Winter Math Camp: Algebra Problems

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1. Let $f_j(x)$ be a polynomial of degree j. Suppose $\sum_{j=0}^{n-1} f_j(y_i)z_j = 0$ for $0 \le i \le n-1$ and y_j are all distinct. Show that $z_i = 0$.

2. Let $d_0 = 1$, $d_1 = 0$, and $d_n = (n-1)(d_{n-1} + d_{n-2})$. Show $d_n = [n!/e]$ for $n > 1$. & closest integer

3. Octahedral dice, each with its 8 sides labelled 0,1,1,1,2,2,2 and 3, rolled 2004 times total to a multiple of 3 with what probability?

4. Does the arrangement of nine squares in Figure 1 form a tenth square? (Bonus: Find another arrangement of noncongruent squares that makes a square.)

5. Let $0 \le p_i \le 1$ with $\sum_{j=1}^{n} p_j = 1$. Show $\prod_{j=1}^{n}(1-p_j) < 1/e$.

6. Show $\sum_{i=1}^{n} \frac{1}{p_i} \ge n^2$, with p_i as above.

7. Let $f(r) = \sqrt[r]{\sum_{j=1}^{n} p_i^{r+1}}$ for $r > 0$, with p_j as above. Show $f(r) \ge f(s)$ for $r \ge s > 0$.

8. Let $S \subset \mathbb{Q}$ satisfy: $\forall b, d \in S, \exists a, c, e \in S : a < b < c < d < e$. Prove or disprove: $\exists$ increasing bijection $f : S \to \mathbb{Q}$.

9. Let $b_0 = 2$, $b_1 = 3$, and $b_n = 3b_{n-1} - b_{n-2}$. Let $f_n(x) = x^{2n} - b_n x^n + 1$. Show $f_m(x) | f_{dm}(x)$ for integers d, m.

10. Let a, b be integers. Let c, d be integers such that $1 + c + d\sqrt{2} = (a + b\sqrt{2})^{120}$. Prove or disprove: c and d are divisible by 11. $11 \nmid a$ or $11 \nmid b$

11. Prove or disprove: $\sin(\cos(x)) < \cos(\sin(x))$ for all x.

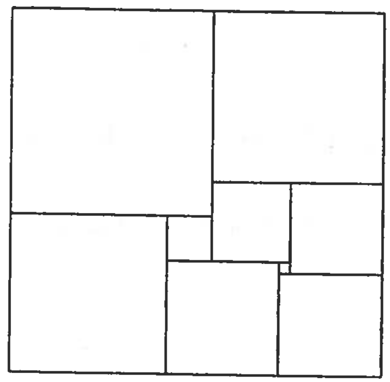

Figure 1: An arrangement of nine squares

12. Let $U_n = \{(a,b,c,d) : a,b,c,d \in \mathbb{Z}, 1 \le a,b,c \le d \le n\}$. Let $V_n = \{(e,f,g,h) : e,f,g,h \in \mathbb{Z}, 1 \le e \le f \le n, 1 \le g \le h \le n\}$. Show $|U_n| = |V_n|$.

13. Let f be a polynomial with integer coefficients. For $i \in \{1,\ldots,10\}$, let $x_i \in \mathbb{Z}$ be distinct with $f(x_i) = 2004$. If $f(y) = -2004$, show $y \notin \mathbb{Z}$.

14. Let $A(x) = \sum_{1\le i,j,k\le n} x_i^{9}x_j^3x_k^2$ and $B(x) = \sum_{1\le i,j,k \le n} x_i^6x_j^5x_k$, both summed over distinct i,j,k only. Show there exists $y,z \in \{(x_1,\ldots,x_n) : 0 < x_i \in \mathbb{R}\}$ with $A(y) < B(y)$ and $A(z) > B(z)$.

15. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \ge 0\}$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an ~~invertible~~ bijective and convex function. Is there an increasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(g(x)) = f(x)$?

16. Show $\sum_{j=1}^{n} \frac{1}{j(j+1)(j+2)(j+3)} < \frac{1}{18}$.

$g(g(x)) = 2^x$?

$g(x) = a^x$?

$g(g(x)) = 3^x$

$g(0) = a,\ g(a) = 1$

$g(1) = f(a) = 3^a$

$g(3^a) = f(1) = 3$

$g(3) = f(3^a) = 3^{3^a}$

$g(3^{3^a}) = f(3) = 27$

$\Rightarrow a < 1$

$g(g(x)) = \frac{1}{x+1}$

$g(0) = a$

$g(a) = 1$

$g(1) = \frac{1}{a+1} < 1$

$\Rightarrow a < 1$

$2^{2^a} < 4$

$\Rightarrow 2^a < 2$

$\Rightarrow a < 1$

$g(2^{2^a}) = f(2) = 4$

$g(0) = a,\ g(a) = 1$

$g(1) = f(a) = 2^a$

$g(2^a) = f(1) = 2$

$g(2) = f(2^a) = 2^{2^a}$

$a > 1$

$a < 1$