Number Theory – Constructions

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A certain class of number theory problems require you to find some set of numbers that satisfy some conditions: one or more equations, a divisibility condition, or something else. You are often given a number of degrees of freedom in the problem, and there can several different ways to make a construction that works; some of them are usually nicer than others. Try to make guesses that make the problem nicer (i.e. make terms cancel each other out) but be brave when expanding whatever algebra you need to see if your guess works. Because there are so many ways to approach these problems, the first method you think of may very well not work, but keep trying!

Warm-up problem

1 (USAJMO 2010)**.** A triangle is called a parabolic triangle if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.

Some general tips

1. Look at prime factorizations: try to make terms cancel each other out or have large common factors.

2. Try to find small solutions: you might see a pattern. If you need to find infinitely many solutions or large solutions to a given equation, see whether you can take a small solution and scale it up to find a larger solution.

3. Find a family of guesses depending on one or two parameters that satisfies some of your conditions, and see if you can pick the parameters appropriately to meet the other conditions.

4. For Diophantine-type problems: the easiest Diophantine equations to solve are ones of low degree in a small number of variables (eg linear equations, Pell equations, Pythagorean triples). Try to guess solutions as above to transform your equation into one of that form.

5. Your problem might really be more of a combinatorics problem, so try thinking of it combinatorially.

6. Inductive constructions can be useful; once you find one solution to your equation, try seeing if you can "bootstrap" off it to find more.

7. Sometimes a "just do it" approach works; approach the problem in a way where you can make choices to make each condition satisified one at a time.

Problems

2 (Yugoslavia 07)**.** Let a, b, c be natural numbers and $a^2 + b^2 + c^2 = n$. Prove that there exist constants p_i, q_i, r_i, $(i = 1, 2, 3)$ independent of a, b, c such that

$$(p_1 a + q_1 b + r_1 c)^2 + (p_2 a + q_2 b + r_2 c)^2 + (p_2 a + q_2 b + r_2 c)^2 = 9n.$$

Furthermore, if a, b, c are not all divisible by 3, show that $9n$ can be expressed as $x^2 + y^2 + z^2$ for some natural numbers x, y, z not divisible by 3.

3. The positive integer m has a prime divisor greater than $\sqrt{2m} + 1$. Find the smallest positive integer M such that there exists a finite set T of distinct positive integers satisfying (i) m and M are the least and greatest elements, respectively, in T, and (ii) the product of all the numbers in T is a perfect square.

4. Let $x < y$ be positive integers and

$$P = \frac{x^3 - y}{1 + xy}.$$

Find all integer values that P can take.

5 (MOP 2007)**.** Find a set of positive integers $n_1, n_2, \ldots, n_{2006}$, each greater than one billion, satisfying the equation

$$n_1^2 + n_2^3 + n_3^4 + \cdots + n_{2005}^{2006} = n_{2006}^{2007}$$

6. Show that there exists a sequence $a_1, a_2, a_3, \ldots,$ of positive integers such that

(i) the sequence $a_1, a_2, a_3, \ldots,$ contains each positive integer exactly once

(ii) the sequence $|a_1 - a_2|, |a_2 - a_3|, \ldots$ also contains each positive integer exactly once.

7 (USAMO 98)**.** Prove that for each integer $n \geq 2$, there is a set S of n integers such that ab is divisible by $(a - b)^2$ for all distinct $a, b \in S$.

8 (USAMO 2002)**.** Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \leq i < k)$.

9 (CGMO 2006)**.** An integer is called *good* if it can be written as the sum of three positive cubes. Prove that for every $i \in \{0, 1, 2, 3\}$ there are infinitely many n such that there are exactly i good numbers among n, $n + 2$, and $n + 28$.

10 (CGMO 2006)**.** Let p be a prime greater than 3. Prove that there exist integers $a_1, a_2, \ldots, a_n$ with

$$-\frac{p}{2} < a_1 < a_2 < \cdots < a_n < \frac{p}{2}$$

such that

$$\frac{(p - a_1)(p - a_2)\cdots(p - a_n)}{|a_1 a_2 \cdots a_n|}$$

is a perfect power of 3.

11. Prove that every positive rational number can be represented in the form $\frac{a^3+b^3}{c^3+d^3}$ where a, b, c, d are positive integers.

12 (MOP 02)**.** Show that there are infinitely many ordered quadruples of integers (x, y, z, w) such that all six of

$$xy + 1, xz + 1, xw + 1, yz + 1, yw + 1, zw + 1$$

are perfect squares.