2003 Winter Camp Mock Olympiad

1. Let ABC be a triangle and D be a point on the side AB. The incircles of the triangles ACD and CDB touch each other on CD. Prove that the incircle of ABC touches AB at D.

2. As usual, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

   Let $n \geq 2$ be a positive integer. Prove that

$$\sum_{k=2}^{n} \left\lfloor \frac{n^2}{k} \right\rfloor = \sum_{k=n+1}^{n^2} \left\lfloor \frac{n^2}{k} \right\rfloor$$

3. Let a, b, c be the sides of a triangle.

   (a) Determine the largest real number m for which

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq m$$

   (b) Determine the smallest real number M for which

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} < M$$

4. There were n people present at a New Year's Party. The following facts are known about the party:

   (a) n is not a multiple of 11.

   (b) n is at least 5, and at most 30.

   (c) Each pair of strangers had exactly two common acquaintances, and each pair of acquaintances had no common acquaintances.

   Determine the value of n, and prove that your answer is unique.

5. Let D be a point inside an acute triangle ABC such that

$$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA = AB \cdot BC \cdot CA.$$

   Prove that D is the orthocentre of $\triangle ABC$.