2018 Special Camp - FE pset

1. Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.

2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x)f(y) = f(x+y) + xy$$

for all real x and y.

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a+b+c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x) + f(y) = f(x+y) \quad \text{and} \quad f(xy) = f(x)f(y)$$

for all $x, y \in \mathbb{R}$.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is the greatest integer not greater than a.

7. Let k be a real number. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$|f(x) - f(y)| \leq k(x-y)^2$$

for all real x and y.

8. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and suppose that positive integers k and c satisfy

$$f^k(n) = n + c$$

for all $n \in \mathbb{N}$, where f^k denotes f applied k times. Show that $k \mid c$.

9. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$

for every positive integer n.

10. Let S be the set of integers greater than 1. Find all functions $f : S \to S$ such that (i) $f(n) \mid n$ for all $n \in S$, (ii) $f(a) \geq f(b)$ for all $a, b \in S$ with $a \mid b$.

11. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

12. Let T denote the set of all ordered triples (p, q, r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ satisfying

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}(f(p+1,q-1,r) + f(p-1,q+1,r) & \\ +f(p-1,q,r+1) + f(p+1,q,r-1) & \\ +f(p,q+1,r-1) + f(p,q-1,r+1)) & \text{otherwise} \end{cases}$$

for all nonnegative integers p, q, r. 13. Determine all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $nf(f(n)) = f(n)^2$ for all positive integers n.

14. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

15. Find all real-valued functions f defined on pairs of real numbers, having the following property: for all real numbers a, b, c, the median of $f(a, b), f(b, c), f(c, a)$ equals the median of a, b, c.

16. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that, for all positive integer n, we have $f(f(n)) < f(n + 1)$.

17. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that, for any $w, x, y, z \in N$,

$$f(f(f(z)))f(wxf(yf(z))) = z^2 f(xf(y))f(w).$$

Show that $f(n!) \geq n!$ for every positive integer n.

18. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

19. Find all functions f from the reals to the reals such that

$$(f(a) + f(b))(f(c) + f(d)) = f(ac + bd) + f(ad - bc)$$

for all real a, b, c, d.

20. Determine all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \pmod{f(p)}$$

for all $n \in \mathbb{N}$ and all prime numbers p.

21. Let $n \geq 4$ be an integer, and define $[n] = \{1, 2, \ldots, n\}$. Find all functions $W : [n]^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,

$$\sum_{a \in A} \sum_{b \in B} \sum_{c \in C} W(a, b)W(b, c) = |A||B||C|.$$

22. Find all infinite sequences $a_1, a_2, \ldots$ of positive integers satisfying the following properties: (a) $a_1 < a_2 < a_3 < \cdots$, (b) there are no positive integers i, j, k, not necessarily distinct, such that $a_i + a_j = a_k$, (c) there are infinitely many k such that $a_k = 2k - 1$.

23. Show that there exists a bijective function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that for all $m, n \in \mathbb{N}_0$,

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n)$$

24. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(f(m) + n) + f(m) = f(n) + f(3m) + 2014$$

for all integers m and n.

25. Let $n \geq 3$ be a given positive integer. We wish to label each side and each diagonal of a regular n-gon $P_1 \ldots P_n$ with a positive integer less than or equal to r so that:

(i) every integer between 1 and r occurs as a label; (ii) in each triangle $P_iP_jP_k$ two of the labels are equal and greater than the third.

Given these conditions:

(a) Determine the largest positive integer r for which this can be done. (b) For that value of r, how many such labellings are there?

26. Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integer n. Prove that $f(n) = g(n)$ for all positive integer n.

27. Find all the functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

for all $n \in \mathbb{N}_0$. 28. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x+y)) + f(xy) = x + f(x+y) + yf(x)$$

for all real numbers x and y.

29. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \qquad \text{and} \qquad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

30. Find all functions f from $\mathbb{N}_0$ to itself such that

$$f(m + f(n)) = f(f(m)) + f(n)$$

for all $m, n \in \mathbb{N}_0$.

31. Consider a function $f : \mathbb{N} \to \mathbb{N}$. For any $m, n \in \mathbb{N}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that f has the following two properties:

(i) if $m, n \in \mathbb{N}$, then $\frac{f^n(m)-m}{n} \in \mathbb{N}$; (ii) The set $\mathbb{N} \setminus \{f(n) \mid n \in \mathbb{N}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

32. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the equation

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c$$

for all $a, b, c \geq 2$.

33. Let $2\mathbb{Z}+1$ denote the set of odd integers. Find all functions $f : \mathbb{Z} \to 2\mathbb{Z}+1$ satisfying

$$f(x + f(x) + y) + f(x - f(x) - y) = f(x+y) + f(x-y)$$

for every $x, y \in \mathbb{Z}$.