Complex Facts

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Abstract

Often we are stuck with hard geo-problems.It's very clever to plung in complex numbers that time.In this note my aim is to provide the proofs of most-used facts of complex number based geometry.

Background

Here I'll use many pure geometrical concepts.So please read *Geometry revisited* before reading it.I hope my readers are familiar with the basic concepts of complex numbers.(at least about their Cartesian and Polar re-presentation.)

Through this note, upper case and lower case of any letter will respectively denote any point and its complex value.

Definitions

- If $z = a + bi$, then $\bar{z} = a - bi$
- $|z|$ is the distance from the origin to z. Mathematically, if $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.
- If $z = a + bi$, then $\Re(z) = a$ and $\Im(z) = b$
- If $z = re^{i\theta}$, then $\arg(z) = \theta$.
- $\overrightarrow{AB}$ is represented by the complex number $b - a$.
- If O is the origin, then $\angle AOB$ is called the angle from A to B.

  And $\angle AOB$=$\arg(b) - \arg(a)$

Conclutions

1.$z = \bar{z}$ iff $z \in \mathbb{R}$ and $z = -\bar{z}$ iff $zi \in \mathbb{R}$ i.e. z is purely imaginary.

2.$z\bar{z} = |z|^2$ and $|z| = |\bar{z}|$

3. $\Re(z) \leq |z|$ equality iff z is positive real.

$\Im(z) \leq |z|$ equality iff z is 'positive' imaginary.

4. $\overline{a+b} = \bar{a} + \bar{b}$ and $\overline{ab} = \bar{a}\bar{b}$

5. $|ab| = |a|\,|b|$ But hey, don't assume $|a+b| = |a| + |b|$.That's not true always! We'll look at this stuff later.

6. $|a| = |b|$ and $\arg(a) = \arg(b) \Longleftrightarrow a = b$

7. $\arg(ab) = \arg(a) + \arg(b)$ and $\arg(a/b) = \arg(a) - \arg(b)$ (actually *arg* does the job of *ln*)

8. $\overrightarrow{AB} = \overrightarrow{CD} \Longleftrightarrow b - a = d - c$

9.The oriented angle from $\overrightarrow{BA}$ to $\overrightarrow{CA}$ is $\angle ABC = \arg(c-b) - \arg(a-b) = \arg\left(\frac{c-b}{a-b}\right)$.

Also notice that $\angle CBA = -\angle ABC$ i.e. in complex plane, angle is direction-sensitive.

1.Point, Line and Angle

Theorem 1.1: $AB \| CD \Longleftrightarrow \frac{a-b}{c-d} = \frac{\bar{a}-\bar{b}}{\bar{c}-\bar{d}}$

Proof: $AB \| CD$ implies $\overrightarrow{AB} = \lambda \overrightarrow{CD}$ where $\lambda \in \mathbb{R}$.

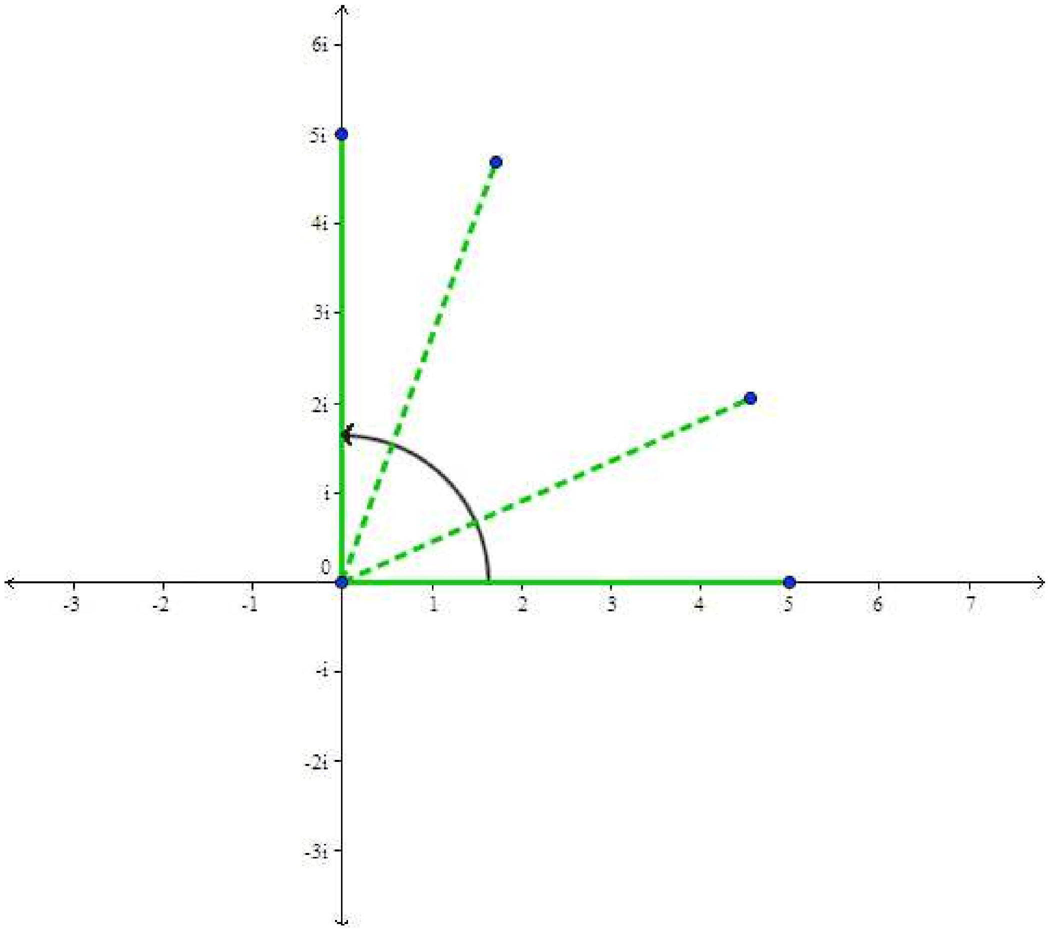
So $a - b = \lambda(c - d) \Longrightarrow \frac{a-b}{c-d} = \lambda \in \mathbb{R}$.

Therefore, from the first and the third conclution $\lambda = \frac{a-b}{c-d} = \frac{\bar{a}-\bar{b}}{\bar{c}-\bar{d}} = \bar{\lambda}$.

Corollary 1.1.1: A,B,C collinear iff $\frac{a-b}{b-c} = \frac{\bar{a}-\bar{b}}{\bar{b}-\bar{c}}$.

Theorem 1.2: $AB \perp CD \Longleftrightarrow \frac{a-b}{c-d} = -\frac{\bar{a}-\bar{b}}{\bar{c}-\bar{d}}$

Proof: You might not have noticed that i has an amazing property.It can rotate any complex number $\left(\frac{\pi}{2}\right)^c$.For example, if 5 is multiplied with i, it just rotates $\left(\frac{\pi}{2}\right)^c$ in the positive direction and we get $5i$.



In fact this is also true for vectors.In particular, if $\overrightarrow{A'B'} = i(b-a)$, then $\overrightarrow{A'B'}$ must be perpendicular to $\overrightarrow{AB}$ implying $A'B' || CD$.So from theorem 1.1 and conclution 1,

$$\frac{a'-b'}{c-d} = \frac{\bar{a}'-\bar{b}'}{\bar{c}-\bar{d}} \Longrightarrow \frac{i(a-b)}{c-d} = \frac{\bar{i}(\bar{a}-\bar{b})}{\bar{c}-\bar{d}} \Longrightarrow \frac{a-b}{c-d} = -\frac{\bar{a}-\bar{b}}{\bar{c}-\bar{d}}$$

Now comes the point...

Theorem 1.3: $P = AB \cap CD$ has the complex value

$$p = \frac{(\bar{a}b - a\bar{b})(c-d) - (\bar{c}d - c\bar{d})(a-b)}{(\bar{a}-\bar{b})(c-d) - (a-b)(\bar{c}-\bar{d})}$$

Proof: $P = AB \cap CD \Longleftrightarrow A, P, B$ as well as C, P, D are collinear.Hence from corollary 1.1.1,

$$\frac{a-p}{p-b} = \frac{\bar{a}-\bar{p}}{\bar{p}-\bar{b}} \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

$$\frac{c-p}{p-d} = \frac{\bar{c}-\bar{p}}{\bar{p}-\bar{d}} \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

By solving last two equations, we are deduced to

$$p=\frac{(\bar{a}b-a\bar{b})(c-d)-(\bar{c}d-c\bar{d})(a-b)}{(\bar{a}-\bar{b})(c-d)-(a-b)(\bar{c}-\bar{d})}$$

Corollary 1.3.1: If AB and CD are chords of unit circle, then

$$p=\frac{ab(c+d)-cd(a+b)}{ab-cd}$$

Theorem 1.4: If Q is the foot of perpendicular from P to AB, then

$$q=\frac{1}{2}\left[p+\frac{\bar{p}(a-b)+\bar{a}b-a\bar{b}}{\bar{a}-\bar{b}}\right]$$

Proof: Here A,Q,B collinear and $PQ\perp AB$.So

$$\frac{a-q}{q-b}=\frac{\bar{a}-\bar{q}}{\bar{q}-\bar{b}} \qquad \cdots \qquad \cdots \qquad \cdots \qquad (i)$$

$$\frac{a-b}{p-q}=-\frac{\bar{a}-\bar{b}}{\bar{p}-\bar{q}} \qquad \cdots \qquad \cdots \qquad \cdots \qquad (ii)$$

By solving last two equations, we are deduced to what is given.

Corollary 1.4.1: If P is the origin, then $q=\frac{1}{2}\cdot\frac{\bar{a}b-a\bar{b}}{\bar{a}-\bar{b}}$ (Of course, much more good-looking)

Corollary 1.4.2: The equation of the perpendicular bisector of segment AB is

$$|a|^2-|b|^2=p(\bar{a}-\bar{b})+\bar{p}(a-b)$$

Corollary 1.4.3: If AB is a chord of unit circle, then

$$q=\frac{1}{2}(a+b+p-ab\bar{p})$$

Theorem 1.5: If $\delta\in\mathbb{R}\setminus-1$, then $P\in$ line AB and $\overrightarrow{AP}:\overrightarrow{PB}=\delta\Longleftrightarrow p=\frac{a+\delta b}{1+\delta}$

Proof: $\delta \in \mathbb{R} \setminus -1$ implies $AP || PB \Longleftrightarrow P \in AB$.

So $\overrightarrow{AP} : \overrightarrow{PB} = \delta \Longleftrightarrow \dfrac{p-a}{b-p} = \delta \Longleftrightarrow p = \dfrac{a+\delta b}{1+\delta}$

Corollary 1.5.1: M is the mid-point of segment $AB \Longleftrightarrow m = \dfrac{a+b}{2}$

Angles

At the conclution part a very useful information was given about the angles.(Conclution 7)We can analize the angles only using this fact.The following theorem is nothing but a popular formulation of it.

Theorem 1.6: $\theta = \angle ABC \Longleftrightarrow \dfrac{c-b}{a-b} = e^{i\theta}\left|\dfrac{c-b}{a-b}\right|$

Proof: From conclution 7, $\theta = \arg\left(\dfrac{c-b}{a-b}\right)$ and so the ***theorem*** actually means $\dfrac{c-b}{a-b} = e^{i\arg\left(\frac{c-b}{a-b}\right)}\left|\dfrac{c-b}{a-b}\right|$ which is obvious.

And...

The triangle inequality

Theorem 1.7: For any two non-zero complex numbers a and b,

$$|a| + |b| \geq |a+b|$$

with equality iff $\dfrac{a}{b}$ is a positive real.

Proof: $|a+b|^2 = (a+b)(\bar{a} + \bar{b}) = |a|^2 + 2\Re(a\bar{b}) + |b|^2$ (check by yourself)

$$\begin{aligned} &\leq |a|^2 + 2|a\bar{b}| + |b|^2 \quad \text{(conclution 3)} \\ &= |a|^2 + 2|a| \cdot |b| + |b|^2 \quad \text{(conclution 2)} \\ &= (|a| + |b|)^2 \end{aligned}$$

Since $|a| + |b|$ and $|a+b|$ both are positive reals, so it follows that

$$|a| + |b| \geq |a+b|$$

Now we'll look at the equality case. From conclution 3, we must have $a\bar{b} \geq 0 \Longleftrightarrow a\dfrac{|b|^2}{b} \geq 0 \Longleftrightarrow \dfrac{a}{b} \geq 0$ i.e. $\dfrac{a}{b}$ is a positive real.

2.The Second Dimension

Triangle

Bellow we are talking about $\triangle ABC$ which has orthocenter H, circumcenter O and centroid G.When there are more than one triangle in a problem, you have to be a bit careful.Because complex numbers are direction-sensitive.(In other words, here writing $\triangle ABC$ and $\triangle ACB$ aren't the same.In fact, $\triangle i, 0, 1$, $\triangle 0, 1, i$ and $\triangle 1, i, 0$ are identical, all directed counterclockwisely, but $\triangle i, 1, 0$ is different from them as it is directed clockwisely.)So in a problem you must define all triangles in the same direction – either clockwise or counterclockwise.Here I am writing all the triangles in positive direction (counterclockwisely).

Theorem 2.1: For the circumcenter O, it follows

$$o = \frac{|a|^2(b-c)+|b|^2(c-a)+|c|^2(a-b)}{\bar{a}\,(b-c)+\bar{b}\,(c-a)+\bar{c}\,(a-b)}$$

Proof: We know that O is on the perpendicular bisector of segments AB, BC and CA. So from corollary 1.4.2, it follows that

$$|a|^2-|b|^2 = o(\bar{a}-\bar{b})+\bar{o}\,(a-b) \quad \ldots \quad \ldots \quad (i)$$

$$|c|^2-|b|^2 = o(\bar{c}-\bar{b})+\bar{o}\,(c-b) \quad \ldots \quad \ldots \quad (ii)$$

Solving these equations we find what is given.

Theorem 2.2: G is the centroid $\Longleftrightarrow g = \dfrac{a+b+c}{3}$

Proof: Let M is the mid-point of AB.So from 1.5.1, $m = \frac{a+b}{2}$.Also we know that $\overrightarrow{CG} = 2\overrightarrow{GM} \Longrightarrow g-c = 2(m-g) \Longrightarrow g = \dfrac{a+b+c}{3}$.

Theorem 2.3: $h+2o = a+b+c$

Proof: From the concept of Euler Line, we know that

$\overrightarrow{HG} = 2\overrightarrow{GO} \Longrightarrow g-h = 2(o-g) \Longrightarrow h+2o = 3g = a+b+c$.

Similarity

It was said before here angles are direction-sensitive. Thus being a relation regarding angles, here similarity is also orientation-sensitive.

We'll say $\triangle ABC \sim \triangle DEF$ iff

They are of same direction; both either oriented clockwisely or anticlockwisely. And at the same time, all of their angles are pairwisely equal.(Like $\angle ABC = \angle DEF$, $\angle BAC = \angle EDF$ etc)

Theorem 2.4: $\triangle ABC \sim \triangle DEF \iff \dfrac{a-b}{d-e} = \dfrac{b-c}{e-f} = \dfrac{c-a}{f-d}$

Proof: From elementary geometry we know that

$\triangle ABC \sim \triangle DEF \Longleftrightarrow \dfrac{CB}{BA} = \dfrac{FE}{ED} \Longleftrightarrow \left|\dfrac{c-b}{a-b}\right| = \left|\dfrac{f-e}{d-e}\right|$.As the triangles are similar and of same direction, we can say $\angle ABC = \angle DEF \Longleftrightarrow \arg\left(\dfrac{c-b}{a-b}\right) = \arg\left(\dfrac{f-e}{d-e}\right)$. Now from conclution 5, it follows that $\dfrac{c-b}{a-b} = \dfrac{f-e}{d-e} \iff \dfrac{a-b}{d-e} = \dfrac{b-c}{e-f}$. By the same method we can prove $\dfrac{b-c}{e-f} = \dfrac{c-a}{f-d}$. So it immediately follows that

$\dfrac{a-b}{d-e} = \dfrac{b-c}{e-f} = \dfrac{c-a}{f-d}$

Regular Polygons

Firstly we start by solving this equation: $x^n = 1$.Surely $x = 1$ is a solution, so we had better look for the others. Being roots of unity, each of the roots must must have modulus 1 (Why?). Let one of these roots is ω_n and $\omega_n = e^{i\theta}$. Also define a variable k such that $1 \leq k \leq n-1$.

$\omega_n^n = 1 \Longrightarrow \omega_n^{nk} = 1 \Longrightarrow (\omega_n^k)^n = 1$. So all the nth roots of unity are $\omega_n^0, \omega_n^1, \ldots, \omega_n^{n-1}$. Again notice that $|\omega_n^{k+1} - \omega_n^k| = |\omega_n^k|\,|1-\omega_n| = |1-\omega_n|$ and

$\angle \omega_n^{k-1}\omega_n^k\omega_n^{k+1} = \arg\left(\dfrac{\omega_n^{k+1} - \omega_n^k}{\omega_n^{k-1} - \omega_n^k}\right) = \arg\left(\dfrac{\omega_n^2 - \omega_n}{1-\omega_n}\right)$. Hence actually they form a regular n-gon inscribed in the unit circle.

Now you can find information about any regular poygon using similar triangles. Bellow I am giving an example for the equilateral triangle:

Theorem 2.5: $\triangle ABC$ is an equilateral triangle $\Longleftrightarrow a + \omega_3 b + \omega_3^2 c = 0$ or $a + \omega_3^2 b + \omega_3 c = 0$

Proof:

$$\triangle ABC \sim 1\omega_3\omega_3^2 \Longleftrightarrow \frac{a-b}{a-c} = \frac{1-\omega_3}{1-\omega_3^2} \Longleftrightarrow a + \omega_3 b + \omega_3^2 c = 0$$

$$\triangle ABC \sim 1\omega_3^2\omega_3 \Longleftrightarrow \frac{a-b}{a-c} = \frac{1-\omega_3^2}{1-\omega_3} \Longleftrightarrow a + \omega_3^2 b + \omega_3 c = 0$$

Cyclic Quadrilateral

Theorem 2.6: $ABCD$ is a cyclic quad. $\Longleftrightarrow \frac{a-b}{a-d} : \frac{c-b}{c-d} \in \mathbb{R}$

Proof: $ABCD$ is a cyclic quad $\Longleftrightarrow \angle DAB + \angle BCD \equiv \pi \pmod{2\pi}$

$\Longleftrightarrow \angle DAB - \angle DCB \equiv \pi \pmod{2\pi} \Longleftrightarrow \arg\left(\frac{b-a}{d-a}\right) - \arg\left(\frac{b-c}{d-c}\right) \equiv \pi \pmod{2\pi}$

$\Longleftrightarrow \arg\left(\frac{a-b}{a-d} : \frac{c-b}{c-d}\right) \equiv \pi \pmod{2\pi} \Longleftrightarrow \frac{a-b}{a-d} : \frac{c-b}{c-d} \in \mathbb{R}$

3.Special Situations

In previous sections, you have seen that most of the complex equations are complex themselves. Therefore, it is worth trying to reduce calculations. To do so, we set the figure (or a portion of it) in some fixed places like at the origin, on the unit circle etc.

The Unit Circle

From now we'll call unit circle by τ.See corollary 1.3.1 and 1.4.3 for two important properties of unit circle.

Fact 3.1: For $a \in \tau$, $|a| = 1$ so $\bar{a} = \frac{1}{a}$

Theorem 3.1: For chord AB of τ, $\frac{a-b}{\bar{a} - \bar{b}} = -ab$

Proof: Using fact 3.1 it is nothing but a muscle exercise.

Theorem 3.2: $C \in \text{chord } AB \Longleftrightarrow \bar{c} = \frac{a+b-c}{ab}$

Proof: The same as theorem 3.1

Theorem 3.3: If PA and PB are tangents from P, then $p = \frac{2ab}{a+b}$

Proof: $OA \perp AP$ and $OB \perp BP$ imply together

$$\frac{o-a}{a-p} = -\frac{\bar{o} - \bar{a}}{\bar{a} - \bar{p}} \Longleftrightarrow \frac{a}{a-p} = -\frac{\bar{a}}{\bar{a} - \bar{p}}$$

$$\frac{o-b}{b-p} = -\frac{\bar{o} - \bar{b}}{\bar{b} - \bar{p}} \Longleftrightarrow \frac{b}{b-p} = -\frac{\bar{b}}{\bar{b} - \bar{p}}$$

Solving these two we find $p=\dfrac{2ab}{a+b}$.

•Midpoint of arc:

One of the drawbacks with the complex numbers is we don't have much facilities with 'curves'. It is quite difficult to work with arc midpoints. Sometimes it is quite puzzling, too.

Theorem 3.4: The perpendicular bisector of AB (a chord of τ) meets τ at $p=\sqrt{ab}$ and $q=-\sqrt{ab}$

Proof: Let the perpendicular bisector of AB meets τ at P and Q. So

$p\cdot\bar{p}=1$ and from colorally 1.4.2, $p(\bar{a}-\bar{b})+\bar{p}(a-b)=0\Rightarrow p^2=ab=q^2$

Therefore $p=\sqrt{ab}$ or $-\sqrt{ab}$. But unfortunately we can't figure it out which one is p and which one is q.

When AB is a side of a triangle incribed in τ, we may need the arc midpoint to find the incenter of the triangle or for some special properties. In that case, this problem can be resolved by using the following technique:

WLOG, we may assume segment AB is parallel to the real axis. Now if it lies in the first and the second quadrant, i is the midpoint od the minor arc $\widehat{AB}$ and $-i$ is that of major arc $\widehat{AB}$. If segment AB is in the third and the fourth, the opposite will be true. Hopfully, we can find the incenter of that triangle.

Theorem 3.5:

Triangles circumscribed about τ

Let $\triangle ABC$ is circumscribed about τ and BC, CA, AB touch τ at P, Q, R, respectively. So from the previous theorem, $a=\dfrac{2qr}{q+r}$, $b=\dfrac{2rp}{r+p}$, $c=\dfrac{2pq}{p+q}$.

Theorem 3.6: If O is the circumcenter of $\triangle ABC$, then

$$o=\frac{2pqr(p+q+r)}{(p+q)(q+r)(r+p)}$$

Proof: It's direct consequence of theorem 2.1.

Theorem 3.7: If h is the orthocenter of $\triangle ABC$, then

$$h=\frac{2(p^2q^2+q^2r^2+r^2p^2)}{(p+q)(q+r)(r+p)}$$

Proof: Corollary of theorem 2.3 and 3.6.