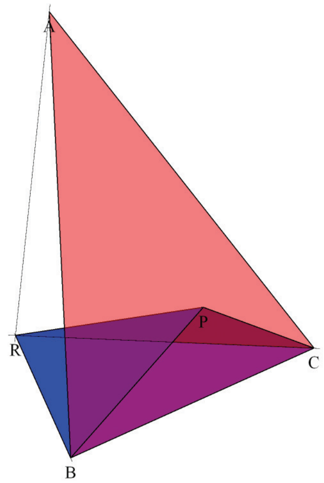
Areal coordinates

A large quantity of problems are concerned with a triangle ABC, known as the reference triangle. It is particularly useful, in these instances, to apply a special type of projective homogeneous coordinates, namely *areal coordinates*. The vertices A, B and C are given by the coordinates (1, 0, 0), (0, 1, 0) and (0, 0, 1), respectively. The line at infinity is given by $x + y + z = 0$. This exploits the symmetry of the triangle in a way that Cartesian coordinates do not.

Areas and lines

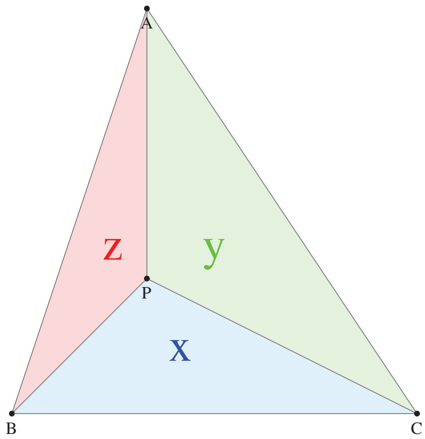
We have already defined areal coordinates as a special case of projective homogeneous coordinates. However, there are several other equivalent definitions explored later, explaining the synonyms ‘areal’ and ‘barycentric’.



We can *normalise* the areal coordinates (x, y, z) in the plane (not on the line at infinity) by assuming that $x + y + z = 1$. To convert unnormalised areals into their normalised counterparts, simply apply the map $(x, y, z) \to \frac{(x,y,z)}{x+y+z}$.

1. If the point P is represented by normalised areal coordinates (x, y, z), prove that $x = \frac{[PBC]}{[ABC]}$. *(Hint: consider the volumes of tetrahedra $RABC$ and $RPBC$.)*

This gives us one definition of areal coordinates, namely the ratio between the areas $[PBC]$, $[PCA]$ and $[PAB]$. If the triangle $[ABC]$ has unit area, then the areas of these triangles are equal to the normalised areal coordinates. This is encapsulated by the following diagram from Tom Lovering’s excellent introduction to areal coordinates (available at http://www.bmoc.maths.org/home/areals.pdf).



As areal coordinates can be defined in terms of ratios of areas of triangles (which are unchanged by affine transformations), the areal coordinates of a point remain invariant when an affine transformation is applied.

2. Deduce that the lines BC, CA and AB correspond to the equations $x = 0$, $y = 0$ and $z = 0$, respectively.

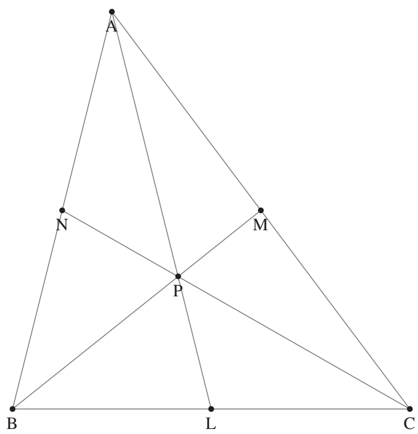
3. Show that the centroid, G, has normalised areal coordinates $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

4. Let the points P, Q and S be represented by normalised areal coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), respectively. Show that $\det\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = \frac{[PQS]}{[ABC]}$.

Ceva's theorem and cevians

5. Show that the equation of the line AP, where $P = (x_1, y_1, z_1)$, is given by $y_1 z = z_1 y$. Hence find the coordinates of the intersection point, $L = AP \bigcap BC$.

The line AP is known as a *cevian* through A, named after Ceva's theorem. This can easily be proved using areal coordinates.



6. Let L, M and N lie on sides BC, CA and AB, respectively, of a triangle ABC. Show that AL, BM and CN are concurrent if and only if $\frac{\overrightarrow{BL}}{\overrightarrow{LC}} \cdot \frac{\overrightarrow{CM}}{\overrightarrow{MA}} \cdot \frac{\overrightarrow{AN}}{\overrightarrow{NB}} = 1$. **[Ceva's theorem]**

7. Suppose we have a point P, and draw the three cevians through it to meet the sides BC, CA and AB at L, M and N, respectively. Reflect L in the perpendicular bisector of BC to obtain L', and define M' and N' similarly. Prove that AL', BM' and CN' are concurrent. **[Existence of isotomic conjugates]**

8. Let L, M and N lie on sides BC, CA and AB, respectively, of a triangle ABC. Show that AL, BM and CN are concurrent if and only if $\frac{\sin \angle LAB}{\sin \angle CAL} \cdot \frac{\sin \angle MBC}{\sin \angle ABM} \cdot \frac{\sin \angle NCA}{\sin \angle BCN} = 1$. **[Trigonometric Ceva's theorem]**

9. Suppose we have a point P, and draw the three cevians through it. Reflect the cevian through A in the line AI, and repeat for the other two cevians. Prove that these three new lines are concurrent. **[Existence of isogonal conjugates]**

In unnormalised areal coordinates, the isotomic conjugate of (x, y, z) is given by $(1/x, 1/y, 1/z)$ and the isogonal conjugate of (x, y, z) is $\left(a^2/x, b^2/y, c^2/z\right)$. The *symmedian point* (intersection of the reflections of the medians in the corresponding angle bisectors) is defined as the isogonal conjugate of the centroid, giving it unnormalised areal coordinates $\left(a^2, b^2, c^2\right)$. Here are the unnormalised coordinates of common triangle centres:

Point	*Unnormalised* areal coordinates of point		
	x	y	z
Vertex A	1	0	0
Centroid	1	1	1
Incentre	a	b	c
Excentre opposite A	$-a$	b	c
Nagel point	$s-a$	$s-b$	$s-c$
Gergonne point	r_A	r_B	r_C
Symmedian point	a^2	b^2	c^2
Circumcentre	$\sin(2A)$	$\sin(2B)$	$\sin(2C)$
Orthocentre	$\tan(A)$	$\tan(B)$	$\tan(C)$
Nine−point centre	$\sin(2B)+\sin(2C)$	$\sin(2A)+\sin(2C)$	$\sin(2A)+\sin(2B)$
First Brocard point	$\frac{1}{b^2}$	$\frac{1}{c^2}$	$\frac{1}{a^2}$
Second Brocard point	$\frac{1}{c^2}$	$\frac{1}{a^2}$	$\frac{1}{b^2}$

The circumcentre, orthocentre and nine-point centre also have non-trigonometric forms (expressed in terms of a^2, b^2 and c^2 alone). However, they are even more complicated than the trigonometrical expressions here, so their practical use is unrecommended. If these points are involved, it is better to use the parametrisation involving complex numbers.

By analogy with the first and second Brocard points, the triangle centre with areal coordinates $\left(\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}\right)$ is known as the *third Brocard point*. Apart from being the isotomic conjugate of the symmedian point, it is completely boring.

10. Using the formulae for isogonal conjugates, prove that the incentre and excentres indeed have the coordinates shown in the table.

11. Prove that the orthocentre has normalised coordinates $(\cot B \cot C, \ \cot C \cot A, \ \cot A \cot B)$.

12. Show that the circumcentre and orthocentre are isogonal conjugates.

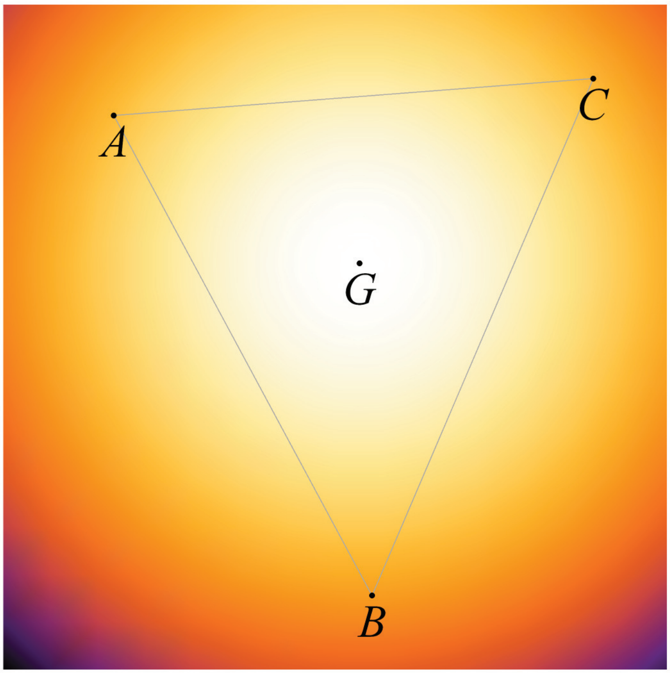
Barycentres and Huygens-Steiner

Another interpretation of the point with coordinates (x, y, z) is the *barycentre* (centre of mass) of the system where masses of x, y and z are placed at the vertices A, B and C, respectively. Hence, areal coordinates are occasionally known as *barycentric coordinates*.

13. Suppose we have a set S of masses in the plane, with total mass $m_1 + m_2 + \cdots + m_n = 1$. The mass m_i is located at the point A_i, and the barycentre is denoted P. For any point Q in the plane, define the *weighted mean square distance* $\mathbb{M}(S, Q) = \sum m_i (A_i Q)^2$. Prove that $P Q^2 = \mathbb{M}(S, Q) - \mathbb{M}(S, P)$. **[Huygens-Steiner theorem]**

This theorem is named after Jakob Steiner and Christiaan Huygens. The latter is famous for inventing the pendulum clock, proposing a wave theory of light, and discovering Titan (the largest of Saturn's many moons) with a telescope he built.

There are equivalent formulations of the Huygens-Steiner theorem in mechanics (the parallel-axis theorem) and statistics ($\sigma^2 = E(X^2) - E(X)^2$).



As a corollary of this theorem, the centroid of a set of points minimises the sum of squared distances to each of those points. This is demonstrated by the density plot of the function $A P^2 + B P^2 + C P^2$, which has a global minimum at $P = G$.

By repeated application of Huygens-Steiner, we can determine the weighted mean square distance between two sets. The ordinary version can be regarded as the case where one of the sets has a single element.

- Suppose we have two sets, S_1 and S_2, each with unit total mass. Every mass $m_i \in S_1$ is located at the point A_i; every mass $n_j \in S_2$ is located at the point B_j. The barycentres of S_1 and S_2 are denoted P_1 and P_2, respectively. We define the *weighted mean square distance* $\mathbb{M}(S_1, S_2) = \sum m_i n_j (A_i B_j)^2$. Then we have $P_1 P_2{}^2 + \mathbb{M}(S_1, P_1) + \mathbb{M}(S_2, P_2) = \mathbb{M}(S_1, S_2)$. **[Generalised Huygens-Steiner theorem]**

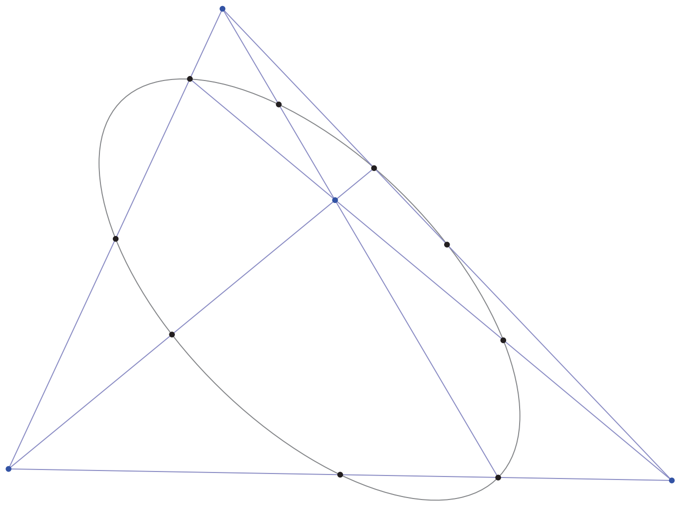
It is particularly relevant to our exploration of barycentric coordinates to consider Huygens-Steiner where $n = 3$ and the masses are positioned at the vertices of the reference triangle $A B C$.

14. If O is the circumcentre of the reference triangle ABC and P has normalised areal coordinates (x, y, z), show that $OP^2 = R^2 - \left(x\,AP^2 + y\,BP^2 + z\,CP^2\right)$.

15. Hence show that $OG^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right)$. **[Circumcentre-centroid distance]**

16. Prove that $OI^2 = R^2 - 2Rr$. **[Euler's formula]**

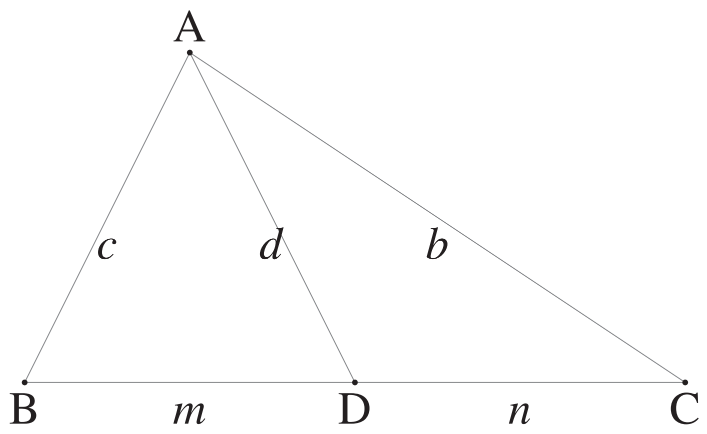
17. For every $n \geq 3$, determine all the configurations of n distinct points $X_1, X_2, \ldots, X_n$ in the plane, with the property that for any pair of distinct points X_i, X_j there exists a permutation σ of the integers $\{1, 2, \ldots, n\}$ such that $d(X_i, X_k) = d\left(X_j, X_{\sigma(k)}\right)$ for all $k \in \{1, 2, \ldots, n\}$, where $d(A, B)$ denotes the distance between A and B. **[RMM 2011, Question 5, Alexander (formerly, at the time he composed the problem, known as Luke) Betts]**



18. A quadrilateral $ABCD$ is drawn in the plane. Show that the midpoints of the four sides, midpoints of the two diagonals, intersections of opposite sides, and intersection of the diagonals all lie on a single conic. Show further that this conic cannot be a parabola. **[Nine-point conic]**

Distance geometry

19. In a triangle ABC (with the side lengths labelled in the usual way), we choose a point D on BC such that $BD = m$, $CD = n$ and $AD = d$. Prove that $man + dad = bmb + cnc$. **[Stewart's theorem]**



Stewart's theorem is particularly attractive, as it is defined solely in terms of distances and nothing else. It can be derived through simple application of the cosine rule; however, the derivation using the Huygens-Steiner theorem remains firmly within the realms of distance geometry. The statement of Stewart's theorem can be remembered with the mnemonic 'a man and his dad put a bomb in the sink'.

If $m = n$, then Stewart's theorem reduces to a special case called *Apollonius' theorem*.

- Suppose we have a triangle $A B C$, where M is the midpoint of $B C$. Then $A M^2 = \frac{1}{2} b^2 + \frac{1}{2} c^2 - \frac{1}{4} a^2$. **[Apollonius' theorem]**

A much more impressive theorem in distance geometry is that of the Cayley-Menger determinant. If $A_1 A_2 \ldots A_n A_{n+1}$ is a n-simplex with volume V, then the following identity applies.

- $-(-2)^n (n!)^2 V^2 = \det\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & A_1 A_2{}^2 & A_1 A_3{}^2 & \cdots & A_1 A_{n+1}{}^2 \\ 1 & A_2 A_1{}^2 & 0 & A_2 A_3{}^2 & \cdots & A_2 A_{n+1}{}^2 \\ 1 & A_3 A_1{}^2 & A_3 A_2{}^2 & 0 & \cdots & A_3 A_{n+1}{}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{n+1} A_1{}^2 & A_{n+1} A_2{}^2 & A_{n+1} A_3{}^2 & \cdots & 0 \end{pmatrix}$. **[Cayley-Menger determinant]**

For $n = 2$, this is equivalent to Heron's formula. For $n = 3$, this is known as *Tartaglia's formula* (remember that angry guy who solved the cubic equation?) for the volume of a tetrahedron. Equating this to zero gives an equation relating the squared distances between four coplanar points, which can itself be considered to be a generalisation of Stewart's theorem.

20. Prove that $R^2 = \frac{a^2 b^2 c^2}{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$.

Circles in areal coordinates

21. Let $A B C$ be the reference triangle, with side lengths $B C = a$, $C A = b$ and $A B = c$. Show that if two points have normalised areal coordinates $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, then $P Q^2 = -a^2 v w - b^2 w u - c^2 u v$, where $u = x_1 - x_2$, $v = y_1 - y_2$ and $w = z_1 - z_2$. **[Areal distance formula]**

By considering a circle to be the locus of points of a particular distance from a given point, we obtain the general formula for a circle.

- A circle has the equation $a^2 y z + b^2 z x + c^2 x y + (x + y + z)(A x + B y + C z) = 0$, where A, B and C are constants. **[Equation of a circle]**

The $(x + y + z)$ bracket is included to make the equation homogeneous, so that it is compatible with unnormalised coordinates. From this, we obtain the equation of a circle through three given points.

- The circle through points $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$ and $R = (x_3, y_3, z_3)$ is given by equating the determinant of the following matrix to zero: **[Concyclicity condition]**

$$\begin{pmatrix} x(x+y+z) & x_1(x_1+y_1+z_1) & x_2(x_2+y_2+z_2) & x_3(x_3+y_3+z_3) \\ y(x+y+z) & y_1(x_1+y_1+z_1) & y_2(x_2+y_2+z_2) & y_3(x_3+y_3+z_3) \\ z(x+y+z) & z_1(x_1+y_1+z_1) & z_2(x_2+y_2+z_2) & z_3(x_3+y_3+z_3) \\ a^2 yz + b^2 zx + c^2 xy & a^2 y_1 z_1 + b^2 z_1 x_1 + c^2 x_1 y_1 & a^2 y_2 z_2 + b^2 z_2 x_2 + c^2 x_2 y_2 & a^2 y_3 z_3 + b^2 z_3 x_3 + c^2 x_3 y_3 \end{pmatrix}$$

This is itself a special case of a variant of Goucher's theorem applicable to areal coordinates.

- Let $S = (x_4, y_4, z_4)$, $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$ and $R = (x_3, y_3, z_3)$. Then the determinant of the following matrix:

$$\begin{pmatrix} x_4(x_4+y_4+z_4) & x_1(x_1+y_1+z_1) & x_2(x_2+y_2+z_2) & x_3(x_3+y_3+z_3) \\ y_4(x_4+y_4+z_4) & y_1(x_1+y_1+z_1) & y_2(x_2+y_2+z_2) & y_3(x_3+y_3+z_3) \\ z_4(x_4+y_4+z_4) & z_1(x_1+y_1+z_1) & z_2(x_2+y_2+z_2) & z_3(x_3+y_3+z_3) \\ a^2 y_4 z_4 + b^2 z_4 x_4 + c^2 x_4 y_4 & a^2 y_1 z_1 + b^2 z_1 x_1 + c^2 x_1 y_1 & a^2 y_2 z_2 + b^2 z_2 x_2 + c^2 x_2 y_2 & a^2 y_3 z_3 + b^2 z_3 x_3 + c^2 x_3 y_3 \end{pmatrix}$$

  is equal to $(x_1 + y_1 + z_1)^2 (x_2 + y_2 + z_2)^2 (x_3 + y_3 + z_3)^2 (x_4 + y_4 + z_4)^2 \frac{[PQR]}{[ABC]}$ Power(S, PQR). **[Goucher's theorem for areal coordinates]**

22. Show that the power of a point $P = (x, y, z)$ (in normalised areal coordinates) with respect to the circumcircle of the reference triangle ABC is given by Power$(P, ABC) = -a^2 yz - b^2 zx - c^2 xy$. **[Power with respect to circumcircle]**

We can combine this with Huygens-Steiner to yield the following equation:

- $R^2 - OP^2 = a^2 yz + b^2 zx + c^2 xy = xAP^2 + yBP^2 + zCP^2$, where $P = (x, y, z)$ in normalised areals.

This enables us to calculate the distances between the circumcentre and several other points.

23. Hence show that $R^2 - OI^2 = 2Rr$. **[Euler's formula]**

24. Prove similarly that $R^2 - OI_A{}^2 = -2Rr_A$. **[Excentral analogue of Euler's formula]**

25. Hence prove that $OI^2 + OI_A{}^2 + OI_B{}^2 + OI_C{}^2 = 12R^2$.

26. Demonstrate also that $R^2 - OH^2 = 4[ABC] \cot A \cot B \cot C = 8R^2 \cos A \cos B \cos C$. **[Power of the orthocentre]**

As areal coordinates are projective homogeneous coordinates, conics have the general form $Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy = 0$, where A, B, C, D, E and F are constants. It is easy to see that a circle is thus a special case of a conic.

27. Let P be chosen randomly in the interior of triangle ABC, such that equal areas have equal probabilities of containing P. Find the probability that $\sqrt{[ABP]} \geq \sqrt{[BCP]} + \sqrt{[CAP]}$. **[Adapted from RMM 2008]**

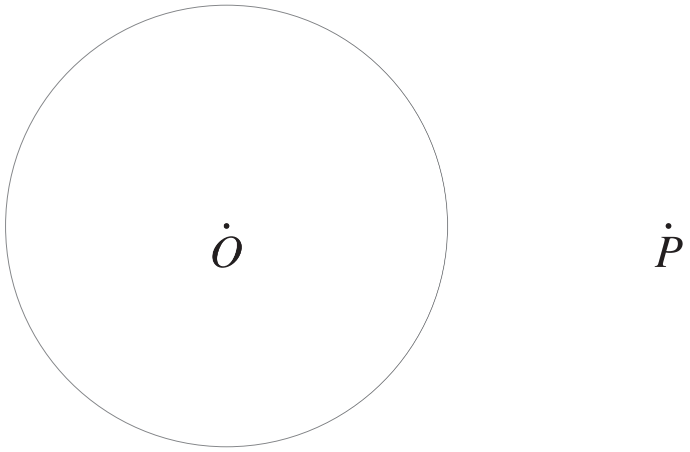
Barycentric combinations of circles

The theory of barycentric combinations of circles is a relatively recent one, emerging from the following problem:

28. Let Γ_1, Γ_2 and Γ_3 each pass through fixed points A and B. Let a line l pass through B and meet the circles again at P_1, P_2 and P_3. Prove that the ratio $P_1P_2 : P_2P_3$ is independent of l. **[Adapted from APMO 2012, Question 4]**

The original problem was solved in many unique ways by members of the British IMO squad, using techniques such as spiral similarity, vectors, coordinates, inversion, trigonometry and similar triangles.

Let Γ be a circle with centre O and radius r. We uniformly distribute a unit mass around the circumference of Γ. By applying the Huygens-Steiner theorem, we can deduce that $\mathbb{M}(\Gamma, P) = \mathbb{M}(\Gamma, O) + OP^2 = r^2 + OP^2 = 2r^2 + \text{Power}(P, \Gamma)$.



Suppose we have n circles in the plane, $\{\Gamma_1, \ldots, \Gamma_n\}$, each considered to have a mass m_i such that $m_1 + m_2 + \ldots + m_n = 1$. We then define an 'average circle' F such that $\text{Power}(P, \text{F}) = \sum(m_i \text{Power}(P, \Gamma_i))$. This is possible by considering the equation for the power of a point in either Cartesian or areal coordinates.

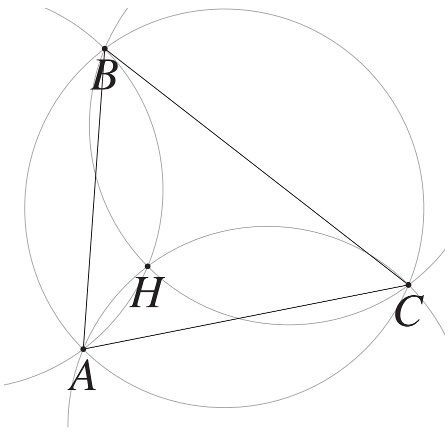
29. Prove that the centre of F is the weighted centroid $G = \sum(m_i O_i)$, where O_i is the centre of Γ_i. **[Barycentric combination of circles]**

It is now possible to determine the radius of F.

- $R^2 = -\text{Power}(G, \text{F}) = \sum m_i(r_i^2 - GO_i^2)$, where the circle Γ_i has centre O_i and radius r_i. **[Radius of barycentric circle]**

Certain barycentric combinations of circles are interesting.

30. Let ABC be a triangle with circumcircle Γ. Γ_A, Γ_B and Γ_C are the reflections of Γ in the sides BC, CA and AB, respectively. Show that the average of the three circles Γ_A, Γ_B and Γ_C is the Euler-Apollonius lollipop.



31. What is the average of the four circles Γ, Γ_A, Γ_B, Γ_C?

If we have four points A, B, C, D which do not form a cyclic quadrilateral, then every circle in the plane can be expressed uniquely as a barycentric combination of the four circles BCD, CDA, DAB, ABC. In other words, the set of circles on the plane is isomorphic to a subset of projective three-space $\mathbb{P}^3$. This idea of giving things other than points coordinates is not a new one; Plücker created a geometry based on the four-dimensional space of lines in $\mathbb{R}^3$.

Solutions

1. By using $V = \frac{1}{3}A\,h$, where A is the base area and h is the perpendicular height (i.e. distance from the origin to the reference plane), we have $[P\,B\,C] = \frac{3}{h}[R\,P\,B\,C] = \frac{1}{2\,h}\det\begin{pmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{x}{2\,h}$. For the point A, $x = 1$ so this area is equal to $\frac{1}{2\,h}$. Hence, $[P\,B\,C] = x\,[A\,B\,C]$.

2. If P lies on the line $B\,C$, the area x is obviously zero (as $X\,B\,C$ is a straight line). By symmetry, we obtain the equations of the other two lines.

3. For the centroid, we have $[G\,B\,C] = [G\,C\,A] = [G\,A\,B]$. When normalised so these areas sum to unity, they must each equal $\frac{1}{3}$.

4. This is the same argument as in the first question, but with the tetrahedra $S\,P\,Q\,R$ and $A\,B\,C\,R$.

5. We can scale the coordinates of P such that $x_1 = 1$. As the line passes through $\left(1, \frac{y_1}{x_1}, \frac{z_1}{x_1}\right)$ and $(1, 0, 0)$, it must obviously be the equation $y_1\,z = z_1\,y$. Hence, the intersection point has unnormalised coordinates $(0,\ y_1,\ z_1)$.

6. Assume P exists and $P = (x_1,\ y_1,\ z_1)$. By the previous result, we have $\frac{\overrightarrow{B\,L}}{\overrightarrow{L\,C}} = \frac{z_1}{y_1}$. The cyclic product is $\frac{\overrightarrow{B\,L}}{\overrightarrow{L\,C}} \cdot \frac{\overrightarrow{C\,M}}{\overrightarrow{M\,A}} \cdot \frac{\overrightarrow{A\,N}}{\overrightarrow{N\,B}} = \frac{x_1\,y_1\,z_1}{x_1\,y_1\,z_1} = 1$. For the converse result, we know that there must exist precisely one point L on $B\,C$ such that $A\,L$, $B\,M$ and $C\,N$ are concurrent, and it must thus be the case where $\frac{\overrightarrow{B\,L}}{\overrightarrow{L\,C}} \cdot \frac{\overrightarrow{C\,M}}{\overrightarrow{M\,A}} \cdot \frac{\overrightarrow{A\,N}}{\overrightarrow{N\,B}} = 1$.

7. This process effectively 'flips' each fraction in Ceva's theorem, so the product remains equal to unity and thus the cevians are concurrent.

8. By the sine rule, we have $\frac{\sin\angle L\,A\,B}{\sin B} = \frac{\overrightarrow{B\,L}}{L\,A}$ and $\frac{\sin\angle C\,A\,L}{\sin C} = \frac{\overrightarrow{L\,C}}{L\,A}$. Dividing one by the other results in $\frac{\sin\angle L\,A\,B}{\sin\angle C\,A\,L} = \frac{\overrightarrow{B\,L}}{\overrightarrow{L\,C}}$. Substituting cyclic permutations of this into the Ceva equation yields $\frac{\sin\angle L\,A\,B}{\sin\angle C\,A\,L} \cdot \frac{\sin\angle M\,B\,C}{\sin\angle A\,B\,M} \cdot \frac{\sin\angle N\,C\,A}{\sin\angle B\,C\,N} = 1$.

9. This process effectively 'flips' each fraction in trigonometric Ceva's theorem, so the product remains equal to unity and thus the Cevians are concurrent.

10. The incentre and excentres must be their own isogonal conjugates, thus have unnormalised areal coordinates $(\pm a,\ \pm b,\ \pm c)$. The incentre is the only one with a symmetrical expression, $(a,\ b,\ c)$.

11. Using the identity $\tan A \tan B \tan C = \tan A + \tan B + \tan C$, we can divide each term in the expression $(\tan A,\ \tan B,\ \tan C)$ by $\tan A \tan B \tan C$ to obtain $(\cot B \cot C,\ \cot C \cot A,\ \cot A \cot B)$.

12. $B\,O\,A$ is isosceles, so $\angle O\,A\,B = \frac{\pi}{2} - C$. As $A\,H$ is perpendicular to $B\,C$, we have $\angle C\,A\,O = \frac{\pi}{2} - C$. By symmetry, we are done.

13. In Cartesian coordinates, let mass m_1 be placed at $A_1 = (x_1,\ y_1)$, *et cetera*, and let $\sum m_i = 1$. Let $P = (u,\ v)$ be the barycentre, and $Q = (x,\ y)$ be a variable point. By Pythagoras' theorem, we have

$A_1 Q^2 = (x - x_1)^2 + (y - y_1)^2 = x^2 + y^2 - 2 x_1 x - 2 y_1 y + c_1$, where c_1 is a constant term that doesn't really matter. Repeat for all points in this manner, and calculate the weighted sum. The weighted mean square distance is given by $\mathbb{M}(S, Q) = x^2 + y^2 - 2 u x - 2 v y + c$, where c is another unimportant constant. But this is just $(x - u)^2 + (y - v)^2 + k$, for some constant k, or $P Q^2 + k$. Substituting $P = Q$ gives $k = \mathbb{M}(S, P)$. *(Normally, one would not use Cartesian coordinates to solve a problem. However, in RMM 2011, I was under the influence of alcohol, so actually successfully performed this derivation.)*

14. Invoking the Huygens-Steiner theorem once again, we obtain $O P^2 = x R^2 + y R^2 + z R^2 - \left(x A P^2 + y B P^2 + z C P^2\right) = R^2 - \left(x A P^2 + y B P^2 + z C P^2\right)$.

15. G has normalised areals $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, giving us $O G^2 = R^2 - \frac{1}{3}\left(A G^2 + B G^2 + C G^2\right)$. If D is the midpoint of $B C$, we obtain $A D^2 = \frac{1}{2} b^2 + \frac{1}{2} c^2 - \frac{1}{4} a^2$ from Stewart's theorem. Multiplying by $\frac{4}{9}$ results in $A G^2 = \frac{2}{9} b^2 + \frac{2}{9} c^2 - \frac{1}{9} a^2$, hence the cyclic sum $A G^2 + B G^2 + C G^2 = \frac{1}{3}\left(a^2 + b^2 + c^2\right)$. Substituting this into the expression for $O G^2$ yields the desired formula.

16. I has normalised areals $\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)$, giving us $O I^2 = R^2 - \frac{1}{a+b+c}\left(a A I^2 + b B I^2 + c C I^2\right)$. Applying Pythagoras' theorem yields $A I^2 = r^2 + (s - a)^2 = r^2 + s^2 - 2 a s + a^2$. Hence, $a A I^2 + b B I^2 + c C I^2 = (a + b + c) r^2 + (a + b + c) s^2 - 2\left(a^2 + b^2 + c^2\right) s + \left(a^3 + b^3 + c^3\right)$. We can then convert this into an expression in terms of R, r and s, namely $2 s r^2 + 2 s^3 + 4 s\left(r^2 + 4 R r - s^2\right) + 2 s\left(s^2 - 3 r^2 - 6 R r\right) = 4 R r s$. Hence, $O I^2 = R^2 - \frac{4 R r s}{2 s} = R^2 - 2 R r$.

17. The sum of squared distances from each point to the other points is constant. Hence, using the Huygens-Steiner theorem, all points must be concyclic. By considering the closest pairs of points, the points must be the vertices of a regular polygon or truncated regular polygon.

18. Apply an affine transformation to make D the orthocentre of $A B C$. Then, those nine points lie on a conic (the nine-point circle), and T is the barycentre of $A B C D$. Reversing the affine transformation results in a conic passing through those nine points; the centre of the conic is the barycentre of $A B C D$. However, a parabola has no centre, so the conic cannot possibly be a parabola.

19. D has normalised areals $\left(0, \frac{n}{a}, \frac{m}{a}\right)$. Using the Huygens-Steiner theorem, we have $A D^2 = \frac{n}{a} A B^2 + \frac{m}{a} A C^2 - \frac{n}{a} D B^2 - \frac{m}{a} D C^2$, or $d^2 = \frac{c^2 n}{a} + \frac{b^2 m}{a} - \frac{m^2 n}{a} - \frac{n^2 m}{a}$. Multiplying through by a gives us the theorem, $m a n + d a d = b m b + c n c$.

20. By considering the circumcentre and three vertices, $\det\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & R^2 & R^2 & R^2 \\ 1 & R^2 & 0 & a^2 & b^2 \\ 1 & R^2 & a^2 & 0 & c^2 \\ 1 & R^2 & b^2 & c^2 & 0 \end{pmatrix} = 0$. We now subtract the last row from the second, third and fourth rows to obtain $\det\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -R^2 & R^2 - b^2 & R^2 - c^2 & R^2 \\ 0 & 0 & -b^2 & a^2 - c^2 & b^2 \\ 0 & 0 & a^2 - b^2 & -c^2 & c^2 \\ 1 & R^2 & b^2 & c^2 & 0 \end{pmatrix} = 0$. We can now use the recursive formula to reduce this to the 4×4 determinant $\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ -R^2 & R^2 - b^2 & R^2 - c^2 & R^2 \\ 0 & -b^2 & a^2 - c^2 & b^2 \\ 0 & a^2 - b^2 & -c^2 & c^2 \end{pmatrix} = 0$.
Subtract the fourth column from the other three columns, giving the equivalent equation

$\det\begin{pmatrix} 0 & 0 & 0 & 1 \\ -2R^2 & -b^2 & -c^2 & R^2 \\ -b^2 & -2b^2 & a^2-b^2-c^2 & b^2 \\ -c^2 & a^2-b^2-c^2 & -2c^2 & c^2 \end{pmatrix} = 0$. We can then apply the recursive formula to give the 3×3 determinant $\det\begin{pmatrix} 2R^2 & b^2 & c^2 \\ b^2 & 2b^2 & b^2+c^2-a^2 \\ c^2 & b^2+c^2-a^2 & 2c^2 \end{pmatrix} = 0$. It is now convenient to use the Rule of Sarrus to evaluate this directly, resulting in the equation $8R^2 b^2 c^2 + 2b^2 c^2(b^2+c^2-a^2) = 2R^2(b^2+c^2-a^2)^2 + 2b^2c^4 + 2c^2b^4$. Dividing throughout by two and rearranging gives $R^2\left((2bc)^2 - (b^2+c^2-a^2)^2\right) = a^2b^2c^2$. Applying the difference of two squares to the left hand side yields $R^2(a^2-(b-c)^2)\left((b+c)^2-a^2\right) = a^2b^2c^2$. Another couple of applications enables further factorisation to $R^2(a+b+c)(a+b-c)(a-b+c)(-a+b+c) = a^2b^2c^2$.

21. Represent A, B and C with complex numbers l, m and n, respectively, where $l\,l^* + m\,m^* + n\,n^* = R^2$. Then we have $p = x_1 l + y_1 m + z_1 n$ and $q = x_2 l + y_2 m + z_2 n$. Subtracting them results in $p - q = u\,l + v\,m + w\,n$. Multiplying by its complex conjugate gives the squared modulus $(p-q)(p^*-q^*) = (u^2+v^2+w^2)R^2 + u\,v\,l\,m^* + u\,v\,m\,l^* + v\,w\,m\,n^* + v\,w\,n\,m^* + w\,u\,n\,l^* + w\,u\,l\,n^*$. As $u+v+w=0$, $u^2+v^2+w^2 = -(2uv+2vw+2wu)$. Applying this substitution gives $(p-q)(p^*-q^*) = \sum_{\text{cyc}}\left(u\,v(l\,m^* + m\,l^* - 2R^2)\right) = -\sum_{\text{cyc}}(u\,v(l-m)(l^*-m^*))$. The final expression is equal to $-u\,v\,c^2 - v\,w\,a^2 - w\,u\,b^2$, as required.

22. Using Goucher's theorem, we have $\det\begin{pmatrix} x(x+y+z) & 1 & 0 & 0 \\ y(x+y+z) & 0 & 1 & 0 \\ z(x+y+z) & 0 & 0 & 1 \\ a^2yz+b^2zx+c^2xy & 0 & 0 & 0 \end{pmatrix} = \frac{[ABC]}{[ABC]}\,\text{Power}(P, ABC)$. This neatly multiplies out to give $-a^2yz - b^2zx - c^2xy = \text{Power}(P, ABC)$.

23. $OI^2 - R^2$ is minus the power of I with respect to the circumcircle of ABC, so is equal to $\frac{a^2yz+b^2zx+c^2xy}{(x+y+z)^2} = \frac{a^2bc+b^2ca+c^2ab}{(a+b+c)^2} = \frac{abc}{a+b+c}$. We can express this in terms of R, r and s, obtaining $\frac{4Rrs}{2s} = 2Rr$.

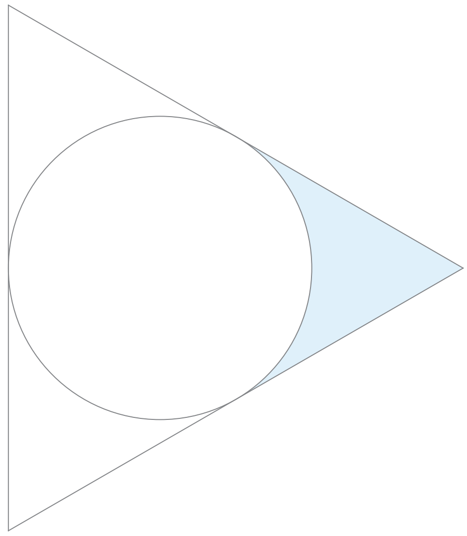
24. $\frac{a^2yz+b^2zx+c^2xy}{(x+y+z)^2} = \frac{a^2bc-b^2ca-c^2ab}{(-a+b+c)^2} = -\frac{abc}{(-a+b+c)} = -\frac{2Rrs}{s-a} = -2Rr_A$.

25. Using Euler's formula on each tritangential circle, this is equal to $4R^2 + 2R(r_A + r_B + r_C - r)$. It was proved in an earlier exercise that $r_A + r_B + r_C - r = 4R$.

26. The orthocentre has unnormalised areal coordinates $(\tan A, \tan B, \tan C)$. Hence, minus the power of H with respect to the circumcircle is $\frac{a^2yz+b^2zx+c^2xy}{(x+y+z)^2} = \frac{a^2\tan B\tan C + b^2\tan C\tan A + c^2\tan A\tan B}{\tan^2 A\tan^2 B\tan^2 C} = \frac{a^2\cot A + b^2\cot B + c^2\cot C}{\tan A\tan B\tan C}$. Remembering that $\cot A = \frac{b^2+c^2-a^2}{4[ABC]}$, the numerator is equal to $\frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{4[ABC]} = 4[ABC]$. (The last step is from squaring Heron's formula.) This results in $R^2 - OH^2 = 4[ABC]\cot A\cot B\cot C$. For the second part of the problem, we use $[ABC] = \frac{abc}{4R} = 2R^2\sin A\sin B\sin C$. Substituting this into the previous formula gives us $8R^2\cos A\cos B\cos C$, as required.

27. As the question is a homogeneous function in areas, we can apply an affine transformation and consider the case of the equilateral triangle. In areal coordinates, the inequality becomes $\sqrt{c} \geq \sqrt{a} + \sqrt{b}$. The set of points for which $\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(-\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}-\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)$ is non-negative is the interior of the conic $a^2+b^2+c^2 \leq 2ab+2bc+2ca$. As the conic passes through $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$,

$\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$ and cyclic permutations thereof, it must be the incircle. Using the formula for the area of a circle, we have that the probability that it lands within the conic is $\frac{\pi\sqrt{3}}{9}$. Hence, the probability that it lands outside the conic must be $\frac{9-\pi\sqrt{3}}{9}$. However, this could occur if any of $\sqrt{c} \geq \sqrt{a} + \sqrt{b}$, $\sqrt{a} \geq \sqrt{b} + \sqrt{c}$ or $\sqrt{b} \geq \sqrt{c} + \sqrt{a}$ are true. By symmetry, we actually want one-third of this, namely $\frac{9-\pi\sqrt{3}}{27}$.



28. Invert about B. The three circles are mapped to lines through A' (the inverse of A), and the line l remains invariant. The cross-ratio $(\infty', P_2'; P_1', P_3')$ is independent of l, as we can view A' as a projector. This is the same as the original cross-ratio $(\infty, P_2; P_1, P_3)$, so that must also be independent of l. As one of the points is infinity, the simple ratio $P_1 P_2 : P_2 P_3$ also remains constant.

29. By Huygens-Steiner, $\sum(m_i \mathbb{M}(\Gamma_i, P)) = \sum(m_i \mathbb{M}(\Gamma_i, G)) + P G^2$. Subtracting $2\sum\left(m_i r_i^2\right)$ from each side gives us $\sum(m_i \operatorname{Power}(P, \Gamma_i)) = \sum(m_i \operatorname{Power}(G, \Gamma_i)) + P G^2$, or $\operatorname{Power}(P, \mathrm{F}) = \operatorname{Power}(G, \mathrm{F}) + P G^2$, so G must be the centre of F.

30. The three circles all pass through the orthocentre H, so H must lie on F. $O_A O_B O_C$ is a dilated copy of the medial triangle $L M N$, which has orthocentre O, therefore we can deduce that O is the orthocentre of $O_A O_B O_C$. As H is the circumcentre of $O_A O_B O_C$, the centroid of $O_A O_B O_C$ (and thus centre of F) must be the point Q on the Euler line of $A B C$ halfway between G and H. As F also passes through H, it must necessarily be the Euler-Apollonius lollipop.

31. This is the weighted barycentre of the Euler-Apollonius lollipop and the circumcircle, where the former has mass $\frac{3}{4}$ and the latter has mass $\frac{1}{4}$. Hence, F must have centre T, *i.e.* the centre of the nine-point circle. Let X be the radius of the barycentric circle F. Using the radius formula, we have
$X^2 = \frac{3}{4}\left(G Q^2 - T Q^2\right) + \frac{1}{4}\left(R^2 - O T^2\right)$. Let $p = T Q$. By the basic ratios along the Euler line, this is
$\frac{3}{4}\left(4 p^2 - p^2\right) + \frac{1}{4}\left(R^2 - 9 p^2\right) = \frac{1}{4} R^2$. Hence, $X = \frac{1}{2} R$ and thus F is the nine-point circle.