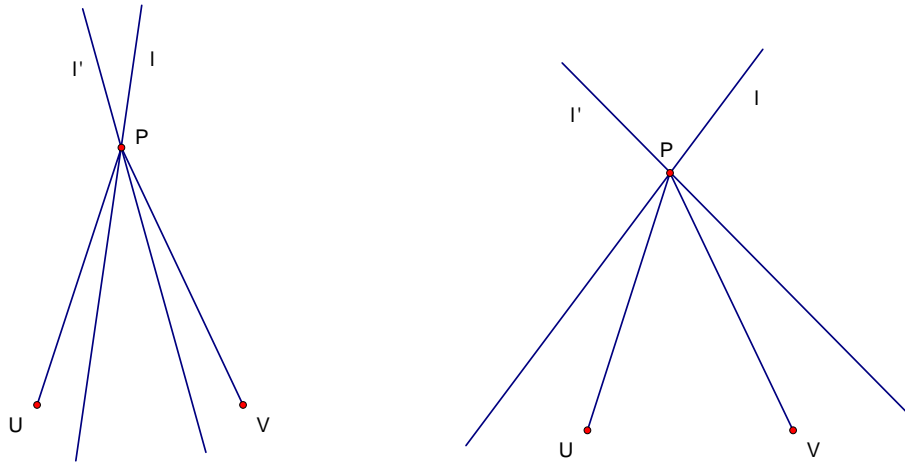
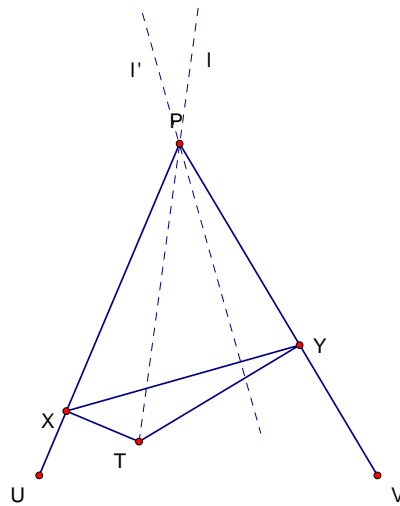


## Isogonal Conjugate

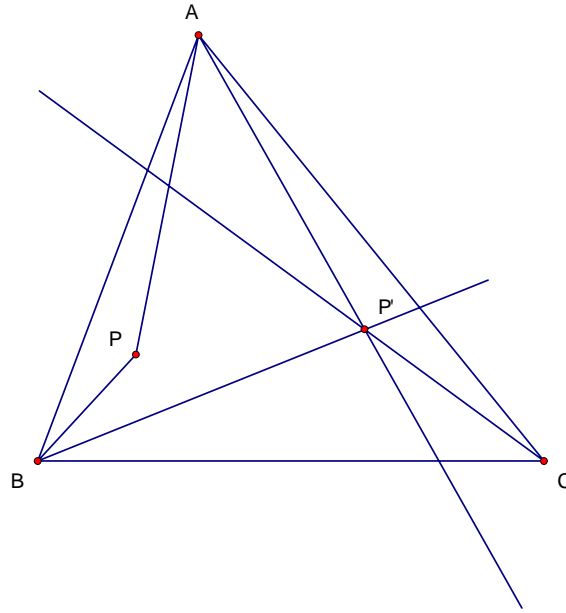
**Definition 1.** Given an acute angle  $\angle UPV$  and a line  $l$  passing through  $P$ , the isogonal of  $l$  wrt  $\angle UPV$  is defined to be the line  $l'$  symmetric to  $l$  about the internal angle bisector of  $\angle UPV$ .



**Theorem 1.** If  $X, Y$  is the orthogonal projections of  $T$  on lines  $UP, VP$  respectively, then the isogonal of line  $TP$  is perpendicular to line  $XY$ .



**Theorem 2. (Isogonal Conjugate Theorem)** Let  $ABC$  be a triangle and  $P$  be a point distinct from  $A, B, C$ . Then the isogonal of lines  $AP, BP, CP$  wrt  $\angle CAB, \angle ABC, \angle BCA$  respectively are concurrent. The concurrent point  $P'$  is called the isogonal conjugate of  $P$  wrt to  $\triangle ABC$ .

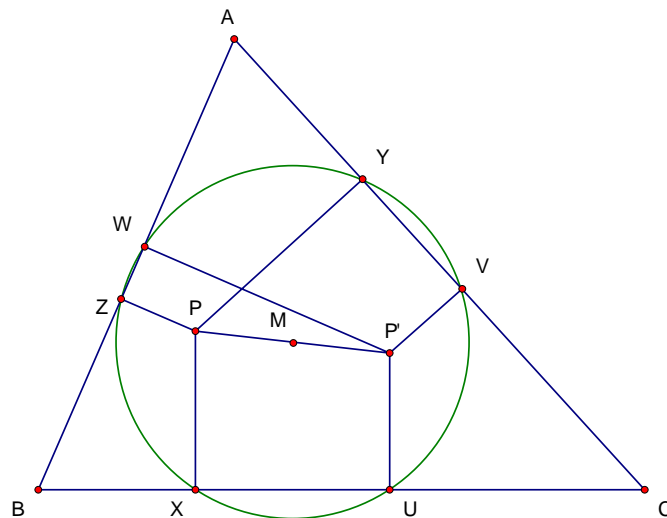


**Theorem 3.** The isogonal conjugate  $P'$  of  $P$  is the circumcenter of  $\triangle XYZ$  where  $X, Y, Z$  are the reflections of  $P$  about  $BC, CA, AB$  respectively.

**Theorem 4.**  $P$  is a point at infinity if and only if  $P'$  lies on the circumcircle of  $\triangle ABC$ .

**Theorem 5. (Pedal Circle Theorem)** Let  $M$  be the midpoint of  $PP'$  and its isogonal conjugate  $P'$ . Let  $X, Y, Z$  be the orthogonal projections of  $P$  and  $U, V, W$  be the orthogonal projections of  $P'$  on  $BC, CA, AB$  respectively. Then  $U, V, W, X, Y, Z$  all lie on a circle centered at  $M$ . Moreover

$$PX \cdot P'U = PY \cdot P'V = PZ \cdot P'W.$$



**Theorem 6.** Let  $P$  and  $P'$  be two isogonal conjugates wrt  $\triangle ABC$ . Then the isogonal of  $AP$  wrt  $\angle BPC$  and the isogonal of  $AP'$  wrt  $\angle BP'C$  are symmetric to each other about  $BC$ .

**Theorem 7.** Let  $P$  be a point in the plane of  $\triangle ABC$ , and let  $D, E, F$  be the reflections of  $P$  about the perpendicular bisectors of  $BC, CA, AB$  respectively. Then the points  $P, D, E, F$  lie on a circle centered at the circumcenter  $O$  of  $\triangle ABC$ . Moreover  $\triangle DEF \sim \triangle ABC$ .

**Definition 2.** If  $G$  is the centroid of  $\triangle ABC$  and  $T$  is an arbitrary point in the plane, then the image of the point  $T$  under the homothety with center  $G$  and factor  $-\frac{1}{2}$  is called the complement of  $T$  wrt  $\triangle ABC$ .

**Theorem 8.** Let  $P$  be a point in the plane of  $\triangle ABC$ , and let  $D, E, F$  be the reflections of  $P$  about the perpendicular bisectors of  $BC, CA, AB$  respectively. Denote by  $A_M, B_M, C_M$  the midpoints of  $BC, CA, AB$  and by  $D_M, E_M, F_M$  the midpoints of  $EF, FD, DE$ . Let  $Q'$  be the complement of  $P'$  wrt  $\triangle ABC$ . Then the lines  $A_M D_M, B_M E_M, C_M F_M$  pass through the point  $Q'$ .

*Reference:*

1. *Isogonal Conjugation with respect to a triangle* --- Darij Grinberg