Summer 2003 IMO Camp

Solution to Open Contest

Algebra.

1. Determine the minimum value of

$$\sum_{i=1}^{10}\sum_{j=1}^{10}\sum_{k=1}^{10}|k(x+y-10i)(3x+6y-36j)(19x+95y-95k)|$$

where x and y range over all real numbers.

Solution:

Let $P_1,\ P_2,\ \ldots,\ P_n$ be points on a line in that order, where $P_i = P_{i+1}$ is permitted. We wish to minimize $XP_1 + XP_2 + \cdots + XP_n$ where X ranges over all points on that line. Note that $XP_1 + XP_n = P_1P_n$ if X is on the segment $P+1P_n$, and greater otherwise. Similarly, the minimum value of XP_2+XP_{n-1} is P_2P_{n-1}, attained if and only if X is on the segment P_2P_{n-1}. It follows that if $n = 2m+1$, the overall minimum is attained if and only if $X = P_{m+1}$, and this minimum is $P_1P_n + P_2P_{n-1} + \cdots + P_mP_{m+2}$. If $n = 2m$, then X can be any point on the segment P_mP_{m+1}, and the minimum sum is $P_1P_n + P_2P_{n-1} + \cdots + P_mP_{m+1}$. Consider now $f(x,y) = \sum_{i=1}^{10}|x+y-10i|$. We have ten points 10, 20, ..., 100. The minimum value of $f(x,y)$ is

$$10((10+9+8+7+6)-(1+2+3+4+5)) = 250,$$

attained if and only if $50 \leq x+y \leq 60$. For $g(x,y) = \sum_{j=1}^{10}|3x-6y-36|$, we have ten points 12, 24, ..., 120. The minimum value of $g(x,y)$ is

$$36((10+9+8+7+6)-(1+2+3+4+5)) = 900,$$

attained if and only if $60 \leq x-2y \leq 72$. Finally, for $\sum_{k=1}^{10}k|19x-95y-95k|$, we have fifty-five points consisting of one at 5, two at 10, three at 15 and so on, to ten at 50. The twenty-eighth point is the last one at 7. Hence the minimum value of $h(x,y)$ is

$$95((10^2+9^2+8^2)-(6\cdot 7+6^2+5^2+4^2+3^2+2^2+1^2)) = 10640,$$

attained if and only if $x+5y = 35$. It is easy to verify that $(55,-4)$ satisfies all three conditions. Hence these minimum values can be attained simultaneously, so that the minimum value of $f(x,y)g(x,y)h(x,y)$ is $250\cdot 900\cdot 10640 = 2394000000$.

2. Let $n \geq 2$ be an integer. Let $x_1,\ x_2,\ \ldots,\ x_n$ be real numbers such that

$$\sum_{i=1}^{n}x_i^2 + \sum_{i=1}^{n-1}x_ix_{i+1} = 1.$$

For any fixed k, $1 \leq k \leq n$, determine the maximum value of $|x_k|$.

First Solution:

We have $2 = x_1^2 + (x_1+x_2)^2 + (x_2+x_3)^2 + \cdots + (x_{n-1}+x_n)^2 + x_n^2$. For a fixed k, $1 \le k \le n$, it follows from the Arithmetic-Geometric Means Inequality that

$$
\begin{aligned}
& \sqrt{\frac{x_1^2 + (x_1+x_2)^2 + \cdots + (x_{k-1}+x_k)^2}{k}} \\
\ge\ & \frac{|x_1| + |x_1+x_2| + \cdots + |x_{k-1}+x_k|}{k} \\
\ge\ & \frac{|x_1 - (x_1+x_2) + \cdots + (-1)^{k-1}(x_{k-1}+x_k)|}{k} \\
=\ & \frac{|x_k|}{k}.
\end{aligned}
$$

This is equivalent to $x_1^2 + (x_1+x_2)^2 + \cdots + (x_{k-1}+x_k)^2 \ge \frac{x_k^2}{k}$. In the same way, we can prove that $(x_k+x_{k+1})^2 + \cdots + (x_{n-1}+x_n)^2 + x_n^2 \ge \frac{x_k^2}{n-k+1}$. Addition yields $2 \ge (\frac{1}{k} + \frac{1}{n-k+1})x_k^2$ or $|x_k| \le \sqrt{\frac{2k(n-k+1)}{n+1}}$ for $1 \le k \le n$. In the above inequalities, equality holds if and only if $x_1 = -(x_1+x_2) = \cdots = (-1)^{k-1}(x_{k-1}+x_k)$ and $x_k+x_{k+1} = -(x_{k+1}+x_{k+2}) = \cdots = (-1)^{n-k}x_n$. In other words, the maximum value of x_k is $\sqrt{\frac{2k(n-k+1}{n+1}}$ if $x_i = (-1)^{k-i}\frac{i}{k}x_k$ for $1 \le i \le k-1$ and $x_j = (-1)^{j-k}\frac{n_j+1}{n-k+1}x_k$ for $k+1 \le j \le n$.

Second Solution:

We first determine the maximum value of x_n. Rewrite the given equation as

$$1 = \sum_{i=1}^{n-1}(\sqrt{a_i}x_i + \sqrt{1-a_{i+1}}x_{i+1})^2 + (1-(1-a_n))x_n^2$$

for suitably chosen constants $a_1,\ a_2,\ \ldots,\ a_n$. Taking into consideration that the coefficient of x_ix_{i+1} for all i is 1 in the original equation, we have $a_1 = 1$ and $\sqrt{a_i}\sqrt{1-a_{i+1}} = 1$ for $1 \le i \le n-1$. In other words, $a_{i+1} = 1 - \frac{1}{4a_i}$ so that $a_i\frac{i+1}{2i}$ for $1 \le i \le n$. From the displayed equation, we have $1 \ge (1-(1-a_n))x_n^2 = \frac{n+1}{2n}x_n^2$ or $x_n \le \sqrt{\frac{2n}{n+1}}$. By symmetry, $x_1 \le \sqrt{\frac{2n}{n+1}}$ also. For $2 \le k \le n-1$, we have

$$
\begin{aligned}
1 \ =\ & \sum_{i=1}^{k-1}(\sqrt{a_i}x_i + \sqrt{1-a_{i+1}}x_{i+1})^2 + \sum_{i=1}^{n-k}(\sqrt{a_i}x_{n-i+1} + \sqrt{1-a_{i+1}}x_{n-i})^2 \\
& +(1-(1-a_k)-(1-a_{n-k+1}))x_k^2.
\end{aligned}
$$

From $1 \ge (1-(1-a_k)-(1-a_{n-k+1}))x_k^2$, we have $x_k \le \sqrt{\frac{2kn-k+1}{n+1}}$. Note that this coincides with previous results when $k = n$ or 1. In all cases, the maximum can be attained when

$$
\begin{aligned}
0 &= \sqrt{a_1}x_1 + \sqrt{1-a_2}x_2 \\
&= \sqrt{a_2}x_2 + \sqrt{1-a_3}x_3 \\
&= \cdots \\
&= \sqrt{a_{k-1}}x_{k-1} + \sqrt{1-a_k}x_k \\
&= \sqrt{a_{n-k}}x_{k+1} + \sqrt{1-a_{n-k+1}}x_k \\
&= \cdots \\
&= \sqrt{a_1}x_n + \sqrt{1-a_2}x_{n-1}.
\end{aligned}
$$

Starting with x_k in the middle, we can work our way outward and choose $x_{k-1},\ x_{k-2},\ \ldots,\ x_1$ as well as $x_{k+1},\ x_{k+2},\ \ldots,\ x_n$ which satisfy the above equations.

3. Let n be a positive integer. Take $x_0 = 0$. For $1 \le i \le n$, let x_i be a positive real number, where $x_1 + x_2 + \cdots + x_n = 1$. Prove that

$$1 \le \sum_{i=1}^{n} \frac{x_i}{\sqrt{1 + x_0 + x_1 + \cdots + x_{i-1}}\sqrt{x_i + x_{i+1} + \cdots + x_n}} < \frac{\pi}{2}.$$

Solution:

By the Arithmetic-Geometric Mean Inequality, we have

$$\sqrt{1 + x_0 + x_1 + \cdots + x_{i-1}}\sqrt{x_1 + x_{i+1} + \cdots + x_n} \le \frac{1}{2}(1 + x_0 + x_1 + \cdots + x_n) = 1$$

for $1 \le i \le n$. Hence

$$\sum_{i=1}^{n} \frac{x_i}{\sqrt{1 + x_0 + x_1 + \cdots + x_{i-1}}\sqrt{x_i + x_{i+1} + \cdots + x_n}} \ge \sum_{i=1}^{n} x_i = 1.$$

For $0 \le i \le n$, since $0 \le x_0 + x_1 + \cdots + x_i \le 1$, we may let $\theta_i = \arcsin(x_0 + x_1 + \cdots + x_n)$. Then $0 = \theta_0 < \theta_1 < \cdots < \theta_n = \frac{\pi}{2}$. Now

$$\begin{aligned}
\cos\theta_{i-1} &= \sqrt{1 - \sin^2\theta_{i-1}} \\
&= \sqrt{1 + (x_0 + x_1 + \cdots + x_{i-1})^2} \\
&= \sqrt{1 + x_0 + x_1 + \cdots + x_{i-1}}\sqrt{x_i + x_{i+1} + \cdots + x_n}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
x_i &= \sin\theta_i - \sin\theta_{i-1} \\
&= 2\cos\frac{\theta_i + \theta_{i-1}}{2}\sin\frac{\theta_i - \theta_{i-1}}{2} \\
&< 2\cos\theta_{i-1}\sin\frac{\theta_i - \theta_{i-1}}{2} \\
&< 2\cos\theta_{i-1}\left(\frac{\theta_i - \theta_{i-1}}{2}\right).
\end{aligned}$$

The last step follows from $\sin x < x$ whenever $0 \le x \le \frac{\pi}{2}$. Finally,

$$\sum_{i=1}^{n} \frac{x_i}{\sqrt{1 + x_0 + x_1 + \cdots + x_{i-1}}\sqrt{x_i + x_{i+1} + \cdots + x_n}} = \sum_{i=1}^{n} \frac{x_i}{\cos\theta_{i-1}} < \sum_{i=1}^{n}(\theta_i - \theta_{i-1}) = \frac{\pi}{2}.$$

4. Let $\{a_1, a_2, \ldots\}$ be a sequence of non-negative numbers such that $a_{n+m} \le a_n + a_m$ for all n and m. Prove that $a_n \le ma_1 + (\frac{n}{m} - 1)a_m$ for all $n \ge m$.

First Solution:

Note that for any k, we have $0 \le a_k \le a_{k-1} + a_1 \le a_{k-2} + 2a_1 \le \cdots \le ka_1$. If $n = m$, then $a_n \le na_1 = ma_1 + (\frac{n}{m} - 1)a_m$. Suppose $n > m$. Then

$$\frac{a_n}{n} - \frac{a_m}{m} \le \frac{a_{n-m} + a_m}{n} - \frac{a_m}{m} = \frac{ma_{n-m} - (n-m)a_m}{mn} = \frac{n-m}{m}\left(\frac{a_{n-m}}{n-m} - \frac{a_m}{m}\right).$$

If $n - m > m$, we can iterate this process. Eventually, there exists s, $1 \le s \le m$, such that $\frac{a_n}{n} - \frac{a_m}{m} \le \frac{s}{n}(\frac{a_s}{s} - \frac{a_m}{m})$. Since $a_s \le sa_1$ and $a_1 - \frac{a_m}{m} \ge 0$, we have

$$\frac{a_n}{n} - \frac{a_m}{m} \le \frac{s}{n}\left(1 - \frac{a_m}{m}\right) \le \frac{m}{n}\left(a_1 - \frac{a_m}{m}\right).$$

This is equivalent to $a_n \le ma_1 + (\frac{n}{m} - 1)a_m$.

Second Solution:

Note that for any k, we have $0 \le a_k \le a_{k-1} + a_1 \le a_{k-2} + 2a_1 \le \cdots \le ka_1$. Let m be fixed. We drop the condition $n \ge m$ and use induction to prove that the desired result holds for all n. For $n = 1$, we have $a_1 \le ma_1 + (\frac{1}{m} - 1)a_m$ is equivalent to $a_m \le ma_1$, which we have already proved. Suppose $a_n \le ma_1 + (\frac{n}{m} - 1)a_m$ for some $n \ge 1$. Consider a_{n+1}. Suppose $n < m$. Since $a_{n+1} \le (n+1)a_1$ and $a_1 - \frac{a_m}{m} \ge 0$, we have $\frac{a_{n+1}}{n+1} - \frac{a_m}{m} \le a_1 - \frac{a_m}{m} \le \frac{m}{m+1}(a_1 - \frac{a_m}{m})$. This is equivalent to $a_{n+1} \le ma_1 - (\frac{n+1}{m} - 1)a_m$. Suppose $n \ge m$. Then $n + 1 - m \ge 1$ and $a_{n+1} \le a_{n+1-m} + a_m$. By the induction hypothesis,

$$a_{n+1} \le ma_1 + \left(\frac{n+1-m}{m} - 1\right)a_m + a_m = ma_1 + \left(\frac{n+1}{m} - 1\right)a_m.$$

This completes the induction argument.

5. Let $x_1, x_2, \ldots, x_{1997}$ be real numbers such that $-\frac{1}{\sqrt{3}} \le x_i \le \sqrt{3}$ for $1 \le i \le 1997$ and $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$. Determine the maximum value of $x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}$.

Solution:

For positive real numbers m and h, $(m+h)^{12} + (m-h)^{12} = \sum_{k=0}^{12}(1+(-1)^k)\binom{12}{k}m^{12-k}h^k$ is an increasing function of h. If we have $-\frac{1}{\sqrt{3}} < x_i \le x_j < \sqrt{3}$, we can increase the value of $x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}$ without changing the value of $x_1 + x_2 + \cdots + x_{1997}$ by decreasing x_i and increasing x_j by the same amount, until either the smaller one becomes $-\frac{1}{\sqrt{3}}$ or the larger one becomes $\sqrt{3}$. It follows that if $x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}$ is maximum, then at most one of the numbers is strictly between $-\frac{1}{\sqrt{3}}$ and $\sqrt{3}$). Let there be u copies of $-\frac{1}{\sqrt{3}}$, v copies of $\sqrt{3}$ and w copies of numbers in between. We have already proved that $w = 0$ or 1, and if $w = 1$, let that number be t. Then $u + v + w = 1997$ and $-\frac{u}{\sqrt{3}} + \sqrt{3}v = wt = -318\sqrt{3}$. Eliminating u, we have $4v + (\sqrt{3}t + 1)w = 1043$. Since $0 \le (\sqrt{3}t + 1)w < 4$ and $1043 = 4(260) + 3$, we must have $v = 260$, $w = 1$ and $t = \frac{2}{\sqrt{3}}$, so that $u = 1716$. Thus the maximum value of $x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}$ is $(-\frac{1}{\sqrt{3}})^{12}u + (\sqrt{3})^{12}v + t^{12} = 189548$.

6. Let $n \ge 3$ be an integer and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$, $0 < a_1 = a_2$, $0 < b_1 \le b_2$ and for $1 \le i \le n-2$, $a_i + a_{i+1} \le a_{i+2}$ and $b_i + b_{i+1} < b_{i+2}$. Prove that $a_{n-1} + a_n \le b_{n-1} + b_n$.

Solution:

Let $c_i = b_i - a_i$ for $1 \le i \le n$. Then $c_1 + c_2 + \cdots + c_n = 0$, $c_1 \le c_2$ and $c_i + c_{i+1} \le c_{i+2}$ for $1 \le i \le n-2$. We prove by induction on n that $c_{n-1} + c_n \ge 0$, weakening the condition $c_1 + c_2 + \cdots + c_n = 0$ to $c_1 + c_2 + \cdots + c_n \ge 0$. For $n = 2$, this yields $c_1 + c_2 \ge 0$ immediately. For $n = 3$, if $c_1 < 0$, then $c_2 + c_3 = (c + 1 + c_2 + c_3) - c_1 > 0$. If $c_1 \ge 0$, then $c_2 \ge c_1 \ge 0$ and $c_3 \ge c_1 + c_2 \ge 0$, so that $c_2 + c_3 \ge 0$. Suppose that the result holds for $n - 1$ and $n \ge 3$.

Consider the next case with the numbers $c_1, c_2, \ldots, c_{n+1}$. If $c_1 + c_2 + \cdots + c_{n-1} < 0$, then $c_n + c_{n+1} = (c_1 + c_2 + \cdots + c_{n+1}) - (c_1 + c_2 + \cdots + c_{n-1}) > 0$. If $c_1 + c_2 + \cdots + c_{n-1} \geq 0$, then $c_n \geq c_{n-2} + c_{n-1} \geq 0$ by the induction hypothesis. Hence $c_1 + c_2 + \cdots + c_n \geq 0$, and $c_{n+1} \geq c_{n-1} + c_n \geq 0$ by the induction hypothesis. It follows that $c_n + c_{n+1} \geq 0$.

7. Let f be a function from the set of positive integers to itself such that $f(1) = 1$ and, for each positive integer n, $f(2n) < 6f(n)$ and $3f(n)f(2n+1) = f(2n)(1+3f(n))$. Determine all pairs (k, ℓ) such that $f(k) + f(\ell) = 293$ and $k < \ell$.

   Solution:

   We have $3f(n)(f(2n+1) - f(2n)) = f(2n) < 6f(n)$. Hence $f(2n+1) - f(2n) < 2$. Since $f(n)$ and $f(2n)$ are positive, $f(2n+1) - f(2n) \geq 1$. It follows that $f(2n+1) = f(2n) + 1$ and $f(2n) = 3f(n)$ for all positive integers n. We now prove by induction on n that if $n = 2^{a_r} + 2^{a_{r-1}} + \cdots + 2^{a_0}$, then $f(n) = 3^{a_r} + 3^{a_{r-1}} + \cdots + 3^{a_0}$. For $n = 1$, we have $f(2^0) = f(1) = 1 = 3^0$. Suppose the result holds for $1, 2, \ldots, n-1$ for some $n \geq 2$. Consider $n = 2^{a_r} + 2^{a_{r-1}} + \cdots + 2^{a_0}$. If $a_0 \geq 1$, let $m = 2^{a_r - 1} + 2^{a_{r-1}-1} + \cdots + 2^{a_0 - 1}$. Then

$$\begin{aligned} f(n) &= f(2m) \\ &= 3f(m) \\ &= 3(3^{a_r-1} + 3^{a_{r-1}-1} + \cdots + 3^{a_0-1}) \\ &= 3^{a_r} + 3^{a_{r-1}} + \cdots + 3^{a_0}. \end{aligned}$$

If $a_0 = 0$, let $m = 2^{a_r-1} + 2^{a_{r-1}-1} + \cdots + 2^{a_1-1}$. Then

$$\begin{aligned} f(n) &= f(2m+1) \\ &= 3f(m) + 1 \\ &= 3(3^{a_r-1} + 3^{a_{r-1}-1} + \cdots + 3^{a_1-1}) + 3^0 \\ &= 3^{a_r} + 3^{a_{r-1}} + \cdots + 3^{a_0}. \end{aligned}$$

This completes the induction argument. Let $k = \sum_{b \in B} 2^b$ and $\ell = \sum_{c \in C} 2^c$. Then

$$f(k) + f(\ell) = 2 \sum_{a \in B \cap C} 3^a + \sum_{b \in B-C} 3^b + \sum_{c \in C-B} 3^c.$$

If $f(k) + f(\ell) = 293 = 3^5 + 3^3 + 2 \cdot 3^2 + 3^1 + 2 \cdot 3^0$, we must have $B \cap C = \{0, 2\}$. Since $k < \ell$, we must have $5 \in C$. There are four ways to distribute 1 and 3 between B and C, yielding the following four solutions:

$$\begin{array}{ll} k = 2^2 + 2^0 = 5, & \ell = 2^5 + 2^3 + 2^2 + 2^1 + 2^0 = 47, \\ k = 2^2 + 2^1 + 2^0 = 7, & \ell = 2^5 + 2^3 + 2^2 + 2^0 = 45, \\ k = 2^3 + 2^2 + 2^0 = 13, & \ell = 2^5 + 2^2 + 2^1 + 2^0 = 39, \\ k = 2^3 + 2^2 + 2^1 + 2^0 = 15, & \ell = 2^5 + 2^2 + 2^0 = 37. \end{array}$$

8. Let $f(x) = x^3 + ax^2 + bx + c$ be any cubic polynomial.

   (a) Determine the maximum value of λ if $f(x) \geq \lambda(x-a)^3$ whenever $f(x)$ has three non-negative roots.

   (b) Determine when $f(x) = \lambda(x-a)^3$ for the maximum value of λ.

Solution:

Let the roots be $0 \leq \alpha \leq \beta \leq \gamma$. Then $f(x) = (x-\alpha)(x-\beta)(x-\gamma)$ with $\alpha = \beta + \gamma = -a$.

(a) When $0 \leq x \leq \alpha$, the Arithmetic-Geometric Means Inequality yields

$$\begin{aligned} -f(x) &= (\alpha - x)(\beta - x)(\gamma - x) \\ &\leq \left(\frac{(\alpha - x) + (\beta - x) + (\gamma - x)}{3}\right)^3 \\ &= \frac{1}{27}(-3x - a)^3 \\ &\leq \frac{1}{27}(x-a)^3. \end{aligned}$$

Hence $f(x) \geq -\frac{1}{27}(x-a)^3$. When $\beta \leq x \leq \gamma$. we have

$$\begin{aligned} -f(x) &= (x-\alpha)(x-\beta)(\gamma - x) \\ &\leq \left(\frac{(x-\alpha) + (x-\beta) + (\gamma - x)}{3}\right)^3 \\ &\leq -\frac{1}{27}(x + \alpha + \beta + \gamma)^2 \\ &= -\frac{1}{27}(x-a)^3. \end{aligned}$$

Hence $f(x) \geq -\frac{1}{27}(x-a)^3$. When $\alpha < x < \beta$ or $\gamma < x$, $f(x) > 0 \geq -\frac{1}{27}(x-a)^3$. In summary, we can take $\lambda = -\frac{1}{27}$. The cases of equality in (b) will show that no larger value of λ is possible.

(b) When $0 \leq x \leq \alpha$, the necessary and sufficient conditions for equality are $\alpha - x = \beta - x = \gamma - x$ and $-3x = x$, or equivalently $\alpha = \beta = \gamma$ and $x = 0$. When $\beta \leq x \leq \gamma$, the necessary and sufficient conditions for equality are $x - \alpha = x - \beta = \gamma - x$ and $\alpha + \beta = -\alpha - \beta$, or equivalently $\alpha = \beta = 0$ and $2\gamma = x$. When $\alpha < x < \beta$ or $\gamma < x$, equality cannot occur. In summary, we have two cases of equality, $f(0) = \frac{a^3}{27} = -\frac{1}{27}(0-a)^3$ when $\alpha = \beta = \gamma$, and $f(\frac{\gamma}{2}) = -\frac{\gamma^3}{8} = -\frac{1}{27}(\frac{\gamma}{2} + \gamma)^3$.

9. Determine all functions $f : [1, \infty) \to [1, \infty)$ such that for all $x \geq 1$, $f(x) \leq 2(x+1)$ and $f(x+1) = \frac{1}{x}((f(x))^2 - 1)$.

First Solution:

We have $(f(x))^2 = xf(x+1) + 1$. Subtracting $(x+1)^2$ from both sides,

$$(f(x) - (x+1))(f(x) + x + 1) = x(f(x+1) - (x+2)).$$

Since $1 \leq f(x) \leq 2(x+1)$ and $1 \leq f(x+n) \leq 2(x+n+1)$, we have

$$\begin{aligned} |f(x) - (x+1)| &\leq \frac{x}{x+2}|f(x+1) - (x+2)| \\ &\leq \frac{x}{x+2} \cdot \frac{x+1}{x+3}|f(x+2) - (x+3)| \\ &\leq \cdots \\ &\leq \frac{x(x+1)}{(x+n)(x+n+1)}|f(x+n) - (x+n+1)| \\ &\leq \frac{x(x+1)}{x+n}. \end{aligned}$$

Since n may be arbitrarily large, we must have $f(x) = x + 1$, and it is easy to verify that this function has all the desired properties.

Second Solution:

Let $g(x) = \frac{f(x)}{x+1}$. Then $\frac{1}{x+1} \le g(x) \le 2$ and

$$\begin{aligned}
g(x+1) - (g(x))^2 &= \frac{f(x+1)}{x+2} - \left(\frac{f(x)}{x+1}\right)^2 \\
&= \frac{(f(x))^2 - 1}{x(x+2)} - \left(\frac{f(x)}{x+1}\right)^2 \\
&= \frac{(f(x))^2 - (x+1)^2}{x(x+2)(x+1)^2} \\
&= \frac{(g(x))^2 - 1}{x(x+2)}.
\end{aligned}$$

If $g(x) > 1$ for some $x \ge 1$, then $g(x+1) > (g(x))^2 > 1$. Iteration yields $g(x+n) > (g(x))^{2^n}$. Since n may be arbitrarily large, this contradicts $g(x) \le 2$. If $g(x) < 1$ for some $x \ge 1$, then $g(x+1) < (g(x))^2 < 1$. Iteration yields $g(x+n) < (g(x))^{2^n}$ so that

$$\left(\frac{1}{x+n+1}\right)^{2^{-k}} < (g(x+n))^{2^{-k}} < g(x) < 1.$$

However, the leftmost term may be made arbitrarily close to 1 by increasing n, and we have a contradiction. It follows that $g(x) = 1$ for all $x \ge 1$. Hence $f(x) = x + 1$, and it is easy to verify that this function has all the desired properties.

10. A function f from the set of real numbers to itself satisfies

$$f(x^3 + y^3) = (x+y)((f(x))^2 - f(x)f(y) + (f(y))^2),$$

where x and y are arbitrary real numbers. Prove that $f(1996x) = 1996f(x)$ for any real number x.

Solution:

We have $f(0) = 0$ by taking $x = y = 0$, and $f(x^3) = x(f(x))^2$ by taking $y = 0$. This may be rewritten as $f(x) = \sqrt[3]{x}(f(\sqrt[3]{x}))^2$, showing that x and $f(x)$ have the same sign. Let S be the set of real numbers k such that $f(kx) = kf(x)$. Clearly, $1 \in S$. If $k \in S$, then

$$kx(f(x))^2 = kf(x^3) = f(kx^3) = f((\sqrt[3]{kx})^3) = \sqrt[3]{kx}(f(\sqrt[3]{kx}))^2,$$

which is equivalent to $(\sqrt[3]{k}f(x))^2 = (f(\sqrt[3]{k}x))^2$. By the sign consideration discussed above, we have $\sqrt[3]{k}f(x) = f(\sqrt{k}x)$, so that $\sqrt[3]{k} \in S$. We claim that if $h, k \in S$, then $h + k \in S$. Indeed,

$$\begin{aligned}
f((h+k)x) &= f((\sqrt[3]{hx})^3 + (\sqrt[3]{kx})^3) \\
&= (\sqrt[3]{hx} + \sqrt[3]{kx})((f(\sqrt[3]{hx})^2 - f(\sqrt[3]{hx})f(\sqrt[3]{kx}) + (f(\sqrt[3]{kx}))^2) \\
&= (\sqrt[3]{h} + \sqrt[3]{k})\sqrt[3]{x}(\sqrt[3]{h^2} - \sqrt[3]{hk} + \sqrt[3]{k^2})(f(\sqrt[3]{x}))^2 \\
&= (h+k)f(x).
\end{aligned}$$

Since $1 \in S$, $1 + 1 = 2 \in S$. It follows that S contains all positive integers. In particular, $1996 \in S$ and $f(1996x) = 1996(f(x))$ for all x.

11. Let a be a fixed real number. A sequence of polynomials $\{f_n(x)\}$ is defined by $f_0(x) = 1$ and for $n = 0, 1, 2, \ldots,$

$$f_{n+1}(x) = xf_n(x) + f_n(ax).$$

(a) Prove that $f_n(x) = x^n f(\frac{1}{x})$ for $n = 0, 1, 2, \ldots$.

(b) Find an explicit expression for $f_n(x)$.

Solution:

(a) We have $f_1(x) = xf_0(x) + f_0(ax) = x + 1$, so that $_1(x) - f_0(x) = x$. We claim in general that $f_n(x) - f_{n-1}(x) = a^{n-1}xf_{n-1}(\frac{x}{a})$. We use induction on n, and the basis $n = 1$ holds since $a^{1-1}xf_0(\frac{x}{a}) = x$. Suppose the claim holds for some $n \geq 1$. Then

$$\begin{aligned}
f_{n+1}(x) - f_n(x) &= (xf_n(x) + f_n(ax)) - (xf_{n-1}(x) + f_{n-1}(ax)) \\
&= x(f_n(x) - xf_{n-1}(x)) + (f_n(ax) - f_{n-1}(ax)) \\
&= xa^{n-1}xf_{n-1}\left(\frac{x}{a}\right) + a^{n-1}(ax)f_{n-1}(x) \\
&= a^n x(\frac{x}{a}f_{n-1}\left(\frac{x}{a}\right) + f_{n-1}\left(a \cdot \frac{x}{a}\right) \\
&= a^n x f_n\left(\frac{x}{a}\right).
\end{aligned}$$

We now prove by induction on n that $f_n(x) = x^n f_n(\frac{1}{x})$. For $n = 0$, we have $f_0(x) = 1$ and $x^0 f_0(1x) = 1$. Suppose the result holds for some $n \geq 0$. Then

$$\begin{aligned}
f_{n+1}(x) &= f_n(x) + a^n x f_n\left(\frac{x}{a}\right) \\
&= x^n f_n\left(\frac{1}{x}\right) + a^n x \left(\frac{x}{a}\right)^n f_n\left(\frac{a}{x}\right) \\
&= x^{n+1}\left(\frac{1}{x}f_n\left(\frac{1}{x}\right) + f_n\left(a \cdot \frac{1}{x}\right)\right) \\
&= x^{n+1} f_{n+1}\left(\frac{1}{x}\right).
\end{aligned}$$

(b) Let $f(x) = \sum_{i=1}^{n} b_i^{(n)} x^i$. From the given conditions, we have $b_n^{(n)} = b_{n-1}^{(n-1)} = \cdots = b_0^{(0)} = 1$. Comparing the coefficient of x^i in $f_n(x) = x^n f_n(\frac{1}{x})$, we have $b_i^{(n)} = b_{n-i}^{(n)}$. It follows that $b_0^{(n)} = b_0^{(n-1)} = \cdots = b_0^{(0)} = 1$. From x^i in $f_n(x) = xf_{n-1}(x) + f_{n-1}(ax)$, we have

$$b_i^{(n)} = b_{i-1}^{(n-1)} + a^i b_i^{(n-1)}.$$

Similarly, $b_{n-i}^{(n)} = b_{n-i-1}^{(n-1)} + a^{n-i} b_{n-i}^{(n-1)}$, which is equivalent to

$$a^i b_i^{(n)} = a^i b_i^{(n-1)} + a^n b_{i-1}^{(n-1)}.$$

From the two displayed equations, we have $(a^i - 1)b_i^{(n)} = (a^n - 1)b_{i-1}^{(n-1)}$. It follows that

$$b_i^{(n)} = \frac{a^n - 1}{a^i - 1} b_{i-1}^{(n-1)}$$

$$
\begin{aligned}
&= \frac{(a^n-1)(a^{n-1}-1)}{(a^i-1)(a^{i-1}-1)} b_{i-2}^{(n-2)} \\
&= \cdots \\
&= \frac{(a^n-1)(a^{n-1}-1)\cdots(a^{n-i+1}-1)}{(a^i-1)(a^{i-1}-1)\cdots(a-1)} b_0^{(n-i)} \\
&= \frac{(a^n-1)(a^{n-1}-1)\cdots(a^{n-i+1}-1)}{(a^i-1)(a^{i-1}-1)\cdots(a-1)}.
\end{aligned}
$$

Hence $f(x) = \sum_{i=0}^{n} \frac{(a^n-1)(a^{n-1}-1)\cdots(a^{n-i+1}-1)}{(a^i-1)(a^{i-1}-1)\cdots(a-1)} x^i$.

12. The coefficient of the n-th degree polynomial

$$f(z) = c_0 z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_{n-1} z + c_n$$

are complex numbers. Prove that there exists a complex number z_0 such that $|z_0| \le 1$ while $|f(z_0)| \ge |c_0| + |c_n|$.

First Solution:

Let $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. We have $1 + \omega^j + \cdots + \omega^{(n-1)j} = n$ for $j = 0$ or n, but this sum is zero for $j = 1, 2, \ldots, n-1$. For any complex number λ, $\sum_{k=0}^{n-1} f(\lambda\omega^k) = n(c_0\lambda + c_n)$. Choose λ so that $|\lambda| = 1$ and $c_0\lambda^n$ has the same argument as c_n. Then

$$\frac{1}{n}\sum_{k=0}^{n-1} |f(\lambda\omega^k)| \ge \frac{1}{n}\left|\sum_{k=0}^{n-1} f(\lambda\omega^k)\right| = |c_0\lambda^n + c_n| = |c_0| + |c_n|.$$

It follows that for some k, $0 \le k \le n-1$, $z_0 = \lambda\omega^k$ satisfies $|f(z_0)| \ge |c_0| + |c_n|$.

Second Solution:

First we suppose that $c_n \ne 0$. Let $g(z) = c_0 z^n + c_1 z^{n-1} + \cdots + c_{n-1} z - \frac{|c_0|c_n}{|c_n|}$. Let its roots be $z_1, z_2, \ldots, z_n$. Then $|z_1 z_2 \cdots z_n| = |\frac{|c_0|c_n}{|c_n|c_0}| = 1$. It follows that at least one root z_0 satisfies $|z_0| \le 1$ and $f(z_0) = g(z_0) + (\frac{|c_0|}{|c_n|} + 1)c_n$. Hence $|f(z_0)| = |c_0| + |c_n|$. We now suppose that $c_n = 0$. Let $g(z) = c_0 z^n + c_1 z^{n-1} + \cdots + c_{n-1} z - c_0$. Let its roots be $z_1, z_2, \ldots, z_n$. Then $|z_1 z_2 \cdots z_n| = |\frac{c_0}{c_0}| = 1$. It follows that at least one root z_0 satisfies $|z_0| \le 1$ and $|f(z_0)| = |g(z_0) + c_0| = |c_0| = |c_0| + |c_n|$.

Number Theory

13. Let a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ be seven distinct prime numbers such that $a+b = 800$. Determine the maximum value of the difference between the largest and the smallest of these seven numbers.

Solution:

We may assume that $a < b$. Note that $a+b \equiv 2 \pmod 3$. Suppose $a \equiv 0 \pmod 3$ and $b \equiv 2 \pmod 3$. Then $a = 3$ and $c \not\equiv 0 \pmod 3$. If $c \equiv 1 \pmod 3$, then $a+b+c \equiv 0 \pmod 3$ is composite since it is distinct from 3. If $c \equiv 2 \pmod 3$, then $a+b-c \equiv 0 \pmod 3$ is also composite. Hence we must have $a \equiv b \equiv 1 \pmod 3$. If $c \equiv 1 \pmod 3$, then $a+b+c \equiv 0$

(mod 3). Since $a+b+c$ is the largest of the 7 distinct primes, it cannot be equal to 3. If $c \equiv 0 \pmod 3$, then $c = 3$ and $c+a-b \equiv 0 \pmod 3$ is composite. Hence we must have $c \equiv 2 \pmod 3$. Now $a+b-c \equiv 0 \pmod 3$ is prime. Hence it is equal to 3, and is the smallest of the 7 distinct primes. We have $c = 800 - 3 = 797$ and the desired difference is $(a+b+c)-(a+b-c) = 2c = 1594$. Note that $a = 7$ and $b = 793$ yield the distinct primes 3, 7, 11, 793, 797, 1571 and 1597.

14. Let $a_1, a_2, \ldots, a_{10}$ be ten distinct positive integers whose sum is 1995. Determine the minimum value of

$$a_1a_2 + a_2a_3 + \cdots + a_9a_{10} + a_{10}a_1.$$

Solution:

Let $\langle a_1, a_2, \ldots, a_{10}\rangle$ be the permutation of the fixed positive integers $x_1 < x_2 < \cdots < x_{10}$ which minimizes $a_1a_2 + a_2a_3 + \cdots + a_9a_{10} + a_{10}a_1$. We may assume that $a_{10} = x_{10}$. We claim that either a_1 or a_9 is x_1. Suppose on the contrary that $x_1 = a_j$ for some j, $1 < j < 9$. Define $b_i = a_{j+1-i}$ for $1 \le i \le j$ and $b_i = a_i$ for $j+1 \le i \le 10$. Then

$$\begin{aligned}
& (a_1a_2 + a_2a_3 + \cdots + a_9a_{10} + a_{10}a_1) - (b_1b_2 + b_2b_3 + \cdots + b_9b_{10} + b_{10}b_1) \\
= \; & (a_ja_{j+1} + a_{10}a_1) - (a_1a_{j+1} + a_ja_{10}) \\
= \; & (a_{10} - a_j)(a_1 - a_j) \\
> \; & 0.
\end{aligned}$$

This contradicts the minimality assumption on $\langle a_1, a_2, \ldots, a_n\rangle$, so that the claim is justified. By symmetry, we may take $a_1 = x_1$. The same argument yields $a_9 = x_2$, $a_2 = x_9$, $a_8 = x_8$, $a_3 = x_3$, $a_7 = x_4$, $a_4 = x_7$, $a_6 = x_6$ and $a_5 = x_5$. We now determine positive integers $x_1 < x_2 < \cdots < x_{10}$ with sum 1995 such that

$$x_1x_9 + x_9x_3 + x_3x_7 + x_7x_5 + x_5x_6 + x_6x_4 + x_4x_8 + x_8x_2 + x_2x_{10} + x_{10}x_1$$

is minimum. We claim that $x_i = i$ for $1 \le i \le 9$ and $x_{10} = 1950$. Suppose on the contrary that for some j, $1 \le j \le 9$, $x_i = i$ for $1 \le i \le j-1$ but $x_j \ne j$. Let the two terms in the sum involving x_j be x_px_j and x_jx_q. Then $x_p + x_q > x_1 + x_2$. Define $y_i = x_i$ for $1 \le i \le 9$ except for $y_j = x_j - 1$ and define $y_{10} = x_{10} + 1$. Then

$$\begin{aligned}
& (x_1x_9 + x_9x_3 + \cdots + x_2x_{10} + x_{10}x_1) - (y_1y_9 + y_9y_3 + \cdots + y_2y_{10} + y_{10}y_1) \\
= \; & (x_p + x_q) - (x_1 + x_2) \\
> \; & 0.
\end{aligned}$$

This is a contradiction, and the claim is justified. It follows that the desired minimum is $1\cdot 9 + 9\cdot 3 + 3\cdot 7 + 7\cdot 5 + 5\cdot 6 + 6\cdot 4 + 4\cdot 8 + 8\cdot 2 + 2\cdot 1950 + 1950\cdot 1 = 6044$.

15. Let n be an integer greater than 1. Do there always exist $2n$ distinct positive integers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ such that $a_1 + a_1 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$ and

$$n - 1 > \sum_{i=1}^{n} \frac{a_i - b_i}{a_i + b_i} > n - 1 - \frac{1}{1998}?$$

Solution:

We first prove that the upper bound holds for any $2n$ distinct positive integers satisfying the hypothesis. Note that we must have $a_j < b_j$ for some j, $1 \le j \le n$, so that $\frac{2b_j}{a_j+b_j} > 1$. It follows that

$$\begin{aligned}
\sum_{i=1}^{n} \frac{a_i - b_i}{a_i + b_i} &= \sum_{i=1}^{n} \left(1 - \frac{2b_i}{a_i + b_i}\right) \\
&= n - \sum_{i=1}^{n} \frac{2b_i}{a_i + b_i} \\
&\ge n - \frac{2b_j}{a_j + b_j} \\
&> n - 1.
\end{aligned}$$

For $1 \le i \le n-1$, choose $a_i = 2Mi$ and $b_i = 2i$ so that $\frac{a_i-b_i}{a_i+b_i} = \frac{M-1}{M+1} = 1 - \frac{2}{M+1}$. Here M is a sufficiently large integer. Choose $a_n = (M-1)^2 n(n-1)$ and $b_n = M(M-1)n(n-1)$ so that

$$a_1 + a_2 + \cdots + a_n = (M^2 - M + 1)n(n-1) = b_1 + b_2 + \cdots + b_n.$$

Then $\sum_{i=1}^{n} \frac{a_i - b_i}{a_i + b_i} = (n-1)\left(1 - \frac{2}{M+1}\right) - \frac{1}{2M-1}$. Since $\frac{1}{1998} > \frac{2(n-1)}{M+1} + \frac{1}{2M-1}$ when M is sufficiently large, we have the desired lower bound.

16. Let m and n be positive integers such that $m < 4002$, $n^2 - m^2 + 2mn \le 4002(n-m)$ and $4002m - m^2 - n^2$ is divisible by $2n$.

(a) Determine the minimum value of $\frac{4002m - m^2 - mn}{n}$.

(b) Determine the maximum value of $\frac{4002m - m^2 - mn}{n}$.

Solution:

(a) Let $k = \frac{4002m - m^2 + n^2}{2n}$. Then $n(2k - n) = m(4002 - m)$. If either m or n is odd, all four factors are odd. It follows that $m \equiv n \pmod 2$. Now

$$\begin{aligned}
k &= \frac{4002m - 2mn + n^2 - m^2 + 2mn}{2n} \\
&\le \frac{4002m - 2mn + 4002(n-m)}{2n} \\
&= 2001 - m.
\end{aligned}$$

Hence $2k - n < 2k \le 4002 - 2m < 4002 - m$, so that $n > m$. Now

$$\begin{aligned}
2mn &\le 4002(n-m) - (n^2 - m^2) \\
&= (n-m)(4002 - n - m).
\end{aligned}$$

Hence $4002 - n - m > 0$, so that the given expression $E(m,n) = \frac{m(4002-m-n)}{n}$ is positive. Also, $E(m,n) = \frac{4002m - m^2 + n^2 - n^2 - mn}{n} = 2k - n - m \equiv 0 \pmod 2$. It follows that $E(m,n) \ge 2$. We have $E(2,n) = \frac{8000}{n} - 2$, and the minimum value is attained at $E(2, 2000) = 2$. It is easy to verify that $(m,n) = (2, 2000)$ satisfies the hypothesis.

(b) Recall that $n > m$ and $m \equiv n \pmod 2$. Consider first

$$E(n-2,n) = \frac{4002(n-2)-(n-2)^2-n(n-2)}{n} = 4008 - 2(n+\frac{4004}{n}).$$

To maximize $E(n-2,n)$, we want n and $\frac{4004}{n}$ to be as close to each other as possible. Since n must be a factor of $4004 = 52 \times 77$, we can take $n = 52$. It is easy to verify that $(m,n) = (50,52)$ satisfies the hypothesis while $(m,n) = (75,77)$ does not. We claim that the maximum value occurs at $E(50,52) = 3750$. In other words, $E(m,n) < 3750$ for all $n \geq m+4$. In such cases, we have

$$\begin{aligned} E(m,n) &= \frac{(4002-m)m}{n} - m \\ &\leq \frac{(4002-m)m}{m+4} - m \\ &= 3998 - 2\left(m+4+\frac{16024}{m+4}\right) \\ &\leq 3998 - 2\sqrt{16024} \\ &\leq 3998 - 2(126) \\ &= 3746. \end{aligned}$$

This justifies the claim.

17. Find all positive integers n for which there exist k integers n_1, n_2, ..., n_k, each greater than 3, such that

$$n = n_1 n_2 \cdots n_k = \sqrt[2^k]{2^{(n_1-1)(n_2-1)\cdots(n_k-1)}} - 1.$$

Solution:

Denote $\frac{1}{2^k}(n_1-1)(n_2-1)\cdots(n_k-1)$ by m. Since $n = 2^m - 1$ is odd, each n_i, $1 \leq i \leq k$, is odd and hence at least 5, so that $(\frac{n_i-1}{2})^3 \geq 4 \cdot \frac{n_i-1}{2} > n_i$. Suppose $m \geq 10$. It is easy to show then that $2^m - 1 > m^3$. Hence $n \geq (\frac{n_1-1}{2})^2 + (\frac{n_2-1}{2})^3 + \cdots + (\frac{n_k-1}{2})^3 > n_1 n_2 \cdots n_k$, which is a contradiction. It follows that $m \leq 9$, and it is routine to check that only when $m = 3$ do we have a number $n = 7$ with the desired property.

18. For any integer m, prove that $2m$ can be expressed in the form $a^{19} + b^{99} + k \cdot 2^{1999}$, where a and b are odd integers and k is a non-negative integer.

Solution:

Let $n = 2^{1999}$. We claim that the integers 1^{19}, 3^{19}, ..., $(2n-1)^{19}$ are not congruent modulo n to one another. Let x and y be any odd numbers with $x \not\equiv y \pmod n$. We have $x^{19} - y^{19} = (x-y)(x^{18} + x^{17}y + \cdots + xy^{17} + y^{18})$. The second factor being odd, $x^{19} \equiv y^{19} \pmod n$ would imply $x \equiv y \pmod n$. This justifies the claim. It follows that for any integer m, $2m - 1 \equiv a^{19} \pmod n$ for some odd integer a, so that $2m = a^{19} + 1^{19} + kn$ for some integer k. If $k \geq 0$, we can simply take $b = 1$. If not, take $b_1 = 1$, $a_1 = a - hn$ and $q_1 = k + \frac{a^{19} - a_1^{19}}{n}$. Then $2m = a_1^{19} + b_1^{19} + k_1 n$, and if h is sufficiently large, we will have $k_1 \geq 0$.

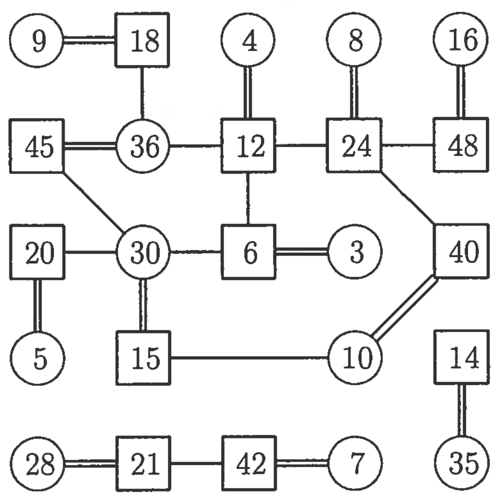
19. Let $S = \{1, 2, \ldots, 50\}$. Determine the smallest positive integer k such that for any k-element subset of S, there are two different elements a and b for which $a + b$ divides ab.

Solution:

Suppose a and b are positive integers such that $a+b$ divides ab. Let their greatest common divisor be d. Then $a = dk$ and $b = d\ell$ for some positive integers k and ℓ which are relatively prime. Hence $k+\ell$ is relatively prime to $k\ell$. However, $d(k+\ell)$ divides $d^2k\ell$. Hence $k+\ell$ must divide d, so that $k + ell \leq d$. Since $a, b \in S$, $d(k+\ell) = a+b \leq 99$. It follows that $3 \leq k+\ell \leq 9$. The following 23 paris of distinct positive integers (a, b) in S have the desired property:

$k+\ell$	(a,b)
3	(3,6), (6,12), (9,18), (12,24), (15,30), (18,36), (21,42), (24,48);
4	(4,12), (8,24), (12,36), (16,48);
5	(5,20), (10,40), (10,15), (20,30), (30,45);
6	(6,36);
7	(7,42), (14,35), (21,28);
8	(24,40);
9	(36,45).

We construct a graph where the 23 edges represent the above pairs and the 24 vertices represent the individual numbers involved.



20. Let $S = \{1, 2, \ldots, 98\}$. Determine the smallest positive integer n for which any subset of S of size n contains 10 elements such that no matter how they are divided into two subsets of size 5, one subset contains an element relatively prime to each of the other four, while the other subset contains an element not relatively prime to any of the other four.

First Solution:

Since S contains 49 even numbers, a subset with $n \leq 49$ elements may consists only of even numbers, and will not satisfy the desired conditions. It follows that $n \geq 50$. We now prove that any subset T with 50 elements contains a subset A of size 10 which satisfies the desired conditions. Let e denote the number of even elements in T. For any odd number x in S, let $f(x)$ denote the number of even elements in S which are not relatively prime to x. In particular, if x is prime, then $f(x) = \lfloor \frac{49}{x} \rfloor$. We claim that if $f(x) \leq e - 9$ for any odd number x in T, then the subset A exists. This is because we can take x and 9 of the $e - f(x)$ even numbers in T relatively prime to x to form A. Thus the claim is justified. Now in S there are

$f(3) = 16$ odd numbers having 3 as the smallest prime divisor. There are $f(5) - 2 = 7$ odd numbers having 5 as the smallest prime divisor, the adjustment -2 reflecting the exclusion of $5 \cdot 3$ and $5 \cdot 3^2$. Similarly, there are $f(7) - 3 = 4$ odd numbers having 7 as the smallest prime divisor, and only 1 odd number having p as the smallest prime divisor for any $p > 7$, namely, p itself. We consider five cases.

Case 1. $e \geq 25$.

For any odd number x in T, we have $f(x) \leq f(3) = 16 = 25 - 9 \leq e - 9$. The desired conclusion follows from our earlier claim.

Case 2. $16 \leq e \leq 24$.

The number of odd elements in T is at least $50 - 24 = 26 > 1 + 16 + 7$, the first term in the sum accounting for the number 1. Hence T contains an odd number x whose smallest prime divisor is at least 7. It follows that $f(x) \leq f(7) = 7 = 16 - 9 \leq e - 9$, and we may appeal to our earlier claim again.

Case 3. $10 \leq e \leq 15$.

The number of odd elements in T is at least $50 - 15 = 35 > 1 + 16 + 7 + 4 + 6 \cdot 1$. Hence T contains an odd number x whose smallest prime divisor is at least 31. It follows that $f(x) \leq f(31) = 1 = 10 - 9 \leq e - 9$.

Case 4. $e = 9$.

The number of odd elements in T is $50 - 9 = 41 > 1 + 16 + 7 + 4 + 12 \cdot 1$. Hence T contains an odd number x whose smallest prime divisor is at least 59. It follows that $f(x) \leq f(59) = 0 = e - 9$.

Case 5. $e \leq 8$

The number of odd elements in T is at least $50 - 8 = 42 > 1 + 16 + 7 + 4 + 13 \cdot 1$. Hence T contains an odd number x whose smallest prime divisor is at least 61. This means that $x \geq 61$ is a prime. At most $49 - (50 - 8) = 7$ odd numbers in S do not belong to T. Hence T contains at least 9 odd multiples of 3. They along with x form A.

Second Solution:

Since S contains 49 even numbers, a subset with $n \leq 49$ elements may consists only of even numbers, and will not satisfy the desired conditions. It follows that $n \geq 50$. We now prove that any subset T with 50 elements contains a subset A of size 10 which satisfies the desired conditions. Partition the odd numbers in S into the following five sets:

$$\begin{aligned}
O_1 &= \{1, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}, \\
O_2 &= \{13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}, \\
O_3 &= \{49, 55, 65, 77, 85, 91, 95\}, \\
O_4 &= \{25, 33, 35, 39, 51, 57, 69, 87, 93\}, \\
O_5 &= \{3, 5, 7, 9, 11, 15, 21, 45, 63, 75, 81\}.
\end{aligned}$$

We consider five cases.

Case 1. $T \cap O_1 \neq \emptyset$.

In S, there are 33 odd numbers not divisible by 3. Hence T contains at least 17 numbers which are either even numbers or odd multiples of 3. By the Pigeonhole Principle, T contains either 9 even numbers or 9 odd multiples of 3. We can take A to consist of these 9 numbers together with an element of $T \cap O_1$.

Case 2. $T \cap O_1 = \emptyset$ but $T \cap O_2 \neq \emptyset$.

T contains at most 38 odd numbers and hence at least 12 even numbers. Let $x \in T \cap O_2$. Then at most 3 of these 12 even numbers are multiples of 13, the worst case. Hence we can take A to consist of x and 9 even numbers which are not multiples of x.

Case 3. $T \cap (O_1 \cup O_2) = \emptyset$ but $T \cap O_3 \neq \emptyset$.

T contains at most 28 odd numbers and hence at least 22 even numbers. Let $x \in T \cap O_3$. Since $\lfloor\frac{49}{5}\rfloor + \lfloor\frac{49}{11}\rfloor - \lfloor\frac{49}{55}\rfloor = 13$, at least $22 - 13 = 9$ even numbers in T are relatively prime to 55, the worst case. Hence we may take A to consist of x and 9 of these even numbers.

Case 4. $T \cap (O_1 \cup O_2 \cup O_3) = \emptyset$ and $T \cap O_4 \neq \emptyset$.

T contains at most 21 odd numbers and hence at least 29 even numbers. Let $x \in T \cap O_4$. Since $\lfloor\frac{49}{3}\rfloor + \lfloor\frac{49}{11}\rfloor - \lfloor\frac{49}{33}\rfloor = 19$, at least $29 - 19 = 10$ even numbers in T are relatively prime to 33, the worst case. Hence we may take A to consist of x and 9 of these even numbers.

Case 5. $T \cap (O_1 \cup O_2 \cup O_3 \cup O_4) = \emptyset$.

T contains at most 12 odd numbers and hence at least 38 even numbers. Let $x \in T \cap O_5$. Since $\lfloor\frac{49}{3}\rfloor + \lfloor\frac{49}{5}\rfloor - \lfloor\frac{49}{15}\rfloor = 22$, at least $38 - 22 = 16$ even number in T are relatively prime to 15, the worst case. Hence we may take A to consist of x and 9 of these even numbers.

21. Determine the smallest positive integer m such that every subset of $\{1, 2, \ldots, 2001\}$ of size m contains two elements, not necessarily distinct, such that their sum is a power of 2.

Solution:

The smallest power of 2 greater than 2001 is $2048 = 2001+47 = 2000+48 = \cdots = 1025+1023$. Leaving out 1024 for now, the next largest number not accounted for is 46. The smallest power of 2 greater than 46 is $64 = 46 + 18 = 45 + 19 = \cdots = 33 + 31$. Leaving out 32 for now, we have $32 = 17 + 15$. Leaving out 16 for now, we have $16 = 14 + 2 = 13 + 3 = \ldots = 9 + 7$. Only 1 and 8 are left. Thus we have divided the first 2001 positive integers into 998 pairs plus the five single elements 1, 8, 16, 32 and 1024. In any subset with 999 elements, if it contains any of these five, then it will have two identical elements whose sum is a power of 2. Otherwise, by the Pigeonhole Principle, it must contain both elements of one of the 998 pairs. Hence it will have two distinct elements whose sum is a power of 2. To prove that 999 is indeed the minimum, we construct a set of 998 elements no two of which add up to a power of 2. We simply take the larger element from each of the above 998 pairs. In other words, we take $\{9, 10, \ldots, 14, 17, 33, 34, \ldots, 46, 1025, 1026, \ldots, 2001\}$. Consider any two elements in this set. If both are at least 1025, their sum is greater than 2048 but less than 2096. If only one is at least 1025, their sum is greater than 1024 but less than 2048, If neither is, their sum is greater than 16 and less than 128, and it is easy to verify that it cannot be 32 or 64.

22. Prove that there exist infintely many positive integers n for which the integers $1, 2, \ldots, 3n$ can be arranged in a $3 \times n$ array such that all rows have the same sum, all columns have the same sum, and both sums are divisible by 6.

First Solution:

Let S be the set of positive integers n with the desired property. For such a $3 \times n$ array, let the sum of each row be s and that of each column be t. Then $18s = \frac{3n(3n+1)}{2} = 6nt$, so that $n(3n + 1) = 12s$ and $3n + 1 = 4t$. Hence $n \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 4$, and it follows that $n \equiv 9 \pmod{12}$. We prove that $9 \in S$. Let $\alpha(1) = \langle 1 \rangle$, $\beta(1) = \langle 2 \rangle$ and $\gamma(1) = \langle 3 \rangle$. Consider the 3×3 array

$$\begin{bmatrix} \alpha(1) & \beta(1)+6 & \gamma(1)+3 \\ \beta(1)+3 & \gamma(1) & \alpha(1)+6 \\ \gamma(1)+6 & \alpha(1)+3 & \beta(1) \end{bmatrix} = \begin{bmatrix} 1 & 8 & 6 \\ 5 & 3 & 7 \\ 9 & 4 & 2 \end{bmatrix}.$$

Note that $3 \notin S$ since the row sum 15 is not a multiple of 6. Let $\alpha(3) = \langle 1,8,6\rangle$, $\beta(3) = \langle 5,3,7\rangle$ and $\gamma(3) = \langle 9,4,2\rangle$ be the rows of this magic square. Consider the 3×9 array

$$\begin{bmatrix} \alpha(3) & \beta(3)+18 & \gamma(3)+9 \\ \beta(3)+9 & \gamma(3) & \alpha(3)+18 \\ \gamma(3)+18 & \alpha(3)+9 & \beta(3) \end{bmatrix} = \begin{bmatrix} 1 & 8 & 6 & 23 & 21 & 25 & 18 & 13 & 11 \\ 14 & 12 & 16 & 9 & 4 & 2 & 19 & 26 & 24 \\ 27 & 22 & 20 & 10 & 17 & 15 & 5 & 3 & 7 \end{bmatrix}.$$

It is easy to verify that it has all the desired properties. We claim that if $m \in S$, then $9m \in S$. It will then follow that since $9 \in S$, we have $9^k \in S$ for all positive integer k, so that S is indeed infinite. To justify the claim, let $\alpha(m)$, $\beta(m)$ and $\gamma(m)$ be the rows of a $3 \times m$ array with all the desired properties. As before, we first construct a $3 \times 3m$ array

$$\begin{bmatrix} \alpha(m) & \beta(m)+6m & \gamma(m)+3m \\ \beta(m)+3m & \gamma(m) & \alpha(m)+6m \\ \gamma(m)+6m & \alpha(m)+3m & \beta(m) \end{bmatrix}.$$

It has all the desired properties except that the row and column sums are not divisible by 6. Let $\alpha(3m)$, $\beta(3m)$ and $\gamma(3m)$ be the rows of this array. It is easy to verify that the $3 \times 9m$ array

$$\begin{bmatrix} \alpha(3m) & \beta(3m)+18m & \gamma(3m)+9m \\ \beta(3m)+9m & \gamma(3m) & \alpha(3m)+18m \\ \gamma(3m)+18m & \alpha(3m)+9m & \beta(3m) \end{bmatrix}$$

has all the desired properties. Thus the claim is justified.

Second Solution:

Let S be the set of positive integers with the desired property. As in the First Solution, we focus on those of the form $12k+9$. We add the condition that $k \equiv 2 \pmod 9$, the reason for which will soon become clear. We first construct the following $3 \times (4k+3)$ array

$$\begin{bmatrix} 1 & 4 & 7 & 10 & \ldots & 12k-2 & 12k+1 & 12k+4 & 12k+7 \\ 6k+5 & 12k+8 & 6k+2 & 12k+5 & \ldots & 6k+11 & 5 & 6k+8 & 2 \\ 12k+9 & 6k+3 & 12k+6 & 6k & \ldots & 6 & 6k+9 & 3 & 6k+6 \end{bmatrix}.$$

Each column has sum $18k+15$. The sums of the three rows are $(4k+3)(6k+4)$, $(4k+3)(6k+5)$ and $(4k+3)(6k+6)$ respectively. Let $\ell = \frac{2k+5}{9}$. Since $k \equiv 2 \pmod 9$, ℓ is a positive integer. The 2ℓ-th term in the first row is $1+3(2\ell-1) = \frac{4k+4}{3}$ while that in the third row is $6k+3-3(\ell-1) = \frac{16k+13}{3}$. Switching these two terms will not change any column sum, but will change all three row sums to $(4k+3)(6k+5)$. Let $\alpha(4k+3)$, $\beta(4k+3)$ and $\gamma(4k+3)$ be the rows of this adjusted array. It is easy to verify that the $3 \times (12k+9)$ array

$$\begin{bmatrix} \alpha(4k+3) & \beta(4k+3)+6(4k+3) & \gamma(4k+3)+3(4k+3) \\ \beta(4k+3)+3(4k+3) & \gamma(4k+3) & \alpha(4k+3)+6(4k+3) \\ \gamma(4k+3)+6(4k+3) & \alpha(4k+3)+3(4k+3) & \beta(4k+3) \end{bmatrix}$$

has all the desired properties.

23. Let $n > 1$ be an odd integer. Suppose

$$X_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) = (1,0,0,\ldots,0,1).$$

For $1 \le k \le n$, let

$$x_i^{(k)} = \begin{cases} 0 & \text{if } x_i^{(k-1)} = x_{i+1}^{(k-1)}; \\ 1 & \text{if } x_i^{(k-1)} \ne x_{i+1}^{(k-1)}. \end{cases}$$

We take $x_{n+1}^{(k-1)} = x_1^{(k-1)}$. Let

$$X_k = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}).$$

If the positive integer m satisfies $X_m = X_0$, prove that m is a multiple of n.

Solution:

Let $A_1^{(0)}A_2^{(0)}\ldots A_n^{(0)}$ be a regular n-gon with centre O. Label $A_i^{(0)}$ with $x_i^{(0)}$ for $1 \le i \le n$. The numerical pattern has a unique axis of symmetry passing through O and perpendicular to $A_1^{(0)}A_n^{(0)}$. For $k \ge 1$, let $A_i^{(k)}$ be the midpoint of $A_i^{(k-1)}A_{i+1}^{(k-1)}$ for $1 \le i \le n$, interpreting $A_{n+1}^{(k-1)}$ as $A_1^{(k-1)}$. Label $A_i^{(k)}$ with $x_i^{(k)}$, $1 \le i \le n$, according to the given rule. Then the new numerical pattern has the same unique axis of symmetry. Now rotate the n-gon $A_1^{(k)}A_2^{(k)}\ldots A_n^{(k)}$ about O through an angle of $\frac{\pi}{n}$ so that $A_i^{(k)}$ is collinear with O and $A_i^{(k-1)}$, $1 \le i \le n$. Then delete the n-gon $A_1^{(k-1)}A_2^{(k-1)}\ldots A_n^{(k-1)}$ along with its associated labels. Now the new numerical pattern still has a unique axis of symmetry, obtained from the old one by a rotation about O through an angle of $\frac{\pi}{n}$. Suppose $X_m = X_0$ for some positive integer m. Then the unique axis of symmetry must coincide with its initial position. It follows that $m\frac{\pi}{n} = \ell\pi$ for some positive integer ℓ, so that $m = \ell n$ as desired.

24. Let $A = \{0, 1, \ldots, 16\}$. For any mapping $f : A \to A$, define $f^{(1)}(x) = f(x)$ and for any $n \ge 1$, $f^{(n+1)}(x) = f(f^{(n)}(x))$. Interpret $f^{(n)}(17)$ as $f^{(n)}(0)$. Suppose that for a bijection $f : A \to A$, there exists a positive integer M such that

$$f^{(M)}(i+1) - f^{(M)}(i) \equiv \pm 1 \pmod{17}$$

for $0 \le i \le 16$ and for $m < M$, we have

$$f^{(m)}(i+1) - f^{(m)}(i) \not\equiv \pm 1 \pmod{17}$$

for $0 \le i \le 16$. Determine the maximum value of M taken over all bijections $f : A \to A$ with the above properties.

Solution:

We first give an example where $M = 8$. Take $f(i) \equiv 3i - 2 \pmod{17}$, with i and $3i - 2$ both in A. If $f(i) = f(j)$, then $i \equiv j \pmod{17}$, so that f is indeed a bijection. Iteration yields $f^{(n)}(i) \equiv 3^n i - 3^n + 1 \pmod{17}$. Hence $f^{(n)}(i+1) - f^{(n)}(i) \equiv 3^n \pmod{17}$. In modulo 17, $3^1 \equiv 3$, $3^2 \equiv -8$, $3^3 \equiv -7$, $3^4 \equiv -4$, $3^5 \equiv 5$, $3^6 \equiv -2$, $3^7 \equiv -6$ and $3^8 \equiv -1$. Hence $M = 8$. Now let $f : A \to A$ be a bijection with the largest possible M. Let $P_0P_1\ldots P_{16}$ be a convex 17-gon. For any m, $1 \le m < M$, if $f^{(m)}(i+1) = a$ and $f^{(m)}(i) = b$, we connect P_a and P_b. Since $a - b \ne \pm 1$, P_aP_b is a diagonal. For each m, exactly 17 diagonals are drawn, and no two are identical. We claim that the diagonals are still distinct even for different values of m. Suppose on the contrary there exist p and q, $1 \le p < q < M$, such that $f^{(p)}(i) = f^{(q)}(j)$ and $f^{(p)}(i \pm 1) = f^{(q)}(j+1)$ for some i and j. From $f^{(p)}(i) = f^{(q)}(j) = f^{(p)}(f^{(q-p)}(j))$, we have $f^{(q-p)}(j) \equiv i \pmod{17}$. Similarly, $f^{(q-p)}(j+1) \equiv i \pm 1 \pmod{17}$. Since $1 \le q - p < M$, this contradicts the definition of M. Thus the claim is justified. Now there are $M - 1$ values of m satisfying $1 \le m < M$, so that $17(M-1)$ diagonals are drawn. The total number of diagonals, on the other hand, is $17 \cdot 7$. From $17(M-1) \le 17 \cdot 7$, we have $M \le 8$.

Combinatorics

25. Let $\langle a_1, a_2, \ldots, a_n\rangle$ be any permutation of 1, 2, ..., n. Define $b_k = \max\{a_i : 1 \leq i \leq k\}$ for $k = 1, 2, \ldots, n$. Determine the average value of the first term a_1 of all permutations for which the sequence $\{b_1, b_2, \ldots, b_n\}$ takes on exactly two distinct values.

First Solution:

Note that $\{b_1, b_2, \ldots, b_n\}$ is non-descending and $b_{k+1} = b_{k+2} = \cdots = b_n = n$ for some k. Hence we must have $b_1 = b_2 = \cdots = b_k = m$ for some $m < n$. Clearly, $a_1 = m$, $a_{k+1} = n$ and $a_i < m$ for $2 \leq i \leq k$. It follows that $k \leq m$. For fixed m and k, $\langle a_2, a_3, \ldots, a_k\rangle$ is permutation of $k-1$ of the elements in $\{1, 2, \ldots, m-1\}$. These elements can be chosen in $\binom{m-1}{k-1}$ ways and then permuted in $(k-1)!$ ways. On the other hand, $\langle a_{k+2}, a_{k+3}, \ldots, a_n\rangle$ is a permutation of the remaining elements, and there are $(n-k-1)!$ such permutations. It follows that the total number of permutations $\langle a_1, a_2, \ldots, a_n\rangle$ with the desired property is

$$\begin{aligned}
&\sum_{m=1}^{n-1}\sum_{k=1}^{m}\frac{(m-1)!(n-k-1)!}{(m-k)!}\\
=\ &\sum_{m=1}^{n-1}(m-1)!(n-m-1)!\sum_{k=1}^{m}\binom{n-k-1}{n-m-1}\\
=\ &\sum_{m=1}^{n-1}(m-1)!(n-m-1)!\binom{n-1}{n-m}\\
=\ &(n-1)!\sum_{m=1}^{n-1}\frac{1}{n-m}.
\end{aligned}$$

Similarly, the total value of the first term of these permutations is

$$\begin{aligned}
&\sum_{m=1}^{n-1}\sum_{k=1}^{m}\frac{m!(n-k-1)!}{(m-k)!}\\
=\ &\sum_{m=1}^{n-1}m!(n-m-1)!\binom{n-1}{n-m}\\
=\ &(n-1)!\sum_{m=1}^{n-1}\frac{m}{n-m}.
\end{aligned}$$

It follows that the desired average is

$$\sum_{m=1}^{n-1}\frac{m}{n-m}\left(\sum_{m=1}^{n-1}\frac{1}{n-m}\right)^{-1} = n-(n-1)\left(\sum_{m=1}^{n-1}\frac{1}{n-m}\right)^{-1}.$$

Second Solution:

We have $a_1 = m$ where can be any value less than n. We can then choose $\binom{n-1}{n-m}$ positions for the numbers $m+1$, $m+2$, ..., n. Among these numbers, n must come first, while the remaining ones can be permuted in $(m-1)!$ ways. The numbers 1, 2, $k-1$, $k+1$, ..., $m-1$ fill in the remaining positions, in $(n-m-1)!$ ways. Thus the number of permutations with the desired property and $a_1 = m$ is given by $(m-1)!(n-m-1)!\binom{n-1}{n-m}$ as in the First Solution. We can continue as before.

26. Determine all integers $n > 3$ such that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$ divides 2^{2000}.

Solution:

Note that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} = \frac{(n+1)(n^2-n+6)}{6} = \frac{m(m^2-3m+8)}{6}$ by setting $m = n+1 > 4$. In order to divide 2^{2000}, we must have $\frac{m(m^2-3m+8)}{6} = 2^k$ for some integer $k > 3$. Hence $m(m^2-3m+8) = 3 \cdot 2^{k+1}$. Suppose first that $m = 2^u$ for some integer $u \geq 3$, so that $m^2 - 3m + 8 = 3 \cdot 2^v$ where $u + v = k + 1$. If $u \geq 4$, then $8 \equiv 3 \cdot 2^v \equiv 2^v \pmod{16}$. This implies that $v = 3$ so that $m^2 - 3m + 8 = 24$ or $m(m-3) = 16$. This is impossible. Hence $u = 3$, $m = 8$ and $n = 7$. Indeed, $\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} = 2^6$ divides 2^{2000}. Suppose now that $m = 3 \cdot 2^v$ for some positive integer v. Then $m^2 - 3m + 8 = 2^v$ where $u + v = k + 1$. If $u \geq 4$, we have $8 \equiv 2^v \pmod{16}$. This implies that $v = 3$ so that $m^2 - 3m + 8 = 8$ or $m(m-3) = 0$. This contradicts $m > 4$. If $u = 1$, then $m = 6$ but $m^2 - 3m + 8 = 26$ is not a power of 2. If $u = 2$, then $m = 12$ but $m^2 - 3m + 8 = 116$ is not a power of 2 either. Hence $u = 3$, $m = 24$ and $n = 23$. Indeed, $\binom{23}{0} + \binom{23}{1} + \binom{23}{2} + \binom{23}{3} = 2^{11}$ divides 2^{2000}.

27. Prove that $\displaystyle\sum_{k=0}^{n} 2^k \binom{n}{k} \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor} = \binom{2n+1}{n}$ for any positive integer n.

First Solution:

From the Binomial Theorem, we have

$$\begin{aligned}
\sum_{i=0}^{2n} \binom{2n}{i} x^i &= (1+x)^{2n} \\
&= (x^2 + (1+2x))^n \\
&= \sum_{i=0}^{n} \binom{n}{i} x^{2i} (1+2x)^{n-i} \\
&= \sum_{i=0}^{n} \binom{n}{i} x^{2i} \sum_{j=0}^{n-i} 2^j \binom{n-i}{j} x^j.
\end{aligned}$$

Comparing the coefficients of x^n on both sides, we have

$$\binom{2n}{n} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2i} \binom{n}{i} \binom{n-i}{n-2i} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2i} \binom{n}{2i} \binom{2i}{i}.$$

Comparing the coefficients of x^{n-1} on both sides, we have

$$\binom{2n}{n-1} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-2i-1} \binom{n}{i} \binom{n-i}{n-2i-1} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-2i-1} \binom{n}{2i+1} \binom{2i+1}{i}.$$

Adding the last two equations, we have

$$\binom{2n+1}{n} = \sum_{k=0}^{n} 2^{n-k} \binom{n}{k} \binom{k}{\lfloor \frac{k}{2} \rfloor} = \sum_{k=0}^{n} 2^k \binom{n}{n-k} \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor} = \sum_{k=0}^{n} 2^k \binom{n}{k} \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}.$$

Second Solution:

Consider the number of ways of choosing n of the $2n+1$ objects a_1, b_1, a_2, b_2, $\ldots$, a_n, b_n and c. First we choose k of the pairs $\{a_i, b_i\}$ and take exactly one from each pair. The number of ways of doing this is $2^k\binom{n}{k}$. The remaining $n-k$ objects are chosen as pairs $\{a_i, b_i\}$ plus c if $n-k$ is odd. This can be done in $\binom{n-k}{\lfloor\frac{n-k}{2}\rfloor}$ ways. Since the total number of ways is obviously $\binom{2n+1}{n}$, we have

$$\binom{2n+1}{n} = \sum_{k=0}^{n} 2^k \binom{n}{k}\binom{n-k}{\lfloor\frac{n-k}{2}\rfloor}.$$

28. The sequence $\{a_n\}$ is defined by $a_1 = 0$, $a_2 = 1$ and $a_n = \frac{n}{2}a_{n-1} + \frac{n(n-1)}{2}a_{n-2} + (-1)^n(1-\frac{n}{2})$ for $n \geq 3$. Simplify

$$a_n + 2\binom{n}{1}a_{n-1} + 3\binom{n}{2}a_{n-2} + \cdots + (n-1)\binom{n}{n-2}a_2 + n\binom{n}{n-1}a_1.$$

First Solution:

We may define $a_0 = 1$. Then $a_1 = 0 = a_0 - 1$. We prove by induction on n that $a_n = na_{n-1} + (-1)^n$ for all $n \geq 1$. We have

$$\begin{aligned} a_{n+1} &= \frac{n+1}{2}a_n + \frac{n(n+1)}{2}a_{n-1} + (-1)^{n+1}\left(1 - \frac{n+1}{2}\right) \\ &= \frac{n+1}{2}a_n + \frac{n+1}{2}(a_n - (-1)^n) + (-1)^{n+1}\left(1 - \frac{n+1}{2}\right) \\ &= (n+1)a_n + (-1)^{n+1}. \end{aligned}$$

Let $S_n = \sum_{i=0}^{n}(i+1)\binom{n}{i}a_{n-i}$. Then the desired expression is equal to $S_n - (n+1)$. Note that $S_1 = 2$ while $S_2 = 4 = 2S_1$. We now prove by induction on n that $S_n = nS_{n-1}$ for all $n \geq 2$. We have

$$\begin{aligned} S_{n+1} &= \sum_{i=0}^{n+1}(i+1)\binom{n+1}{i}a_{n+1-i} \\ &= \sum_{i=0}^{n+1}(i+1)\binom{n+1}{i}((n+1-i)a_{n-i} + (-1)^{n+1-i}) \\ &= (n+1)\sum_{i=0}^{n}(i+1)\binom{n}{i}a_{n-i} + \sum_{i=0}^{n+1}(i+1)\binom{n+1}{i}(-1)^{n+1-i} \\ &= (n+1)S_n + \sum_{i=0}^{n+1}\binom{n+1}{i}(-1)^{n+1-i} + (n+1)\sum_{i=1}^{n+1}\binom{n}{i-1}(-1)^{n+1-i} \\ &= (n+1)S_n + (1+(-1))^{n+1} + (n+1)\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i} \\ &= (n+1)S_n + (n+1)(1+(-1))^n \\ &= (n+1)S_n. \end{aligned}$$

It follows that $S_n = 2n!$ and that the desired expression is equal to $2n! - (n+1)$.

Second Solution:

In a permutations of $\{1, 2, \ldots, n\}$, i is said to be fixed point if it is in the i-th place. The derangement number d_n counts the number of permutations without fixed points. We have

$d_0 = 1$ and $d_1 = 0$. Suppose $n \geq 2$, There are $n-1$ places where 1 can be. Suppose it is in the k-th place for some $k > 1$. If k is in the first place, then the remaining $n-2$ numbers can be deranged in d_{n-2} ways. Suppose k is not in the first place. We can pretend that it is 1. Apart from the real 1 staying put in the k-th place, the remaining $n-1$ numbers can be deranged in d_{n-1} ways. It follows that $d_n = (n-1)(d_{n-1} + d_{n-2})$. This may be rewritten as

$$\begin{aligned}
d_n - nd_{n-1} &= -(d_{n-1} - (n-1)d_{n-2}) \\
&= d_{n-2} - (n-2)d_{n-3} \\
&= \cdots \\
&= (-1)^{n-1}(d_1 - d_0) \\
&= (-1)^n.
\end{aligned}$$

In the First Solution, we have shown that the sequence $\{a_n\}$ satisfies the same recurrence relation and has the same initial values. It follows that $a_n = d_n$ for all n. We claim that

$$d_n + \binom{n}{1}d_{n-1} + \binom{n}{2}d_{n-2} + \cdots + \binom{n}{n-2}d_2 + \binom{n}{n-1}d_1 = n! - 1.$$

This is because the first term d_n counts all permutations of $\{1, 2, \ldots, n\}$ with 0 fixed points, the second term $\binom{n}{1}d_{n-1}$ counts all those with 1 fixed point, and so on. Since there are $n!$ permutations over all and the only one not counted is the one with all n points fixed, we have justified the claim. Similarly,

$$\begin{aligned}
&\binom{n}{1}d_{n-1} + 2\binom{n}{2}d_{n-2} + \cdots + (n-1)\binom{n}{n-1}d_1 \\
= \;& n\left(d_{n-1} + \binom{n-1}{1}d_{n-2} + \cdots + \binom{n-1}{n-2}d_1\right) \\
= \;& n((n-1)! - 1).
\end{aligned}$$

It follows that the desired expression is equal to $n! - 1 + n((n-1)! - 1) = 2n! - (n+1)$.

29. There are at least 4 smarties randomly distributed among at least 4 boxes. In each move, remove 1 smarty from each of 2 boxes and put both of them into a third box. Is it always possible to have all the smarties in 1 box?

First Solution:

The answer is affirmative because we just have to merge the contents of the boxes 3 at a time. We may need a fourth box, but will leave its content unchanged. Let the contents of the four boxes be (a, b, c, x), with $0 \leq a \leq b \leq c$. Suppose $a = b$. Then we perform the sequence $(a, a, c, x) \to (a-1, a-1, c+2, x) \to \cdots \to (0, 0, c+2a, x)$. Similarly, if $b = c$, we have $(a, b, b, x) \to (a+2b, 0, 0, x)$. Suppose $a > 0$, $b = a+1$ and $c = a+2$. Then

$$\begin{aligned}
(a, a+1, a+2, x) &\to (a, a, a+1, x+2) \\
&\to (a+2, a, a, x+1) \\
&\to (a+2, a+2, a-1, x) \\
&\to (0, 0, 3a+3, x).
\end{aligned}$$

Note that $a > 0$ is crucial as (0,0,1,2) is an unsolvable distribution. This is why we need at least 4 smarties. Finally, if $0 \leq a < b < c$ and $c - a > 2$, we take from the "rich" and give to the "poor" among these 3 boxes, starting with $(a,b,c,x) \longrightarrow (a+2,b-1,c-1,x)$. This reduces the difference between the richest and the poorest. Eventually, we will arrive at one of the cases considered above.

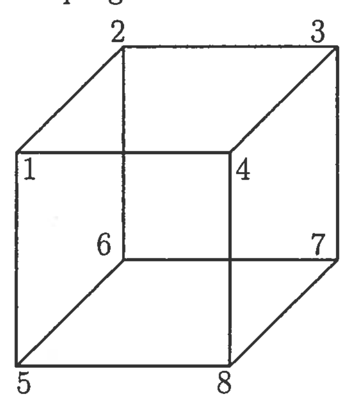
Second Solution:

We use induction on the number n of smarties to prove that the answer is affirmative. For $n=4$, we have $(1,1,1,1) \longrightarrow (0,0,3,1) \longrightarrow (0,2,2,0) \longrightarrow (0,1,1,2) \longrightarrow (0,0,0,4)$. Since this sequence includes all possible distributions of 4 smarties, the basis is established. Suppose the result holds for some $n \geq 4$. Consider the next case with $n+1$ smarties. For now, treat one of them as non-existent. By the induction hypothesis, the other n can be put into 1 box. If the $(n+1)$-st smarty is there as well, we have nothing further to do. Otherwise, we perform $(0,0,1,n) \longrightarrow (2,0,0,n-1) \longrightarrow (1,2,0,n-2) \longrightarrow (0,2,2,n-3) \longrightarrow (0,1,1,n-1) \longrightarrow (0,0,0,n+1)$. This completes the induction argument.

30. Eight singers take part in a festival. The organizer wants to plan a number of concerts with four singers performing in each. The number of concerts in which a pair of singers performs together is the same for every pair. Determine the minimum number of concerts.

Solution:

Let b be the number of concerts and λ be the number of concerts in which each pair of singers performs together. Counting the total number of appearances of such pairs in two ways, we have $\lambda\binom{8}{2} = b\binom{4}{2}$ or $14\lambda = 3b$. Since 3 and 14 are relatively prime, b must be divisible by 14, so that $b \geq 14$. We now construct a program which shows that $b = 14$ is sufficient.



Represent the 8 singers by the vertices of a cube. The 14 concerts are represented by the faces (1,2,3,4), (5,6,7,8), (1,2,5,6), (3,4,7,8), (1,4,5,8) and (2,3,6,7), the cross-sections (1,3,5,7), (2,4,6,8), (1,2,7,8), (3,4,5,6), (1,4,6,7) and (2,3,5,8), along with the tetrahedra (1,3,6,6) and (2,4,5,7).

31. A multiple-choice examination has 5 questions, each with 4 choices. Each of 2000 students picks exactly 1 choice for each question. Among any n students for some positive integer n, there exist 4 such that any 2 of them give the same answers to at most 3 questions. Determine the minimun value of n.

Solution:

We first prove that $n \geq 25$. Divide the 2000 students into $4^4 = 256$ groups according to how they answer the first four questions. Since $7 \times 256 = 1792 < 2000$, at least one group consists of at least 8 students. Take 8 of them aside. Since $1792 < 1984$, we can take aside two more sets of 8 students, each set from one of the groups. Among any 4 of these 24 students, 2 of them must be from the same group and can answer differently at most one question. We now give an example to show that we can have $n = 25$. Let the groups be as before, and let the multiple choices be 0, 1, 2 and 3. We discard 6 groups at random and put 8 of the 2000 students in each of the remaining groups. Within each group, we insist that the last question be answered in the same way, so that the total of the five answers is a multiple of 4. Among any 25 students, there will always be 4 who are from different groups, and they must answer at least one question differently. However, if they answer exactly one question differently, at least one of the totals of their answers will not be a multiple of 4. It follows that the desired minimum value is $n = 25$.

32. A space city consists of 99 space stations. Every two stations are connected by a space highway. All highways are one-way except for 99 which are two-way. Design such a space city so that the number of *groups* is as large as possible, where a group is defined as a set of four stations such that we can travel from any one to any other of the four along the highways.

Solution:

If a set of four space stations is not a group, it must consist of two non-empty subsets X and Y such that all space-highways between X and Y are one-way going from X to Y. Such a set may be classified as type I, II or III according to whether $|X| = 1$, 2 or 3. Note that this classification is not mutually exclusive, so that a set may be of two or even all three types. We solve a more general problem with n space stations, where $n > 3$ is an odd integer. For $1 \le i \le n$, let s_i be the number of one-way space highways from the i-th space station. Then the number of type I sets with this space station as the sole member of X is $\binom{s_i}{3}$, so that the number of sets of type I is $T = \sum_{i=1}^{n} \binom{s_i}{3}$. Suppose a and b are positive integers such that $a > b + 1$. By Pascal's Formula, $\binom{a}{3} - \binom{a-1}{3} = \binom{a-1}{2} < \binom{b}{2} = \binom{b+1}{3} - \binom{b}{3}$. In other words, $\binom{a}{3} + \binom{b}{3} < \binom{a-1}{3} + \binom{b+1}{3}$. It follows that to minimize T, we should make s_i, $1 \le i \le n$, equal to one another if possible. Now $\frac{1}{n}\sum_{i=1}^{n} s_i = \frac{1}{n}\left(\binom{n}{2} - n\right) = \frac{n-3}{2}$. Denote this value by m. If $s_1 = s_2 = \cdots = s_n = m$, then $T = n\binom{m}{3}$ and the number of groups is at most $\binom{n}{4} - n\binom{m}{3}$. For $n = 99$, we have $m = 48$ and $\binom{n}{4} - n\binom{m}{3} = 2052072$. We now show that this maximum can be attained. Let the space stations be 1, 2, ..., n clockwise round a circle. The space highways joining two adjacent space stations are two-way. For non-adjacent space stations i and j, the one-way space highway goes from i to j if and only if in going from i to j clockwise round the circle, the number of other space stations passed over is odd. Since n is odd, this scheme can be applied consistently. By symmetry, there are exactly m one-way space highways from each space station. It follows that the number of sets of type I is exactly $n\binom{m}{3}$. To complete the argument, we prove that all sets not of type I are groups. Suppose in $\{A, B, C, D\}$, A and B are joined by a two-way space highway. Suppose the space highways joining C to A and B are both one-way. Then the space highway between A to C goes from A to C if and only if the one between B and C goes from C to B. If either is two-way, it makes things even simpler. In the same way, we can also go from D to A and B and vice versa. Suppose $\{A < B < C < D\}$ is of type II, with $X = \{A < B\}$ and $Y = \{C < D\}$. If X and Y separate each other round the circle, we may assume that A, C, B and D are in clockwise order. Then the numbers of space stations from A to C and from B to D must be odd. Since n is odd, either the number of space stations from C to B or that from D to A

must be even. This will contradict the direction of the one-way space highway between B and C or that between A and D. It follows that X and Y cannot separate each other round the circle, so that we may assume that A, B, C and D are in clockwise order. Now the number of space stations from A to B, from B to C and from C to D must all be odd. This means that we may redefine $X = \{A\}$ and $Y = \{B < C < D\}$, so that the set is also of type I. Finally, suppose $\{A, B, C, D\}$ is of type III. In order for this not to be of type I, we must have exactly one one-way space highway from each of A, B and C to the other two. We may assume that A, B and C are in clockwise order. If the space highway between A and B goes from A to B, then the numbers of space stations from A to B, from B to C and from C to A must all be odd. However, this is impossible since n is odd. Hence the space highway goes from B to A, and the numbers of space stations from A to B, from B to C and from C to A are all even. By symmetry, we may assume that D is one of those between A and B. Since a one-way space highway goes from A to D, the number of space stations from A to D is odd. However, this implies that the number of space stations from D to B is even, contradicting the fact that a one-way space highway goes from B to D. This completes the proof that all sets which are not of type I are groups.

33. In a table-tennis tournament, all games are between pairs of participants. Each participant is a member of at most two pairs. No participant ever plays against another if the two form a pair. Two pairs play exactly once against each other as long as the preceding rule is not violated. A set $\{a_1, a_2, \ldots, a_k\}$ is given, where k is a positive integer and $0 < a_1 < a_2 < \cdots < a_k$ are multiples of 6. What is the minimum number of participants so that at the end of the tournament, the number of games played by each participant is a_i for some i, and for each i, at least one participant has played exactly a_i games?

Solution:

We first prove that the number of participants is at least $\frac{a_k}{2}+3$. Consider a participant X who has played a_k games. If X is in only one pair, then there are a_k other pairs who have played this pair. Since each participant is in at most two pairs, there are at least a_k participants in those a_k pairs, yielding a total of at least $a_k + 2 > \frac{a_k}{2} + 3$. If X is in two pairs, then there are a_k other pairs who have played either of these two pairs, so that one of them has played $\frac{a_k}{2}$ pairs. It follows that the total number of participants is at least $\frac{a_k}{2} + 3$. We now give a construction that $\frac{a_k}{2} + 3$ participants are sufficient. We use induction on k. For $k = 1$, divide the $\frac{a_1}{2} + 3$ participants into sets of three, and any two of the three form a pair. Pairs in different sets play each other. The number of games each participant plays is $2\frac{a_1}{2} = a_1$ as desired. For $k = 2$, divide the $\frac{a_2}{2} + 3$ participants into two subsets, with $\frac{a_1}{2}$ in S and the other $\frac{a_2-a_1}{2} + 3$ in T. Each of S and T is divided into sets of three, and any two of the three form a pair. Pairs in different sets play each other unless both are in T. The number of games each participant in T plays is $2\frac{a_1}{2} = a_1$ while the number of games each participant in S plays is $2(\frac{a_1}{2} - 3) + 2(\frac{a_2-a_1}{2} + 3) = a_2$ as desired. Assume that such a tournament exists for some $k-1$ with $k \geq 2$. Consider a tournament with $\frac{a_{k+1}}{2} + 3$ players. Divide them into three subsets, with $\frac{a_1}{2}$ in S, $\frac{a_k-a_1}{2} + 3$ in T and $\frac{a_{k+1}-a_k}{2}$ in U. Each of S, T and U is divided into sets of three, and any two of the three form a pair. Pairs in S play all pairs not in the same set. By the induction hypothesis, there exists a mini-tournament within T for the set $\{a_2 - a_1, a_3 - a_1, \ldots, a_k - a_1\}$. The number of games each participants in U plays is $2\frac{a_1}{2} = a_1$. The number of games played by a participant in T who has played $a_i - a_1$ games in the mini-tournament within T is $2\frac{a_1}{2} + (a_i - a_1) = a_i$ for $i = 2, \ldots, k$. The number of games each participant in S plays is $2(\frac{a_1}{2} - 3) + 2(\frac{a_k-a_1}{2} + 3) + 2(\frac{a_{k+1}-a_k}{2}) = a_{k+1}$. This completes the inductive argument.

34. On the circumference of a circle are 24 points which divide it into 24 arcs of length 1. In how many ways can we choose 8 of these points such that neither arc determined by any two chosen points has length 3 or 8.

Solution:

Label the points 0 to 23 in cyclic order and arrange the labels in a 3×8 array as shown below.

$$\begin{array}{cccccccc} 0 & 3 & 6 & 9 & 12 & 15 & 18 & 21 \\ 8 & 11 & 14 & 17 & 20 & 23 & 2 & 5 \\ 16 & 19 & 22 & 1 & 4 & 7 & 10 & 13 \end{array}$$

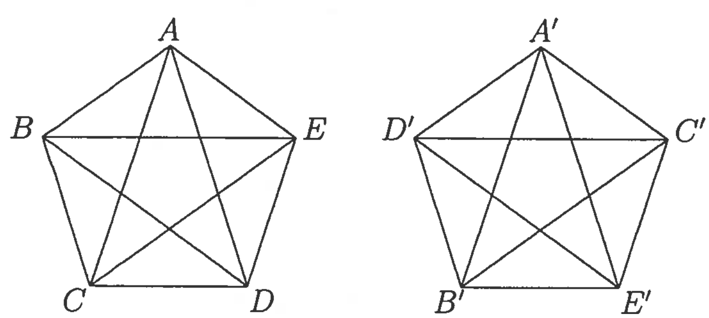
Two adjacent labels in the same row (the first and the last labels being considered adjacent) differ by 3 while two adjacent labels in the same column (the top and the bottom labels being considered adjacent) differ by 8. Thus the problem is equivalent to choosing 8 mutually non-adjacent labels. Sioce we can choose at most one from each column, we have to take exactly one from each. In general, let x_n be the number of valid choices from a $3 \times n$ array where $n \geq 2$. There are 3 choices in the first column and 2 in each subsequent columns. However, it may happen that the labels in the first and the last columns are adjacent. In this case, omission of the last column yields a valid choise for the $3 \times (n-1)$ array. It follows that $x_n + x_{n-1} = 3 \cdot 2^{n-1}$ for $n \geq 3$, with $x_2 = 6$. Hence

$$\begin{aligned} x_8 &= (x_8 + x_7) - (x_7 + x_6) + (x_6 + x_5) - (x_5 + x_4) + (x_4 + x_3) - (x_3 + x_2) + x_2 \\ &= 3(2^7 - 2^6 + 2^5 - 2^4 + 2^3 - 2^2 + 2) \\ &= 258. \end{aligned}$$

35. On each vertex of a regular n-gon is a blue jay. They fly away and then return, again one blue jay on each vertex, but not necessarily to their original positions. Prove that there exist three blue jays such that the triangle determined by their earlier positions and the triangle determined by their later positions are of the same type, that is, both acute, both right or both obtuse.

Solution:

Identify each blue jay by its initial position J and denote its new position by J'. Consider first a regular $2n$-gon for $n \geq 2$. Let A and B be two blue jays which are diametrically opposite. If A' and B' are still diametrically opposite, then any third blue jay C will work since $\angle ACB = 90° = \angle A'C'B'$. Otherwise, let C be the blue jay such that C' is diametrically opposite to A'. Then $\angle ACB = 90° = \angle A'B'C'$. Note that the result is trivially true for an equilateral triangle, while the following example shows that it is false for a regular pentagon.



Consider now a regular $2n+1$-gon for $n \geq 3$. Clearly there are no right triangles. The number of obtuse triangles with a particular diagonal as the longest side is equal to the number of vertices between the endpoints of this diagonal, going the shorter way. Since there are $2n+1$ diagonals of each length, the total number of obtuse triangles is $(2n+1)(1+2+\cdots+(n-1)) = \frac{1}{2}(n-1)n(2n+1)$. The total number of triangles is $\binom{2n+1}{3} = \frac{1}{3}(2n-1)n(2n+1)$. Since $\frac{\frac{1}{2}(n-1)}{\frac{1}{3}(2n-1)} = \frac{1}{2} + \frac{n-2}{4n-2} > \frac{1}{2}$ for $n \geq 3$, there are more obtuse triangles than acute ones. By the Pigeonhole Principle, there exist three blue jays whose initial and final positions both determine obtuse triangles.

36. Determine the number of ways of constructing a $4 \times 4 \times 4$ block from 64 unit cubes, exactly 16 of which are red, so that there is exactly one red cube within each $1 \times 1 \times 4$ subblock in any orientation.

Solution:

Clearly, each vertical $1 \times 1 \times 4$ stack contains exactly one red cube. All we have to decide is whether it is on level 1, 2, 3 or 4. If we represent the base of the cube as a 4×4 table, we fill in each square with a number indicating the level of the red cube on that stack. Also, each horizontal $1 \times 1 \times 4$ slab contains exactly one red cube, and for this to happen, no two numbers in each row and each column in the table can be the same. The first row can be filled in 4! ways, and by symmetry we only need consider the case when the entries are 1, 2, 3 and 4 in that order. The remaining entries of the first column are 2, 3 and 4 in some order. They can be permuted in 3! ways, and we need only consider the permutation 2, 3 and 4 in that order. Now the entry in the second row and second column is 1, 3 or 4. If it is 3 or 4, the remaining entries are determined uniquely. If it is 1, the remaining entries in the second row and second column are determined uniquely, but there are two ways to complete the table. Thus there are four different tables with the elements in the first row and the first column being 1, 2, 3 and 4 in that order, and these are shown in the diagram below. The number of different tables is therefore $4! \times 3! \times 4 = 576$, which is also the number of different ways of assembling the cube.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

Geometry

37. Let p be a prime. Determine the number of right triangles such that the incentre is (0,0), the vertex of the right angle is $(1994p, 7 \cdot 1994p)$, and the other two vertices have integer coordinates.

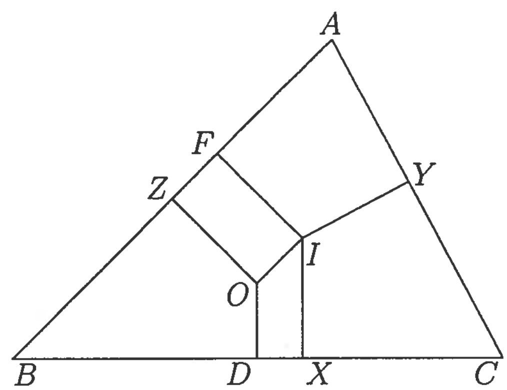
Solution:

Performing a half-turn about the point $(997p, 7 \cdot 997p)$, we may instead take $I(1994p, 7 \cdot 1994p)$ as the incentre and $O(0,0)$ as the vertex of the right angle of triangle OAB. Since the slope of OI is 7, the slope of one side, say OA, is $\tan(\arctan 7 - 45°) = \frac{7-1}{1+7} = \frac{3}{4}$. Then the slope of the other side OB is $-\frac{4}{3}$. From $(3k)^2 + (4k)^2 = (5k)^2$, we see that the distance of any lattice point on OA or OB from O is a multiple of 5. Rotate about O so that OA falls on the positive

x-axis and OB on the positive y-axis. Take as a new unit of length 5 times the old one. Then the new coordinates of I are (r, r) where $r = 1994p$. Let the new coordinates of A and B be $(r+s, 0)$ and $(0, r+t)$, respectively. Then $AB = s+t$ and we have $(r+s)^2+(r+t)^2 = (s+t)^2$. This is equivalent to $2r^2 = (s-r)(t-r)$. Since the hypotenuse is the longest side in a right triangle, both $u = s-r$ and $v = t-r$ are positive and we have $2r^2 = uv$. We claim that for any pair (u, v) of positive integers, the triangle OAB with O at (0,0), A at $(2r+u, 0)$ and B at $(0, 2r+v)$ has incentre $I(r, r)$. Since $(2r+u)^2 + (2r+v)^2 = (2r+u+v)^2$, we have $AB = 2r+u+v$. The inradius is given by $\frac{1}{2}(OA+OB-AB) = r$. This justifies the claim. The problem now becomes finding the number of pairs (u, v) of positive integers such that $uv = 2r^2 = 2^3 997^2 p^2$. For $p = 2$, the number of pairs of positive divisors of $2^5 997^2$ is $(5+1)(2+1) = 18$. For $p = 997$, the number of pairs of positive divisors of $2^3 997^4$ is $(3+1)(4+1) = 20$. For any prime p other than 2 and 5, the number of pairs of positive divisors of $2^3 997^2 p^2$ is $(3+1)(2+1)(2+1) = 36$.

38. Let ABC be a non-obtuse triangle with circumcentre O and incentre I. Determine $\sin A$ if $AB > AC$, $\angle B = 45°$ and $\sqrt{2}OI = AB - AC$.

First Solution:



Let the incircle of triangle ABC touch BC at X, CA at Y and AB at Z. Let D be the midpoint of BC and F be the midpoint of AB. Then

$$\begin{aligned}
\sqrt{2}OI &= AB - AC \\
&= (AZ + ZB) - (AY + YC) \\
&= ZB - YC \\
&= XB - XC \\
&= (\frac{1}{2}BC + XD) - (\frac{1}{2} - XD) \\
&= 2XD.
\end{aligned}$$

Hence $OI = \sqrt{2}XD$. Since XD is the projection of OI onto BC, the angle between OI and BC is 45°. Since $\angle ABC = 45°$, OI is either perpendicular or parallel to AB. In the first case, Z and F coincide. By symmetry, $AC = BC$. Hence $\angle CAB = \angle ABC = 45°$ so that $\sin A = \frac{1}{\sqrt{2}}$. In the second case, as illustrated in the diagram above, let r be the inradius and R the circumradius. Then $\angle AOF = \angle ACB$, $OF = IZ = r$ and $AO = R$. Hence

$$\begin{aligned}
\cos C &= \frac{r}{R} \\
&= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}
\end{aligned}$$

$$\begin{aligned}
&= 2\sin\frac{\pi}{8}\left(\cos\frac{C-A}{2} - \cos\frac{C+A}{2}\right)\\
&= 2\sin\frac{\pi}{8}\left(\cos\left(\frac{3\pi}{8}\right) - \sin\frac{\pi}{8}\right)\\
&= \sin\left(\frac{\pi}{2} - C\right) + \sin\left(C - \frac{\pi}{4}\right) + \cos\frac{\pi}{4} - 1\\
&= \cos C + \sin\left(C - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}} - 1.
\end{aligned}$$

Hence $\sin(C - \frac{\pi}{4}) = 1 - \frac{1}{\sqrt{2}}$ and

$$\begin{aligned}
\sin A &= \sin\left(\frac{\pi}{2} + \left(\frac{\pi}{4} - C\right)\right)\\
&= \cos\left(\frac{\pi}{4} - C\right)\\
&= \sqrt{1 - \sin^2\left(\frac{\pi}{4} - C\right)}\\
&= \sqrt{1 - \left(1 - \frac{1}{\sqrt{2}}\right)^2}\\
&= \sqrt{\sqrt{2} - \frac{1}{2}}.
\end{aligned}$$

Second Solution:

Let r be the inradius and R be the circumradius of triangle ABC, By Euler's Formula and the Law of Sines, $2(R^2 - 2Rr) = (\sqrt{2}OI)^2 = (AB - AC)^2 = 4R^2(\sin C - \sin B)^2$. Also, $r = \frac{AB+BC-CA}{2}\tan\frac{B}{2} = R(\sqrt{2} - 1)(\sin C + \sin A - \sin B)$. It follows that

$$1 - 2(\sqrt{2} - 1)\left(\sin C + \sin A - \frac{1}{\sqrt{2}}\right) = 2\left(\sin C - \frac{1}{\sqrt{2}}\right)^2.$$

Since $\sin C = \sin(\frac{3\pi}{4} - A) = \frac{1}{\sqrt{2}}(\sin A + \cos A)$, we have

$$1 - (2 - \sqrt{2})((\sqrt{2} + 1)\sin A + \cos A - 1) = (\sin A + \cos A - 1)^2.$$

This is equivalent to $(\sqrt{2}\sin A - 1)(\sqrt{2}\cos A - \sqrt{2} + 1) = 0$. Hence either $\sin A = \frac{1}{2}$ or $\cos A = 1 - \frac{1}{\sqrt{2}}$. The latter yields $\sin A = \sqrt{1 - (1 - \frac{1}{\sqrt{2}})^2} = \sqrt{\sqrt{2} - \frac{1}{2}}$.

39. In triangle ABC, $\angle C = 90°$, $\angle A = 30°$ and $BC = 1$. Determine the minimum length of the longest sides of all triangles whose vertices lie respectively on the three sides of triangle ABC.

Solution:

Set up a coordinate system with C at (0,0), A at $(\sqrt{3}, 0)$ and B at (0,1). Let D be any point on BC and let $BD = d$. Take E on AC such that $CE = \frac{\sqrt{3}d}{2}$, and take F on AB such that $BF = 1 - \frac{d}{2}$. By Pythagoras' Theorem, $DE^2 = (1-d)^2 + (\frac{\sqrt{3}d}{2})^2 = \frac{7}{4}d^2 - 2d + 1$. By the Law of Cosines, $DF^2 = d^2 + (1 - \frac{d}{2})^2 - 2d(1 - \frac{d}{2})\cos 60° = \frac{7}{4}d^2 + 2d - 1$. Similarly, $EF^2 = (\sqrt{3} - \frac{\sqrt{3}d}{2})^2 + (1 + \frac{d}{2})^2 - 2(\sqrt{3} - \frac{\sqrt{3}d}{2})(1 + \frac{d}{2})\cos 30° = \frac{7}{4}d^2 + 2d - 1$. Since d can take any value from 0 to 1, this means that we can always inscribe an equilateral triangle DEF in ABC from any point F on AB such that $\frac{1}{2} \le BF \le 1$. Now $\frac{7}{4}d^2 + 2d - 1 = \frac{7}{4}(d - \frac{4}{7})^2 + \frac{3}{7}$. It follows that the minimum value of $DE = EF = FD$ is $\sqrt{\frac{3}{7}}$. Let P be the point on AB with y-coordinate $\sqrt{\frac{3}{7}}$ and Q be the point on AB with x-coordinate $\sqrt{\frac{3}{7}}$. Then P lies on the segment BQ. Let XYZ be any triangle inscribed in ABC, with X on BC, Y on CA and Z on AB. If Z lies on the segment BP, then $ZY \ge \sqrt{\frac{3}{7}}$. If Z lies on the segment AQ, then $ZX \ge \sqrt{\frac{3}{7}}$. Suppose Z lies on the segment PQ. Then $\frac{1}{2} < ZB < 1$. Hence we may inscribe an equilateral triangle DEZ in ABC as before. Now the y-coordinate of Z is greater than that of D, and the x-coordinate of Z is greater than of E. If X lies on the segment CD, then $XZ \ge DZ \ge \sqrt{\frac{3}{7}}$. If Y lies on the segment CE, then $YZ \ge EZ \ge \sqrt{\frac{3}{7}}$. If X lies on the segment BD and Y lies on the segment AE, then $XY \ge DE \ge \sqrt{\frac{3}{7}}$. Hence $\sqrt{\frac{3}{7}}$ is the desired minimum.

40. In triangle ABC, $a \le b \le c$ where $a = BC$, $b = CA$ and $c = AB$. The circumradius is R and the inradius is r. What can be said about $\angle C$ if $a + b - 2R - 2r$ is

(a) positive;

(b) zero;

(c) negative?

Solution:

We have

$$\begin{aligned}
\frac{a+b-2R-2r}{2R} &= \sin A + \sin B - 1 - 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\
&= 2\sin\frac{B+A}{2}\cos\frac{B-A}{2} - 1 - 2\left(\cos\frac{B+A}{2} - \cos\frac{B-A}{2}\right)\sin\frac{C}{2} \\
&= 2\cos\frac{B-A}{2}\left(\sin\frac{\pi-C}{2} - \sin\frac{C}{2}\right) - 1 + 2\cos\frac{\pi-C}{2}\sin\frac{C}{2} \\
&= 2\cos\frac{B-A}{2}\left(\cos\frac{C}{2} - \sin\frac{C}{2}\right) - \left(\cos^2\frac{C}{2} - \sin^2\frac{C}{2}\right) \\
&= \left(\cos\frac{C}{2} - \sin\frac{C}{2}\right)\left(2\cos\frac{B-A}{2} - \cos\frac{C}{2} - \sin\frac{C}{2}\right).
\end{aligned}$$

Since $0 \le B - A < B \le C$ and $0 \le B - A < B + A$, we have $cos\frac{B-A}{2} > \cos\frac{C}{2}$ and $\cos\frac{B-A}{2} > \cos\frac{B+A}{2} = \sin\frac{C}{2}$. It follows that the second factor in the last displayed expression is always positive.

(a) If $a + b - 2R - 2r > 0$, then $\cos\frac{C}{2} > \sin\frac{C}{2}$ or $\angle C < 90°$.

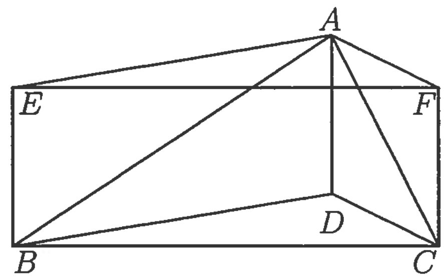
(b) If $a + b - 2R - 2r = 0$, then $\cos\frac{C}{2} = \sin\frac{C}{2}$ or $\angle C = 90°$.

(c) If $a + b - 2R - 2r < 0$, then $\cos\frac{C}{2} < \sin\frac{C}{2}$ or $\angle C > 90°$.

41. Let D be a point inside an acute triangle ABC. Characterize geometrically the set of possible locations of the point D if it satisfies

$$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA = AB \cdot BC \cdot CA.$$

Solution:



We prove a stronger results that the left side of the given equation is always greater than or equal to the right side, with equality if and only if D is the orthocentre of triangle ABC. Complete the parallelograms $ABCE$ and $ADCF$. Then $BCFE$ is also a parallelogram. Applying Ptolemy's Inequality to the quadrilaterals $ABCF$ and $AEBF$, we have

$$DC \cdot BC + AB \cdot DA = AF \cdot BC + AB \cdot CF \geq AC \cdot BF$$

and

$$DB \cdot BF + DC \cdot DA = AE \cdot BF + AF \cdot BE \geq AB \cdot EF = AB \cdot BC.$$

It follows that

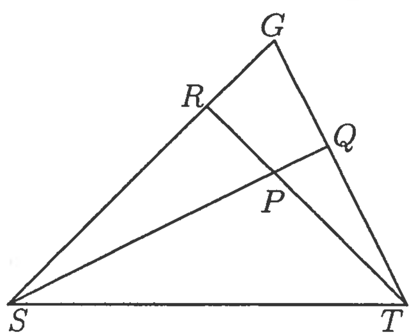
$$\begin{aligned} & DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA \\ = \ & DB(AB \cdot DA + BC \cdot DC) + DC \cdot DA \cdot CA \\ \geq \ & DB \cdot AC \cdot BF + DC \cdot DA \cdot CA \\ = \ & AC(DB \cdot BF + DC \cdot DA) \\ \geq \ & AC \cdot AB \cdot BC. \end{aligned}$$

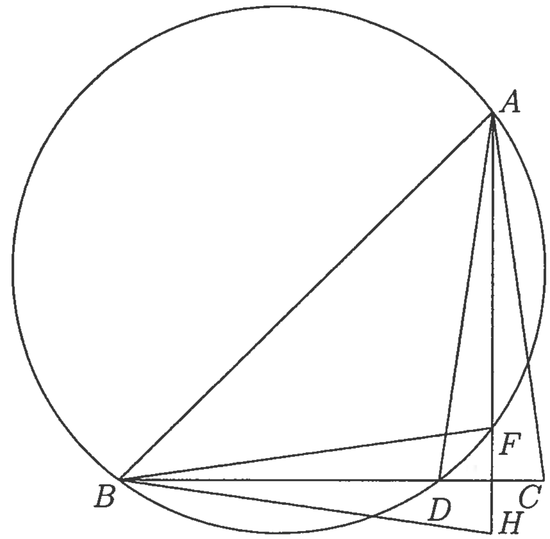
Equality holds if and only if both quadrilaterals are cyclic, that is, $AEBCF$ is a cyclic pentagon. Since $BCFE$ is a parallelogram, it must be a rectangle, so that BF and CE are diameters of the circumcircle. Hence $\angle BAF = \angle CAE = 90°$, so that AB is perpendicular to CD and AC is perpendicular to BD. This means that D is the orthocentre of triangle ABC.

42. In an acute triangle ABC, $\angle C > \angle B$. D is a point on BC such that $\angle ADB$ is obtuse. H is the orthocentre of triangle BAD. F is a point inside triangle ABC and on the circumcircle of triangle BAD. Prove that F is the orthocentre of triangle ABC if and only if CF is parallel to HD and H is on the circumcircle of triangle ABC.

Solution:

We first prove a preliminary result. Let GST be a triangle with acute angles at S and T. Let P be a point on the same side of ST as G such that GP is perpendicular to ST. Then P is the orthocentre of GST if and only if $\angle SPT + \angle SGT = 180°$. Suppose P is the orthocentre. First assume that it is inside GST. Let SP cut GT at Q, and let TP cut GS at R. Then $\angle SQT = 90° = \angle GRT$. Hence $\angle SPT = 90° + \angle GTR = 180° - \angle SGT$. If P is outside GST, then the same argument applies with the roles of G and P interchanged. Conversely, suppose that $\angle SPT + \angle SGT = 180°$. Let the orthocentre of GST be some point P' which must lie on the line GP. Since $\angle GST$ and $\angle GTS$ are both acute, P' is on the same side of ST as G and P. By what we have proved, $\angle SP'T = 180° - \angle SGT = \angle SPT$. This is only possible if $P' = P$.

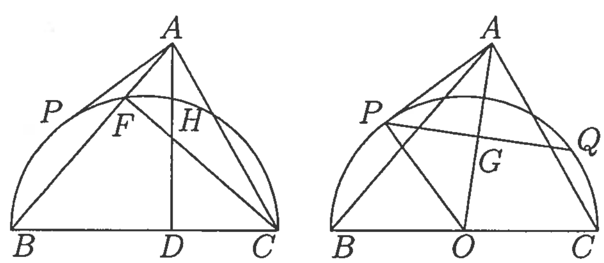




Suppose F is the orthocentre of triangle ABC. Then CF is perpendicular to AB, and hence parallel to DH. By our preliminary result, we have $\angle ACB = 180° - \angle AFB$ and $\angle AHB = 180° - \angle ADB$. Since $ABDF$ is cyclic, $\angle AFB = \angle ADB$ by Thales' Theorem. Hence $\angle ACB = \angle AHB$. By the converse of Thales' Theroem, $ABHC$ is cyclic. Conversely, suppose CF is parallel to DH and $ABHC$ is cyclic. Then CF is perpendicular to AB. By Thales' Theorem and our preliminary result, $\angle ACB = \angle AHB = 180° - \angle ADB = 180° - \angle AFB$. By the converse of our preliminary result, F is the orthocentre of ABC.

43. Let H be the orthocentre of an acute triangle ABC. From A draw two tangent lines AP and AQ to the circle whose diameter is BC, the points of tangency being P and Q respectively. Prove that P, H and Q are collinear.

Solution:

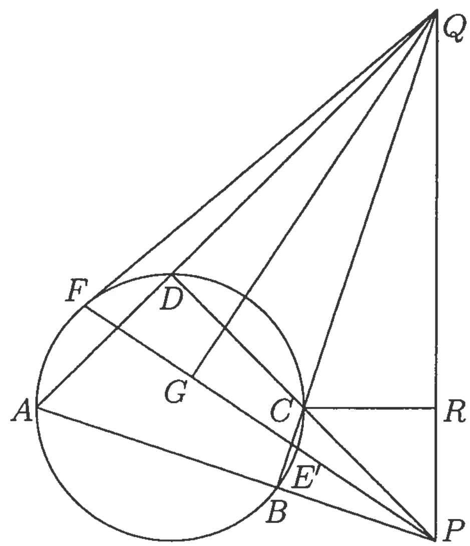


Let AH cut BC at D and let AB cut the circle at F. Then $\angle BFC = 90° = \angle ADB$. Hence $BDHF$ is a cyclic quadrilateral, so that $AH \cdot AD = AF \cdot AB = AP^2$. Let O be the midpoint of BC and let AO cut PQ at G. Then $\angle APO = 90° = \angle AGP$. Hence triangles APO and AGP are simialr, so that $AP^2 = AG \cdot AO$. From $AG \cdot AO = AH \cdot AD$, $DHGO$ is a cyclic quadrilateral. Since $\angle HDO = 90°$, we have $\angle HGO = 90° = \angle QGO$. It follows that H lies on PQ.

The twelve edges marked by double lines form an independent set, that is, no two share a vertex. If we take any subset of S of size $k \geq 39$, we will only be missing at most eleven numbers. Hence this subset must contain both vertices of one of the twelve independent edges, which means that it contains two distinct positive integers a and b such that $a + b$ divides ab. On the other hand, the twelve vertices marked by squares form a covering set, that is, every edge has at least one of them as a vertex. If we remove these 12 elements from S, we will leave behind a subset of S of size 38 in which $a + b$ does not divide ab for any two distinct elements a and b . It follows that the desired minimum value is $k = 39$.

44. $ABCD$ is a quadrilateral inscribed in a circle. The extensions of AB and DC meet at P, and the extensions of AD and BC meet at Q. The tangents from Q to the circle touch it at E and F. Prove that P, E and F are collinear.

Solution:



Let PF intersect the circle again at E' and let G be the foot of perpendicular from Q to PF. Since $ABCD$ is cyclic, $\angle ADC = \angle PBC$. Let R be the point on PQ such that $\angle QRC$ has the same measure. Then both $PRCB$ and $QDCR$ are cyclic quadrilaterals. Hence $PQ \cdot PR = PC \cdot PD = PE' \cdot PF$ and $PQ \cdot QR = QB \cdot QC = QF^2$. Addition yields $PQ(PR + QR) = PE' \cdot PF + QF^2$. Hence

$$PE' \cdot PF = PQ^2 - QF^2 = PG^2 - GF^2 = PF(PG - GF).$$

It follows that $PE' = PG - GF$ or $GF = PG - PE' = GE'$. This means that GQ passes through the centre of the circle and E' is symmetric to F with respect to GQ. Hence $E' = E$ and P, E and F are indeed collinear.

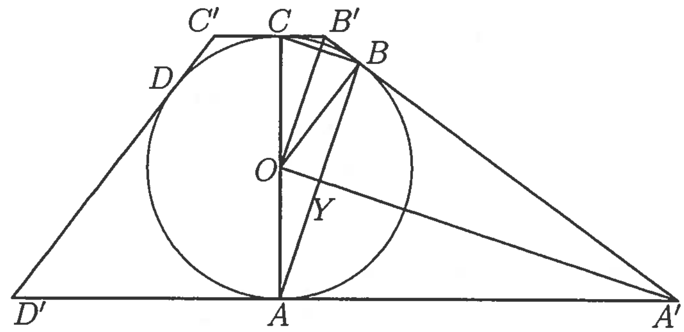
45. The quadrilateral $ABCD$ is inscribed in the unit circle such that it contains the centre of the circle and the length of its shortest side is a, where $\sqrt{2} < a < 2$. Let $A'B'C'D'$ be the quadrilateral determined by the tangents to the circle at A, B, C and D.

(a) Determine the minimum value of the ratio of the area of $A'B'C'D'$ to that of $ABCD$.

(b) Determine the maximum value of the ratio of the area of $A'B'C'D'$ to that of $ABCD$.

Solution:

Let O be the centre of the unit circle. If we cut up $A'B'C'D'$ along the radii OA, OB, OC and OD, we can rearrange the four pieces in any order without affecting the result of the problem. Thus we may assume that $AB = a$ while $BC = \sqrt{4 - a^2}$. Let AX be a diameter of the circle. Then $BX^2 = AX^2 - AB^2 = 4 - a^2 = BC^2$. Hence $C = X$.



Let AB cut OA' at Y. Then $OY = \frac{BC}{2}$. Since triangles $AA'Y$ and OBY are similar, we have $AA' = \frac{AY \cdot OB}{OY} = \frac{a}{\sqrt{4-a^2}}$. Now $AA' \cdot CB' = BA' \cdot BB' = OB^2 = 1$. Hence $CB' = \frac{\sqrt{4-a^2}}{a}$. The area of $AA'B'C$ is given by $AA' + CB' = \frac{a}{\sqrt{4-a^2}} + \frac{\sqrt{4-a^2}}{a} = \frac{4}{a\sqrt{4-a^2}}$ while the area of ABC is given by $\frac{AB \cdot BC}{2} = \frac{a\sqrt{4-a^2}}{2}$. Consider now $AD'C'C$ and ADC. The area of the former is given by $AD' + CC'$. Since $AD' \cdot CC' = 1$, its value increases as D moves away from the midpoint of the semicircular AC not passing through B. On the other hand, the area of ADC, considering AC as its base, decreases as D moves away from the midpoint of this arc.

(a) The minimum ratio between the areas of $A'B'C'D'$ and $ABCD$ occurs when $AD = CD$, and we have

$$\frac{\frac{4}{a\sqrt{4-a^2}} + 2}{\frac{a\sqrt{4-a^2}}{2} + 1} = \frac{4}{a\sqrt{4-a^2}}.$$

(b) Since we must have $\min\{AD, CD\} \geq BC$, the maximum ratio occurs when $AD = BC$ or $CD = BC$, and we have

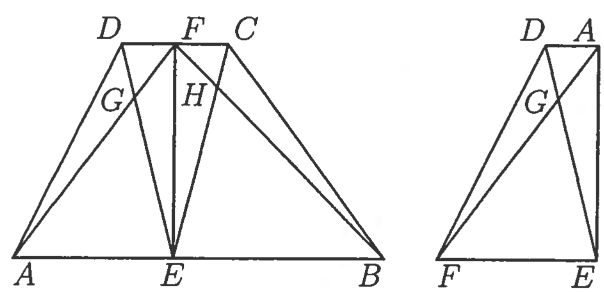
$$\frac{\frac{4}{a\sqrt{4-a^2}}}{\frac{a\sqrt{4-a^2}}{2}} = \frac{8}{a^2(4-a^2)}.$$

46. Let $ABCD$ be a quadrilateral with AB parallel to DC. Let E be a point on AB and F a point on CD. The segments AF and DE intersect at G, while the segments BF and CE intersect at H.

(a) Prove that the area of $EGFH$ is at most one-quarter that of $ABCD$.

(b) Is this conclusion still valid if $ABCD$ is an arbitrary convex quadrilateral?

Solution:



Denote the area of a polygon P by $[P]$.

(a) In the diagram on the left, let $\frac{FD}{EA} = \frac{1}{k}$ for some $k > 0$. If we take $[DFG] = 1$, then $[EFG] = [DAG] = k$ and $[AGE] = k^2$. By the Arithmetic-Geometric Mean Inequality,

$\frac{1}{2}(k^2+1) \geq k$. Hence $[EFG] \leq \frac{1}{4}[ADFE]$. Similarly, we have $[EFH] \leq \frac{1}{4}[BCFE]$, so that $[EGFH] \leq \frac{1}{4}[ABCD]$.

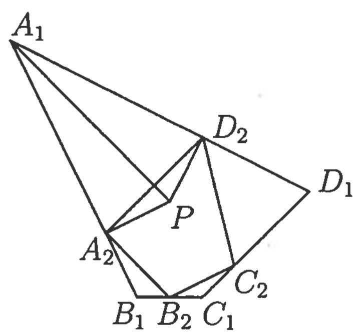
(b) The answer is negative and we construct a counter-example as follows. In the diagram on the right, we have switched the labels A and F in $ADFE$. Then $[EFG] > \frac{1}{4}[ADFE]$. Extend AE to B and DF to C so that $\frac{1}{4}[BCFE] < [EFG] - \frac{1}{4}[ADFE]$. Then we have $[EGFH] > [EFG] > \frac{1}{4}[ABCD]$.

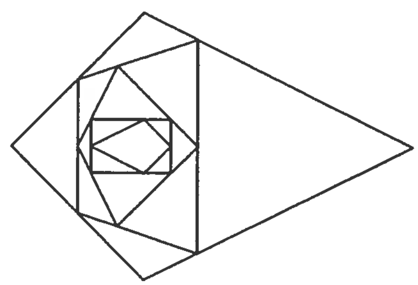
47. $A_1B_1C_1D_1$ is any convex quadrilateral. P is a point inside such that any line joining P to a vertex forms an acute angle with each of the two sides meeting at that vertex. Suppose A_{k-1}, B_{k-1}, C_{k-1} and D_{k-1} have been defined. Let A_k, B_k, C_k and D_k be the respective reflecitons of P across $A_{k-1}B_{k-1}$, $B_{k-1}C_{k-1}$, $C_{k-1}D_{k-1}$ and $D_{k-1}A_{k-1}$.

(a) Which of $A_iB_iC_iD_i$, $1 \leq i \leq 12$, is necessarily similar to $A_{1997}B_{1997}C_{1997}D_{1997}$?

(b) Which of $A_iB_iC_iD_i$, $1 \leq i \leq 12$, is necessarily cyclic if $A_{1997}B_{1997}C_{1997}D_{1997}$ is?

Solution:





Instead of reflecting P across the sides, we simply project it onto the sides since the two quadrilateral so obtained are homothetic to each other.

(a) The quadrilateral $A_1A_2PD_2$ is cyclic since $\angle PA_2A_1 = 90° = \angle PD_2A_1$, as illustrated by the diagram on the left. Hence $\angle PA_1D_1 = \angle PA_2D_2 = \cdots = \angle PA_{1997}D_{1997}$. On the other hand,

$$\angle PA_1B_1 = \angle PD_2A_2 = \angle PC_3D_3 = \angle PD_4A_4$$
$$= \angle PA_5B_5 = \cdots = \angle PA_9B_9 = \cdots = \angle PA_{1997}B_{1997}.$$

It follows that $\angle B_1A_1D_1 = \angle B_5A_5D_5 = \angle B_9A_9D_9 = \cdots = \angle B_{1997}A_{1997}D_{1997}$. The same applies to the other three angles of the quadrilaterals. Thus $A_1B_1C_1D_1$, $A_5B_5C_5D_5$ and $A_9B_9C_9D_9$ are similar to $A_{1997}B_{1997}C_{1997}D_{1997}$ via spiral homothety from P. The diagram on the right, generated by the point of intersection of the outermost kite, shows that $A_iB_iC_iD_i$ need not be similar to $A_{1997}B_{1997}C_{1997}D_{1997}$ for $i = 2, 3, 4, 6, 7, 8, 10, 11, 12$.

(b) As in (a), we have

$$\begin{aligned}
& \angle B_1A_1D_1 + \angle D_1C_1B_1 \\
= & \angle B_3A_3D_3 + \angle D_3C_3B_3 \\
= & \cdots \\
= & \angle B_{11}A_{11}D_{11} + \angle D_{11}C_{11}B_{11} \\
= & \cdots \\
= & \angle B_{1997}A_{1997}D_{1997} + \angle D_{1997}C_{1997}B_{1997}.
\end{aligned}$$

If $A_{1997}B_{1997}C_{1997}D_{1997}$ is cyclic, so is $A_iB_iC_iD_i$ for $i = 1, 3, 5, 7, 9, 11$. That this is not necessarily so for $i = 2, 4, 6, 8, 10, 12$ is justified by the same counter-example in (a).

48. The radii of four spheres are 2, 2, 3 and 3 respectively. Each is externally tangent to the three others. If a smaller sphere is tangent to each of these 4 spheres, determine the radius of the smaller sphere.

Solution:

Let the centre of the small sphere be O and its radius be r. Let A and B be the centres of the spheres of radii 3, and C and D be the centres of the spheres of radii 2. Let E be the midpoint of AB and F be the midpoint of CD. Now A and B are symmetric to each other with respect to the plane CDE, while C and D are symmetric with respect to the plane ABF. These two planes intersect along EF. By symmetry, O lies on the segment EF. Now $CE = \sqrt{AC^2 - AE^2} = \sqrt{5^2 - 3^2} = 4$ and $EF = \sqrt{CE^2 - CF^2} = \sqrt{4^2 - 2^2} = 2\sqrt{3}$. We also have

$$OE = \sqrt{OA^2 - AE^2} = \sqrt{(r+3)^2 - 3^2} = \sqrt{r^2 + 6r}$$

and

$$OF = \sqrt{OC^2 - CE^2} = \sqrt{(r+2)^2 - 2^2} = \sqrt{r^2 + 4r}.$$

Squaring both sides of $\sqrt{r^2 + 6r} = 2\sqrt{3} - \sqrt{r^2 + 4r}$, we have $12 - 2r = 4\sqrt{3(r^2 + 4r)}$. Squaring both sides again, we have $11r^2 - 60r - 36 = (11r - 6)(r + 6) = 0$. Hence $r = \frac{6}{11}$.