The Pigeonhole Principle

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1 Introduction

There is a fundamental principle of discrete mathematics called the *pigeonhole principle*, or for those with a historical bent *Dirichlet's pigeonhole principle*.

> If n letters are distributed among m pigeonholes, and $n > m$ then some pigeonhole must contain more than one letter.

Or slightly more generally:

> If n letters are distributed among m pigeonholes, and $n > km$ then some pigeonhole must contain at least $k+1$ letters.

That's a useful name, and a reasonable, if somewhat strained metaphor but I think it sometimes limits ones horizons concerning applications. I prefer:

> *Mum's theorem*: There was enough cake to go around, and Mary didn't get her share, so someone took more than they should.

This also reminds me of the rule

> Not everyone can be of above average intelligence.

What I'm getting at is that the pigeonhole principle talks about the consequences of dividing some collection of things up into a number of categories, and about how such divisions can be arranged. But it is much more general than saying "If 25 letters are delivered into 20 pigeonholes then there must be a pigeonhole with more than one letter", or even "If 125 letters are delivered into 20 pigeonholes then there must be a pigeonhole with more than six letters". In a general context it can be applied when we know that some condition is met by the distribution, or even when dividing continuous quantities up into parts.

2 Example 1

In problem solving just like in sports it's useful to begin with a warm up – if nothing else it gives a sense of confidence. Let's try:

Thirty one balls, some black, and some white are divided into five jars. Does one of the jars necessarily contain at least four balls of the same colour?

Well of course it does. There must be a jar with at least seven balls (since the average number of balls per jar is $6\,1/5$. In a jar with at least seven balls, the colour which occurs more often (or either colour if they occur the same number of times) must occur at least four times.

Obviously the particular numbers chosen in this problem were critical. With only 30 balls, we could have five jars each with three balls of each colour in it. Problem setters are fond of choosing the numbers in pigeonhole principle problems so that they come out "just right" and often provide a clue as a result.

3 Example 2

Now something a little harder.

A disc is divided into 100 equal segments, each of which is either black or white. There are 60 black segments. Show that there is a section of 40 consecutive segments exactly 24 of which are black.

Well, let's look at the numbers again. Obviously 60% of the segments are black, and 24 is 60% of 40. So we seem to be asked to show that there is a section with exactly the average number of black segments. Why should this be so? Certainly it's not true that in any group of numbers there must be one which equals their average. For that matter, is it even true that "the average number of black segments in a 40 segment section" is the same as $40\times$ (the proportion of black segments)? Well it is true – we'll get that out of the way later, once we're sure it's helpful.

So, suppose there were no section with exactly 24 segments. They can't all have less (or the average wouldn't be 24), nor can they all have more (same reason). So some sections have fewer than 24 black segments, and some more. So what? The key in this case is the solidity of the disc – how do we get from that section with fewer to that one with more? Imagine advancing from one to the other, one segment at a time. Each time we move a segment we "lose" one segment and "gain" one segment. At most the number of black segments in the section can increase by 1. Yet it starts below 24, and finishes above. So, at some point in between we must have hit 24 segments exactly!

Let's clear up that other little issue. Why is the average number of black segments in the 40 segment sections equal to 24? Suppose that we add up the number of black segments over the 100 different 40 segment sections. We'll get some big number S. On the other hand, each black segment belongs to 40 different 40 segment sections (starting from the one where it's the furthest clockwise segment through to the one where the rest of the section is clockwise of it). So S must just be 40 times the number of black segments, that is 2400. Thus the average number of black segments per 40 segment section is indeed 24.

There's a lot more that could be done with that example. What happens say in 37 segment sections? What might we be able to do with more colours?

The last example introduced another principle of discrete mathematical proof – the *double counting argument.* We wanted to know the average number of black segments per section. We knew how many sections there were (100), and that we'd work out this average by adding up the number of black segments in each section. What we didn't know was how to find that sum directly. So we counted it in a different way (that's the double counting), namely by working out how many sections each black segment contributed to.

Another double counting example for you to work out (it's easy).

At a party, a number of the guests shake hands. Show that the total number of handshakes is one half of the sum of the number of times each guest has shaken hands.

(No self-handshakes allowed here!) In particular note that this also implies that the sum in question is necessarily even, and so there must be an even number of people who have shaken hands an odd number of times.

4 Example 3

Let's close with another classic example – where the "holes" aren't so obvious.

Ten sheep are scattered around a 90m by 90m paddock. Show that there are two sheep that are less than 45m apart.

This is a problem where it's easy to get trapped into the following sort of argument:

> The best way to distribute 9 sheep in the paddock is to put them at the corners, the midpoint of each side, and one in the middle. Then some pairs are 45m apart (but not less than 45m apart) but there is no way to add a 10th sheep.

The argument may be correct (I don't know off hand), or I should say correctable, but it is highly incomplete as it stands. The tip off is the word "best". How do we know that is the best way? If the claim is that the only way to distribute 9 sheep in a paddock without two of them being closer than 45m together is as specified, then that needs to be proven, and it hasn't been. Proofs that contain words like "in the best case" or "in the worst case" or "it can't be good to do ..." always come under heavy scrutiny, because they usually contain some fairly major holes.

Let's try a different approach. Ten sheep, and we want two of them in a "hole". So, we should make 9 holes. Nine holes in a square – how about dividing the paddock up into 30m by 30m squares. At least two of the sheep will be in one of the squares, and their distance apart is at most $30\sqrt{2}$ metres (I will allow you to claim without proof that the two points of a square which are furthest apart are diagonally opposite corners!), and $30\sqrt{2}$ is indeed less than 45.

5 Exercises

1. In a 2003 segment disk, no three consecutive segments are black. What is the maximum possible number of black segments?
2. Show that for any positive integer n there exists a multiple of n containing only the digits 3 and 0.
3. Seventy seven people each choose a different square in a 15×15 grid. Show that there are at least six of them whose squares do not share a common row or column.
4. Colour the plane with two colours. Show that some equilateral triangle has all of its vertices the same colour.

5. Let a positive integer n and a real number a be given. Find a *good* upper bound on the smallest of the numbers:

$$\langle a\rangle \ \langle 2a\rangle \ \langle 3a\rangle \ \ldots \langle na\rangle$$

where $\langle x\rangle$ denotes the fractional part of x. What, if anything, does this say about the best possible rational approximation to a with denominator less than or equal to n?[1]

[1]Dirichlet's theorem on rational approximation used, guess what?