2001 Winter Camp

1 Arithmetic Mean Geometric Mean Inequality

Recall that:

AM-GM *Let $x_1, x_2, \cdots, x_n$ be positive real numbers. Then*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Use this inequality to prove the following:

Example 1 Let $a_1, a_2, \ldots, a_n$ be a sequence of positive numbers. Show that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2,$$

with equality holding if and only if the a_i are equal. [**Hint:** Prove this first for $n = 2$ and then reduce the general case to this case.]

The following problem needs some 'massaging' (change the terms a little bit so that they become easier to deal with).

Example 2 Let $A = \sum_{n=1}^{10000} \frac{1}{\sqrt{n}}$. Find $\lfloor A \rfloor$ without a calculator. [**Hint:** Use telescoping.]

Example 3 Let $r_1, r_2, \ldots, r_n$ be a sequence of positive numbers. Prove:

$$\left(\sum_{i=1}^{n} \frac{1}{r_i}\right)\left(\prod_{i=1}^{n}(1 + r_i)\right) \geq \frac{n^{n+1}}{(n-1)^{n-1}}.$$

Example 4 Let $a_1, a_2, \ldots, a_n$ be a sequence of positive numbers, and let S_k be the sum of all k-fold products. Prove:

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \cdots a_n \quad (1 \leq k \leq n-1)$$

Example 5 Let $0 \leq x_1, x_2, \cdots, x_n \leq 1$, and $\sum_{i=1}^{n} x_i = m + r$ where $m = \mathbb{Z}$ and $0 \leq r < 1$. Prove:

$$\sum_{i=1}^{n} x_i^2 \leq m + r^2.$$

2 Cauchy-Schwarz

Recall the following very useful inequality which is a simple consequence of AM-GM:

Cauchy-Schwarz *Let* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$ *be sequences of real numbers. The* Cauchy-Schwarz inequality *states that*

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right),$$

with equality holding only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

Here are some straight forward applications of this inequality. If we let $a = b = c = 1$, we obtain:

$$\frac{(x+y+z)^2}{3} \leq x^2 + y^2 + z^2.$$

If all the variables in C-S are positive, then

$$(\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \cdots + \sqrt{a_n b_n})^2 \leq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \quad (1)$$

Here is a more interesting example that uses this last inequality:

Example 6 (Titu Andreescu) Let P be a polynomial with positive coefficients. Prove that if

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

holds for $x = 1$ then it holds for every $x > 0$.

Solution Write $P(x) = u_0 + u_1 x + u_2 x^2 + \cdots + u_n x^n$. When $x = 1$, the inequality is just $P(1) \geq 1/P(1)$, or

$$(u_0 + u_1 + u_2 + \cdots + u_n)^2 \geq 1. \quad (2)$$

We need to show that

$$\left(u_0 + \frac{u_1}{x} + \cdots + \frac{u_n}{x^n}\right)(u_0 + u_1 x + \cdots + u_n x^n) \geq 1$$

for all positive x. This can be proved using (1) when we make the following substitutions:

$$a_0 = u_0,\ a_1 = u_1/x,\ \ldots,\ a_n = u_n/x^n$$

and

$$b_0 = u_0,\ b_1 = u_1 x,\ \ldots,\ b_n = u_n x^n$$

With these choices for a_i and b_i, the inequality (1) reads

$$\begin{aligned}(u_0 + u_1 + \cdots + u_n)^2 &\leq (u_0 + u_1/x + u_2/x^2 + \cdots + u_n/x^n)(u_0 + u_1 x + \cdots + u_n x^n) \\ &= P\left(\frac{1}{x}\right) P(x)\end{aligned}$$

With inequality (2) above we conclude that

$$P\left(\frac{1}{x}\right) P(x) \geq 1.$$

In the solution above we saw the use of substitution. For the following problem substitution is again an important tool. Also be aware of amount of freedom in the Cauchy-Schwarz inequality.

Example 7 (IMO1995) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution We begin by substituting $x = 1/a$, $y = 1/b$ and $z = 1/c$ (this is often a useful tool!). This transforms the original problem into showing that

$$\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq \frac{3}{2}, \tag{3}$$

where $xyz = 1$. Write S for the left hand side of this inequality. Notice that

$$S = \left(\frac{x}{\sqrt{y+z}}\right)^2 + \left(\frac{y}{\sqrt{x+z}}\right)^2 + \left(\frac{z}{\sqrt{y+x}}\right)^2,$$

Now Cauchy-Schwarz implies that

$$S(u^2+v^2+w^2) \geq \left(\frac{xu}{\sqrt{y+z}} + \frac{yv}{\sqrt{z+x}} + \frac{zw}{\sqrt{x+y}}\right)^2, \tag{4}$$

for *any* choice of u, v, w. Here is the freedom in applying C-S! What would be the most helpful choice?

Certainly $u = \sqrt{y+z}$, $v = \sqrt{z+x}$, $w = \sqrt{x+y}$ is a natural choice, since it will simplify the right hand side of the inequality considerably. And better still, the left hand side will also be simplified:

$$u^2 + v^2 + w^2 = 2(x+y+z).$$

SO the inequality (4) reduces to

$$2S(x+y+z) \geq (x+y+z)^2,$$

and this is equivalent to

$$2S \geq (x+y+z).$$

Now by AM-GM, we have

$$x+y+z \geq 3\sqrt[3]{xyz} = 3,$$

since $xyz = 1$. We conclude that $S \geq 3/2$ as required.

In conclusion here is a list of techniques/approaches to solve inequality problems:

1. Use substitution to transform the problem into a nicer form.
2. Try small cases ($n = 1, 2, 3$).
3. Almost every inequality problem can be solved using AM-GM and Cauchy-Schwarz.
4. Exploit symmetry.
5. Use massage: change the terms slightly so they become nicer.
6. Try different approaches.
7. Don't give up.

3 Problems

1. Prove that
$$0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$$
where $x, y, z \geq 0$ with $x + y + z = 1$.

2. Show that
$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots < 3.$$
(Don't use calculus!)

3. For which n is $1/n$ closest to
$$\sqrt{1,000,000} - \sqrt{999,999}\,?$$

4. Prove that
$$n! < \left(\frac{n+1}{2}\right)^n, \text{ for } n = 2, 3, 4, \ldots$$

5. (IMO1976) Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

6. Show that
$$\frac{1}{\sqrt{4n}} \leq \frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n}}.$$

7. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive numbers. Show that for all positive x,
$$(x + a_1)(x + a_2)\cdots(x + a_n) \leq \left(x + \frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n.$$

8. Find all ordered pairs of positive real numbers (x, y) such that $x^y = y^x$. Notice that the set of pairs of the form (t, t) where t is any positive number is *not* the full solution, since $2^4 = 4^2$.

9. Show that
$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_n^2 + b_n^2} \geq \sqrt{(a_1 + a_2 + \cdots + a_n)^2 + (b_1 + b_2 + \cdots + b_n)^2}$$

for all real values of the variables, and give a condition for equality to hold. Algebraic methods will certainly work, but there must be a better way...

10. Let $a_1, a_2, \ldots, a_n$ be positive, with a sum of 1. Show that
$$\sum_{i=1}^{n} a_i^2 \geq 1/n.$$

11. If $a, b, c > 0$, prove that
$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2.$$

12. Let $a, b, c \geq 0$. Prove that
$$\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

13. Let $a, b, c, d > 0$. Prove that
$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

14. (USAMO 1983) Prove that the zeros of
$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$
cannot *all* be real if $2a^2 < 5b$.

15. (IMO1984) Prove that
$$0 \leq yz + zx + xy - 2xyz \leq 7/27,$$
where x, y and z are non-negative real numbers for which $x+y+z = 1$.

16. (Putnam 1968) Determine all polynomials that have only real roots and all coefficients are equal to ± 1.

17. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive numbers, and let $b_1, b_2, \ldots, b_n$ be any permutation of the first sequence. Show that
$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n.$$

18. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be increasing sequences of real numbers and let $x_1, x_2, \ldots, x_n$ be any permutation of $b_1, b_2, \ldots, b_n$. Show that
$$\sum a_i b_i \geq \sum a_i x_i.$$