Functional Equations

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Introduction

This is a brief set of notes on functional equations. It is one of the harder and less popular areas among Olympiad problems, but yet, it is very important to know. This is mainly because the solution to a functional equation problem depends heavily on what is given about the function in question. Sometimes, it is not easy to determine where to begin.

We will nevertheless look at some universal concepts and strategies which may be useful. A certain method may work in one problem but not in another one.

Therefore, the best way to get better at solving functional equations is just to do more and more of them!

Continuity

Usually, a function defined on an interval $I \subseteq \mathbb{R}$ has many different characteristics than one which is defined on $\mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{N}$. When $f(x)$ is defined on an interval $I \subseteq \mathbb{R}$, one thing that we can talk about is the concept of *continuity* (another concept is *differentiability*, but we will not talk about this here).

Definition 1 *Let $I \subseteq \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ be a function. Then*

- *$f(x)$ is* continuous at $a \in I$ *if $f(x) \to f(a)$ as $x \to a$.*
- *$f(x)$ is* continuous on I *if it is continuous at every $a \in I$.*

Intuitively, a continuous function is function whose graph does not "break up". But one should only view this idea informally, since there are functions whose graphs cannot be drawn!

If we know that a function $f(x)$ is continuous on some interval I, then the following theorem may be useful.

Theorem 1 (Intermediate Value Theorem) *Let $I \subseteq \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ be a continuous function on I. Suppose that $a, b \in I$, with $a < b$. Then, for every t lying between $f(a)$ and $f(b)$ (including $f(a)$ and $f(b)$), there exists c with $a \leq c \leq b$, such that $f(c) = t$.*

Other Properties of Functions

Definition 2 *In the following, let $f : S \to T$, where $S, T \subseteq \mathbb{R}$.*

- $f(x)$ *is* increasing on S *if* $a < b$ *implies that* $f(a) \leq f(b)$, *and* decreasing on S *if* $a < b$ *implies that* $f(a) \geq f(b)$.

- $f(x)$ *is* strictly increasing on S *if* $a < b$ *implies that* $f(a) < f(b)$, *and* strictly decreasing on S *if* $a < b$ *implies that* $f(a) > f(b)$.

- $f(x)$ *is* (strictly) monotonic *if it is either (strictly) increasing or (strictly) decreasing.*

- $f(x)$ *is* one-to-one, *or* injective, *if* $f(a) = f(b)$ *implies that* $a = b$ *(In other words, no value of* T *may be taken by* $f(x)$ *more than once).*

- $f(x)$ *is* onto, *or* surjective, *if for every* $t \in T$, *there exists* $a \in S$ *such that* $f(a) = t$.

- $f(x)$ *is* bijective *if it is both injective and surjective.*

- $a \in S$ *is a* fixed point of $f(x)$ *if* $f(a) = a$.

- *Suppose that* S *is symmetric about* 0. *That is,* $x \in S$ *if and only if* $-x \in S$. *Then,* $f(x)$ *is an* even function *if* $f(-x) = f(x)$ *for all* $x \in S$, *and an* odd function *if* $f(-x) = -f(x)$ *for all* $x \in S$. *Note that if* $f(x)$ *is an odd function and* $0 \in S$, *then* $f(0) = 0$.

Roughly speaking, a bijective function "pairs off" each element of S with an element of T, in such a way that every element of T does get paired up. If $f : S \to T$ is bijective, then its inverse $f^{-1} : T \to S$ exists, is unique, and satisfies $(f^{-1} \circ f)(x) = x$ for all $x \in S$ and $(f \circ f^{-1})(x) = x$ for all $x \in T$.

Some Well-known Examples

The following is a list of some simple, well-known examples of functional equations.

(a) $f(x + y) = f(x) + f(y)$ (Cauchy's equation)

(b) $f(x + y) = f(x)f(y)$

(c) $f(xy) = f(x) + f(y)$

(d) $f(xy) = f(x)f(y)$

(e) $f\Big(\dfrac{x+y}{2}\Big) = \dfrac{1}{2}(f(x) + f(y))$ (Jensen's equation)

Let us solve equation (a).

Firstly, setting $x = y = 0$ gives $f(0) = 2f(0)$, so $f(0) = 0$.

Next, setting $x = y$, we get $f(2x) = 2f(x)$. An easy induction argument shows that

$$f(nx) = nf(x) \quad \text{for } n \in \mathbb{N}. \tag{1}$$

Setting $x = 1$ in (1) gives $f(n) = an$ for all $n \in \mathbb{N}$, where $a = f(1)$. Now, for $x \in \mathbb{Q}^+$, where $\mathbb{Q}^+$ denotes the positive rational numbers, setting $x = \frac{m}{n}$ in (1) gives $nf(x) = f(nx) = f(m) = am$, so $f(x) = ax$ for $x \in \mathbb{Q}^+$. For negative rationals, if $x \in \mathbb{Q}^+$, $f(-x) + f(x) = f(-x + x) = f(0) = 0$, so $f(-x) = -f(x) = a(-x)$. Hence $f(x) = ax$ for all $x \in \mathbb{Q}$.

If $f(x)$ is assumed to be continuous on $\mathbb{R}$, then taking $\alpha \in \mathbb{R}$ and a sequence of rationals α_n converging to α, we have $f(\alpha) = f(\lim\limits_{n\to\infty} \alpha_n) = \lim\limits_{n\to\infty} a\alpha_n = a\alpha$. So $f(x) = ax$ for all $x \in \mathbb{R}$ if $f(x)$ is continuous.

The solutions to the others are: (b) $f(x) = a^x$ and $f(x) \equiv 0$, (c) $f(x) = a\log x$ for $x > 0$, (d) $f(x) = x^k$ and $f(x) \equiv 0$, (e) $f(x) = ax + b$. Here, a, b and k are arbitrary constants.

More Examples of Functional Equations

1. Functions on $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ (or other similar sets)

If a functional equation involves a function $f(x)$ which has $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ (or something similar) as its domain, then the following could be possible strategies.

- We can try and work out $f(0)$, $f(1), \dots$ (provided 0, 1, etc. are in the domain of $f(x)$).
- Then we may try to proceed and prove things about $f(x)$, possibly by induction.

Example 1. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a function such that $f(n+1) > f(f(n))$ for all $n \in \mathbb{N}$. Prove that $f(n) = n$ for all $n \in \mathbb{N}$.

Solution. We first claim that $f(1) < f(2) < f(3) < \cdots$. This follows if we can show that, for every $n \geq 1$, $f(n)$ is the unique smallest element of $\{f(n), f(n+1), f(n+2), \dots\}$. We proceed by induction on n. Firstly, for $m \geq 2$, $f(m) > f(f(m-1))$. Since $f(m-1) \in \{1, 2, 3, \dots\}$, this means that $f(m)$ cannot be the smallest of $\{f(1), f(2), f(3), \dots\}$. Since

$\{f(1), f(2), f(3), \dots\}$ is bounded below by 1, it follows that $f(1)$ must be the unique smallest element of $\{f(1), f(2), f(3), \dots\}$.

Now, suppose that $f(n)$ is the smallest element of $\{f(n), f(n+1), f(n+2), \dots\}$. Let $m > n+1$. By the induction hypothesis, $f(m-1) > f(n)$. Since $f(n) > f(n-1) > \cdots > f(1) \geq 1$, we have $f(n) \geq n$, and so $f(m-1) \geq n+1$, so $f(m-1) \in \{n+1, n+2, \dots\}$. But $f(m) > f(f(m-1))$. So $f(m)$ is not the smallest in $\{f(n+1), f(n+2), \dots\}$. Since $\{f(n+1), f(n+2), \dots\}$ is bounded below, it follows that $f(n+1)$ is the unique smallest element of $\{f(n+1), f(n+2), \dots\}$.

Now, since $1 \leq f(1) < f(2) < f(3) < \cdots$, clearly we have $f(n) \geq n$ for all n. But if $f(n) > n$ for some n, then $f(f(n)) \geq f(n+1)$, a contradiction. So we have $f(n) = n$ for all $n \in \mathbb{N}$.

2. Functions on $\mathbb{R}$ or an Interval $I \subseteq \mathbb{R}$

If a functional equation involves a function $f(x)$ which has $\mathbb{R}$ or an interval $I \subseteq \mathbb{R}$ as its domain, then there are several things that we may be able to do.

- If the functional equation is true for any two variables x and y, then we may try to let $x = y$ and get a functional equation which is true for all x.
- We may also be able to make a clever choice for x or y to make the functional equation become pleasant.
- Again, we can try and determine $f(0)$, $f(1), \dots$ (if 0 or 1, etc. are in the domain).
- Sometimes, one may be able to determine the fixed points of $f(x)$.

Example 2. Let $\mathbb{R}^+ = (0, +\infty)$. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that (i) $f(xf(y)) = yf(x)$ for all $x, y \in \mathbb{R}^+$, and (ii) $f(x) \to 0$ as $x \to +\infty$.

Solution. We first establish that $f(1) = 1$. Let $f(1) = a$. Setting $x = y = 1$ in the given functional equation, we have $f(a) = a$. Now set $x = 1$ and $y = a$. Then $f(f(a)) = a^2$, so $a = a^2$, so $a = 1$ since $a > 0$.

Next, suppose that b is a fixed point of $f(x)$. By easy induction, one has $f(b^k) = b^k$ for $k = 1, 2, 3, \dots$. Also, letting $x = \frac{1}{b}$ and $y = b$, we have $bf(\frac{1}{b}) = f(\frac{1}{b}f(b)) = 1$, so $f(\frac{1}{b}) = \frac{1}{b}$. Again by induction, we have $f(b^k) = b^k$ for $k = -1, -2, -3, \dots$. So $f(b^k) = b^k$ for all $k \in \mathbb{Z}$. Now, if $b > 1$, for any $\varepsilon > 0$, $f(b^k) > \varepsilon$ for infinitely many k, contradicting condition (ii). A similar argument holds if $b < 1$. It follows that $b = 1$.

Finally, setting $y = x$ in the functional equation, we have $f(xf(x)) = xf(x)$ for all $x \in \mathbb{R}^+$. But since 1 is the unique fixed point, we have $xf(x) = 1$, or $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}^+$.

It is easy to check that $f(x) = \frac{1}{x}$ does satisfy both conditions (i) and (ii). Indeed, if $f(x) = \frac{1}{x}$, then

$$f(xf(y)) = f\left(\frac{x}{y}\right) = \frac{y}{x} \quad \text{and} \quad yf(x) = \frac{y}{x}.$$

Clearly, condition (ii) holds as well.

Example 3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xf(x) + f(y)) = f(x)^2 + y$ for all $x, y \in \mathbb{R}$.

Solution. We first show that $f(0) = 0$. Setting $x = 0$ gives $f(f(y)) = f(0)^2 + y$. Now setting $y = -f(0)^2$ and $k = f(y) = f(-f(0)^2)$ in this gives $f(k) = 0$. Next, setting $x = y = k$ in the given equation gives $f(0) = k$. Now setting $y = k$ and $x = 0$ in the original equation gives $k = 0$, so $f(0) = 0$.

Now, setting $x = 0$ in the original equation gives

$$f(f(y)) = y \quad \text{for all } y. \tag{2}$$

Setting $y = 0$ in the original equation gives

$$f(xf(x)) = f(x)^2 \quad \text{for all } x. \tag{3}$$

Now, for any z, setting $x = f(z)$ in (3), and using (2), $f(zf(z)) = z^2$ for all z. Using (3) again gives $f(z)^2 = z^2$ for all $z \in \mathbb{R}$. Hence $f(z) = z$ for all $z \in \mathbb{R}$, or $f(z) = -z$ for all $z \in \mathbb{R}$.

We now check that both of these functions are indeed solutions.

If $f(x) = x$, then

$$f(xf(x) + f(y)) = x^2 + y \quad \text{and} \quad f(x)^2 + y = x^2 + y.$$

If $f(x) = -x$, then

$$f(xf(x) + f(y)) = x^2 + y \quad \text{and} \quad f(x)^2 + y = x^2 + y.$$

3. Polynomial Functional Equations

If our task is to determine all polynomial solutions of a functional equation, then we can try many things which are exclusive to polynomials.

- We may use a theorem which applies to polynomials. For example, the Remainder Theorem, or the Fundamental Theorem of Algebra.

- We may also use properties of polynomials. For example, the fact that a polynomial has a degree, or the fact that a polynomial has finitely many zeros (namely, at most the degree of the polynomial).

- If we are asked to find all the polynomials of a given functional equation with domain $\mathbb{R}$, we may be better off by assuming that they are defined on $\mathbb{C}$. We can then deduce the solutions which are defined on $\mathbb{R}$.

Example 4. Find all polynomials $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2) = f(x)f(x+1)$ for all $x \in \mathbb{R}$.

Solution. We will actually find the solutions $f : \mathbb{C} \to \mathbb{C}$. It is easy to check that if $f(x)$ is constant, then $f(x) \equiv 0$ or $f(x) \equiv 1$. Now suppose that $f(x)$ is not constant. Let $a \in \mathbb{C}$ be a zero of $f(x)$, that is, $f(a) = 0$. Then $f(a^2) = f(a^4) = f(a^8) = \cdots = 0$. If $|a| > 1$, then a, a^2, a^4, $a^8, \ldots$ are distinct since they have strictly increasing moduli, and this implies that $f(x) \equiv 0$, contradiction. A similar argument holds if $0 < |a| < 1$. It follows that either $a = 0$ or $|a| = 1$ (so a is a complex number lying on the unit circle in the Argand diagram in this latter case). Now, suppose that $b + 1$ is a zero of $f(x)$. Then b^2 is also a zero of $f(x)$, and hence, so are b^4, $b^8, \ldots$. It follows that both b and $b + 1$ must either have modulus 1, or is equal to 0. So $b = -1$, 0, $e^{2\pi i/3}$ or $e^{-2\pi i/3}$, so $b+1 = 0$, 1, $e^{\pi i/3}$ or $e^{-\pi i/3}$. But if $b+1 = e^{\pi i/3}$, if we set $x = e^{2\pi i/3}$ in the original equation, we get that $(e^{2\pi i/3})^2 = e^{-2\pi i/3}$ is also a zero of $f(x)$, a contradiction (as this needs to be one of 0, 1, $e^{\pi i/3}$ or $e^{-\pi i/3}$). A similar argument shows that we may not have $b + 1 = e^{-\pi i/3}$ (whence $e^{2\pi i/3}$ is also a zero of $f(x)$).

Hence we can at least say at this stage that $f(x) = cx^m(x-1)^n$, where $c \in \mathbb{R}$ and m, $n \in \mathbb{Z}$ with $m, n \geq 0$ (by the Remainder Theorem). Now,

$$\begin{aligned} f(x^2) &= cx^{2m}(x^2-1)^n \\ f(x)f(x+1) &= c^2x^m(x+1)^m x^n(x-1)^n \end{aligned}$$

Equating the leading coefficients gives $c = c^2$, so $c = 0$ or $c = 1$. If $c = 1$, then equating the lowest power of x gives $2m = m + n$, so $m = n$. So the class of polynomials which satisfy the given equation is

$$f(x) \equiv 0 \quad \text{or} \quad f(x) = x^n(x-1)^n \quad \text{where } n \in \mathbb{Z},\, n \geq 0$$

We can easily check that these are all indeed solutions of the original functional equation.

Problems

Here are some problems. They are separated according to the three areas that we have discussed. These problems are not generally ordered by difficulty.

1. Functions with Domain $\mathbb{N}$ or $\mathbb{Q}$

1. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) + f(n) = 2n + 2004$ or $2n + 2005$ for all $n \in \mathbb{N}$.

2. Find all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n + f(n)) = 2f(n)$ for all $n \in \mathbb{N}$.

3. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = 2n$ for all $n \in \mathbb{N}$?

4. Let $\mathbb{Q}^+$ denote the positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $f(x + 1) = f(x) + 1$, and $f(x^3) = f(x)^3$ for all $x \in \mathbb{Q}^+$.

5. Let $\mathbb{N}_0$ denote the non-negative integers. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $xf(y) + yf(x) = (x + y)f(x^2 + y^2)$ for all $x,\ y \in \mathbb{N}_0$.

6. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x+y) = f(x)f(y) - f(xy)+1$ for all $x,\ y \in \mathbb{Q}$.

2. Functions with domain $\mathbb{R}$ or an Interval $I \subseteq \mathbb{R}$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(2005) = 2004$ and $f(x)f(f(x)) = 1$ for all $x \in \mathbb{R}$. Find the value of $f(1000)$.

8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x) + y) = 2x + f(f(y) - x)$ for all $x,\ y \in \mathbb{R}$.

9. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$ for all $x,\ y \in \mathbb{R}$.

10. Find all functions $f : (-1, 1) \to \mathbb{R}$ such that $(1 - x^2)f\left(\dfrac{2x}{1 + x^2}\right) = (1 + x^2)^2 f(x)$ for all $x \in \mathbb{R}$.

11. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) + f(y+z) + f(z+x) \geq 3f(x+2y+3z)$ for all $x,\ y,\ z \in \mathbb{R}$.

12. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x)f(y)f(xy)$ for all x, $y \in \mathbb{R}$.

13. Show that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ such that (i) $f(x+y) \geq f(x) + yf(f(x))$ for all x, $y \in \mathbb{R}$, and (ii) $f(0) > 0$.

14. Let $\mathbb{R}^+$ denote the positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x)f(yf(x)) = f(x+y)$ for all x, $y \in \mathbb{R}^+$.

15. Let $\mathbb{R}^+$ denote the positive real numbers. Find all strictly monotonic functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f\left(\dfrac{x^2}{f(x)}\right) = x$ for all $x \in \mathbb{R}^+$.

16. Let $\mathbb{R}^+$ denote the positive real numbers. Find all monotonic functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(xy)f\left(\dfrac{f(y)}{x}\right) = 1$ for all x, $y \in \mathbb{R}^+$.

17. Let $\mathbb{R}^+$ denote the positive real numbers. Show that there does not exist a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x)^2 \geq f(x+y)(f(x)+y)$ for all x, $y \in \mathbb{R}^+$.

18. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xy)(f(x) - f(y)) = (x-y)f(x)f(y)$ for all x, $y \in \mathbb{R}$.

19. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 - y^2) = xf(x) - yf(y)$ for all x, $y \in \mathbb{R}$.

3. Polynomial Functional Equations

20. Find all polynomials $f : \mathbb{R} \to \mathbb{R}$ such that $2(1 + f(x)) = f(x-1) + f(x+1)$ for all $x \in \mathbb{R}$.

21. Find all polynomials $f : \mathbb{R} \to \mathbb{R}$ such that $(x-16)f(2x) = 16(x-1)f(x)$ for all $x \in \mathbb{R}$.

22. Find all polynomials $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) + f(x) = x^4 + 3x^2 + 3$ for all $x \in \mathbb{R}$.

23. Find all polynomials $f : \mathbb{R} \to \mathbb{R}$ such that $f(x-1)f(x+1) = f(x^2+1)$ for all $x \in \mathbb{R}$.

24. Find all polynomials $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)^2 + 2f(x)f\left(\dfrac{1}{x}\right) + f\left(\dfrac{1}{x}\right)^2 = f(x^2)f\left(\dfrac{1}{x^2}\right)$ for all $x \in \mathbb{R} \setminus \{0\}$.