Inequalities: Mixing Variables

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I will assume that prior to reading these notes that you know the QM-AM-GM-HM Inequality, Jensen's Inequality, Cauchy-Schwarz Inequality, Rearrangement Inequality, Chebyshev Inequality, Holder Inequality, Muirhead Inequality and Schur's Inequality very well. Not to mention substitution techniques such as Ravi Substitution and Trignometric Substitution.

Warm-Up Problems

1. Let $x, y, z \geq 0$ with $x + y + z = 1$. Prove that $0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$.
2. Let $x, y, z \geq 0$ with $x + y + z = 3$. Prove that
$$\frac{x^5}{y^3 + z^3} + \frac{y^5}{z^3 + x^3} + \frac{z^5}{x^3 + y^3} \geq \frac{3}{2}.$$
3. Let $x, y, z > 0$ such that
$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 2.$$
Prove that
$$\frac{1}{1+4x} + \frac{1}{1+4y} + \frac{1}{1+4z} \geq 1.$$
4. Let $x, y, z \geq 0$ and $n \in \mathbb{N}$. Prove that
$$a^n + b^n + c^n \geq \left(\frac{a+2b}{3}\right)^n + \left(\frac{b+2c}{3}\right)^n + \left(\frac{c+2a}{3}\right)^n.$$
5. Let $x, y, z > 0$ with $x + y + z = 3xyz$. Prove that
$$\frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} \geq 3.$$
6. Let $x, y, z \geq 0$ such that $x^2 + y^2 + z^2 + xyz = 4$. Prove that $x + y + z \leq 3$.
7. Let $x, y, z \geq 0$ with $xy + yz + zx = 2$. Prove that
$$7(x+y+z)^3 - 9(x^3 + y^3 + z^3) \leq 108.$$
8. Let $a, b, c \geq 0$. Prove that $a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca)$.
9. Let ABC be a triangle with circumradius R and inradius r. Prove that $R \geq 2r$. Determine when equality holds.
10. Let ABC be a triangle with side lengths a, b, c, circumcentre O, centroid G and circumradius R. Prove that
$$|OG|^2 \leq R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right).$$

Erdos-Mordell Inequality: Let ABC be a triangle with an interior point P. Let D, E, F be the feet of the perpendicular from P onto BC, CA, AB, respectively. Then

$$|PA| + |PB| + |PC| \geq 2(|PD| + |PE| + |PF|).$$

Mixing Variable Technique

What do the following inequalities have in common?

1. Let $x, y, z \in \mathbb{R}$ with $x^2 + y^2 + z^2 = 9$. Prove that $2(x + y + z) - xyz \leq 10$.
2. Let $x, y, z \geq 0$ such that $xy + yz + zx = 1$. Prove that
$$\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \geq \frac{9}{4}.$$
3. Let $x, y, z \geq 0$. Prove that
$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+z}} + \frac{z}{\sqrt{z+x}} \leq \frac{5}{4}\sqrt{x+y+z}.$$

All of these inequalities are symmetric or cyclic, but have non-trivial equality cases. Can you find out what the equality cases are? Equality cases come in the following categories for three-variable inequalities.

1. All three variables are equal.
2. Exactly two of the variables are equal.
3. At least one variable is the boundary of the domain of the variables. For example, if $x, y, z \geq 0$, then at least one of x, y, z is equal to 0 in the equality case.

We will first handle inequalities that satisfies one of the first two cases. (i.e. at least two variables are equal in the equality case). Let's revisit a very simple inequality.

Problem 1: Let $x, y, z \geq 0$. Prove that $x + y + z \geq 3\sqrt[3]{xyz}$.

Proof: Let $f(x, y, z) = x + y + z - 3\sqrt[3]{xyz}$. Let $t = \frac{x+y}{2}$. Prove that $f(x, y, z) \geq f(t, t, z) \geq 0$. You can prove this also for $t = \sqrt{xy}$. $\square$

How does this help us to solve inequalities with non-trivial equality cases? Suppose equality holds for $x = y$. Then when we set t to be equal to one of the means of x and y. Then prove $f(x, y, z) \geq f(t, t, z) \geq 0$. The whole equality case will then depend on the steps in the proof of $f(t, t, z) \geq 0$. Of course, we want to check the equality cases first prior to attempting to solve the problem.

Let's dissect an equality problem.

Problem 2: Let $a, b, c > 0$. Prove that
$$\frac{1}{2a^2 + bc} + \frac{1}{2b^2 + ca} + \frac{1}{2c^2 + ab} \geq \frac{8}{(a+b+c)^2}.$$

Proof: We conjecture that $(a, b, c) = (t, t, 0)$ for some $t > 0$ and its permutations is when the equality case holds. Note that $a = b = c$ does not give equality. Of course, there is likely a way to solve this problem using Schur's Inequality. But I am presenting this proof as a means of presenting a technique, and will not argue one way or another that this method is cleaner than the other methods.

Let
$$f(x,y,z)=\frac{1}{2x^2+yz}+\frac{1}{2y^2+zx}+\frac{1}{2z^2+xy}-\frac{8}{(x+y+z)^2}.$$
Again, we conjecture that our equality case occurs at $(a,b,c)=(t,t,0)$. i.e. $a=b$ and $c=0$. Hence, let's assume that $c=\min\{a,b,c\}$. Most likely, this fact will become important to solve this problem.

Let $t=\frac{a+b}{2}$. We will need to prove that $f(a,b,c)\geq f(t,t,c)$ with equality if and only if $a=b=t$ and $f(t,t,c)\geq 0$ with equality if and only if $c=0$. Use the fact that $a\geq c, b\geq c$ to prove this and establish the equality case for good. □

Other minimum and maximum cases can be done using elementary differential calculus. We present the MV technique for another inequality problem. However, this problem contains an initial condition, which influences are choice of t (the mean of two of the variables) when applying the MV method.

Problem 3: Let x,y,z be real numbers such that $x^2+y^2+z^2=9$. Prove that
$$2(x+y+z)-xyz\leq 10.$$
Playing around with the equality case will yield that $(x,y,z)=(2,2,-1)$ and its permutations are equality cases for this inequality. This problem also allows x,y,z to be negative. Let $t=\sqrt{\frac{x^2+y^2}{2}}$. Why did we choose this t? This is because $(x,y,z)=(t,t,z)$ needs to satisfy $x^2+y^2+z^2=9$ for the MV method to work.

Let $z=min\{x,y,z\}$. We may want to try to prove that $f(x,y,z)\leq f(t,t,z)$ and $f(t,t,z)\leq 10$. Note that by basic algebraic manipulation,
$$f(x,y,z)-f(t,t,z)=2(x+y-2t)-z(xy-t^2).$$
By $QM-AM$ inequality, we have $x+y-2t\leq 0$. But $xy\leq t^2$. However, $f(x,y,z)\leq f(t,t,z)$ is therefore true if $z\leq 0$. This is a good observation since if $z\geq 0$, then $x,y\geq z\geq 0$ implying $x,y,z\geq 0$. This case will be left for the reader.

Now suppose $z\leq 0$. Then $f(x,y,z)\leq f(t,t,z)$. We will proceed now to prove that $f(t,t,z)\leq 10$. By algebraic manipulation,
$$f(t,t,z)=4t+2z-t^2z.$$
Since $2t^2+z^2=9$, let $t^2=\sqrt{(9-z^2)/2}$. Then we have
$$g(z)=f(t,t,z)=4\sqrt{\frac{9-z^2}{2}}+2z-\frac{9-z^2}{2}\cdot z=2\sqrt{18-2z^2}+\frac{z^3-5z}{2}.$$
We want to prove that the maximum of $g(z)$ is 10 in the range $-3\leq z\leq 0$. Hence,
$$g'(z)=\frac{-4z}{\sqrt{18-2z^2}}+\frac{3z^2-5}{2}.$$
Before you think that solving for $g'(z)=0$ is a lot of work, remember that we "know" that $z=-1$ is a solution. (Remember the conjectured equality case.) This will help in the factoring when we solve for $g'(z)$. We will leave to the reader to prove that in the domain of $z\in[-3,0]$ that $z=-1$ is the only solution to $g'(z)=0$. We will then see that $g(-3)<g(-1)$ and $g(0)<g(-1)$. This implies $z=-1$ is the maximum of

g in the domain $z \in [-3, 0]$. We certainly hope at this point that $g(-1) = 10$. And yes this is true. Hence, for $z \le 0$, $f(x, y, z) \le f(t, t, z) \le 10$.

The only remaining case is $x, y, z \ge 0$. We will leave for the reader to complete this case. $\square$

We will present one more example.

Problem 4: Let $a, b, c \ge 0$ such that $ab + bc + ca = 1$. Prove that

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \ge \frac{9}{4}.$$

Note that $(x, y, z) = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and $(1, 1, 0)$ are both equality cases. So we need to choose t such that $f(a, b, c) \ge f(t, t, c) \ge 9/4$. But what is t this time so that $ab + bc + ca = 1$ is satisfied? Well, clearly, we want $a = b = t$ to work. Hence, we want $t \in \mathbb{R}$ such that $t^2 + 2tc = 1$.

Now, prove that $f(a, b, c) \ge f(t, t, c) \ge 9/4$. A few tips to make the following to make things better. These tips work for any variables such that $ab + bc + ca = 1 = 2t^2 + ct$.

- $t^2 + 2ct = 1 = ab + bc + ca \Leftrightarrow (t + c)^2 = (a + c)(b + c)$
- $a + b - 2t = (a + c) + (b + c) - 2(c + t) = (\sqrt{a+c} - \sqrt{b+c})^2 = (a - b)^2/(\sqrt{a+c} + \sqrt{b+c})$.

Then

$$\begin{aligned} f(a,b,c) - f(t,t,c) &= \tfrac{1}{(a+b)^2} - \tfrac{1}{4t^2} + \tfrac{1}{(b+c)^2} + \tfrac{1}{(a+c)^2} - \tfrac{2}{(a+c)(b+c)} \\ &= \tfrac{4t^2-(a+b)^2}{4(a+b)^2t^2} + \tfrac{(a-b)^2}{(a+c)^2(b+c)^2} \\ &= \tfrac{(2t-a-b)(2t+a+b)}{4(a+b)^2t^2} + \tfrac{(a-b)^2}{(a+c)^2(b+c)^2} \\ &= (a-b)^2\left(\tfrac{-(2t+a+b)}{4(\sqrt{a+c}+\sqrt{b+c})^2(a+b)^2t^2} + \tfrac{1}{(a+c)^2(b+c)^2}\right). \end{aligned}$$

Now, use we assume that $a \ge c, b \ge c$ (and therefore $t \ge c$) to finish the proof. $\square$

If an initial condition is say $x + y + z = 3$, then a way to prove that $f(x, y, z) \le f(t, t, z)$ (or $\ge$) where $t = (x + y)/2$ is to fix z, and prove using calculus that $f(x, y, z)$ is maximized (or minimized) when $x = y$.

Exercises

1. Let $a, b, c \ge 0$ such that $a + b + c = 1$. Prove that
$$\sqrt{a + (b-c)^2} + \sqrt{b + (c-a)^2} + \sqrt{c + (a-b)^2} \ge \sqrt{3}.$$
2. Let $a, b, c \ge 0$ such that $ab + bc + ca = 1$. Prove that
$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{5}{2}.$$
3. Let A, B, C be angles of a non-obtuse triangle. Prove that
$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} \le 1 + \frac{\sqrt{2}}{2}.$$