Art of Problem Solving

WOOT 2010–11

Complex Numbers in Geometry

1 Definitions

A *complex number* is a number of the form $z = a + bi$, where a and b are real numbers, and $i^2 = -1$. The numbers a and b are the *real* and *imaginary* parts of z, respectively. A number of the form bi is said to be *pure imaginary*.

Given $z = a + bi$, the complex number $\overline{z} = a - bi$ is known as the *conjugate* of z. Note that

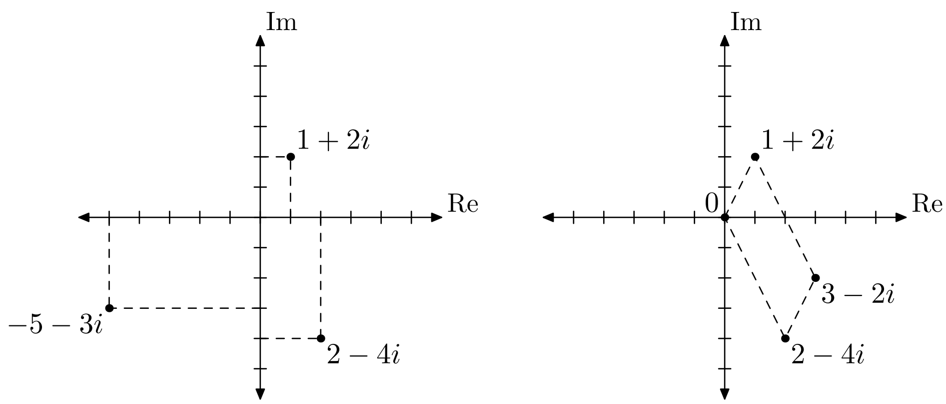
$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2.$$

Also, $\overline{z} = z$ if and only z is a real number, and $\overline{z} = -z$ if and only if z is pure imaginary. It is easy to verify that $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$ for all complex numbers z, w.

The *absolute value* (or norm, magnitude) of $z = a + bi$ is given by $|z| = \sqrt{a^2 + b^2}$, so we can also write

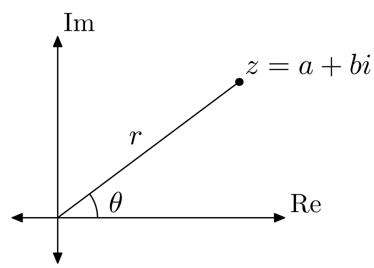
$$z\overline{z} = |z|^2.$$

We can plot complex numbers in the *complex plane*, just as we can plot points in the coordinate plane.



When we add $1 + 2i$ and $2 - 4i$, we get $3 - 2i$. If we plot the complex numbers 0, $1 + 2i$, $2 - 4i$, and $3 - 2i$, we find that they form the vertices of a parallelogram. Thus, even an operation as simple as addition has a geometric interpretation. Many geometric concepts can be elegantly described using complex numbers.

If we plot a complex number $z = a + bi$ in the complex plane, then we can measure its distance from the origin and the angle that z makes with the positive x-axis, denoting them by r and θ, respectively. Note that $r = |z|$. The angle θ is known as the *argument* of z. Thus, the parameters r and θ give us an alternative way of specifying complex numbers.



Furthermore, $a = r\cos\theta$ and $b = r\sin\theta$, so we can write

$$\begin{aligned} z &= a + bi \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$

The expression $\cos\theta + i\sin\theta$ is sometimes abbreviated as $\operatorname{cis}\theta$, so we can also write $z = r\operatorname{cis}\theta$. This form of a complex number is known as *polar form.* (The form $z = a + bi$ is known as *rectangular form.*)

Multiplying two complex numbers in polar form is simple:

$$\begin{aligned} & r(\cos\theta + i\sin\theta)\cdot s(\cos\psi + i\sin\psi) \\ &= rs(\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi) \\ &= rs[(\cos\theta\cos\psi - \sin\theta\sin\psi) + i(\cos\theta\sin\psi + \sin\theta\cos\psi)] \\ &= rs[\cos(\theta+\psi) + i\sin(\theta+\psi)]. \end{aligned}$$

Thus, the absolute value of a product is the **product** of the absolute values, and the argument of a product is the **sum** of the arguments. Hence, if we let $f(\theta) = \cos\theta + i\sin\theta$, then

$$f(\theta)f(\psi) = f(\theta+\psi) \tag{$*$}$$

for all angles θ and ψ. By a straightforward induction argument, $f(\theta)^n = f(n\theta)$ for any integer n. This gives us the following result.

Theorem. (De Moivre's Theorem) For any angle θ and integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

The only "nice" functions that satisfy $(*)$ are exponential functions, i.e. functions of the form $f(x) = b^x$. The following result confirms that the function $f(\theta)$ is of this form.

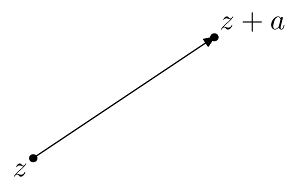
Theorem. (Euler's Formula) For any angle θ,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

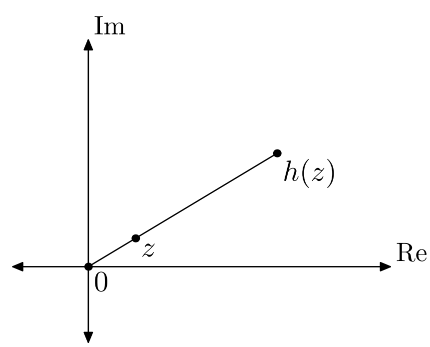
This gives us another way of expressing complex numbers, known as *exponential form*. Note that De Moivre's Theorem is an immediate consequence of Euler's Theorem.

2 Transformations

In this section, we discuss how to express geometric transformations using complex numbers, namely translation, homothety, rotation, and spiral similarity. (We will discuss reflection in the next section.) A translation corresponds to adding a fixed complex number. Hence, every translation is of the form $t(z) = z + a$, where a is the complex number we are translating by.

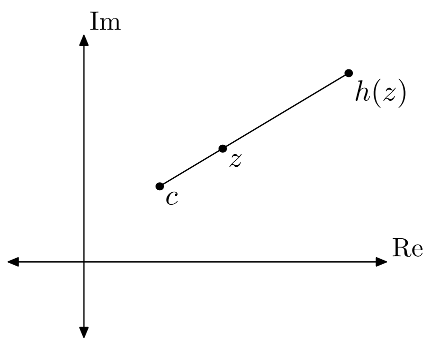


A homothety is specified by a point c (the center of homothety) and a scale factor r. If the center of homothety is the origin, then the homothety is given by $h(z) = rz$.

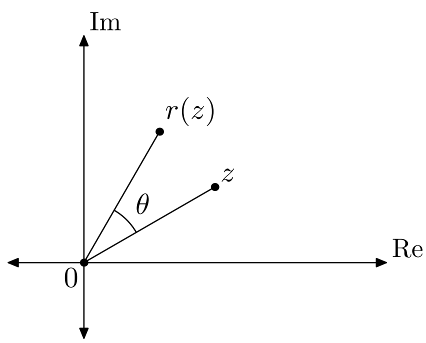


For a homothety centered at an arbitrary complex number c, we see that $h(z) - c = r(z - c)$, so

$$h(z) = r(z - c) + c.$$

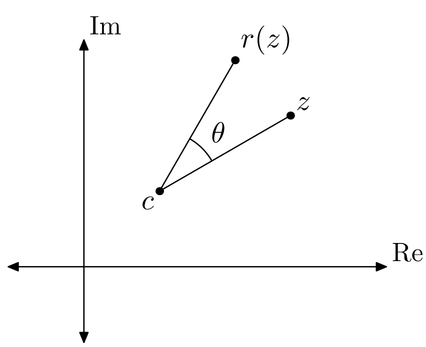


A rotation is specified by a point c (the center of rotation), an angle θ, and an orientation (clockwise or counter-clockwise). If the center of rotation is the origin, then the rotation by an angle of θ counter-clockwise is given by $r(z) = e^{i\theta}z$. (The rotation by an angle of θ clockwise is given by $r(z) = e^{-i\theta}z$. Alternatively, we can agree that the angle is positive if the rotation is counter-clockwise, and negative if the rotation is clockwise.)



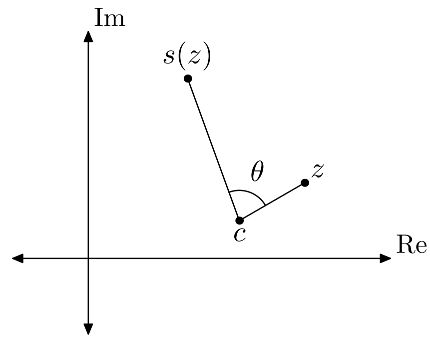
For a rotation centered at an arbitrary complex number c, we see that $r(z) - c = e^{i\theta}(z - c)$, so

$$r(z) = e^{i\theta}(z - c) + c.$$



We can combine a homothety and a rotation to obtain a transformation called a *spiral similarity*. A spiral similarity is specified by a point c (the center of the spiral similarity), a scale factor r, an angle θ, and an orientation (clockwise or counter-clockwise). From our formulas above, the spiral similarity is given by

$$s(z) = re^{i\theta}(z - c) + c.$$



We will use the convention that each point will be denoted by an upper case letter, and the affix of that point (i.e. the complex number that corresponds to that point) will be denoted by a lower case letter.

Problem. Let A, B, and C be three points. Show that triangle ABC is equilateral if and only if

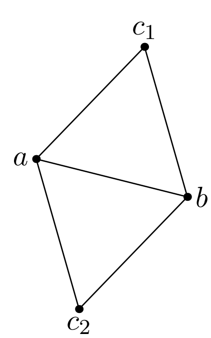
$$a^2 + b^2 + c^2 = ab + ac + bc.$$

Solution. Let $\omega = e^{i\pi/3}$. Then $\omega^3 = e^{i\pi} = -1$, so $\omega^3 + 1 = 0$, which factors as

$$(\omega + 1)(\omega^2 - \omega + 1) = 0.$$

Since $\omega \neq -1$, we have that $\omega^2 - \omega + 1 = 0$.

Given complex numbers a and b, there are only two possible locations for the complex number c, which are shown as c_1 and c_2 in the figure below:



The complex number c_1 can be obtained by rotating b around a counter-clockwise by $\pi/3$, so

$$c_1 = \omega(b-a) + a.$$

Similarly, the complex number c_2 can be obtained by rotating a around b counter-clockwise by $\pi/3$, so

$$c_2 = \omega(a-b) + b.$$

Then

$$c_1 + c_2 = \omega(b-a) + a + \omega(a-b) + b = a + b,$$

and

$$\begin{aligned} c_1c_2 &= [\omega(b-a)+a][\omega(a-b)+b] \\ &= -\omega^2(a-b)^2 + \omega a(a-b) + \omega b(b-a) + ab \\ &= (1-\omega)(a-b)^2 + \omega(a^2 - 2ab + b^2) + ab \\ &= a^2 - ab + b^2. \end{aligned}$$

Hence, by Vieta's Formulas, c_1 and c_2 are the roots of the quadratic

$$c^2 - (a+b)c + (a^2 - ab + b^2) = 0,$$

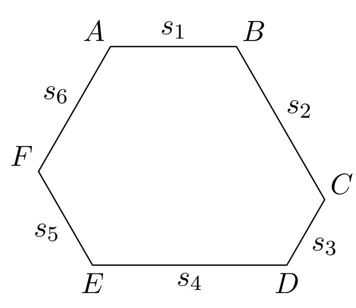
which becomes

$$a^2 + b^2 + c^2 = ab + ac + bc.$$

Problem. The angles of the convex hexagon $ABCDEF$ are equal. Prove that

$$|AB - DE| = |BC - EF| = |CD - FA|.$$

Solution. All the angles of the hexagon are equal to $120°$. Without loss of generality, we may orient hexagon $ABCDEF$ as shown below, so that the vertices are labeled clockwise, and sides AB and DE are parallel to the real axis. Let $s_1 = AB$, $s_2 = BC$, $s_3 = CD$, $s_4 = DE$, $s_5 = EF$, and $s_6 = FA$.



Then

$$\begin{aligned} b - a &= s_1, \\ c - b &= s_2 e^{5\pi i/3}, \\ d - c &= s_3 e^{4\pi i/3}, \\ e - d &= s_4 e^{3\pi i/3}, \\ f - e &= s_5 e^{2\pi i/3}, \\ a - f &= s_6 e^{\pi i/3}. \end{aligned}$$

Adding these equations, we get

$$0 = s_1 + s_2 e^{5\pi i/3} + s_3 e^{4\pi i/3} + s_4 e^{3\pi i/3} + s_5 e^{2\pi i/3} + s_6 e^{\pi/3},$$

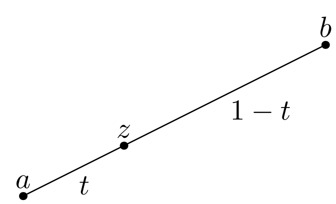
which expands as

$$\begin{aligned} 0 &= s_1 + s_2 e^{5\pi i/3} + s_3 e^{4\pi i/3} + s_4 e^{3\pi i/3} + s_5 e^{2\pi i/3} + s_6 e^{\pi/3} \\ &= s_1 + s_2 \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) + s_3 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) - s_4 + s_5 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i \right) + s_6 \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right) \\ &= \left(s_1 + \frac{1}{2} s_2 - \frac{1}{2} s_3 - s_4 - \frac{1}{2} s_5 + \frac{1}{2} s_6 \right) + \frac{\sqrt{3}}{2} (s_5 + s_6 - s_2 - s_3) i. \end{aligned}$$

In particular, the imaginary part must be 0, so $s_5 + s_6 = s_2 + s_3$, or $s_2 - s_5 = s_6 - s_3$. In other words, $BC - EF = AF - CD$, so $|BC - EF| = |AF - CD|$. By symmetry, $|AB - DE| = |BC - EF|$, and the conclusion follows.

3 Lines

Let A and B be points, with affixes a and b, respectively, and let Z be the point dividing segment AB in the ratio $t : (1 - t)$, with affix z.



We can obtain Z from B by applying the homothety centered at A with scale factor t, so

$$z - a = t(b - a) = tb - ta.$$

Hence,

$$z = (1-t)a + tb.$$

(Note that this formula holds for all real numbers t, not just those between 0 and 1.)

Solving for t, we find

$$t = \frac{z-a}{b-a}.$$

Since every point on AB corresponds to some real number t, we conclude that the complex number z lies on the line joining a and b if and only if

$$\frac{z-a}{b-a}$$

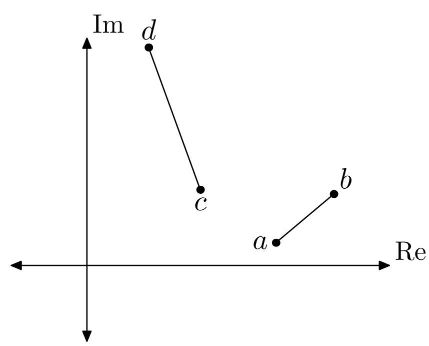
is real. Since z is a real number if and only if $z = \overline{z}$, and

$$\overline{\frac{z-a}{b-a}} = \frac{\overline{z}-\overline{a}}{\overline{b}-\overline{a}},$$

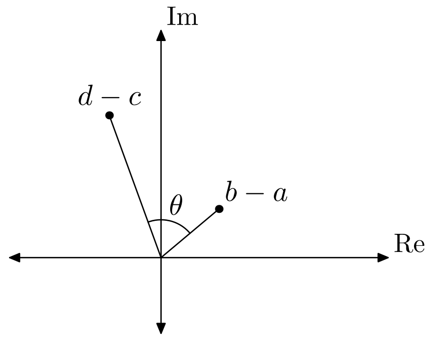
we conclude that z lies on the line joining a and b if and only if

$$\frac{z-a}{b-a} = \frac{\overline{z}-\overline{a}}{\overline{b}-\overline{a}} \quad \Leftrightarrow \quad (\overline{b}-\overline{a})z - (b-a)\overline{z} = a\overline{b} - \overline{a}b.$$

Let A, B, C, and D be four points, with affixes a, b, c, and d, respectively.



To compare AB and CD, we apply translations so that each has the origin as an endpoint. Specifically, we apply the translation that takes A to the origin to AB, so that a and b go to 0 and $b-a$, respectively, and we apply the translation that takes C to the origin to CD, so that c and d go to 0 and $d-c$, respectively.



Then we can obtain $d-c$ from $b-a$ by applying a spiral similarity centered at the origin, say with scale factor r and angle θ. Hence,

$$d-c = re^{i\theta}(b-a),$$

or

$$\frac{d-c}{b-a} = re^{i\theta} = r(\cos\theta + i\sin\theta).$$

Geometrically, $r = CD/AB$, and θ is the angle between AB and CD.

We see that AB and CD are parallel if and only if θ is a multiple of π. But θ is a multiple of π if and only if $\sin\theta = 0$. Hence, AB and CD are parallel if and only if

$$\frac{d-c}{b-a}$$

is a real number. Since z is a real number if and only if $z = \overline{z}$, we conclude that AB is parallel to CD if and only if

$$\frac{d-c}{b-a} = \frac{\overline{d}-\overline{c}}{\overline{b}-\overline{a}} \quad \Leftrightarrow \quad (b-a)(\overline{d}-\overline{c}) = (\overline{b}-\overline{a})(d-c).$$

Similarly, AB and CD are perpendicular if and only if θ is an odd multiple of $\frac{\pi}{2}$. But θ is an odd multiple of $\frac{\pi}{2}$ if and only if $\cos\theta = 0$. Hence, AB and CD are perpendicular if and only if

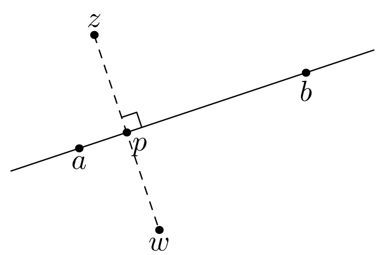
$$\frac{d-c}{b-a}$$

is pure imaginary. Since z is pure imaginary if and only if $z + \overline{z} = 0$, we conclude that AB and CD are perpendicular if and only if

$$\frac{d-c}{b-a} + \frac{\overline{d}-\overline{c}}{\overline{b}-\overline{a}} = 0 \quad \Leftrightarrow \quad (b-a)(\overline{d}-\overline{c}) + (\overline{b}-\overline{a})(d-c) = 0.$$

Problem. Let a, b, and z be complex numbers, and let w be the reflection of z in the line joining a and b. Express w in terms of a, b, and z.

Solution. Let p be the projection of z onto the line joining a and b.



Since p lies on the line joining a and b, $\frac{p-a}{b-a}$ is a real number. In other words,

$$\frac{p-a}{b-a} = \frac{\overline{p}-\overline{a}}{\overline{b}-\overline{a}}.$$

Also, the line joining z and p is perpendicular to the line joining a and b, so $\frac{p-z}{b-a}$ is pure imaginary. In other words,

$$\frac{p-z}{b-a} + \frac{\overline{p}-\overline{z}}{\overline{b}-\overline{a}} = 0.$$

We can view the two equations above as a system of equations in p and $\overline{p}$. Solving for p and $\overline{p}$, we find

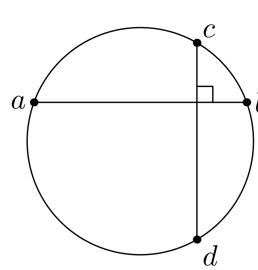
$$\begin{aligned} p &= \frac{(\overline{a}-\overline{b})z + (a-b)\overline{z} + \overline{a}b - a\overline{b}}{2(\overline{a}-\overline{b})}, \\ \overline{p} &= \frac{(a-b)\overline{z} + (\overline{a}-\overline{b})z + a\overline{b} - \overline{a}b}{2(a-b)}. \end{aligned}$$

Note that these two expressions are conjugates.

Finally, since p is the midpoint of z and w, $p = \frac{z+w}{2}$, so

$$w = 2p - z = \frac{(a-b)\overline{z} + \overline{a}b - a\overline{b}}{\overline{a}-\overline{b}}.$$

Problem. Let A, B, C, D be four points on the unit circle, such that chords AB and CD are perpendicular. Find d in terms of a, b, and c.



Solution. Since A lies on the unit circle, $|a| = 1$. Then $a\overline{a} = |a|^2 = 1$, so $\overline{a} = 1/a$. Similarly, $\overline{b} = 1/b$, $\overline{c} = 1/c$, and $\overline{d} = 1/d$.

Since AB and CD are perpendicular,

$$(b-a)(\overline{d}-\overline{c}) + (\overline{b}-\overline{a})(d-c) = 0,$$

which becomes

$$\begin{aligned} (b-a)\left(\frac{1}{d}-\frac{1}{c}\right) + \left(\frac{1}{b}-\frac{1}{a}\right)(d-c) &= 0 \\ \Rightarrow \quad \frac{(b-a)(c-d)}{cd} + \frac{(a-b)(d-c)}{ab} &= 0 \\ \Rightarrow \quad (b-a)(c-d)\left(\frac{1}{cd}+\frac{1}{ab}\right) &= 0. \end{aligned}$$

Since $b \neq a$ and $c \neq d$, we may divide both sides by $(b-a)(c-d)$, to get

$$\frac{1}{cd} + \frac{1}{ab} = 0.$$

Solving for d, we find $d = -ab/c$.

4 Roots of Unity

For a positive integer n, the set of complex numbers z satisfying $z^n = 1$ are known as the n^{th} *roots of unity*. To find these roots of unity, let $z = re^{i\theta}$. Then the equation $z^n = 1$ becomes

$$r^n e^{ni\theta} = 1.$$

Taking the absolute value of both sides, we get $|r^n e^{ni\theta}| = 1$, which simplifies to $|r|^n = 1$. Since r is positive, $r = 1$, so

$$e^{ni\theta} = 1.$$

We can rewrite this equation as

$$\cos n\theta + i \sin n\theta = 1,$$

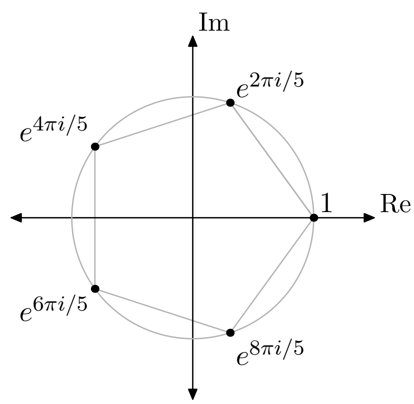
so $\cos n\theta = 1$ and $\sin n\theta = 0$. This implies $n\theta$ is a multiple of 2π, so let $n\theta = 2k\pi$. Then

$$\theta = \frac{2k\pi}{n},$$

where k is an integer. Therefore, $z^n = 1$ if and only if $z = e^{2k\pi i/n}$ for some integer k. But since $e^{2\pi i} = 1$, we can restrict our attention to the values of k where $0 \leq k \leq n-1$. Hence, the n^{th} roots of unity are

$$1,\ e^{2\pi i/n},\ e^{4\pi i/n},\ \ldots,\ e^{2(n-1)\pi i/n}.$$

In the complex plane, for $n \geq 3$, the n^{th} roots of unity form the vertices of a regular n-gon. The 5^{th} roots of unity are shown below.



Problem. Let $n \geq 3$ be a positive integer, and let $P_0P_1 \cdots P_{n-1}$ be a regular n-gon inscribed in the unit circle. Compute $P_0P_1 \cdot P_0P_2 \cdots P_0P_{n-1}$.

Solution. Let $\omega = e^{2\pi i/n}$. Without loss of generality, let the affix of P_i be ω^i. Then

$$\begin{aligned} P_0P_1 \cdot P_0P_2 \cdots P_0P_{n-1} &= |1-\omega| \cdot |1-\omega^2| \cdots |1-\omega^{n-1}| \\ &= |(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1})|. \end{aligned}$$

Since $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the roots of $x^n - 1 = 0$,

$$x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1}).$$

Dividing both sides by $x - 1$, we get

$$(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1}) = \frac{x^n-1}{x-1} = x^{n-1} + x^{n-2} + \cdots + 1.$$

Setting $x = 1$, we get

$$(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1}) = \underbrace{1+1+\cdots+1}_{n \text{ 1s}} = n.$$

Therefore, $P_0P_1 \cdot P_0P_2 \cdots P_0P_{n-1} = n$.

5 Exercises

1. Given $|z| = 1$, find $|z-1|^2 + |z+1|^2$.
2. Let a and b be distinct complex numbers. Show that z lies on the perpendicular bisector of a and b if and only if
$$(\overline{a} - \overline{b})z + (a-b)\overline{z} = |a|^2 - |b|^2.$$

3. Describe all triangles whose vertices a, b, and c satisfy

$$\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}=0.$$

4. The points $(0,0)$, $(a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of ab. (AIME, 1994)

5. Let v and w be distinct, randomly chosen roots of the equation $z^{1997}-1=0$. Let m/n be the probability that $\sqrt{2+\sqrt{3}} \le |v+w|$, where m and n are relatively prime positive integers. Find $m+n$. (AIME, 1997)

6. Regular decagon $P_1P_2\cdots P_{10}$ is drawn in the coordinate plane, with $P_1=(1,0)$ and P_6 at $(3,0)$. If P_n is the point (x_n,y_n), compute the numerical value of the product

$$(x_1+y_1i)(x_2+y_2i)(x_3+y_3i)\cdots(x_{10}+y_{10}i).$$

(ARML, 1994)

7. In the complex plane, z, z^2, z^3 form, in some order, three of the vertices of a non-degenerate square. Let a and b represent the smallest and largest possible areas of the squares, respectively. Compute the ordered pair (a,b). (ARML, 2008)

8. Let A, B, and C be three points, with respective affixes a, b, and c. Show that the signed area of triangle ABC is given by

$$\frac{i}{4}(a\overline{b}+b\overline{c}+c\overline{a}-\overline{a}b-\overline{b}c-\overline{c}a).$$

(The area is signed because this formula returns a positive real number when triangle ABC is oriented counter-clockwise, and a negative real number when triangle ABC is oriented clockwise.)