Problem Set: Functional Equations

Problem 1. Find $f : \mathbb{R} \to \mathbb{R}$ if $f(x)f(y) - f(xy) = x + y$ for all x, $y \in \mathbb{R}$.

Problem 2. Find all functions f, defined on the nonnegative real numbers and taking nonnegative real values, such that:

(i) $f(xf(y))f(y) = f(x + y)$ for all x, $y \geq 0$,

(ii) $f(2) = 0$,

(iii) $f(x) \neq 0$ for $0 \leq x < 0$.

Problem 3. Let $(f(n))$ be a strictly increasing sequence of positive integers such that $f(2) = 2$ and $f(mn) = f(m)f(n)$ for every relatively prime pair of positive integers m and n (the greatest common divisor of m and n is equal to 1). Prove that $f(n) = n$ for every positive integer.

Problem 4. Find, with proof, all real-valued functions $f(x)$ satisfying the equation

$$xf(x) - yf(y) = (x - y)f(x + y)$$

for all real numbers x, y.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

(i) For all x, $y \in \mathbb{R}$,

$$f(x) + f(y) + 1 \geq f(x + y) \geq f(x) + f(y),$$

(ii) For all $x \in [0, 1)$, $f(0) \geq f(x)$,

(iii) $-f(-1) = f(1) = 1$.

Find all such functions f.

Problem 6. Find all functions f from the positive reals to the positive reals such that

$$f(x + f(y)) = \frac{y}{xy + 1}$$

for all x, $y > 0$.

Problem 7. Find all functions f from the reals to the reals such that

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real x, y, z, t.

Problem 8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x + y) + f(x)f(y) = f(xy) + 2xy + 1$$

for all real x and y.

Problem 9. Let $\mathbb{R}^*$ be the set of nonzero real numbers. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ such that

$$f(x^2 + y) = f^2(x) + \frac{f(xy)}{f(x)}$$

for all x, $y \in \mathbb{R}^*$, $y \neq -x^2$.

Problem 10. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) - y^2) = f(x)^2 - 2f(x)y^2 + f(f(y))$$

for all x, $y \in \mathbb{R}$.