Art of Problem Solving

WOOT

Class Transcript 03/17 - Continuity

nsato 7:30:26 pm
WOOT 2013-14: Continuity

nsato 7:30:30 pm
Today, we will look at problems that involve the concept of continuity.

nsato 7:30:39 pm
Qualitatively speaking, a function is continuous if its graph can drawn without lifting the pencil from the paper. That is, the graph does not have any breaks or sudden jumps.

nsato 7:30:52 pm
For completeness, we give a rigorous definition of continuity.

nsato 7:31:04 pm
A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $x = x_0$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

mathcool2009 7:31:19 pm
scary!!!

nsato 7:31:24 pm
This is the infamous delta-epsilon definition of continuity. Loosely speaking, this definition says that small changes in x result in small changes in $f(x)$. We won't be referring to this definition again, but if you end up taking a particularly rigorous class in calculus, then you will get to know this definition very well.

nsato 7:31:48 pm
For the purpose of today's class, the most important property about continuous functions is a result known as the Intermediate Value Theorem.

nsato 7:32:02 pm
(Intermediate Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for any value y between $f(a)$ and $f(b)$, there exists an $x \in [a, b]$ such that $f(x) = y$.

nsato 7:32:59 pm
In other words, if we are given a continuous function f, and we know that f takes on two certain values, then f takes on any value in between those two values.

nsato 7:33:16 pm
(We're not going to prove it, because it's a standard result.)

nsato 7:33:26 pm
One practical application of the Intermediate Value Theorem is finding roots of functions.

nsato 7:33:34 pm
For example, suppose we have a continuous function f such that $f(0) = 1$ and $f(1) = -1$. What can we say?

Cosmynx 7:34:06 pm
there is a root between 0 and 1

NextEinstein 7:34:06 pm
there is a root between 0 and 1

Arithmophobia 7:34:06 pm
there is a root between 0 and 1

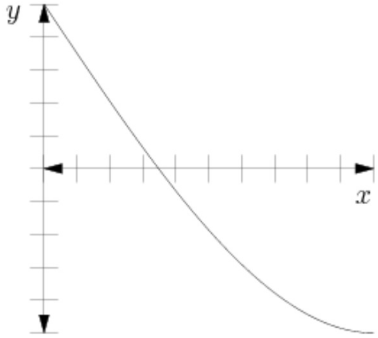
codyj 7:34:06 pm
$f(x) = 0$ for some $0 < x < 1$

TheStrangeCharm 7:34:06 pm
There must be a root between 0 and 1.

DaChickenInc 7:34:06 pm
There is a root of f between 0 and 1

nsato 7:34:13 pm
We can say that f has a root between 0 and 1.

nsato 7:34:17 pm



nsato 7:34:45 pm
Suppose we want to zoom in on this root. What value of $f(x)$ can we check?

DaChickenInc 7:35:23 pm
$f(\frac{1}{2})$

eccfcco15 7:35:23 pm
f(.5)

ABCDE 7:35:23 pm
f(1/2)

MSTang 7:35:23 pm
$f(0.5)$ [like binary search]

Cosmynx 7:35:23 pm
f(0.5)

nsato 7:35:34 pm
We can compute $f(0.5)$. What do we want to know about $f(0.5)$?

Tuxianeer 7:36:14 pm
positive or negative (or 0)

ABCDE 7:36:14 pm
whether it's positive or negative

Cosmynx 7:36:14 pm
whether it's positive or negative

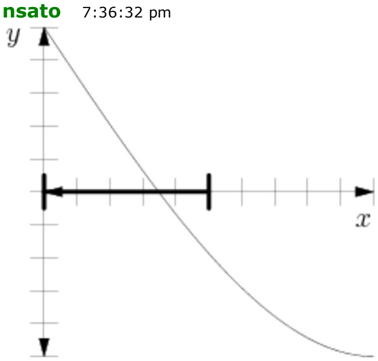
forthegreatergood 7:36:14 pm
It is either positive or negative

Rogman 7:36:14 pm
positive or negative

heron 7:36:14 pm
Is it greater or less than 0

nsato 7:36:25 pm
If $f(0.5)$ is positive, then the root lies between 0.5 and 1. If $f(0.5)$ is negative, then the root lies between 0 and 0.5. In this case, $f(0.5)$ is negative, so we can restrict our attention to the interval $[0, 0.5]$.

nsato 7:36:32 pm



nsato 7:36:44 pm
Now what?

va2010 7:37:22 pm
Then f(0.25)

fclvbfm934 7:37:22 pm
f(0.25)

soy_un_chemisto 7:37:22 pm
f(0.25)

minimario 7:37:22 pm
$f(0.25)$

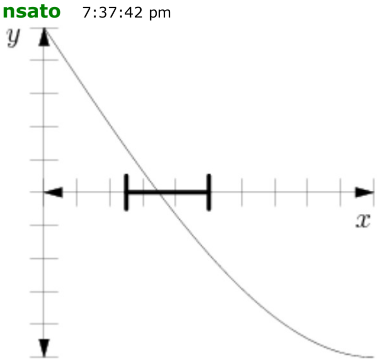
64138luc 7:37:22 pm
then find $f(1/4)$

thkim1011 7:37:22 pm
f(0.25)

zqjx 7:37:22 pm
f(0.25)

nsato 7:37:26 pm
Now we can check the value of $f(0.25)$, since 0.25 is the midpoint of this interval.

nsato 7:37:34 pm
If $f(0.25)$ is positive, then the root lies between 0.25 and 0.5. If $f(0.25)$ is negative, then the root lies between 0 and 0.25. In this case, $f(0.25)$ is positive, so we can restrict our attention to the interval $[0.25, 0.5]$.

nsato 7:37:42 pm


codyj 7:37:45 pm
as long as one endpoint is positive and the other is negative, there's a root

joshxiong 7:37:51 pm
repeat, halving the interval each time

forthegreatergood 7:37:51 pm
rinse and repeat

nsato 7:37:56 pm
It is clear that we can repeat this procedure as many times as we want to.

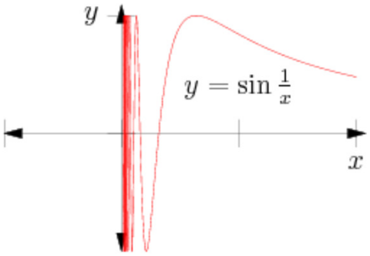
nsato 7:38:08 pm
This method of finding roots is called the Bisection method, because at each iteration, the length of the interval is reduced by half. Hence, we can repeat until we have the desired level of accuracy.

nsato 7:38:38 pm
In our first few problems, we will be working with closed curves in the plane. But what is a curve?

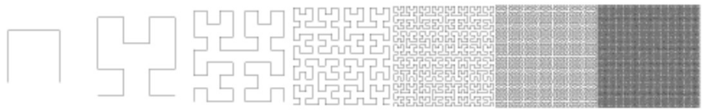
nsato 7:38:47 pm
For example, this is a curve.

nsato 7:38:52 pm


nsato 7:39:17 pm
That's a normal looking curve. But consider the graph of $y = \sin(1/x)$. As x approaches 0, the graph starts to oscillate wildly. Is this a curve?

nsato 7:39:28 pm


nsato 7:39:43 pm
We can also draw a graph that fills a square. Is this a curve?

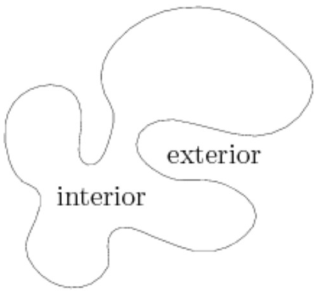
nsato 7:39:53 pm


nsato 7:40:22 pm
These examples show us that curves may not be as straight-forward as we thought.

nsato 7:40:31 pm
Again, for completeness, we give a rigorous definition.

nsato 7:40:40 pm
A (simple, closed) curve is a continuous function f from the interval $[0, 1]$ to the plane, such that $f(0) = f(1)$, and the restriction of f to $[0, 1)$ is injective. The first condition says that the curve is a loop (closed), and the second condition says that the curve does not intersect itself (simple).

nsato 7:41:05 pm
The Jordan Curve Theorem asserts that every simple, closed curve has a well-defined interior and exterior.

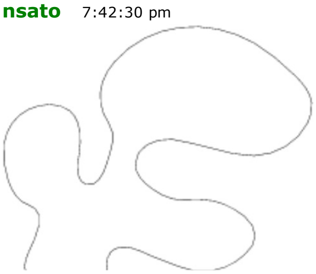
nsato 7:41:15 pm


nsato 7:41:26 pm
It actually took mathematicians a long time to find a rigorous proof of this result. This result may seem completely obvious, but if you think about the strange behavior that curves can exhibit, as in the examples given above, then you should realize that this result is not as obvious as it first seems.

nsato 7:42:07 pm
Now that we know a bit more about curves, we can start asking problems about them.

nsato 7:42:14 pm
Given a curve, prove that there exists a line that bisects the area of the curve.

nsato 7:42:24 pm
First, let's draw our curve.

nsato 7:42:30 pm


nsato 7:42:48 pm
Given the generality of the problem, it's not likely that we can find a formula that will give us such a line. But we do not need to find the line explicitly - we just need to show that it exists.

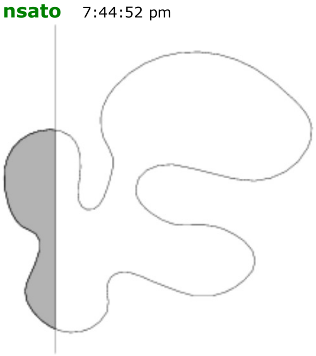
nsato 7:43:17 pm
How can we do that?

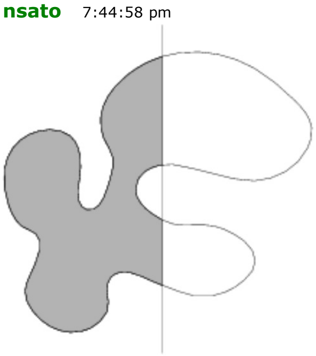
willwang123 7:44:07 pm
draw a vertical line on the left of the curve then slide it over toward the right

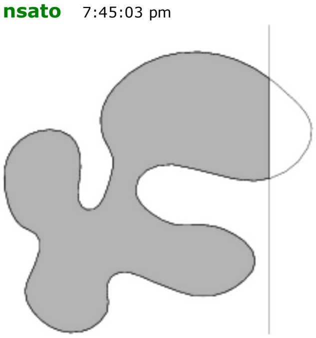
TheStrangeCharm 7:44:07 pm
Consider any line ℓ and start sliding it around but keep the same slope and use intermediate value theorem.

nsato 7:44:34 pm
We can start with a line that sits outside the curve, and let it pass over the curve gradually, until it reaches the other side.

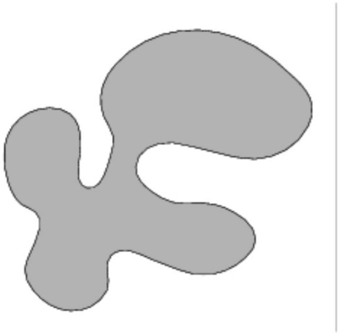
nsato 7:44:47 pm


nsato 7:44:52 pm


nsato 7:44:58 pm


nsato 7:45:03 pm


nsato 7:45:09 pm



nsato 7:45:12 pm
Thus, the area of the curve that is passed over (shaded in grey) is a function of the position of the line.

nsato 7:45:38 pm
What can we say about the area as a function of the position of the line?

ahaanomegas 7:46:29 pm
Importantly, this function is continuous.

zhuangzhuang 7:46:29 pm
it waries continuously from 0 to A (total area, grey)

anwang16 7:46:29 pm
it is continuous.

RocketSingh 7:46:29 pm
were done if we prove that function is continuous

64138luc 7:46:29 pm
it changes continuously from 0 to 1

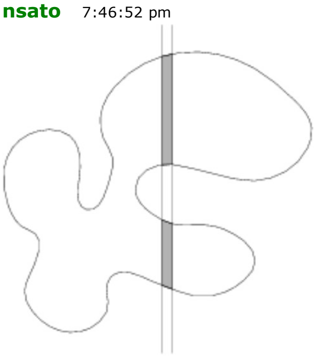
mathcool2009 7:46:29 pm
it takes on f(0) = 0 and f(1) = 1 and is continuoous and all that good stuff

avery 7:46:29 pm
It is continuous and ranges from 0 to the whole area of the closed curve

nsato 7:46:36 pm
The area varies continuously as a function of the position of the line.

nsato 7:46:46 pm
To see why this is true, consider a small change in the position of the line.

nsato 7:46:52 pm



nsato 7:46:58 pm
Qualitatively, a small change in the position of the line leads to a small change in the area passed over.

nsato 7:47:14 pm
In fact, we can make the change in the area passed over as small as we want to. For example, if we want the change in the area to be less than 0.01 square units, we can always move the line some sufficiently small amount for this to happen. This is what it means for area to be a continuous function of the position of the line.

minimario 7:47:46 pm
$f(\text{start}) = 0, f(\text{end}) = \text{area}_{\text{curve}}$

MSTang 7:47:46 pm

Starts at 0, finishes at the whole area

Bg1 7:47:46 pm
starts at 0 on the left and ends at the area of the curve on the right

willwang123 7:47:46 pm
it goes from 0% to 100%

lucylai 7:47:46 pm
it is 0 on the left and a on the right, where a is the area

nsato 7:47:50 pm
Also, at the start, the line has passed over none of the area, and at the end, the line has passed over all of the area.

Rogman 7:48:16 pm
intermediate value theorem!

ahaanomegas 7:48:16 pm
This area function is continuous, has a minimum of 0 and a maximum of 1. Thus, it must have crossed the 1/2 point due to IVT.

TheStrangeCharm 7:48:16 pm
We know it is 0 and then later it is 1, so if we can show that the area we are considering is changing continuously, then we are done because by IVT it must be $\frac{1}{2}$ somewhere in-between.

ProbaBillity 7:48:19 pm
Thus, at some point in time, it must have passed over half the area.

nsato 7:48:29 pm
Therefore, at some point, the line passes over exactly half the area. This gives us the line we are looking for.

nsato 7:48:40 pm
Note that we can replace half with any fraction between 0 and 1.

nsato 7:48:57 pm
Also, we can choose the direction of the line to be any direction we want.

nsato 7:49:18 pm
This degree of freedom turns out to be important.

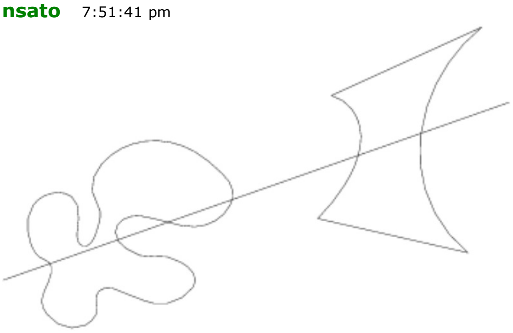
mota60ceng 7:50:55 pm
how can we rigorously prove that the function for areas is continuous?

nsato 7:50:58 pm
For this lecture, we're not going to worry about making our arguments ideas rigorous, because we don't have the right tools. Mostly, the point is to introduce you to the ideas behind continuity.

nsato 7:51:14 pm
That was easy, so let's raise the stakes a little.

nsato 7:51:20 pm
Given two curves, prove that there exists a line that simultaneously bisects the area of both curves.

nsato 7:51:41 pm



nsato 7:52:19 pm
What can we do here?

nsato 7:52:55 pm
We know that there are infinitely many lines that bisect the area of the first curve, each for a different direction. How can we use this?

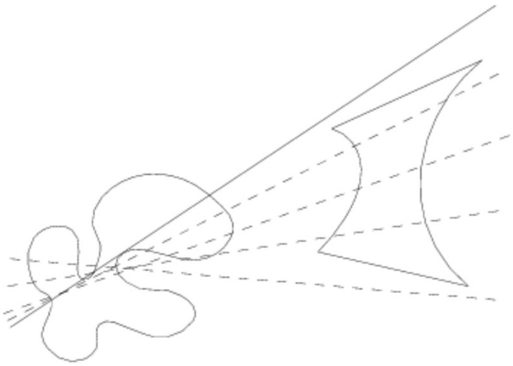
anwang16 7:53:22 pm
Go for one curve first, then adjust slope and location according to the other curve.

mathcool2009 7:53:59 pm

consider all the lines that bisect the area of the first curve, take that set of lines and apply it to the second curve with IVTF

nsato 7:54:27 pm
For each direction, we can draw the line that bisects the area of the first curve. We can then change this direction gradually, so that the line passes over the second curve.

nsato 7:54:44 pm



nsato 7:55:00 pm
At some point, the line bisects the area of the second curve. This gives us the line we are looking for.

nsato 7:55:09 pm
However, the diagram shows a somewhat special case, where there are bisection lines of the first curve that don't touch the second.

nsato 7:55:21 pm
What if one curve sits inside the other for example? What can we do then?

nsato 7:56:13 pm
We can argue this case as follows.

nsato 7:56:19 pm
Start with any line that bisects the area of the first curve. We color one side white, and the other side black. Suppose that the areas of the second curve that are white and black are X and Y, respectively.

nsato 7:56:37 pm
What happens as we start varying the angle of the bisecting line?

mathcool2009 7:57:10 pm
X and Y change

nilaisarda 7:57:10 pm
X and Y begin to change

nsato 7:57:43 pm
They do. Is there anything else we can say?

Cosmynx 7:58:19 pm
when we turn 180 degrees, X and Y flip

joshxiong 7:58:19 pm
if you rotate the line 180, X and Y swap

nsato 7:58:27 pm
When we've rotated an angle of 180 degrees, we get back to the bisecting line we started with.

nsato 7:58:39 pm
Also, what was black before is now white, and vice versa.

nsato 7:58:52 pm
So the white and black areas of the second curve are now Y and X, respectively.

DaChickenInc 7:59:06 pm
$X < \frac{1}{2} < Y = X'$ use IVT

nsato 7:59:17 pm
Assume without loss of generality that $X > 1/2$ and $Y < 1/2$. In other words, the white part is more than half, at the beginning.

nsato 7:59:26 pm
At the end, the white part is now less than half.

MSTang 7:59:37 pm
white-black went from X-Y to Y-X so it switched sign

nsato 7:59:42 pm
So again, at some point, the white part and black part have equal areas, which gives us our line.

nsato 8:00:18 pm
This problem generalizes as a result known as the Ham Sandwich Theorem: Given n objects in n-dimensional space, there exists an $(n-1)$-dimensional hyperplane that bisects the n-dimensional volume of all n objects.

nsato 8:00:39 pm
For example, for $n=3$, you can think of the three objects as two pieces of bread and a slice of ham between them. The theorem asserts that there is a single cut that bisects both pieces of bread and the ham simultaneously; hence, the name of the theorem. (The two-dimensional version is sometimes called the Pancake Theorem.)

nsato 8:01:10 pm
By the way, what is wrong with the following argument: The area inside each curve has a center of gravity (or mass). The line passing through both centers of gravity bisects each area.

nsato 8:01:33 pm
(You can assert that both areas have a center of gravity. That's not the issue.)

fclvbfm934 8:02:42 pm
line passing through center of gravity doesnt necessarily bisect area

ithinksomuch 8:02:42 pm
thats not necessarily half the area just because it passes through the center of mass

mathcool2009 8:02:42 pm
the bisecting lines don't pass through the center of gravity, as contrary to popular belief

ProbaBillity 8:02:42 pm
it is not true that any line passing through the center of gravity of a curve will bisect its area.

joshxiong 8:02:42 pm
a line passing through that center may not bisect the area

ABCDE 8:02:52 pm
equilateral triangle - contradiction

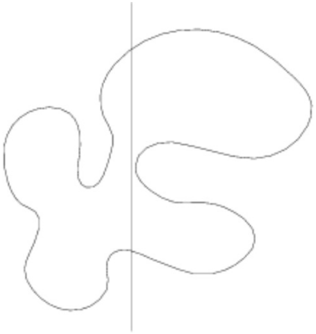
nsato 8:02:54 pm
The problem is that a line passing through a center of gravity does not necessarily bisect the area. For example, given an equilateral triangle, draw the line passing through the centroid parallel to a side of the triangle. This line divides the area of the triangle in the ratio $4:5$, not $1:1$.

nsato 8:03:22 pm
(In fact, that's true for any triangle.)

nsato 8:03:29 pm
For the next problem, we return to one curve.

nsato 8:03:33 pm
Given a curve with finite perimeter, prove that there exists a line that simultaneously bisects the area and perimeter of the curve.

nsato 8:04:04 pm
Here is our curve again, with a line that bisects the area.

nsato 8:04:09 pm


nsato 8:04:14 pm
What can we do here?

ahaanomegas 8:05:27 pm
Rotate

Cosmynx 8:05:27 pm
"rotate" our line bisecting the area

eccfcco15 8:05:27 pm
Rotate the curve. Use the color argument above, but with perimeter instead. It works.

bengals 8:05:28 pm
rotate

nsato 8:05:39 pm
We can use the same idea as in the last problem - we can change the direction of this line, so that it always bisects the area of the curve.

nsato 8:06:02 pm
But how do we know that there is such a line that bisects the perimeter as well? Since the line always bisects the area, we cannot move the line so that all of the perimeter lies on one side of the line. So what can we do?

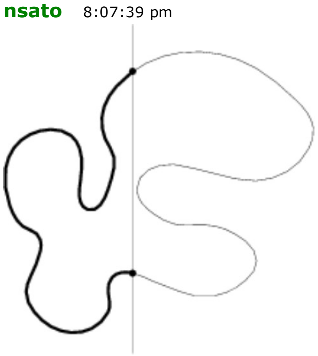
ABCDE 8:07:02 pm
black and white side - consider the continuous function f=B-W

RocketSingh 8:07:02 pm
let the right perimenter be X and the left be Y and do what we did for two curves

joshxiong 8:07:04 pm
consider perimeter to the left and right of the line

nsato 8:07:33 pm
We can keep track of how much perimeter lies on one side of the line.

nsato 8:07:39 pm



nsato 8:07:52 pm
As the direction of the line changes, so does the portion of the perimeter.

nsato 8:08:04 pm
What can we do with this?

Bg1 8:08:45 pm
rotate it until 180, then use ivt

zqjx 8:08:45 pm
after rotating through 180 degrees, X and Y flip

pickten 8:08:45 pm
rotate 180 degrees, the colors swap

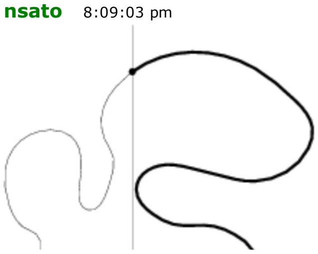
codyj 8:08:45 pm
rotate 180, notice that X and Y flip

RocketSingh 8:08:45 pm
rotate the lines 180 degrees so that X and Y flip

sammyMaX 8:08:51 pm
Let X be the perimeter on one side, Y on the other. Rotate line 180 and use IVT

nsato 8:08:56 pm
We can move the line until it coincides with its initial position.

nsato 8:09:03 pm



nsato 8:09:09 pm
We know that eventually, the line returns to its original position, because if half the area is on one side of the line, then half the area is on the other side of the line.

nsato 8:09:30 pm
However, the portion of the perimeter that we are keeping track of is now all on the other side of the line. How does that help?

NextEinstein 8:10:13 pm
If the portion was less then half the perimeter before, then now it's greater than half, and vice versa

pickten 8:10:13 pm
the fraction has gone from x to 1-x continuously, so there was a 1/2 in there somewher

chenjamin 8:10:20 pm
if it was once larger than half, it is now smaller than half, and vice versa

64138luc 8:10:20 pm
use IVT, since their sum is P(perimeter), then one must be larger or equal to 0.5P and one must be smaller

neutrinonerd3333 8:10:22 pm
If we rotate the line 180 degrees, the fraction of the perimeter on one side of the line goes from under half to over half (or vice versa). So IVT guarantees existence of a direction that exactly bisects perimeter.

MSTang 8:10:22 pm
X-Y becomes Y-X, so it goes from positive to negative

nsato 8:10:31 pm
By the Intermediate Value Theorem, the portion of the perimeter that we are keeping track of must have been exactly half the perimeter at some point.

nsato 8:10:38 pm
To be more precise, consider the portion of the perimeter that we are keeping track of as a function of the direction of the line. This function is continuous.

nsato 8:11:02 pm
We start with a line (that bisects the area) in some direction. If this line also bisects the perimeter, then we are done. Otherwise, the portion of the perimeter we are keeping track of is either less than half or greater than half the perimeter. Without loss of generality, assume that it is less than half.

nsato 8:11:19 pm
When the line returns to its original position, the portion of the perimeter that we are keeping track of is now the complement of what it was originally, so it is greater than half the perimeter.

nsato 8:11:34 pm
Then by continuity, it must have been exactly half the perimeter at some point.

nsato 8:12:12 pm
Given a curve, show that there are three points on the curve that form the vertices of an equilateral triangle.

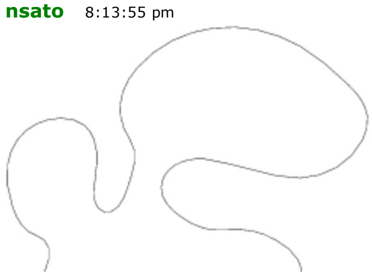
nsato 8:12:19 pm
How can we start?

willwang123 8:13:38 pm
pick two points

eyzhang 8:13:38 pm
draw a curve and pick two points

nsato 8:13:48 pm
As a first guess, we can pick two points on the curve as the vertices of an equilateral triangle, with the third vertex inside the curve.

nsato 8:13:55 pm



nsato 8:14:04 pm
How can we use continuity?

NextEinstein 8:14:44 pm
fix the first point

MSTang 8:14:44 pm
Vary one of the points on the curve

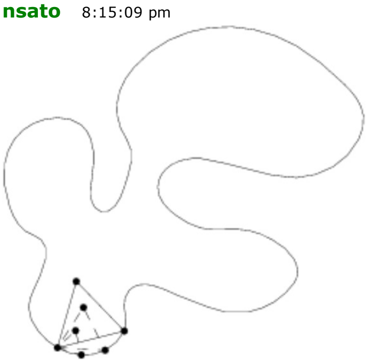
Rogman 8:14:44 pm
if we keep one point fixed and move the second continuously along the perimeter, the third point will be of interest

pickten 8:14:44 pm
let the second point travel the curve. the third begins inside, ends outside, so it crossed the curve

joshxiong 8:14:52 pm
try moving one of the points that in on the curve while fixing the other

nsato 8:15:04 pm
We can fix one point on the curve, and let the other point vary along the curve, maintaining the equilateral triangle.

nsato 8:15:09 pm



nsato 8:15:25 pm
As the point that's varying approaches its initial position, what happens?

willwang123 8:16:09 pm
it eventually goes outside the curve

patchosaur 8:16:09 pm
the third point is on the outside of the curve

MSTang 8:16:09 pm
It goes outside the curve

willwang123 8:16:09 pm
the 3rd point goes outside the curve

eyzhang 8:16:12 pm
The second third point will eventually go out of the circle

Rogman 8:16:12 pm
it is outside the curve!

ProbaBillity 8:16:22 pm
the triangle flips so the third point goes outside

nsato 8:16:31 pm
The third vertex of the equilateral triangle is now on the outside of the curve.

nsato 8:16:38 pm

nsato 8:16:40 pm
So what can we conclude?

ABCDE 8:17:36 pm
it intersected the curve at least once, so that's our third point

mathcool2009 8:17:36 pm
it crossed the curve at some point

SuperSnivy 8:17:36 pm
at some point, the third vertex was on the curve

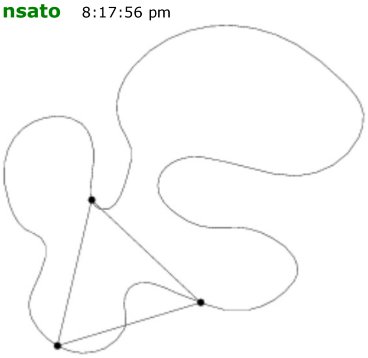
giratina150 8:17:36 pm
at some point the third point crosses the figure so there are 3 vertices on the figure

Cosmynx 8:17:36 pm
at some point the third point must have been on the curve

mlcindy 8:17:36 pm
there is some position that the third vertex is also on curve

nsato 8:17:46 pm
Since the third vertex is initially inside the curve, and moves outside the curve, at some point it must have been actually on the curve. This gives us three points on the curve that form an equilateral triangle.

nsato 8:17:56 pm

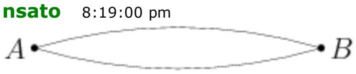


nsato 8:18:24 pm
This looks very convincing, but there are some details that we need to address.

nsato 8:18:35 pm
For example, we actually made a critical assumption in the beginning when we supposed our third triangle vertex was inside the curve. We never showed this was always possible.

nsato 8:18:48 pm
The argument above seems to show that any point on the curve can be a vertex of the equilateral triangle. However, this is not actually true. For example, in the following curve, both the points A and B cannot serve as vertices of an equilateral triangle.

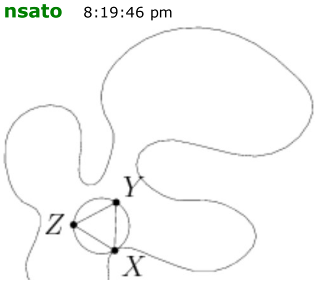
nsato 8:19:00 pm



nsato 8:19:18 pm
This means we need a slightly more sophisticated argument, to deal with these bad points.

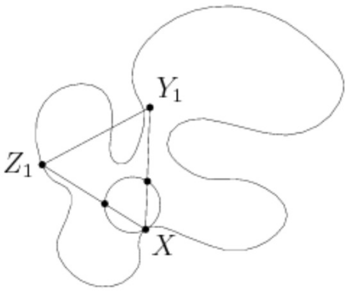
nsato 8:19:27 pm
Let W be a point inside the curve, and let C be the smallest circle centered at W that meets the curve. Let C touch the curve at X, and let Y and Z be points on C so that $\triangle XYZ$ is equilateral.

nsato 8:19:46 pm



nsato 8:19:58 pm
We then dilate points Y and Z to Y_1 and Z_1 around X outward until one of them hits the curve.

nsato 8:20:08 pm

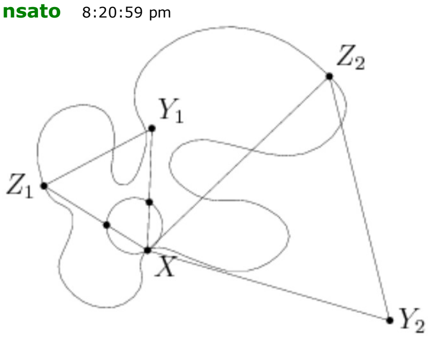


nsato 8:20:17 pm
For this diagram, point Z_1 is on the curve, so the other point Y_1 is still inside the curve.

nsato 8:20:32 pm
(If Y_1 lies on the curve too, then we're done.)

nsato 8:20:42 pm
We then let Z_1 vary on the curve until it reaches Z_2, the point on the curve that is farthest from X. (There is a result from calculus that assures us that such a point exists.)

nsato 8:20:59 pm



nsato 8:21:20 pm
Since Y_2 has the same distance from X as Z_2, and Z_2 is the point on the curve farthest from X, Y_2 must be either on the curve or outside.

nsato 8:21:35 pm
Hence, at some point where Y was moving from Y_1 to Y_2, it must intersect the curve at some point. This gives us an equilateral triangle.

Cpi2728 8:21:54 pm
Does this hold for a square?

nsato 8:21:58 pm
It is natural to ask whether any curve contains four points that form the vertices of a square.

nsato 8:22:07 pm
This problem is unsolved! We have partial results, that involve conditions on the curve, but mathematicians do not know if the general result is true. You can find out more information here:

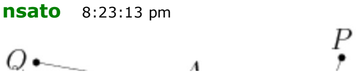
nsato 8:22:16 pm
http://www.webpages.uidaho.edu/~markn/squares/

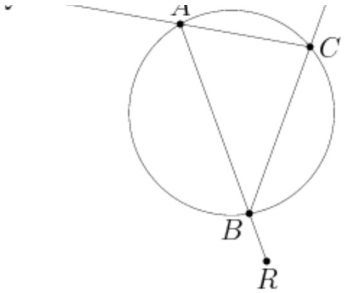
nsato 8:22:43 pm
The Intermediate Value Theorem, and the notion of continuity can be used to solve a wide variety of problems, particularly existential-type problems.

nsato 8:22:54 pm
Points P, Q, and R lie outside a circle. Show that there exist points A, B, and C on the circle such that P lies on BC, Q lies on CA, and R lies on AB.

nsato 8:23:08 pm
Here is what the diagram looks like:

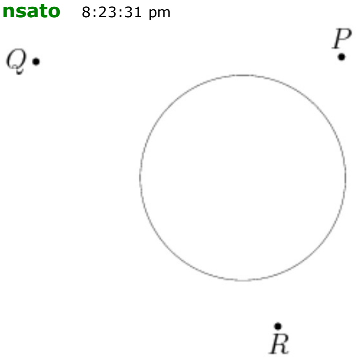
nsato 8:23:13 pm





nsato 8:23:25 pm
How would you locate A, B, and C, without the benefit of knowing where they are supposed to go?

nsato 8:23:31 pm



nsato 8:23:45 pm
As in the previous problems with the curves, it may help to fix certain objects, while letting others vary. With this in mind, how can we start?

va2010 8:24:28 pm
arbitrary A?

sammyMaX 8:24:28 pm
Place A somewhere

nsato 8:24:35 pm
As an initial guess, we can start by picking an arbitrary point on the circle to be A. Then where does B go?

MSTang 8:25:47 pm
intersection of RA with the circle

ABCDE 8:25:47 pm
intersection of circle and RA

Cosmynx 8:25:47 pm
the second intersection of RA with the circle

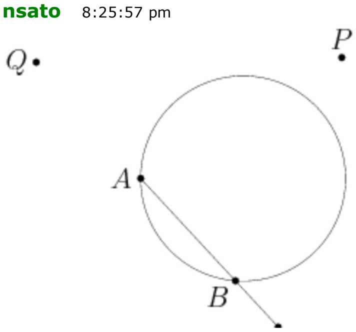
noodleeater 8:25:47 pm
intersection of AR and the circle

mentalgenius 8:25:47 pm
AR intersect with circle

patchosaur 8:25:47 pm
RA intersects the circle

nsato 8:25:51 pm
B is the intersection of AR with the circle.

nsato 8:25:57 pm



nsato 8:26:01 pm
Then where does C go?

minimario 8:26:53 pm
Intersection of PB and the circle

codyj 8:26:53 pm
intersection of BP and circle

mlcindy 8:26:53 pm
intersection of PB and the circle

ws5188 8:26:53 pm
intersection of BP with the circle

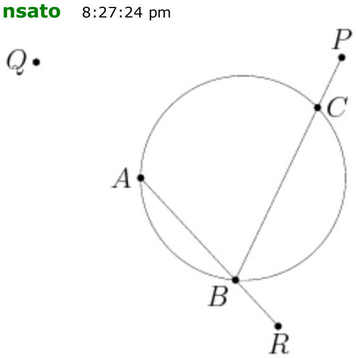
Rogman 8:26:53 pm
intersection of circle and BP

joshxiong 8:26:53 pm
intersection of PB and circle

ithinksomuch 8:26:53 pm
when BP intersects the circle

nsato 8:27:19 pm
C is the intersection of BP with the circle.

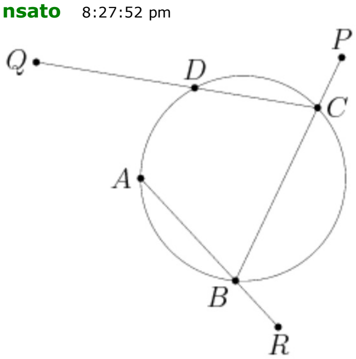
nsato 8:27:24 pm



nsato 8:27:27 pm
If we are very lucky and A happens to lie on CQ, then we are done. However, in general, this won't happen.

nsato 8:27:46 pm
For reference, let D be the intersection of CQ with the circle.

nsato 8:27:52 pm



nsato 8:28:10 pm
We want D to coincide with A. How can we get these points to coincide?

willwang123 8:28:39 pm
slide around A

fclvbfm934 8:28:39 pm
now you can vary A

AkshajK 8:28:39 pm
vary A

jeff10 8:28:39 pm
Move A

nsato 8:28:47 pm
We can let point A vary on the circle. More precisely, we can let point A move clockwise on the circle.

nsato 8:28:57 pm
As point A moves clockwise, what happens to point B?

mathcool2009 8:29:40 pm
counterclockwise

SuperSnivy 8:29:40 pm
moves counterclockwise

chenjamin 8:29:40 pm
B moves counterclockwise

DaChickenInc 8:29:40 pm
B goes counter-clockwise

64138luc 8:29:40 pm
move counter clock wise

nsato 8:29:49 pm
As point A moves clockwise, point B moves counter-clockwise (since AB passes through point R, which is outside the circle).

nsato 8:30:07 pm
What happens to point C?

noodleeater 8:30:58 pm
moves clockwise

ithinksomuch 8:30:58 pm
clockwise

sammyMaX 8:30:58 pm
Moves clockwise

DaChickenInc 8:30:58 pm
C moves clockwise

chenjamin 8:30:58 pm
C moves clockwise

henrydu 8:30:58 pm
clockwise

SuperSnivy 8:31:00 pm
moves clockwise

nsato 8:31:05 pm
Point C moves clockwise.

nsato 8:31:09 pm
Finally, what happens to point D?

Cosmynx 8:31:48 pm
D moves counterclockwise

anonymous0 8:31:48 pm
moves counterclockwise

eyzhang 8:31:48 pm
counter clockwise

patchosaur 8:31:48 pm
counterclockwise

neutrinonerd3333 8:31:48 pm
moves counterclockwies

ABCDE 8:31:48 pm
counterclockwise

nilaisarda 8:31:48 pm
Goes counterclockwise

nsato 8:31:54 pm
Point D moves counter-clockwise.

nsato 8:32:02 pm
Furthermore, when A returns to its initial position, so do B, C, and D.

nsato 8:32:20 pm
What can we conclude?

Rogman 8:33:14 pm
since A moves clockwise and D moves counterclockwise, they must coincide by IVT at some point. QED

chenjamin 8:33:14 pm
A meets D somewhere

sammyMaX 8:33:14 pm
At some position for A, D coincides

Cosmynx 8:33:14 pm
at some point A and D must be the same point, since they move in opposite directions

nsun48 8:33:14 pm
D=A at some point

joshxiong 8:33:14 pm
A and D coincide at some point

mlcindy 8:33:14 pm
A D will meet some time since they move in different directions

nsato 8:33:20 pm
Since point A moves clockwise and point D moves counter-clockwise, at some point they must meet. These positions of points A, B, and C give a solution.

nsato 8:33:40 pm
In fact, points A and D must meet exactly twice. Hence, our argument shows that there are exactly two solutions.

nsato 8:39:41 pm
Here's a nice, quick problem that involves continuity.

nsato 8:39:51 pm
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(x) \cdot f(f(x)) = 1$$

for all $x \in \mathbb{R}$ and $f(1000) = 999$. Find $f(500)$. (Poland, 1993)

nsato 8:40:07 pm
We are given $f(1000) = 999$. How can we use that?

soy_un_chemisto 8:40:33 pm
plug in x = 1000

giratina150 8:40:33 pm
x=1000

NextEinstein 8:40:33 pm
plug in x=1000

eccfcco15 8:40:33 pm
Try letting x=1000.

Arithmophobia 8:40:46 pm
f(1000)f(999)=1

NonEuclideanCow 8:40:54 pm
f(999)=1/999

64138luc 8:40:54 pm
999*$f(999)$=1

minimario 8:40:54 pm
$f(1000) \cdot f(f(1000)) = 1 \Rightarrow f(999) = 1/999$

fclvbfm934 8:40:54 pm
f(999) = 1/999

nsato 8:40:57 pm

Setting $x = 1000$, the given functional equation becomes

$$f(1000)f(f(1000)) = 1,$$

or

$$999f(999) = 1.$$

Hence, $f(999) = 1/999$.

nsato 8:41:19 pm
This example suggests a more general result.

nsato 8:41:28 pm
We see that $f(x)$ appears twice in the given functional equation, so let $y = f(x)$. What do we get?

16navidr 8:42:08 pm
y*f(y)=1

lucylai 8:42:08 pm
$yf(y) = 1$

joshxiong 8:42:08 pm
f(y) = 1/y

chenjamin 8:42:08 pm
y*f(y) = 1

Tuxianeer 8:42:14 pm
f(y)=1/y

ws5188 8:42:14 pm
f(y)=1/y

nsato 8:42:27 pm
We get $yf(y) = 1$, so $f(y) = 1/y$.

neutrinonerd3333 8:42:52 pm
$f(y) = \frac{1}{y}$ for all y in the range of f.

mentalgenius 8:42:52 pm
f(f(x)) = 1/f(x). Thus, if y is in the range of f, f(y) = 1/y

nsato 8:42:55 pm
However, we can only say this when y is in the range of f.

nsato 8:43:09 pm
How can we use this to find $f(500)$?

giratina150 8:44:06 pm
prove that 500 is in the range

64138luc 8:44:06 pm
prove for some x, f(x)=500

16navidr 8:44:06 pm
test if 500 is in the range of f

mlcindy 8:44:08 pm
if 500 is in the range we can say f(500)=1/500

nsato 8:44:12 pm
If we can show that 500 is in the range of the function f, then we would be done. How can we show that 500 is in the range of the function f?

MSTang 8:45:23 pm
1/999 and 999 are in the range of f, so by IVT, 500 is in the range of f

Cosmynx 8:45:23 pm
f ranges from 1/999 to 999, so 500 is in the range of f

va2010 8:45:23 pm
there exists a real such that f(x)=500 by IVT on 1/999 and 999

neutrinonerd3333 8:45:23 pm
Since $\frac{1}{999}$ and 999 are both in the range of f, 500 is in the range of f as well, by IVT. So $f(500) = \frac{1}{500}$.

sammyMaX 8:45:23 pm

We know 999 and $1/999$ are in range, so 500 is in range

nsato 8:45:40 pm
Since the function f is continuous, to show that 500 is in the range of f, it suffices to show that f takes on some value less than 500, and some value greater than 500.

nsato 8:45:51 pm
Since $f(999) = 1/999$ and $f(1000) = 999$, the function f takes on values less than 500 and greater than 500.

nsato 8:46:07 pm
Therefore, by the Intermediate Value Theorem, f takes on the value 500, which means $f(500) = 1/500$.

nsato 8:46:15 pm
Now we look at problems involving discrete continuity. Discrete continuity arises from phenomena that change in small, discrete quantities.

nsato 8:46:22 pm
Our first result is a discrete version of the Intermediate Value Theorem.

nsato 8:46:27 pm
(Discrete Intermediate Value Theorem) Let

$$a_1, a_2, \dots, a_n$$

be a sequence of integers, such that any two consecutive terms differ by at most 1. Then every integer between a_1 and a_n appears in the sequence.

Rogman 8:46:53 pm
seems trivial

nsato 8:46:57 pm
This result is "obvious," but we can give a rigorous proof for it, using the Extremal Principle.

nsato 8:47:10 pm
Without loss of generality, we can assume that $a_1 \le a_n$. We want to show that for every integer a such that $a_1 \le a \le a_n$, there exists an i such that $a_i = a$.

nsato 8:47:21 pm
There are some cases we can dispose of immediately.

Rogman 8:47:31 pm
a_1 and a_n

noodleeater 8:47:32 pm
i = 1 and n

nsato 8:47:34 pm
If $a = a_1$, then we can take $i = 1$, and if $a = a_n$, then we can take $i = n$, so assume that $a_1 < a < a_n$.

nsato 8:47:48 pm
Let S be the set of indices k such that $a_k < a$. What can we say about the set S?

nsato 8:48:13 pm
(Think about what we need to apply the Extremal Principle.)

DaChickenInc 8:48:39 pm
nonempty

nsato 8:48:45 pm
First, we know that S is non-empty because $a_1 < a$.

MSTang 8:48:59 pm
Finite and thus has a maximum element

Cosmynx 8:48:59 pm
finite set

nsato 8:49:02 pm
Second, we know that S is finite, because S is a subset of $\{1, 2, \dots, n\}$.

nsato 8:49:09 pm
Therefore, S has a maximal element.

nsato 8:49:19 pm
Let j be the largest element in S. In particular, $a_j < a$. Also, $j \le n - 1$.

nsato 8:49:40 pm
What can we say about a_{j+1}?

MSTang 8:50:28 pm
Cannot be less than a still

infinity1 8:50:28 pm
>= a

nsato 8:50:35 pm
By maximality of j, $a_{j+1} \geq a$.

nsato 8:50:46 pm
However, a_{j+1} and a_j differ by at most 1.

nsato 8:50:54 pm
What can we conclude?

neutrinonerd3333 8:51:33 pm
$a_{j+1} = a$

noodleeater 8:51:33 pm
a_(j+1) = a

MSTang 8:51:33 pm
So $a_j = a - 1$ and $a_{j+1} = a$ after all

Tuxianeer 8:51:33 pm
$a_{j+1} = a$

joshxiong 8:51:33 pm
$a_{j+1} = a$

nsato 8:51:37 pm
If $a_j < a$, $a_{j+1} \geq a$, and a_{j+1} and a_j differ by at most 1, then we must have $a_j = a - 1$ and $a_{j+1} = a$.

nsato 8:51:46 pm
Hence, the integer a appears in the sequence.

nsato 8:51:53 pm
This useful result has many applications.

nsato 8:52:00 pm
Prove that there exists a set of 1000 consecutive positive integers that contains exactly 10 primes.

nsato 8:53:00 pm
Considering that we do not have any exact formulas regarding the distribution of primes, it seems unlikely that we can find an explicit example. Our best bet is to go with an existential approach.

nsato 8:53:12 pm
For a positive integer n, let $f(n)$ be the number of primes among the 1000 consecutive positive integers

$$n, n+1, \dots, n+999.$$

nsato 8:53:20 pm
What can we say about $f(1)$?

jeff10 8:54:00 pm
Contains more than 10 primes

Arithmophobia 8:54:00 pm
a lot more than 10

ahaanomegas 8:54:00 pm
It is definitely way above 10. :P

fclvbfm934 8:54:00 pm
certaintly bigger than 10

64138luc 8:54:00 pm
$f(1)$>10

nsato 8:54:06 pm
We know that $f(1)$ is greater than 10. (Quick Fact: There are 25 primes from 1 to 100.)

nsato 8:54:18 pm
Is there anything we can say about the behavior of the function f?

nsato 8:55:17 pm
We know that $f(n)$ is the number of primes among the numbers

$$n, n+1, \dots, n+999,$$

and $f(n+1)$ is the number of primes among the numbers

$$n+1, n+2, \dots, n+1000.$$

thkim1011 8:55:41 pm
consecutive terms differ by at most 1

chenjamin 8:55:41 pm
it either goes up one , down one, or stays the same

mssmath 8:55:41 pm
It can change by at most 1

ABCDE 8:55:41 pm
either increases by one, decreases by one, or stays the same when n is increased by 1

nsato 8:55:49 pm
Going from the first set to the second set, the set drops at most one prime (namely n) and picks up at most one prime (namely $n+1000$), so the difference $f(n+1) - f(n)$ must be between 1 and -1.

nsato 8:56:03 pm
In other words, $f(n+1)$ and $f(n)$ differ by at most 1.

va2010 8:56:30 pm
discrete IVT !

nsato 8:56:41 pm
Exactly. What do we need to finish the problem?

RocketSingh 8:57:12 pm
so we need f(n)<10 and we're done

64138luc 8:57:12 pm
find n such that f(n)<10, use IVT

Tuxianeer 8:57:12 pm
a set with <10 primes

Cosmynx 8:57:12 pm
find some k such that f(k)<10

fclvbfm934 8:57:12 pm
find some value of n that gives f(n) < 10

nsato 8:57:26 pm
We need to find an n such that $f(n)$ is less than 10. Can we find such an n?

vincenthuang75025 8:58:36 pm
But the numbers $1001!+2, 1001!+3, \dots 1001!+1001$ all are not prime so $f(1001!+2) = 0$

zhuangzhuang 8:58:36 pm
Note that 1001!+2...1001!+1002 is a set with 1001 consecutive non primes

neutrinonerd3333 8:58:36 pm
$n = 1001!+2$ -- $f(n) = 0$ in this case

lucylai 8:58:46 pm
the numbers $1000!+2, 1000!+3, \dots 1000!+1001$

nsato 8:58:51 pm
For $n = 1001!+2$, the 1000 consecutive positive integers become

$$1001!+2,\ 1001!+3,\ \dots,\ 1001!+1001,$$

which are clearly all composite. Hence, for this n, $f(n) = 0$.

nsato 8:59:43 pm
(Some people chose $n = 1000!$. This should still work, but it's not obvious if $1000!+1$ is prime or not.)

ksun48 8:59:56 pm
Well, if there's 1 prime it's okay.

nsun48 8:59:59 pm
but f(n) still smaller than 10

nsato 9:00:12 pm
So we know there exists an n such that $f(n) < 10$, an n such that $f(n) > 10$, and consecutive values of f differ by at most 1. Therefore, there exists an n such that $f(n) = 10$.

nsato 9:00:27 pm
In other words, there exists a set of 1000 consecutive positive integers that contains exactly 10 primes.

nsato 9:00:41 pm
With a discrete version of the Intermediate Value Theorem, we can also look at discrete versions of the curve problems we looked at above.

nsato 9:00:53 pm
Given $2n$ points in the plane, prove that there exists a line so that there are exactly n points on either side of the line.

nsato 9:01:06 pm
What can we do here?

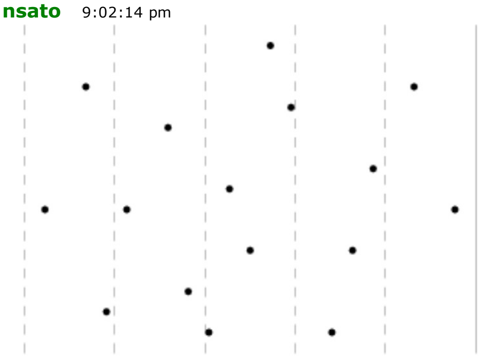
ABCDE 9:01:56 pm
slide a line across

Tuxianeer 9:01:56 pm
move a line through

MSTang 9:01:56 pm
Start a line to the left, move towards the right

brainiac1 9:01:56 pm
isn't this similar to the first problem?

nsato 9:02:02 pm
As in the continuous version, we can pass a line over the points, from one end to another.

nsato 9:02:14 pm


nsato 9:02:29 pm
The number of points passed over by the line increases from 0 points to $2n$ points. But what do we have to be careful of?

Cosmynx 9:03:12 pm
it might pass through more than 1 point at a time

patchosaur 9:03:12 pm
if the line passes through 2 points at once

thkim1011 9:03:12 pm
2 point on one line

ahaanomegas 9:03:12 pm
Two points on the same vertical line.

64138luc 9:03:16 pm
if line slide through 2 points at one time

giratina150 9:03:16 pm
the lines pass over 2+ points

nsato 9:03:20 pm
We have to make sure that the line never passes over two or more points at the same time. How can we make sure of this?

noodleeater 9:04:21 pm
take all 2n choose 2 possible slopes, and just make the line a slope that is none of these 2n choose 2 slopes

DaChickenInc 9:04:21 pm
Draw every line connecting two points. There are a finite number of slopes drawn, but there are infinitely many slopes. Therefore there exists a such slope.

sammyMaX 9:04:21 pm
There are a finite number of line segments you can draw with the points. Make the line parallel to none of those segments

MSTang 9:04:21 pm
pick the line not to be parallel to any of the $\binom{2n}{2}$ possible lines

jeff10 9:04:21 pm
Draw every single line you can connecting 2 points, since there are infinitely many slopes for a line, there has to be a way to view the points so that no two points are on the same vertical line

nsato 9:04:30 pm
Since there are only a finite number of points, there are only a finite number of lines that pass through two or more points.

nsato 9:04:58 pm
(In other words, there are only a finite number of "bad" directions.)

nsato 9:05:04 pm
However, there are an infinite number of directions to choose from, so we can always choose a direction that is not parallel to any of these lines.

nsato 9:05:13 pm
Once we have chosen an appropriate direction, we see that the number of points passed over increases from 0 points to $2n$ points, one at a time, so it hits every integer in between.

nsato 9:05:30 pm
In particular, the number of points passed over must be exactly n points at some time; this gives us the desired line.

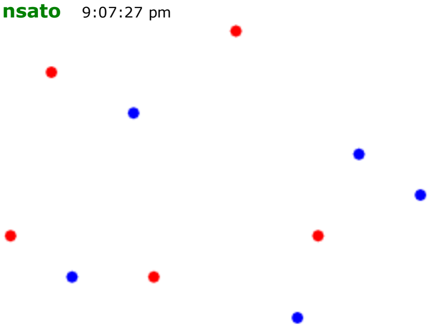
nsato 9:06:07 pm
We can also look at the discrete version of the Ham Sandwich Theorem.

nsato 9:06:11 pm
Given $2r+1$ red points and $2b+1$ blue points in the plane, so that no three points of any color are collinear, prove that there exists a line that passes through a red point and a blue point, so that there are exactly r red points and b blue points on either side of the line.

nsato 9:06:44 pm
There is a cute solution that uses the continuous version of the Ham Sandwich Theorem.

nsato 9:06:53 pm
First, we replace every point with a small disk of the same color. (A disk is the boundary of a circle along with the interior.) Since no three points are collinear, we can make the radius of each disk sufficiently small so that no line intersects more than two disks.

nsato 9:07:27 pm



nsato 9:07:33 pm
What can we do with this diagram?

nsato 9:08:36 pm
By the continuous version of the Ham Sandwich Theorem, there exists a line that simultaneously bisects the area of the red set and the area of the blue set. Here, the red set refers to the set consisting of all red disks. This set is not connected, but this property is not required, and we can still apply the Ham Sandwich Theorem.

nsato 9:08:53 pm
What can we say about this line?

nsato 9:09:34 pm
Is it possible that the line does not intersect any red disks?

Cosmynx 9:10:17 pm
no, since the number of disks is odd

neutrinonerd3333 9:10:17 pm
No; otherwise there will be more red disks on one side of the line, as there are an odd number of red disks.

Rogman 9:10:19 pm
no! since the number of red disks are odd!

nsato 9:10:30 pm
If this line does not intersect any red disks, then every red disk lies completely on one side of the line. But there are an odd number of red disks, so the line cannot bisect the red area, contradiction.

Rogman 9:10:40 pm
and similarly for the number of blue disks!

nsato 9:10:43 pm
Similarly, the line must intersect at least one blue disk.

nsato 9:10:48 pm
Then what can we say?

nsato 9:11:43 pm
Remember that we made the disks sufficiently small so that no line intersects more than two disks. Therefore, the line intersects exactly one red disk and one blue disk.

nsato 9:11:55 pm
What else can we say?

Tuxianeer 9:12:27 pm
it goes through the center of a red and blue disk

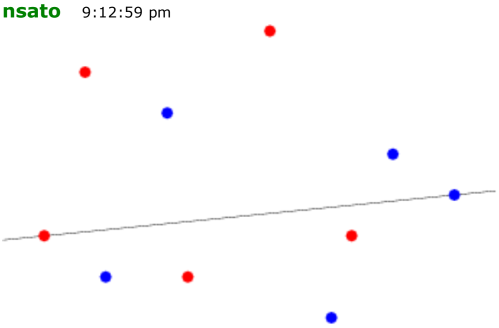
pickten 9:12:27 pm
it intersects exactly one of each by the condition we imposed, so it bisects each

joshxiong 9:12:32 pm
it must pass through the centers of both thos disks

nsato 9:12:41 pm
Since the line bisects the red area and intersects only one red disk, the line must pass through the center of the red disk.

nsato 9:12:51 pm
Similarly, the line must pass through the center of the blue disk. This is the line we seek.

nsato 9:12:59 pm



nsato 9:13:14 pm
There is a more general discrete version of the Ham Sandwich Theorem which is true.

nsato 9:13:23 pm
Let R be a finite set of red points, and let B be a finite set of blue points in the plane, so that no three points of any color are collinear. Then there exists a line such that there are at least $\lfloor |R|/2 \rfloor$ red points and at least $\lfloor |B|/2 \rfloor$ blue points on either side of the line, so that the line passes through at most one red point and at most one blue point.

nsato 9:13:49 pm
We can use the same idea to prove this version, namely replacing points with disks and applying the continuous version. However, surprisingly, there does not seem to be a simple proof of this result - there are some subtleties that can only be overcome by fairly advanced mathematics.

nsato 9:14:35 pm
We should have time for one more problem.

nsato 9:14:37 pm
Let n, p, q be positive integers with $n > p + q$. Let $x_0, x_1, \dots, x_n$ be integers satisfying the following conditions:
(a) $x_0 = x_n = 0$;
(b) for each integer i with $1 \le i \le n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.
Show that there exists a pair (i, j) of indices with $i < j$ and $(i, j) \neq (0, n)$ such that $x_i = x_j$.

nsato 9:16:11 pm
This was problem #6 from the 1996 IMO. It looks hard, but if you've been following everything, then it's actually not that difficult.

nsato 9:16:40 pm
The first thing is to break down the conditions.

MSTang 9:17:36 pm
So either we go up by p or go down by q

nsato 9:17:43 pm
We have a sequence $x_0, x_1, \dots, x_n$, where the first and last terms are 0. Also, each term is obtained by adding either p or $-q$ to the previous term. Hence, each term can be thought of as a sum of ps and $-q$s.

nsato 9:18:00 pm
We want to show that some value is attained twice in the sequence, apart from the case $x_0 = x_n = 0$.

nsato 9:18:19 pm
Let a and b be the number of differences that are equal to p and $-q$, respectively. Then what can we say?

Tuxianeer 9:19:09 pm
ap=bq

noodleeater 9:19:09 pm
ap = bq

brainiac1 9:19:09 pm
ap=bq

giratina150 9:19:09 pm
ap=qb

Tuxianeer 9:19:09 pm
$ap = bq$

ahaanomegas 9:19:09 pm
$ap = bq$

AkshajK 9:19:13 pm
$ap = bq$

nsato 9:19:21 pm
The first term x_0 is 0 and the last term x_n is 0, so the sum of all these differences is 0. In other words, $ap = bq$.

nsato 9:19:29 pm
What does this equation tell us about a and b?

nsato 9:20:19 pm
It depends on the greatest common divisor of p and q. To keep things simple, let's first look at the case where p and q are relatively prime.

nsato 9:20:29 pm
If p and q are relatively prime, what can we say?

Cosmynx 9:21:07 pm
a is a multiple of q, b is a multiple of p

sammyMaX 9:21:07 pm
a is a multiple of q, b is a multiple of p

joshxiong 9:21:07 pm
q|a , p|b

minimario 9:21:07 pm
$p|b, q|a$

ABCDE 9:21:07 pm
a is a multiple of q and b is a multiple of p

nsato 9:21:12 pm
If $ap = bq$, and p and q are relatively prime, then a must be divisible by q and b must be divisible by p.

delta1 9:21:30 pm
a=qk, b=pk

nsato 9:21:36 pm
Let $a = kq$ and $b = kp$.

nsato 9:21:45 pm
Then $n = a + b = k(p + q)$. What does that say about k?

va2010 9:22:32 pm
k>=2

Tuxianeer 9:22:32 pm
$k > 1$

DaChickenInc 9:22:32 pm
$k > 1$

bengals 9:22:32 pm
k>1

noodleeater 9:22:32 pm
k > 1

nsato 9:22:34 pm
We are given that $n > p + q$, so k is at least 2.

nsato 9:22:42 pm
We want to find a pair of indices i and j with $i < j$, other than 0 and n, such that $x_i = x_j$.

nsato 9:22:51 pm
To get from x_i to x_j, we must add a number of ps and $-q$s. What can we say about the number of ps and $-q$s if $x_i = x_j$?

MSTang 9:23:41 pm
the total is 0

ProbaBillity 9:23:43 pm
ps and qs weigh each other out between i and j

nsato 9:23:52 pm
Since $x_i = x_j$, the ps and $-q$s must cancel. But since p and q are relatively prime, the number of ps must be a multiple of q, and the number of $-q$s must be a multiple of p.

nsato 9:24:00 pm
Hence, by the same reasoning as above, the total number of ps and $-q$s must be a multiple of $p + q$.

nsato 9:24:09 pm
Therefore, if $x_i = x_j$, then $j - i$ must be a multiple of $p + q$. So to find a pair of indices i and j such that $x_i = x_j$, let's look at the differences of terms that are $p + q$ terms apart.

nsato 9:24:23 pm
Let

$$y_i = x_{i+p+q} - x_i$$

for $0 \le i \le n - p - q$.

nsato 9:24:32 pm
Thus, if $y_i = 0$ for some i, then we are done.

nsato 9:24:37 pm
To get from x_i to x_{i+p+q}, we add a number of ps and $-q$s.

nsato 9:24:47 pm
Let u and v be the number of ps and $-q$s, respectively, so $u + v = p + q$.

nsato 9:24:58 pm
Then

$$\begin{aligned} y_i &= x_{i+p+q} - x_i \\ &= up - vq \\ &= up - (p + q - u)q \\ &= up - (p + q)q + uq \\ &= (u - q)(p + q). \end{aligned}$$

nsato 9:25:17 pm
Thus, y_i is always a multiple of $p + q$.

nsato 9:25:25 pm
Any ideas how we can show that $y_i = 0$ for some i?

zhuangzhuang 9:26:13 pm
continuity now!!!!!!!

nsato 9:26:22 pm
We can try applying the Discrete Intermediate Value Theorem. What do we need to check?

Tuxianeer 9:27:06 pm
it only changes by p+q

nsato 9:27:10 pm
We need to check the differences of consecutive terms.

nsato 9:27:22 pm
We have that

$$y_{i+1} - y_i = (x_{i+p+q+1} - x_{i+1}) - (x_{i+p+q} - x_i),$$

which we can write as

$$y_{i+1} - y_i = (x_{i+p+q+1} - x_{i+p+q}) - (x_{i+1} - x_i).$$

nsato 9:27:37 pm
So what are the possible values of $y_{i+1} - y_i$?

joshxiong 9:28:46 pm
0, p+q, -p-q

Tuxianeer 9:28:46 pm
p+q,0,-p-q

DaChickenInc 9:28:51 pm
$\{0, p+q, -p-q\}$

nsato 9:28:57 pm
The difference $x_{i+p+q+1} - x_{i+p+q}$ is p or $-q$, and the difference $x_{i+1} - x_i$ is p or $-q$, so the possible values of

$$y_{i+1} - y_i = (x_{i+p+q+1} - x_{i+p+q}) - (x_{i+1} - x_i)$$

are 0, $p+q$, and $-(p+q)$.

nsato 9:29:07 pm
Hence, in the sequence

$$\frac{y_0}{p+q}, \frac{y_1}{p+q}, \dots, \frac{y_{n-p-q}}{p+q},$$

each term is an integer, and any two consecutive terms differ by at most one.

nsato 9:29:26 pm
Suppose that none of the y_i are equal to 0. Then what must happen?

noodleeater 9:30:35 pm
either all positive or all negative

david_sun 9:30:44 pm
all of them must be positive or all negative

nsato 9:30:46 pm
If none of the y_i are equal to 0, then they are all positive, or all negative. How does this lead to a contradiction?

joshxiong 9:31:00 pm
all are positive or all are negative, but this is a contradiction because $x_0 = x_n = 0$

nsato 9:31:05 pm
We know that $x_0 = x_n = 0$. Is this useful?

mssmath 9:31:30 pm
sum of the y_i's is 0

Tuxianeer 9:31:30 pm
the sum must be 0

nsato 9:31:36 pm
We have that

$$\begin{aligned}&y_0 + y_{p+q} + \dots + y_{n-p-q} \\ &= (x_{p+q} - x_0) + (x_{2p+2q} - x_{p+q}) + \dots + (x_n - x_{n-p-q}) \\ &= x_n - x_0 \\ &= 0.\end{aligned}$$

nsato 9:31:52 pm
Therefore, not all the y_i can be positive, and not all the y_i can be negative.

nsato 9:32:07 pm
We conclude that some y_i is equal to 0.

nsato 9:32:18 pm

But we are not done! We only looked at the case where p and q are relatively prime. What can we do if p and q are not relatively prime?

sammyMaX 9:32:55 pm
Divide by GCD

MSTang 9:32:55 pm
Divide by their GCD and repeat

lucylai 9:32:55 pm
divide everything by gcd

pickten 9:32:55 pm
divide all terms by the gcf

64138luc 9:32:55 pm
divide each term by gcd(p,q)

Rogman 9:32:55 pm
factor out the gcd

nsato 9:33:01 pm
If p and q are not relatively prime, then we can divide them by their greatest common divisor, say d, to get $p' = p/d$ and $q' = q/d$. Then p' and q' are relatively prime.

nsato 9:33:11 pm
We also divide each x_i by d to get x_i'.

nsato 9:33:17 pm
We see that $x_0' = x_n' = 0$, and $x_i - x_{i-1} = p'$ or $-q'$ for all i.

nsato 9:33:26 pm
Finally, is it true that $n > p' + q'$?

sammyMaX 9:34:00 pm
Yes, division makes things smaller

ws5188 9:34:00 pm
yes, since n>p+q

codyj 9:34:00 pm
yes because we divided

va2010 9:34:00 pm
n>p+q>p'+q'

MSTang 9:34:00 pm
$p > p'$ and $q > q'$, so yes

lucylai 9:34:02 pm
yes because $p/d + q/d < p + q$

nsato 9:34:07 pm
Since $n > p + q$, it follows that

$$n > \frac{p+q}{d} = p' + q'.$$

nsato 9:34:19 pm
Therefore, by the relatively prime case, there exist i and j (other than 0 and n) such that $x_i' = x_j'$, which means $x_i = x_j$.

nsato 9:34:35 pm
SUMMARY

nsato 9:34:41 pm
Although many of the functions that we see are continuous, continuity is a strong condition, which means they satisfy nice properties like the Intermediate Value Theorem. You may recall that continuity was used to solve Cauchy's Functional Equation, showing that we can exploit continuity in many ways.

nsato 9:34:55 pm
We also saw how we can apply the discrete version of continuity to solve problems involving phenomena that change in discrete steps. If you want to prove the existence of something, or that something must occur, under a certain process, there is a good chance that you can find it using a continuity argument.

nsato 9:35:09 pm
That's it for today's class.