$$P(x, y) \Longrightarrow f(f(xy)) = f(xf(y) + yf(y))$$

In below, **n:** means **Case n:** and **n.m:*Statement*** means **in step n.m, we'll prove/find *Statement***.

And we'll use the fact f is continuous many times without saying it.

We'll also use the easy-to-check result that, if f is constant in (a, b), then it is constant in $[a, b]$.

Define:

$S_t = \left\{ \frac{f(x)}{x} + \frac{f\left(\frac{t}{x}\right)}{\frac{t}{x}} \middle| x \in \mathbb{R}^+ \right\}$

$a_t = \max \{a : x \geq a \text{ for all } x \in S_t\}$

$b_t = \min \{a : x \leq a \text{ for all } x \in S_t\}$

(for clarity, if $S_t = [a_t, b_t]$)

By definition, f is constant in each interval S_t as, $f(f(t)) = f(s) \forall s \in S_t$

0: $f(x) = c$ for all $x \in \mathbb{R}^+$

1: $\max \left\{ \frac{f(x)}{x} \middle| x \in \mathbb{R}^+ \right\} = \infty$

By definition of b_t, we have $b_t = \infty$ for all $t > 0$

Define $r = \max \{r' : 2\,x\,f(x) \geq r' \text{ for all } x \in \mathbb{R}^+\}$

We have, $f(x) \geq \frac{r}{2\,x}$ for all $x \in \mathbb{R}^+$

1.1: f is constant in the interval $[r, \infty)$

$b_t = \infty$ for all $t \in \mathbb{R}^+$, $a_t \leq 2\,t\,f(t)$ so, f is constant in the interval $[2\,t\,f(t), \infty)$

As, by definition of r, $(\forall \varepsilon > 0)(\exists t)$ such that, $2\,t\,f(t) - r < \varepsilon$.

Take ε arbitrarily small, we have, f is constant in the interval $[r, \infty)$

$r = 0$ is same as **Case 0**, so let, $r > 0$

Define $a = \min \{a' : f(x) = c \text{ for all } x \geq a'\}$

Obviously, $a \leq r$

1.2: $f(f(t)) = c$ for all $t > 0$

$f(f(t)) = f\left(2\sqrt{t}\, f\left(\sqrt{t}\right)\right) = c$ as $2\sqrt{t}\, f\left(\sqrt{t}\right) \geq r \geq a$

1.3: $c \geq a$, $f(x) \geq \max\left(a, \frac{r}{2\,x}\right)$ for all $x > 0$

we have $\lim_{x \longrightarrow 0} f(x) \geq \lim_{x \longrightarrow 0} \frac{r}{2\,x} = \infty$

Let, $\mathcal{F} = \{f(x) \,|x > 0\}$

but, by 1.2, $f(x) = c$ for all $x \in \mathcal{F}$

thus, minimality of a implies, $z \geq a$ for all $z \in \mathcal{F}$

So, $c = f(a) \geq a$

By definition of r, we had, $f(x) \geq \frac{r}{2\,x}$

So, $f(x) \geq \max\left(a, \frac{r}{2\,x}\right)$

1.4: $2\,c^2 \geq 2\,c\,a \geq r \geq a$

we proved, $c \geq a$ and $r \geq a$, it remains to prove, $2\,c\,a \geq r$,

which is obvious since, $2\,c\,a = 2\,a\,f(a) \geq r$

1.∞: Finding solutions

we have,

i. $f(x) = c \geq \max\left(a, \frac{r}{2x}\right) = a$ for all $x \geq a$

ii. $f(x) \geq \max\left(a, \frac{r}{2\,x}\right) = a$ for all $c \geq x \geq \frac{r}{2\,a}$

iii. $f(x) \geq \max\left(a, \frac{r}{2\,x}\right) = \frac{r}{2\,x}$ for all $\frac{r}{2\,a} \geq x$

Now, define, $h\colon (0, a] \longrightarrow \mathbb{R}$, such that,

$h(x) = f(x) - a$ for all $c \geq x \geq \frac{r}{2\,a}$ and,

$h(x) = f(x) - \frac{r}{2\,x}$ for all $\frac{r}{2\,a} \geq x$

It's easy to check that h is continuous, $h(a) = c - a$ and $h(x) \geq 0$ for all $x \in (0, a]$

Where, $2\,c^2 \geq 2\,c\,a \geq r \geq a$ and $h\colon (0, a] \longrightarrow [0, \text{infty})$ is any continuous function with $h(a) = c - a$.

Checking:

We proved, $f\,(f\,(x\,y)) = c$. So it is sufficient to prove, $x\,f(y) + y\,f(x) \geq a$ for all $x, y \in \mathbb{R}^+$

Indeed, $f(t) \geq \frac{r}{2\,t}$ for all $t > 0$.

So, $x\,f(y) + y\,f(x) \geq \frac{r\,x}{2\,y} + \frac{r\,y}{2\,x} \geq r \geq a$ by A.M-G.M inequality. Hence f is a valid solution.

(Note: **Case 0** is nothing but $a = 0$)

2: $\max\left\{\frac{f(x)}{x}\middle| x \in \mathbb{R}^+\right\}$ is finite

Let, $M = \min\left\{M'\middle|\frac{f(x)}{x} \leq M' \text{for all} x \in \mathbb{R}^+\right\}$,

$m = \max\left\{m'\middle|\frac{f(x)}{x} \geq m' \text{for all} x \in \mathbb{R}^+\right\}$

2.1: $M + m \in S_t$ for all $t \in \mathbb{R}^+$

from the definition of M and m, we have,

$(\forall \varepsilon > 0)\,(\exists u, v, \delta, \varphi \in \mathbb{R}^+, \delta, \varphi \leq \varepsilon)$ such that $\frac{f(u)}{u} = m + \delta$ and $\frac{f(v)}{v} = M - \varphi$

So, $a_t \leq \frac{f(u)}{u} + \frac{f\left(\frac{t}{u}\right)}{\frac{t}{u}} \leq M + m + \delta$ and $b_t \geq \frac{f(v)}{v} + \frac{f\left(\frac{t}{v}\right)}{\frac{t}{v}} \geq M + m - \varphi$

as, δ and φ can be arbitrarily small, we have, indeed, $a_t \leq M + m \leq b_t$

So, $M + m \in S_t$

2.2: $m = 0$, $f(f(t)) = f\,(M t)$

By definition of M, $(\forall \varepsilon > 0)\,(\exists x \in \mathbb{R}^+)$ such that, $\frac{f(x)}{x} = M - \varepsilon$

Now, we have, $M^2 \frac{x}{M + m} \geq f\left(f\left(\frac{x}{M + m}\right)\right) = f(x) = (M - \varepsilon)\,x$

$\Longrightarrow \varepsilon \geq \frac{m\,M}{m + M}$

As ε can be arbitrarily small, we have $m = 0$

So, $f(f(t)) = f\,(M t)$

2.3: $M \geq 1$ and $\exists L', c \in \mathbb{R}^+$ such that $f(x) = c \forall x \geq L'$

$P(t, t) \Longrightarrow 2\,t\,f(t) \in S_{t^2} \Longrightarrow a_{t^2} \leq 2\,t\,f(t) \leq b_{t^2}$

As, $m = 0$, we have,

$(\forall \varepsilon > 0)\,(\exists x, \delta, \varepsilon \geq \delta \geq 0)$ such that, $f(x) = \delta\,x \leq \varepsilon\,x$

we have, $f\,(M x) = f(f(x)) = f\,(\delta\,x) \leq M\,\delta\,x \leq M\,\varepsilon\,x$

So, by induction, $f\,(M^n\,x) \leq M^n\,\varepsilon\,x$

Take, ε small enough such that, $2\,\varepsilon < M$

$a_{M^{2n} x^2} \leq 2\,M^n\,x\,f\,(M^n\,x) \leq 2\,M^{2n}\,x^2\,\varepsilon < M^{2n+1}\,x^2 \leq b_{M^{2n} x^2}$

$\Longrightarrow [2\,\varepsilon\,M^{2n}\,x^2, M^{2n+1}\,x^2] \subseteq S_{M^{2n} x^2}$

Define, $X = \bigcup_{n \geq 0}\,[2\,\varepsilon\,M^{2n}\,x^2, M^{2n+1}\,x^2]$

if $M < 1$, take ε small enough such that, $2\,\varepsilon \leq M^3 \Longrightarrow X = (0, M\,x^2]$

as, f is constant in each interval $[2\,\varepsilon\,M^{2n}\,x^2, M^{2n+1}\,x^2]$, $f(x) = c > 0$ for all $x \in X$

So, $\lim_{x \longrightarrow 0^+} f(x) = c$, but $\lim_{x \longrightarrow 0^+} f(x) \leq \lim_{x \longrightarrow 0^+} M\,x = 0 \Longrightarrow c = 0$,

a contradiction!

So, $M \geq 1$, and take ε small enough such that, $2\,\varepsilon\,M \leq 1 \Longrightarrow X = [2\,\varepsilon\,x^2, \infty)$

as, f is constant in each interval $[2\,\varepsilon\,M^{2n}\,x^2, M^{2n+1}\,x^2]$, $f(x) = c$ for all $x \in X$

Take, $L' = 2\,\varepsilon\,x^2$ and we are done!

Define $L = \min\,\{L' \colon f(x) = c \text{ for all } x \geq L'\}$

2.4: $S_{\frac{L}{M}} = \{M\}$ i.e. $\frac{f(x)}{x} + \frac{f\left(\frac{L}{M\,x}\right)}{\frac{L}{M\,x}} = M$ for all x

Consider the functions $g_x(t) = \frac{t}{M}\left(\frac{f(x)}{x} + \frac{f\left(\frac{t}{M\,x}\right)}{\frac{t}{M\,x}}\right) > \frac{t}{M}\,\frac{f(x)}{x}$ for all x

Note that,

$$f\left(f\left(\frac{r}{M}\right)\right) = f(r) = f\left(\frac{r}{M}\left(\frac{f(x)}{x} + \frac{f\left(\frac{r}{M\,x}\right)}{\frac{r}{M\,x}}\right)\right) = f(g_x(r))$$

$$f\left(f\left(\frac{R}{M}\right)\right) = f(R) = f\left(\frac{R}{M}\left(\frac{f(x)}{x} + \frac{f\left(\frac{R}{M\,x}\right)}{\frac{R}{M\,x}}\right)\right) = f(g_x(R))$$

Now, by definition of L, we have $(\forall \varepsilon > 0)\,(\exists r < L, L - r < \varepsilon)$ such that, $f(r) \neq c$

So, we have, $g_x(r) < L$

As we can take ε arbitrarily small, we have $g_x(L) \leq L$

Let, $g_x(L) < L$

Note that, for sufficiently large $R' > L$, $g_x(R') > \frac{R'}{M}\frac{f(x)}{x} \geq L$

So, as g is continuous (which can easily be checked), there must be some $R'' > L$, such that, $g_x(R'') = L$, and let, $R = \min\{R''\}$ (existence of R can easily be checked)

Then, we have $g_x(t) < L$ for all $L \leq t < R$

But then, $f(g_x(t)) = f(t) = c$ for all $L \leq t$

Let, $g_x(L) = L_0 < L$, $T = \{g_x(t) \colon t \geq L\} \Longrightarrow [L_0, \infty) \subseteq T \Longrightarrow f(t) = c$ for all $t \geq L_0$, contradicting the minimality of L.

So, $g_x(L) = L \Longrightarrow \frac{f(x)}{x} + \frac{f\left(\frac{L}{M\,x}\right)}{\frac{L}{M\,x}} = M$

2.∞: No solution here!

$x \leq \frac{1}{M} \Longrightarrow \frac{L}{M\,x} \geq L$

So, $\frac{f(x)}{x} + \frac{f\left(\frac{L}{M\,x}\right)}{\frac{L}{M\,x}} = M \Longrightarrow f(x) = M\,x - \frac{c\,M\,x^2}{L} < M\,x$ for all $x \leq \frac{1}{M}$

Now, take, $x \leq \frac{1}{M^2} \leq \frac{1}{M} \Longrightarrow f(x) \leq M\,x \leq \frac{1}{M}$

So, $f(f(x)) = f\,(M\,x) \Longrightarrow M\,f(x) - \frac{c\,M\,f(x)^2}{L} = f(f(x)) = f\,(M\,x) = M^2\,x - \frac{c\,M^3\,x^2}{L}$ for all $x \leq \frac{1}{M^2}$

$\Longrightarrow (f(x) - Mx)\left(f(x) + Mx - \frac{L}{c}\right) = 0$

$\Longrightarrow f(x) = \frac{L}{c} - M\,x = M\,x - \frac{c\,M\,x^2}{L}$ for all $x \leq \frac{1}{M^2}$

$x \longrightarrow 0^+ \Longrightarrow \frac{L}{c} = 0 \Longrightarrow L = 0$, Contradiction!

Hence, no solution is this case.

So, the solutions are:

$$f(x) = \begin{cases} c & \text{if } x \geq a \\ a + h(x) & \text{if } a \geq x \geq \frac{r}{2\,a} \\ \frac{r}{2\,x} + h(x) & \text{if } \frac{r}{2\,a} \geq x \end{cases}$$

Where, $2\,c^2 \geq 2\,c\,a \geq r \geq a \geq 0$ and $h \colon (0, a] \longrightarrow [0, \infty)$ is any continuous function with $h(a) = c - a$.