CANADA
1995

PROBLEMS 1999 Winter Camp

SUI 3

Writing

1. n^2 people, all of different heights, are arranged in a $n \times n$ square array. The tallest is selected from each row, and the shortest of those tallest people is selected; call this person A. In each column, the shortest is selected, and the tallest of these shortest people is selected; call this person B.

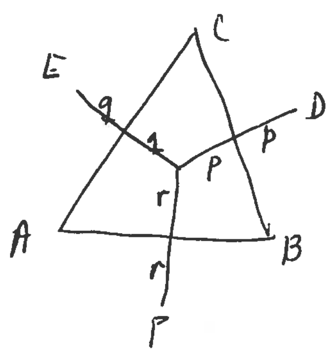
   (a) Can A and B be the same person?

   (b) If A and B are different, which is taller and why?

2. 250 scientist attend a conference. If A and B are any two of them, A speaks a language not known by B, and B speaks a language not known by A. What is the smallest number of languages that at least one of the scientists knows?

3. In an apartment building there are 7 elevators, each stopping at no more than 6 floors. It is possible to go from any one floor to any other floor without changing elevators. What is the maximum number of floors in the building?

4. Let P be any point inside an equilateral triangle ABC and let D, E, F be the reflections of P in the sides BC, AC and AB respectively. Which has larger area, ΔABC or ΔDEF?



5. Determine for which positive integers k the set

$$S = \{1990, 1991, 1992, \ldots, 1990+k\}$$

can be partitioned into disjoint subsets A and B such that the sum of the members of A is the same as the sum of the members of B.

6. Around a circle $4n$ points are chosen and alternately coloured red and green. The $2n$ red points are divided arbitrarily into n pairs and the members of each pair are joined by a red chord. Similarly, n green chords are drawn. If no three of the chords are concurrent, prove that there are at least n green-red points of intersection determined by a green and a red chord.

7. At an arbitrary selection of n different points $s_1, s_2, \dots, s_n$ around a circular track, there are respectively n cars $c_1, c_2, \dots, c_n$, ready to start. The cars are not about to race, for they all go at the same speed, namely one circuit of the track per hour. All the cars start off at the same moment, at which time each driver independently selects a direction and proceeds around the track. These cars are like bumper-cars, for whenever two cars meet, both of them instantly reverse direction and proceed without loss of speed.

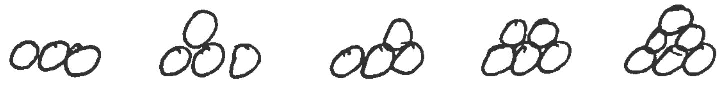
Prove that, at some moment in the future, each car will be at its own starting point going in its original direction.

8. Determine the number of ways of

(a) arranging exactly n A's and at most m B's in a row;

(b) arranging at most n A's and at most m B's in a row.

9. Let n be a positive integer. Arrange n coppers in a straight line. In how many ways can you construct a "tower" of coppers consisting of rows in which each copper above the first row has exactly two adjacent coppers from the next lower row. For example, when $n=3$, the answer is 5.



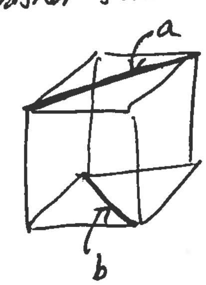
D.1. The interior of a wine glass is a right circular cone. The glass is half filled with water and is then slowly tilted so that the water reaches a point P on the rim. If the glass is further tilted (so that water spills out), what fraction of the conical interior is occupied by water when the horizontal plane of the water bisects the generator of the cone farthest from P?

D.2. Prove that if x, y, z are positive real numbers, then

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}$$

D.3. Three coppers and two nickels can be arranged in a circular array in essentially two ways according as the nickels are adjacent or not. Suppose for each configuration, all five coins are tangent to an interior circular washer. For which configuration is the interior washer smaller?

D.4.



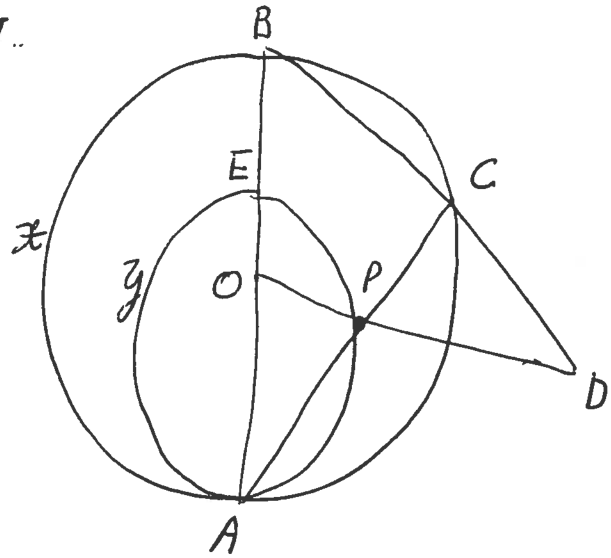
a and b are two skew face diagonals of a cube (diagrammed). Determine the locus of midpoints of segments XY with $X \in a$ and $Y \in b$.

D.5. Let P be a convex polygon which does not contain any triangle of area $\frac{1}{4}$. Prove that P itself is contained in some triangle of area 1.

D.6. Let $a_1, a_2, \ldots, a_n$ be positive reals for which $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)\cdots\left(1+\frac{1}{a_n}\right) \geq (n+1)^n.$$

D.7.



E trisects diameter AB of a circle $\mathcal{X}$ with centre O, and $\mathcal{Y}$ is the circle on diameter AE. C is an arbitrary point on $\mathcal{X}$ and AC crosses $\mathcal{Y}$ at P. Suppose that OP and BC meet at D.

Prove that C is the midpoint of BD.

D.8. In a certain library, there are n shelves, each holding at least one book. k new shelves are acquired and the books are rearranged on the $n+k$ shelves, again with at least one book on each shelf.

A book is said to be <u>privileged</u> if it is on a shelf with fewer books in the new arrangement than in the original arrangement.

Prove that there are at least $k+1$ privileged books in the rearranged library.

D.9. Suppose $x_1, x_2, \dots, x_7$ are numbers such that

$$x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 = 1$$
$$4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 = 12$$
$$9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 = 123.$$

Determine
$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7 .$$

D.10. A $7\times7\times7$ box is packed with 114 $1\times1\times3$ parallelepipeds and a $1\times1\times1$ cube. Show that, if the $1\times1\times1$ cube is not on the surface of the $7\times7\times7$ packing, it must go right into the centre.

D.11. A polynomial $f(x)$ of degree n satisfies $f(k) = \dfrac{k}{k+1}$ $(0 \le k \le n)$. Determine $f(n+1)$.

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PROBLEMS
Discussion

NET 3

D. 12. Prove that every selection of 1325 integers from $\{1, 2, \ldots, 1987\}$ must contain three numbers $\{a, b, c\}$ such that

$$\gcd(a, b) = \gcd(b, c) = \gcd(c, a) = 1,$$

but that there exist 1324 integers from $\{1, 2, \ldots, 1987\}$ for which such a selection of 3 numbers is impossible.

D. 13. Let $a, b, c > 0$. What is minimum value of

$$f(x) = \sqrt{a^2 + x^2} + \sqrt{(b-x)^2 + c^2}$$