1 Green Homework 2010

1.1 Algebra

1. (Ireland 2009 Paper 2 - 1) Let $p(x)$ be a polynomial with rational coefficients. Prove that there exists a positive integer n such that the polynomial $q(x)$ defined by

$$q(x) = p(x+n) - p(x)$$

has integer coefficients.

2. (British 09/10 Round 1-1) Find all integers x, y and z such that

$$x^2 + y^2 + z^2 = 2(yz+1) \text{ and } x + y + z = 4018.$$

3. (Turkey TST 2009-1) Find all functions $f : Q^+ \to Z$ that satisfy $f\left(\frac{1}{x}\right) = f(x)$ and $(x+1)f(x-1) = xf(x)$ for all rational numbers $x > 1$.

4. (Canada 2009-3) Define

$$f(x,y,z) = \frac{(xy+yz+xz)(x+y+z)}{(x+y)(x+z)(y+z)}.$$

Determine the set of real numbers r for which there exists a triplet of real numbers (x, y, z) satisfying $f(x, y, z) = r$.

5. (Canada 2010-5) Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Let $a_n = n! + n$. Show that if $P(a_n)/Q(a_n)$ is an integer for every n, then $P(n)/Q(n)$ is an integer for every integer n such that $Q(n) \neq 0$.

1.2 Combinatorics

1. (British 09/10 Round 1-3) Isaac attempts all six questions on an Olympiad paper in order. Each question is marked on a scale from 0 to 10. He never scores more in a later question than in any earlier question. How many different possible sequences of six marks can be achieved?

2. (Canada 2009-2) Two circles of different radii are cut out of cardboard. Each circle is subdivided into 200 equal sectors. On each circle 100 sectors are painted white and the other 100 are painted black. The smaller circle is then placed on top of the larger circle, in such a way that their centers coincide. Show that one can rotate the small circle so that the sectors on the two circles line up and at least 100 sectors on the small circle lie over sectors of the same color on the big circle.

3. (Ireland 2009 Paper 1 - 4) Given an n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ where each $x_i = +1$ or -1, form a new n-tuple

$$(x_1x_2, x_2x_3, x_3x_4, \ldots, x_nx_1),$$

and continue to repeat this operation. Show that if $n = 2^k$ for some integer $k \geq 1$, then after a certain number of repetitions of the operation, we obtain the n-tuple

$$(1, 1, 1, \ldots, 1).$$

4. (British 09/10 Round 2-1) There are 2010^{2010} children at a mathematics camp. Each has at most three friends at the camp and if A is friends with B, then B is friends with A. The camp leader would like to line the children up so that there are at most 2010 children between any pair of friends. Is it always possible to do this?

1.3 Number Theory

1. (Ireland 2009 Paper 1 - 3) Find all positive integers n for which $n^8 + n + 1$ is a prime number.

2. (Japan 2009-1) Find all positive integers n such that $8^n + n$ is divisible by $2^n + n$.

3. (Canada 2009-4) Find all pairs (a, b) of integers such that $3^a + 7^b$ is a perfect square.

4. (Ireland 2009 Paper 2 - 2) For any positive integer n define

$$E(n) = n(n+1)(2n+1)(3n+1)\cdots(10n+1).$$

   Find the greatest common divisor of $E(1), E(2), E(3), \ldots, E(2009)$.

1.4 Geometry

1. (Irish 2009 Paper 2 - 5) In the triangle ABC we have $|AB| < |AC|$. The bisectors of the angles at B and C meet AC and AB at D and E respectively. BD and CE intersect at the incenter I of triangle ABC. Prove that $\angle BAC = 60^\circ$ if and only if $|IE| = |ID|$.

2. (Canada 2010 - 2) Let A, B, P be three points on a circle. Prove that if a and b are the distances from P to the tangents at A and B and c is the distance from P to the chord AB, then $c^2 = ab$.

3. (British 09/10 Round 2-2) In triangle ABC the centroid is G and D is the midpoint of CA. The line through G parallel to BC meets AB at E. Prove that $\angle AEC = \angle DGC$ if and only if $\angle ACB = 90^\circ$. *The centroid of a triangle is the intersection of the three medians, the lines which join each vertex to the midpoint of the opposite side.*

4. (British 09/10 Round 1-4) Two circles of different radius with centers at B and C touch externally at A. A common tangent, not through A touches the first circle at D and the second at E. The line through A which is perpendicular to DE and the perpendicular bisector of BC meet at F. Prove that $BC = 2AF$.