

Algebra

- [1] Let a_{ij} (with the indices i and j from the set $\{1, 2, 3\}$) be real numbers such that $a_{ij} > 0$ for $i = j$; $a_{ij} < 0$ for $i \neq j$.
Prove the existence of positive real numbers c_1, c_2, c_3 such that the numbers $a_{11}c_1 + a_{12}c_2 + a_{13}c_3, a_{21}c_1 + a_{22}c_2 + a_{23}c_3, a_{31}c_1 + a_{32}c_2 + a_{33}c_3$ are either all negative, or all zero, or all positive.
- [2] Find all nondecreasing functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that (i) $f(0) = 0, f(1) = 1$; (ii) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a, b such that $a < 1 < b$.
- [3] Consider two monotonically decreasing sequences (a_k) and (b_k) , where $k \geq 1$, and a_k and b_k are positive real numbers for every k . Now, define the sequences $c_k = \min(a_k, b_k)$; $A_k = a_1 + a_2 + \dots + a_k$; $B_k = b_1 + b_2 + \dots + b_k$; $C_k = c_1 + c_2 + \dots + c_k$ for all natural numbers k .
(a) Do there exist two monotonically decreasing sequences (a_k) and (b_k) of positive real numbers such that the sequences (A_k) and (B_k) are not bounded, while the sequence (C_k) is bounded?
(b) Does the answer to problem (a) change if we stipulate that the sequence (b_k) must be $b_k = \frac{1}{k}$ for all k ?
- [4] Let n be a positive integer and let $x_1 \leq x_2 \leq \dots \leq x_n$ be real numbers. Prove that

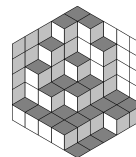
$$\left(\sum_{i,j=1}^n |x_i - x_j| \right)^2 \leq \frac{2(n^2 - 1)}{3} \sum_{i,j=1}^n (x_i - x_j)^2.$$

Show that the equality holds if and only if x_1, \dots, x_n is an arithmetic sequence.

- [5] Let \mathbb{R}^+ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ that satisfy the following conditions:
- $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;
- $f(x) < f(y)$ for all $1 \leq x < y$.
- [6] Let n be a positive integer and let $(x_1, \dots, x_n), (y_1, \dots, y_n)$ be two sequences of positive real numbers. Suppose (z_2, \dots, z_{2n}) is a sequence of positive real numbers such that $z_{i+j}^2 \geq x_i y_j$ for all $1 \leq i, j \leq n$.



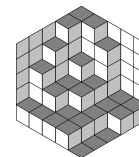
IMO Shortlist 2003



Let $M = \max\{z_2, \dots, z_{2n}\}$. Prove that

$$\left(\frac{M + z_2 + \dots + z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1 + \dots + x_n}{n}\right)\left(\frac{y_1 + \dots + y_n}{n}\right).$$

comment: *Edited by Orl.*



Combinatorics

- [1] Let A be a 101-element subset of the set $S = \{1, 2, \dots, 1000000\}$. Prove that there exist numbers t_1, t_2, \dots, t_{100} in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \quad j = 1, 2, \dots, 100$$

are pairwise disjoint.

- [2] Let D_1, D_2, \dots, D_n be closed discs in the plane. (A closed disc is the region limited by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs D_i . Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1 = 14020$ other discs D_i .

- [3] Let $n \geq 5$ be an integer. Find the maximal integer k such that there exists a polygon with n vertices (convex or not, but not self-intersecting!) having k internal 90° angles.

- [4] Given n real numbers x_1, x_2, \dots, x_n , and n further real numbers y_1, y_2, \dots, y_n . The elements a_{ij} (with $1 \leq i, j \leq n$) of an $n \times n$ matrix are defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } x_i + y_j \geq 0; \\ 0 & \text{if } x_i + y_j < 0. \end{cases}$$

Further, let B be an $n \times n$ matrix whose elements are numbers from the set $0; 1$ satisfying the following condition: The sum of all elements of each row of B equals the sum of all elements of the corresponding row of A ; the sum of all elements of each column of B equals the sum of all elements of the corresponding column of A . Show that in this case, $A = B$.

comment: (This one is from the ISL 2003, but in any case, [[url=http://www.bundeswettbewerb-mathematik.de/imo/aufgaben/aufgaben.htm](http://www.bundeswettbewerb-mathematik.de/imo/aufgaben/aufgaben.htm)]the official problems and solutions - in German -[/url] are already online, hence I take the liberty to post it here.)

Darij

- [5] Regard a plane with a Cartesian coordinate system; for each point with integer coordinates, draw a circular disk centered at this point and having the radius $\frac{1}{1000}$.

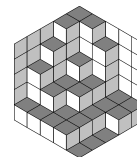
a) Prove the existence of an equilateral triangle whose vertices lie in the interior of different disks;

b) Show that every equilateral triangle whose vertices lie in the interior of different disks has a sidelength ≤ 96 .

Radu Gologan, Romania

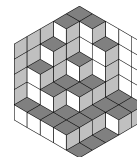


IMO Shortlist 2003



- 6 Let $f(k)$ be the number of all non-negative integers n satisfying the following conditions:
- (1) The integer n has exactly k digits in the decimal representation (where the first digit is not necessarily non-zero!), i. e. we have $0 \leq n < 10^k$.
 - (2) These k digits of n can be permuted in such a way that the resulting number is divisible by 11.

Show that for any positive integer number m , we have $f(2m) = 10f(2m - 1)$.



Geometry

- 1 Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB , respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC .
- 2 Given three pairwise distinct points A, B, C lying on one straight line in this order. Let G be a circle through A and C with center not on the line AC . The tangents to G at A and C meet at P ; the segment PB meets G at Q .
Show that the point of intersection of the angle bisector of the angle AQC with the line AC does not depend on the choice of the circle G .
- 3 Let ABC be a triangle, and P a point in the interior of this triangle. Let D, E, F be the feet of the perpendiculars from the point P to the lines BC, CA, AB , respectively. Assume that $AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2$.
Furthermore, let I_a, I_b, I_c be the excenters of triangle ABC . Show that the point P is the circumcenter of triangle $I_a I_b I_c$.
- 4 Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be distinct circles such that Γ_1, Γ_3 are externally tangent at P , and Γ_2, Γ_4 are externally tangent at the same point P . Suppose that Γ_1 and Γ_2 ; Γ_2 and Γ_3 ; Γ_3 and Γ_4 ; Γ_4 and Γ_1 meet at A, B, C, D , respectively, and that all these points are different from P . Prove that

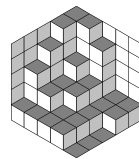
$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

- 5 Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I . Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC . The lines through P parallel to CA and CB meet AB at D and E , respectively. The line through P parallel to AB meets CA and CB at F and G , respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC .

comment: (According to my team leader, last year some of the countries wanted a geometry question that was even easier than this...that explains IMO 2003/4...)

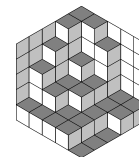
[Note by Darij: This was also Problem 6 of the German pre-TST 2004, written in December 03.]

Edited by Orl.



- [6] Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\frac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.
- [7] Let ABC be a triangle with semiperimeter s and inradius r . The semicircles with diameters BC , CA , AB are drawn on the outside of the triangle ABC . The circle tangent to all of these three semicircles has radius t . Prove that

$$\frac{s}{2} < t \leq \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right)r.$$



Number Theory

- [1] Let m be a fixed integer greater than 1. The sequence x_0, x_1, x_2, \dots is defined as follows:

$$x_i = 2^i \text{ if } 0 \leq i \leq m-1 \text{ and } x_i = \sum_{j=1}^m x_{i-j}, \text{ if } i \geq m.$$

- [2] Each positive integer a is subjected to the following procedure, yielding the number $d = d(a)$:

(a) The last digit of a is moved to the first position. The resulting number is called b . (b) The number b is squared. The resulting number is called c . (c) The first digit of c is moved to the last position. The resulting number is called d .

(All numbers are considered in the decimal system.) For instance, $a = 2003$ gives $b = 3200$, $c = 10240000$ and $d = 02400001 = 2400001 = d(2003)$.

Find all integers a such that $d(a) = a^2$.

- [3] Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

- [4] Let b be an integer greater than 5. For each positive integer n , consider the number

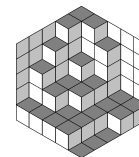
$$x_n = \underbrace{11 \dots 1}_{n-1} \underbrace{22 \dots 2}_n 5,$$

written in base b .

Prove that the following condition holds if and only if $b = 10$:

there exists a positive integer M such that for any integer n greater than M , the number x_n is a perfect square.

- [5] An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property: m can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.
- [6] Let p be a prime number. Prove that there exists a prime number q such that for every integer n , the number $n^p - p$ is not divisible by q .



- 7 The sequence a_0, a_1, a_2, \dots is defined as follows: $a_0 = 2$, $a_{k+1} = 2a_k^2 - 1$ for $k \geq 0$. Prove that if an odd prime p divides a_n , then 2^{n+3} divides $p^2 - 1$.

comment: Hi guys ,

Here is a nice problem:

Let be given a sequence a_n such that $a_0 = 2$ and $a_{n+1} = 2a_n^2 - 1$. Show that if p is an odd prime such that $p|a_n$ then we have $p^2 \equiv 1 \pmod{2^{n+3}}$

Here are some further question proposed by me : Prove or disprove that : 1) $\gcd(n, a_n) = 1$ 2) for every odd prime number p we have $a_m \equiv \pm 1 \pmod{p}$ where $m = \frac{p^2 - 1}{2^k}$ where $k = 1$ or 2

Thanks kiu si u

Edited by Orl.

- 8 Let p be a prime number and let A be a set of positive integers that satisfies the following conditions: **(1)** the set of prime divisors of the elements in A consists of $p - 1$ elements; **(2)** for any nonempty subset of A , the product of its elements is not a perfect p -th power. What is the largest possible number of elements in A ?