Winter Camp 2008 Buffet Contest

Saturday, January 5, 2008

Solutions

A1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$f(xf(y) + x) = xy + f(x).$$

Solution: Putting $x = 1$, $y = -1 - f(1)$ and letting $a = f(y) + 1$, we get

$$f(a) = f(f(y) + 1) = y + f(1) = -1.$$

Putting $y = a$ and letting $b = f(0)$, we get

$$b = f(xf(a) + x) = ax + f(x),$$

so $f(x) = -ax + b$. Putting this into the equation, we have

$$a^2xy - abx - ax + b = xy - ax + b.$$

Equating coefficients, we get $a = \pm 1$ and $b = 0$, so $f(x) = x$ or $f(x) = -x$. We can easily check both are solutions.

A2. Let x, y, z be positive real numbers. Prove that

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(y+z)(y+x)}} + \frac{z}{z + \sqrt{(z+x)(z+y)}} \leq 1.$$

Solution: By Cauchy, we have $(x + y)(x + z) \geq (\sqrt{xy} + \sqrt{xz})^2$. Hence,

$$\sum_{\text{cyc}} \frac{x}{x + \sqrt{(x+y)(x+z)}} \leq \sum_{\text{cyc}} \frac{x}{x + \sqrt{xy} + \sqrt{xz}} = \sum_{\text{cyc}} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{y} + \sqrt{z}} = 1.$$

A3. Let $p(x)$ be a polynomial with integer coefficients. Does there always exist a positive integer k such that $p(x) - k$ is irreducible?

(An integer polynomial is *irreducible* if it cannot be written as a product of two nonconstant integer polynomials.)

Solution: Yes. Choose k such that the constant term of $p(x) - k$ is $-q$, where q is some large prime (to be specified later). Now, suppose $p(x) - k = f(x)g(x)$ for some nonconstant integer polynomials f, g. Since q is prime, looking at the constant term, we must have that either $|f(0)| = 1$ or $|g(0)| = 1$. Assume without loss of generality $|f(0)| = 1$. Then if $r_1, \ldots, r_k$ are the roots of f, we have $|r_1 \cdots r_k| = 1/|a| \leq 1$, where a is the leading coefficient of f. Therefore, there is some r_i such that $|r_i| \leq 1$.

Consequently, we must have that $p(r_i) - k = 0$. Let $p(x) - k = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x - q$, then

$$q = \left|\sum_{k=1}^{n} a_k r_i^k\right| \leq \sum_{k=1}^{n} |a_k||r_i|^k \leq \sum_{k=1}^{n} |a_k|.$$

Therefore, if we pick k large enough so that $q > \sum_{k=1}^{n} |a_k|$, then we have a contradiction, which means that $p(x) - k$ is irreducible.

Source: MOP 2007

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C1. Let X be a finite set of positive integers and A a subset of X. Prove that there exists a subset B of X such that A equals the set of elements of X which divide an odd number of elements of B.

First solution: We construct B in stages. Set $B = \emptyset$ and consider every number in X, starting with the largest and going down. For each element $x \in X$, see whether it divides the correct parity of elements in B. (That is, if $x \in A$, x divides an odd number of elements in B; if $x \in X - A$, x divides an even number of elements in B.) If it does not, add it to B. Thus the first element added to B is the largest element of A. Now, this procedure will not change the divisibility condition for any element greater than x, and will fulfill the condition for x. Thus when all elements of X have been examined, the divisibility conditions will be satisfied by all elements of X, and B will be as desired.

Second solution: Given B a subset of X, it is clear that there is a unique A such that A equals the set of elements of X which divide an odd number of elements of B. Now, given two distinct subsets, $B_1, B_2 \subseteq X$, let us show that the corresponding subsets $A_1, A_2 \subseteq X$ are distinct too. If B_1 and B_2 are not disjoint, then we can simply replace B_1 by $B_1 - B_1 \cap B_2$ and replace B_2 by $B_2 - B_1 \cap B_2$, as this would not change the distinctness of the two resulting A_1, A_2. So we assume that B_1 and B_2 are disjoint. Let n be the largest element in $B_1 \cup B_2$. Say that $n \in B_1$. Then $n \in A_1$ but $n \notin A_2$. It follows that $B_1 \neq B_2$ implies $A_1 \neq A_2$.

It follows that the map sending B to A is a bijection from the subsets of X to itself. Thus for every A, we can find a corresponding B.

Source: 102 Combinatorics Problems

C2. Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm1, \pm1, \ldots, \pm1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

Solution: Let S be the set of points with all coordinates equal to ±1. For each $P \in B$, let S_P be the set of points in S which differ from P in exactly one coordinate. Since there are more than $2^{n+1}/n$ points in B, and each S_P has n elements, the cardinalities of the sets S_P sum to more than 2^{n+1}, which is to say, more than twice the number of points in S. By the Pigeonhole Principle, there must be a point of S in at least three of the sets, say in S_P , S_Q, S_R. But then any two of P, Q, R differ in exactly two coordinates, so PQR is an equilateral triangle of side length $2\sqrt{2}$, by the Pythagorean Theorem.

Source: Putnam 2000

C3. Let S be a set of n points on a plane, no three collinear. A subset of these points is called *polite* if they are the vertices of a convex polygon with no points of S in the interior. Let c_k denote the number of polite sets with k points. Show that the sum

$$\sum_{i=3}^{n} (-1)^i c_i$$

depends only on n and not on the configuration of the points.

Solution: Consider the sum

$$\sum_{T \subseteq S, |T| \geq 3} (-1)^{|T|}. \tag{†}$$

This clearly does not depend on the configuration of the points, and so we may call it $f(n)$, where $n = |S|$. (It is easy to see that $f(n) = -\frac{1}{2}(n-1)(n-2)$, but we will not need this.)

Alternatively, let us calculate (†) by first grouping subsets of S into collection of subsets that share the same convex hull. If the convex hull has t vertices on the boundary, and k points of S inside, then its contribution to the sum in (†) will be

$$(-1)^t \sum_{i=0}^{k} (-1)^i \binom{k}{i},$$

which equals to zero if $k \geq 1$ and $(-1)^t$ otherwise (consider the expansion of $(1-1)^k$). In other words, if T is a subset of S whose vertices form a convex polygon, then the collection that T belongs to contributes $(-1)^t$ to the sum if T is polite and 0 otherwise. Therefore,

$$\sum_{i=3}^{n}(-1)^i c_i = \sum_{T\subset S, |T|\geq 3}(-1)^{|T|} = f(n).$$

Source: Iran 2006 Round 3

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G1. Let ABC be an acute triangle. The points M and N are taken on the sides AB and AC, respectively. The circles with diameters BN and CM intersect at points P and Q respectively. Prove that P, Q and the orthocenter H are collinear.

Solution: We need to show that H lies on the radical axis of the two circles, and thus it suffices to show that it is equal powers with respect to the two circles. Let Y and Z be the feet of the altitudes from B and C, respectively. Since $\angle BYN = 90°$, Y lies on the circle with diameter BN. Since BY passes through H, it follows that the power of H with respect to this circle is $HB \cdot HY$. Similarly, the power of H with respect to the other circle is $HC \cdot HZ$. On the other hand, $HB \cdot HY = HC \cdot HZ$ since B, C, Y, Z are concyclic. Thus, H has equal powers with respect to the two circles.

Source: Leningrad 1988

G2. Let ABC be a triangle with $AC \neq AB$, and select point B_1 on ray AC such that $AB = AB_1$. Let ω be the circle passing through C, B_1, and the foot of the internal bisector of angle CAB. Let ω intersect the circumcircle of triangle ABC again at Q. Prove that AC is parallel to the tangent to ω at Q.

Solution: Let the angle bisector of $\angle BAC$ meet BC at E and ω again at D. We have $\angle ADB_1 = \angle ADB = \angle ACB$ (this is true in both configurations) and it follows that C, B_1, E, D are concyclic. Thus $Q = D$. Let ℓ be the line tangent to ω at Q. Then we have $\angle(\ell, DA) = \angle ECD = \angle BCD = \angle BAD = \angle CAD$. It follows that ℓ is parallel to AC.

Source: Russia 2001

G3. Let OAB and OCD be two directly similar triangles (i.e., CD can be obtained from AB by some rotation and dilatation both centered at O). Suppose their incircles meet at E and F. Prove that $\angle AOE = \angle DOF$.

Solution: Let Ω_1 be the incircle of OAB and Ω_2 the incircle of OCD. Suppose that Ω_1 touches OA at X, and Ω_2 touches OC at Y. Consider an inversion about O with radius $\sqrt{|OX| \cdot |OY|}$. Suppose that Ω_1 gets sent to Ω_1' and Ω_2 gets sent to Ω_2'. Note that the choice of the radius of inversion implies that the radii of Ω_1 and Ω_2' are equal, and that the radii of Ω_1' and Ω_2 are equal. It follows that Ω_1' is the reflection of Ω_2 about the angle bisector of $\angle AOD$, and likewise with Ω_1 and Ω_2'.

Let E' denote the image of E, so that E' is an intersection point of Ω_1' and Ω_2'. Then, E' is the reflection of F across the angle bisector of $\angle AOD$. But O, E, E' are collinear. It follows that $\angle AOE = \angle AOE' = \angle DOF$.

Source: Tournament of Towns 2004 Fall

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N1. Let $n > 1$ be an odd integer. Prove that n does not divide $3^n + 1$.

Solution: Assume to the contrary that there is a positive odd integer n that divides $3^n + 1$. Let p be the smallest prime divisor of n. Then p divides $3^n + 1$; that is, $3^n \equiv -1 \pmod{p}$, so $3^{2n} \equiv 1 \pmod{p}$. By Fermat's little theorem, we also have $3^{p-1} \equiv 1 \pmod{p}$. It follows that

$$3^{\gcd(2n,p-1)} \equiv 1 \pmod{p}.$$

Since p is the smallest prime divisor of n, $\gcd(n, p-1) = 1$. Because n is odd, $p-1$ is even. Hence $\gcd(2n, p-1) = 2$. It follows that $3^2 \equiv 1 \pmod{p}$, or p divides 8, which is impossible as p is odd.

N2. Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.

Solution: Let n be the smallest positive integer such that $\gcd(s, n) > 1$ for all s in n; note that n has no repeated prime factors. By the condition on S, there exists $s \in S$ which divides n.

On the other hand, if p is a prime divisor of s, then by the minimality of n, n/p is relatively prime to some element t of S. Since n cannot be relatively prime to t, t is divisible by p, but not by any other prime divisor of s (any such prime divides n/p). Thus $\gcd(s, t) = p$, as desired.

Source: Putnam 1999

N3. Let a positive integer k be given. Prove that there are infinitely many pairs of integers (a, b) with $|a| > 1$ and $|b| > 1$ such that $ab + a + b$ divides $a^2 + b^2 + k$.

Solution: By inspection, we see that if $(a, b) = (0, 1)$, then $a^2 + b^2 + k$ is divisible by $ab + a + b$:

$$\frac{a^2 + b^2 + k}{ab + a + b} = k + 1. \qquad (\dagger)$$

Let us rearrange this as a quadratic in a:

$$a^2 - (k + 1)(b + 1)a + (b - 1)(b - k) = 0.$$

As a quadratic in a, the sum of the roots is $(k + 1)(b + 1)$. Hence, if (a, b) is a solution to ($\dagger$), then so is $((k + 1)(b + 1) - a, b)$, and hence by symmetry of ($\dagger$), so is $(b, (k + 1)(b + 1) - a)$.

Define a sequence (a_n) as follows: $a_1 = 0$, $a_2 = 1$, and

$$a_n = (k + 1)(a_{n-1} + 1) - a_{n-2}$$

for all $n \geq 2$. Then by the above reasoning, $(a, b) = (a_n, a_{n+1})$ is a solution of ($\dagger$) for all $n \geq 0$. Furthermore, the sequence (a_n) is increasing, giving an infinite number of positive integer solutions.