Summer 2005 IMO Camp

Trigonometry & Algebra

Problem 1.
Prove that if $0° < A < 180°$, then $\sin A + \frac{1}{2}\sin 2A + \frac{1}{3}\sin 3A > 0$.

Problem 2.
Prove that if A is an acute angle, then $(1 + \sec A)(1 + \csc A) > 5$.

Problem 3.
Prove that if a, b and c are positive real numbers, each less than 1, then the products $(1-a)b$, $(1-b)c$ and $(1-c)a$ cannot all be greater than $\frac{1}{4}$.

Problem 4.
Determine the maximum value of the expression $|a_1 - 1| + |a_2 - 2| + \cdots + |a_n - n|$, where n is a positive integer and $\langle a_1, a_2, \ldots, a_n\rangle$ is any permutation of $1, 2, \ldots, n$.

Problem 5.
Let a_1, b_1, c_1, a_2, b_2 and c_2 be real numbers such that for any integers x and y, at least one of $a_1x + b_1y + c_1$ and $a_2x + b_2y + c_2$ is an even integer. Prove that either all of a_1, b_1 and c_1 are integers or all of a_2, b_2 and c_2 are integers.

Problem 6.
Let x, y and z be distinct integers and let n be a non-negative integer. Prove that

$$\frac{x^n}{(x-y)(x-z)} + \frac{y^n}{(y-z)(y-x)} + \frac{z^n}{(z-x)(z-y)}$$

is an integer.

Number Theory

Problem 1.
Prove that if n is a positive odd integer, then $46^n + 296 \cdot 13^n$ is divisible by 1947.

Problem 2.
For which positive integers m is $(m-1)!$ divisible by m?

Problem 3.
Let u and v be integers such that $u^2 + uv + v^2$ is divisible by 9. Prove that each of u and v is divisible by 3.

Problem 4.
Prove that from any set of n positive integers, a non-empty subset can be chosen such that the sum of the numbers in the subset is divisible by n. The subset may be equal to the whole set.

Problem 5.
How many five-digit multiples of 3 contains the digit 6?

Problem 6.
Let n be a positive integer. Consider all ordered pairs (u, v) of positive integers such that the least common multiple of u and v is n. Prove that the number of such pairs is equal to the number of positive divisors of n^2.

Combinatorial Number Theory

Problem 1.
Let n be a positive integer and let d be a positive divisor of $2n^2$. Prove that $n^2 + d$ is not a perfect square.

Problem 2.
Which positive integers cannot be expressed as sums of two or more consecutive positive integers?

Problem 3.
In the infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers, $a_1 = 1$ and $a_k \leq 1 + a_1 + a_2 + \cdots + a_{k-1}$ for $k > 1$. Prove that every positive integer either appears in this sequence or can be expressed as the sum of distinct members of the sequence.

Problem 4.
Let n be an integer greater than 2. Two subsets of $\{1, 2, \ldots, n-1\}$ are chosen arbitrarily. Prove that if the total number of elements in the two sets is at least n, then there is one element from each subset such that their sum is n.

Problem 5.
Let n be an integer greater than 1. From the integers from 1 to $3n$, $n+2$ of them are chosen arbitrarily. Prove that among the chosen numbers, there exist two of them whose difference is strictly between n and $2n$.

Problem 6.
It was Saturday on the 23rd October, 1948. Can one conclude that New Year falls more often on Sundays than on Mondays?

Combinatorics

Problem 1.
A factory manufactures several kinds of cloth, using for each of them exactly two of six different colours of silk. Each colour appears on at least three kinds of cloth, each with a distinct second colour. Prove that there exist three kinds of cloth such that between them, all six colours are represented.

Problem 2.
Prove that in a round-robin tournament without ties, there must be a contestant who will list all of his opponents when he lists the ones whom he beats as well as the ones beaten by those whom he beats.

Problem 3.
Prove that in any group of six people, either there are three people who know one another or three people who do not know one another. Assume that "knowing" is a symmetric relation.

Problem 4.
Among any four members of a group of travellers, there is one who knows all the other three. Prove that among each foursome, there is one who knows all of the other travellers. Assume that "knowing" is a symmetric relation.

Problem 5.
On a certain day, a number of readers visited a library. Each went only once. Among any three readers, two of them met at the library on that day. Prove that there were two particular instants such that each reader was in the library at one of the two instants.

Problem 6.
On a certain day, three men visited a friend who is hospitalized. On the same day, their wives did likewise. None of the six visitors went to the sickroom more than once. Each man met the wives of the other two in the sickroom. Prove that at least one of them met his own wife in the sickroom.

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Combinatorial Geometry

Problem 1.
Let p and q be integers greater than 1. There are pq chairs, arranged in p rows and q columns. Each chair is occupied by a student of different height. A teacher chooses the shortest student in each row; among these the tallest one is of height a. The teacher then chooses the tallest student in each column; among these the shortest one is of height b. Determine which of $a < b, a = b$ and $a > b$ are possible by an appropriate rearrangement of the seating of the students.

Problem 2.
Prove that it is impossible to choose more than n diagonals of a convex polygon with n sides, such that every pair of them having a common point.

Problem 3.
Prove that except for any tetrahedron, no convex polyhedron has the property that every two vertices are connected by an edge. Degenerate polyhedra are not considered.

Problem 4.
A plane can be covered completely by four half-planes. Prove that three of these four half-planes are sufficient for covering the plane completely.

Problem 5.
The radius of each small disc is half that of the large disc. How many small discs are needed to cover the large disc completely?

Problem 6.
The three vertices of a certain triangle are lattice points. There are no other lattice points on its perimeter but there is exactly one lattice point in its interior. Prove that this lattice point is the centroid of the triangle.

Circle Geometry

Problem 1.
Let P be any point on the base of a given isosceles triangle. Let Q and R be the intersections of the equal sides with lines drawn through P parallel to these sides. Prove that the reflection of P about the line QR lies on the circumcircle of the given triangle.

Problem 2.
$ABCD$ is a square of side a. E is a point on BC with $BE = \frac{a}{3}$. F is a point on DC extended with $CF = \frac{a}{2}$. Prove that the point of intersection of AE and BF lies on the circumcircle of $ABCD$.

Problem 3.
In the square $ABCD$, E is the midpoint of AB, F is on the side BC and G on CD such that AG and EF are parallel. Prove that FG is tangent to the circle inscribed in the square.

Problem 4.
Three circles k_1, k_2 and k_3 on a plane are mutually tangent at three distinct points. The point of tangency of k_1 and k_2 is joined to the other two points of tangency. Prove that these two segments or their extensions intersect k_3 at the endpoints of one of its diameters.

Problem 5.
Given two circles, exterior to each other, a common inner and a common outer tangents are drawn. The resulting points of tangency define a chord in each circle. Prove that the point of intersection of these two chords or their extensions is collinear with the centres of the circles.

Problem 6.
The centres of three mutually disjoint circles lie on a line. Prove that if a fourth circle is tangent to each of them, then its radius is at least the sum of the radii of the other three circles.

Geometric Inequalities

Problem 1.
Prove that if a triangle is not obtuse, then the sum of the lengths of its medians is not less than or equal to four times the radius of the triangle's circumcircle.

Problem 2.
Let $\frac{1}{2} < k < 1$. Let A', B' and C' be points on the sides BC, CA and AB, respectively, of the triangle ABC such that $BA' = kBC, CB' = kCA$ and $AC' = kAB$. Prove that the length of the perimeter of the triangle $A'B'C'$ does not exceed k times the length of the perimeter of ABC.

Problem 3.
In a convex quadrilateral $ABCD$, $AB + BD \leq AC + CD$. Prove that $AB \leq AC$.

Problem 4.
Prove that if the base angles of a trapezium are unequal, then the diagonal passing through the vertex of the smaller base angle is longer than the other diagonal.

Problem 5.
Six points are given on the plane, no three collinear. Prove that three of these points determine a triangle with an interior angle not less than $120°$.

Problem 6.
Consider the six distances determined by four points in a plane. Prove that the ratio of the largest of these distances to the smallest cannot be less than $\sqrt{2}$.

Hexagons & Solid Geometry

Problem 1.
In the convex hexagon $ABCDEF$, the sum of the interior angles at A, C and E is equal to the sum of the interior angles at B, D and F. Prove that opposite angles of the hexagon are equal.

Problem 2.
$ABCDEF$ is a convex hexagon in which opposite edges are parallel. Prove that triangles ACE and BDF have equal area.

Problem 3.
A vertical pole stands on a horizontal plane. The distances from the base of the pole to three other points on the plane are 100, 200 and 300 metres respectively. The sum of the angles of elevation from these three points to the top of the pole is $90°$. What is the height of the pole?

Problem 4.
Let ABC be an acute triangle. Consider the set of all tetrahedra with ABC as base such that all lateral faces are acute triangles. Find the locus of the projection onto the plane of ABC of the vertex of the tetrahedron which ranges over the above set.

Problem 5.
Let P be a point in or on a tetrahedron $ABCD$ which does not coincide with D. Prove that at least one of the distances PA, PB and PC is shorter than at least one of the distances DA, DB and DC.

Problem 6.
Prove that if every planar section of a three-dimensional solid is a circle, then the solid is a sphere.