Functional Equations

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Given some weird relationship between $f(x)$ and x for various values of x, can you completely determine f? This is what functional equations are all about, and they are very popular on the IMO these days. (6 years in a row with a functional equation!)

Functional equations can be intimidating just because they are so unlike other problems. However, once you learn a few basic tricks, you will find even the hard ones are pretty approachable.

In these notes, I will go over a few of the most basic techniques that everyone should know. Before we start, here is a quick reminder on some common notation:

- $\mathbb{Z}$ is the set of integers, and $\mathbb{N}$ is the set of natural numbers (aka the positive integers).
- $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{Q}^+$ is the set of positive rational numbers.
- $\mathbb{R}$ is the set of real numbers, and $\mathbb{R}^+$ is the set of positive real numbers.

1 Get one term to appear in two ways

When you start working on a functional equation, it is always a good idea to plug in small values like $x = 0$ and see what comes out. However, this will not usually be enough to solve the whole problem. You will need to move on to bigger values and then it becomes helpful to have a plan.

In general, I would say you should start by looking for multiple ways to get a single term to show up. Think about the functional equation as a giant system of equations, and try to find a way to make some things cancel out.

Example 1: Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x, y,

$$f(x^2 - y^2) = (x - y)(f(x) + f(y))$$

(Source: Korea 2000)

Solution. The equation is almost but not quite symmetrical. This means that if we reverse the order of x and y, we will get almost but not quite what we started with. We can then compare with what we started with to get some useful information.

Let z be an arbitrary non-negative real number. Setting $x = \sqrt{z}, y = 0$, we have

$$f(z) = \sqrt{z} \cdot \left(f(\sqrt{z}) + f(0)\right).$$

Setting $x = 0, y = \sqrt{z}$, we have

$$f(-z) = -\sqrt{z} \cdot \left(f(0) + f(\sqrt{z})\right).$$

Comparing these formulas gives us $f(z) = -f(-z)$.

The next idea is similar: can we vary one value in a way that keeps other things constant as much as possible?

Set $y = -z$. Then we have

$$f(x^2 - z^2) = (x + z)(f(x) + f(-z)) = (x + z)(f(x) - f(z)).$$

However, we also know from the the original equation that $f(x^2 - z^2) = (x - z)(f(x) + f(z))$. Therefore,

$$\begin{aligned} & (x - z)(f(x) + f(z)) = (x + z)(f(x) - f(z)) \\ \implies \quad & xf(z) = zf(x). \end{aligned}$$

Let $C = f(1)$ and set $z = 1$ to get $f(x) = Cx$. Conversely, it is easy to check that any such function satisfies the given equation. □

Example 2: Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$. *(Source: Iran 1999)*

Solution Sketch. We have three separate f terms. Wouldn't it be nice if we could just cancel two of them out? We can! Set $y = \frac{x^2 - f(x)}{2}$. Then we get $f\left(\frac{x^2+f(x)}{2}\right) = f\left(\frac{x^2+f(x)}{2}\right) + 4f(x) \cdot \frac{x^2-f(x)}{2}$, which implies $f(x) = 0$ or $f(x) = x^2$.

We still have a bunch more work to do though. For each x independently, we have shown that either $f(x) = 0$ or $f(x) = x^2$, but there are many functions with these properties (e.g. $f(x) = \max(x, 0)^2$). That part is longer but more straightforward. □

Summarizing, here are a few variations on the same theme:

- **Pseudo-symmetry:** If an equation is almost but not completely symmetrical, what happens if you change the order of the variables and compare with what you started with?
- **Fudging:** Can you change one variable so as to alter the equation only slightly? If so, compare with what you started with.
- **Self-cancelation:** Can you make two terms in the same functional equation cancel each other out?
- These are the most mechanical ways of getting the same value to show up multiple times, but each problem has its own tricks. If you see an interesting expression pop up, always ask yourself whether you can get it to pop up in a slightly different way too.

Problems

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x+y)) = x + f(y)$ for all $x, y \in \mathbb{R}$.
2. Complete the solution for Example 2.
3. (a) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $2f(x) + f(1-x) = x^2$ for all $x \in \mathbb{R}$.
   (b) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f\left(\frac{x-3}{x+1}\right) + f\left(\frac{3+x}{1-x}\right) = x$ for all $x \in \mathbb{R}$.
4. Let $f(x)$ be a real-valued function defined for all positive x, satisfying $f(x+y) = f(xy)$ for all positive x, y. Prove that f is a constant function.
5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds: $f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$.

2 Induction, squeezing, and Cauchy

On some problems, you cannot get a handle on every value of f in just one or two steps. For these problems, induction comes to the rescue. In many ways, induction is the most natural approach to functional equations. You play around with the equation a little bit until you can see a pattern, and then you try to prove it.

For problems on the integers or the rational numbers, induction should almost always be a top choice. For problems on the real numbers, it can be more tricky...

Example 3: Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 + y) = f(x)^2 + f(y)$$

for all $x, y \in \mathbb{R}$. *(Source: IMO 1992 sort of)*

Solution. Setting $x = y = 0$, we have $f(0) = f(0)^2 + f(0)$, so $f(0) = 0$.

Setting $y = 0$, we have $f(x^2) = f(x)^2$. For any real numbers a, b with $a \geq 0$, we can then set $x = \sqrt{a}, y = b$ to obtain

$$f(a+b) = f(\sqrt{a})^2 + f(b) = f(a) + f(b). \qquad (*)$$

Now if you start playing with the equation, you might find it is easy to calculate individual values like $f(1), f(2)$, etc. But how do we turn that into a general solution? The key is induction.

Key Lemma #1: For any real number $a \geq 0$ and any integer $n \geq 0$, we have $f(na) = nf(a)$.
Proof: If $n = 0$, the claim follows from the fact that $f(0) = 0$. Otherwise, suppose the result holds for $n = k$. Then, we know $f((k+1)a) = f(ka) + f(a) = kf(a) + f(a) = (k+1)f(a)$ by (*). Thus, the result also holds for $n = k+1$ and hence the entire lemma follows from mathematical induction.

Thus, if we let $C = f(1)$, we have $f(n) = Cn$ for all positive integers n. In fact, we can go one step further. Let q be a positive rational number. Then $q = \frac{n}{m}$ for some positive integers n, m, and Key Lemma #1 implies $f(q) = n \cdot f\left(\frac{1}{m}\right)$ while $f(1) = m \cdot f\left(\frac{1}{m}\right)$, so $f(q) = \frac{n}{m} \cdot f(1) = Cq$. Furthermore, $0 = f(q - q) = f(q) + f(-q)$ by (*), so we also have have $f(-q) = -Cq$.

This means $f(x) = Cx$ for all rational numbers x. This might seem like big progress, but the really clever step is still ahead. How do we calculate f on the real numbers?

Key Lemma #2: f is non-decreasing.
Proof: Let c, d be real numbers with $c \leq d$. Then, we can set $x = \sqrt{d-c}$ and $y = c$ in the original equation to obtain $f(d) = f(\sqrt{d-c}^2 + c) = f(\sqrt{d-c})^2 + f(c)$. Since $f(\sqrt{d-c})^2 \geq 0$, it follows that $f(d) \geq f(c)$.

Now, let x be an arbitrary real number, and let $\epsilon > 0$. Then there exists a rational number q such that $q \in [x - \epsilon, x]$. Assuming $C \geq 0$ (the exact same argument holds in reverse if $C < 0$), we have $f(x) \geq f(q) = Cq \geq C(x - \epsilon) = Cx - C\epsilon$. Since this is true for *all* $\epsilon > 0$, it must in fact be that $f(x) \geq Cx$. Similarly, choosing a rational number just larger than x shows that $f(x) \leq Cx$.

Thus, $f(x) = Cx$ for all real numbers x as well. It is then straightforward to plug this back into the equation and find that the two possible solutions are $f(x) = 0$ and $f(x) = x$. $\square$

The functional equation

$$f(a + b) = f(a) + f(b)$$

is super-important and is called Cauchy's equation. Our induction argument showed that there exists some C such that $f(x) = Cx$ for rational numbers x. However, there can be many other solutions over the real numbers, and they are too weird and random to try to describe without university math. For example, for arbitrary C_1 and C_2, we can find f such that $f(a + b\sqrt{2}) = aC_1 + bC_2$ holds for all rational a, b. To solve Cauchy's equation on the real numbers, you need some extra piece of information: f is continuous, $f(x) \geq 0$ for $x \geq 0$, or something similar.

There is only one example in this section, but it has a lot of different ideas, and all of them are important! Here is a short summary:

- If you want to solve a functional equation over the integers or over the rationals, it often helps to inductively calculate something like $f(nx)$ in terms of $f(x)$.

- A slight variation: if the domain or range of f is $\mathbb{N}$, definitely look at induction! In addition to asking what is $f(1)$, you can also ask when is $f(n) = 1$.

- If you want to show $f(x) \geq y$, it suffices to show $f(x) \geq y - \epsilon$ for all $\epsilon > 0$. Can you find progressively tighter ways of bounding $f(x)$ and then apply this argument? Many of the hardest functional equations use this kind of idea.

Problems

1. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(1) = 1$ and $f(m + n) = f(m) + f(n) + mn$ for all $m, n \in \mathbb{N}$. Can you still solve it if we drop the condition that $f(1) = 1$?

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ has the property that there exists a constant K such that $|f(x) - f(y)| \leq K(x-y)^2$. Prove that f is a constant function.

3. A function f from the set of real numbers to itself satisfies
$$f(x^3 + y^3) = (x + y)(f(x)^2 - f(x)f(y) + f(y)^2),$$
for arbitrary real numbers x and y. Prove that $f(2014x) = 2014f(x)$ for any real number x.

4. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $f\left(x + \frac{y}{x}\right) = f(x) + \frac{f(y)}{f(x)} + 2y$ for all $x, y \in \mathbb{Q}^+$.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 + f(y)) = y + f(x)^2$ for all $x, y \in \mathbb{R}$.

6. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that
$$f(a + b + c) + f(a) + f(b) + f(c) = f(a + b) + f(b + c) + f(c + a) + f(0)$$
for all $a, b, c \in \mathbb{Q}$.

3 Injectivity and surjectivity

Olympiad problems are tricky and clever, and sometimes more indirect methods are called for. The two most common approaches are showing a function is *injective* or *surjective*. In case you are not familiar with this terminology:

- A function f is said to be *injective* if $f(x) = f(y)$ only when $x = y$.
- A function f is said to be *surjective* if for every y, there exists x such that $f(x) = y$.
- A function f is said to be *bijective* if it is both injective and surjective.

Let's see how proving f is injective or surjective can be helpful.

Example 4: Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for all $x, y \in \mathbb{R}^+$ we have
$$(x + y)f(f(x)y) = x^2 f(f(x) + f(y)).$$
(Source: Iran 2006)

Solution. Once you get over the sheer randomness of this functional equation and start planning your line of attack, you should probably start by trying to see if you can find x, y such that $f(x)y = f(x) + f(y)$. Then you would have $f(f(x)y) = f(f(x) + f(y))$ and hence $x + y = x^2$, which would be pretty revealing. Unfortunately, the only such pair x, y that is easy to find is $x = y = 2$, and that doesn't tell us anything useful.

Well, if we can't make the part inside the f really nice, at least we can do the reverse. If we choose x, y such that $x + y = x^2$, we can then deduce $f(f(x)y) = f(f(x) + f(y))$. If f is injective, then we would be in good shape.

So, let's suppose $f(a) = f(b)$ for $a, b \in \mathbb{R}^+$. Setting $x = a$ and $x = b$, we have

$$\frac{a+y}{a^2} = \frac{f(f(a)+f(y))}{f(f(a)y)} = \frac{f(f(b)+f(y))}{f(f(b)y)} = \frac{b+y}{b^2}.$$

Multiplying this out gives $ab^2 + yb^2 = ba^2 + ya^2 \implies (b-a)(ab+ya+yb) = 0$. However, $ab + ya + yb \neq 0$ since the domain and range of f is $\mathbb{R}^+$, and hence $a = b$. Thus, f is indeed injective.

Now set $y = x^2 - x$ and the equation reduces to $x^2 f(f(x)\cdot(x^2-x)) = x^2 f(f(x)+f(x^2-x))$. Since f is injective, it follows that $f(x)\cdot(x^2-x) = f(x)+f(x^2-x) \implies f(x)\cdot(x^2-x-1) = f(x^2-x)$. And now we let $x = 1.5$. Then $x^2 - x = 0.75 > 0$ so this is a valid substitution. However, we are left with $f(x^2-x) = f(x)\cdot(-0.25) < 0$, which is impossible since f only takes on positive values.

Therefore, there are no solutions for f. $\square$

Example 5: Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(y) + x) = xy + f(x)$$

for all real numbers x and y. *(Source: Mathlinks)*

Solution. The main difficulty here is we want both $f(x)$ and $f(xf(y)+x)$ to be nice at the same time, but how do we choose x and y? If we don't know anything about $f(x)$, there is not much we can do. We can set $x = 0$, but that turns out not to be too helpful. But what if we could choose the value of $f(y)$? Then we could ensure $xf(y) + x = 0$, which would help a lot.

This means we would love to show f is surjective, and in fact, it is pretty easy to do so. Set $x = 1$ and let y vary, and you can see the right-hand side varies over all real numbers. More rigorously, let z be an arbitrary real number. Set $x = 1$ and $y = z - f(1)$. Then

$$f(f(z - f(1)) + 1) = z - f(1) + f(1) = z,$$

and so f is surjective.

In particular, there exists some a such that $f(a) = -1$. Setting $y = a$, we have $f(0) = f(xf(a)+x) = xa + f(x)$, and so $f(x) = -ax + b$ for some real number b. Plugging this back into the original equation, we see the two valid solutions are $f(x) = x$ and $f(x) = -x$. $\square$

A few parting comments on injectivity and surjectivity:

- There are many variations on injectivity and you should not be too fixated on the form used here. The main idea is this: if you can show a relationship between $f(x)$ and $f(y)$, what can conclude about x and y?
  - If f is injective and $f(x) = f(y)$, then $x = y$.
  - If f is increasing and $f(x) > f(y)$, then $x > y$.
  - Often it helps to start with a weaker version of injectivity: if $f(x) = f(y) = 0$, then $x = y$.

  - Even if f is not injective, we can often still end up with something useful. For example, if $f(x) = x^2$ is a valid solution, we will not be able to show $f(x)$ is injective, but perhaps we can show that if $f(x) = f(y)$, then $x = \pm y$. That is almost as good.
  - If you can show any kind of injectivity results, it is often useful to set $x = f(z)$ for some arbitrary z.

- Surjectivity is a little less common but it still comes in a couple flavours. The main idea is this: is there some nasty expression in your equation that you wish could be replaced by x? If so, prove that expression is surjective, and you are good to go.
  - Usually the nasty expression will be f itself.
  - It does not have to be though. For example, if you could show $f(\text{blah})$ has some nice property, then a good follow-up would be to show that blah is surjective.

Problems

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(2f(x)) = x$ for all x. Calculate $f(f(2014))$.
2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f\Big(f(f(x)) + y\Big) = f(f(x - y)) + 2y$ for all $x, y \in \mathbb{R}$.
3. Let $f : \mathbb{R} \to (0, 1)$ be a continuous function. If $g : \mathbb{R} \to \mathbb{R}$ satisfies $g(x + f(y)) = g(y + f(x))$ for all $x, y \in \mathbb{R}$, prove that $g(x)$ does not depend on x.
4. Do there exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^3$ for all $x \in \mathbb{R}$?
5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f\left(f(x) + y\right) = 2x + f\left(f(y) - x\right)$ for all $x, y \in \mathbb{R}$.

4 More Problems

The following problems are general Olympiad functional equations. Some are related to the ideas in these notes, and some are not. The last few problems are about as hard as functional equations get, so good luck!

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) - f(x - y) = 4xy$ for all $x, y \in \mathbb{R}$.
2. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x, y \in \mathbb{Q}$.
3. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy the following conditions:
   (a) $f(f(n)) = n$ for all $n \in \mathbb{Z}$;
   (b) $f(f(n + 2) + 2) = n$ for all $n \in \mathbb{Z}$;
   (c) $f(0) = 1$.
4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) \leq x$ for all x, and $f(x + y) \leq f(x) + f(y)$ for all x, y.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x+y) + f(y+z) + f(z+x) \geq 3f(x+2y+3z)$ for all x, y, z.

6. Let S be the set of non-negative integers.

   (a) Find surjective functions f, g from S to itself such that $f(n)g(n) = n$ for all $n \in S$.

   (b) Let $h : S \to S$ be a bijective function. Prove that there do not exist functions f, g from S to itself, f injective and g surjective, such that $f(n)g(n) = h(n)$ for all $n \in S$.

7. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(f(a) + f(b) + f(c)) = a + b + c$ for all $a, b, c \in \mathbb{N}$.

8. Find all functions $u : \mathbb{R} \to \mathbb{R}$ for which there exists a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x+y) = f(x)u(y) + f(y) \text{ for all } x, y \in \mathbb{R}.$$
(Note: A function is said to be *strictly monotonic* if it is either strictly increasing or strictly decreasing.)

9. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that
$$f(m - n + f(n)) = f(m) + f(n)$$
for all $m, n \in \mathbb{Z}$.

10. Let S be the set of positive real numbers. Find all functions $f : S^3 \to S$ such that, for all positive real numbers x, y, z and k, the following three conditions are satisfied:

    (a) $xf(x, y, z) = zf(z, y, x)$,

    (b) $f(x, ky, k^2z) = kf(x, y, z)$,

    (c) $f(1, k, k+1) = k + 1$.

11. Find all functions $\mathbb{R} \to \mathbb{R}$ such that
$$f(xf(y) + f(x)) = 2f(x) + xy$$
for all $x, y \in \mathbb{R}$.

12. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(f(x) + f(y) + f(z)) = f(f(x) - f(y)) + f(2xy + f(z)) + 2f(xz - yz)$$
for all $x, y, z \in \mathbb{R}$.

13. Determine the least possible value of $f(1998)$, where $f : \mathbb{N} \to \mathbb{N}$ satisfies
$$f\left(n^2 f(m)\right) = m\left[f(n)\right]^2$$
for all $m, n \in \mathbb{N}$.

14. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the property that for all $n \in \mathbb{N}$,

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \ldots + \frac{1}{f(n)f(n+1)} = \frac{f(f(n))}{f(n+1)}.$$

15. Let $\mathbb{R}^*$ be the set of non-zero real numbers. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ such that

$$f(x^2 + y) = f(x)^2 + \frac{f(xy)}{f(x)}$$

for all $x, y \in \mathbb{R}^*$ and $y \neq -x^2$.

16. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + yf(x)) = f(x) + xf(y)$$

for all x and y.