Harmonic Bundles

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DGW-HARMONIC

§1 Reading

Cross Ratios (MOP 2016), §1, §2. Or, EGMO §9.2, §9.3.

§2 Lecture notes

Lemma 2.1 (Midpoints and Parallel Lines)

Given points A and B, let M be the midpoint of $\overline{AB}$ and P_∞ the point at infinity of line AB. Then $(A, B; M, P_\infty)$ is a harmonic bundle.

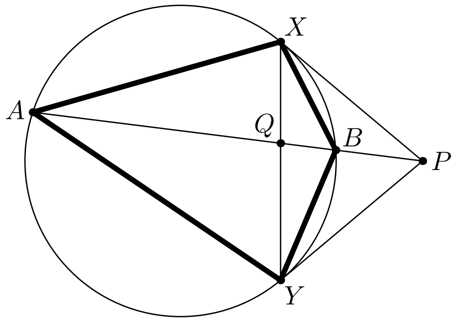
Lemma 2.2 (Harmonic Quadrilaterals)

Let γ be a nondegenerate conic, and P a point with tangents PX, PY to γ. Consider another line through P meeting γ at A and B. Then

(a) $(A, B; P, \overline{AB} \cap \overline{XY}) = -1$ and

(b) $AXBY$ is harmonic.

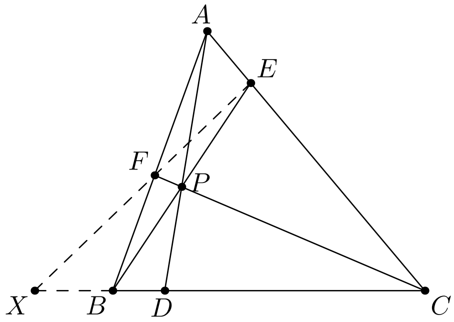
Proof. $(A, B; P, Q) \stackrel{X}{=} (A, B; X, Y) \stackrel{Y}{=} (A, B; Q, P)$. □



Lemma 2.3 (Cevians Induce Harmonic Bundles)

Let ABC be a triangle with concurrent cevians $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Line EF meets BC at X. Then $(X, D; B, C)$ is a harmonic bundle.

Proof. $(B, C; X, D) \stackrel{A}{=} \left(F, E; X, \overline{AD} \cap \overline{EF}\right) \stackrel{P}{=} (B, C; D, X)$. □



Lemma 2.4 (Right Angles and Bisectors)
Let X, A, Y, B be collinear points in that order and let C be any point not on this line. Then any two of the following conditions implies the third condition.

(i) $(A, B; X, Y)$ is a harmonic bundle.

(ii) $\angle XCY = 90^\circ$.

(iii) $\overline{CY}$ bisects $\angle ACB$.

Proof. Amounts to the fact that if a, b meeting at C are two lines and x, y are their internal/external bisectors, then $(a, b; x, y) = -1$. □

Problem 2.5 (JMO 2011/5)**.** Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$.
Prove that $\overline{BE}$ bisects $\overline{AC}$.

Problem 2.6 (Brazil 2011/5)**.** Let ABC be an acute triangle with orthocenter H and altitudes $\overline{BD}$, $\overline{CE}$. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on $\overline{BC}$.

Problem 2.7 (Taiwan TST 2014/1J/3)**.** In $\triangle ABC$ with incenter I, the incircle is tangent to $\overline{CA}$, $\overline{AB}$ at E, F. The reflections of E, F across I are G, H. Let Q be the intersection of $\overline{GH}$ and $\overline{BC}$, and let M be the midpoint of $\overline{BC}$. Prove that $\overline{IQ}$ and $\overline{IM}$ are perpendicular.

§3 Practice problems

Problem 3.1 (Canada 1994/5)**.** Let ABC be an acute triangle. Let $\overline{AD}$ be the altitude on $\overline{BC}$, and let H be any interior point on $\overline{AD}$. Lines BH and CH, when extended, intersect $\overline{AC}$, $\overline{AB}$ at E and F respectively.
Prove that $\angle EDH = \angle FDH$.

Problem 3.2 (IMO 2014/4)**.** Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on $\overline{AP}$ and $\overline{AQ}$, respectively, such that P is the midpoint of $\overline{AM}$ and Q is the midpoint of $\overline{AN}$. Prove that $\overline{BM} \cap \overline{CN}$ is on the circumference of triangle ABC.

Problem 3.3 (APMO 2013/5)**.** Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of $\overline{AC}$ such that $\overline{PB}$ and $\overline{PD}$ are tangent to ω. The tangent at C intersects $\overline{PD}$ at Q and the line AD at R. Let E be the second point of intersection between $\overline{AQ}$ and ω. Prove that B, E, R are collinear.

Problem 3.4 (TSTST 2015/2)**.** Let ABC be a scalene triangle. Let K_a, L_a and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.

Problem 3.5 (Shortlist 2004 G8)**.** Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$.

Prove that the points E, F, N are collinear, where $E = \overline{AC} \cap \overline{BD}$ and $F = \overline{BC} \cap \overline{DA}$.