1 Problems

1.1 Algebra

A1. Positive real numbers x, y, z are given such that the difference between any two of them is less than 2. Prove that:

$$\sqrt{xy+1}+\sqrt{yz+1}+\sqrt{zx+1}>x+y+z$$

A2. Let M be a set of $n \geq 4$ points in the plane, no three of which are collinear, and not all lying on a circle. Suppose that f is a function assigning a real number to each point in M such that for any circle C passing through at least three points of M,

$$\sum_{P \in M \cap C} f(P)=0.$$

Prove that $f(P)=0$ for all points P.

A3. A sequence of non-negative real numbers $a_1, a_2, \ldots, a_n$ is given. For $k=1,2\ldots,n$, let m_k be equal to

$$\max_{1 \leq i \leq k} \frac{a_{k-l+1}+a_{k-l+2}+\ldots+a_k}{l}.$$

Prove that for any $x>0$, the number of integers k such that $m_k \geq x$, is less than or equal to $\frac{a_1+a_2+\ldots+a_n}{x}$.

1.2 Combinaotircs

C1. All entries of an 8×8 matrix are positive integers. One may repeatedly transform the entries of the matrix according to the following rules:

(a) Multiply all entries in the same row by 2.

(b) Subtract 1 from all entries in the same column.

Prove that it is possible to transform the given matrix into the zero matrix.

C2. Let n, k be positive integers and $I_1, I_2, \ldots, I_n$ be n closed intervals on a line such that among any k of the intervals, there are two with non-empty intersection. Prove that one can choose $k-1$ points on the line such that any of the intervals contains at least one of the chosen points.

C3. In a certain country, every town is connected by a road to exactly three other towns. A tourist traveling by roads visited each town exactly once and returned to the initial town. Next year he comes back for a round trip different from the last years trip (not the original path in either regular or reverse order), again visiting each town exactly once. Prove that he can always do so.

1.3 Geometry

G1. Let ABC be a triangle. Let P be the point on line BC such that B is between P and C, and $BP = BA$. Similarly, let Q be the point on line BC such that C is between Q and B, and $CQ = CA$. If R is the second intersection of the circumcircles of $\triangle ACP$ and $\triangle ABQ$, prove that $\triangle PQR$ is isosceles.

G2. Points K, L, M, N are respectively the midpoints of sides AB, BC, CD, DA in a convex quadrilateral. Line KM meets diagonals AC and BD at points P and Q, respectively. Line LN meets diagonals AC and BD at points R and S, respectively. Prove that if $AP{\cdot}PC = BQ{\cdot}QD$, then $AR \cdot RC = BS \cdot SD$.

G3. Let $ABCD$ be a convex quadrilateral whose opposite sides are not parallel. Let E, F be the intersections of the opposite sides of $ABCD$, P the intersection of AC and BD, and J the foot of the perpendicular from P to EF. Prove that $\angle AJD = \angle BJC$.

1.4 Number Theory

N1. Show that the equation $3y^2 = x^4 + x$ has no solutions in positive integers.

N2. Prove that for any positive integer n greater than 10000, there is a positive integer m that can be written as a sum of two squares, such that $0 < m - n < 3\sqrt[4]{n}$.

N3. k is a given natural number. Find all functions f mapping natural numbers to natural numbers, such that for all pairs of natural numbers m, n, we have

$$f(m) + f(n) | (m + n)^k.$$

2 Solutions

2.1 Algebra

A1.

$$|x-y|<2 \Rightarrow (x-y)^2<4 \Rightarrow x^2+2xy+y^2<4+4xy \Rightarrow (\frac{x+y}{2})^2<xy+1$$

Hence $\frac{x+y}{2}<\sqrt{xy+1}$. Similarly $\frac{y+z}{2}<\sqrt{yz+1}$ and $\frac{z+x}{2}<\sqrt{zx+1}$. Adding these three inequalities we get the result.

Source: Russia 2004.

A2. Label all possible pairs of points in M from 1 to $K=\binom{n}{2}$. Look at an arbitrary pair i of points A_i, B_i in M and consider the set C_i of all circles passing through the two points in the pair and through at least one other point in M. The number of circles in C_i is $m_i \geq 2$, since not all points lie on a circle. Label the circles from ω_1^i to $\omega_{m_i}^i$.

Let $S_i=f(A_i)+f(B_i)$ and $S=\sum_{P\in M} f(P)$. We have:

$$S=[\sum_{j=1}^{m_i}\sum_{P\in\omega_{m_j}^i} f(P)]-(m_i-1)S_i=(1-m_i)S_i \Rightarrow S_i=\frac{S}{1-m_i} \tag{1}$$

Assume $S\neq 0$. Equation (1) holds for all i hence $n-1=\sum_{i=1}^{K}\frac{S_i}{S}=\sum_{i=1}^{K}\frac{1}{1-m_i}$. But $1-m_i<0$ for all i which gives $n-1<0$, a contradiction. Hence $S=0$, and equation (1) implies $S_i=0$ for all i, hence $f(P)=0$ for all points P.

Source: Romania 1998.

A3. Let $b_i=a_1+...+a_i$. Then $b_1\leq b_2\leq ...\leq b_n$. For any positive integers $l<k$ with $m=k-l$:

$$\frac{a_{l+1}+a_{l+2}+\ldots+a_k}{m}=\frac{b_k-b_l}{k-l}$$

Consider points $B_0(0,0), B_1(1,b_1), B_2(2,b_2)..., B_n(n,b_n)$ in the coordinate plane. Then $\frac{b_k-b_l}{k-l}$ is equal to the tangent of the angle formed by line B_kB_l and the $x-$axis. The condition $m_k>x$ is equivalent to the condition that the line l_k passing through B_k with the slope angle of $\tan^{-1}(x)$ lies above at least one point B_l for $l<k$. We will call such a point B_k *good*. Also $\frac{a_1+a_2+\ldots+a_n}{x}=\frac{b_n}{x}$, which is the distance between the point $(n,0)$ and the point of intersection of line l_n with the $x-$axis.

Let us prove by induction on n that this distance is greater than the number of good points. The base case is clear. If point B_n is not good, remove it; then the number of good points does not change and the distance decreases, since $b_{n-1}\leq b_n$. If B_n is good, let k be the largest integer such that B_k lies below l_n. Remove points B_{k+1} through B_n; the number of good points will then decrease by $n-k$, and the distance will decrease by more than $n-k$, which finishes the induction step.

Source: Russia 2000.

2.2 Combinatorics

C1. Repeat the following procedure on the left-most column. If all integers in the left-most column are greater than 1, subtract 1 from all entries in the column. Otherwise, multiply by 2 all entries in every row containing a 1 in the column. Repeat this procedure until all entries in the left-most column are 1. The process will eventually stop since the difference between the smallest and the greatest entries in the column is non-decreasing, and it cannot stay constant forever. Once all entries in the left-most column are 1, subtract 1 from every entry in the column, so that the left-most column contains all zeroes. Once all entries in the left-most column are 0, perform the same procedure on the next column, and so on.

Source: Netherlands 1999.

C2. Denote by P_1 the left-most right endpoint of an interval. Throw away all intervals containing P_1, and let P_2 be the left-most right endpoint of the remaining intervals. Define $P_3, P_4 \ldots$ in the same way until all intervals have been thrown away. Let P_m be the last point defined. The intervals with the right endpoints $P_1, P_2, \ldots, P_m$ cannot intersect, hence $m \leq k-1$. Every interval that has been thrown away contains at least one of the chosen points, hence every interval in the collection contains one of the chosen points.

C3. Consider a graph where the vertices are the towns, and edges are the roads. We call a directed path $P_1, P_2, \ldots, P_n$ *Hamiltonian* if the path contains every vertex in the graph exactly once. We call such a Hamiltonian path *adjacent* to another Hamiltonian path, if the latter path can be obtained from the first one by inserting edge P_nP_i and deleting edge P_iP_{i+1} for $i > 1$, so that the latter path is $P_1, P_2, \ldots, P_i, P_n, P_{n-1}, \ldots, P_{i+1}$. Since P_n is adjacent to exactly 3 other vertices, every Hamiltonian path is adjacent to two paths or to one path, the latter case happening only if P_n is adjacent to P_1. (1)

Consider a graph G where vertices are the paths, and edges join adjacent paths. Look at the cycle along which the tourist traveled the first time. Remove one edge in the cycle to get a Hamiltonian path $X = S_1, S_2, \ldots, S_n$. Look at the subgraph of G which contains path X and all paths that can be reached from X by following the edges of G. This subgraph must be itself a path because of (1). Let $Y = T_1, T_2, \ldots, T_n$ be the endpoint of this path different from X. Then T_n is adjacent to T_1, so path Y can be extended to a Hamiltonian cycle. By definition of adjacency of Hamiltonian paths, $T_1 = S_1, T_2 = S_2$, hence the new cycle is not the reverse of the old one. Since $X \neq Y$, the new cycle is different from the old one, hence it satisfies the required conditions.

Source: Japan 2004

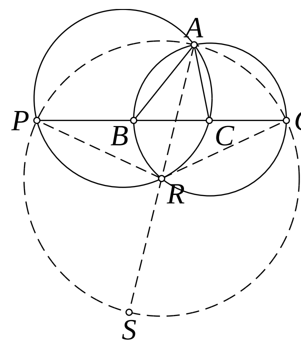
Comment: The result also holds if we only assume that all vertices in the graph have odd degree. Think about how the above solution needs to be modified for this more general problem.

2.3 Geometry

G1. *Solution 1:* Let S be the intersection of AR with the circumcircle of $\triangle PAQ$. Then:

$$\angle PSA = \angle AQP = \frac{1}{2}\angle ACP = \frac{1}{2}\angle ARP \Rightarrow \angle PSA = \angle SPR \Rightarrow RP = RS$$

Similarly $RQ = RS$. Hence $RP = RQ$.



Solution 2: Since $PACR$ and $QABR$ are cyclic, it follows that:

$$\angle ARC = \angle APC = \frac{\angle ABC}{2}; \angle ARB = \angle AQB = \frac{\angle BCA}{2}$$

$$\angle PRB = 180^\circ - \angle PAC - \angle CRB = 180^\circ - \frac{\angle ABC}{2} -$$

$$-\angle CAB - \frac{\angle BCA}{2} - \frac{\angle ABC}{2} = \frac{\angle BCA}{2} = \angle BRA$$

Consider $\triangle PAR$. By Ceva's Theorem in Sine Form applied to concurrent cevians PB, AB, RB,

$$\frac{\sin(\angle RPB)}{\sin(\angle APB)} \cdot \frac{\sin(\angle PAB)}{\sin(\angle RAB)} \cdot \frac{\sin(\angle ARB)}{\sin(\angle PRB)} = 1$$

Since $\angle PAB = \angle BPA, \angle PRB = \angle BRA$, it follows that $\sin(\angle BPR) = \sin(\angle BAR)$. We cannot have $\angle BPR + \angle BAR = 180^\circ$, hence $\angle BPR = \angle BAR \Rightarrow \angle PAR = \angle APR$ and $PR = AR$. Similarly $RQ = AR$, so $RP = RQ$.

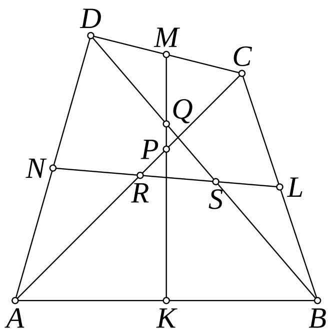
G2. Let PQ and AD intersect at T. By Menelaus Theorem in $\triangle ABD$ applied to points T, K, Q:

$$\frac{AK}{KB} \cdot \frac{BQ}{QD} \cdot \frac{DT}{TA} = -1 \Rightarrow \frac{BQ}{QD} \cdot \frac{DT}{TA} = -1 \tag{2}$$

By Menelaus Theorem in $\triangle ACD$ applied to collinear points T, P, M:

$$\frac{AP}{PC} \cdot \frac{CM}{MD} \cdot \frac{DT}{TA} = -1 \Rightarrow \frac{AP}{PC} \cdot \frac{DT}{TA} = -1 \tag{3}$$

(We use directed lengths). From equations (3), (4) it follows that $\frac{BQ}{QD} = \frac{AP}{PC}$. Similarly it follows that $\frac{AR}{RC} = \frac{DS}{SB}$.



Since $\frac{BQ}{QD} = \frac{AP}{PC}$, $AP \cdot PC = BQ \cdot QD$, we have $|AP| = |BQ|, |PC| = |QD|$, and $|AC| = |BD|$.

Since $|AC| = |BD|$, $\frac{BQ}{QD} = \frac{AP}{PC}$ we have $|AR| = |DS|, |RC| = |SB|$. Then $AR \cdot RC = BS \cdot SD$, as required.

Source: Mathlinks.

G3. *Lemma:* Points A, C, B, D lie on a line in this order. P is a point not on on this line. Then any two of the following conditions imply the third:
1. $\dfrac{CA}{CB} \cdot \dfrac{DB}{DA} = -1$.
2. PB is the angle bisector of $\angle CPD$.
3. $AP \perp PB$.

Proof: Assume 2 and 3 hold. Then PA, PB are the external and internal angle bisectors of $\angle CPD$, hence:

$$\frac{|CA|}{|DA|} = \frac{|CP|}{|DP|} = \frac{|CB|}{|DB|} \Rightarrow \frac{CA}{CB} \cdot \frac{DB}{DA} = -1$$

Assume 1 and 2 hold. Let A' be a point on line CD such that $A'C < A'D$ and $A'P \perp PB$. Then $\frac{CA'}{CB} \cdot \frac{DB}{DA'} = -1$. Since 1 holds, it follows that $A \equiv A'$ and $AP \perp PB$.

Assume 1 and 3 hold. Let C' be on line segment AB such that PB is the angle bisector of $\angle C'PD$. Since 3 holds, it follows that $\frac{C'A}{C'B} \cdot \frac{DB}{DA} = -1$. Since 1 holds, it follows that $C \equiv C'$ and PB is the angle bisector of $\angle CPD$. The lemma is proved.

Now back to the problem. It suffices to show that JP is the angle bisector of $\angle AJC$ and $\angle BJD$. Let $Q = AP \cap EF, Q' = BD \cap EF$. Using the lemma, it suffices to prove

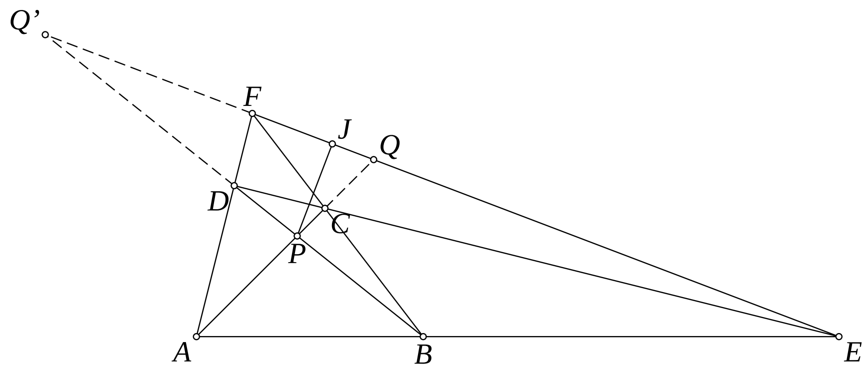
$$\frac{AQ}{QC} = \frac{AP}{PC}, \frac{BP}{PD} = \frac{BQ'}{Q'D}$$

By Menelaus' Theorem on ΔACF via the three collinear points B, P, D, we have

$$\frac{AP}{PC} \cdot \frac{CB}{BF} \cdot \frac{FD}{DA} = 1.$$

By Ceva's Theorem on ΔACF via the three concurrent lines AB, CD, FQ, we have

$$\frac{AQ}{QC} \cdot \frac{CB}{BF} \cdot \frac{FD}{DA} = 1.$$



Hence, $\frac{AP}{PC} = \frac{AQ}{QC}$. Similarly, $\frac{BP}{PD} = \frac{BQ'}{Q'D}$. This completes the problem.

Source: Mathlinks.

Comment: Consider four points A, C, B, D, occurring on a line in this order. These points are called *harmonic* iff $(A, B; C, D) = \dfrac{CA}{CB} \cdot \dfrac{DB}{DA} = -1$. Let P be a point not collinear with A, B, C, D; we define the *pencil* $P(A, B, C, D)$ to be made up of 4 lines PA, PB, PC, PD. There are a few useful results involving harmonic points that make them a powerful geometry tool.

Fact: A pencil $P(A, B, C, D)$ is given. The lines PA, PB, PC, PD intersect a line l at A', B', C', D' respectively. Then $(A', B'; C', D') = (A, B; C, D)$.

Fact: In $\triangle ABC$, points D, E, F are on sides BC, CA, AB. Let FE intersect BC at G. Then $(B, C; D, G)$ is harmonic iff AD, BE, CF are concurrent.

Fact: The lemma used in the solution. Condition 1 is equivalent to $(A, B; C, D) = -1$.

2.4 Number Theory

N1. Factor the right side: $3y^2 = x(x+1)(x^2-x+1)$. Then every two of the terms $x, x+1, x^2-x+1$ are either coprime, or share a factor of 3. Hence each of the terms is a perfect square or 3 times a perfect square. Since $x, x+1$ cannot both be perfect squares, it follows that exactly one of these two terms is 3 times a perfect square, and the other term is not divisible by 3. Hence $x^2 - x + 1$ must be a perfect square. But $(x-1)^2 < x^2 - x + 1 \leq x^2$, with equality only when $x = 1$. However, $x = 1$ does not yield a positive integer solution for y.

Source: South Korea 2004.

N2. Let $a = \lfloor \sqrt{n} \rfloor, b = \lfloor \sqrt{n - a^2} \rfloor + 1$. We claim that $m = a^2 + b^2$ works.

$$a^2 \leq n < (a+1)^2 \Rightarrow n - a^2 \leq 2a \leq 2\sqrt{n}$$

$$(b-1)^2 \leq n - a^2 \Rightarrow b - 1 \leq \sqrt{n - a^2} \leq \sqrt{2\sqrt{n}}$$

$$(b-1)^2 \leq n - a^2 < b^2 \Rightarrow m - n = b^2 - (n - a^2) \leq 2b - 1 \leq 2\sqrt{2}\sqrt[4]{n} + 1$$

It is clear that $2\sqrt{2}\sqrt[4]{n} + 1 < 3\sqrt[4]{n}$ for $n > 10000$, and we are done.

Source: Russia 2002.

N3. We first prove that f is injective. Assume $f(a) = f(b)$ for $a \neq b$, then for every positive integer n, we have $f(a)+f(n)|(a+n)^k$, $f(b)+f(n)|(b+n)^k$. Since $f(a)+f(n) = f(b)+f(n)$, it follows that $f(a)+f(n)$ is a common divisor of $(a+n)^k$ and $(b+n)^k$. Since $\gcd(a+n, b+n) = \gcd(a+n, b-a)$, we can take n such that $a+n$ is a prime greater than $b-a$, then $\gcd(a+n, b+n) = 1$, and $f(a)+f(n)$ cannot be a common divisor of $(a+n)^k$ and $(b+n)^k$.

Fix a positive integer m. For every n, $f(n)+f(m)|(n+m)^k$, $f(n)+f(m+1)|(n+m+1)^k$. Since $\gcd(m+n, m+n+1) = 1$, it follows that

$$\gcd(f(n)+f(m), f(n)+f(m+1)) = \gcd(f(n)+f(m), f(m+1)-f(m)) = 1 \qquad (4)$$

Assume there is a prime p dividing $f(m+1)-f(m)$. Let a be a positive integer such that $p^a > m$. Let $n = p^a - m$, then $f(n)+f(m)|(n+m)^k = p^{ak}$, hence $p|f(n)+f(m)$. But then $p|\gcd(f(n)+f(m), f(m+1)-f(m))$, contradicting (4). Hence $f(m+1)-f(m) = \pm 1$. Since f is injective, $f(m+1)-f(m)$ is always 1 or always -1 for all m. Since f only takes positive values, $f(m+1)-f(m) = 1$ for all positive integers m.

Therefore for some non-negative integer c, $f(n) = n + c$, for all n. If $c > 0$, let p be a prime greater than $2c$. Then $f(1)+f(p-1)|p^k$, hence $p|f(1)+f(p-1) = 2c+p$, which is impossible since $0 < 2c < p$. Hence $c = 0$ and the only solution is $f(n) = n$ for all positive integers n.

Source: Iran 2008.