{3} \geq G\left(\frac{u + v + w}{3}\right) = G(1) = 1.$$

Similarly, we may deduce that M is monotone increasing on $(-\infty, 0)$. $\square$

We've learned that the convexity of $f(x) = x^\lambda$ $(\lambda \geq 1)$ implies the monotonicity of the power means. Now, we shall show that the convexity of $x \ln x$ also implies The Power Mean Inequality.

Second Proof of the Monotonicity. Write $f(x) = M_{(a,b,c)}(x)$. We use the increasing function theorem. It's enough to show that $f'(x) \geq 0$ for all $x \neq 0$. Let $x \in \mathbb{R} - \{0\}$. We compute

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}(\ln f(x)) = -\frac{1}{x^2} \ln\left(\frac{a^x + b^x + c^x}{3}\right) + \frac{1}{x} \frac{\frac{1}{3}\left(a^x \ln a + b^x \ln b + c^x \ln c\right)}{\frac{1}{3}(a^x + b^x + c^x)}$$

or

$$\frac{x^2 f'(x)}{f(x)} = -\ln\left(\frac{a^x + b^x + c^x}{3}\right) + \frac{a^x \ln a^x + b^x \ln b^x + c^x \ln c^x}{a^x + b^x + c^x}.$$

To establish $f'(x) \geq 0$, we now need to establish that

$$a^x \ln a^x + b^x \ln b^x + c^x \ln c^x \geq (a^x + b^x + c^x) \ln \left(\frac{a^x + b^x + c^x}{3} \right).$$

Let us introduce a function $f : (0, \infty) \longrightarrow \mathbf{R}$ by $f(t) = t \ln t$, where $t > 0$. After the substitution $p = a^x$, $q = a^y$, $r = a^z$, it becomes

$$f(p) + f(q) + f(r) \geq 3f \left(\frac{p + q + r}{3} \right).$$

Since f is convex on $(0, \infty)$, it follows immediately from Jensen's Inequality. $\square$

86 Let x, y, z be nonnegative real numbers. Then, we have

$$3xyz + x^3 + y^3 + z^3 \geq 2\left((xy)^{\frac{3}{2}} + (yz)^{\frac{3}{2}} + (zx)^{\frac{3}{2}}\right).$$

Second Solution. After employing the substitution

$$x = e^{\frac{p}{3}},\ y = e^{\frac{q}{3}},\ z = e^{\frac{r}{3}},$$

the inequality becomes

$$3e^{\frac{p+q+r}{3}} + e^p + e^q + e^r \geq 2\left(e^{\frac{q+r}{2}} + e^{\frac{r+p}{2}} + e^{\frac{p+q}{2}}\right)$$

It is a straightforward consequence of Popoviciu's Inequality. $\square$

87 Let ABC be an acute triangle. Show that

$$\cos A + \cos B + \cos C \geq 1.$$

Proof. Observe that $\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ majorize (A, B, C). Since $-cosx$ is convex on $\left(0, \frac{\pi}{2}\right)$, The Hardy-Littlewood-Pólya Inequality implies that

$$\cos A + \cos B + \cos C \geq \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) + \cos 0 = 1.$$

□

88 Let ABC be a triangle. Show that

$$\tan^2\left(\frac{A}{4}\right)+\tan^2\left(\frac{B}{4}\right)+\tan^2\left(\frac{C}{4}\right)\le 1.$$

Proof. Observe that $(\pi,0,0)$ majorizes (A,B,C). The convexity of $\tan^2\left(\frac{x}{4}\right)$ on $[0,\pi]$ yields the estimation:

$$\tan^2\left(\frac{A}{4}\right)+\tan^2\left(\frac{B}{4}\right)+\tan^2\left(\frac{C}{4}\right)\le \tan^2\left(\frac{\pi}{4}\right)+\tan^2 0+\tan^2 0=1.$$

$\square$

89 Use The Hardy-Littlewood-Pólya Inequality to deduce Popoviciu's Inequality.

Proof. [NP, p.33] Since the inequality is symmetric, we may assume that $x \geq y \geq z$. We consider the two cases. In the case when $x \geq \frac{x+y+z}{3} \geq y \geq z$, the majorization

$$\left(x, \frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}, y, z\right) \succ \left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{z+x}{2}, \frac{z+x}{2}, \frac{y+z}{2}, \frac{y+z}{2}\right)$$

yields Popoviciu's Inequality. In the case when $x \geq y \geq \frac{x+y+z}{3} \geq z$, the majorization

$$\left(x, y, \frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}, z\right) \succ \left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{z+x}{2}, \frac{z+x}{2}, \frac{y+z}{2}, \frac{y+z}{2}\right)$$

yields Popoviciu's Inequality. $\square$

90 **[IMO 1999/2 POL]** Let n be an integer with $n \geq 2$.

Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.

Second Solution. (Kin Y. Li[11]) As in the first solution, after normalizing $x_1 + \cdots + x_n = 1$, we maximize

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) = \sum_{i=1}^{n} f(x_i),$$

where $f(x) = x^3 - x^4$ is a convex function on $[0, \frac{1}{2}]$. Since the inequality is symmetric, we can restrict our attention to the case $x_1 \geq x_2 \geq \cdots \geq x_n$. If $\frac{1}{2} \geq x_1$, then we see that $\left(\frac{1}{2}, \frac{1}{2}, 0, \cdots 0\right)$ majorizes $(x_1, \cdots, x_n)$. Hence, by The Hardy-Littlewood-Pólya Inequality, the convexity of f on $[0, \frac{1}{2}]$ implies that

$$\sum_{i=1}^{n} f(x_i) \leq f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + f(0) + \cdots + f(0) = \frac{1}{8}.$$

We now consider the case when $\frac{1}{2} \geq x_1$. Write $x_1 = \frac{1}{2} - \epsilon$ for some $\epsilon \in \left[0, \frac{1}{2}\right]$. We find that $(1 - x_1, 0, \cdots 0)$ majorizes $(x_2, \cdots, x_n)$. The Hardy-Littlewood-Pólya Inequality shows that

$$\sum_{i=2}^{n} f(x_i) \leq f(1 - x_1) + f(0) + \cdots + f(0) = f(1 - x_1)$$

so that

$$\begin{aligned} \sum_{i=1}^{n} f(x_i) &\leq f(x_1) + f(1 - x_1) \\ &= x_1(1 - x_1)[{x_1}^2 + (1 - x_1)^2] \\ &= \left(\frac{1}{4} - \epsilon^2\right)\left(\frac{1}{2} + 2\epsilon^2\right) \\ &= 2\left(\frac{1}{16} - \epsilon^4\right) \\ &\leq \frac{1}{8}. \end{aligned}$$

□

My brain is open - P. Erdős

[11] I slightly modified his solution in [KYL].

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