Mock Olympiad #2 Solutions

July 4, 2009

1. *(IMO Short list 1988, #22)*
   Suppose integers $x_1, x_2, \ldots, x_p$ exist. We will show that $p = 2$ or 6. Note that:

$$\begin{aligned} LHS &= \sum_{i=1}^{p} x_i^2 - \frac{4}{4p+1} \cdot \left(\sum_{i=1}^{p} x_i \right)^2 \\ &= \frac{1}{4p+1} \cdot \sum_{i=1}^{p} x_i^2 + \frac{4}{4p+1} \cdot \left(p \cdot \sum_{i=1}^{p} x_i^2 - \left(\sum_{i=1}^{p} x_i \right)^2 \right) \\ &= \frac{1}{4p+1} \cdot \sum_{i=1}^{p} x_i^2 + \frac{4}{4p+1} \cdot \left(\sum_{1 \le i < j \le p} (x_i - x_j)^2 \right). \end{aligned}$$

   Suppose x_i takes on at least 3 values, with a numbers taking on the minimum value, c numbers taking on the maximum value, and b numbers taking on intermediate values. Then, $\sum_{1 \le i < j \le p} (x_i - x_j)^2 \ge ac \cdot 2^2 + ab \cdot 1^2 + bc \cdot 1^2 \ge 4a + b + c \ge p + 3$, and $LHS > 1$. Therefore, x_i can take on at most 2 different values.

   If all the x_i are equal to some value n, then $LHS = \frac{pn^2}{4p+1}$. If $p = 1$, then this has no solutions because 5 is not a square. If $p > 1$, then the factor of p in the numerator can never be canceled out, so there are no solutions in this case either. Assume now that $p > 2$.

   Let a numbers take the value A, and b numbers take the value B. Suppose that $|A - B| > 1$, then

$$LHS \ge \frac{4}{4p+1} \cdot (4ab) \ge \frac{16(p-1)}{4p+1} > 1$$

   since $p > 1$. So we can assume that $|A - B| = 1$.

   Now assume that neither A nor B are equal to 0. Then

$$LHS \ge \frac{(p-1)+4}{4p+1} + \frac{4(p-1)}{4p+1} = \frac{5p-1}{4p+1}$$

   which is bigger than 1 since $p > 2$.

   So we can further assume $A = 0$. Then

$$LHS = \frac{b + 4ab}{4p+1}.$$

If $b = 1$, then $a = p - 1$ and $LHS = \frac{4p-3}{4p+1} \neq 1$. If $b \in [2, p-2]$, then $p \geq 4, ab \geq 2p - 4$, and $LHS \geq \frac{8p-14}{4p+1} > 1$. If $b = p - 1$, then $a = 1$ and $LHS = \frac{5p-5}{4p+1}$, which is 1 only if $p = 6$.

This proves that p must equal 2 or 6. Conversely, if $p = 2$, we can take $\{x_1, x_2\} = \{1, 2\}$, and if $p = 6$, we can take $\{x_1, x_2, \ldots, x_6\} = \{0, 1, 1, 1, 1, 1\}$.

2. *(IMO Short list 2008, C5)*
If $k = l = 1$, the claim is trivial, so we will assume that $k + l > 2$. Consider a permutation $\{y_1, y_2, \ldots, y_{k+l}\}$ of S. Now look at the $k + l$ k-element subsets

$$A_i = \{y_i, y_{i+1}, \ldots, y_{i+k-1}\}, i = \{1, 2, ..., k + l\}$$

where all indices are taken mod $k + l$.

Claim 1: At least 2 of the A_i are nice.

Define $f(A_i) = \frac{1}{k} \sum\limits_{x_j \in A_i} x_j - \frac{1}{l} \sum\limits_{x_j \in S \setminus A_i} x_j$.

Notice that $f(A_1) + f(A_2) + \cdots + f(A_{k+l}) = 0$. (This is because each element x_j appears in A_i for k different values of i, and it appears in $S \setminus A_i$ for l different values of i.) Also,

$$|f(A_{i+1}) - f(A_i)| = \left| \frac{x_{i+k} - x_i}{k} + \frac{x_{i+k} - x_i}{l} \right| \leq \frac{1}{k} + \frac{1}{l}.$$

Therefore if A_i and A_{i+1} are of different signs[1], then either $|f(A_i)|$ or $|f(A_{i+1}|$ is at most $\frac{1}{2} \cdot \left(\frac{1}{k} + \frac{1}{l}\right) = \frac{k+l}{2kl}$, and therefore one of A_i, A_{i+1} is nice.

Since the sum of the $f(A_i)$ is 0, we must have at least 1 negative and 1 positive sign (unless they're all 0 which is silly). If there exist 2 disjoint sets $\{i, i + 1\}, \{j, j + 1\}$ such that $f(A_i), f(A_{i+1})$ and $f(A_j), f(A_{j+1})$ are of opposite signs, then by above we have at least two nice sets. Otherwise, exactly one $f(A_i)$ is of a different sign from the rest. Assume wlog that $f(A_1) \geq 0$ and for $i \neq 1$, $f(A_i) < 0$. If A_1 is not nice, than both A_{k+l} and A_2 are nice, so we have found our two nice sets. Otherwise, $f(A_1) \leq \frac{k+l}{2kl}$, and $\sum_{i \neq 1} |f(A_i)| = f(A_1) \leq \frac{k+l}{2kl}$, so every set must be nice. This finishes the proof of Claim 1.

Now, consider choosing a random permutation $\{y_1, y_2, \ldots, y_{k+l}\}$, and then choosing a random A_i corresponding to this permutation. By Claim 1, this chooses a nice set with probability at least $\frac{2}{k+l}$. On the other hand, this is equivalent to first choosing the shift i and then the permutation $\{y_1, y_2, \ldots, y_{k+l}\}$, but once i is fixed, we will be equally likely to choose any possible set. Therefore, this entire process chooses a set uniformly at random, so it follows that at least $\frac{2}{k+l} \cdot \binom{k+l}{k}$ sets are nice.

3. *Ukraine 2008,11.8*
Solution 1:

Denote the angles of triangle ABC by a, b, c, and let $\angle A_1BC = \angle A_1AB = x$,$\angle A_1CB = \angle A_1AC = y$. Then

$\angle ABA_1 = b - x$, and so $\angle BA_1A = 180 - b$. Similarly, $\angle CA_1A = 180 - c$, and so $\angle BA_1C = 180 - a$. Therefore, if we let H be the orthocenter of triangle ABC, CBA_1H are concyclic.

[1]We consider 0 to be of positive sign.

Denote the circle they lie on by S_1. Do a dilation centered at A with factor $\frac{1}{2}$, and let S_1 transform to S_2. Then denoting the midpoints of ABC by A', B', C', we know that S_2 contains B', C', and the midpoint of AH, so it must be the nine-point circle. So S_2 passes through B_0, B', C_0 and C' as well. Notice that since S_1 passed through A_1, A_2 lies on the nine-point circle.

We will prove A_2A_0, B_2B_0, C_2C_0 are concurrent using Sine-Ceva's theorem on triangle $A_0B_0C_0$. Let B'' denote the image of B_0 under the dilation centered at A with factor 2. We have

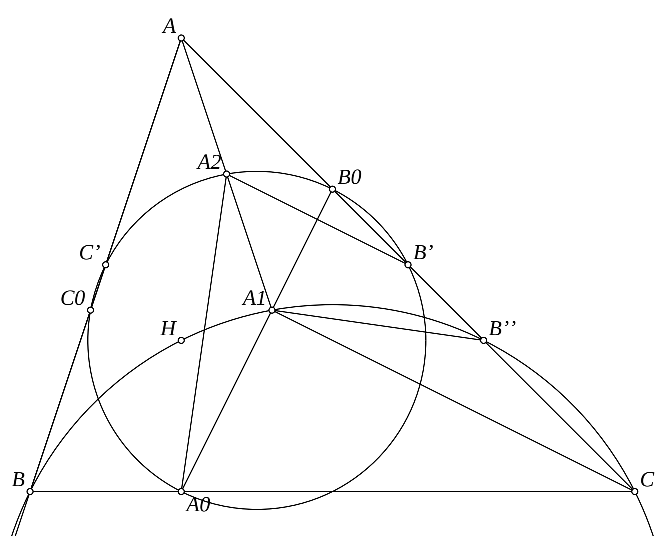
$$\begin{aligned}
\sin\angle A_2A_0B_0 &= \sin\angle A_2B'B_0 && \text{by concyclicity}\\
&= \sin\angle A_1CB'' && \text{by dilating around A}\\
&= \sin\angle A_1CA \\
&= \frac{AA_1 \cdot \sin\angle AA_1C}{AC} && \text{by sine law for } AA_1C\\
&= \frac{AA_1}{AC}\sin c
\end{aligned}$$

Similarly, $\sin\angle A_2A_0C_0 = \frac{AA_1}{AB}\sin b$.

Applying Sine-Ceva to $A_0B_0C_0$,

$$\frac{\sin\angle A_2A_0B_0 \sin\angle B_2B_0C_0 \sin\angle C_2C_0A_0}{\sin\angle A_2A_0C_0 \sin\angle B_2B_0A_0 \sin\angle C_2C_0B_0} = 1,$$

so the 3 lines are concurrent and we're done.



Solution 2:

Let A' denote the intersection of AA_2 and BC. The given condition implies that the circumcircle of $\triangle AA_1B$ is tangent to BC at B, and the circumcircle of $\triangle AA_1C$ is tangent to BC at

C. Since A' is on the radical axis of these two circles, it follows that $BA' = CA'$, and hence A' is the midpoint of BC. Also let B' and C' denote the midpoints of AC and AB.

Now, as in the other solution, note that A_2 lies on the nine-point circle, and hence C_0, A_2, B_0, A', and A_0 are concyclic. Therefore, $\angle C_0A_0A_2 = \angle C_0A'A_2 = \angle C_0A'A$. By the sine law, $\sin\angle C_0A'A = AC_0 \cdot \frac{\sin\angle BAA'}{A'C_0}$. Now, BC_0C is a right triangle with circumcenter A' so $A'C_0 = A'B$, and $\sin\angle C_0A'A = (AC\cos A) \cdot \frac{\sin\angle BAA'}{A'B} = (AC\cos A) \cdot \frac{\sin\angle AA'B}{AB}$.

Similarly, $\sin\angle B_0AA' = (AB\cos A) \cdot \frac{\sin\angle AA'C}{AC}$, so $\frac{\sin\angle C_0A'A}{\sin\angle B_0A'A} = \frac{AC^2}{AB^2}$. Therefore,

$$\frac{\sin\angle C_0A_0A_2}{\sin\angle B_0A_0A_2} \cdot \frac{\sin\angle A_0C_0C_2}{\sin\angle C_0B_0B_2} \cdot \frac{\sin\angle B_0B_0B_2}{\sin\angle A_0C_0C_2} = \frac{AC^2}{AB^2} \cdot \frac{AB^2}{BC^2} \cdot \frac{BC^2}{AC^2} = 1.$$

The result now follows from Sine Ceva on $\triangle A_0B_0C_0$.