1998 IMO Camp

Scrumptious Sequences

Served with your choice of rice, fries,
salad, $soup_0$, $soup_1$, $soup_2$, $soup_3$, ...

1. The sequence $\{u_n\}$ of real numbers is defined by $u_0 = 2$, $u_1 = 5$, and

$$u_{n+1}u_{n-1} - u_n^2 = 6^{n-1} \qquad n \geq 1$$

Prove that each u_n is an integer.

2. An integer sequence is defined by $a_n = 2a_{n-1} + a_{n-1}$ $(n > 1)$, $a_0 = 0$ and $a_1 = 1$. Prove that 2^k divides a_n if and only if 2^k divides n. (IMO 1988 shortlist)

3. Let a be the greatest positive root of the equation $x^3 - 3x^2 + 1 = 0$. Show that $\lfloor a^{1788} \rfloor$ and $\lfloor a^{1988} \rfloor$ are both divisible by 17, where $\lfloor x \rfloor$ denotes the integer part of x. (IMO 1988 shortlist)

4. The sequence $\{a_n\}$ of integers is defined by $a_1 = 2$, $a_2 = 7$ and

$$-\frac{1}{2} < a_{n+1} - \frac{a_n^2}{a_{n-1}} \leq \frac{1}{2}$$

for $n \geq 2$. Prove that a_n is odd for all $n > 1$. (IMO 1988 shortlist)

5. If $u_0 = -\frac{14}{3}$ and $u_{n+1} = \frac{u_n + 6}{u_n + 2}$ for $n = 0, 1, 2, \ldots$ determine the value of u_{1984}.

6. A sequence $\{a_n\}$ of real numbers is defined by $a_1 = A$, $a_2 = B$ and

$$a_n = \frac{a_{n-1}^2 + C}{a_{n-2}}$$

for all $n > 2$. If A, B and $\frac{A^2 + B^2 + C}{AB}$ are all integers, show that a_n is an integer for all n.

7. A sequence $\{a_n\}$ is defined by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{16}\left(1 + 4a_n + \sqrt{1 + 24a_n}\right)$$

Find an explicit formula for a_n.

8. Define a sequence $\{a_n\}$ by $a_n = \sqrt{p_n p_{n+1} + 1}$ where p_n denotes the n^{th} prime number. Show that a_n is an integer for infinitely many values of n.

9. Let $\{X_n\}$ and $\{Y_n\}$ be two sequences defined as follows:

$$X_0 = 1, \qquad X_1 = 1, \qquad X_{n+1} = X_n + 2X_{n-1} \quad (n \geq 1)$$

$$Y_0 = 1, \qquad Y_1 = 7, \qquad Y_{n+1} = 2Y_n + 3Y_{n-1} \quad (n \geq 1)$$

Prove that, except for the '1', there is no integer which appears in both sequences. (USAMO 1973 #2).

10. Let $a_1, a_2, \ldots, a_8$ be eight real numbers, not all equal to zero. Define a sequence $\{c_n\}$ by

$$c_n = a_1^{\,n} + a_2^{\,n} + \ldots + a_8^{\,n} \quad (n \geq 1)$$

If an infinite number of terms of the sequence $\{c_n\}$ are zero, then find all natural numbers n for which $c_n = 0$. (IMO 1967 # 5).

11. Let $a_0 = 0$ and

$$a_{n+1} = k(a_n + 1) + (k+1)a_n + 2\sqrt{k(k+1)a_n(a_n+1)} \qquad \text{n} = 0, 1, 2, \ldots$$

where k is a positive integer. Prove that a_n is a positive integer for all $n \geq 1$.

12. For every real number x, construct the sequence $x_1, x_2, \ldots$ by setting

$$x_{n+1} = x_n\left(x_n + \frac{1}{n}\right)$$

for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which

$$0 < x_n < x_{n+1} < 1$$

for all n. (IMO 1985 # 6).