(Adrian)

IMO Training 2007: WarmUp

1. Find all solutions (including non-real ones) to the following system of equations.

$$\begin{array}{ccccccc} x & + & y & + & z & = & 3 \\ x^2 & + & y^2 & + & z^2 & = & 3 \\ x^3 & + & y^3 & + & z^3 & = & 3 \end{array}$$

2. Let f be a polynomial with integer coefficients. Prove that there does not exist three distinct integers a, b, c such that $f(a) = b,\ f(b) = c,\ f(c) = a$.

3. A polynomial with real coefficients with degree 2007 satisfies $f(n) = 2^n$ for $n = 0, 1, \cdots, 2007$. Find $f(2008)$.

4. Let a, b, c, d be any positive real numbers. Prove that

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}$$

5. A sequence of integers $a_0, a_1, \cdots$ satisfies $a_0 = 0, a_1 = 1$ and $a_{n+2} = 2a_{n+1} + a_n$ for $n \geq 0$. Prove that 2^k divides a_n if and only if 2^k divides n.

6. Let ABC be a triangle. Prove that there exists a unique point P such that

$$PA^2 + PB^2 + AB^2 = PB^2 + PC^2 + BC^2 = PC^2 + PA^2 + CA^2$$

7. ABC is a triangle with $\angle A = 60^o$ and incentre I. P lies on BC with $3BP = BC$. The point F lies on AB and IF is parallel to AC. Show that $\angle BFP = \angle FBI$.

8. Let $n > 2$ be any fixed positive integer and f be a function that maps points on the Cartesian plane to the real numbers such that any n points, $P_1, \cdots, P_n$ that form a regular n-gon satisfy

$$f(P_1) + \cdots + f(P_n) = 0$$

   Prove that $f(P) = 0$ for all points P in the Cartesian plane.

9. Let n be a positive integer. The squares of a $n \times n$ chessboard are each coloured black or white such that every white square is adjacent to a black square and for any two distinct black squares, there exists a chain of black squares joining these two squares so that any two consecutive squares in the chain are adjacent. (Two squares are adjacent if they share a common side.) Prove that there are at least $(n^2 - 2)/3$ black squares on the board.

10. For any positive integer n, a partition of n is defined to be a finite sequence of positive integers $a_1, a_2, \cdots, a_t$ such that $a_1 + a_2 + \cdots + a_t = n$ and $a_1 \leq a_2 \leq \cdots \leq a_t$. The integers $a_1, a_2, \cdots a_t$ are each called parts of the partition.Let $a(n)$ be the number of partitions of n consisting of only odd parts. Let $b(n)$ be the number of partitions of n consisting of pairwise distinct parts. Prove that $a(n) = b(n)$ for all positive integers n.