Math Olympiad Problem Solving
Stanford University EPGY Summer Institutes 2008
Problem Set: The Pigeonhole Principle

1. (1986 AHSME #17) A drawer in a darkened room contains 100 red socks, 80 green socks, 60 blue socks, and 40 black socks. Socks are randomly selected one at a time from the drawer, without replacement. What is the smallest number of socks that must be selected to ensure that the selection contains at least 10 pairs?

   (A) 21 (B) 23 (C) 24 (D) 30 (E) 50

2. (1994 AHSME #19) Label one disk "1", two disks "2", three disks "3",..., and fifty disks "50". Put these $1 + 2 + 3 + \cdots + 50 = 1275$ labeled disks in a box. Disks are then drawn from the box at random without replacement. What is the minimum number of disks that must be drawn to ensure drawing at least ten disks with the same label?

   (A) 10 (B) 51 (C) 415 (D) 451 (E) 501

3. (1997 AHSME #16) The three row sums and the column sums of the array

$$\begin{bmatrix} 4 & 9 & 2 \\ 8 & 1 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

   are the same. What is the least number of entries that must be altered to make all six sums different from one another?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

4. (1991 AHSME) A circular table has exactly sixty chairs around it. There are N people seated at this table in such a way that the next person to be seated must sit next to someone. What is the smallest possible value of N?

   (A) 15 (B) 20 (C) 30 (D) 40 (E) 50

5. Show that if any five points are all in, or on, a square of side 1, then some pair of them will be at most at distance $\sqrt{2}/2$.

6. Show that in any sum of nonnegative real numbers there is always one number which is at least the average of the numbers and that there is always one member that it is at most the average of the numbers.

7. (1947 Hungarian Math Olympiad) Prove that amongst six people in a room there are at least three who know one another, or at least three who do not know one another.

8. We call a set "sum free" if no two elements of the set add up to a third element of the set. What is the maximum size of a sum free subset of $\{1, 2, \ldots, 2n-1\}$. Hint: Observe that the set $\{n+1, n+2, \ldots, 2n-1\}$ of $n+1$ elements is sum free. Show that any subset with $n+2$ elements is not sum free.

9. Suppose that the letters of the English alphabet are listed in an arbitrary order.

   (a) Prove that there must be four consecutive consonants.

   (b) Give a list to show that there need not be five consecutive consonants.

   (c) Suppose that all the letters are arranged in a circle. Prove that there must be five consecutive consonants.

10. Bob has ten pockets and forty four silver dollars. He wants to put his dollars into his pockets so distributed that each pocket contains a different number of dollars.

    (a) Can he do so?

    (b) Generalize the problem, considering p pockets and n dollars. The problem is most interesting when

    $$n = \frac{(p-1)(p-2)}{2}.$$

    Why?

11. Let M be a seventeen-digit positive integer and let N be the number obtained from M by writing the same digits in reversed order. Prove that at least one digit in the decimal representation of the number $M + N$ is even.

12. No matter which fifty five integers may be selected from

    $$\{1, 2, \ldots, 100\},$$

    prove that you must select some two that differ by 9, some two that differ by 10, some two that differ by 12, and some two that differ by 13, but that you need not have any two that differ by 11.

13. If the points of the plane are colored with three colors, show that there will always exist two points of the same color which are one unit apart.

14. Inside a 1×1 square, 101 points are placed. Show that some three of them form a triangle with area no more than 0.01.

15. Choose any $(n+1)$-element subset from $\{1, 2, \ldots, 2n\}$. Show that this subset must contain two integers that are relatively prime.

16. Show that if the points of the plane are colored with two colors, there will always exist an equilateral triangle with all its vertices of the same color. There is, however, a coloring of the points of the plane with two colors for which no equilateral triangle of side 1 has all its vertices of the same color.

17. Show that if the points of the plane are colored with two colors, there will always exist a rectangle with all its vertices of the same color.

18. Consider a sequence of N positive integers containing n distinct integers. If $N \geq 2^n$, show that there is a consecutive block of integers whose product is a perfect square. Is this inequality the best possible?

19. (1979 USAMO) Nine mathematicians meet at an international conference and discover that amongst any three of them, at least two speak a common language. If each of the mathematicians can speak at most three languages, prove that there are at least three of the mathematicians who can speak the same language.

20. (1982 USAMO) In a party with 1982 persons, amongst any group of four there is at least one person who knows each of the other three. What is the minimum number of people in the party who know everyone else?

21. (1986 USAMO) During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that, at some moment, some three were sleeping simultaneously.

22. Given any 8 integers between 1 and 100 (inclusive), prove that it is possible to choose two whose ratio lies between 1 and 2.

23. Seven points are placed inside a regular hexagon with side length 1. Show that at least two points are at most 1 unit apart.

24. Show that for any positive integer n, there exists a positive multiple of n that contains only the digits 7 and 0.

25. Forty-one rooks are placed on a 10×10 chessboard. Prove that there must exist 5 rooks, none of which attack each other. (Recall that rooks attach each piece located on its row or column).