SOME PROBLEMS IN PROBABILITY

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A. BALLS AND BOXES

You have two identical empty boxes. You also have 50 white balls and 50 black balls, which you must put into the two boxes in any distribution you like.

You will then be blindfolded and asked to do the following:
(*i*) pick a box at random, and then
(*ii*) pick a ball at random from the box you have chosen.
If you pick a black ball, you will win $100.

How should you distribute the balls between the two boxes in order to maximize your chances of picking a black ball? In particular, does it make a difference how you distribute them?

B. SOME PROBABILITY PROBLEMS BY LEWIS CARROLL

Lewis Carroll, the author of *Alice in Wonderland* and *Through the Looking Glass*, compiled 72 original mathematical puzzles of varying degrees of difficulty, together with his solutions, in a collection called *Pillow Problems* (1893; reprinted by Dover, 1958). There were many problems of geometry, equations, and number theory, in addition to several problems of probabilty.

Probability theory had been around for many years before Carroll, but it's "modern" history only began in the early 1900's with its axiomatization by A.N. Kolmogorov. Carroll's view of probability was somewhat naive by today's standards. Some of his probability problems were not well-defined, and other solutions exhibit logical fallacies. In at least one case, however, he gave a clever false solution in full knowledge that he was pulling the reader's leg.

C. COIN TOSSING PATTERNS

$p = \frac{1}{2} + \frac{1}{4}p$

$p = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3}$ ✓

Dana and Pat watch while I flip a fair coin repeatedly. Dana is waiting for the pattern HHT, and Pat is waiting for the pattern HTT. On any given sequence of three tosses, each pattern clearly has a probability 1/8 of occurring. Indeed, in 8 million consecutive tosses, one would expect that HHT would occur about 1 million times, as would HTT.

However, we are waiting for the *first* occurrence of HHT or HTT. Dana wins if HHT occurs first, and Pat wins if HTT occurs first. One of the players has an advantage. Which one? What is each person's probability of winning?

(You may assume that there is a probability of 0 that neither HHT nor HTT ever occurs. That is, with probability 1 the game will eventually end.)

D. A PROBLEM FROM THE CRUX OLYMPIAD CORNER (No. 143)

A set of $n(n+1)/2$ distinct numbers is arranged at random in n rows so that for every $1 \le i \le n$, there are exactly i numbers in the i^{th} row. Let M_i be the largest number in the i^{th} row. Find the probability that $M_1 < M_2 < \cdots < M_n$.

E. BERTRAND'S PARADOX

What is the probability that a chord drawn at random on a circle will be longer than the side of an inscribed equilateral triangle? It all depends on how you interpret the phrase "at random". Picking a point at random is pretty unambiguous, but there are several plausible interpretations of what it means to pick a chord at random. Thus one can get at least three different answers: 1/4, 1/3, and 1/2. (At least two of these require no calculus.)

PILLOW-PROBLEMS

THOUGHT OUT DURING WAKEFUL HOURS

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5. (19, 31)

A bag contains one counter, known to be either white or black. A white counter is put in, the bag shaken, and a counter drawn out, which proves to be white. What is now the chance of drawing a white counter? [8/9/87

16. (20, 40)

$\frac{2}{3}\left(\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}\right)$
vs
$\frac{2}{3}$ $\frac{2}{4}$

There are two bags, one containing a counter, known to be either white or black; the other containing 1 white and 2 black. A white is put into the first, the bag shaken, and a counter drawn out, which proves to be white. Which course will now give the best chance of drawing a white—to draw from one of the two bags without knowing which it is, or to empty one bag into the other and then draw? [10/87

41. (23, 62)

$+\frac{1}{3}\left(\frac{1}{2}\cdot\frac{1}{3}\right)$
$\frac{4}{9}+\frac{1}{18}=\frac{1}{2}$
$\left(\frac{2}{4}\right)+\frac{1}{3}\left(\frac{1}{4}\right)=\frac{5}{12}$

My friend brings me a bag containing four counters, each of which is either black or white. He bids me draw two, both of which prove to be white. He then says "I meant to tell you, before you began, that there was at least *one* white counter in the bag. However, you know it now, without my telling you. Draw again."

(1) What is now my chance of drawing white?

(2) What would it have been, if he had not spoken? [9/87

45. (23, 67)

If an infinite number of rods be broken: find the chance that one at least is broken in the middle. [5/84

58. (25, 83)

Three Points are taken at random on an infinite Plane. Find the chance of their being the vertices of an obtuse-angled Triangle. [20/1/84

72. (27, 109)

A bag contains 2 counters, as to which nothing is known except that each is either black or white. Ascertain their colours without taking them out of the bag. [8/9/87

Carroll's Solutions to Pillow-Problems

5. (2, 19)

At first sight, it would appear that, as the state of the bag, *after* the operation, is necessarily identical with its state *before* it, the chance is just what it then was, viz. $\frac{1}{2}$. This, however, is an error.

The chances, *before* the addition, that the bag contains (*a*) 1 white (*b*) 1 black, are (*a*) $\frac{1}{2}$ (*b*) $\frac{1}{2}$. Hence the chances, *after* the addition, that it contains (*a*) 2 white (*b*) 1 white, 1 black, are the same, viz. (*a*) $\frac{1}{2}$ (*b*) $\frac{1}{2}$. Now the probabilities, which these 2 states give to the observed event, of drawing a white counter, are (*a*) certainty (*b*) $\frac{1}{2}$. Hence the chances, after drawing the white counter, that the bag, before drawing, contained (*a*) 2 white, (*b*) 1 white, 1 black, are proportional to (*a*) $\frac{1}{2} \cdot 1$ (*b*) $\frac{1}{2} \cdot \frac{1}{2}$; i.e. (*a*) $\frac{1}{2}$ (*b*) $\frac{1}{4}$; i.e. (*a*) 2 (*b*) 1. Hence the chances are (*a*) $\frac{2}{3}$ (*b*) $\frac{1}{3}$. Hence, after the removal of a white counter, the chances, that the bag now contains (*a*) 1 white (*b*) 1 black, are for (*a*) $\frac{2}{3}$ and for (*b*) $\frac{1}{3}$.

Thus the chance, of now drawing a white counter, is $\frac{2}{3}$.

Q. E. F.

16. (4, 20)

The 'a priori' chances of possible states of first bag are '*W*, $\frac{1}{2}$; *B*, $\frac{1}{2}$'. Hence chances, after putting *W* in, are '*WW*, $\frac{1}{2}$; *WB*, $\frac{1}{2}$'. The chances, which these give to the 'observed event', are 1, $\frac{1}{2}$. Hence chances of possible states '*W*, *B*', after the event, are proportional to 1, $\frac{1}{2}$; i.e. to 2, 1; i.e. their actual values are $\frac{2}{3}$, $\frac{1}{3}$.

Now, in first course, chance of drawing *W* is $\frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3}$; i.e. $\frac{1}{2}$.

And, in second course, chances of possible states '*WWBB*, *WBBB*' are $\frac{2}{3}$, $\frac{1}{3}$: hence chance of drawing *W* is $\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{4}$; i.e. $\frac{5}{12}$.

Hence *first* course gives best chance.

Q. E. F.

41. (9, 23)

(1) As there was certainly at least one W in the bag at first, the 'a priori' chances for the various states of the bag, '$WWWW$, $WWWB$, $WWBB$, $WBBB$,' were '$\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{8}$'.

These would have given, to the observed event, the chances '1, $\frac{1}{2}$, $\frac{1}{6}$, 0'.

Hence the chances, after the event, for the various states, are proportional to '$\frac{1}{8} \cdot 1$, $\frac{3}{8} \cdot \frac{1}{2}$, $\frac{3}{8} \cdot \frac{1}{6}$'; i.e. to '$\frac{1}{8}$, $\frac{3}{16}$, $\frac{1}{16}$'; i.e. to '2, 3, 1'. Hence their actual values are '$\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{6}$'.

Hence the chance, of now drawing W, is '$\frac{1}{3} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2}$'; i.e. it is $\frac{7}{12}$.

Q. E. F.

(2) If he had not spoken, the 'a priori' chances for the states '$WWWW$, $WWWB$, $WWBB$, $WBBB$, $BBBB$', would have been $\dfrac{\text{'}1, 4, 6, 4, 1\text{'}}{16}$.

These would have given, to the observed event, the chances '1, $\frac{1}{2}$, $\frac{1}{6}$, 0, 0'.

Hence the chances, after the event, for the various states, are proportional to '$\frac{1}{16} \cdot 1$, $\frac{1}{4} \cdot \frac{1}{2}$, $\frac{1}{6} \cdot \frac{3}{8}$'; i.e. to '1, 2, 1'. Hence their actual values are '$\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$'.

Hence the chance, of now drawing W, is '$\frac{1}{4} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2}$'; i.e. it is $\frac{1}{2}$.

Q. E. F.

45. (10, 23)

Divide each rod into $(n+1)$ parts, where n is assumed to be odd, and the n points of division are assumed to be the only points where the rod will break, and to be equally frangible.

The chance of one failure is $\dfrac{n-1}{n}$;

$\therefore$ " " n failures is $\left(\dfrac{n-1}{n}\right)^n$

$$=\left(1-\frac{1}{n}\right)^n.$$

Now, if $m=\dfrac{1}{n}$; then, when $n=\dfrac{1}{0}$, $m=0$;

$\therefore$ the chance that no rod is broke in the middle $=(1-m)^{\frac{1}{m}}$, when $m=0$;

i.e. it approaches the limit $(1-0)^{\frac{1}{0}}$.

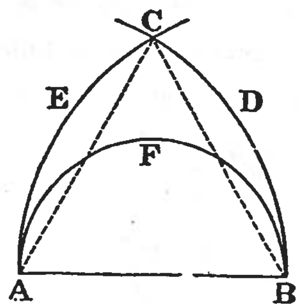
And Ans. $=1-(1-0)^{\frac{1}{0}}$.

Now $(1+0)^{\frac{1}{0}}=e$. Hence if, in the series for e, we call the sum of the odd terms 'a', and of the even terms 'b'; then $e=a+b$; and $(1-0)^{\frac{1}{0}}=a-b=2a-e$.

Q. E. F.

58. (14, 25)

It may be assumed that the 3 Points form a Triangle, the chance of their lying in a straight Line being (practically) *nil.*



Take the longest side of the Triangle, and call it '*AB*': and, on that side of it, on which the Triangle lies, draw the semicircle *AFB*. Also, with centres *A*, *B*, and distances *AB*, *BA*, draw the arcs *BDC*, *AEC*, intersecting at *C*.

Then it is evident that the vertex of the Triangle cannot fall outside the Figure *ABDCE*.

Also, if it fall inside the semicircle, the Triangle is obtuse-angled: if outside it, acute-angled. (The chance, of its falling *on* the semicircle, is practically *nil.*)

Hence required chance $= \dfrac{\text{area of semicircle}}{\text{area of fig. } ABDCE}$.

Now let $AB = 2a$: then area of semicircle $= \dfrac{\pi a^2}{2}$; and area of Fig. $ABDCE = 2 \times \text{sector } ABDC - \text{Triangle } ABC$;

$$= 2 \cdot \frac{4\pi a^2}{6} - \sqrt{3} \cdot a^2 = a^2 \cdot \left(\frac{4\pi}{3} - \sqrt{3}\right);$$

$$\therefore \quad \text{chance} = \frac{\frac{\pi}{2}}{\frac{4\pi}{3} - \sqrt{3}} = \frac{3}{8 - \frac{6\sqrt{3}}{\pi}}.$$

Q. E. F.

72. (18, 27)

We know that, if a bag contained 3 counters, 2 being black and one white, the chance of drawing a black one would be $\frac{2}{3}$; and that any *other* state of things would *not* give this chance.

Now the chances, that the given bag contains (α) BB, (β) BW, (γ) WW, are respectively $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$.

Add a black counter.

Then the chances, that it contains (α) BBB, (β) BWB, (γ) WWB, are, as before, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$.

Hence the chance, of now drawing a black one,

$$= \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{2}{3}.$$

Hence the bag now contains BBW (since any *other* state of things would *not* give this chance).

Hence, before the black counter was added, it contained BW, i.e. one black counter and one white.

Q. E. F.

THE END.