Spiral Similarity and Miquel Points

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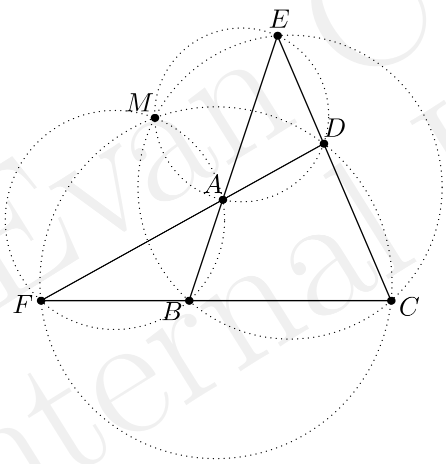
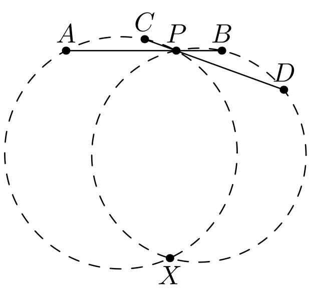
§1 Lecture Notes

Spiral similarity lemma, and Miquel points.

Theorem 1.1 (The Miquel Point)
Let $ABCDEF$ be a complete quadrilateral, with E, F as shown. Then the circumcircles of triangles EAD, EBC, FAB, FCD are concurrent at the Miquel point M.



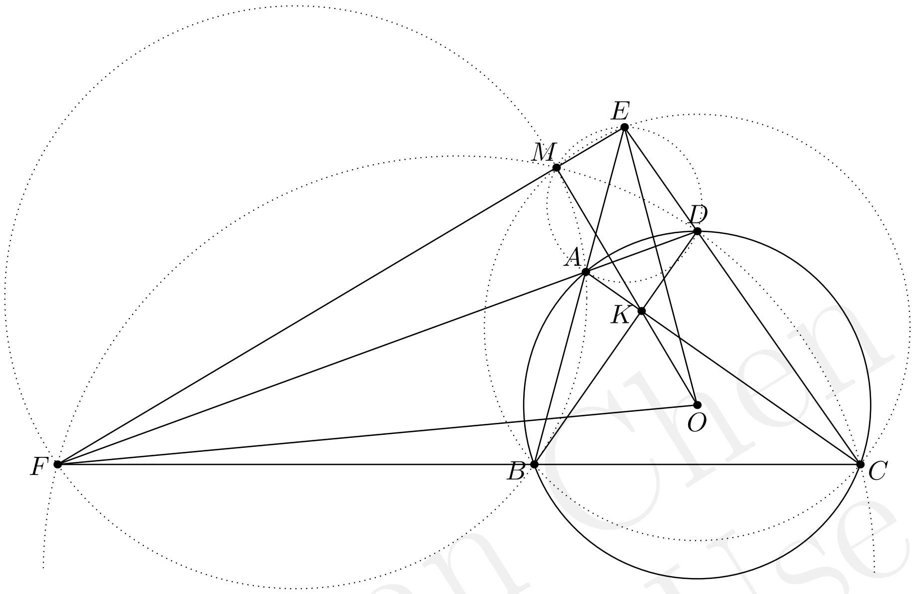
Theorem 1.2 (The Spiral Similarity Center)
Consider arbitrary AB and CD as shown. X is the center of two spiral similarities ($AB \mapsto CD$ and $AC \mapsto BD$).



The most interesting special case is when $ABCD$ is cyclic. See `http://yufeizhao.com/olympiad/cyclic_quad.pdf` for the authoritative reference on this situation.

Theorem 1.3 (Miquel Points of Cyclic Quadrilaterals)

Let $ABCDEF$ be a complete quadrilateral as shown and let K be the intersection of the diagonals. If $ABCD$ is cyclic with center O, then K is the orthocenter of triangle OEF and $M = \overline{OK} \cap \overline{EF}$ is the Miquel point. The point M also lies on the circumcircles of triangles AOC and BOD.



Problem 1.4 (Brazil 2011/5)**.** Let ABC be an acute triangle with orthocenter H and altitudes $\overline{BD}$, $\overline{CE}$. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on $\overline{BC}$.

Problem 1.5 (Shortlist 2015 G3)**.** Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q.

Prove that the lines CQ and AD meet on ω.

§2 Problems

Problem 2.1 (IMO 1985/5)**.** Let $ABCD$ be a cyclic quadrilateral with center O. Lines AB and CD meet at P, while lines AD and BC meet at Q. The circumcircles of triangles PAB and PDC meet at M. Prove that $\angle OMP = 90^\circ$.

Problem 2.2 (USAMO 2013/1)**.** In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively, prove that $YX/XZ = BP/PC$.

Problem 2.3 (Russia 1995 et al)**.** Quadrilateral $ACDB$ is inscribed in a semicircle with diameter AB and point O is the midpoint of AB. Let K be the intersection of the circumcircles of AOC and BOD. Lines AB and CD intersect at M. Prove that $\angle OKM = 90^\circ$.

Problem 2.4 (USAMO 2006/6)**.** Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

Problem 2.5 (TSTST 2012/7)**.** Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

Problem 2.6 (IMO 2005/2)**.** Let $a_1, a_2, \dots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \dots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \dots$.