Inequalities

1 The AM-GM Inequality

For any real numbers $x_1, x_2, \ldots, x_n \geq 0$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

The left-hand side is known as the *arithmetic mean (AM)*, and the right-hand side is known as the *geometric mean (GM)*. Equality occurs if and only if $x_1 = x_2 = \cdots = x_n$.

There is a weighted version of the AM-GM inequality: Let $w_1, w_2, \ldots, w_n > 0$ be real numbers (known as *weights*) such that $w_1 + w_2 + \cdots + w_n = 1$. Then for any real numbers $x_1, x_2, \ldots, x_n \geq 0$,

$$w_1 x_1 + w_2 x_2 + \cdots + w_n x_n \geq x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}.$$

Equality occurs if and only if $x_1 = x_2 = \cdots = x_n$. Taking $w_i = 1/n$ for all i recovers the usual AM-GM inequality.

Problem 1. Show that for all $a, b, c \geq 0$,

$$a^2 + b^2 + c^2 \geq ab + ac + bc.$$

Solution. By the AM-GM inequality,

$$\frac{a^2 + b^2}{2} \geq ab, \quad \frac{a^2 + c^2}{2} \geq ac, \quad \text{and} \quad \frac{b^2 + c^2}{2} \geq bc.$$

Adding these inequalities gives $a^2 + b^2 + c^2 \geq ab + ac + bc$. ■

Problem 2. Let $0 \leq x \leq 4$.

(a) Find the maximum value of $f(x) = x(4 - x)$.

(b) Find the maximum value of $g(x) = x^3(4 - x)$.

Solution. (a) By the AM-GM inequality,

$$x(4 - x) \leq \left[\frac{x + (4 - x)}{2}\right]^2 = 4.$$

Equality occurs if and only if $x = 4 - x$, or $x = 2$. Indeed, $f(2) = 4$, so the maximum value is 4.

(b) The AM-GM inequality was successful in part (a) because we were able to compare $f(x)$ to a constant; we can try the same strategy here.

The function $g(x)$ is the product of the factors x, x, x, and $4-x$. Their sum $x+x+x+4-x = 2x+4$ is not constant. However, the sum $x+x+x+3(4-x) = 12$ is constant. Thus, by the AM-GM inequality,

$$3g(x) = x^3(12-3x) \le \left[\frac{x+x+x+(12-3x)}{4}\right]^4 = 81,$$

so $g(x) \le 27$. Equality occurs if and only if $x = 12-3x$, or $x = 3$. Indeed, $g(3) = 27$, so the maximum value is 27. ∎

Problem 3. Nonnegative real numbers a, b, x, y satisfy $a^5+b^5 \le 1$, $x^5+y^5 \le 1$. Show that $a^2x^3+b^2y^3 \le 1$. (Austrian-Polish Mathematics Competition, 1983)

Solution. By the weighted AM-GM inequality,

$$a^2x^3 \le \frac{2}{5}a^5 + \frac{3}{5}x^5$$

and

$$b^2y^3 \le \frac{2}{5}b^5 + \frac{3}{5}y^5.$$

Hence,

$$a^2x^3 + b^2y^3 \le \frac{2}{5}(a^5+b^5) + \frac{3}{5}(x^5+y^5) \le \frac{2}{5} + \frac{3}{5} = 1.$$

∎

Exercises

1. Prove the AM-GM inequality using the following steps:
   (1) Prove that the inequality holds for two variables.
   (2) Prove that if the inequality holds for k variables, then it holds for $2k$ variables.
   (3) Prove that if the inequality holds for k variables, then it holds for $k-1$ variables.
2. Show that for any positive integer $n \ge 1$,
$$1 \cdot 3 \cdot 5 \cdots (2n-1) \le n^n.$$
3. Let m and n be positive integers. Find the minimum value of
$$x^m + \frac{1}{x^n}$$
   for $x > 0$.
4. Prove that among all triangles of a given perimeter, the equilateral triangle has maximum area.

2 The Power Mean Inequality

Let $x_1, x_2, \dots, x_n > 0$ be real numbers. For any real number $r \neq 0$, let

$$M(r) = \left(\frac{x_1^r + x_2^r + \cdots + x_n^r}{n} \right)^{1/r},$$

and set $M(0) = \sqrt[n]{x_1 x_2 \cdots x_n}$. Then $M(r)$ is an increasing function of r. In other words, if $r < s$, then $M(r) \le M(s)$, and equality occurs if and only if $x_1 = x_2 = \cdots = x_n$.

The Power Mean inequality most often manifests itself as the QM-AM-GM-HM inequality: From $M(2) \ge M(1) \ge M(0) \ge M(-1)$, we get

$$\sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} \ge \frac{x_1 + x_2 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 x_2 \cdots x_n} \ge \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

(The first and fourth quantities are known as the *quadratic mean (QM)* and *harmonic mean (HM)*, respectively.)

As with the AM-GM inequality, there is a weighted version of the Power Mean inequality: Let $x_1, x_2, \dots, x_n > 0$ be real numbers, and let $w_1, w_2, \dots, w_n > 0$ be real numbers such that $w_1 + w_2 + \cdots + w_n = 1$. For any real number $r \neq 0$, let

$$M(r) = (w_1 x_1^r + w_2 x_2^r + \cdots + w_n x_n^r)^{1/r},$$

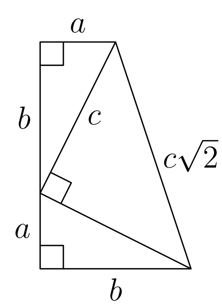
and set $M(0) = x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}$. Then $M(r)$ is an increasing function of r. In other words, if $r < s$, then $M(r) \le M(s)$, and equality occurs if and only if $x_1 = x_2 = \cdots = x_n$.

Problem 4. Let c be the length of the hypotenuse of a right angle triangle whose other two sides have lengths a and b. Prove that $a + b \le \sqrt{2}c$. When does equality hold? (Canada, 1969)

Solution. By Pythagoras, $c^2 = a^2 + b^2$, so the inequality can be re-written as

$$\frac{a+b}{2} \le \sqrt{\frac{a^2 + b^2}{2}},$$

which follows from the QM-AM inequality. Equality holds if and only if $a = b$. Alternatively, the following diagram gives a quick "proof without words".



■

Problem 5. Let x_1, x_2, x_3, x_4 be positive real numbers such that $x_1x_2x_3x_4 = 1$. Prove that

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \geq \max\left\{x_1 + x_2 + x_3 + x_4, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right\}.$$

(Iran, 1998)

Solution. We must prove that

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \geq x_1 + x_2 + x_3 + x_4 \tag{1}$$

and

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \geq \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}. \tag{2}$$

(1) By the Power Mean inequality,

$$\left(\frac{x_1^3 + x_2^3 + x_3^3 + x_4^3}{4}\right)^{1/3} \geq \frac{x_1 + x_2 + x_3 + x_4}{4},$$

so

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \geq \frac{(x_1 + x_2 + x_3 + x_4)^3}{16}.$$

By the AM-GM inequality, $x_1 + x_2 + x_3 + x_4 \geq 4\sqrt[4]{x_1x_2x_3x_4} = 4$, so

$$\frac{(x_1 + x_2 + x_3 + x_4)^3}{16} \geq x_1 + x_2 + x_3 + x_4.$$

Hence,

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \geq x_1 + x_2 + x_3 + x_4.$$

(2) Since $x_1x_2x_3x_4 = 1$,

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4.$$

By the AM-GM inequality,

$$\begin{aligned}
x_1^3 + x_2^3 + x_3^3 &\geq 3x_1x_2x_3, \\
x_1^3 + x_2^3 + x_4^3 &\geq 3x_1x_2x_4, \\
x_1^3 + x_3^3 + x_4^3 &\geq 3x_1x_3x_4, \\
x_2^3 + x_3^3 + x_4^3 &\geq 3x_2x_3x_4.
\end{aligned}$$

Adding these four inequalities and dividing by 3 gives

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 \geq x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}.$$

■

Exercises

5. Prove the QM-AM-GM-HM inequality. (Since we already have AM-GM, it suffices to show QM-AM and GM-HM).

6. For positive real numbers a, b, c, show that
$$\frac{ab}{a+b} + \frac{ac}{a+c} + \frac{bc}{b+c} \le \frac{a+b+c}{2}.$$

7. Let a_1, a_2, $\ldots$, a_n be positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that
$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \ge n^2.$$

8. Prove that
$$(ab + ac + bc)(a + b + c)^4 \le 27(a^3 + b^3 + c^3)^2$$
for a, b, $c \ge 0$.

3 The Triangle Inequality

For any real numbers x_1, x_2, $\ldots$, x_n,
$$|x_1| + |x_2| + \cdots + |x_n| \ge |x_1 + x_2 + \cdots + x_n|.$$
Equality occurs if and only if $x_i \ge 0$ for all i, or $x_i \le 0$ for all i.

Exercises

9. Prove the Triangle inequality.

10. Show that for all real numbers x and y, $|x - y| \ge |x| - |y|$.

11. What is the minimum value of $f(x) = |x - 1| + |2x - 1| + |3x - 1| + \cdots + |119x - 1|$? (2010 AMC 12A)

12. For a positive integer n, define S_n to be the minimum value of the sum
$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2},$$
where a_1, a_2, $\ldots$, a_n are positive real numbers whose sum is 17. There is a unique positive integer n for which S_n is also an integer. Find this n. (1991 AIME)

4 The Cauchy-Schwarz Inequality

For any real numbers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$,

$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) \ge (x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2.$$

Equality occurs if and only if there exist constants λ and μ such that $\lambda x_i = \mu y_i$ for all i. (Are two constants necessary? Why is one not enough?)

Problem 6. Let x, y, and z be real numbers satisfying $x^2 + y^2 + z^2 = 1$. Find the maximum value of $3x + 4y + 12z$.

Solution. By the Cauchy-Schwarz inequality,

$$(3x + 4y + 12z)^2 \le (x^2 + y^2 + z^2)(3^2 + 4^2 + 12^2) = 169,$$

so $|3x + 4y + 12z| \le 13$. Equality occurs if and only if $x/3 = y/4 = z/12$.

Let $k = x/3 = y/4 = z/12$. Then

$$x^2 + y^2 + z^2 = (3k)^2 + (4k)^2 + (12k)^2 = 169k^2 = 1,$$

so $k = \pm 1/13$. Taking $k = 1/13$ gives $(x, y, z) = (3/13, 4/13, 12/13)$, for which $3x + 4y + 12z = 13$, so the maximum value is 13. Taking $k = -1/13$ gives $(x, y, z) = (-3/13, -4/13, -12/13)$, for which $3x+4y+12z = -13$, so the minimum value is -13. ■

Problem 7. Given that a, b, c, d, e are real numbers such that

$$a + b + c + d + e = 8,$$
$$a^2 + b^2 + c^2 + d^2 + e^2 = 16.$$

Determine the maximum value of e. (USAMO, 1978)

Solution. From the given equations, $a + b + c + d = 8 - e$ and $a^2 + b^2 + c^2 + d^2 = 16 - e^2$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} (1^2 + 1^2 + 1^2 + 1^2)(a^2 + b^2 + c^2 + d^2) &\ge (a + b + c + d)^2 \\ \Rightarrow \quad 4(16 - e^2) &\ge (8 - e)^2 \\ \Rightarrow \quad -5e^2 + 16e &\ge 0 \\ \Rightarrow \quad e(16 - 5e) &\ge 0. \end{aligned}$$

Hence, $0 \le e \le 16/5$. Since $(a, b, c, d, e) = (6/5, 6/5, 6/5, 6/5, 16/5)$ satisfies the given system, the maximum value of e is $16/5$. ■

Problem 8. Let a, b, c, and d be positive numbers whose sum is 1. Prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \ge \frac{1}{2},$$

with equality if and only if $a = b = c = d = 1/4$. (Ireland, 1999)

Solution. By the Cauchy-Schwarz inequality,

$$[(a+b)+(b+c)+(c+d)+(d+a)]\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\right) \geq (a+b+c+d)^2,$$

so

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a} \geq \frac{(a+b+c+d)^2}{2(a+b+c+d)} = \frac{a+b+c+d}{2} = \frac{1}{2}.$$

Equality occurs if and only if

$$\frac{a^2}{(a+b)^2} = \frac{b^2}{(b+c)^2} = \frac{c^2}{(c+d)^2} = \frac{d^2}{(d+a)^2},$$

or equivalently

$$\frac{a+b}{a} = \frac{b+c}{b} = \frac{c+d}{c} = \frac{d+a}{d}.$$

Let

$$k = \frac{a+b}{a} = \frac{b+c}{b} = \frac{c+d}{c} = \frac{d+a}{d}.$$

Then $b = (k-1)a$, $c = (k-1)b$, $d = (k-1)c$, and $a = (k-1)d$. Adding, we get

$$a+b+c+d = (k-1)(a+b+c+d),$$

so $k = 2$, which implies that $a = b = c = d$. Since $a + b + c + d = 1$, equality occurs if and only if $a = b = c = d = 1/4$. ■

Exercises

13. Prove the Cauchy-Schwarz inequality using one of the following methods:

    (1) Let

    $$f(t) = \sum_{i=1}^{n} (x_i t - y_i)^2.$$

    We see that $f(t)$ is a quadratic in t. What is the discriminant of $f(t)$?

    (2) Let $\vec{v} = (x_1, x_2, \ldots, x_n)$ and $\vec{w} = (y_1, y_2, \ldots, y_n)$, and let θ be the angle between $\vec{v}$ and $\vec{w}$. Then

    $$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}.$$

    What is the range of $\cos\theta$?

(3) Prove Lagrange's identity:

$$\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right) - \left(\sum_{i=1}^n x_i y_i\right)^2 = \sum_{1\le i<j\le n} (x_i y_j - x_j y_i)^2.$$

14. Let a_1, a_2, $\dots$, a_n be real numbers, and let b_1, b_2, $\dots$, b_n be positive real numbers. Show that

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \ge \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

(This inequality has recently come to be known as the "Engel form" of the Cauchy-Schwarz inequality.)

15. Let a and b be positive real numbers with $a + b = 1$. Prove that

$$\frac{a^2}{a+1} + \frac{b^2}{b+1} \ge \frac{1}{3}.$$

(Hungary, 1996)

16. Let x_1, x_2, $\dots$, $x_n > 0$, and $s = x_1 + x_2 + \cdots + x_n$. Prove that

$$\frac{s}{s - x_1} + \frac{s}{s - x_2} + \cdots + \frac{s}{s - x_n} \ge \frac{n^2}{n-1},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

5 Miscellaneous Problems

17. Suppose that $|x_i| < 1$ for $i = 1$, 2, $\dots$, n. Suppose further that

$$|x_1| + |x_2| + \cdots + |x_n| = 19 + |x_1 + x_2 + \cdots + x_n|.$$

What is the smallest possible value of n? (1988 AIME)

18. For $a > b > 0$, find the minimum value of

$$a + \frac{1}{(a-b)b}.$$

19. Show that if a, b, c are the lengths of the sides of a triangle, then

$$3(ab + ac + bc) \le (a + b + c)^2 < 4(ab + ac + bc).$$

20. Let a_1, a_2, $\dots$, a_n be nonnegative real numbers. Let a and g be the arithmetic and geometric mean of the a_i, respectively. Prove that for all $x \ge 0$,

$$(x + g)^n \le (x + a_1)(x + a_2)\cdots(x + a_n) \le (x + a)^n.$$

21. (Nesbitt's Inequality) Show that for a, b, $c > 0$,

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Furthermore, show that if a, b, and c are the sides of a triangle, then

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} < 2.$$

22. Let a, b, c, d, e be positive real numbers such that $abcde = 1$. Prove that

$$a^4 + b^4 + c^4 + d^4 + e^4 \geq a + b + c + d + e.$$

23. Let a_1, a_2, $\dots$, a_n be fixed, positive real numbers, and let x_1, x_2, $\dots$, x_n be nonnegative real numbers such that $x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n$. Prove that the maximum value of

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

occurs when $x_i = a_i$ for all i.

24. Let a, b, c be the sides of a triangle, and T its area. Prove: $a^2 + b^2 + c^2 \geq 4\sqrt{3}T$. In what case does equality hold? (IMO, 1961)

25. Show that if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then

    (a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$, and

    (b) $(n-1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.

    (Putnam, 1975)

26. Show that if x and y are nonnegative real numbers such that

$$x + y + \sqrt{2x^2 + 2xy + 3y^2} = 4,$$

then $x^2y < 4$.