Combinatorics II

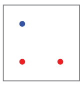
This chapter discusses Ramsey theory, graph theory and topology. The principal principle of Ramsey theory is that 'sufficiently large objects contain arbitrarily large homogeneous objects'. For example, *Ramsey's theorem* in graph theory states that one can find arbitrarily large monochromatic cliques in a sufficiently large complete graph coloured with c colours.

Gallai-Witt theorem

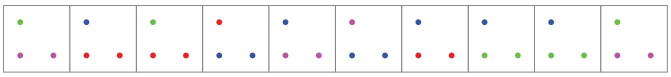
- Suppose we have a d-dimensional hypercube divided into g^d elements, each of which is coloured with one of c colours. If $g \geq G(d, c)$, where G is a function of d and c, then there exists some monochromatic (irregular) d-simplex homothetic to $\{(0, 0, 0, \ldots, 0), (1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)\}$. **[Lemma 1]**

Proof:

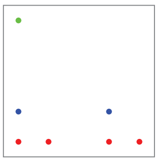
To prove this, we induct on the number of dimensions. For $d = 1$, this is trivially true by the pigeonhole principle: $G(1, c) = c + 1$, as any set of $c + 1$ elements must contain two of the same colour. We can use this as a starting point for proving the case for $d = 2$. Firstly, we can guarantee the existence of things like this, known as $(1, c, 1)$-objects:



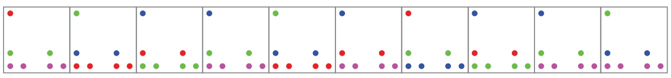
We assume that the top vertex is a different colour to either of the bottom vertices of the triangle, since otherwise we are done. We consider a strip of squares, each of size $G(1, c, 1) = G(1, c)$. They must each contain at least one $(1, c, 1)$-object, like so:



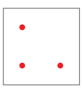
Moreover, each square can only have $c^{G(1,c,1)^2}$ possible states. Consider a row of $G\left(1, c^{G(1,c,1)^2}\right)$ such squares. At least two of them must be identical, so we can guarantee the existence of things like this, known as $(1, c, 2)$-objects:



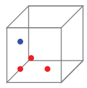
All three rows of points must necessarily have different colours. We define $G(1, c, 2) = G\left(1, c^{G(1,c,1)^2}\right) G(1, c, 1)$ to be an upper bound on the size of a box containing such an object. We then repeat the argument, considering a row of $G\left(1, c^{G(1,c,2)^2}\right)$ boxes of size $G(1, c, 2)$:



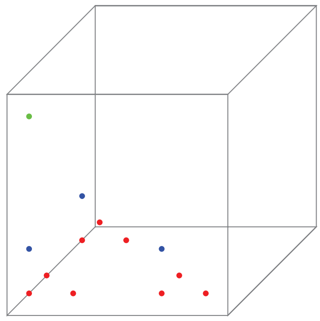
This gives us $G(1, c, f + 1) = G\left(1, c^{G(1,c,f)^2}\right) G(1, c, f)$. Now consider $(1, c, c)$-objects. They must have $c + 1$ rows of points, two of which must be the same colour by the pigeonhole principle. This means that we have a monochromatic isosceles right-angled triangle:



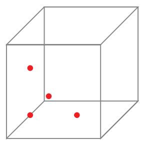
In other words, $G(2, c) = G(1, c, c)$. Now that we have tackled the two-dimensional case, we can begin work on three dimensions. We use the two-dimensional result $G(2, c, 1) = G(2, c)$ as a base case, and perform an identical inductive argument. Firstly, we can guarantee the existence of $(2, c, 1)$-objects within a box of side length $G(2, c, 1)$.



Now, consider a plane of $G\left(2, c^{G(2,c,f)^3}\right)$ such boxes. They must necessarily contain an isosceles right-angled triangle of identical boxes, by the theorem for $G(2, c)$, so we can guarantee a $(2, c, 2)$-object.



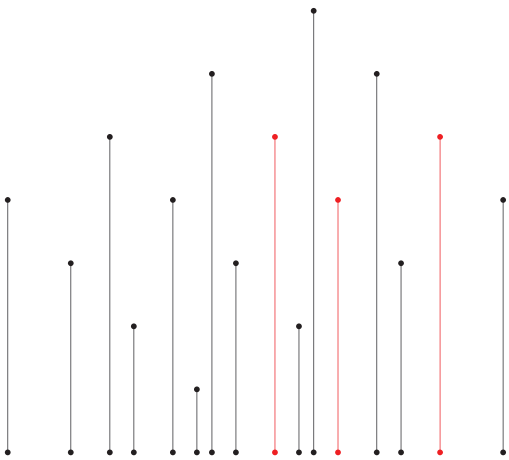
Proceeding in this manner, we can guarantee the existence of a $(2, c, c)$-object, therefore a monochromatic tetrahedron.



By repeating this argument and inducting on the number of dimensions, we can find upper bounds for $G(d, c)$ for all integers d and c. This concludes the proof of Lemma 1.

- Suppose we colour the elements of $\mathbb{Z}^n$ with c colours. Then, given a set of $d + 1$ vectors $\{0, a_1, a_2, \ldots, a_d\} \subset \mathbb{Z}^n$, we can find an integer $\lambda \in \mathbb{N}$ and vector $v \in \mathbb{Z}^n$ such that the points $\{v, v + \lambda a_1, v + \lambda a_2, \ldots, v + \lambda a_d\}$ are monochromatic. **[Gallai-Witt theorem]**

This theorem also holds if $\mathbb{Z}^n$ is replaced with $\mathbb{Q}^n$ or $\mathbb{R}^n$. To prove the Gallai-Witt theorem, we 'project' Lemma 1 from d dimensions onto a n-dimensional subplane by using a degenerate affine transformation.



For example, the existence of the monochromatic triangle in the diagram above proves the existence of a set of reals homothetic to $(0, a, b)$ on the line below. This argument generalises very easily to prove the Gallai-Witt theorem.

1. Suppose c and n are integers. Prove that there exists an integer $w = W(c, n)$ such that any c-colouring of the integers $\{1, 2, \ldots, w\}$ contains a monochromatic arithmetic progression of length n. **[Van der Waerden's theorem]**

By the pigeonhole principle, at least one of these c subsets has a 'density' $\delta \geq \frac{1}{c}$. A generalised version of van der Waerden's theorem states that if $s \geq S(\delta, n)$, then any subset $A \subset \{1, 2, \ldots, s\}$ with $|A| \geq \delta s$ must contain an arithmetic progression of length n. This is known as *Szemeredi's theorem*. Allowing s to approach infinity, we can apply this to the set of positive integers and obtain the following theorem:

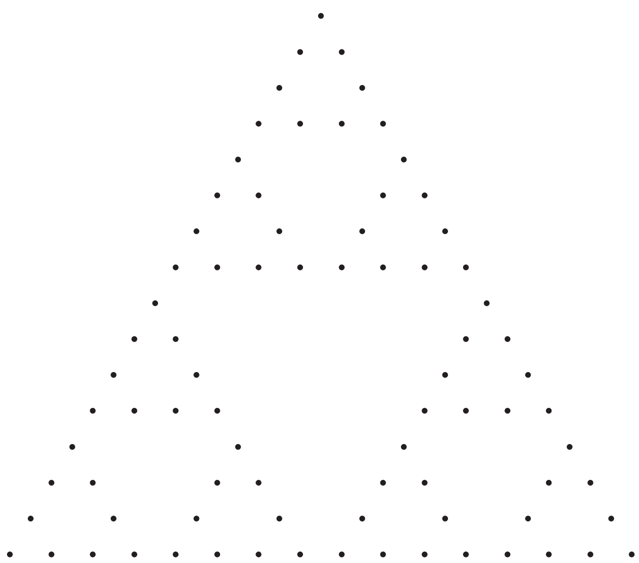
- If some subset of the positive integers has a non-zero asymptotic density, then it contains arbitrarily long arithmetic progressions.

Some other subsets of the positive integers contain arbitrarily long arithmetic progressions. Ben Green and Terence Tao proved that the prime numbers exhibit this property, despite having zero asymptotic density due to the prime number theorem. An unproven conjecture by Paul Erdős is that any set $\{a_1, a_2, a_3, \ldots\} \subset \mathbb{N}$ such that $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots$ diverges to infinity contains arbitrarily long arithmetic progressions. Of course, Szemeredi's theorem and the Green-Tao theorem are both special cases of this conjecture, since the prime harmonic series $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \ldots$ diverges (albeit very slowly).

Our argument gives very weak upper bounds for the van der Waerden numbers (minimal values of $W(c, n)$). By considering Szemeredi's theorem, Tim Gowers currently has the tightest known upper bound, which is $W(c, n) \leq 2^{2^{c^{2^{2^{n+9}}}}}$. The lower bounds are merely exponential, so very little is known about the asymptotics of van der Waerden numbers.

Hales-Jewett theorem

Observe that in no part of our proof of Lemma 1 did we use the entire $\mathbb{Z}^n$. For the two-dimensional case, we only used a set of points corresponding to approximations to a fractal known as the *Sierpinski triangle*. The Sierpinski triangle is generated by beginning with a single point, then repeatedly placing three copies of it at the vertices of an equilateral triangle and scaling by $\frac{1}{2}$. Repeating this process four times, we obtain the order-4 approximation to the Sierpinski triangle, with $3^4 = 81$ points (shown below). The Sierpinski triangle is the limit, when this process is repeated infinitely.



We can associate points in the order-3 Sierpinski triangle with words of length 3 from the alphabet $\{A, B, C\}$, like so:

CCC

CCB CCA

CBC CAC

CBB CBA CAB CAA

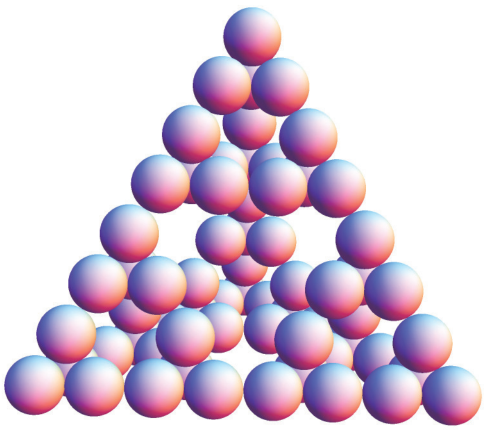
BCC ACC

BCB BCA ACB ACA

BBC BAC ABC AAC

BBB BBA BAB BAA ABB ABA AAB AAA

Suppose we introduce an additional symbol, $*$, which is considered to be a 'variable'. A word containing at least one asterisk is known as a *root*. The root $A\,B\,A * * C$ corresponds to the three words $A\,B\,A\,A\,A\,C$, $A\,B\,A\,B\,B\,C$ and $A\,B\,A\,C\,C\,C$, where $*$ successively takes on each of the three possible values. This set of three words is known as a *combinatorial line*. Note that combinatorial lines correspond to (upright) equilateral triangles of points in the Sierpinski triangle.



More generally, we can associate points in the order-h Sierpinski $(n-1)$-simplex with words from Σ^h, where Σ is an alphabet of n symbols.

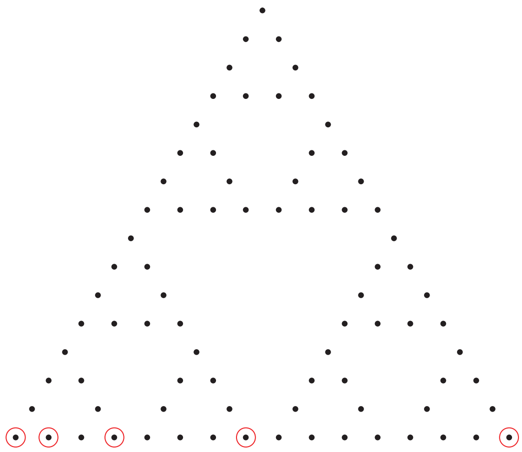
- Let Σ be an alphabet of n symbols. We colour each word of Σ^h with one of c colours. If $h \geq H(n, c)$, then there exists a monochromatic combinatorial line. **[Hales-Jewett theorem]**

So, it is equivalent to the following alternative formulation.

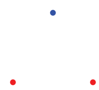
- Suppose we colour the vertices of an order-h Sierpinski $(n-1)$-simplex with c colours. If $h \geq H(n, c)$, there exists a monochromatic (upright) equilateral $(n-1)$-simplex. **[Hales-Jewett theorem]**

Proof:

The proof of Lemma 1 requires some slight refinement before it can be applied to prove Hales-Jewett. If we choose two generic points on the base of the Sierpinski triangle, then we cannot guarantee that there is a third vertex capable of completing the equilateral triangle. For example, $B\,A\,B$ and $A\,B\,A$ do not belong to a combinatorial line. So, we cannot merely apply the pigeonhole principle to the 2^h points on the base of the Sierpinski triangle. Instead, however, we *can* apply it to the $h+1$ points corresponding to 'powers of two'.



If we select any two of the $h+1$ circled vertices, there is a third point capable of completing the equilateral triangle. Applying the Pigeonhole principle, we can let $h = c$ and there must be two circled vertices of the same colour. This proves the existence of $(1, c, 1)$-objects, *i.e.* equilateral triangles with two vertices of the same colour.

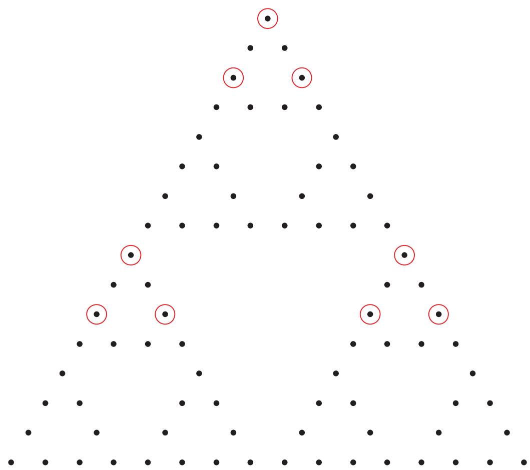


The remainder of the argument is identical to that of Lemma 1, so there is no need to repeat it here.

The Hales-Jewett theorem has a generalisation. If we allow roots with two variable symbols, such as $*$ and $\odot$, then we can create a *combinatorial* 2-*plane* by considering the set of n^2 words formed by replacing each variable with each of the letters in Σ. For example, if $\Sigma = \{A, B\}$, the root $A * B B \odot A *$ would correspond to the combinatorial 2-plane $\{A A B B A A A,\ A B B B A A B,\ A A B B B A A,\ A B B B B A B\}$. More generally, if we have roots with p variable symbols, then we generate a combinatorial p-plane of n^p words.

2. Let Σ be a finite alphabet of n symbols, and colour the words of Σ^j with c colours. Prove that if $j > J(n, c, p)$, then there exists a monochromatic combinatorial p-plane. **[Generalised Hales-Jewett]**

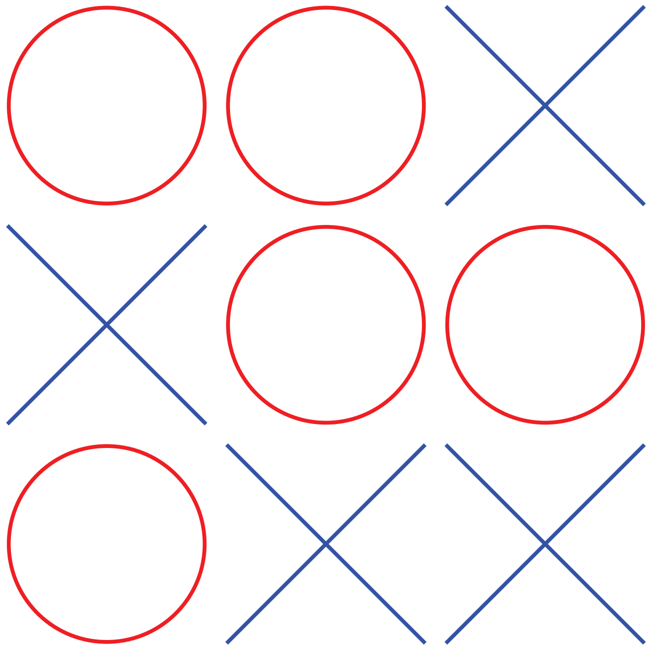
Here is an example of a combinatorial 2-plane on the order-4 Sierpinski triangle:



Just as van der Waerden's theorem has a stronger 'density' version (namely Szemeredi's theorem), so does the Hales-Jewett theorem and its generalisation. Using rather advanced methods, Furstenberg and Katznelson proved the theorem in 1991. More recently, a large collaborative effort (the *Polymath* project) led by Gowers resulted in an elementary combinatorial proof of the theorem, and thus Szemeredi's theorem and its multidimensional extension (the density version of Gallai-Witt).

Noughts and crosses

Effectively, the ordinary Hales-Jewett theorem states that in a c-player, h-dimensional game of tic-tac-toe on a board of size n, where $h \geq H(n, c)$, the game cannot terminate in a draw. Hence, one player has a winning strategy. The second player cannot have a winning strategy, as the first player can play randomly on the first move and emulate the winning strategy of the second player, knowing that owning an extra square cannot possibly be detrimental. This means that the first player can always win if the dimension is sufficiently large.



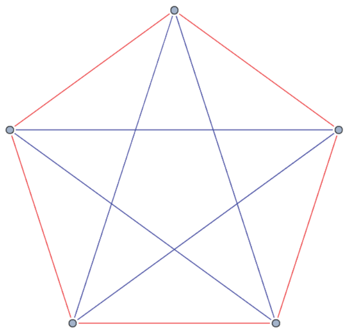
For ordinary 'noughts and crosses' where $c = 2$, $n = 3$ and $h = 2$, it is well known that neither player has a winning strategy. A typical drawing pattern is displayed above. It has been proved that $H(3, 2) = 4$, so it is impossible to draw in a four-dimensional game of tic-tac-toe. This does not necessarily mean that three-dimensional games terminate in a draw; certain diagonal lines are not considered to be combinatorial lines.

3. Does the first player have a winning strategy for $c = 2$, $n = 3$ and $h = 3$? **[Three-dimensional noughts and crosses]**

4. A game is played between two players on a 1 by 2010 grid. Taking it in turns, they place either an S or an O in an empty square. The game ends when three consecutive squares spell out $S\ O\ S$, at which point the player who has just played wins. If the grid fills up without this happening, the game is a draw. Prove that the second player has a winning strategy. **[Advanced Mentoring Scheme, November 2010, Question 2]**

Ramsey's theorem

Colour each of the edges of the complete graph K_n either red or blue. Let $R_2(r, b)$ be the smallest value of n such that there must be either a red K_r or blue K_b contained within the graph. For example, $R_2(3, 3) = 6$, as all colourings of K_6 contain a monochromatic triangle, whereas the following colouring of K_5 does not:



5. Prove that $R_2(r+1, b+1) \leq R_2(r+1, b) + R_2(r, b+1)$. **[Bicoloured Ramsey's theorem]**

This argument generalises. If we colour the edges of K_n red, blue and green, where $n \geq R_2(r, g, b)$, then there must be either a red K_r, a green K_g or a blue K_b.

A further generalisation is by considering *hypergraphs* instead of graphs. Edges can be considered to be unordered pairs of vertices; if, instead, we colour unordered sets of k vertices, we obtain a *complete k-hypergraph*. It transpires that Ramsey's theorem generalises to hypergraphs.

- Let $\{C_1, C_2, \ldots, C_c\}$ be a set of c colours. Colour each unordered k-tuple of $\{1, 2, 3, \ldots, n\}$ with one of $\{C_1, C_2, \ldots, C_c\}$. Then, if $n \geq R_k(a_1, a_2, \ldots, a_c)$, there is some $1 \leq i \leq c$ and some subset of a_i vertices, all k-tuples of which are coloured with C_i. **[Generalised Ramsey's theorem]**

Proof:

We induct on the value of k. For $k = 1$, this reduces to the pigeonhole principle: $R_1(a_1, a_2, \ldots, a_c) = 1 + (a_1 - 1) + (a_2 - 1) + \ldots + (a_c - 1)$. Suppose we are trying to prove the existence of $R_k(a_1, a_2, \ldots, a_c)$. Let $n = 1 + R_{k-1}(R_k(a_1 - 1, a_2, \ldots, a_c), R_k(a_1, a_2 - 1, \ldots, a_c), \ldots, R_k(a_1, a_2, \ldots, a_c - 1))$. Select an arbitrary vertex V. By Ramsey's theorem for $(k-1)$-hypergraphs, we can guarantee that there must be, for some colour C_i, a set of $b = R_k(a_1, a_2, \ldots, a_i - 1, \ldots, a_c)$ vertices $\{W_1, W_2, \ldots, W_b\}$ such that every unordered k-tuple containing V and $k-1$ elements of $\{W_1, W_2, \ldots, W_b\}$ is coloured with C_i. Amongst those b vertices, there must either be a set of a_j vertices, all k-tuples of which are coloured with C_j (in which case we are done), or a set of $a_i - 1$ vertices, all k-tuples of which are coloured with C_i. Consider those $a_i - 1$ vertices together with V. All k-tuples of those a_i vertices are coloured with C_i, so the inductive step is complete. As the base case $R_k(0, a_2, \ldots, a_c)$ is trivial, we are done.

6. Prove that, for every $n \geq 3$, there exists an integer $k = K(n)$ such that every set S of k points in the plane in general position must contain a convex n-gon formed from n points of S. **[Happy ending problem]**

This problem is so named as it lead to the eventual marriage of George Szekeres and Esther Klein. Klein was responsible for discovering that $K(4) = 5$, and the result was generalised by Erdős and Szekeres.

Dilworth's theorem

The case of Ramsey's theorem for two colours and ordinary graphs gives exponential bounds on the number of vertices. With the base case of $R(2, n) = n$, it is evident that $R(n, m)$ is bounded above by the binomial coefficient $\frac{(n+m)!}{n!\,m!}$. If we make additional constraints on how the edges are allowed to be coloured, then we obtain a much stronger (indeed, optimal) bound.

- Suppose we define a relation $\succeq$ on the elements of a set S, such that, for all $a, b, c \in S$:
  - $a \succeq a$; **[Reflexivity]**
  - If $a \succeq b$ and $b \succeq a$ then $a = b$; **[Antisymmetry]**
  - If $a \succeq b$ and $b \succeq c$ then $a \succeq c$; **[Transitivity]**

  Then, $\succeq$ is known as a *partial order* on the elements of S. **[Definition of partial order]**

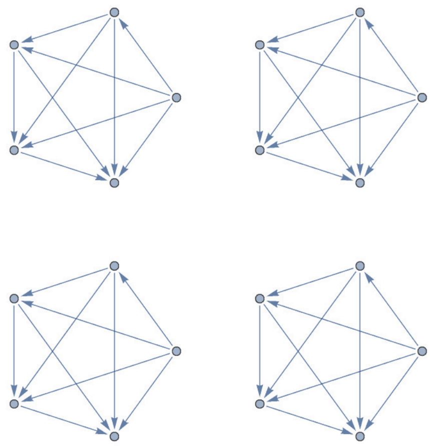
Common examples of partial orders include the relation $a \mid b$ on the set of positive integers and the relation $a \leq b$ on the set of real numbers.

- If neither $a \succeq b$ nor $b \succeq a$, then a and b are said to be *incomparable*. A set $\{c_1, c_2, \ldots, c_n\} \subset S$ such that $c_1 \succeq c_2 \succeq \ldots \succeq c_n$ is known as a *chain* of length n. A set $\{a_1, a_2, \ldots, a_m\} \subset S$ such that a_i and a_j are incomparable for all $i \neq j$ is known as an *antichain* of length m. **[Definition of chains and antichains]**

We can interpret the elements of S as the vertices of a complete graph, joined with a blue edge if the elements are incomparable and a red edge otherwise. Then, Ramsey's theorem guarantees that if $\left|S\right| \geq R(n, m) \leq \frac{(n+m)!}{n!\,m!}$, there must be either a chain of length n or an antichain of length m. However, it is possible to prove much tighter bounds than those applicable to general graphs.

7. Prove that if $|S| \geq (n-1)(m-1)+1$, then there is either a chain of length n or antichain of length m. **[Dilworth's theorem]**

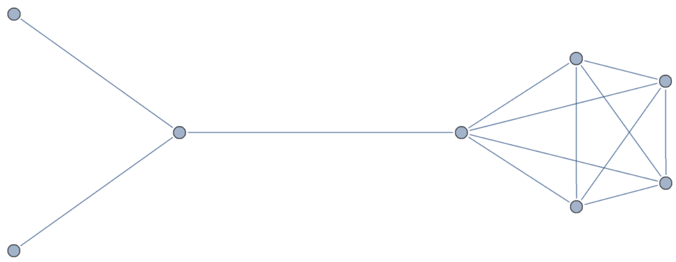
In general, $(n-1)(m-1)+1$ is much smaller than $\frac{(n+m)!}{n!\,m!}$. Note that Dilworth's theorem cannot be improved upon, as it is easy to define sets of $(n-1)(m-1)$ elements where the longest chain is length $n-1$ and the longest antichain is length $m-1$. For example, here is a partially ordered set of 20 elements where there are no chains of length 6 or antichains of length 5.



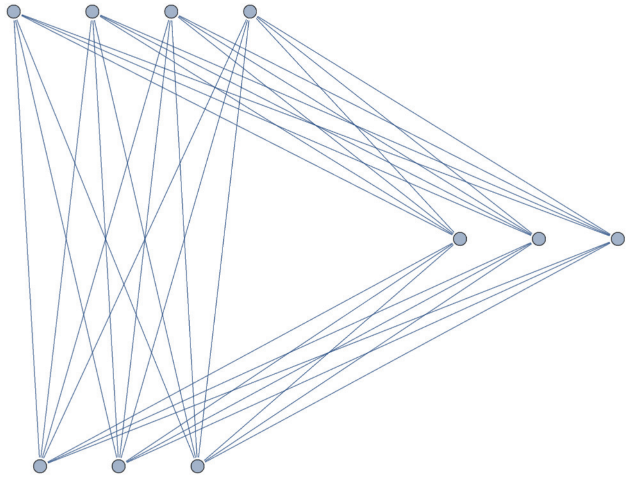
8. Show that a sequence of length $n\,m - n - m$ has either a monotonically increasing subsequence of length n or a monotonically decreasing subsequence of length m. **[Erdős-Szekeres theorem]**

Turán's theorem

Suppose we have a graph G of n vertices. If there exists a set of k vertices $K = \{A_1, A_2, \ldots, A_k\} \subseteq G$ such that every pair of vertices $A_i\,A_j$ is connected by an edge, then K is described as a *k-clique*. For example, the following graph contains a 5-clique:

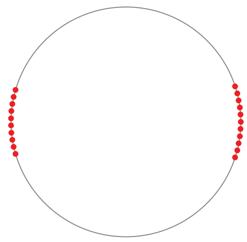


What is the maximum number of edges G can have such that there are no $(r+1)$-cliques? For $r = 1$, there can be no edges, since an edge is a 2-clique. For $r = n-1$, there can be $\binom{n}{2} - 1$ edges, since we can delete a single edge from the complete graph K_n. For other r, it turns out that the maximum number of edges is uniquely achieved by the *Turán graph* $T(n, r)$, which is constructed by partitioning the n vertices into r subsets of almost equal size (differing by at most 1) and joining two vertices if and only if they inhabit different subsets. For instance, the tetrahedron-free graph on 10 vertices with the most edges is shown below:



9. 21 apples are placed on the unit circle. Show that there are at least 100 line segments of length $\leq \sqrt{3}$ with Rosaceae endpoints. **[Ross Atkins, Trinity 2012]**

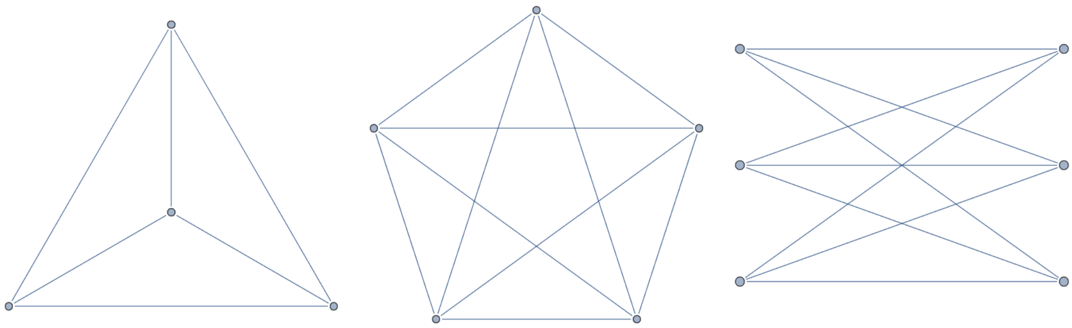
Again, this bound is attained by the configuration equivalent to the Turán graph, by separating the 21 apples into two groups of roughly equal size, situated near diametrically opposite points on the unit circle.



10. Given nine points in space, no four of which are coplanar, find the minimal natural number n such that for any colouring with red or blue of n edges drawn between these nine points there always exists a monochromatic triangle. **[IMO 1992, Question 3]**

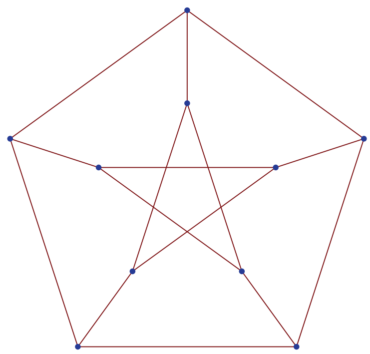
Planar graphs

We describe a graph as *planar* if it can be drawn in the plane without any edges crossing. For example, the complete graph K_4 is planar, whereas K_5 and the complete bipartite graph $K_{3,3}$ are not.

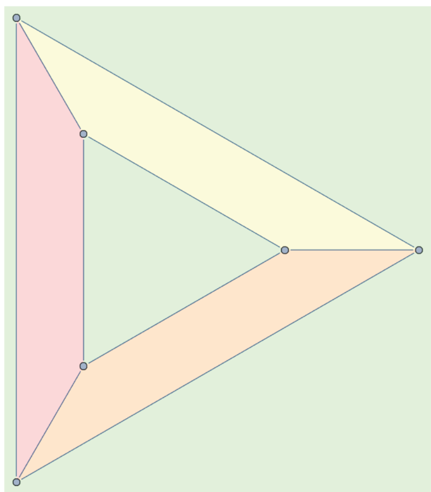


planar *non−planar* *non−planar*

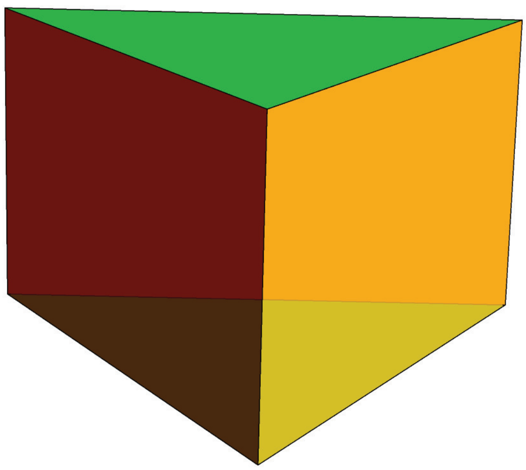
Indeed, a graph is planar if and only if it contains either K_5 or $K_{3,3}$ as a *minor*, *i.e.* can be reduced to one of these graphs by a combination of deleting edges, deleting vertices and contracting edges. For example, the (non-planar) *Petersen graph* below can be reduced to K_5 by contracting the five 'shortest' edges.



A planar graph divides the plane into well-defined regions (or *faces*). The following graph has five faces, four of which are bounded. We have used four colours such that neighbouring faces are different colours; in general, this is possible with any planar graph.



Indeed, it is best to append a point at infinity (this will become familiar to you later when we explore inversion) to convert the plane into a topological sphere. In this case, our planar graph is equivalent to the triangular prism.

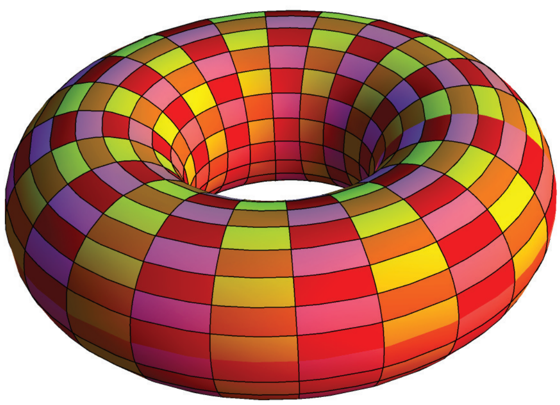


There is a useful invariant applying to graphs drawn on some surface. Suppose we have the following two elementary operations:

- A: inserting a vertex somewhere on an edge;
- B: drawing an edge between two vertices, ensuring the graph remains planar.

We are also allowed their inverse operations, A^{-1} and B^{-1}. The value of $\chi = V + F - E$ (where V, F, E are the numbers of vertices, faces and edges, respectively) is referred to as the *Euler characteristic*, and is unaffected by these elementary operations. As we can obtain all planar graphs (or, equivalently, polyhedra with no holes) from these operations, then the Euler characteristic is constant. It is a trivial exercise to verify for a simple polyhedron (such as the tetrahedron, with $(V, F, E) = (4, 4, 6)$), that the Euler characteristic must be 2.

Assuming no 'funny business' such as faces containing holes, the Euler characteristic is constant for all graphs drawn on a particular surface. Equivalently, it is constant for all polyhedra with a certain number of holes. For example, the Euler characteristic of a torus is 0. Every new hole decreases the Euler characteristic by 2, so by induction we have $\chi = 2 - 2H$, where H is the number of holes. More remarkably, an unbounded surface can be identified simply by its Euler characteristic and *orientability* (whether indirect isometries exist). For instance, the torus and *Klein bottle* are the orientable and unorientable surfaces, respectively, with an Euler characteristic of zero.



11. The *Klein quartic* is a surface topologically equivalent to a multi-holed torus, which can be tiled by 24 heptagons, where three heptagons meet at each vertex. What is its Euler characteristic, and thus how many holes does it have?

This description of the Klein quartic may remind you of the {7, 3} tiling of hyperbolic space. Indeed, it is obtained by 'rolling up' a finite patch of the hyperbolic tiling into a surface in the same way that a chessboard (finite patch of the square tiling {4, 4}) can be rolled up into an ordinary torus as in the diagram. Similarly, the Platonic solids are obtained by rolling up a finite patch (namely all) of a spherical tiling into a sphere. The Klein quartic has a symmetry group of order 336, far exceeding that of the most symmetrical Platonic solids (the icosahedron and dodecahedron, with 120 symmetries).

Solutions

1. This is a trivial corollary of Gallai-Witt in one dimension.

2. As the base case, $J(n, c, 1) = H(n, c)$. For the inductive step, we let $j = J\left(n,\ c^{n^{J(n,c,k)}},\ 1\right)$. This means that there must be a monochromatic line of identical objects, each of which must contain a monochromatic k-plane. In total, this gives us a monochromatic $(k + 1)$-plane. Note that this is a ridiculously fast-growing function.

3. Yes. Place a 'nought' in the central cube. Assume the opponent plays a 'cross' in a cube C. Choose a cube A which is not collinear with C, and place a 'nought' in the cube diametrically opposite to A. This forces your opponent to place a 'cross' in A. As the two 'crosses' are non-collinear, you now have a free move. Place a 'nought' in a position coplanar with your existing two 'noughts'. This creates two partial lines; your opponent can only block one of them.

4. Define an 'unsafe move' to be one that results in the opponent winning on the subsequent move. The only unsafe move for placing an O is $S\,\_\,\_ \rightarrow S\ O\ \_$. Define an 'unsafe square' to be one where placing either an S or an O is an unsafe move. The only unsafe squares are of the form $S\,\_\,\_\,S$, which occur in pairs due to the bilateral symmetry. Hence, there is always an even number of unsafe squares. So, if there is a nonzero number of unsafe squares, the second player has a winning strategy as her opponent must eventually place a letter in an unsafe square, resulting in a win for the second player. To force a win, therefore, she must simply create an arrangement of the form $S\,\_\,\_\,S$. Immediately after the first player moves, the second player places a S sufficiently far from the first move. If the first player tries to block by playing close to the S, simply place another S on the opposite side, resulting in $S\,\_\,\_\,S$. As soon as the first player makes an unsafe move (which he inevitably will), the second player can immediately win.

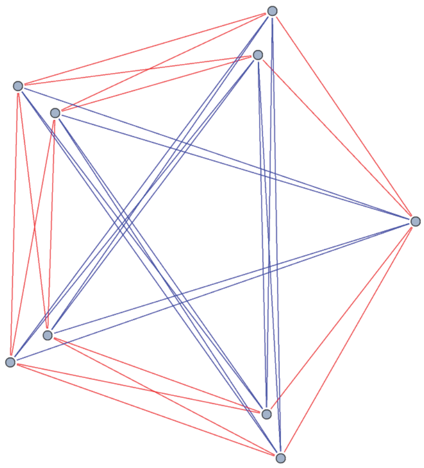
5. Consider a vertex V from the graph. It must have either at least $R(r, b + 1)$ red edges or $R(r + 1, b)$ blue edges connected to it. Assume, without loss of generality, that the former is true. Consider the set S of $R(r, b + 1)$ vertices connected to V via red edges. The subgraph on the vertices of S must then either contain a blue K_{b+1} (in which case we are done) or a red K_r. V is connected to each of the vertices of the K_r by a red edge, resulting in a red K_{r+1}.

6. It is straightforward to show that $K(4) = 5$, *i.e.* that every set of five points contains a convex quadrilateral, by considering all possible diagrams. This acts as a 'base case' to apply Ramsey's theorem for $K(n)$. Colour each 4-tuple of points blue if they are convex, or red if they are non-convex. By Ramsey's theorem for $R_4(5, n)$, there must be either a set of n points that form a convex polygon or a set of 5 points, no 4 of which form a convex polygon. Due to the base case, the latter is impossible, so the former must invariably be true.

7. Let $f : S \rightarrow \mathbb{Z}^+$ be a function mapping each element of S to the length of the longest chain terminating in S. If there are no n-chains, then the values may only range from 1 to $n - 1$. Similarly, if there are no m-antichains, then only $m - 1$ elements are allowed to take each value. So, there are at most $(n - 1)(m - 1)$ elements in S.

8. Consider the sequence $\{a_1, a_2, \ldots, a_{nm-n-m}\}$. For each a_i and a_j, we say that $a_i \geq a_j$ if both $i \geq j$ and $a_i \geq a_j$. Then, Dilworth's theorem guarantees that either a chain (increasing subsequence) or antichain (decreasing subsequence) exists.

9. Let G be the graph on 21 vertices, where two vertices share an edge if and only if they are separated by a distance greater than $\sqrt{3}$. As any three apples must form the vertices of a triangle, and one angle must be at most $\frac{\pi}{3}$, we can use the sine rule to deduce that one of the edges must be less than or equal to $\sqrt{3}$. Hence, G is triangle-free and has at most 110 edges by Turán's theorem. As we have a total of $\binom{21}{2} = 210$ pairs of apples, there are at least $210 - 110 = 100$ line segments of length $\leq \sqrt{3}$.



10. To prove that $n > 32$, consider the Turán graph above on nine vertices containing no 6-cliques. It has 32 edges. We call the five subsets of vertices A_1, A_2, A_3, A_4, A_5, and join vertices $X \in A_i$ and $Y \in A_j$ with a red edge if $(i - j) \equiv \pm 1 \pmod 5$, a blue edge if $(i - j) \equiv \pm 2 \pmod 5$, and no edge if $i = j$. To prove that $n \leq 33$, note that all graphs with 33 edges must contain a 6-clique by Turán's theorem. This 6-clique must contain a monochromatic triangle by Ramsey's theorem.

11. Each vertex is adjacent to three heptagons, and each heptagon has seven vertices. Hence, there must be $24 \times \frac{7}{3} = 56$ vertices. Similarly, each heptagon has seven edges, and each edge is adjacent to two heptagons, so there are 84 edges. The Euler characteristic is $24 + 56 - 84 = -4$, so the Klein quartic is topologically a three-holed torus.