WOOT 2010–11

Practice AIME 2 Solutions

Answer Key

1. 633	6. 060	11. 510
2. 800	7. 227	12. 211
3. 990	8. 441	13. 901
4. 096	9. 567	14. 666
5. 137	10. 084	15. 736

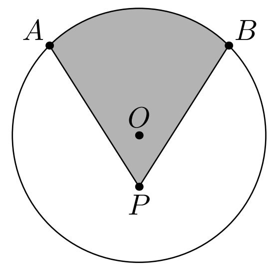
1. The ratio of the surface area of a cube with side length 2011 to the perimeter of one of its faces can be written in the form m/n, where m and n are relatively prime positive integers. Find $m - 2700n$.

   Solution. Let $s = 2011$ denote the side length of the cube. Then the ratio of the surface area of the cube to the perimeter of one of its faces is

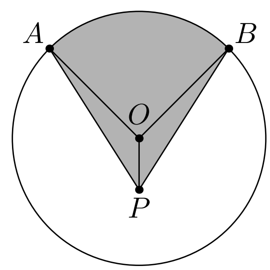
$$\frac{6s^2}{4s} = \frac{3s}{2} = \frac{6033}{2},$$

   so the final answer is $6033 - 2700 \cdot 2 = 633$.

2. Points A and B lie on a circle of radius 42, centered at O, such that $\angle AOB = 90^\circ$. Point P is located inside the circle such that $PA = PB$, $OP = 17$, and O lies inside triangle ABP. The area of the shaded region can be expressed in the form $m\pi + n\sqrt{p}$, where m, n, and p are positive integers, and p is not divisible by the square of any prime. Find $m + n + p$.



   Solution. Drawing line segments AO, BO, and OP, we see that the shaded region consists of one quarter of the circle, and triangles AOP and BOP.



The area of one quarter of the circle is $\pi \cdot 42^2/4 = 441\pi$. By symmetry, $\angle AOP = (360^\circ - 90^\circ)/2 = 135^\circ$, so the area of triangle AOP is

$$\frac{1}{2} \cdot AO \cdot OP \cdot \sin \angle AOP = \frac{1}{2} \cdot 42 \cdot 17 \cdot \frac{\sqrt{2}}{2} = \frac{357\sqrt{2}}{2}.$$

This is also the area of triangle BOP, so the area of the shaded region is $441\pi + 357\sqrt{2}$. The final answer is then $441 + 357 + 2 = 800$.

3. Evaluate

$$\frac{21^3 + 22^3 + 23^3 + \cdots + 39^3}{21 + 22 + 23 + \cdots + 39}.$$

Solution. From the formula

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = \left[\frac{n(n+1)}{2}\right]^2,$$

the numerator is equal to

$$\begin{aligned} 21^3 + 22^3 + 23^3 + \cdots + 39^3 &= (1^3 + 2^3 + \cdots + 39^3) - (1^3 + 2^3 + \cdots + 20^3) \\ &= \left(\frac{39 \cdot 40}{2}\right)^2 - \left(\frac{20 \cdot 21}{2}\right)^2. \end{aligned}$$

Also, the denominator is equal to

$$\begin{aligned} 21 + 22 + 23 + \cdots + 39 &= (1 + 2 + \cdots + 39) - (1 + 2 + \cdots + 20) \\ &= \frac{39 \cdot 40}{2} - \frac{20 \cdot 21}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{21^3 + 22^3 + 23^3 + \cdots + 39^3}{21 + 22 + 23 + \cdots + 39} &= \frac{(\frac{39 \cdot 40}{2})^2 - (\frac{20 \cdot 21}{2})^2}{\frac{39 \cdot 40}{2} - \frac{20 \cdot 21}{2}} \\ &= \frac{39 \cdot 40}{2} + \frac{20 \cdot 21}{2} \\ &= 780 + 210 = 990. \end{aligned}$$

4. Let n be a positive integer. When the fraction

$$\frac{n}{n+42}$$

is reduced, the result is the fraction $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the sum of all possible values of $q - p$.

Solution. Let $d = \gcd(n, n+42)$, so d divides n, and d divides $n+42$. Then d divides $(n+42)-n = 42$. Furthermore,

$$\frac{n}{n+42} = \frac{n/d}{(n+42)/d}.$$

We have that

$$\gcd\left(\frac{n}{d}, \frac{n+42}{d}\right) = \frac{\gcd(n, n+42)}{d} = \frac{d}{d} = 1,$$

so n/d and $(n+42)/d$ are relatively prime. Hence, $p = n/d$ and $q = (n+42)/d$, and

$$q - p = \frac{n+42}{d} - \frac{n}{d} = \frac{42}{d}.$$

Thus, $q - p$ must be a divisor of 42.

Conversely, let a be a divisor of 42, and let $n = 42/a$. Then

$$\frac{n}{n+42} = \frac{42/a}{42/a+42} = \frac{42}{42a+42} = \frac{1}{a+1},$$

so $p = 1$ and $q = a+1$, and $q - p = a$.

Therefore, the possible values of $q - p$ are the divisors of 42. The sum of the divisors of $42 = 2 \cdot 3 \cdot 7$ is $(1+2)(1+3)(1+7) = 96$.

5. Let N be the number of ways to divide 5050 students into 100 teams, so that the first team has one student, the second team has two students, and so on. (The order of the students on a team is irrelevant.) Compute the number of zeros at the end of the decimal representation of N.

Solution. The number of ways to divide 5050 students into 100 such teams is given by

$$N = \frac{5050!}{1!2!3!\cdots 100!}.$$

To find the number of zeros at the end of the decimal representation of N, we count the number of factors of 2 and 5 in the prime factorization of N.

First, we start with the factors of 5. The number of factors of 5 in the prime factorization of 5050! is

$$\left\lfloor \frac{5050}{5} \right\rfloor + \left\lfloor \frac{5050}{5^2} \right\rfloor + \left\lfloor \frac{5050}{5^3} \right\rfloor + \cdots = 1010 + 202 + 40 + 8 + 1 = 1261.$$

We can write the denominator as

$$1!2!3!\cdots 100! = 1 \cdot (1 \cdot 2) \cdot (1 \cdot 2 \cdot 3) \cdots (1 \cdot 2 \cdots 100) = 1^{100} \cdot 2^{99} \cdot 3^{98} \cdots 100^1.$$

In this product, the multiples of 5 contribute $96 + 91 + \cdots + 1 = 970$ factors of 5, and the multiples of 25 contribute another $76 + 51 + 26 + 1 = 154$ factors of 5. Therefore, N has $1261 - 970 - 154 = 137$ factors of 5.

Let $A(n)$ denote the number of factors of 2 in the prime factorization of $n!$, so

$$A(n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{2^3} \right\rfloor + \cdots.$$

The number of factors of 2 in the prime factorization of 5050! is

$$\begin{aligned} A(5050) &= \left\lfloor \frac{5050}{2} \right\rfloor + \left\lfloor \frac{5050}{2^2} \right\rfloor + \left\lfloor \frac{5050}{2^3} \right\rfloor + \cdots \\ &= 2525 + 1262 + 631 + 315 + 157 + 78 + 39 + 19 + 9 + 4 + 2 + 1 \\ &= 5042. \end{aligned}$$

The number of factors of 2 in the prime factorization of $1!2! \cdots 100!$ is

$$\begin{aligned} A(1) + A(2) + \cdots + A(100) = &\left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{1}{2^2} \right\rfloor + \left\lfloor \frac{1}{2^3} \right\rfloor + \cdots \\ &+ \left\lfloor \frac{2}{2} \right\rfloor + \left\lfloor \frac{2}{2^2} \right\rfloor + \left\lfloor \frac{2}{2^3} \right\rfloor + \cdots \\ &+ \cdots \\ &+ \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{2^2} \right\rfloor + \left\lfloor \frac{100}{2^3} \right\rfloor + \cdots. \end{aligned}$$

To help compute this sum, let

$$f(k) = \left\lfloor \frac{1}{2^k} \right\rfloor + \left\lfloor \frac{2}{2^k} \right\rfloor + \cdots + \left\lfloor \frac{100}{2^k} \right\rfloor.$$

Then

$$A(1) + A(2) + \cdots + A(100) = f(1) + f(2) + f(3) + f(4) + f(5) + f(6).$$

(Since $2^7 > 100$, $f(k) = 0$ for $k \ge 7$.)

Given $k \ge 1$, let $100 = 2^k q + r$, where q and r are nonnegative integers, and $0 \le r \le 2^k - 1$. Then the sequence $\lfloor 1/2^k \rfloor$, $\lfloor 2/2^k \rfloor$, $\dots$, $\lfloor 100/2^k \rfloor$ is of the form

$$\underbrace{0, 0, \dots, 0}_{2^k - 1 \text{ 0s}}, \underbrace{1, 1, \dots, 1}_{2^k \text{ 1s}}, \underbrace{2, 2, \dots, 2}_{2^k \text{ 2s}}, \dots, \underbrace{q-1, q-1, \dots, q-1}_{2^k \ (q-1)\text{s}}, \underbrace{q, q, \dots, q}_{r+1 \ q\text{s}}.$$

Hence,

$$\begin{aligned} f(k) &= 2^k[0+1+\cdots+(q-1)]+(r+1)q \\ &= \frac{q(q-1)}{2}\cdot 2^k+(r+1)q \\ &= \frac{q(q-1)}{2}\cdot\frac{100-r}{q}+(r+1)q \\ &= \frac{(q-1)(100-r)}{2}+(r+1)q \\ &= \frac{100q-qr-100+r+2qr+2q}{2} \\ &= \frac{qr+102q+r-100}{2} \\ &= \frac{(q+1)(r+102)-202}{2} \\ &= \frac{(q+1)(r+102)}{2}-101. \end{aligned}$$

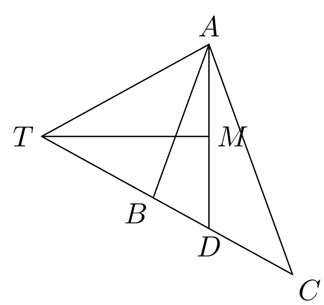
We can then compute $f(k)$ for $1 \le k \le 6$:

k	q	r	$f(k)$
1	50	0	2500
2	25	0	1225
3	12	4	588
4	6	4	270
5	3	4	111
6	1	36	37

Hence, the number of factors of 2 in N is $5042-2500-1225-588-270-111-37=311$. Therefore, the number of zeros at the end of the decimal representation of N is $\min(137,311)=137$.

6. Let ABC be a triangle, and let D be the point on BC such that AD bisects $\angle BAC$. Let T be the intersection of line BC with the perpendicular bisector of AD. If $BD=20$ and $CD=30$, then find TD.

Solution. Let M be the midpoint of AD. We see that triangles ATM and DTM are congruent.



Also, $\angle TAB = \angle TAD - \angle TAB = \angle TDA - \angle CAD = \angle TCA$. Hence, triangles TAB and TCA are similar, so

$$\frac{BT}{AT} = \frac{AT}{CT}.$$

Let $x = BT$, so

$$\begin{aligned} \frac{x}{x+20} &= \frac{x+20}{x+50} \\ \Rightarrow \quad x^2 + 50x &= x^2 + 40x + 400 \\ \Rightarrow \quad 10x &= 400 \\ \Rightarrow \quad x &= 40. \end{aligned}$$

Then $TD = TB + BD = 40 + 20 = 60$.

7. Gary rolls three standard, six-sided dice and discards the die with the lowest number. The expected value of the sum of the remaining two dice can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

   Solution. The expected value of a single roll is $(1 + 2 + \cdots + 6)/6 = 7/2$, so the expected value of the sum of three rolls is $3 \cdot 7/2 = 21/2$.

   Next, for $1 \le k \le 6$, we consider the number of possible rolls of three dice where the lowest number is k. For $k = 6$, there is only one possible roll, namely $(6, 6, 6)$.

   For $k = 5$, each roll must be a 5 or a 6, and there are 2^3 such rolls. However, we must exclude the roll $(6, 6, 6)$, for a total of $2^3 - 1$.

   For $k = 4$, each roll must be a 4, 5, or 6, and there are 3^3 such rolls. However, we must exclude the rolls where all the rolls are 5 or 6, for a total of $3^3 - 2^3$.

   In general, the number of rolls of three dice where the lowest number is k is $(7-k)^3 - (6-k)^3$. Hence, the expected value of the lowest number is

$$\begin{aligned} &\frac{(6^3 - 5^3) \cdot 1 + (5^3 - 4^3) \cdot 2 + (4^3 - 3^3) \cdot 3 + (3^3 - 2^3) \cdot 4 + (2^3 - 1^3) \cdot 5 + 1^3 \cdot 6}{6^3} \\ &= \frac{91 + 122 + 111 + 76 + 35 + 6}{216} \\ &= \frac{441}{216} = \frac{49}{24}. \end{aligned}$$

   Therefore, the expected value of the sum of the other two rolls, once the lowest number has been discarded, is

$$\frac{21}{2} - \frac{49}{24} = \frac{203}{24},$$

   and the final answer is $203 + 24 = 227$.

8. In triangle ABC, let D, E, and F be points on sides BC, AC, and AB, respectively such that AD, BE, and CF are the three angle bisectors. If $AF = 7$, $BC + CE = 21$, and $BC + BF = 27$, then the

length of AE can be expressed in the form m/n, where m and n are relatively prime positive integers. Find mn.

Solution. Let $a = BC$, $b = AC$, and $c = AB$. By the angle bisector theorem,

$$AF = \frac{bc}{a+b}, \quad CE = \frac{ab}{a+c}, \quad BF = \frac{ac}{a+b},$$

so from the given equations,

$$\frac{bc}{a+b} = 7, \tag{1}$$
$$a + \frac{ab}{a+c} = 21, \tag{2}$$
$$a + \frac{ac}{a+b} = 27. \tag{3}$$

We seek

$$AE = \frac{bc}{a+c}.$$

Adding equations (1) and (3), we get $a + c = 34$. Multiplying both sides of equation (2) by $a + c$ and both sides of equation (3) by $a + b$, respectively, we get

$$a^2 + ab + ac = 21(a + c),$$
$$a^2 + ab + ac = 27(a + b).$$

Hence,

$$a + b = \frac{21}{27}(a + c) = \frac{7}{9} \cdot 34 = \frac{238}{9}.$$

Then from equation (1), $bc = 7(a + b) = 7 \cdot 238/9 = 1666/9$, so

$$AE = \frac{bc}{a+c} = \frac{1666/9}{34} = \frac{49}{9}.$$

The final answer is $49 \cdot 9 = 441$.

9. Find the number of ten-digit numbers that satisfy the following properties:

- All the digits 0, 1, 2, ..., 9 appear exactly once (and 0 cannot be the first digit).
- The digits 1, 2, 3, 4, 5, 6, and 7 are in order, reading from left to right.
- The 8 does *not* appear anywhere after the 7, reading from left to right.

For example, the number 1823405967 satisfies these conditions, but the number 1239456708 does not.

Solution. We count the number of such ten-digit numbers by counting the number of ways of placing the digit 0, the digit 9, and the remaining digits, in that order.

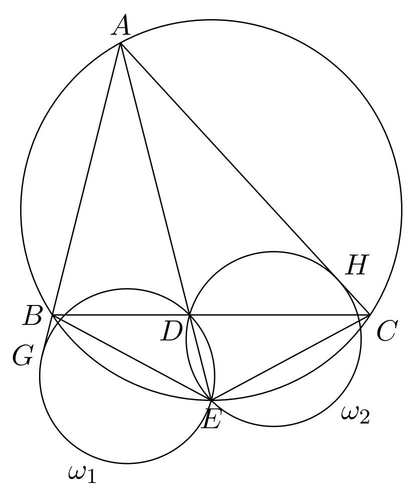
First, we can place the digit 0 in 9 possible positions. (There are a total of ten positions, but 0 cannot be the first digit.) We can then place the digit 9 in any of the 9 remaining positions. Then we must place the digits 1, 2, …, 8 in the remaining 8 positions, so that the digits 1 through 7 are in order, but the 8 does not appear anywhere after the 7. There are 7 possible such orderings:

$$\begin{gathered} 8,1,2,3,4,5,6,7 \\ 1,8,2,3,4,5,6,7 \\ 1,2,8,3,4,5,6,7 \\ 1,2,3,8,4,5,6,7 \\ 1,2,3,4,8,5,6,7 \\ 1,2,3,4,5,8,6,7 \\ 1,2,3,4,5,6,8,7 \end{gathered}$$

Therefore, the number of such ten-digit numbers is $9 \cdot 9 \cdot 7 = 567$.

10. In triangle ABC, $AB = 13$, $BC = 14$, and $AC = 15$. Let the bisector of $\angle BAC$ intersect BC at D and the circumcircle of triangle ABC at E. Let ω_1 be the circle that passes through D and E and is tangent to AB; let G be the point of tangency. Similarly, let ω_2 be the circle that passes through D and E and is tangent to AC; let H be the point of tangency. Determine the area of triangle AGH.

Solution. By power of a point with respect to circle ω_1, $AG^2 = AD \cdot AE$. Similarly, $AH^2 = AD \cdot AE$, so $AG = AH$.



Since AD is the angle bisector of $\angle BAC$, $\angle BAD = \angle EAC = \angle BAC/2$. Also, $\angle ABD = \angle ABC = \angle AEC$. Hence, triangles ABD and AEC are similar. Then

$$\frac{AB}{AE} = \frac{AD}{AC},$$

so $AD \cdot AE = AB \cdot AC$, which means $AG \cdot AH = AB \cdot AC$. Hence, the area of triangle AGH is

$$\frac{1}{2} AG \cdot AH \cdot \sin \angle GAH = \frac{1}{2} AB \cdot AC \cdot \sin \angle BAC = [ABC].$$

By Heron's formula, the area of triangle ABC is 84, so the area of triangle AGH is also 84.

11. Let z_1, z_2, $\dots$, z_9 be distinct complex numbers that form a regular nonagon in the complex plane. If $z_1 + z_2 + \dots + z_9 = 9$ and $z_1 z_2 \cdots z_9 = 512$, then find $(z_1 - 2)(z_2 - 2) \cdots (z_9 - 2)$.

Solution. Let $\omega = e^{2\pi i/9}$. Then the vertices of the nonagon are $a + b$, $a + b\omega$, $\dots$, $a + b\omega^8$, for some complex numbers a and b. Let $z_k = a + b\omega^{k-1}$. Then $z_k - a = b\omega^{k-1}$, so

$$(z_k - a)^9 = (b\omega^{k-1})^9 = b^9 (\omega^9)^{k-1} = b^9.$$

Hence, the vertices of the nonagon are the roots of the equation $(z - a)^9 - b^9 = 0$, which expands as

$$z^9 - 9az^8 + \dots - a^9 - b^9 = 0.$$

By Vieta's Formulas, $z_1 + z_2 + \dots + z_9 = 9a$ and $z_1 z_2 \cdots z_9 = a^9 + b^9$, so $9a = 9$ and $a^9 + b^9 = 512$. Then $a = 1$, so $b^9 = 512 - a^9 = 511$. Since z_1, z_2, $\dots$, z_9 are the roots of $(z - a)^9 - b^9 = 0$, by the Factor Theorem,

$$(z - z_1)(z - z_2) \cdots (z - z_9) = (z - a)^9 - b^9$$

for all complex numbers z. Setting $z = 2$, we get

$$(2 - z_1)(2 - z_2) \cdots (2 - z_9) = (2 - a)^9 - b^9 = 1^9 - 511 = -510,$$

so

$$(z_1 - 2)(z_2 - 2) \cdots (z_9 - 2) = 510.$$

12. The sum

$$\sum_{k=0}^{12} \frac{(-1)^k \binom{12}{k}}{k^2 - 29k + 210}$$

can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

Solution. First, we can write

$$\begin{aligned}\sum_{k=0}^{12} \frac{(-1)^k \binom{12}{k}}{k^2-29k+210} &= \sum_{k=0}^{12} \frac{(-1)^k \binom{12}{k}}{(14-k)(15-k)} \\ &= \sum_{k=0}^{12} \frac{(-1)^k 12!}{k!(12-k)!(14-k)(15-k)} \\ &= \sum_{k=0}^{12} \frac{(-1)^k 12!(13-k)}{k!(12-k)!(13-k)(14-k)(15-k)} \\ &= \sum_{k=0}^{12} \frac{(-1)^k 12!(13-k)}{k!(15-k)!} \\ &= \frac{1}{13\cdot 14\cdot 15} \sum_{k=0}^{12} \frac{(-1)^k 15!(13-k)}{k!(15-k)!} \\ &= \frac{1}{13\cdot 14\cdot 15} \sum_{k=0}^{12} (-1)^k (13-k)\binom{15}{k}.\end{aligned}$$

We must find

$$\sum_{k=0}^{12} (-1)^k (13-k)\binom{15}{k} = 13\binom{15}{0} - 12\binom{15}{1} + 11\binom{15}{2} - \dots + \binom{15}{12}.$$

From the identity

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1},$$

we get

$$\begin{aligned}&13\binom{15}{0} - 12\binom{15}{1} + 11\binom{15}{2} - \dots + \binom{15}{12} \\ &= 13\binom{14}{0} - 12\left[\binom{14}{0} + \binom{14}{1}\right] + 11\left[\binom{14}{1} + \binom{14}{2}\right] - \dots + \left[\binom{14}{11} + \binom{14}{12}\right] \\ &= \binom{14}{0} - \binom{14}{1} + \binom{14}{2} - \dots + \binom{14}{12}.\end{aligned}$$

We know that

$$\binom{14}{0} - \binom{14}{1} + \binom{14}{2} - \dots + \binom{14}{12} - \binom{14}{13} + \binom{14}{14} = 0,$$

so

$$\binom{14}{0} - \binom{14}{1} + \binom{14}{2} - \dots + \binom{14}{12} = \binom{14}{13} - \binom{14}{14} = 13.$$

Hence,

$$\frac{1}{13\cdot 14\cdot 15}\sum_{k=0}^{12}(-1)^k(13-k)\binom{15}{k} = \frac{13}{13\cdot 14\cdot 15} = \frac{1}{14\cdot 15} = \frac{1}{210},$$

so the final answer is $1 + 210 = 211$.

13. Let $N = 10203\ldots979899$ be the positive integer formed by concatenating all the positive integers from 1 to 99 inclusive, with the single-digit numbers using two digits (so 4 is represented as 04, for example). Let r be the remainder when 100^{100} is divided by N. Find the last three digits of r.

Solution. We have that

$$100N = 10203\ldots97989900.$$

Subtracting $N = 10203\ldots979899$, we get

$$\begin{array}{rcr} 100N & = & 1020304\ldots989900 \\ N & = & -10203\ldots979899 \\ \hline 99N & = & 1010101\ldots010001 \end{array}$$

Also, N has $2\cdot 99 - 1 = 197$ digits, so $99N$ has $197 + 2 = 199$ digits.

Multiplying again by 99, we get

$$99^2N = 999999\ldots990099.$$

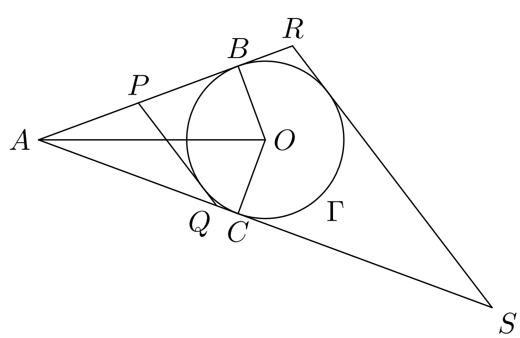
This last number has 200 digits, so it is equal to $10^{200} - 9901 = 100^{100} - 9901$. In other words, $99^2N = 100^{100} - 9901$, or

$$100^{100} = 99^2N + 9901.$$

Therefore, when 100^{100} is divided by N, the remainder is $r = 9901$. The last three digits of r are 901.

14. Let Γ be a circle with center O and radius $\sqrt{3}$, and let A be a point such that $AO = 15$. Let AB and AC be the tangents from A to Γ, and let P be a point on AB, other than A or B. Let Q be the point on AC such that PQ is tangent to Γ. Let R and S be points on AB and AC, respectively, such that RS is parallel to PQ and tangent to Γ (and does not coincide with PQ). Find $[ARS]\cdot[APQ]$.

Note: $[XYZ]$ denotes the area of triangle XYZ.



Solution. Let s_1 and s_2 denote the semi-perimeters of triangles APQ and ARS, respectively. Let r_1 and r_2 denote the inradii of triangles APQ and ARS, respectively.

Triangles APQ and ARS are similar, so

$$\frac{s_1}{s_2} = \frac{r_1}{r_2}.$$

Note that $s_1 = AB = \sqrt{15^2 - 3} = \sqrt{222}$. Also, $r_2 = \sqrt{3}$. Hence, $r_1 s_2 = s_1 r_2 = \sqrt{666}$.

Then

$$K_1 K_2 = [APQ] \cdot [ARS] = r_1 s_1 \cdot r_2 s_2 = r_1 s_2 \cdot r_2 s_1 = \sqrt{666} \cdot \sqrt{666} = 666.$$

15. Let a_0, a_1, $\ldots$ be the sequence sequence defined by $a_0 = 1$ and $a_n = 2^{a_{n-1}}$ for all $n \geq 1$. Find the remainder when a_{2011} is divided by 1000.

Solution. We have that $\phi(125) = 100$, so by Euler's Theorem, $2^{100} \equiv 1 \pmod{125}$. Also, for any positive integer $n \geq 3$, 2^n is divisible by 8, so $2^n(2^{100} - 1) = 2^{n+100} - 2^n$ is divisible by 1000. In other words,

$$2^{n+100} \equiv 2^n \pmod{1000}$$

for all $n \geq 3$. Hence, if we want to compute the remainder of 2^n when it is divided by 1000 for large n, we can find the unique positive integer m such that $3 \leq m \leq 102$ and $m \equiv n \pmod{100}$. Then $2^n \equiv 2^m \pmod{1000}$.

We compute the first few values of a_n:

n	a_n
0	1
1	2
2	4
3	16
4	65536

Then $a_5 = 2^{65536}$. From our work above,

$$a_5 = 2^{65536} \equiv 2^{36} \pmod{1000}.$$

We know that $2^{12} = 4096$, so

$$\begin{aligned} 2^{36} &\equiv 2^{12} \cdot 2^{12} \cdot 2^{12} \\ &\equiv 96 \cdot 96 \cdot 96 \\ &\equiv 9216 \cdot 96 \\ &\equiv 216 \cdot 96 \\ &\equiv 20736 \\ &\equiv 736 \pmod{1000}. \end{aligned}$$

Then

$$a_6 \equiv 2^{736} \equiv 2^{36} \equiv 736 \pmod{1000}.$$

It follows that $a_n \equiv 736 \pmod{1000}$ for all $n \geq 5$, so the answer is 736.

Art of Problem Solving

WOOT 2010–11

Practice AIME 2 Solutions

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