

2. DESARGUES' THEOREM

§2.1. Orthogonality

Recall that two vectors \mathbf{a}, \mathbf{b} in \mathbf{R}^3 are **orthogonal** if $\mathbf{a} \cdot \mathbf{b} = 0$. Recall, too, that the orthogonal complement of a subspace U is $U^\perp = \{\mathbf{v} \mid \mathbf{u} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{u} \in U\}$. In other words U^\perp is the set of all vectors that are orthogonal to every vector in U .

Now $U + U^\perp$ is always the total vector space, which in this case is \mathbf{R}^3 , and clearly $U \cap U^\perp = \{0\}$ since the only vector that is orthogonal to itself is the zero vector.

Since $\dim(U + U^\perp) = \dim(U) + \dim(U^\perp) - \dim(U \cap U^\perp)$ we have

$$3 = \dim(U) + \dim(U^\perp) - 0 \text{ so } \dim(U) + \dim(U^\perp) = 3.$$

This means that if U is a projective point (dimension 1) then U^\perp is a projective line (dimension 2) and vice versa. So $U \leftrightarrow U^\perp$ is a 1-1 correspondence between the projective points and the projective lines in the real projective plane.

This means that, just as we can represent any projective point as $\langle \mathbf{p} \rangle$ using one vector, so any projective line can be represented in the form $\langle \mathbf{q} \rangle^\perp$. Geometrically you can think of $\langle \mathbf{p} \rangle$ as the line through the origin that passes through \mathbf{p} and $\langle \mathbf{q} \rangle^\perp$ as the plane through the origin that is perpendicular to \mathbf{q} . So the projective point $\langle \mathbf{p} \rangle$ lies on the projective line $\langle \mathbf{q} \rangle^\perp$ if and only if \mathbf{p} and \mathbf{q} are orthogonal.

Theorem 1:

(i) If $P = \langle \mathbf{p} \rangle$ and $Q = \langle \mathbf{q} \rangle$ are two distinct projective points then the projective line PQ is $\langle \mathbf{p} \times \mathbf{q} \rangle^\perp$.

(ii) If $h = \langle \mathbf{a} \rangle^\perp$ and $m = \langle \mathbf{b} \rangle^\perp$ are two distinct projective lines then $h \cap m = \langle \mathbf{a} \times \mathbf{b} \rangle$.

Proof: (i) Since \mathbf{p} and \mathbf{q} are each orthogonal to $\mathbf{p} \times \mathbf{q}$ then $\langle \mathbf{p} \rangle$ and $\langle \mathbf{q} \rangle$ both lie on $\langle \mathbf{p} \times \mathbf{q} \rangle^\perp$.

(ii) Since $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} , $\langle \mathbf{a} \times \mathbf{b} \rangle \leq \langle \mathbf{a} \rangle^\perp \cap \langle \mathbf{b} \rangle^\perp$. But both of these subspaces are projective points and so have dimension 1, which means they must be equal.

Example 1: Does $\langle (1, -3, 2) \rangle$ lie on the line $\langle (-11, 1, 7) \rangle^\perp$?

Solution: Yes, since $(1, -3, 2) \cdot (-11, 1, 7) = -11 - 3 + 14 = 0$.

Example 2: If $A = \langle (1, 3, 2) \rangle$ and $B = \langle (5, 1, 9) \rangle$ find the line AB in the form $\langle \mathbf{p} \rangle^\perp$.

Solution: $AB = \langle (1, 3, 2), (5, 1, 9) \rangle = \langle \mathbf{p} \rangle^\perp$ where \mathbf{p} is a non-zero vector orthogonal to both $(1, 3, 2)$ and $(5, 1, 9)$. Clearly such a vector is the cross-product.

$$(1, 3, 2) \times (5, 1, 9) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 2 \\ 5 & 1 & 9 \end{vmatrix} = (25, 1, -14) \text{ so } AB = \langle (25, 1, -14) \rangle^\perp.$$

Example 3: If $h = \langle (1, 3, 2) \rangle^\perp$ and $k = \langle (5, 1, 9) \rangle^\perp$ find $h \cap k$.

Solution: We need to find a vector that is orthogonal to both vectors $(1, 3, 2)$ and $(5, 1, 9)$. Once again we may take the cross-product $(1, 3, 2) \times (5, 1, 9) = (25, 1, -14)$. So $h \cap k = \langle (25, 1, -14) \rangle$.

Example 4: Suppose $A = \langle(1, 3, 5)\rangle$, $B = \langle(1, -1, 1)\rangle$ and $h = \langle(2, 1, 0)\rangle^\perp$. Find $AB \cap h$.

Solution: $(1, 3, 5) \times (1, -1, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 5 \\ 1 & -1 & 1 \end{vmatrix} = (8, 4, -4)$ so $AB = \langle(8, 4, -4)\rangle^\perp = \langle(2, 1, -1)\rangle^\perp$.

$(2, 1, -1) \times (2, 1, 0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 2 & 1 & 0 \end{vmatrix} = (1, 2, 0)$ so $AB \cap h = \langle(1, 2, 0)\rangle$.

The following table translates properties of projective points and lines into the concepts of linear algebra and back again. In proving theorems of projective geometry by linear algebra techniques we convert our assumptions into the language of linear algebra, work with them using the standard techniques of linear algebra and then convert our conclusions back to the language of projective geometry.

Let $P = \langle\mathbf{p}\rangle$, $Q = \langle\mathbf{q}\rangle$, $R = \langle\mathbf{r}\rangle$ be projective points and $h = \langle\mathbf{u}\rangle^\perp$, $k = \langle\mathbf{v}\rangle^\perp$, $m = \langle\mathbf{w}\rangle^\perp$ be projective lines (where all these vectors are non-zero).

projective point	1-dimensional subspace
projective line	2-dimensional subspace
$P = Q$	$\mathbf{p} = \lambda\mathbf{q}$ for some real $\lambda \neq 0$
P lies on h	$\mathbf{p} \cdot \mathbf{u} = 0$
P, Q, R are collinear	$\mathbf{p}, \mathbf{q}, \mathbf{r}$ are linearly dependent
h, k, m are concurrent	$\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent
projective line PQ	2-dimensional subspace $P + Q = \langle\mathbf{p} \times \mathbf{q}\rangle$
intersection of projective lines h, k	1-dimensional subspace $h \cap k = \langle\mathbf{u} \times \mathbf{v}\rangle$

§2.2. The Collinearity Lemma

Definition: If $P = \langle(x, y, z)\rangle$ is a projective point, we say that x, y, z are **homogeneous coordinates** for P .

Of course they're not unique. For example $\langle(1, 2, 3)\rangle = \langle(2, 4, 6)\rangle$. This fact is exploited by the following lemma which provides a particularly simple set of homogeneous coordinates for three or four collinear points.

Theorem 2: If $P = \langle\mathbf{p}\rangle$, Q, R, S are collinear projective points such that P, Q, R are distinct and $P \neq S$, then for a suitably chosen vector \mathbf{q} and scalar λ we may express the four points as: $P = \langle\mathbf{p}\rangle$, $Q = \langle\mathbf{q}\rangle$, $R = \langle\mathbf{p} + \mathbf{q}\rangle$, $S = \langle\lambda\mathbf{p} + \mathbf{q}\rangle$.

Proof: We break the proof into 12 separate steps.

(1) Since $P = \langle\mathbf{p}\rangle$ has dimension 1, $\mathbf{p} \neq \mathbf{0}$.

(2) Since P lies on $QR = Q + R$ we may write $\mathbf{p} = \mathbf{q}_1 + \mathbf{r}$ for some $\mathbf{q}_1 \in Q$, $\mathbf{r} \in R$.

(3) Now if $\mathbf{r} = \mathbf{0}$ we would have $\mathbf{p} = \mathbf{q}_1$ and so $P = Q$, a contradiction. Hence $\mathbf{r} \neq \mathbf{0}$.

(4) Thus $R = \langle\mathbf{r}\rangle$.

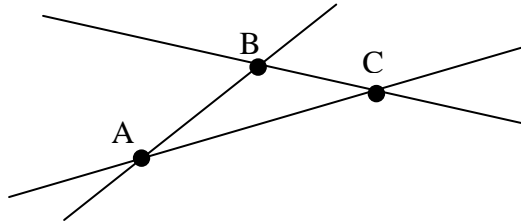
(5) Now $\mathbf{r} = \mathbf{p} - \mathbf{q}_1 = \mathbf{p} + \mathbf{q}$ if we define $\mathbf{q} = -\mathbf{q}_1$.

(6) If $\mathbf{q} = \mathbf{0}$ then $\mathbf{r} = \mathbf{p}$ and so $R = P$, a contradiction. Hence $\mathbf{q} \neq \mathbf{0}$.

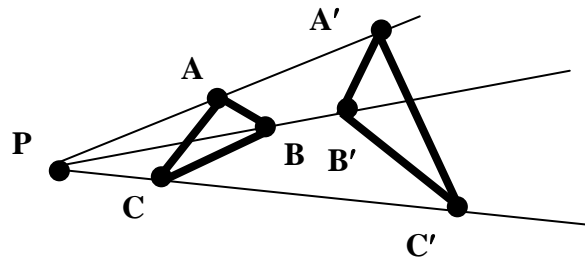
- (7) Since any non-zero vector in a 1-dimensional subspace spans that subspace we must have $Q = \langle \mathbf{q} \rangle$ and $R = \langle \mathbf{r} \rangle = \langle \mathbf{p} + \mathbf{q} \rangle$.
- (8) Since Q lies on PS , $\mathbf{q} \in P + S = \langle \mathbf{p} \rangle + S$.
- (9) Thus $\mathbf{q} = \lambda_1 \mathbf{p} + \mathbf{s}$ for some $\lambda_1 \in \mathbf{R}$ and $\mathbf{s} \in S$.
- (10) Now $\mathbf{s} = (-\lambda_1) \mathbf{p} + \mathbf{q} = \lambda \mathbf{p} + \mathbf{q}$ if we define $\lambda = -\lambda_1$.
- (11) If $\mathbf{s} = \mathbf{0}$ then $\mathbf{q} = \lambda_1 \mathbf{p}$ and so $Q = P$, a contradiction. Hence $\mathbf{s} \neq \mathbf{0}$.
- (12) Therefore $S = \langle \mathbf{s} \rangle = \langle \lambda \mathbf{p} + \mathbf{q} \rangle$.

§2.3. Perspective Triangles

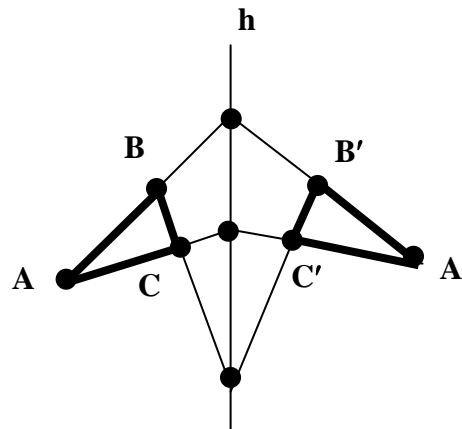
Definition: A triangle $\triangle ABC$ is a set of distinct projective points A, B, C together with the projective lines AB, AC, BC .



Definition: $\triangle ABC$ is in perspective with $\triangle A'B'C'$ from the point P if: $\{P, A, A'\}$, $\{P, B, B'\}$, $\{P, C, C'\}$ are three sets of collinear points.



Definition: $\triangle ABC$ is in perspective with $\triangle A'B'C'$ from the line h if: $\{h, AB, A'B'\}$, $\{h, AC, A'C'\}$, $\{h, BC, B'C'\}$ are three sets of concurrent lines.

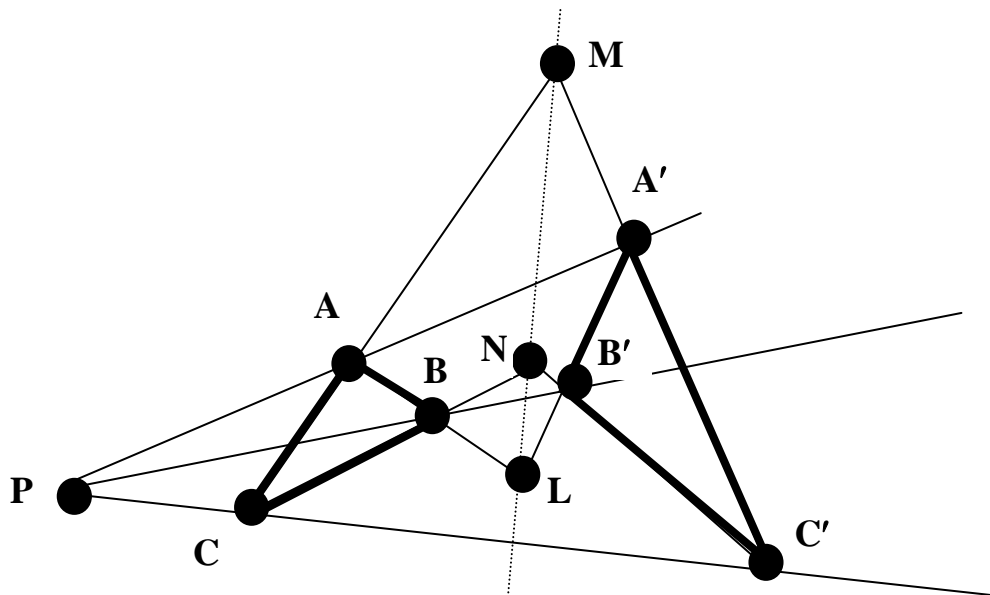


§2.4. Desargues' Theorem

Girard Desargues (1591-1661), a French architect and mathematician who lived in Lyons and Paris, was one of the founders of projective geometry. In 1639 he introduced many of the basic concepts projective geometry.

Theorem 3: (Desargues) Two triangles are in perspective from a point if and only if they are in perspective from a line.

Proof:



Suppose $\{P, A, A'\}$, $\{P, B, B'\}$ and $\{P, C, C'\}$ are three collinear sets of points on distinct lines. (The result is trivial if we allow two of the lines to coincide.)

Let $L = AB \cap A'B'$, $M = AC \cap A'C'$ and $N = BC \cap B'C'$.

We must show that L, M, N are collinear.

By the collinearity lemma we may write, for suitable vectors $\mathbf{p}, \mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$P = \langle \mathbf{p} \rangle, A = \langle \mathbf{a} \rangle, A' = \langle \mathbf{p} + \mathbf{a} \rangle,$$

$$B = \langle \mathbf{b} \rangle, B' = \langle \mathbf{p} + \mathbf{b} \rangle,$$

$$C = \langle \mathbf{c} \rangle, C' = \langle \mathbf{p} + \mathbf{c} \rangle$$

Now $\mathbf{a} - \mathbf{b} \in A + B$ and $\mathbf{a} - \mathbf{b} = (\mathbf{p} + \mathbf{a}) - (\mathbf{p} + \mathbf{b}) \in A' + B'$. Since $\mathbf{a} \neq \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ is a non-zero vector. Let $L = \langle \mathbf{a} - \mathbf{b} \rangle$. Since L is a subspace of $A + B$ as a projective point L lies on the projective line AB . Similarly L lies on $A'B'$ and so $L = \langle \mathbf{a} - \mathbf{b} \rangle = AB \cap A'B'$.

By defining M, N similarly we have:

$$M = \langle \mathbf{c} - \mathbf{a} \rangle = AC \cap A'C' \text{ and}$$

$$N = \langle \mathbf{b} - \mathbf{c} \rangle = BC \cap B'C'.$$

Now $(\mathbf{a} - \mathbf{b}) + (\mathbf{b} - \mathbf{c}) + (\mathbf{c} - \mathbf{a}) = \mathbf{0}$ so $\mathbf{a} - \mathbf{b}, \mathbf{b} - \mathbf{c}$ and $\mathbf{c} - \mathbf{a}$ are linearly dependent.

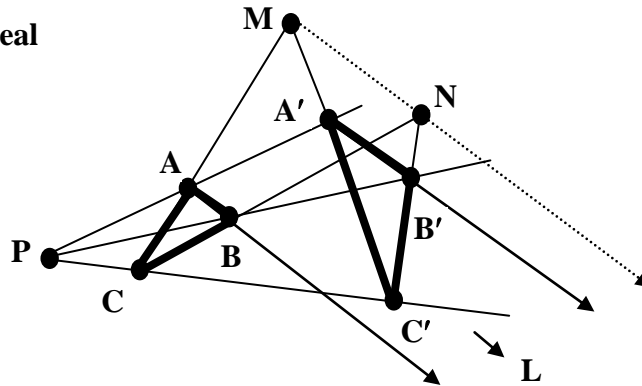
Hence L, M, N are collinear.

This proves only one half of the theorem, namely that if two triangles are in perspective from a point they are in perspective from a line. We ought now to prove the converse. But shortly we'll develop the very powerful Principle of Duality which will enable us to do that with virtually no further effort!

§2.5. Euclidean Interpretation of Desargues' Theorem

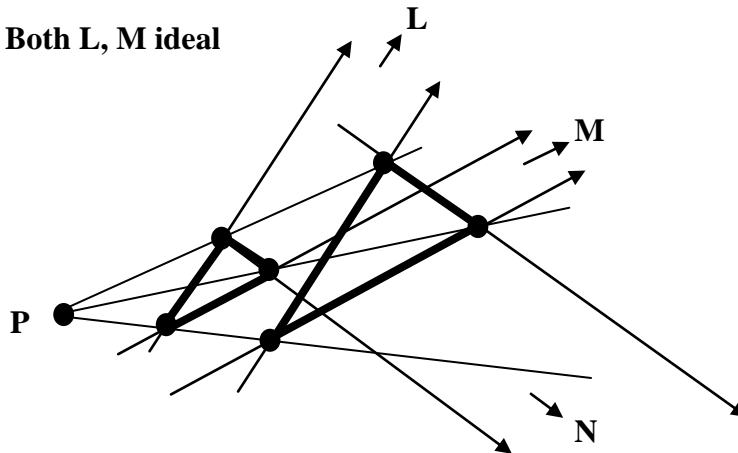
Since Desargues' Theorem is true for the Real Projective Plane it must hold for the Real Affine Plane inside it, that is, ordinary points and ordinary lines relative to some embedding of the Real Affine Plane in the Real Projective Plane. This is the case illustrated by the above diagram. However if some of the points are taken to be ideal points we obtain different affine interpretations of the same theorem. If we had attempted to prove them just for the affine plane we'd need separate proofs for each of them. By working in the real projective plane we get them all with no extra effort!

Interpretation 1: L ideal



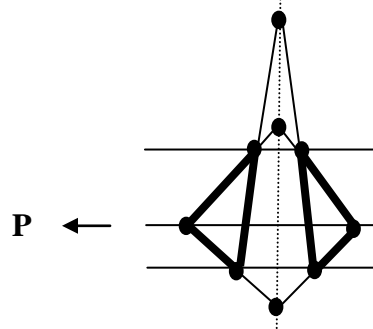
If $AB \parallel A'B'$ and $M = AC \cap A'C'$ and $N = BC \cap B'C'$ then $MN \parallel AB$.

Interpretation 2: Both L, M ideal



Since L, M, N are collinear N is an ideal point.
So if $AB \parallel A'B'$ and $AC \parallel A'C'$ then $BC \parallel B'C'$.

Interpretation 3: P is ideal



If AA' , BB' and CC' are parallel then $L = AB \cap A'B'$, $M = AC \cap A'C'$ and $N = BC \cap B'C'$ are collinear.

The great power of Projective Geometry is illustrated here. Quite apart from the fact that algebraic methods are simpler than geometric ones, and what could be simpler than $(\mathbf{a} - \mathbf{b}) + (\mathbf{b} - \mathbf{c}) + (\mathbf{c} - \mathbf{a}) = \mathbf{0}$, we can prove several affine theorems by taking suitable interpretations of a single projective theorem.

But wait! There's more! The Principle of Duality means that every time we prove a projective theorem we get a second one for free. And this, too, will have several affine interpretations.

EXERCISES FOR CHAPTER 2

Exercise 1: Let $A = \langle(1, 2, 4)\rangle$, $B = \langle(5, -3, 2)\rangle$, $C = \langle(-3, 7, 6)\rangle$ and $D = \langle(13, -13, -2)\rangle$.

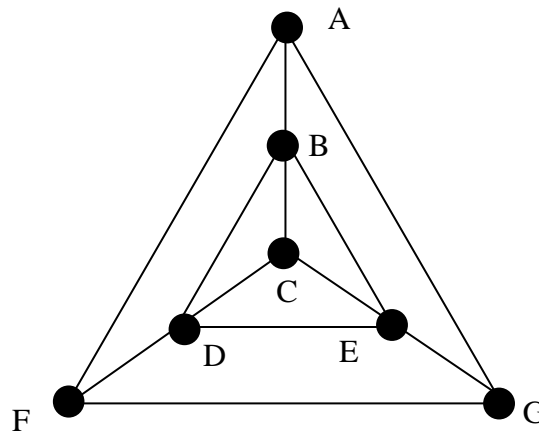
(a) Find AB and show that C, D both lie on AB .

(b) Use the proof of the Collinearity Lemma to find vectors \mathbf{a}, \mathbf{b} and a scalar λ such that $A = \langle\mathbf{a}\rangle$, $B = \langle\mathbf{b}\rangle$, $C = \langle\mathbf{a} + \mathbf{b}\rangle$ and $D = \langle\lambda\mathbf{a} + \mathbf{b}\rangle$.

Exercise 2: If $P = \langle(1, 2, 8)\rangle$, $Q = \langle(4, 1, 5)\rangle$, $R = \langle(-5, 4, 14)\rangle$, $S = \langle(14, 7, 31)\rangle$, find vectors \mathbf{p}, \mathbf{q} and a scalar λ such that $P = \langle\mathbf{p}\rangle$, $Q = \langle\mathbf{q}\rangle$, $R = \langle\mathbf{p} + \mathbf{q}\rangle$, $S = \langle\lambda\mathbf{p} + \mathbf{q}\rangle$.

Exercise 3:

(a) In the following diagram AF is parallel to BD and AG is parallel to BE . Find two triangles that are in perspective from a point.



(b) State the Euclidean interpretation of Desargues' Theorem for this configuration.

Exercise 4: Let $A = \langle(1, 0, 0)\rangle$, $B = \langle(0, 1, 0)\rangle$, $C = \langle(0, 0, 1)\rangle$,

$A' = \langle(2, 1, 1)\rangle$, $B' = \langle(2, 3, 2)\rangle$, $C' = \langle(3, 3, 4)\rangle$.

Let $L = AB \cap A'B'$, $M = AC \cap A'C'$ and $N = BC \cap B'C'$.

Find L, M and N and verify that they are collinear.

Exercise 5: In each of the following cases find a point O and a line h such that the triangles ABC and $A'B'C'$ are in perspective from the point O and from the line h .

(For simplicity corresponding points have corresponding labels.)

(i) $A = (0, 2)$, $B = (1, 1)$, $C = (1, 0)$, $A' = (1, 2)$, $B' = (2, 1)$, $C' = (0, 0)$.

These points are in the Euclidean plane embedded in the real projective plane.

(ii) $A = \langle(1, 4, 7)\rangle$, $B = \langle(2, -1, 3)\rangle$, $C = \langle(3, 5, 3)\rangle$,

$A' = \langle(2, 3, 5)\rangle$, $B' = \langle(1, 1, -1)\rangle$, $C' = \langle(1, -4, -2)\rangle$.

Exercise 6: In the following diagram A, B, P are *not* collinear while $ACXYZB$, AQP , RQX , RSZ , RPB , QSB are straight lines.

Let

$A = \langle \mathbf{a} \rangle$,

$B = \langle \mathbf{b} \rangle$,

$C = \langle \mathbf{a} + \mathbf{b} \rangle$,

$X = \langle \mathbf{a} + x\mathbf{b} \rangle$,

$Y = \langle \mathbf{a} + y\mathbf{b} \rangle$,

$Z = \langle \mathbf{a} + z\mathbf{b} \rangle$,

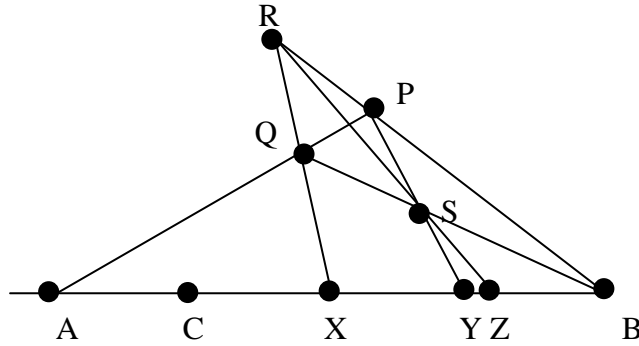
$P = \langle \mathbf{p} \rangle$,

$Q = \langle \mathbf{a} + \mathbf{p} \rangle$.

(i) Prove that $R = \langle \mathbf{p} - x\mathbf{b} \rangle$;

(ii) Find S in terms of $\mathbf{a}, \mathbf{b}, \mathbf{p}, x, y, z$;

(iii) Use the fact that RSZ are collinear to prove that $x + y = z$.



Exercise 7:

Let A, B, C and D be four distinct points, no three of which are collinear. Let S and R be points on AB and AD respectively, and let $K = AC \cap BD$, $M = KS \cap BC$ and $N = KR \cap CD$. Prove that the lines BD , MN and RS are concurrent.

SOLUTIONS FOR CHAPTER 2

Exercise 1: $AB = \langle(1, 2, 4) \times (5, -3, 2)\rangle^\perp = \langle(16, 18, -13)\rangle^\perp$. Since $(-3, 7, 6) \cdot (16, 18, -13) = 0$ and $(13, -13, -2) \cdot (16, 18, -13) = 0$, C, D lie on AB .

Suppose $(-3, 7, 6) = x(1, 2, 4) + y(5, -3, 2)$.

$$\text{Then } \begin{cases} x + 5y = -3 \\ 2x - 3y = 7 \\ 4x + 2y = 6 \end{cases}.$$

Solving, we get $x = 2$, $y = -1$.

Hence $(-3, 7, 6) = 2(1, 2, 4) - (5, -3, 2)$. Therefore we let $\mathbf{a} = 2(1, 2, 4) = (2, 4, 8)$ and $\mathbf{b} = -(5, -3, 2) = (-5, 3, -2)$. Then $A = \langle \mathbf{a} \rangle$, $B = \langle \mathbf{b} \rangle$ and $C = \langle \mathbf{a} + \mathbf{b} \rangle$.

Suppose $(13, -13, 2) = x(1, 2, 4) + y(5, -3, 2)$.

$$\text{Then } \begin{cases} x + 5y = 13 \\ 2x - 3y = -13 \\ 4x + 2y = 2 \end{cases}.$$

Solving, we get $x = -2$, $y = 3$.

Hence $(-3, 7, 6) = -2(1, 2, 4) + 3(5, -3, 2) = -\mathbf{a} - 3\mathbf{b}$.

Let $\mathbf{c} = (-1/3)(-3, 7, 6)$. Then $\mathbf{c} = (1/3)\mathbf{a} + \mathbf{b}$.

Hence $\mathbf{a} = (2, 4, 8)$, $\mathbf{b} = (-5, 3, -2)$ and $\lambda = 1/3$.

Exercise 2:

Let $x(1, 2, 8) + y(4, 1, 5) + z(-5, 4, 14) = (0, 0, 0)$.

$$\begin{pmatrix} 1 & 4 & -5 \\ 2 & 1 & 4 \\ 8 & 5 & 14 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & -5 \\ 0 & -7 & 14 \\ 0 & -27 & 54 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}. \text{ Let } z = 1. \therefore y = 2, x = -3.$$

$$\therefore -3(1, 2, 8) + 2(4, 1, 5) + (-5, 4, 14) = (0, 0, 0).$$

$$\therefore (-5, 4, 14) = 3(1, 2, 8) - 2(4, 1, 5).$$

Let $\mathbf{p} = (3, 6, 24)$ and $\mathbf{q} = (-8, -2, -10)$.

$$\therefore P = \langle \mathbf{p} \rangle, Q = \langle \mathbf{q} \rangle, R = \langle \mathbf{p} + \mathbf{q} \rangle.$$

Let $x(1, 2, 8) + y(4, 1, 5) + z(14, 7, 31) = (0, 0, 0)$.

$$\begin{pmatrix} 1 & 4 & 14 \\ 2 & 1 & 7 \\ 8 & 5 & 31 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 14 \\ 0 & -7 & -21 \\ 0 & -27 & -81 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 14 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}. \text{ Let } z = 1. \therefore y = -3, x = -2.$$

$$\therefore (14, 7, 31) = 2(1, 2, 8) + 3(4, 1, 5) = (2/3)\mathbf{p} - (3/2)\mathbf{q}$$

$$\therefore -(3/2)(14, 7, 31) = (-4/9)\mathbf{p} + \mathbf{q}.$$

So $S = \langle \lambda\mathbf{p} + \mathbf{q} \rangle$ where $\lambda = -4/9$.

Exercise 3: Triangles AFG and BDE are in perspective from C.

Let $L = AF \cap BD$, $M = AG \cap BE$ and $N = FG \cap DE$. Then L, M, N are collinear. Since AF is parallel to BD and AG is parallel to BE , the points L, M are ideal points and so LM is the ideal line. Thus N must also be an ideal point. It follows that FG is parallel to DE .

Exercise 4:

$$AB = \langle (0, 0, 1) \rangle^\perp, A'B' = \langle (-1, -2, 4) \rangle^\perp \text{ so } L = \langle (2, -1, 0) \rangle.$$

$$AC = \langle (0, -1, 0) \rangle^\perp, A'C' = \langle (1, -5, 3) \rangle^\perp \text{ so } M = \langle (-3, 0, 1) \rangle.$$

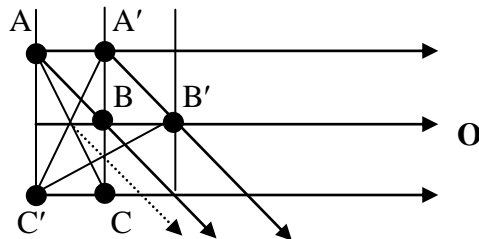
$$BC = \langle (1, 0, 0) \rangle^\perp, B'C' = \langle (6, -2, -3) \rangle^\perp \text{ so } N = \langle (0, 3, -2) \rangle.$$

$$LM = \langle (-1, -2, -3) \rangle^\perp = \langle (1, 2, 3) \rangle^\perp, MN = \langle (-3, -6, -9) \rangle^\perp = LM.$$

[We could have simply observed that $3(2, -1, 0) + 2(-3, 0, 1) + (0, 3, -2) = (0, 0, 0)$ and so L, M, N are collinear.]

Exercise 5:

(i)



(ii) $(1, 4, 7) \times (2, 3, 5) = (-1, 9, -5)$ so $\mathbf{AA}' = \langle (-1, 9, -5) \rangle^\perp$.
 $(2, -1, 3) \times (1, 1, -1) = (-2, 5, 3)$ so $\mathbf{BB}' = \langle (-2, 5, 3) \rangle^\perp$.
 $\mathbf{O} = \langle (-1, 9, -5) \times (-2, 5, 3) \rangle^\perp = \langle (52, 13, 13) \rangle = \langle (4, 1, 1) \rangle$.
 $\mathbf{CC}' = \langle (3, 5, 3) \times (1, -4, -2) \rangle^\perp = \langle (2, 9, -17) \rangle^\perp$.
Since $(2, 9, -17) \cdot (4, 1, 1) = 0$, \mathbf{O} lies on \mathbf{CC}' .
Thus the triangles are in perspective from $\mathbf{O} = \langle (4, 1, 1) \rangle$.

Thus h is $\langle(4, 7, 17) \times (1, 6, 0)\rangle^\perp = \langle(-102, 17, 17)\rangle^\perp = \langle(-\mathbf{6}, \mathbf{1}, \mathbf{1})\rangle^\perp$.

$AC = \langle (-23, 18, -7) \rangle^\perp$ and $A'C' = \langle (14, 9, -11) \rangle^\perp$ so $AC \cap A'C' = \langle (-135, -351, -459) \rangle^\perp = \langle (5, 13, 17) \rangle^\perp$. Since $(5, 13, 17) \cdot (-6, 1, 1) = 0$ $AC \cap A'C'$ lies on h .

Now $\beta \neq 0$ so $x + y = z$.

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