Summer 2005 IMO Camp

Number Theory Problem Set 1

1. Define the Fibonacci sequence by $F_1 = 1, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that $gcd(F_k, F_{k+2}) = 1$ for all $k \in \mathbb{N}$.

2. Let $m, n \in \mathbb{N}$. Then prove $gcd(m, n) = 1$ if and only if $gcd(2005^m - 1, 2005^n - 1) = 2004$.

3. Let $a, b, n \in \mathbb{N}$ such that $gcd(a, b) = 1$ and $n \overset{>}{\geq} ab - a - b$. Prove that the line $ax + by = n$ contains a lattice point in the first quadrant of the Cartesian plane.

4. Generalize the result in Question 3 to the case $gcd(a, b) = d$ for an arbitrary $d \in \mathbb{N}$.

5. (USAMO 1972) Let $(a_1, \cdots, a_n) = gcd(a_1, \cdots, a_n)$ and $[a_1, \cdots, a_n] = lcm(a_1, \cdots, a_n)$. Prove that $\frac{[a,b,c]^2}{[a,b][a,c][b,c]} = \frac{(a,b,c)^2}{(a,b)(a,c)(b,c)}$

6. (USAMO 1993) Let a, b be odd positive integers. Define a sequence (f_n) as follows: Let $f_1 = a$, $f_2 = b$ and f_n be the largest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n becomes constant for a sufficiently large n and determine this constant.

7. Find all positive integers n such that $n | 2^n - 1$.

8. Prove that for any positive integer $n \in \mathbb{N}$, there exists n consecutive positive integers such that none of these integers are prime.

9. (1984 IMO Shortlist) Suppose that $a_1, a_2, \cdots, a_{2n}$ are distinct integers such that $(x - a_1)(x - a_2) \cdots (x - a_{2n}) - (-1)^n (n!)^2 = 0$ has an integer solution r. Show that $r = \frac{a_1 + a_2 + \cdots, a_{2n}}{2n}$.

10. Let $F_n = 2^{2^n} + 1$. Given $a, b \in \mathbb{N}$, find $gcd(F_a, F_b)$.

11. (IMO 1978/1) $m, n \in \mathbb{N}$, $1 \leq m < n$ & 1978^m, 1978^n have the same last 3 (decimal) digits. Find m, n minimizing $m + n$. Felix!