Solutions to Exercises

1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = 1 - x - y$$

for all real numbers x, y. (Slovenia, 1999)

Solution. Setting $y = 0$, we get

$$f(x - f(0)) = 1 - x$$

for all x.

Let $c = f(0)$, and let $t = x - c$, so $x = t + c$, and

$$f(t) = 1 - (t + c) = 1 - t - c$$

for all t. Setting $t = 0$, we get $f(0) = 1 - c$. But $f(0) = c$, so $1 - c = c$, which means $c = \frac{1}{2}$. Therefore, the solution is $f(x) = \frac{1}{2} - x$. It is easy to verify that this solution works.

Note. Another substitution that works nicely is setting $x = f(y)$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f(x) + f(1 - x) = 2x - x^4$$

for all $x \in \mathbb{R}$.

Solution. Substituting $1 - x$ for x, we get

$$(1 - x)^2 f(1 - x) + f(x) = 2(1 - x) - (1 - x)^4 = -x^4 + 4x^3 - 6x^2 + 2x + 1.$$

Thus, we have the following system of equations in $f(x)$ and $f(1 - x)$:

$$\begin{aligned} x^2 f(x) + f(1 - x) &= -x^4 + 2x, \\ f(x) + (1 - x)^2 f(1 - x) &= -x^4 + 4x^3 - 6x^2 + 2x + 1. \end{aligned}$$

Multiplying the first equation by $(1 - x)^2$, we get

$$\begin{aligned} x^2 (x - 1)^2 f(x) + (1 - x)^2 f(1 - x) &= (1 - x)^2 (-x^4 + 2x) \\ &= -x^6 + 2x^5 - x^4 + 2x^3 - 4x^2 + 2x. \end{aligned}$$

Subtracting the equation $f(x) + (1 - x)^2 f(1 - x) = -x^4 + 4x^3 - 6x^2 + 2x + 1$, we get

$$[x^2 (x - 1)^2 - 1] f(x) = -x^6 + 2x^5 - 2x^3 + 2x^2 - 1,$$

which factors as

$$(x^2 - x - 1)(x^2 - x + 1)f(x) = -(x^2 - x - 1)(x^2 - x + 1)(x^2 - 1).$$

Now, the quadratic equation $x^2 - x + 1 = 0$ has no real roots, so we may safely divide both sides by $x^2 - x + 1$. However, the quadratic equation $x^2 - x - 1 = 0$ does have real roots, namely $x = \frac{1 \pm \sqrt{5}}{2}$. So, if x is not a root of $x^2 - x - 1 = 0$, then we may divide both sides by $x^2 - x - 1$ to get

$$f(x) = -(x^2 - 1) = 1 - x^2.$$

We then check this solution: Let x be a real number that is not a root of $x^2 - x - 1 = 0$. Since the roots of $x^2 - x - 1 = 0$ add up to 1, $1 - x$ is also not a root of $x^2 - x - 1 = 0$. Therefore, $f(x) = 1 - x^2$ and $f(1 - x) = 1 - (1 - x)^2$, so

$$\begin{aligned} x^2 f(x) + f(1 - x) &= x^2(1 - x^2) + 1 - (1 - x)^2 \\ &= x^2 - x^4 + 1 - 1 + 2x - x^2 \\ &= 2x - x^4. \end{aligned}$$

Thus, the function $f(x) = 1 - x^2$ works, when x is not a root of $x^2 - x - 1 = 0$. But what if x is a root of $x^2 - x - 1 = 0$?

To take a closer look at this case, take $x = \frac{1+\sqrt{5}}{2}$ and $x = \frac{1-\sqrt{5}}{2}$, respectively, in the given functional equation. This gives us the system of equations

$$\begin{aligned} \frac{3 + \sqrt{5}}{2} f\left(\frac{1 + \sqrt{5}}{2}\right) + f\left(\frac{1 - \sqrt{5}}{2}\right) &= \frac{-5 - \sqrt{5}}{2}, \\ \frac{3 - \sqrt{5}}{2} f\left(\frac{1 - \sqrt{5}}{2}\right) + f\left(\frac{1 + \sqrt{5}}{2}\right) &= \frac{-5 + \sqrt{5}}{2}. \end{aligned}$$

To make these equations easier to work with, let $a = f(\frac{1+\sqrt{5}}{2})$ and $b = f(\frac{1-\sqrt{5}}{2})$, so these equations become

$$\begin{aligned} \frac{3 + \sqrt{5}}{2} a + b &= \frac{-5 - \sqrt{5}}{2}, \\ \frac{3 - \sqrt{5}}{2} b + a &= \frac{-5 + \sqrt{5}}{2}. \end{aligned}$$

From the first equation,

$$b = \frac{-5 - \sqrt{5}}{2} - \frac{3 + \sqrt{5}}{2} a.$$

Substituting into the second equation, the left-hand side becomes

$$\begin{aligned}\frac{3-\sqrt{5}}{2}b+a &= \frac{3-\sqrt{5}}{2}\left(\frac{-5-\sqrt{5}}{2}-\frac{3+\sqrt{5}}{2}a\right)+a\\ &= \frac{(3-\sqrt{5})(-5-\sqrt{5})}{4}-\frac{(3-\sqrt{5})(3+\sqrt{5})}{4}a+a\\ &= \frac{-15-3\sqrt{5}+5\sqrt{5}+5}{4}-\frac{9-5}{4}a+a\\ &= \frac{-10+2\sqrt{5}}{4}\\ &= \frac{-5+\sqrt{5}}{2},\end{aligned}$$

which is the right-hand side. Thus, the second equation is in fact equivalent to the first equation. There are no other conditions on the values a and b, so we are free to choose a, which then determines b. Hence, the complete solution is as follows:

$$f(x)=\begin{cases} 1-x^2 & \text{if } x\neq \frac{1+\sqrt{5}}{2},\ x\neq\frac{1-\sqrt{5}}{2},\\ a & \text{if } x=\frac{1+\sqrt{5}}{2},\\ \frac{-5-\sqrt{5}}{2}-\frac{3+\sqrt{5}}{2}a & \text{if } x=\frac{1-\sqrt{5}}{2},\end{cases}$$

where a is any constant.

3. Let $F(x)$ be a real valued function defined for all real x except for $x=0$ and $x=1$ and satisfying the functional equation

$$F(x)+F\left(\frac{x-1}{x}\right)=1+x.$$

Find all functions $F(x)$ satisfying these conditions. (Putnam, 1971)

Solution. Substituting $(x-1)/x$ for x, we get

$$F\left(\frac{x-1}{x}\right)+F\left(\frac{(x-1)/x-1}{(x-1)/x}\right)=1+\frac{x-1}{x},$$

which simplifies as

$$F\left(\frac{x-1}{x}\right)+F\left(\frac{1}{1-x}\right)=\frac{2x-1}{x}.$$

Substituting $1/(1-x)$ for x in the given functional equation, we get

$$F\left(\frac{1}{1-x}\right)+F\left(\frac{1/(1-x)-1}{1/(1-x)}\right)=1+\frac{1}{1-x},$$

which simplifies as

$$F\left(\frac{1}{1-x}\right)+F(x)=\frac{2-x}{1-x}.$$

Thus, we have the system of equations

$$\begin{aligned} F(x)+F\left(\frac{x-1}{x}\right) &= 1+x, \\ F\left(\frac{x-1}{x}\right)+F\left(\frac{1}{1-x}\right) &= \frac{2x-1}{x}, \\ F\left(\frac{1}{1-x}\right)+F(x) &= \frac{2-x}{1-x}. \end{aligned}$$

Adding the first and third equation and subtracting the second equation, we get

$$2F(x)=1+x+\frac{2-x}{1-x}-\frac{2x-1}{x}=\frac{1+x^2-x^3}{x(1-x)},$$

so

$$F(x)=\frac{1+x^2-x^3}{2x(1-x)}.$$

We check that this solution works. Substituting into the given functional equation, we get

$$\begin{aligned} F(x)+F\left(\frac{x-1}{x}\right) &= \frac{1+x^2-x^3}{2x(1-x)}+\frac{1+(\frac{x-1}{x})^2-(\frac{x-1}{x})^3}{2\cdot\frac{x-1}{x}\cdot(1-\frac{x-1}{x})} \\ &= \frac{1+x^2-x^3}{2x(1-x)}+\frac{x^3+x(x-1)^2-(x-1)^3}{2x(x-1)} \\ &= \frac{1+x^2-x^3}{2x(1-x)}+\frac{x^3+x^2-2x+1}{2x(x-1)} \\ &= \frac{1+x^2-x^3}{2x(1-x)}+\frac{-x^3-x^2+2x-1}{2x(1-x)} \\ &= \frac{2x-2x^3}{2x(1-x)} \\ &= \frac{2x(1+x)(1-x)}{2x(1-x)} \\ &= 1+x, \end{aligned}$$

so our solution works.

4. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$x^2(f(x)+f(y))=(x+y)f(f(x)y)$$

holds for any positive real numbers x and y. (Slovenia, 2005)

Solution. Let a be a fixed point of f (assuming such a fixed point exists). Setting $x = a$ and $y = 1$ into the given functional equation, we get

$$a^2(a + f(1)) = (a + 1)a.$$

Since a is positive, we can divide both sides by a, and the equation simplifies to

$$a^2 + [f(1) - 1]a - 1 = 0.$$

Since this is a quadratic, a can have at most two values. In fact, it can only have one positive value since the product of the two roots is -1. So f has at most one fixed point.

Setting $y = x$ in the given functional equation, we get

$$2x^2 f(x) = 2xf(xf(x)).$$

Since x is positive, we can divide both sides by $2x$ to get

$$xf(x) = f(xf(x))$$

for all $x > 0$.

Hence, $xf(x)$ is a fixed point of f for all $x > 0$, which means that f has at least one fixed point. Furthermore, if a is a fixed point, then $af(a) = a^2$ is also a fixed point. So, a, a^2, a^4, ... are all fixed points. However, we found earlier that f has at most one fixed point, so we must have $a = 1$. Therefore, the only possible fixed point f can have is 1, which means $xf(x) = 1$, or $f(x) = 1/x$ for all $x > 0$. It is easy to verify that this solution works.

5. Let $g : S \to S$ be a function such that g has exactly two fixed points, and $g \circ g$ has exactly four fixed points. Prove that there is no function $f : S \to S$ such that $g = f \circ f$.

   Solution. Let a and b be the fixed points of g, so $g(a) = a$ and $g(b) = b$. Then $(g \circ g)(a) = g(g(a)) = g(a) = a$ and $(g \circ g)(b) = g(g(b)) = g(b) = b$, so a and b are also two of the fixed points of $g \circ g$. Let c and d be the other two fixed points of $g \circ g$, so $(g \circ g)(c) = c$ and $(g \circ g)(d) = d$.

   We compute $(g \circ g \circ g)(c)$ in two different ways. First, $(g \circ g \circ g)(c) = g((g \circ g)(c)) = g(c)$. Also, $(g \circ g \circ g)(c) = (g \circ g)(g(c))$, so
$$(g \circ g)(g(c)) = g(c).$$

   In other words, $g(c)$ is a fixed point of $g \circ g$. Therefore, $g(c) \in \{a, b, c, d\}$.

   We claim that $g(c)$ cannot be a. If $g(c) = a$, then $g(a) = g(g(c)) = c$. But $g(a) = a$, contradiction, so $g(c)$ cannot be a. Similarly, $g(c)$ cannot be b. Also, $g(c)$ cannot be c (because then c would be a fixed point of g), so $g(c) = d$. Then $g(d) = g(g(c)) = c$.

   For the sake of contradiction, suppose there exists a function $f : S \to S$ such that $g = f \circ f$.

Then $f \circ g = g \circ f$, so in particular

$$(f \circ g)(a) = (g \circ f)(a).$$

But $(f \circ g)(a) = f(g(a)) = f(a)$, and $(g \circ f)(a) = g(f(a))$, so $g(f(a)) = f(a)$. Hence, $f(a)$ is a fixed point of g, which means $f(a)$ is a or b. Similarly, $f(b)$ is a or b.

Also,

$$(f \circ g)(c) = (g \circ f)(c).$$

But $(f \circ g)(c) = f(g(c)) = f(d)$, and $(g \circ f)(c) = g(f(c))$, so $g(f(c)) = f(d)$. Similarly, $g(f(d)) = f(c)$. Then $(g \circ g)(f(c)) = g(g(f(c))) = g(f(d)) = f(c)$, so $f(c)$ is a fixed point of $g \circ g$, which means $f(c) \in \{a, b, c, d\}$.

We consider the possible values of $f(c)$.

Case 1: $f(c) = a$.

If $f(c) = a$, then $g(c) = (f \circ f)(c) = f(f(c)) = f(a)$, which is a or b. But $g(c) = d$, contradiction.

Case 2: $f(c) = b$.

If $f(c) = b$, then $g(c) = (f \circ f)(c) = f(f(c)) = f(b)$, which is a or b. But $g(c) = d$, contradiction.

Case 3: $f(c) = c$.

If $f(c) = c$, then $g(c) = (f \circ f)(c) = f(f(c)) = f(c) = c$, so c is a fixed point of g. But the only fixed points of g are a and b, contradiction.

Case 4: $f(c) = d$.

If $f(c) = d$, then $g(c) = (f \circ f)(c) = f(f(c)) = f(d)$. But $g(c) = d$, so $f(d) = d$. Then $g(d) = (f \circ f)(d) = f(f(d)) = f(d) = d$, so d is a fixed point of g. But the only fixed points of g are a and b, contradiction.

Every possible case leads to a contradiction, so there is no function $f : S \to S$ such that $g = f \circ f$.

6. Let $S = \{0, 1, 2, \dots\}$. Find all functions defined on S taking their values in S such that

$$f(m + f(n)) = f(f(m)) + f(n)$$

for all m and n in S. (IMO, 1996)

Solution. Setting $m = n = 0$ in the given functional equation, we get

$$f(f(0)) = f(f(0)) + f(0),$$

so $f(0) = 0$.

Setting $m = 0$ in the given functional equation, we get

$$f(f(n)) = f(f(0)) + f(n),$$

which simplifies to $f(f(n)) = f(n)$ for all n. Hence, the given functional equation implies that

$$f(m + f(n)) = f(m) + f(n) \tag{$*$}$$

for all m and n.

We see that the function $f(n) = 0$ satisfies the given functional equation, so assume that f is not identically 0.

Since $f(f(n)) = f(n)$ for all n, $f(n)$ is a fixed point of f for any n. Let a be the smallest nonzero fixed point of f, so $f(a) = a$. Setting $m = n = a$ in $(*)$, we get

$$f(a + f(a)) = 2f(a),$$

which simplifies to $f(2a) = 2a$.

Setting $m = 2a$ and $n = a$ in $(*)$, we get

$$f(2a + f(a)) = f(2a) + f(a),$$

which simplifies to $f(3a) = 3a$. By a straightforward induction argument, $f(ka) = ka$ for all positive integers k, i.e. ka is a fixed point of f for all $k \geq 1$. We claim that every fixed point of f is of the form ka, where $k \geq 1$.

Let b be a fixed point of f, and write

$$b = ka + r,$$

where $0 \leq r < a$. Then

$$f(b) = f(ka + r).$$

Setting $m = r$ and $n = ka$ in $(*)$, we get

$$f(r + f(ka)) = f(r) + f(ka).$$

But ka is a fixed point of f, i.e. $f(ka) = ka$, so this equation becomes $f(ka + r) = ka + f(r)$. Hence,

$$f(b) = f(ka + r) = ka + f(r).$$

Also, b is a fixed point of f, so $f(b) = b = ka + r$, and

$$ka + f(r) = ka + r,$$

which means $f(r) = r$. By definition, a is the smallest nonzero fixed point of f, and $r < a$. Therefore, $r = 0$, so $b = ka$. We conclude that every fixed point of f is a multiple of a.

Since $f(n)$ is a fixed point of f for any n, $f(n)$ is a multiple of a for all n. Let $f(n) = g(n)a$, so $g(n)$ is a positive integer for all n.

Let N be a nonnegative integer, and write

$$N = ka + r,$$

where $0 \le a < r$. Setting $m = r$ and $n = ka$ in $(*)$, we get

$$f(r + f(ka)) = f(r) + f(ka),$$

which simplifies to $f(ka + r) = ka + f(r)$. Since $N = ka + r$ and $f(r) = g(r)a$,

$$f(N) = ka + g(r)a, \qquad (**)$$

where k and r are the quotient and remainder when N is divided by a, respectively. If $r = 0$, then $N = ka$ is divisible by a, which means N is a fixed point, so $f(N) = ka$. Then

$$ka = ka + g(0)a,$$

so $g(0) = 0$.

Thus, the function f is determined by:

(i) A positive integer a, and
(ii) the positive integers $g(1)$, $g(2)$, $\dots$, $g(a-1)$. (We set $g(0) = 0$.)

Conversely, we claim that any function so determined (by these values and $(**)$) satisfies the given functional equation.

To see this, let N and M be arbitrary nonnegative integers, and write $N = ka + r$ and $M = la + s$, where $0 \le r$, $s < a$. Then

$$\begin{aligned} f(N) &= ka + g(r)a, \\ f(M) &= la + g(s)a, \end{aligned}$$

so

$$\begin{aligned} f(M + f(N)) &= f(la + s + ka + g(r)a) \\ &= f((k + l + g(r))a + s) \\ &= [k + l + g(r)]a + g(s)a \\ &= [la + g(s)a] + [ka + g(r)a] \\ &= f(M) + f(N). \end{aligned}$$

Also, $f(M) = la + g(s)a$ is a multiple of a, and since $g(0) = 0$,

$$f(f(M)) = f(la + g(s)a) = la + g(s)a = f(M).$$

Therefore,

$$f(M + f(N)) = f(M) + f(N) = f(f(M)) + f(N),$$

so the given functional equation is always satisfied.

In summary, all the solutions can be described as follows: Either $f(n) = 0$ for all n, or there exists a positive integer a and positive integers $g(0) = 0$, $g(1)$, $g(2)$, $\dots$, $g(a-1)$ such that if $N = ka + r$, where k and r are the quotient and remainder when N is divided by a, respectively, then

$$f(N) = ka + g(r)a.$$

7. A function f is *multiplicative* if $f(xy) = f(x)f(y)$ for all x and y. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that are both additive and multiplicative.

   Solution. Since f is multiplicative,
   $$f(xy) = f(x)f(y)$$
   for all x and y. Setting $y = x$, we get
   $$f(x^2) = f(x)^2$$
   for all x.

   This equation tells us that for any nonnegative real number x,
   $$f(x) = f(\sqrt{x})^2 \geq 0.$$

   Since f is also additive, f is of the form $f(x) = cx$, where c is a constant.

   By the multiplicative condition,
   $$c^2xy = cxy$$
   for all x and y, so $c^2 = c$, which means $c = 0$ or $c = 1$.

   Therefore, the only functions $f : \mathbb{R} \to \mathbb{R}$ that are both additive and multiplicative are $f(x) = 0$ and $f(x) = x$. It is easy to verify that both solutions work.

8. Find all continuous functions $f : (0, \infty) \to (0, \infty)$ such that
   $$f(xy) = f(x)f(y)$$
   for all x, $y > 0$.

   Solution. If we could turn the products in our functional equation into sums, then we could apply Cauchy. So, we seek a substitution that turns products into sums. This gets us thinking about logarithms and exponentials. Let's try the substitution $f(x) = e^{g(x)}$, which will produce a sum of functions on the right-hand side:
   $$e^{g(xy)} = e^{g(x)}e^{g(y)} = e^{g(x)+g(y)},$$
   so $g(xy) = g(x) + g(y)$. That takes care of the right side, but the left side is still a product. If we could replace xy with $\log xy$, then we could produce a sum. That suggests trying $f(x) = e^{h(\log x)}$. Let's give that a try in our original functional equation:
   $$e^{h(\log xy)} = e^{h(\log x)}e^{h(\log y)} = e^{h(\log x)+h(\log y)},$$
   so $h(\log xy) = h(\log x) + h(\log y)$, from which we have
   $$h(\log x + \log y) = h(\log x) + h(\log y).$$

   Success!

Let $a = \log x$ and $b = \log y$, so

$$h(a+b) = h(a) + h(b)$$

for all a and b. Solving $f(x) = e^{h(\log x)}$ for $h(x)$ gives $h(x) = \log f(e^x)$. Since f is continuous, so is h. Finally, since h is continuous and $h(a+b) = h(a) + h(b)$, we have $h(x) = cx$ for some constant c. Then

$$f(t) = e^{h(\log t)} = e^{c \log t} = (e^{\log t})^c = t^c.$$

Therefore, the solutions are of the form $f(x) = x^c$, where c is a constant. It is easy to verify that any solution of this form works.

9. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y) + xy$$

for all x, $y \in \mathbb{R}$.

Solution. Setting $y = 1$, we get

$$f(x+1) = f(x) + f(1) + x$$

for all x. Let $a = f(1)$, so

$$f(x+1) = f(x) + x + a \tag{$*$}$$

for all x.

Setting $x = 1$ in $(*)$, we get

$$f(2) = f(1) + 1 + a = 2a + 1.$$

Setting $x = 2$ in $(*)$, we get

$$f(3) = f(2) + 2 + a = 3a + 3.$$

Setting $x = 3$ in $(*)$, we get

$$f(4) = f(3) + 3 + a = 4a + 6.$$

Setting $x = 4$ in $(*)$, we get

$$f(5) = f(4) + 4 + a = 5a + 10.$$

By a straightforward induction argument,

$$f(n) = an + \frac{n(n-1)}{2} = \frac{1}{2}n^2 + \left(a - \frac{1}{2}\right)n$$

for all positive integers n, so let

$$g(x) = f(x) - \frac{1}{2}x^2.$$

Then

$$f(x) = g(x) + \frac{1}{2}x^2.$$

Substituting into the given functional equation, we get

$$g(x+y) + \frac{1}{2}(x+y)^2 = g(x) + \frac{1}{2}x^2 + g(y) + \frac{1}{2}y^2 + xy,$$

which simplifies as

$$g(x+y) = g(x) + g(y)$$

for all x and y. Since g is continuous, $g(x) = cx$ for some constant c. Then

$$f(x) = \frac{1}{2}x^2 + cx,$$

where c is a constant. It is easy to verify that any solution of this form works.

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