Summer 2005 IMO Camp

GPT SU-26

a) Considering extreme values:

1. Show that in any set consisting of exactly $n \geq 2$ points in the plane, not all collinear, there are always two points such that the line that passes through these two points does not contain any of the other points in the set.
2. Let A be a set of $2n$ points in the plane, no three of which are collinear. Suppose that n of them are coloured red and the remaining n are coloured blue. Show that there are n segments such that
   a) the endpoints of each segment are points of A,
   b) no two of the segments have a common point, and
   c) the endpoints of each segment have different colors.
3. At a party, no boy dances with every girl, but every girl dances with at least one boy. Prove that there are two boys b_1 and b_2 and two girls g_1 and g_2 such that b_1 dances with g_1 and b_2 dances with g_2, whereas b_1 does not dance with g_2 nor b_2 dances with g_1.
4. Show that there exists a rational number c/d with $d < 100$, such that $[kc/d] = [73k/100]$ for every positive integer $k \leq 99$.
5. Let S be a nonempty set of integers such that
   a) the difference $x - y$ is in S whenever x and y are in S, and
   b) all multiples of x are in S whenever x is in S.

   Show that there is an element d in S such that S consists of all multiples of d.
6. a) Show that there are an infinite number of primes of the form $4n - 1$.
   b) Show that there are an infinite number of primes of the form $6n - 1$.
7. Can you find 2005 pairwise distinct positive integers, all of them less than 100000 and such that no three are in arithmetic progression?
8. A closed and bounded figure F with the following property is given in a plane. Any two points of F can be connected by a half-circle lying completely in F. Find the figure F.
9. Suppose that $2n$ ambassadors are invited to a banquet and that every ambassador has at most $n - 1$ enemies. Prove that the ambassadors can be seated around a round table, so that nobody sits next to an enemy.
10. Find all positive solutions of the system of equations:
    $p+q=r^2$, $q+r=s^2$, $r+s=t^2$, $s+t=p^2$, $t+p=q^2$.
11. Several positive real numbers are written on a paper. The sum of their pairwise products is 1. Show that we can cross out one number, so that the sum of the remaining numbers is less than $\sqrt{2}$.
12. The sum of 2005 nonnegative real numbers is 3, and the sum of their squares is > 1. Prove that it is possible to choose three of these numbers with sum > 1.
13. Suppose that each of 30 students in a class has the same number of friends among their classmates and that of any two of them there is always one which is a better student. What is the highest possible number of students, who are better than the majority of their friends?
14. Fifty segments are given on a line. Prove that some eight of the segments have a common point or eight of the segments are pairwise disjoint.
15. It is known that the numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are both permutations of $1, 1/2, 1/3, \ldots, 1/n$. In addition, we know that $a_1+b_1 \geq a_2+b_2 \geq \ldots \geq a_n+b_n$. Prove that $a_k+b_k \leq 4/k$ for all k from 1 to n.

b) Considering invariances:

16. Take 25 sweets and place them in a table in two piles: the first one with 12 sweets and the second one with 13 sweets. You play with your friend following the rule that at each step each of you in turn either eats two sweets from one pile or move one sweet from the first pile to the second one. The loser is the one who cannot make her/his next step. Prove that the player who starts first always loses the game.
17. A disk is divided into n sectors. We put n coins in an arbitrary manner into the sectors. At each step we are allowed to take any two coins (possibly from different sectors) and move each to another sector: one of them into the next sector clockwise and the other into the next sector anti-clockwise. Find the values of n for which it is possible to get from the situation when every sector contains a single coin into the situation when all the coins are in the same sector
18. A rectangular $m \times n$ table is filled with numbers. At any time we are allowed to change the signs of all numbers that stay in the same row or column. Prove that by repeating this operation we can get a configuration such that all sums of numbers in each row and each column are non negative.
19. Consider n points on a plane. On the same plane we are given n lines in a general position, namely, no two of them are parallel to each other. Prove that it is always possible to drop non intersecting perpendicular segments from the given points on the given lines possible so that there is a perpendicular on each of the lines and there is a perpendicular from every point.
20. a) Is it possible to arrange the numbers $1, 1, 2, 2, \ldots, 7, 7$ such that there are exactly $i-1$ other numbers between any two i's ?
    a) Is it possible to arrange the numbers $1, 1, 2, 2, \ldots, 8, 8$ such that there are exactly $i-1$ other numbers between any two i's ?
21. There are a white, b black and c red chips on a table. In one step you may choose two chips of different colors and replace each one by a chip of the third color. Find conditions for all chips to become of the same color.
22. There is a positive integer in each square of a rectangular table. In each move, you may double each number in a row or subtract one from each number of a column. Prove that you can reach a table of zeros by a sequence of these permitted moves.
23. Let S be the sum of all the integer numbers from 2005^2 to 2006^2. Suppose that the number S is repeatedly replaced by its digital sum until it is reduced to a one digit number d. Find d.
24. The vertices of an n-gon are labelled by real numbers $x_1, x_2, \ldots, x_n$. Let a, b, c, d be four successive labels. If $(a-d)(b-c)<0$, then we may switch b with c. Decide is this switching operation can be performed infinitely often.
25. To each vertex of a pentagon we assign an integer a_i with sum $s = a_1 + \ldots + a_5 > 0$. If x, y and z are the numbers assigned to three successive vertices and if $y<0$, then we replace (x, y, z) by $(x+y, -y, y+z)$. This step is repeated as long as there is a $y<0$. Decide if the algorithm always stops.
26. Suppose that not all four integers a, b, c, d are equal. Start with (a, b, c, d) and repeatedly replace (a, b, c, d) with $(a-b, b-c, c-d, d-a)$. Show that at least one number of the quadruple will eventually become arbitrarily large.