Art of Problem Solving

WOOT 2010–11

Practice Olympiad 2

Instructions

- You should take the test under "olympiad conditions," meaning that the test should be completed in one sitting, with handwritten solutions (just like on an actual olympiad exam). Take the test using only the resources that would be available to you on an actual olympiad, meaning you should use scrap paper, a ruler, a compass, etc., but no calculators and no reference materials.
- You should allot 3 hours to take the test.
- Completely fill out the cover sheet and make sure it is the first page of your solutions.
- On the WOOT Home Page, there is a **WOOT Practice Olympiad Answer Sheet**. Print out (or copy) several blank copies of the answer sheets, and write all of your work on these sheets. Use only black pen or very dark pencil. Make sure that the top of every answer sheet page is completely filled out. Each problem's solution should start on a new page, along with new page numbering.
- Do not discuss the problems on or before the due date of Wednesday, November 3, 2010.

How to submit your solutions

You can submit solutions by upload, email, or by fax. DO NOT MAIL YOUR SOLUTIONS!

By Upload: In your "My Classes" area, follow the link that says "Submit."

By email: Scan your solutions as a single PDF file. **Check to make sure that the file is legible before emailing it!** Email to `woot@artofproblemsolving.com`. Put "WOOT Practice Olympiad 2" in the subject line, attach your solutions as a single PDF file, and write something in the message body - if you leave the message body blank, it will get blocked by our spam filters.

By fax: Fax to (619)659-8146.

Solutions are due by Wednesday, November 3, 2010

Late submissions will not be accepted except under extraordinary circumstances

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WOOT Practice Olympiad Cover Sheet

Username: ____

Class ID: ___________

User ID: ______________________
(Your Class ID and User ID can be found in the "My Classes" section of the website)

Practice Olympiad Number: 2

Beginning	
Intermediate	
Advanced	

Number of pages (including cover sheet): ____

Practice Olympiad 2

1. A set M containing four positive integers is called *connected* if for every element x in M, at least one of the numbers $x-1$, $x+1$ also belongs to M. For $n \geq 4$, let U_n denote the number of connected subsets of the set $\{1, 2, \ldots, n\}$.

   (a) Evaluate U_7.

   (b) Determine the least positive integer n for which $U_n \geq 2010$.

2. The incircle of triangle ABC touches sides BC, AC, and AB at A', B', and C', respectively. The line $A'C'$ meets the angle bisector of $\angle A$ at D. Prove that $\angle ADC = 90°$.

3. The sequence x_n of integers is defined by $x_1 = 1$, $x_2 = a$, and
$$x_n = (2n+1)x_{n-1} - (n^2 - 1)x_{n-2}$$
for $n \geq 3$, where a is a positive integer. For which values of a does this sequence have the property that x_i divides x_j whenever $i \leq j$?

4. Let ABC be a triangle with $AB = AC$. Let t be the length of the interior angle bisector from B. Find the greatest real number m and the least real number M such that
$$m < \frac{BC}{t} < M$$
for all such triangles ABC.

5. We say that a positive integer N is *delightful* if, when one of its digits is deleted (when written in decimal), the result is equal to $N/9$, and $N/9$ is divisible by 9. For example, the number 70875 is delightful, because if we delete the digit 0, we get 7875, which is equal to 70875/9, and 7875 is divisible by 9.

   (a) Prove that if the positive integer N is delightful, then one of the digits of $N/9$ may be deleted, resulting in a number that is equal to $N/81$.

   (b) Find the smallest delightful positive integer.

6. Let m, n, and k be positive integers such that $n \geq m \geq 2$. Prove that
$$\binom{n}{m}\binom{k}{0} + \binom{n-1}{m-1}\binom{k+1}{1} + \binom{n-2}{m-2}\binom{k+2}{2} + \cdots + \binom{n-m}{0}\binom{k+m}{m} = \binom{n+k+1}{m}.$$

7. There exist 5 points in the plane, such that the area of any triangle formed by three of these points is at least 2. Prove that one of these triangles has area at least 3.

Remember . . .

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- Complete the WOOT Practice Olympiad Cover Sheet (included in this document), and make it the first page of your solutions. We will not accept your solutions without this cover sheet.
- Do not mail your solutions. Use only upload, e-mail, or fax. If you use e-mail, send your solutions to

  woot@artofproblemsolving.com,

  NOT **classes**@artofproblemsolving.com. If you use e-mail, we will only accept solutions in PDF format. In particular, solutions in JPG or Word format will not be accepted.