Here I'll use an inequality and will call "by cauchy".

The inequality is $\sum \frac{x_i^2}{a_i} \geq \frac{(\sum x_i)^2}{\sum a_i}$

Which is a direct result by applying cuchy after multiplying both side by $\sum a_i$

Problem 1:

Dividng both side by abc we get,

$\sum_{\text{cyc}} a \leq \sum_{\text{cyc}} \frac{a^2}{b} \Longleftrightarrow (a+b+c)^2 \leq \left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right)(b+c+a)$

The later is is true by Cauchy.

Problem 2:

The function $f(x) = \frac{x}{x+1}$ is increasing as $f'(x) = \frac{1}{(x+1)^2} \geq 0$

So, $\frac{|a|}{|a|+1} + \frac{|b|}{|b|+1} \geq \frac{|a|}{|a|+|b|+1} + \frac{|b|}{|a|+|b|+1} = f(|a|+|b|) \geq f(|a+b|) = \frac{|a+b|}{|a+b|+1}$

as, $|a| + |b| \geq |a+b|$

Problem 3:

$\sum_{\text{cyc}} \frac{a}{a+b+c} > \sum_{\text{cyc}} \frac{a}{a+b+c+d} = 1$

The function $f(x) = \frac{x}{x+k}$ is concave as, $f''(x) = -\frac{2k}{(x+k)^3}$

So, $\frac{x}{x+k} + \frac{y}{y+k} \leq \frac{2(x+y)}{x+y+2k}$

So, $\sum_{\text{cyc}} \frac{a}{a+b+d} = \left(\frac{a}{a+(b+d)} + \frac{c}{c+(b+d)}\right) + \left(\frac{b}{b+(c+a)} + \frac{d}{d+(c+a)}\right)$

$\leq \frac{2(a+c)}{a+c+2(b+d)} + \frac{2(b+d)}{b+d+2(a+c)} < \frac{2(a+c)}{a+c+b+d} + \frac{2(b+d)}{a+c+b+d} = 2$

Problem 4:

for $n=1$, it's true. We'll prove it by induction. Let it's true for n, Now let's prove for $n+1$

Now, as, $\frac{2n+1}{2n+2} \prod_{1 \leq i \leq n} \frac{2i-1}{2i} \leq \frac{1}{\sqrt{3n+1}} . \frac{2\text{n}+1}{2\text{n}+2}$

it's enough to prove $\frac{2n+1}{2(n+1)} \frac{1}{\sqrt{3n+1}} \leq \frac{1}{\sqrt{3n+4}}$

By squaring and simplifing,

$12n^3 + 28n^2 + 19n + 4 \leq 12n^3 + 28n^2 + 20n + 4$

Which is obvious.

Problem 5:

$(a+b)(a+c) = a^2 + ab + ac + bc = a(a+b+c) + bc \geq 2\sqrt{abc(a+b+c)}$

by AM-GM

Problem 6:

Let, $a=x^3, b=y^3, c=z^3$, so, $xyz=1$

Then, we need to prove, $\sum_{\text{cyc}} \frac{x^3}{y^3} \geq \sum_{\text{cyc}} x^3$

$\Longleftrightarrow \sum_{\text{cyc}} x^6 z^3 \geq \sum_{\text{cyc}} x^5 y^2 z^2 = \sum_{\text{cyc}} x^4 z^2 \cdot y^2 x$

Let, $p=x^2 z, q=y^2 x, r=z^2 y$

Then, we need to prove $p^3+q^3+r^3 \geq p^2 q + q^2 r + r^2 p$

Which is obvious using rearrangement inequality

Problem 7:

Let, both are false, that is,

$\sum \frac{a_i}{b_i}, \sum \frac{b_i}{a_i} < n \Longrightarrow \left(\sum \frac{a_i}{b_i}\right)\left(\sum \frac{b_i}{a_i}\right) < n^2$

from AM-HM inequality, we get, $\frac{\sum \frac{a_i}{b_i}}{n} \geq \frac{n}{\sum \frac{b_i}{a_i}}$

So, $\left(\sum \frac{a_i}{b_i}\right)\left(\sum \frac{b_i}{a_i}\right) \geq n^2$, contradiction!

Problem 8:

If the inequality holds for $(a,b,c)=(x,y,z)$, then it also holds for $(a,b,c)=(y,z,x)$

So, WLOG we can let either $a \geq b \geq c$ or $a \leq b \leq c$

$\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a} \implies \frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$

again $\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c} \Longrightarrow \frac{1}{a+b} \geq \frac{1}{c+a} \geq \frac{1}{b+c}$

So, in by rearrangement inequality, for both cases

$2\sum_{\text{cyc}} \frac{1}{c}\frac{1}{b+c} \geq 2\sum_{\text{cyc}} \frac{1}{a(b+c)}$

$=\frac{2}{a+b+c}\sum_{\text{cyc}} \left(\frac{1}{a}+\frac{1}{b+c}\right)$

Now, by cauchy $\sum \frac{1}{a} \geq \frac{9}{a+b+c}$ and $\sum \frac{1}{b+c} \geq \frac{9}{2(a+b+c)}$

So,$2\sum_{\text{cyc}} \frac{1}{c}\frac{1}{b+c} \geq \frac{2}{a+b+c}\left(\frac{9}{a+b+c}+\frac{1}{2}\frac{9}{a+b+c}\right) = \frac{27}{(a+b+c)^2}$

Problem 9:

Let, $(x,y)=(e^u,e^v)$

Now the function $f(x)=\frac{1}{\sqrt{1+e^{2x}}}$ is convex as $f''(x)$

Problem 10:

We'll prove it by induction. for $n=1$, it's given.Now let,

$\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \geq 2^n|x-y|$ for all x, y

Now, using this 3 times for, $(x,y) \longrightarrow \left(x, \frac{x+y}{2}\right), \left(\frac{x+y}{2}, y\right), \left(\frac{3x+y}{4}, \frac{x+3y}{4}\right)$, we get

$$\frac{f(x)+f\left(\frac{x+y}{2}\right)}{2}-f\left(\frac{3x+y}{4}\right)\geq 2^{n-1}|x-y|$$

$$\frac{f\left(\frac{x+y}{2}\right)+f(y)}{2}-f\left(\frac{x+3y}{4}\right)\geq 2^{n-1}|x-y|$$

$$f\left(\frac{3x+y}{4}\right)+f\left(\frac{x+3y}{4}\right)-2f\left(\frac{x+y}{2}\right)\geq 2^{n}|x-y|$$

Adding them, we get, $\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\geq 2^{n+1}|x-y|$, which is true for all $n\in\mathbb{N}$ by induction.

But, here, if we fix x and y, such that $x\neq y$, we see $L.H.S$ is fixed where $R.H.S$ can be arbitary large, so such a function can't exist.

Problem 11:

by Cauchy, $\sum_{\text{cyc}}\frac{a^2x^2}{(by+cz)(bz+cy)}\geq\sum_{\text{cyc}}\frac{a^2x^2}{\left(\frac{by+cz+bz+cy}{2}\right)^2}$

$=4\sum_{\text{cyc}}\left(\frac{a}{b+c}\right)^2\left(\frac{x}{y+z}\right)^2$

given, $a\geq b\geq c$, and $x\geq y\geq z$

So, $\frac{a}{b+c}\geq\frac{b}{c+a}\geq\frac{c}{a+b}$ and $\frac{x}{y+z}\geq\frac{y}{z+x}\geq\frac{z}{x+y}$

So, by techbyshev's, QM-AM and nesbit's inequality, we get

$\sum_{\text{cyc}}\frac{a^2x^2}{(by+cz)(bz+cy)}\geq 4\sum_{\text{cyc}}\left(\frac{a}{b+c}\right)^2\left(\frac{x}{y+z}\right)^2$

$\geq\frac{4}{3}\left(\sum_{\text{cyc}}\frac{a}{b+c}\frac{x}{y+z}\right)^2\geq\frac{4}{3}\left(\frac{1}{3}\left(\sum_{\text{cyc}}\frac{a}{b+c}\right)\left(\sum_{\text{cyc}}\frac{x}{y+z}\right)\right)^2\geq\frac{4}{3}\left(\frac{1}{3}\frac{3}{2}\frac{3}{2}\right)^2=\frac{3}{4}$

Problem 12:

The function $f(x)=\sqrt{x}$ is concave.

So, $\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}$

$=\sqrt{a}+2\sqrt{\frac{b}{4}}+3\sqrt{\frac{c}{9}}+4\sqrt{\frac{d}{16}}$

$=f(a)+2f\left(\frac{b}{4}\right)+3f\left(\frac{c}{9}\right)+4f\left(\frac{d}{16}\right)$

$\leq 10f\left(\frac{1}{10}a+\frac{2}{10}\frac{b}{4}+\frac{3}{10}\frac{c}{9}+\frac{4}{10}\frac{d}{16}\right)$

$=10\sqrt{\frac{12a+6b+4c+3d}{120}}$

$=10\sqrt{\frac{3(a+b+c+d)+(a+b+c)+2(a+b)+6a}{120}}$

$\leq 10\sqrt{\frac{3\times 30+14+2\times 5+6}{120}}=10$

Dhananjoy Biswas

Chittagong College

Class XII

Mobile: 01682290016

Forum Nic: Corei13

ebappa1971@gmail.com