Summer 2003 IMO Camp

Graph Theory Problems

1. What is the largest number of towns that can meet the following criteria. Each pair is directly linked by just one of air, bus or train. At least one pair is linked by air, at least one pair by bus and at least one pair by train. No town has an air link, a bus link and a trian link. No three towns, A, B, C are such that the links between AB, AC and BC are all air, all bus or all train. (1981 USAMO)

2. A graph has 1982 points. Given any four points, there is at least one joined to the other three. What is the smallest number of points which are joined to 1981 points? (1982 USAMO)

3. A graph has $n > 2$ points. Show that we can find two points A and B such that at least $\left\lfloor \frac{n}{2} \right\rfloor - 1$ of the remaining points are joined to either both or neither of A and B. (1985 USAMO)

4. A graph with n points and k edges has no triangles. Show that it has a point P such that there are at most $k(1 - \frac{4k}{n^2})$ edges between points not joined to P (by an edge). (1995 USAMO)

5. A space station has 99 pods. Every pair of pods is connected by a tube. 99 tubes are two-way, the others are all one-way. What is the largest number of sets of 4 pods that can be fully communicating (so that one can go between any two pods of the set without going outside the four pods and their six tubes)? (1999 China)

6. A graph has 1991 points and every point has degree at least 1593. Show that there are six points, each of which is joined to the others. Is 1593 the best possible? (1991 IMO Shortlist)

7. A finite graph is connected. A positive real number is assigned to each point. Each point is coloured red or blue and at least one point is coloured red. Show that, given a knowledge of (1) the points and edges of the graph, (2) the number assigned to each red point, and (3) for each blue point the average of the numbers ~~for~~ at it and the points joined to it, one can find the number assigned to each point. (1994 IMO Shortlist)

8. A graph has $12k$ points. Each point has $3k + 6$ edges. For any two points the number of points joined to both is the same. Find k. (1995 IMO Shortlist)

9. G is the complete graph on 10 points (so that there is an edge between each pair of points). Show that we can colour each edge with one of 5 colours so that for any 5 points we can find 5 differently coloured edges all of whose endpoints are amongst the 5 points. Show that we cannot colour each edge with one of 4 colours so that for any 4 points we can find 4 differently coloured edges with all endpoints amongst the 4 points. (1998 IMO Shortlist)

10. ? A finite graph is such that there is no complete graph on 5 vertices, and every two triangles have at least one point in common. Show that there are at most two points X such that removing X leaves no triangles. (2001 IMO Shortlist)