Problems on recurrences

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1. Show that each of the following sequences of objects is counted by the Fibonacci numbers:

   (a) tilings of a $2 \times n$ rectangle by 1×2 dominos;

   (b) ordered sequences of 1's and 2's which sum to n (e.g., for $n = 4$, there are 5 such sequences: $4 = 1+1+1+1 = 2+1+1 = 1+2+1 = 1+1+2 = 2+2$);

   (c) ways to select some of n objects in a row so that no two adjacent objects are selected.

   For "extra credit", find bijections between corresponding sets with equal numbers of elements.

2. If you draw lines through the entries of Pascal's triangle with the right slope, the sum of the numbers they pass through will be Fibonacci numbers. Find this slope, and use recurrence relations to prove this result. Extra credit: Relate this fact to the previous problem.

3. Show that each of the following sequences of objects is counted by the Catalan numbers:

   (a) properly matching strings of n left parentheses and n right parentheses (e.g., for $n = 3$, there are 5 such strings: ((())), (())(), ()(()), (()()), ()()());

   (b) divisions of a regular n-gon into $n - 2$ triangles;

   (c) ways to connect $2n$ points around the circumference of a circle with n disjoint line segments.

   Again, there are combinatorial relationships between these sets as well.

4. (Poland '96)

   (a) Find the number of permutations f of $\{1, 2, \ldots, n\}$ that satisfy the conditions
   $$f(i) \geq i - 1, \quad i = 1, 2, \ldots, n.$$

   (b) Find the number of permutations f of $\{1, 2, \ldots, n\}$ that satisfy the conditions
   $$i - 1 \leq f(i) \leq i + 1, \quad i = 1, 2, \ldots, n.$$

5. (Canada '96) Let $f(n)$ be the number of permutations $a_1, \ldots, a_n$ of the integers $1, \ldots, n$ such that

   (i) $a_1 = 1$;

   (ii) $|a_i - a_{i+1}| \leq 2$, $i = 1, \ldots, n - 1$.

   Determine whether $f(1996)$ is divisible by 3.

6. (USA '96) An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a *binary sequence of length n*. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

Some extra problems on sequences

1. (Part of Putnam '99) The sequence a_1, a_2, $\ldots$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

   Find a simple expression for a_n.

2. Show that if

$$x \geq \sqrt{2}, \quad x \geq \sqrt{2+\sqrt{2}}, \quad x \geq \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

   then $x \geq 2$.

3. Let a be a real number not equal to 0, $\pm\frac{1}{\sqrt{3}}$, or $\pm\sqrt{3}$, and define a sequence x_1, x_2, x_3, $\ldots$ by

$$x_{n+1} = \frac{\sqrt{3}\,x_n + 1}{\sqrt{3} - x_n} \quad \text{for } n \geq 1.$$

   Prove that this sequence is periodic, that is, there exists a positive integer k such that $x_{n+k} = x_n$ for all positive integers n.

4. Consider the sequence x_1, x_2, x_3, $\ldots$ defined by

$$x_1 = a, \quad x_{n+1} = 4x_n(1 - x_n) \quad \text{for } n \geq 1.$$

   Show that

   (a) if $0 \leq a \leq 1$, then $0 \leq x_n \leq 1$ for all positive integers n;

   (b) there exists $0 \leq a \leq 1$ such that $x_{2005} = x_1$, and $x_n \neq x_1$ for $2 \leq n \leq 2004$.