Mock Olympiad #4

July 9, 2009

1. Let a, b, c, d be positive real numbers such that

$$abcd = 1 \text{ and } a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}.$$

   Prove that

$$a+b+c+d < \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}.$$

2. In an acute triangle ABC, segments BE and CF are altitudes. Two circles passing through the points A and F are tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that the lines PE and QF intersect on the circumcircle of triangle AEF.

3. For every $n \in \mathbb{N}$, let $d(n)$ denote the number of (positive) divisors of n. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the following properties:

   (a) $d(f(x)) = x$ for all $x \in \mathbb{N}$;

   (b) $f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all $x, y \in \mathbb{N}$.

1 Solutions

1. *(IMO 2008 Short list, A5)*

   We will prove that

$$2a + 2b + 2c + 2d \leq \sum_{cyc} \left(\frac{a}{b} + \frac{b}{a} \right). \tag{1}$$

   This is clearly sufficient. Using $abcd = 1$ to homogenize, (1) becomes

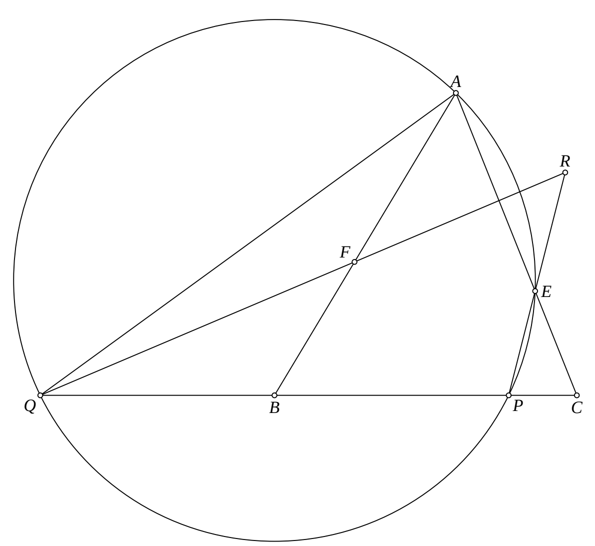
$$\frac{2a^{\frac{3}{4}}}{b^{\frac{1}{4}}c^{\frac{1}{4}}d^{\frac{1}{4}}} + \frac{2b^{\frac{3}{4}}}{a^{\frac{1}{4}}c^{\frac{1}{4}}d^{\frac{1}{4}}} + \frac{2c^{\frac{3}{4}}}{a^{\frac{1}{4}}b^{\frac{1}{4}}d^{\frac{1}{4}}} + \frac{2d^{\frac{3}{4}}}{b^{\frac{1}{4}}c^{\frac{1}{4}}a^{\frac{1}{4}}} \leq \sum_{cyc} \left(\frac{a}{b} + \frac{b}{a} \right)$$

   Using AM-GM, we get

$$\begin{aligned} \frac{a}{b} + \frac{a}{b} + \frac{b}{c} + \frac{a}{d} &\geq \frac{4a^{\frac{3}{4}}}{b^{\frac{1}{4}}c^{\frac{1}{4}}d^{\frac{1}{4}}}, \text{ and} \\ \frac{a}{d} + \frac{a}{d} + \frac{d}{c} + \frac{a}{b} &\geq \frac{4a^{\frac{3}{4}}}{b^{\frac{1}{4}}c^{\frac{1}{4}}d^{\frac{1}{4}}}. \end{aligned}$$

   Adding these inequalities cyclically, we get the desired result.

2. *(IMO 2008 Short list, G4)*



   We claim that $AEPQ$ is a cyclic quadrilateral. Let a, b, c denote the angles of $\triangle ABC$. By power of a point, $BP^2 = BF \cdot BA = BQ^2$. Therefore, $CP \cdot CQ = (CB + BQ) \cdot (CB -$

$BP) = CB^2 - BF \cdot BA = CB^2 - (CB \cdot \cos b) \cdot \left(\frac{CB \sin c}{\sin a}\right) = CB^2 \cdot \left(\frac{\sin(b+c) - (\cos b)(\sin c)}{\sin a}\right) = CB^2 \cdot \frac{(\cos c)(\sin b)}{\sin a} = (CB \cos c) \cdot \left(\frac{CB \sin b}{\sin a}\right) = CE \cdot CA$. This proves $AEPQ$ is cyclic by power of a point.

Now, $\angle FRE = \angle QRP = 180^\circ - \angle RPQ - \angle RQP = \angle QAC - \angle QAB$. Here, we used the fact that $AEPQ$ is cyclic and the circumcircle of AFQ is tangent to BC at Q. Therefore, $\angle FRE = \angle BAC = \angle FAE$, which completes the proof.

Remark: Here is another way to prove $AEPQ$ is cyclic. Let H be the orthocentre of ABC. Note that $AFHE$ is cyclic. By power of a point, $BP^2 = BQ^2 = BF \cdot BA = BH \cdot BE$, so the circumcircle of QEH is tangent to line BC. Likewise, the circumcircle of PEH is tangent to line BC. Now invert around C, preserving circle $AFHE$. The circles passing through AF tangent to line BC goes to the circle passing through EH tangent to line BC, so P and Q go to each other.

3. *(IMO 2008 Short list, N5)*

Recall that if the prime factorization of n is

$$p_1^{e_1} \cdot p_2^{e_2} \cdots p_m^{e_m},$$

then $d(n) = (e_1 + 1)(e_2 + 1) \cdots (e_m + 1)$.

Now, if p is a prime, then by above there must exist a prime q such that $f(p) = q^{p-1}$. Suppose $q \neq p$. Taking $x = q, y = p$ gives $f(pq) | (q-1)p^{pq-1} f(q)$. There exists some prime r with $f(q) = r^{q-1}$, so there are at most $q - 1$ factors of q in $(q-1)p^{pq-1} f(q)$, and hence also in $f(pq)$.

Taking $x = p, y = q$ gives $f(pq) | (p-1)q^{pq-1} f(p) = (p-1)q^{pq+p-2}$, and since there are at most $q - 1$ factors of q in $f(pq)$, we get $f(pq) | (p-1)q^{q-1}$. Now, $d(p-1) \leq p - 1 < p$, and $d(q^{q-1}) = q$, so

$$d(f(pq)) \leq d((p-1)q^{q-1}) \leq d(p-1)d(q^{q-1}) < pq,$$

which contradicts condition (a) for pq. Hence, $f(p) = p^{p-1}$ for all primes p.

Now let n be a positive integer, and let p be the smallest prime dividing n. Taking $x = p, y = \frac{n}{p}$, we get

$$f(n) | (p-1) \cdot \left(\frac{n}{p}\right)^{n-1} f(p).$$

Now, since p is the smallest prime dividing n and $d(f(n)) = n$, the exponent of any prime dividing $f(n)$ must be at least $(p - 1)$. Suppose a prime power r^k divides $f(n)$ but not n. Then r doesn't divide $\left(\frac{n}{p}\right)^{n-1} f(p)$, so r^k divides $p - 1$. But $k \geq p - 1$, and so $r^k \leq p - 1 \leq k$, which is impossible. So all primes dividing $f(n)$ must divide n. In particular, it follows immediately that if $n = p^k$ is a prime power, then $f(n) = p^{n-1}$.

We claim that if $n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_m^{e_m}$, then $f(n) = p_1^{p_1^{e_1}-1} \cdot p_2^{p_2^{e_2}-1} \cdots p_m^{p_m^{e_m}-1}$. We proceed by induction on n. $d(f(1)) = 1$, so $f(1) = 1$. Now assume it is true for all numbers less than n. If n is a prime power, then we've already proven the claim. So assume n has at least 2 prime factors. Let r be a prime dividing $f(n)$, and let $p \neq r$ be a prime dividing n. Taking $x = \frac{n}{p}, y = p$, we get $f(n) | \left(\frac{n}{p} - 1\right) p^{n-1} f\left(\frac{n}{p}\right)$. We know r must divide n, so it doesn't divide $\left(\frac{n}{p} - 1\right) p^{n-1}$, and hence it divides $f(\frac{n}{p})$. By our induction claim, the exponent of r dividing $f(n)$ is bounded by $r^k - 1$, where $r^k || n$. Hence, $f(n)$ divides $p_1^{p_1^{e_1} - 1} \cdot p_2^{p_2^{e_2} - 1} \cdots p_m^{p_m^{e_m} - 1}$. The claim now follows from the fact that $d(f(n)) = n$.

It's straightforward to check this f actually satisfies the two conditions.