2015 Winter Camp Warm-Up Set

December 22, 2014

1 Algebra

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y we have the identity

$$f(x^3) + f(y^3) = (x + y) \left(f(x^2) + f(y^2) - f(xy)\right).$$

2. Let a, b, c be positive real numbers with $a + b + c = 3$. Prove that

$$\frac{a^2}{a + \sqrt[3]{bc}} + \frac{b^2}{b + \sqrt[3]{ac}} + \frac{c^2}{c + \sqrt[3]{ab}} \geq \frac{3}{2}$$

and determine the ases of equality.

3. Let $\mathbb{R}^+$ denote the set of positive reals. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}^+$ we have

$$f(x) + f(y) \leq \frac{f(x + y)}{2}, \frac{f(x)}{x} + \frac{f(y)}{y} \geq \frac{f(x + y)}{x + y}$$

4. $\mathbb{Q}$ is the set of rational numbers. Find all functions $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ such that for all $x, y, z \in \mathbb{Z}$ we have

$$f(x, y) + f(y, z) + f(z, x) = f(0, x + y + z).$$

2 Combinatorics

1. There are 100 cards with numbers from 1 to 100 on the table. Joan and Jill took the same number of cards in a way such that the following condition holds:if Joan has a card with a number n then Jill has a card with a number $2n + 2$.What is the maximal number of cards that could be taken by the two girls?

2. There is an integer in each cell of a $2m \times 2n$ table. The following operation is allowed: Choose three cells forming an L-tromino (a 2×2 subsquare with one coroner removed) and add 1 to each integer in the three chosen cells. For which m, n is it always possible to perform a finite number of such operations such that afterwards all the integers are the same?

3. There are n intersecting convex k-gons on the plane. Any of them can be transferred to any other by a homothety with a positive coefficient. Prove that there is a point in a plane belonging to at least $1 + \frac{n-1}{2k}$ of these k-gons.

4. Jacob and David play a game. There are N gummy bears in a bag of candy. Jacob goes first, and takes k gummy bears, where k must be between 1 and $N - 1$ from the bag. For each subsequent move, if r gummy bears were removed the previous move, then on this move any number l of gummy bears can be removed, so long as $1 \leq l \leq 2r$. So for instance, if Jacob takes 1 gummy bear to begin with, and then David takes 2 gummy bears, Jacob must now take between 1 and 4 gummy bears.

   The play who takes the last gummy bear win. Find all N such that Jacob has a winning strategy.

3 Geometry

1. Let ABC be a triangle where $AC \neq BC$. Let P be the foot of the altitude taken from C to AB; and let H be the orthocentre, O the circumcentre of ABC, and D the point of intersection between the radius OC and the side AB. The midpoint of CD is E. In what ratio does the segment EP divide the segment OH?

2. Let M be the midpoint of the side BC of acute triangle ABC and H be the orthocenter of ABC. Prove that if D is the base of the perpendicular dropped from the vertex A to the line HM, then the intersection point of the bisectors of the angles DBH and DCH lie on the line HM.

3. A straight needle with length 2015 is placed in a vertical position in a co-ordinate grid, not passing through any lattice point. It is then rotated $90°$ around some point in the plane, such that the needle ends up in a horizontal position. What is the smallest possible number of lattice points the needle could have passed through?

4. Acute triangle ABC is inscribed in a circle ω_1, and the incircle of ABC touch side BC at N.Let ω_2 be a circle tangent to BC at N, and tangent to ω_1 such that ω_2 is on the same side of BC as A. Let O be the center of ω_2 and J be the center of the excircle ABC opposite angle $\angle BAC$. Prove that AO and JN are parallel.

4 Number Theory

1. Find all primes p such that $p + 2$ and $p^2 + 2p - 8$ are also primes.

2. For even positive integer n we put all numbers $1, 2, \ldots, n^2$ into the squares of an $n \times n$ chessboard (each number appears once and only once). Let S_B be the sum of the numbers put in the black squares and S_W be the sum of the numbers put in the white squares. Find all n such that we can achieve

$$\frac{S_W}{S_B} = \frac{39}{64}.$$

3. For each natural number n, define the set S_n as

$$S_n = \left\{ \binom{n}{n}, \binom{2n}{n}, \ldots, \binom{n^2}{n} \right\}$$

   (a) Prove there are infinitely many composite (not prime) numbers n such that S_n is a complete residue system mod n.

   (b) Prove there are infinitely many composite numbers n such that S_n is **not** a complete residue system mod n.

4. Let $P(x)$ be a degree n polynomial all of whose coefficients are equal to ± 1, and divisible by $(x - 1)^m$. Prove that if $m \geq 2^k, k \geq 2$, then $n \geq 2^{k+1} - 1$.