Art of Problem Solving

WOOT

Class Transcript 11/04 - Functional Equations

markan 7:30:00 pm
Hello everyone!

markan 7:30:07 pm
WOOT 2013-14: Functional Equations

markan 7:30:18 pm
In a functional equation, we are given some property of the function, and we are asked to find all functions that satisfy the given property. The problems we will discuss will demonstrate some of the main techniques for solving functional equation.

markan 7:30:27 pm
We start with a simple example of a functional equation.

markan 7:30:40 pm
Find all functions $f: \mathbb{R} \to \mathbb{R}$ **such that** $f(a+b) - f(a-b) = 4ab$ **for all** a **,** $b \in \mathbb{R}$ **.**

markan 7:30:59 pm
How shall we start?

brian22 7:31:27 pm
a=0: f is even

cerberus88 7:31:27 pm
a+b=x, a-b=y?

RocketSingh 7:31:27 pm
Let a =b

thkim1011 7:31:27 pm
let x = a+b and y = a-b

fclvbfm934 7:31:27 pm
substitute

fprosk 7:31:27 pm
substitution

markan 7:31:33 pm
All good ideas.

markan 7:31:37 pm
The main technique for solving a functional equation is substitution. Typically, the most useful substitutions involve setting the variables to 0, 1, **or each other, or anything else that simplifies the functional equation.**

markan 7:31:47 pm
Setting $a = 0$ **, we get** $f(b) - f(-b) = 0$ **for all** b **.**

markan 7:31:48 pm
In other words, $f(b) = f(-b)$ **for all** b **, so** f **is even.**

markan 7:32:02 pm
What if we set b=0? Does that buy us anything?

bryanxqchen 7:32:19 pm
no

delta1 7:32:19 pm
no

lazorpenguin27143 7:32:19 pm
no we get 0=0

sunny2000 7:32:19 pm
nope

vincenthuang75025 7:32:19 pm
0=0

bryanxqchen 7:32:19 pm
We get $0 = 0$

SuperSnivy 7:32:19 pm
It just tells us that 0=0.

markan 7:32:22 pm
Setting $b = 0$, we get $0 = 0$. This is always a true equation, so this does not tell us anything about the function f.

markan 7:32:29 pm
What other substitutions can we make?

delta1 7:32:34 pm
set a=b

vincenthuang75025 7:32:34 pm
$a = b$

markan 7:32:36 pm
We can set $b = a$.

markan 7:32:42 pm
Setting $b = a$, we get

$$f(2a) - f(0) = 4a^2.$$

Then

$$f(2a) = 4a^2 + f(0)$$

for all a.

markan 7:33:05 pm
So what is $f(x)$?

thkim1011 7:33:31 pm
f(x) = x^2 + c

fclvbfm934 7:33:31 pm
$f(a) = a^2 + c$ for some constant c

vincenthuang75025 7:33:31 pm
$x^2 + c$

pier17 7:33:31 pm
x^2+f(0)

lucylai 7:33:31 pm
let $f(0) = c$, we have $f(x) = x^2 + c$

hardmath123 7:33:31 pm
f(x) = x^2 + k

markan 7:33:33 pm
$f(x) = x^2 + f(0)$ **for all x.**

markan 7:33:37 pm
How can we try to find the value of $f(0)$?

fclvbfm934 7:34:24 pm
substitute that back in

ssk9208 7:34:24 pm
Substitute to the original eq

thkim1011 7:34:24 pm
substitute the function and solve for c if possible

pier17 7:34:24 pm
plug the expression for f(x) into the functional equation

markan 7:34:30 pm
We can substitute into the given functional equation.

markan 7:34:44 pm
If $f(x) = x^2 + f(0)$ for all x, then

$$\begin{aligned} f(a+b) - f(a-b) &= (a+b)^2 + f(0) - (a-b)^2 - f(0) \\ &= 4ab. \end{aligned}$$

markan 7:34:53 pm
So our original functional equation then says 4ab=4ab.

markan 7:34:59 pm
What does this mean?

lucylai 7:35:18 pm
$f(0)$ can be anything

fclvbfm934 7:35:18 pm
$f(0)$ can be anything, all functions satisfy the desired condition

hardmath123 7:35:18 pm
So f(0) could be any constant.

patchosaur 7:35:18 pm
any f(0) will work

pier17 7:35:18 pm
f(0) can be any constant

TheStrangeCharm 7:35:18 pm
$f(0)$ can be anything

Bzhaothecow 7:35:18 pm
all functions of the form f(x)=x^2+k work

markan 7:35:20 pm
This tells us that $f(0)$ can be any constant.

markan 7:35:22 pm
Therefore, the solutions are of the form $f(x) = x^2 + c$, where c is any constant.

markan 7:35:29 pm
The solution to a functional equation is not necessarily unique, so when you have narrowed down the function to a particular form, try substituting into the given functional equation, because the functional equation is ultimately what you are trying to solve.

markan 7:35:37 pm
Testing the given functional equation may tell you that you have found a family of functions that work, or help you narrow down the solutions.

markan 7:35:48 pm
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) - f(f(y))) = x - y$$

for all $x, y \in \mathbb{R}$.

markan 7:35:59 pm
How should we start?

ssk9208 7:36:24 pm
$x = y = 0$?

vincenthuang75025 7:36:24 pm
Plug $y = 0$?

markan 7:36:30 pm
Sure, let's try some substitutions first.

markan 7:36:34 pm
Setting $x = 0$, we get

$$f(f(0) - f(f(y))) = -y$$

for all y.

markan 7:36:43 pm
Setting $y = 0$, we get

$$f(f(x) - f(f(0))) = x$$

for all x .

markan 7:37:13 pm
Nothing really jumps out at us about those equations. What else can we try?

lazorpenguin27143 7:37:56 pm
let x = f(y)

chenjamin 7:37:56 pm
plug in x = f(y)

brian22 7:37:56 pm
x=f(y)!

delta1 7:37:56 pm
plug in x=f(y)

chenjamin 7:37:56 pm
plug in x = f(y)

patchosaur 7:37:56 pm
x=f(y)

markan 7:38:12 pm
Yes, let's try x=f(y), since that will create nice cancellation on the LHS.

markan 7:38:19 pm
We can set $x = f(y)$ to make the expression $f(x) - f(f(y))$ on the left side equal to 0.

markan 7:38:26 pm
Setting $x = f(y)$, we get

$$f(f(f(y)) - f(f(y))) = f(y) - y,$$

which simplifies to

$$f(0) = f(y) - y.$$

Hence, $f(y) = y + f(0)$ for all y.

zhuangzhuang 7:39:06 pm
plug it back in

lazorpenguin27143 7:39:06 pm
plug back into the original equation to find possible f(0)

sunny2000 7:39:06 pm
try to get f(0) or plug in

pier17 7:39:06 pm
plug it back in

ssk9208 7:39:06 pm
Substitute back

markan 7:39:12 pm
We now substitute into the given functional equation.

markan 7:39:14 pm
Let $c = f(0)$, so $f(y) = y + c$ for all y.

markan 7:39:27 pm
Then

$$\begin{aligned} f(f(x) - f(f(y))) &= f((x + c) - (y + 2c)) \\ &= f(x - y - c) \\ &= x - y. \end{aligned}$$

Therefore, the solutions are of the form $f(x) = x + c$, where c is any constant.

markan 7:40:02 pm
Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f\left((x - y)^2\right) = f(x)^2 - 2xf(y) + y^2$$

for all $x, y \in \mathbb{R}$. (Slovenia, 1998)

markan 7:40:35 pm
Ideas?

zhuangzhuang 7:40:56 pm
x=y?

noodleeater 7:40:56 pm
x = y

chenjamin 7:40:56 pm
x=y?

Cosmynx 7:40:56 pm
let x=y?

markan 7:41:03 pm
Sure, let's try it.

markan 7:41:06 pm
That would give us

markan 7:41:26 pm
$f(0) = f(x)^2 - 2xf(x) + x^2$

TheStrangeCharm 7:41:50 pm
quadratic in $f(x)$

thkim1011 7:41:50 pm
hm.. we can factor

sunny2000 7:41:50 pm
quadratic in $f(x)$

RocketSingh 7:41:55 pm
RHS = (f(x)-x)^2

markan 7:42:10 pm
$f(0) = (f(x) - x)^2$

RocketSingh 7:42:32 pm
f(0)>= 0

markan 7:42:43 pm
Yes, f(0) would have to be nonnegative.

markan 7:43:12 pm
So what else can we conclude?

hardmath123 7:44:10 pm
f(x) - x = sqrt(f(0))

RocketSingh 7:44:10 pm
f(x) = sqrt(x+c)

zhuangzhuang 7:44:10 pm
f(x)=x+c for constant c

ssk9208 7:44:10 pm
So $f(x) - x$ is constant in absolute value

lucylai 7:44:10 pm
$f(x) = x + c$

markan 7:44:27 pm
Yeah, we can just take the square root of both sides to obtain

markan 7:44:40 pm
f(x) = x + sqrt(f(0))

sunny2000 7:45:12 pm
how about negative square root?

markan 7:45:20 pm
Ah yes, that is possible as well.

markan 7:45:32 pm
So for each individual x, we could have

markan 7:45:38 pm
f(x) = x + sqrt(f(0))

markan 7:45:38 pm
or

markan 7:46:03 pm
f(x) = x - sqrt(f(0))

markan 7:46:41 pm
So what can we do next?

markan 7:47:43 pm
Many people want to plug back in.

markan 7:47:51 pm
The trouble with this is that we don't quite know the form of f(x) yet.

markan 7:48:00 pm
It could be that for some x's, f(x) = x + c, but for other x's, f(x) = x + b.

markan 7:48:18 pm
(Due to our two solutions.)

noodleeater 7:49:03 pm
plug in x = 0 to the original equation

markan 7:49:11 pm
Yeah, let's take a step back and try something else.

markan 7:49:23 pm
Setting $x = 0$, we get

$$f(y^2) = f(0)^2 + y^2$$

for all y.

markan 7:49:48 pm
So then what?

patchosaur 7:50:28 pm
let y=0

markan 7:50:39 pm
Sure, we can do that, and we will find that f(0) = 0 or 1.

RocketSingh 7:50:52 pm
for all positive x f(x) = c + x where c is also positive

markan 7:51:35 pm
Yup, we can now see what f(x) is for positive x.

TheStrangeCharm 7:51:42 pm
So for all $x \geq 0$, we must have either $f(x) = x$ or $f(x) = x + 1$.

markan 7:52:00 pm
Any ideas on dealing with negative numbers?

vincenthuang75025 7:54:19 pm
Replace all the x's with -x's?

noodleeater 7:54:19 pm
plug in -x for x and 0 for y?

markan 7:55:08 pm
We could try this, but using -x for x shouldn't give us anything we can't tell by looking at the equation with plain x.

markan 7:55:12 pm
(Just assume x is negative.)

markan 7:55:27 pm
But I don't think we've tried letting y=0 yet, so we could see what that gets us.

markan 7:55:44 pm
f(x^2) = f(x)^2 - 2xf(0)

markan 7:56:39 pm
But this is a little concerning since f(x) is squared, so we can't solve it and determine the sign of f(x).

markan 7:57:16 pm
Let's go back to here:

markan 7:57:17 pm
Setting $x = 0$, we get

$$f(y^2) = f(0)^2 + y^2$$

for all y.

markan 7:57:34 pm
We seem to be getting frustrated by the presence of squares inside f.

markan 7:57:51 pm
So how might we use what we just found and combine it with the original functional equation?

nilaisarda 7:58:36 pm
set x-y = a, then f(a^2) might give us something?

noodleeater 7:58:36 pm
plug in x-y for y?

vincenthuang75025 7:58:36 pm
$LHS = f(0)^2 + (x-y)^2$?

markan 7:58:50 pm
Yeah let's take x-y and put it into y in that last equation.

markan 7:58:56 pm
We can say that

$$f((x-y)^2) = (x-y)^2 + f(0)^2.$$

Hence, the given functional equation becomes

$$(x-y)^2 + f(0)^2 = f(x)^2 - 2xf(y) + y^2$$

for all x, y.

markan 7:59:27 pm
This simplifies to

$$x^2 - 2xy + f(0)^2 = f(x)^2 - 2xf(y).$$

What we can we do with this equation?

markan 8:00:43 pm
Hint: we're almost done.

Cpi2728 8:01:32 pm
solve for f(x)

pier17 8:01:32 pm
y=0?

RocketSingh 8:01:32 pm
x = 0

lucylai 8:01:32 pm
set $y = 0$

markan 8:01:47 pm
Yeah, notice that we can solve our equation for f(y)!

markan 8:02:11 pm
Then we can just plug in something for x and we should have the functional form of f(y)!

markan 8:02:36 pm
(We could also try to solve for f(x), but the fact that it's squared makes this less promising.)

markan 8:02:46 pm
So let's go ahead and isolate f(y).

markan 8:03:02 pm
We have

$$2xf(y) - 2xy = f(x)^2 - x^2 - f(0)^2,$$

so

$$2x(f(y) - y) = f(x)^2 - x^2 - f(0)^2.$$

markan 8:03:17 pm
Assuming $x \neq 0$, we can divide both sides by $2x$ to get

$$f(y) - y = \frac{f(x)^2 - x^2 - f(0)^2}{2x}.$$

markan 8:03:45 pm
So what does this tell us?

pier17 8:04:20 pm
f(y) = y + c

noodleeater 8:04:20 pm
f(y) = y + c

fclvbfm934 8:04:20 pm
$f(y) = y + c$

markan 8:04:22 pm
Right.

markan 8:04:53 pm
The equation holds for all x and y, so therefore both sides must be constants.

nilaisarda 8:05:05 pm
Isn't that assuming x ≠ 0?

markan 8:05:19 pm
(Ok, technically it holds for all x not equal to 0, but this doesn't affect the argument.)

noodleeater 8:05:26 pm
then plug in and we're done...

markan 8:05:33 pm
Now we can just plug into the original functional equation to figure out what c is.

fclvbfm934 8:05:38 pm
c = 0 or 1

markan 8:05:52 pm
Indeed, the algebra at this point just boils down to

markan 8:05:54 pm
c^2 = c

markan 8:06:03 pm
(Try it yourself later if you like.)

markan 8:06:06 pm
Then $c = 0$ or $c = 1$.

markan 8:06:11 pm
Therefore, the solutions are $f(x) = x$ and $f(x) = x + 1$.

markan 8:06:22 pm
Any questions so far?

markan 8:06:38 pm
Find all functions f, defined on the real numbers and taking real values, which satisfy the equation

$$f(x)f(y) = f(x + y) + xy$$

for all real numbers x and y. (British Mathematical Olympiad, 2009)

chenjamin 8:07:36 pm
y = 0

noodleeater 8:07:36 pm
plug in y = 0

brian22 8:07:36 pm
x=y=0! my favorite!

zhuangzhuang 8:07:36 pm
plug in y=0?

markan 8:07:41 pm
Yeah, let's plug some things in as usual.

markan 8:08:04 pm
Setting $y = 0$ **, we get**

$$f(x)f(0) = f(x)$$

for all x.

markan 8:08:07 pm
Moving all the terms to one side, we get $f(x)(f(0) - 1) = 0$ **for all** x.

markan 8:08:19 pm
What happens if $f(0) \neq 1$ **?**

JoetheFixer 8:08:30 pm
f(x)=0

werdnerd360 8:08:30 pm
f(x)=0

SuperSnivy 8:08:30 pm
f(x)=0

mathcool2009 8:08:38 pm
then f(x) = 0 for all x

markan 8:08:41 pm
If $f(0) \neq 1$, **then we can divide both sides by** $f(0) - 1$ **to get** $f(x) = 0$ **for all** x.

markan 8:08:44 pm
But does this give us a solution?

pier17 8:09:11 pm
but that doesn't satisfy the functional equation

Cpi2728 8:09:11 pm
No.

sunny2000 8:09:11 pm
no

RocketSingh 8:09:11 pm
plug it back in

chenjamin 8:09:11 pm
no, it doesn't work with the original equation

markan 8:09:17 pm
No, we can plug in to see that $f(x) = 0$ **is not a solution to the given functional equation.**

markan 8:09:26 pm
Therefore, $f(0) = 1$.

markan 8:09:51 pm
What can we try next?

soy_un_chemisto 8:10:15 pm
try x = -y

brian22 8:10:15 pm
x=-y

chenjamin 8:10:15 pm
x = -y

JoetheFixer 8:10:15 pm
y=-x

zhuangzhuang 8:10:15 pm
x=-y

markan 8:10:25 pm
Yes, we like this idea because it will make x+y become 0.

markan 8:10:30 pm
Setting $y = -x$ in the given functional equation, we get

$$f(x)f(-x) = f(0) - x^2 = 1 - x^2$$

for all x.

markan 8:10:44 pm
Now what?

zhuangzhuang 8:11:29 pm
let x=1??

brian22 8:11:29 pm
x=1

brian22 8:11:29 pm
f(1)=0 or f(-1)=0

markan 8:11:33 pm
We can substitute $x = 1$.

markan 8:11:35 pm
Setting $x = 1$, we get $f(1)f(-1) = 0$.

markan 8:11:38 pm
Hence, $f(1) = 0$ or $f(-1) = 0$.

markan 8:11:43 pm
First, we take the case $f(1) = 0$. What can we do in this case?

anwang16 8:12:13 pm
put this in the original

Michelangelo 8:12:13 pm
plug in x=1 to the original

markan 8:12:15 pm
We can substitute $y = 1$ in the given functional equation.

markan 8:12:20 pm
Setting $y = 1$ in the given functional equation, we get

$$f(x)f(1) = f(x + 1) + x,$$

which simplifies to

$$f(x + 1) = -x$$

for all x.

markan 8:12:51 pm
So what is the function f ?

fclvbfm934 8:13:15 pm
f(x) = 1-x

zhuangzhuang 8:13:15 pm
f(x)=1-x

chenjamin 8:13:15 pm
f(x) = 1-x

pier17 8:13:15 pm
so f(x) = 1 - x

JoetheFixer 8:13:15 pm
f(x)=1-x

steve123456 8:13:15 pm
f(x) = 1-x

RocketSingh 8:13:15 pm
f(x) = 1-x

distortedwalrus 8:13:15 pm
-x+1 = f(x)

markan 8:13:17 pm
Let $t = x + 1$, so $x = t - 1$. Then $f(t) = -x = 1 - t$ for all t.

markan 8:13:21 pm
Does this give us a solution?

lucylai 8:13:42 pm
yes

anwang16 8:13:42 pm
substitute in the original

RocketSingh 8:13:42 pm
Yes

brian22 8:13:42 pm
yuup

noodleeater 8:13:44 pm
yes

markan 8:13:45 pm
Substituting into the given functional equation, we get

$$f(x)f(y) = (1 - x)(1 - y) = 1 - x - y + xy,$$

and

$$f(x + y) + xy = 1 - x - y + xy,$$

so $f(x) = 1 - x$ is a solution.

sunny2000 8:13:58 pm
now consider f(-1)=0

RocketSingh 8:13:58 pm
f(-1) = 0

markan 8:14:01 pm
Next, we take the case $f(-1) = 0$. What can we do in this case?

chenjamin 8:14:23 pm
same thing, and it gives us f(x) = 1+x

anwang16 8:14:23 pm
put back in original again

noodleeater 8:14:23 pm
plug in -1 as x

distortedwalrus 8:14:23 pm
same thing

ssk9208 8:14:25 pm
$y = -1$

anwang16 8:14:25 pm
substitute

markan 8:14:37 pm
We can go through the exact same process again, plugging in y=-1 to the original equation.

markan 8:14:44 pm
This gives us f(x) = 1+x.

noodleeater 8:15:05 pm
that works

markan 8:15:05 pm
And just like before, we want to plug it into the original equation to verify.

markan 8:15:16 pm
I'll spare you the details, since it's just simple algebra.

markan 8:15:21 pm
It does end up working.

distortedwalrus 8:15:24 pm
and we're done

markan 8:15:26 pm
Therefore, the solutions are $f(x) = 1 + x$ and $f(x) = 1 - x$.

markan 8:15:41 pm
Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ that satisfy (i) $f(1) = 2$, and (ii) $f(xy) = f(x)f(y) - f(x + y) + 1$ for all $x, y \in \mathbb{Q}$.

Cpi2728 8:15:56 pm
Q?

markan 8:16:00 pm
Q = the set of rational numbers.

markan 8:16:18 pm
So as usual, let's try some substitutions.

markan 8:16:24 pm
Setting $y = 0$, we get

$$f(0) = f(x)f(0) - f(x) + 1.$$

What can we do with this equation?

noodleeater 8:17:03 pm
solve for f(x)

fclvbfm934 8:17:03 pm
solve for $f(x)$

JoetheFixer 8:17:03 pm
solve for f(x)

markan 8:17:14 pm
Moving all the terms to one side, we get

$$f(x)f(0) - f(x) - f(0) + 1 = 0.$$

markan 8:17:20 pm
This equation factors as

$$(f(x) - 1)(f(0) - 1) = 0.$$

What does this equation tell us?

anwang16 8:17:53 pm
EITHER f(x)=1 or f(0)=1

RocketSingh 8:17:53 pm
f(0) =1 or f(x) =1

lucylai 8:17:53 pm
$f(0) = 1$

sunny2000 8:17:53 pm
$f(0) = 1$

JoetheFixer 8:17:53 pm
f(x)=1 or f(0)=1

steve314 8:17:53 pm
either f(0)=1 or f(x)=1

soy_un_chemisto 8:17:53 pm
f(0) = 1

kad2361 8:17:53 pm
f(x)=1 or f(0)=1

markan 8:18:03 pm
Yeah, either f(0)=1 or f(x)=1 for all x.

markan 8:18:08 pm
Of course, the latter would imply the former.

markan 8:18:15 pm
So all we really know is f(0) = 1.

brian22 8:18:17 pm
f(0)=1 either way!

markan 8:18:32 pm
We could also substitute $x = 1$ to get $(f(1) - 1)(f(0) - 1) = 0$. Since $f(1) = 2$, this equation becomes $f(0) - 1 = 0$, so $f(0) = 1$.

lucylai 8:18:45 pm
but the latter is impossible because $f(1) = 2$

markan 8:18:55 pm
Good point.

markan 8:18:59 pm
So what next?

fclvbfm934 8:19:18 pm
x = 1

nilaisarda 8:19:18 pm
substitute more things!

markan 8:19:24 pm
Since we know $f(1) = 2$, we could substitute $y = 1$. (Again, our strategy of focusing on unused given information.)

markan 8:19:27 pm
Setting $y = 1$, we get

$$f(x) = f(x)f(1) - f(x+1) + 1.$$

Since $f(1) = 2$, we get

$$f(x) = 2f(x) - f(x+1) + 1,$$

so

$$f(x+1) = f(x) + 1$$

for all $x \in \mathbb{Q}$.

markan 8:19:55 pm
Is this useful?

sunny2000 8:20:50 pm
induction! f(1)=2 => f(2)=3 etc.....

ssk9208 8:20:50 pm
For integers, $f(x) = x + 1$ since $f(0) = 1$

vincenthuang75025 8:20:50 pm
its useful for integers...

lucylai 8:20:50 pm
for integers $f(x) = x + 1$

fclvbfm934 8:20:50 pm
f(x) = x+1 for integers

markan 8:21:07 pm
By a straightforward induction argument,

$$f(x+n) = f(x) + n$$

for all rational numbers x and nonnegative integers n.

markan 8:21:26 pm
(In fact, it works for all integers n, not just nonnegative, since you can do the induction the other way as well.)

markan 8:21:53 pm
And as you point out, since $f(0) = 1$, we conclude that $f(n) = n + 1$ for all integers n.

markan 8:22:04 pm
So we know what $f(x)$ is for all integers x. How can we use this to determine $f(x)$ for all rational numbers x ?

chenjamin 8:22:51 pm
more substitution

markan 8:22:56 pm
Yeah, what's a good thing to substitute?

fclvbfm934 8:23:44 pm
y = a/b, x = b

fprosk 8:23:44 pm
x=a/b

RocketSingh 8:23:44 pm
We know f(xy) we could use it backward to get some division

markan 8:23:53 pm
Right, we know we want to learn about what f does to rational numbers.

markan 8:23:59 pm
So let's just plug in an arbitrary rational number!

markan 8:24:04 pm
Let $x = \frac{m}{n}$ be an arbitrary rational number, where m and n are integers. We can assume that n is positive.

markan 8:24:11 pm
What's a good choice for y then?

noodleeater 8:24:25 pm
let y =n

hardmath123 8:24:25 pm
n

brian22 8:24:25 pm
n

distortedwalrus 8:24:25 pm
y=n

ssk9208 8:24:25 pm
n

sharonmath 8:24:25 pm
n

markan 8:24:27 pm
We can set $y = n$, because this makes $xy = \frac{m}{n} \cdot n = m$ an integer.

markan 8:24:31 pm
Hence, setting $x = \frac{m}{n}$ and $y = n$, we get

$$f(m) = f\left(\frac{m}{n}\right) f(n) - f\left(\frac{m}{n} + n\right) + 1.$$

Let's fill in what we know. What is $f(m)$?

fclvbfm934 8:24:47 pm
m+1

ssk9208 8:24:47 pm
$m+1$

steve314 8:24:49 pm
m+1

markan 8:24:51 pm
Since m is an integer, $f(m) = m+1$.

markan 8:24:53 pm
Similarly, $f(n) = n+1$.

markan 8:24:57 pm
What can we say about $f(\frac{m}{n} + n)$?

hardmath123 8:25:21 pm
f(m/n) + n?

fclvbfm934 8:25:21 pm
f(m/n) + n

mathcool2009 8:25:21 pm
=f(m/n) + n

RocketSingh 8:25:21 pm
f(m/n) +n

noodleeater 8:25:21 pm
f(m/n + n) = f(m/n) + n

markan 8:25:23 pm
We derived that $f(x+n) = f(x) + n$ for all rational numbers x and nonnegative integers n, so $f(\frac{m}{n} + n) = f(\frac{m}{n}) + n$.

markan 8:25:30 pm
This gives us

$$m + 1 = f\left(\frac{m}{n}\right)(n+1) - f\left(\frac{m}{n}\right) - n + 1.$$

So what is $f(\frac{m}{n})$?

noodleeater 8:26:12 pm
(m+n)/n

patchosaur 8:26:12 pm
m/n+1

JoetheFixer 8:26:12 pm
(m+n)/n

fclvbfm934 8:26:12 pm
$f(\frac{m}{n}) = \frac{m}{n} + 1$

ssk9208 8:26:12 pm
$\frac{m}{n} + 1$

Bzhaothecow 8:26:14 pm
(m+n)/n

markan 8:26:15 pm
Solving for $f(\frac{m}{n})$, we find

$$f\left(\frac{m}{n}\right) = \frac{m+n}{n} = \frac{m}{n} + 1.$$

Therefore, $f(x) = x + 1$ for all rational numbers x.

markan 8:26:23 pm
Are we done?

sunny2000 8:26:38 pm
NOOO CHECK ANSWERS

RocketSingh 8:26:38 pm
plug it in

zhuangzhuang 8:26:38 pm
check it

noodleeater 8:26:38 pm
plug back in?

markan 8:26:45 pm
We need to check that this solution works. Does it?

distortedwalrus 8:26:58 pm
yes.

noodleeater 8:26:58 pm
yep

zhuangzhuang 8:26:58 pm
yes🙂

markan 8:27:00 pm
If $f(x) = x + 1$ for all $x \in \mathbb{Q}$, then $f(1) = 2$, and

and

$$\begin{aligned} f(xy) &= xy+1, \\ f(x)f(y) - f(x+y) + 1 &\\ &= (x+1)(y+1) - (x+y+1) + 1 \\ &= xy+1, \end{aligned}$$

so $f(x) = x+1$ **works.**

markan 8:27:10 pm
This problem illustrates another important concept in solving functional equations: Often, we need to solve a functional equation in steps, gathering bits and pieces about the function until we fill in the whole picture. In this case, we found the values of the functional equation on all integers, and then all rational numbers. So when solving functional equations, you must be prepared to be persistent, and revisit results that you have already derived.

markan 8:34:26 pm
So far, we have seen how the right substitutions can successfully solve functional equations. However, substitutions are not always enough. In some problems, we must use other properties of functions, such as being injective, surjective, increasing, decreasing, odd, or even. We may even have to draw on topics from other areas. In the next few problems, we will use these properties to solve functional equations.

markan 8:34:41 pm
If $f : \mathbb{R} \to \mathbb{R}$ **satisfies**

$$f(x^2 + f(y)) = y + xf(x)$$

for all $x, y \in \mathbb{R}$, **find** $f(x)$.

markan 8:34:57 pm
How shall we start?

fclvbfm934 8:35:24 pm
substitute x = 0

Cosmynx 8:35:24 pm
x=0?

eyzhang 8:35:24 pm
sub x=0?

vincenthuang75025 8:35:25 pm
Plug x=0?

markan 8:35:28 pm
Setting $x = 0$, **we get**

$$f(f(y)) = y$$

for all y.

RocketSingh 8:35:39 pm
Period = 2

RocketSingh 8:35:43 pm
Bijective

markan 8:35:46 pm
This is a very useful property for a function to have.

markan 8:35:52 pm
First, it tells us that the function f **is injective; that is, if** $f(a) = f(b)$, **then** $a = b$.

markan 8:36:34 pm
Why does $f(f(y)) = y$ **tell us that** f **is injective?**

noodleeater 8:36:57 pm
if f(a) = f(b), a = f(f(a)) = f(f(b)) = b

markan 8:37:01 pm
To see why f **is injective in this case, let** a **and** b **be real numbers such that** $f(a) = f(b)$.

markan 8:37:05 pm
Then $f(f(a)) = f(f(b))$. **But** $f(f(a)) = a$ **and** $f(f(b)) = b$, **so** $a = b$.

markan 8:37:11 pm
Therefore, $f(a) = f(b)$ **implies** $a = b$, **which means** f **is injective.**

markan 8:37:24 pm
Second, our equation tells us that the function f is surjective; that is, for any real number b, there exists a real number a such that $f(a) = b$.

brian22 8:37:45 pm
namely, f(b)

markan 8:37:47 pm
(To prove this, just let a = f(b).)

markan 8:37:56 pm
Hence, f is both injective and surjective, which means f is bijective.

markan 8:38:13 pm
Let's go back to the given functional equation. What should we try next?

chenjamin 8:38:48 pm
y=0

avery 8:38:48 pm
y=0?

markan 8:38:50 pm
We've already tried $x = 0,$ so let's see what happens when we set $y = 0.$

markan 8:38:56 pm
Setting $y = 0$, we get

$$f(x^2 + f(0)) = xf(x)$$

for all $x.$

markan 8:39:21 pm
Then what?

RocketSingh 8:40:19 pm
x^2 + c = f(xf(x))?

chenjamin 8:40:19 pm
x^2 + f(0) = f(xf(x))

distortedwalrus 8:40:20 pm
take f of both sides

markan 8:40:26 pm
Sure, that's one option.

RocketSingh 8:40:32 pm
x = 1

RocketSingh 8:41:12 pm
1 + f(0) = 1 --> f(0) = 0 --> Fixed Point!

markan 8:41:14 pm
We could also try that, and we would obtain f(1 + f(0)) = f(1), so 1+f(0)=1 and f(0) = 0.

markan 8:41:44 pm
What else?

markan 8:42:11 pm
I know taking f of both sides was suggested, but that leaves f(xf(x)), which looks hard to deal with.

brian22 8:43:10 pm
x=f(y)

noodleeater 8:43:10 pm
plug in f(x)

zhuangzhuang 8:43:10 pm
plug in x=f(x)!!!

markan 8:43:21 pm
Yeah, let's mess around with plugging in f(x) or f(y) for some of our variables.

markan 8:43:27 pm
We used that strategy earlier when we had nested functions.

markan 8:43:38 pm
For example, if we replace y with $f(y)$ in the given functional equation, then we get

Since $f(f(y)) = y$, this simplifies to

$$f\left(x^2 + f(f(y))\right) = f(y) + xf(x).$$

$$f(x^2 + y) = f(y) + xf(x)$$

for all x and y.

markan 8:44:04 pm
That's kind of nice because at least the nesting is gone.

markan 8:44:25 pm
We can also replace x with $f(x)$. If nothing else, we just want to see what happens.

markan 8:44:36 pm
If we replace x with $f(x)$, then we get

$$f\left(f(x)^2 + f(y)\right) = y + f(x)f(f(x)),$$

which simplifies to

$$f\left(f(x)^2 + f(y)\right) = y + xf(x).$$

fclvbfm934 8:45:37 pm
f(x)^2 = x^2

noodleeater 8:45:37 pm
f(x)^2 = x^2

RocketSingh 8:45:37 pm
f(x)^2 + f(y) =x^2 + f(y) --> f(x)^2 = f(x^2)

vincenthuang75025 8:45:37 pm
So $f(x)^2 + f(y) = x^2 + f(y)$

markan 8:45:50 pm
Good observation.

markan 8:46:06 pm
We can notice that in that last equation, the RHS is the same as the RHS of the original equation.

markan 8:46:17 pm
Therefore,

$$f\left(f(x)^2 + f(y)\right) = f\left(x^2 + f(y)\right).$$

What does this equation tell us?

markan 8:46:42 pm
Well, I guess you already told me :)

lucylai 8:46:46 pm
$f(x)^2 = x^2$

chenjamin 8:46:47 pm
f(x)^2 = x^2

markan 8:46:50 pm
Since f is injective,

$$f(f(x)^2 + f(y)) = f(x^2 + f(y))$$

implies

$$f(x)^2 + f(y) = x^2 + f(y).$$

Therefore,

$$f(x)^2 = x^2$$

for all x.

Fibonacci97 8:47:01 pm
f(x)=+/-x for each x

markan 8:47:08 pm
So can we say that the only two possible solutions for f are $f(x) = x$ or $f(x) = -x$?

RocketSingh 8:47:34 pm
Nooo!

fclvbfm934 8:47:34 pm
no, because some interval may have f(x) = x, and another f(x)=-x

pier17 8:47:34 pm
but that's for each individual x

markan 8:47:36 pm
No, we can only conclude that $f(x) = x$ or $f(x) = -x$ for each individual value of x. For example, notice that $f(x) = |x|$ satisfies $f(x)^2 = x^2$, even though the function is not x or $-x$.

markan 8:47:49 pm
But we can determine that $f(0) = 0.$

markan 8:48:13 pm
So what should we do now?

markan 8:49:52 pm
We could do some more experimentation with plugging things in, which may or may not lead anywhere.

RocketSingh 8:50:14 pm
f(x^2) = xf(x)

markan 8:50:27 pm
Yup, that is true (setting y=0).

markan 8:50:40 pm
But at this point we do have more tools at our disposal.

markan 8:50:49 pm
Since we know so much about f(x) already, we might just try conjecturing the answer.

markan 8:51:04 pm
For instance, we could just conjecture that f(x) = x and f(x) = -x are the only solutions.

markan 8:51:12 pm
And then we could try to prove it.

markan 8:51:22 pm
How might we go about such a proof?

noodleeater 8:51:50 pm
contradiction?

markan 8:51:54 pm
The typical approach to getting rid of the bizarre solutions here that mix these two values is by contradiction. How should we start such a proof by contradiction?

Fibonacci97 8:52:08 pm
Prove that if x and y are both positive and f(x)=x and f(y)=-y thenther is a contradition

brian22 8:52:08 pm
f(x)=x, f(y)=-y

mathocean97 8:52:08 pm
Assume $f(a) = a, f(b) = -b$

markan 8:52:11 pm
We're trying to eliminate any f that mixes up the solutions x and $-x$. So let us suppose for some solution f to the equation, there exists nonzero reals a, b such that $f(a) = a$ and $f(b) = -b$. (The nonzero assumption is important. There will not be a contradiction if either one is 0.)

markan 8:52:29 pm
Next step?

markan 8:52:42 pm
Of course, we are just guessing at this point, but what's a good guess?

noodleeater 8:52:46 pm
plug in?

delta1 8:52:46 pm
plug them in

markan 8:52:59 pm
Yeah, let's just put them into the original equation and see what happens.

markan 8:53:06 pm
Let us try plugging in $x = a$ and $y = b$ into the original equation.

markan 8:53:11 pm
Since $f(a) = a$ and $f(b) = -b$, we get

$$f(a^2 - b) = a^2 + b.$$

Is this possible?

cerberus88 8:53:38 pm
no

noodleeater 8:53:38 pm
nope

chenjamin 8:53:38 pm
nope, doesn't work either way

SuperSnivy 8:53:38 pm
no

zhuangzhuang 8:53:38 pm
No, unless a or b is 0

markan 8:53:40 pm
No, it isn't. We know $f(a^2 - b)$ is either $a^2 - b$ or $-a^2 + b$, so one of the following is true:

$$a^2 - b = a^2 + b, \qquad -a^2 + b = a^2 + b.$$

But in the first case, we get $b = 0$, and in the second case, we get $a = 0$. This contradicts our assumption that a, b were both nonzero.

markan 8:53:46 pm
We conclude that if $f(x) = x$ for any nonzero x, then $f(x) = x$ for all nonzero x. Likewise, if $f(x) = -x$ for any nonzero x, then $f(x) = -x$ for all nonzero x.

markan 8:54:06 pm
That is, $f(x) = x$ and $f(x) = -x$ are the only solutions to our original equation.

markan 8:54:34 pm
There are some functional equations that end in this sort of step, where you reduce each individual value $f(x)$ to one of two possible values and then have to prove that no solutions can mix the values. Typically the end of the solution will proceed as above. The fact that every $f(x)$ can only be two values almost always makes the contradiction immediate to find.

markan 8:54:56 pm
A sequence of real numbers a_0, a_1, a_2, ... satisfies the condition

$$a_{m+n} + a_{m-n} - m + n - 1 = \frac{a_{2m} + a_{2n}}{2}$$

for all $m \geq n \geq 0$. If $a_1 = 3$, determine a_{2004}.

markan 8:55:47 pm
You might be thinking that this doesn't look like a functional equation problem. But we can view a sequence as a function whose domain is nonnegative integers. And the fact that we have two variables involved is reminiscent of many functional equations.

ultrasonic360 8:55:52 pm
plug in m=1, n=0

RocketSingh 8:55:52 pm
m =n = 0

noodleeater 8:55:52 pm
plug in m = n

markan 8:56:02 pm
Yes, let's start plugging things in.

markan 8:56:12 pm
Setting $n=0$, we get

$$2a_m - m - 1 = \frac{a_{2m} + a_0}{2}$$

for all m .

markan 8:56:23 pm
Then

$$4a_m - 2m - 2 = a_{2m} + a_0,$$

so

$$a_{2m} = 4a_m - 2m - a_0 - 2$$

for all m .

markan 8:56:47 pm
Might as well try m=0 as well.

markan 8:56:52 pm
Setting $m=0$ and $n=0$ in the given functional equation, we get

$$2a_0 - 1 = a_0,$$

so $a_0 = 1.$

markan 8:57:03 pm
Then

$$a_{2m} = 4a_m - 2m - 3$$

for all $m.$

markan 8:57:14 pm
What next?

cerberus88 8:58:02 pm
set n=1?

zhuangzhuang 8:58:02 pm
let n=1??

markan 8:58:15 pm
I'll do that in a moment, but first, can we exploit our formula for a_{2m}?

delta1 8:58:35 pm
plug that in for a_2m

Cpi2728 8:58:35 pm
RHS of the original

markan 8:58:40 pm
We can substitute into the given functional equation.

markan 8:58:43 pm
We have that $a_{2m} = 4a_m - 2m - 3$ and $a_{2n} = 4a_n - 2n - 3,$ so the given functional equation becomes

$$a_{m+n} + a_{m-n} - m + n - 1 = \frac{4a_m - 2m - 3 + 4a_n - 2n - 3}{2}.$$

This simplifies to

$$a_{m+n} + a_{m-n} - m + n - 1 = 2a_m + 2a_n - m - n - 3,$$

so

$$a_{m+n} + a_{m-n} + 2n + 2 = 2a_m + 2a_n$$

for all m, n .

Fibonacci97 8:59:19 pm
make n=1

delta1 8:59:19 pm
now plug in n=1

markan 8:59:32 pm
We can substitute $n = 1,$ since we are given $a_1 = 3.$ (When you're stuck on a problem, look back at the problem for information you haven't used yet. Here, we haven't touched the $a_1 = 3$ yet, so we try focusing on that.) Also, we might notice that letting n=1 will give us a nice recursive definition.

markan 8:59:37 pm
Setting $n = 1,$ we get

$$a_{m+1} + a_{m-1} + 4 = 2a_m + 2a_1 = 2a_m + 6,$$

so

$$a_{m+1} + a_{m-1} = 2a_m + 2$$

for all $m \ge 1.$

markan 8:59:53 pm
We can rewrite this equation as

$$a_{m+1} - 2a_m + a_{m-1} = 2$$

for all $m \ge 1.$

markan 9:00:05 pm
Now what?

RocketSingh 9:00:22 pm
linear recurrence?

Bzhaothecow 9:00:22 pm
characteristic polynomial

Cosmynx 9:00:22 pm
shift the index and subtract

brian22 9:00:22 pm
we did this a few handouts ago

markan 9:00:40 pm
Yup, let's just do the standard method for solving linear recurrences.

markan 9:00:46 pm
Shifting the index m by $1,$ we get

$$\begin{aligned} a_{m+1} - 2a_m + a_{m-1} &= 2, \\ a_{m+2} - 2a_{m+1} + a_m &= 2. \end{aligned}$$

Subtracting these equations, we get

$$a_{m+2} - 3a_{m+1} + 3a_m - a_{m-1} = 0$$

for all $m \ge 1.$

markan 9:00:58 pm
So $\{a_m\}$ satisfies a linear recurrence. What is the characteristic polynomial of this sequence?

Fibonacci97 9:01:21 pm
(x-1)^3

thkim1011 9:01:21 pm
(x-1)^3

vincenthuang75025 9:01:21 pm
$(x-1)^3$

lucylai 9:01:21 pm
$(x-1)^3$

markan 9:01:26 pm
The characteristic polynomial of this sequence is $x^3 - 3x^2 + 3x - 1,$ which factors as $(x - 1)^3.$ So what can we say about the sequence?

pier17 9:02:56 pm
c+dn+en^2 for constants c,d,e

markan 9:03:00 pm
We can say that there exist constants A, B, and C such that

$$a_n = An^2 + Bn + C$$

for all n.

markan 9:03:05 pm
Now what do we do?

Cosmynx 9:04:03 pm
plug in 0, 1, 2

lucylai 9:04:03 pm
solve for A, B, C because we already know a_0, a_1, a_2

cerberus88 9:04:03 pm
a_0, a_1, a_2 to find A,B,C

nilaisarda 9:04:03 pm
plug in a1 = 3?

pier17 9:04:03 pm
get 3 equations: m=1,2,3

zhuangzhuang 9:04:03 pm
plug in known values for a0, a1, and solve for a2

markan 9:04:22 pm
Yup, we can plug in the known values of a_0, a_1, and a_2.

markan 9:04:38 pm
(Or we could even take the whole thing and plug it into the original functional equation, though that would probably be more work.)

markan 9:04:52 pm
I'll spare you the algebra and tell you the final answer.

markan 9:04:55 pm
$a_n = n^2 + n + 1$ **for all n.**

markan 9:05:00 pm
In particular, $a_{2004} = 2004^2 + 2004 + 1$.

markan 9:05:19 pm
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(y)) + f(f(x) + f(y)) = yf(x) + f(x + f(y))$$

for all $x, y \in \mathbb{R}$.

ultrasonic360 9:05:55 pm
plug in 0, 0!

cerberus88 9:05:55 pm
x=0, y=0?

markan 9:05:59 pm
We can try the usual substitutions.

markan 9:06:02 pm
Setting $x = 0$, we get

$$f(0) + f(f(0) + f(y)) = yf(0) + f(f(y))$$

for all y.

markan 9:06:08 pm
Setting $y = 0$, we get

$$f(xf(0)) + f(f(x) + f(0)) = f(x + f(0))$$

for all x.

markan 9:06:14 pm
Setting $x = 0$ and $y = 0$, we get

$$f(0) + f(2f(0)) = f(f(0)).$$

None of these equations looks useful.

markan 9:06:48 pm
We have four terms in our functional equation, two on each side. Which one is qualitatively very different from the others?

yangwy 9:07:09 pm
yf(x)

noodleeater 9:07:09 pm
yf(x)

ultrasonic360 9:07:09 pm
yf(x)

markan 9:07:15 pm
Every term in the given functional equation is the result of evaluating f, except for the factor y in $yf(x)$. Moreover, in the other three terms, y only appears as a direct input to f.

markan 9:07:30 pm
Let's see what we can do with this observation.

markan 9:07:37 pm
On the actual test, you'll probably try a bunch of stuff before stepping back and thinking, "Instead of finding f, let's see if I can learn something general about f. " Here are some candidates:

markan 9:07:41 pm
- **f is increasing**
- **f is decreasing**
- **f is odd**
- **f is even**
- **f is injective ($f(x) = f(y)$ implies $x = y$)**
- **f is surjective (for all c there's a b such that $f(b) = c$)**
- **f is bijective (f is injective and surjective)**

markan 9:08:13 pm
Which of these might we try to prove?

yangwy 9:08:45 pm
f is injective

nilaisarda 9:08:45 pm
injective, surjective, bijective?

ultrasonic360 9:08:45 pm
f is surjective

chenjamin 9:08:45 pm
bijective?

eyzhang 9:08:45 pm
injective

markan 9:08:52 pm
Yeah, injective and surjective seem like good options.

markan 9:09:28 pm
If we proved f(y) was surjective, then we could pick a y to make f(y) anything we want.

markan 9:09:48 pm
We might also go after "f is injective", because if we start with $f(y_1) = f(y_2)$ for some y_1 and y_2, then three of the terms in the original functional equation are identical for $y = y_1$ and $y = y_2$.

markan 9:10:17 pm
We'll look at injectivity first.

markan 9:10:23 pm
Again, to set up the argument, we let y_1 and y_2 be two values such that $f(y_1) = f(y_2)$.

markan 9:10:32 pm
Setting $y = y_1$ in the given functional equation, we get

$$f(xf(y_1)) + f(f(x) + f(y_1)) = y_1 f(x) + f(x + f(y_1))$$

for all x.

markan 9:10:36 pm
Setting $y = y_2$ in the given functional equation, we get

$$f(xf(y_2)) + f(f(x) + f(y_2)) = y_2 f(x) + f(x + f(y_2))$$

for all x.

markan 9:10:45 pm
Then what?

pier17 9:11:53 pm
y_2f(x) = y_1f(x) ->y_1 = y_2

Cpi2728 9:11:53 pm
cancel, cancel, cancel

yangwy 9:11:53 pm
f(y1) =f(y2) so by our previous observation y1f(x)=y2f(x) so if f is nonzero y1=y2

noodleeater 9:11:53 pm
we get y1 = y2

markan 9:12:13 pm
If $f(y_1) = f(y_2)$, then all the corresponding terms involving $f(y_1)$ and $f(y_2)$ are equal, and we are left with

$$y_1 f(x) = y_2 f(x).$$

If $f(x)$ is non-zero, then we can divide both sides by $f(x)$ to get $y_1 = y_2$.

markan 9:12:26 pm
So if we can find a value c such that $f(c)$ is not equal to 0, then f is injective. What if we can't find such a value c ?

zhuangzhuang 9:12:59 pm
then f(c)=0 for all c.

distortedwalrus 9:12:59 pm
then we've found a function: f(x) = 0.

RocketSingh 9:12:59 pm
F(x) always = 0

markan 9:13:01 pm
If there is no value c such that $f(c)$ is not equal to 0, then f is identically 0.

markan 9:13:22 pm
If $f(x) = 0$ for all x, then clearly the given functional equation is satisfied, so from now on we'll assume that $f(x)$ is nonzero for at least one value of x.

markan 9:13:27 pm
Then there exists a c such that $f(c)$ is not 0, and we conclude that f is injective.

markan 9:13:38 pm
Now we have to find a way to use this property.

markan 9:14:10 pm
So let's look back at our earlier work.

markan 9:14:16 pm
When we set $x = 0$, we got

$$f(0) + f(f(0) + f(y)) = yf(0) + f(f(y))$$

for all y.

markan 9:14:33 pm
Anything further we might like to do here?

markan 9:15:37 pm
We actually could have done this next thing earlier, we just never got around to it.

Cosmynx 9:16:03 pm
let y=1

markan 9:16:08 pm
Yeah, let's try y=1.

markan 9:16:16 pm
That gets us some cancellation.

markan 9:16:21 pm
Setting $x = 0$ and $y = 1$, we get

$$f(0) + f(f(0) + f(1)) = f(0) + f(f(1)),$$

which simplifies as

$$f(f(0) + f(1)) = f(f(1)).$$

RocketSingh 9:16:43 pm
f(0) = 0

zhuangzhuang 9:16:43 pm
so f(0)=0!

Bzhaothecow 9:16:43 pm
f(0)=0

noodleeater 9:16:43 pm
f(0) + f(1) = f(1), so f(0) = 0

cerberus88 9:16:43 pm
implies that f(0)+f(1) = f(1)

markan 9:16:45 pm
Since f is injective, $f(0) + f(1) = f(1)$, so $f(0) = 0$.

markan 9:16:55 pm
Now what?

RocketSingh 9:17:13 pm
Plug in 0 again

NT2048 9:17:13 pm
sub in y=0 to original

markan 9:17:27 pm
Yeah, now that we know f(0) = 0, we should definitely take another look at plugging in y=0.

zhuangzhuang 9:17:33 pm
so f(x)=f(f(x))

markan 9:17:38 pm
Now, setting $y = 0$ in the given functional equation, we get

$$f(xf(0)) + f(f(x) + f(0)) = f(x + f(0)),$$

which simplifies to

$$f(f(x)) = f(x)$$

for all x.

cerberus88 9:17:49 pm
f(x)=x

pier17 9:17:52 pm
so f(x)=x

markan 9:17:54 pm
Since f is injective, $f(f(x)) = f(x)$ implies $f(x) = x$, for all x.

markan 9:17:57 pm
And it is easy to check that $f(x) = x$ works.

markan 9:18:01 pm
Therefore, the solutions are $f(x) = 0$ and $f(x) = x$.

markan 9:18:23 pm
Ok, last problem!

markan 9:18:24 pm
Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

markan 9:18:35 pm
Let's start with the usual substitutions.

markan 9:18:41 pm
Setting $x = 0$, we get

$$f(-f(y)) = f(f(y)) + f(0) - 1$$

for all y.

markan 9:18:46 pm
Setting $y = 0$, we get

$$f(x - f(0)) = f(f(0)) + xf(0) + f(x) - 1$$

for all x.

markan 9:18:53 pm
Setting $x = 0$ and $y = 0$, we get

$$f(-f(0)) = f(f(0)) + f(0) - 1.$$

markan 9:19:00 pm
All of these are messy.

markan 9:19:08 pm
What else can we try?

patchosaur 9:19:24 pm
x=f(y)?

Fibonacci97 9:19:24 pm
x=1

ssk9208 9:19:33 pm
$x = f(y)$

markan 9:19:38 pm
Both seem reasonable, but I'll start with x=f(y).

markan 9:19:41 pm
This gives us

$$f(0) = f(f(y)) + f(y)^2 + f(f(y)) - 1,$$

or

$$f(f(y)) = \frac{f(0) + 1 - f(y)^2}{2}$$

for all y.

markan 9:20:04 pm
Why is this equation interesting?

nilaisarda 9:20:35 pm
if we prove surjectivity we get the answer?

NT2048 9:20:35 pm
if we can prove f is surjective, we're done

brian22 9:20:35 pm
if f is surjective, we're done

markan 9:20:38 pm
This equation is interesting, because if we set $a = f(y)$, then we get

$$f(a) = \frac{f(0) + 1 - a^2}{2}.$$

Hence, we have an expression for $f(a)$ whenever a is in the range of f. This means we should be on the look-out for anything that can tell us about the range of f.

markan 9:20:45 pm
(If we could prove that every real number is in the range of f, then we would be set. Unfortunately, this will not turn out to be the case.)

markan 9:21:14 pm
Let's now focus on f(0) and see if we can figure that out.

markan 9:21:19 pm
To make things easier, let $c = f(0)$, so

$$f(a) = \frac{c + 1 - a^2}{2}$$

for all a in the range of f.

markan 9:21:33 pm
Recall that by setting $x = 0$ and $y = 0$, we got

$$f(-f(0)) = f(f(0)) + f(0) - 1,$$

so

$$f(-c) = f(c) + c - 1.$$

Does this equation tell us anything about c ?

cerberus88 9:22:36 pm
not 0

markan 9:22:40 pm
If $c = 0$, then $f(0) = f(0) + 0 - 1$, which is a contradiction. Therefore, c cannot be equal to 0.

markan 9:23:06 pm
Apart from that, it seems hard to tell much from this equation, so let's go back to just setting y=0.

markan 9:23:12 pm
Recall that by setting $y = 0$, we got

$$f(x - f(0)) = f(f(0)) + xf(0) + f(x) - 1,$$

so

$$f(x - c) = f(c) + cx + f(x) - 1.$$

Then

$$f(x - c) - f(x) = cx + f(c) - 1$$

for all x.

markan 9:23:31 pm
This equation is interesting, because the left-hand side is the difference of two values in the range of f.

markan 9:23:39 pm
What can we say about the right-hand side?

markan 9:24:22 pm
What are the possible values of the right-hand side, as x ranges over all real numbers?

brian22 9:24:39 pm
EVERYTHING!!!!!!!

distortedwalrus 9:24:39 pm
all real numbers.

markan 9:24:42 pm
Since c is nonzero, we can make the right-hand side $cx + f(c) - 1$ any value we want to (with the appropriate value of x).

markan 9:24:56 pm
So this means that for any real number a, there exist real numbers b_1 and b_2 such that

$$a = f(b_1) - f(b_2).$$

Then

$$f(a) = f(f(b_1) - f(b_2)).$$

markan 9:25:41 pm
What can we do with this?

markan 9:25:53 pm
In particular, what can we do with the right hand side?

markan 9:27:11 pm
Remember earlier in the class we noticed a similarity between an expression we got and an expression in the original functional equation. This led us to match them up.

markan 9:27:22 pm
Looking at f(f(b1) - f(b2)), what can I match that to?

noodleeater 9:27:37 pm
f(x-f(y))

nilaisarda 9:27:37 pm
set the rhs = f(x-f(y))

ultrasonic360 9:27:37 pm
f(x-f(y))

cerberus88 9:27:37 pm
f(x-f(y))

markan 9:27:44 pm
Setting $x = f(b_1)$ and $y = b_2$, in the given functional equation, we get

$$\begin{aligned} f(a) &= f(f(b_1) - f(b_2)) \\ &= f(f(b_2)) + f(b_1)f(b_2) + f(f(b_1)) - 1. \end{aligned}$$

markan 9:28:01 pm
Can we simplify this?

markan 9:28:44 pm
Think back to what we derived early on.

markan 9:29:01 pm
And people said "if f is surjective we're done".

nilaisarda 9:29:30 pm
f(f(b2)) and f(f(b1)) can be simplified

chenjamin 9:29:30 pm
use formula for f(f(x))

markan 9:29:33 pm
We derived that

$$f(f(y)) = \frac{c + 1 - f(y)^2}{2}$$

for all y.

markan 9:29:38 pm
So let's plug that in.

markan 9:29:49 pm
This is a big mountain of algebra, which I will do for you.

markan 9:29:57 pm
Hence,

$$f(f(b_1)) = \frac{c + 1 - f(b_1)^2}{2}$$

and

$$f(f(b_2)) = \frac{c+1-f(b_2)^2}{2}.$$

Then

$$\begin{aligned} f(a) &= f(f(b_2)) + f(b_1)f(b_2) + f(f(b_1)) - 1 \\ &= \frac{c+1-f(b_2)^2}{2} + f(b_1)f(b_2) + \frac{c+1-f(b_1)^2}{2} - 1 \\ &= -\frac{f(b_1)^2 - 2f(b_1)f(b_2) + f(b_2)^2}{2} + c. \end{aligned}$$

markan 9:30:09 pm
Can we simplify further?

RocketSingh 9:30:20 pm
Perfect Square

noodleeater 9:30:20 pm
square in the numerator

markan 9:30:23 pm
We can write

$$\begin{aligned} f(a) &= -\frac{f(b_1)^2 - 2f(b_1)f(b_2) + f(b_2)^2}{2} + c \\ &= -\frac{(f(b_1) - f(b_2))^2}{2} + c. \end{aligned}$$

What is this expression equal to?

cerberus88 9:30:53 pm
-f(a)^2/2+c

zhuangzhuang 9:30:53 pm
thisis -a^2/2 +c

markan 9:30:57 pm
We have that $a = f(b_1) - f(b_2)$, so

$$f(a) = -\frac{a^2}{2} + c.$$

This equation holds for all real numbers a.

markan 9:31:19 pm
Now we're pretty much done.

markan 9:31:26 pm
We just need to solve for c and verify our answer.

markan 9:31:30 pm
Earlier we derived that

$$f(a) = \frac{c+1-a^2}{2}$$

for all a in the range of f.

noodleeater 9:31:46 pm
2c = c+1

markan 9:31:52 pm
Letting a=0,

markan 9:31:59 pm
we get c = (c+1)/2.

hardmath123 9:32:01 pm
c = 1

JoetheFixer 9:32:01 pm
c=1

markan 9:32:04 pm
So c=1.

markan 9:32:15 pm
Now we plug back in.

markan 9:32:24 pm
You can do the algebra on your own; it does end up working.

markan 9:32:42 pm
Therefore,

$$f(x) = 1 - \frac{x^2}{2}$$

is our only solution.

markan 9:32:52 pm
SUMMARY

markan 9:32:56 pm
In today's class, we have seen how to solve functional equations using substitutions, including finding key values (like $f(0)$ and $f(1)$), using cyclic functions, and building the function gradually (such as going from the integers to all rational numbers). We also saw how to use injectivity and surjectivity of functions to solve functional equations.

markan 9:33:03 pm
That's it for today's class.