New Zealand Mathematical Olympiad Committee

Equations in Integers

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1 Introduction

In this article we will consider methods that are often used for solving equations in integers. Of course, they are not completely independent and many problems can be solved by using several methods.

2 Uniqueness of prime factorization

This is a powerful method which we illustrate by the following example:

Example 1 (New-York, 1977)**.** Solve the Diophantine equation $2^x + 1 = y^2$.

Solution: We write $2^x = y^2 - 1 = (y-1)(y+1)$. Since the prime factorization is unique, we get $y - 1 = 2^k$ and $y + 1 = 2^m$ with $m > k > 0$. But the $2^m - 2^k = (y+1) - (y-1) = 2$, whence $2^k(2^{m-k} - 1) = 2$ and $k = 1$ and $m = 2$. Finally we get $x = y = 3$. □

Example 2. (Moscow, 1945) Find all integer solutions of the equation

$$xy + 3x - 5y = -3.$$

Solution: Let us express y as a function of x:

$$y = \frac{-3-3x}{x-5} = \frac{15-3x-18}{x-5} = -3 - \frac{18}{x-5}.$$

Thus $x - 5$ divides 18, hence $x - 5$ can take any of the following values: $\pm1, \pm2, \pm3, \pm6, \pm9, \pm18$. Hence we will have 12 solutions shown in the following table:

$x-5$	-1	1	-2	2	-3	3	-6	6	-9	9	-18	18
x	4	6	3	7	2	8	-1	11	-4	14	-13	23
y	15	-21	6	-12	3	-9	0	-6	-1	-5	-2	-4

□

3 Proving that there are no large solutions

Example 3 (Moscow, 1944)**.** Find all integer solutions of

$$x + y = x^2 - xy + y^2.$$

Solution: We write this equation as

$$\frac{1}{2}x^2 - xy + \frac{1}{2}y^2 + \frac{1}{2}x^2 - x + \frac{1}{2} + \frac{1}{2}y^2 - y\frac{1}{2} = 1.$$

or

$$((x-y)^2+(x-1)^2+(y-1)^2=2.$$

Now it is clear that $|x-y|\le 1$, $(x-1)\le 1$, $(y-1)\le 1$, and we find all solutions by inspection: $(0,0)$, $(1,0)$, $(0,1)$, $(2,1)$, $(1,2)$, $(2,2)$. ☐

This method often involves using inequalities.

Example 4. Find all pairs positive integers (x,y) which satisfy the equation

$$x^3-y^3=xy+61.$$

Solution: Let (x,y) be a solution. By AM-GM inequality $x^2+y^2\ge 2xy$ and hence

$$x^3-y^3=(x-y)(x^2+xy+y^2)\ge 3(x-y)xy.$$

Hence $3(x-y)xy\ge xy+61$ or $(3(x-y)-1)xy\le 61$. It is clear that $x\ne y$. Let us consider the case, when $x-y=1$. Then substituting $x=y+1$ into the original equation we get $2y^2+2y+1=61$ from which we deduce that $y=5$, $x=6$. If $x-y=2$, we obtain the equation $5y^2+10y+8=61$, which has no integer solutions. Finally, let us suggest that $x-y\ge 3$. Then $3(x-y)-1\ge 8$, hence $xy\le 7$. Since $x-y\ge 3$, the only possibility is $x=7$ and $y=1$ which do not satisfy the equation. Hence $(x,y)=(6,5)$ is the only solution. ☐

Solution: Alternatively, we can prove that the difference $d=x-y$ is small by substituting $x=y+d$ into the equation. We obtain

$$(3d-1)y^2+(3d^2-d)y+d^3=61$$

from which we deduce that $d^3\le 61$ or $d\le 3$. We then need to consider three cases: $d=1$, $d=2$, and $d=3$. We leave this to the reader. ☐

4 Squeeze principle

Squeezing an unknown between two consequtive squares is a method which is surprisingly often used. In its simplest form it works as follows:

Example 5. Prove that the equation

$$m^2=n^4+n^2+1$$

does not have integer solutions.

Solution: We note that

$$(n^2)^2=n^4<m^2<n^4+2n^2+1=(n^2+1)^2$$

which is impossible as the square m^2 is squezed between two consequtive squares $(n^2)^2$ and $(n^2+1)^2$. ☐

This argument can be more elaborate as in the following problem.

Example 6 (GDR, 1973)**.** Find all integer solutions to the equation

$$x(x+1)(x+7)(x+8)=y^2.$$

Solution: Let (x,y) be any solution. Then

$$y^2=x(x+8)(x+1)(x+7)=(x^2+8x)(x^2+8x+7)=z^2+7z,$$

where $z=x^2+8x$. If $z>9$, then

$$(z+3)^2=z^2+6z+9<z^2+7z=y^2<z^2+8z+16=(z+4)^2,$$

which is impossible. Hence $x^2+8x=z\le 9$ which implies $-9\le x\le 1$. Trying these numbers one by one, we find all solutions: $(-9,\pm 12)$, $(-8,0)$, $(-7,0)$, $(-4,\pm 12)$, $(-1,0)$, $(0,0)$, $(1,\pm 12)$. ☐

5 Infinite descent

This is a very well-known technique, used for instance in proving that $\sqrt{2}$ is irrational. We reformulate irrationality of $\sqrt{2}$ in terms of the existence of solutions of a Diophantine equation.

Example 7. Prove that the equation $x^2 - 2y^2 = 0$ has no integer solutions apart from $(0, 0)$.

Solution: Suppose that an integer solution (x, y) exits. Then x must be even, hence $x = 2x_1$ and $(2x_1)^2 - 2y^2 = 0$ which implies $2x_1^2 - y^2 = 0$. Now we see that y must be also even, hence $y = 2y_1$. Substituting this expression into the equation we get $2x_1^2 - (2y_1)^2 = 0$ or $x_1^2 - 2y_1^2 = 0$. Hence we obtained another solution (x_1, y_1) of the original equation with $|x_1| < |x|$. What is wrong with that? Well, we can now repeat everything for the new solution and get a solution (x_2, y_2) with $|x_2| < |x_1|$. And we can repeat it again. Therefore we obtain an infinite sequence of solutions $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n), \ldots$ with $|x_1| > |x_2| > \ldots > |x_n| > \ldots$. This is impossible since all the $|x_i|$ are natural numbers. $\square$

6 Quadratic residues and residues of higher order

Let p be an odd prime. The nonzero squares modulo p are called the quadratic residues $\pmod{p}$. If $n \geq p$, then $n = kp + r$, where $0 \leq r < p$ and $n^2 \equiv r^2 \pmod{p}$, hence it is enough to consider the squares of the numbers from 1 to $p - 1$. Moreover since $(p - a)^2 \equiv (-a)^2 = a^2 \pmod{p}$, it is enough to consider the squares

$$1^2,\ 2^2,\ \ldots,\ \left(\frac{p-1}{2}\right)^2 \pmod{p}$$

These are distinct since if $i^2 \equiv j^2 \pmod{p}$ for $i > j$ and $1 \leq i, j \leq \frac{1}{2}(p - 1)$, then $i^2 - j^2 = (i - j)(i + j)$ is divisible by p, which is only possible, when $i = j$, which is a contradiction. Therefore, among the numbers $0, 1, \ldots, p - 1$ there are $\frac{1}{2}(p - 1)$ quadratic residues $\pmod{p}$. The $\frac{1}{2}(p - 1)$ remaining nonzero numbers are called quadratic nonresidues. Zero is neither a residue nor a nonresidue.

Example 8. If $p = 11$, the quadratic residues $\pmod{11}$ are 1, 3, 4, 5, 9 since

$$1^2 = 1, \quad 2^2 = 4, \quad 3^2 = 9, \quad 4^2 = 16 \equiv 5, \quad 5^2 = 25 \equiv 3.$$

We can now apply this to some Diophantine equations. If $p = 3$, then 1 is a quadratic residue and 2 is a nonresidue. This means that the equation $x^2 = 2003$ cannot have a solution because $2003 \equiv 2 \pmod{3}$ and 2 is a quadratic nonresidue. The next example is the equation $x^2 + y^2 = z^2$. We can easily prove that it does not have any solutions with odd x and y. In such case $x^2 \equiv 1 \pmod{3}$ and $y^2 \equiv 1 \pmod{3}$ leading to the conclusion that $z^2 \equiv 2 \pmod{3}$, which we know is impossible.

Example 9. Prove that there are no positive integers a, b such that

$$a^2 - 3b^2 = 8.$$

Solution: Let us consider this equation modulo 3. Then we get $a^2 \equiv 2 \pmod{3}$ which we know means no solution. $\square$

Sometimes it is useful to consider residues for composite numbers as well.

Example 10 (USA, 79)**.** Find all integer solutions of the equation

$$x_1^4 + x_2^4 + \cdots + x_{14}^4 = 1599. \tag{1}$$

Solution: We claim that x^4 can have only two residues $\pmod{16}$, namely 0 and 1. Indeed, if x is even, then $x^4 \equiv 0 \pmod{16}$. Suppose $x = 2k + 1$, where k is an integer. Then

$$x^4 = (2k + 1)^4 = (4k^2 + 4k + 1)^2 = (8s + 1)^2 = 64s^2 + 16s + 1 \equiv 1 \pmod{16}$$

Here we denoted $k^2 + k = 2s$, because $k^2 + k = k(k + 1)$ is even.

Since $1599 \equiv 15 \pmod{16}$, the equation (1) does not have solutions at all. $\square$

7 Problems

1. Prove that there does not exist pairwise distinct positive integers x, y, z, t such that
$$x^x + y^y = z^z + t^t.$$
2. x, y and z are positive integers such that $x^2 + y^2 = z^2$. Prove that xy is divisible by 12.
3. (Croatia, 97) Let x, y, z, a, b, c be integers such that
$$\begin{aligned} x^2 + y^2 &= a^2 \\ x^2 + z^2 &= b^2 \\ y^2 + z^2 &= c^2. \end{aligned}$$
Prove that the number xyz is divisible by 55.
4. (Moscow, 1963) Find all integer solutions to
$$\frac{xy}{z} + \frac{xz}{y} + \frac{yz}{x} = 3.$$
5. Prove that the equations
   (a) $6x^3 + 3 = y^6$,
   (b) $x^3 + y^3 + 4 = z^3$.

   does not have solutions for which x, y, z are positive integers.
6. (Romania, 81) Find all integer solutions of the equation
$$x^6 + x^3 + 1 = y^4.$$
7. (Moscow, 1963) Prove that for an odd n the equation
$$x^n + y^n = z^n$$
does not have integer solutions (x, y) such that $x + y$ is a prime.
8. (Hungary, 83) Prove that the equation
$$x^3 + 3y^3 + 9z^3 - 9xyz = 0$$
has the only rational solution $x = y = z = 0$.
9. (USA, 76) Find all integer solutions of the equation
$$x^2 + y^2 + z^2 = x^2y^2.$$
10. (GDR, 81) Prove that the equation
$$x_1^2 + x_2^2 + \cdots + x_n^2 = y^2$$
has a natural solution for every $n \in N$.
11. (Moscow, 1949) prove that the only integer solution of the equation
$$x^2 + y^2 + z^2 = 2xyz$$
is $x = y = z = 0$.
12. (Bulgaria, 65) Prove that there exists only one triple of positive integers greater than 1 with the property that the product of any two of these integers plus 1 is divisible by the third integer.

January 24, 2009
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