Graph Theory

First, some definitions.

- A *graph* is a collection of points (called *vertices*) with some of them connected by links (called *edges*), so that every link connects exactly two points.
- The *degree* of a vertex is the number of edges coming out of that vertex.
- A *cycle* is a closed path along the edges which doesn't pass through any edge twice.
- A graph is *connected* if it is possible to "travel" from any vertex to any other vertex along the edges.

Some results relevant specifically to graphs:

- The sum of the degrees of the vertices in a graph is twice the number of edges.
- Therefore the sum of the degrees of the vertices in a graph is even.
- If a graph with n vertices has no cycles, but there is a path connecting any two vertices, the graph has at least $n - 1$ edges.
- The set of vertices in a graph can be partitioned into two sets A and B such that no two vertices in A are connected by an edge, and no two vertices in B are connected by an edge, if and only if there are no odd cycles in this graph.

Some useful tricks:

- Use induction (e.g. consider a graph with one less vertex than the given one and assume a certain property holds for this graph.)
- Count the edges or vertices of the graph in a clever way.
- Take a vertex A and consider the disjoint sets $A_1, A_2, ..., A_n$ so that all vertices in A_1 are connected by an edge to A; all vertices in A_{i+1} are connected by an edge to some vertex in A_i, and not connected by an edge to any of the vertices in $\{A\}, A_1, A_2, ...,$ or A_{i-1}.
- Look at specific parts of the graph satisfying various properties.
- Color vertices or edges in the graph.
- Define your graph, vertices, and edges in a non-standard way.

Problems

1. (a) There are 25 students in the class. Is it possible that 6 of them have 9 friends each, 8 of them have 8 friends each, and 11 of them have 7 friends each? (friendships are mutual.)
(b) Is it possible to draw 7 line segments on a sheet of paper, so that each of them intersects exactly 3 others?
2. There are 2001 roads exiting the capital of a certain country. There is exactly one road exiting a small village X. There are 1000 roads exiting each of the other cities in this country. (All these roads are within the country). Prove that it is possible to get from X to the capital by roads without leaving the country.
3. In a graph every vertex has even degree. One of the edges is removed. Prove that the graph is still connected.

4. In a country every two cities are connected either by a road, or by an air route. Prove that one of these methods of travel allows one to get from every city to every other city using only that method of travel.

5. Prove that if a graph contains n vertices and at least n edges, it contains a cycle.

6. In a certain city, for any three road intersections A, B, C, it is possible to drive along the roads of the city from A to B without passing through the intersection C. Prove that for any two intersections P and Q in the city, there exist two routes from P to Q that do not intersect each other. (An intersection is a place where at least two roads meet.)

7. A graph is *bipartite* if its vertices can be partitioned into two disjoint sets X and Y so that no two vertices in X are connected by an edge and no two vertices in Y are connected by an edge. Prove that a graph is bipartite if and only if it has no cycles with an odd number of edges. [This is useful for hard problem 5.]

Hard Problems

1. (Moscow 1996) A math contest with 8 problems was written by 8 students. Each problem was solved by 5 students. Prove that there exist two students such that each problem was solved by at least one of these students.

2. The inhabitants of a village start getting sick with the flu. One day in the morning some of them ate too much ice cream and got sick; and after that day the only way a healthy person would get sick is if they visited a sick friend. Every person in the village is sick for exactly 1 day, and the next day he is immune to the flu virus - he cannot get sick that day. Despite the pandemic, every day a healthy person visits all his sick friends. After the pandemic started, nobody got vaccines. Prove that:
(a) If some people got a vaccine before the first day when the pandemic started, and were immune to the flu on the first day, the pandemic can last forever.
(b) If nobody was immune to the flu on the first day, eventually the pandemic will end.

3. Let n be a positive integer. In a group of $2n + 1$ people, each pair is classified as friends or strangers. For every set S of at most n people, there is one person outside of S who is friends with everyone in S. Prove that at least one person is friends with everyone else.

4. (Russia 1993) In a country with 1993 cities, at least 93 roads exit out of each city. It is possible to travel from any city to any other city along the roads. Prove that it is possible to do this while visiting at most 62 other cities along the way (e.g. a route from city A to city E that includes cities B, C, D visits 3 cities along the way.)

5. (Russia 2000) (a) In a country with 2000 cities, some cities are connected by roads. It is known that through every city there are at most N non-self-intersecting cycles of odd length. Prove that the country can be divided into $2N + 2$ provinces so that no two cities from the same province are connected by a road.
(b) Same problem statement as above, but now prove that the country can be divided into $N + 2$ provinces so that no two cities from the same province are connected by a road.
Hint: remove one edge from every odd cycle in the graph. Color the vertices in the graph in 2 ways - using 2 colors, and using $N + 1$ colors.

A great resource for olympiad graph theory training - *IMO Training 2008: Graph Theory* by Adrian Tang: `http://web.mit.edu/yufeiz/www/imo2008/tang-graph.pdf`