2005 Winter Camp

Answers and Solutions for CIS Olympiads

1st CIS 1992

Problem 1

Show that $x^4 + y^4 + z^2 \geq xyz\sqrt{8}$ for all positive reals x, y, z.

Solution

By AM/GM $x^4 + y^4 \geq 2x^2y^2$. Then by AM/GM again $2x^2y^2 + z^2 \geq (\sqrt{8})xyz$.

<u>1st CIS 1992</u>

Problem 7

Find all real x, y such that $(1 + x)(1 + x^2)(1 + x^4) = 1+ y^7$, $(1 + y)(1 + y^2)(1 + y^4) = 1+ x^7$?

Answer

$(x,y) = (0,0)$ or $(-1,-1)$

Solution

If $x = y$, then clearly $x \neq 1$, so we have $(1-x^8) = (1-x)(1+x^7) = 1-x^8-x+x^7$, so $x = 0$ or $x^6 = 1$, whose only real root (apart from the $x = 1$ we have discarded) is $x = -1$. That gives the two solutions above. So assume $x \neq y$. wlog $x > y$.

So $(1+x) > (1+y)$ and $(1+x^7) > (1+y^7)$. So we must have $(1+x^2)(1+x^4) < (1+y^2)(1+y^4)$ and hence $y < 0$. If $x > 0$, then $(1 + x)(1 + x^2)(1 + x^4) > 1 > 1+y^7$, so $x < 0$ also.

Multiplying the first equ by (1-x) and the second by (1-y) and subtracting: $y^8 - x^8 = (y-x) + (y^7 - x^7) + xy(x^6 - y^6)$. But lhs > 0 and each term on rhs < 0. Contradiction. So there are no more solutions.

1st CIS 1992

Problem 9

Show that for any real numbers x, y > 1, we have $x^2/(y - 1) + y^2/(x - 1) \geq 8$.

Solution

We have $(x-2)2 \geq 0$, so $x2 \geq 4(x-1)$. Hence $x/\sqrt{(x-1)} \geq 2$. Now by AM/GM, $x^2/(y - 1) + y^2/(x - 1) \geq 2xy/\sqrt{((x-1)(y-1))}$. But rhs $\geq 2 \cdot 2 \cdot 2$.

1st CIS 1992

Problem 23

If $a > b > c > d > 0$ are integers such that $ad = bc$, show that $(a - d)^2 \geq 4d + 8$.

Solution

We need first that $a + d > b + c$. Put $a = m + h$, $d = m - h$, $b = m' + k$, $c = m' - k$. Then since $a - d > b - c$, we have $h > k$. But $m^2 - h^2 = ad = bc = m'^2 - k^2$, so $m > m'$ and hence $a + d > b + c$. Since a, b, c, d are integers it follows that $(a + d - b - c) \geq 1$.

Now $(a - d)^2 = (a + d)^2 - 4ad = (a + d)^2 - 4bc > (a + d)^2 - (b + c)^2$ (AM/GM) $= (a + b + c + d)(a + d - b - c) \geq (a + b + c + d)$. But $a \geq d + 3$, $b \geq d + 2$, $c \geq d + 1$, so $(a - d)^2 \geq 4d + 6$. But a square cannot = 2 or 3 mod 4, so $(a - d)^2 \geq 4d + 8$.

1st CIS 1992

Problem 1

Find all integers a, b, c, d such that $ab - 2cd = 3$, $ac + bd = 1$.

Answer

$(a,b,c,d) = (1,3,1,0), (-1,-3,-1,0), (3,1,0,1), (-3,-1,0,-1)$

Solution

$11 = (ab - 2cd)^2 + 2(ac + bd)^2 = (a^2 + 2d^2)(b^2 + 2c^2)$, so we must have either (1) $a^2 + 2d^2 = 1$, $b^2 + 2c^2 = 11$, or (2) $a^2 + 2d^2 = 11$, $b^2 + 2c^2 = 1$.

(1) gives $a = \pm 1$, $d = 0$, $b = \pm 3$, $c = \pm 1$. If $a = 1$ and $d = 0$, then $ac + bd = 1$ implies $c = 1$, and $ab - 2cd = 3$ implies $b = 3$. Similarly, if $a = -1$, then $c = -1$, and $b = -3$. Similarly, (2) gives $(a,b,c,d) = (3,1,0,1), (-3,-1,0,-1)$.

Thanks to Suat Namli

25th ASU 1991

Problem 9

Show that $(x + y + z)^2/3 \geq x\sqrt{(yz)} + y\sqrt{(zx)} + z\sqrt{(xy)}$ for all non-negative reals x, y, z.

Solution

By AM/GM $xy + yz \geq 2x\sqrt{(yz)}$. Adding the similar results gives $2(xy + yz + zx) \geq 2(x\sqrt{(yz)} + y\sqrt{(zx)} + z\sqrt{(xy)})$.

By AM/GM $x^2 + x^2 + y^2 + z^2 \geq 4x\sqrt{(yz)}$. Adding the similar results gives $x^2 + y^2 + z^2 \geq x\sqrt{(yz)} + y\sqrt{(zx)} + z\sqrt{(xy)}$. Adding the first result gives $(x + y + z)^2/3 \geq x\sqrt{(yz)} + y\sqrt{(zx)} + z\sqrt{(xy)}$.

Thanks to Suat Namli

25th ASU 1991

Problem 18

$p(x)$ is the cubic $x^3 - 3x^2 + 5x$. If h is a real root of $p(x) = 1$ and k is a real root of $p(x) = 5$, find $h + k$.

Solution

Put $y = 2-h$, where $p(h) = 1$, then $(2-y)^3 - 3(2-y)^2 + 5(2-y) - 1 = 0$, so $8-12y+6y^2-y^3 - 12+12y-3y^2 + 10-5y - 1 = 0$, or $y^3 - 3y^2 + 5y = 5$, or $p(y) = 5$. So if h is a root of $p(h) = 1$, then there is a root k of $p(k) = 5$ such that $h+k = 2$. To complete the proof we have to show that $p(x) = 5$ has only one real root.

But $x^3 - 3x^2 + 5x = (x-1)^3 + 2(x-1) + 3$ which is a strictly increasing function of x-1 and hence of x. So $p(x) = k$ has only one real root.

Thanks to Suat Namli

25th ASU 1991

Problem 1

Show that $x^4 > x - 1/2$ for all real x.

Solution

$x^4 - x + 1/2 = (x^2 - 1/2)^2 + (x - 1/2)^2 \geq 0$. We could only have equality if $x^2 = x = 1/2$, which is impossible, so the inequality is strict.

24th ASU 1990

Problem 10

Let $x_1, x_2, \ldots , x_n$ be positive reals with sum 1. Show that $x_1^2/(x_1 + x_2) + x_2^2/(x_2 + x_3) + \ldots + x_{n-1}^2/(x_{n-1} + x_n) + x_n^2/(x_n + x_1) \geq 1/2$.

Solution

$\sum x_i^2/(x_i+x_{i+1}) - \sum x_{i+1}^2/(x_i+x_{i+1}) = \sum (x_i - x_{i+1}) = 0$. Hence $\sum x_i^2/(x_i+x_{i+1}) = ½ \sum (x_i^2+x_{i+1}^2)/(x_i+x_{i+1}) \geq ¼ \sum (x_i + x_{i+1}) = ½$.

Thanks to Suat Namli

24th ASU 1990

Problem 7

If rationals x, y satisfy $x^5 + y^5 = 2 x^2 y^2$ show that $1 - x y$ is the square of a rational.

Solution

Put $y = kx$, then $x^5(1 + k^5) = 2k^2x^4$, so $x = 2k^2/(1 + k^5)$, $y = 2k^3/(1 + k^5)$ and $1 - xy = (1 - k^5)^2/(1 + k^5)^2$. x and y are rational, so $(1 - k^5)/(1 + k^5)$ is rational.

Problem 9

Find all positive integers n satisfying $(1 + 1/n)^{n+1} = (1 + 1/1998)^{1998}$.

Solution

Answer: no solutions.

We have $(1 + 1/n)^{n+1} > e > (1 + 1/n)^n$.

<u>22nd ASU 1988</u>

<u>22nd ASU 1988</u>

Problem 15

What is the minimal value of $b/(c + d) + c/(a + b)$ for positive real numbers b and c and non-negative real numbers a and d such that $b + c \geq a + d$?

Solution

Answer: $\sqrt{2} - 1/2$.

Obviously $a + d = b + c$ at the minimum value, because increasing a or d reduces the value. So we may take $d = b + c - a$. We also take $b >= c$ (interchanging b and c if necessary). Dividing through by b/2 shows that there is no loss of generality in taking $b = 2$, so $0 < c <= 2$. Thus we have to find the minimum value of $2/(2c - a + 2) + c/(a + 2)$. We show that it is $\sqrt{2} - 1/2$.

This is surprisingly awkward. Note first that $(c - (h - k))^2 >= 0$, so $c^2 + c(2k - 2h) + h^2 - 2hk + k^2 >= 0$. Hence $c^2 + ck + h^2 >= (2h - k)(c + k)$. Hence $c/h^2 + 1/(c+k) >= 2/h - k/h^2$ with equality iff $c = h - k$. Applying this to $c/(a+2) + 1/(c+1-a/2)$ where $h = \sqrt{(a+2)}$, $k = 1-a/2$, we find that $c/(a+2) + 2/(2c+2-a) >= 2/\sqrt{(a+2)} + (a-2)/(2a+4)$.

The allowed range for c is $0 \le c \le 2$ and $0 \le a \le c+2$, hence $0 \le a \le 4$. Put $x = 1/\sqrt{(a+2)}$, so $1/\sqrt{6} <= x <= 1/\sqrt{2}$. Then $2/\sqrt{(a+2)} + (a-2)/(2a+4) = 2x + 1/2 - 2x^2 = 1 - (2x-1)^2/2$. We have $-0.184 = (2/\sqrt{6} - 1) \le 2x-1 \le \sqrt{2} - 1 = 0.414$. Hence $c/(a+2) + 2/(2c+2-a) \ge 1 - (\sqrt{2} - 1)^2/2 = \sqrt{2} - 1/2$.

We can easily check that the minimum is achieved at $b = 2$, $c = \sqrt{2} - 1$, $a = 0$, $d = \sqrt{2} + 1$.

22nd ASU 1988

Problem 18

Find the minimum value of $xy/z + yz/x + zx/y$ for positive reals x, y, z with $x^2 + y^2 + z^2 = 1$.

Solution

Answer: min $\sqrt{3}$ when all equal.

Let us consider z to be fixed and focus on x and y. Put $f(x, y, z) = xy/z + yz/x + zx/y$. We have $f(x, y, z) = p/z + z(1-z^2)/p = (p + k^2/p)/z$, where $p = xy$, and $k = z\sqrt{(1-z^2)}$. Now p can take any value in the range $0 < p \le (1-z^2)/2$. The upper limit is achieved when $x = y$.

We have $p + k^2/p = (p - k)^2/p$. For $p \le k$, $(p - k)$ and $1/p$ are both decreasing functions of p, so $p + k^2/p$ is a decreasing function of p. Thus if p is restricted to the interval $(0, h]$, then for $k \le h$ the minimum value of $p + k^2/p$ is 2k and occurs at $p = k$. For $k \ge h$ the minimum is $h + k^2/h$ and occurs at $p = h$.

We have $h = (1-z^2)/2$, $k = z\sqrt{(1-z^2)}$. So $k \le h$ iff $z <= 1/\sqrt{5}$. So if $z \le 1/\sqrt{5}$, then $f(x, y, z) \ge 2k/z = 2\sqrt{(1-z^2)} \ge 27\sqrt{(1-1/5)} = 4/\sqrt{5} > \sqrt{3}$.

If $z > 1/\sqrt{5}$, then the minimum of $f(x, y, z)$ occurs at $x = y$ and is $x^2/z + z + z = (1-z^2)/(2z) + 2z = 3z/2 + 1/(2z) = (\sqrt{3})/2\ (z\sqrt{3} + 1/(z\sqrt{3}) \ge \sqrt{3}$ with equality at $z = 1/\sqrt{3}$ (and hence $x = y = 1/\sqrt{3}$ also).

22nd ASU 1988

Problem 22

What is the smallest n for which there is a solution to $\sin x_1 + \sin x_2 + \dots + \sin x_n = 0$, $\sin x_1 + 2 \sin x_2 + \dots + n \sin x_n = 100$?

Solution

Put $x_1 = x_2 = \dots = x_{10} = 3\pi/2$, $x_{11} = x_{12} = \dots = x_{20} = \pi/2$. Then $\sin x_1 + \sin x_2 + \dots + \sin x_{20} = (-1 - 1 - 1 - \dots - 1) + (1 + 1 + \dots + 1) = 0$, and $\sin x_1 + 2 \sin x_2 + \dots + 20 \sin x_{20} = -(1 + 2 + \dots + 10) + (11 + 12 + \dots + 20) = 100$. So there is a solution with $n = 20$. If there is a solution with $n < 20$, then there must be a solution for $n = 19$ (put any extra $x_i = 0$). But then $100 = (\sin x_1 + 2 \sin x_2 + \dots + 19 \sin x_{19}) - 10 (\sin x_1 + \sin x_2 + \dots + \sin x_{19}) = -9 \sin x_1 - 8 \sin x_2 - 7 \sin x_3$

$- \dots - \sin x_9 + \sin x_{11} + 2 \sin x_{12} + \dots + 9 \sin x_{19}$. But $|\text{rhs}| \le (9 + 8 + \dots + 1) + (1 + 2 + \dots + 9) = 90$. Contradiction. So there is no solution for $n < 20$.

<u>22nd ASU 1988</u>

Problem 1

The quadratic $x^2 + ax + b + 1$ has roots which are positive integers. Show that $(a^2 + b^2)$ is composite.

Solution

Let the roots be c, d, so $c + d = -a$, $cd = b+1$. Hence $a^2 + b^2 = (c^2 + 1)(d^2 + 1)$.

Thanks to Suat Namli

<u>20th ASU 1986</u>

Problem 20

x is a real number. Define $x_0 = 1 + \sqrt{(1 + x)}$, $x_1 = 2 + x/x_0$, $x_2 = 2 + x/x_1$, ... , $x_{1985} = 2 + x/x_{1984}$. Find all solutions to $x_{1985} = x$.

Answer

3

Solution

If $x = 0$, then $x^{1985} = 2 \neq x$. Otherwise we find $x_1 = 2 + x/(1+\sqrt{(1+x)}) = 2 + (\sqrt{(1+x)} - 1) = 1 + \sqrt{(1+x)}$. Hence $x_{1985} = 1 + \sqrt{(1+x)}$. So $x - 1 = \sqrt{(1+x)}$. Squaring, $x = 0$ or 3. We have already ruled out $x = 0$. It is easy to check that $x = 3$ is a solution.

Thanks to Suat Namli

19th ASU 1985

Problem 2

Show that $(a + b)^2/2 + (a + b)/4 \geq a\sqrt{b} + b\sqrt{a}$ for all positive a and b.

Answer

By AM/GM $\sqrt{(ab)} \leq (a+b)/2$, so $\frac{1}{2}(a+b) + \sqrt{(ab)} \leq a + b$. Hence $\sqrt{(2a+2b)} \geq \sqrt{a} + \sqrt{b}$ (*).

By AM/GM $(a + b) \geq 2\sqrt{(ab)}$ and $2(a+b) + 1 \geq 2\sqrt{(2a+2b)}$. Multiplying, $(a+b)(2a+2b+1) \geq 4\sqrt{(ab)}\sqrt{(2a+2b)}$. Then using (*) $\geq 4\sqrt{(ab)}(\sqrt{a} + \sqrt{b})$.

Thanks to Suat Namli

Solution

18th ASU 1984

Problem 2

The sequence a_n is defined by $a_1 = 1$, $a_2 = 2$, $a_{n+2} = a_{n+1} + a_n$. The sequence b_n is defined by $b_1 = 2$, $b_2 = 1$, $b_{n+2} = b_{n+1} + b_n$. How many integers belong to both sequences?

Answer

1,2,3 only

Solution

The first few terms are:

n	1	2	3	4	5	6	7
a_n	1	2	3	5	8	13	21
b_n	2	1	3	4	7	11	18

Note that for n = 4, 5 we have $a_{n-1} < b_n < a_n$. So by a trivial induction, the inequality holds for all n ≥ 4.

Thanks to Suat Namli

16th ASU 1982

Problem 15

x is a positive integer. Put $a = x^{1/12}$, $b = x^{1/4}$, $c = x^{1/6}$. Show that $2^a + 2^b \geq 2^{1+c}$.

Solution

Put $x = r^{12}$. Since x is a positive integer, we have $r \geq 1$. We have to show that $(2^r + 2^{r^3})/2 \geq 2^{r^2}$. But this follows immediately from AM/GM.

Thanks to Suat Namli

16th ASU 1982

Problem 13

Find all solutions (x, y) in positive integers to $x^3 - y^3 = xy + 61$.

Answer

$(6,5)$

Solution

Put $x = y + a$. Then $(3a-1)y^2 + a(3a-1)y + (a^3-61) = 0$. The first two terms are positive, so the last term must be negative, so $a = 1, 2, 3$. Trying each case in turn, we get $(y+6)(y-5) = 0$, $5y^2+10y-53 = 0$, $4y^2+12y-17 = 0$. The last two equations have no integers solutions.

Thanks to Suat Namli

15th ASU 1981

Problem 16

The positive reals x, y satisfy $x^3 + y^3 = x - y$. Show that $x^2 + y^2 < 1$.

Solution

Since x, y are positive, so is $x^3 + y^3$, and hence $x > y$. So $(x^2 + y^2)(x - y) = (x^3 - y^3) - xy(x - y) < x^3 - y^3 = x - y$. Hence $x^2 + y^2 < 1$.

Thanks to Suat Namli

15th ASU 1981 Problem 5

Are there any solutions in positive integers to $a^4 = b^3 + c^2$?

Solution

We have $b^3 = (a^2 - c)(a^2 + c)$, so one possibility is that a2 ± c are both cubes. So we want two cubes whose sum is twice a square. Looking at the small cubes, we soon find $8 + 64 = 2 \cdot 36$ giving $6^4 = 28^2 + 8^3$. Multiplying through by k^{12} gives an infinite family of solutions. Note that the question does not ask for all solutions.

14th ASU 1980

Problem 16

A rectangular box has sides $x < y < z$. Its perimeter is $p = 4(x + y + z)$, its surface area is $s = 2(xy + yz + zx)$ and its main diagonal has length $d = \sqrt{(x^2 + y^2 + z^2)}$. Show that $3x < (p/4 - \sqrt{(d^2 - s/2)}$ and $3z > (p/4 + \sqrt{(d^2 - s/2)}$.

Solution

We have $3(y-x)(z-x) > 0$, so $3x^2 + 3yz > 3xy + 3xz$. Hence $y^2 + z^2 + 4x^2 +2yz - 4xy - 4xz > x^2 + y^2 + z^2 - xy - yz - xz$ or $(y + z - 2x)^2 > (d^2 - s/2)$. Hence $(x + y + z) > 3x + \sqrt{(d^2 - s/2)}$. So $3x < p/4 - \sqrt{(d^2 - s/2)}$.

Similarly, $3(z-x)(z-y) > 0$, so $x^2 + y^2 + 4z^2 > x^2 + y^2 + z^2 + 3zx + 3zy - 3xy$, so $(2z - x - y)^2 > x^2 + y^2 + z^2 - xy - yz - zx$ or $(3z - p/4)^2 > (d^2 - s/2)$. Hence $3z > p/4 + \sqrt{(d^2 - s/2)}$.

14th ASU 1980