WOOT 2010–11
Inequalities Solutions

1. Prove the AM-GM inequality using the following steps:

   (1) Prove that the inequality holds for two variables.

   (2) Prove that if the inequality holds for k variables, then it holds for $2k$ variables.

   (3) Prove that if the inequality holds for k variables, then it holds for $k-1$ variables.

Solution. (1) We want to prove that

$$\frac{x_1+x_2}{2} \ge \sqrt{x_1x_2}$$

for all x_1, $x_2 \ge 0$. We can re-write this inequality as $x_1 - 2\sqrt{x_1x_2} + x_2 = (\sqrt{x_1} - \sqrt{x_2})^2 \ge 0$, which is clearly true. Equality occurs if and only if $x_1 = x_2$.

(2) Assume that the AM-GM inequality holds for k variables. Let x_1, x_2, $\dots$, $x_{2k} \ge 0$. By the AM-GM inequality for two variables,

$$\begin{aligned}\frac{x_1+x_2+\cdots+x_{2k}}{2k} &\ge \frac{2\sqrt{x_1x_2}+2\sqrt{x_3x_4}+\cdots+2\sqrt{x_{2k-1}x_{2k}}}{2k} \\ &= \frac{\sqrt{x_1x_2}+\sqrt{x_3x_4}+\cdots+\sqrt{x_{2k-1}x_{2k}}}{k}.\end{aligned}$$

We know that the AM-GM inequality holds for k variables, so

$$\begin{aligned}\frac{\sqrt{x_1x_2}+\sqrt{x_3x_4}+\cdots+\sqrt{x_{2k-1}x_{2k}}}{k} &\ge \sqrt[k]{\sqrt{x_1x_2}\sqrt{x_3x_4}\cdots\sqrt{x_{2k-1}x_{2k}}} \\ &= \sqrt[2k]{x_1x_2\cdots x_{2k}}.\end{aligned}$$

Hence,

$$\frac{x_1+x_2+\cdots+x_{2k}}{2k} \ge \sqrt[2k]{x_1x_2\cdots x_{2k}}.$$

Equality holds if and only if $x_1 = x_2$, $x_3 = x_4$, $\dots$, $x_{2k-1} = x_{2k}$, and (by the induction hypothesis) $\sqrt{x_1x_2} = \sqrt{x_3x_4} = \cdots = \sqrt{x_{2k-1}x_{2k}}$. Thus, equality holds if and only if $x_1 = x_2 = \cdots = x_{2k}$.

(3) Assume that the AM-GM inequality holds for k variables. Let x_1, x_2, $\dots$, $x_{k-1} \ge 0$, and let $x = (x_1+x_2+\cdots+x_{k-1})/(k-1)$. Then by the AM-GM inequality on the k variables x_1, x_2, $\dots$, x_{k-1}, and x,

$$\frac{x_1+x_2+\cdots+x_{k-1}+x}{k} \ge \sqrt[k]{x_1x_2\cdots x_{k-1}x}.$$

Substituting $x = (x_1+x_2+\cdots+x_{k-1})/(k-1)$, we get

$$\frac{x_1+x_2+\cdots+x_{k-1}}{k-1} \ge \sqrt[k]{x_1x_2\cdots x_{k-1}\cdot\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}}.$$

Raising both sides to the power of k, we get

$$\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)^k \ge x_1x_2\cdots x_{k-1}\cdot\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}.$$

Then

$$\left(\frac{x_1 + x_2 + \cdots + x_{k-1}}{k-1}\right)^{k-1} \geq x_1 x_2 \cdots x_{k-1},$$

so

$$\frac{x_1 + x_2 + \cdots + x_{k-1}}{k-1} \geq \sqrt[k-1]{x_1 x_2 \cdots x_{k-1}}.$$

By the induction hypothesis, equality occurs if and only if $x_1 = x_2 = \cdots = x_{k-1}$.

We can then finish the proof as follows. From (1), the AM-GM inequality holds for two variables. Then from (2), the AM-GM inequality holds for n variables whenever n is a power of 2. Every positive integer is less than some power of 2, so from (3), the AM-GM inequality holds for any number of variables.

2. Show that for any positive integer $n \geq 1$,

$$1 \cdot 3 \cdot 5 \cdots (2n-1) \leq n^n.$$

Solution. By the AM-GM inequality,

$$1 \cdot 3 \cdot 5 \cdots (2n-1) \leq \left[\frac{1 + 3 + 5 + \cdots + (2n-1)}{n}\right]^n = \left(\frac{n^2}{n}\right)^n = n^n.$$

3. Let m and n be positive integers. Find the minimum value of

$$x^m + \frac{1}{x^n}$$

for $x > 0$.

Solution. By the weighted AM-GM inequality,

$$\begin{aligned}
x^m + \frac{1}{x^n} &= \frac{n}{m+n} \cdot \frac{(m+n)x^m}{n} + \frac{m}{m+n} \cdot \frac{m+n}{mx^n} \\
&\geq \left[\frac{(m+n)x^m}{n}\right]^{n/(m+n)} \cdot \left(\frac{m+n}{mx^n}\right)^{m/(m+n)} \\
&= \left[\frac{(m+n)^n x^{mn}}{n^n} \cdot \frac{(m+n)^m}{m^m x^{mn}}\right]^{1/(m+n)} \\
&= \left[\frac{(m+n)^{m+n}}{m^m n^n}\right]^{1/(m+n)} \\
&= \frac{m+n}{\sqrt[m+n]{m^m n^n}}.
\end{aligned}$$

Equality occurs if and only if

$$\begin{aligned} \frac{(m+n)x^m}{n} &= \frac{m+n}{mx^n} \\ \Leftrightarrow \quad x^{m+n} &= \frac{n}{m} \\ \Leftrightarrow \quad x &= \sqrt[m+n]{\frac{n}{m}}. \end{aligned}$$

Hence, the minimum value is

$$\frac{m+n}{\sqrt[m+n]{m^m n^n}}.$$

4. Prove that among all triangles of a given perimeter, the equilateral triangle has maximum area.

Solution. Let a, b, and c be the sides of the triangle, and let $s = (a+b+c)/2$ be the semi-perimeter. Since the perimeter is fixed, so is s. By Heron's formula, the area of the triangle is given by

$$K = \sqrt{s(s-a)(s-b)(s-c)}.$$

Maximizing K is equivalent to maximizing

$$K^2 = s(s-a)(s-b)(s-c).$$

By the AM-GM inequality,

$$\begin{aligned} (s-a)(s-b)(s-c) &\le \left[\frac{(s-a)+(s-b)+(s-c)}{3}\right]^3 \\ &= \left[\frac{3s-(a+b+c)}{3}\right]^3 \\ &= \left(\frac{3s-2s}{3}\right)^3 \\ &= \frac{s^3}{27}. \end{aligned}$$

Equality occurs if and only if $s-a = s-b = s-c$, or $a = b = c$, i.e. the triangle is equilateral.

5. Prove the QM-AM-GM-HM inequality. (Since we already have AM-GM, it suffices to show QM-AM and GM-HM).

Solution. The QM-AM inequality states that

$$\sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} \ge \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

for all $x_1, x_2, \ldots, x_n \ge 0$, with equality if and only if $x_1 = x_2 = \cdots = x_n$. Squaring both sides, we get

$$\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} \ge \frac{(x_1 + x_2 + \cdots + x_n)^2}{n^2},$$

which is equivalent to

$$n(x_1^2 + x_2^2 + \cdots + x_n^2) \geq (x_1 + x_2 + \cdots + x_n)^2.$$

Clearly,

$$\sum_{1 \leq i < j \leq n} (x_i - x_j)^2 \geq 0,$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$. Expanding, this expression becomes

$$(n-1) \sum_{i=1}^n x_i^2 - 2 \sum_{1 \leq i < j \leq n} x_i x_j \geq 0,$$

so

$$n \sum_{i=1}^n x_i^2 \geq \sum_{i=1}^n x_i^2 + 2 \sum_{1 \leq i < j \leq n} x_i x_j.$$

In other words,

$$n(x_1^2 + x_2^2 + \cdots + x_n^2) \geq (x_1 + x_2 + \cdots + x_n)^2,$$

so the QM-AM inequality holds.

The GM-HM inequality states that

$$\sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

for all x_1, x_2, $\dots$, $x_n > 0$, with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Let $y_i = 1/x_i$ for all i. Then we get

$$\frac{1}{\sqrt[n]{y_1 y_2 \cdots y_n}} \geq \frac{n}{y_1 + y_2 + \cdots + y_n},$$

which is equivalent to

$$\frac{y_1 + y_2 + \cdots + y_n}{n} \geq \sqrt[n]{y_1 y_2 \cdots y_n}.$$

Thus, the GM-HM inequality follows from the AM-GM inequality.

6. For positive real numbers a, b, c, show that

$$\frac{ab}{a+b} + \frac{ac}{a+c} + \frac{bc}{b+c} \leq \frac{a+b+c}{2}.$$

Solution. By the AM-HM inequality,

$$\frac{a+b}{2} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a+b},$$

so

$$\frac{ab}{a+b} \le \frac{a+b}{4}.$$

Similarly,

$$\frac{ac}{a+c} \le \frac{a+c}{4},$$
$$\frac{bc}{b+c} \le \frac{b+c}{4}.$$

Adding all three inequalities, we get

$$\frac{ab}{a+b} + \frac{ac}{a+c} + \frac{bc}{b+c} \le \frac{a+b+c}{2}.$$

7. Let a_1, a_2, …, a_n be positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \ge n^2.$$

Solution. By the AM-HM inequality,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \ge \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}},$$

so

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \ge \frac{n^2}{a_1 + a_2 + \cdots + a_n} = n^2.$$

8. Prove that

$$(ab + ac + bc)(a + b + c)^4 \le 27(a^3 + b^3 + c^3)^2$$

for a, b, $c \ge 0$.

Solution. By the Power Mean inequality,

$$\sqrt[3]{\frac{a^3 + b^3 + c^3}{3}} \ge \frac{a+b+c}{3},$$

so $9(a^3 + b^3 + c^3) \ge (a + b + c)^3$, and

$$27(a^3 + b^3 + c^3)^2 \ge \frac{(a+b+c)^6}{3}.$$

As shown in the handout, $a^2 + b^2 + c^2 \ge ab + ac + bc$. Adding $2(ab + ac + bc)$ to both sides, we get

$$(a + b + c)^2 \ge 3(ab + ac + bc).$$

Hence,

$$\frac{(a+b+c)^6}{3} = \frac{(a+b+c)^2 \cdot (a+b+c)^4}{3} \ge (ab + ac + bc)(a + b + c)^4.$$

9. Prove the Triangle inequality.

Solution. We want to prove that for any real numbers $x_1, x_2, \dots, x_n$,

$$|x_1| + |x_2| + \cdots + |x_n| \ge |x_1 + x_2 + \cdots + x_n|,$$

and that equality occurs if and only if $x_i \ge 0$ for all i, or $x_i \le 0$ for all i.

Squaring both sides, since $|x|^2 = x^2$ for all x, we get

$$\sum_{i=1}^n x_i^2 + 2 \sum_{1 \le i < j \le n} |x_i||x_j| \ge \sum_{i=1}^n x_i^2 + 2 \sum_{1 \le i < j \le n} x_i x_j,$$

which is equivalent to

$$\sum_{1 \le i < j \le n} |x_i x_j| \ge \sum_{1 \le i < j \le n} x_i x_j.$$

This inequality follows from the fact that $x \le |x|$ for all x. Furthermore, equality occurs if and only if $x_i x_j \ge 0$ for all $1 \le i < j \le n$. In other words, equality occurs if and only if $x_i \ge 0$ for all i, or $x_i \le 0$ for all i.

10. Show that for all real numbers x and y, $|x - y| \ge |x| - |y|$.

Solution. By the Triangle inequality, $|x - y| + |y| \ge |(x - y) + y| = |x|$, so $|x - y| \ge |x| - |y|$.

11. What is the minimum value of $f(x) = |x - 1| + |2x - 1| + |3x - 1| + \cdots + |119x - 1|$? (2010 AMC 12A)

Solution. More generally, let $x_1 \le x_2 \le \cdots \le x_n$. We claim that the minimum value of

$$f(x) = |x - x_1| + |x - x_2| + \cdots + |x - x_n|$$

occurs at $x = x_{\lceil n/2 \rceil}$.

First, we take the case where n is even. Let $n = 2k$, where k is a positive integer. Then by the Triangle inequality,

$$\begin{aligned}
f(x) &= |x - x_1| + |x - x_2| + \cdots + |x - x_{2k}| \\
&= (|x_{2k} - x| + |x - x_1|) + (|x_{2k-1} - x| + |x - x_2|) + \cdots + (|x_{k+1} - x| + |x - x_k|) \\
&\ge |x_{2k} - x_1| + |x_{2k-1} - x_2| + \cdots + |x_{k+1} - x_k| \\
&= (x_{2k} - x_1) + (x_{2k-1} - x_2) + \cdots + (x_{k+1} - x_k).
\end{aligned}$$

for all x. Also,

$$\begin{aligned}
f(x_{\lceil n/2 \rceil}) &= f(x_k) \\
&= |x_k - x_1| + |x_k - x_2| + \cdots + |x_k - x_{2k}| \\
&= (x_k - x_1) + (x_k - x_2) + \cdots + (x_k - x_{k-1}) + (x_{k+1} - x_k) + \cdots + (x_{2k} - x_k) \\
&= x_{2k} + x_{2k-1} + \cdots + x_{k+1} - x_k - x_{k-1} - \cdots - x_1.
\end{aligned}$$

Hence, $f(x) \geq f(x_{\lceil n/2 \rceil})$ for all x.

Next, we take the case where n is odd. Let $n = 2k - 1$, where k is a positive integer. Then by the Triangle inequality,

$$\begin{aligned} f(x) &= |x - x_1| + |x - x_2| + \cdots + |x - x_{2k-1}| \\ &= (|x_{2k-1} - x| + |x - x_1|) + (|x_{2k-2} - x| + |x - x_2|) + \cdots + (|x_{k+1} - x| + |x - x_{k-1}|) + |x - x_k| \\ &\geq |x_{2k-1} - x_1| + |x_{2k-2} - x_2| + \cdots + |x_{k+1} - x_{k-1}| + 0 \\ &= (x_{2k-1} - x_1) + (x_{2k-2} - x_2) + \cdots + (x_{k+1} - x_{k-1}). \end{aligned}$$

for all x. Also,

$$\begin{aligned} f(x_{\lceil n/2 \rceil}) &= f(x_k) \\ &= |x_k - x_1| + |x_k - x_2| + \cdots + |x_k - x_{2k-1}| \\ &= (x_k - x_1) + (x_k - x_2) + \cdots + (x_k - x_{k-1}) + (x_{k+1} - x_k) + \cdots + (x_{2k-1} - x_k) \\ &= x_{2k-1} + x_{2k-2} + \cdots + x_{k+1} - x_{k-1} - x_{k-2} - \cdots - x_1. \end{aligned}$$

Hence, $f(x) \geq f(x_{\lceil n/2 \rceil})$ for all x.

In either case, we see that the minimum value of $f(x)$ occurs at $x = x_{\lceil n/2 \rceil}$.

Now, we can re-write the function given in the problem as

$$\begin{aligned} f(x) &= |x - 1| + |2x - 1| + |3x - 1| + \cdots + |119x - 1| \\ &= |x - 1| + 2\left|x - \frac{1}{2}\right| + 3\left|x - \frac{1}{3}\right| + \cdots + 119\left|x - \frac{1}{119}\right|, \end{aligned}$$

Hence, we can take 119 of the first x_i to be $1/119$, the next 118 to be $1/118$, and so on, for a total of $1+2+\cdots+119 = 119 \cdot 120/2 = 7140$ x_i. Hence, the minimum value of $f(x)$ occurs at $x = x_{7140/2} = x_{3570}$.

Since $119 + 118 + \cdots + 85 = 84 + 83 + \cdots + 1 = 3570$, we have that $x_{3570} = 1/85$. Hence, the minimum value of $f(x)$ is

$$\begin{aligned} f\left(\frac{1}{85}\right) &= \left|\frac{1}{85} - 1\right| + \left|\frac{2}{85} - 1\right| + \cdots + \left|\frac{119}{85} - 1\right| \\ &= \left(1 - \frac{1}{85}\right) + \left(1 - \frac{2}{85}\right) + \cdots + \left(1 - \frac{84}{85}\right) + \left(\frac{86}{85} - 1\right) + \left(\frac{87}{85} - 1\right) + \cdots + \left(\frac{119}{85} - 1\right) \\ &= \frac{84}{85} + \frac{83}{85} + \cdots + \frac{1}{85} + \frac{1}{85} + \frac{2}{85} + \cdots + \frac{34}{85} \\ &= \frac{84 \cdot 85/2 + 34 \cdot 35/2}{85} \\ &= \frac{4165}{85} \\ &= 49. \end{aligned}$$

12. For a positive integer n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2},$$

where $a_1, a_2, \dots, a_n$ are positive real numbers whose sum is 17. There is a unique positive integer n for which S_n is also an integer. Find this n. (1991 AIME)

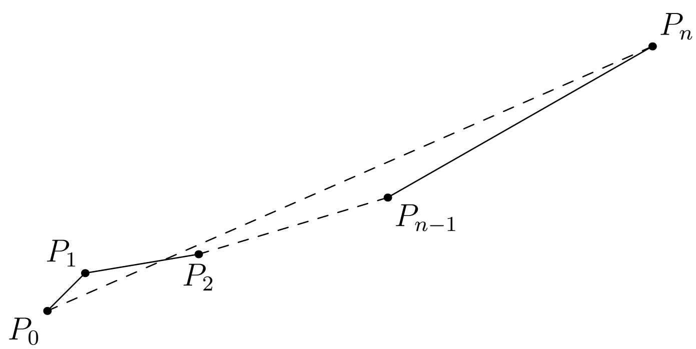
Solution. In the coordinate plane, let $P_0 = (0,0)$, and let

$$P_k = (k^2, a_1 + a_2 + \cdots + a_k)$$

for $1 \le k \le n$, so

$$P_{k-1}P_k = \sqrt{[k^2 - (k-1)^2]^2 + a_k^2} = \sqrt{(2k-1)^2 + a_k^2}$$

for all $1 \le k \le n$.



Then by the Triangle inequality,

$$\begin{aligned} \sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2} &= \sum_{k=1}^{n} P_{k-1}P_k \\ &\ge P_0P_n \\ &= \sqrt{(n^2)^2 + (a_1 + a_2 + \cdots + a_n)^2} \\ &= \sqrt{n^4 + 289}. \end{aligned}$$

Equality occurs when $P_0, P_1, \dots, P_n$ are collinear (which is achievable), so the minimum value of S_n is $\sqrt{n^4 + 289}$. Then $S_n^2 = n^4 + 289$, so

$$S_n^2 - n^4 = (S_n^2 + n^2)(S_n^2 - n^2) = 289.$$

If S_n is an integer, then we must have $S_n^2 + n^2 = 289$ and $S_n - n^2 = 1$. Subtracting these equations, we get $2n^2 = 288$, so $n^2 = 144$, and $n = 12$.

13. Prove the Cauchy-Schwarz inequality using one of the following methods:

(1) Let

$$f(t) = \sum_{i=1}^{n} (x_i t - y_i)^2.$$

We see that $f(t)$ is a quadratic in t. What is the discriminant of $f(t)$?

(2) Let $\vec{v} = (x_1, x_2, \dots, x_n)$ and $\vec{w} = (y_1, y_2, \dots, y_n)$, and let θ be the angle between $\vec{v}$ and $\vec{w}$. Then

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}.$$

What is the range of $\cos\theta$?

(3) Prove Lagrange's identity:

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2 = \sum_{1 \le i < j \le n} (x_i y_j - x_j y_i)^2.$$

Solution. (1) If $x_i = 0$ for all i, then the Cauchy-Schwarz inequality holds trivially, so assume that $x_i \neq 0$ for some i. Expanding, we get

$$f(t) = \sum_{i=1}^{n} (x_i t - y_i)^2 = \left(\sum_{i=1}^{n} x_i^2\right) t^2 - 2\left(\sum_{i=1}^{n} x_i y_i\right) t + \sum_{i=1}^{n} y_i^2.$$

We see that $f(t) \ge 0$ for all t. Then as a quadratic in t, the discriminant of $f(t)$ must be nonpositive, i.e.

$$4\left(\sum_{i=1}^{n} x_i y_i\right)^2 - 4\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \le 0,$$

or

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \ge \left(\sum_{i=1}^{n} x_i y_i\right)^2.$$

Equality occurs if and only if $f(t) = 0$ for some t. Then we must have $x_i t = y_i$ for all i.

(2) Since $|\cos\theta| \le 1$, $|\vec{v} \cdot \vec{w}| \le |\vec{v}||\vec{w}|$. In other words,

$$\left|\sum_{i=1}^{n} x_i y_i\right| \le \sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} y_i^2},$$

or

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \ge \left(\sum_{i=1}^{n} x_i y_i\right)^2.$$

Equality occurs if and only if the vectors $\vec{v} = (x_1, x_2, \dots, x_n)$ and $\vec{w} = (y_1, y_2, \dots, y_n)$ are collinear, i.e. there exist constants λ and μ such that $\lambda\vec{v} = \mu\vec{w}$, which means $\lambda x_i = \mu y_i$ for all i.

(3) Expanding

$$\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right) - \left(\sum_{i=1}^n x_i y_i\right)^2,$$

we see that all terms of the form $x_i^2 y_i^2$ cancel, and we are left with

$$\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right) - \left(\sum_{i=1}^n x_i y_i\right)^2 = \sum_{\substack{1\le i,j\le n\\ i\ne j}} x_i^2 y_j^2 - 2\sum_{1\le i<j\le n} x_i x_j y_i y_j.$$

Expanding

$$\sum_{1\le i<j\le n} (x_i y_j - x_j y_i)^2,$$

we get

$$\begin{aligned}\sum_{1\le i<j\le n} (x_i y_j - x_j y_i)^2 &= \sum_{1\le i<j\le n} (x_i^2 y_j^2 - 2x_i x_j y_i y_j + x_j^2 y_i^2) \\ &= \sum_{\substack{1\le i,j\le n\\ i\ne j}} x_i^2 y_j^2 - 2\sum_{1\le i<j\le n} x_i x_j y_i y_j,\end{aligned}$$

so the two expressions are equal.

Equality occurs if and only if $x_i y_j = x_j y_i$ for all $1 \le i < j \le n$. Taking $\lambda = y_1$ and $\mu = x_1$, we see that $\lambda x_i = \mu y_i$ for all $i \ne 1$, but this also holds for $i = 1$.

14. Let a_1, a_2, $\dots$, a_n be real numbers, and let b_1, b_2, $\dots$, b_n be positive real numbers. Show that

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \ge \frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n}.$$

(This inequality has recently come to be known as the "Engel form" of the Cauchy-Schwarz inequality.)

Solution. Taking $x_i = \sqrt{b_i}$ and $y_i = a_i/\sqrt{b_i}$ in the Cauchy-Schwarz inequality, we get

$$(b_1 + b_2 + \dots + b_n)\left(\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n}\right) \ge (a_1 + a_2 + \dots + a_n)^2,$$

so

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \ge \frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n}.$$

15. Let a and b be positive real numbers with $a+b=1$. Prove that

$$\frac{a^2}{a+1}+\frac{b^2}{b+1}\geq\frac{1}{3}.$$

(Hungary, 1996)

Solution. By the Cauchy-Schwarz inequality,

$$[(a+1)+(b+1)]\left(\frac{a^2}{a+1}+\frac{b^2}{b+1}\right)\geq(a+b)^2,$$

so

$$\frac{a^2}{a+1}+\frac{b^2}{b+1}\geq\frac{(a+b)^2}{(a+1)+(b+1)}=\frac{1}{3}.$$

16. Let x_1, x_2, ..., $x_n>0$, and $s=x_1+x_2+\cdots+x_n$. Prove that

$$\frac{s}{s-x_1}+\frac{s}{s-x_2}+\cdots+\frac{s}{s-x_n}\geq\frac{n^2}{n-1},$$

with equality if and only if $x_1=x_2=\cdots=x_n$.

Solution. By the Cauchy-Schwarz inequality,

$$[s(s-x_1)+s(s-x_2)+\cdots+s(s-x_n)]\left(\frac{s}{s-x_1}+\frac{s}{s-x_2}+\cdots+\frac{s}{s-x_n}\right)\geq(ns)^2,$$

so

$$\begin{aligned}\frac{s}{s-x_1}+\frac{s}{s-x_2}+\cdots+\frac{s}{s-x_n}&\geq\frac{(ns)^2}{s(s-x_1)+s(s-x_2)+\cdots+s(s-x_n)}\\&=\frac{n^2s}{(s-x_1)+(s-x_2)+\cdots+(s-x_n)}\\&=\frac{n^2s}{ns-(x_1+x_2+\cdots+x_n)}\\&=\frac{n^2s}{ns-s}\\&=\frac{n^2}{n-1}.\end{aligned}$$

Equality occurs if and only if $(s-x_1)^2=(s-x_2)^2=\cdots=(s-x_n)^2$, or $x_1=x_2=\cdots=x_n$.

17. Suppose that $|x_i|<1$ for $i=1$, 2, ..., n. Suppose further that

$$|x_1|+|x_2|+\cdots+|x_n|=19+|x_1+x_2+\cdots+x_n|.$$

What is the smallest possible value of n? (1988 AIME)

Solution. Since $|x_i| < 1$ for all i,

$$|x_1| + |x_2| + \cdots + |x_n| < n.$$

But

$$|x_1| + |x_2| + \cdots + |x_n| = 19 + |x_1 + x_2 + \cdots + x_n| \geq 19,$$

so $n \geq 20$.

For $n = 20$, we can take $x_1 = x_2 = \cdots = x_{10} = 19/20$ and $x_{11} = x_{12} = \cdots = x_{20} = -19/20$. Then

$$|x_1| + |x_2| + \cdots + |x_{20}| = 20 \cdot \frac{19}{20} = 19,$$

and

$$19 + |x_1 + x_2 + \cdots + x_{20}| = 19.$$

Hence, the smallest possible value of n is 20.

18. For $a > b > 0$, find the minimum value of

$$a + \frac{1}{(a-b)b}.$$

Solution. By the AM-GM inequality,

$$a + \frac{1}{(a-b)b} = (a-b) + b + \frac{1}{(a-b)b} \geq 3\sqrt[3]{(a-b) \cdot b \cdot \frac{1}{(a-b)b}} = 3.$$

Taking $a = 2$ and $b = 1$, we get

$$a + \frac{1}{(a-b)b} = 2 + \frac{1}{1 \cdot 1} = 3,$$

so the minimum value is 3.

19. Show that if a, b, c are the lengths of the sides of a triangle, then

$$3(ab + ac + bc) \leq (a + b + c)^2 < 4(ab + ac + bc).$$

Solution. As shown in the handout, $a^2 + b^2 + c^2 \geq ab + ac + bc$. Adding $2(ab + ac + bc)$ to both sides, we get

$$(a + b + c)^2 \geq 3(ab + ac + bc).$$

By the Triangle inequality,

$$a + b > c,$$
$$a + c > b,$$
$$b + c > a.$$

These inequalities are awkward to work with, so to make them easier to work with, let

$$\begin{aligned} x &= \frac{b+c-a}{2}, \\ y &= \frac{a+c-b}{2}, \\ z &= \frac{a+b-c}{2}. \end{aligned}$$

From the inequalities above, x, y, and z are all positive. Furthermore,

$$\begin{aligned} y+z &= \frac{a+c-b}{2} + \frac{a+b-c}{2} = a, \\ x+z &= \frac{b+c-a}{2} + \frac{a+b-c}{2} = b, \\ x+y &= \frac{b+c-a}{2} + \frac{a+c-b}{2} = c. \end{aligned}$$

Thus, we can express a, b, and c in terms of x, y, and z. Geometrically, x, y, and z are the lengths of the tangents from the vertices to the incircle. (This technique is known as the *Ravi Substitution*, named after Canadian IMO medallist Ravi Vakil.)

Thus, the inequality

$$(a+b+c)^2 < 4(ab+ac+bc)$$

becomes

$$(2x+2y+2z)^2 < 4[(x+y)(x+z) + (x+y)(y+z) + (x+z)(y+z)].$$

This simplifies to

$$4xy + 4xz + 4yz > 0,$$

which is clear.

20. Let a_1, a_2, $\dots$, a_n be nonnegative real numbers. Let a and g be the arithmetic and geometric mean of the a_i, respectively. Prove that for all $x \ge 0$,

$$(x+g)^n \le (x+a_1)(x+a_2)\cdots(x+a_n) \le (x+a)^n.$$

Solution 1. By the AM-GM inequality,

$$\begin{aligned} (x+a_1)(x+a_2)\cdots(x+a_n) &\le \left[\frac{(x+a_1)+(x+a_2)+\cdots+(x+a_n)}{n}\right]^n \\ &= \left[\frac{nx+(a_1+a_2+\cdots+a_n)}{n}\right]^n \\ &= \left(\frac{nx+na}{n}\right)^n \\ &= (x+a)^n. \end{aligned}$$

For $1 \le k \le n$, let s_k be the sum of the products of x_1, x_2, ..., x_n, taken k at a time. Then

$$(x+a_1)(x+a_2)\cdots(x+a_n) = x^n + s_1 x^{n-1} + s_2 x^{n-2} + \cdots + s_n.$$

The sum s_k contains $\binom{n}{k}$ distinct terms. Each variable x_i appears in $\binom{n-1}{k-1}$ of these terms, so by the AM-GM inequality,

$$\frac{s_k}{\binom{n}{k}} \ge (x_1 x_2 \cdots x_n)^{\binom{n-1}{k-1}/\binom{n}{k}} = (x_1 x_2 \cdots x_n)^{k/n} = g^k,$$

and

$$s_k \ge \binom{n}{k} g^k.$$

Then

$$\begin{aligned}(x+a_1)(x+a_2)\cdots(x+a_n) &= x^n + s_1 x^{n-1} + s_2 x^{n-2} + \cdots + s_n \\ &\ge x^n + \binom{n}{1} x^{n-1} g + \binom{n}{2} x^{n-2} g^2 + \cdots + g^n \\ &= (x+g)^n.\end{aligned}$$

Solution 2. We give another proof of the the left inequality. By the AM-GM inequality,

$$\begin{aligned}\frac{x}{x+a_1} + \frac{x}{x+a_2} + \cdots + \frac{x}{x+a_n} &\ge n\left[\frac{x^n}{(x+a_1)(x+a_2)\cdots(x+a_n)}\right]^{1/n} \\ &= \frac{nx}{\sqrt[n]{(x+a_1)(x+a_2)\cdots(x+a_n)}},\end{aligned}$$

and

$$\begin{aligned}\frac{a_1}{x+a_1} + \frac{a_2}{x+a_2} + \cdots + \frac{a_n}{x+a_n} &\ge n\left[\frac{a_1 a_2 \cdots a_n}{(x+a_1)(x+a_2)\cdots(x+a_n)}\right]^{1/n} \\ &= n\left[\frac{g^n}{(x+a_1)(x+a_2)\cdots(x+a_n)}\right]^{1/n} \\ &= \frac{ng}{\sqrt[n]{(x+a_1)(x+a_2)\cdots(x+a_n)}}.\end{aligned}$$

Adding these inequalities, we get

$$n \ge \frac{n(x+g)}{\sqrt[n]{(x+a_1)(x+a_2)\cdots(x+a_n)}},$$

which implies

$$(x+g)^n \le (x+a_1)(x+a_2)\cdots(x+a_n).$$

21. (Nesbitt's Inequality) Show that for a, b, $c > 0$,

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Furthermore, show that if a, b, and c are the sides of a triangle, then

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} < 2.$$

Solution. By the Cauchy-Schwarz inequality,

$$[a(b+c) + b(a+c) + c(a+b)] \left(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \right) \geq (a+b+c)^2,$$

so

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{(a+b+c)^2}{2(ab+ac+bc)}.$$

As shown in the handout, $a^2 + b^2 + c^2 \geq ab + ac + bc$. Adding $2(ab + ac + bc)$ to both sides, we get

$$(a+b+c)^2 \geq 3(ab+ac+bc).$$

Hence,

$$\frac{(a+b+c)^2}{2(ab+ac+bc)} \geq \frac{3}{2}.$$

If a, b, and c are the sides of a triangle, then we can use the Ravi Substitution. Thus, the given inequality becomes

$$\frac{x+y}{x+y+2z} + \frac{x+z}{x+2y+z} + \frac{y+z}{2x+y+z} < 2.$$

This inequality is true, because

$$\begin{aligned} \frac{x+y}{x+y+2z} + \frac{x+z}{x+2y+z} + \frac{y+z}{2x+y+z} &< \frac{x+y}{x+y+z} + \frac{x+z}{x+y+z} + \frac{y+z}{x+y+z} \\ &= \frac{2(x+y+z)}{x+y+z} \\ &= 2. \end{aligned}$$

22. Let a, b, c, d, e be positive real numbers such that $abcde = 1$. Prove that

$$a^4 + b^4 + c^4 + d^4 + e^4 \geq a + b + c + d + e.$$

Solution 1. By the Power Mean inequality,

$$\sqrt[4]{\frac{a^4 + b^4 + c^4 + d^4 + e^4}{5}} \geq \frac{a+b+c+d+e}{5},$$

so

$$a^4 + b^4 + c^4 + d^4 + e^4 \geq \frac{(a+b+c+d+e)^4}{125}.$$

By the AM-GM inequality,

$$a + b + c + d + e \geq 5\sqrt[5]{abcde} = 5,$$

so

$$\frac{(a+b+c+d+e)^4}{125} = \frac{(a+b+c+d+e)^3}{125} \cdot (a+b+c+d+e) \geq a+b+c+d+e.$$

Solution 2. We can try using the weighted AM-GM inequality. We have an expression with degree 4 on the left-hand side, so we want an equivalent expression with degree 4 on the right-hand side. The current degree of the right-hand side is 1, so we must add 3 to the degree of every term. We are given that $abcde = 1$, and the degree of $abcde$ is 5, so to obtain an expression with degree 3, we raise both sides to the power of 3/5. This gives us $a^{3/5}b^{3/5}c^{3/5}d^{3/5}e^{3/5} = 1$, so multiplying the right-hand side by $a^{3/5}b^{3/5}c^{3/5}d^{3/5}e^{3/5}$ adds 3 to the degree of every term, without changing its value. Hence,

$$\begin{aligned} a+b+c+d+e &= (a+b+c+d+e)(a^{3/5}b^{3/5}c^{3/5}d^{3/5}e^{3/5}) \\ &= a^{8/5}b^{3/5}c^{3/5}d^{3/5}e^{3/5} + a^{3/5}b^{8/5}c^{3/5}d^{3/5}e^{3/5} + a^{3/5}b^{3/5}c^{8/5}d^{3/5}e^{3/5} \\ &\quad + a^{3/5}b^{3/5}c^{3/5}d^{8/5}e^{3/5} + a^{3/5}b^{3/5}c^{3/5}d^{3/5}e^{8/5}, \end{aligned}$$

giving us an equivalent expression with degree 4. (This technique of setting every term in the inequality to the same degree is known as *homogenizing*.) We now want to prove that

$$\begin{aligned} a^4 + b^4 + c^4 + d^4 + e^4 \geq\ & a^{8/5}b^{3/5}c^{3/5}d^{3/5}e^{3/5} + a^{3/5}b^{8/5}c^{3/5}d^{3/5}e^{3/5} + a^{3/5}b^{3/5}c^{8/5}d^{3/5}e^{3/5} \\ & + a^{3/5}b^{3/5}c^{3/5}d^{8/5}e^{3/5} + a^{3/5}b^{3/5}c^{3/5}d^{3/5}e^{8/5}. \end{aligned}$$

Let's look at the first term in the right-hand side, namely $a^{8/5}b^{3/5}c^{3/5}d^{3/5}e^{3/5}$. By the weighted AM-GM inequality,

$$w_1a^4 + w_2b^4 + w_3c^4 + w_4d^4 + w_5e^4 \geq a^{4w_1}b^{4w_2}c^{4w_3}d^{4w_4}e^{4w_5}.$$

for any weights w_1, w_2, w_3, w_4, and w_5. Since we want the right-hand side to be $a^{8/5}b^{3/5}c^{3/5}d^{3/5}e^{3/5}$, there is only one choice for the set of weights, namely $w_1 = 2/5$ and $w_2 = w_3 = w_4 = w_5 = 3/20$. We also need these weights to sum to 1, and they do: $w_1 + w_2 + w_3 + w_4 + w_5 = 2/5 + 4 \cdot 3/20 = 1$. This is not luck – this is precisely because we have homogenized both sides to have the same degree. Thus, we have that

$$\frac{2}{5}a^4 + \frac{3}{20}b^4 + \frac{3}{20}c^4 + \frac{3}{20}d^4 + \frac{3}{20}e^4 \geq a^{8/5}b^{3/5}c^{3/5}d^{3/5}e^{3/5}.$$

Similarly,

$$\begin{aligned}
\frac{3}{20}a^4 + \frac{2}{5}b^4 + \frac{3}{20}c^4 + \frac{3}{20}d^4 + \frac{3}{20}e^4 &\ge a^{3/5}b^{8/5}c^{3/5}d^{3/5}e^{3/5}, \\
\frac{3}{20}a^4 + \frac{3}{20}b^4 + \frac{2}{5}c^4 + \frac{3}{20}d^4 + \frac{3}{20}e^4 &\ge a^{3/5}b^{3/5}c^{8/5}d^{3/5}e^{3/5}, \\
\frac{3}{20}a^4 + \frac{3}{20}b^4 + \frac{3}{20}c^4 + \frac{2}{5}d^4 + \frac{3}{20}e^4 &\ge a^{3/5}b^{3/5}c^{3/5}d^{8/5}e^{3/5}, \\
\frac{3}{20}a^4 + \frac{3}{20}b^4 + \frac{3}{20}c^4 + \frac{3}{20}d^4 + \frac{2}{5}e^4 &\ge a^{3/5}b^{3/5}c^{3/5}d^{3/5}e^{8/5}.
\end{aligned}$$

We can then add up all these inequalities.

Having gone through all the calculations, we can present our solution succinctly as follows: By the weighted AM-GM inequality,

$$\begin{aligned}
\frac{2}{5}a^4 + \frac{3}{20}b^4 + \frac{3}{20}c^4 + \frac{3}{20}d^4 + \frac{3}{20}e^4 &\ge a^{8/5}b^{12/20}c^{12/20}d^{12/20}e^{12/20} \\
&= a^{8/5}b^{3/5}c^{3/5}d^{3/5}e^{3/5} \\
&= a(a^{3/5}b^{3/5}c^{3/5}d^{3/5}e^{3/5}) \\
&= a(abcde)^{3/5} \\
&= a.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{3}{20}a^4 + \frac{2}{5}b^4 + \frac{3}{20}c^4 + \frac{3}{20}d^4 + \frac{3}{20}e^4 &\ge b, \\
\frac{3}{20}a^4 + \frac{3}{20}b^4 + \frac{2}{5}c^4 + \frac{3}{20}d^4 + \frac{3}{20}e^4 &\ge c, \\
\frac{3}{20}a^4 + \frac{3}{20}b^4 + \frac{3}{20}c^4 + \frac{2}{5}d^4 + \frac{3}{20}e^4 &\ge d, \\
\frac{3}{20}a^4 + \frac{3}{20}b^4 + \frac{3}{20}c^4 + \frac{3}{20}d^4 + \frac{2}{5}e^4 &\ge e.
\end{aligned}$$

Adding all these inequalities, we get

$$a^4 + b^4 + c^4 + d^4 + e^4 \ge a + b + c + d + e.$$

23. Let a_1, a_2, $\dots$, a_n be fixed, positive real numbers, and let x_1, x_2, $\dots$, x_n be nonnegative real numbers such that $x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n$. Prove that the maximum value of

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

occurs when $x_i = a_i$ for all i.

Solution. Let $s = a_1 + a_2 + \cdots + a_n$. Then by the weighted AM-GM inequality,

$$\begin{aligned}\left(\frac{x_1}{a_1}\right)^{a_1/s}\left(\frac{x_2}{a_2}\right)^{a_2/s}\cdots\left(\frac{x_n}{a_n}\right)^{a_n/s} &\le \frac{a_1}{s}\cdot\frac{x_1}{a_1}+\frac{a_2}{s}\cdot\frac{x_2}{a_2}+\cdots+\frac{a_n}{s}\cdot\frac{x_n}{a_n}\\ &= \frac{x_1+x_2+\cdots+x_n}{s}\\ &= 1.\end{aligned}$$

Then

$$\left(\frac{x_1}{a_1}\right)^{a_1}\left(\frac{x_2}{a_2}\right)^{a_2}\cdots\left(\frac{x_n}{a_n}\right)^{a_n} \le 1,$$

which implies that

$$x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \le a_1^{a_1}a_2^{a_2}\cdots a_n^{a_n}.$$

Equality occurs if and only if $x_1/a_1 = x_2/a_2 = \cdots = x_n/a_n = 1$, or $x_i = a_i$ for all i.

24. Let a, b, c be the sides of a triangle, and T its area. Prove: $a^2 + b^2 + c^2 \ge 4\sqrt{3}T$. In what case does equality hold? (IMO, 1961)

Solution 1. Using the Ravi Substitution, the inequality becomes

$$(x+y)^2 + (x+z)^2 + (y+z)^2 \ge 4\sqrt{3}\cdot\sqrt{xyz(x+y+z)},$$

which simplifies to

$$x^2+y^2+z^2+xy+xz+yz \ge 2\sqrt{3xyz(x+y+z)}.$$

We claim that

$$xy+xz+yz \ge \sqrt{3xyz(x+y+z)}.$$

Squaring both sides, we get

$$x^2y^2+x^2z^2+y^2z^2+2(x^2yz+xy^2z+xyz^2) \ge 3(x^2yz+xy^2z+xyz^2),$$

which simplifies to

$$x^2y^2+x^2z^2+y^2z^2 \ge x^2yz+xy^2z+xyz^2.$$

As shown in the handout, $a^2+b^2+c^2 \ge ab+ac+bc$ for all a, b, $c \ge 0$. Taking $a = xy$, $b = xz$, and $c = yz$ gives us this inequality.

By the same token,

$$x^2+y^2+z^2 \ge xy+xz+yz \ge \sqrt{3xyz(x+y+z)}.$$

Hence,

$$x^2+y^2+z^2+xy+xz+yz \ge 2\sqrt{3xyz(x+y+z)}.$$

Equality occurs if and only if $x = y = z$ or $a = b = c$, i.e. the triangle is equilateral.

Solution 2. By Problem 4, among all triangles with perimeter $a+b+c$, the equilateral triangle has maximum area. This equilateral triangle has area

$$\frac{\sqrt{3}}{4} \cdot \left(\frac{a+b+c}{3} \right)^2 = \frac{\sqrt{3}(a+b+c)^2}{36},$$

so

$$T \le \frac{\sqrt{3}(a+b+c)^2}{36},$$

which means

$$4\sqrt{3}T \le \frac{(a+b+c)^2}{3}.$$

Hence, it suffices to prove that

$$a^2+b^2+c^2 \ge \frac{(a+b+c)^2}{3}.$$

By the QM-AM inequality,

$$\sqrt{\frac{a^2+b^2+c^2}{3}} \ge \frac{a+b+c}{3},$$

which implies

$$a^2+b^2+c^2 \ge \frac{(a+b+c)^2}{3}.$$

Equality occurs if and only if $a=b=c$, i.e. the triangle is equilateral.

25. Show that if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then

(a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$, and

(b) $(n-1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.

(Putnam, 1975)

Solution. (a) We have that

$$\begin{aligned} s_n + n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + n \\ &= (1+1) + \left(\frac{1}{2} + 1 \right) + \left(\frac{1}{3} + 1 \right) + \cdots + \left(\frac{1}{n} + 1 \right) \\ &= 2 + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n}. \end{aligned}$$

By the AM-GM inequality,

$$2 + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} > n \left(2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n} \right)^{1/n} = n(n+1)^{1/n}.$$

(b) We have that

$$\begin{aligned} n - s_n &= n - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \\ &= (1-1) + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{3}\right) + \cdots + \left(1 - \frac{1}{n}\right) \\ &= \frac{1}{2} + \frac{2}{3} + \cdots + \frac{n-1}{n}. \end{aligned}$$

By the AM-GM inequality,

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n-1}{n} > (n-1)\left(\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n-1}{n}\right)^{1/(n-1)} = (n-1)n^{-1/(n-1)}.$$

26. Show that if x and y are nonnegative real numbers such that

$$x + y + \sqrt{2x^2 + 2xy + 3y^2} = 4,$$

then $x^2y < 4$.

Solution. We claim that $x^2y < 1$. From the given equation

$$\sqrt{2x^2 + 2xy + 3y^2} = 4 - x - y.$$

Squaring both sides and completing the square, we get

$$(x+4)^2 + 2(y+2)^2 = 40.$$

Hence,

$$\frac{2}{3}\left[\sqrt{\frac{3}{2}}(x+4)\right]^2 + \frac{1}{3}\left[\sqrt{6}(y+2)\right]^2 = 40.$$

By the weighted QM-AM inequality,

$$\sqrt{\frac{2}{3}\left[\sqrt{\frac{3}{2}}(x+4)\right]^2 + \frac{1}{3}\left[\sqrt{6}(y+2)\right]^2} \geq \frac{2}{3} \cdot \sqrt{\frac{3}{2}}(x+4) + \frac{1}{3} \cdot \sqrt{6}(y+2),$$

so

$$\frac{2}{3} \cdot \sqrt{\frac{3}{2}}(x+4) + \frac{1}{3} \cdot \sqrt{6}(y+2) \leq \sqrt{40},$$

which simplifies to

$$x + y \leq \sqrt{60} - 6.$$

Then by the AM-GM inequality,

$$x + y = \frac{x}{2} + \frac{x}{2} + y \geq 3\sqrt[3]{\frac{x}{2} \cdot \frac{x}{2} \cdot y} = 3\sqrt[3]{\frac{x^2y}{4}},$$

so

$$3\sqrt[3]{\frac{x^2y}{4}} \le \sqrt{60} - 6.$$

Hence,

$$x^2y \le 4\left(\frac{\sqrt{60}-6}{3}\right)^3.$$

Then

$$\begin{aligned}
4\left(\frac{\sqrt{60}-6}{3}\right)^3 &= 4\left[\frac{(\sqrt{60}-6)(\sqrt{60}+6)}{3(\sqrt{60}+6)}\right]^3 \\
&= 4\left[\frac{24}{3(\sqrt{60}+6)}\right]^3 \\
&= 4\left(\frac{8}{\sqrt{60}+6}\right)^3 \\
&< 4\left(\frac{8}{7+6}\right)^3 \\
&= \frac{2048}{2197} \\
&< 1.
\end{aligned}$$

Therefore, $x^2y < 1$.