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IMO Winter Camp 2007: Number Theory

Euclid's Theorem: Let a, b be positive integers. Then there exist unique positive integers q, r such that $a = qb + r$ and $0 \leq r < b$.

Fermat's Little Theorem: Let p be a prime and a be an integer relatively prime to p. (i.e. $gcd(a, p) = 1$). Then

$$a^{p-1} \equiv 1 \bmod p$$

Let $\varphi(n)$ be the number of positive integers at most n that are relatively prime to n.

For example, $\varphi(12) = 4$ since $1, 5, 7, 11$ are the only positive integers at most 12 that are relatively prime to 12.

Another example is that $\varphi(p) = p - 1$ for all primes p.

Euler's Theorem: Let n be a positive integer and a be an integer such that $gcd(a, n) = 1$. Then

$$a^{\varphi(n)} \equiv 1 \bmod n$$

Wilson's Theorem: Let p be a prime. Then

$$(p-1)! \equiv -1 \bmod p$$

Definition of Factorial: Let n be a non-negative integer. $n!$ is defined to be the product $n! = n \cdot (n-1) \cdot 2 \cdot 1$.

Warm-Up Problems

1. Prove that there are no integer solutions to $x^2 + y^2 = 2007$.

2. What are the two rightmost digits of the sum $1! + 2! + 3! + \cdots 2007!$?

3. Find all positive integers n such that $n^4 + n^2 + 1$ is prime.

4. Prove that for all positive integers n, $gcd(3n + 7, n + 2) = 1$.

5. Find all positive integer solutions to $a! + b! + c! = n!$.

6. Find all integer solutions to $x^2 = 2y^2$.

Lecture Problems

1. Find all positive integers n such that $n + 2 | 5n + 13$.

2. A store sells 4-cent stamps and 7-cent stamps. What is the largest total value of stamps that cannot be purchased at this store ?

3. Let p be prime. Find all solutions to $x^2 \equiv 1 \mod p$ where $x \in \{0, 1, \cdots, p-1\}$.

4. Let p be prime and $a \in \{1, 2, \cdots, p-1\}$. Prove that the set $\{a, 2a, 3a, \cdots, (p-1)a\}$ consists of the elements $\{1, 2, \cdots, p-1\}$ modulo p.

5. Let p be prime. Which elements (modulo p) are its own inverse modulo p ? (Hint: You've done this problem already, just worded differently.)

6. Suppose that n is not divisible by 2 or 3. Prove that $2^{-1} + 3^{-1} + 6^{-1} \equiv 1 \mod n$.

7. Let p be a prime larger than 3. Prove that $2^{p-2} + 3^{p-2} + 6^{p-2} \equiv 1 \mod p$

8. Let p be an odd prime and a be a positive integer. Prove that $a^{\frac{p-1}{2}} \equiv 0, -1, 1 \mod p$.

9. Prove that there are no integer solutions to $200\cancel{7}m = n^2 + 1$.

   3

   $1995m = n^2 + 1$

Problem Set

1. Find all integer solutions to the following equations.
   (a) $a^2 + b^2 + c^2 = 2007$
   (b) $a^2 + b^2 = n! + 3$
   (c) $a^5 + b^5 + c^5 + d^5 = 2007$
   (d) $x_1^4 + x_2^4 + \cdots + x_{14}^4 = 1599$

2. A set of integers S satisfies the following properties.
   - If $a, b \in S$, then $a - b \in S$.
   - S contains the integers 3141 and 5926.

   Prove that S contains every integer.

3. Find all primes p, q such that $p^4 + 4^p = q$.

4. An integer n is said to be *powerful* if every positive integer less than or equal to n can be written as the sum of distinct divisors of n.

   For example, 6 is powerful since $1 = 1$, $2 = 2$, $3 = 3$, $4 = 1+3$, $5 = 2+3$ and $6 = 1+2+3$.

   Prove that the product of two powerful numbers is also powerful.

5. A triple (a, b, c) is called a *Pythagorean Triple* if $a^2 + b^2 = c^2$. Suppose (a, b, c) is a Pythagorean Triple such that $gcd(a, b, c) = 1$.

   (a) Prove that a, b cannot be both odd or both even.
   (b) Prove that exactly one of a, b, c is divisible by 5.
   (c) Suppose that a is odd and b is even. Prove that there exists integers m, n such that $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$.

6. Prove that there exists an integer n such that 2007^n ends with 001.

7. How many pairs of integers (x, y) are there such that $gcd(x, y) = 5!$ and $lcm(x, y) = 50!$?

8. Prove that there are infinitely many positive integer solutions to $4ab - a - b = c^2 - 1$ but no positive integer solutions to $4ab - a - b = c^2$.

9. Determine all pairs of integers (x, y) such that $1 + 2^x + 2^{2x+1} = y^2$.

10. Alice and Bob are playing the following game. A stack of n chips (with $n \geq 2$) are on a table. Alice and Bob alternate taking chips off the table. On each player's turn, the number of chips removed by this player must be a divisor of the number of chips on

the table. The player who removes the last chip off the table loses. If Alice goes first, for which n can Alice always win ?

11. Find all integer solutions to the equation $(m^2-mn-n^2)^2 = 1$ such that $0 \leq m, n \leq 100$.

12. Find all positive integers n such that $gcd(n! + 1, n + 1) = 1$.

$\sqrt{a_n}$

13. Let a be a positive integer. Define a sequence $a_0 = a$ and $a_{n+1} = a_n + \lfloor \cancel{a_n} \rfloor$ where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Prove that the sequence $a_0, a_1, a_2, \cdots$ contains a perfect square.

14. Let f_n be the right-most non-zero digit of $n!$. Prove that the sequence $f_1, f_2, f_3, \cdots$ is not periodic.

15. Let p be a prime larger than 3. Let $\frac{m}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$ where m, n are relatively prime. Prove that m is divisible by p^2. (For those who wrote the APMO last year, does this look familiar?)

16. Prove that every integer in the sequence $10001, 100010001, 1000100010001, \cdots$ is composite.

17. A *wobbly* number is a positive integer whose digits are alternating zero and non-zero with the right-most digit non-zero. Find all positive integers n such that there exists a wobbly number divisible by n.

18. Let p be a prime and let $n = 11\cdots122\cdots2\cdots99\cdots9 - 123456789$ where the dots indicate that the corresponding digit appears p times consecutively. Prove that n is divisible by p.

19. Prove that $n^{n^{n^n}} - n^{n^n}$ is divisible by 1989 for all integers $n > 2$.

20. (a) Let n be a positive integer. Prove that there exists an integer whose digits are all 0's and 1's that is divisible by n.

    (b) If n is not divisible by 2 or 5, prove that there exists an integer whose digits are all $1's$ that is divisible by n.