Algebra Mock Olympiad

July 3^{rd}, 2003

1. (a) Find all strictly monotonic functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(x+f(y)) = f(x) + y$ for all $x, y \in \mathbb{R}$.
   (b) Prove that for every integer $n > 1$, there do not exist strictly monotonic $f: \mathbb{R} \to \mathbb{R}$ such that $f(x+f(y)) = f(x) + y^n$ for all $x, y \in \mathbb{R}$.

2. Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers for which there exists a real number c with $0 \le a_i \le c$ for all i, such that $|a_i - a_j| \ge \frac{1}{i+j}$ for all i, j with $i \ne j$. Prove that $c \ge 1$.

3. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(f(x)+y) = 2x + f(f(y) - x)$ for all real x, y.

4. Let A be a nonempty set of positive integers. Suppose that there are positive integers $b_1, \ldots, b_n, c_1, \ldots, c_n$ such that

   i) for each i, the set $b_iA + c_i = \{b_i \cdot a + c_i : a \in A\}$ is a subset of A,
   ii) the sets $b_iA + c_i$ and $b_jA + c_j$ are disjoint whenever $i \ne j$.

   Prove that $\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \le 1$.