2018 Special Camp - Polynomials

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1 Important

All solutions are stolen without consent.

2 Problems

Problem 1. *Prove that for any integer n, there exists a unique polynomial Q with coefficients in $\{0, 1, \ldots, 9\}$ such that $Q(-2) = Q(-5) = n$.*

Solution. $Q(x) = (x+2)(x+5)(r_0 + r_1x + r_2x^2 + \ldots) + n \Rightarrow 10r_0 + n, 10r_k + 7r_{k-1} + r_{k-2} \in [9] \cup \{0\}$, So $r_0 = -\lfloor \frac{n}{10} \rfloor, r_k = -\lfloor \frac{7r_{k-1}+r_{k-2}}{10} \rfloor$ $[r_{-1} = 0]$. So if the polynomial exists, it'd be unique. If we prove that $\{r_k\}$ terminates, we're done.

If $r_{k-1}, r_{k-2} \le m, r_k \ge -\frac{4m}{5}$. If $r_{k-1}, r_{k-2} \ge l, r_k \le -\frac{8l-9}{10}$. So the sequence is bounded.

Let $l_k = \min(r_k, r_{k+1}, \ldots)$, and $m_k = \max(r_k, r_{k+1}, \ldots)$. Note that $l_k \le m_k$, $\{l_k\}$ is non-decreasing, $\{m_k\}$ is non-increasing. So these sequences will stabilize at L and M.

Then we'll have $5L \ge -4M$, $10M \le -8L + 9$, $M \ge 0$ since no three consecutive terms can be zero. But then we have $M = L = 0$. So we're done.

Algo: $Q(x) = \sum_{i=0}^{m} a_i x^i, m_0 = n_0 = n$

$$\begin{cases} m_i \overset{0 \le r < 2}{\equiv} r \pmod 2 \\ n_i \overset{0 \le s < 5}{\equiv} s \pmod 5 \end{cases} \implies a_i = \begin{cases} s, & r - s \equiv 0 \pmod 2 \\ s + 5, & r - s \equiv 1 \pmod 2 \end{cases}$$

$m_{i+1} = (m_i - a_i)/(-2), \quad n_{i+1} = (n_i - a_i)/(-5)$

Problem 2. *Let $P(z)$ be a polynomial with complex coefficients which is of degree 2018 and has distinct zeros. Prove that there exist complex numbers $a_1, a_2, \ldots, a_{2018}$ such that $P(z)$ divides the polynomial*

$$\left(\cdots \left((z - a_1)^2 - a_2 \right)^2 \cdots - a_{2017} \right)^2 - a_{2018}.$$

Solution. Can be rephrased: Let n be a positive integer, and z_1, z_2, ..., z_n be n complex numbers. Then, there exist n complex numbers a_1, a_2, ..., a_n such that the polynomial $Q(z) = \left(... \left((z - a_1)^2 - a_2 \right)^2 ... - a_{n-1} \right)^2 - a_n$ satisfies $Q(z_i) = 0$ for every $i \in \{1, 2, ..., n\}$.

Induct.

$$R(z) = \left(... \left(\left((z - a_1)^2 - a_2 \right)^2 ... - a_{k-1} \right)^2 - \left(a_k + \tfrac{1}{2} Q(z_{k+1}) \right) \right)^2 - \tfrac{1}{4} \left(Q(z_{k+1}) \right)^2$$

$$= \left(\left(... \left(\left((z - a_1)^2 - a_2 \right)^2 ... - a_{k-1} \right)^2 - a_k \right) - \tfrac{1}{2} Q(z_{k+1}) \right)^2 - \tfrac{1}{4} \left(Q(z_{k+1}) \right)^2$$

$$= \left(Q(z) - \tfrac{1}{2} Q(z_{k+1}) \right)^2 - \tfrac{1}{4} \left(Q(z_{k+1}) \right)^2 = Q(z) \left(Q(z) - Q(z_{k+1}) \right).$$

Problem 3. *Let $P(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$ be a polynomial in the complex variable z, with real coefficients c_k. Suppose that $|P(i)| < 1$. Prove that there exist real numbers a and b such that $P(a + bi) = 0$ and $(a^2 + b^2 + 1)^2 < 4b^2 + 1$.*
Solution. $P(i) = \prod(i - r_k) \Rightarrow \prod |i - r_k| < 1$. When r is real $|i - r| = \sqrt{r^2 + 1} \geq 1$. Pairing conjugates of the complex roots, there is at least one root such that $|i - r||i - \bar{r}| < 1$. Let $r = a + ib$ and we're done.
Problem 4. *The cubic polynomial $x^3 + ax^2 + bx + c$ has real coefficients and three real roots $r \geq s \geq t$. Show that $k = a^2 - 3b \geq 0$ and that $\sqrt{k} \leq r - t$.*
Solution. By Vieta's formulas,

$$k = a^2 - 3b = (r + s + t)^2 - 3(rs + st + tr) = \tfrac{1}{2}(r - s)^2 + \tfrac{1}{2}(s - t)^2 + \tfrac{1}{2}(t - r)^2 \geq 0,$$

with equality when $r = s = t$.

Also,

$$r \geq s \geq t \implies (s - r)(s - t) \leq 0$$

$$\implies s^2 - rs - st + rt \leq 0$$

$$\implies (r^2 + s^2 + t^2) - (rs + st + tr) \leq r^2 + t^2 - 2rt$$

$$\implies k \leq (r - t)^2$$

$$\implies \sqrt{k} \leq r - t,$$

with equality when $s = r$ and/or $s = t$.
Problem 5. *A polynomial product of the form*

$$(1 - z)^{b_1}(1 - z^2)^{b_2}(1 - z^3)^{b_3}(1 - z^4)^{b_4}(1 - z^5)^{b_5} \cdots (1 - z^{32})^{b_{32}},$$

where the b_k are positive integers, has the surprising property that if we multiply it out and discard all terms involving z to a power larger than 32, what is left is just $1 - 2z$. Determine, with proof, b_{32}.
Solution. First generalize the problem. Turn $b_1, b_2, \ldots$ into an infinite sequence such that for all $k \in \mathbb{N}$, the polynomial

$$P_k(z) := (1 - z)^{b_1}(1 - z^2)^{b_2} \cdots (1 - z^k)^{b_k}$$

has the property that if we multiply it out and discard all terms involving z to a power larger than k, what is left is just $1 - 2z$. It is easy to see that by induction this is unique. Indeed, if $P_k(z) = 1 - 2z + a_{k+1}z^{k+1} + \cdots$, then $P_{k+1}(z) = P_k(z)(1 - z^{k+1})^{a_{k+1}}$, i.e. $b_k = a_{k+1}$. Further remark that each P_k has the property that $P_k(\frac{1}{z}) = (-z)^{\deg P_k} P_k(z)$ - this will be useful later.

Now let $Q_{n,k}$ denote the sum of the k^{th} powers of the roots of P_n. I claim that $Q_{n,n} = 2^n$ for all n. We prove this by induction. The base case of $n = 1$ is easy, since we need $b_1 = 2$, and so the sum of the roots of P_1 is two. Now assume the result holds for some n. Note that the sum of the n^{th} powers of the roots of P_{n+1} is equal to the sum of the n^{th} powers of P_n plus the sum of the n^{th} powers of the roots of $(1 - z^{n+1})^{b_{n+1}}$. But upon letting ω be a primitive $(n + 1)^{\text{st}}$ root of unity, we see that

$$1 + \omega^n + \omega^{2n} + \cdots + \omega^{n^2} = \overline{1 + \omega + \omega^2 + \cdots + \omega^n} = 0,$$

and so $Q_{n+1,n} = Q_{n,n}$. Now by Newton's Sums, since all coefficients of P_n smaller than n (not including the $1 - 2z$ terms) are zero, we obtain

$$0 = Q_{n+1,n+1} - 2Q_{n+1,n} = Q_{n+1,n+1} - 2Q_{n,n} \quad \Rightarrow \quad Q_{n+1,n+1} = 2Q_{n,n} \overset{\text{(IH)}}{=} 2^{n+1}$$

as desired. (Note that here we use the antisymmetry result from earlier, since it says that we can focus on the lower terms as opposed to the higher terms to perform the Newton's sum calculations.)

Now we need another expression for $Q_{n,n}$ in terms of the b_i. This is actually not so bad. Indeed, note that $Q_{n,n}$ is the sum of the n^{th} powers of the roots of $(1 - z^k)^{b_k}$ for all k. A quick check shows that this equals kb_k if $k \mid n$ and 0 otherwise, and so we obtain the formula

$$2^n = \sum_{d|n} db_d.$$

It follows by Mobius Inversion that

$$b_n = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)2^d \qquad \text{for all } n \geq 1,$$

and so $b_{32} = \frac{1}{32}(2^{32} - 2^{16}) = \boxed{2^{27} - 2^{11}}$.

Newton sums part: the polynomial is like $\ldots - 2z + 1$. If we wrote $P_{n+1}(x) = a_0 + a_1x + \ldots$, then $a_2 = a_3 = \ldots = a_{n+1} = 0, a_0 = 1, a_1 = -2$. So $a_{n+1}S_0 + a_nS_1 + \ldots + a_1S_{S_n} + a_0S_{n+1} = 0 \Rightarrow -2S_n + S_{n+1} = 0$.

Mobius Inversion: $g(n) = \sum_{d|n} f(d) \Rightarrow f(n) = \sum_{d|n}\mu(n/d)g(d)$

$$\mu(n) = \begin{cases} 0 & d^2|n, \\ (-1)^k & n = p_1p_2\cdots p_k. \end{cases}$$

In addition, $\mu(1) = 1$

Problem 6. *Prove that the roots of $P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ cannot all be real if $2a^2 < 5b$. [All roots are distinct.]*

Solution. 5 real roots mean 4 turns in the graph, so $P'(x)$ has 4 real roots. Reiterate.

$5.4.3x^2 + 4.3.2ax + 3.2.1b = 0 \Rightarrow 10x^2 + 4ax + b = 0$, $\Delta = 16a^2 - 40b \geq 0 \Rightarrow 2a^2 \geq 5b$.

Alternatively, use AM-GM with vieta.

Problem 7. *Determine all pairs of positive integers (m, n) such that $(1 + x^n + x^{2n} + \cdots + x^{mn})$ is divisible by $(1 + x + x^2 + \cdots + x^m)$.*

Solution. Let $x^{m+1} = 1$ with $x \neq 1$. Then, as the second polynomial is equal to zero, we must also have the first polynomial equal to zero.

For that to be true, we must have $n, 2n, ...mn$ be an arrangement of $1, 2, ...m$ in $\pmod{m+1}$; in other words, none of $n, 2n, ...mn$ can be divisible by $m + 1$.

We shall show that n and $m + 1$ must be relatively prime. Suppose that there is a common divisor of both, k, that is greater than 1. Then clearly $\frac{m+1}{k} \leq m$, meaning that there will be at least one number among $n, 2n, ...mn$ that is divisible by $m + 1$, which we cannot have. Thus, $m + 1$ and n must be relatively prime.

Problem 8. *If $P(x)$ denotes a polynomial of degree n such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$, determine $P(n + 1)$.*

Solution. Let $Q(x) = (x + 1)P(x) - x$. So $Q(x) = cx(x - 1)(x - 2)\ldots(x - n)$, $Q(-1) = 1 \Rightarrow c = \frac{(-1)^{n+1}}{(n+1)!}$.

So $P(n+1) = \frac{Q(n+1)+(n+1)}{n+2} = \frac{1}{n+2}\left(\frac{(-1)^{n+1}}{(n+1)!}(n + 1)n(n - 1)\ldots 1 + (n + 1)\right)$

So, ans is $\frac{n}{n+2}$ when n is even, 1 when n is odd.

Problem 9. *Let $p(x) = x^n + ax + p$ be a polynomial where a, n are natural numbers and p is prime, $p > a + 1$. Prove that $p(x)$ cannot be written as a product of two non-constant polynmials with integer coefficients.*

Solution. We prove by contradiction.

Suppose that $p(x) = x^n + ax + p$ is not irreducible, then $p(x) = f(x)g(x)$, where f and g are polynomials with integer coefficients and nonzero degree. Notice that the constants of $f(x)$ and $g(x)$ are ± 1 and $\pm p$ (or vice versa). WIthout loss of generality, let the constant of $f(x)$ be ± 1. Let $r_1, r_2, r_3, ..., r_k$ be the roots of f, then $f(x) = (x - r_1)(x - r_2)...(x - r_k)$ so $|r_1r_2...r_k| = 1$. There must be at least one root r_i such that $|r_i| \leq 1$. Hence,

$p(r_i) = r_i^n + ar_i + p = 0 \implies |p| = |r_i^n + ar_i| \leq |r_i^n| + |a||r_i| \leq 1 + |a|$

$\implies p \leq 1 + a$

contradiction.

Therefore, $p(x)$ is irreducible.

Problem 10. *Show that there exists a degree 58 monic polynomial $P(x) = x^{58} + a_1x^{57} + \cdots + a_{58}$ such that $P(x)$ has exactly 29 positive real roots and 29 negative real roots and that $\log_{2017}|a_i|$ is a positive integer for all $1 \leq i \leq 58$.*

Solution. Let us prove a more general claim.

For any $a \in \mathbb{R}$, $a > 1$ and every $n \in \mathbb{N}$, there exists a monic polynomial $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$, for which coefficients holds $|a_i| = a^{k_i}$, $k_i \in \mathbb{N}$, $i = 1, 2, \ldots, n$, such that all its roots are real and different, and the number of its negative and positive roots are equal (or almost equal) to $n/2$. (It means if n is odd and $n = 2n_1 + 1$, there exists such polynomials with n_1 negative and $n_1 + 1$ positive roots and also with $n_1 + 1$ negative and n_1 positive roots).

Proof goes by induction on n. It's trivial for $n = 1$. The induction step slightly differs depending to the parity of n. Let's consider first n is even, i.e. even $\rightarrow$ odd.

We take a monic polynomial $Q(x)$ of degree n with $n/2$ negative and $n/2$ positive different roots. Let $x_1 < x_2 < \cdots < x_{n/2} < x_{n/2+1} < 0$ and $y_1 > y_2 > \cdots > y_{n/2} > y_{n/2+1} > 0$ be two sequences of negative (corr. positive) points where the sign of Q of alternatively changes as $+, -, +, -, \ldots$.

Consider $P(x) := x^{n+1} + a^k Q(x)$, where $k \in \mathbb{N}$. Taking k large enough, we can ensure that the sign of $P(x_i), P(y_i)$ is the same as the sign correspondingly of $Q(x_i), Q(y_i)$, since for large enough k, Q will dominate in these points. Since the degree of P is odd $\lim_{x\to-\infty} = -\infty, \lim_{x\to\infty} = +\infty$, implying P has $n/2+1$ negative and $n/2$ positive roots. Similarly, considering $P(x) := x^{n+1} - a^k Q(x)$, yields P has $n/2$ negative and $n/2 + 1$ positive roots.

The inductive step $n =$ odd $\rightarrow$ even: As above, we take Q to have $\lfloor n \rfloor$ negative and $\lfloor n \rfloor + 1$ positive roots and consider $P(x) := x^{n+1} + a^k Q(x)$ for large enough k.

Problem 11. *Let $\varphi(x)$ be a cubic polynomial with integer coefficients. Given that $\varphi(x)$ has have 3 distinct real roots u, v, w and u, v, w are not rational number. there are integers a, b, c such that $u = av^2 + bv + c$. Prove that $b^2 - 2b - 4ac - 7$ is a square number.*

Solution. As pointed out above, the polynomial φ is irreducible and is the minimal polynomial for u, v, w. Since $\varphi(ax^2 + bx + c)$ has v as its root, u, w must be the roots as well. Hence, $aw^2 + bw + c \in \{u, v, w\}$. We can rule out $aw^2 + bw + c = w$ because w has degree 3. Moreover if $aw^2 + bw + c = u = av^2 + bv + c$, then $(v - w)(a(v + w) + b) = 0$, which implies $v + w$ is rational. But then u would be rational, contradicting the assumption. Thus we are left with $v = aw^2 + bw + c$ and similarly $w = au^2 + bu + c$.

Next, to simply the problem, we may add a common rational number to u, v, w and assume wlog that $u+v+w = 0$. We may also consider au, av, aw in place to u, v, w to assume that $a = 1$. Finally, it will turn out to be helpful to work with $2c$ instead of c. With these transformations we have three equations:

$$u = v^2 + bv + 2c$$

$$v = w^2 + bw + 2c$$

$$w = u^2 + bu + 2c$$

Summing them and using $u + v + w = 0$ gives $u^2 + v^2 + w^2 = -6c$, and so

$$uv + vw + wu = 3c.$$

But using the RHS of these equations, we can also obtain

$$uv + vw + wu = \sum u^2 v^2 \ + \ b \sum u^2 v \ + \ 4c \sum u^2 \ + \ b^2 \sum uv \ + \ 4bc \sum u \ + \ 12c^2.$$

We have $\sum u^2 v^2 = (\sum uv)^2 - 2uvw(u+v+w) = 9c^2$ and $\sum u^2 v = (u+v+w)(uv+vw+wu) - 3uvw = -3uvw$. Simplifying, the above equation gives us the key expressions

$$uvw = \frac{b^2 c - c^2 - c}{b}.$$

Next, consider the difference between the first two equations: $u - v = (v - w)(v + w + b) = (v - w)(b - u)$. Multiplying such relations yields

$$(b - u)(b - v)(b - w) = 1.$$

Plugging in the expressions derived earlier, we conclude that

$$c^2 + (1 + 2b^2)c + (b^4 - b) = 0.$$

Simple factorization gives $c = -(b^2 + b + 1)$ or $c = -(b^2 - b)$. In the former case, $b^2 - 2b - 8ac - 7 = 9b^2 + 6b + 1 = (3b + 1)^2$ (note that we have $8ac$ here because we are working with $2c$). The latter case gives $uv + vw + wu = -3(b^2 - b)$ and $uvw = -2b^3 + 3b^2 - 1$. Thus $\varphi(x) = x^3 - 3(b^2 - b)x + (2b^3 - 3b^2 + 1)$, which admits a rational root $x = b - 1$ and leads to a contradiction. The proof is completed.

$u = av^2 + bv + c$

$u + t = a'(v + t)^2 + b'(v + t) + c'$

$u = a'v^2 + (2a't + b')v + (a't^2 + b't + c' - t)$

$a = a', b = 2at + b', c = at^2 + (b - 2at - 1)t + c'$

$a' = a, b' = b - 2at, c' = c - at^2 + (2at + 1 - b)t = c - at^2 + 2at^2 + t - tb = c + at^2 + t - tb$

$b'^2 - 2b' - 4a'c' - 7 = b^2 - 4abt + 4a^2t^2 - 2b + 4at - 4ac - 4a^2t^2 - 4at + 4atb$

$= b^2 - 2b - 4ac - 7$

So we can shift everything by a rational number.

Minimal Polynomial of ξ means a polynomial of lowest degree where ξ is a root. The relevant proposition can be proved using Euclid's algo.

Problem 12. *Consider solutions to the equation*

$$x^2 - cx + 1 = \frac{f(x)}{g(x)},$$

where f and g are polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.

Solution. First, if $c \geq 2$ then we claim no such f and g exist. Indeed, one simply takes $x = 1$ to get $f(1)/g(1) \leq 0$, impossible.

For $c < 2$, let $c = 2\cos\theta$. We claim that f exists and has minimum degree equal to n, where n is defined as the smallest integer satisfying $\sin n\theta \leq 0$. In other words

$$n = \left\lceil \frac{\pi}{\arccos(c/2)} \right\rceil.$$

First we show that this is necessary. To see it, write explicitly

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-2}x^{n-2}$$

with each $a_i \geq 0$, and $a_{n-2} \neq 0$. Assume that n is such that $\sin(k\theta) \geq 0$ for $k = 1, \ldots, n-1$. Then, we have the following system of inequalities:

$$\begin{aligned}
a_1 &\geq 2\cos\theta \cdot a_0 \\
a_0 + a_2 &\geq 2\cos\theta \cdot a_1 \\
a_1 + a_3 &\geq 2\cos\theta \cdot a_2 \\
&\vdots \\
a_{n-5} + a_{n-3} &\geq 2\cos\theta \cdot a_{n-4} \\
a_{n-4} + a_{n-2} &\geq 2\cos\theta \cdot a_{n-3} \\
a_{n-3} &\geq 2\cos\theta \cdot a_{n-2}.
\end{aligned}$$

Now, multiply the first equation by $\sin\theta$, the second equation by $\sin 2\theta$, et cetera, up to $\sin((n-1)\theta)$. This choice of weights is selected since we have

$$\sin(k\theta) + \sin((k+2)\theta) = 2\sin((k+1)\theta)\cos\theta$$

so that summing the entire expression cancels nearly all terms and leaves only

$$\sin((n-2)\theta)\, a_{n-2} \geq \sin((n-1)\theta) \cdot 2\cos\theta \cdot a_{n-2}$$

and so by dividing by a_{n-2} and using the same identity gives us $\sin(n\theta) \leq 0$, as claimed.

Chebyshev polynomials of the second kind is : $U_n(\cos\theta) = \dfrac{\sin(n+1)\theta}{\sin\theta}$

One may imagine trying to construct a polynomial recursively / greedily by making all inequalities above hold (again the "sharpest situation" in which f has few coefficients). If one sets $c = 2t$, then we have

$$a_0 = 1, \quad a_1 = 2t, \quad a_2 = 4t^2 - 1, \quad a_3 = 8t^2 - 4t, \quad \dots$$

which are the Chebyshev polynomials of the second type. This means that trigonometry is essentially mandatory. (One may also run into this when by using standard linear recursion techniques, and noting that the characteristic polynomial has two conjugate complex roots.)

3 Problems for self study

1. Find all polynomials $P(x)$ such that $P(P(x)) = [P(x)]^k$ where $k \in \mathbb{N}$ and is fixed.
2. For three real roots a, b, c of $x^3 - 3x + 1 = 0$, $(a < b < c)$, calculate $20(a^2 + b^2 - a - c)$.
3. $\cos^{10}\frac{\pi}{5} + \cos^{10}\frac{2\pi}{5} + \cos^{10}\frac{3\pi}{5} + \cos^{10}\frac{4\pi}{5}$ can be written as a rational p/q in lowest form, determine $p + q$.
4. Find all polynomials P with real coefficients such that for all $x, y \in \mathbb{R}$, $P(x^{2018} + y^{2018}) = P(x)^{2018} + P(y)^{2018}$
5. Find all polynomials P, Q with real coefficients so that for all $x \in \mathbb{R}$, $P(x)Q(x+1) - P(x+1)Q(x) = 1$.
6. $P(x)$ is a polynomial of degree 2011 such that $P(n) = 2^n$ for n in $[2011] \cup 0$. Find the coeficient of x in $P(x)$.
7. Find all second degree polynomial $d(x) = x^2 + ax + b$ with integer coefficients so that there exists a polynomial with integer coefficients $p(x)$ and a polynomial with non-zero integer coefficients $q(x)$ that satisfy $p(x)^2 - d(x)q(x)^2 = 1$ for all $x \in \mathbb{R}$.
8. Given an arbitrary positive integer a larger than 1, show that for any positive integer n, there always exists a n-degree polynomial with integer coefficients $p(x)$, such that $p(0), p(1), \dots, p(n)$ are pairwise distinct positive integers, and all have the form of $2a^k + 3$, where k is also an integer.
9. Let $P, Q \not\equiv 0$ be two polynomials with integer coefficients, and $(P, Q) = 1$. We call a pair of positive integers (a, b) sweet if $X_a = \{m | P(n) \equiv 0 (\text{mod } a), n \in \mathbb{Z}\}$ is a non-empty set, and $n \in X_a \implies b | Q(n)$. Prove or disprove: The set $B = \{b | (a, b) \text{ is sweet}\}$ is bounded.
10. Find all the real coefficient polymials satisfy that for all real number x, y, z satisfy $x + y + z = 0$ so $(x, P(x)); (y, P(y)); (z, P(z))$ are straight in coordinate plane.
11. P, Q, R are non-zero polynomials that for each $z \in \mathbb{C}$, $P(z)Q(\bar{z}) = R(z)$.
    - If $P, Q, R \in \mathbb{R}[x]$, prove that Q is constant polynomial.
    - Is the above statement correct for $P, Q, R \in \mathbb{C}[x]$?
12. Find all pairs of integers a, b for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form

    $$x^n + c_{n-1}x^{n-1} + \dots + c_1 x + c_0$$

    where each of $c_0, c_1, \dots, c_{n-1}$ is equal to 1 or -1.
13. Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

    $$P(x)^{10} + P(x)^9 = Q(x)^{21} + Q(x)^{20}.$$

14. Determine all positive integers n satisfying the following condition: for every monic polynomial P of degree at most n with integer coefficients, there exists a positive integer $k \leq n$ and $k+1$ distinct integers $x_1, x_2, \cdots, x_{k+1}$ such that

$$P(x_1) + P(x_2) + \cdots + P(x_k) = P(x_{k+1})$$

15. Let

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \text{ and} \\ g(x) &= c_{n+1}x^{n+1} + c_n x^n + \cdots + c_0 \end{aligned}$$

be non-zero polynomials with real coefficients such that $g(x) = (x+r)f(x)$ for some real number r. If $a = \max(|a_n|, \ldots, |a_0|)$ and $c = \max(|c_{n+1}|, \ldots, |c_0|)$, prove that $\frac{a}{c} \leq n+1$.

16. Find the smallest possible value of a real number c such that for any 2012-degree monic polynomial $P(x) = x^{2012} + a_{2011}x^{2011} + \ldots + a_1 x + a_0$ with real coefficients, we can obtain a new polynomial $Q(x)$ by multiplying some of its coefficients by -1 such that every root z of $Q(x)$ satisfies the inequality: $|\Im(z)| \leq c|\Re(z)|$[1]

17. Determine all positive integers n for which there exists a polynomial $f(x)$ with real coefficients, with the following properties:

- for each integer k, the number $f(k)$ is an integer if and only if k is not divisible by n;
- the degree of f is less than n.

18. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, where $a_0, \ldots, a_n$ are integers, $a_n > 0$, $n \geq 2$. Prove that there exists a positive integer m such that $P(m!)$ is a composite number

[1] $\Im(z)$ and $\Re(z)$ denote the imaginary part and the real part of a complex number respectively.