Algebra

A1. Find all monic integer polynomials $p(x)$ of degree two for which there exists an integer polynomial $q(x)$ such that $p(x)q(x)$ is a polynomial having all coefficients ± 1.

First solution. We show that the only polynomials $p(x)$ with the required property are $x^2 \pm x \pm 1$, $x^2 \pm 1$ and $x^2 \pm 2x + 1$.

Let $f(x)$ be any polynomial of degree n having all coefficients ± 1. Suppose that z is a root of $f(x)$ with $|z| > 1$. Then

$$|z|^n = | \pm z^{n-1} \pm z^{n-2} \pm \cdots \pm 1| \le |z|^{n-1} + |z|^{n-2} + \cdots + 1 = \frac{|z|^n - 1}{|z| - 1}.$$

This leads to $|z|^n(|z| - 2) \le -1$; hence $|z| < 2$. Thus, all the roots of $f(x) = 0$ have absolute value less than 2.

Clearly, a polynomial $p(x)$ with the required properties must be of the form $p(x) = x^2 + ax \pm 1$ for some integer a. Let x_1 and x_2 be its roots (not necessarily distinct). As $x_1 x_2 = \pm 1$, we may assume that $|x_1| \ge 1$ and $|x_2| \le 1$. Since x_1, x_2 are also roots of $p(x)q(x)$, a polynomial with coefficients ± 1, we have $|x_1| < 2$, and so $|a| = |x_1 + x_2| \le |x_1| + |x_2| < 2 + 1$. Thus, $a \in \{\pm 2, \pm 1, 0\}$.

If $a = \pm 1$, then $q(x) = 1$ leads to a solution.

If $a = 0$, then $q(x) = x + 1$ leads to a solution.

If $a = \pm 2$, both polynomials $x^2 \pm 2x - 1$ have one root of absolute value greater than 2, so they cannot satisfy the requirement. Finally, the polynomials $p(x) = x^2 \pm 2x + 1$ do have the required property with $q(x) = x \mp 1$, respectively.

Comment. By a "root" we may mean a "complex root," and then nothing requires clarification. But complex numbers need not be mentioned at all, because $p(x) = x^2 + ax \pm 1$ has real roots if $|a| \ge 2$; and the cases of $|a| \le 1$ must be handled separately anyway.

The proposer remarks that even if $p(x)q(x)$ is allowed to have zero coefficients, the conclusion $|z| < 2$ about its roots holds true. However, extra solutions appear: x^2 and $x^2 \pm x$.

Second solution. Suppose that the polynomials $p(x) = a_0 + a_1x + x^2$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$ are such that $p(x)q(x) = c_0 + c_1x + \cdots + c_{n+2}x^{n+2}$ with all $c_k = \pm 1$. Then $|a_0| = |b_0| = |b_n| = 1$ and

$$a_0b_1 = c_1 - a_1b_0, \qquad a_0b_k = c_k - a_1b_{k-1} - b_{k-2} \quad \text{for } k = 2, \dots, n,$$

whence

$$|b_1| \ge |a_1| - 1, \qquad |b_k| \ge |a_1b_{k-1}| - |b_{k-2}| - 1 \quad \text{for } k = 2, \dots, n.$$

Assume $|a_1| \ge 3$. Then clearly $q(x)$ cannot be a constant, so $n \ge 1$, and we get

$$|b_1| \ge 2, \qquad |b_k| \ge 3|b_{k-1}| - |b_{k-2}| - 1 \quad \text{for } k = 2, \dots, n.$$

Recasting the last inequality into

$$|b_k| - |b_{k-1}| \ge 2|b_{k-1}| - |b_{k-2}| - 1 \ge 2\big(|b_{k-1}| - |b_{k-2}|\big) - 1$$

we see that the sequence $d_k = |b_k| - |b_{k-1}|$ $(k = 1, \dots, n)$ obeys the recursive estimate $d_k \ge 2d_{k-1} - 1$ for $k \ge 2$. As $d_1 = |b_1| - 1 \ge 1$, this implies by obvious induction $d_k \ge 1$ for $k = 1, \dots, n$. Equivalently, $|b_k| \ge |b_{k-1}| + 1$ for $k = 2, \dots, n$, and hence $|b_n| \ge |b_0| + n$, in contradiction to $|b_0| = |b_n| = 1$, $n \ge 1$.

It follows that $p(x)$ must be of the form $a_0 + a_1x + x^2$ with $|a_0| = 1$, $|a_1| \le 2$. If $|a_1| \le 1$ or $|a_1| = 2$ and $a_0 = 1$, then the corresponding $q(x)$ exists; see the eight examples in the first solution.

We are left with the case $|a_1| = 2$, $a_0 = -1$. Assume $q(x)$ exists. There is no loss of generality in assuming that $b_0 = 1$ and $a_1 = 2$ (if $b_0 = -1$, replace $q(x)$ by $-q(x)$; and if $a_1 = -2$, replace $q(x)$ by $q(-x)$). With $b_0 = 1$, $a_0 = -1$, $a_1 = 2$ the initial recursion formulas become

$$b_1 = 2 - c_1, \qquad b_k = 2b_{k-1} + b_{k-2} - c_k \quad \text{for } k = 2, \dots, n.$$

Therefore $b_1 \ge 1$, $b_2 \ge 2b_1 + 1 - c_2 \ge 2$, and induction shows that $b_k \ge 2$ for $k = 2, \dots, n$, again in contradiction with $|b_n| = 1$. So there are no "good" trinomials $p(x)$ except the eight mentioned above.

A2. Let $\mathbb{R}^+$ denote the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x)f(y) = 2f\big(x + yf(x)\big)$$

for all positive real numbers x and y.

Solution. The answer is the constant function $f(x) = 2$ which clearly satisfies the equation.

First, we show that a function f satisfying the equation is nondecreasing. Indeed, suppose that $f(x) < f(z)$ for some positive real numbers $x > z$. Set $y = (x - z)(f(z) - f(x)) > 0$, so that $x + yf(x) = z + yf(z)$. The equation now implies

$$f(x)f(y) = 2f(x + yf(x)) = 2f(z + yf(z)) = f(z)f(y),$$

therefore $f(x) = f(z)$, a contradiction. Thus, f is nondecreasing.

Assume now that f is not strictly increasing, that is, $f(x) = f(z)$ holds for some positive real numbers $x > z$. If y belongs to the interval $(0, (x - z)/f(x)]$ then $z < z + yf(z) \le x$. Since f is nondecreasing, we obtain

$$f(z) \le f(z + yf(z)) \le f(x) = f(z),$$

leading to $f(z+yf(z)) = f(x)$. Thus, $f(z)f(y) = 2f(z+yf(z)) = 2f(x) = 2f(z)$. Hence, $f(y) = 2$ for all y in the above interval.

But if $f(y_0) = 2$ for some y_0 then

$$2 \cdot 2 = f(y_0)f(y_0) = 2f(y_0 + y_0 f(y_0)) = 2f(3y_0); \quad \text{therefore} \quad f(3y_0) = 2.$$

By obvious induction, we get that $f(3^k y_0) = 2$ for all positive integers k, and so $f(x) = 2$ for all $x \in \mathbb{R}^+$.

Assume now that f is a strictly increasing function. We then conclude that the inequality $f(x)f(y) = 2f(x + yf(x)) > 2f(x)$ holds for all positive real numbers x, y. Thus, $f(y) > 2$ for all $y > 0$. The equation implies

$$2f(x + 1 \cdot f(x)) = f(x)f(1) = f(1)f(x) = 2f(1 + xf(1)) \qquad \text{for } x > 0,$$

and since f is injective, we get $x + 1 \cdot f(x) = 1 + x \cdot f(1)$ leading to the conclusion that $f(x) = x(f(1) - 1) + 1$ for all $x \in \mathbb{R}^+$. Taking a small x (close to zero), we get $f(x) < 2$, which is a contradiction. (Alternatively, one can verify directly that $f(x) = cx + 1$ is not a solution of the given functional equation.)

A3. Four real numbers p, q, r, s satisfy

$$p+q+r+s=9 \qquad \text{and} \qquad p^2+q^2+r^2+s^2=21.$$

Prove that $ab-cd \geq 2$ holds for some permutation (a,b,c,d) of (p,q,r,s).

First solution. Up to a permutation, we may assume that $p \geq q \geq r \geq s$. We first consider the case where $p+q \geq 5$. Then

$$p^2+q^2+2pq \geq 25 = 4+(p^2+q^2+r^2+s^2) \geq 4+p^2+q^2+2rs,$$

which is equivalent to $pq-rs \geq 2$.

Assume now that $p+q<5$; then

$$4 < r+s \leq p+q < 5. \tag{1}$$

Observe that

$$(pq+rs)+(pr+qs)+(ps+qr) = \frac{(p+q+r+s)^2-(p^2+q^2+r^2+s^2)}{2} = 30.$$

Moreover,

$$pq+rs \geq pr+qs \geq ps+qr,$$

because $(p-s)(q-r) \geq 0$ and $(p-q)(r-s) \geq 0$.

We conclude that $pq+rs \geq 10$. From (1), we get $0 \leq (p+q)-(r+s) < 1$, therefore

$$(p+q)^2-2(p+q)(r+s)+(r+s)^2 < 1.$$

Adding this to $(p+q)^2+2(p+q)(r+s)+(r+s)^2 = 9^2$ gives

$$(p+q)^2+(r+s)^2 < 41.$$

Therefore

$$\begin{aligned} 41 = 21+2\cdot 10 &\leq (p^2+q^2+r^2+s^2)+2(pq+rs) \\ &= (p+q)^2+(r+s)^2 < 41, \end{aligned}$$

which is a contradiction.

Second solution. We first note that $pq + pr + ps + qr + qs + rs = 30$, as in the first solution. Thus, if $(a,\ b,\ c,\ d)$ is any permutation of (p, q, r, s), then

$$bc + cd + db = 30 - a(b + c + d) = 30 - a(9 - a) = 30 - 9a + a^2,$$

while

$$bc + cd + db \leq b^2 + c^2 + d^2 = 21 - a^2.$$

Hence $30 - 9a + a^2 \leq 21 - a^2$, leading to $a \in [3/2, 3]$. Thus the numbers p, q, r and s are in the interval $[3/2, 3]$.

Assume now that $p \geq q \geq r \geq s$. Note that $q \geq 2$ because otherwise $p = 9 - (q + r + s) \geq 9 - 3q > 9 - 6 = 3$, which is impossible.

Write $x = r - s,\ y = q - r$ and $z = p - q$. On the one hand,

$$\begin{aligned}(p - q)^2 + (p - r)^2 + (p - s)^2 + (q - r)^2 + (q - s)^2 + (r - s)^2 \\ = 3(p^2 + q^2 + r^2 + s^2) - 2(pq + pr + ps + qr + qs + rs) = 3.\end{aligned}$$

On the other hand, this expression equals

$$\begin{aligned}z^2 + (z + y)^2 &+ (z + y + x)^2 + y^2 + (y + x)^2 + x^2 \\ &= 3x^2 + 4y^2 + 3z^2 + 4xy + 4yz + 2zx.\end{aligned}$$

Hence,

$$3x^2 + 4y^2 + 3z^2 + 4xy + 4yz + 2zx = 3. \tag{2}$$

Furthermore,

$$pq - rs = q(p - s) + (q - r)s = q(x + y + z) + ys.$$

If $x + y + z \geq 1$ then, in view of $q \geq 2$, we immediately get $pq - rs \geq 2$.

If $x + y + z < 1$ then (2) implies

$$3x^2 + 4y^2 + 3z^2 + 4xy + 4yz + 2zx > 3(x + y + z)^2.$$

It follows that $y^2 > 2xy + 2yz + 4zx \geq 2y(x + z)$, so that $y > 2(x + z)$ and hence $3y > 2(x + y + z)$. The value of the left-hand side of (2) obviously does not exceed $4(x + y + z)^2$, so that $2(x + y + z) \geq \sqrt{3}$. Eventually, $3y > \sqrt{3}$ and recalling that $s \geq 3/2$, we obtain

$$pq - rs = q(x + y + z) + ys \geq 2 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3} \cdot \frac{3}{2} = \frac{3\sqrt{3}}{2} > 2.$$

A4. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x+y) + f(x)f(y) = f(xy) + 2xy + 1$$

for all real numbers x and y.

Solution. The solutions are $f(x) = 2x - 1$, $f(x) = -x - 1$ and $f(x) = x^2 - 1$. It is easy to check that these functions indeed satisfy the given equation.

We begin by setting $y = 1$ which gives

$$f(x+1) = af(x) + 2x + 1, \tag{1}$$

where $a = 1 - f(1)$. Then we change y to $y+1$ in the equation and use (1) to expand $f(x+y+1)$ and $f(y+1)$. The result is

$$a\,(f(x+y) + f(x)f(y)) + (2y+1)(1 + f(x)) = f(x(y+1)) + 2xy + 1,$$

or, using the initial equation again,

$$a\,(f(xy) + 2xy + 1) + (2y+1)(1 + f(x)) = f(x(y+1)) + 2xy + 1.$$

Let us now set $x = 2t$ and $y = -1/2$ to obtain

$$a\,(f(-t) - 2t + 1) = f(t) - 2t + 1.$$

Replacing t by $-t$ yields one more relation involving $f(t)$ and $f(-t)$:

$$a\,(f(t) + 2t + 1) = f(-t) + 2t + 1. \tag{2}$$

We now eliminate $f(-t)$ from the last two equations, leading to

$$(1 - a^2)f(t) = 2(1-a)^2 t + a^2 - 1.$$

Note that $a \neq -1$ (or else $8t = 0$ for all t, which is false). If additionally $a \neq 1$ then $1 - a^2 \neq 0$, therefore

$$f(t) = 2\left(\frac{1-a}{1+a}\right)t - 1.$$

Setting $t = 1$ and recalling that $f(1) = 1 - a$, we get $a = 0$ or $a = 3$, which gives the first two solutions.

The case $a = 1$ remains, where (2) yields

$$f(t) = f(-t) \qquad \text{for all } t \in \mathbb{R}. \tag{3}$$

Now set $y = x$ and $y = -x$ in the original equation. In view of (3), we obtain, respectively,

$$f(2x) + f(x)^2 = f(x^2) + 2x^2 + 1, \quad f(0) + f(x)^2 = f(x^2) - 2x^2 + 1.$$

Subtracting gives $f(2x) = 4x^2 + f(0)$. Set $x = 0$ in (1). Since $f(1) = 1 - a = 0$, this yields $f(0) = -1$. Hence $f(2x) = 4x^2 - 1$, i. e. $f(x) = x^2 - 1$ for all $x \in \mathbb{R}$. This completes the solution.

A5. Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove the inequality

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

First solution. Standard recasting shows that the given inequality is equivalent to

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{z^2 + x^2 + y^2}{z^5 + x^2 + y^2} \leq 3.$$

In view of the Cauchy-Schwarz inequality and the condition $xyz \geq 1$, we have

$$(x^5 + y^2 + z^2)(yz + y^2 + z^2) \geq \left(x^{5/2}(yz)^{1/2} + y^2 + z^2\right)^2 \geq (x^2 + y^2 + z^2)^2,$$

or

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} \leq \frac{yz + y^2 + z^2}{x^2 + y^2 + z^2}.$$

Taking the cyclic sum and using the fact that $x^2 + y^2 + z^2 \geq yz + zx + xy$ gives

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \leq 2 + \frac{yz + zx + xy}{x^2 + y^2 + z^2} \leq 3,$$

which is exactly what we wished to show.

Comment. The way the Cauchy-Schwarz inequality is used is the crucial point in the solution; it is not at all obvious! The condition $xyz \geq 1$ (which might as well have been $xyz = 1$) allows to transform the expression to a homogeneous form. The smart use of Cauchy-Schwarz inequality has the effect that the common *numerators* of the three fractions become common *denominators* in the transformed expression.

Second solution. We shall prove something more, namely that

$$\frac{x^5}{x^5 + y^2 + z^2} + \frac{y^5}{y^5 + z^2 + x^2} + \frac{z^5}{z^5 + x^2 + y^2} \geq 1, \tag{1}$$

and

$$1 \geq \frac{x^2}{x^5 + y^2 + z^2} + \frac{y^2}{y^5 + z^2 + x^2} + \frac{z^2}{z^5 + x^2 + y^2}. \tag{2}$$

We first prove (1). We have $yz(y^2 + z^2) = y^3z + yz^3 \leq y^4 + z^4$; the latter inequality holds because $y^4 - y^3z - yz^3 + z^4 = (y^3 - z^3)(y - z) \geq 0$. Therefore $x(y^4 + z^4) \geq xyz(y^2 + z^2) \geq y^2 + z^2$, or

$$\frac{x^5}{x^5 + y^2 + z^2} \geq \frac{x^5}{x^5 + xy^4 + xz^4} = \frac{x^4}{x^4 + y^4 + z^4}.$$

Taking the cyclic sum, we get the desired inequality.

The proof of (2) is based on exactly the same ideas as in the first solution. From the Cauchy-Schwarz inequality and the fact that $xyz \geq 1$, we have

$$(x^5 + y^2 + z^2)(yz + y^2 + z^2) \geq (x^2 + y^2 + z^2)^2,$$

implying

$$\frac{x^2}{x^5 + y^2 + z^2} \leq \frac{x^2(yz + y^2 + z^2)}{(x^2 + y^2 + z^2)^2}.$$

Taking the cyclic sum, we have

$$\begin{aligned} & \frac{x^2}{x^5 + y^2 + z^2} + \frac{y^2}{y^5 + z^2 + x^2} + \frac{z^2}{z^5 + x^2 + y^2} \\ \leq \quad & \frac{2(x^2y^2 + y^2z^2 + z^2x^2) + x^2yz + y^2zx + z^2xy}{(x^2 + y^2 + z^2)^2} \\ = \quad & \frac{(x^2 + y^2 + z^2)^2 - (x^4 + y^4 + z^4) + (x^2yz + y^2zx + z^2xy)}{(x^2 + y^2 + z^2)^2}. \end{aligned}$$

Thus we need to show that $x^4 + y^4 + z^4 \geq x^2yz + y^2zx + z^2xy$; and this last inequality holds because

$$\begin{aligned} x^4 + y^4 + z^4 &= \frac{x^4 + y^4}{2} + \frac{y^4 + z^4}{2} + \frac{z^4 + x^4}{2} \geq x^2y^2 + y^2z^2 + z^2x^2 \\ &= \frac{x^2y^2 + y^2z^2}{2} + \frac{y^2z^2 + z^2x^2}{2} + \frac{z^2x^2 + x^2y^2}{2} \\ &\geq y^2zx + z^2xy + x^2yz. \end{aligned}$$

Combinatorics

C1. A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

Solution. Two lamps sharing a switch will be called *twins*. A room will be called *normal* if some of its lamps are on and some are off. We devise an algorithm that increases the number of normal rooms in the house. After several runs of the algorithm we arrive at the state with all rooms normal.

Choose any room R_0 which is not normal, assuming without loss of generality that all lamps in R_0 are off. If there is a pair of twins in R_0, we switch them on and stop. Saying *stop* means that we have achieved what we wanted: there are more normal rooms than before the algorithm started.

So suppose there are no twins in R_0. Choose any lamp $a_0 \in R_0$ and let $b_0 \in R_1$ be its twin. Change their states. After this move room R_0 becomes normal. If R_1 also becomes (or remains) normal, then stop. Otherwise all lamps in R_1 are in equal state; as before we can assume that there are no twins in R_1. Choose any lamp $a_1 \in R_1$ other than b_0 and let $b_1 \in R_2$ be its twin. Change the states of these two twin lamps. If R_2 becomes (or stays) normal, stop.

Proceed in this fashion until a repetition occurs in the sequence $R_0, R_1, R_2, \ldots$. Thus assume that the rooms $R_0, R_1, \ldots, R_m$ are all distinct, each R_i connected to R_{i+1} through a twin pair $a_i \in R_i$, $b_i \in R_{i+1}$ $(i = 0, \ldots, m-1)$, and there is a lamp $a_m \in R_m$ $(a_m \neq b_{m-1})$ which has its twin b_m in some room R_k visited earlier $(0 \leq k \leq m-1)$. If the algorithm did not stop after we entered room R_m, we change the states of the lamps a_m and b_m; room R_m becomes normal.

If $k \geq 1$, then there are two lamps in R_k touched previously, b_{k-1} and a_k. They are the twins of a_{k-1} and b_k, so neither of them can be b_m (twin to a_m). Recall that the moment we entered room R_k the first time, by pressing the b_{k-1} switch, this room became "abnormal" only until we touched lamp a_k. Thus b_{k-1} and a_k are in different states now. Whatever the new state of lamp b_m, room R_k remains normal. Stop.

Finally, if $k = 0$, then $b_m \in R_0$ and $b_m \neq a_0$ because the twin of a_0 is b_0. Each room has at least three lamps, so there is a lamp $c \in R_0$, $c \neq a_0$, $c \neq b_m$. In the first move lamp a_0 was put on while c stayed off. Whatever the new state of b_m, room R_0 stays normal. Stop.

So, indeed, after a single run of this algorithm, the number of normal rooms increases at least by 1. This completes the proof.

Comment. The problem was submitted in the following formulation:

> A school has an even number of students, each of whom attends exactly one of its (finitely many) classes. Each class has at least three students, and each student has exactly one "best friend" in the same school such that, whenever B is A's "best friend", then A is B's "best friend". Furthermore, each student prefers either apple juice over orange juice or orange juice over apple juice, but students change their preferences from time to time. "Best friends", however, will change their preferences (which may or may not be the same) always together, at the same moment.
>
> Whatever preference each student may initially have, prove that there is always a sequence of changes of preferences which will lead to a situation in which no class will have students all of whom have the same preference.

C2. Let k be a fixed positive integer. A company has a special method to sell sombreros. Each customer can convince two persons to buy a sombrero after he/she buys one; convincing someone already convinced does not count. Each of these new customers can convince two others and so on. If each one of the two customers convinced by someone makes at least k persons buy sombreros (directly or indirectly), then that someone wins a free instructional video. Prove that if n persons bought sombreros, then at most $n/(k+2)$ of them got videos.

First solution. Consider the problem in reverse: If w persons won free videos, what is the least number n of persons who bought sombreros? One can easily compute this minimum for small values of w: for $w = 1$ it is $2k+3$, and for $w = 2$ it is $3k+5$. These can be rewritten as $n \geq 1 \cdot (k+2) + (k+1)$ and $n \geq 2(k+2) + (k+1)$, leading to the conjecture that

$$n \geq w(k+2) + (k+1). \tag{1}$$

Let us say that a person P *influenced* a person Q if P made Q buy a sombrero directly or indirectly, or if $Q = P$. A *component* is the set of persons influenced by someone who was influenced by no one else but himself. No person from a component influenced another one from a different component. So it suffices to prove (1) for each component. Indeed, if (1) holds for r components of size n_i with w_i winners, $i = 1, \ldots, r$, then

$$n = \sum n_i \geq \sum (w_i(k+2) + (k+1)) = \left(\sum w_i\right)(k+2) + r(k+1),$$

implying (1) for $n = \sum n_i$, $w = \sum w_i$.

Thus one may assume that the whole group is a single component, i. e. all customers were influenced by one person A (directly or indirectly).

Moreover, it suffices to prove (1) for a group G with w winners and of *minimum* size n. Notice that then A is a video winner. If not, imagine him removed from the group. A video winner from the original group is also a winner in the new one. So we have decreased n without changing w, a contradiction.

Under these assumptions, we proceed to prove (1) by induction on $w \geq 1$. For $w = 1$, the group of customers contains a single video winner A, the two persons B and C he/she convinced directly to buy sombreros, and two nonintersecting groups of k persons, the ones persuaded by B and C (directly or indirectly). This makes at least $2k+3$ persons, as needed.

Assume the claim holds for groups with less than w winners, and consider a group with n winners where everyone was influenced by some person A. Recall that A is a winner. Let B and C be the persons convinced directly by A to buy sombreros. Let n_B be the number of people influenced by B, and w_B the number of video winners among them. Define n_C and w_C analogously.

We have $n_B \geq w_B(k+2)+(k+1)$, by the induction hypothesis if $w_B > 0$ and because A is a winner if $w_B = 0$. Analogously $n_C \geq w_C(k+2)+(k+1)$. Adding the two inequalities gives us $n \geq w(k+2)+(k+1)$, since $n = n_B + n_C + 1$ and $w = w_B + w_C + 1$. This concludes the proof.

Second solution. As in the first solution, we say that a person P *influenced* a person Q if P made Q buy a sombrero directly or indirectly, or if $Q = P$. Likewise, we keep the definition of a component. For brevity, let us write winners instead of video winners.

The components form a partition of the set of people who bought sombreros. It is enough to prove that in each component the fraction of winners is at most $1/(k+2)$.

We will minimise the number of people buying sombreros while keeping the number of winners fixed.

First, we can assume that no winners were convinced (directly) by a nonwinner. Indeed, if a nonwinner P convinced a winner Q, remove all people influenced by P but not by Q and let whoever convinced P (if anyone did) now convince Q. Observe that no winner was removed, hence the new configuration has fewer people, but the same winners.

Thus, indeed, there is no loss of generality in assuming that:

The set of all buyers makes up a single component. (2)

Every winner could have been convinced only by another winner. (3)

Now remove all the winners and consider the new components. We claim that

Each new component has at least $k+1$ persons. (4)

Indeed, let $\mathcal{C}$ be a new component. In view of (2), there is a member C of $\mathcal{C}$ who had been convinced by some removed winner W. Then C must have influenced at least $k+1$ people (including himself), but all the people influenced by C are in $\mathcal{C}$. Therefore $|\mathcal{C}| \geq k+1$.

Now return the winners one by one in such a way that when a winner returns, the people he convinced (directly) are already present. This is possible because of (3). In that way the number of components decreases by one with each winner, thus the number of components with all winners removed is equal to $w+1$, where w is the number of winners. It follows from (4) that the number of nonwinners satisfies the estimate

$$n - w \geq (w+1)(k+1).$$

This implies the desired bound.

C3. In an $m \times n$ rectangular board of mn unit squares, *adjacent* squares are ones with a common edge, and a *path* is a sequence of squares in which any two consecutive squares are adjacent. Each square of the board can be coloured black or white. Let N denote the number of colourings of the board such that there exists at least one black path from the left edge of the board to its right edge, and let M denote the number of colourings in which there exist at least two non-intersecting black paths from the left edge to the right edge. Prove that $N^2 \geq M \cdot 2^{mn}$.

Solution. We generalise the claim to the following. Suppose that a two-sided $m \times n$ board is considered, where some of the squares are transparent and some others are not. Each square must be coloured black or white. However, a transparent square needs to be coloured only on one side; then it looks the same from above and from below. In contrast, a non-transparent square must be coloured on both sides (in the same colour or not).

Let A (respectively B) be the set of colourings of the board with at least one black path from the left edge to the right edge if one looks from above (respectively from below).

Let C be the set of colourings of the board in which there exist two black paths from the left edge to the right edge of the board, one on top and one underneath, not intersecting at any transparent square.

Let D be the set of all colourings of the board.

We claim that

$$|A| \cdot |B| \geq |C| \cdot |D|. \tag{1}$$

Note that this implies the original claim in the case where all squares are transparent: one then has $|A| = |B| = N$, $|C| = M$, $|D| = 2^{mn}$.

We prove (1) by induction on the number k of transparent squares. If $k = 0$ then $|A| = |B| = N \cdot 2^{mn}$, $|C| = N^2$ and $|D| = (2^{mn})^2$, so equality holds in (1). Suppose the claim is true for some k and consider a board with $k+1$ transparent squares. Let A, B, C and D be the sets of colourings of this board as defined above. Choose one transparent square ϑ. Now, convert ϑ into a non-transparent square, and let A', B', C' and D' be the respective sets of colourings of the new board. By the induction hypothesis, we have:

$$|A'| \cdot |B'| \geq |C'| \cdot |D'|. \tag{2}$$

Upon the change made, the number of all colourings doubles. So $|D'| = 2|D|$.

To any given colouring in A, there correspond two colourings in A', obtained by colouring ϑ black and white from below. This is a bijective correspondence, so $|A'| = 2|A|$. Likewise, $|B'| = 2|B|$. In view of (2), it suffices to prove that

$$|C'| \geq 2|C|. \tag{3}$$

Make ϑ transparent again and take any colouring in C. It contains two black paths (one seen from above and one from below) that do not intersect at transparent squares. Being transparent, ϑ can therefore lie on at most one of them, say on the path above. So when we make ϑ non-transparent, let us keep its colour on the side above but colour the side below in the two possible ways. The two colourings obtained will be in C'. It is easy to see that when doing so, different colourings in C give rise to different pairs of colourings in C'. Hence (3) follows, implying (2). As already mentioned, this completes the solution.

Comment. A more direct approach to the problem may go as follows. Consider two $m \times n$ boards instead of one. Let $\mathcal{A}$ denote the set of all colourings of the two boards such that there are at least two non-intersecting black paths from the left edge of the first board to its right edge. Clearly, $|\mathcal{A}| = M \cdot 2^{mn}$: we can colour the first board in M ways and the second board in an arbitrary fashion.

Let $\mathcal{B}$ denote the set of all colourings of the two boards such that there is at least one black path from the left edge of the first board to its right edge, and at least one black path from the left edge of the second board to its right edge. Clearly, $|\mathcal{B}| = N^2$.

It suffices to find an injective function $f : \mathcal{A} \hookrightarrow \mathcal{B}$.

Such an injection can indeed be constructed. However, working it out in all details seems to be a delicate task.

C4. Let $n \geq 3$ be a given positive integer. We wish to label each side and each diagonal of a regular n-gon $P_1, \ldots, P_n$ with a positive integer less than or equal to r so that:

(i) every integer between 1 and r occurs as a label;

(ii) in each triangle $P_iP_jP_k$ two of the labels are equal and greater than the third.

Given these conditions:

(a) Determine the largest positive integer r for which this can be done.

(b) For that value of r, how many such labellings are there?

Solution. A labelling which satisfies condition (ii) will be called *good*. A labelling which satisfies both given conditions (i) and (ii) will be called *very good*. Let us try to understand the structure of good labellings.

Sides and diagonals of the polygon will be called just *edges*. Let AB be an edge with the maximum label m. Let X be any vertex different from A and B. Condition (ii), applied to triangle ABX, implies that one of the segments AX, BX has label m, and the other one has a label smaller than m. Thus we can split all vertices into two disjoint groups 1 and 2; group 1 contains vertices X such that AX has label m (including vertex B) and group 2 contains vertices X such that BX has label m (including vertex A). We claim that the edges labelled m are exactly those which join a vertex of group 1 with a vertex of group 2.

First consider any vertex $X \neq B$ in group 1 and any vertex $Y \neq A$ in group 2. In triangle AXY, we already know that the label of AX (which is m) is larger than the label of AY (which is not m). Therefore the label of XY also has to be equal to m, as we wanted to show.

Now consider any two vertices X, Y in group 1. In triangle AXY, the edges AX and AY have the same label m. So the third edge must have a label smaller than m, as desired. Similarly, any edge joining two vertices in group 2 has a label smaller than m.

We conclude that a good labelling of an n-gon consists of:

• a collection of edges with the maximum label m; they are the ones that go from a vertex of group 1 to a vertex of group 2,

• a good labelling of the polygon determined by the vertices of group 1, and

• a good labelling of the polygon determined by the vertices of group 2.

(a) The greatest possible value of r is $n-1$. We prove this by induction starting with the degenerate cases $n = 1$ and $n = 2$, where the claim is immediate. Assusme it true for values less than n, where $n \geq 3$, and consider any good labelling of an n-gon P.

Its edges are split into two groups 1 and 2; suppose they have k and $n-k$ vertices, respectively. The k-gon P_1 formed by the vertices in group 1 inherits a good labelling. By the induction hypothesis, this good labelling uses at most $k-1$ different labels. Similarly, the $(n-k)$-gon P_2 formed by the vertices in group 2 inherits a good labelling which uses at most $n-k-1$ different labels. The remaining segments, which join a vertex of group 1 with a vertex of group 2, all have the same (maximum) label. Therefore, the total number of different labels in our good labelling is at most $(k-1) + (n-k-1) + 1 = n-1$. This number can be easily achieved, as long as we use different labels in P_1 and P_2.

(b) Let $f(n)$ be the number of very good labellings of an n-gon P with labels $1, \ldots, n-1$. We will show by induction that

$$f(n) = n!(n-1)!/2^{n-1}.$$

This holds for $n = 1$ and $n = 2$. Fix $n \geq 3$ and assume that $f(k) = k!(k-1)!/2^{k-1}$ for $k < n$.

Divide the n vertices into two non-empty groups 1 and 2 in any way. If group 1 is of size k, there are $\binom{n}{k}$ ways of doing that. We must label every edge joining a vertex of group 1 and a vertex of group 2 with the label $n-1$. Now we need to choose which $k-1$ of the remaining labels $1, 2, \ldots, n-2$ will be used to label the k-gon P_1; there are $\binom{n-2}{k-1}$ possible choices. The remaining $n-k-1$ labels will be used to label the $(n-k)$-gon P_2. Finally, there are $f(k)$ very good labellings of P_1 and $f(n-k)$ very good labellings of P_2.

Now we sum the resulting expression over all possible values of k, noticing that we have counted each very good labelling twice, since choosing a set to be group 1 is equivalent to choosing its complement. We have:

$$\begin{aligned}
f(n) &= \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\binom{n-2}{k-1} f(k) f(n-k) \\
&= \frac{n!(n-1)!}{2(n-1)}\sum_{k=1}^{n-1}\frac{f(k)}{k!(k-1)!}\cdot\frac{f(n-k)}{(n-k)!(n-k-1)!} \\
&= \frac{n!(n-1)!}{2(n-1)}\sum_{k=1}^{n-1}\frac{1}{2^{k-1}}\cdot\frac{1}{2^{n-k-1}} = \frac{n!(n-1)!}{2^{n-1}},
\end{aligned}$$

which is what we wanted to show.

C5. There are n markers, each with one side white and the other side black, aligned in a row so that their white sides are up. In each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it and reverse the closest marker to the left and the closest marker to the right of it. Prove that one can achieve the state with only two markers remaining if and only if $n-1$ is not divisible by 3.

First solution. Given a particular chain of markers, we call white (resp. black) markers the ones with the white (resp. black) side up. Note that the parity of the number of black markers remains unchanged during the game. Hence, if only two markers remain, these markers must have the same colour.

Next, we define an invariant. To a white marker with t black markers to its left we assign the number $(-1)^t$. Only white markers have numbers assigned to them. Let S be the residue class modulo 3 of the sum of all numbers assigned to the white markers.

It is easy to check that S is an invariant under the allowed operations. Suppose, for instance, that a white marker W is removed, with t black markers to the left of it, and that the closest neighbours of W are black. Then S increases by $-(-1)^t + (-1)^{t-1} + (-1)^{t-1} = 3(-1)^{t-1}$. The other three cases are analogous.

If the game ends with two black markers, the number S is zero; if it ends with two white markers, then S is 2. Since we start with n white markers and in this case $S \equiv n \pmod 3$, a necessary condition for the game to end is $n \equiv 0,\ 2 \pmod 3$.

If we start with $n \geq 5$ white markers, taking the leftmost allowed white markers in three consecutive moves, we obtain a row of $n-3$ white markers without black markers. Since the goal can be reached for $n = 2,\ 3$, we conclude that the game can end with two markers for every positive integer n satisfying $n \equiv 0,\ 2 \pmod 3$.

Second solution. Denote by L the leftmost and by R the rightmost marker, respectively. To start with, note again that the parity of the number of black-side-up markers remains unchanged. Hence, if only two markers remain, these markers must have the same colour up.

We will show by induction on n that the game can be succesfully finished if and only if $n \equiv 0,\ 2 \pmod 3$ and that the upper sides of L and R will be black in the first case and white in the second case.

The statement is clear for $n = 2$ and 3. Assume that we finished the game for some n, and denote by k the position of the marker X (counting from the left) that was last removed. Having finished the game, we have also finished the subgames with the k markers from L to X and with the $n-k+1$ markers from X to R (inclusive). Thereby, by the induction hypothesis, before X was removed, the upper side of L had been black if $k \equiv 0 \pmod 3$, and white if $k \equiv 2 \pmod 3$, while the upper side of R had been black if $n-k+1 \equiv 0 \pmod 3$, and white if $n-k+1 \equiv 2 \pmod 3$. Markers R and L were reversed upon the removal of X. Therefore, in the final position, R and L are white if and only if $k \equiv n-k+1 \equiv 0 \pmod 3$, which yields $n \equiv 2 \pmod 3$, and black if and only if $k \equiv n-k+1 \equiv 2 \pmod 3$, which yields $n \equiv 0 \pmod 3$.

On the other hand, a game with n markers can be reduced to a game with $n-3$ markers by removing the second, fourth and third marker in this order. This finishes the induction.

C6. In a mathematical competition in which 6 problems were posed to the participants, every two of these problems were solved by more than $2/5$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

First solution. Assume there were n contestants. Let us count the number N of ordered pairs (C, P), where P is a pair of problems solved by contestant C. On the one hand, for every one of the 15 pairs of problems, there are at least $(2n+1)/5$ contestants who solved both problems in the pair. Therefore

$$N \geq 15 \cdot \frac{2n+1}{5} = 6n + 3. \tag{1}$$

On the other hand, assume k contestants solved 5 problems. Each of them solved 10 pairs of problems, whereas each of the $n-k$ remaining contestants solved at most 6 pairs of problems. Thus

$$N \leq 10k + 6(n-k) = 6n + 4k. \tag{2}$$

From these two estimates we immediately get $k \geq 1$. If $(2n+1)/5$ were not an integer, there would be, for every pair of problems, at least $(2n+1)/5$ contestants who solved both problems in the pair (rather than $(2n+1)/5$). Then (1) would improve to $N \geq 6n+6$ and this would yield $k \geq 2$. Alternatively, had some student solved less than 4 problems, he would have solved at most 3 pairs of problems (rather than 6), and our second estimate would improve to $N \leq 6n + 4k - 3$, which together with $N \geq 6n+3$ also gives $k \geq 2$.

So we are left with the case where 5 divides $2n+1$ and every contestant has solved 4 or 5 problems. Let us assume $k = 1$ and let us call the contestant who solved 5 problems the 'winner'. We must then have $N = 6n+4$ (the winner solved 10 pairs of problems, and the rest of the contestants solved exactly 6 pairs of problems each). Let us call a pair of problems 'special' if more than $(2n+1)/5$ contestants solved both problems of the pair. If there were more than one special pair of problems, our first estimate would be improved to

$$N \geq 13 \cdot \frac{2n+1}{5} + 2\left(\frac{2n+1}{5} + 1\right) = 6n + 5,$$

which is impossible. Similarly, if a special pair of problems exists, no more than $(2n+1)/5 + 1$ contestants could have solved both problems in the pair, because otherwise

$$N \geq 14 \cdot \frac{2n+1}{5} + \left(\frac{2n+1}{5} + 2\right) = 6n + 5.$$

Let us now count the number M of pairs (C, P) where the 'tough' problem (the one not solved by the winner) is one of the problems in P. For each of the 5 pairs of problems containing the tough problem, there are either $(2n+1)/5$ or $(2n+1)/5+1$ contestants who solved both problems of the pair. We then get $M = 2n+1$ or $M = 2n+2$; the latter is possible only if there is a special pair of problems and this special pair contains the tough problem. On the other hand, assume m contestants solved the tough problem. Each of them solved 3 other problems and therefore solved 3 pairs of problems containing the tough one. We can then write $M = 3m$. Hence $2n+1 \equiv 0$ or $2 \pmod 3$.

Finally, let us chose one of the problems other than the tough one, say p, and count the number L of pairs (C, P) for which $p \in P$. We can certainly chose p such that the special pair of problems, if it exists, does not contain p. Then we have $L = 2n+1$ (each of the 5 pairs of problems containing p have exactly $(2n+1)/5$ contestants who solved both problems of the pair). On the other hand, if l is the number of contestants, other than the winner, who solved problem p, we have $L = 3l+4$ (the winner solved problem p and other 4 problems, so she solved 4 pairs of problems containg p, and each of the l students who solved p, solved other 3 problems, hence each of them solved 3 pairs of problems containing p). Therefore $2n+1 \equiv 1 \pmod 3$, which is a contradiction.

Second solution. This is basically the same proof as above, written in symbols rather than words. Suppose there were n contestants. Let p_{ij} be the number of contestants who solved both problem i and problem j ($1 \le i < j \le 6$) and let n_r be the number of contestants who solved exactly r problems ($0 \le r \le 6$). Clearly, $\sum n_r = n$.

By hypothesis, $p_{ij} \ge (2n+1)/5$ for all $i < j$, and so

$$S = \sum_{i<j} p_{ij} \ge 15 \cdot \frac{2n+1}{5} = 6n+3.$$

A contestant who solved exactly r problems contributes a '1' to $\binom{r}{2}$ summands in this sum (where as usual $\binom{r}{2} = 0$ for $r < 2$). Therefore

$$S = \sum_{r=0}^{6} \binom{r}{2} n_r.$$

Combining this with the previous estimate we obtain

$$3 \le S - 6n = \sum_{r=0}^{6} \left(\binom{r}{2} - 6 \right) n_r, \tag{3}$$

which rewrites as

$$4n_5 + 9n_6 \geq 3 + 6n_0 + 6n_1 + 5n_2 + 3n_3.$$

If no contestant solved all problems, then $n_6 = 0$, and we see from the above that n_5 must be positive. To show that $n_5 \geq 2$, assume the contrary, i. e., $n_5 = 1$. Then all of n_0, n_1, n_2, n_3 must be zero, so that $n_4 = n - 1$. The right equality of (3) reduces to $S = 6n + 4$.

Each one of the 15 summands in $S = \sum p_{ij}$ is at least $(2n+1)/5 = \lambda$. Because $S = 6n + 4$, they cannot be all equal ($6n + 4$ is not divisible by 15); therefore 14 of them are equal to λ and one is $\lambda + 1$.

Let (i_0, j_0) be this specific pair with $p_{i_0 j_0} = \lambda + 1$. The contestant who solved 5 problems will be again called the winner. Assume, without loss of generality, that it was problem 6 at which the winner failed, and that problem 1 is not in the pair (i_0, j_0); that is, $2 \leq i_0 < j_0 \leq 6$. Consider the sums

$$S' = p_{16} + p_{26} + p_{36} + p_{46} + p_{56} \qquad \text{and} \qquad S'' = p_{12} + p_{13} + p_{14} + p_{15} + p_{16}.$$

Suppose that problem 6 has been solved by x contestants (each of them contributes a '3' to S') and problem 1 has been solved by y contestants other than the winner (each of them contributes a '3' to S'', and the winner contributes a '4'). Thus $S' = 3x$ and $S'' = 3y + 4$.

The pair (i_0, j_0) does not appear in the sum S'', which is therefore equal to $5\lambda = 2n + 1$. The sum S' is either 5λ or $5\lambda + 1$. Hence $3x \in \{2n+1,\, 2n+2\}$ and $3y + 4 = 2n + 1$, which is impossible, as examination of remainders (mod 3) shows. Contradiction ends the proof.

Comment. The problem submitted by the proposer consisted of two parts which were found to be two independent problems by the PSC.

Part (a) asked for a proof that if every problem has been solved by more than 2/5 of the contestants then there exists a set of 3 problems solved by more than 1/5 of the contestants and a set of 4 problems solved by more than 1/15 of the contestants.

The arguments needed for a proof of (a) seem rather standard, giving advantage to students who practised those techniques at training courses. This is much less the case with part (b), which was therefore chosen to be Problem **C6** on the shortlist.

The proposer remarks that there exist examples showing the bound 2 can be attained for the number of contestants solving 5 problems, and that the problem would become harder if it asked to find one such example.

C7. Let $n > 1$ be a given integer, and let $a_1, \dots, a_n$ be a sequence of integers such that n divides the sum $a_1 + \dots + a_n$. Show that there exist permutations σ and τ of $1, 2, \dots, n$ such that $\sigma(i) + \tau(i) \equiv a_i \pmod{n}$ for all $i = 1, \dots, n$.

Solution. Suppose that there exist suitable permutations σ and τ for a certain integer sequence $a_1, \dots, a_n$ of sum zero modulo n. Let $b_1, \dots, b_n$ be another integer sequence with sum divisible by n, and let $b_1, \dots, b_n$ differ modulo n from $a_1, \dots, a_n$ only in two places, i_1 and i_2. Based on the fact that $\sigma(i) + \tau(i) \equiv b_i \pmod{n}$ for each $i \neq i_1, i_2$, one can transform σ and τ into suitable permutations for $b_1, \dots, b_n$. All congruences below are assumed modulo n.

First we construct a three-column rectangular array

$$\begin{array}{ccc}
\sigma(i_1) & -b_{i_1} & \tau(i_1) \\
\sigma(i_2) & -b_{i_2} & \tau(i_2) \\
\sigma(i_3) & -b_{i_3} & \tau(i_3) \\
\vdots & \vdots & \vdots \\
\sigma(i_{p-1}) & -b_{i_{p-1}} & \tau(i_{p-1}) \\
\sigma(i_p) & \boxed{-b_{i_p}} & \boxed{\tau(i_p)} \\
\sigma(i_{p+1}) & -b_{i_{p+1}} & \boxed{\tau(i_{p+1})} \\
\vdots & \vdots & \vdots \\
\boxed{\sigma(i_{q-1})} & -b_{i_{q-1}} & \tau(i_{q-1}) \\
\boxed{\sigma(i_q)} & \boxed{-b_{i_q}} & \tau(i_q)
\end{array}$$

whose rows are some of the ordered triples $T_i = \big(\sigma(i), -b_i, \tau(i)\big)$, $i = 1, \dots, n$. In the first two rows, write the triples T_{i_1} and T_{i_2}, respectively. Since σ and τ are permutations of $1, \dots, n$, there is a unique index i_3 such that $\sigma(i_1) + \tau(i_3) \equiv b_{i_2}$. Write the triple T_{i_3} in row 3. There is a unique i_4 such that $\sigma(i_2) + \tau(i_4) \equiv b_{i_3}$; write the triple T_{i_4} in row 4, and so on. Stop the first moment a number from column 1 occurs in this column twice, as i_p in row p and i_q in row q, where $p < q$.

We claim that $p = 1$ or $p = 2$. Assume on the contrary that $p > 2$ and consider the subarray containing rows p through q. Each of these rows has sum 0 modulo n, because $\sigma(i) + \tau(i) \equiv b_i$ for $i \neq i_1, i_2$, as already mentioned. On the other hand, by construction the sum in each downward right diagonal of the original array is also 0 modulo n. It follows that the six boxed entries add up to 0 modulo n, i. e.

$$-b_{i_p} + \tau(i_p) + \tau(i_{p+1}) + \sigma(i_{q-1}) + \sigma(i_q) - b_{i_q} \equiv 0.$$

Now, $i_p = i_q$ gives $b_{i_q} \equiv \sigma(i_q) + \tau(i_p)$, so that the displayed formula becomes $-b_{i_p} + \tau(i_{p+1}) + \sigma(i_{q-1}) \equiv 0$. And since $\sigma(i_{p-1}) - b_{i_p} + \tau(i_{p+1}) \equiv 0$ by the remark about diagonals, we obtain $\sigma(i_{p-1}) = \sigma(i_{q-1})$. This implies $i_{p-1} = i_{q-1}$, in contradiction with the definition of p and q. Thus $p = 1$ or $p = 2$ indeed.

Now delete the repeating qth row of the array. Then shift cyclically column 1 and column 3 by moving each of their entries one position down and one position up, respectively. The sum in each row of the new array is 0 modulo n, except possibly in the first and the last row ("most" of the new rows used to be diagonals of the initial array). For $p = 1$, the last row sum is also 0 modulo n, in view of $i_p = i_q = i_1$ and $\sigma(i_{q-2}) - b_{i_{q-1}} + \tau(i_q) \equiv 0$ (see the array on the left below). A single change is needed to accomodate the case $p = 2$: in column 3, interchange the top entry $\tau(i_2)$ and the bottom entry $\tau(i_1)$ (see the array on the right). The last row sum becomes 0 modulo n since $i_p = i_q = i_2$.

$$\begin{array}{ccc}
\sigma(i_{q-1}) & -b_{i_1} & \tau(i_2) \\
\sigma(i_1) & -b_{i_2} & \tau(i_3) \\
\sigma(i_2) & -b_{i_3} & \tau(i_4) \\
\vdots & \vdots & \vdots \\
\sigma(i_{q-3}) & -b_{i_{q-2}} & \tau(i_{q-1}) \\
\sigma(i_{q-2}) & -b_{i_{q-1}} & \tau(i_1)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\sigma(i_{q-1}) & -b_{i_1} & \tau(i_1) \\
\sigma(i_1) & -b_{i_2} & \tau(i_3) \\
\sigma(i_2) & -b_{i_3} & \tau(i_4) \\
\vdots & \vdots & \vdots \\
\sigma(i_{q-3}) & -b_{i_{q-2}} & \tau(i_{q-1}) \\
\sigma(i_{q-2}) & -b_{i_{q-1}} & \tau(i_2)
\end{array}$$

$$(p = 1) \qquad\qquad\qquad\qquad (p = 2)$$

For both $p = 1$ and $p = 2$, column 1 and column 3 are permutations the numbers of $\sigma(i_1), \dots, \sigma(i_{q-1})$ and $\tau(i_1), \dots, \tau(i_{q-1})$, respectively. So, adding the triples T_i not involved in the construction above, we obtain permutations σ' and τ' of $1, \dots, n$ in column 1 and column 3 such that $\sigma'(i) + \tau'(i) \equiv b_i$ for all $i \neq i_1$. Finally, the relation $\sigma'(i_1) + \tau'(i_1) \equiv b_{i_1}$ follows from the fact that $\Sigma\big(\sigma'(i) + \tau'(i)\big) \equiv 0 \equiv \Sigma b_i$.

We proved that the statement remains true if we change elements of the original sequence $a_1, \dots, a_n$ two at a time. However, one can obtain from any given $a_1, \dots, a_n$ any other zero-sum sequence by changing two elements at a time. (The condition that the sequence has sum zero modulo n is used here again.) And because the claim is true for any constant sequence, the conclusion follows.

C8. Let M be a convex n-gon, $n \geq 4$. Some $n-3$ of its diagonals are coloured green and some other $n-3$ diagonals are coloured red, so that no two diagonals of the same colour meet inside M. Find the maximum possible number of intersection points of green and red diagonals inside M.

Solution. We start with some preliminary observations. It is well-known that $n-3$ is the maximum number of nonintersecting diagonals in a convex n-gon and that any such $n-3$ diagonals partition the n-gon into $n-2$ triangles. It is also known (and not hard to show by induction) that at least two nonadjacent vertices are then left free; that is, there are at least two diagonals cutting off triangles from the n-gon.

Passing to the conditions of the problem, for any diagonal d, denote by $f(d)$ the number of green/red inresections lying on d. Take any pair of green diagonals d, d' and suppose there are k vertices, including the endpoints of d and d', of the part of M between d and d'. The remaining $n-k$ vertices span a convex polygon $A \ldots BC \ldots D$; here A and B are the vertices of M, adjacent to the endpoints of d, outside the "part of M" just mentioned, and C and D are the vertices adjacent to the endpoints of d', also outside that part. (A, B can coincide, as well as C, D.)

Let m be the number of red segments in the polygon $A \ldots BC \ldots D$. Since this $(n-k)$-gon has at most $n-k-3$ nonintersecting diagonals, we get

$$m \leq (n-k-3) + 2;$$

the last '2' comes from the segments AD and BC, which also can be red.

Each one of these m red segments intersects both d and d'. Each one of the remaining $n-3-m$ red segments can meet at most one of d, d'. Hence follows the estimate

$$f(d) + f(d') \leq 2m + (n-3-m) = n - 3 + m \leq n - 3 + (n-k-1) = 2n - k - 4.$$

Now we pair the green diagonals in the following way: we choose any two green diagonals that cut off two triangles from M; they constitute the first pair d_1, d_2. Then we choose two green diagonals that cut off two triangles from the residual $(n-2)$-gon, to make up the second pair d_3, d_4, and so on; d_{2r-1}, d_{2r} are the two diagonals in the r-th pairing. If $n-3$ is odd, the last green diagonal remains unpaired.

The polygon obtained after the r-th pairing has $n-2r$ vertices. Two sides of that polygon are the two diagonals from that pairing; its remaining sides are either sides of M or some of the green diagonals $d_1, \ldots, d_{2r}$. There are at most $2r$ vertices of M outside the part of M between d_{2r-1} and d_{2r}. Thus, denoting by k_r the number of vertices of that part, we have $k_r \geq n - 2r$.

In view of the previous estimates, the number of intersection points on those two diagonals satisfies the inequality

$$f(d_{2r-1}) + f(d_{2r}) \leq 2n - k_r - 4 \leq n + 2r - 4.$$

If $n-3$ is even, then $d_1, d_2, \ldots, d_{n-3}$ are all the green diagonals; and if $n-3$ is odd, the last unpaired green diagonal can meet at most $n-3$ (i.e., all) red ones. Thus, writing $n-3 = 2\ell + \varepsilon$, $\varepsilon \in \{0, 1\}$, we conclude that the total number of intersection points does not exceed the sum

$$\begin{aligned}\sum_{r=1}^{\ell}(n + 2r - 4) + \varepsilon \cdot (n-3) &= \ell(2\ell + \varepsilon - 1) + \ell(\ell+1) + \varepsilon(2\ell + \varepsilon) \\ &= 3\ell^2 + \varepsilon(3\ell+1) = \left\lceil \frac{3}{4}\big(n-3\big)^2 \right\rceil,\end{aligned}$$

where $\lceil t \rceil$ is the least integer not less than t. (For $n = 4$ the void sum $\sum_{r=1}^{0}$ evaluates to 0.)

The following example shows that this value can be attained, for both n even and n odd. Let PQ and RS be two sides of M chosen so that the diagonals QR and SP do not meet and, moreover, so that: if U is the part of the boundary of M between Q and R, and V is the part of the boundary of M between S and P ($S, P \notin U$, $Q, R \notin V$), then the numbers of vertices of M on U and on V differ by at most 1.

Colour in green: the diagonal PR, all diagonals connecting P to vertices on U and all diagonals connecting R to vertices on V.

Colour in red: the diagonal QS, all diagonals connecting Q to vertices on V and all diagonals connecting S to vertices on U.

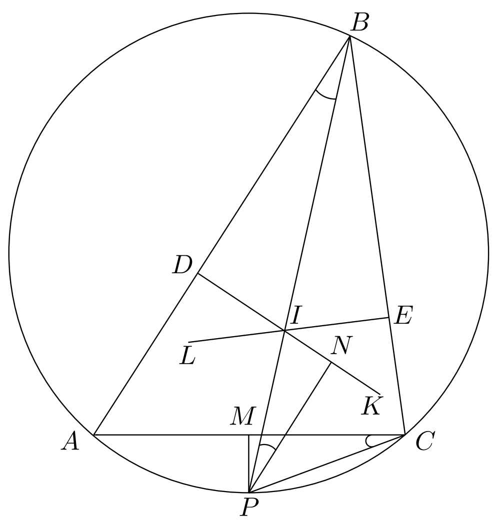
Then equality holds in the estimate above. In conclusion, $\lceil \frac{3}{4}\big(n-3\big)^2 \rceil$ is the greatest number of intersection points available.

Comment. It seems that the easiest way to verify that these examples indeed yield equality in the estimates obtained is to draw a diagram and visualise the process of detaching the corner triangles in appropriate pairings; all inequalities that appear in the arguments above turn into equalities. This is also the way (by inspecting the detaching procedure) in which it is expected that the solver can construct these examples.

Geometry

G1. In a triangle ABC satisfying $AB + BC = 3AC$ the incircle has centre I and touches the sides AB and BC at D and E, respectively. Let K and L be the symmetric points of D and E with respect to I. Prove that the quadrilateral $ACKL$ is cyclic.

Solution. Let P be the other point of intersection of BI with the circumcircle of triangle ABC, let M be the midpoint of AC and N the projection of P to IK. Since $AB + BC = 3AC$, we get $BD = BE = AC$, so $BD = 2CM$. Furthermore, $\angle ABP = \angle ACP$, therefore the triangles DBI and MCP are similar in ratio 2.



It is a known fact that $PA = PI = PC$. Moreover, $\angle NPI = \angle DBI$, so that the triangles PNI and CMP are congruent. Hence $ID = 2PM = 2IN$; i. e. N is the midpoint of IK. This shows that PN is the perpendicular bisector of IK, which gives $PC = PK = PI$. Analogously, $PA = PL = PI$. So P is the centre of the circle through A, K, I, L and C.

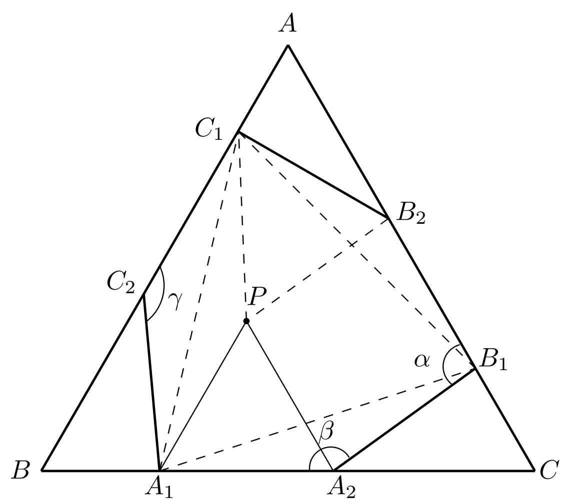
Comment. Variations are possible here. One might for instance *define* N to be the midpoint of IK and apply Ptolemy's theorem to the quadrilateral $BAPC$ and deduce that the triangles NPI and DBI are similar in ratio 2 to conclude that $PN \perp IK$.

G2. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA, and C_1, C_2 on AB, so that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

First solution. Let P be the point inside triangle ABC such that the triangle A_1A_2P is equilateral. Note that $A_1P \parallel C_1C_2$ and $A_1P = C_1C_2$, therefore $A_1PC_1C_2$ is a rhombus. Similarly, $A_2PB_2B_1$ is also a rhombus. Hence, the triangle C_1B_2P is equilateral. Let $\alpha = \angle B_2B_1A_2$, $\beta = \angle B_1A_2A_1$ and $\gamma = \angle C_1C_2A_1$. Then α and β are external angles of the triangle CB_1A_2 with $\angle C = 60^\circ$, and hence $\alpha + \beta = 240^\circ$. Note also that $\angle B_2PA_2 = \alpha$ and $\angle C_1PA_1 = \gamma$. So,

$$\alpha + \gamma = 360^\circ - (\angle C_1PB_2 + \angle A_1PA_2) = 240^\circ.$$

Hence, $\beta = \gamma$. Similarly, $\angle C_1B_2B_1 = \beta$. Therefore the triangles $A_1A_2B_1$, $B_1B_2C_1$ and $C_1C_2A_1$ are congruent, which implies that the triangle $A_1B_1C_1$ is equilateral. This shows that B_1C_2, A_1B_2 and C_1A_2 are the perpendicular bisectors of A_1C_1, C_1B_1 and B_1A_1; hence the result.



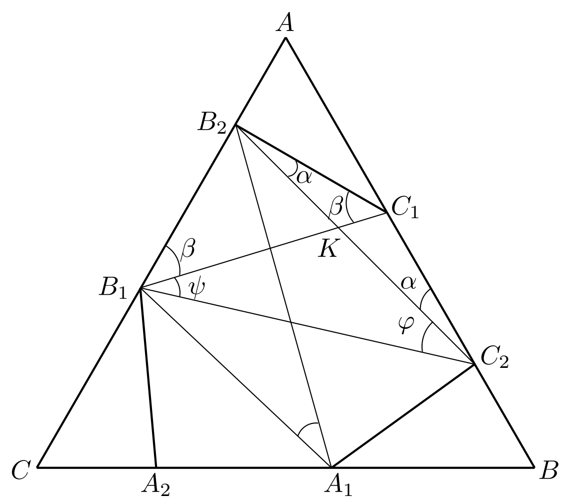
Second solution. Let $\alpha = \angle AC_2B_2$, $\beta = \angle AB_1C_1$ and K be the intersection of B_1C_1 with B_2C_2. The triangles $B_1B_2C_1$ and $B_2C_1C_2$ are isosceles, so $\angle B_1C_1B_2 = \beta$ and $\angle C_2B_2C_1 = \alpha$.

Denoting further $\angle B_1C_2B_2 = \varphi$ and $\angle C_1B_1C_2 = \psi$ we get (from the triangle AB_1C_2) $\alpha + \beta + \varphi + \psi = 120^\circ$; and (from the triangles KB_1C_2, KC_1B_2) $\alpha + \beta = \varphi + \psi$. Then $\alpha + \beta = 60^\circ$, $\angle C_1KB_2 = 120^\circ$, and so the quadrilateral

AB_2KC_1 is cyclic. Hence $\angle KAC_1 = \alpha$ and $\angle B_2AK = \beta$. From $KC_2 = KA = KB_1$ and $\angle B_1KC_2 = 120°$ we get $\varphi = \psi = 30°$.

In the same way, one shows that $\angle B_2A_1B_1 = \angle C_2B_1A_1 = 30°$. It follows that $A_1B_1B_2C_2$ is a cyclic quadrilateral and since its opposite sides A_1C_2 and B_1B_2 have equal lengths, it is an isosceles trapezoid. This implies that A_1B_1 and C_2B_2 are parallel lines, hence $\angle A_1B_1C_2 = \angle B_2C_2B_1 = 30°$.

Thus, B_1C_2 bisects the angle $C_1B_1A_1$. Similarly, by cyclicity, C_1A_2 and A_1B_2 are the bisectors of the angles $A_1C_1B_1$ and $B_1A_1C_1$, therefore they are concurrent.



Third solution. Consider the six vectors of equal lengths, with zero sum:

$$\mathbf{u} = \overrightarrow{B_2C_1},\ \mathbf{u}' = \overrightarrow{C_1C_2},\ \mathbf{v} = \overrightarrow{C_2A_1},\ \mathbf{v}' = \overrightarrow{A_1A_2},\ \mathbf{w} = \overrightarrow{A_2B_1},\ \mathbf{w}' = \overrightarrow{B_1B_2}.$$

Since $\mathbf{u}', \mathbf{v}', \mathbf{w}'$ clearly add up to zero vector, the same is true of $\mathbf{u}, \mathbf{v}, \mathbf{w}$. So $\mathbf{u} + \mathbf{v} = -\mathbf{w}$.

The sum of two vectors of equal lengths is a vector of the same length only if they make an angle of 120°. This follows e. g. from the parallelogram interpretation of vector addition or from the law of cosines. Therefore the three lines B_2C_1, C_2A_1, A_2B_1 define an equilateral triangle.

Consequently the "corner" triangles AC_1B_2, BA_1C_2, CB_1A_2 are similar, and in fact congruent, as $B_2C_1 = C_2A_1 = A_2B_1$. Thus the whole configuration is invariant under rotation through 120° about O, the centre of the triangle ABC.

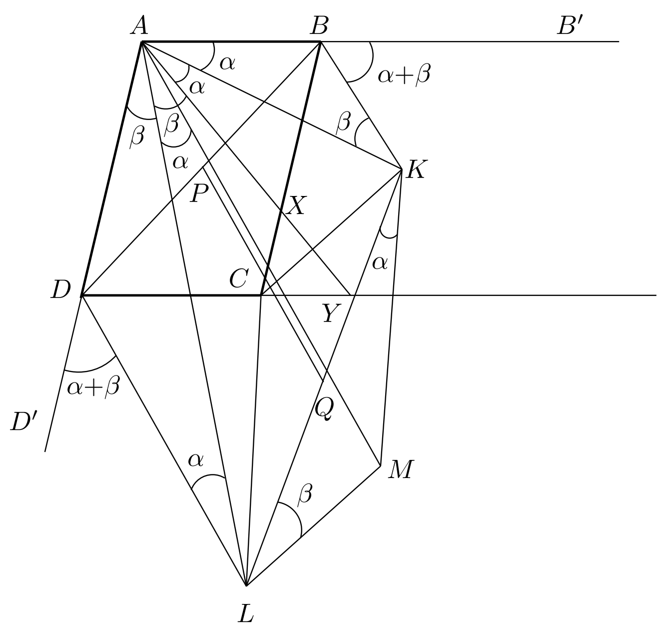
In view of the equalities $\angle B_2C_1C_2 = \angle C_2A_1A_2$ and $\angle A_1A_2B_1 = \angle B_1B_2C_1$ the line B_1C_2 is a symmetry axis of the hexagon $A_1A_2B_1B_2C_1C_2$, so it must pass through the rotation centre O. In conclusion, the three lines in question concur at O.

G3. Let $ABCD$ be a parallelogram. A variable line ℓ passing through the point A intersects the rays BC and DC at points X and Y, respectively. Let K and L be the centres of the excircles of triangles ABX and ADY, touching the sides BX and DY, respectively. Prove that the size of angle KCL does not depend on the choice of the line ℓ.

First solution. Let $\angle BAX = 2\alpha$, $\angle DAY = 2\beta$. The points K and L lie on the internal bisectors of the angles A in triangles ABX, ADY and on the external bisectors of their angles B and D. Taking B' and D' to be any points on the rays AB and AD beyond B and D, we have

$$\angle KAB = \angle KAX = \alpha, \qquad \angle LAD = \angle LAY = \beta,$$
$$\angle KBB' = \frac{1}{2}\angle BAD = \alpha + \beta = \angle LDD', \quad \text{so} \quad \angle AKB = \beta, \quad \angle ALD = \alpha.$$

Let the bisector of angle BAD meet the circumcircle of triangle AKL at a second point M. The vectors $\overrightarrow{BK}$, $\overrightarrow{AM}$, $\overrightarrow{DL}$ are parallel and equally oriented.



Since K and L lie on distinct sides of AM, we see that $AKML$ is a cyclic convex quadrilateral, and hence

$$\angle MKL = \angle MAL = \angle MAD - \angle LAD = \alpha; \qquad \text{likewise}, \quad \angle MLK = \beta.$$

Hence the triangles AKB, KLM, LAD are similar, so $AK \cdot LM = KB \cdot KL$ and $KM \cdot LA = KL \cdot LD$. Applying Ptolemy's theorem to the cyclic quadrilateral

$AKLM$, we obtain

$$AM \cdot KL = AK \cdot LM + KM \cdot LA = (KB + LD) \cdot KL,$$

implying $AM = BK + DL$.

The convex quadrilateral $BKLD$ is a trapezoid. Denoting the midpoints of its sides BD and KL respectively by P and Q, we have

$$2 \cdot PQ = BK + DL = AM;$$

notice that the vector $\overrightarrow{PQ}$ is also parallel to the three vectors mentioned earlier, in particular to $\overrightarrow{AM}$, and equally oriented.

Now, P is also the midpoint of AC. It follows from the last few conclusions that Q is the midpoint of side CM in the triangle ACM. So the segments KL and CM have a common midpoint, which means that $KCLM$ is a parallelogram. Thus, finally,

$$\angle KCL = \angle KML = 180^\circ - (\alpha + \beta) = 180^\circ - \frac{1}{2}\angle BAD,$$

which is a constant value, depending on the parallelogram $ABCD$ alone.

Second solution. Let the line AK meet DC at E, and let the line AL meet BC at F. Denote again $\angle BAX = 2\alpha$, $\angle DAY = 2\beta$. Then $\angle BFA = \beta$. Moreover, $\angle KBF = (1/2)\angle BAD = \alpha + \beta = \angle KAF$. Since the points A and B lie on the same side of the line KF, we infer that $ABKF$ is a cyclic quadrilateral.

Speaking less rigourously, the points A, K, B, F are concyclic. The points E and C lie on the lines AK and BF, and the segment EC is parallel to AB. Therefore the points E, K, C, F lie on a circle, too; this follows easily from an inspection of angles—one just has to consider three cases, according as two, one or none of the points E, C lie(s) on the same side of line KF as the segment AB does.

Analogously, the points F, L, C, E lie on a circle. Clearly C, K, L are three distinct points. It follows that all five points C, E, F, K, L lie on a circle Ω.

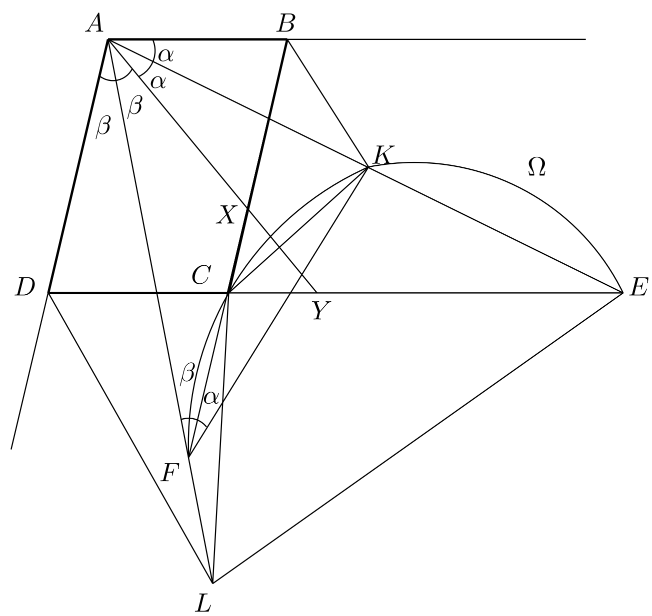
From the cyclic quadrilateral $ABKF$ we have $\angle BFK = \angle BAK = \alpha$, which combined with $\angle BFA = \beta$ implies $\angle KFA = \alpha + \beta$. Since the points A, F, L are in line, $\angle KFL$ is either $\alpha + \beta$ or $180^\circ - (\alpha + \beta)$; and since K, C, F, L are concyclic, also $\angle KCL$ is either $\alpha + \beta$ or $180^\circ - (\alpha + \beta)$.

All that remains is to eliminate one of these two possibilities. To this effect, we will show that the points A and C lie on the same side of the line KL.

Assume without loss of generality that Y, the point where ℓ cuts the ray DC, lies beyond C on that ray. Then so does E.

If also F lies on the ray BC beyond C then Ω does not penetrate the interior of $ABCD$. Hence the line KL does not separate A from C. And if F lies on the segment BC then L lies in the half-plane with edge BC, not containing A. Since K also lies in that half-plane, and since L lies on the opposite side of the line DC than A, this again implies that the line KL does not separate A from C.

Notice that the circle Ω intersects each one of the rays AK, AL at two points (K, E, resp. L, F), possibly coinciding. Thus A lies outside this circle. Knowing that C and A lie on the same side of the line KL, we infer that $\angle KCL > \angle KAL = \alpha + \beta$. This leaves the other possibility as the unique one: $\angle KCL = 180^\circ - (\alpha + \beta)$.



Comment. Alternatively, continuity argument could be applied. If $\angle KCL$ takes on only two values, it must be a constant.

In our attempt to stay within the realm of classical geometry, we were forced to investigate the disposition of the points and lines in question. Notice that the first solution is case-independent.

Other solutions are available by calculation, be it with complex numbers or linear transformations in the coordinate plane; but no one of such approaches seems to be straightforward.

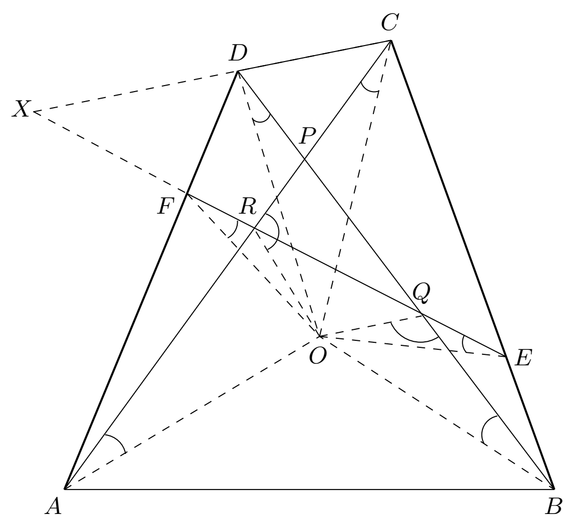
G4. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel to DA. Let two variable points E and F lie on the sides BC and DA, respectively, and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of triangles PQR, as E and F vary, have a common point other than P.

First solution. Let the perpendicular bisectors of the segments AC and BD meet at O. We show that the circumcircles of triangles PQR pass through O, which is fixed.

It follows from the equalities $OA = OC$, $OB = OD$ and $DA = BC$ that the triangles ODA and OBC are congruent. So the rotation about the point O through the angle BOD takes the point B to D and the point C to A. Since $BE = DF$, the same rotation takes the point E to F. This implies that $OE = OF$ and

$$\angle EOF = \angle BOD = \angle COA \text{ (= the angle of rotation)}.$$

These equalities imply that the isosceles triangles EOF, BOD and COA are similar.



Suppose first that the three lines AB, CD and EF are not all parallel. Assume without loss of generality that the lines EF and CD meet at X. From the Menelaus theorem, applied to the triangles ACD and BCD, we obtain

$$\frac{AR}{RC} = \frac{AF}{FD} \cdot \frac{DX}{XC} = \frac{CE}{EB} \cdot \frac{DX}{XC} = \frac{DQ}{QB}.$$

In the case $AB \parallel EF \parallel CD$, the quadrilateral $ABCD$ is an isosceles trapezoid, and E, F are the midpoints of its lateral sides. The equality $AR/RC = DQ/QB$ is then obvious.

It follows from the this equality and the similitude of triangles BOD and COA that the triangles BOQ and COR are similar. Thus $\angle BQO = \angle CRO$, which means that the points P, Q, R and O are concyclic.

Second solution. This is just a variation of the preceding proof. As in the first solution, we show that the triangles EOF, BOD and COA are similar. Denote by K, L, M the feet of the perpendiculars from the point O onto the lines EF, BD, AC, respectively. In view of the similarity just mentioned,

$$\frac{OK}{OE} = \frac{OL}{OB} = \frac{OM}{OC} = \lambda \quad \text{and} \quad \angle EOK = \angle BOL = \angle COM = \varphi\,.$$

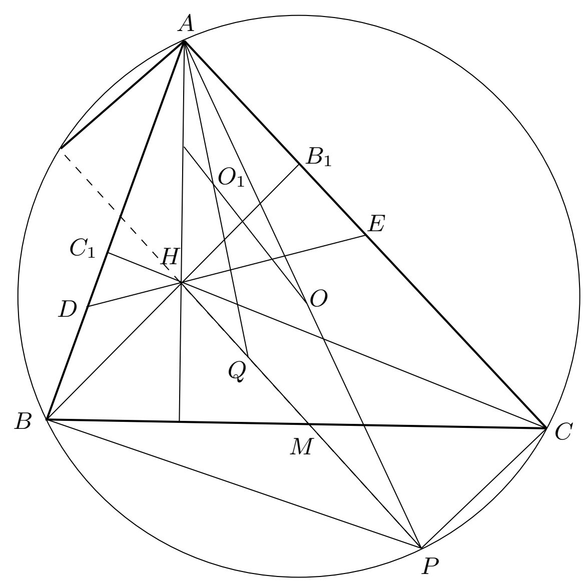
Therefore the rotation about the point O through the angle φ, composed with the homothety with centre O and ratio λ, takes the points B, E, C to the points L, K, M, respectively. This implies that the points L, K, M are collinear. Hence by the theorem about the Simson line we conclude that the circumcircle of PQR passes through O.

Comment. The proposer observes that (as can be seen from the above solutions) the point under discussion can also be identified as the second common point of the circumcircles of triangles BCP and DAP.

G5. Let ABC be an acute-angled triangle with $AB \neq AC$, let H be its orthocentre and M the midpoint of BC. Points D on AB and E on AC are such that $AE = AD$ and D, H, E are collinear. Prove that HM is orthogonal to the common chord of the circumcircles of triangles ABC and ADE.

Solution. Let O and O_1 be the circumcentres of the triangles ABC and ADE, respectively. Since the radical axis of two circles is perpendicular to their line of centres, we have to prove that OO_1 is parallel to HM.

Consider the diameter AP of the circumcircle of ABC and let B_1 and C_1 be the feet of the altitudes from B and C in the triangle ABC. Since $AB \perp BP$ and $AC \perp CP$, it follows that $HC \parallel BP$ and $HB \parallel CP$. Thus $BPCH$ is a parallelogram; as a consequence, HM cuts the circle at P.



The triangle ADE is isosceles, so its circumcentre O_1 lies on the bisector of the angle BAC. We shall prove that the intersection Q of AO_1 with HP is the symmetric of A with respect to O_1. The rays AH and AO are isogonal conjugates, so the line AQ bisects $\angle HAP$. Then the bisector theorem in the triangle AHP yields

$$\frac{QH}{QP} = \frac{AH}{AP}.$$

Because ADE is an isosceles triangle, an easy angle computation shows that HD bisects $\angle C_1HB$. Hence the bisector theorem again gives

$$\frac{DC_1}{DB} = \frac{HC_1}{HB}.$$

Applying once more the fact that AH and AP are isogonal lines, we see that the right triangles C_1HA and CPA are similar, so

$$\frac{AH}{AP} = \frac{C_1H}{CP} = \frac{C_1H}{BH};$$

the last equality holds because $BPCH$ is a parallelogram, so that $PC = BH$.

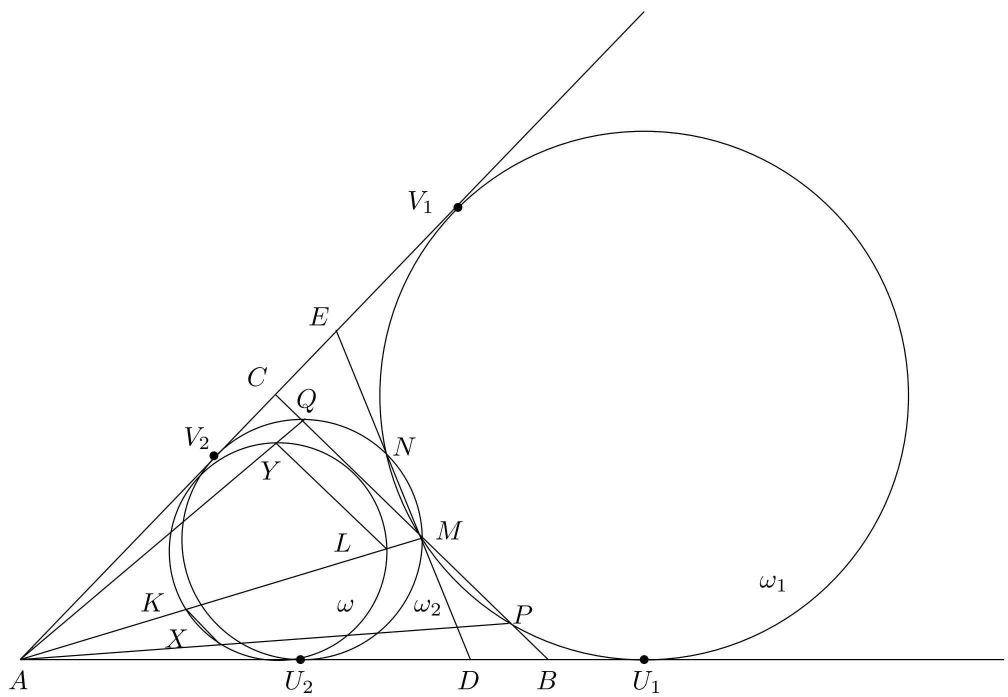
Summarizing, we conclude that

$$\frac{DC_1}{DB} = \frac{QH}{QP},$$

that is, $QD \parallel HC_1$. In the same way we obtain $QE \parallel HB_1$. As a consequence, AQ is a diameter of the circumcircle of triangle ADE, implying that O_1 is the midpoint of AQ. Thus $OO_1 \parallel PQ$; that is, OO_1 is parallel to HM.

G6. The median AM of a triangle ABC intersects its incircle ω at K and L. The lines through K and L parallel to BC intersect ω again at X and Y. The lines AX and AY intersect BC at P and Q. Prove that $BP = CQ$.

First solution. Without loss of generality, one can assume the notation in the figure. Let ω_1 be the image of ω under the homothety with centre A and ratio AM/AK. This homothety takes K to M and hence X to P, because $KX \parallel BC$. So ω_1 is a circle through M and P inscribed in $\angle BAC$. Denote its points of tangency with AB and AC by U_1 and V_1, respectively. Analogously, let ω_2 be the image of ω under the homothety with center A and ratio AM/AL. Then ω_2 is a circle through M and Q also inscribed in $\angle BAC$. Let it touch AB and AC at U_2 and V_2, respectively. Then $U_1U_2 = V_1V_2$, as U_1U_2 and V_1V_2 are the common external tangents of ω_1 and ω_2.



By the power-of-a-point theorem in ω_1 and ω_2, one has $BP = BU_1^2/BM$ and $CQ = CV_2^2/CM$. Since $BM = CM$, it suffices to show that $BU_1 = CV_2$.

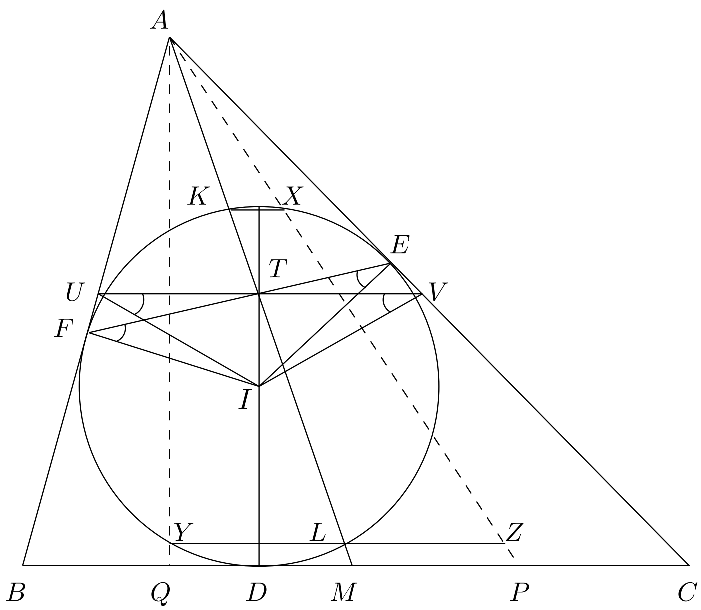
Consider the second common point N of ω_1 and ω_2 (M and N may coincide, in which case the "line MN" is the common tangent). Let the line MN meet AB and AC at D and E, respectively. Clearly D is the midpoint of U_1U_2 because $DU_1^2 = DM \cdot DN = DU_2^2$ by the power-of-a-point theorem again. Likewise, E is the midpoint of V_1V_2. Note that B and C are on different sides of DE, which reduces the problem to proving that $BD = CE$.

Since DE is perpendicular to the line of centres of ω_1 and ω_2, we have $\angle ADM = \angle AEM$. Then the law of sines for triangles BDM and CEM gives

$$BD = \frac{BM \sin\angle BMD}{\sin\angle BDM} = \frac{BM \sin\angle BMD}{\sin\angle ADM}, \qquad CE = \frac{CM \sin\angle CME}{\sin\angle AEM}.$$

Because $BM = CM$ and $\angle BMD = \angle CME$, the conclusion follows.

Second solution. Let ω touch BC, CA and AB at D, E and F, respectively, and let I be the incentre of triangle ABC. The key step of this solution is the observation that the lines AM, EF and DI are concurrent.



Indeed, suppose that EF and DI meet at T. Let the parallel through T to BC meet AB and AC at U and V, respectively. One has $IT \perp UV$, and since $IE \perp AC$, it follows that the points I, T, V and E are concyclic. Moreover, V and E lie on the same side of the line IT, so that $\angle IVT = \angle IET$. By symmetry, $\angle IUT = \angle IFT$. But $\angle IET = \angle IFT$, hence UVI is an isosceles triangle with altitude IT to its base UV. So T is the midpoint of UV, implying that AT meets BC at its midpoint M.

Now observe that EF is the polar of A with respect to ω, therefore

$$\frac{AK}{AL} = \frac{TK}{TL}.$$

Furthermore, let LY meet AP at Z. Then

$$\frac{KX}{LZ} = \frac{AK}{AL}.$$

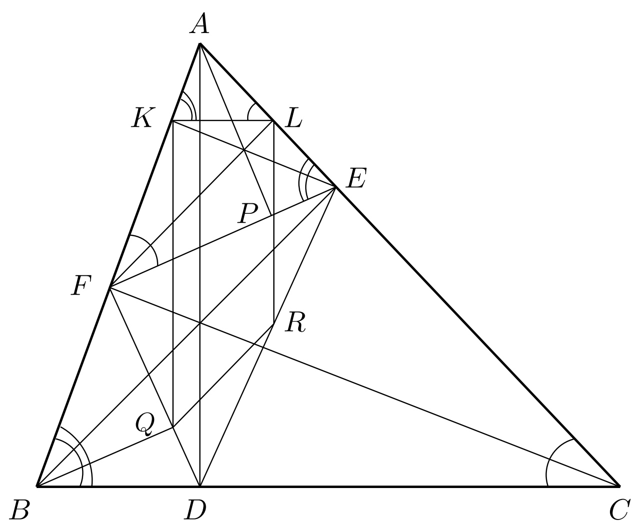
The line IT is the common perpendicular bisector of KX and LY. As we have shown, T lies on AM, i. e. on KL. Hence

$$\frac{KX}{LY} = \frac{TK}{TL}.$$

The last three relations show that L is the midpoint of YZ, so M is the midpoint of PQ.

G7. In an acute triangle ABC, let D, E, F, P, Q, R be the feet of perpendiculars from A, B, C, A, B, C to BC, CA, AB, EF, FD, DE, respectively. Prove that $p(ABC)p(PQR) \ge p(DEF)^2$, where $p(T)$ denotes the perimeter of the triangle T.

First solution. The points D, E and F are interior to the sides of triangle ABC which is acute-angled. It is widely known that the triangles ABC and AEF are similar. Equivalently, the lines BC and EF are antiparallel with respect to the sides of $\angle A$. Similar conclusions hold true for the pairs of lines CA, FD and AB, DE. This is a general property related to the feet of the altitudes in every triangle. In particular, it follows that P, Q and R are interior to the respective sides of triangle DEF.



Let K and L be the feet of the perpendiculars from E and F to AB and AC, respectively. By the remark above, KL and EF are antiparallel with respect to the sides of the same $\angle A$. Therefore $\angle AKL = \angle AEF = \angle ABC$, meaning that $KL \parallel BC$.

Now, EK and BQ are respective altitudes in the similar triangles AEF and DBF, so they divide the opposite sides in the same ratio:

$$\frac{AK}{KF} = \frac{DQ}{QF}.$$

This implies $KQ \parallel AD$. By symmetry, $LR \parallel AD$. Since KL is parallel to BC, it is perpendicular to AD. It follows that $QR \ge KL$.

From the similar triangles AKL, AEF, ABC we obtain

$$\frac{KL}{EF} = \frac{AK}{AE} = \cos\angle A = \frac{AE}{AB} = \frac{EF}{BC}.$$

Hence $QR \geq EF^2/BC$. Likewise, $PQ \geq DE^2/AB$ and $RP \geq FD^2/CA$.

Therefore it suffices to show that

$$(AB+BC+CA)\left(\frac{DE^2}{AB}+\frac{EF^2}{BC}+\frac{FD^2}{CA}\right) \geq (DE+EF+FA)^2,$$

which is a direct consequence of the Cauchy-Schwarz inequality.

Second solution. Let $\alpha = \angle A$, $\beta = \angle B$, $\gamma = \angle C$. There is no loss of generality in assuming that triangle ABC has circumradius 1. The triangles AEF and ABC are similar in ratio $\cos\alpha$, so $EF = BC\cos\alpha = \sin 2\alpha$. By symmetry, $FD = \sin 2\beta$, $DE = \sin 2\gamma$. Next, since $\angle BDF = \angle CDE = \alpha$, it follows that $DQ = BD\cos\alpha = AB\cos\beta\cos\alpha = 2\cos\alpha\cos\beta\sin\gamma$.

Similarly, $DR = 2\cos\alpha\sin\beta\cos\gamma$. Now the law of cosines for triangle DQR gives after short manipulation

$$QR = \sin 2\alpha\sqrt{1-\sin 2\beta\sin 2\gamma}.$$

Likewise, $RP = \sin 2\beta\sqrt{1-\sin 2\gamma\sin 2\alpha}$, $PQ = \sin 2\gamma\sqrt{1-\sin 2\alpha\sin 2\beta}$.

Therefore the given inequality is equivalent to

$$2\sum\sin\alpha\sum\sin 2\alpha\sqrt{1-\sin 2\beta\sin 2\gamma} \geq \left(\sum\sin 2\alpha\right)^2,$$

where Σ means a cyclic sum over α, β, γ, the angles of an acute triangle. In view of this, all trigonometric functions below are positive. To eliminate the square roots, observe that

$$1-\sin 2\beta\sin 2\gamma = \sin^2(\beta-\gamma)+\cos^2\alpha \geq \cos^2\alpha.$$

Hence it suffices to establish $2\sum\sin\alpha\sum\sin 2\alpha\cos\alpha \geq (\sum\sin 2\alpha)^2$. This is yet another immediate consequence of the Cauchy-Schwarz inequality:

$$\sum 2\sin\alpha\sum\sin 2\alpha\cos\alpha \geq \left(\sum\sqrt{2\sin\alpha}\sqrt{\sin 2\alpha\cos\alpha}\right)^2 = \left(\sum\sin 2\alpha\right)^2.$$

Third solution. A stronger conclusion is true, namely:

$$\frac{p(ABC)}{p(DEF)} \geq 2 \geq \frac{p(DEF)}{p(PQR)}.$$

The left inequality is a known fact, so we consider only the right one.

It is immediate that the points A, B and C are the excentres of triangle DEF. Therefore P, Q and R are the tangency points of the excircles of this triangle with its sides. For the sake of clarity, let us adopt the notation $a = EF$, $b = FD$, $c = DE$, $\alpha = \angle D$, $\beta = \angle E$, $\gamma = \angle F$ now for the sides and angles of triangle DEF. Also, let $s = (a+b+c)/2$. Then $ER = FQ = s - a$, $FP = DR = s - b$, $DQ = EP = s - c$.

Now we regard the line DE as an axis by choosing the direction from D to E as the positive direction. The signed length of a line segment UV on this axis will be denoted by $\overline{UV}$. Let X and Y be the orthogonal projections onto DE of P and Q, respectively. On one hand, $\overline{DE} = \overline{DY} + \overline{YX} + \overline{XE}$. On the other hand,

$$\overline{DY} = DQ\cos\alpha, \qquad \overline{XE} = EP\cos\beta.$$

Observe that these inequalities hold true in all cases, regardless of whether or not α and β are acute. Finally, it is clear that $\overline{YX} \le PQ$. In conclusion,

$$DE = (s-c)(\cos\alpha + \cos\beta) + \overline{YX} \le (s-c)(\cos\alpha + \cos\beta) + PQ.$$

By symmetry,

$$EF \le (s-a)(\cos\beta + \cos\gamma) + QR, \qquad FD \le (s-b)(\cos\gamma + \cos\alpha) + RP.$$

Adding up yields $p(DEF) \le \sum(s-c)(\cos\alpha + \cos\beta) + p(PQR)$, where again Σ denotes a cyclic sum over α, β, γ. This sum is equal to $a\cos\alpha + b\cos\beta + c\cos\gamma$, since $(s-b) + (s-c) = a$, $(s-c) + (s-a) = b$, $(s-a) + (s-b) = c$.

Now it suffices to show that $a\cos\alpha + b\cos\beta + c\cos\gamma \le (1/2)p(DEF)$. Suppose that $a \le b \le c$; then $\cos\alpha \ge \cos\beta \ge \cos\gamma$, so one can apply Chebyshev's inequality to the triples (a, b, c) and $(\cos\alpha, \cos\beta, \cos\gamma)$. This gives

$$a\cos\alpha + b\cos\beta + c\cos\gamma \le \frac{1}{3}(a+b+c)(\cos\alpha + \cos\beta + \cos\gamma).$$

But $\cos\alpha + \cos\beta + \cos\gamma \le 3/2$ for every triangle, and the result follows.

Comment. This last solution shows that the proposed inequality splits into two independent ones, which can be expressed in words:

> In every triangle, the perimeter of its orthic triangle is not greater than half the perimeter of the triangle itself, and the perimeter of its Nagel triangle is not smaller than half the perimeter of the triangle itself.

Whereas the first of these inequalities is indeed a very well-known fact, this seems not to be the case with the second one.

Number Theory

N1. Determine all positive integers relatively prime to all terms of the infinite sequence $a_n = 2^n + 3^n + 6^n - 1$ $(n = 1, 2, 3, \dots)$.

Solution. We claim that 1 is the only such number. This amounts to showing that every prime p is a divisor of a certain a_n. This is true for $p = 2$ and $p = 3$ as $a_2 = 48$.

Fix a prime $p > 3$. All congruences that follow are considered modulo p. By Fermat's little theorem, one has $2^{p-1} \equiv 1$, $3^{p-1} \equiv 1$, $6^{p-1} \equiv 1$. Then the evident congruence $3 + 2 + 1 \equiv 6$ can be written as

$$3 \cdot 2^{p-1} + 2 \cdot 3^{p-1} + 6^{p-1} \equiv 6, \quad \text{or} \quad 6 \cdot 2^{p-2} + 6 \cdot 3^{p-2} + 6 \cdot 6^{p-2} \equiv 6.$$

Simplifying by 6 shows that $a_{p-2} = 2^{p-2} + 3^{p-2} + 6^{p-2} - 1$ is divisible by p, and the proof is complete.

N2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and infinitely many negative terms. Suppose that for every positive integer M the numbers $a_1, a_2, \ldots, a_M$ leave different remainders upon division by M. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

Solution. The hypothesis of the problem can be reformulated by saying that for every positive integer M the numbers $a_1, a_2, \ldots, a_M$ form a complete system of residue classes modulo M. Note that if $i < j$ then $a_i \neq a_j$, otherwise the set $\{a_1, \ldots, a_j\}$ would contain at most $j-1$ distinct residues modulo j. Furthermore, if $i < j \leq n$, then $|a_i - a_j| \leq n-1$, for if $m = |a_i - a_j| \geq n$, then the set $\{a_1, \ldots, a_m\}$ would contain two numbers congruent modulo m, which is impossible.

Given any $n \geq 1$, let $i(n)$, $j(n)$ be the indices such that $a_{i(n)}$, $a_{j(n)}$ are respectively the smallest and the largest number among $a_1, \ldots, a_n$. The above arguments show that $|a_{i(n)} - a_{j(n)}| = n - 1$, therefore the set $\{a_1, \ldots, a_n\}$ consists of all integers between $a_{i(n)}$ and $a_{j(n)}$.

Now let x be an arbitrary integer. Since $a_k < 0$ for infinitely many k and the terms of the sequence are distinct, we conclude that there exists i such that $a_i < x$. By a similar argument, there exists j such that $x < a_j$. Hence, if $n > \max\{i, j\}$, we conclude that every number between a_i and a_j (x in particular) is in $\{a_1, \ldots, a_n\}$.

Comment. Proving that for every M the set $\{a_1, \ldots, a_M\}$ is a block of consecutive integers can be also achieved by induction.

N3. Let a, b, c, d, e and f be positive integers. Suppose that the sum $S = a + b + c + d + e + f$ divides both $abc + def$ and $ab + bc + ca - de - ef - fd$. Prove that S is composite.

Solution. By hypothesis, all coefficients of the quadratic polynomial

$$\begin{aligned} f(x) &= (x + a)(x + b)(x + c) - (x - d)(x - e)(x - f) \\ &= Sx^2 + (ab + bc + ca - de - ef - fd)x + (abc + def) \end{aligned}$$

are multiples of S. Evaluating f at d we get that $f(d) = (a + d)(b + d)(c + d)$ is a multiple of S. This readily implies that S is composite because each of $a + d$, $b + d$ and $c + d$ is less than S.

N4. Find all positive integers $n > 1$ for which there exists a unique integer a with $0 < a \leq n!$ such that $a^n + 1$ is divisible by $n!$.

Solution. The answer is "n is prime."

If $n = 2$, the only solution is $a = 1$. If $n > 2$ is even, then a^n is a square, therefore $a^n + 1$ is congruent to 1 or 2 modulo 4, while $n!$ is divisible by 4. So there is no appropriate a in this case.

From now on, n is odd. Assume that $n = p$ is a prime and that $p! \mid a^p + 1$ for some a, $0 < a \leq p!$. By Fermat's little theorem, $a^p + 1 \equiv a + 1 \pmod{p}$. So, if p does not divide $a + 1$, then $a^{p-1} + \cdots + a + 1 = (a^p + 1)/(a + 1) \equiv 1 \pmod{p}$, which is a contradiction. Thus, $p \mid a + 1$.

We shall show that $(a^p + 1)/(a + 1)$ has no prime divisors $q < p$. This will be enough to deduce the uniqueness of a. Indeed, the relation

$$(p-1)! \mid (a+1)\left(\frac{a^p+1}{a+1}\right)$$

forces $(p-1)! \mid a + 1$. Combined with $p \mid a + 1$, this leads to $p! \mid a + 1$, and hence showing $a = p! - 1$.

Suppose on the contrary that $q \mid (a^p + 1)/(a + 1)$, where $q < p$ is prime. Note that q is odd. We get $a^p \equiv -1 \pmod{q}$, therefore $a^{2p} \equiv 1 \pmod{q}$. Clearly, q is coprime to a, so $a^{q-1} \equiv 1 \pmod{q}$. Writing $d = \gcd(q - 1, 2p)$, we obtain $a^d \equiv 1 \pmod{q}$. Since $q < p$, we have $d = 2$. Hence, $a \equiv \pm 1 \pmod{q}$. The case $a \equiv 1 \pmod{q}$ gives $(a^p + 1)/(a + 1) \equiv 1 \pmod{q}$, which is impossible. The case $a \equiv -1 \pmod{q}$ gives

$$\begin{aligned}\frac{a^p+1}{a+1} &\equiv a^{p-1} - a^{p-2} + \cdots + 1 \\ &\equiv (-1)^{p-1} - (-1)^{p-2} + \cdots + 1 \equiv p \pmod{q},\end{aligned}$$

leading to $q \mid p$ which is not possible as $q < p$. So, we see that primes fulfill the conditions under discussion.

It remains to deal with the case when n is odd and composite. Let $p < n$ be the least prime divisor of n. Let p^α be the highest power of p which divides $n!$. Since $2p < p^2 \leq n$, we have $n! = 1 \dots p \dots (2p) \dots$, so $\alpha \geq 2$. Write $m = n!/p^\alpha$, and take any integer a satisfying

$$a \equiv -1 \left(\bmod\ p^{\alpha-1}m\right). \tag{1}$$

Write $a = -1 + p^{\alpha-1}k$. Then

$$a^p = (-1 + p^{\alpha-1}k)^p = -1 + p^\alpha k + p^\alpha \sum_{j=2}^{p} (-1)^{p-j} \binom{p}{j} p^{j(\alpha-1)} k^j = -1 + p^\alpha M,$$

where M is an integer because $j(\alpha - 1) \geq \alpha$ for all $j \geq 2$ and $\alpha \geq 2$. Thus p^α divides $a^p + 1$, and hence also $a^n + 1$, because $p \mid n$ and n is odd. Furthermore, m too is a divisor of $a + 1$, and hence of $a^n + 1$. Since m is coprime to p, $(a^n + 1)/n!$ is an integer for all a satisfying congruence (1). Since it is clear that there are $p > 2$ integers in the interval $[1, n!]$ satisfying (1), we conclude that composite values of n do not satisfy the condition given in the problem.

Comment. The fact that no prime divisor of $(a^p + 1)/(a + 1)$ is smaller than p is not a mere curiosity. More is true and can be deduced easily from the above proof, namely that if q is a prime factor of the above number, then either $q = p$ (and this happens if and only if $p \mid a + 1$) or $q \equiv 1 \pmod{p}$.

N5. Denote by $d(n)$ the number of divisors of the positive integer n. A positive integer n is called *highly divisible* if $d(n) > d(m)$ for all positive integers $m < n$. Two highly divisible integers m and n with $m < n$ are called consecutive if there exists no highly divisible integer s satisfying $m < s < n$.

(a) Show that there are only finitely many pairs of consecutive highly divisible integers of the form (a, b) with $a \mid b$.

(b) Show that for every prime number p there exist infinitely many positive highly divisible integers r such that pr is also highly divisible.

Solution. This problem requires an analysis of the structure of the highly divisible integers. Recall that if n has prime factorization

$$n = \prod_{p^{\alpha_p(n)} || n} p^{\alpha_p(n)},$$

where p stands for a prime, then $d(n) = \prod_{p^{\alpha_p} || n} (\alpha_p(n) + 1)$.

Let us start by noting that since $d(n)$ takes arbitrarily large values (think of $d(m!)$, for example, for arbitrary large m's), there exist infinitely many highly divisible integers. Furthermore, it is easy to see that if n is highly divisible and

$$n = 2^{\alpha_2(n)} 3^{\alpha_3(n)} \dots p^{\alpha_p(n)},$$

then $\alpha_2(n) \geq \dots \geq \alpha_p(n)$. Thus, if $q < p$ are primes and $p \mid n$, then $q \mid n$.

We show that for every prime p all but finitely many highly divisible integers are multiples of p. This is obviously so for $p = 2$. Assume that this were not so, that p is the rth prime ($r > 1$), and that n is one of the infinitely many highly divisible integers whose largest prime factor is less than p. For such an n, $(\alpha_2(n) + 1)^{r-1} \geq d(n)$, therefore $\alpha_2(n)$ takes arbitrarily large values. Let n be such that $2^{\alpha_2(n)-1} > p^2$ and look at $m = np/2^{\lfloor \alpha_2(n)/2 \rfloor}$. Clearly, $m < n$, while

$$d(m) = 2d(n) \frac{\alpha_2(n) - \lfloor \alpha_2(n)/2 \rfloor + 1}{\alpha_2(n) + 1} > d(n)$$

contradicting the fact that n is highly divisible.

We now show a stronger property, namely that for any prime p and constant κ, there are only finitely many highly divisible positive integers n such that $\alpha_p(n) \leq \kappa$. Indeed, assume that this were not so. Let κ be a constant such that $\alpha_p(n) \leq \kappa$ for infinitely many highly divisible n. Let q be a large prime satisfying $q > p^{2\kappa+1}$. All but finitely many such positive integers n are multiples of q. Look at the number $m = p^{\alpha_p(n)\alpha_q(n) + \alpha_p(n) + \alpha_q(n)} n / q^{\alpha_q(n)}$. An immediate calculation shows that $d(m) = d(n)$, therefore $m > n$. Thus,

$$p^{2\alpha_p(n)\alpha_q(n) + \alpha_q(n)} \geq p^{\alpha_p(n)\alpha_q(n) + \alpha_q(n) + \alpha_p(n)} > q^{\alpha_q(n)},$$

giving $p^{2\alpha_p(n)+1} > q > p^{2\kappa+1}$, and we get a contradiction with the fact that $\alpha_p(n) \leq \kappa$.

We are now ready to prove both (a) and (b). For (a), let n be highly divisible and such that $\alpha_3(n) \geq 8$. All but finitely many highly divisible integers n have this property. Now $8n/9$ is an integer and $8n/9 < n$, therefore $d(8n/9) < d(n)$. This implies

$$(\alpha_2(n)+4)(\alpha_3(n)-1) < (\alpha_2(n)+1)(\alpha_3(n)+1),$$

which is equivalent to

$$3\alpha_3(n) - 5 < 2\alpha_2(n). \tag{1}$$

Assume now that $n \mid m$ are consecutive and highly divisible. Since already $d(2n) > d(n)$, we get that there must be a highly divisible integer in $(n, 2n]$. Thus $m = 2n$, leading to $d(3n/2) \leq d(n)$ (or else there must be a highly divisible number between n and $3n/2$). This gives

$$\alpha_2(n)(\alpha_3(n)+2) \leq (\alpha_2(n)+1)(\alpha_3(n)+1),$$

which is equivalent to

$$\alpha_2(n) \leq \alpha_3(n) + 1,$$

which together with $\alpha_3(n) \geq 8$ contradicts inequality (1). This proves (a).

For part (b), let k be any positive integer and look at the smallest highly divisible positive integer n such that $\alpha_p(n) \geq k$. All but finitely many highly divisible integers n satisfy this last inequality. We claim that n/p is also highly divisible. If this were not so, then there would exist a highly divisible positive integer $m < n/p$ with $d(m) \geq d(n/p)$. Note that, by assumption, $\alpha_p(m) < \alpha_p(n)$. Then,

$$d(mp) = d(m)\frac{\alpha_p(m)+2}{\alpha_p(m)+1} \geq d(n/p)\frac{\alpha_p(n)+1}{\alpha_p(n)} = d(n),$$

where for the above inequality we used the fact that the function $(x+1)/x$ is decreasing. However, $mp < n$, so the above inequality contradicts the fact that n is highly divisible. This contradiction shows that n/p is highly divisible, and since k can be taken to be arbitrarily large, we get infinitely many examples of highly divisible integers n such that n/p is also highly divisible.

Comment. The notion of a highly divisible integer first appeared in a paper of Ramanujan in 1915. Eric Weinstein's *World of Mathematics* has one web page mentioning some properties of these numbers (called highly composite) and giving some bibliographical references, while Ross Honsberger's *Mathematical*

Gems (Third Edition) has a chapter dedicated to them. In spite of all these references, the properties of these numbers mentioned in the above sources have little relevance for the problem at hand and we believe that if given to the exam, the students who have seen these numbers before will not have any significant advantage over the ones who encounter them for the first time.

N6. Let a and b be positive integers such that $a^n + n$ divides $b^n + n$ for every positive integer n. Show that $a = b$.

Solution. Assume that $b \neq a$. Taking $n = 1$ shows that $a + 1$ divides $b + 1$, so that $b \geq a$. Let $p > b$ be a prime and let n be a positive integer such that

$$n \equiv 1 \pmod{p-1} \quad \text{and} \quad n \equiv -a \pmod{p}.$$

Such an n exists by the Chinese remainder theorem. (Without the Chinese remainder theorem, one could notice that $n = (a+1)(p-1)+1$ has this property.)

By Fermat's little theorem, $a^n = a(a^{p-1} \cdots a^{p-1}) \equiv a \pmod{p}$, and therefore $a^n + n \equiv 0 \pmod{p}$. So p divides the number $a^n + n$, hence also $b^n + n$. However, by Fermat's little theorem again, we have analogously $b^n + n \equiv b - a \pmod{p}$. We are therefore led to the conclusion $p \mid b - a$, which is a contradiction.

Comment. The first thing coming to mind is to show that a and b share the same prime divisors. This is easily established by using Fermat's little theorem or Wilson's theorem. However, we know of no solution which uses this fact in any meaningful way.

For the conclusion to remain true, it is not sufficient that $a^n + n \mid b^n + n$ holds for infinitely many n. Indeed, take $a = 1$ and any $b > 1$. The given divisibility relation holds for all positive integers n of the form $p - 1$, where $p > b$ is a prime, but $a \neq b$.

N7. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_0, \ldots, a_n$ are integers, $a_n > 0$, $n \geq 2$. Prove that there exists a positive integer m such that $P(m!)$ is a composite number.

Solution. We may assume that $a_0 = \pm 1$, otherwise the conclusion is immediate. Observe that if $p > k \geq 1$ and p is a prime then

$$(p-k)! \equiv (-1)^k \left((k-1)!\right)^{-1} \pmod{p}, \tag{1}$$

where t^{-1} denotes the multiplicative inverse (mod p) of t. Indeed, this is proved by writing

$$(p-1)! = (p-k)![p-(k-1)][p-(k-2)]\cdots(p-1),$$

reducing modulo p and using Wilson's theorem. With (1) in mind, we see that it might be worth looking at the rational numbers

$$P\left(\frac{(-1)^k}{(k-1)!}\right) = \frac{(-1)^{kn}}{((k-1)!)^n} Q((-1)^k(k-1)!),$$

where $Q(x) = a_n + a_{n-1}x + \cdots + a_0 x^n$.

If $k-1 > a_n^2$, then $a_n \mid (k-1)!$ and $(k-1)!/a_n = 1 \cdot 2 \cdots (a_n^2/a_n) \cdots (k-1)$ is divisible by all primes $\leq k-1$. Hence, for such k we have $Q((k-1)!) = a_n b_k$, where $b_k = 1 + a_{n-1}(k-1)!/a_n + \cdots$ has no prime factors $\leq k-1$. Clearly, $Q(x)$ is not a constant polynomial, because its leading term is $a_0 = \pm 1$. Therefore $|Q((k-1)!)|$ becomes arbitrarily large when k is large, and so does $|b_k|$. In particular, $|b_k| > 1$ if k is large enough.

Take such an even k and choose any prime factor p of b_k. The above argument, combined with (1), shows that $p > k$ and that $P((p-k)!) \equiv 0 \pmod{p}$.

In order to complete the proof, we only need to ensure that k can be chosen so that $|P((p-k)!)| > p$. We do not know p, but we know that $p \geq k$. Our best bet is to take k such that the first possible prime following k is "far away" from it; i. e., $p-k$ is large. For this, we may choose $k = m!$, where $m = q-1 > 2$ and q is a prime. Then $m!$ is composite, $m!+1$ is also composite (because $m!+1 > m+1 = q$ and $m!+1$ is a multiple of q by Wilson's theorem), and $m! + \ell$ is also composite for all $\ell = 2, \ldots, m$. So, $p = m! + m + t$ for some $t \geq 1$, therefore $p - k = m + t$. For large m,

$$P((p-k)!) = P((m+t)!) > \frac{(m+t)!}{2},$$

because $a_n > 0$. So it suffices to observe that

$$\frac{(m+t)!}{2} > m! + m + t,$$

which is obviously true for m large enough and $t \geq 1$.