Functional Equations

Functional equations are equations for unknown functions instead of unknown numbers. In this chapter, we will try to explore how we can find the unknown function when we know that the conditions it satisfies.

1. Functional equations in one variable

Functional equations in one variable are usually easier to solve. Although there is no definite method to solve functional equations, there are some tips.

Transformation of variables

It is one of the most common approaches to solve a functional equation. When we apply this technique, we replace one variable by another (*remember that the domain of the original variable should NOT be affected*) so that a new functional equation is obtained. Sometimes it is easier for us to find the unknown function.

Example 1.1.

If $f(x+7) = x^2 - 5x + 2$, find $f(x)$.

Solution.

Let $t = x + 7$, then $x = t - 7$. Directly substitution yields,

$f(t) = (t-7)^2 + 5(t-7) + 2 = t^2 - 9t + 16$. Thus $f(x) = x^2 - 9x + 16$.

Example 1.2.

If $f\left(\frac{x+1}{x}\right) = \frac{x^2+1}{x} + \frac{1}{x}$, find $f(x)$.

Solution.

Let $t = \frac{x+1}{x}$, then $x = \frac{1}{t-1}$. Directly substitution yields,

$$f(t) = \frac{\left(\frac{1}{t-1}\right)^2 + 1}{\left(\frac{1}{t-1}\right)^2} + \frac{1}{\frac{1}{t-1}} = t^2 - t + 1.$$

Thus $f(x) = x^2 - x + 1$.

Example 1.3.

If $f(\ln x) = x^2 + x + 1$, where $x > 0$, find $f(x)$.

Solution.

Let $t = \ln x$, then $x = e^t$. Directly substitution yields,

$f(t) = \left(e^t\right)^2 + e^t + 1$. Thus $f(x) = e^{2x} + e^x + 1$.

In general, if we have $f(g(x)) = h(x)$ and $g(x)$ has an inverse function, then we may replace x by $g^{-1}(x)$ and get $f(x) = h(g^{-1}(x))$.

Solving equation

Sometimes after performing transformation of variables, we can arrive at simultaneous equations. We can find the unknown function after solving the simultaneous equations. We may also treat the unknown function as a variable in ordinary equations and solve it.

Example 1.4.

If $\frac{f(x)}{3+f(x)} = \frac{4+x^2}{x^2}$, find $f(x)$.

Solution.

It is equivalent to $x^2 f(x) = (4+x^2)(3+f(x))$. Simplifying it, we have

$$x^2 f(x) = 3(4+x^2) + (4+x^2) f(x)$$
$$-4 f(x) = 3(4+x^2)$$
$$f(x) = -\frac{3(4+x^2)}{4}$$

Example 1.5.

If $f(x) + f\left(\frac{x-1}{x}\right) = 1+x$, find $f(x)$.

Solution.

Let $t = \frac{x-1}{x}$, then $x = \frac{1}{1-t}$. Direct substitution yields

$$f\left(\frac{1}{1-t}\right) + f(t) = 1 + \frac{1}{1-t}$$

(1.1) $$f\left(\frac{1}{1-x}\right) + f(x) = 1 + \frac{1}{1-x}$$

Let $t = \frac{1}{1-x}$, then $x = \frac{t-1}{t}$. Direct substitution yields

$$f\left(\frac{t-1}{t}\right) + f\left(\frac{1}{1-t}\right) = 1 + \frac{t-1}{t}$$

(1.2) $$f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-x}\right) = 1 + \frac{x-1}{x}$$

On the other hand, by substituting it into (1), we have

(1.3) $$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x$$

((1.1) + (1.2) + (1.3))/2 :

(1.4) $$f(x) + f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-x}\right) = \frac{1}{2}\left(3 + \frac{1}{1-x} + \frac{x-1}{x} + x\right)$$

By subtracting (1.2) from (1.4), we have

$$f(x) = \frac{1}{2}\left(3 + \frac{1}{1-x} + \frac{x-1}{x} + x\right) - \left(1 + \frac{x-1}{x}\right) = \frac{1}{2}\left(1 + \frac{1}{1-x} + x - \frac{x-1}{x}\right) = \frac{-x^3 + x^2 + 1}{2x(1-x)}$$

Method of undetermined coefficients

When we know that the unknown function satisfies certain conditions, say it is a quadratic or cubic function, we can immediately set up variables (e.g. let $f(x)=ax^2+bx+c$ if $f(x)$ is a quadratic polynomial) and solve for them.

Example 1.6.

If $f(x)$ is a quadratic function such that $f(x+1)-f(x)=8x+3$ and $f(0)=5$, find $f(x)$.

Solution.

Let $f(x)=ax^2+bx+c$, then $a(x+1)^2+b(x+1)+c-ax^2-bx-c=8x+3$.

Simplifying gives

$$2ax+a+b=8x+3.$$

After solving, we have $a=4$ and $b=-1$.

Putting $x=0$, we have $c=5$.

Therefore $f(x)=4x^2-x+5$.

2. Functional equations in more than one variable

For functional equations with more than one variable, we can also apply the methods mentioned above. Besides we can also try to substitute some special values, say $x=y=0$ into the given condition given to obtain some results. As it is very difficult to describe these techniques in words, we will try to see their application in various examples below.

Example 2.1.

If $f:\mathbb{Q}\to\mathbb{Q}$ satisfies

1. $f(1)=2$,

2. For all $x,y\in\mathbb{Q}$, $f(xy)=f(x)f(y)-f(x+y)+1$, find $f(x)$.

Solution.

Putting $y=1$, then

$$\begin{aligned} f(x) &= f(x)f(1)-f(x+1)+1 \\ &= 2f(x)-f(x+1)+1 \\ f(x+1) &= f(x)+1 \end{aligned}$$

Therefore applying condition 1 and by mathematical induction, for all integer *x*, we have $f(x)=x+1$.

For any rational number, let $x=\frac{m}{n}$ where *m*, *n* are integers and *n* is not zero. Putting $x=\frac{m}{n}, y=n$, then $f(m)=f\left(\frac{m}{n}\right)(n+1)-f\left(\frac{m}{n}+n\right)+1$.

Since $f(x+1)=f(x)+1$ for $\forall x\in\mathbb{Q}$, we have $f\left(\frac{m}{n}+n\right)=f\left(\frac{m}{n}\right)+n$.

Substituting this into the original equation, we have $m+1=f\left(\frac{m}{n}\right)(n+1)-f\left(\frac{m}{n}\right)-n+1$. Thus

$f\left(\frac{m}{n}\right)=\frac{m}{n}+1$.

So we have $f(x)=x+1 \ \forall x\in\mathbb{Q}$.

Remark.

In this question we can see that we first solve the functional equation for a special case (we found $f(x)$ when $x\in\mathbb{Z}$), then we solve for the more general case. (we found $f(x)$ when $x\in\mathbb{Q}$). Indeed, this method is used in many occasions. For $f:\mathbb{R}\to\mathbb{R}$, we can apply this method similarly. First, find $f(x)$ when $x\in\mathbb{Z}$. Then, find $f(x)$ when $x\in\mathbb{Q}$ by substituting $x=\frac{m}{n}$. Finally, find $f(x)$ when $x\in\mathbb{R}$ by the density of rational numbers (for continuous functions only). We shall see more about this later.

Example 2.2.

If $(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$ for all x, y, find $f(x)$.

Solution.

The given condition is equivalent to

$$\frac{f(x+y)}{x+y}-\frac{f(x-y)}{x-y}=4xy=(x+y)^2-(x-y)^2$$

$$\frac{f(x+y)}{x+y}-(x+y)^2=\frac{f(x-y)}{x-y}-(x-y)^2 \text{ for all } x, y.$$

Thus $\frac{f(x)}{x}-x^2$ is a constant. Let $\frac{f(x)}{x}-x^2=k$, then $f(x)=x^3+kx$.

Remark.

In this question, we have a symmetric condition. By using the symmetry, we reduce the equation to a one-variable functional equation. This is a useful technique for symmetric functional equations.

Example 2.3.

If $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(x^2+f(y))=y+xf(x)$ for all $x, y\in\mathbb{R}$, find $f(x)$.

Solution.

Putting $x=0$, then $f(f(y))=y$. Thus we have,

(2.1) $$f(y+xf(x))=f(f(x^2+f(y)))=x^2+f(y)$$

Now replace x by $f(x)$,

$f(y+(f(x)f(f(x)))=(f(x))^2+f(y)$. Remember that $f(f(y))=y$, so

(2.2) $$f(y+xf(x))=(f(x))^2+f(y)$$

Comparing (2.1) and (2.2), we have

(2.3) $$\left(f(x)\right)^2=x^2$$

Now replace y by $f(y)$ in the original equation, we have

$f(x^2+y)=f(y)+xf(x)$. Squaring both sides, we have

$$(x^2+y)^2=\left(f(x^2+y)\right)^2=f(y)^2+x^2f(x)^2+2xf(x)f(y)=x^4+y^2+2xf(x)f(y)$$

From this, we have

(2.4) $$xy=f(x)f(y)$$

From (2.3), we have $f(x)=x$ or $f(x)=-x$.

If $f(x)=x$, from (2.4) we have $f(x)=x$ for $x\in\mathbb{R}$.

If $f(x) = -x$, from (2.4) we have $f(x) = -x$ for $x \in \mathbb{R}$.

Therefore $f(x) = x$ or $f(x) = -x$.

Remark.

In this question, we replace (or substitute) variables by other things. (we replaced x by $f(x)$ in the question). This is a very useful technique. Some common replacements and substitutions include: replacing *x* by $f(x)$; replacing *x* by $f(f(x))$; substituting *x* = 0; substituting *x* = *y* = 0; substituting *x* = 1, etc.

Example 2.4.

If $f: \mathbb{R} \to \mathbb{R}^+$ satisfies

1. $f(xf(y)) = yf(x)$ for all $x, y \in \mathbb{R}^+$
2. $f(x) \to 0$ as $x \to \infty$, find $f(x)$.

Solution.

First, let us show that $f(x)$ is surjective.

Let $y = \dfrac{x}{f(x)}$, then $f\left(xf\left(\dfrac{x}{f(x)}\right)\right) = \dfrac{x}{f(x)} f(x) = x$. Therefore $f(x)$ is surjective, i.e. for all y there exists an *x* such that $f(x) = y$.

Assume $f(y) = 1$. Putting $x = 1$, then

$f(1) = f(1f(y)) = yf(1)$. So $y = 1$ and $f(1) = 1$.

Now putting $x = y$, then $f(xf(x)) = xf(x)$. Thus $xf(x)$ is a fixed point for the function. Let us show two things about fixed point now.

(i) If *a*, *b* are fixed points of *f*, then $f(ab) = f(af(b)) = bf(a) = ab$. Thus *ab* is also a fixed point.

(ii) If *a* is a fixed point of *f*, then $1 = f(1) = f\left(a \bullet \dfrac{1}{a}\right) = f\left(\dfrac{1}{a} f(a)\right) = af\left(\dfrac{1}{a}\right)$. Thus $f\left(\dfrac{1}{a}\right) = \dfrac{1}{a}$, which means $\dfrac{1}{a}$ is also a fixed point.

Now, if $xf(x) > 1$, then $f((xf(x))^n) = (xf(x))^n$. As $n \to \infty$, $\left[xf(x)\right]^n \to \infty$ while $f((xf(x))^n) = (xf(x))^n$. This contradicts with condition 2.

If $xf(x)<1$, then by (ii) we have $\frac{1}{xf(x)}$ is a fixed point larger than 1. This again contradicts with condition 2.

Thus, we must have $xf(x)=1$. This means $f(x)=\frac{1}{x}$.

Remark.

The concept of fixed point is introduced in this question. If $f(x)=x$, then *x* is called a fixed point of *f*. It is not so commonly used but it is also a useful technique.

3. Some famous functional equations

In this part, we will introduce some famous functional equations. We may quote them directly in competitions.

Theorem 3.1. (Cauchy equation)

If *f* is a continuous function such that $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbb{R}$, then $f(x)=cx$ where c is a constant.

Proof.

We may use the technique mentioned in example 2.1.

First, put $y=1$ and let x be a positive integer. Then $f(x+1)=f(x)+c$, where $c=f(1)$. Thus $f(x)=cx$ for positive integers *x*. It is easy to verify that $f(0)=0$ by putting $x=y=0$ into the original equation. For negative integers, we replace *x* by –*x* and get $f(-x+1)=f(-x)+c$. Thus $f(-x)=-cx$ for positive integers *x*. Therefore we conclude that $f(x)=cx$ for integers x.

Let $x=\frac{m}{n}$, where *m*, *n* are integers. Then we have

$f\left(\frac{m+1}{n}\right)=f\left(\frac{m}{n}\right)+f\left(\frac{1}{n}\right)$, thus $f\left(\frac{m}{n}\right)=mf\left(\frac{1}{n}\right)$. However, $c=f(\frac{1}{n}\cdot n)=nf\left(\frac{1}{n}\right)$. So $f\left(\frac{1}{n}\right)=\frac{c}{n}$ and we have $f\left(\frac{1}{n}\right)=\frac{c}{n}$. This implies $f(x)=cx$ for rational numbers *x*.

Now, as f is continuous, we can always bound an irrational number by two rational numbers. (For example, we may bound π by 3, 3.1, 3.14, 3.141 and 4, 3.2, 3.15, 3.142 respectively). Let $\{x_n\}, x_n \in \mathbb{Q}$ be such a sequence with $\lim_{n\to\infty} x_n = x$, then by continuity of f, we have $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} cx_n = cx$. Therefore $f(x) = cx$ for all $x \in \mathbb{R}$.

Q.E.D.

Corollary 3.2.

If f is a continuous function and for all $x, y \in \mathbb{R}$,

(i) $f(x+y) = f(x)f(y)$

then $f(x) = c^x$

(ii) $f(xy) = f(x) + f(y)$

then $f(x) = c\ln x$

(iii) $f(xy) = f(x)f(y)$

then $f(x) = x^c$

where c is a constant.

Example 3.3.

If $f:(1,+\infty) \to \mathbb{R}$ is a continuous function such that $f(xy) = xf(y) + yf(x)$ for all $1 < x, y \in \mathbb{R}$, find $f(x)$.

Solution.

It is equivalent to $\frac{f(xy)}{xy} = \frac{f(x)}{x} + \frac{f(y)}{y}$. If we let $g(x) = \frac{f(x)}{x}$, then the equation becomes $g(xy) = g(x) + g(y)$ which is simply (ii) of corollary 1.32 (although there is a minor difference between them). So we have $g(x) = c\ln x$ and $f(x) = xg(x) = cx\ln x$.

4. Exercises

1. Solve the functional equation $f(x+2)=x^2+4x+6$.
2. If $3f(x)+2f\left(\frac{1}{x}\right)=4x$, find $f(x)$.
3. If $af(x-1)+bf(1-x)=cx$ where *a*, *b*, *c* are constants, find $f(x)$.
4. Solve the functional equation $xf(x)+2xf(-x)=-1$.
5. Find all continuous functions *f* for $x>0$ such that $f(x+y)=\dfrac{f(x)f(y)}{f(x)+f(y)}$.
6. Find all continuous functions *f* such that $f(x+y)=f(x)+f(y)+f(x)f(y)$.
7. If $f(x+y)+f(x-y)=2f(x)\cos y$, find $f(x)$.

Harder problems

8. (IMO 1982) The function $f(n)$ is defined for all positive integers *n* and takes on non-negative integral values.

   Also, for all *m*, *n*

$$f(m+n)-f(m)-f(n)=0 \text{ or } 1$$

$$f(2)=0,\ f(3)>0, \text{ and } f(9999)=3333.$$

   Determine $f(1982)$.

9. Find all pairs of functions $f,g:\mathbb{R}\to\mathbb{R}$ such that
   (a) if $x<y$, then $f(x)<f(y)$
   (b) for all $x,y\in\mathbb{R}$, $g(y)f(x)+f(y)=f(xy)$.
10. Find all $f:\mathbb{Z}\to\mathbb{Z}$ such that
   (a) $f(1)=1$
   (b) for all $m,n\in\mathbb{Z}$, $f(m+n)(f(m)-f(n))=f(m-n)(f(m)+f(n))$.

References:

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