2007 Mock Thurs Jan 4

1. Find all positive real numbers x such that

$$\frac{x}{x+4} = \frac{5\lfloor x\rfloor-7}{7\lfloor x\rfloor-5}$$

where $\lfloor x\rfloor$ is the largest integer less than or equal to x.

2. A square and an equilateral triangle are inscribed in the same circle. All seven vertices are distinct. Prove that at least one of the seven arcs does not exceed $1/24$ of the circumference of the circle.

3. P and Q are points on the equal sides AB and AC respectively of an isosceles triangle ABC such that $AP = CQ$. Moreover, neither P nor Q is a vertex of ABC. Prove that the circumcircle of triangle APQ passes through the circumcentre of the triangle ABC.

4. A 3×4 grid is given. Find the number of possible ways to write a number among $1, 2, 3$ or 4 in each square so that no number appears twice (or more) in the same row and no number appears twice (or more) in the same column.

5. Find the set of all positive integers n with the property that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

6. There were n people present at a New Year's party. Determine the value of n, given the following information:

- n is not a multiple of 11, and $5 \leq n \leq 30$
- Each pair of strangers had exactly two common acquaintances and each pair of acquaintances had NO common acquaintances.

7. On a plane, a line l and two circles c_1 and c_2 of different radii are given such that l touches both circles at point P. Point $M \neq P$ on l is chosen so that the angle Q_1MQ_2 is as large as possible where Q_1, Q_2 are the tangency points of the tangent lines drawn from M to c_1 and c_2, respectively, differing from l. Find $\angle PMQ_1 + \angle PMQ_2$.

8. Let ABC be a triangle and let X be a point on the side AB that is not A or B. Let P be the incentre of the triangle ACX, Q the incentre of the triangle BCX and M the midpoint of the segment PQ. Show that $|MC| > |MX|$.

9. Let x_i be positive real numbers. Prove that

$$\frac{1}{1+x_1} + \frac{1}{1+x_1+x_2} + \cdots + \frac{1}{1+x_1+x_2+\cdots+x_n} < \sqrt{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}$$

10. The arithmetic mean of a number of mutually distinct primes equals 27. Determine the biggest prime that can occur amongst them.

11. For any positive integer n, denote by $f(n)$ the highest power of 2 which divides $n!$ and by $g(n)$ the number of 1's in the base-2 representation of n. Prove that $f(n)+g(n) = n$.

12. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x)^2 + 2yf(x) + f(y) = f(y + f(x))$$