Art of Problem Solving

WOOT

Class Transcript 03/03 - Extremal Principle

joshuazucker 7:30:41 pm
WOOT 2013-14: Extremal Principle

joshuazucker 7:30:59 pm
Hi folks! I've met most of you before, but I usually teach Thursday WOOT, so I figure I'll introduce myself briefly.

joshuazucker 7:31:41 pm
I'm Joshua Zucker, and among other things I was on the first US team to the International Physics Olympiad, helped start the Julia Robinson Mathematics Festivals and the Math Teachers' Circles, and recently I was on the US Sudoku team at the World Sudoku Championships.

joshuazucker 7:31:49 pm
Today, we will look at the Extremal Principle, which depends on the following simple, yet useful fact: Every non-empty, finite set of real numbers has a smallest (and largest) element.

joshuazucker 7:32:00 pm
In problems where we seek objects that have specific properties, looking at extreme cases gives us a concrete starting point and can often take us in the right direction, depending what we choose to maximize or minimize.

joshuazucker 7:32:42 pm
Some of you might enjoy taking a look at http://www.maa.org/node/129121/ later -- I enjoy James Tanton's writing.

joshuazucker 7:32:47 pm
In our first problem, we see how playing with the problem leads us naturally to the function that we want to maximize or minimize.

joshuazucker 7:33:03 pm
There are n red points and n blue points in the plane, so that no three points are collinear. Show that we can draw n line segments so that every line segment connects a red point and a blue point, and no two line segments intersect.

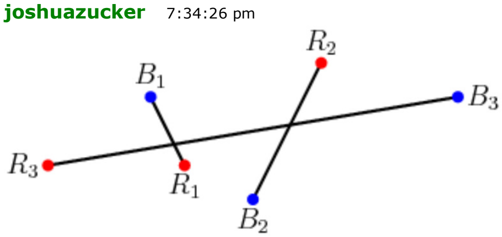
joshuazucker 7:33:42 pm
How can we try to draw these line segments so that no two line segments intersect?

zhuangzhuang 7:34:06 pm
Consider the pair that is closest??

mathcool2009 7:34:10 pm
start drawing one at random and then go through and eliminate intersections

joshuazucker 7:34:21 pm
We can try taking a red point and connecting it to the closest blue point, then doing the same with another red point, and so on. Unfortunately, this does not always work.

joshuazucker 7:34:26 pm



ProbaBillity 7:35:25 pm
pick the farthest-to-the-right red point and the farthest-to-the-right blue point and then connect them, then keep going

anwang16 7:35:25 pm
Connect the two on the top, then the second-to-top, etc.

aty1998 7:35:56 pm
still doesnt always work

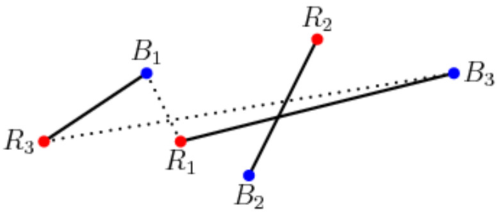
joshuazucker 7:35:57 pm
These are good ideas, but you can probably see how to move some points around to make them not always work.

joshuazucker 7:36:35 pm
However, let's not abandon this diagram right away. We may be able to fix it by getting rid of the intersections.

lawrenceli 7:37:07 pm
try to "fix" two segments if they intersect

sujaykazi 7:37:08 pm
If R1B1 and R2B2 intersect, then connect R1B2 and R2B1.

joshuazucker 7:37:20 pm
Right, for example we can replace the line segments R_1B_1 and R_3B_3 with the line segments R_1B_3 and R_3B_1.

joshuazucker 7:37:25 pm


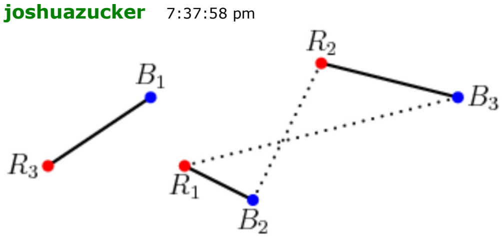
joshuazucker 7:37:28 pm
Now what?

Petaminx 7:37:46 pm
Continue doing that until there are no more intersections.

ProbaBillity 7:37:46 pm
Now we fix R2B2 and R1B3

aty1998 7:37:50 pm
do the same for all intersecting pairs?

joshuazucker 7:37:53 pm
The line segments R_1B_3 and R_2B_3 intersect. We can get rid of this intersection by replacing them with line segments R_1B_2 and R_2B_3.

joshuazucker 7:37:58 pm


joshuazucker 7:38:01 pm
Now we have what we want: We have three line segments with no intersections.

ProbaBillity 7:38:19 pm
but how do we know we can do this in the first place? how do we know that fixing one pair of lines doesn't mess up another?

joshuazucker 7:38:22 pm
Yeah, that's a good point.

joshuazucker 7:38:34 pm
By getting rid of an intersection by uncrossing the line segments, we may think that the number of intersections always decreases, which would give us the result. Is this true?

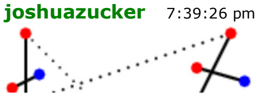
Petaminx 7:39:13 pm
No

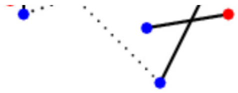
SuperSnivy 7:39:13 pm
no, it can create more intersections with other segments

ProbaBillity 7:39:13 pm
no. for example, if r_1 was a bit lower in the original diagram, then fixing R3B3 and B2R2 would result in R3B2 intersecting R1B1

noodleeater 7:39:21 pm
no we could make more intersection with other

joshuazucker 7:39:23 pm
It is not true that the number of intersections always decreases. Here is an example, where by removing one intersection, we introduce three more:

joshuazucker 7:39:26 pm


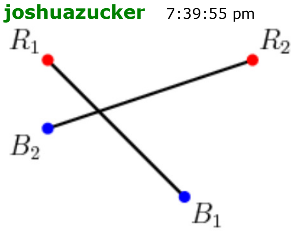


joshuazucker 7:39:33 pm
Uh oh.

joshuazucker 7:39:44 pm
Is it even always possible to remove the one crossing we are focused on?

joshuazucker 7:39:52 pm
Suppose we have two line segments that intersect.

joshuazucker 7:39:55 pm



joshuazucker 7:40:01 pm
Can we always uncross them to remove the intersection?

mathcool2009 7:40:27 pm
YES

Petaminx 7:40:27 pm
Yes

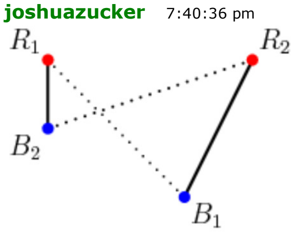
noodleeater 7:40:27 pm
yes

baldcypress 7:40:27 pm
yes

Michelangelo 7:40:27 pm
yes

joshuazucker 7:40:32 pm
We can replace line segments R_1B_1 and R_2B_2 with line segments R_1B_2 and R_2B_1. In fact, this is the only way to replace these line segments. (We can't connect points R_1 and R_2, for example, because they are both red.)

joshuazucker 7:40:36 pm



joshuazucker 7:40:46 pm
So if we want to immediately get rid of any particular intersection, then there is only one way.

joshuazucker 7:41:02 pm
The diagram looks pretty convincing, anyway. I worry a bit about concave quadrilaterals ... but wait, for them there's no chance of intersection to begin with, is there?

joshuazucker 7:41:29 pm
We need to show that eventually, by repeatedly uncrossing line segments, we always reach a diagram where no two line segments intersect.

joshuazucker 7:41:37 pm
But uncrossing segments may make more intersections!

joshuazucker 7:41:44 pm
So the number of intersections doesn't necessarily decrease.

joshuazucker 7:41:49 pm
Is there anything else that is always decreasing?

TheStrangeCharm 7:42:39 pm
consider the sum of the lengths of all of the segments we draw.

noodleeater 7:42:39 pm
find the configuration with the minimum possible total length

Arithmophobia 7:42:39 pm
the sum of the lengths

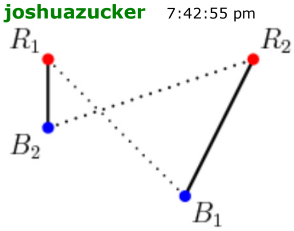
lucylai 7:42:39 pm
distance between points

noodleeater 7:42:39 pm
total sum of lengths

Lord.of.AMC 7:42:39 pm
sum of length of line segments?

lawrenceli 7:42:39 pm
sum of lengths of segments

joshuazucker 7:42:55 pm



joshuazucker 7:42:55 pm
We have that $R_1B_2 + R_2B_1 < R_1B_1 + R_2B_2$. Why is this true?

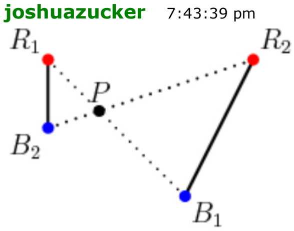
zhuangzhuang 7:43:31 pm
Triangle inequalty

shandongboy 7:43:31 pm
triangle inequality?

eyzhang 7:43:31 pm
triangle inequality

joshuazucker 7:43:34 pm
This follows from the Triangle Inequality. Let P be the intersection of R_1B_1 and R_2B_2.

joshuazucker 7:43:39 pm



joshuazucker 7:43:41 pm
Then by the Triangle Inequality,

$$R_1B_2 < R_1P + PB_2$$

and

$$R_2B_1 < R_2P + PB_1.$$

joshuazucker 7:43:49 pm
Adding these, we get

$$R_1B_2 + R_2B_1 < R_1B_1 + R_2B_2.$$

joshuazucker 7:43:55 pm
Therefore, when we uncross two line segments, the total length of the line segments always decreases. (In other words, we have found a *monovariant*).

ksun48 7:44:00 pm
use the extremal Principal

joshuazucker 7:44:07 pm
We can use this observation to repeatedly uncross line segments, until we reach a diagram that has no intersections. But by using the Extremal Principle, we can go for broke!

joshuazucker 7:44:14 pm

Instead of repeatedly uncrossing line segments, which decreases the total length of the line segments, what can we do?

yangwy 7:44:42 pm
take a minimal configuration

noodleeater 7:44:42 pm
find configuration with minimum total length

baldcypress 7:44:42 pm
pick the arrangement with minimum length

Arithmophobia 7:44:42 pm
find the arrangement that has the least total length

joshuazucker 7:44:46 pm
We can look at the diagram such that the total length of the line segments is the smallest.

joshuazucker 7:44:50 pm
We claim that in this diagram, no two line segments intersect. Why is this true?

ProbaBillity 7:45:47 pm
since it has the minimum possible sum of lengths, we can't uncross anymore, which means there are no more intersections

Cosmynx 7:45:47 pm
If any two segments intersected, we could eliminate the intersection and decrease the length

Bg1 7:45:47 pm
you cannot not uncross anymore, meaning no intersections

TheStrangeCharm 7:45:47 pm
For if they did, we could perform this operation to make the quantity smaller, which is a contradiction.

chenjamin 7:45:47 pm
if they do, then by uncrossing the intersection we make the length smaller

joshuazucker 7:45:53 pm
In this diagram, if two line segments intersect, then we can uncross them. From our work above, this decreases the total length of the line segments.

joshuazucker 7:46:00 pm
But this is a contradiction, because we deliberately chose the diagram where the total length of the line segments is the smallest. Hence, no two line segments intersect, as desired.

joshuazucker 7:46:07 pm
There is nothing wrong with the approach of repeatedly uncrossing line segments, until we reach a diagram that has no intersections. But by using the Extremal Principle, we get there in one direct, simple step.

joshuazucker 7:46:19 pm
Before we move on, we make a small, but important point: In our solution, we chose the diagram such that the total length of the line segments is the smallest. How do we know that there is a diagram where the total length of the line segments is the smallest?

joshuazucker 7:46:32 pm
This is a non-trivial point, because there are sets that do not have a minimal element. For example, the set of integers does not have a minimal element.

joshuazucker 7:46:38 pm
Even if we restrict ourselves to positive real numbers, it is still not true that a set must have a minimal element. For example, the set {1, 1/2, 1/3, 1/4, ...} does not have a minimal element.

baldcypress 7:47:24 pm
there are a finite number of arrangements

hardmath123 7:47:24 pm
It's a finite set,

joshuazucker 7:47:34 pm
In our problem, we know that there is a diagram where the total length of the line segments is the smallest because there are only a finite number of ways to connect the $2n$ points with n line segments. We can compute this number exactly, if we want to, but the actual number is not important - it only matters that the number is finite. And every finite set of real numbers has a minimal element.

joshuazucker 7:47:50 pm
Also, if we are dealing with an infinite set, and we know that every element in the set is a nonnegative integer, then this set contains a minimal element as well. This fact is sometimes called the *Well-ordering Principle*.

joshuazucker 7:48:02 pm
So when applying the Extremal Principle, keep in mind the following: First, make sure that you are dealing with a non-empty finite set, or a non-empty set of nonnegative integers. If you are not, then it may take more work to apply the Extremal Principle. It may

also be an indication that you can't use the Extremal Principle at all.

joshuazucker 7:48:15 pm
Second, if you are dealing with a non-empty finite set of real numbers, then it is easy to choose a minimal (or maximal) element. However, if you are dealing with objects other than real numbers, such as geometry diagrams, then you need to find a way to assign a real number to each object. It may take some experimentation to find the right way to do this.

joshuazucker 7:48:31 pm
Find all positive solutions to the system of equations

$$\begin{aligned} x_1 + x_2 &= x_3^2, \\ x_2 + x_3 &= x_4^2, \\ x_3 + x_4 &= x_5^2, \\ x_4 + x_5 &= x_1^2, \\ x_5 + x_1 &= x_2^2. \end{aligned}$$

joshuazucker 7:48:59 pm
That looks scary.

joshuazucker 7:49:04 pm
We can try solving the system using the usual algebraic techniques, say by adding all the equations, or substitution. However, this will lead to very complicated equations. We should find a simpler way.

ProbaBillity 7:50:34 pm
the chances are that the only solution is when they're all 2

thkim1011 7:50:34 pm
maybe (2,2,2,2,2) is the only solution?

joshuazucker 7:50:39 pm
That's certainly a good guess.

ProbaBillity 7:51:06 pm
looks like the equality case of something

Arithmophobia 7:51:34 pm
assume there is a smallest one out of the five. let that be a_1.

joshuazucker 7:51:47 pm
Let x_i be the smallest variable. What can we say about x_i?

joshuazucker 7:52:00 pm
(We could call it x_1 if we wanted, since the equations are symmetric.)

lawrenceli 7:52:52 pm
x_1 > 1 since x_4 + x_5 = x_1^2 and x_1 < x_4, x_5

ProbaBillity 7:53:04 pm
we know $x_1 > 2$ since $x_1^2 = x_4 + x_5 > 2x_1$

joshuazucker 7:53:17 pm
We can say that

$$x_i^2 \geq 2x_i,$$

because $x_i^2 = x_j + x_k$ for some j and k, and by assumption about x_i, $x_j \geq x_i$ and $x_k \geq x_i$.

joshuazucker 7:53:52 pm
Dividing both sides by x_i, we get

$$x_i \geq 2.$$

joshuazucker 7:54:05 pm
So the smallest one of the five variables is at least 2.

joshuazucker 7:54:08 pm
What should we do next?

thkim1011 7:54:14 pm
what about the largest?

ProbaBillity 7:54:43 pm
If we can show that the largest is at most 2, then we are done.

Naysh 7:56:19 pm
If x_k is the largest, then $x_k^2 = x_{k-1} + x_{k-2} \leq 2x_k$ means that $x_k \leq 2$, so done.

Cosmynx 7:56:29 pm
If the largest (say x_k) is greater than 2, then (x_k)^2=x_(k-1)+x_(k+1)<2x_k which gets a contradiction

Lord.of.AMC 7:56:31 pm
if x_5 is the largest $x_3 + x_4 = x_5^2 \leq 2x_5$

joshuazucker 7:56:34 pm
We know that

$$x_1^2 = x_4 + x_5.$$

joshuazucker 7:56:47 pm
If (WLOG) x_1 is the largest of the five variables,

$$x_1^2 = x_4 + x_5 \leq 2x_1.$$

joshuazucker 7:56:49 pm
We are only seeking positive solutions, so we can divide both sides by x_1, to get $x_1 \leq 2$. Therefore, all the variables are at most 2.

joshuazucker 7:56:59 pm
Why are we done now?

anwang16 7:57:25 pm
Because they can only be 2!

SuperSnivy 7:57:25 pm
they must be at least 2 and at most 2, so they can only be 2

baldcypress 7:57:25 pm
all variables must be between 2 and 2, so they are all 2

noodleeater 7:57:27 pm
we've narrowed it down to only one solution

hardmath123 7:57:29 pm
Because the only numbers between 2 and 2 are 2.

joshuazucker 7:57:32 pm
If the largest of the variables is at most 2, and the smallest of the variables is at least 2, then all the variables must be equal to 2. And we see that this solution works.

joshuazucker 7:58:05 pm
A finite set of points S in the plane has the property that the triangle determined by any three points in S has area at most 1. Prove that there exists a triangle of area 4 that contains all the points in S.

zhuangzhuang 7:59:10 pm
Consider the triangle of maximal area!

joshuazucker 7:59:20 pm
We want to find a triangle that contains all the points in the set S.

joshuazucker 7:59:23 pm
We can consider the triangle that has the largest area, among all the triangles formed by points in the set S. Let this triangle be $\triangle ABC$.

joshuazucker 7:59:40 pm
(We know that such a triangle exists, because there are only a finite number of triangles that can be formed by the points in the set S. Be sure to state that if you're solving an olympiad problem with this kind of method!)

joshuazucker 8:00:00 pm
What can we say about $\triangle ABC$?

lucylai 8:00:33 pm
$[ABC] \leq 1$

mlcindy 8:00:33 pm
area<=1

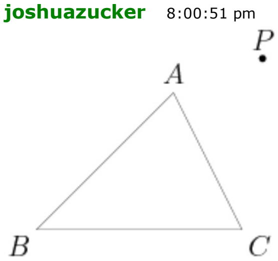
noodleeater 8:00:33 pm
it has area <= 1, and is the maximum area

joshuazucker 8:00:37 pm
From the condition in the problem, $\triangle ABC$ has area at most 1.

joshuazucker 8:00:41 pm
It would be nice if triangle ABC happens to contain every point in the set S, but this probably won't happen, which is fine because we are looking for a triangle of area 4. So let's see if we can find a triangle of area 4 based on $\triangle ABC$.

joshuazucker 8:00:48 pm
Intuitively, points in S can't be "too far" from $\triangle ABC$. For example, can a point P in the set S be located as follows?

joshuazucker 8:00:51 pm



sonamu8snow 8:02:00 pm
No, the area of PBC would be greater than the area of ABC

chenjamin 8:02:00 pm
no, [PBC]>[ABC]

patchosaur 8:02:00 pm
no, because PBC would have larger area

Petaminx 8:02:00 pm
No

avery 8:02:00 pm
no

pier17 8:02:00 pm
no, since distance from P to BC is greater than distance from A to BC

baldcypress 8:02:00 pm
No, cuz PBC is too big

eyzhang 8:02:00 pm
No, triangle ABC is the biggest area

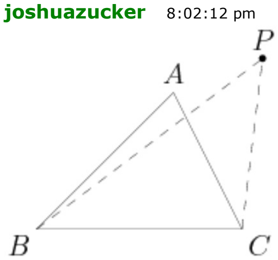
Bg1 8:02:00 pm
no because [PBC]>1

yangwy 8:02:00 pm
no, since the perpindicular from A to BC is smaller than the perpindicular from P to BC, so the area of PBC is greater

ultrasonic360 8:02:00 pm
no, height from BC is greater than that of A

mathcool2009 8:02:05 pm
triangle PBC is greater than the area of triangle ABC and that is a contradiction

joshuazucker 8:02:12 pm



joshuazucker 8:02:15 pm
Triangles PBC and ABC have the same base BC, but triangle PBC has a greater height, so the area of triangle PBC is greater than the area of triangle ABC. This contradicts the maximality of the area of triangle ABC.

joshuazucker 8:02:19 pm
So what does that say about points in S?

Lord.of.AMC 8:02:29 pm
medial triangle?

Naysh 8:02:29 pm
It's area is at most 1. Now, take the antimedial triangle.

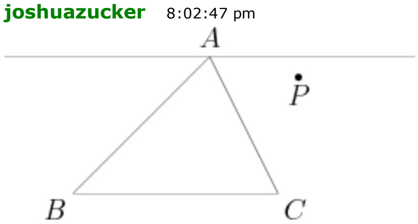
zhuangzhuang 8:02:29 pm
ABC is the medial triangle of the locus!!

lucylai 8:02:31 pm
P must lie in the triangle whose medial triangle is ABC

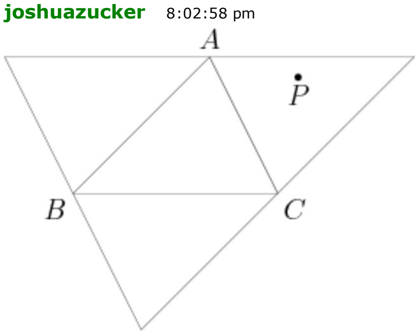
ProbaBillity 8:02:35 pm
No point P can be located above the line through A parallel to BC, since then the altitude from P to BC would be greater than that of A, and then $[PBC] > [ABC]$ which contradicts our assumption.

ProbaBillity 8:02:37 pm
We can thus conclude that all points P must lie within the area bounded by the line parallel to BC passing through A, the line parallel to AB passing through C, and the line parallel to CA passing through B.

joshuazucker 8:02:40 pm
Draw a line through A parallel to BC. Then every point in S must be on the same side of this line as B and C.

joshuazucker 8:02:47 pm


joshuazucker 8:02:54 pm
By the same argument, we can draw the corresponding lines through B and C.

joshuazucker 8:02:58 pm


joshuazucker 8:03:01 pm
How do we finish the problem?

thkim1011 8:03:39 pm
area of 4.

thkim1011 8:03:39 pm
area less than 4

distortedwalrus 8:03:39 pm
the medial triangle has area 1/4 the whole thing

Michelangelo 8:03:39 pm
all the triangles creates are congruent to ABC so area is 4

joshuazucker 8:03:45 pm
(Mathematicians have the habit of saying "and so we're done" at this point. Olympiad competitors lose lots of points by saying that.)

joshuazucker 8:03:55 pm
Instead say something like this:

joshuazucker 8:03:58 pm
These three lines determine a triangle T that is similar to $\triangle ABC$. Furthermore, the sides of triangle T are twice as long as the sides of ABC, so T has area at most 4. Since triangle T contains all the points in S, we are done.

ProbaBillity 8:04:22 pm
This "big" triangle is composed of four smaller ones, each congruent to $\triangle ABC$. Thus, its area is four times the area of ABC, which is at most four. We can always draw a triangle of area 4 around this (quite trivial to see); thus, we are done.

joshuazucker 8:04:45 pm
Good point, the problem says are exactly 4, not at most 4. So it might be worth mentioning that every triangle is contained in a triangle of any larger area you care to pick.

joshuazucker 8:04:57 pm
This is sufficiently obvious that I doubt you'd lose a point for it, though.

joshuazucker 8:05:11 pm
King Arthur summoned $2n$ knights to his court. Each knight has no more than $n-1$ enemies among the knights present. Prove that the knights can sit at the Round Table so that no two enemies sit next to each other.

joshuazucker 8:05:35 pm
If we want to show that there is a seating where no two enemies sit next to each other, then what can we consider?

baldcypress 8:06:46 pm
the set of all possible arrngements

pier17 8:06:50 pm
sum of number of times 2 enemies sit next to each other

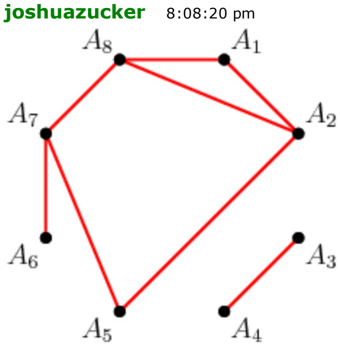
joshuazucker 8:07:33 pm
We can consider the seating that has the least number of pairs of adjacent enemies. (This exists, but may not be unique, because there are only a finite number of different seatings, and anyway the number of pairs of enemies is a nonnegative integer.)

ProbaBillity 8:07:39 pm
Just to confirm, if A is the enemy of B, then B is also the enemy of A, right?

joshuazucker 8:07:46 pm
Yes, we're assuming this is symmetric.

joshuazucker 8:07:54 pm
To make things easier, we represent enemies with a graph. Each vertex corresponds to a knight. A red edge indicates enemies. We show some such example graph for $n = 4$.

joshuazucker 8:08:17 pm
(Note that they are *not* seated in a way that gives the minimal number of pairs of adjacent enemies.)

joshuazucker 8:08:20 pm


joshuazucker 8:09:05 pm
For the sake of contradiction, assume we have a minimal configuration and suppose that two enemies are sitting next to each other, say A_1 and A_{2n}.

joshuazucker 8:09:29 pm
Somehow we need to decrease the total number of pairs of enemies sitting next to each other.

joshuazucker 8:09:56 pm
We can try switching A_1 or A_{2n} with another vertex, but this gets messy, because switching a vertex with another vertex can easily introduce more enemies sitting next to each other. Is there a way of changing the edge between A_1 and A_{2n} that doesn't change many other pairs of adjacent neighbors?

joshuazucker 8:11:30 pm
There's a lot of suggestions here but most of them I think don't work necessarily.

joshuazucker 8:12:14 pm
If we pick someone up and stick them between ... then we need to know that the person we're picking up is friends with both A_1 and A_{2n}, and also that the two people that person used to be between aren't enemies.

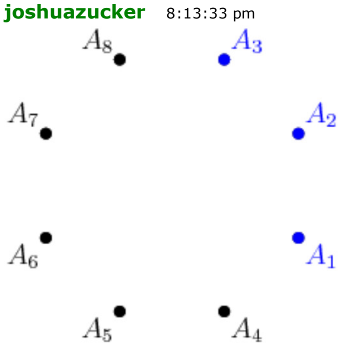
joshuazucker 8:12:37 pm
There's a problem with picking up a person from elsewhere and moving them around: they may have been necessary where they were to keep two enemies separated.

joshuazucker 8:13:18 pm
We could switch A_1 and A_2 but then if A_2 and A_{2n} are enemies, or A_1 and A_3 are enemies, we'd still be in trouble.

ProbaBillity 8:13:22 pm
maybe we find someone who a_2n is friends with and the guy next to him is friends with a_1, then we cut the round table into two pieces, flip one of them, and join them together again (if that makes any sense)

joshuazucker 8:13:30 pm
We can take a block of vertices $A_1, A_2, \dots, A_i$, and reverse them. We show the case for $i = 3$.

joshuazucker 8:13:33 pm



joshuazucker 8:13:43 pm
This transformation preserves all the pairs among $A_1, A_2, \ldots, A_i$.

joshuazucker 8:14:18 pm
So, we're eliminating the enemy pair of A_1 and A_{2n}, but we have to show that we can do this in such a way that the two new pairs of neighbors are not enemies either.

joshuazucker 8:14:28 pm
Then we'll have eliminated one pair of neighboring enemies.

joshuazucker 8:14:38 pm
In other words, we need A_1 and A_{i+1} to be friends (A_1 and A_4 in our example), and we need A_i and A_{2n} to be friends (A_3 and A_8 here). If we can find such an i, then we are done.

joshuazucker 8:14:48 pm
Can we show that such an i exists?

noodleeater 8:15:11 pm
pigeonhole on the number of friends each person has

joshuazucker 8:15:23 pm
Yeah. How many indices i do we have to choose from?

ProbaBillity 8:16:01 pm
$2n-2$

pier17 8:16:01 pm
2n-2

anwang16 8:16:01 pm
from 2 to 2n-1

joshuazucker 8:17:02 pm
OK, so we have $2n-2$ possible indices i to choose from.

joshuazucker 8:17:28 pm
How many can be bad because they would put A_1 next to an enemy? And how many can be bad because they would put A_{2n} next to an enemy? Be careful here...

zhuangzhuang 8:18:16 pm
At most n-2 for each--> 2n-4 total

delta1 8:18:16 pm
n-2, n-2

pier17 8:18:16 pm
A_1 and A_2n together have at most 2(n-2) enemies since they are enemies with each other

Cosmynx 8:18:16 pm
n-2 each, since each can have at most n-1 enemies (but they a_1 and a_2n are enemies)

joshuazucker 8:18:40 pm
But wait, we were allowing index $2n-1$ which puts A_1 next to A_{2n} on the other side, right?

distortedwalrus 8:18:44 pm
so A1 has at least n friends, so there are n possibilities for i, so A2n must be friends with one of them

joshuazucker 8:18:49 pm
That looks like a clearer way to say it.

joshuazucker 8:19:28 pm
We can count i from 1 to $2n-1$ and have $n-1$ enemies for each being ruled out.

joshuazucker 8:19:41 pm
Or we can count i from 2 to $2n-2$ and have $n-2$ other enemies being ruled out.

joshuazucker 8:20:03 pm
Either way there's guaranteed to be one choice for i (at least) that does not put any enemies next to each other that weren't already adjacent before the swap.

joshuazucker 8:20:07 pm
Thus, for this index i, reversing the block $A_1, A_2, \dots, A_i$ reduces the number of enemies sitting next to each other.

joshuazucker 8:20:19 pm
What can we conclude?

mathcool2009 8:20:46 pm
and so we're done :)

distortedwalrus 8:20:46 pm
that it's always possible to decrease the number of pairs as long as it's positive

Cosmynx 8:20:46 pm
there must be no enemies sitting next to each other in the minimal configuration

Lord.of.AMC 8:20:51 pm
=> we must be able to find a way to continually reduce

joshuazucker 8:20:53 pm
We chose a seating arrangement where the number of adjacent enemies was minimal. If there are any enemies sitting next to each other, then by our work above, we can get rid of an adjacent pair of enemies, contradicting minimality. Hence, in this optimal seating, no two enemies sit next to each other.

joshuazucker 8:20:57 pm
This problem is actually an important result in graph theory, in disguise.

joshuazucker 8:21:00 pm
(Dirac's Theorem) Let G be a simple graph on n vertices, where $n \geq 3$. If every vertex in G has degree at least $n/2$, then G contains a Hamiltonian cycle.

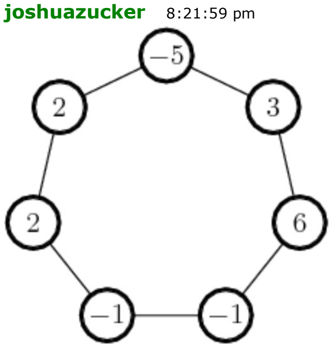
joshuazucker 8:21:13 pm
(A Hamiltonian cycle is a circular path that visits all the vertices once. In this problem the graph is the graph of *friends* and the cycle is the seating around the table.)

joshuazucker 8:21:45 pm
Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \dots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k-1 \quad \text{for} \quad k = 1, 2, \dots, n$$

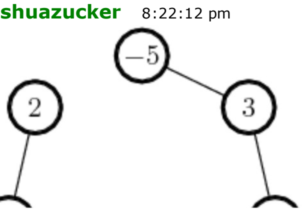
joshuazucker 8:21:56 pm
Here's an example:

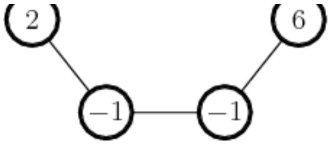
joshuazucker 8:21:59 pm



joshuazucker 8:22:09 pm
If we cut it like this:

joshuazucker 8:22:12 pm





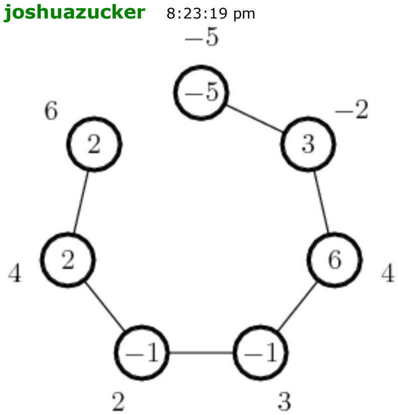
joshuazucker 8:22:17 pm
What are the sums here?

noodleeater 8:23:14 pm
-5, -2, 4, 3, 2, 4, 6

mentalgenius 8:23:14 pm
-5, -2, 4, 3, 2, 4, 6

zhuangzhuang 8:23:14 pm
-5,-2,4,3,2,4,6

joshuazucker 8:23:19 pm



joshuazucker 8:23:26 pm
The sums here are

$$-5, -2, 4, 3, 2, 4, 6.$$

Unfortunately $4 > 2$ so this cutting fails.

joshuazucker 8:24:01 pm
We have to compare these with 0,1,2,3,4,5,6 and have the sums be less than or equal to those numbers at every step. (We're given that the final one will be 6.)

joshuazucker 8:24:24 pm
If we cut the original thing between the two 2s, we're in trouble right away since 2 > 0.

joshuazucker 8:24:33 pm
Is there a way to cut this to give a good clockwise sum?

mentalgenius 8:24:42 pm
we need to cut right before a negative number

ultrasonic360 8:25:03 pm
cut it between 6 and -1

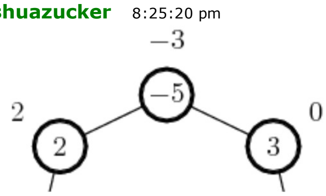
mentalgenius 8:25:03 pm
cut between 6 and -1

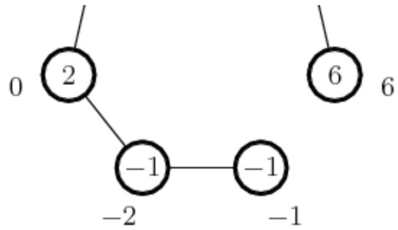
Cosmynx 8:25:03 pm
cut between -1 and 6

Lord.of.AMC 8:25:03 pm
cut between 6 and -1

joshuazucker 8:25:09 pm
If we cut between 6 and -1 we are safe.

joshuazucker 8:25:20 pm





joshuazucker 8:25:31 pm
The sums are now

$$-1, -2, 0, 2, -3, 0, 6.$$

These satisfy all of our inequalities.

joshuazucker 8:27:08 pm
How does this sequence change when we cut in different places? We're still summing the same numbers, so there ought to be some relationship.

joshuazucker 8:27:15 pm
We had

joshuazucker 8:27:20 pm
-5,-2,4,3,2,4,6

joshuazucker 8:27:25 pm
-1,-2,0,2,-3,0,6

joshuazucker 8:27:31 pm
and if I cut elsewhere I get

joshuazucker 8:27:43 pm
2,-3,0,6,5,4,6 for instance (cutting between the two 2s).

Cosmynx 8:28:27 pm
it always ends in 6

ProbaBillity 8:28:27 pm
well it always ends in $n-1$, that's pretty clear

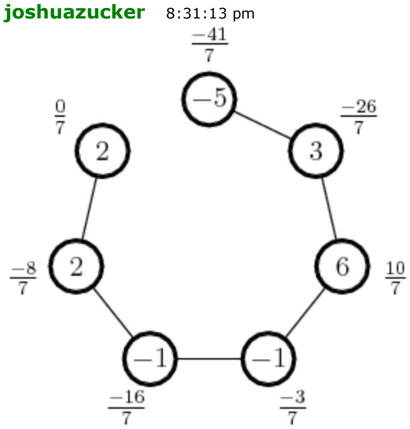
joshuazucker 8:30:04 pm
It always ends up at 6 after 7 steps. It wiggles up and down, but on average it goes up that far.

joshuazucker 8:30:26 pm
In other words, it increases on average 6/7 per step. It's often a good idea to measure deviations from an overall trend like that.

joshuazucker 8:30:52 pm
In other words, there's a linear part that increases by $n-1=6$ every $n=7$ steps you make. Therefore we could subtract $\frac{6}{7}$ from the first, $\frac{12}{7}$ from the second, $\frac{18}{7}$ from the third, etc.

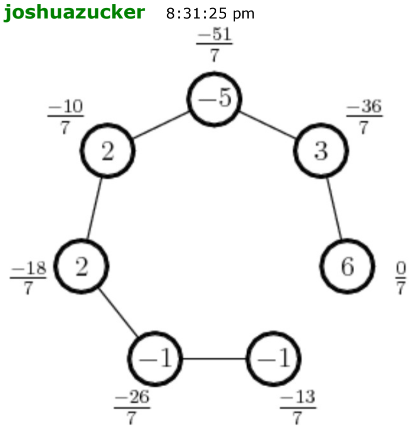
joshuazucker 8:31:10 pm
Our first example becomes

joshuazucker 8:31:13 pm



joshuazucker 8:31:20 pm
And our successful example becomes

joshuazucker 8:31:25 pm



zhuangzhuang 8:31:49 pm
All numbers must be nneg

distortedwalrus 8:31:49 pm
here, all the sums are non positive

Lord.of.AMC 8:31:49 pm
looks like it always has to be negative

joshuazucker 8:31:50 pm
Now we know we'll always end at 0, because we subtracted 6/7 per step for 7 steps.

joshuazucker 8:32:34 pm
And since we end up at 0 at the end, if we kept going we'd just repeat the same sums around and around.

joshuazucker 8:32:54 pm
Is the nonpositivity here equivalent to what we're trying to show?

mentalgenius 8:33:31 pm
yes

ProbaBillity 8:33:31 pm
yes, since it is analogous to the $\leq$

zhuangzhuang 8:33:31 pm
not quite

joshuazucker 8:33:32 pm
The fact that they're all non-positive tells us that $s_k - \dfrac{k(n-1)}{n} \leq 0$. That simplifies to

$$s_k \leq \frac{k(n-1)}{n} = k - \frac{k}{n}.$$

joshuazucker 8:33:42 pm
But we want it to be less than or equal to $k-1$.

Cosmynx 8:33:52 pm
yes, because the numbers are integers

joshuazucker 8:33:56 pm
Since k and s_k are positive integers, we get $s_k \leq k-1$ which was exactly our goal.

joshuazucker 8:34:05 pm
Now we need to show that whatever the necklace is, we can cut it somewhere so that all the sums $t_k = s_k - k\dfrac{n-1}{n}$ are non-positive.

joshuazucker 8:34:55 pm
Scroll up a bit and look at the two diagrams.

joshuazucker 8:35:10 pm
Remember that if we keep going around, we'll get the same sums: it's periodic, now that the sum of all the numbers is 0 instead of $n-1$.

joshuazucker 8:35:27 pm
How do the two sets of sums relate to each other?

pier17 8:36:00 pm

each pair differs by 10/7

joshuazucker 8:36:11 pm
Wherever we cut, the set of t_k we get will be the same up to adding some constant.

joshuazucker 8:37:28 pm
So we can cut anywhere and get the set of t_k.

joshuazucker 8:37:43 pm
But some of them might be positive! How can we look at the t_k values to know where we should have cut?

joshuazucker 8:38:11 pm
This one should give a big hint.

Cosmynx 8:38:35 pm
cut to the right of the largest positive value?

delta1 8:38:35 pm
cut after largest t_k

joshuazucker 8:38:41 pm
First we cut the necklace anywhere and determine the set of t_k. Then we start over and cut again so that the largest t_k is the last one.

joshuazucker 8:39:16 pm
Because we know the last one will be 0!

joshuazucker 8:39:19 pm
After this new cut, $t_n = 0$ and all other t_k are smaller (so non-positive). Therefore we are done.

joshuazucker 8:44:59 pm
Show that if $2n + 1$ real numbers have the property that the sum of any n is less than the sum of the remaining $n + 1$, then all the numbers are positive.

joshuazucker 8:45:45 pm
What can we do with the $2n + 1$ numbers to get more of a handle on them?

distortedwalrus 8:46:22 pm
choose the smallest number

pier17 8:46:22 pm
put them in increasing order

ultrasonic360 8:46:25 pm
designate a largest and smallest one

joshuazucker 8:46:29 pm
We can arrange them in order.

joshuazucker 8:46:30 pm
Let the $2n + 1$ numbers be

$$a_1 \le a_2 \le \cdots \le a_{2n+1}.$$

joshuazucker 8:46:40 pm
So now how can we restate our goal?

delta1 8:47:15 pm
a_1>0

vincenthuang75025 8:47:15 pm
$a_1 > 0$

joshuazucker 8:47:17 pm
To show that all the numbers are positive, it suffices to show that a_1 is positive.

joshuazucker 8:47:31 pm
And what's the strongest use of the given information we can make?

willwang123 8:48:59 pm
sum of the last n is less than the sum of the first n+1

zhuangzhuang 8:48:59 pm
Focus on the largest

SuperSnivy 8:48:59 pm
a_2n+1 + ... + a_n+2 < a_n+1 + ... + a_1

chenjamin 8:48:59 pm

a_n+2 + a_n+3 +.... a_2n+1 > a_1 + a_2 + ... a_n+1

pier17 8:48:59 pm
a_1 + ... + a_n+1 > a_n+2 + ... + a_2n+1

Naysh 8:48:59 pm
$a_{n+2} + a_{n+3} + \ldots + a_{2n+1} < a_1 + a_2 + \ldots + a_{n+1}$.

Cosmynx 8:48:59 pm
oh wait a_1+...a_(n+1)>a_(n+2)...+a_(2n+1)

noodleeater 8:48:59 pm
a1+...+a_(n+1) > a_(n+2) + ... + a_(2n+1)

joshuazucker 8:49:12 pm
Yeah, we need to be careful to read the problem.

joshuazucker 8:49:22 pm
We are given that the sum of any n numbers is less than the sum of the remaining $n+1$ numbers.

joshuazucker 8:49:38 pm
If this is true even when we choose the largest n numbers, then it is always true.

joshuazucker 8:49:40 pm
So, we should look at the n largest numbers.

joshuazucker 8:49:54 pm
In other words, we need to use that

$$a_{n+2} + a_{n+3} + \cdots + a_{2n+1} < a_1 + a_2 + \cdots + a_{n+1}.$$

joshuazucker 8:50:09 pm
The sum of the n largest numbers is less than the sum of the remaining $n+1$ numbers.

lawrenceli 8:50:19 pm
pair up all numbers from a_2 ... a_2n+1 then a_1 has to be positive to make the sum of a_1 a_2 + ... + a_n+1 > a_n+2 + ... + a_2n+1

Naysh 8:50:19 pm
But now everything is clear, since

$$a_1 > (a_{n+2} - a_2) + \ldots + (a_{2n+1} - a_{n+1}) > 0.$$

Cosmynx 8:50:21 pm
We can pair them up accordingly, and there's one term left

zhuangzhuang 8:50:23 pm
Pair up a_2 to a_n+1 with a_n+2 to a_2n+1

joshuazucker 8:50:41 pm
We can isolate a_1 to get

$$a_1 > a_{n+2} + a_{n+3} + \cdots + a_{2n+1} - a_2 - a_3 - \cdots - a_{n+1}.$$

joshuazucker 8:50:53 pm
Then we pair terms and find

$$\begin{aligned} a_1 &> a_{n+2} + a_{n+3} + \cdots + a_{2n+1} - a_2 - a_3 - \cdots - a_{n+1} \\ &= (a_{n+2} - a_2) + (a_{n+3} - a_3) + \cdots + (a_{2n+1} - a_{n+1}) \\ &\geq 0. \end{aligned}$$

baldcypress 8:51:14 pm
i think the contrapositive is trivially easy to prove... it will always fail if there's a negative number

joshuazucker 8:51:44 pm
True, take the negative along with the n smallest among the rest, and it'll be less than the n largest of the other $2n$ numbers.

joshuazucker 8:51:53 pm
But the given information says the sum of any $n+1$ has to be more.

mathcool2009 8:51:58 pm
Another approach: split into three sets of size n, n, and 1. (This is a completely arbitrary decision.) the set with size 1 , let its only member be x. Let the sum of the members of one of the other sets be S and let the sum of the members of the only remaining set be T. Then we have S + x > T and T + x > S, or -S > -T - x. Adding the two we have x > -x and 2x > 0 and x > 0 so x

joshuazucker 8:52:36 pm
Good point, we didn't have to use the ordering to get this pairing. One pairing or its reverse has to be a positive number!

joshuazucker 8:53:06 pm
Let's try another one.

joshuazucker 8:53:09 pm
Let m and n be positive integers. Let $a_1, a_2, \dots, a_m$ be distinct elements of $\{1, 2, \dots, n\}$ such that whenever $a_i + a_j \le n$ for some i, j, $1 \le i \le j \le m$, there exists k, $1 \le k \le m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \ge \frac{n+1}{2}.$$

joshuazucker 8:53:55 pm
Can anyone state that "such that" part in English for us?

pier17 8:55:34 pm
the sum of any two elements is in the set (given that the sum is <= n)

ProbaBillity 8:55:34 pm
Given any two elements among the a_i which sum to less than n, there is another element among the a_i which is their sum

hardmath123 8:55:34 pm
If you take two elements whose sum is less than n, then there is another element which is that sum.

shandongboy 8:55:34 pm
whenever 2 terms in a subset are less than or equal to the largest possible element, there is a term within that subset that is equal to the sum of the 2 terms

distortedwalrus 8:55:34 pm
whenever the sum of two elements is less than or equal to n, their sum is in fact another element of the a's

noodleeater 8:55:34 pm
for an m-element subset of numbers from 1 to n, if two elements sum to <= n, there exists another element of this subset that equals the sum. prove the average >= (n+1)/2

joshuazucker 8:55:41 pm
The condition states that if we take any two elements in S (not necessarily distinct), and their sum is less than or equal to n, then their sum is another element in S.

joshuazucker 8:56:19 pm
We want to prove the average of the elements of S is more than the average of the set 1 through n. (At least to me, that makes sense as where the $\frac{n+1}{2}$ would come from.)

anwang16 8:57:28 pm
What is S?

joshuazucker 8:57:35 pm
Let S be the set of these numbers $a_1, \dots, a_m$.

joshuazucker 8:57:45 pm
Sorry, should have said that earlier.

pier17 8:57:52 pm
given a number in the set, all of its multiples that are less or equal to n are in the set

joshuazucker 8:57:56 pm
That's an interesting observation.

joshuazucker 8:59:55 pm
I think in general when we have an arbitrary finite set, putting them in order is often a good technique.

ProbaBillity 9:00:21 pm
Assume WLOG that $a_1 \le a_2 \le \cdots \le a_n$

joshuazucker 9:00:27 pm
They're distinct, so we can assume that

$$a_1 < a_2 < \cdots < a_m.$$

Cosmynx 9:00:33 pm
If 1 is in the set, we're trivially done. Else, taking the "multiples" of some number and averaging them gives over the average of 1 to n

joshuazucker 9:00:53 pm
Yeah, but if say 5 and 7 are in the set then it's not just multiples of 5 and 7 but also 12 and all the other linear combinations ...

joshuazucker 9:01:07 pm
It's less immediate to prove that the average of them must be at least the average of 1 through n, I think.

joshuazucker 9:01:50 pm
In other words, I worry about the overcounting of 35 ... the average of 5,10,...,35 and 7,14,..,35 might both be big enough, but when we have only one copy of 35, are we still OK?

joshuazucker 9:02:00 pm
So I'm nervous about finishing that approach.

ProbaBillity 9:02:03 pm
a_1 is the key...

joshuazucker 9:02:12 pm
Look at the extremes!

joshuazucker 9:02:47 pm
We can look at a_1 or a_m to start with.

ProbaBillity 9:03:23 pm
$a_1 + a_m$ must be more than n

joshuazucker 9:03:52 pm
The sum $a_1 + a_m$ is greater than or equal to $n + 1$. Otherwise there would be an element larger than a_m in the set.

joshuazucker 9:04:06 pm
So the average of those two elements is big enough.

joshuazucker 9:04:35 pm
This is feeling like a pairing trick now. Can we prove that $a_2 + a_{m-1}$ is also more than n?

Naysh 9:05:50 pm
Also, $a_2 + a_{m-1} > n$, since if it isn't, then $a_m = a_2 + a_{m-1}$, but then we need $a_{m-1} < a_1 + a_{m-1} < a_m < n$, which doesn't work.

pier17 9:05:50 pm
if it weren't, then a_1 + a_m-1 and a_2 + a_m-1 would both be in the set but there is only one number in the set larger than a_m-1

joshuazucker 9:07:48 pm
Either $a_1 + a_{m-1}$ = a_m or it's already greater than n. Either way, $a_2 + a_{m-1}$ is strictly larger than a_m and thus not in S and thus greater than n.

zhuangzhuang 9:08:40 pm
Do this argument all the way down! If n odd,however, we will have a parity issue.

joshuazucker 9:08:58 pm
OK, so how do we induct?

zhuangzhuang 9:10:59 pm
on pairs with indices adding to m+1

baldcypress 9:10:59 pm
extend pier17's argument using a_1 thru a_k and k-1 instead of 1

joshuazucker 9:11:01 pm
For a_{m-k+1}, we have the k sums $a_1 + a_{m-k+1}, a_2 + a_{m-k+1}, \dots, a_k + a_{m-k+1}$ that are strictly increasing. But there are only $k - 1$ elements of S larger than a_{m-k+1}, so the last one must be greater than n.

joshuazucker 9:11:50 pm
I think this is OK whether m is odd or even, because the two elements we sum do not have to be distinct.

joshuazucker 9:12:06 pm
So if there's a middle element, it plus itself turns out to be greater than n as well.

joshuazucker 9:12:37 pm
Or we can even cruise right past the middle!

joshuazucker 9:12:53 pm
The last sum in each case has to be more than n, which means

$$a_i + a_{m-i+1} \geq n + 1.$$

joshuazucker 9:13:08 pm
Now sum over all i from 1 through m -- this is like the forwards/backwards arithmetic sequence method.

joshuazucker 9:13:14 pm
Summing over all $1 \leq i \leq m$, we get

$$2a_1 + 2a_2 + \cdots + 2a_m \geq m(n + 1),$$

so

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \geq \frac{n+1}{2}.$$

joshuazucker 9:13:39 pm
Or we can look at the average of all the terms directly, and the average of every pair is big enough too.

joshuazucker 9:13:51 pm
Looks like time for one more problem? Maybe two if we're lucky.

joshuazucker 9:13:54 pm
300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 3/2 times any other group.

joshuazucker 9:14:24 pm
Again, we can put them in order.

joshuazucker 9:14:31 pm
Let the weights of the apples be

$$a_1 \le a_2 \le \cdots \le a_{300}.$$

joshuazucker 9:14:52 pm
This often helps and rarely hurts, so you might as well do this whenever you have a big collection of numbers in arbitrary order.

Lord.of.AMC 9:14:59 pm
a_300 <= 3*a_1

shandongboy 9:14:59 pm
less than 3*a1?

joshuazucker 9:15:03 pm
We want to divide the apples into groups of 4 so that no group weighs more than 3/2 times any other group.

joshuazucker 9:15:11 pm
Does anyone have the same (wrong) intuition I do about how to do this?

joshuazucker 9:16:09 pm
Huh. Sounds like you have better intuitions than I do.

joshuazucker 9:16:32 pm
My intuition was to take the biggest two and smallest two, and then keep working in from the ends like that.

joshuazucker 9:16:44 pm
Can you find a counterexample to that strategy?

delta1 9:17:14 pm
1,1,..1,1,3,3

Arithmophobia 9:17:14 pm
two apples that weigh 3 and the rest weigh 1

joshuazucker 9:17:23 pm
If the two lightest apples weigh 1 and 1, and the two heaviest apples weigh 3 and 3, then the first group weighs $1 + 1 + 3 + 3 = 8$.

joshuazucker 9:17:27 pm
But the next group could weigh as little as $1 + 1 + 1 + 1 = 4$, **and** $8/4 = 2 > 3/2$, **so this kind of grouping does not always work.**

patchosaur 9:17:36 pm
pair the heaviest and the lightest

distortedwalrus 9:17:36 pm
pair up elements on the ends?

joshuazucker 9:17:42 pm
Instead of forming groups of 4, we can pair the apples. In particular, we can pair the lightest apples with the heaviest apples.

joshuazucker 9:17:44 pm
This gives us the 150 weights

$$a_1 + a_{300},\ a_2 + a_{299},\ \dots,\ a_{150} + a_{151}.$$

joshuazucker 9:18:04 pm
Let $w_i = a_i + a_{301-i}$ **for** $1 \le i \le 150.$

lawrenceli 9:18:30 pm
then sort them?

zhuangzhuang 9:18:30 pm
pair likewise again!!!

zhuangzhuang 9:18:30 pm
order w_i and pair again!!

joshuazucker 9:18:35 pm
OK!

joshuazucker 9:18:41 pm
Like before, we can pair the lightest weights w_i with the heaviest weights w_i.

joshuazucker 9:18:59 pm
What do we know about the w_i? If $a_1 = 1$, what's the biggest and smallest possible w_i?

Cpi2728 9:20:42 pm
smallest = 2, biggest = 6 (not from same set)

lawrenceli 9:20:42 pm
biggest is 6, smallest is 2?

shandongboy 9:20:42 pm
biggest possible is 4a1, smallest is 2a1?

fprosk 9:20:56 pm
biggest is 4, smallest is 2

chenjamin 9:20:56 pm
biggest is 4, smallest is 2

SuperSnivy 9:20:56 pm
biggest is 4, smallest is 2

noodleeater 9:20:58 pm
smallest = 2, biggest = 6

Lord.of.AMC 9:20:59 pm
3,3 darn

joshuazucker 9:21:14 pm
Hm, this is tough. Yeah. There could be 1,1 pairs if all the first pile of apples are 1s.

joshuazucker 9:21:20 pm
There could be 3,3 pairs if almost all the apples are 3s.

joshuazucker 9:21:24 pm
So the range is from 2 to 6.

joshuazucker 9:21:31 pm
But we can't have both a 2 and a 6, can we?

joshuazucker 9:22:13 pm
So what we'd like to know is what we can get from the maximum ratio of the a_i -- not what's the biggest and smallest possible w_i, but what's the biggest possible ratio given one set of apple weights.

Cpi2728 9:22:21 pm
Nope. A 2 and a 4, or a 4 and a 6.

Lord.of.AMC 9:22:21 pm
if there's a 6, lowest is 4, otherwise lowest is 2

joshuazucker 9:22:27 pm
Hm, but could there be a 5.5 and a 2.5, or what?

joshuazucker 9:22:31 pm
I think we need to be careful.

joshuazucker 9:22:53 pm
Then $w_i = a + d$ and $w_j = b + c$, where a, b, c, and d are the weights of some apples with $a \le b \le c \le d$.

joshuazucker 9:23:14 pm
(I'm assuming $i < j$.)

joshuazucker 9:23:25 pm
We want to place bounds on

$$\frac{w_i}{w_j} = \frac{a+d}{b+c}.$$

joshuazucker 9:24:01 pm
This might be more or less than 1: we could have a 1+3 followed by a 1+1 or by a 3+3, for instance.

joshuazucker 9:24:06 pm
First, let's find the minimum value of this expression.

joshuazucker 9:24:16 pm
Given the ordering of a, b, c, d, what values of b and c minimize this expression?

anwang16 9:25:06 pm
are we saying the basic apple is weight 1?

joshuazucker 9:25:15 pm
Yeah, we were taking the lightest apple to be weight 1 for convenience.

lawrenceli 9:25:24 pm
b = c = d?

sujaykazi 9:25:24 pm
b=c=d=3a, then w(i)/w(j)=2/3

joshuazucker 9:25:28 pm
This expression is minimized when $b = c = d$, so

$$\frac{w_i}{w_j} \geq \frac{a+d}{2d} = \frac{a/d+1}{2}.$$

joshuazucker 9:25:31 pm
We know that $a/d \geq 1/3$.

joshuazucker 9:25:33 pm
Therefore,

$$\frac{w_i}{w_j} \geq \frac{1/3+1}{2} = \frac{2}{3}.$$

joshuazucker 9:25:44 pm
Now we find the maximum value of

$$\frac{w_i}{w_j} = \frac{a+d}{b+c}.$$

joshuazucker 9:25:48 pm
What values of b and c maximize this expression?

sujaykazi 9:26:39 pm
a=b=c=d/3, then w(i)/w(j)=2

mlcindy 9:26:39 pm
a a

Cosmynx 9:26:39 pm
a=b=c=1

distortedwalrus 9:26:39 pm
b=c = 1, d = 3

Lord.of.AMC 9:26:39 pm
a=b=c=d/3

joshuazucker 9:26:41 pm
This expression is maximized when $b = c = a$, so

$$\frac{w_i}{w_j} \leq \frac{a+d}{2a} = \frac{1+d/a}{2}.$$

joshuazucker 9:26:46 pm
And $d/a \leq 3$, so

$$\frac{w_i}{w_j} \leq \frac{1+3}{2} = 2.$$

joshuazucker 9:26:58 pm
The bounds

$$\frac{2}{3} \leq \frac{w_i}{w_j} \leq 2$$

tell us that every w_i is at most twice any other w_i.

joshuazucker 9:27:24 pm
So we went from 300 apples with a maximum ratio of 3:1 to 150 apple-pairs (apple-pears?) with a maximum ratio of 2:1.

joshuazucker 9:27:47 pm
Now we can go and use the strategy you suggested a while ago. Sort them and pair them again.

joshuazucker 9:27:51 pm
Let $v_1, v_2, \ldots, v_{150}$ be the values $w_1, w_2, \ldots, w_{150}$ in increasing order, so

$$v_1 \le v_2 \le \cdots \le v_{150}.$$

joshuazucker 9:27:55 pm
Then to get groups of 4, we pair the lightest v_i with the heaviest v_i.

joshuazucker 9:28:03 pm
Let $x_i = v_i + v_{151-i}$ for $1 \le i \le 75$.

joshuazucker 9:28:07 pm
We want to show that the ratio between any two x_i is at most $\frac{3}{2}$.

joshuazucker 9:28:28 pm
We can repeat similar arithmetic to what we did above, only now the maximum ratio is 2:1 instead of 3:1.

joshuazucker 9:28:31 pm
Consider two x_i and x_j, where $i < j$.

joshuazucker 9:28:32 pm
Then $x_i = p + s$ and $x_j = q + r$, where p, q, r, and s are all among the v_i. We can assume that $p \le q \le r \le s$.

joshuazucker 9:28:35 pm
We want to place bounds on

$$\frac{x_i}{x_j} = \frac{p+s}{q+r}.$$

joshuazucker 9:28:38 pm
This expression is minimized when $q = r = s$, so

$$\frac{x_i}{x_j} \ge \frac{p+s}{2s} = \frac{p/s+1}{2}.$$

joshuazucker 9:28:43 pm
We know that $\frac{p}{s} \ge \frac{1}{2}$.

joshuazucker 9:28:45 pm
Therefore,

$$\frac{x_i}{x_j} \ge \frac{1/2+1}{2} = \frac{3}{4}.$$

joshuazucker 9:28:53 pm
Now we find the maximum value of

$$\frac{x_i}{x_j} = \frac{p+s}{q+r}.$$

joshuazucker 9:28:57 pm
This expression is maximized when $q = r = p$, so

$$\frac{x_i}{x_j} \le \frac{p+s}{2p} = \frac{1+s/p}{2}.$$

joshuazucker 9:29:04 pm
But $s/p \le 2$, so

$$\frac{w_i}{w_j} \le \frac{1+2}{2} = \frac{3}{2}.$$

joshuazucker 9:29:11 pm
The bounds

$$\frac{3}{4} \le \frac{x_i}{x_j} \le \frac{3}{2}$$

tell us that every x_i is at most 3/2 times any other x_i, as desired.

joshuazucker 9:29:48 pm
SUMMARY

joshuazucker 9:29:49 pm
In today's class, we have seen how the Extremal Principle can be used effectively to solve problems across a broad variety of

subjects. Whenever you are dealing with a large number of objects and you don't know where to start, reaching for the Extremal Principle can provide a good start by giving you a handle on the problem.

joshuazucker 9:29:55 pm
The Extremal Principle is also useful in cases where you have a set of objects, and there is a transformation that can turn one object into another, such as the problem with the 2n red and blue points, and the Rearrangement Inequality. In such cases, applying the transformation to the maximal (or minimal) element often tells you something very important about the element.

joshuazucker 9:30:07 pm
That's it for today's class. Thanks for the fun! I hope to see you again soon.

joshuazucker 9:30:18 pm
Are there any last questions?

shandongboy 9:31:17 pm
Is there anywhere we can read more into this?

joshuazucker 9:31:25 pm
I mentioned that Tanton piece at the beginning of class.

joshuazucker 9:32:06 pm
There's also a nice piece of Paul Zeitz's _Art and Craft of Problem Solving_ book on the extreme principle.

joshuazucker 9:32:23 pm
Is it a conflict of interest that he's a friend of mine?

joshuazucker 9:32:28 pm
I'll recommend it anyway.

joshuazucker 9:32:31 pm
Anyone else have suggestions?