New Zealand Mathematical Olympiad Committee

2012 Squad Assignment Two

Algebra

Due: Monday 27th February 2012

1. Does there exist a positive real number C such that the inequality
$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \le C(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1)$$
holds for arbitrary positive real numbers x_1, x_2, x_3, x_4?

2. Let $n \ge 2$ be a natural number, and suppose that positive numbers $a_0, a_1, \ldots, a_n$ satisfy the equality
$$(a_{k-1} + a_k)(a_k + a_{k+1}) = a_{k-1} - a_{k+1}$$
for each $k = 1, 2, \ldots, n-1$. Prove that $a_n < \dfrac{1}{n-1}$.

3. For a positive integer d define the sequence
$$a_0 = 1, \qquad\qquad a_{n+1} = \begin{cases} \dfrac{a_n}{2}, & \text{if } a_n \text{ is even,} \\ a_n + d, & \text{if } a_n \text{ is odd.} \end{cases}$$
Determine all d such that $a_n = 1$ for some $n > 0$.

4. Find all triples (x, y, z) of real numbers that satisfy
$$x^2 + y^2 + z^2 + 1 = xy + yz + zx + |x - 2y + z|.$$

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 + f(y)) = y - x^2$$
for all $x, y \in \mathbb{R}$.

6. If k is an integer, let $c(k)$ denote the largest cube that is less than or equal to k. Find all positive integers p for which the following sequence is bounded:
$$a_0 = p \qquad \text{and} \qquad a_{n+1} = 3a_n - 2c(a_n) \quad \text{for } n \ge 0.$$
(A sequence $a_0, a_1, \ldots$ of real numbers is said to be bounded *if there exists an $M \in \mathbb{R}$ such that, for all $n \ge 0$, $|a_n| \le M$.)*

7. For each integer $n \ge 2$, determine the largest real constant C_n such that for all positive real numbers $a_1, \ldots, a_n$ we have
$$\frac{a_1^2 + \cdots + a_n^2}{n} \ge \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 + C_n \cdot (a_1 - a_n)^2.$$

6th February 2012
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