Generating Functions

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1 How do we "know" a sequence?

Suppose we are given the first few terms of a sequence, $a_0, a_1, a_2, \ldots$ like $1, 2, 4, 8, \ldots$, then we don't really "know" the sequence unless we have a general rule that identifies every term of the sequence.

For example,

1. Our $a_n =$ _______________, called:

2. Or $a_0 = 1$, $a_n =$ ______________________, called:

3. Or $a_n =$ _____________________________, called:

Plan:

1. We first study how to take a recurrence relation and get its generating function.

2. Then we will learn how to take objects and find a recurrence relation for them.

3. Finally, we will use these techniques creatively on problems.

2 How to find generating functions

EXAMPLE 1 A starter. Find the generating function that satisfies the recurrence $a_0 = 1$, $a_{n+1} = a_n$, for $n \geq 0$.

Step 1: Always look at a few terms of the sequence to get insight.
Step 2: State the objective: To find the generating function

$$A(x) = \sum_{n\geq 0} a_n x^n.$$

Step 3: Take the recurrence and multiply both sides by x^n.

Step 4: Take $\sum_{n\geq 0}$ on both sides.

Step 5: Relate $\sum$ to $A(x)$ or other known generating functions.

Step 6: Solve for $A(x)$.

Can you think of another way?

EXAMPLE 2 An easy two-term recurrence. Consider the sequence $a_0, a_1, \ldots$ satisfying the conditions $a_{n+1} = 2a_n + 1$, for $n \geq 0$, and $a_0 = 0$. Find its generating function.

Now, state the steps listed above to see if we can find the generating function.

EXAMPLE 3 Another useful two-term recurrence. Find the generating function for $a_0 = 0$, $a_{n+1} = a_n + 1$, for $n \geq 0$.

What are the terms?

EXAMPLE 4 Find the generating function for $a_{n+1} = a_n + n + 1$, $a_0 = 1$, $n \geq 0$.

$$\sum a_{n+1} x^n = \sum a_n x^n + \sum n x^n + \sum 1 x^n$$

$$\sum a_{n+1} x^n = A(x) + \frac{x}{(1-x)^2} + \frac{1}{1-x}$$

$$\sum_{n \geq 0} a_{n+1} x^{n+1} = x A(x) + \frac{x^2}{(1-x)^2} + \frac{x}{1-x}$$

$$A(x) - a_0 = x A(x) + \frac{x^2 + x(1-x)}{(1-x)^2}$$

$$A(x)(1-x) = 1 + \frac{x}{(1-x)^2} = \frac{x^2 - x + 1}{(1-x)^2}$$

$$A(x) = \frac{x^2 - x + 1}{(1-x)^3}$$

What does $\dfrac{1}{1-x}$ do when we multiply it to a generating function $A(x)$?

EXAMPLE 5 Can you find the generating function to the following three-term recurrence?

$$a_{n+1} = a_n + a_{n-1}, \quad (n \geq 1, a_0 = a_1 = 1)$$

3 Match the right problem with its recurrence

EXAMPLE 6 A line separates a plane into two regions. Two intersecting lines divide a plane into four regions. Determine the number of regions into which a plane is separated by n lines in general position, that is, no two lines parallel and no three lines concurrent.

EXAMPLE 7 The tower of Hanoi puzzle (1883) consisted of three pegs with eight circular rings of tapering sizes placed in order (the largest on the bottom) on one of the pegs. These rings could be moved one at a time from one peg to another with the proviso that a ring never be placed on top of a smaller-sized ring. The object of the game was to move all the rings to one of the other two pegs, that is, to transfer the "tower" of rings to another peg. Determine the minimum number of moves required to transfer a tower of n rings.

EXAMPLE 8 Leonardo Fibonacci (1180–1228) was fond of problems, his most famous of which is concerned with rabbits! Suppose that newborn rabbits start producing offspring by the end of their second month of life and that after this point, they produce a pair a month (one male, one female). Assuming just one pair of rabbits initially, how many pairs of rabbits will be alive after n months?

4 Recurrence with two independent variables

Until now, we have studied sequences $\{a_n\}$ in *one* independent variable n. In this section, we will consider sequences in two independent variables, such as $\{a_{m,n}\}_{m,n\geq 0}$. To avoid writing too many indices, we will use the function notation $f(m,n)$ $(m,n \geq 0)$ instead of $\{a_{m,n}\}_{m,n\geq 0}$.

EXAMPLE 9 Prove the identity

$$f(n,m) = f(n-1,m) + f(n-1,m-1)$$

$$((n,m) \neq (0,0); f(0,0) = 1)$$

where $f(n,m)$ is the number of ways of choosing a subset of m objects from a set of n distinct objects.

An algebraic proof:

A combinatorial proof:

Can you find a generating function for the $f(n, m)$ above?
Let us go through the same steps as in the one variable case.

5 Set partition

By a *partition of a set* S, we mean a collection of non-empty, pairwise disjoint sets whose union is S. No significance attaches to the order of the elements within the disjoint sets, nor to the order of the sets. All we care is which elements are together, and which are apart.

EXAMPLE 10 Try to partition $\{1, 2, 3, 4\}$ into 2 classes (or disjoint subsets).

EXAMPLE 11 Another two-variable case. Let $\left\{ {n \atop k} \right\}$ denote the number of partitions of an n-element set into k classes. Prove the recurrence

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

for $(n, k) \neq (0, 0)$; $\left\{ {0 \atop 0} \right\} = 1$.

How do you construct a combinatorial proof?

Problem 2 (From 1987 Austrian Olympiad)

This problem concerns sequences $x_1x_2\ldots x_n$ in which each x_i is either a, b, or c. Determine the number of these sequences

- which have length n,
- begin and end with the letter a, and
- in which adjacent terms are always different letters.

PROBLEM 3 (FROM M16, PROPOSED BY MAYHEM STAFF)
Can an 8×9 checkerboard be completely covered by twelve 1×6 rectangles?

Problem 4 (From 14th IMO, 1972/3)

Let m and n be arbitrary non-negative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer. $(0! = 1.)$

$\binom{9}{2}$

PROBLEM 5 (FROM A191, CRUX)

Taken over all ordered partitions of n, show that

$$\sum_{k_1+k_2+\cdots+k_m=n} k_1 k_2 \dots k_m = \binom{m+n-1}{2m-1}$$

$(1+x+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)$

Problem 6 (45th AHSME, 1994)
When n standard 6-sided dice are rolled, the probability of obtaining a sum of 1994 is greater than zero and is the same as the probability of obtaining a sum of S. Find the smallest possible value of S.

PROBLEM 7 (62ND PUTNAM, 2001, A2)

You have coins C_1, C_2, ... , C_n. For each k, coins C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

PROBLEM 8 (1988 USA OLYMPIAD)

Let S be the set $\{1, 2, \ldots, 20\}$, and let each 9-element subset of S be assigned a label from S itself. Thus, for example,

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

There are $\binom{20}{9} = 167960$ different 9-element subsets of S, and so, on the average, each of the 20 labels gets used 8398 times.

Prove that no matter how these labels might be assigned, there always exists a 10-element subset T of S with the property that no element of T is the label for the subset determined by the other 9 elements of T.

PROBLEM 9 (1983 KŰRSCHÁK COMPETITION, HUNGARY)
Consider the polynomial $f(x)$ whose first and last coefficients are 1 and whose intervening coefficients a_i are all non-negative:

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + 1$$

If the equation $f(x) = 0$ happens to have n real roots, is it not remarkable that the value of $f(2)$ must then be at least 3^n? Prove this unlikely consequence: $f(2) \geq 3^n$.

Problem 10 (1994 IMO Mock Test from Hong Kong)

A function $f(n)$, defined on the natural numbers, satisfies:

$$f(n) = \begin{cases} n - 12, & n > 2000, \\ f(f(n+16)), & n \leq 2000. \end{cases}$$

1. Find $f(n)$.

2. Find all solutions to $f(n) = n$.