Mock Exam #2,
Jan 7, 2002

#1. Let $P(x)$ be a polynomial of degree at most n, such that

$$P(k) = 2^k, \quad k = 0, 1, 2, \ldots, n.$$

Find $P(n+1)$.

#2. An $m \times n$ checkerboard is painted red and black with the following property: For every black square, the number of adjacent (share an edge) black squares is odd. Prove that the number of black squares is even.

#3. Let $a_1 = \alpha$ and $b_1 = \beta$, where $\alpha, \beta \in \mathbb{R}$. Define

$$a_{n+1} = \alpha a_n - \beta b_n$$
$$b_{n+1} = \beta a_n + \alpha b_n$$

Find the number of pairs (α, β) such that

$$a_{1997} = a_1 \quad \text{and} \quad b_{1997} = b_1$$

#4. The quadrilateral $ABCD$ circumscribes a circle of radius r. The points of tangency are E, F, G, and H, on AB, BC, CD, and DA, respectively. Let r_1, r_2, r_3, and r_4 be the inradii of $\triangle EBF$, $\triangle FCG$, $\triangle GDH$, and $\triangle HAE$, respectively. Prove

$$r_1 + r_2 + r_3 + r_4 \geq 2(2 - \sqrt{2})r.$$

5. Let x, y and z be positive integers such that

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$$

and $\gcd(x, y, z)=1$. Prove that both $x-z$ and xyz are perfect squares.

6. Let a, b, c, α, β, and γ be positive real numbers such that $\alpha+\beta+\gamma=1$. Prove

$$\alpha a+\beta b+\gamma c+2\sqrt{(\alpha\beta+\beta\gamma+\gamma\alpha)(ab+bc+ca)} \leq a+b+c.$$

7. In an acute $\triangle ABC$, $|AC|>|BC|$ and C' is the midpoint of AB. Let AD be the altitude from A, BE the altitude from B, H the orthocenter, and let AB and DE intersect at R. Prove RH and CA' are perpendicular.