Concurrency and Collinearity

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1 Elementary Tools

Here are some tips for concurrency and collinearity questions:

1. You can often restate a concurrency question as a collinearity question, and vice versa. For example, proving that AB, CD and EF are concurrent is equivalent to proving that E, F and $AB \cap CD$ are collinear.

2. A common way of proving concurrency is to consider the pairwise intersections of the lines, and then show that they are the same. A common way of proving collinearity is to show that the three points form an angle of 180^o.

3. A special case of concurrency is parallel lines meeting at the point at infinity. Make sure to be mindful of this case in your solutions of contest problems.

We will be discussing several powerful tools in this lecture: Pappus', Pascal's and Desargues' Theorems. However, you should remember that in questions on concurrency and collinearity, your best friends are good old Ceva and Menelaus. They are universal, and in fact the proofs of Pascal and Desargues consist of repeated applications of Menelaus.

Theorem 1. ***(Ceva's Theorem.)*** *In $\triangle ABC$, let D, E, F be points on lines BC, AC, AB respectively. Then AD, BE, CF are concurrent iff*

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1$$

Theorem 2. ***(Ceva's Theorem, trig version.)*** *In $\triangle ABC$, let D, E, F be points on lines BC, AC, AB respectively. Then AD, BE, CF are concurrent iff*

$$\frac{\sin(\angle BAD)}{\sin(\angle CAD)} \cdot \frac{\sin(\angle ACF)}{\sin(\angle BCF)} \cdot \frac{\sin(\angle CBE)}{\sin(\angle ABE)} = 1$$

Theorem 3. ***(Menelaus Theorem.)*** *In $\triangle ABC$, let D, E, F be points on lines BC, AC, AB respectively. Then AD, BE, CF are concurrent iff*

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1$$

Problems

1. (Korea 1997) In an acute triangle ABC with $AB \neq AC$, let V be the intersection of the angle bisector of A with BC, and let D be the foot of the perpendicular from A to BC. If E and F are the intersections of the circumcircle of $\triangle AVD$ with AC and AB, respectively, show that the lines AD, BE, CF are concurrent.

2. (Iran 1998) Let ABC be a triangle and D be the point on the extension of side BC past C such that $CD = AC$. The circumcircle of $\triangle ACD$ intersects the circle with diameter BC again at P. Let BP meet AC at E and CP meet AB at F. Prove that the points D, E, F are collinear.

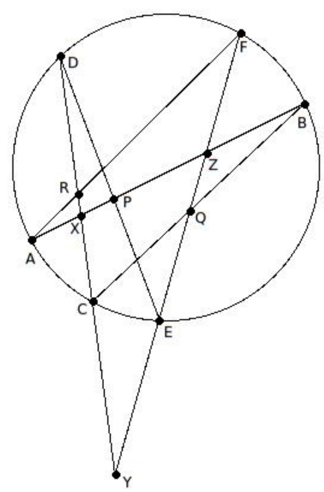
3. (Turkey 1996) In a parallelogram $ABCD$ with $\angle A < 90^o$, the circle with diameter AC meets the lines CB and CD again at E and F, respectively, and the tangent to this circle at A meets BD at P. Show that P, F, E are collinear.

4. (IMO SL 1995) Let ABC be a triangle. A circle passing through B and C intersects the sides AB and AC again at C' and B', respectively. Prove that BB', CC', and HH' are concurrent, where H and H' are the orthocenters of triangles ABC and $AB'C'$ respectively.

5. (IMO SL 2000) Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that $OD + DH = OE + EH = OF + FH$ and the lines AD, BE, and CF are concurrent.

2 Power Tools

Theorem 4. ***(Pascal's theorem)*** *Let A, B, C, D, E, F be points on a circle, in some order. Then if $P = AB \cap DE$, $Q = BC \cap EF$, $R = CD \cap FA$, then P, Q, R are collinear. In other words, if $ABCDEF$ is a cyclic (not necessarily convex) hexagon, then the intersections of the pairs of opposite sides are collinear.*



Proof. Let $X = AB \cap CD$, $Y = CD \cap EF$, $Z = EF \cap AB$. We apply Menelaus three times, to lines BC, DE, FA cutting the sides (possibly extended) of $\triangle XYZ$:

$$\frac{XB}{BZ}\frac{ZQ}{QY}\frac{YC}{CX} = -1$$
$$\frac{YD}{DX}\frac{XP}{PZ}\frac{ZE}{EY} = -1$$
$$\frac{ZF}{FY}\frac{YR}{RX}\frac{XA}{AZ} = -1$$

We multiply the three equations, and observe that by Power of a Point

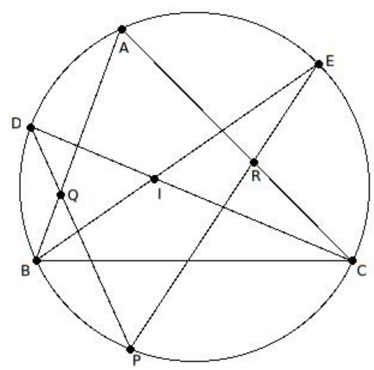
$$XA \cdot XB = CX \cdot DX,\ YC \cdot YD = EY \cdot FY,\ ZE \cdot ZF = AZ \cdot BZ,$$

so after cancellation the product becomes

$$\frac{YR}{RX}\frac{XP}{PZ}\frac{ZQ}{QY} = -1$$

which implies by Menelaus in $\triangle XYZ$ that P, Q, R are collinear. □

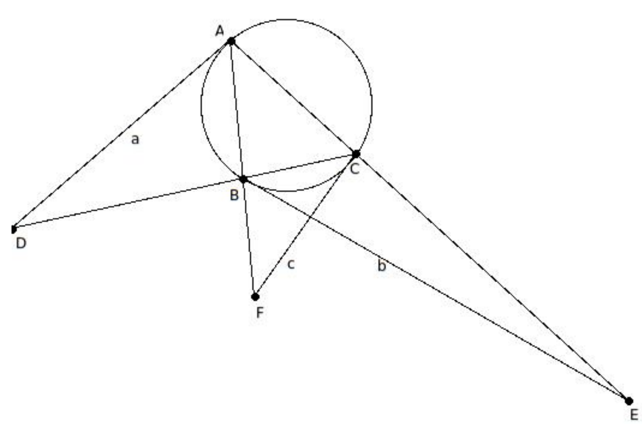
Example 1. On the circumcircle of triangle ABC, let D be the midpoint of the arc AB not containing C, and E be the midpoint of the arc AC not containing B. Let P be any point on the arc BC not containing A, $Q = DP \cap AB$ and $R = EP \cap AC$. Prove that Q, R and the incenter of triangle ABC are collinear.



Solution. Since CD and BE are angle bisectors in $\triangle ABC$, they intersect at the incenter I. Now we want to apply Pascal to points A, B, C, D, E, F in such an order that Q, R, I are the intersections of pairs of opposite sides of the resulting hexagon. This is accomplished using the hexagon $ABEPDC$. □

In many problems, you don't have a configuration with six points around a circle. But it is still possible to apply the degenerate case of Pascal's Theorem, where some of the adjacent vertices of the "hexagon" coincide. In the limiting case of vertex A approaching vertex B, the line AB becomes the tangent line at B. We illustrate this with the following simple example.

Example 2. (Macedonian MO 2001) Let ABC be a scalene triangle. Let a, b, c be tangent lines to its circumcircle at A, B, C, respectively. Prove that points $D = AB \cap c$, $E = AC \cap b$, and $F = BC \cap a$ exist, and that they are collinear.



Solution. Apply Pascal to the degenerate hexagon $AABBCC$. Then the sides (in order) are lines a, AB, b, BC, c, CA. The desired conclusion follows. $\square$

Theorem 5. ***(Pappus' Theorem)*** *Points A, C, E lie on line l_1, and points B, D, F lie on line l_2. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.*

Proof. Exercise (Use Menelaus lots of times). $\square$

Theorem 6. ***(Desargues' Theorem)*** *Given triangles ABC and $A'B'C'$, let $P = BC \cap B'C'$, $Q = CA \cap C'A'$, $R = AB \cap A'B'$. Then AA', BB', CC' concur iff P, Q, R are collinear. In other words, the lines joining the corresponding vertices are concurrent iff the intersections of pairs of corresponding sides are collinear.*

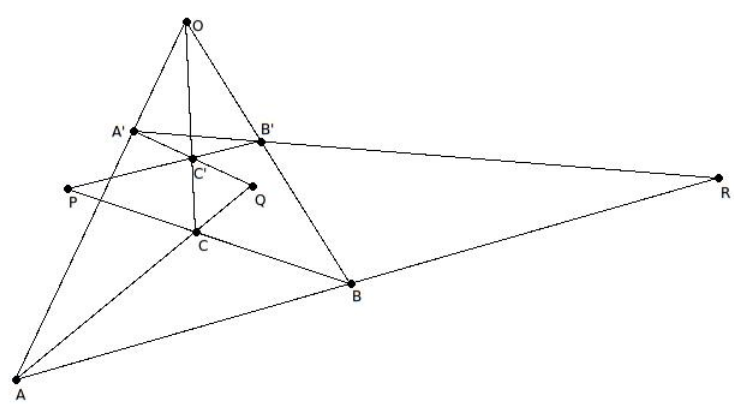
Proof. $\underline{(\Rightarrow)}$ Suppose AA', BB', CC' concur at a point O. Apply Menelaus to lines $A'B'$, $B'C'$, $C'A'$ cutting triangles ABO, BCO, CAO respectively:

$$\frac{AA'}{A'O}\frac{OB'}{B'B}\frac{BR}{RA} = -1$$
$$\frac{BB'}{B'O}\frac{OC'}{C'C}\frac{CP}{PB} = -1$$
$$\frac{CC'}{C'O}\frac{OA'}{A'A}\frac{AQ}{QC} = -1$$

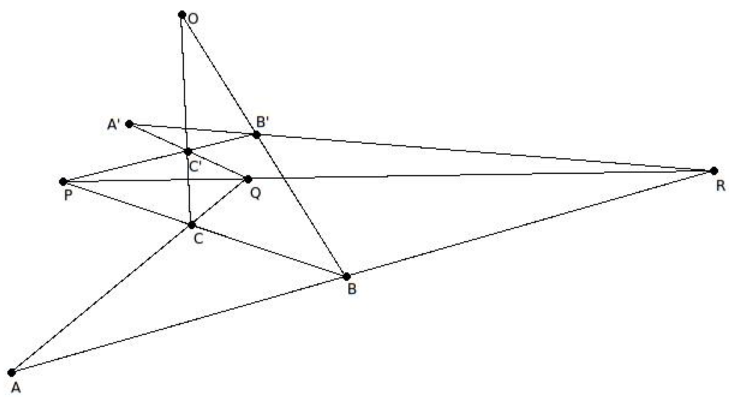
Multiplying the three equations together, we obtain

$$\frac{AQ}{QC}\frac{CP}{PB}\frac{BR}{RA} = -1$$

and so by Menelaus in triangle ABC, we have that P, Q, R are collinear.



(⇐) Suppose P, Q, R are collinear. Consider $\triangle QCC'$ and $\triangle RBB'$, and let B correspond to C, B' correspond to C', R correspond to Q. Since the lines through corresponding vertices concur, we apply (⇒) and obtain that $O = BB' \cap CC'$, $A' = QC' \cap RB'$ and $A = QC \cap RB$ are collinear. Therefore, AA', BB', CC' concur at O.



□

Example 3. In a quadrilateral ABCD, $AB \cap CD = P$, $AD \cap BC = Q$, $AC \cap BD = R$, $QR \cap AB = K$, $PR \cap BC = L$, $AC \cap PQ = M$. Prove that K, L, M are collinear.

Solution. Since AQ, BR, CP intersect at D, we apply (⇒) of Desargues' Theorem to triangles ABC and QRP. Then $K = AB \cap QR$, $L = BC \cap RP$, $M = AC \cap QP$ are collinear. □

The great thing about these theorems is that there are no configuration issues whatsoever. Pascal in particular is very versatile - given six points around a circle, the theorem can be applied to them in many ways (depending on the ordering), and some of them are bound to give you useful information.

The following problems use Pascal, Pappus, and Desargues, sometimes repeatedly or in combination. I have attempted to arrange them roughly in order of difficulty.

Problems

1. Points A_1 and A_2 that lie inside a circle centered at O are symmetric through point O. Points P_1, P_2, Q_1, Q_2 lie on the circle such that rays A_1P_1 and A_2P_2 are parallel and in the same direction, and rays A_1Q_1 and A_2Q_2 are also parallel and in the same direction. Prove that lines P_1Q_2, P_2Q_1 and A_1A_2 are concurrent.

2. (Australian MO 2001) Let A, B, C, A', B', C' be points on a circle, such that $AA' \perp BC$, $BB' \perp CA$, $CC' \perp AB$. Let D be an arbitrary point on the circle, and let $A'' = DA' \cap BC$, $B'' = DB' \cap CA$ and $C'' = DC' \cap AB$. Prove that A'', B'', C'' and the orthocenter of $\triangle ABC$ are collinear.

3. The extensions of sides AB and CD of quadrilateral $ABCD$ meet at point P, and the extensions of sides BC and AD meet at point Q. Through point P a line is drawn that intersects sides BC and AD at points E and F. Prove that the intersection points of the diagonals of quadrilaterals $ABCD$, $ABEF$ and $CDFE$ lie on a line that passes through point Q.

4. (IMO SL 1991) Let P be a point inside $\triangle ABC$. Let E and F be the feet of the perpendiculars from the point P to the sides AC and AB respectively. Let the feet of the perpendiculars from point A to the lines BP and CP be M and N respectively. Prove that the lines ME, NF, BC are concurrent.

5. Quadrilateral $ABCD$ is circumscribed about a circle. The circle touches the sides AB, BC, CD, DA at points E, F, G, H respectively. Prove that AC, BD, EG and FH are concurrent.

6. (Bulgaria 1997) Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O denote the orthocenter and circumcenter of the triangle ABC. Prove that D, O, H are collinear.

7. In triangle ABC, let the circumcenter and incenter be O and I, and let P be a point on line OI. Given that A' is the midpoint of the arc BC containing A, let A'' be the intersection of $A'P$ and the circumcircle of ABC. Similarly construct B'' and C''. Prove that AA'', BB'', CC'' are concurrent.

8. In triangle ABC, heights AA_1 and BB_1 and angle bisectors AA_2 and BB_2 are drawn. The inscribed circle is tangent to sides BC and AC at points A_3 and B_3, respectively. Prove that lines A_1B_1, A_2B_2 and A_3B_3 are concurrent.

9. (China 2005) A circle meets the three sides BC, CA, AB of a triangle ABC at points D_1, D_2; E_1, E_2; F_1, F_2 respectively. Furthermore, line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the lines AL, BM, CN are concurrent.

10. (IMO SL 1997) Let $A_1A_2A_3$ be a non-isosceles triangle with incenter I. Let ω_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (the addition of indices being mod 3). Let B_i, $i = 1, 2, 3$, be the second point of intersection of ω_{i+1} and ω_{i+2}. Prove that the circumcentres of the triangles A_1B_1I, A_2B_2I, A_3B_3I are collinear.

3 References

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