Art of Problem Solving

WOOT 2010–11

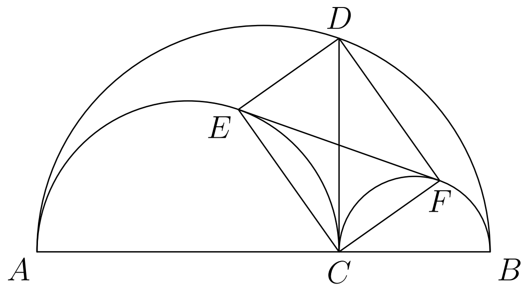
Practice Olympiad 6 Solutions

1. Let x and y be positive real numbers such that x^3, y^3, and $x+y$ are all rational. Show that both x and y are also rational.

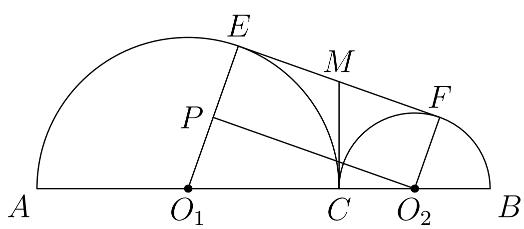
   Solution. Since x^3 and y^3 are rational,
   $$x^3+y^3=(x+y)(x^2-xy+y^2)$$
   is also rational. Since $x+y$ is rational, x^2-xy+y^2 is rational. But $(x+y)^2=x^2+2xy+y^2$ is also rational, so $(x^2+2xy+y^2)-(x^2-xy+y^2)=3xy$ is rational, which means xy is rational. Then $(x^2+2xy+y^2)-xy=x^2+xy+y^2$ is also rational.

   Since x^3 and y^3 are rational,
   $$x^3-y^3=(x-y)(x^2+xy+y^2)$$
   is also rational, so $x-y$ is rational. Therefore, $(x+y)+(x-y)=2x$ is rational, which means x is rational, and so y is rational.

2. Let C be a point on line segment AB. We construct semicircles with diameters AB, AC, and BC, all on the same side of AB. Let D be the point on the semicircle with diameter AB such that AB and CD are perpendicular. Let E be a point on the semicircle with diameter AC, and let F be a point on the semicircle with diameter BC, such that EF is the common external tangent to these semicircles. Prove that quadrilateral $CEDF$ is a rectangle.

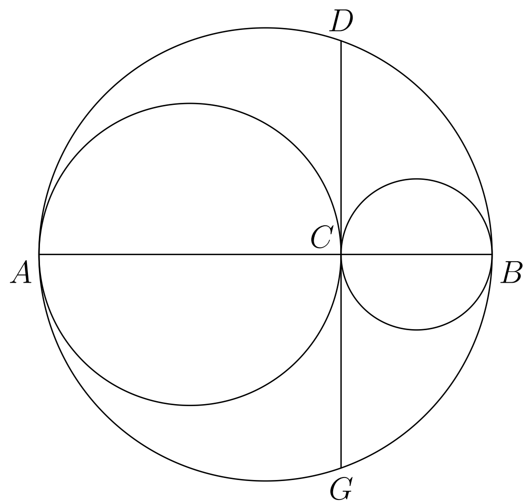


Solution. Without loss of generality, assume that $AC \geq AB$. Let O_1 and O_2 be the midpoints of AC and BC, respectively. Let r_1 and r_2 be the radii of semicircles AC and BC, respectively, so the radius of semicircle AB is $r_1 + r_2$.



Let P be the projection of O_2 onto O_1E, so quadrilateral $PEFO_2$ is a rectangle. Then $O_1O_2 = r_1 + r_2$, and $O_1P = O_1E - PE = O_1E - O_2F = r_1 - r_2$. Then by Pythagoras on right triangle O_1O_2P, $EF = PO_2 = \sqrt{O_1O_2^2 - O_1P^2} = \sqrt{(r_1 + r_2)^2 - (r_1 - r_2)^2} = \sqrt{4r_1r_2} = 2\sqrt{r_1r_2}$.

Let M be the intersection of EF and CD. Then as tangents from the same point to the same circle, $ME = MC$ and $MF = MC$, so $ME = MF$, i.e. M is the midpoint of EF, and $ME = MF = EF/2 = \sqrt{r_1r_2}$.



Let G be the reflection of D in AB. Then by power of a point on C, $CD \cdot CG = CA \cdot CB$. But $CD = CG$, so $CD = \sqrt{CA \cdot CB} = \sqrt{2r_1 \cdot 2r_2} = 2\sqrt{r_1 r_2}$, which means that M is also the midpoint of CD, and $EF = CD$. Hence, quadrilateral $CEDF$ is a rectangle.

3. There are n tennis players in a tournament, where $n \geq 4$, and certain pairs of players played games against each other. In every group of four players, there exist three players, say A, B, and C, such that A and B played a game, A and C played a game, and B and C played a game. What is the minimum number of games that could have been played in the tournament?

   Solution. We claim that the minimum number of games that could have been played is $\binom{n-1}{2}$.

   First, we show that there exists such a tournament in which $\binom{n-1}{2}$ games were played. Of the n players, take one player away, leaving $n-1$ players. We let every pair among these $n-1$ players play, for a total of $\binom{n-1}{2}$ games. It is clear that this tournament has the given property.

   Now we show that in any such tournament, at least $\binom{n-1}{2}$ games were played. If every pair of players played a game against each other, then we are done, so assume that there were two players that did not play against each other, say A and B.

   Let C and D be any other two players. We know that among the four players A, B, C, and D, there exists three players such that every pair among these three players played a game against each other. Since A and B did not play against each other, the three players must be either A, C, and D, or B, C, and D. In either case, C and D played against each other, and C played against either A or B, and D played against either A or B. Hence, among all $n-2$ players other than A or B, every pair of such players played against each other, and each such player played against A or B, for a total of at least

$$\binom{n-2}{2} + n - 2 = \frac{(n-2)(n-3)}{2} + n - 2 = \frac{(n-2)(n-1)}{2} = \binom{n-1}{2}$$

   games.

4. Let a_1, a_2, $\ldots$, a_n be distinct real numbers. Show that

$$\min_{1 \le i,j \le n} (a_i - a_j)^2 \le \frac{12}{n(n^2-1)} (a_1^2 + a_2^2 + \cdots + a_n^2).$$

Solution. First, we state a lemma.

Lemma. For any real numbers x and y,

$$x^2 + y^2 \ge \frac{(x-y)^2}{2}.$$

Proof. The inequality is equivalent to $(x+y)^2 \ge 0$. ■

Without loss of generality, assume that $a_1 < a_2 < \cdots < a_n$, and let $d = \min_{i<j}(a_j - a_i)$. Then $a_n - a_1 \ge (n-1)d$, $a_{n-1} - a_2 \ge (n-3)d$, and so on. Then by the lemma,

$$\begin{aligned} a_n^2 + a_1^2 &\ge \frac{(n-1)^2}{2} \cdot d^2, \\ a_{n-1}^2 + a_2^2 &\ge \frac{(n-3)^2}{2} \cdot d^2, \end{aligned}$$

and so on.

If n is even, then let $n = 2k$. Then

$$\begin{aligned} (n-1)^2 + (n-3)^2 + \cdots + 1^2 &= 1^2 + 3^2 + \cdots + (2k-1)^2 \\ &= [1^2 + 2^2 + \cdots + (2k)^2] - [2^2 + 4^2 + \cdots + (2k)^2] \\ &= [1^2 + 2^2 + \cdots + (2k)^2] - 4(1^2 + 2^2 + \cdots + k^2) \\ &= \frac{(2k)(2k+1)(4k+2)}{6} - 4 \cdot \frac{k(k+1)(2k+1)}{6} \\ &= \frac{k(2k-1)(2k+1)}{3} \\ &= \frac{n(n-1)(n+1)}{6}. \end{aligned}$$

If n is odd, then let $n = 2k-1$. Then

$$\begin{aligned} (n-1)^2 + (n-3)^2 + \cdots + 2^2 &= 2^2 + 4^2 + \cdots + (2k-2)^2 \\ &= 4[1^2 + 2^2 + \cdots + (k-1)^2] \\ &= 4 \cdot \frac{(k-1)(k)(2k-1)}{6} \\ &= \frac{2(k-1)(k)(2k-1)}{3} \\ &= \frac{(2k-2)(2k)(2k-1)}{6} \\ &= \frac{n(n-1)(n+1)}{6}. \end{aligned}$$

Therefore, in either case,

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{n(n-1)(n+1)/6}{2} \cdot d^2 = \frac{n(n^2-1)}{12} \cdot d^2,$$

so

$$d^2 \leq \frac{12}{n(n^2-1)}(a_1^2 + a_2^2 + \cdots + a_n^2).$$

5. Let $\mathcal{P}$ be the set of all primes, and let M be a subset of $\mathcal{P}$ containing at least three elements. For any proper subset A of M, all of the prime factors of the number

$$-1 + \prod_{p \in A} p$$

are in M. Prove that $M = \mathcal{P}$.

Solution. First, we show that the primes 2, 3, and 5 are all in M. If 2 is not in M, then let p be any prime in M, so p is odd, and take $A = \{p\}$. Then $p - 1$ is even, so 2 is in M, contradiction. Hence, 2 is in M.

Next, we show that 3 is in M. Let p be a prime in M, other than 2. If $p = 3$, then we are done. If $p \equiv 1 \pmod 3$, then taking $A = \{p\}$, we see that 3 is in M. If $p \equiv 2 \pmod 3$, then $2p - 1 \equiv 0 \pmod 3$, so taking $A = \{2, p\}$, we see that 3 is in M. Taking $A = \{2, 3\}$, we see that 5 is in M.

Suppose that M contains a finite number of elements. Let $M = \{p_1, p_2, \dots, p_k\}$, where $p_1 < p_2 < \dots < p_k$, so $p_1 = 2$ and $p_2 = 3$, and $k \ge 3$. Let $P = p_3 p_4 \cdots p_k$. Taking $A = \{p_2, p_3, \dots, p_k\}$, we have that all the prime factors of

$$p_2 p_3 \cdots p_k - 1 = 3P - 1$$

are in M. But $3P - 1$ is relatively prime to all the elements in M, except $p_1 = 2$, so $3P - 1$ must be a power of 2. Let

$$3P - 1 = 2^c.$$

Taking $A = \{p_3, p_4, \dots, p_k\}$, we have that all the prime factors of

$$p_3 p_4 \cdots p_k - 1 = P - 1$$

are in M. But $P - 1$ is relatively prime to all the elements in M, except $p_1 = 2$ and $p_2 = 3$, so $P - 1$ must be the product of a power of 2 and a power of 3. Let

$$P - 1 = 2^a \cdot 3^b.$$

From the equations $3P - 1 = 2^c$ and $P - 1 = 2^a \cdot 3^b$, we get

$$2^a \cdot 3^{b+1} = 2^c - 2.$$

We have that $2^a \cdot 3^{b+1} \ge 3$, so $2^c - 2 \ge 3$, which means $c \ge 3$. Then $2^c - 2$ is even, so $a \ge 1$. Dividing both sides by 2, we get

$$2^{a-1} \cdot 3^{b+1} = 2^{c-1} - 1.$$

Then $2^{c-1} - 1$ is odd, so $a = 1$. Hence,

$$3^{b+1} = 2^{c-1} - 1.$$

Taking this equation modulo 3, we get $(-1)^{c-1} \equiv 1 \pmod 3$, so $c - 1$ is even. Let $c - 1 = 2k$, so

$$3^{b+1} = 2^{2k} - 1 = (2^k - 1)(2^k + 1).$$

Hence, both $2^k - 1$ and $2^k + 1$ are powers of 3, that differ by $(2^k + 1) - (2^k - 1) = 2$. But the only powers of 3 that differ by 2 are 3 and 1, so $2^k = 2$, which means $k = 1$, $c = 3$, and $b = 0$. Then $P = 2^a \cdot 3^b + 1 = 3$. But $P \geq 5$, contradiction. Therefore, M contains an infinite number of elements.

Let
$$M = \{p_1, p_2, p_3, \dots\},$$
and let q be an arbitrary prime. We claim that q is in M.

Consider the $q+1$ numbers $p_1 - 1$, $p_1p_2 - 1$, $\dots$, $p_1p_2 \cdots p_{q+1} - 1$. By the Pigeonhole principle, two of these numbers are congruent modulo q, say $p_1p_2 \cdots p_i - 1$ and $p_1p_2 \cdots p_j - 1$, where $1 \leq i < j \leq q+1$. We have that
$$p_1p_2 \cdots p_i - 1 \equiv p_1p_2 \cdots p_j - 1 \pmod{q}.$$
Then
$$p_1p_2 \cdots p_i - p_1p_2 \cdots p_j \equiv 0 \pmod{q},$$
and
$$p_1p_2 \cdots p_i(p_{i+1} \cdots p_j - 1) \equiv 0 \pmod{q}.$$
If q appears among p_1, p_2, $\dots$, p_i, then we are done. Otherwise, $p_1p_2 \cdots p_i$ is relatively prime to q, so
$$p_{i+1} \cdots p_j - 1 \equiv 0 \pmod{q},$$
which means q is in M. We conclude that all primes are in M.

6. Let x_1, x_2, ..., x_n be real numbers. Prove that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |x_i + x_j| \geq n \sum_{i=1}^{n} |x_i|.$$

Solution 1. Let

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= \sum_{i,j=1}^{n} |x_i + x_j|, \\ g(x_1, x_2, \dots, x_n) &= n \sum_{i=1}^{n} |x_i|, \\ h(x_1, x_2, \dots, x_n) &= f(x_1, x_2, \dots, x_n) - g(x_1, x_2, \dots, x_n). \end{aligned}$$

We want to prove that $h(x_1, x_2, \dots, x_n) \geq 0$.

Without loss of generality, assume that x_1, x_2, ..., $x_k \geq 0$ and x_{k+1}, x_{k+2}, ..., $x_n < 0$. Let $m = (x_1 + x_2 + \cdots + x_k)/k$. We claim that if we replace each of x_1, x_2, ..., x_k with m, then $h(x_1, x_2, \dots, x_n)$ does not increase.

Clearly, this operation does not change the value of $g(x_1, x_2, \dots, x_n)$. We can express $f(x_1, x_2, \dots, x_n)$ as

$$f(x_1, x_2, \dots, x_n) = \sum_{i,j=1}^{k} |x_i + x_j| + \sum_{i,j=k+1}^{n} |x_i + x_j| + 2 \sum_{i=1}^{k} \sum_{j=k+1}^{n} |x_i + x_j|.$$

The values of the first two terms do not change.

For a fixed value of j, the function $|x + x_j|$ is convex. Hence, by Jensen's theorem,

$$\sum_{i=1}^{k} |x_i + x_j| = |x_1 + x_j| + |x_2 + x_j| + \cdots + |x_k + x_j| \geq k|m + x_j|.$$

This holds for all $k + 1 \leq j \leq n$. We conclude that if we replace each of x_1, x_2, ..., x_k with m, then $f(x_1, x_2, \dots, x_n)$ does not increase, so $h(x_1, x_2, \dots, x_n)$ does not increase.

By a similar argument, if we replace each of x_{k+1}, x_{k+2}, ..., x_n with their arithmetic mean, then again $h(x_1, x_2, \dots, x_n)$ does not increase. Hence, it suffices to prove that $h(x_1, x_2, \dots, x_n) \geq 0$ in the case where $x_1 = x_2 = \cdots = x_k = a$ and $x_{k+1} = x_{k+2} = \cdots = x_n = -b$, where a and b are nonnegative real numbers. In this case, the inequality becomes

$$2k^2 a + 2(n-k)^2 b + 2k(n-k)|a - b| \geq kna + (n-k)nb.$$

Without loss of generality, assume that $a \geq b$, so

$$2k^2 a + 2(n-k)^2 b + 2k(n-k)(a - b) \geq kna + (n-k)nb.$$

This inequality simplifies to
$$kna + (n^2 - 5kn + 4k^2)b \geq 0.$$

Since $a \geq b$,
$$kna + (n^2 - 5kn + 4k^2)b \geq (n^2 - 4kn + 4k^2)b = (n - 2k)^2 b \geq 0,$$

as desired.

Solution 2. First, we prove some lemmas.

Lemma 1. For any real numbers a and b,
$$\min(a, b) = \frac{a + b - |a - b|}{2}.$$

Proof. The expressions on both sides are symmetric in a and b, so without loss of generality, assume that $a \leq b$. Then $\min(a, b) = a$ and
$$\frac{a + b - |a - b|}{2} = \frac{b + a - (b - a)}{2} = a.$$

Hence, the identity holds. ■

Lemma 2. For any real numbers a and b,
$$|a| + |b| - |a + b| = \begin{cases} 0 & \text{if } ab \geq 0, \\ 2\min(|a|, |b|) & \text{if } ab < 0. \end{cases}$$

Proof. If $a = 0$ or $b = 0$, then $|a| + |b| - |a + b| = 0$. If $a > 0$ and $b > 0$, then $|a| + |b| - |a + b| = a + b - (a + b) = 0$. If $a < 0$ and $b < 0$, then $|a| + |b| - |a + b| = (-a) + (-b) - [-(a + b)] = 0$. Hence, $|a| + |b| - |a + b| = 0$ if $ab \geq 0$.

Otherwise, $ab < 0$. Without loss of generality, assume that $a > 0$ and $b < 0$. Then $|a| + |b| - |a + b| = a - b - |a + b|$. By Lemma 1, $a - b - |a + b| = 2\min(a, -b) = 2\min(|a|, |b|)$. ■

Let P be the set of indices p such that $x_p \geq 0$, and let Q be the set of indices q such that $x_q < 0$. Then by Lemma 2,
$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n (|x_i| + |x_j| - |x_i + x_j|) &= \sum_{x_i x_j < 0} 2\min(|x_i|, |x_j|) \\ &= 2 \sum_{p \in P, q \in Q} 2\min(|x_p|, |x_q|) \\ &= 4 \sum_{p \in P, q \in Q} \min(|x_p|, |x_q|). \end{aligned}$$

But $\min(|x|, |y|) \le \sqrt{|x||y|}$ for all real numbers x and y, so

$$\begin{aligned} 4 \sum_{p\in P, q\in Q} \min(|x_p|, |x_q|) &\le 4 \sum_{p\in P, q\in Q} \sqrt{|x_p||x_q|} \\ &= 4 \left(\sum_{p\in P} \sqrt{|x_p|} \right) \left(\sum_{q\in Q} \sqrt{|x_q|} \right). \end{aligned}$$

By the AM-GM inequality,

$$\begin{aligned} 4 \left(\sum_{p\in P} \sqrt{|x_p|} \right) \left(\sum_{q\in Q} \sqrt{|x_q|} \right) &\le \left(\sum_{p\in P} \sqrt{|x_p|} + \sum_{q\in Q} \sqrt{|x_q|} \right)^2 \\ &= \left(\sum_{i=1}^n \sqrt{|x_i|} \right)^2. \end{aligned}$$

Finally, by the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^n \sqrt{|x_i|} \right)^2 \le n \sum_{i=1}^n |x_i|.$$

Hence,

$$\sum_{i=1}^n \sum_{j=1}^n (|x_i| + |x_j| - |x_i + x_j|) \le n \sum_{i=1}^n |x_i|.$$

Then

$$n \sum_{i=1}^n |x_i| + n \sum_{i=1}^n |x_i| - \sum_{i=1}^n \sum_{j=1}^n |x_i + x_j| \le n \sum_{i=1}^n |x_i|,$$

so

$$\sum_{i=1}^n \sum_{j=1}^n |x_i + x_j| \ge n \sum_{i=1}^n |x_i|.$$

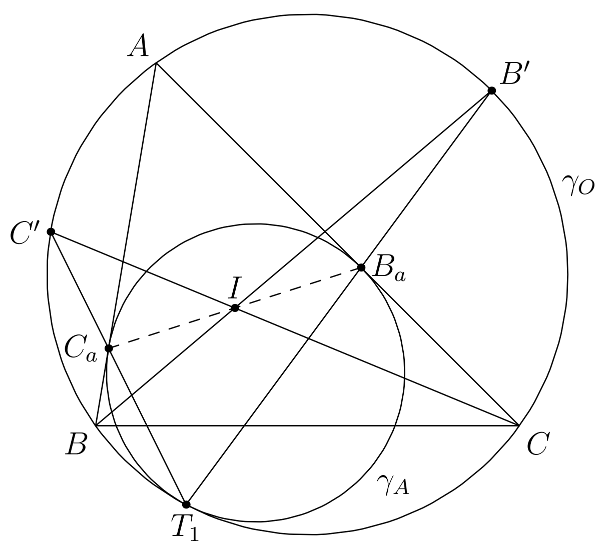
7. Let I be the incenter of triangle ABC. Let A_1 and A_2 be points on side BC such that $\angle BIA_1 = \angle CIA_2 = 90^\circ$, let B_1 and B_2 be points on side AC such that $CIB_1 = \angle AIB_2 = 90^\circ$, and let C_1 and C_2 be points on side AB such that $\angle AIC_1 = \angle BIC_2 = 90^\circ$.

   Let A', B', and C' be the midpoints of arcs BC, AC, and AB on the circumcircle of triangle ABC. Let $A'A_1$ intersect AC at A_1', let $A'A_2$ intersect AB at A_2', let $B'B_1$ intersect AB at B_1', let $B'B_2$ intersect BC at B_2', let $C'C_1$ intersect BC at C_1', and let $C'C_2$ intersect AC at C_2'. Prove that $A_1'A_2'$, $B_1'B_2'$, and $C_1'C_2'$ are concurrent.

   Solution. Let γ_O be the circumcircle of triangle ABC. First, we prove a lemma.

   Lemma. Let γ_A be the circle that is tangent to sides AB and AC, and internally tangent to γ_O. Then γ_A is tangent to AB and AC at C_1 and B_2, respectively.

   Proof. Let γ_A be tangent to AB, AC, and γ_O at C_a, B_a, and T_1, respectively.

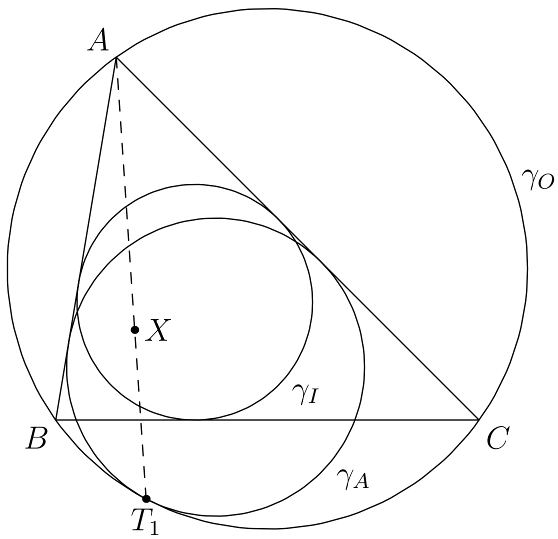


   There exists a homothety, centered at T_1, that takes γ_A to γ_O. This homothety takes the tangent to γ_A at B_a (namely AC) to a tangent to γ_O. These tangents must be parallel, so the image of B_a under the homothety must be the midpoint of arc AC, namely B'. Hence, T_1, B_a, and B' are collinear. Similarly, T_1, C_a, and C' are collinear. Note that BB' and CC' intersect at I.

   By Pascal's theorem on hexagon $ABB'T_1C'C$, I lies on B_aC_a. But $AB_a = AC_a$, so B_aC_a is perpendicular to AI. Therefore, $B_a = B_2$ and $C_a = C_1$. ■

   Similarly, we can define γ_B as the circle that is tangent to sides AB, BC, and internally tangent to γ_O, and γ_C as the circle that is tangent to sides AC, BC, and internally tangent to γ_O. Then γ_B is tangent to AB and BC at C_2 and A_1, respectively, and γ_C is tangent to AC and BC at B_1 and A_2, respectively. Let γ_B and γ_C be tangent to γ_O at T_2 and T_3, respectively.

Let γ_I be the incircle of triangle ABC, and let X be the external center of similitude of γ_O and γ_I. Note that A is the external center of similitude of γ_I and γ_A, and T_1 is the external center of similitude of γ_A and γ_O, so by Monge's theorem, X lies on AT_1. Similarly, X lies on BT_2 and CT_3.



Finally, by Pascal's theorem on hexagon $ABT_2A'T_3C$, X lies on $A_1'A_2'$. Similarly, X lies on $B_1'B_2'$ and $C_1'C_2'$. Thus, $A_1'A_2'$, $B_1'B_2'$, and $C_1'C_2'$ are all concurrent at X.