Combinatorial Geometry

Maria Monks

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1 What is combinatorial geometry?

Combinatorial geometry is the general term used to describe geometrical problems or topics that have a discrete, combinatorial flavor, along with combinatorial problems that have a geometric flavor.[1] Many combinatorial geometry problems are questions about arrangements of a finite set of points, lines, planes, etc.

Here is a classic example of a combinatorial geometry problem:

Example. There are $2n + 2$ points in the plane, no three of which are collinear. Show that two of them determine a line that separates n of the points from the other n. (From Math Olympiad Challenges.)

One nice way to approach this problem is to use the notion of the *convex hull* of the set of points. Since the convex hull comes up so often in combinatorial geometry problems, we list here several ways of thinking about the convex hull:

Definition. The *convex hull* of a set of points $x_1, \ldots, x_n$ is

- The smallest convex[2] polygon containing the n points, along with its interior.
- The region

  $$\left\{ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \mid \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0 \text{ for } i = 1, \ldots, n \right\}$$

  where we think of the x_i as vectors.
- The polygon (along with its interior) formed by "snapping a rubber band" around all the points.
- The polygon (along with its interior) whose vertices are precisely the points $x \in \{x_1, \ldots, x_n\}$ for which there is a line L passing through x that defines a half-plane (including L) that contains all n points.

[1] A *combinatorial geometry* is in fact a well-defined mathematical object (also known as a *simple matroid*), but this is beyond the scope of this handout.

[2] Recall that a region is *convex* if and only if the line segment joining any two points in the region is also contained in the region.

As an exercise, see if you can see why these are all equivalent ways of defining the convex hull.

Going back to the problem, we choose a point among the n points that lies on the boundary of the convex hull. Call this point x. Since x lies on the boundary of the convex hull, we can consider a line passing through x for which all n points lie on one side of the line (or on the line). Rotate this line until it has swept past n points, and continue until it hits the next point. Then n of the points are on one side of the line determined by these two points! ☺

You can think of this trick as the *sweeping technique*, a method that comes up in many combinatorial geometry problems. Here are some other strategies to keep in mind when solving combinatorial geometry problems:

- Use extremal arguments.
- Use the pigeonhole principle in coloring problems.
- If stuck on the combinatorial aspect, try purely geometric techniques (clever constructions, computing side lengths and angles, etc) to better understand the geometry of the figure.
- When all else fails, try induction!

Now, try your hand at these...

2 Problems

These problems are listed roughly in increasing order of difficulty.

1. (Proposed for IMO, 1993.) Given $2n + 3$ points in the plane, no 3 collinear and no 4 on a circle, show that there exists a circle containing 3 of the points such that exactly n of the remaining points are in its interior.

2. (Math Olympiad Challenges.) Given $4n$ points in the plane, no 3 of which are collinear, show that one can form n nonintersecting quadrilateral surfaces (not necessarily convex) with vertices at these points.

3. Define an n-collection to be an arrangement of n points in the plane such that no three are collinear and each is colored either red or blue. What is the smallest value of n such that in any n-collection, there will always exist two monochromatic triangles (triangles having either all blue points or all red points as vertices) which do not intersect?

4. (USAMO 2005.) Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines. ☺

5. (APMO 1999.) Let S be a set of $2n+1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called *good* if it has 3 points of S on its circumference, $n-1$ points in its interior and $n-1$ points in its exterior. Prove that the number of good circles has the same parity as n.

6. (Putnam 1994.) Prove that the points of a right isosceles triangle whose equal sides have length 1 cannot be colored in four colors such that no two points at a distance at least $2-\sqrt{2}$ from each other recieve the same color.

7. (IMO 2002.) Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x+y < n$. Each point of T is colored red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \leq x$ and $y' \leq y$. Define an X-set to be a set of n blue points having distinct x-coordinates, and a Y -set to be a set of n blue points having distinct y-coordinates. Prove that the number of X-sets is equal to the number of Y-sets.

8. (From Art and Craft of Problem Solving.)

   (a) Color the plane in 3 colors. Prove that there are two points of the same color 1 unit apart.

   (b) Color the plane in 2 colors. Prove that one of these colors contains pairs of points at every mutual distance r.

   (c) Color the plane in 2 colors. Prove that there will always exist an equilateral triangle with all its vertices of the same color.

   (d) Show that it is possible to color the plane in 2 colors in such a way that there cannot exist an equilateral triangle of side length 1 with all vertices the same color.

   (e) Color the plane in 2 colors. Show that there exists a rectangle, all of whose vertices are the same color.

9. Given a unit square, show that if five points are placed anywhere inside or on this square, then two of them must be at most $\sqrt{2}/2$ units apart.

10. Consider 9 lattice points in three-dimensional space. Show that there must be a lattice point on the interior of one of the line segments joining two of these points.

11. (IMO Shortlist 2007.) A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that in this dissection, there exists a rectangle having no point on the boundary of the square.

12. (From Po-Shen Loh's 2009 Combinatorial Geometry handout.) The vertices of a convex polygon are colored by at least three colors such that no two consecutive vertices have the same color. Prove that one can dissect the polygon into triangles by diagonals that do not cross and whose endpoints have different colors.

13. **Sylvester's Theorem.** Show that, if n points in the plane do not all lie on the same line, then there is a line containing exactly two of the points.

14. **The Braid Arrangement.** Consider the hyperplanes in n-dimensional space $\mathbb{R}^n$ defined by the equations $x_i = x_j$ for $1 \leq i < j \leq n$. How many distinct regions do these hyperplanes divide $\mathbb{R}^n$ into? (Each hyperplane $x_i = x_j$ divides the plane into two regions: the points having $x_i < x_j$, and those having $x_i > x_j$).

15. **Pick's Theorem.** Given a polygon whose vertices are lattice points, let I be the number of lattice points on the interior of the polygon and B the number of lattice points on its boundary. Show that the area of this polygon is $I + \frac{1}{2}B - 1$.

16. **Euler's Formula.** Show that in any (three-dimensional) polyhedron, if f is the number of faces, v the number of vertices, and e the number of edges, then $f+v-e = 2$.