2010 BdMO Summer Math Camp Combinatorics Exercises and Hints

1. **(IMO 2009: Shortlist C1)**: Consider 2009 cards, each having one gold side and one black side, lying in parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now shows black and vice versa. The last player who can make a legal move wins. (A) Does the game necessarily end? (B) Does there exist a winning strategy for the starting player?

   Hint: (a) Let black = 0, gold = 1, does each move increase/decrease, etc this binary number? (b) Label the cards from right to left by 1,...,2009. Let S set of cards with labels $50i$ $(i = 1, \ldots, 40)$. Let g_n be the number of cards from S showing gold after n moves. What is g_0? If the game goes on what is the relation between g_n and g_{n+1}? Thus after how many moves would a non-starting player find a card from S showing gold?

2. **(Singapore Mathematical Olympiad 2008, Round 2)**: There are 11 committees in a club. Each committee has 5 members and every two committees have a member in common. Show that there is a member who belongs to 4 committee.

   Hint: Form an incidence matrix A where the rows are indexed by the committees and the columns are indexed by the members where the (i, j) entry is $a_{ij} = 1$ if member j is in committee i. There there are five 1's in A. Show there is a column with four 1's.

3. **(IMO 2005: Shortlist C2)**: Let k be a fixed positive integer. A company has a special method to sell sombreros. Each customer can convince two persons to a buy a sombrero after he/she buys one; convincing someone already convinced does not count. Each of these new customers can convince two others and so on. If each one of the two customers convinced by someone makes at least k people buy sombreros (directly or indirectly), then that someone wins a free instructional video. Prove that if n persons bought sombreros, then at most $n/(k+2)$ of them got videos.

   Solution: Let v be the number of video winners. One easily nds that for $v = 1$ and $v = 2$, the number n of customers is at least $2k + 3$ and $3k + 5$ respectively. We prove by induction on v that if $n \geq k + 1$ then $n \geq (k + 2)(v + 1) - 1$. We can assume w.l.o.g. that the total number n of customers is minimum possible for given $v > 0$. Consider a person P who was convinced by nobody but himself. Then P must have won a video; otherwise P could be removed from the group without decreasing the number of video winners. Let Q and R be the two persons convinced by P. We denote by C the set of persons made by P through Q to buy a sombrero, including Q, and by D the set of all other customers excluding P. Let x be the number of video winners in C . Then there are $v - x - 1$ video winners in D . We have $|C| \geq (k + 2)(x + 1)?1$, by induction hypothesis if $x > 0$ and because P is a winner if $x = 0$. Similarly, $|D| \geq (k+2)(v-x)?1$. Thus $n \geq 1 + (k + 2)(x + 1) - 1 + (k + 2)(v - x)?1$, i.e. $n \geq (k + 2)(v + 1) - 1$.

4. **(MOP 2006)**: Let M be a subset of the set $S = \{1, 2, ..., n\}$ with $m \geq 3$ elements. Prove that there are sets $A \subseteq M$ and $B = \{b, c\} \in S$ such that A has at least $\frac{m(m-1)(m-2)}{3(n-1)(n-2)}$ elements and that for each element a in A, all three numbers $a, a + b$ and $a + c$ belong to M.

   Hint: Given some subset of the first n natural numbers pick 2 numbers and try to find a set of the given size such that for each element picked, if you add one the 2 numbers it is still in M.

A nicer way to write the size of the set is $\binom{m}{3}/\binom{n-1}{2}$, or since the the answer must be an integer, the ceiling of that. This looks like Pigeonhole Bhai. Now try some examples for small values of m and n. There are $\binom{m}{3}$ ways to pick triples, How many possible sets B are there? Can you partition the triples such that it is a possible choice of B?

5. (**Italy TST 2009, Day 1**): Let n and k be integers such that $n \geq k \geq 1$. There are n light bulbs placed in a circle. They are all turned off. Each turn, you change the state of any set of k consecutive light bulbs. How many of the 2^n possible combinations can be reached if: (a) if k is an odd prime; (b) if k is an odd integer; (c) if k is an even integer?

   Hint: Try a number of examples with small n and k. Find the pattern, and form a induction hypothesis. Then prove the induction hypothesis.

6. Prove the following well known result: If more than half of the subsets of an n-element set are selected, then some two of the selected subsets have the property that one is a subset of the other. (Hint, use the pigeon hole principle.)

   Hint: Look up Sperner's Theorem on the Cut the Knot website.

7. Now prove the stronger version of the previous theorem which shows that one doesn't need to take half of all the subsets for one subset to contain some other subset. *Sperner's Theorem; Let E be a set with n elements, and let S be a set of its subsets such that A is not contained in B for each subset A and B in S. Then $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$.*

   Hint: The proof is on many websites. Think of counting in 2 ways.

8. Let $I_1, ..., I_n$ be n closed intervals of the real numbers such that among k of them there are 2 with nonempty intersection. Prove that one can choose $k-1$ points on the real line such that any of the intervals contains at least one of the chosen points.

9. **(International Zhautykov Olympiad in Mathematics 2007)**: We have 111 coins. They are to be placed in unit cells of an $n \times n$ table such that the number of coins in any two neighboring (i.e sharing a common side) cells differs exactly by 1 (a cell may contain several coins or may be empty). For which maximal n is this possible?

10. **(Romanian National Olympiad 2005)**: Consider $n \geq 2$ finite sets $A_1, A_2, ...A_n$ such that

    (a) $|A_i| \geq 2$ for $1 \leq i \leq n$

    (b) $|A_i \cap A_j| \neq 1$ for $1 \leq i < j \leq n$.

    Prove that the elements of $A_1 \cup A_2 \cup \cdots \cup A_n$ can be coloroured with two colours so that no A_i is coloured in only one colour. Use the method of contradiction.

11. **(C1 IMO2008)**: In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a box. Two boxes *intersect* if they have a common point in their interior or on their boundary. Find the largest n for which there exist n boxes $B_1, ..., B_n$ such that B_i and B_j intersect if and only if $i \neq j \pm 1 \pmod{n}$.

12. **(C1 IMO2007)**: Let's do this problem again. Let $n > 1$ be an integer. Find all sequences $a_1, a_2, ..., a_{n^2+n}$ satisfying the following conditions

(a) $a_i \in \{0,1\} \quad \forall \quad 1 \leq i \leq n^2+n$;

(b) $a_{i+1} + a_{i+2} + \cdots a_{i+n} < a_{i+n+1} + a_{i+n+2} + \cdots a_{i+2n} \quad \forall \quad 0 \leq i \leq n^2 - n$.

13. **(USAMO 2005:4)**: Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i $(i = 1, 2, 3, 4)$ from the end of the leg L_i and still have a stable table? (The table is stable if it can be placed so that all 4 legs touch the floor; not a cut length of 0 is permitted).

14. **(USA TST Day 1:1)**: Let n be an integer greater than 1. For a positive integer m, let $S_m = \{1, 2, ..., mn\}$. Suppose that there exists a $2n$-element set T such that

    (a) each element T is an m-element subset of S_m

    (b) each pair of elements of T shares at most one common element and

    (c) each element of S_m is contained in exactly 2 elements of T.

    Determine the maximum possible value of m in terms of n.

15. **(USAMO 2005:1)**: Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.