IMO Winter Camp 2006

More Problems! – January 7, 2006

Naoki Sato

1. Prove that

$$\sum_{a+b+c=n} \binom{2a}{a}\binom{2b}{b}\binom{2c}{c} = (2n+1)\binom{2n}{n}.$$

The sum is taken over all triples of non-negative integers (a,b,c) such that $a+b+c=n$.

2. Prove that

$$\sum_{\{i,j,k\}=\{1,2,3\}} \csc^{13}\frac{2^i\pi}{7}\csc^{14}\frac{2^j\pi}{7}\csc^{2005}\frac{2^k\pi}{7}$$

is rational. The sum is taken over all permutations $\{i,j,k\}$ of $\{1,2,3\}$.

3. Let $a < b < c$ be the roots of $x^3 - 3x + 1 = 0$. Compute $T = a/b + b/c + c/a$.

4. There are ten people in a room. Two of any three of them know each other. Show that four people can be found in the room such that any two of them know each other. Show that the same holds if there are nine people.

5. For a positive integer n, let $P(n)$ denote the set of all partitions of n, and for a partition S, let $f(S)$ be the product of the elements of S and the factorials of the number of times each distinct element appears in S. For example,

$$f((5,3,3,2,2,2,1,1)) = (5\cdot 3\cdot 3\cdot 2\cdot 2\cdot 2\cdot 1\cdot 1)\cdot(1!\cdot 2!\cdot 3!\cdot 2!).$$

Show that for any n,

$$\sum_{S\in P(n)} \frac{1}{f(S)} = 1.$$

6. The integers 1, 2, …, n each appear n times in an $n\times n$ array. Show that there is a row or column which contains at least $\sqrt{n}$ distinct integers.

7. On a 7×7 square piece of graph paper, the centres of k of the 49 squares are chosen. No four of the chosen points are the vertices of a rectangle whose sides are parallel to those of the paper. What is the largest k for which this is possible?

8. Two circles touch each other internally at a point A. A secant intersects the circles at points M, N, P, and Q, situated in consecutive order. Prove that $\angle MAN = \angle PAQ$.

9. Two circles C_1 (centre R, radius r) and C_2 (centre P, radius s), where $r > s$, touch externally at A. Their direct common tangent touches C_1 at B and C_2 at C. The line RP extended meets C_2 again at D and BC produced at E. Prove that if $BC = 6DE$, then (a) the lengths of the sides of triangle REB are in arithmetic progression, and (b) $AB = 2AC$.

10. In triangle ABC, let B_1 be on AC, E on AB, G on BC, and let EG be parallel to AC. Furthermore, EG is tangent to the inscribed circle of the triangle ABB_1 and intersects BB_1 at F. Let r, r_1, and r_2 be the inradii of the triangles ABC, ABB_1, and BFG, respectively. Prove that $r = r_1 + r_2$.

11. Find all functions $f : [a, b] \to \mathbb{R}$ such that for all x, $y \in [a, b]$, $|f(x) - f(y)| \leq |x - y|^2$.

12. Let a, b, c, and d be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2,$$

and find when equality occurs.

13. The sequence a_0, a_1, ..., a_n is defined by $a_0 = 1/2$ and $a_{k+1} = a_k + a_k^2/n$ for $k = 0$, 1, ..., $n-1$. Prove that

$$1 - \frac{1}{n} < a_n < 1.$$

14. Prove that

$$N = \left(\frac{N}{2}\right) + \left(\frac{N}{4}\right) + \left(\frac{N}{8}\right) + \cdots + \left(\frac{N}{2^n}\right) + \cdots$$

for all positive integers N, where (x) denote the nearest integer to x rounded up.

15. Show that if x and y are positive integers such that $x^2 + y^2 - x$ is divisible by $2xy$ then x is a perfect square.

16. Let p be a prime of the form $4n + 3$. Prove that

$$\prod_{k=1}^{p-1} (k^2 + 1) \equiv 4 \pmod{p}.$$