Algebra

1. We can write $p(x) = P(x)/k$ for some integer k so that $P(x)$ has integer coefficients. Then we can choose $n = k$ so that
$$q(x) = p(x+n) - p(x) = \frac{P(x+k) - P(x)}{k},$$
which has integer coefficients because $k|P(x+k) - P(x)$.■

2. The first equation is equivalent to $x^2 + (y-z)^2 = 2$, so $x, y-z = \pm 1$. With the second equation, these four cases yield the triples $(1, 2008, 2009), (1, 2009, 2008), (-1, 2009, 2010), (-1, 2010, 2009)$.■

3. First note that
$$f(x) = \frac{(x+1)f(x-1)}{x}$$
for all rational $x > 1$. We prove by strong induction on $p + q \geq 2$ that for all rational $x = p/q$ with $\gcd(p, q) = 1$, $f(x) = f(1)(p+q)/2$. This is clearly true for $p + q = 2 \implies p = q = 1$. If $p + q \geq 3$, we obtain from the inductive hypothesis for $p = (p-q) + q$ (since we have $\gcd(p-q, q) = \gcd(p, q) = 1$ and $p - q > 0$) that
$$f\left(\frac{q}{p}\right) = f\left(\frac{p}{q}\right) = \frac{\left(\frac{p}{q}+1\right) f\left(\frac{p}{q}-1\right)}{\frac{p}{q}} = \left(\frac{p+q}{p}\right) f\left(\frac{p-q}{q}\right) = \frac{1}{2} f(1)(p+q),$$
completing the induction. Finally, since $f(x)$ only outputs integers, and $p + q$ can be odd, $f(1)$ must be even so we can set the integer $C = f(1)/2$ so that
$$f\left(\frac{p}{q}\right) = C(p+q), \qquad \gcd(p, q) = 1,$$
which fulfils the problem's conditions.■

4. First assume $x, y, z \geq 0$. Let $x + y + z = 3u$, $xy + yz + zx = 3v^2$, and $xyz = w^3$. Then
$$\begin{aligned} f(x, y, z) &= \frac{(xy + yz + zx)(x + y + z)}{(x+y)(y+z)(z+x)} \\ &= \frac{(3v^2)(3u)}{(3u-x)(3u-y)(3u-z)} \\ &= \frac{9uv^2}{(3u)^3 - (3u)(3u)^2 + (3v^2)(3u) - w^3} \\ &= 1 + \frac{w^3}{9uv^2 - w^3}. \end{aligned}$$
Now note that $u, v \geq w$ by AM-GM, so $9uv^2 - w^3 \geq 9w^3 - w^3 = 8w^3$. Hence
$$1 \leq f(x, y, z) = 1 + \frac{w^3}{9uv^2 - w^3} \leq 1 + \frac{w^3}{8w^3} = \frac{9}{8}.$$
To prove that all values in $[1, 9/8]$ are achievable, set $y = z = 1$ so
$$f(x, y, z) = \frac{(xy + yz + zx)(x + y + z)}{(x+y)(y+z)(z+x)} = \frac{(2x+1)(x+2)}{2(x+1)^2} = 1 + \frac{x}{2(x+1)^2}.$$
This means that $f(0, 1, 1) = 1$ and $f(1, 1, 1) = 9/8$. By the continuity of $f(x, 1, 1)$, all values in $[1, 9/8]$ are achievable.

Now we consider the case where x, y, z need not be positive. Recall that
$$f(x, 1, 1) = 1 + \frac{x}{2(x+1)^2},$$

so
$$f\left(-1+\frac{1}{k},1,1\right)=1+\frac{-1+\frac{1}{k}}{2\left(-1+\frac{1}{k}+1\right)^2}=1-\frac{k(k-1)}{2}.$$
Then $k=1 \implies f=1$, and $\lim_{k\to+\infty} f=-\infty$, so all values in $(-\infty,1]$ are achievable since f is continuous. Next, we find that
$$f\left(x,y,-\frac{x+y}{2}\right)=\frac{\left(xy+y\left(-\frac{x+y}{2}\right)+x\left(-\frac{x+y}{2}\right)\right)\left(x+y+\left(-\frac{x+y}{2}\right)\right)}{(x+y)\left(y+\left(-\frac{x+y}{2}\right)\right)\left(x+\left(-\frac{x+y}{2}\right)\right)}=\frac{x^2+y^2}{(x-y)^2}.$$
Therefore
$$f\left(k+1,k,-\frac{2k+1}{2}\right)=(k+1)^2+k^2,$$
with $k=0 \implies f=1$ and $\lim_{k\to+\infty} f=+\infty$. Again by the continuity of f, $[1,+\infty)$ is in the range of f, so we conclude that if x,y,z can be negative, then all real numbers are in the range of f.■

5. Via polynomial division we can write
$$\frac{P(x)}{Q(x)}=D(x)+\frac{R(x)}{Q(x)},$$
where either $R(x)=0$ for all x or $\deg R<\deg Q$. Assume for the sake of contradiction that $R(x)$ is not the zero polynomial, so $\deg R<\deg Q$. Because $P(x)$ and $Q(x)$ have integer coefficients, $D(x)$ has rational coefficients and for some integer k we have $D(x)=d(x)/k$ where $d(x)$ has integer coefficients. Because $\deg R<\deg Q$, for $x>C$ for some sufficiently large positive constant C we have
$$-\frac{1}{k}<\frac{R(x)}{Q(x)}<\frac{1}{k} \implies \frac{d(x)-1}{k}<\frac{P(x)}{Q(x)}=D(x)+\frac{R(x)}{Q(x)}<\frac{d(x)+1}{k}.$$
Note that a_n is unbounded, so there are infinitely many values of n for which $a_n>C$. But $P(a_n)/Q(a_n)$ is always an integer, so for infinitely many values of n we must have
$$\frac{P(a_n)}{Q(a_n)}=\frac{d(a_n)}{k} \implies R(a_n)=0$$
since
$$\frac{d(a_n)-1}{k}<\frac{P(a_n)}{Q(a_n)}<\frac{d(a_n)+1}{k}$$
and $d(a_n)$ only outputs integers. But if $R(x)=0$ for infinitely many values of x, it must be 0 for all x since it has finite degree. Thus
$$\frac{P(x)}{Q(x)}=\frac{d(x)}{k}$$
for all x with $Q(x)\neq 0$, and when $x=a_n$ we have to have $k|d(a_n)$. Now, observe that for $0\le i\le k-1$,
$$0\equiv d(a_{k+i})=d((k+i)!+k+i)=d\left(i+k\left(1+\frac{(k+i)!}{k}\right)\right)\equiv d(i) \pmod k$$
since $k|d(i+jk)-d(i)$ for all integers j. Now for arbitrary integers n, write $n=qk+r$ with $0\le r\le k-1$. Then
$$d(n)=d(r+qk)\equiv d(r)\equiv 0 \pmod k$$
for all n with $Q(n)\neq 0$, and we are done.■

Combinatorics

1. Let a_i be the number of questions on which Isaac receives a score of i. Then $a_0 + \cdots + a_{10} = 6$, and for each solution in nonnegative integers $(a_0, \ldots, a_{10})$, there is precisely one sequence of six marks that can be achieved (the scores in nonincreasing order). A simple application of balls and urns tells us that the answer is
$$\binom{6+11-1}{11-1} = \binom{16}{10} = 8008.\blacksquare$$

2. When the sectors of the two circles line up, define a *good sector* on the small circle as one that lies over a sector of the same color on the big circle. Since there are an equal number of white and black sectors on both circles, the expected number of good sectors is $E = \sum pv = 200(1/2)(1) = 100$. This means that at least one configuration has at least 100 good sectors, for otherwise each configuration would have at most 99 good sectors, contradicting the fact that the expected number of good sectors is 100.$\blacksquare$

3. In this solution, reduce all indices i of x_i to the smallest possible positive integer modulo n. We shall prove by induction on $r \geq 0$ that all n-tuples of the form
$$\left(\prod_{i=1}^{2^r} x_i, \prod_{i=2}^{2^r+1} x_i, \ldots, \prod_{i=n}^{2^r+n-1} x_i\right)$$
will eventually be obtained. Since we start with $(x_1, \ldots, x_n)$, this is clearly true for $r = 0$. Call the operation
$$(x_1, x_2, \ldots) \mapsto \left(\prod_{i=1}^{2^r} x_i, \prod_{i=2}^{2^r+1} x_i, \ldots\right)$$
the r-operation on $(x_1, x_2, \ldots)$. Assume it exists for $r = s$. Then through the 1-operation on $(x_1, x_2, \ldots)$ and the fact that $x_i^2 = 1$, we get
$$\left(\prod_{i=1}^{2^s} x_i, \prod_{i=2}^{2^s+1} x_i, \ldots\right) \mapsto (x_1 x_{2^s+1}, x_2 x_{2^s+2}, \ldots).$$
Now through the s-operation on $(x_1 x_{2^s+1}, x_2 x_{2^s+2}, \ldots)$, we get
$$(x_1 x_{2^s+1}, x_2 x_{2^s+2}, \ldots) \mapsto \left(\prod_{i=1}^{2^s} x_i x_{2^s+i}, \prod_{i=2}^{2^s+1} x_i x_{2^s+i}, \ldots\right) = \left(\prod_{i=1}^{2^{s+1}} x_i, \prod_{i=2}^{2^{s+1}+1} x_i, \ldots\right),$$
completing the induction. Finally, if $n = 2^k$ for some integer $k \geq 1$, a k-operation on $(x_1, x_2, \ldots)$ will yield
$$\left(\prod_{i=1}^{2^k} x_i, \prod_{i=2}^{2^k+1} x_i, \ldots\right) = \left(\prod_{i=1}^{2^k} x_i, \prod_{i=1}^{2^k} x_i, \ldots\right),$$
the right hand side of which consists of 2^k equal numbers. A 1-operation on this will give
$$\left(\prod_{i=1}^{2^k} x_i^2, \prod_{i=1}^{2^k} x_i^2, \ldots\right) = (1, 1, \ldots),$$
as desired.$\blacksquare$

4. We provide a construction such that it is not possible to arrange them in a desired line with at most 2010 children between any two friends. Define the sets $A_i,\ 0 \le i \le n$ as follows:

   - Let n be the unique positive integer such that $1 + 3(2^{n-1} - 1) < 2010^{2010}$ and $1 + 3(2^n - 1) \ge 2010^{2010}$.
   - A_0 contains a single person who has 3 friends. These 3 friends are the members of A_1.
   - For $2 \le i \le n-1$, A_i consists of the friends of the members of A_{i-1}. Each person X in A_{i-1} has 2 friends in A_i. Both of these friends have no friends in a set $A_j,\ j \le i$ besides X.
   - In this way, the set $A_i,\ 1 \le i \le n-1$ has $3 \cdot 2^{i-1}$ members so

     $$|A_0| + |A_1| + \cdots + |A_{n-1}| = 1 + 3(2^{n-1} - 1) < 2010^{2010}.$$

     If A_n were defined in the same way as $A_i,\ 2 \le i \le n-1$, we would have

     $$|A_0| + |A_1| + \cdots + |A_{n-1}| + |A_n| = 1 + 3(2^n - 1) \ge 2010^{2010}.$$

     Hence we can just take away $1 + 3(2^n - 1) - 2010^{2010}$ members from A_n so that

     $$|A_0| + |A_1| + \cdots + |A_{n-1}| + |A_n| = 2010^{2010}.$$

     This gives a valid construction of 2010^{2010} children. Also note that A_n is not empty, or else there would not be enough children.

   Number the positions of the 2010^{2010} children in line from 1 to 2010^{2010}. Let $X_{A_i},\ 0 \le i \le n$ be an arbitrary member of A_i. Define $d(X_{A_i}, X_{A_j})$ as the absolute value of the difference of the numerical positions of X_{A_i} and X_{A_j}. Through a simple induction utilizing the condition that there are at most 2010 children between any pair of friends, we can prove that

   $$d(X_{A_0}, X_{A_i}) \le 2011i, \qquad 0 \le i \le n.$$

   Hence

   $$d(X_{A_0}, X_{A_i}) \le d(X_{A_0}, X_{A_n}) \le 2011n, \qquad 0 \le i \le n,$$

   which means that the entire line of children can have length at most $1 + 2(2011n) = 1 + 4022n$, so

   $$2010^{2010} \le 1 + 4022n \implies n \ge \frac{2010^{2010} - 1}{4022} > \frac{2010^{2009}}{2010^2} = 2010^{2007}.$$

   Recall, however, that

   $$1 + 3(2^{n-1} - 1) < 2010^{2010},$$

   so

   $$2010^{2010} > 1 + 3(2^{n-1} - 1) > 2^n > 2^{2010^{2007}} = (2^{11})^{\frac{2010^{2007}}{11}} > 2010^{\frac{2010^{2007}}{11}} > 2010^{2010},$$

   a clear contradiction.■

Number Theory

1. Let $\omega = e^{2\pi i/3}$. Then $\omega^8 + \omega + 1 = \omega^2 + \omega + 1 = 0$, so we conclude that $n^2 + n + 1 | n^8 + n + 1$. Hence for $n^8 + n + 1$ to be prime, we must have $n^2 + n + 1 = n^8 + n + 1 \implies n = 1$, which does indeed yield the prime $n^8 + n + 1 = 3$.$\blacksquare$

2. Because $2^n + n | 8^n + n$, we have
$$n^3 - n \equiv (-2^n)^3 - (-8^n) \equiv 0 \pmod{2^n + n}.$$
From this, we find that either $n = 1$, which works since $3 = 2^1 + 1 | 8^1 + 1 = 9$, or $n^3 - n \geq 2^n + n \implies n^3 > n^3 - n \geq 2^n + n > 2^n$. A simple induction shows that $2^n \geq n^3$ for integers $n \geq 10$. Testing $n < 10$, we find that the solutions are $n = 1, 2, 4, 6$.$\blacksquare$

3. Clearly $a, b \geq 0$. Let
$$n^2 = 3^a + 7^b \equiv (-1)^a + (-1)^b \equiv 0, 2 \pmod 4 \implies n^2 \equiv 0 \pmod 4,$$
so a and b must be of opposite parity since 2 is not a quadratic residue modulo 4. We have two cases.

Case 1: a is even and b is odd. We have
$$\left(n - 3^{\frac{a}{2}}\right)\left(n + 3^{\frac{a}{2}}\right) = 7^b,$$
so both $n - 3^{a/2}$ and $n + 3^{a/2}$ are powers of 7. Their difference is not divisible by 7, so we must have $n - 3^{a/2} = 1$ and $n + 3^{a/2} = 7^b$, so
$$7^b - 1 = 2 \cdot 3^{\frac{a}{2}}.$$
If $a = 2$, we get the solution $(2, 1)$. Otherwise, $a \geq 4$, so
$$7^b \equiv 1 \pmod 9 \implies 3 | b.$$
Thus $b/3$ is an integer, and
$$\left(7^{\frac{b}{3}} - 1\right)\left(7^{\frac{2b}{3}} + 7^{\frac{b}{3}} + 1\right) = 2 \cdot 3^{\frac{a}{2}},$$
from which it is clear that for nonnegative integers u, v with $u + v = a/2 \geq 2$, we have $7^{b/3} - 1 = 2 \cdot 3^u$ and $7^{2b/3} + 7^{b/3} + 1 = 3^v$. If $u \geq 2$, then
$$7^{\frac{b}{3}} \equiv 1 \pmod 9 \implies 7^{\frac{2b}{3}} + 7^{\frac{b}{3}} + 1 \equiv 3 \pmod 9,$$
so $v = 1 \implies b/3 = 0 \implies 0 = 2 \cdot 3^u$ which is impossible. If $u = 1$, we arrive at $b = 3 \implies 7^2 + 7^1 + 1 = 57 = 3^v$, with no solutions. Finally, if $u = 0$, then $7^{b/3} - 1 = 2$, also without solutions, so we see that this case only provides the solution $(2, 1)$.

Case 2: a is odd and b is even. We have
$$\left(n - 7^{\frac{b}{2}}\right)\left(n + 7^{\frac{b}{2}}\right) = 3^a,$$
so both $n - 7^{b/2}$ and $n + 7^{b/2}$ are powers of 3. However, their difference is not divisible by 3, so $n - 7^{b/2} = 1$ and $n + 7^{b/2} = 3^a$, and we get
$$3^a - 1 = 2 \cdot 7^{\frac{b}{2}} \implies 3^a \equiv 1 \pmod 7 \implies 6 | a,$$
contradicting the fact that a is odd. This case has no solutions.

In conclusion, the only solution is $(2, 1)$.$\blacksquare$

4. **Lemma:** For all primes p and integers n,

$$p|n\prod_{k=1}^{p-1}(kn+1).$$

Proof: If $p|n$, we are done. Otherwise, $\gcd(p,n)=1$ and n has an inverse modulo p. Define $1\le u\le p-1$ so that

$$u\equiv -n^{-1}\pmod p \implies p|un+1\Bigg|n\prod_{k=1}^{p-1}(kn+1),$$

as desired.■

Observe that

$$E(1)=11!=2^{a_1}3^{a_2}5^{a_3}7^{a_4}11^{a_5},\qquad a_i\ge 1.$$

Now note that for all integers n,k, we have $\gcd(n,kn+1)=1$. Hence for $p=2,3,5,7,11$,

$$v_p(E(p))=v_p\left(p\prod_{k=1}^{10}(kp+1)\right)=1,$$

so

$$\gcd(E(1),\ldots,E(2009))|2\cdot 3\cdot 5\cdot 7\cdot 11.$$

By the lemma, all five primes $2,3,5,7,11$ divide $E(n)$ for all n, so we conclude that

$$\gcd(E(1),\ldots,E(2009))=2\cdot 3\cdot 5\cdot 7\cdot 11=2310.■$$

Geometry

1. Let $\alpha = \angle A/2$, $\beta = \angle B/2$, and $\gamma = \angle C/2$, so $\alpha + \beta + \gamma = 90^\circ$ with $0^\circ < \alpha, \beta, \gamma < 90^\circ$. Simple angle chasing with exterior angles gives us

$$\angle AEI = \angle B + \angle BCE = 2\beta + \gamma = 90^\circ - \alpha + \beta$$

and

$$\angle ADI = \angle C + \angle CBD = 2\gamma + \beta = 90^\circ - \alpha + \gamma.$$

By the Law of Sines on $\angle IAE = \alpha$, we find that

$$\frac{IE}{\sin \angle IAE} = \frac{IE}{\sin \alpha} = \frac{AI}{\sin \angle AEI} = \frac{AI}{\sin(90^\circ - \alpha + \beta)},$$

and from another application on $\angle IAD = \alpha$, we find

$$\frac{ID}{\sin \angle IAD} = \frac{ID}{\sin \alpha} = \frac{AI}{\sin \angle ADI} = \frac{AI}{\sin(90^\circ - \alpha + \gamma)},$$

so dividing the two equations yields

$$\frac{IE}{ID} = \frac{\sin(90^\circ - \alpha + \gamma)}{\sin(90^\circ - \alpha + \beta)}.$$

Hence $IE = ID$ is equivalent to at least one of (recall that $0^\circ < \alpha, \beta, \gamma < 90^\circ$)

$$(90^\circ - \alpha + \gamma) \equiv (90^\circ - \alpha + \beta) \pmod{360^\circ} \iff \beta = \gamma \iff AB = AC$$

and

$$(90^\circ - \alpha + \gamma) + (90^\circ - \alpha + \beta) \equiv 180^\circ \pmod{360^\circ} \iff \alpha = 30^\circ \iff \angle BAC = 60^\circ.$$

The first is impossible as we are given $AB < AC$, so

$$IE = ID \iff \angle BAC = 60^\circ,$$

as desired.$\blacksquare$

2. We use directed angles modulo 180°, so A, B, C, D are concyclic, regardless of order, if and only if $\angle ABC = \angle ADC = \angle AOC/2$ (if O is the center of the circle). Let the center of the circle be O and let T be the intersection of the tangents at A and B. Now let X, Y, Z be the feet from P to AT, BT, AB respectively so that $PX = a$, $PY = b$, and $PZ = c$. We shall use the facts that $AT = BT$, $OB = OP$, $AZPX$ is cyclic, and $BYPZ$ is cyclic. Then

$$\angle ZPX = \angle ZAX = \angle YBZ = \angle YPZ,$$

and

$$\angle PXZ = \angle PAZ = \angle PAB = \frac{1}{2}\angle POB = 90^\circ - \angle OBP = \angle PBY = \angle PZY.$$

Thus $\triangle PXZ \sim \triangle PZY$, which means that

$$\frac{a}{c} = \frac{PX}{PZ} = \frac{PZ}{PY} = \frac{c}{b} \implies c^2 = ab,$$

as desired.$\blacksquare$

3. Extend CG to hit AB at point N, the midpoint of AB. First, if $\angle ACB = 90°$, then N is the circumcenter of $\triangle ABC$ so $NC = NB$. By definition, $EG \| BC$ so $NE = NG$, giving us

$$\triangle CNE \sim \triangle BNG \implies \angle DGC = \angle NGB = \angle NEC = \angle AEC,$$

as desired. Now, for the other direction, note that

$$\angle CEB = 180° - \angle AEC = 180° - \angle DGC = \angle CGB.$$

This means that $CGEB$ is cyclic, so using the fact that $EG \| BC$, we obtain

$$\angle NBC = \angle EBC = 180° - \angle EGC = \angle NGE = \angle NCB \implies NC = NB = NA,$$

where $NB = NA$ because N is the midpoint of AB. This shows that

$$\angle ACB = \angle ACN + \angle NCB = \angle CAN + \angle NBC = 180° - \angle ACB \implies \angle ACB = 90°,$$

and we are done.$\blacksquare$

4. Let the midpoint of BC be M, which is also the foot from F to BC by definition, and let FA hit DE at P. Without loss of generality, assume the radius of the circle with center B is less than that of the circle with center C. Let their radii be $r < s$, and let the foot from B to CE be N. Note that $BD \| FA \| CNE$, and $BN \| DE$. We can easily find that $AM = CA - CM = s - (r+s)/2 = (s-r)/2$ and $CN = CE - NE = CE - BD = s - r$, so $CN/AM = 2$. Clearly $\angle AMF = \angle CNB = 90°$ and $\angle FAM = \angle BAP = \angle BCE = \angle BCN$, so

$$\triangle AMF \sim \triangle CNB \implies \frac{CB}{AF} = \frac{CN}{AM} = 2,$$

which is what we wanted to prove.$\blacksquare$