Mock Olympiad Solutions

1 Problems

1. Let a_1 be a natural number not divisible by 5. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_{n+1} = a_n + b_n$, where b_n is the last digit of a_n. Prove that the sequence contains infinitely many powers of 2.

2. Let ABC be a triangle and let D and E be points on the sides AB and AC respectively such that DE is parallel to BC. Let P be any point interior to triangle ADE and let F and G be the intersections of DE with the lines BP and CP respectively. Let Q be the second intersection point of the circumcircles of triangles PDG and PFE. Prove that the points A, P, and Q lie on a straight line.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ and $y \in f(\mathbb{R})$

   $$f(x - y) = f(x) + xy + f(y).$$

   Note: y is said to be in $f(\mathbb{R})$ if there exists $z \in \mathbb{R}$ such that $f(z) = y$.

4. There are a number of markets in a city, joined by one-way streets. Every market has exactly two one-way streets going out of it, and there is at most one street joining any two markets. Prove that the city may be partitioned into 2014 districts such that:

   (a) No street joins two markets in the same district.

   (b) For every pair of districts, the streets joining the two districts all go in the same direction (from the first district to the second, or from the second district to the first).

2 Solutions

1. Checking each possible last digit for a_1, we can see there must always exist some k such that a_k has last digit equal to 2.

   Focusing on that k, we have $a_k = 10m + 2$ for some non-negative integer m, and hence

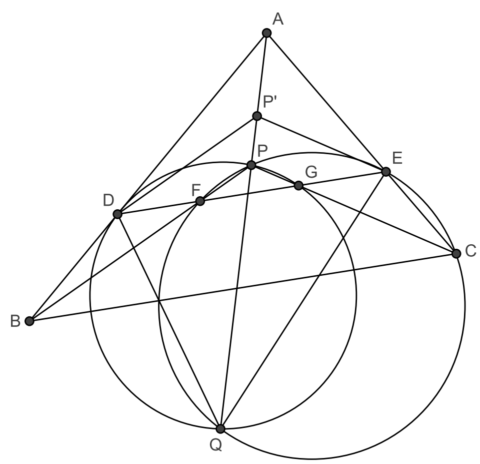
   - $a_{k+1} = 10m + 4$
   - $a_{k+2} = 10m + 8$
   - $a_{k+3} = 10(m + 1) + 6$
   - $a_{k+4} = 10(m + 2) + 2$.

   At this point, the process repeats.

   If m is even, then (a_n) includes every sufficiently large number that is 4 mod 20. In particular, it includes $4 \cdot 16^t = 2^{4t+2}$ for all sufficiently large t.

   If m is odd, then (a_n) includes every sufficiently large number that is 12 mod 20. In particular, it includes $32 \cdot 16^t = 2^{4t+5}$ for all sufficiently large t.

   Therefore, (a_n) includes infinitely many powers of 2 in either case. *(Russia 1994)*



2. Let P' be the image of P under the dilation about A that takes BC to DE. Then A lies on line PP'.

   Now,

$$\begin{aligned} \angle DQE &= \angle DQP + \angle EQP \\ &= \angle DGP + \angle EFP \text{ since } DQGP \text{ and } EQFP \text{ are cyclic} \\ &= \angle DEP' + \angle EDP' \text{ since } DP'||FP \text{ and } EP'||GP \text{ by the definition of } P' \\ &= 180^\circ - \angle DP'E. \end{aligned}$$

   Therefore, $DP'EQ$ is cyclic. It follows that $\angle DQP' = \angle DEP' = \angle DGP = \angle DQP$, and so Q lies on line PP'.

   We have now shown A and Q both lie on line PP', which implies A, P, Q must be collinear. *(India 1995)*

3. It is easy to check that $f(x) = 0$ for all x is a solution. We focus on the other case, in which there exists some $a \in f(\mathbb{R})$ with $a \neq 0$.

   Setting $x = y$, we have $f(0) = 2f(y) + y^2$ for all $y \in f(\mathbb{R})$. (*) Let z be an arbitrary real number, and set $x = \frac{z-f(a)}{a}, y = a$. Then we have

   $$f\left(\frac{z-f(a)}{a} - a\right) = f\left(\frac{z-f(a)}{a}\right) + z.$$

   Therefore, there exist real numbers u, v such that $f(u) - f(v) = z$. Setting $x = f(u), y = f(v)$, we have

   $$\begin{aligned} f(z) = f(f(u) - f(v)) &= f(f(u)) + f(u)f(v) + f(f(v)) \\ &= -\frac{f(u)^2}{2} + f(u)f(v) - \frac{f(v)^2}{2} + f(0) \text{ by (*)} \\ &= -\frac{1}{2} \cdot (f(u) - f(v))^2 + f(0) \\ &= -\frac{z^2}{2} + f(0). \end{aligned}$$

   We have thus shown there exists a constant C such that $f(z) = \frac{z^2}{2} + C$ for all $z \in \mathbb{R}$. Setting $x = y = C$, we have $C = -\frac{C^2}{2} + C + C^2 - \frac{C^2}{2} + C$, so $C = 0$ and $f(z) = -\frac{z^2}{2}$ for all z.

   Checking, we see this does indeed satisfy the given equation. *(Korea 2002 and IMO 1999)*

4. Let G be the graph with vertices corresponding to markets and directed edges corresponding to one-way streets. We are given that every vertex has out-degree equal to 2.

   Lemma: Let us say two vertices are "separated" if there is no path from one to the other of length 1 or 2. Then we can assign every vertex a "colour" in $\{1, 2, \ldots, 13\}$ such that if vertices u, v have the same colour (possibly with $u = v$), then they are separated.

   Proof: We prove this by induction on the number of vertices, relaxing the conditions to say each vertex has out-degree *at most* two.

   If there is only one vertex, the claim is trivial. Now suppose it has been shown for $n-1$ vertices, and consider a graph with n vertices. From every vertex, there are 2 paths going out of length 1 and 4 paths going out of length 2. Therefore, the total number of unordered vertex pairs (u, v) that are *not* separated is at most $6n$. By the pigeonhole principle, there exists a vertex v that belongs to at most 12 of these pairs.

   Using the inductive hypothesis, all vertices except v can be coloured in the required manner. Then there are up to 12 vertices that v cannot be the same colour as, so we can choose a valid colour for v as well, completing the inductive proof.

   Using the colouring described above, associate to each vertex a triple (a, b, c) where a is the colour of the vertex, and $b \leq c$ are the colours of the vertices that the vertex has an edge leading out to. We create one district for each possible value of this triple, and use that to assign vertices into districts. There are 13 possible values for a. Once a is fixed, (b, c) can be any pair chosen from the remaining colours, so there are $12 + \binom{12}{2} = 78$ possibilities for them, giving a total of $13 \cdot 78 < 2014$ districts.

If two vertices are in the same district, then they have the same colour and the colouring guarantees there is no edge between them.

Next suppose there is an edge from vertex u in district (a_1, b_1, c_1) to vertex v in district (a_2, b_2, c_2). If b_2 or c_2 equals a_1, then we can follow a second edge from v to another vertex w with colour a_1. However, u and w then have the same colour but are not separated, which is a contradiction. Thus, b_2 and c_2 are both distinct from a_1, and so there is no edge from district (a_2, b_2, c_2) to district (a_1, b_1, c_1).

Therefore, our district division satisfies the required conditions, and the problem is solved. *(Russia 2002)*