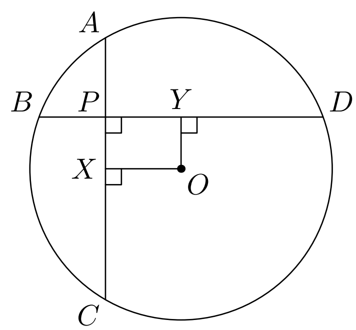
WOOT 2010–11

Practice Olympiad 4 Solutions

1. Let AC and BD be two perpendicular chords of a circle with radius r, and let the two chords intersect at P. Express $PA^2 + PB^2 + PC^2 + PD^2$ in terms of r.

Solution 1. Let O be the center of the circle, and let X and Y be the feet of the perpendiculars from O to AC and BD, respectively. Then X and Y are the midpoints of chords AC and BD, respectively, so $AX = CX$ and $BY = DY$.



Hence,

$$\begin{aligned} PA^2 + PC^2 &= (AX - PX)^2 + (CX + PX)^2 \\ &= (CX - PX)^2 + (CX + PX)^2 \\ &= 2(CX^2 + PX^2), \end{aligned}$$

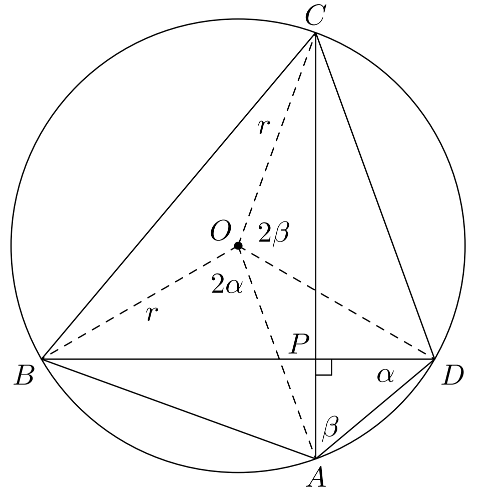
and similarly, $PB^2 + PD^2 = 2(DY^2 + PY^2)$. Adding, we get

$$PA^2 + PB^2 + PC^2 + PD^2 = 2(CX^2 + PX^2 + DY^2 + PY^2).$$

Since AC and BD are perpendicular, quadrilateral $OXPY$ is a rectangle, so $PX = OY$ and $PY = OX$. Therefore,

$$\begin{aligned} 2(CX^2 + PX^2 + DY^2 + PY^2) &= 2(CX^2 + OY^2 + DY^2 + OX^2) \\ &= 2(CX^2 + OX^2 + DY^2 + OY^2) \\ &= 2(OC^2 + OD^2) \\ &= 2(r^2 + r^2) \\ &= 4r^2. \end{aligned}$$

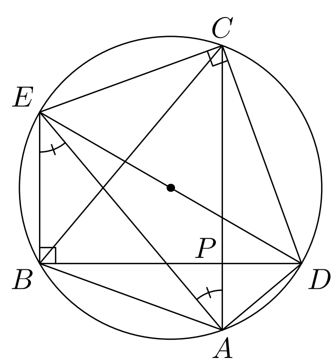
Solution 2. Let $\angle AOB = 2\alpha$ and $\angle COD = 2\beta$. Then $\angle CAD = \angle COD/2 = \beta$ and $\angle ADB = \angle AOB/2 = \alpha$. From right triangle APD, $\alpha + \beta = \pi/2$.



By sine law, $AB = 2r \sin\alpha$ and $CD = 2r \sin\beta$. Therefore,

$$\begin{aligned} PA^2 + PB^2 + PC^2 + PD^2 &= AB^2 + CD^2 \\ &= (2r\sin\alpha)^2 + (2r\sin\beta)^2 \\ &= 4r^2(\sin^2\alpha + \sin^2\beta) \\ &= 4r^2(\sin^2\alpha + \cos^2\alpha) \\ &= 4r^2. \end{aligned}$$

Solution 3. Let E be the point diametrically opposite D on the circle. Then $DB \perp BE$ and $DC \perp CE$. But we are also given that $AC \perp DB$, so AC and BE are parallel. Then $\angle CAE = \angle AEB$. It follows that $AB = CE$.



By the Pythagorean Theorem, $PA^2 + PB^2 = AB^2$, $PC^2 + PD^2 = DC^2$, and $CE^2 + DC^2 = DE^2 = (2r)^2 = 4r^2$. Hence,

$$\begin{aligned} PA^2 + PB^2 + PC^2 + PD^2 &= AB^2 + DC^2 \\ &= CE^2 + DC^2 \\ &= 4r^2. \end{aligned}$$

2. The coefficients of x^{r-1}, x^r, and x^{r+1} in the binomial expansion of $(1+x)^n$ are three consecutive terms in an arithmetic sequence, where n and r are positive integers.

   (a) Prove that $(n-2r)^2 = n+2$.

   (b) Determine all pairs of positive integers (n, r) that satisfy the equation in part (a).

Solution. (a) From the given information about the coefficients of x^{r-1}, x^r, and x^{r+1},

$$\binom{n}{r-1} + \binom{n}{r+1} = 2\binom{n}{r},$$

which we can re-write as

$$\frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{(r+1)!(n-r-1)!} = \frac{2n!}{r!(n-r)!}.$$

Multiplying both sides by $(r+1)!(n-r+1)!/n!$, we get

$$\begin{aligned} & r(r+1) + (n-r)(n-r+1) = 2(r+1)(n-r+1) \\ \Rightarrow \quad & r^2 + r + n^2 - nr + n - nr + r^2 - r = 2nr - 2r^2 + 2r + 2n - 2r + 2 \\ \Rightarrow \quad & n^2 - 4nr + 4r^2 = n + 2 \\ \Rightarrow \quad & (n-2r)^2 = n+2. \end{aligned}$$

(b) Since $(n-2r)^2$ is a square, $n+2$ must also be a square. Then $n = k^2 - 2$ for some positive integer $k \geq 2$. But for $k = 2$, $n = 2^2 - 2 = 2$, in which case $r = 0$ or $r = 2$, which is not allowed since r must satisfy $1 \leq r \leq n-1$. Hence, $n = k^2 - 2$ for some positive integer $k \geq 3$.

Substituting into the equation $(n-2r)^2 = n+2$, we get

$$(k^2 - 2 - 2r)^2 = k^2,$$

so

$$k^2 - 2 - 2r = k \quad \Rightarrow \quad r = \frac{(k-2)(k+1)}{2},$$

or

$$k^2 - 2 - 2r = -k \quad \Rightarrow \quad r = \frac{(k+2)(k-1)}{2}.$$

Therefore, all the pairs of positive integers (n, r) that satisfy the equation in part (a) are of the form

$$\left(k^2 - 2, \frac{(k-2)(k+1)}{2}\right) \quad \text{and} \quad \left(k^2 - 2, \frac{(k+2)(k-1)}{2}\right),$$

where k is a positive integer, $k \geq 3$.

3. Find all non-empty sets A of real numbers satisfying the following property: For all real numbers x and y, if $x + y \in A$, then $xy \in A$.

   Solution. Let A be a non-empty set with the given property, and let a be an element in A. Then $a + 0 = a \in A$, so $a \cdot 0 = 0 \in A$. Then for any real number x, $x + (-x) = 0 \in A$, so $x \cdot (-x) = -x^2 \in A$. Hence, all negative real numbers are in A.

   Let y be a positive real number. Then $(-y) + (-1) = -y - 1 < 0 \in A$, so $(-y) \cdot (-1) = y \in A$. Hence, all positive real numbers are in A.

   Therefore, the only non-empty set that satisfies the given property is the set of all real numbers.

4. Let
$$P(x) = 2x^{2010} + x^{2009} + 2x^{2008} + x^{2007} + \cdots + 2x^2 + x + 2.$$
Prove that if z is a complex number such that $P(z) = 0$, then $|z| = 1$.

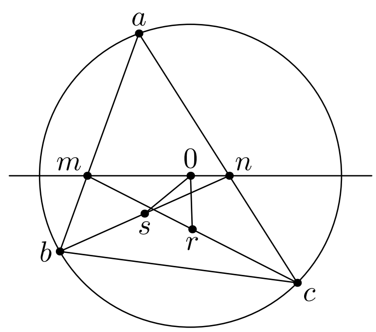
Solution. Note that
$$x^2 P(x) = 2x^{2012} + x^{2011} + 2x^{2010} + x^{2009} + \cdots + 2x^4 + x^3 + 2x^2,$$
so
$$\begin{aligned}(x^2 - 1)P(x) &= (2x^{2012} + x^{2011} + 2x^{2010} + x^{2009} + \cdots + 2x^4 + x^3 + 2x^2) \\ &\quad - (2x^{2010} + x^{2009} + 2x^{2008} + x^{2007} + \cdots + 2x^2 + x + 2) \\ &= 2x^{2012} + x^{2011} - x - 2.\end{aligned}$$
Let z be a complex number such that $P(z) = 0$. Then $2z^{2012} + z^{2011} - z - 2 = 0$, so
$$z^{2011}(2z + 1) = z + 2.$$
If $2z+1 = 0$, then $z = -1/2$ and $z^{2011}(2z+1) = 0$. But then $z+2 = 3/2$, contradiction, so $2z+1 \neq 0$, and we can divide both sides by $2z + 1$, to get
$$z^{2011} = \frac{z + 2}{2z + 1}.$$
Hence,
$$|z^{2011}| = \left|\frac{z + 2}{2z + 1}\right|,$$
or
$$|z|^{2011} = \frac{|z + 2|}{|2z + 1|},$$
so
$$|z|^{4022} = \frac{|z + 2|^2}{|2z + 1|^2}.$$
If $z = a + bi$, where a and b are real numbers, then
$$\begin{aligned}\frac{|z + 2|^2}{|2z + 1|^2} &= \frac{|a + 2 + bi|^2}{|2a + 1 + 2bi|^2} \\ &= \frac{(a + 2)^2 + b^2}{(2a + 1)^2 + (2b)^2} \\ &= \frac{(a^2 + b^2) + 4a + 4}{4(a^2 + b^2) + 4a + 1} \\ &= \frac{|z|^2 + 4a + 4}{4|z|^2 + 4a + 1},\end{aligned}$$

so
$$|z|^{4022} = \frac{|z|^2 + 4a + 4}{4|z|^2 + 4a + 1}.$$
If $|z| > 1$, then $|z|^{4022} > 1$. But $(4|z|^2 + 4a + 1) - (|z|^2 + 4a + 4) = 3(|z|^2 - 1) > 0$, so
$$\frac{|z|^2 + 4a + 4}{4|z|^2 + 4a + 1} < 1,$$
contradiction.

Similarly, if $|z| < 1$, then $|z|^{4022} < 1$. But $(|z|^2 + 4a + 4) - (4|z|^2 + 4a + 1) = 3(1 - |z|^2) > 0$, so
$$\frac{|z|^2 + 4a + 4}{4|z|^2 + 4a + 1} > 1,$$
again a contradiction. Therefore, z must satisfy $|z| = 1$.

5. Let O be the circumcenter of triangle ABC. A line through O intersects sides AB and AC at M and N, respectively. Let S and R be the midpoints of BN and CM, respectively. Prove that $\angle ROS = \angle BAC$.

Solution 1. We use complex numbers. Let a be the complex number corresponding to point A, and so on. Without loss of generality, assume that a, b, and c lie on the unit circle. Also, assume that the line through the center of the circle is the real axis, so in particular, m and n are real.



Since m lies on the line connecting a and b,

$$\frac{m-b}{a-b} = \overline{\frac{m-b}{a-b}} = \frac{\overline{m}-\overline{b}}{\overline{a}-\overline{b}}.$$

Since a and b lie on the unit circle, $\overline{a} = 1/a$ and $\overline{b} = 1/b$. Also, m is real, so $\overline{m} = m$. Hence,

$$\frac{m-b}{a-b} = \frac{m-1/b}{1/a-1/b}.$$

Solving for m, we find

$$m = \frac{a+b}{ab+1}.$$

Then

$$r = \frac{c+m}{2} = \frac{c+(a+b)/(ab+1)}{2} = \frac{abc+a+b+c}{2(ab+1)}.$$

Similarly,

$$n = \frac{a+c}{ac+1},$$

and

$$s = \frac{b+n}{2} = \frac{b+(a+c)/(ac+1)}{2} = \frac{abc+a+b+c}{2(ac+1)}.$$

We see that $\angle BAC$ is the argument of

$$\frac{c-a}{b-a},$$

and $\angle ROS$ is the argument of

$$\frac{r}{s} = \frac{ac+1}{ab+1}.$$

Hence, to show that $\angle BAC = \angle ROS$, it suffices to show that

$$z = \frac{(c-a)/(b-a)}{r/s} = \frac{(c-a)(ab+1)}{(b-a)(ac+1)}$$

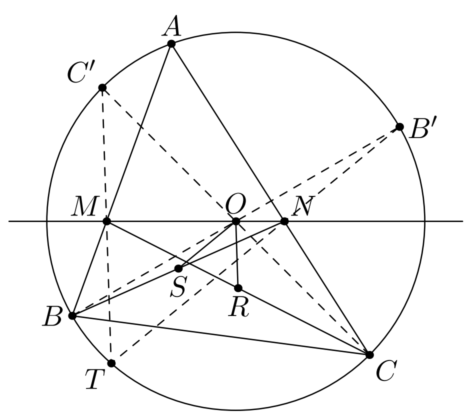
is real.

We have that

$$\begin{aligned}\overline{z} &= \frac{(\overline{c}-\overline{a})(\overline{a}\overline{b}+1)}{(\overline{b}-\overline{a})(\overline{a}\overline{c}+1)} \\ &= \frac{(1/c-1/a)(1/a\cdot 1/b+1)}{(1/b-1/a)(1/a\cdot 1/c+1)} \\ &= \frac{(a-c)(1+ab)}{(a-b)(1+ac)} \\ &= z.\end{aligned}$$

Since z is equal to its own conjugate, z is real. Therefore, $\angle ROS = \angle BAC$.

Solution 2. Let B' and C' be the points on the circumcircle diametrically opposite points B and C, respectively. We see that M lies on AB, N lies on AC, and $O = BB' \cap CC'$. Since M, O, and N are collinear, by Pascal's theorem, $C'M$ and $B'N$ concur at a point T on the circumcircle.



Since O is the midpoint of BB', and S is the midpoint of BN, OS is parallel to $B'N$. Similarly, OR is parallel to $C'M$. Hence, $\angle ROS = \angle B'TC'$. But arcs BC and $B'C'$ are equal, so $\angle ROS = \angle B'TC' = \angle BAC$.

6. Let $n \geq 4$ be a given integer. For every integer $m \geq 0$, let S_m denote the set $\{m+1, m+2, \dots, m+n\}$. Let $f(n)$ be the smallest positive integer such that for every m and for every set $T \subseteq S_m$ with $|T| = f(n)$, T contains at least three (distinct) pairwise relatively prime elements. Determine $f(n)$.

Solution. Let us call a set *good* if it contains three (distinct) elements that are pairwise relatively prime.

First, we give a lower bound for $f(n)$. For $n = 4$, the subset $T = \{1, 2, 4\}$ of $S_0 = \{1, 2, 3, 4\}$ shows that $f(4) \geq 4$. For $n = 5$, the subset $T = \{2, 3, 4, 6\}$ of $S_1 = \{2, 3, 4, 5, 6\}$ shows that $f(5) \geq 5$. Assume that $n \geq 6$.

Consider the set $S_1 = \{2, 3, \dots, n+1\}$, and let T be the set of all elements in S_1 that are multiples of either 2 or 3 (or both). By the Pigeonhole Principle, for any three elements in T, two of them must have a common factor of either 2 or 3, so the set T is not good, which means $f(n) \geq |T| + 1$. By the Principle of Inclusion-Exclusion,

$$|T| = \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor - \left\lfloor \frac{n+1}{6} \right\rfloor,$$

so

$$f(n) \geq \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor - \left\lfloor \frac{n+1}{6} \right\rfloor + 1. \qquad (*)$$

This formula is consistent with the lower bounds $f(4) \geq 4$ and $f(5) \geq 5$ that we found above. We claim that this lower bound is in fact the exact value of $f(n)$. We will establish this with a series of results.

First, we claim that

$$f(n) \leq n$$

for all $n \geq 4$. Since $n \geq 4$, we know that $m+1$, $m+2$, $m+3$, and $m+4$ are distinct elements in S_m. If m is even, then the three elements $m+1$, $m+2$, $m+3$ are pairwise relatively prime, and if m is odd, then the three elements $m+2$, $m+3$, $m+4$ are pairwise relatively prime. Hence, the n-element set S_m is good for all $m \geq 0$, which shows that $f(n) \leq n$. It follows that $f(4) = 4$ and $f(5) = 5$.

Next, we claim that

$$f(n+1) \leq f(n) + 1 \qquad (**)$$

for all $n \geq 4$. To prove this, let T be a subset of $\{m+1, m+2, \dots, m+n+1\}$ containing $f(n)+1$ elements. If T does not contain the element $m+n+1$, then T is a subset of $\{m+1, m+2, \dots, m+n\}$ containing $f(n)+1$ elements, so the set T is good. If T does contain the element $m+n+1$, then let $T' = T \setminus \{m+n+1\}$. Then T' is a subset of $\{m+1, m+2, \dots, m+n\}$ containing $f(n)$ elements, so the set T' is good, which means that the set T is good. Hence, $f(n+1) \leq f(n) + 1$.

From $(*)$, $f(6) \geq 5$. We claim that $f(6) = 5$. It suffices to show that any 5-element subset T of a set of six consecutive positive integers is good. Among these six positive integers, there are three odd consecutive positive integers (which are pairwise relatively prime), and three even consecutive positive integers. If T contains all three odd positive integers, then T is good. Otherwise, T must contain all the even positive integers, and two of the three odd positive integers. If the two odd positive integers in T are consecutive (of the form $2x+1$ and $2x+3$), then T contains the three elements $2x+1$, $2x+2$,

$2x+3$, which are pairwise relatively prime, so T is good. Otherwise, the two odd positive integers in T are of the form $2x+1$ and $2x+5$. At least one of these elements is not divisible by 3. If $2x+1$ is not divisible by 3, then the three elements $2x+1$, $2x+4$, $2x+5$ are pairwise relatively prime. If $2x+5$ is not divisible by 3, then the three elements $2x+1$, $2x+2$, $2x+5$ are pairwise relatively prime. Hence, T is good, so $f(6) = 5$.

Then from $(**)$, $f(7) \le 6$, $f(8) \le 7$, and $f(9) \le 8$. But from $(*)$, $f(7) \ge 6$, $f(8) \ge 7$, and $f(9) \ge 8$. Therefore, $f(7) = 6$, $f(8) = 7$, and $f(9) = 8$.

We now prove that

$$f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor - \left\lfloor \frac{n+1}{6} \right\rfloor + 1$$

by induction on n. We have already shown that the result holds for $4 \le n \le 9$, so assume that the result holds for all positive integers $n = 4, 5, \dots, k$, where $k \ge 9$.

For $n = k+1$, note that

$$\begin{aligned} S_m &= \{m+1, m+2, \dots, m+k+1\} \\ &= \{m+1, m+2, \dots, m+k-5\} \cup \{m+k-4, \dots, m+k+1\}. \end{aligned}$$

Let T be a subset of $\{m+1, m+2, \dots, m+k+1\}$ containing $f(k-5) + f(6) - 1$ elements, and let

$$\begin{aligned} T_1 &= T \cap \{m+1, m+2, \dots, m+k-5\}, \\ T_2 &= T \cap \{m+k-4, \dots, m+k+1\}, \end{aligned}$$

so $T = T_1 \cup T_2$. Then either T_1 contains at least $f(k-5)$ elements, in which case T_1 is good, or T_2 contains at least $f(6)$ elements, in which case T_2 is good, so T is good. (If neither were true, then $T = T_1 \cup T_2$ would contain at most $f(k-5) - 1 + f(6) - 1 = f(k-5) + f(6) - 2$ elements.) Therefore,

$$f(k+1) \le f(k-5) + f(6) - 1.$$

Then by the induction hypothesis,

$$\begin{aligned} f(k+1) &\le f(k-5) + f(6) - 1 \\ &= \left\lfloor \frac{k-4}{2} \right\rfloor + \left\lfloor \frac{k-4}{3} \right\rfloor - \left\lfloor \frac{k-4}{6} \right\rfloor + 5 \\ &= \left\lfloor \frac{(k+2)-6}{2} \right\rfloor + \left\lfloor \frac{(k+2)-6}{3} \right\rfloor - \left\lfloor \frac{(k+2)-6}{6} \right\rfloor + 5 \\ &= \left\lfloor \frac{k+2}{2} \right\rfloor - 3 + \left\lfloor \frac{k+2}{3} \right\rfloor - 2 - \left\lfloor \frac{k+2}{6} \right\rfloor + 1 + 5 \\ &= \left\lfloor \frac{k+2}{2} \right\rfloor + \left\lfloor \frac{k+2}{3} \right\rfloor - \left\lfloor \frac{k+2}{6} \right\rfloor + 1. \end{aligned}$$

This bound, with $(*)$, shows that the result holds for $n = k+1$. By induction, the result holds for all positive integers $n \ge 4$.

7. Let a, b, c, and d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{a}{\sqrt{15+a^2}} + \frac{b}{\sqrt{15+b^2}} + \frac{c}{\sqrt{15+c^2}} + \frac{d}{\sqrt{15+d^2}} \geq 1.$$

Solution 1. Let

$$w = \frac{a}{\sqrt{15+a^2}}, \quad x = \frac{b}{\sqrt{15+b^2}}, \quad y = \frac{c}{\sqrt{15+c^2}}, \quad z = \frac{d}{\sqrt{15+d^2}}.$$

Then

$$w^2 = \frac{a^2}{15+a^2} \quad \Rightarrow \quad a^2 = \frac{15w^2}{1-w^2}.$$

Similarly,

$$b^2 = \frac{15x^2}{1-x^2}, \quad c^2 = \frac{15y^2}{1-y^2}, \quad d^2 = \frac{15y^2}{1-z^2}.$$

Since $a^2b^2c^2d^2 = 1$, we get

$$\begin{aligned} &\frac{15w^2}{1-w^2} \cdot \frac{15x^2}{1-x^2} \cdot \frac{15y^2}{1-y^2} \cdot \frac{15z^2}{1-z^2} = 1 \\ &\Rightarrow \quad 15^4 w^2x^2y^2z^2 = (1-w^2)(1-x^2)(1-y^2)(1-z^2). \end{aligned} \tag{$*$}$$

For the sake of contradiction, suppose that

$$w + x + y + z < 1, \tag{$**$}$$

which means w, x, y, $z < 1$, so $1 - w$ is positive. Now by $(**)$ and AM-GM,

$$\begin{aligned} 1 - w^2 &= (1+w)(1-w) \\ &> (w+w+x+y+z)(x+y+z) \\ &\geq 5(w^2xyz)^{1/5} \cdot 3(xyz)^{1/3} \\ &= 15(w^2xyz)^{1/5} \cdot (xyz)^{1/3}. \end{aligned}$$

Similarly,

$$\begin{aligned} 1 - x^2 &> 15(wx^2yz)^{1/5} \cdot (wyz)^{1/3}, \\ 1 - y^2 &> 15(wxy^2z)^{1/5} \cdot (wxz)^{1/3}, \\ 1 - z^2 &> 15(wxy^2z)^{1/5} \cdot (wxz)^{1/3}. \end{aligned}$$

Hence,

$$\begin{aligned} (1-w^2)(1-x^2)(1-y^2)(1-z^2) &> 15(w^2xyz)^{1/5} \cdot (xyz)^{1/3} \cdot 15(wx^2yz)^{1/5} \cdot (wyz)^{1/3} \\ &\quad \cdot 15(wxy^2z)^{1/5} \cdot (wxz)^{1/3} \cdot 15(wxy^2z)^{1/5} \cdot (wxz)^{1/3} \\ &= 15^4 w^2x^2y^2z^2. \end{aligned}$$

But this contradicts $(*)$, so $(**)$ is false, which means

$$w+x+y+z = \frac{a}{\sqrt{15+a^2}} + \frac{b}{\sqrt{15+b^2}} + \frac{c}{\sqrt{15+c^2}} + \frac{d}{\sqrt{15+d^2}} \geq 1.$$

Solution 2. By AM-GM,

$$a^2+15 = a^2 + \underbrace{1+1+\cdots+1}_{15\ 1\text{'s}} \geq 16\sqrt[16]{a^2} = 16a^{1/8},$$

and

$$\begin{aligned} a+\sqrt{a^2+15} &\geq a+4a^{1/16} \\ &= a+a^{1/16}+a^{1/16}+a^{1/16}+a^{1/16} \\ &\geq 5\sqrt[5]{a\cdot a^{1/16}\cdot a^{1/16}\cdot a^{1/16}\cdot a^{1/16}} \\ &= 5a^{1/4}. \end{aligned}$$

It follows that

$$\frac{a}{\sqrt{a^2+15}} = \frac{a}{a+\frac{15}{a+\sqrt{a^2+15}}} \geq \frac{a}{a+\frac{15}{5a^{1/4}}} = \frac{a}{a+3a^{-1/4}}.$$

Now, since $abcd=1$,

$$\frac{a}{a+3a^{-1/4}} = \frac{a^{15/16}}{a^{15/16}+3a^{-5/16}} = \frac{a^{15/16}}{a^{15/16}+3b^{5/16}c^{5/16}d^{5/16}}.$$

By AM-GM,

$$b^{15/16}+c^{15/16}+d^{15/16} \geq 3\sqrt[3]{b^{15/16}c^{15/16}d^{15/16}} = 3b^{5/16}c^{5/16}d^{5/16},$$

so

$$\frac{a^{15/16}}{a^{15/16}+3b^{5/16}c^{5/16}d^{5/16}} \geq \frac{a^{15/16}}{a^{15/16}+b^{15/16}+c^{15/16}+d^{15/16}}.$$

Therefore,

$$\frac{a}{\sqrt{15+a^2}} \geq \frac{a^{15/16}}{a^{15/16}+b^{15/16}+c^{15/16}+d^{15/16}}.$$

Similarly,

$$\begin{aligned} \frac{b}{\sqrt{15+b^2}} &\geq \frac{b^{15/16}}{a^{15/16}+b^{15/16}+c^{15/16}+d^{15/16}}, \\ \frac{c}{\sqrt{15+c^2}} &\geq \frac{c^{15/16}}{a^{15/16}+b^{15/16}+c^{15/16}+d^{15/16}}, \\ \frac{d}{\sqrt{15+d^2}} &\geq \frac{d^{15/16}}{a^{15/16}+b^{15/16}+c^{15/16}+d^{15/16}}. \end{aligned}$$

Adding, we get

$$\frac{a}{\sqrt{15+a^2}} + \frac{b}{\sqrt{15+b^2}} + \frac{c}{\sqrt{15+c^2}} + \frac{d}{\sqrt{15+d^2}} \geq \frac{a^{15/16}+b^{15/16}+c^{15/16}+d^{15/16}}{a^{15/16}+b^{15/16}+c^{15/16}+d^{15/16}} = 1,$$

as desired.

Solution 3. The function $f(x) = 1/\sqrt{x}$ is convex, so by Jensen's inequality,

$$\begin{aligned}
&\frac{af(15+a^2)+bf(15+b^2)+cf(15+c^2)+df(15+d^2)}{a+b+c+d} \\
&\geq f\left(\frac{a(15+a^2)+b(15+b^2)+c(15+c^2)+d(15+d^2)}{a+b+c+d}\right) \\
&= \sqrt{\frac{a+b+c+d}{a^3+b^3+c^3+d^3+15(a+b+c+d)}},
\end{aligned}$$

so

$$\frac{a}{\sqrt{15+a^2}} + \frac{b}{\sqrt{15+b^2}} + \frac{c}{\sqrt{15+c^2}} + \frac{d}{\sqrt{15+d^2}} \geq \sqrt{\frac{(a+b+c+d)^3}{a^3+b^3+c^3+d^3+15(a+b+c+d)}}.$$

Hence, it suffices to show that

$$(a+b+c+d)^3 \geq a^3+b^3+c^3+d^3+15(a+b+c+d).$$

Expanding and homogenizing with the constraint $abcd = 1$, this inequality becomes

$$3\sum_{\text{sym}} a^2b + 2\sum_{\text{sym}} abc \geq 5\sum_{\text{sym}} a^{3/2}b^{1/2}c^{1/2}d^{1/2}.$$

By the AM-GM inequality,

$$\begin{aligned}
a^2b + acd &\geq 2a^{3/2}b^{1/2}c^{1/2}d^{1/2}, \\
a^2c + abd &\geq 2a^{3/2}b^{1/2}c^{1/2}d^{1/2}, \\
a^2d + abc &\geq 2a^{3/2}b^{1/2}c^{1/2}d^{1/2},
\end{aligned}$$

and so on. Adding, we get

$$\sum_{\text{sym}} a^2b + \sum_{\text{sym}} abc \geq 2\sum_{\text{sym}} a^{3/2}b^{1/2}c^{1/2}d^{1/2},$$

so

$$2\sum_{\text{sym}} a^2b + 2\sum_{\text{sym}} abc \geq 4\sum_{\text{sym}} a^{3/2}b^{1/2}c^{1/2}d^{1/2}. \tag{1}$$

By Muirhead's inequality,

$$\sum_{\text{sym}} a^2 b \ge \sum_{\text{sym}} a^{3/2} b^{1/2} c^{1/2} d^{1/2}. \tag{2}$$

Adding (1) and (2) gives

$$3 \sum_{\text{sym}} a^2 b + 2 \sum_{\text{sym}} abc \ge 5 \sum_{\text{sym}} a^{3/2} b^{1/2} c^{1/2} d^{1/2},$$

as desired.

Solution 4. By Hölder's inequality,

$$\left(\sum_{\text{cyclic}} \frac{a}{\sqrt{15 + a^2}} \right)^2 \left(\sum_{\text{cyclic}} a(15 + a^2) \right) \ge (a + b + c + d)^3,$$

so it suffices to prove that

$$(a + b + c + d)^3 \ge a^3 + b^3 + c^3 + d^3 + 15(a + b + c + d).$$

Define

$$f(a, b, c, d) = (a + b + c + d)^3 - a^3 - b^3 - c^3 - d^3 - 15(a + b + c + d).$$

Then

$$\begin{aligned}
&f(a, b, c, d) - f(a, b, \sqrt{cd}, \sqrt{cd}) \\
&= (a + b + c + d)^3 - a^3 - b^3 - c^3 - d^3 - 15(a + b + c + d) \\
&\quad - (a + b + \sqrt{cd} + \sqrt{cd})^3 + a^3 + b^3 + (\sqrt{cd})^3 + (\sqrt{cd})^3 + 15(a + b + \sqrt{cd} + \sqrt{cd}) \\
&= 3a^2c + 3ac^2 + 3a^2d + 3ad^2 + 3b^2c + 3bc^2 + 3b^2d + 3bd^2 + 3c^2d + 3cd^2 \\
&\quad + 6abc + 6abd - 6acd - 6bcd - 6a^2\sqrt{cd} - 6b^2\sqrt{cd} - 12ab\sqrt{cd} - 6cd\sqrt{cd} - 15c - 15d + 30\sqrt{cd} \\
&= (3a^2c - 6a^2\sqrt{cd} + 3a^2d) + (6abc - 12ab\sqrt{cd} + 6abd) + (3b^2c - 6b^2\sqrt{cd} + 3b^2d) \\
&\quad + (3ac^2 - 6acd + 3ad^2) + (3bc^2 - 6bcd + 3bd^2) + (3c^2d - 6cd\sqrt{cd} + 3cd^2) - (15c - 30\sqrt{cd} + 15d) \\
&= 3a^2(\sqrt{c} - \sqrt{d})^2 + 6ab(\sqrt{c} - \sqrt{d})^2 + 3b^2(\sqrt{c} - \sqrt{d})^2 \\
&\quad + 3a(c - d)^2 + 3b(c - d)^2 + 3cd(\sqrt{c} - \sqrt{d})^2 - 15(\sqrt{c} - \sqrt{d})^2 \\
&= (3a^2 + 6ab + 3b^2)(\sqrt{c} - \sqrt{d})^2 + 3(a + b)(c - d)^2 + 3cd(\sqrt{c} - \sqrt{d})^2 - 15(\sqrt{c} - \sqrt{d})^2 \\
&= 3(a + b)^2(\sqrt{c} - \sqrt{d})^2 + 3(a + b)(c - d)^2 + 3cd(\sqrt{c} - \sqrt{d})^2 - 15(\sqrt{c} - \sqrt{d})^2 \\
&= 3(\sqrt{c} - \sqrt{d})^2[(a + b)^2 + (a + b)(\sqrt{c} + \sqrt{d})^2 + cd - 5].
\end{aligned}$$

By AM-GM,

$$\begin{aligned}(a+b)^2 + (a+b)(\sqrt{c}+\sqrt{d})^2 + cd - 5 &\geq (a+b)(\sqrt{c}+\sqrt{d})^2 - 5 \\ &\geq 2\sqrt{ab}\cdot 4\sqrt{cd} - 5 \\ &= 8-5 \\ &= 3.\end{aligned}$$

We conclude that $f(a,b,c,d) \geq f(a,b,\sqrt{cd},\sqrt{cd})$ for all positive real numbers a, b, c, d such that $abcd = 1$. Furthermore, the function f is symmetric. Hence,

$$\begin{aligned} f(a,b,c,d) &\geq f(a,b,\sqrt{cd},\sqrt{cd}) \\ &= f(\sqrt{cd},\sqrt{cd},a,b) \\ &\geq f(\sqrt{cd},\sqrt{cd},\sqrt{ab},\sqrt{ab}) \\ &= f(\sqrt{ab},\sqrt{cd},\sqrt{ab},\sqrt{cd}) \\ &\geq f(\sqrt{ab},\sqrt{cd},\sqrt[4]{abcd},\sqrt[4]{abcd}) \\ &= f(\sqrt[4]{abcd},\sqrt[4]{abcd},\sqrt{ab},\sqrt{cd}) \\ &\geq f(\sqrt[4]{abcd},\sqrt[4]{abcd},\sqrt[4]{abcd},\sqrt[4]{abcd}) \\ &= f(1,1,1,1) = 0, \end{aligned}$$

as desired.