Exam on Electrostatics

2 hours :: 25 marks

March 27, 2014

Problem 1: KVL

Derive Kirchoff's voltage law (which states that the algebraic sum of voltage rise and voltage drops in a closesd circuit is zero) from one of the Maxwell's equations. Just give arguments.

(2 marks)

Problem 2: Cylindrical as if parallel plates

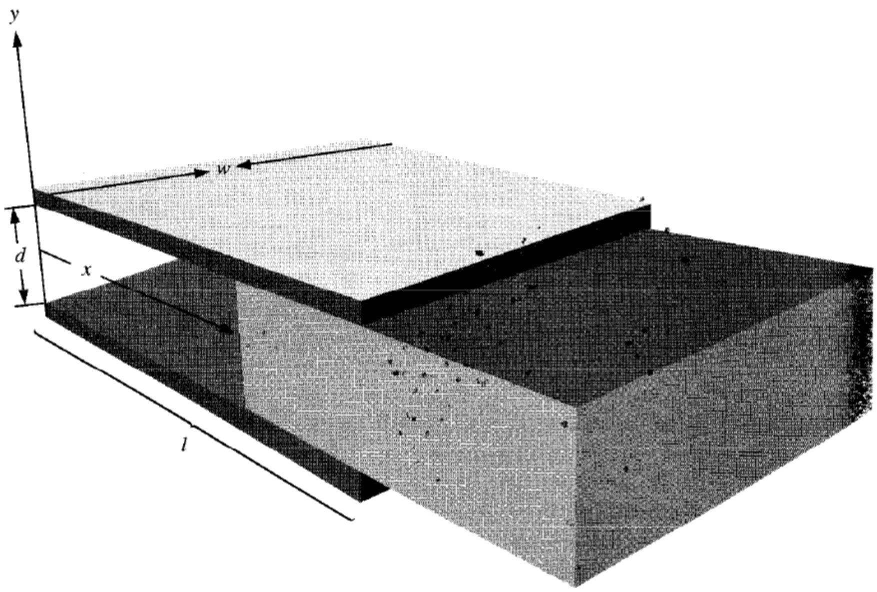
A long cylindrical capacitor is given (inner radius a, outer radius b) with a dielectric inside. Its dielectric constant varies as

$$k = \frac{T}{r}$$

where r is the perpendicular distance from the common axis. Find the capacitance per unit length.

(3 marks)

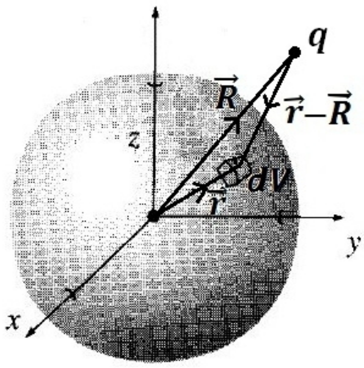
Problem 3: Dielectric oscillation



A parallel plate capacitor (plate separation d) is charged with a dielectric slab (mass m, dielectric constant k) perfectly inside it so that each plate (length l, width w) contains charge of magnitude Q. The battery is then disconnected and a tiny nudge is applied on the slab (the figure is exaggerated). Find the oscillation frequency.

(4 marks)

Problem 4: Averages



Show that the average electric field over the volume of a sphere due to all charges outside is the same as the field they would produce at the center. Do this by the following steps:

(a) Take the co-ordinate system so that the sphere, say of radius a, is centered at the origin. Now show that the average field due to a single point charge q at a point $\vec{R}$ outside the sphere ($R > a$) is the same as the field that the sphere would produce at $\vec{R}$ had it been uniformly charged with volume charge density

$$\rho = -\frac{q}{\frac{4}{3}\pi a^3}.$$

(2 marks)

(b) Now use the Gauss' law to complete the result for a single point charge.

(1 mark)

(c) Give arguments how you can extend the proof for an arbitrary charge distribution that lie outside the sphere.

(1 mark)

(d) While you are at it, show similarly that the average field due to all charges within the sphere is given by

$$\vec{E}_{ave} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{a^3}$$

where $\vec{p} = \int_{sphere} \rho\vec{r}\, dV$ is the total electric dipole moment for the charges located inside the sphere with respect to the co-ordinate system we chose in part (a).

(2 marks)

Problem 5: Images

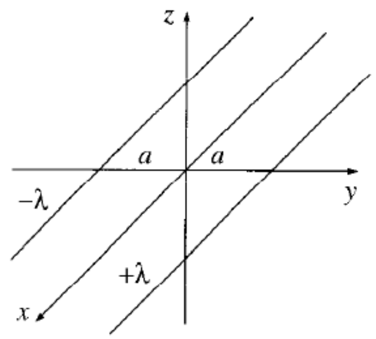


Figure 1: Two parallel wires

Two infinitely long wires running parallel to the x axis carry uniform charge densities $+\lambda$ and $-\lambda$ (Fig. 1).

(a) Find the potential at any point (x, y, z), using origin as your reference.

(2 marks)

(b) Show that the equipotential surfaces are circular cylinders, and locate the axis and radius of the cylinder corresponding to a given potential V_0.

(3 marks)

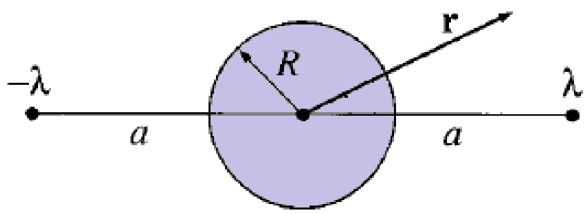


Figure 2: Two parallel wires with a conductor between them

(c) Now a long conducting cylinder is placed between them in parallel (Fig. 2). The conductor is not grounded, carries no net charge and has a radius R. Find the potential $V(\vec{r})$ now.

(5 marks)

Solution:

Problem 1: KVL

$\oint \vec{E}.\vec{dl} = 0$ means sum of voltage drops and rises are zero. In case there is a change in flux we introduce it as a back emf of magnitude $N\dfrac{d\phi}{dt} = L\dfrac{di}{dt}$ so as to make the KVL still valid.

Problem 2: Cylindrical as if parallel plates

- Taking a cylindrical gaussian surface of radius r $(a < r < b)$ and length l we get

  $$\oint \vec{D}.d\vec{A} = \oint \epsilon\vec{E}.d\vec{A} = \epsilon_0 kE \times 2\pi rl = \epsilon_0 \frac{T}{r} E \times 2\pi rl = q_{free} = 2\pi al\sigma$$

  or,

  $$E = \frac{a\sigma}{T\epsilon_0}$$

  Note that the $\vec{E}$ is uniform !

- Find the potential

  $$V = E(b - a)$$

- And then the capacitance per unit length

  $$c = \frac{C}{l} = \frac{Q}{Vl} = \frac{2\pi T\epsilon_0}{b - a}.$$

Problem 3: Dielectric oscillation

- Find the capacitance first in terms of the displacement:

  $$C = \frac{\epsilon_0 w}{d} \left(|x|(1 - k) + kl\right)$$

- Use $U = \dfrac{Q^2}{2C}$ to express U in terms of x and find

  $$\left.\frac{d^2U}{dx^2}\right|_{x=0} = \frac{Q^2 d(1-k)^2}{\epsilon_0 w(kl)^3} = K, \text{ say}$$

- And now it's pretty straight forward to find the oscillation frequency

  $$\omega = \sqrt{\frac{K}{m}}$$

Problem 4: Averages

(a)

$$\vec{E}_{ave} = \frac{1}{\frac{4}{3}\pi a^3}\int \vec{E}\, dV = \frac{1}{\frac{4}{3}\pi a^3}\int \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{R}|^3}(\vec{r}-\vec{R})dV = \int \frac{1}{4\pi\epsilon_0}\rho\frac{\vec{R}-\vec{r}}{|\vec{R}-\vec{r}|^3}dV = \vec{E}_{sphere}.$$

(b) Now according Gauss' law, the whole sphere acts like a point charge to the point $\vec{R}$. Therefore—

$$\vec{E}_{sphere} = \frac{1}{4\pi\epsilon_0}\frac{\int\limits_{0<r<a} \rho\, dV}{R^3}\vec{R} = \frac{1}{4\pi\epsilon_0}\frac{-q}{R^3}(\vec{R}) = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}(-\vec{R}) = \vec{E}_{center}.$$

(c) Using superposition, the above proposition can easily be extended for any arbitrary charge distribution exterior to the sphere.

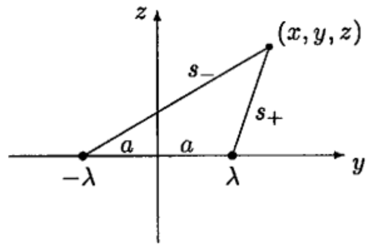
(d) From part (a) we still have $\vec{E}_{ave} = \vec{E}_{sphere}$. But now, using Gauss' law we have—

$$\vec{E}_{sphere} = \frac{1}{4\pi\epsilon_0}\frac{\int\limits_{0<r<R} \rho\, dV}{R^3}\vec{R} = \frac{1}{4\pi\epsilon_0}\frac{-q\frac{R^3}{a^3}}{R^3}\vec{R} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{a^3};$$

where $\vec{p} = q\vec{R}$. Using superposition principle, the above claim can be easily generalized.

Problem 5: Images

(a) Potential of $+\lambda$ is $V_+ = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_+}{a}\right)$, where s_+ is distance from λ_+ (Prob. 2.22).
Potential of $-\lambda$ is $V_- = +\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{a}\right)$, where s_- is distance from λ_-.



$\therefore$ Total $\boxed{V = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{s_+}\right).}$

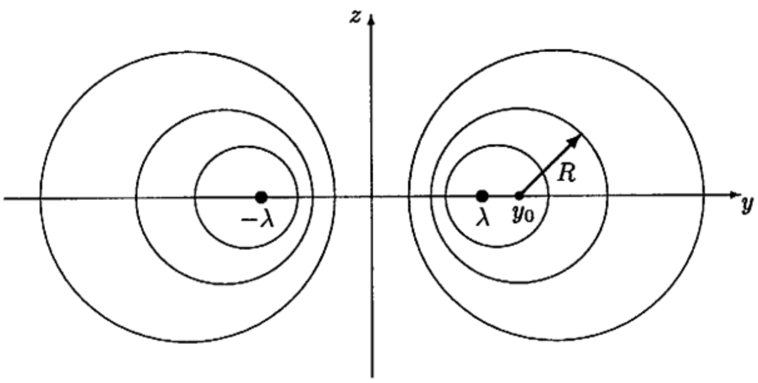
Now $s_+ = \sqrt{(y-a)^2+z^2}$, and $s_- = \sqrt{(y+a)^2+z^2}$, so

$V(x,y,z) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\sqrt{(y+a)^2+z^2}}{\sqrt{(y-a)^2+z^2}}\right) = \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right].}$

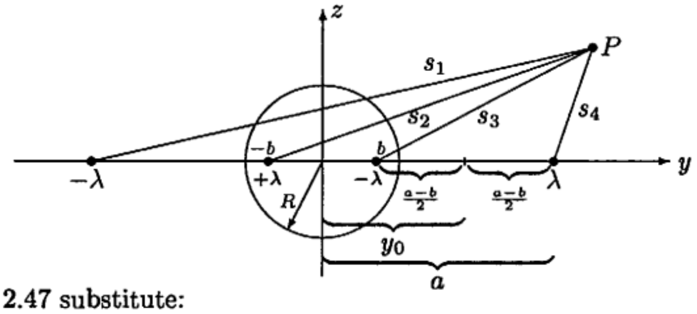
(b) Equipotentials are given by $\frac{(y+a)^2+z^2}{(y-a)^2+z^2} = e^{(4\pi\epsilon_0 V_0/\lambda)} = k =$ constant. That is:
$y^2+2ay+a^2+z^2 = k(y^2-2ay+a^2+z^2) \Rightarrow y^2(k-1)+z^2(k-1)+a^2(k-1)-2ay(k+1) = 0$, or
$y^2+z^2+a^2-2ay\left(\frac{k+1}{k-1}\right) = 0$. The equation for a *circle*, with center at $(y_0, 0)$ and radius R, is
$(y-y_0)^2+z^2 = R^2$, or $y^2+z^2+(y_0^2-R^2)-2yy_0 = 0$.
Evidently the equipotentials *are* circles, with $y_0 = a\left(\frac{k+1}{k-1}\right)$ and
$a^2 = y_0^2 - R^2 \Rightarrow R^2 = y_0^2 - a^2 = a^2\left(\frac{k+1}{k-1}\right)^2 - a^2 = a^2\frac{(k^2+2k+1-k^2+2k-1)}{(k-1)^2} = a^2\frac{4k}{(k-1)^2}$, or
$R = \frac{2a\sqrt{k}}{|k-1|}$; or, in terms of V_0:

$$y_0 = a\frac{e^{4\pi\epsilon_0 V_0/\lambda}+1}{e^{4\pi\epsilon_0 V_0/\lambda}-1} = a\frac{e^{2\pi\epsilon_0 V_0/\lambda}+e^{-2\pi\epsilon_0 V_0/\lambda}}{e^{2\pi\epsilon_0 V_0/\lambda}-e^{-2\pi\epsilon_0 V_0/\lambda}} = \boxed{a\coth\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right).}$$

$$R = 2a\frac{e^{2\pi\epsilon_0 V_0/\lambda}}{e^{4\pi\epsilon_0 V_0/\lambda}-1} = a\frac{2}{(e^{2\pi\epsilon_0 V_0/\lambda}-e^{-2\pi\epsilon_0 V_0/\lambda})} = \frac{a}{\sinh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)} = \boxed{a\,\mathrm{csch}\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right).}$$



(c) we place image line charges $-\lambda$ at $y = b$ and $+\lambda$ at $y = -b$ (here y is the horizontal axis, z vertical).



In the solution to Prob. 2.47 substitute:

$$a \to \frac{a-b}{2},\ y_0 \to \frac{a+b}{2} \text{ so } \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2 - R^2 \Rightarrow b = \frac{R^2}{a}.$$

$$\begin{aligned}
V &= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{s_3^2}{s_4^2}\right) + \ln\left(\frac{s_1^2}{s_2^2}\right)\right] = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_1^2 s_3^2}{s_4^2 s_2^2}\right) \\
&= \frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{[(y+a)^2+z^2][(y-b)^2+z^2]}{[(y-a)^2+z^2][(y+b)^2+z^2]}\right\}, \quad \text{or, using } y = s\cos\phi,\ z = s\sin\phi, \\
&= \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{(s^2+a^2+2as\cos\phi)[(as/R)^2+R^2-2as\cos\phi]}{(a^2+a^2-2as\cos\phi)[(as/R)^2+R^2+2as\cos\phi]}\right\}}.
\end{aligned}$$