2006 MOSP Homework

Congratulations on your excellent performance on the AMC, AIME, and USAMO tests, which has earned you an invitation to attend the Math Olympiad Summer Program! This program will be an intense and challenging opportunity for you to learn a tremendous amount of mathematics.

To better prepare yourself for MOSP, you need to work on the following homework problems, which come from last year's National Olympiads, from countries all around the world. Even if some may seem difficult, you should dedicate a significant amount of effort to think about them—don't give up right away. All of you are highly talented, but you may have a disappointing start if you do not put in enough energy here. At the beginning of the program, written solutions will be submitted for review by MOSP graders, and you will present your solutions and ideas during the first few study sessions.

You are encouraged to use the email list to discuss these and other interesting math problems. Also, if you have any questions about these homework problems, please feel free to contact us.

Zuming Feng, zfeng@exeter.edu

Alex Saltman, saltman@stanford.edu

1
Problems for the Red Group

Algebra

1.1. Let a, b, and c be positive real numbers. Prove that

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

1.2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+f(y)) = x + f(f(y))$$

for all real numbers x and y, with the additional constraint that $f(2004) = 2005$.

1.3. Prove that for every irrational real number a, there are irrational real numbers b and b' such that $a+b$ and ab' are both rational while ab and $a+b'$ are both irrational.

1.4. Let n be a positive integer. Solve the system of equations

$$\begin{cases} x_1 + x_2^2 + \cdots + x_n^n = n \\ x_1 + 2x_2 + \cdots + nx_n = \frac{n(n+1)}{2} \end{cases}$$

for n-tuples $(x_1, x_2, \ldots, x_n)$ of nonnegative real numbers.

1.5. Real numbers x and y satisfy the system of equalities

$$\begin{cases} \sin x + \cos y = 1 \\ \cos x + \sin y = -1. \end{cases}$$

Prove that $\cos 2x = \cos 2y$.

Combinatorics

1.6. Determine if there is a way to tile a 5×6 unit square board by dominos such that one can not use a needle to peer through the tiling? Determine if there is a way to tile a 5×6 unit square board by dominos such that one can use a needle to through the tiling? What if it is a 6×6 board?

1.7. Mykolka the numismatist possesses 241 coins, each worth an integer number of turgiks. The total value of the coins is 360 turgiks. Is it necessarily true that the coins can be divided into three groups of equal total value?

1.8. In a volleyball tournament for the Euro-African cup, there were nine more teams from Europe than from Africa. Each pair of teams played exactly once and the Europeans teams won precisely nine times as many matches as the African teams, overall. What is the maximum number of matches that a single African team might have won?

1.9. The squares of an $n \times n$ chessboard ($n \geq 2$) are filled with 1s and -1s. A series of steps are performed. For each step, the number in each square is replaced with the product of the numbers that were in the squares adjacent. Find all values of n for which, starting from an arbitrary collection of numbers, after finitely many steps one obtains a board filled with 1s.

1.10. Find all pairs of positive integers (m, n) for which it is possible to paint each unit square of an $m \times n$ chessboard either black or white in such way that, for any unit square of the board, the number of unit squares which are painted the same color as that square and which have at least one common vertex with it (including the square itself) is even.

Geometry

1.11. ABC is an acute triangle. The points B' and C' are the reflections of B and C across the lines AC and AB respectively. Suppose that the circumcircles of triangles ABB' and ACC' meet at A and P. Prove that the line PA passes through the circumcenter of triangle ABC.

1.12. Let ABC be a right triangle with $\angle A = 90°$. Point D lies on side BC such that $\angle BAD = \angle CAD$. Point I_a is the excenter of the triangle opposite A. Prove that

$$\frac{AD}{DI_a} \leq \sqrt{2} - 1.$$

1.13. In triangle ABC, $\angle BAC = 120°$. Let the angle bisectors of angles A, B and C meet the opposite sides at D, E and F respectively. Prove that the circle on diameter EF passes through D.

1.14. Let P and Q be interior points of triangle ABC such that $\angle ACP = \angle BCQ$ and $\angle CAP = \angle BAQ$. Denote by D, E and F the feet of the perpendiculars from P to the lines BC, CA and AB, respectively. Prove that if $\angle DEF = 90°$, then Q is the orthocenter of triangle BDF.

1.15. Show that among the vertices of any area 1 convex polygon with $n \geq 4$ sides there exist four such that the quadrilateral formed by these four has area at least $1/2$.

Number Theory

1.16. Show that for all integers $a_1, a_2, ..., a_n$ where $n \geq 2$, the product

$$\prod_{1 \leq i \leq j \leq n} (a_j - a_i)$$

is divisible by the product

$$\prod_{1 \leq i \leq j \leq n} (j - i).$$

1.17. Determine all pairs of positive integers (m, n) such that m is but divisible by every integer from 1 to n (inclusive), but not divisible by $n + 1$, $n + 2$, and $n + 3$.

1.18. Let $X = \{A_1, A_2, \ldots, A_n\}$ be a set of distinct 3-element subsets of the set $\{1, 2, \ldots, 36\}$ such that

(a) A_i and A_j have non-empty intersections for every i and j;

(b) The intersection of all elements of X is the empty set.

Show that $n \leq 100$. Determine the number of such sets X when $n = 100$.

1.19. Find all pairs (a, b) of positive real numbers such that $\lfloor a \lfloor bn \rfloor \rfloor = n - 1$ for all positive integers n. (For real number x, $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.)

1.20. Let n be a nonnegative integer, and let p be a prime number that is congruent to 7 modulo 8. Prove that

$$\sum_{k=1}^{p-1} \left\{ \frac{k^{2^n}}{p} - \frac{1}{2} \right\} = \frac{p-1}{2}.$$

2
Problems for the Blue Group

Algebra

2.1. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(m)+f(n)$ divides $m+n$ for all positive integers m and n.

2.2. Prove that

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4},$$

where a, b and c are positive real numbers satisfying $abc = 1$.

2.3. Let $a_1, a_2, \ldots, a_n$ be positive real numbers with $a_1 \leq a_2 \leq \cdots \leq a_n$ such that the arithmetic mean of $a_1^2, a_2^2, \ldots, a_n^2$ is 1. If the arithmetic mean of $a_1, a_2, \ldots, a_n$ is m. Prove that if $a_i \leq m$ for some $1 \leq i \leq n$, then $n - i \geq n(m - a_i)^2$.

2.4. Assume that $f : [0, \infty) \to \mathbb{R}$ is a function such that $f(x) - x^3$ and $f(x) - 3x$ are both increasing functions. Determine if $f(x) - x^2 - x$ is also an increasing function.

2.5. Let $a_1, a_2, \ldots, a_{2005}, b_1, b_2, \ldots, b_{2005}$ be real numbers such that

$$(a_i x - b_i)^2 \geq \sum_{j \neq i, j=1}^{2005} (a_j x - b_j)$$

for all real numbers x and every integer i with $1 \leq i \leq 2005$. What is maximal number of positive a_i's and b_i's?

2.6. Let n be an integer greater than 3. Prove that all the roots of the polynomial

$$P(x) = x^n - 5x^{n-1} + 12x^{n-2} - 15x^{n-3} + a_{n-4}x^{n-4} + \cdots + a_0$$

cannot be both real and positive.

2.7. Let a, b, and c be real numbers in the interval $(0, 1]$. Prove that

$$\frac{a}{bc+1} + \frac{b}{ca+1} + \frac{c}{ab+1} \leq 2.$$

Combinatorics

2.8. In how many ways can the set $N = \{1, 2, \ldots, n\}$ be partitioned in the form $p(N) = A_1 \cup A_2 \cup \cdots \cup A_k$, where A_i consists of elements forming arithmetic progressions, all with the same common positive difference d_p and of length at least one? at least two?

2.9. Determine the number of subset S of the set $T = \{1, 2, \ldots, 2005\}$ such that the sum of elements in s is congruent to 2006 modulo 2048.

2.10. Let P_n denote the number of paths in the coordinate plane traveling from $(0, 0)$ to $(n, 0)$ with three kinds of moves: *upstep* $u = [1, 1,]$, *downstep* $d = [1, -1]$, and *flatstep* $f = [1, 0]$ with the path always staying above the line $y = 0$. Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the n^{th} Catalan number. Prove that

$$P_n = \sum_{i=0}^{\infty} \binom{n}{2i} C_i \quad \text{and} \quad C_n = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} p_{2n-i}.$$

2.11. For positive integers t, a, and b, Lucy and Windy play the (t, a, b)-*game* defined by the following rules. Initially, the number t is written on a blackboard. On her turn, a player erases the number on the board and writes either the number $t - a$ or $t - b$ on the board. Lucy goes first and then the players alternate. The player who first reaches a negative losses the game. Prove that there exist infinitely many values of t in which Lucy has a winning strategy for all pairs (a, b) with $a + b = 2005$.

2.12. Smallville is populated by unmarried men and women, some of which are acquainted. The two City Matchmakers know who is acquainted with whom. One day, one of the matchmakers claimed: "I can arrange it so that every red haired man will marry a woman with who he is acquainted." The other matchmaker claimed: "I can arrange it so that every blonde woman will marry a man with whom she is acquainted." An amateur mathematician overheard this conversation and said: "Then it can be arranged so that every red haired man will marry a woman with whom he is acquainted and at the same time very blonde woman will marry a man with who she is acquainted." Is the mathematician right?

2.13. The transportation ministry has just decided to pay 80 companies to repair 2400 roads. These roads connects 100 cities. Each road is between two cities and there is at most one road between any two cities. Each company must repair exactly 30 roads, and each road is repaired by exactly one company. For a company to repair a road, it is necessary for the company to set up stations at the both cities on its endpoints. Prove that there are at least 8 companies stationed at one city.

2.14. Two concentric circles are divided by n radii into $2n$ parts. Two parts are called neighbors (of each other) if they share either a common side or a common arc. Initially, there are $4n+1$ frogs inside the parts. At each second, if there are three or more frogs inside one part, then three of the frogs in the part will jump to its neighbors, with one to each neighbor. Prove that in a finite amount of time, for any part either there are frogs in the part or there are frogs in each of its neighbors.

Geometry

2.15. Triangle ABC is inscribed in circle ω. Line ℓ_1 bisects $\angle BAC$ and meets segment BC and ω in D and M, respectively. Let γ denote the circle centered at M with radius BM. Line ℓ_2 passes through D and meets circle γ at X and Y. Prove that line ℓ_1 also bisects $\angle XAY$.

2.16. Points P and Q lies inside triangle ABC such that $\angle ACP = \angle BCQ$ and $\angle CAP = \angle BAQ$. Denote by $D, E,$ and F the feet of perpendiculars from P to lines $BC, CA,$ and AB, respectively. Prove that if $\angle DEF = 90^\circ$, then Q is the orthocenter of triangle BDF.

2.17. Let ABC be a triangle with $AB \neq AC$, and let $A_1B_1C_1$ be the image of triangle ABC through a rotation $\mathbf{R}$ centered at C. Let $M, E,$ and F be the midpoints of the segments BA_1, AC, and BC_1, respectively. Given that $EM = FM$, compute $\angle EMF$.

2.18. Let $ABCD$ be a tetrahedron, and let $H_a, H_b, H_c,$ and H_d be the orthocenters of triangles $BCD, CDA, DAB,$ and ABC, respectively. Prove that lines $AH_a, BH_b, CH_c,$ and DH_d are concurrent if and only if

$$AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2.$$

2.19. Let $ABCD$ be a convex quadrilateral. Lines AB and CD meet at P, and lines AD and BC meet at Q. Let O be a point in the interior of $ABCD$ such that $\angle BOP = \angle DOQ$. Prove that $\angle AOB + \angle COD = 180^\circ$.

2.20. In triangle ABC, $AB \neq AC$. Circle ω passes through A and meets sides AB and AC at M and N, respectively, and the side BC at P and Q such that Q lies in between B and P. Suppose that $MP \parallel AC$, $NQ \parallel AB$, and $BP \cdot AC = CQ \cdot AB$. Find $\angle BAC$.

2.21. In acute triangle ABC, $CA \neq BC$. Let I denote the incenter of triangle ABC. Points A_1 and B_1 lie on rays CB and CA, respectively, such that $2CA_1 = 2CB_1 = AB + BC + CA$. Line CI intersects the circumcircle of triangle ABC again at P (other than C). Point Q lies on line AB such that $PQ \perp CP$. Prove that $QI \perp A_1B_1$.

Number Theory

2.22. Let S be a set of rational numbers with the following properties:

(a) $\frac{1}{2}$ is an element in S;

(b) if x is in S, then both $\frac{1}{x+1}$ and $\frac{x}{x+1}$ are in S.

Prove that S contains all rational numbers in the interval $(0, 1)$.

2.23. Determine all unordered triples (x, y, z) of positive integers for which the number

$$\sqrt{\frac{2005}{x+y}} + \sqrt{\frac{2005}{y+z}} + \sqrt{\frac{2005}{z+x}}$$

is an integer.

2.24. For positive integer k, let $p(k)$ denote the greatest odd divisor of k. Prove that for every positive integer n,

$$\frac{2n}{3} < \frac{p(1)}{1} + \frac{p(2)}{2} + \cdots + \frac{p(n)}{n} < \frac{2(n+1)}{3}.$$

2.25. Let p be an odd prime, and let $S = \{n_1, n_2, \ldots, n_k\}$ be an arbitrary set of perfect squares relatively prime to p. Compute the smallest k such that there necessarily exists a subset of S with the product of its elements one more than a multiple of p.

2.26. Set X has 56 elements. Determine the least positive integer n such that for any 15 subsets of X, if the union of any 7 of the subsets has at least n elements, then 3 of the subsets have nonempty intersection.

2.27. Let m and n be positive integers with $m > n \geq 2$. Set $S = \{1, 2, \ldots, m\}$, and set $T = \{a_1, a_2, \ldots, a_n\}$ is a subset of S such that every element of S is not divisible by any pair of distinct elements of T. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \frac{m+n}{m}.$$

2.28. Let n be a given integer greater than two, and let $S = \{1, 2, \ldots, n\}$. Suppose the function $f : S^k \to S$ has the property that $f(a) \neq f(b)$ for every pair a and b of elements in S^k with a and b differ in all components. Prove that f is a function of one of its elements.

3
Problems for the Black Group

Algebra

3.1. Determine all positive real numbers a such that there exists a positive integer n and partition $A_1, A_2, \ldots, A_n$ of infinity sets of the set of the integers satisfying the following condition: for every set A_i, the positive difference of any pair of elements in A_i is at least a^i.

3.2. Let $a, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n$ be real numbers such that

$$\begin{aligned} & x^{2n} + ax^{2n-1} + ax_{2n-2} + \cdots + ax + 1 \\ = & (x^2 + b_1x + c_1)(x^2 + b_2x + c_2)\cdots(x^2 + b_nx + c_n) \end{aligned}$$

for all real numbers x. Prove that $c_1 = c_2 = \cdots = 1$.

3.3. Find the number of all infinite sequence $a_1, a_2, \ldots$ of positive integers such that

$$a_n + a_{n+1} = 2a_{n+2}a_{n+3} + 2005$$

for all positive integers n.

3.4. Determine if there exists a strictly increasing sequence of positive integers $a_1, a_2, \ldots$ such that $a_n \le n^3$ for every positive integer n and that every positive integer can be written uniquely as the difference of two terms in the sequence.

3.5. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequence of real numbers such that $a_{n+1} = 2b_n - a_n$ and $b_{n+1} = 2a_n - b_n$ for every positive integer n. Prove that if $a_n > 0$ for all n, then $a_1 = b_1$.

3.6. Let $\mathbb{R}^*$ denote the set of nonzero real numbers. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ such that

$$f(x^2 + y) = f(f(x)) + \frac{f(xy)}{f(x)}$$

for every pair of nonzero real numbers x and y with $x^2 + y \ne 0$.

3.7. Let S denote the set of rational numbers in the interval $(0, 1)$. Determine, with proof, if there exists a subset T of S such that every element in S can be uniquely written a the sum of finitely many distinct elements in T.

Combinatorics

3.8. There are 2005 players in a chess tournament played a game. Every pair of players played a game against each other. At the end of the tournament, it turned out that if two players A and B drew the game between them, then every other player either lost to A or to B. Suppose that there are at least two draws in the tournament. Prove that all players can be lined up in a single file from left to right in the such a way that every play won the game against the person immediately to his right.

3.9. Let m be a positive integer, and let $S = \{a_1 = 1, a_2, \dots, a_m\}$ be a set of positive integers. Prove that there exists a positive integer n with $n \le m$ and a set $T = \{b_1, b_2, \dots, b_n\}$ of positive integers such that

(a) all the subsets of T have distinct sums of elements;

(b) each of the numbers $a_1, a_2, \dots, a_m$ is the sum of the elements of a subset of T.

3.10. There are b boys and g girls, with $g \ge 2b - 1$, at presence at a party. Each boy invites a girl for the first dance. Prove that this can be done in such a way that either a boy is dancing with a girl he knows or all the girls he knows are not dancing.

3.11. A *k-coloring* of a graph G is a coloring of its vertices using k possible colors such that the end points of any edge have different colors. We say a graph G is *uniquely k-colorable* if one hand it has a k-coloring, on the other hand there do not exist vertices u and v such that u and v have the same color in one k-coloring and u and v have different colors in another k-coloring. Prove that if a graph G with n vertices ($n \ge 3$) is uniquely 3-colorable, then it has at least $2n - 3$ edges.

3.12. For a triple (m, n, r) of integers with $0 \le r \le n \le m - 2$, define

$$p(m, n, r) = \sum_{k=0}^{r} (-1)^k \binom{m + n - 2(k+1)}{n} \binom{r}{k}.$$

Prove that $p(m, n, r)$ is positive and that

$$\sum_{r=0}^{n} p(m, n, r) = \binom{m+n}{n}.$$

3.13. Suppose there are 18 light houses on the Mexican gulf. Each of the lighthouses lightens an angle with size 20 degrees. Prove that we can choose the directions of the lighthouses such that the whole gulf is lightened.

3.14. Let $A_{n,k}$ denote the set of lattice paths in the coordinate plane of *upsteps* $u = [1, 1,]$, *downsteps* $d = [1, -1]$, and *flatsteps* $f = [1, 0]$ that contain n steps, k of which are slanted (u or d). A *sharp turn* is a consecutive pair ud or du. Let $B_{n,k}$ denote the set of paths in $A_{n,k}$ with no upsteps among the first $k-1$ steps, and let $C_{n,k}$ denote the set of paths in $A_{n,k}$ with no sharps anywhere. (For example, fdu is in $B_{3,2}$ but not in $C_{3,2}$, while ufd is in $C_{3,2}$ but not $B_{3,2}$.) For $1 \le k \le n$, prove that the sets $B_{n,k}$ and $C_{n,k}$ contains the same number of elements.

Geometry

3.15. In isosceles triangle ABC, $AB = AC$. Extend segment BC through C to P. Points X and Y lie on lines AB and AC, respectively, such that $PX \parallel AC$ and $PY \parallel AB$. Point T lies on the circumcircle of triangle ABC such that $PT \perp XY$. Prove that $\angle BAT = \angle CAT$.

3.16. Let ABC be a acute triangle. Determine the locus of points M in the interior of the triangle such that

$$AB - FG = \frac{MF \cdot AG + MG \cdot BF}{CM},$$

where F and G are the feet of the perpendiculars from M to lines BC and AC, respectively.

3.17. There are n distinct points in the plane. Given a circle in the plane containing at least one of the points in its interior. At each step one moves the center of the circle to the baricenter of all the points in the interior of the circle. Prove that this moving process terminates in the finite number of steps.

3.18. Let ABC be a triangle with circumcenter O. Let A_1 be the midpoint of side BC. Ray AA_1 meet the circumcircle of triangle ABC again at A_2 (other than A). Let Q_a be the foot of the perpendicular from A_1 to line AO. Point P_a lies on line Q_aA_1 such that $P_aA_2 \perp A_2O$. Define points P_b and P_c analogously. Prove that points P_a, P_b, and P_c lie on a line.

3.19. Let ABC be a acute triangle with $AC \neq BC$. Points H and I are the orthocenter and incenter of the triangle, respectively. Line CH and CI meet the circumcircle of triangle ABC again at D and L (other than C), respectively. Prove that $\angle CIH = 90^\circ$ if and only if $\angle IDL = 90^\circ$.

3.20. Let $\mathcal{P}$ be a convex polygon in the plane. A real number is assigned to each point in the plane so that the sum of the numbers assigned to the vertices of any polygon similar to $\mathcal{P}$ is equal to 0. Prove that all the assigned numbers are equal to 0.

3.21. Circle ω_1 and ω_2 are externally tangent to each other at T. Let X be a point on circle ω_1. Line ℓ_1 is tangent to circle ω_1 and X, and line ℓ intersects circle ω_2 at A and B. Line XT meet circle ω at S. Point C lies on arc $\widehat{TS}$ (of circle ω_2, not containing points A and B). Point Y lies on circle ω_1 and line YC is tangent to circle ω_1. let I be the intersection of lines XY and SC. Prove that

(a) points C, T, Y, I lie on a circle; and

(b) I is an excenter of triangle ABC.

Number Theory

3.22. Let n be an integer greater than 1, and let $a_1, a_2, \ldots, a_n$ be not all identical positive integers. Prove that there are infinitely many primes p such that p divides $a_1^k + a_2^k + \cdots + a_n^k$ for some positive integer k.

3.23. Let n be an integer greater than 1, and let $a_1, a_2, \ldots, a_n$ be not all identical positive integers. Prove that there are infinitely many primes p such that p divides $a_1^k + a_2^k + \cdots + a_n^k$ for some positive integer k.

3.24. Let c be a fixed positive integer, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $a_n < a_{n+1} < a_n + c$ for every positive integer n. Let s denote the infinite string of digits obtained by writing the terms in the sequence consecutively from left to right, starting from the first term. For every positive integer k, let s_k denote the number whose decimal representation is identical to the k most left digits of s. Prove that for every positive integer m there exists a positive integer k such that s_k is divisible by m.

3.25. Prove that the following inequality holds with the exception of finitely many positive integers n:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\gcd(i,j) > 4n^2.$$

3.26. Let n be a positive integer, and let p be a prime number. Prove that if $p^p \mid n!$ then $p^{p+1} \mid n!$.

3.27. Let a, b, and c be positive integers such that the product ab divides the product $c(c^2-c+1)$ and the sum $a+b$ is divisible the number c^2+1. Prove that the sets $\{a, b\}$ and $\{c, c^2-c+1\}$ coincide.

3.28. Find all integers n for which there exists an equiangular n-gon whose side lengths are distinct rational numbers.

4
Selected Problems from MOSP Tests in 2003 and 2004

4.1. Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be real numbers such that

$$(a_1^2 + a_2^2 + \cdots + a_n^2 - 1)(b_1^2 + b_2^2 + \cdots + b_n^2 - 1)$$
$$> (a_1b_1 + a_2b_2 + \cdots + a_nb_n - 1)^2.$$

Show that $a_1^2 + a_2^2 + \cdots + a_n^2 > 1$ and $b_1^2 + b_2^2 + \cdots + b_n^2 > 1$.

4.2. Let a, b and c be positive real numbers. Prove that

$$\frac{a^2 + 2bc}{(a+2b)^2 + (a+2c)^2} + \frac{b^2 + 2ca}{(b+2c)^2 + (b+2a)^2} + \frac{c^2 + 2ab}{(c+2a)^2 + (c+2b)^2}$$
$$\le \frac{1}{2}.$$

4.3. Let a, b, and c be positive real numbers. Prove that

$$\left(\frac{a+2b}{a+2c}\right)^3 + \left(\frac{b+2c}{b+2a}\right)^3 + \left(\frac{c+2a}{c+2b}\right)^3 \ge 3.$$

4.4. Prove that for any nonempty finite set S, there exists a function $f : S \times S \to S$ satisfying the following conditions:

(a) for all $a, b \in S$, $f(a, b) = f(b, a)$;

(b) for all $a, b \in S$, $f(a, f(a, b)) = b$;

(c) for all $a, b, c, d \in S$, $f(f(a, b), f(c, d)) = f(f(a, c), f(b, d))$.

4.5. Find all pairs (x, y) of real numbers with $0 < x < \frac{\pi}{2}$ such that

$$\frac{(\sin x)^{2y}}{(\cos x)^{y^2/2}} + \frac{(\cos x)^{2y}}{(\sin x)^{y^2/2}} = \sin(2x).$$

4.6. Prove that in any acute triangle ABC,

$$\cot^3 A + \cot^3 B + \cot^3 C + 6 \cot A \cot B \cot C \ge \cot A + \cot B + \cot C.$$

4.7. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \ge \frac{3}{\sqrt[3]{abc}\left(1 + \sqrt[3]{abc}\right)}.$$

4.8. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(m+n)f(m-n) = f(m^2)$$

for $m, n \in \mathbb{N}$.

4.9. Let A, B, C be real numbers in the interval $(0, \frac{\pi}{2})$. Prove that

$$\frac{\sin A \sin(A-B)\sin(A-C)}{\sin(B+C)} + \frac{\sin B \sin(B-C)\sin(B-A)}{\sin(C+A)}$$
$$+\frac{\sin C \sin(C-A)\sin(C-B)}{\sin(A+B)} \geq 0.$$

4.10. For a pair of integers a and b, with $0 < a < b < 1000$, set $S \subseteq \{1, 2, \ldots, 2003\}$ is called a *skipping set* for (a, b) if for any pair of elements $s_1, s_2 \in S$, $|s_1 - s_2| \notin \{a, b\}$. Let $f(a, b)$ be the maximum size of a skipping set for (a, b). Determine the maximum and minimum values of f.

4.11. Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(yf(x) - 1) = x^2 f(y) - f(x)$$

for all real numbers x and y.

4.12. Show that there is an infinite sequence of positive integers

$$a_1, a_2, a_3, \ldots$$

such that

(i) each positive integer occurs exactly once in the sequence, and

(ii) each positive integer occurs exactly once in the sequence $|a_1 - a_2|, |a_2 - a_3|, \ldots$.

4.13. Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be real numbers in the interval $[1, 2]$ with $a_1^2 + a_2^2 + \cdots + a_n^2 = b_1^2 + b_2^2 + \cdots + b_n^2$. Determine the minimum value of constant c such that

$$\frac{a_1^3}{b_1} + \frac{a_2^3}{b_2} + \cdots + \frac{a_n^3}{b_n} \leq c(a_1^2 + a_2^2 + \cdots + a_n^2).$$

4.14. Let x, y, z be nonnegative real numbers with $x^2 + y^2 + z^2 = 1$. Prove that

$$1 \leq \frac{z}{1+xy} + \frac{x}{1+yz} + \frac{y}{1+zx} \leq \sqrt{2}.$$

4.15. Let n be a positive number, and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that

$$x_1 + x_2 + \cdots + x_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}.$$

Prove that

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \le 1.$$

4.16. Let x, y, and z be real numbers. Prove that

$$xyz(2x+2y-z)(2y+2z-x)(2z+2x-y)+$$
$$[x^2+y^2+z^2-2(xy+yz+zx)](xy+yz+zx)^2 \ge 0.$$

4.17. Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(yf(x)-1) = x^2 f(y) - f(x)$$

for all real numbers x and y.

4.18. Let $\mathbb{N}$ denote the set of positive integers, and let S be a set. There exists a function $f : \mathbb{N} \to S$ such that if x and y are a pair of positive integers with their difference being a prime number, then $f(x) \ne f(y)$. Determine the minimum number of elements in S.

4.19. Let n be a integer with $n \ge 2$. Determine the number of non-congruent triangles with positive integer side lengths two of which sum to n.

4.20. Jess has 3 pegs and disks of different sizes. Jess is supposed to transfer the disks from one peg to another, and the disks have to be sorted so that for any peg the disk at the bottom is the largest on that peg. (Discs above the bottom one may be in any order.) There are n disks sorted from largest on bottom to smallest on top at the start. Determine the minimum number of moves (moving one disk at a time) needed to move the disks to another peg sorted in the same order.

4.21. Let set $S = \{1, 2, \ldots, n\}$ and set T be the set of all subsets of S (including S and the empty set). One tries to choose three (not necessarily distinct) sets from the set T such that either two of the chosen sets are subsets of the third set or one of the chosen set is a subset of both of the other two sets. In how many ways can this be done?

4.22. Let $\mathbb{N}$ denote the set of positive integers, and let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(m) > f(n)$ for all $m > n$. For positive integer m, let $g(m)$ denote the number of positive integers n such that $f(n) < m$. Express

$$\sum_{i=1}^{m} g(i) + \sum_{k=1}^{n} f(k)$$

in closed form.

4.23. Given that it is possible to use 501 discs of radius two to cover a rectangular sheet of paper, determine if 2004 discs of radius one can always cover the same piece of paper.

4.24. [7pts] The usual procedure for shuffling a deck of n cards involves first splitting the deck into two blocks and then merging the two blocks to form a single deck in such a way that the order of the cards within each block is not changed. A trivial cut in which one block is empty is allowed.

(a) How many different permutations of a deck of n cards can be produced by a single shuffle?

(b) How many of these single shuffle permutations can be inverted by another such single shuffle, putting the deck back in order?

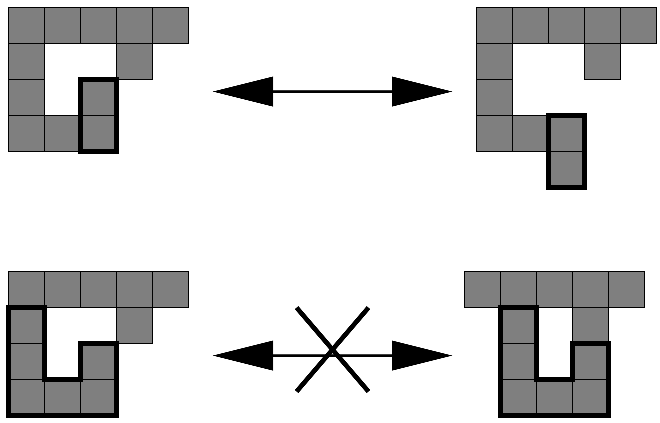
4.25. A robot is placed on an infinite square grid; it is composed of a (finite) connected block of units occupying one square each. A *valid subdivision* of the robot is a partition of its units into two connected pieces which meet along a single unbroken line segment. The robot moves as follows: it may divide into a valid subdivision, then one piece may slide one square sideways so that the result is again a valid subdivision, at which point the pieces rejoin. (See diagram for examples.)

We say a position of the robot (i.e., a connected block of squares in the plane) is *row-convex* if

(a) the robot does not occupy only a single row or only a single column, and

(b) no row meets the robot in two or more separate connected blocks.

Prove that from any row-convex position in the plane, the robot can move to any other row-convex position in the plane.



4.26. Let n be a positive integer, and let S_n be the set of all positive integer divisors of n (including 1 and n). Prove that at most half of the elements of S_n end in the digit 3.

4.27. Let $\mathcal{S}$ denote the set of points inside and on the boundary of a regular hexagon with side length 1 unit. Find the smallest value of r such that the points of $\mathcal{S}$ can be colored in three colors in such a way that any two points with the same color are less than r units apart.

4.28. Let m be a positive integer. In the coordinate plane, we call a polygon $A_1A_2\cdots A_n$, *admissible* if

(i) each side is parallel to one of the coordinate axes;

(ii) its perimeter $A_1A_2 + A_2A_3 + \cdots + A_nA_1$ is equal to m.

(iii) for $1 \le i \le n$, $A_i = (x_i, y_i)$ is a lattice point;

(iv) $x_1 \le x_2 \le \cdots \le x_n$; and

(v) $y_1 = y_n = 0$, and there is k with $1 < k < m$ such that $y_1 \le y_2 \cdots \le y_k$ and $y_k \ge y_{k+1} \ge \cdots \ge y_n$.

Determine the number of admissible polygons as a function in m. (Two admissible polygons are consider distinct if one can be obtained by the other via a composition of reflections and rotations.)

4.29. In the coordinate plane, color the lattice points which have both coordinates even black and all other lattice points white. Let P be a polygon with black points as vertices. Prove that any white point on or inside P lies halfway between two black points, both of which lie on or inside P.

4.30. Let n be a positive integer greater than or equal to 4. Let S be a sphere with center O and radius 1, and let $H_1, H_2, \ldots H_n$ be n hemispheres with center O and radius 1. Sphere S is covered by hemispheres $H_1, H_2, \ldots, H_n$. Prove that one can find positive integers i_1, i_2, i_3, i_4 such that sphere S is covered by $H_{i_1}, H_{i_2}, H_{i_3}, H_{i_4}$.

4.31. Let n be a positive integer. Consider sequences $a_0, a_1, \ldots, a_n$ such that $a_i \in \{1, 2, \ldots, n\}$ for each i and $a_n = a_0$.

(a) Call such a sequence *good* if for all $i = 1, 2, \ldots, n$, $a_i - a_{i-1} \not\equiv i \pmod{n}$. Suppose that n is odd. Find the number of good sequences.

(b) Call such a sequence *great* if for all $i = 1, 2, \ldots, n$, $a_i - a_{i-1} \not\equiv i, 2i \pmod{n}$. Suppose that n is an odd prime. Find the number of great sequences.

4.32. For each positive integer n, let D_n be the set of all positive divisors of $2^n 3^n 5^n$. Find the maximum size of a subset S of D_n in which no element of S is a proper divisor of any other.

4.33. Let $A = (0, 0, 0)$ be the origin in the three dimensional coordinate space. The *weight* of a point is the sum of the absolute values of its coordinates. A point is a *primitive lattice point* if all its coordinates are integers with their greatest common divisor equal to 1. A square $ABCD$ is called a *unbalanced primitive integer square* if it has integer side length and the points B and D are primitive lattice points with different weights.

Show that there are infinitely many unbalanced primitive integer squares $AB_iC_iD_i$ such that the plane containing the squares are not parallel to each other.

4.34. A 2004×2004 array of points is drawn. Find the largest integer n such that it is possible to draw a convex n-sided polygon whose vertices lie on the points of the array.

4.35. Let ABC be an acute triangle. Let A_1 be the foot of the perpendicular from A to side BC, and let A_B and A_C be the feet of the perpendiculars from A_1 to sides AB and AC, respectively. Line ℓ_A passes through A and is perpendicular to line A_BA_C. Lines ℓ_B and ℓ_C are defined analogously. Prove that lines ℓ_A, ℓ_B, and ℓ_C are concurrent.

4.36. Let n be a positive integer. Given n non-overlapping circular discs on a rectangular piece of paper, prove that one can cut the piece of paper into convex polygonal pieces each of which contains exactly one disc.

4.37. Let ABC be an acute-angled scalene triangle, and let H, I, and O be its orthocenter, incenter, and circumcenter, respectively. Circle ω passes through points H, I, and O. Prove that if one of the vertices of triangle ABC lies on circle ω, then there is one more vertex lies on ω.

4.38. A circle is inscribed in trapezoid $ABCD$, with $AD \parallel BC$ and $AD > BC$. Diagonals AC and BD meet at P. Prove that $\angle APD$ is obtuse.

4.39. Let ABC be a triangle, and let M be the midpoint of side BC. The circumcircle of triangle ACM meets side AB again at D (other than A). Points E and F lie on segments CB and CA, respectively, such that $CE = EM$ and $CF = 3FA$. Suppose that $EF \perp BC$. Prove that $\angle ABC = \angle DEF$.

4.40. Let $ABCD$ be a tetrahedron such that triangles ABC, BCD, CDA, and DAB all have the same inradius. Is it necessary that all four triangles be congruent?

4.41. A convex polygon is called *balanced* if for any interior point P, the sum of distance from P to the lines containing the sides of the polygon is a constant. Two convex polygons $A_1A_2\ldots A_n$ and $B_1B_2\ldots B_n$ have mutually parallel sides. Prove that $A_1A_2\ldots A_n$ is balanced if and only if $B_1B_2\ldots B_n$ is balanced.

4.42. Let ABC be a triangle with circumcircle ω, and let P be a point inside triangle ABC. Line ℓ is tangent to circle ω at C. Points D, E, F are the feet of perpendiculars from P to lines ℓ, AC, BC, respectively. Prove that

$$PD \cdot AB = PE \cdot BC + PF \cdot CA.$$

4.43. Let ABC be a triangle and let D be a point in its interior. Construct a circle ω_1 passing through B and D and a circle ω_2 passing through C and D such that the point of intersection of ω_1 and ω_2 other than D lies on line AD. Denote by E and F the points where ω_1 and ω_2 intersect side BC, respectively, and by X and Y the intersections of lines DF, AB and DE, AC, respectively. Prove that $XY \parallel BC$.

4.44. Let $ABCD$ be a convex cyclic quadrilateral, and let bisectors of $\angle A$ and $\angle B$ meet at point E. Points P and Q lie on sides AD and BC, respectively, such that PQ passes through E and $PQ \parallel CD$. Prove that $AP + BQ = PQ$.

4.45. Let ABC be an acute triangle with O and I as its circumcenter and incenter, respectively. The incircle of triangle ABC touches its sides at D, E, and F respectively. Let G be the centroid of triangle DEF. Prove that $OG \geq 7IG$.

4.46. Let ABC be an acute triangle with $AB \neq AC$, and let D be the foot of perpendicular from A to line BC. Point P is on altitude AD. Rays BP and CP meet sides AC and AB at E and F, respectively. If $BFEC$ is cyclic, prove that P is the orthocenter of triangle ABC.

4.47. Let $ABCD$ be a cyclic quadrilateral. Diagonals AC and BD meet at P. Points E, F, G, and H are the feet of perpendiculars from P to sides AB, BC, CD, and DA, respectively. Prove that lines BD, EH, and FG are either concurrent or parallel to each other.

4.48. Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, PC intersect sides BC, CA, AB at D, E, F, respectively. Prove that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$

if and only if P lies on at least one of the medians of triangle ABC. (Here $[XYZ]$ denotes the area of triangle XYZ.)

4.49. In triangle ABC three distinct triangles are inscribed, similar to each other, but not necessarily similar to triangle ABC, with corresponding points on corresponding sides of triangle ABC. Prove that if two of these triangles share a vertex, than the third one does as well.

4.50. Let ABC be an isosceles right triangle with $\angle A = 90^\circ$ and $AB = 1$, D the midpoint of $\overline{BC}$ and E and F two other points on the side BC. Let M be the second point of intersection of the circumcircles of triangles ADE and ABF. Denote by N the second point of intersection of the line AF with the circumcircle of triangle ACE and by P the second point of intersection of the line AD with the circumcircle of triangle AMN. Find with proof the distance from A to P.

4.51. Let ABC be a triangle. Circle ω passes through B and C and meet sides AB and AC again at C' and B', respectively. Let H and H' be the orthocenters of triangles ABC and $AB'C'$, respectively. Prove that lines BB', CC', and HH' are concurrent.

4.52. Let $\mathbb{T}$ be the set of triangles ABC for which there is a point D on BC such that segments AB, BD, AD, DC and AC have integral lengths and $\angle ACD = \frac{1}{2}\angle ABC = \frac{1}{3}\angle ADB$.

(a) Characterize all triples $\{a, b, c\}$ that are sets of side lengths of triangles in $\mathbb{T}$.

(b) Find the triangle of minimum area in $\mathbb{T}$.

5.53. Let ABC be a triangle. Points D, E, F are on sides BC, CA, AB, respectively, such that $DC+CE = EA+AF = FB+BD$. Prove that

$$DE + EF + FD \geq \frac{1}{2}(AB + BC + CA).$$

4.54. Let ABC be a triangle with ω and I with incircle and incenter, respectively. Circle ω touches the sides AB, BC, and CA at points C_1, A_1, and B_1, respectively. Segments AA_1 and BB_1 meet at point G. Circle ω_A is centered at A with radius AB_1. Circles ω_B and ω_C are defined analogously. Circles ω_A, ω_B, and ω_C are externally tangent to circle ω_1. Circles ω_A, ω_B, and ω_C are internally tangent to circle ω_2. Let O_1 and O_2 be the centers of ω_1 and ω_2, respectively. Lines A_1B_1 and AB meet at C_2, and lines A_1C_1 and AC meet at B_2. Prove that points I, G, O_1, and O_2 lie on a line ℓ that is perpendicular to line B_2C_2.

4.55. Convex quadrilateral $ABCD$ is inscribed in circle ω. The bisector of $\angle ADC$ of passes through the incenter of triangle ABC. Let M be an arbitrary point on arc ADC of ω. Denote by P and Q the incenters of triangles ABM and BCM.

(a) Prove that all triangles DPQ are similar, regardless of the position of point M.

(b) Suppose that $\angle BAC = \alpha$ and $\angle BCA = \beta$. Express $\angle PDQ$ and the ratio DP/DQ in terms of α and β.

4.56. Let $ABCD$ be a convex quadrilateral inscribed in circle ω, which has center at O. Lines BA and DC meet at point P. Line PO intersects segments AC and BD at E and F, respectively.

Construct point Q in a such way that $QE \perp AC$ and $QF \perp BD$. Prove that triangles ACQ and BDQ have the same area.

4.57. Let $\overline{AH_1}, \overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute scalene triangle ABC. The incircle of triangle ABC is tangent to $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at T_1, T_2, and T_3, respectively. For $k = 1, 2, 3$, let P_i be the point on line H_iH_{i+1} (where $H_4 = H_1$) such that $H_iT_iP_i$ is an acute isosceles triangle with $H_iT_i = H_iP_i$. Prove that the circumcircles of triangles $T_1P_1T_2, T_2P_2T_3, T_3P_3T_1$ pass through a common point.

4.58. Let r be an integer with $1 < r < 2003$. Prove that the arithmetic progression

$$\{2003n + r \mid n = 1, 2, 3, \ldots\}$$

contains infinitely many perfect power integers.

4.59. For positive integers m and n define

$$\phi_m(n) = \begin{cases} \phi(n) & \text{if } n \text{ divides } m; \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi(n)$ counts the number of positive integers between 1 and n which are relatively prime to n. Show that $\sum_{d|n} \phi_m(d) = \gcd(m, n)$.

4.60. Let $\{a_1, a_2, \ldots, a_{\phi(2005)}\}$ be the set of positive integers less than 2005 and relatively prime to 2005. Compute

$$\left| \prod_{k=1}^{\phi(2005)} \cos\left(\frac{a_k \pi}{2005}\right) \right|.$$

4.61. Find all triples of nonnegative integers (x, y, z) for which $4^x + 4^y + 4^z$ is the square of an integer.

4.62. Given eight distinct positive integers not exceeding 2004, prove that there are four of them, say a, b, c, and d, such that

$$4 + d \leq a + b + c \leq 4d.$$

4.63. Find all polynomials $p(x)$ with integer coefficients such that for each positive integer n, the number $2^n - 1$ is divisible by $p(n)$.

4.64. Determine if there exists a polynomial $Q(x)$ of degree at least 2 with nonnegative integer coefficients such that for each prime p, $Q(p)$ is also a prime.

4.65. Let a, b, c be nonzero integers such that both

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \quad \text{and} \quad \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$$

are integers. Prove that $|a| = |b| = |c|$.

4.66. Let $a_1, a_2, \ldots, a_n$ be a finite integer geometrical sequence and let k a positive integer, relatively prime to n. Prove that $a_1^k + a_2^k + \cdots + a_n^k$ is divisible by $a_1 + a_2 + \cdots + a_n$.

4.67. Let m and n be positive integers such that 2^m divides the number $n(n+1)$. Prove that 2^{2m-2} divides the number $1^k + 2^k + \cdots + n^k$, for all positive odd numbers k with $k > 1$.

4.68. Find all ordered triples of primes (p, q, r) such that

$$p \mid q^r + 1, \quad q \mid r^p + 1, \quad r \mid p^q + 1.$$

4.69. Let $\mathbb{N}_0^+$ and $\mathbb{Q}$ be the set of nonnegative integers and rational numbers, respectively. Define the function $f : \mathbb{N}_0^+ \to \mathbb{Q}$ by $f(0) = 0$ and

$$f(3n + k) = -\frac{3f(n)}{2} + k, \quad \text{for } k = 0, 1, 2.$$

Prove that f is one-to-one, and determine its range.

4.70. Let $a_1, a_2, \ldots, a_n$ be integers such that all the subset sums $a_{i_1} + a_{i_2} + \cdots + a_{i_k}$, for $1 \le i_1 < i_2 < \cdots < i_k \le n$, are nonzero. Prove that it is possible to partition the set of positive integers into finitely many subsets $S_1, S_2, \ldots, S_m$ in a such way that if S_i $(1 \le i \le m)$ has at least n elements, then $a_1x_1 + a_2x_2 + \cdots + a_nx_n \neq 0$, where $x_1, x_2, \ldots, x_n$ are arbitrary distinct elements in S_i.

5
Selected Problems from MOSP Tests in 2005

5.1. Let a and b be the distinct roots of quadratic equation $x^2-x-1=0$, and let

$$c_k = \frac{a^k - b^k}{a-b} \quad \text{for } k = 1, 2, \ldots.$$

Determine all the pairs of positive integers (x, y) with $x < y$ such that y divides $c_n - 2nx^n$ for all positive integers n.

5.2. Let a, b, and c be positive real numbers with $a + b + c = 1$, and let p, q, and r be the sides of a triangle with perimeter 1. Prove that

$$8abc \le ap + bq + cr.$$

5.3. Let a, b, and c be positive real numbers, prove that

$$1 \le \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{b^2+c^2}} + \frac{c}{\sqrt{c^2+a^2}} \le \frac{3\sqrt{2}}{2}.$$

5.4. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$, for $n = 1, 2, \ldots$. Prove that

$$1 - \frac{1}{2005^{2005}} < \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2005}} < 1.$$

5.5. Let x, y, z be positive real numbers. Prove that

$$\sum_{\text{cyc}} \frac{3y^2 - 2xz + 2z^2}{x^2 + 2xz} \ge 3.$$

5.6. Let a, b, c, and d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \ge 1.$$

5.7. Given real numbers $a_1, a_2, \ldots, a_n$, where n is an integer greater than 1, prove that there exist real numbers $b_1, b_2, \ldots, b_n$ satisfying the following conditions:

(a) $a_i - b_i$ are positive integers for $1 \le i \le n$; and

(b) $\sum_{1 \le i < j \le n} (b_i - b_j)^2 \le \frac{n^2-1}{12}$.

5.8. Fish Nemo has been caught and put in a large circular pool $\mathcal{P}$ by Doctor Dentist. Pool $\mathcal{P}$ is in the middle of the ocean. In order to prevent Nemo to get free by jumping from the pool to the ocean, Dentist attaches a circular ring-shape pool $\mathcal{R}$ around pool $\mathcal{P}$ and puts shark Shark in pool $\mathcal{R}$. (Shark can only swim in pool $\mathcal{R}$, and the edges of pool $\mathcal{R}$ are designed to prevent Shark jump into either pool $\mathcal{P}$ and the ocean.) Nemo works hard to improve its jumping skills so that if it can reach the edge of pool $\mathcal{P}$ without Shark being there, it can jump through pool $\mathcal{R}$ to get into the free water in the ocean. But on the other hand, if Shark is at the point of jumping, Nemo will be eaten by Shark. Shark can swim four as times fast as Nemo. Determine if it is possible for Nemo to develop a strategy to get back into the ocean safely.

5.9. At each corner of a cube, an integer is written. A *legal transition* of the cube consists of picking any corner of the cube and adding the value written at that corner to the value written at some adjacent corner (i.e. pick a corner with some value x written at it, and an adjacent corner with some value y written at it, and replace y by $x+y$). Prove that there is a finite sequence of legal transitions of the given cube so that the eight integers written are all the same modulo 2005.

5.10. Prove that any given triangle can be dissected into n isosceles triangles, for every positive integers n greater than three.

5.11. Two squares of side 0.9 are drawn inside a disk of radius 1. Prove that the squares overlap.

5.12. The positive integer n satisfies the following property: It is possible to draw n lines in 3-space passing through a fixed point, such that the non-obtuse angles formed by any two lines are equal to each other. Determine the maximum value of n and determine the size of the angle for this n.

5.13. A deck of 32 cards has 3 suits and 2 jokers. Each suit has cards numbered from 1 through 10, and each joker is numbered 0. One chooses a hand of cards from the deck. A card numbered k has value 2^k, and the value of a hand is the sum of the values of the cards in the hand. Determine the number of hands of value 2004.

5.14. A finite set S of natural numbers is called *obese* if every element is at least twice as large as the number of elements in the set. Determine the number of nonempty obese subsets of $\{1, 2, \ldots, n\}$ in which all elements have the same parity.

5.15. Tiankai makes some random moves along the lines parallel to the x- and y-axes. He starts at $(0,0)$, and on each of his moves he may move one unit along a line parallel to one of the axes. He cannot land on points below the x-axis. Determine the number of distinct paths he can make after n moves.

5.16. Each of 2^n computers ($n \geq 2$) is labeled with an n-tuple $(e_1, e_2, \dots, e_n)$ consisting of 0's and 1's, and two computers can directly communicate if and only if their corresponding n-tuples differ in exactly one place. (The resulting network is an n-dimensional cube.) A hostile group wants to disable enough computers so that no computer will be able to communicate with two others. Determine the least number of computers that need to be disabled to achieve this goal.

5.17. A square array of numbers is called a *binary square* if the entries are either 0 or 1. A binary square is called *good* if all the entries above its main diagonal (the diagonal from top left to bottom right) are the same, and all the entries below its main diagonal are also the same. Let m be a given positive integer. Prove that there exists a positive integer M such that for any integer n with $n > M$ and any given $n \times n$ binary array A_n, there exist integers $1 \leq i_1 < i_2 < \cdots < i_{n-m} \leq n$ such that the binary array B_m, obtained by deleting rows and columns $i_1, i_2, \dots, i_{n-m}$ from A_n, is good.

5.18. Suppose n coins have been placed in piles on the integers on the real line. (A "pile" may contain zero coins.) Let T denote the following sequence of operations.

(a) Move piles $0, 1, 2, \dots$ to $1, 2, 3, \dots$, respectively.

(b) Remove one coin from each nonempty pile from among piles $1, 2, 3, \dots$, then place the removed coins in pile 0.

(c) Swap piles i and $-i$ for $i = 1, 2, 3, \dots$.

Prove that successive applications of T from any starting position eventually lead to some sequence of positions being repeated, and describe all possible positions that can occur in such a sequence.

5.19. Let ABC be an obtuse triangle inscribed in a circle of radius 1. Prove that triangle ABC can be covered by an isosceles right triangle with hypotenuse $\sqrt{2} + 1$.

5.20. Let $ABCDEF$ be a convex cyclic hexagon. Prove that segments AD, BE, and CF are concurrent if and only if

$$\frac{AB \cdot CD \cdot EF}{BC \cdot DE \cdot FA} = 1.$$

5.21. Let $ABCD$ be a convex quadrilateral such that $\angle C = \angle D =$

120°. Prove that

$$(AD + CD)^3 + (BC + CD)^3 \le 2AB^3.$$

When does the equality hold?

5.22. Circle ω_1 and ω_2 intersect at points P and Q. Line ℓ_1 is tangent to ω_1 and ω_2 at A and B, respectively; line ℓ_2 is tangent to ω_1 and ω_2 at C and D respectively. Line ℓ passes through the centers of the two circles, and intersects segments AC and BD at M and N, respectively. Show that $PMQN$ is a rhombus.

5.23. Let ABC be a triangle. Circle ω passes through A and B and meets sides AC and BC at D and E, respectively. Let F be the midpoint of segment AD. Suppose that there is a point G on side AB such that $FG \perp AC$. Prove that $\angle EGF = \angle ABC$ if and only if $AF/FC = BG/GA$.

5.24. Let ABC be a triangle. Let A_1 and A_2 be the feet of perpendiculars from A to the bisectors of angles B and C. Define points B_1 and B_2; C_1 and C_2 analogously. Prove that

$$2(A_1A_2 + B_1B_2 + C_1C_2) = AB + BC + CA.$$

5.25. Let ABC be an acute triangle. Points M and H lie on side AB such that $AM = MB$ and $CH \perp AB$. Points K and L lie on the opposite side of line AB from C, such that $AK \perp CK$, $BL \perp CL$, and $\angle ACK = \angle LCB$. Prove that points K, H, M, and L lie on a circle.

5.26. Let ABC be a triangle, and let P be a point on side BC. The incircle of triangle ABC touches side BC at D. The excircle of triangle ABC opposite A touches side BC at D_1. Let Q and R be the incenters of triangles ABP and ACP, respectively. Let Q_1 and R_1 be the excenters opposite A of triangles ABP and ACP, respectively. Prove that

(a) lines BC, QR, and Q_1R_1 are either parallel to each other or concurrent; and that

(b) triangles DQR and $D_1Q_1R_1$ are similar (not necessarily in that order).

5.27. Triangle ABC is inscribed in circle ω. Points D, E, and F are the midpoints of arcs $\widehat{BC}$ (not containing A), $\widehat{CA}$ (not containing B), and $\widehat{AB}$ (not containing C), respectively. Segment DE meet segments BC and CA at X and Y, respectively. Segment DF meet segment AB and BC at S and T, respectively. Segments AD and EF meet at P. Let M and N be the midpoints of segment ST and XY, respectively.

(a) Prove that lines AD, BM, and CN are concurrent.

(b) The circumcircle of triangle MNP passes through the circumcenter of triangle DMN.

5.28. Triangle ABC is inscribed in circle ω. Circle γ is tangent to sides AB and AC and circle ω at P, Q, and S, respectively. Segments AS and PQ meet at T. Prove that $\angle BTP = \angle CTQ$.

5.29. Let ABC be an acute triangle, and let H and O denote its orthocenter and circumcenter, respectively. Points D, E, and F lie on side BC such that $AD \perp BC$, $\angle BAE = \angle EAC$, and $BF = FC$. Point T is the foot of the perpendicular from H to line AE. Prove that $AO \parallel TF$.

5.30. Let $ABCDEF$ be a cyclic hexagon such that $AB = CD = EF$. Let P, Q, R be the intersections of diagonals AC and BD, CE and DF, and EA and FB, respectively. Prove that the triangles PQR and BDF are similar.

5.31. Given a circle ω and two fixed points A and B on the circle. Assume that there is a point C on ω such that $AC + BC = 2AB$. Show how to construct point C with a compass and a straightedge.

5.32. Let ABC be an acute triangle. For a point P on side AB, points P_A and P_B lie on sides CA and CB, respectively, such that CP_APP_B is a parallelogram. Show how to construct the point P, using a compass and straightedge, for which P_AP_B is minimal.

5.33. Let ABC be a scalene acute-angled triangle, and let A_1, B_1, and C_1 are the midpoints of sides BC, CA, and AB, respectively. Circle ω_a passes through A, A_1, and the circumcenter of triangle ABC. Circles ω_b and ω_c are defined analogously. Prove that circles ω_a, ω_b, and ω_c pass through two common points.

5.34. Let ABC be an acute triangle, and let $A = \angle A$, $B = \angle B$, and $C = \angle C$. Prove that

$$\sum_{\text{cyc}} \frac{1}{\cot^2 A + 2\cot B \cot C + 3\cot A(\cot B + \cot C)} \leq 1.$$

5.35. Find a set of 2005 composite integers in arithmetic progression, each pair of which is relatively prime.

5.36. Find all integers n such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square number.

5.37. Let n be a positive integer, and let S_n be the set of all positive integer divisors of n (including 1 and itself). Prove that at most half of the elements of S_n have their last digit equal to 3.

5.38. Let S be the set of nonnegative rational numbers, and let n be a positive integer. Suppose T is a subset of S with the following properties.

(a) $0 \in T$.

(b) If $i \in T$, then $j \in T$ for all $j \in S$ with $j < i$.

(c) If a, b are nonnegative integers (not both zero) with $\frac{a}{b} \in T$, then $\frac{a}{b} + \frac{1}{bn} \in T$.

(d) If $i \in S$ satisfies $j \in T$ for all $j \in S$ with $j < i$, then $i \in T$.

Prove that $T = S$.

5.39. Let n be a positive integer. Prove that

$$(2004n + 2)(2004n + 3) \cdots (2005n - 1)(2005n)$$

is divisible by $n!$.

5.40. Let p be an odd prime, and let $P(x) = a_{p-1}x^{p-1} + a_{p-2}x^{p-2} + \cdots + a_1 x + a_0$ be a polynomial of degree $p-1$ with integer coefficients. Suppose that p does not divide the number $P(a) - P(b)$ whenever p does not divide the number $a - b$. Prove that p divides a_{p-1}.

5.41. Let p and q be two distinct positive integers, and let n be a nonnegative integer. Determine the number of integers that can be written in the form $ip + jq$, where i and j are nonnegative integers with $i + j \le n$.

5.42. Determine all ordered triples (n, i, j) of integers with $n \ge 1$ and $0 \le i < j \le n$ such that $\binom{n}{i}$ and $\binom{n}{j}$ are relatively prime.

5.43. Let n be an integer greater than 1. For a positive integer m, let $S_m = \{1, 2, \ldots, mn\}$. Suppose that there exists a $2n$-element set T such that

(a) each element of T is an m-element subset of S_m;

(b) each pair of elements of T shares at most one common element; and

(c) each element of S_m is contained in exactly two elements of T.

Determine the maximum possible value of m in terms of n.

5.44. Let $A_1A_2A_3$ be an acute triangle, and let O and H be its circumcenter and orthocenter, respectively. For $1 \le i \le 3$, points P_i and Q_i lie on lines OA_i and $A_{i+1}A_{i+2}$ (where $A_{i+3} = A_i$), respectively, such that OP_iHQ_i is a parallelogram. Prove that

$$\frac{OQ_1}{OP_1} + \frac{OQ_2}{OP_2} + \frac{OQ_3}{OP_3} \ge 3.$$

5.45. For a positive integer n, let S denote the set of polynomials $P(x)$ of degree n with positive integer coefficients not exceeding $n!$. A polynomial $P(x)$ in set S is called *fine* if for any positive integer k, the sequence $P(1), P(2), P(3), \ldots$ contains infinitely many integers relatively prime to k. Prove that at least 71% of the polynomials in the set S are fine.

5.49. Consider the polynomials

$$f(x) = \sum_{k=1}^{n} a_k x^k \quad \text{and} \quad g(x) = \sum_{k=1}^{n} \frac{a_k}{2^k - 1} x^k,$$

where $a_1, a_2, \ldots, a_n$ are real numbers and n is a positive integer. Show that if 1 and 2^{n+1} are zeros of g then f has a positive zero less than 2^n.

5.47. Find all finite sets S of points in the plane with the following property: for any three distinct points A, B, C in S, there is a fourth point D in S such that A, B, C, D are the vertices of a parallelogram (in some order).

5.48. Let ABC be a acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.

5.49. Let $A_1, A_2, \ldots, B_1, B_2, \ldots$ be sets such that $A_1 = \emptyset$, $B_1 = \{0\}$,

$$A_{n+1} = \{x + 1 \mid x \in B_n\}, \quad B_{n+1} = A_n \cup B_n - A_n \cap B_n,$$

for all positive integers n. Determine all the positive integers n such that $B_n = \{0\}$.

6
Final Round of 2004 Russia Mathematics Olympiad

6.1. Each lattice point in the plane is colored with one of three colors, and each of the three colors is used at least once. Show that there exists a right-angled triangle whose vertices have three different colors.

6.2. Let $ABCD$ be a quadrilateral with an inscribed circle. The external angle bisectors at A and B intersect at K, the external angle bisectors at B and C intersect at L, the external angle bisectors at C and D intersect at M, and the external angle bisectors at D and A intersect at N. Let K_1, L_1, M_1 and N_1 be the orthocenters of triangles ABK, BCL, CDM and DAN respectively. Show that quadrilateral $K_1L_1M_1N_1$ is a parallelogram.

6.3. There are 2004 boxes on a table, each containing a single ball. I know that some of the balls are white and that the number of white balls is even. I am allowed to indicate any two boxes and ask whether at least one of them contains a white ball. What is the smallest number of questions needed to determine two boxes, each of which is guaranteed to contain a white ball?

6.4. Let $n > 3$ and let $x_1, x_2, \ldots, x_n$ be positive real numbers whose product is 1. Prove that

$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \cdots + \frac{1}{1+x_n+x_nx_1} > 1.$$

6.5. Are there pairwise distinct positive integers m, n, p, q satisfying $m+n = p+q$ and $\sqrt{m} + \sqrt[3]{n} = \sqrt{p} + \sqrt[3]{q} > 2004$?

6.6. There are 2004 telephones in a cabinet. Each pair of telephones is connected by a cable, which is colored in one of four colors. Each of the four colors appears on at least one cable. Can one always select some of the telephones so that among their pairwise cable connections exactly three different colors occur?

6.7. The natural numbers from 1 to 100 are arranged on a circle in such a way that each number is either larger than both of its neighbors or smaller than both of its neighbors. A pair of adjacent numbers is called "good" if when it is removed the circle of remaining numbers still has the above property. What is the smallest possible number of good pairs?

6.8. Let ABC be an acute triangle with circumcenter O. Let T be the circumcenter of triangle AOC and let M be the midpoint of segment AC. Points D and E are selected on lines AB and BC respectively such that $\angle BDM = \angle BEM = \angle ABC$. Show that lines BT and DE are perpendicular.

6.9. Let $ABCD$ be a quadrilateral with both an inscribed circle and a circumscribed circle. The incircle of quadrilateral $ABCD$ touches the sides AB, BC, CD and DA at points K, L, M and N respectively. The external angle bisectors at A and B intersect at K', the external angle bisectors at B and C intersect at L', the external angle bisectors at C and D intersect at M', and the external angle bisectors at D and A intersect at N'. Prove that the lines KK', LL', MM' and NN' pass through a common point.

6.10. A sequence of nonnegative rational numbers $a_1, a_2, \ldots$ satisfies $a_m + a_n = a_{mn}$ for all m, n. Show that not all elements of the sequence can be distinct.

6.11. A country has 1001 cities, each pair of which is connected by a one-way street. Exactly 500 roads begin in each city and exactly 500 roads end in each city. Now an independent republic containing 668 of the 1001 cities breaks off from the country. Prove that it is possible to travel between any two cities in the republic without leaving the republic.

6.12. A triangle T is contained inside a polygon M which has a point of symmetry. Let T' be the reflection of T through some point P inside T. Prove that at least one vertex of T' lies in or on the boundary of M.

6.13. Does there exist a natural number $n > 10^{1000}$ such that $10 \nmid n$ and it is possible to exchange two distinct nonzero digits in the decimal representation of n, leaving the set of prime divisors the same?

6.14. Let I_a and I_b be the centers of the excircles of triangle ABC opposite A and B, respectively. Let P be a point on the circumcircle ω of ABC. Show that the midpoint of the segment connecting the circumcenters of triangles I_aCP and I_bCP is the center of ω.

6.15. The polynomials $P(x)$ and $Q(x)$ satisfy the property that for a certain polynomial $R(x, y)$, the identity $P(x)-P(y) = R(x, y)(Q(x)-Q(y))$ holds. Prove that there exists a polynomial $S(x)$ such that

$P(x) = S(Q(x))$.

6.16. The cells of a 9×2004 rectangular array contain the numbers 1 to 2004, each 9 times. Furthermore, any two numbers in the same column differ by at most 3. Find the smallest possible value for the sum of the numbers in the first row.

6.17. Let $M = \{x_1, \ldots, x_{30}\}$ be a set containing 30 distinct positive real numbers, and let A_n denote the sum of the products of elements of M taken n at a time, $1 \le n \le 30$. Prove that if $A_{15} > A_{10}$, then $A_1 > 1$.

6.18. Prove that for $N > 3$, there does not exist a finite set S containing more than $2N$ pairwise non-collinear vectors in the plane satisfying:

(i) for any N vectors in S, there exist $N - 1$ more vectors in S such that the sum of the $2N - 1$ vectors is the zero vector;

(ii) for any N vectors in S, there exist N more vectors in S such that the sum of the $2N$ vectors is the zero vector.

6.19. A country contains several cities, some pairs of which are connected by airline flight service (in both directions). Each such pair of cities is serviced by one of k airlines, such that for each airline, all flights that the airline offers contain a common endpoint. Show that it is possible to partition the cities into $k + 2$ groups such that no two cities from the same group are connected by a flight path.

6.20. A parallelepiped is cut by a plane, giving a hexagon. Suppose there exists a rectangle R such that the hexagon fits in R; that is, the rectangle R can be put in the plane of the hexagon so that the hexagon is completely covered by the rectangle. Show that one of the faces of the parallelepiped also fits in R.

7
Selected problems from the 2005 Russia Mathematics Olympiad

7.1. Lesha puts the numbers from 1 to 22^2 into the cells of a 22×22 board. Can Lego always choose two cells, sharing at least one vertex, which contain numbers whose sum is divisible by 4?

7.2. Ten distinct nonzero reals are given such that for any two, either their sum or their product is rational. Prove that the squares of all these numbers are rational.

7.3. Positive numbers are written in the cells of a $2 \times n$ board such that the sum of the numbers in each of the n columns equals 1. Prove that it is possible to select one number from every column such that the sum of selected numbers in each of two rows does not exceed $\frac{n+1}{4}$.

7.4. Let a_1, a_2, and a_3 be real numbers greater than 1, and let $S = a_1 + a_2 + a_3$. Assume that $a_i^2 > S(a_i - 1)$ for $i = 1, 2, 3$. Prove that

$$\frac{1}{a_1 + a_2} + \frac{1}{a_2 + a_3} + \frac{1}{a_3 + a_1} > 1.$$

7.5. Let $ABCD$ be a cyclic quadrilateral, and let H_A, H_B, H_C and H_D be the orthocenters of triangles BCD, CDA, DAB, and ABC, respectively. Prove that quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent.

7.6. Is there a bounded function $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) > 0$ and

$$f^2(x + y) \geq f^2(x) + 2f(xy) + f^2(y)$$

for all real numbers x and y?

7.7. Let ABC be a triangle, and let A', B', C' be the points of tangency of the respective excircles with sides BC, CA, AB. The circumcircles of triangles $A'B'C$, $AB'C'$, $A'BC'$ intersect the circumcircle of ABC at points $C_1 \neq C$, $A_1 \neq A$, $B_1 \neq B$ respectively. Prove that $\triangle A_1B_1C_1$ is similar to the triangle formed by points of tangency of the incircle of $\triangle ABC$ with its sides.

7.8. Let $a_1, a_2, \ldots, a_{2005}$ be distinct numbers. We can ask questions of the form "What is the set $\{a_i, a_j, a_k\}$?" for any three integers $1 \leq i < j < k \leq 2005$. Find the minimum number of questions that are necessary to find out all the a_i.

7.9. Find the maximum possible finite number of roots of an equation $|x - a_1| + \cdots + |x - a_{50}| = |x - b_1| + \cdots + |x - b_{50}|$, where $a_1, a_2, \ldots, a_{50}, b_1, \ldots, b_{50}$ are distinct reals.

7.10. In acute triangle ABC, AA' and BB' are its altitudes. A point D is chosen on arc ACB of the circumcircle of ABC. Lines AA' and BD meet at P, and lines BB' and AD meet at Q. Prove that the line $A'B'$ contains the midpoint of PQ.

7.11. Sixteen rooks are placed in different squares on a regular chessboard. Find the minimal number of pairs of rooks that are either in the same row or in the same column.

7.12. Positive integers $x > 1$ and y satisfy the equation $2x^2 - 1 = y^{15}$. Prove that 5 divides x.

7.13. Find all positive integers n such that the number $n \cdot 2^{n-1} + 1$ is a perfect square.

7.14. A quadrilateral $ABCD$ without parallel sides is circumscribed around a circle with center O. Prove that $OA \cdot OC = OB \cdot OD$ if and only if O lies on both lines joining the midpoints of pairs of opposite sides.

7.15. Let ω_B and ω_C be the excircles of triangle ABC opposite vertices B and C, respectively. The circle ω'_B is symmetric to ω_B with respect to the midpoint of AC, and the circle ω'_C is symmetric to ω_C with respect to the midpoint of AB. Prove that the radical axis of ω'_B and ω'_C bisects the perimeter of triangle ABC.

7.16. The U.S. Senators are standing in a circle. Prove that one can partition them into 2 groups so that no two senators from the same state, nor any three consecutive people in the circle, are in the same group.

7.17. In an infinite square grid, a finite number of cells are colored black. Each black cell is adjacent to (shares a side with) an even number of uncolored cells. Prove that one may color each uncolored cell green or red such that every black cell will be adjacent to an equal number of red and green cells.

7.18. Let x, y, and z be integers with $x > 2$, $y > 1$, and $z > 0$ such that $x^y + 1 = z^2$. Let p be the number of distinct prime divisors of x, and let q be the number of distinct prime divisors of y. Prove that $p \geq q + 2$.

7.19. Find the least positive integer that cannot be represented in the form

$$\frac{2^a - 2^b}{2^c - 2^d}$$

for some positive integers a, b, c, and d.

7.20. For a set X, let $S(X)$ denote the sum of elements in X. Let $M = \{2^0, 2^1, 2^2, \ldots, 2^{2005}\}$. Determine the number of ordered pairs (A, B) of non-empty subsets of M such that A and B form a partition of M and the equation $x^2 - S(A)x + S(B) = 0$ has integer roots.

7.21. The *Republic of Fat* has 25 states, and each state has 4 senators. The 100 senators stand in a circle. Prove that one can partition them into 4 groups so that no two senators from the same state, nor any three consecutive people in the circle, are in the same group.

8
Selected Problems from Chinese IMO Team Training in 2003 and 2004

8.1. Let x and y be positive integers with $x < y$. Find all possible integer values of

$$P = \frac{x^3 - y}{1 + xy}.$$

8.2. Let ABC be a triangle, and let x be a nonnegative number. Prove that

$$a^x \cos A + b^x \cos B + c^x \cos C \le \frac{1}{2}(a^x + b^x + c^x).$$

8.3. Given integer n with $n \ge 2$, determine the number of ordered n-tuples of integers $(a_1, a_2, \ldots, a_n)$ such that

(i) $a_1 + a_2 + \cdots + a_n \ge n^2$; and

(ii) $a_1^2 + a_2^2 + \cdots + a_n^2 \le n^3 + 1$.

8.4. Let a, b, and c denote the side lengths of a triangle with perimeter no greater than 2π. Prove that there is a triangle with side lengths $\sin a, \sin b$, and $\sin c$.

8.5. Let n be an integer greater than 1. Determine the largest real number λ, in terms of n, such that

$$a_n^2 \ge \lambda(a_1 + a_2 + \cdots + a_{n-1}) + 2a_n.$$

for all positive integers $a_1, a_2, \ldots, a_n$ with $a_1 < a_2 < \cdots < a_n$.

8.6. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$, for $n = 1, 2, \ldots$. Prove that

$$1 - \frac{1}{2003^{2003}} < \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2003}} < 1.$$

8.7. Let $a_1, a_2, \ldots, a_{2n}$ be real numbers such that $\sum_{i=1}^{2n-1}(a_{i+1} - a_i)^2 = 1$. Determine the maximum value of

$$(a_{n+1} + a_{n+2} + \cdots + a_{2n}) - (a_1 + a_2 + \cdots + a_n).$$

8.8. Let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that there is a k, $1 \le k \le n$, such that

$$\left|\sum_{i=1}^{k} a_i - \sum_{i=k+1}^{n} a_i\right| \le \max\{|a_1|, |a_2|, \ldots, |a_n|\}.$$

8.9. Let n be a fixed positive integer. Determine the smallest positive real number λ such that for any $\theta_1, \theta_2, \ldots, \theta_n$ in the interval

$(0, \frac{\pi}{2})$, if the product

$$\tan\theta_1 \tan\theta_2 \cdots \tan\theta_n = 2^{\frac{n}{2}},$$

then the sum

$$\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n \le \lambda.$$

8.10. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $a_1 = 3$, $a_2 = 7$, and $a_n^2 + 5 = a_{n-1}a_{n+1}$ for $n \ge 2$. Prove that if $a_n + (-1)^n$ is a prime, then $n = 3^m$ for some nonnegative integer m.

8.11. Let k be a positive integer. Prove that $\sqrt{k+1} - \sqrt{k}$ is not the real part of the complex number z with $z^n = 1$ for some positive integer n.

8.12. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that the system of equations

$$\begin{array}{ccccccc} & - & 2x_1 & + & x_2 & = & -a_1x_1, \\ x_1 & - & 2x_2 & + & x_3 & = & -a_2x_2, \\ x_2 & - & 2x_3 & + & x_4 & = & -a_3x_3, \\ & \cdots & & & & \cdots & \\ x_{n-2} & - & 2x_{n-1} & + & x_n & = & -a_{n-1}x_{n-1}, \\ x_{n-1} & - & 2x_n & & & = & -a_nx_n, \end{array}$$

has a solution $(x_1, x_2, \ldots, x_n)$ in real numbers with $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$. Prove that

$$a_1 + a_2 + \cdots + a_n \ge \frac{4}{n+1}.$$

8.13. Let a, b, c, d be positive real numbers with $ab + cd = 1$. For $i = 1, 2, 3, 4$, points $P_i = (x_i, y_i)$ are on the unit circle. Prove that

$$(ay_1+by_2+cy_3+dy_4)^2+(ax_4+bx_3+cx_2+dx_1)^2 \le 2\left(\frac{a^2+b^2}{ab} + \frac{c^2+d^2}{cd}\right).$$

8.14. Determine all functions $f, g\ \mathbb{R} \to \mathbb{R}$ such that

$$f(x + yg(x)) = g(x) + xf(y)$$

for all real numbers x and y.

8.15. Let n be a positive integer, and let $a_0, a_1, \ldots, a_{n-1}$ be complex numbers with

$$|a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2 \le 1.$$

Let $z_1, z_2, \dots z_n$ be the (complex) roots of the polynomial

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Prove that

$$|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 \le n.$$

8.16. Let T be a real number satisfying the property: For any non-negative real numbers a, b, c, d, e with their sum equal to 1, it is possible to arrange them around a circle such that the products of any two neighboring numbers are no greater than T. Determine the minimum value of T.

8.17. Let $A = \{1, 2, \dots, 2002\}$ and $M = \{1001, 2003, 3005\}$. A subset B of A is called M-*free* if the sum of any pairs of elements b_1 and b_2 in B, $b_1 + b_2$ is not in M. An ordered pair of subset (A_1, A_2) is called a M-*partition* of A if A_1 and A_2 is a partition of A and both A_1 and A_2 are M-free. Determine the number of M-partitions of A.

8.18. Let $S = (a_1, a_2, \dots, a_n)$ be the longest binary sequence such that for $1 \le i < j \le n - 4$, $(a_i, a_{i+1}, a_{i+2}, a_{i+3}, a_{i+4}) \ne (a_j, a_{j+1}, a_{j+2}, a_{j+3}, a_{j+4})$. Prove that

$$(a_1, a_2, a_3, a_4) = (a_{n-3}, a_{n-2}, a_{n-1}, a_n).$$

8.19. For integers r, let $S_r = \sum_{j=1}^{n} b_j z_j^r$ where b_j are complex numbers and z_j are nonzero complex numbers. Prove that

$$|S_0| \le n \max_{0<|r|\le n} |S_r|.$$

8.20. Let n be a positive integer. Three letters (x_1, x_2, x_3), with each x_i being an element in $\{R, G, B\}$, is called a *set* if $x_1 = x_2 = x_3$ or $\{x_1, x_2, x_3\} = \{R, G, B\}$. Claudia is playing the following game: She is given a row of n of letters $a_{1,1}, a_{1,2}, \dots, a_{1,n}$ with each letter $a_{1,i}$ being either A, or B, or C. Claudia then builds a triangular array of letter with the given row as the initial row on the top. She writes down $n-1$ letters $a_{2,1}, a_{2,2}, \dots, a_{2,n-1}$ in the second row such that $(a_{1,i}, a_{1,i+1}, a_{2,i}$ form a set for each $1 \le i \le n-1$. She then writes down $n-2$ letters in the third row by applying the similar rules to letter in the second row, and so on. The games ends when she obtain the single letter $a_{n,1}$ in the n^{th} row. For example, Claudia will obtain the following triangular array with

$(a_1, a_2, \ldots, a_6) = (A, C, B, C, B, A))$ as the given initial row:

$$\begin{array}{ccccccccccc}
R & & G & & B & & G & & B & & R \\
& B & & R & & R & & R & & C & \\
& & G & & R & & R & & B & & \\
& & & B & & R & & G & & & \\
& & & & G & & B & & & & \\
& & & & & R & & & & &
\end{array}$$

An initial row is *balanced* if there are equal numbers of A, B, C in the row. An initial row is *good* if $a_{1,1}, a_{1,n}, a_{n,1}$ form a set.

(a) Find all the integers n such that all possible initial sequence are good.

(b) Find all the integers n such that all possible balanced initial sequences are good.

8.21. Let S be a set such that

(a) Each element of S is a positive integer no greater than 100;

(b) For any two distinct elements a and b in S, there exists an element c in S such that a and c, b and c are relatively prime; and

(c) For any two distinct elements a and b in S, there exists a third element d in S such that a and d, b and d are not relatively prime to each other.

Determine the maximum number of elements S can have.

8.22. Ten people are applying for a job. The job selection committee decides to interview the candidates one by one. The order of candidates being interviewed is random. Assume that all the candidates have distinct abilities. For $1 \le k \le 10$. The following policies are set up within the committee:

(i) The first three candidates interviewed will not be fired;

(ii) For $4 \le i \le 9$, if the i^{th} candidate interviewed is more capable than all the previously interviewed candidates, then this candidate is hired and the interview process is terminated;

(iii) The 10^{th} candidate interviewed will be hired.

Let P_k denote the probability that the k^{th} most able person is hired under the selection policies. Show that,

(a) $P_1 > P_2 > \cdots > P_8 = P_9 = P_{10}$; and

(b) that there are more than 70% chance that one of the three most able candidates is hired and there are no more than 10% chance that one of the three least able candidates is hired.

8.23. Let A be a subset of the set $\{1, 2, \ldots, 29\}$ such that for any integer k and any elements a and b in A (a and b are not necessarily distinct), $a+b+30k$ is not the product of two consecutive integers. Find the maximum number of elements A can have.

8.24. Let set $S = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R},\ 1 \le i \le n\}$, and let A be a finite subset of S. For any pair of elements $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in A, define $d(a, b) = (|a_1 - b_1|, |a_2 - b_2|, \ldots, |a_n - b_n|)$ and $D(A) = \{d(a, b) \mid a, b \in A\}$. Prove that the set $D(A)$ contains more elements than the set A does.

8.25. Let n be a positive integer, and let $A_1, A_2, \ldots A_{n+1}$ be nonempty subsets of the set $\{1, 2, \ldots, n\}$. Prove that there exists nonempty and nonintersecting index sets $I_1 = \{i_1, i_2, \ldots, i_k\}$ and $I_2 = \{j_1, j_2, \ldots, j_m\}$ such that

$$A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} = A_{j_1} \cup A_{j_2} \cup \cdots \cup A_{j_m}.$$

8.26. Integers $1, 2, \ldots, 225$ are arranged in a 15×15 array. In each row, the five largest numbers are colored red. In each column, the five largest numbers are colored blue. Prove that there are at least 25 numbers are colored purple (that is, colored in both red and blue).

8.27. Let $p(x)$ be a polynomial with real coefficients such that $p(0) = p(n)$. Prove that there are n distinct pairs of real numbers (x, y) such that $y - x$ is a positive integer and $p(x) = p(y)$.

8.28. Determine if there is an $n \times n$ array with entries $-1, 0, 1$ such that the $2n$ sums of all the entries in each row and column are all different, where (1) $n = 2003$; and (2) $n = 2004$.

8.29. Let P be a 1000-sided regular polygon. Some of its diagonals were drawn to obtain a *triangulation* the polygon P. (The region inside P is cut into triangular regions, and the diagonals drawn only intersect at the vertices of P.) Let n be the number of different lengths of the drawn diagonals. Determine the minimum value of n.

8.30. Let m and n be positive integers with $m \ge n$, and let $ABCD$ be a rectangle with $A = (0, 0)$, $B = (m, 0)$, $C = (m, n)$, and

$D = (0, n)$. Rectangle $ABCD$ is tiled by mn unit squares. There is a bug in each unit square. At a certain moment, Ben shouts at the bugs: "Move!!!" Each bug can choose independently whether or not to follow Ben's order and bugs do not necessarily move to an adjacent square. After the moves are finished, Ben noticed that if bug a and bug b were neighbors before the move, then either they are still neighbors or they are in the same square after the move. (Two bugs are called *neighbors* if the square they are staying share a common edge.) Prove that, after the move, there are n bugs such that the centers of the squares they are staying are on a line of slope 1.

8.31. Quadrilateral $ABCD$ is inscribed in circle with AC a diameter of the circle and $BD \perp AC$. Diagonals AC and BD intersect at E. Extend segment DA through A to F. Extend segment BA through A to G such that $DG \parallel BF$. Extend segment GF through F to H such that $CH \perp GH$. Prove that points B, E, F, H lie on a circle.

8.32. Let ABC be a triangle. Points D, E, and F are on segments AB, AC, and DE, respectively. Prove that

$$\sqrt[3]{[BDF]} + \sqrt[3]{[CEF]} \leq \sqrt[3]{[ABC]}$$

and determine the conditions when the equality holds.

8.33. Circle ω is inscribed in convex quadrilateral $ABCD$, and it touches sides AB, BC, CD, and DA at A_1, B_1, C_1, and D_1, respectively. Let E, F, G, and H be the midpoints of A_1B_1, B_1C_1, C_1D_1, and D_1A_1, respectively. Prove that quadrilateral $EFGH$ is a rectangle if and only if $ABCD$ is cyclic.

8.34. Let $ABCD$ be a cyclic convex quadrilateral with $\angle A = 60°$, $BC = CD = 1$. Rays AB and DC meet E, and rays AD and BC meet at F. Suppose that the the perimeters of triangles BCE and CDF are integers. Compute the perimeter of quadrilateral $ABCD$.

8.35. Let $ABCD$ be a convex quadrilateral. Diagonal AC bisects $\angle BAD$. Let E be a point on side CD. Segments BE and AC intersect at F. Extend segment DF through F to intersect segment BC at G. Prove that $\angle GAC = \angle EAC$.

8.36. Four lines are given in the plane such that each three form a non-degenerate, non-equilateral triangle. Prove that, if it is true that one line is parallel to the Euler line of the triangle formed by the other three lines, then this is true for each of the lines.

8.37. Let ABC be an acute triangle with I and H be its incenter and orthocenter, respectively. Let B_1 and C_1 be the midpoints of $\overline{AC}$ and $\overline{AB}$ respectively. Ray B_1I intersects $\overline{AB}$ at $B_2 \neq B$. Ray C_1I intersects ray AC at C_2 with $C_2A > CA$. Let K be the intersection of $\overline{BC}$ and $\overline{B_2C_2}$. Prove that triangles BKB_2 and CKC_2 have the same area if and only if A, I, A_1 are collinear, where A_1 is the circumcenter of triangle BHC.

8.38. In triangle ABC, $AB \neq AC$. Let D be the midpoint of side BC, and let E be a point on median AD. Let F be the foot of perpendicular from E to side BC, and let P be a point on segment EF. Let M and N be the feet of perpendiculars from P to sides AB and AC, respectively. Prove that M, E, and N are collinear if and only if $\angle BAP = \angle PAC$.

8.39. [7pts] Let ABC be a triangle with $AB = AC$. Let D be the foot of perpendicular from C to side AB, and let M be the midpoint of segment CD. Let E be the foot of perpendicular from A to line BM, and let F be the foot of perpendicular from A to line CE. Prove that

$$AF \le \frac{AB}{3}.$$

8.40. Let ABC be an acute triangle, and let D be a point on side BC such that $\angle BAD = \angle CAD$. Points E and F are the foot of perpendiculars from D to sides AC and AB, respectively. Let H be the intersection of segments BE and CF. The circumcircle of triangle AFH meets line BE again at G. Prove that segments BG, GE, BF can be the sides of a right triangle.

8.41. Let $A_1A_2A_3A_4$ be a cyclic quadrilateral that also has an inscribed circle. Let B_1, B_2, B_3, B_4, respectively, be the points on sides $A_1A_2, A_2A_3, A_3A_4, A_4A_1$ at which the inscribed circle is tangent to the quadrilateral. Prove that

$$\left(\frac{A_1A_2}{B_1B_2}\right)^2 + \left(\frac{A_2A_3}{B_2B_3}\right)^2 + \left(\frac{A_3A_4}{B_3B_4}\right)^2 + \left(\frac{A_4A_1}{B_4B_1}\right)^2 \ge 8.$$

8.42. Triangle ABC is inscribed in circle ω. Line ℓ passes through A and is tangent to ω. Point D lies on ray BC and point P is in the plane such that D, A, P lie on ℓ in that order. Point U is on segment CD. Line PU meets segments AB and AC at R and S, respectively. Circle ω and line PU intersect at Q and T. Prove that if $QR = ST$, then $PQ = UT$.

8.43. Two circles ω_1 and ω_2 (in the plane) meet at A and B. Points P and Q are on ω_1 and ω_2, respectively, such that line PQ is tangent to both ω_1 and ω_2, and B is closer to line PQ than A. Triangle APQ is inscribed in circle ω_3. Point S is such that lines PS and QS are tangent to ω_3 at P and Q, respectively. Point H is the image of B reflecting across line PQ. Prove that A, H, S are collinear.

8.44. Convex quadrilateral $ABCD$ is inscribed in circle ω. Let P be the intersection of diagonals AC and BD. Lines AB and CD meet at Q. Let H be the orthocenter of triangle ADQ. Let M and N be the midpoints of diagonals AC and BD, respectively. Prove that $MN \perp PH$.

8.45. Let ABC be a triangle with I as its incenter. Circle ω is centered at I and lies inside triangle ABC. Point A_1 lies on ω such that $IA_1 \perp BC$. Points B_1 and C_1 are defined analogously. Prove that lines AA_1, BB_1, and CC_1 are concurrent.

8.46. Given integer a with $a > 1$, an integer m is *good* if $m = 200a^k + 4$ for some integer k. Prove that, for any integer n, there is a degree n polynomial with integer coefficients such that $p(0), p(1), \ldots, p(n)$ are distinct good integers.

8.47. Find all the ordered triples (a, m, n) of positive integers such that $a \geq 2$, $m \geq 2$, and $a^m + 1$ divides $a^n + 203$.

8.48. Determine if there exists a positive integer n such that n has exactly 2000 prime divisors, n is not divisible by a square of a prime number, and $2^n + 1$ is divisible by n.

8.49. [7pts] Determine if there exists a positive integer n such that n has exactly 2000 prime divisors, n is not divisible by a square of a prime number, and $2^n + 1$ is divisible by n.

8.50. A positive integer u if called *boring* if there are only finitely many triples of positive integers (n, a, b) such that $n! = u^a - u^b$. Determine all the boring integers.

8.51. Find all positive integers n, $n > 1$, such that all the divisors of n, not including 1, can be written in the form of $a^r + 1$, where a and r are positive integers with $r > 1$.

8.52. Determine all positive integers m satisfying the following property: There exists prime p_m such that $n^m - m$ is not divisible by p_m for all integers n.

8.53. Let m and n be positive integers. Find all pairs of positive integers (x, y) such that

$$(x+y)^m = x^n + y^n.$$

8.54. Let p be a prime, and let $a_1, a_2, \dots, a_{p+1}$ be distinct positive integers. Prove that there are indices i and j, $1 \le i < j \le p+1$, such that

$$\frac{\max\{a_i, a_j\}}{\gcd(a_i, a_j)} \ge p + 1.$$

8.55. For positive integer $n = p_1^{a_1} p_2^{a_2} \cdot p_m^{a_m}$, where $p_1, p_2, \dots, p_n$ are distinct primes and $a_1, a_2, \dots, a_m$ are positive integers, $d(n) = \max\{p_1^{a_1}, p_2^{a_2}, \dots, p_m^{a_m}\}$ is called the *greatest prime power divisor* of n. Let $n_1, n_2, \dots,$ and n_{2004} be distinct positive integers with $d(n_1) = d(n_2) = \cdots = d(n_{2004})$. Prove that there exists integers $a_1, a_2, \dots,$ and a_{2004} such that infinite arithmetic progressions $\{a_i, a_i + n_i, a_i + 2n_i, \dots\}$, $i = 1, 2, \dots, 2004$, are pairwise disjoint.

8.56. Let N be a positive integer such that for all integers $n > N$, the set $S_n = \{n, n+1, \dots, n+9\}$ contains at least one number that has at least three distinct prime divisors. Determine the minimum value of N.

8.57. Determine all the functions from the set of positive integers to the set of real numbers such that

(a) $f(n+1) \ge f(n)$ for all positive integers n; and

(b) $f(mn) = f(m)f(n)$ for all relatively prime positive integers m and n.

8.58. Let n be a positive integer. Prove that there is a positive integer k such that digit 7 appears for at least $\lfloor 2n/3 \rfloor$ times among the last n digits of the decimal representation of 2^k.

8.59. An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \dots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, 2, \dots, n$, the i^{th} row and i^{th} column together contain all elements of S. Determine all the values n for which silver matrices exist.

8.60. Let n be a positive integer. Determine the largest integer $f(n)$

such that

$$\binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}}$$

is divisible by $2^{f(n)}$.

9
Selected Problems from Chinese IMO Team Training in 2005

9.1. Given real numbers a, b, and c with $a + b + c = 1$, prove

$$(ab)^{\frac{5}{4}} + (bc)^{\frac{5}{4}} + (ca)^{\frac{5}{4}} < \frac{1}{4}.$$

9.2. For a given integer n greater than one, prove that there exists a proper subset A of $\{1, 2, \cdots, n\}$, such that

(a) $|A| \le 2\lfloor\sqrt{n}\rfloor + 1$;

(b) $\{|x - y| : x, y \in A, x \neq y\} = \{1, 2, \cdots, n - 1\}$.

9.3. Determine all functions f defined from the set of positive real numbers to itself such that

$$f(g(x)) = \frac{x}{xf(x) - 2} \quad \text{and} \quad g(f(x)) = \frac{x}{xg(x) - 2}.$$

for all positive real numbers x.

9.4. Given real numbers a, b, c, x, y, and z with

$$(a+b+c)(x+y+z) = 3 \quad \text{and} \quad (a^2+b^2+c^2)(x^2+y^2+z^2) = 4,$$

prove that $ax + by + cz \ge 0$.

9.5. Define the sequence $\{a_n\}_{n=1}^{\infty}$ as following: $a_1 = \frac{1}{2}$ and

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1},$$

for all positive integers n. Prove that

$$a_1 + a_2 + \cdots + a_n < 1.$$

for every positive integer n.

9.6. Find all integers n such that the polynomial $p(x) = x^5 - nx - n - 2$ can be written as the product of two non-constant polynomials with integral coefficients.

9.7. Let x, y, and z be real number greater than -1. Prove that

$$\frac{1 + x^2}{1 + y + z^2} + \frac{1 + y^2}{1 + z + x^2} + \frac{1 + z^2}{1 + x + y^2} \ge 2.$$

9.8. Given nonnegative real numbers a, b and c with $ab + bc + ca = \frac{1}{3}$, prove that

$$\frac{1}{a^2 - bc + 1} + \frac{1}{b^2 - ca + 1} + \frac{1}{c^2 - ab + 1} \le 3.$$

9.9. Let $(a_1, a_2, \ldots, a_6)$, $(b_1, b_2, \ldots, b_6)$, and $(c_1, c_2, \ldots, c_6)$ be permutations of $(1, 2, \ldots, 6)$. Determine the minimum value of $a_1b_1c_1 + a_2b_2c_2 + \cdots + a_6b_6c_6$.

9.10. Let $a_1, a_2, \ldots, a_n$ be given positive real numbers, and let

$$A_k = \left(\frac{a_1a_2 \cdots a_k + a_1a_3 \cdots a_{k+1} + a_{n-k+1} \cdots a_{n-1}a_n}{\binom{n}{k}} \right)^{\frac{1}{k}}.$$

(The sum in the numerator of A_k is taken all the k-element products of $a_1, a_2, \ldots, a_n$.) Prove that if $1 \le k_1 < k_2 \le n$, then $A_{k_1} \ge A_{k_2}$.

9.11. Given real numbers $a_1, a_2, \ldots, a_n$ such that $a_1 + a_2 + \cdots + a_n = 0$, prove that

$$\max\{a_1^2, a_2^2, \ldots, a_n^2\} \le \frac{n\left[(a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_{n-1} - a_n)^2\right]}{3}.$$

9.1. Let n be a given integer greater than one, and let $S = \{z_1, z_2, \ldots, z_n\}$ and $T = \{w_1, w_2, \ldots, w_n\}$ be two sets of complex numbers. Prove that

$$\sum_{k=1}^{n} \frac{\prod_{i=1}^{n}(z_k + w_i)}{\prod_{i=1, i\neq k}^{n}(z_k - z_i)} = \sum_{k=1}^{n} \frac{\prod_{i=1}^{n}(w_k + z_i)}{\prod_{i=1, i\neq k}^{n}(w_k - w_i)}.$$

9.13. Let $g(x)$ be a polynomial with integer coefficients. Suppose that $g(x)$ has no nonnegative real roots. Prove that there exists polynomial $h(x)$ with integer coefficients such that the coefficients of $g(x)h(x)$ are all positive.

9.14. Determine all positive real numbers a such that if x, y, z, and w are positive real numbers with $xyzw = a^4$, then

$$\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1+y}} + \frac{1}{\sqrt{1+z}} + \frac{1}{\sqrt{1+w}} \le \frac{4}{\sqrt{1+a}}.$$

9.15. Let n be an integer greater than 2, and let p be a prime. Prove that polynomial

$$p(x) = x^n + p^2x^{n-1} + \cdots + p^2x + p^2$$

cannot be written as the product of two polynomials with integer coefficients.

9.16. For $i = 1, 2, 3, 4$, angle θ_i is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Prove that

$$\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 + \sin^2\theta_4 \le 2(1 + \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4 + \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4)$$

if and only if there exists a real number x such that

$$\cos^2\theta_1\cos^2\theta_2(\sin\theta_1\sin\theta_2 - x)^2 \geq 0 \text{and} \cos^2\theta_3\cos^2\theta_4(\sin\theta_3\sin\theta_4 - x)^2 \geq 0.$$

9.17. Infinite sequence $\{a_k\}_{k=1}^{\infty}$ of real numbers is given with $a_1 = \frac{1}{2}$ and

$$a_{k+1} = -a_k + \frac{1}{2 - a_k}$$

for every positive integer k. Prove that

$$\left(\frac{n}{2(a_1 + a_2 + \cdots + a_n)} - 1\right)^n \leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n \left(\frac{1}{a_1} - 1\right)\left(\frac{1}{a_2} - 1\right)\cdots\left(\frac{1}{a_n} - 1\right).$$

9.18. For a real number k greater than 1, define set S_k be the set of all ordered triples (x, y, z) positive real numbers such that

$$xyz \leq 2 \quad \text{and} \quad \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} < k.$$

Find all real numbers k, such that if (x, y, z) belongs to the set S_k, then x, y, and z are the side lengths of a triangle.

9.19. Let S be a subset of the set $\{1, 2, \ldots, 2002\}$. Suppose that for any (not necessarily) elements a and b in S, their product ab is not in S. Determine the maximal number of elements of S.

9.20. Let n be an integer greater than 14. Let S be the set of n points on the plane such that the distance between any two distinct points in S is at least 1. Prove that there exists a subset T of S with at least $\left\lfloor \frac{n}{7} \right\rfloor$ elements such that the distance between any two distinct points in T is at least $\sqrt{3}$.

9.21. Determine the minimum value of n such that given any n points in the plane, no three of which are collinear, three of them are the vertices of a scalene triangle.

9.22. Two concentric circles are divided by n radii into $2n$ parts. Two parts are called neighbors (of each other) if they share either a common side or a common arc. Initially, there are $4n + 1$ frogs inside the parts. At each second, if there are three or more frogs inside one part, then three of the frogs in the part will jump to its neighbors, with one to each neighbor. Prove that in a finite amount of time, for any part either there are frogs in the part or there are frogs in each of its neighbors.

9.23. Five points $P_1, P_2, \ldots, P_5$, with no three collinear, lie within the boundary of a parallelogram with area four. Determine the minimum number of triangles $P_iP_jP_k$ $(1 \le i < j < k)$ with area no greater than one.

9.24. Set X has 56 elements. Determine the least positive integer n such that for any 15 subsets of X, if the union of any 7 of the subsets has at least n elements, then 3 of the subsets have nonempty intersection.

9.25. Each edge of a graph $\mathcal{G}$ is colored in one of the four colors (red, green, blue, and white) in such a way that . Prove that is possible to color the vertices of $\mathcal{G}$ in one of these four colors such that each edge is in a color other than that of its two end points.

9.26. For every positive integer n, $\mathbf{u}_n = [x_n, y_n]$ is vector with integer components. Prove that there exists a positive integer N such that for every positive integer n,

$$\mathbf{u}_n = k_{n,1}\mathbf{u}_1 + k_{n,2}\mathbf{u}_2 + \cdots + k_{n,N}\mathbf{u}_N$$

for some integers $k_{n,1}, k_{n,2}, \ldots, k_{n,N}$.

9.27. Polygon V is *inscribed* in polygon W if the vertices of V lie on the side of W. Prove that for every convex polygon with area one, there exists a non-degenerated centrally symmetric convex hexagon inscribed in W with its area no less than $\frac{2}{3}$.

9.28. There are n subway lines in a city satisfying the following conditions:

(1) there are at most three lines stop at each subway station;

(2) each line stops at at least two stations;

(3) for any pair of lines, say a and b, there exists the third line, say c, one can transfer from line a to line b via line c.

Prove that there are at least $\frac{5(n-5)}{6}$ stops in the subway system.

9.29. The perimeter of triangle ABC is equal to $3 + 2\sqrt{3}$. In the coordinate plane the, any triangle congruent to triangle ABC has at least one lattice point in its interior or on its sides. Prove triangle ABC is equilateral.

9.30. Let $ABCD$ be a convex quadrilateral. Point M lies in the interior of quadrilateral $ABCD$ such that $AM = MB, CM = MD$ and $\angle AMB = \angle CMD = 120°$. Prove that there exist a point N such that triangles BNC and DNA are equilateral.

9.31. Let ABC be a triangle. Circle ω intersects the side BC at D_1 and D_2, side CA at E_1 and E_2, and side AB at F_1 and F_2, such that $D_1D_2E_1E_2F_1F_2$ is a convex hexagon. Segments D_1E_1 and D_2F_2 meet at L, E_1F_1 and E_2D_2 meet at M, and F_1D_1 and F_2E_2 meet at N. Prove that lines AL, BM, and CN are concurrent.

9.32. In triangle ABC, M and N are the midpoints of sides AB and AC, respectively. Points P and Q lie on sides AB and AC, respectively, such that $\angle ACB$ and $\angle MCP$ share the same angle bisector, and that $\angle ABC$ and $\angle NBQ$ also share the same bisector. Suppose that $AP = AQ$. Determine if it necessarily that triangle ABC is isosceles.

9.33. Point P lies inside triangle ABC. Let D, E, and F be the foot of perpendiculars from P to sides BC, CA, and AB, respectively. Let M and N be the foot of perpendiculars from A to lines BP and CP. Prove that lines BC, ME, and NF are concurrent.

9.34. In acute triangle ABC, $BC > CA > AB$. Let I, O, and H denote the incenter, circumcenter, and orthocenter of triangle ABC, respectively. Point D and E lie on sides BC and CA, respectively, such that $AE = BD$ and $CD+CE = AB$. Segments BE and AD meet at K. Prove that $KH \parallel IO$ and compute the ratio KH/IO.

9.35. Gangsong Convex quadrilateral $ABCD$ inscribed in circle ω. Diagonals AC and BD meet at P. Circle ω_1 passes through points P and A, and circle ω_2 passes through points P and B. Circles ω_1 and ω_2 meet at P and Q. Circles ω and ω_1 meet at A and F, and ω and ω_2 meet at B and E. Prove that lines PQ, CE, and DF are either concurrent or parallel to each other.

9.36. In convex quadrilateral $ABCD$, points E, F, G and H lie on sides AB, BC, CD, and DA, respectively, such that $EFGH$ is a parallelogram. Points E_1, F_1, G_1, B_1 lie on sides AB, BC, CD, and DA, respectively, such that $AE_1 = BE$, $BF_1 = CF$, $CG_1 = DG$, and $DH_1 = AH$. Prove that $E_1F_1G_1H_1$ is a parallelogram that has the same area as $EFGH$.

9.37. Points E and F lie inside angle AOB such that $\angle AOE = \angle BOF$. Let E_1 and E_2 be the foot of perpendiculars from E to lines OA and OB, respectively. Let F_1 and F_2 be the foot of perpendiculars from F to lines OA and OB, respectively. Lines E_1E_2 and F_1F_2 meet at P. Prove that $OP \perp EF$.

9.38. In triangle ABC, $\angle B = 90°$. The incircle ω of triangle ABC touches sides BC, CA, and AB at D, E, and F, respectively. Segment AD meet ω again at P (other than D). Suppose that $PC \perp PF$. Prove that $PE \parallel BC$.

9.39. Acute triangle ABC is inscribed in circle ω. Lines ℓ_B and ℓ_C are tangent to ω at B and C, respectively, and they meet at point A_1. Segments AA_1 and BC intersect at A_2. Points A_b and A_c lie on side AC and AB, respectively, such that $A_2A_b \parallel AB$ and $A_2A_c \parallel AC$.

(a) Prove that points B, C, A_b, and A_c lie on a circle.

(b) Let P_a denote the circumcenter of quadrilateral BCA_bA_c, Likewise, we define P_b and P_c. Prove that lines AP_a, BP_b, and CP_c are concurrent.

9.40. For a given integer n greater than one, n distinct rational numbers are chosen from the interval $(0, 1)$. Prove that the sum of all the denominators of the chosen numbers (when written in the lowest form) is greater then $\frac{n^{\frac{2}{3}}}{3}$.

9.41. Let a and b be distinct positive rational numbers such that there exist infinitely many positive integers n for which $a^n - b^n$ is a positive integer. Prove that a and b are both positive integers.

9.42. Sequence $\{a_n\}_{n=0}^{\infty}$ is given with $a_0 = a_1 = a_2 = a_3 = 1$ and

$$a_{n-4}a_n = a_{n-3}a_{n-1} + a_{n-2}^2$$

for all integers $n \geq 4$, prove that all the terms of this sequence are integers.

9.43. Let n be a given positive integer. Find all integers a and b such that $x^2 + ax + b$ divides $x^{2n} + ax^n + b$.

9.44. Prove that there exists a positive integer m such that leftmost digits in the decimal representation 2004^m are 20042005200620072008.

9.45. For integers n and r greater than 1, integers $a_0, a_1, a_2, \ldots, a_n$, $x_0, x_1, x_2, \ldots, x_n$ are given such that

$$a_0 + a_1x_1^k + a_2x_2^k + \cdots a_nx_n^k = 0$$

for all $k = 1, 2, \ldots, r$. Prove that

$$a_0 + a_1 x_1^m + a_2 x_2^m + \cdots a_n x_n^m \equiv 0 \pmod{m}$$

for all integers $m = r+1, r+2, \ldots, 2r+1$.

9.46. Prove that for all positive integers k it is possible to find a partition $\{x_1, x_2, \ldots, x_{2^k}\}$ and $\{y_1, y_2, \ldots, y_{2^k}\}$ of the set

$$\{0, 1, 2, \ldots, 2^{k+1} - 1\}$$

with

$$x_1^m + x_2^m + \cdots + x_{2^k}^m = y_1^m + y_2^m + \cdots + y_{2^k}^m$$

for all integers m with $1 \le m \le k$.

9.47. For a positive integer n, the set S_n consists of all the 2^n-digit binary sequences. For two elements $a = a_1 a_2 \ldots a_{2^n}$ and $b = b_1 b_2 \ldots b_{2^n}$ in S, define their distance by

$$d(a, b) = \sum_{i=1}^{2^n} |a_i - b_i|.$$

A subset A of S_n is called *remote* if the pairwise distance for every pair of distinct elements in A is at least 2^{n-1}. Determine the maximum number of elements in a remote subset of S_n.

9.48. Determine all ordered pairs (m, n) of positive integers such that

$$\lfloor (m+n)\alpha \rfloor + \lfloor (m+n)\beta \rfloor \ge \lfloor m\alpha \rfloor + \lfloor m\beta \rfloor + \lfloor n(\alpha+\beta) \rfloor$$

for all pairs (α, β) of real numbers. (For real number x, $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.)

9.49. Let p be a prime, and let k be an integer greater than two. There are integers $a_1, a_2, \ldots, a_k$ such that p divides neither a_i $(1 \le i \le k)$ nor $a_i - a_j$ $(1 \le i < j \le k)$. Denote by S the set

$$\{n \mid 1 \le n \le p-1, (na_1)_p < (na_2)_p < \cdots < (na_k)_p\},$$

where $(b)_p$ represents the remainder when b is divided by p. Prove that S contains less than $\frac{2p}{k+1}$ elements.

9.50. Determine all ordered triples (a, m, n) of positive integers with $a > 1$ and $m < n$ such that a prime p divides $a^m - 1$ if and only if p divides $a^n - 1$.

9.51. Let n be a positive integer greater than two, and let $F_n = 2^{2^n} + 1$. Prove that F_n has a prime divisor greater than $2^{n+2}(n+1)$.

9.52. Let n be a given positive integer. Prove that

$$\sum_{k=1}^{n}\left(x\left\lfloor\frac{k}{x}\right\rfloor-(x+1)\left\lfloor\frac{k}{x+1}\right\rfloor\right)\leq n$$

for all positive real numbers x. (For real number x, $\lfloor x\rfloor$ denote the greatest integer less than or equal to x.)

9.53. Let α be a given positive real number. Determine all the functions f defined from the set of positive integers to the set of real numbers such that for every pair (k, m) of positive integers with $\alpha m \leq k < (\alpha+1)m$,

$$f(k+m)=f(k)+f(m).$$

9.54. Determine all polynomials with integer coefficients such that for every positive integer n, the number $f(n)$ divides 2^n-1.

9.55. Let n and k be given nonnegative integers, and let a be a given positive integer. Define function

$$f(x)=\left\lfloor\frac{n+k+x}{a}\right\rfloor-\left\lfloor\frac{n+x}{a}\right\rfloor-\left\lfloor\frac{k+x}{a}\right\rfloor+\left\lfloor\frac{x}{a}\right\rfloor.$$

Prove that for every nonnegative integer m,

$$f(0)+f(1)+\cdots+f(m)\geq 0.$$

(For real number x, $\lfloor x\rfloor$ denote the greatest integer less than or equal to x.)

9.56. Determine all quadruples (x, y, z, w) of nonnegative integers such that

$$2^x\cdot 3^y-5^z\cdot 7^w=1.$$

9.57. Given positive integers m, n, and k such that $mn=k^2+k+3$, prove that there exits a pair (x, y) of odd integers satisfying one of the equations

$$x^2+11y^2=4m \quad \text{and} \quad x^2+11y^2=4n.$$

9.58. Let a and b be integers greater than 1. Sequence $\{x_n\}_{n=1}^{\infty}$ is defined as $x_0=0$, $x_1=1$, and

$$x_{2n}=ax_{2n-1}-x_{2n-2}, \quad x_{2n+1}=bx_{2n}-x_{2n-1}$$

for every positive integer n. Prove that for every pair of positive integers n and m, the product $x_{n+m}x_{n+m-1}\cdots x_{n+1}$ is divisible by the product $x_m x_{m-1}\cdots x_1$.

9.59. Let $a_1, a_2, \ldots, a_k$ be distinct positive integers each of which is not divisible by a square of a prime. Prove that for any nonzero rational numbers $r_1, r_2, \ldots, r_k$,

$$r_1\sqrt{a_1} + r_2\sqrt{a_2} + \cdots + r_k\sqrt{a_k}$$

is not equal to 0.

9.60. Prove that for every positive integer k, there is a positive integer n such that at least half of the right most k digits of the decimal representation of 2^n equal to 9.

10
Romania Team Selections Exams in 2004

10.1. Let a_1, a_2, a_3, a_4 be the sides of an arbitrary quadrilateral of perimeter $2s$. Prove that

$$\sum_{i=1}^{4} \frac{1}{a_i + s} \leq \frac{2}{9} \sum_{1 \leq i \leq j \leq 4} \frac{1}{\sqrt{(s-a_i)(s-a_j)}}.$$

When does equality hold?

10.2. Let $\{R_i\}_{1 \leq i \leq n}$ be a family of disjoint closed rectangular surfaces with total area 4 such that their projections of the Ox axis is an interval. Prove that there exist a triangle with vertices in $\cup_{i=1}^{n} R_i$ which has an area of at least 1.

10.3. Find all one-to-one mappings $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers the following relation holds:

$$f(f(n)) \leq \frac{n + f(n)}{2}.$$

10.4. Let D be a closed disc in the complex plane. Prove that for all positive integers n, and for all complex numbers $z_1, z_2, \ldots, z_n \in D$ there exists a $z \in D$ such that $z^n = z_1 \cdot z_2 \cdots z_n$.

10.5. A disk is partitioned in $2n$ equal sectors. Half of the sectors are colored in blue, and the other half in red. We number the red sectors with numbers from 1 to n in counter-clockwise direction, and then we number the blue sectors with numbers from 1 to n in clockwise direction. Prove that one can find a half-disk which contains sectors numbers with all the numbers from 1 to n.

10.6. Let a, b be two positive integers, such that $ab \neq 1$. Find all the integer values that $f(a, b)$ can take, where

$$f(a, b) = \frac{a^2 + ab + b^2}{ab - 1}.$$

10.7. Let a, b, c be 3 integers, b odd, and define the sequence $\{x_n\}_{n \geq 0}$ by $x_0 = 4, x_1 = 0, x_2 = 2c, x_3 = 3b$ and for all positive integers n we have

$$x_{n+3} = ax_{n-1} + bx_n + cx_{n+1}.$$

Prove that for all positive integers m, and for all primes p the number x_{p^m} is divisible by p.

10.8. Let Γ be a circle, and let $ABCD$ be a square lying inside the circle Γ. Let C_a be a circle tangent interiorly to Γ, and also tangent to the sides AB and AD of the square, and also lying inside the opposite angle of $\angle BAD$. Let A' be the tangency point of the two circles. Define similarly the circles C_b, C_c, C_d and the points B', C', D' respectively.

Prove that the lines AA', BB', CC', and DD' are concurrent.

10.9. Let $n \geq 2$ be a positive integer and X a set with n elements. Let $A_1, A_2, \ldots, A_{101}$ be subsets of X such that the union of any 50 of them has more than $50n/51$ elements.

Prove that among these 101 subsets there exist 3 subsets such that any two have a common element.

10.10. Prove that for all positive integers n, m with m odd, the following number is an integer

$$\frac{1}{3^m n} \sum_{k=0}^{m} \binom{3m}{3k} (3n-1)^k.$$

10.11. Let I be the incenter of the non-isosceles triangle ABC and let A', B', C' be the tangency points of the incircle with the sides BC, CA, AB respectively. The lines AA' and BB' intersect in P, the lines AC and $A'C'$ in M and the lines $B'C'$ and BC intersect in N. Prove that the lines IP and MN are perpendicular.

10.12. Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that for any non-empty subset $S \subset \{1, 2, 3, \ldots, n\}$ we have

$$\left(\sum_{i \in S} a_i \right)^2 \leq \sum_{1 \leq i \leq j \leq n} (a_i + \cdots + a_j)^2.$$

11
Selected Problems from Bulgaria Math Olympiad in 2004 and 2005

11.1. Let I be the incenter of triangle ABC, and let A_1, B_1, and C_1 be arbitrary points lying on segments AI, BI, and CI, respectively. The perpendicular bisectors of segments AA_1, BB_1, and CC_1 form triangles $A_2B_2C_2$. Prove that the circumcenter of triangle $A_2B_2C_2$ coincides with the circumcenter of triangle ABC if and only if I is the orthocenter of triangle $A_1B_1C_1$.

11.2. For any positive integer n, the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is written in the lowest form $\frac{p_n}{q_n}$; that is p_n and q_n are relative prime positive integers. Find all n such that p_n is divisible by 3.

11.3. A group consists of n tourists. Among every three of them there are at least two that are not familiar. For any partition of the group into two busses, there are at least two familiar tourists in one bus. Prove that there is a tourist who is familiar with at most two fifth of all the tourists.

11.4. Consider all binary sequences (sequences consisting of 0's and 1's). In such a sequence the following four types of operation are allowed: (a) $010 \to 1$, (b) $1 \to 010$, (c) $110 \to 0$, and (d) $0 \to 110$. Determine if it is possible to obtain sequence $1\underbrace{00\ldots0}_{2003}$ from the sequence $\underbrace{0\ldots0}_{2003}1$.

11.5. Let a, b, c, and d be positive integers satisfy the following properties:

(a) there are exactly 2004 pairs of real numbers (x, y) with $0 \le x, y \le 1$ such that both $ax + by$ and $cx + dy$ are integers

(b) $\gcd(a, c) = 6$.

Find $\gcd(b, d)$.

11.6. Let p be a prime number, and let $0 \le a_1 < a_2 < \cdots < a_m < p$ and $0 \le b_1 < b_2 < \cdots < b_n < p$ be arbitrary integers. Denote by k the number of different remainders of $a_i + b_j$, $1 \le i \le m$ and $1 \le j \le n$, modulo p. Prove that

(i) if $m + n > p$, then $k = p$;

(ii) if $m + n \le p$, then $k \ge m + n - 1$,

11.7. Let n be a positive integer. Determine all positive integers m for which there exists an polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with integer coefficients such that $a_n \neq 0$, $\gcd(a_0, a_1, \ldots, a_n) = 1$, and $f(k)$ is divisible by m for all positive integers k.

11.8. Find all primes p greater than 2 such that for any prime q with $q < p$, the number

$$p - \left\lfloor \frac{p}{q} \right\rfloor \cdot q$$

is not divisible by the square of any prime.

11.9. The vertices of a triangle lie in the interior or on the boundary of a square with side length one. Determine the maximal possible value of the inradius of the triangle.

11.10. Determine all unordered triples (x, y, z) of positive integers for which the number

$$\sqrt{\frac{2005}{x+y}} + \sqrt{\frac{2005}{y+z}} + \sqrt{\frac{2005}{z+x}}$$

is an integer.

11.11. Circle ω_1 and ω_2 are externally tangent to each other at T. Let X be a point on circle ω_1. Line ℓ_1 is tangent to circle ω_1 and X, and line ℓ intersects circle ω_2 at A and B. Line XT meet circle ω at S. Point C lies on arc $\widehat{TS}$ (of circle ω_2, not containing points A and B). Point Y lies on circle ω_1 and line YC is tangent to circle ω_1. let I be the intersection of lines XY and SC. Prove that

(a) points C, T, Y, I lie on a circle; and

(b) I is an excenter of triangle ABC.

11.12. Let S denote the set of rational numbers in the interval $(0, 1)$. Determine, with proof, if there exists a subset T of S such that every element in S can be uniquely written a the sum of finitely many distinct elements in T.

11.13. Let ABC be a triangle with $AB \neq AC$, and let $A_1B_1C_1$ be the image of triangle ABC through a rotation $\mathbf{R}$ centered at C. Let M, E, and F be the midpoints of the segments BA_1, AC, and BC_1, respectively. Given that $EM = FM$, compute $\angle EMF$.

11.14. For positive integers t, a, and b, Lucy and Windy play the (t, a, b)-*game* defined by the following rules. Initially, the number t is written on a blackboard. On her turn, a player erases the number on the board and writes either the number $t - a$ or $t - b$ on the board. Lucy goes first and then the players alternate. The player who first reaches a negative losses the game. Prove that there exist infinitely many values of t in which Lucy has a winning strategy for all pairs (a, b) with $a + b = 2005$.

11.15. Let a, b, and c be positive integers such that the product ab divides the product $c(c^2 - c + 1)$ and the sum $a + b$ is divisible the number $c^2 + 1$. Prove that the sets $\{a, b\}$ and $\{c, c^2 - c + 1\}$ coincide.

12
Balkan Mathematics Olympiad 2003 and 2005

12.1. Is there a set B of 4004 distinct positive integers, such that for each subset A of B with 2003 elements, the sum of the integers in A is not divisible by 2003?

12.2. Let ABC be a triangle with $AB \neq AC$, and let D be the point where the tangent from A to the circumcircle of triangle ABC meets BC. Point E lies on the perpendicular bisector of segment AB such that $BE \perp BC$. Point F lies on the perpendicular bisector of segment AC such that $CF \perp BC$. Prove that points D, E, and F are collinear.

12.3. Find all functions $f : Q \to R$ for which the following are true:

(a) $f(x+y) - yf(x) - xf(y) = f(x) \cdot f(y) - x - y + xy$ for all rational numbers x and y;

(b) $f(x) = 2f(x+1) + 2 + x$ for all rational numbers x;

(c) $f(1) + 1 > 0$.

12.4. Let m and n be relatively prime odd integers. A rectangle $ABCD$ with $AB = m$ and $AD = n$ is partitioned into mn unit squares. Starting from $A_1 = A$ denote by $A_1, A_2, \ldots A_k = C$ the consequent intersecting points of the diagonal AC with the sides of the unit squares. Prove that

$$\sum_{j=1}^{k-1} (-1)^{j+1} A_j A_{j+1} = \frac{\sqrt{m^2+n^2}}{mn}.$$

12.5. The incircle of triangle ABC is tangent to sides AB and AC at D and E, respectively. Let X and Y be the intersections of line DE with the bisectors of $\angle ACB$ and $\angle ABC$, respectively. Let Z be the midpoint of side BC. Prove that triangle XYZ is equilateral if and only if $\angle A = 60^\circ$.

12.6. Find all primes p such that $p^2 - p + 1$ is a perfect cube.

12.7. Let a, b, and c be positive real numbers. Prove that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \leq a + b + c + \frac{4(a-b)^2}{a+b+c}.$$

12.8. Let n is an integer greater than 1. Let S be a subset of $\{1, 2, \ldots, n\}$ such that S neither contains two elements one of which divides the other nor contains two elements which are relatively prime. What is the maximal number of elements of such a set S?