2004

"Let no one who is ignorant of geometry enter here."

Transformations of the plane: a summary.

Transformation: By a transformation of the plane, we shall mean a mapping of the whole plane onto itself so that every point P has a unique "image" P', and every point P' has a unique "prototype" P.

Translation: A translation is a transformation of the plane which preserves the distance between any two points and the direction of the line through them.

Rotation: A rotation is a transformation that preserves distance by turning the entire plane about some point through a given angle. The point about which the plane is rotated is called the "centre" of the rotation.

Half-turn: A half-turn is a rotation through an angle of 180°.

Reflection: A reflection is a transformation that preserves distance by flipping the points of the plane about a line AB. The points of the line AB are left invariant (*i.e.*, unchanged) by a reflection about AB. The image of any point P not on the line AB is the point P' such that PP' is perpendicular to AB and such that the line segment PP' is bisected by the line AB.

Isometry: An isometry is a transformation of the plane that preserves distance. It is also called a **congruence** or a **rigid motion**. Translations, rotations, and reflections are all examples of isometries.

Dilatation: A dilatation is a transformation which preserves orientation in the following sense. If P is mapped to P' and Q is mapped to Q' then PQ is parallel to $P'Q'$. Every translation is a dilatation, but there are many dilatations which are not translations.

Central dilatation: A dilatation which is not a translation is called a central dilatation. A central dilatation which maps P to P', Q to Q' and R to R' has the property that the lines PP', QQ' and RR' are concurrent at a point C which is called the centre of the dilatation. *Can you prove this?*

Spiral similarity: A spiral similarity can be defined as a transformation produced by first performing a central dilatation about a point and then rotating the plane about the same point.

Similarity: A similarity is a transformation that preserves the ratios of distances in the following sense. Suppose P is mapped to P', Q is mapped to Q', and R is mapped to R'. If $P'Q' = k \cdot PQ$ we also have $P'R' = k \cdot PR$, $Q'R' = k \cdot QR$, and so on for other points. A similarity transformation transforms any triangle to a triangle similar to it. A similarity transformation is also characterised as a transformation that preserves angles: $\angle PQR = \angle P'Q'R'$. Isometries, central dilatations and spiral similarities are all examples of similarity transformations.

Warm-Up Problems

1. A **collineation** is a transformation of the plane which preserves collinearity. That is, if P, Q, and R are collinear, then P', Q' and R' are collinear. Prove that every similarity is a collineation.

2. Suppose the plane is reflected about a line ℓ and then reflected about another line ℓ'. What is the resulting transformation?

3. If an isometry of the plane has two fixed points, prove that it is either the identity transformation or a reflection about the straight line through those points.

4. A **direct isometry** is an isometry which preserves the orientation of angles. (So a reflection is an isometry which is not direct.) Prove that every direct isometry which is not a translation is a rotation about some point.

5. Prove that every dilatation which is not a translation has the stated property for a central dilatation.

6. A point P and two parallel lines are given in the plane. Construct an equilateral triangle with one of its vertices at P and each of the other two vertices lying on each of the other two lines.

Harder Problems in Transformation Geometry

1. An **involution** or **involutory transformation** is a transformation, which when applied twice gives the identity transformation. Prove that the only involutory isometries of the plane are
   - a reflection about a line
   - a half-turn, or
   - the identity.
2. Prove that every isometry of the plane can be obtained by the composition of at most three reflections.
3. A transformation of the plane maps circles to circles. Does it map lines to lines?
4. In an equilateral triangle, what is the shortest path which divides the triangle into two regions of equal area? *Hint: what does this have to do with transformation geometry?*
5. Given an equilateral triangle ABC and a point P which does not lie on the circumcircle through ABC, show that we can construct a triangle with side lengths PA, PB, and PC. In the special case where P lies on the circumcircle, what is the relationship between PA, PB and PC?
6. Two circles are tangent internally at a point A. A secant intersects the two circles in four points labelled M, N, P, Q in consecutive order along the secant. What is the relationship between $\angle MAP$ and $\angle NAQ$?
7. A chord MN is drawn in a circle ω. In one of the circular segments, the circles ω_1, ω_2 are inscribed touching the arc in A and C and the chord in B and D respectively. Show that the point of intersection of AB and CD is independent of the choice of ω_1 and ω_2.
8. Use transformation geometry to construct the **Euler line** of a triangle. That is, show that the **orthocentre**, the **median**, and the **circumcentre** of a general triangle are collinear. The line through these points is called the Euler line.

9. Consider n circles $C_1, \ldots, C_n$ with C_i touching C_{i+1} externally at T_i for $i = 1, \ldots n$. Here $C_{n+1} = C_1$. Start at any point A_1 on C_1 and for $i = 1, \ldots, n$ draw straight lines A_iT_i intersection C_{i+1} at A_{i+1}. What is the relationship between A_1 and A_{n+1}?

10. Suppose a polygon lies in the plane. It is known to have n sides, although the exact positions of its sides and vertices are unknown. However, we are given the locations of $P_1, \ldots, P_n$ which are the mid-points of the line segments which form the sides. Is it possible to determine the polygon from the locations of $P_1, \ldots, P_n$?

11. A collection of n lines are given. When is it possible to construct a polygon in a given circle with n sides parallel to the given straight lines?