



Here's a problem that has already tormented many WOOTers. In this article, we see how it tormented me. I thought I had found a slick, quick solution, but my solution nagged at me and nagged at me. It seemed just a little bit too easy for a USAMO #3. So I was bothered by it. Only when I wrote up my solution for the WOOT Induction Handout did I stumble on the flaw in my reasoning. OK, it was well after I wrote it up, but I finally saw the error. Can you find it?

Problem. The 2010 positive numbers $a_1, a_2, \dots, a_{2010}$ satisfy the inequality $a_i a_j \leq i + j$ for all distinct indices i, j . Determine, with proof, the largest possible value of the product $a_1 a_2 \cdots a_{2010}$. (USAMO, 2010)

Solution. Right away, we at least have an upper bound for the maximum product. We can take inequalities corresponding to 1005 disjoint pairs of the a_i and multiply those to get an upper bound. For example, multiplying

$$\begin{aligned} a_1 a_2 &\leq 3, \\ a_3 a_4 &\leq 7, \\ &\vdots \\ a_{2009} a_{2010} &\leq 4019, \end{aligned}$$

gives

$$a_1 a_2 a_3 a_4 \cdots a_{2010} \leq (3)(7)(11)(15) \cdots (4019).$$

However, this isn't the only way we can combine 1005 pairs to get an upper bound. For example, we could replace the first two inequalities above with $a_1 a_3 \leq 4$ and $a_2 a_4 \leq 6$. But the product of these gives $a_1 a_2 a_3 a_4 \leq 24$, which is a less tight bound than $a_1 a_2 a_3 a_4 \leq 21$. So, we first have to show that our 1005 products shown above give the least possible upper bound from among those we can form by multiplying 1005 of the inequalities.

To prove that this group of 1005 inequalities offers the least upper bound, we must show that if $p < q < r < s$, then the least upper bound of $a_p a_q a_r a_s$ comes from multiplying $a_p a_q \leq p + q$ and $a_r a_s \leq r + s$. The three possible pairings of these four a_i are

$$\begin{aligned} a_p a_q \leq (p + q), a_r a_s \leq (r + s) &\implies a_p a_q a_r a_s \leq (p + q)(r + s), \\ a_p a_r \leq (p + r), a_q a_s \leq (q + s) &\implies a_p a_q a_r a_s \leq (p + r)(q + s), \\ a_p a_s \leq (p + s), a_r a_q \leq (r + q) &\implies a_p a_q a_r a_s \leq (p + s)(r + q). \end{aligned}$$

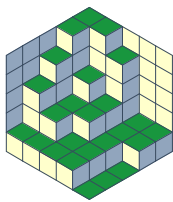
To see that the first gives the least upper bound, we note that

$$\begin{aligned} (p + r)(q + s) - (p + q)(r + s) &= pq + rs - pr - qs = (p - r)(q - s) \geq 0, \\ (p + s)(r + q) - (p + q)(r + s) &= pq + rs - ps - qr = (p - s)(q - r) \geq 0, \end{aligned}$$

where the final step in each chain comes from the fact that $p, q < r, s$. The only 1005 disjoint pairs of the a_i for which every two pairs $a_p a_q$ and $a_r a_s$ satisfy $p, q < r, s$ are the pairs shown above. We have therefore shown that

$$a_1 a_2 a_3 a_4 \cdots a_{2010} \leq (3)(7)(11)(15) \cdots (4019),$$





and we don't have any easy way to tighten the upper bound. We at least now have a target. If we can show that this bound can be achieved, then we have our desired maximum.

We'll try induction. First, we tackle a base case, and thereby make sure we aren't barking up the wrong tree with trying to prove that this particular bound can be achieved. Let's tackle the problem with just a_1, a_2, a_3, a_4 . We seek values such that $a_1a_2 = 3$, $a_3a_4 = 7$, and $a_ia_j \leq i + j$ for the other four possible pairs. Sure enough, we can find lots of values that work, such as $(a_1, a_2, a_3, a_4) = (\frac{3}{2}, 2, \frac{7}{3}, 3)$. Now that we have a base case, on to the inductive step.

We assume that there exist a_1, a_2, \dots, a_{2n} such that $a_{2k-1}a_{2k} = 4k - 1$ for $1 \leq k \leq n$, and that $a_ia_j \leq i + j$ for all $1 \leq i, j \leq 2n$.

We wish to show that this assumption implies that there exist $a_1, a_2, \dots, a_{2n+2}$ such that $a_{2k-1}a_{2k} = 4k - 1$ for $1 \leq k \leq n + 1$, and that $a_ia_j \leq i + j$ for all $i, j \leq 2n + 2$. A natural guess is to start with the values of a_1, a_2, \dots, a_{2n} in our inductive hypothesis, and prove that we can tack on a_{2n+1} and a_{2n+2} such that

$$\begin{aligned} a_{2n+1}a_{2n+2} &= (2n + 1) + (2n + 2), \\ a_{2n+1}a_i &\leq 2n + 1 + i \text{ for all } i \leq 2n, \\ a_{2n+2}a_j &\leq 2n + 2 + j \text{ for all } j \leq 2n. \end{aligned}$$

Our inductive hypothesis assures us that $a_{2k-1}a_{2k} = 4k - 1$ for $1 \leq k \leq n$, and that $a_ia_j \leq i + j$ for all $i, j \leq 2n$, so we only have to worry about satisfying the inequalities for a_{2n+1} and a_{2n+2} above. Since $a_{2n+1}a_{2n+2} = (2n + 1) + (2n + 2)$, we have

$$a_{2n+2} = \frac{(2n + 1) + (2n + 2)}{a_{2n+1}}.$$

So, we have

$$\begin{aligned} &a_{2n+2}a_j \leq 2n + 2 + j \\ \Leftrightarrow &\frac{(2n + 1) + (2n + 2)}{a_{2n+1}} \cdot a_j \leq 2n + 2 + j \\ \Leftrightarrow &\frac{((2n + 1) + (2n + 2))a_j}{2n + 2 + j} \leq a_{2n+1} \end{aligned} \tag{1}$$

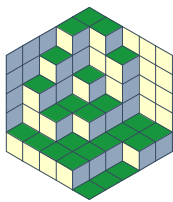
We also have $a_{2n+1}a_i \leq 2n + 1 + i \Leftrightarrow a_{2n+1} \leq \frac{2n+1+i}{a_i}$. Combining this with 1 above, we must now show that there exists an a_{2n+1} such that

$$\frac{((2n + 1) + (2n + 2))a_j}{2n + 2 + j} \leq a_{2n+1} \leq \frac{2n + 1 + i}{a_i}$$

for all $1 \leq i, j \leq 2n$. Such an a_{2n+1} exists if and only if each of these lower bounds is below each of the upper bounds, or

$$\frac{2n + 1 + i}{a_i} - \frac{((2n + 1) + (2n + 2))a_j}{2n + 2 + j} \geq 0.$$





Art of Problem Solving

WOOT 2010–11

Induction FAIL

Now, we just crank through the algebra:

$$\frac{2n+1+i}{a_i} - \frac{((2n+1)+(2n+2))a_j}{2n+2+j} \geq 0$$

$$\Leftrightarrow (2n+2+j)(2n+1+i) - ((2n+1)+(2n+2))a_i a_j \geq 0.$$

By our inductive hypothesis, we have $a_i a_j \leq i + j$, so

$$\begin{aligned} & (2n+2+j)(2n+1+i) - ((2n+1)+(2n+2))a_i a_j \\ & \geq ((2n+2)+j)((2n+1)+i) - ((2n+1)+(2n+2))(i+j) \\ & = (2n+2)(2n+1) - (2n+1)i - (2n+2)j + ij \\ & = ((2n+1)-j)((2n+2)-i) \\ & \geq 0, \end{aligned}$$

where the final step follows from the fact that $i, j \leq 2n$.

This completes the inductive step and shows that the bound $3 \cdot 7 \cdot 11 \cdots 4019$ can indeed be achieved.

Or does it? Where's the flaw?

Moral of the story: if a problem seems too easy for its placement on a particular contest, then you should be extra skeptical about your solution. After you find the flaw, think about how you could avoid making a similar mistake when performing an induction.

