Selected problems from 2012 APMO, Balkan MO, EGMO, KSA TST, and USA TST

1. $\sum(x+y)\sqrt{(z+x)(z+y)} \geq \sum(x+y)(z+\sqrt{xy}) = \sum(x+y)z+(x+y)\sqrt{xy} \geq \sum(x+y)z+2xy = 4\sum xy$

2. We compute metric conditions for $AM \perp CM$, $ADEF$ cyclic, and AN bisects BC independently and then put everything together.

   First note that $FEN \parallel ADC$ by Ceva, and by Menelaus we have $BE/BC = BF/BA = 1/5$. Clearly $AM \perp CM$ iff $DM = DA$, or equivalently, $BD = 3b/4$. By Stewart's theorem, this is equivalent to $13b^2 = 8(a^2 + c^2)$.

   Since $FE \parallel AD$, $ADEF$ is cyclic iff $(AD - FE)/2 = AF \cos A$, so using the law of cosines and the facts that $AF = 4c/5$, $EF = b/5$, $AD = b/2$, we have $ADEF$ cyclic iff $5b^2 + 8c^2 = 8a^2$.

   Finally, note that AN bisects BC iff $NE = 3b/5$ (by the fact that $EN \parallel CA$. On the other hand, we can compute NE only using the facts that $BE = a/5$, $CE = 4a/5$, $\angle NEC = C$, and $\angle BNC = 180° - A$. First it's important to note that these uniquely determine NE. From areas, we get $cx = yz$, where $x = NE$, $y = BN$, $z = CN$. By law of cosines, we have $y^2 + z^2 + 2yz\cos A = a^2$, $a^2/25 + x^2 + 2ax\cos C/5 = y^2$, and $16a^2/25 + x^2 - 8ax\cos C/5 = z^2$, so adding up and using $yz = cx$, we have

   $$2x^2 - \frac{8}{25}a^2 = \frac{x}{5}\frac{8a^2 - 2b^2 - 8c^2}{b}.$$

   Note that this quadratic has a positive root and a negative root; we know x is uniquely determined by our equations, so it must be the positive root. Thus AN bisects BC iff $NE = 3b/5$ iff

   $$2\frac{9}{25}b^2 - \frac{8}{25}a^2 = \frac{3b}{25}\frac{8a^2 - 2b^2 - 8c^2}{b},$$

   which is equivalent to $3b^2 = 4a^2 - 3c^2$.

   But $(8a^2 - 13b^2 + 8c^2) + 7(8a^2 - 5b^2 - 8c^2) - 16(4a^2 - 3b^2 - 3c^2) = 0$, so we're done.

3. Note that the centers of (ABC) and (ADO) lie on opposite sides of AC (consider the perpendicular bisector ℓ of AD and use angle bisector theorem to show that since $AB = BD > BO$, the midpoint of AO lies on the same side of ℓ as A; clearly center of (ABC) lies between AB and AC by symmetry).

   Angle chasing, we have $\angle APO = 90° - \angle ABO/2$, and we need to show $\angle CQO = \angle ABO/2$.

   By easy continuity arguments, the problem can be rephrased as follows: let P lie on arc AC of (ABC) and define $Q = AP \cap BC$. Let X vary on segment AC. If $\angle ABX = 2\theta$, show that $\angle APX = 90° - \theta$ iff $\angle CQX = \theta$. Now let $\alpha = \angle ABC = \angle ACB$ $(0 \leq \theta \leq \alpha/2)$ and $\beta = \angle CAP$. By simple angle chasing and the law of sines, we find that $\angle APX = 90° - \theta$ iff $2\sin\theta\sin\alpha\cos(\theta - \beta) = \sin(\alpha - \beta)\sin(2\alpha - 2\theta)$ and $\angle CQX = \theta$ iff $\sin\theta\cos(\alpha - \theta)\sin\beta = \cos\alpha\sin(\alpha - 2\theta)\sin(\alpha - \beta)$.

   Expanding and expressing in terms of $\sin\theta\cos\theta$ and $\sin^2\theta$, they are clearly equivalent. Alternatively, note that if we subtract the RHS from the LHS for the two expressions and take the first minus two times the second, we get zero from easy sum-to-product and product-to-sum identities.

4. It's easy to compute $a_n = \sum_{r\geq 1}\lfloor n/(r+1)\rfloor + \sum_{r\geq s\geq 2}\lfloor n/(rs)\rfloor$. To show $a_n > n$, we just need $1 \leq r \leq n-1$ from the first part and $r = s = 2$ from the second.

   Now note that

   $$a_n = -n + \sum_{r\geq s\geq 1}\lfloor n/rs\rfloor \leq -n + \sum_{n\geq r\geq s\geq 1}(n/(rs)) < -n + (n/2)(H_n^2 + \pi^2/6) < nH_n^2/2 < n^2,$$

   where it's easy to prove $H_n < 2\sqrt{n}$ by induction on $n \geq 1$.

5. After side chasing, this boils down to $16(a^2 + ab + b^2)(b^2 + bc + c^2) \geq 9(a+b)^2(b+c)^2$ for $a, b, c \geq 0$, so it suffices to show that $4(x^2 + xy + y^2) \geq 3(x+y)^2$, which is trivial inequality.

6. Induct on n, noting that $P_n = 3 \cdot P_{n-1} \cup \{2^n\}$: gaps become at most $3 \cdot 2^{n-1}$, and are within range upon adding 2^n where necessary.

7. $(1,1,1)$ is an initial solution; now consider $(1+\omega\sqrt[3]{2}+\omega^2\sqrt[3]{2}^2)^n$ to generate a recursion.

8. First get rid of $p=2$ by bounding. Now we have $p \ge 3$, so $n^{1/n} \ge p^{1/p}$ means that $n \le p$ using the fact that $x^{1/x}$ is strictly decreasing for $x \ge e$. Since p is odd, n is odd. If $q \mid n^p+1$, then $q \mid (-n)^{(p,q-1)}-1$, so consider primes $q \le p$ such that $q \mid p^n+1$. In particular, since n is odd, just consider primes $q \mid p+1 \mid p^n+1$. It's easy to see, however, that $q \mid n+1 \implies q \nmid (n^p+1)/(n+1) \equiv p \mod q$, so $v_q(p+1) \le v_q(n+1)$ for all $q \mid p+1$. But then $n+1 \ge p+1$, so $n \ge p \implies n = p$.

9. It's enough to show that $\measuredangle GPD = \measuredangle GQF$. But $DE \parallel BA$ and $ABQE$ is cyclic, so $\triangle GDE \sim \triangle GEQ$ with opposite orientation; similarly, $\triangle GEF \sim \triangle GFP$ with opposite orientation. Using complex numbers, set $g=0$ so we have $d+e+f=0$. It's easy to see that $q = e\overline{e}/\overline{d}$ and $p = f\overline{f}/\overline{e}$, so using $d=-e-f$, it's easy to show that

$$\frac{d-p}{g-p}\Big/\frac{f-q}{g-q} = |e|^2/|f|^2 \in \mathbb{R},$$

as desired.

10. Equivalently, show that AF is the A-symmedian, or $\angle BAF = \angle MAC$. Well-known that H_A, reflection of H over D, lies on ω, and A', reflection of A over O, is the other intersection of HM with ω. Hence $AEDM$ is cyclic, so

$$\angle BAF = \angle BAO - \angle FAA' = \angle BAO - \angle DEM = \angle HAC - \angle DAM = \angle MAC,$$

as desired.

11. It would probably be difficult to show that no such set exists, so we should first try to find a construction.

    The dumbest greedy construction is to keep on adding the integer a of smallest absolute value that makes $a_0 - a \in A \cup \{a\}$ for the smallest $a_0 \notin A+A$ while not creating a zero-sum subset. Fortunately, this works: we get $A = \{1,-2,3,-5,8,\ldots\}$, i.e. set $a_i = (-1)^{i+1}F_{i+1}$ for $i \ge 1$.

    Obviously $a_i = a_{i+1}+a_{i+2}$ for all $i \ge 1$, so $A \subseteq A+A$.

    Now we prove by induction on $n \ge 1$ that the sums of the non-empty subsets of $A_n = \{a_1,\ldots,a_n\}$ span exactly $[1, F_{n+2}-1] \cup [-F_{n+1}+1,-1]$ for odd n and $[1,F_{n+1}-1] \cup [-F_{n+2}+1,-1]$ for even n. For $n=1$ and $n=2$ this is trivial. First consider $n \ge 3$ odd; then the sums of the non-empty subsets of A_{n-1} span $[1,F_n-1] \cup [-F_{n+1}+1,-1]$, so when we add in $a_n = (-1)^{n+1}F_{n+1} = F_{n+1}$, we get $[F_{n+1}+1, F_{n+2}-1]$ from the positives, F_{n+1} from a_n itself, and $[1,F_{n+1}-1]$ from the negatives. Otherwise, if $n \ge 3$ is even, then we start with $[1,F_{n+1}-1] \cup [-F_n+1,-1]$ from A_{n-1}; when we add in $a_n = (-1)^{n+1}F_{n+1} = -F_{n+1}$, we get $[-F_{n+1}+1,-1]$ from the positives, $-F_{n+1}$ from a_n itself, and $[-F_{n+2}+1,-F_{n+1}-1]$ from the negatives, as desired. Hence all nonzero integers can be obtained from the subsets of A_n for sufficiently large n, while zero can never be obtained, so we're done.

12. Suppose $Q_n/P_n = f$ is a polynomial; obviously it has rational coefficients. By Gauss's lemma (e.g. viewing as polynomials in x, show that if $p = qr$ for primitive multivariate polynomials q, r with integer coefficients, then p is primitive also), f has integer coefficients. But then $2^{2n}(2^{2n-1}+1)^{2n}/(2^{2n+1}+1) = f(2,1,0) \in \mathbb{Z}$. But $\gcd(2^{2n+1}+1, 2^{2n-1}+1) = \gcd(-3, 2^{2n-1}+1) = 3$, so $2^{2n+1}+1 = 3^m$ for some $m \ge 1$. It's well-known that only $n=1$ works here. Checking, we have $[\sum(x-y)^2]^2/[\sum(x-y)^2(y-z)^2] = [(\sum(x-y))^2 - 2\sum(x-y)(y-z)]^2/[(\sum(x-y)(y-z))^2 - 2(x-y)(y-z)(z-x)\sum(x-y)] = 4$, so we're done.

    Another perspective is to note that if $a+b+c=0$ and $\sum a^{2n}b^{2n} = 0$, then we must have $\sum a^{2n} = 0$, so using $a=-b-c$ and $a^{2n} = -b^{2n}-c^{2n}$ this means that if $(x+1)^{2n}(x^{2n}+1)+x^{2n} = 0$ for some complex x, then $x^{4n}+x^{2n}+1 = 0$. After some manipulation, we find that $(x^2+x)^{2n} = 1$, $(x+1)^{2n}+x^{2n}+1 = 0$, and $(x+1)^{4n}+(x+1)^{2n}+1 = 0$. In any case, we find that x and $x+1$ are both $(6n)^{th}$ roots of unity but not $(2n)^{th}$ roots. Using $e^{it} = \cos t + i\sin t$, we find that $x \in \{\omega, \omega^2\}$, where $\omega = e^{2\pi i/3}$. Since the minimal polynomial of each of ω, ω^2 is x^2+x+1, we must have $(x+1)^{2n}(x^{2n}+1)+x^{2n} = (x^2+x+1)^{2n}$ (by matching degrees). Plugging in $x=1$, this means that $3^{2n} = 1+2^{2n+1}$, so we're done.

13. One way is to represent with $\omega = e^{2\pi i/n}$, noting that if we assign to each letter its position in an arbitrarily ordered alphabet, then $\sum_{k=1}^{n} a_k\omega^k = 0$ iff $a_1 \dots a_n$ is repetitive (i.e. nontrivially periodic). The problem hypothesis implies that $(a_2-a_1)(\omega-\omega^2) = (a_3-a_2)(\omega^2-\omega^3) = \cdots = (a_n-a_{n-1})(\omega^{n-1}-\omega^n)$. If $n \geq 3$, this clearly implies $a_1 = \cdots = a_n$; if $n = 2$, the problem is obvious anyway.

    There is also a stupider solution. Suppose the word (W) has n letters. For $n \leq 6$ the problem is easy to verify (trivial for primes), so we can assume $n \geq 7$.

    If a word has period d (i.e. if it is a concatenation of n/d identical subwords), call it [i]d-periodic[/i] (of course, $d \mid n$). Let a_k be the k^{th} letter from the left (for $1 \leq k \leq n$) and $f(k) \leq n/2$ (for $1 \leq k \leq n-1$) denote the smallest $d < n$ such that W is d-periodic upon switching a_k and a_{k+1}. For convenience, extend indices modulo n (this is compatible with the period condition). If $f(k) = 1$ for some k, the problem is trivial, so suppose that $f(k) \geq 2$ for all k.

    First observe that if $a_k = a_{k+1}$, then W is $f(k)$-periodic itself.

    [b]Lemma:[/b] $a_{k-1} = a_{k+1}$ for $2 \leq k \leq n-1$.

    [i]Proof:[/i] If $\max(f(k-1), f(k)) < n/2$, then using the condition when a_{k-1}, a_k are swapped yields $a_{k-1} = a_{k+f(k-1)}$ (since $k + f(k-1) \notin \{k-1, k\} \pmod n$) and $a_{k+f(k)} = a_{k+f(k)+f(k-1)}$ (since $k+f(k), k+f(k)+f(k-1) \notin \{k-1, k\} \pmod n$, where we use the fact that $f(k)+f(k-1) \leq n/2+n/3 = 5n/6 < n-1$). Similarly, using the condition when a_k, a_{k+1} are swapped yields $a_{k+1} = a_{k+f(k)}$ (since $k+f(k) \notin \{k, k+1\} \pmod n$) and $a_{k+f(k-1)} = a_{k+f(k-1)+f(k)}$ (since $k+f(k-1), k+f(k-1)+f(k) \notin \{k, k+1\} \pmod n$, where we use the fact that $f(k) + f(k-1) < n-1 < n$). Thus

    $$a_{k-1} = a_{k+f(k-1)} = a_{k+f(k-1)+f(k)} = a_{k+f(k)+f(k-1)} = a_{k+f(k)} = a_{k+1},$$

    so we're done in this case.

    Otherwise, $f(k-1) = f(k) = n/2$, so because $n/2 \geq 3$, $\{k-1, k, k+1\} \cap \{k-1+n/2, k+n/2, k+1+n/2\} \pmod n = \emptyset$. Thus $a_{k-1} = a_{k+n/2} = a_{k+1}$ from the a_{k-1}, a_k and a_k, a_{k+1} swaps, as desired.$\blacksquare$

    In light of the lemma, set $a = a_1 = a_3 = \cdots$ and $b = a_2 = a_4 = \cdots$, and suppose that $a \neq b$. Swapping a_1 and a_2, we see that there must be at least two disjoint sets of consecutive a's, contradiction.

14. The key is to remove unnecessary points: all we need is $\triangle PBE \sim \triangle PCF$ (oppositely).

    Using coordinates and cross products, we can set $P = (0, 0)$, $E = (e_1, e_2)$, $F = (f_1, f_2)$, $B = (e_1, e_2) + \alpha(e_2, -e_1)$, $C = (f_1, f_2) - \alpha(f_2, -f_1)$ for some nonzero real α. Solving for $Q = (x, y)$, we get $x(e_2 - f_2 - \alpha f_1) + y(f_1 - \alpha f_2 - e_1) = (e_2 f_1 - e_1 f_2) + \alpha(e_2 f_2 - e_1 f_1)$ and $x(f_2 - e_2 + \alpha e_1) + y(e_1 + \alpha e_2 - f_1) = (e_1 f_2 - e_2 f_1) - \alpha(e_2 f_2 - e_1 f_1)$, so adding the two equations, $x\alpha(e_1 - f_1) + y\alpha(e_2 - f_2) = 0$, as desired.

    This is also easy with complex numbers, if we set $p = 0$ and take a real r such that $(b-e)/e = -(c-f)/f = ri$.

15. The key idea for both of the following solutions is to use a lot of bounding ideas; the former focuses on parity while the latter focuses on discrete continuity (since the slope of the partial sums graph is aproximately 2).

    The answer is $n = 1, 2, 3$, whose constructions are obvious. Assume for the sake of contradiction that there is a counterexample $(a_1, \dots, a_n)$ for some $n \geq 4$.

    We will only use the integer arithmetic mean condition for blocks of 2^k terms, where $k \geq 1$. For $k = 1$, we get $a_{i+1} + a_{i+2} \equiv 1 \pmod 2$; by induction, we get $a_{i+1} + \cdots + a_{i+2^k} \equiv 2^{k-1} \pmod{2^k}$ (assuming all indices are in range). As an easy corollary, $a_i \equiv a_{i+2^k} \pmod{2^k}$ for all $k \geq 1$, and the a_i must alternate in parity. (*)

    First suppose $n = 2n_1$ is even ($n_1 \geq 2$). By (*), we can WLOG assume a_1 is odd (since there are an even number of terms), so set $a_i = 2b_i + 1$ for odd i and $a_i = 2b_i + 2$ for even i ($b_i \geq 0$ for all i). Then

    $$b_1 + \cdots + b_{2n_1} = \frac{2(2n_1) - 1 - (1+2)n_1}{2} = \frac{n_1 - 1}{2} = n_2$$

    for some positive integer n_2. Thus

    $$b_1 + \cdots + b_{4n_2+2} = n_2.$$

By (*), we have $b_i \equiv b_{i+2^k} \pmod{2^{k-1}}$ and $b_{i+1} + \cdots + b_{i+2^k} \equiv 0 \pmod{2^{k-1}}$ for all $k \geq 2$ (since $(1+2)(2^{k-1}) \equiv 2^{k-1} \pmod{2^k}$). In particular, by bounding we know that at most one of $b_1, \ldots, b_4$ can be odd, but because $b_1 + \cdots + b_4$ is even, we must have the b_i all even. Thus we can set $b_i = 2c_i$ for all i and $n_2 = 2n_3$ ($n_3 \geq 1$) so that

$$c_1 + \cdots + c_{8n_3+2} = n_3.$$

Again by (*) (we can just directly extend the congruences for b_i this time, however), we have $c_i \equiv c_{i+2^k} \pmod{2^{k-2}}$ and $c_{i+1} + \cdots + c_{i+2^k} \equiv 0 \pmod{2^{k-2}}$ for all $k \geq 3$. By the same bounding method (this time using $c_1, \ldots, c_8$), we get that all the c_i must be even, so we can set $c_i = 2d_i$ for all i and $n_3 = 2n_4$ ($n_4 \geq 1$). Continuing this descent, we get a contradiction, since n is finite.

Next suppose $n = 4n_1 + 1$ ($n_1 \geq 1$). Since the terms alternate in parity, and $2n-1$ is odd, we must have an odd number of odd terms and so a_i is odd iff i is odd. If we let $a_i = 2b_i + 1$ for odd i and $a_i = 2b_i + 2$ for even i, then

$$b_1 + \cdots + b_{4n_1+1} = \frac{2(4n_1+1) - 1 - (1+2)(2n_1) - 1}{2} = n_1,$$

and we can proceed in the same way as we did for the $n = 2n_1$ case.

Finally, suppose $n = 4n_1 + 3$ ($n_1 \geq 1$). This time, we have a_i odd iff i is even, so set $a_i = 2b_i + 1$ for even i and $a_i = 2b_i + 2$ for odd i. Then

$$b_1 + \cdots + b_{4n_1+3} = \frac{2(4n_1+3) - 1 - (2+1)(2n_1+1) - 2}{2} = n_1,$$

and we can again finish in the same way.

Alternatively (while this solution is more natural, it requires not just powers of 2 for the integer mean condition), note that the integer arithmetic mean property is unaffected by translation; thus it's natural to translate everything down by 2. Note that $a_i' \geq -1$ always, so look at the set of partial sums and use discrete continuity.

16. KEY IDEA: EVERY TROMINO CONTAINS A PAIR (EITHER NORMAL OR DISJOINT)

approach 1:

1. natural to try 3x3 (nice way is to consider first row, then next two add at most 2)

2. extend (1) to 3xn to get an asymptotically 1/3 # squares bound, which allows for an inductive solution

3. alternatively to get to 2, want to understand mxn for small m, n ($m = 1, 2$ are obvious though, and don't allow for each induction if the leading quadratic constant is 1/3)

approach 2:

1. global approaches difficult; note that each tromino has ≥ 2 squares of some color, so because $2 + 2 > 3$, exactly one color with ≥ 2 squares

2. want to analyze each color separately, so consider # x's and # trominos "bearing" x and do some sums (each row, column independently)

3. alternatively to get to 2, consider number of normal pairs and disjoint pairs, then note that to count these pairs we want to sum over triples

17. either use approximation for $1/(n-x)$ for $x \approx 1$ (solves for all but small n, which we do manually), or do standard manipulations so that we want to show $\sum \frac{a_i a_j}{n - a_i a_j} \leq \frac{n}{2}$; then use $a_i a_j \leq (a_i + a_j)^2/4$ and $n - a_i a_j \geq ((n - a_i^2) + (n - a_j^2))/2$, then titu's, then sum over everything

18. One powerful idea is if $r_1 + r_2 = b_1 + b_2$, then eitehr $f(r_1 + r_2) = f(r_1) + f(r_2)$ or $f(b_1 + b_2) = f(b_1) + f(b_2)$. More generally, we want to use $f(x) + f(y) = f(z)$ as much as possible (since $f(z)/z \leq \max\{f(x)/x, f(y)/y\}$, which requires $x, y, x + y$ to be the same color. Obviously this is easiest to do

when we have long strings of the same color. Indeed, suppose $[n, n+mx]$ are all the same color as x; then $f(n+kx) = f(n) + kf(x)$ for $0 \le k \le m$. In particular, consider x, y of the same color; then if $[n, n+xy]$ are all the same color, we have $f(x)/x = f(y)/y$.

Thus we first consider the case where at least one color has unbounded strings of consecutive numbers, WLOG red. Then $f(r) = cr$ for all red r. If there are unbounded strings of blues as well, then $f(b) = c'b$ for all b, so we're done. Otherwise, there exists $C > 0$ such that for any blue b, there exists a red in $[b+1, b+C]$, so we're also done in this case using the nondecreasing condition.

It remains to consider when consecutive instances of a color differ by at most M (some positive integer), i.e. strings of constant color have length at most $M-1$. In particular, there exist infinitely many of each color. WLOG let $1 = r_1 < b_1 < r_2 < b_2 < \cdots$ be the starts of these strings. Now take m large enough so that $b_m - 1 > M$. Then for all $i \ge 2$, $f(r_i + b_m - 1) \in \{f(r_i) + f(b_m - 1), f(r_i - 1) + f(b_m)\}$, so

$$\max\{f(r_{i+1}), f(r_{i+1} - 1)\} \le f(r_i + b_m - 1) \le \max\{f(r_i), f(r_i - 1)\} + \max\{f(b_m - 1), f(b_m)\}.$$

But then there exists a positive constant α such that $f(r_n) \le alphan$ for all $n \ge 1$, so $f(n)/n \le f(r_n)/n \le \alpha$ for all $n \ge 1$, and we're done.

19.

20.