MOCK OLYMPIAD 1

(1) Let $n \geq 2$ be an integer, and let A_n be the set
$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z}, 0 < k < n\}.$$
Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n.

(2) Let $m \geq 1$ be a positive integer. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. At every step, we choose 2 distinct sheets of paper. If the numbers on the sheets of paper are a and b, we erase those numbers and instead write $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

(3) Consider a fixed circle Γ with distinct fixed points A, B, C on it, and let λ be a fixed real number in $(0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of AMP and BMC. Prove that as P varies, Q lies on a fixed circle.