1998 IMO Camp
Intriguing Inequalities

1. Let $\{a_k\}$ $(k = 1, 2, \dots, n \dots)$ be a sequence of distinct positive integers. Prove that for all natural numbers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}$$

(IMO 1978 #5)

2. Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1, \quad \text{and for all } i \geq 0, \quad x_{i+1} \leq x_i$$

a) Prove that for every such sequence, there is an $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \dots + \frac{x_{n-1}^2}{x_n} \geq 3.999$$

b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \dots + \frac{x_{n-1}^2}{x_n} < 4$$

(IMO 1982 #3)

3. Prove that $0 \leq yz + zx + xy - 2xyz \leq 7/27$ where x, y, z are non-negative real numbers for which $x + y + z = 1$ (IMO 1984 #1)

4. Let $x_n = \sqrt[2]{2 + \sqrt[3]{3 + \sqrt[4]{\dots + \sqrt[n]{n}}}}$

Prove that $x_{n+1} - x_n < \frac{1}{n!}$ for $n \geq 2$

5. Prove that for all $n \geq 2$, for positive real numbers x_i,

$$\frac{x_1^2}{x_1^2 + x_2 x_3} + \frac{x_2^2}{x_2^2 + x_3 x_4} + \dots + \frac{x_{n-1}^2}{x_{n-1}^2 + x_n x_1} + \frac{x_n^2}{x_n^2 + x_1 x_2} \leq n - 1$$

6. Prove that if a, b, c are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}$$ (USAMO 1974 #2)

7. Prove that, for numbers a, b, c in the interval $[0, 1]$,

$$\frac{a}{b+c+1} + \frac{b}{a+c+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1$$

(USAMO 1980 #5)

8. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1$$

(IMO 1996 Short list)

9. If a, b, c are the sides of a triangle, show that

$$\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \leq 2$$

10. Let $0 \leq a_i < 1$ $i = 1, 2, \ldots, n$, and $A = \sum_{i=1}^{n} a_i$. Show that

$$\sum_{i=1}^{n} \frac{a_i}{1-a_i} \geq \frac{nA}{n-A}$$ with equality iff all the a_i are equal.

11. Given positive real numbers $x_1, x_2, \ldots, x_{n+1}$ such that

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_{n+1}} = 1,$$ show that

$$x_1 x_2 \cdots x_{n+1} \geq n^{n+1}$$

12. Given positive numbers a, b, c, d, prove that

$$\frac{a^3+b^3+d^3}{a+b+d} + \frac{b^3+c^3+d^3}{b+c+d} + \frac{c^3+d^3+a^3}{c+d+a} + \frac{d^3+a^3+b^3}{d+a+b} \geq a^2+b^2+c^2+d^2$$

13. Let x, y, z be positive reals such that $x+y+z=1$. Prove that

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 64$$

and determine when equality occurs.

14. If a, b, c are real numbers such that $a^2+b^2+c^2=1$, show that

$$-\frac{1}{2} \leq ab+bc+ca \leq 1$$

15. Assume $a_i > 0$ $(i=1,2,\ldots,n)$ with $a_{n+1}=a_1$. Prove or disprove:

$$\sum_{i=1}^{n}\left(\frac{a_i}{a_{i+1}}\right)^n \geq \sum_{i=1}^{n}\frac{a_{i+1}}{a_i}$$

16. The set $\{a_0, a_1, \ldots, a_n\}$ of real numbers satisfies the following conditions:

i) $a_0 = a_n = 0$

ii) For $1 < k < n-1$, $\quad a_k = c + \sum_{i=k}^{n-1} a_{i-k}(a_i + a_{i+1})$ $\quad$ (a, c constant)

Prove that $c < \frac{1}{4n}$

17. If a, b, c are positive reals such that $abc=1$, prove that

$$\frac{1}{a^3+b^3+1} + \frac{1}{b^3+c^3+1} + \frac{1}{c^3+a^3+1} \leq 1$$

18. If a, b, c are positive reals, show that $a^a b^b c^c \geq a^b b^c c^a$, and determine when equality occurs