



Art of Problem Solving

WOOT 2010–11

Pigeonhole Principle

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The **Pigeonhole Principle** states that if $kn + 1$ objects (say, pigeons) are distributed among n holes, then one of the holes must contain at least $k + 1$ objects. Here are a couple of the themes involved in pigeonhole problems:

- The Pigeonhole Principle is most effective in problems that ask you to prove that something exists. If a problem contains numbers of the form n and $kn + 1$, then it is almost certainly a Pigeonhole Principle problem.
- When applying the Pigeonhole Principle, an important step is to identify what are the pigeons and what are the holes. Sometimes it is obvious, but if it is not, then the problem may require some thought before it becomes apparent. You cannot apply the Pigeonhole Principle unless you know exactly what the pigeons and holes are.
- The Pigeonhole Principle can often be used to find solutions to combinatorial problems with a geometric component. For instance, questions which ask you to prove that there can't be too many points in a region with some property often boil down to treating the points as your pigeons and subdividing your region into some number of pigeonholes.

Problem. Around a table are seated representatives of n countries ($n \geq 2$), such that if two representatives are from the same country, then their neighbors on the right are from two different countries. Prove that there are at most n^2 representatives.

Solution. That ‘not more than n^2 representatives from n countries’ points to Pigeonhole. If we had $n^2 + 1$ or more representatives, there would be at least $n + 1$ from the same country for some country. The $n + 1$ representatives to the right of these representatives from the same country must themselves all be from different countries. However, there are only n countries, so we have a contradiction.

Problem. Let S be a set with n integers. Show that there is a subset T of S such that the sum of the elements of T is divisible by n .

Scratchwork. The first thing that we notice is that there are 2^n subsets of S , but only n residue classes modulo n . Therefore we have a large amount of latitude in this problem. (We expect we need no more than $n + 1$ pigeons.) Let's consider how we will be able to use a pigeonhole argument in this proof. If we can find two subsets, T' and T'' of S such that the sum of each is in the same residue class, then the “difference” of the sums will be divisible by n . This is a huge clue for how to build our subsets, since the difference is only a sum of elements of S if T' is a subset of T'' (or vice-versa). With this in mind, we begin.

Solution. Let $S = \{a_1, a_2, \dots, a_n\}$, and let

$$\begin{aligned}s_1 &= a_1, \\s_2 &= a_1 + a_2, \\&\dots, \\s_n &= a_1 + a_2 + \dots + a_n.\end{aligned}$$





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For $1 \leq k \leq n$, let r_k denote the remainder when s_k is divided by n .

If any of the r_k are equal to 0, then we are done. If not, then each r_k must be an element of the set $\{1, 2, \dots, n-1\}$. Since we have n remainders, by the Pigeonhole Principle, two of the r_k must be equal, say r_i and r_j , where $i < j$. In other words, s_i and s_j leave the same remainder when divided by n , so their difference $s_j - s_i = a_{i+1} + a_{i+2} + \dots + a_j$ is divisible by n , which gives us the desired subset.

Comments. The partial sums s_1, s_2, \dots, s_n were useful here because the difference of any two partial sums is always the sum of a subset of elements. The same technique can be applied to other problems that involve sums of elements of subsets. Furthermore, notice that we only had n pigeons and n holes. We were able to rescue ourselves from this problem by eliminating the remainder 0 hole. You'll find that in some pigeonhole problems, there are n holes and nk pigeons. Here there are two cases: either every hole has k pigeons or there is some hole with $k+1$ pigeons, and you need to treat both cases separately.

Problem. Let S be a set of 10 distinct positive real numbers. Show that there exist $x, y \in S$ such that

$$0 < x - y < \frac{(1+x)(1+y)}{9}.$$

(USAMTS, Year 21)

Solution. The first hint we get that this problem might succumb to the Pigeonhole Principle is the word “exists.” It looks as if we want to subdivide the positive real numbers into 9 intervals such that if two numbers are in any interval, this inequality holds. There are actually many manipulations that will get us to the answer, but possibly the prettiest is to realize that we can achieve the number $\frac{1}{9}$ by dividing through:

$$0 < \frac{x - y}{(1+x)(1+y)} < \frac{1}{9}.$$

The reason this is so appealing is that it seems likely that there will be some algebraic manipulation which turns the problem into subdividing some interval into ninths. What we've done here is force the algebraic quantity to be bounded on both sides by constants. This must be helpful.

Next we invoke partial fractions decomposition on the algebraic quantity. We find

$$0 < \frac{1}{1+y} - \frac{1}{1+x} < \frac{1}{9}.$$

Now we're ready to write our proof.

If S is any set of 10 distinct positive real numbers, then the set

$$T = \left\{ \frac{1}{1+x} : x \in S \right\}$$

is a set of 10 real numbers between 0 and 1 (exclusive). Therefore, among the nine intervals $(\frac{n}{9}, \frac{n+1}{9}]$, for $0 \leq n \leq 8$, one interval must contain two elements of T . Specifically, there exist x and $y \in S$, and some integer n such that

$$\frac{n}{9} < \frac{1}{1+x} \leq \frac{n+1}{9}$$





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and

$$\frac{n}{9} < \frac{1}{1+y} \leq \frac{n+1}{9}.$$

Without loss of generality, we can assume that $x > y$, so

$$\begin{aligned} x - y &= \frac{x - y}{(1+x)(1+y)} \cdot (1+x)(1+y) \\ &= \left(\frac{1}{1+y} - \frac{1}{1+x} \right) (1+x)(1+y) \\ &< \frac{1}{9}(1+x)(1+y), \end{aligned}$$

and

$$0 < x - y < \frac{(1+x)(1+y)}{9}.$$

Problem. Let S be a set of 55 distinct positive integers, from 1 to 100 inclusive. Show that there must be two elements of S that differ by 9, two that differ by 10, two that differ by 12, and two that differ by 13, but not necessarily two that differ by 11. (*More Mathematical Morsels*, Ross Honsberger)

Solution. Let the 55 integers in S be $x_1 < x_2 < \dots < x_{55}$. First, we try to find two numbers in S that differ by 9. Let T be set of numbers we obtain when we add 9 to each number in S , so the 55 integers in T are

$$x_1 + 9 < x_2 + 9 < \dots < x_{55} + 9.$$

Furthermore, every number in T lies between 10 and 109. Hence, the 110 numbers in the sets S and T combined all lie between 1 and 109. Then by the Pigeonhole Principle, two of these 110 numbers are equal. In other words, the sets S and T have an element in common. Since this element in T was obtained by adding 9 to some element in S , we conclude that there are two elements in S that differ by 9.

Unfortunately, the same approach does not work for the case of 10, 12, or 13. We must find another approach.

To start, we want to show that there are two numbers in S that differ by 10. For example, the numbers 1 and 11 differ by 10, so we are done if both 1 and 11 are in S . Similarly, we are done if both 2 and 12 are in S , and so on. This gives us the idea of partitioning all the numbers from 1 to 100 into 50 pairs, so that the two numbers in each pair differ by 10:

$$\begin{array}{ccccc} \{1, 11\}, & \{21, 31\}, & \{41, 51\}, & \{61, 71\}, & \{81, 91\}, \\ \{2, 12\}, & \{22, 32\}, & \{42, 52\}, & \{62, 72\}, & \{82, 92\}, \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \{10, 20\}, & \{30, 40\}, & \{50, 60\}, & \{70, 80\}, & \{90, 100\}. \end{array}$$

Since $55 > 50$, by the Pigeonhole Principle, two of the 55 numbers in S belong to the same pair, which provides two numbers that differ by 10.





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We were very fortunate in this case because we were able to partition the entire set for 1 to 100 into pairs that differ by 10. Notice that this was only possible because 100 is a multiple of $2 \cdot 10$. We are going to need to find yet another method for 12 and 13. Notice, however, that 100 is close to a multiple of $2 \cdot 12$ and also to a multiple of $2 \cdot 13$. We might be able to modify this argument slightly in those cases.

For the case of finding two numbers that differ by 12, we can construct a similar argument by partitioning the numbers from 1 to 100 into the following sets:

$$\begin{array}{cccccc} \{1, 13\}, & \{25, 37\}, & \{49, 61\}, & \{73, 85\}, & \{97, 98, 99, 100\}. \\ \{2, 14\}, & \{26, 38\}, & \{50, 62\}, & \{74, 86\}, & \\ \vdots & \vdots & \vdots & \vdots & \\ \{12, 24\}, & \{36, 48\}, & \{60, 72\}, & \{84, 96\}, & \end{array}$$

This gives us 48 pairs, and a set containing 4 elements. At least $55 - 4 = 51$ of the 55 numbers must be distributed among the 48 pairs, and $51 > 48$, so again by the Pigeonhole Principle, two of the 51 numbers belong to the same pair.

For the case of finding two numbers that differ by 13, we partition the numbers from 1 to 100 into the following sets:

$$\begin{array}{cccc} \{1, 14\}, & \{27, 40\}, & \{53, 66\}, & \{79, 92\}, \\ \{2, 15\}, & \{28, 41\}, & \{54, 67\}, & \{80, 93\}, \\ \vdots & \vdots & \vdots & \vdots \\ \{8, 21\}, & \{34, 47\}, & \{60, 73\}, & \{86, 99\}, \\ \{9, 22\}, & \{35, 48\}, & \{61, 74\}, & \{87, 100\}, \\ \{10, 23\}, & \{36, 49\}, & \{62, 75\}, & \{88, 89, 90, 91\}. \\ \vdots & \vdots & \vdots & \\ \{13, 26\}, & \{39, 52\}, & \{65, 78\}, & \end{array}$$

As in the previous case, this gives us 48 pairs, and a set containing 4 elements. At least $55 - 4 = 51$ of the 55 numbers must be distributed among the 48 pairs, and $51 > 48$, so again by the Pigeonhole Principle, two of the 51 numbers belong to the same pair.

Finally, for $n = 11$, among the 55 numbers

$$1, 2, \dots, 11, 23, 24, \dots, 33, 45, 46, \dots, 55, 67, 68, \dots, 77, 89, 90, \dots, 99,$$

no two differ by 11.

Problem. Prove that among any ten points inside a circle of diameter 5, there exist two points whose distance is less than 2. (Japan, 1997)

Solution. We would like to use the Pigeonhole Principle, so with all caution thrown aside, we divide the circle into nine equal sectors. Then there are two points in some sector, and the maximum distance between any two points in the same sector is $\dots 2.5$. But 2.5 is greater than 2, so we need to do better than that. We take a look at where the sectors approach went wrong.



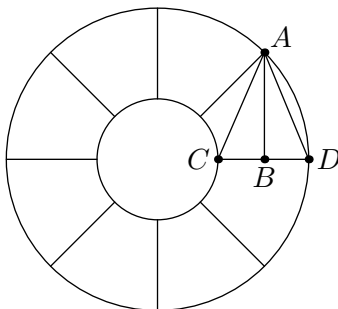


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The main problem with the sectors approach is that the center is 2.5 units away from the circumference. We need to separate the center of the circle from points on the circumference. So, we draw a circle with radius 1 that is concentric with the original circle. We then divide the remaining annulus into 8 congruent regions.



To complete the proof, we need to show that the distance between any two points in the same region is at most 2. We already know that the maximum distance for the inner circle is 2, so we only need to find the maximum distance between two points in one of the outer regions. We only need to consider a chord of the outer circle (\overline{AD}) and a diagonal of the region (\overline{AC}). The length of the chord \overline{AD} is

$$2 \cdot 2.5 \sin \frac{\pi}{8} = 5 \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \frac{5}{2} \sqrt{2 - \sqrt{2}} < 2.$$

For the length of the diagonal \overline{AC} , we use the Pythagorean theorem on triangle ABC . Since $AB = 2.5/\sqrt{2}$ and $BC = 2.5/\sqrt{2} - 1$, we have

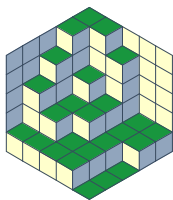
$$\begin{aligned} AC &= \sqrt{AB^2 + BC^2} \\ &= \sqrt{\frac{6.25}{2} + \frac{6.25}{2} - 2 \cdot \frac{2.5}{\sqrt{2}} + 1} \\ &= \sqrt{7.25 - \frac{5}{\sqrt{2}}} \\ &< 2. \end{aligned}$$

Comments. When an approach is close to working, but fails, think about why it fails. Sometimes that will point you to a modification that allows a solution.

Exercises

- Five points are chosen in a 2×2 square. Show that two points are within a distance of $\sqrt{2}$ of each other.





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2. (a) Prove that for any set of five integers, there are three integers whose sum is divisible by 3.
(b) Prove that for any set of 17 integers, there are five integers whose sum is divisible by 5.
3. A tennis star, preparing for a tournament, decides to play at least one match a day, but no more than 20 matches altogether, over a period of two weeks. Show that during some set of consecutive days the star must play exactly 7 matches.
4. Every point in the plane is colored either red, green, or blue. Prove that there exists a rectangle in the plane such that all four of its vertices are the same color. (USAMTS, Year 18)
5. Show that there exists a Fibonacci number F_n , $n \geq 1$, that is divisible by 10^4 .
6. Let n be a positive integer, and let S be a subset of the set $\{1, 2, \dots, 2n\}$ with $n + 1$ elements.
(a) Show that there exist two elements of S that are relatively prime.
(b) Show that there exist two elements of S , one of which divides the other.
7. Mathew has an unusual habit: Much to Amanda's consternation, Mathew likes to shuffle around the items in his refrigerator every day. However, at least he does it in a consistent way.
For example, if he moves whatever is in the egg rack to the vegetable crisper on one day, then he does the same thing every day. (In other words, he applies the same permutation every day.) Prove that eventually, all the food will be returned to its original location.
8. Find the greatest positive integer n for which there exist n nonnegative integers x_1, x_2, \dots, x_n , not all zero, such that for any sequence $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ of elements of $\{-1, 0, 1\}$, not all zero, n^3 does not divide $\epsilon_1 x_1 + \epsilon_2 x_2 + \dots + \epsilon_n x_n$. (Romania, 1996)

FOR NEXT TIME - ADD IBEROAMERICAN AND GRID PROBLEM

