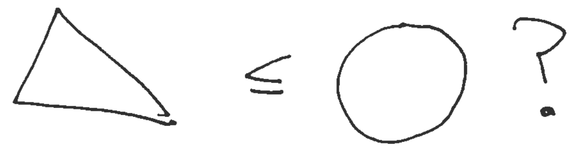
1998 IMO Camp

Geometric Inequalities

Note: (XYZ) denotes the area of triangle XYZ



1) The triangle ABC is inscribed in a circle. The interior bisectors of the angles $A^\circ, B^\circ, C^\circ$ meet the circle again at A', B', C' respectively. Show that $(A'B'C') \geq (ABC)$

2) In a triangle ABC, choose any points $K \in BC$, $L \in AC$, $M \in AB$, $N \in LM$, $R \in MK$ and $F \in KL$. If E_1, E_2, E_3, E_4, E_5 and E_6 denote the areas of triangles AMR, CKR, BKF, ALF, BNM, CLN respectively and $E = (ABC)$, show that

$$E \geq 8(E_1 E_2 E_3 E_4 E_5 E_6)^{1/6}$$

3) If R, r denote the circumradius and inradius of a triangle, show that $R \geq 2r$

4) Suppose that the four vertices of a quadrilateral $P_1P_2P_3P_4$ lie on the sides of ΔABC. Prove that at least one of the four triangles $P_1P_2P_3$, $P_1P_2P_4$, $P_1P_3P_4$ and $P_2P_3P_4$ has area $\leq \frac{1}{4}(ABC)$

5) In a triangle ABC with semiperimeter s, sides of lengths a, b and c and medians of length m_a, m_b, m_c, prove that:

a) There is a triangle with sides $a(s-a)$, $b(s-b)$, $c(s-c)$

b) $\left(\frac{m_a}{a}\right)^2 + \left(\frac{m_b}{b}\right)^2 + \left(\frac{m_c}{c}\right)^2 \geq \frac{9}{4}$, with equality iff ABC is

6) Let the bisector of angle C in triangle ABC intersect side AB in point D. Show that the segment CD is shorter than the geometric mean of the sides CA and CB.

7) Triangle ABC lies entirely inside a polygon. Prove that the perimeter of ΔABC is not greater than that of the polygon.

8) Let a, b, c be the lengths of the sides of a triangle with area K and perimeter P. Prove or disprove that

1) $a^3 + b^3 + c^3 \geq \frac{4\sqrt{3}}{3} KP$

2) $a^4 + b^4 + c^4 \geq 16 K^2$

9) In a triangle ABC for which $6(a+b+c)r^2 = abc$, consider a point M of its inscribed circle and the projections D, E, F of M on the sides BC, AC and AB. Find the maximum and minimum values of $\frac{(ABC)}{(DEF)}$

10) The quadrilateral $ABCD$ has the following properties:

(i) $AB = AD + BC$

(ii) There is a point P inside it at a distance x from the side CD such that $AP = x + AD$ and $BP = x + BC$

Show that $\frac{1}{\sqrt{x}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$

11) A quadrilateral with sides a, b, c, d has one vertex on each side of a square of side length 1. Show that:

12) P is an interior point of the angle whose sides are the rays OA and OB. Locate X on OA and Y on OB so that the line segment XY contains P and so that the product of the distances $PX \cdot PY$ is a maximum.

13) Let P be an interior point of triangle ABC, and let x, y, z denote the distances from P to BC, AC and AB respectively. Where should P be located to maximize the product xyz?

14) Convex pentagon ABCDE is inscribed in a circle having AE as diameter, with AB = a, BC = b, CD = d, DE = e, and AE = 2. Show that $a^2 + b^2 + c^2 + d^2 + abc + bcd < 4$.

15) Prove that for any point P inside a triangle ABC, PA + PB + PC is at least 6 times the inradius of ΔABC.

16) Prove that the product of two sides of a triangle is always greater than the product of the diameters of the inscribed circle and the circumscribed circle.

17) Let K be a convex polygon positioned in the cartesian plane so that exactly one quarter of its area lies in each quadrant. If K contains no non-zero lattice points, show that area(K) < 4.

18) C is a closed plane curve such that the distance between any two points of C is always less than 1. Show that C lies inside a circle of radius $\sqrt{3}$.

19) In ΔABC with inradius r, show that $(s-a)^{-2} + (s-b)^{-2} + (s-c)^{-2} \geq r^{-2}$.