Art of Problem Solving

WOOT 2010–11

Practice Olympiad 1

Instructions

- You should take the test under "olympiad conditions," meaning that the test should be completed in one sitting, with handwritten solutions (just like on an actual olympiad exam). Take the test using only the resources that would be available to you on an actual olympiad, meaning you should use scrap paper, a ruler, a compass, etc., but no calculators and no reference materials.
- You should allot 3 hours to take the test.
- Completely fill out the cover sheet and make sure it is the first page of your solutions.
- On the WOOT Home Page, there is a **WOOT Practice Olympiad Answer Sheet**. Print out (or copy) several blank copies of the answer sheets, and write all of your work on these sheets. Use only black pen or very dark pencil. Make sure that the top of every answer sheet page is completely filled out. Each problem's solution should start on a new page, along with new page numbering.
- Do not discuss the problems on or before the due date of Wednesday, October 13, 2010.

How to submit your solutions

You can submit solutions by upload, email, or by fax. DO NOT MAIL YOUR SOLUTIONS!

By Upload: In your "My Classes" area, follow the link that says "Submit."

By email: Scan your solutions as a single PDF file. **Check to make sure that the file is legible before emailing it!** Email to `woot@artofproblemsolving.com`. Put "WOOT Practice Olympiad 1" in the subject line, attach your solutions as a single PDF file, and write something in the message body - if you leave the message body blank, it will get blocked by our spam filters.

By fax: Fax to (619)659-8146.

Solutions are due by Wednesday, October 13, 2010

Late submissions will not be accepted except under extraordinary circumstances

WOOT Practice Olympiad Cover Sheet

Username: ____________________

Class ID: __________

User ID: ____________________
(Your Class ID and User ID can be found in the "My Classes" section of the website)

Practice Olympiad Number: 1

Beginning	
Intermediate	
Advanced	

Number of pages (including cover sheet): ____

1. For positive integers n, the sequence a_1, a_2, $\ldots$ is defined by $a_1 = 1$,

$$a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + \cdots + a_{n-1})$$

for $n > 1$. Determine the value of a_{2010}.

2. Find all positive integers n such that $1 + 2^2 + 3^3 + 4^n$ is a perfect square.

3. For a partition π of $\{1, 2, 3, \ldots, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. For example, for the partition π given by $\{1, 4, 6, 7\} \cup \{2, 8\} \cup \{3\} \cup \{5, 9\}$, $\pi(5) = 2$, $\pi(6) = 4$, and $\pi(3) = 1$.

   Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, \ldots, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

4. (a) Show that for every positive integer n, there exists a polynomial P_n such that

$$t^n + \frac{1}{t^n} = P_n\left(t + \frac{1}{t}\right)$$

   for all $t \neq 0$.

   (b) Find all positive integers n for which there exists a polynomial Q_n such that

$$t^n - \frac{1}{t^n} = Q_n\left(t - \frac{1}{t}\right)$$

   for all $t \neq 0$.

5. 49 students solve a set of three problems. The score for each problem is an integer between 0 and 7. Prove that there are two students, such that for each problem, the first student scored at least as many as the second student.

6. Let n be a positive integer, and let a_1, a_2, $\ldots$, a_n be distinct positive integers. Prove that

$$(a_1^7 + a_2^7 + \cdots + a_n^7) + (a_1^5 + a_2^5 + \cdots + a_n^5) \geq 2(a_1^3 + a_2^3 + \cdots + a_n^3)^2.$$

7. Show that the set of positive integers that cannot be represented as a sum of distinct perfect squares is finite.

Remember . . .

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- Complete the WOOT Practice Olympiad Cover Sheet (included in this document), and make it the first page of your solutions. We will not accept your solutions without this cover sheet.
- Do not mail your solutions. Use only upload, e-mail, or fax. If you use e-mail, send your solutions to

  woot@artofproblemsolving.com,

  NOT **classes**@artofproblemsolving.com. If you use e-mail, we will only accept solutions in PDF format. In particular, solutions in JPG or Word format will not be accepted.