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Section 1. Introduction, Definitions and Notations

1. A **graph** is a pair of sets $G = (V, E)$ where V is a set of vertices and E is a collection of edges whose endpoints are in V. It is possible that a graph can have infinitely many vertices and edges. Unless stated otherwise, we assume that all graphs are simple. [1]

2. Two vertices v, w are said to be **adjacent** if there is an edge joining v and w. An edge and a vertex are said to be **incident** if the vertex is an endpoint of the edge.

3. Given a vertex v, the **degree** of v is defined to be the number of edges containing v as an endpoint.

4. A **path** in a graph G is defined to be a finite sequence of distinct vertices $v_0, v_1, \cdots, v_t$ such that v_i is adjacent to v_{i+1}. (A graph itself can also be called a path.) The **length** of a path is defined to be the number of edges in the path.

5. A **cycle** in a graph G is defined to be a finite sequence of distinct vertices $v_0, v_1, \cdots, v_t$ such that v_i is adjacent to v_{i+1} where the indices are taken modulo $t + 1$. (A graph itself can also be called a cycle.) The **length** of a cycle is defined to be the number of vertices (or edges) in the path.

6. A graph is said to be **connected** if for any pair of vertices, there exists a path joining the two vertices. Otherwise, a graph is said to be **disconnected**.

7. The **distance** between two vertices u, v in a graph is defined to be the length of the shortest path joining u, v. (In the case the graph is disconnected, this may not be well-defined.)

8. Let $G = (V, E)$ be a graph. The **complement** $\overline{G}$ of G is a graph with the same vertex set as G and $E(\overline{G}) = \{e \notin E(G)\}$. i.e. $\overline{G}$ has edges exactly where there are no edges in G.

9. Let $G = (V, E)$ be a finite graph. A graph G is said to be **complete** if every pair of vertices in G is joined by an edge. A complete graph on n vertices is denoted by K_n.

10. A graph G is said to be **bipartite** if $V(G)$ can be partitioned into two non-empty disjoint sets A, B such that no edge has both endpoints in the same set. A graph is said to be **complete bipartite** if G is bipartite and all possible edges between the two sets A, B are drawn. In the case where $|A| = m, |B| = n$, such a graph is denoted by $K_{m,n}$.

11. Let $k \geq 2$. A graph G is said to be k-partite if $V(G)$ can be partitioned into k pairwise disjoint sets $A_1, \cdots, A_k$ such that no edge has both endpoints in the same set. A **complete k-partite** graph is defined similarly as a complete bipartite. In the case where $|A_i| = n_i$, such a graph is denoted by $K_{n_1,n_2,\cdots,n_k}$. (Note that a 2-partite graph is simply a bipartite graph.)

[1] An edge whose endpoints are the same is called a **loop**. A graph where there is more than one edge joining a pair of vertices is called a **multigraph**. A graph without loops and is not a multigraph is said to be **simple**.

Section 1 Exercises

The exercises in this section, while not of the olympiad nature, will familiarize you with the techniques that might be required to solve olympiad problems. It is important that you know how to solve all of these problems.

1. Let G be a graph with n vertices, m edges and the degrees of the n vertices are $d_1, d_2, \cdots, d_n$. Prove that
$$\sum_{i=1}^{n} d_i = 2m.$$

2. For any graph G, let $\Delta(G)$ be the maximum degree amongst the vertices in G. Characterize all graphs with $\Delta(G) \leq 2$. Characterize all graphs with $\Delta(G) = 2$.

3. Let G be a disconnected graph. Prove that its complement $\overline{G}$ is connected.

4. Let G be a connected graph. Prove that two paths which are both a longest path in the graph, contain at least one vertex in common.

5. Let G be a connected. An edge e is said to be a **cut-edge** if its removal disonnects the graph. Prove that e is not a cut-edge if and only if e is an edge of a cycle.

6. A graph is said to be **planar** if it can be drawn such that a pair of edges can only cross at a vertex.
   (a) Convince yourself that K_5 and $K_{3,3}$ are not planar. [2]
   (b) Suppose a (simple) planar graph G has $n \geq 3$ vertices. Prove that G has at most $3n-6$ edges. (Hint: Doesn't a planar graph look like a polyhedron to you?)

7. Prove that a graph is bipartite if and only if it does not contain an odd cycle.

8. Let G be a connected graph with an even number of vertices. Prove that you can select a subset of edges of G such that each vertex is incident to an odd number of selected edges.

9. (Italy 2007) Let n be a positive odd integer. There are n computers and exactly one cable joining each pair of computers. You are to colour the computers and cables such that no two computers have the same colour, no two cables joined to a common computer have the same colour, and no computer is assigned the same colour as any cable joined to it. Prove that this can be done using n colours.

10. Given a graph G, let $\chi(G)$ be the minimum number of colours required to colour the vertices of G such that no two adjacent vertices are assigned the same colour. Let m be the number of edges in G. Prove that
$$\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}.$$

[2]Kuratowski's Theorem states that a graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor (a certain type of subgraph). The proof of this theorem is beyond the scope of this training.

Section 2: Trees and Balancing

A **tree** is defined to be a connected graph that does not contain any cycles. We will first give characterizations to such graphs.

Lemma: (Characterization of Trees) Let G be a connected graph with n vertices. The following statements are equivalent.

1. G does not contain any cycles
2. G contains exactly $n-1$ edges
3. For any two vertices, there exists exactly one path joining the two vertices
4. The removal of any edge disconnects the graph

Sketch Of Proof: (1) $\Leftrightarrow$ (3) is clear since existence of a cycle means two paths joining any two vertices on the cycle. (1) $\Leftrightarrow$ (4) is clear from Section 1, Exercise 5. To show (1) $\Rightarrow$ (2), start with an empty graph G on n vertices and add one edge at a time to construct G. We start with n components, and every edge we add decreases the number of components by 1 (since no cycles can be formed). Since G is connected, we end up with 1 component, meaning $n-1$ edges is added. To show (2) $\Rightarrow$ (1), suppose G contains a cycle, then using the same construction technique mentioned, the addition of some edge does not decrease the number of components, thus G must contain strictly more than $n-1$ edges. $\square$

Corollary: Let G be a connected graph with n vertices and at least n edges. Then there exists an edge whose removal does not disconnect the graph.

Sketch Of Proof: G has a cycle. Choose edge e on cycle. By Section 1, Exercise 5. Done. $\square$

Corollary: Let G be a connected graph. Then G contains a subgraph which is a tree and containing every vertex of G. Such a subgraph is called a **spanning tree** of G.

Let G be a tree and v be any vertex of G. Let $v_1, v_2, \cdots, v_t$ be the vertices adjacent to v. Let e_i be the edge joining v and v_i. Let T_i be the subtree containing v_i after removing edge e_i. (Draw a diagram for this. It will help.) Let $f(v) = \max_{i=1}^{t} |V(T_i)|$.

Let's use a little intuition here. Since $\sum |V(T_i)| = n-1$, if $f(v)$ is "large", then the tree looks unbalanced. If $f(v) \approx (n-1)/t$, then the tree looks balanced. We want to find the vertex v that minimizes $f(v)$, the vertex that makes the tree the most balanced.

Tree Balancing Exercise: Let G be a tree with n vertices and $\Delta > 1$ be the maximum degree amongst all vertices in G. Using the same function f as defined before, prove that there exists a vertex v such that

$$\frac{1}{\Delta}(n-1) \leq f(v) \leq \frac{\Delta-1}{\Delta}(n-1).$$

Sketch of Proof: Left inequality follows for all v from pigeonhole principle. To prove the right inequality, choose v such that $f(v)$ is minimum. Suppose $f(v) \geq (\Delta-1)/(\Delta)\cdot(n-1)+1$. Let v_i be the neighbour of v with $|T_i| \geq (\Delta-1)/\Delta(n-1)+1$. Let $v = w_1, w_2, \cdots, w_\Delta$ be the neighbours of v_i. Then since the tree containing v after removing $v_i v$ contains at most $1/\Delta(n-1)$ vertices, then $f(v_i) \leq (\Delta-1)/\Delta(n-1) - 1 < f(v)$, contradicting the minimality of $f(v)$. (Draw a diagram to understand this proof better.) $\square$

Now, we are ready to balance graphs in general.

Warm-Up Exercises

1. Let G be a connected graph with n vertices and maximum degree Δ. Prove that G has two vertex-disjoint connected subgraphs containing at least $\lceil (n-1)/\Delta \rceil$ vertices each.

2. Let G be a tree with n vertices with maximum degree Δ. Prove that there exists an edge in G whose removal leaves two trees with at least $\lceil (n-2)/\Delta \rceil$ edges each.

Olympiad Exercises

1. (USAMO 2007) An animal with n cells is a connected figure consisting of n equal-sized cells. A dinosaur is an animal with at least 2007 cells. It is said to be primitive it its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

2. (Iran 2005) A simple polygon is one where the perimeter of the polygon does not intersect itself (but is not necessarily convex). Prove that a simple polygon $\mathcal{P}$ contains a diagonal which is completely inside $\mathcal{P}$ such that the diagonal divides the perimeter into two parts both containing at least $n/3 - 1$ vertices. (Do not count the vertices which are endpoints of the diagonal.)

3. During a lecture, five students were present and each were asleep on exactly two separate occasions during the lecture. For every pair of students, there exist a time during the lecture for which both students were asleep. Prove that there was a time where three students were asleep at that time.

* - Note that the Tree-Balancing Exercise and the Warm-Up problems do not have names. When you write complete solutions to these problems, you would have to re-write the proofs to those problems for the appropriate value of Δ.

Section 3: Friends, Strangers and Cliques

Given a graph G, a **clique** in G is a subset of vertices of G where every pair of vertices in the subset is joined by an edge. This becomes important in certain math olympiad problems involving friends and strangers.

Warm-Up Problems

1. (Alberta 2007) Let n be a positive integer. A test has n problems, and was written by several students. Exactly three students solved each problem, each pair of problems has exactly one student that solved both and no student solved every problem. Find the maximum possible value of n.

2. Let n be a positive integer. In a group of $2n+1$ people, each pair is classified as friends or strangers. For every set S of at most n people, there is one person outside of S who is friends with everyone in S. Prove that at least one person is friends with everyone else.

3. **Turan's Theorem:** Let G be a graph on n vertices and m is a positive integer with $2 \leq m \leq n$. Suppose G does not contain a clique of size m.

   a.) Prove that the number of edges in G is at most
   $$\frac{n^2}{2}\left(1 - \frac{1}{m-1}\right).$$

   b.) Let G be a graph on n vertices with a maximum number of edges such that G is m-clique free. Prove that if u, v, w are three vertices such that u is not adjacent to v and v is not adjacent to w, then u is also not adjacent to w. Conclude that G is complete multi-partite with at most $m-1$ parts. Furthermore, prove that the number of vertices in two parts differ by at most 1.

4. (APMO 1990 Variant) A group of n people at a party has the property that each pair of persons is classifed as friends or strangers. The following properties are also satisfied.

   - Nobody is friends with everyone else.
   - Every pair of strangers has exactly one common friend.
   - No three people are mutually friends.

   Prove that everybody has the same number of friends.

Olympiad Problems

1. Let n be a positive integer. For a set S of $2n$ real numbers, find the maximum possible number of pairwise (positive) differences between two elements in S, that are in the range $(1, 2)$.

2. (IMO 2001 Shortlist) Define a k-clique to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

3. Let G be a graph with n vertices and m edges that does not contain a 4-cycle. Prove that
$$m \leq \frac{n}{4}\left(1+\sqrt{4n-3}\right).$$

4. (APMO 1989) Let G be a graph with n vertices and m edges. Prove that the graph contains
$$\frac{m(4m-n^2)}{3n}$$
cycles of length 3.

5. (Zarankiewicz Theorem:) Let n, k be positive integers. Amongst a group of n people, there are no groups of k people that all mutually are friends with each other. Prove that there is someone who is friends with less than $\lfloor \frac{k-2}{k-1} \cdot n \rfloor$ people.

6. There are $2n$ people at a party where each person has an even number of friends at the party. Prove that there are two people who have an even number of common friends at the party.

7. In a group of n people, each pair are friends or strangers. No set of three people are mutually friends. For any partition of the n people into two groups, there exists two people in a group that are friends. Prove that there exists a person who is friends with at most $2n/5$ people in the group.

8. (IMO 2002 Shortlist) There are a 120 people in a room where each pair of persons are classified are friends or strangers. A weak quartet is defined to be a set of four people where exactly one pair in the set are friends. Find the maximum possible number of weak quartets in the room.

   (Hint: Set this as a graph in the obvious way. Prove that in a graph where the maximum possible number of weak quartets is attained, the graph consists of disjoint union of complete graphs. i.e. If x, y are adjacent and y, z are adjacent, then x, z are adjacent.)

9. (IMO 2007) In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its size. Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

   (Hint: Do this methodically. Let C be the largest clique. Place everyone in one room and start placing people in C to the other room one at a time, until the difference in maximum clique size in the two rooms are either equal, or differs by 1. If the former happens, we are done. Otherwise, ?)

10. (USA TST 2008) For a pair $A = (x_1, y_1), B = (x_2, y_2)$ of points on the coordinate plane, let $d(A, B) = |x_1 - x_2| + |y_1 - y_2|$. We call a pair (A, B) of unordered points harmonic if $1 < d(A, B) \leq 2$. Determine the maximum number of harmonic points amongst 100 points in the plane.

Section 4: Directed Graphs, Lots of Arrows and Tournaments

A **direct graph** is a graph where each edge is oriented with an arrow pointing in exactly one direction. A **directed path** is a path in the graph that moves along with the orientation of the arrow. A **directed cycle** is defined similarly. A **tournament** is defined to be a complete directed graph.

Warm-Up Problems

1. Prove that every tournament has a directed path that contains every vertex. Suppose for every pair of vertices, there exists a directed path to go from one vertex to the other. Such a directed graph is said to be **strongly connected**. Prove that a tournament has a directed cycle that contains every vertex if and only if it is strongly connected.
2. Let G be a connected graph with an even number of edges. Prove that the edges can be assigned an arrow such that the number of arrows leaving each vertex, is even.

Olympiad Problems

1. (Canada 2006) Consider a round-robin tournament with $2n+1$ teams, where each team plays each other team exactly once. We say that three teams X, Y, Z form a cycle triplet if X beats Y, Y beats Z, Z beats X. There are no ties.
   (a) Determine the minimum number of cycle triplets possible.
   (b) Determine the maximum number of cycle triplets possible.
2. (Romania 2006) Each edge of a polyhedron is oriented with an arrow such that at each vertex, there is at least one arrow leaving the vertex and at least one arrow entering the vertex. Prove that there exists a face on the polyhedron such that the edges on its boundary form a directed cycle.
3. Let k, n be positive integers with $k < n$ such that $\binom{n}{k}(1-\frac{1}{2^k})^{n-k} < 1$. Prove that there exists a tournament on n vertices such that for every subset S of vertices of size k, there exists a vertex v outside of S such that vx is a directed edge for all $x \in S$. (Hint: Looking at the weird condition given in the question, what tool does it suggest that you should try?)
4. (Iran 2005) Each edge of a tournament is coloured red or blue. Prove that there exists a vertex v in the tournament such that for every other vertex w, there exists a directed path from v to w of the same colour.

Section 5: Matchings: Pair Them Up

Given a graph G, a matching M is a set of edges in G such that no two edges in M touch. A matching is said to be **perfect** if every vertex in G is incident to an edge in M. A vertex is said to be M-exposed if it is not covered by an edge in M. Clearly, M is a perfect matching if and only if there are no M-exposed vertices. We will now state two important properties regarding matchings.

Hall's Theorem: Let $G = A \cup B, A \cap B = \emptyset$ be a bipartite graph. For $\emptyset \neq S \subseteq A$, let $\Gamma(S)$ be the vertices in B which is adjacent to a vertex in A. Then there is a matching that covers every vertex in A if and only if for all $S \subseteq A$,

$$|\Gamma(S)| \geq |S|.$$

The latter condition is called **Hall's Condition**. In the special case where $|A| = |B|$, then G has a perfect matching if and only if G satisfies Hall's Condition.

Sketch of Proof: Strong induction on $|A|$. If $|A| = 1$, clear. Suppose $|A| = n$. If for all $S \subseteq A$, $|\Gamma(S)| \geq |S| + 1$, then choose $e = uv \in E(G)$ with $u \in A$. Hall's Condition still applies to the graph $G - \{u, v\}$. Done. Otherwise, $|\Gamma(S)| = S$ for some $S \subseteq A$. I claim Hall's condition holds for the graph induced by $(A - S) \cup (B - \Gamma(S))$. For $T \subseteq A - S$, let $\Gamma'(T)$ be the neighbours of T in $B - \Gamma(S)$. Then by Hall's condition on all of G, $\Gamma(S \cup T) \geq S \cup \Gamma'(T)$. Since $|\Gamma(S)| = |S|$, then $|T| \geq |\Gamma'(T)|$. Then strong induction applies. $\square$

One More Tip: You will encounter problems where a grid with numerical entries are involved. Say you are given an $m \times n$ grid with entries which are in the set $\{0, 1\}$. Think about how you can naturally construct a bipartite graph, with the two parts having m, n vertices respectively.

Warm-Up Problems:

1. Let n be a positive integer. Let $S_1, S_2, \cdots, S_n$ be subsets of $\{1, 2, \ldots, n\}$ such that for any $1 \leq k \leq n$, the union of any k of the subsets contains at least k elements. Prove there there is a permutation $(a_1, a_2, \cdots, a_n)$ of $(1, 2, \cdots, n)$ such that $a_i \in S_i$.

2. Let G be a bipartite graph where each vertex have an equal positive degree. Prove that G has a perfect matching.

3. Let G be any graph and N be a matching of G. Prove that N is the matching of maximum size if and only if there does not exist a path that begins and ends at an N-exposed vertex, whose edges are alternating being not in N and in N.

   (Hint: One direction is "easy". For the other, let M be a larger matching than N and consider the graph formed by the edges in $M \cup N$.)

Olympiad Problems:

1. (Canada 2006) In a rectangular array of nonnegative real numbers with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect at a positive element, then the sums of their elements are the same. Prove that $m = n$.

2. A $n \times n$ grid is said to be permutation grid if its entries are all 0's and 1's such that there is exactly one 1 in each row and in each column of the grid. Let G be a n by n grid containing non-negative integer entries such that the sum of each row is equal and the sum of each column is equal. Prove that G can be written as the sum of permutation grids. (Addition on grids is performed component-wise.)

3. (Iran 1998) Let $n \geq 3$ be a positive integer. Let G be a grid whose entries are all $0, 1$ or -1 such that each row and each column contains exactly one 1 and one -1. Prove that the rows and the columns of the grid can be re-ordered such that the resulting grid is the negative of G.

4. A $n \times n$ grid has entries in $\{0, 1\}$ such that any subset of n cells with no two cells in the same row or the same column, contains at least one 1. Prove that there exists i rows and j columns, with $i + j \geq n + 1$, whose intersection contains only 1's.

5. There are $2n$ people in a room where each pair of persons is classified as friends or strangers. Two game players from the outside play a game where they alternate turns picking one person in the room such that this person was not picked before and this person is friends with the person previously picked. The last player who can make a legal move wins. The player that moves first can pick anyone he/she wants. Prove that the player that moves second has a winning strategy if and only if the $2n$ people can be used to form n disjoint pairs such that the two people in each pair are friends.

Section 6: Hamiltonian and Eulerian Paths and Cycles

Given a graph G, an **Eulerian walk** is defined to be a sequence of successive adjacent vertices that encounter every edge in the graph exactly once. An **Eulerian cycle** is a sequence of successive adjacent vertices that begin and ends at the same vertex, that encounter every edge in the graph exactly once.

Eulerian Walk and Cycle Characterization: A connected graph has an Eulerian walk if and only if the number of vertices with odd degree is 0 or 2. A connected graph has an Eulerian cycle if and only if every vertex has even degree.

Proof: You do it. It's fairly easy. $\square$

Now, back to simple graphs. A **Hamiltonian path** is a path that encounters every vertex in the graph. A **Hamiltonian cycle** is a cycle that encounters every vertex in the graph. A graph containing a Hamiltonian cycle is said to be **Hamiltonian**. It is in general difficult to determine whether these paths and cycles exist. The only real useful tool at our disposal is Dirac's Theorem.

Dirac's Theorem: Let $n \geq 3$. Suppose a graph G has n vertices and the degree of each vertex is at least $\lceil n/2 \rceil$. Then G has a Hamiltonian cycle.

Proof of Dirac's Theorem: You can add edges to make G complete. By Exercise 1, since a complete graph is Hamiltonian, then by removing the edges in the reverse order we added them, we get that G is Hamiltonian. $\square$

Olympiad Problems

1. Let G be a graph on n vertices. Let u, v be two non-adjacent vertices such that $deg(u) + deg(v) \geq n$. Then G is Hamiltonian if and only if $G + \{uv\}$ is Hamiltonian.

2. Given a 8×8 board, find all pairs of squares on the board, such that the remaining 62 squares can be tiled using 2×1 tiles.

3. There are $2n$ people in a room where each person is enemies with at most $n - 1$ people in the room. Prove that the $2n$ people can sit at a circular table so that no two enemies are sitting next to each other.

4. (Japan 2004) In a land with a finite number of towns, each town is connected by roads with exactly other three towns. Last year we made a trip using the roads by starting from a town and coming back to the town by visiting every other town in the land exactly once. This year we plan to make a trip in the same way as last year's trip, but without taking the same trip as last year or its reverse. Prove that this is possible. [3]

[3] I do not have a (satisfying) solution to this problem.

Section 7: Miscellaneous Problems Using Graph Theory

1. Given n points in the plane, prove that the number of (unordered) pairs of points which are distance 1 apart is at most $n^2/3$.

2. (Romania) Given n points in the plane with no three collinear, prove that there exists a set of at least $\sqrt{n}$ points such that no three points in the set form an equilateral triangle.

3. (Crux) Consider $n \geq 3$ be a set of distinct points S in the plane with no three collinear and no four concyclic. Let $f(S)$ be the number of (unordered) pairs of points (P, Q), such that there exists a circle containing P, Q in its interior, but no other points in S. Find the maximum possible value of $f(S)$. Express your answer in terms of n.

4. (IMO 1991) Let G be a connected graph with m edges. Prove that the edges can be labelled with the positive integers $1, 2, \cdots, m$ such that for each vertex with degree at least two, the greatest common divisors amongst the labels on the edges incident to this vertex, is 1.

5. (Belarus 2005) Prove that it is not possible colour the squares of a 11 by 11 grid using three colours, such that no four squares whose centres form the vertices of a rectangle with sides parallel to the sides of the grid, have the same colour.

6. (St. Petersberg) Let $n \geq 3, k \geq 2$ be positive integers. Within a group of n students, on each of k days, a group of at least two students go together to buy ice cream, such that each pair of students went together for ice cream exactly once. Prove that $k \geq n$.

7. Let n be a positive integer. Prove that the edges of the complete graph on n vertices can be decomposed into $n-1$ paths of lengths $1, 2, \cdots, n-1$.

8. (IMO 2005) In a mathematical competition, in which six problems were posed to the participants, every two of these problems were solved by more than $2/5$ of the contestants. Moreover, no contestant solved all the six problems. Show that there are at least two contestants who solved exactly five problems each.