Hall's Marriage Lemma

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BCW-HALL

§1 Lecture notes

§1.1 Problems to try

Problem 1.1. A deck of poker cards is arranged in a 13×4 array, face up. Show that it's possible to pick a card from each of the 13 rows so that each rank appears exactly once.

Problem 1.2 (Kazahkstan 2003)**.** Two identical unit squares are given; each has been divided into n disjoint polygons of area $\frac{1}{n}$. We overlay these two sheets of paper on top of each other. Prove that one can puncture the overlaid sheet n times such that every polygon is punctured.

Problem 1.3 (PUMaC 2016, Alex Song)**.** There are twelve candies arranged in a circle, four of which are rare candies. Chad and Eric want to collaborate on a strategy for the following act. First, Eric comes and is told which four candies are rare candies, then removes four non-rare candies from the circle. Then Eric leaves, and Chad comes and must determine which of the four candies (of the eight remaining candies) are rare. Decide whether this is possible or not.

§1.2 A narrative on the poker problem

The first thing we might try is a **greedy algorithm** (always try the dumbest things first!), namely

> Grab any Ace, then any 2, and so on, and hope that this always works.

This, unfortunately, fails. For example, consider the following first five rows.

$$\begin{array}{cccc} A\heartsuit & 3\spadesuit & 4\heartsuit & 4\spadesuit \\ 2\clubsuit & 3\diamondsuit & 3\clubsuit & 3\heartsuit \\ 2\spadesuit & 4\diamondsuit & 5\heartsuit & 5\diamondsuit \\ 2\diamondsuit & 4\clubsuit & 5\clubsuit & 5\spadesuit \\ A\spadesuit & A\diamondsuit & A\clubsuit & 2\heartsuit \end{array}$$

[*]Developed as part of Olympiad Training for Individual Students (OTIS). Internal use only.

As promised, we start by taking the first A we see and the first 2 we see – and lo and behold, we're already stuck!

$\boxed{A\heartsuit}$	$3\spadesuit$	$4\heartsuit$	$4\spadesuit$
$\boxed{2\clubsuit}$	$3\diamondsuit$	$3\clubsuit$	$3\heartsuit$
$2\spadesuit$	$4\diamondsuit$	$5\heartsuit$	$5\diamondsuit$
$2\diamondsuit$	$4\clubsuit$	$5\clubsuit$	$5\spadesuit$
$A\spadesuit$	$A\diamondsuit$	$A\clubsuit$	$2\heartsuit$

We see that we have boxed ourselves in, and we can't pick any 3's. So let's **backtrack**, and reverse our most recent decision, grabbing $3\diamondsuit$ from the second row. We grab a $2\spadesuit$, and then the $4\clubsuit$ (the only remaining 4), and... stuck, no 5.

$\boxed{A\heartsuit}$	$3\spadesuit$	$4\heartsuit$	$4\spadesuit$
$2\clubsuit$	$\boxed{3\diamondsuit}$	$3\clubsuit$	$3\heartsuit$
$\boxed{2\spadesuit}$	$4\diamondsuit$	$5\heartsuit$	$5\diamondsuit$
$2\diamondsuit$	$\boxed{4\clubsuit}$	$5\clubsuit$	$5\spadesuit$
$A\spadesuit$	$A\diamondsuit$	$A\clubsuit$	$2\heartsuit$

Now, by hand I'm sure you can see a way to pick. But there do exist situations in which your first move could fatally sabotage the position, and you wouldn't know until quite a bit later. But how could you show this works in general? You can try for a while, but it turns out to not be easy.

In summary:

- One is trying to create some sort of "pairing": in this case, match each row with a particular rank.
- A "naïve" or "greedy" algorithm does not work: it's possible that you can make a "bad" choice early on, like we did by picking $A\heartsuit$ and $2\clubsuit$ on the offset. In this sense the problem sort of "feels like induction" even though the structure of the problem changes after the first move.
- You feel like you can solve cases by hand, but aren't sure how you might want to prove this always works.

§1.3 Main Result

In many olympiad contexts, if you realize to use Hall's Theorem, you're done. (Hence the need for the "mystery" earlier.)

Definition (Hall's Condition)**.** Consider a bipartite graph of boys and girls. We say that a graph satisfies **Hall's Condition** if for any k boys, there are at least k girls which at least one boy is compatible with.

Theorem 1.4 (Hall's Marriage Theorem)
It is possible to match every boy with a girl if and only if Hall's condition is satisfied.

The most important special case is:

Lemma 1.5 (Regular Hall)

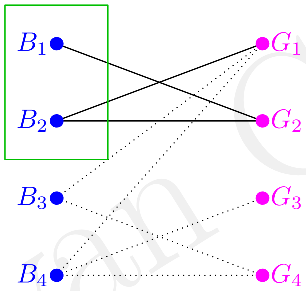
Assume that $|B| = |G|$, and moreover the graph is regular (all boys and girls like exactly m others). Then Hall's Theorem applies and there is a regular matching.

Proof. Suppose the graph is m-regular, meaning all boys and girls have exactly m compatible partners. Consider any random k boys. In the graph, they emit a total of km edges. Each girl absorbs at most m of the edges, so there are at least k girls compatible with at least one of the boys. $\square$

§1.4 The Proof of Hall's Theorem

It's by induction. With B boys and G girls (with $|B| \le |G|$). we'll go by strong induction on $|B|$.

Call a nonempty proper subset $X \subsetneq B$ of the boys *critical* if the boys in X like exactly $|X|$ girls.



Observe that whenever there is a critical set X, the girls who like boys in X *must* be paired with the boys in X if we are to have any chance of succeeding.

Exercise 1.6. Check that Hall's condition is still true after removing the pairs. So, we apply the induction hypothesis again to resolve this case completely.

In the other case, assume there is no critical set. Then we pair B_1 off with any girl, and send those two away.

Exercise 1.7. Check Hall's Condition also remains true in the resulting graph.

(Intuitively, the absence of a critical set means that we have the latitude to make a single arbitrary move while preserving Hall's Condition.) This completes the proof.

Remark 1.8. You might be surprised how simple the proof of the theorem is – if the proof of the theorem is so easy, why does it give people such an upper hand? The reason is that the *hypothesis we are inducting on (Hall's Condition) is actually fairly strong*. Most typical applications of Hall's Theorem satisfy Hall's Condition in a way that doesn't generalize after trying to make a first few "greedy" steps.

§1.5 Complaints about bipartite "graphs"

Bipartite graphs rarely feel like the usual graphs that you encounter in typical graph theory problems. Actually, I think Hall's theorem should explicitly *not* be taught in the context of graph theory.

- Typical olympiad graph theory problems are stated cleanly, using "tennis tournaments" or "schoolchildren with friendships" or "schoolchildren in Russia" and what not. In contrast, problems involving bipartite graphs are stated in completely different terms; the most transparent is "rectangular grids", and several less transparent examples below.

- One reason this is true is that in a bipartite graph setup, the vertices often have different "types": one side is rows and the other is columns, or one side is children and the other is presents, et cetera. In contrast for a typical graph theory problem the vertices are "more or less the same". So standard graphs and bipartite graphs are suited for entirely different classes of problems.

- Pro tip: if you have a rectangular board, consider thinking of it as a bipartite graph.

§2 Practice problems

Problem 2.1. On a 1000×1000 chessboard, we delete some squares from the board so that each row and each column has at most k deleted squares. For which values of k is it always still possible to place 1000 non-attacking rooks on the board?

Problem 2.2 (Birkoff von Neumann)**.** Let A be a square $n \times n$ matrix whose entries are nonnegative and whose rows and columns have sum 1 (such a matrix is *doubly schochastic*). Show that it's possible to write

$$A = c_1 P_1 + \cdots + c_m P_m$$

where each P_i is a permutation matrix, and each c_i is a positive real.

Problem 2.3 (Baltic Way 2013)**.** Santa Claus has some gifts for n children. For $1 \le i \le n$, the i-th child considers $x_i > 0$ of these items to be desirable. Assume that

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} \le 1.$$

Prove that Santa Claus can give each child a gift that this child likes.

Problem 2.4 (Hall's Chessboard)**.** An $n \times n$ chessboard has some of its squares painted blue. Assume that for every n squares chosen, no two in the same row or column, at least one of the squares is blue. Prove that one can find a rows and b columns whose intersection contains only blue squares, so that $a + b \ge n + 1$.

Problem 2.6. A table has m rows and n columns with $m, n > 1$. The following permutations of its mn elements are permitted: any permutation leaving each element in the same row (a "horizontal move"), and any permutation leaving each element in the same column (a "vertical move"). Find the smallest integer k such that any permutation of the mn elements can be realized by at most k permitted moves.

Problem 2.7 (December TST 2014/3)**.** Let n be an even positive integer, and let G be an n-vertex graph with exactly $\frac{n^2}{4}$ edges, where there are no loops or multiple edges (each unordered pair of distinct vertices is joined by either 0 or 1 edge). An unordered pair of distinct vertices $\{x, y\}$ is said to be *amicable* if they have a common neighbor (there is a vertex z such that xz and yz are both edges). Prove that G has at least $2\binom{n/2}{2}$ pairs of vertices which are amicable.