A Powerful Technique for Proving Remarkable Trigonometric Identities

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1 Introductory Examples

Proving trigonometric identities usually requires building certain algebraic equations and using relations between the coefficients and the roots. We provide a simple method for proving these types of identities.

Theorem. If $P_0P_1 \dots P_{n-1}$ is a regular polygon inscribed in a circle of radius $R > 0$, then we have that

$$\prod_{j,j\neq k} P_kP_j = nR^{n-1}$$

Proof. Consider the polynomial $P \in \mathbb{C}[Z], P = Z^n - R^n$. The roots $\omega_j, j = 0, 1, \dots n-1$ of P are the vertices of the polygon in the complex plane. Then we have $|\omega_j| = R, j = \overline{1, n-1}$ and by the decomposition of the polygon:

$$P = Z^n - R^n = \prod_{j=1}^{n-1}(Z - \omega_j)$$

Calculating $P'(\omega_k)$ we find

$$n\omega_k^{n-1} = \prod_{j,j\neq k}(\omega_k - \omega_j)$$

Now we can take the modulus and use the fact that the distance between two points graphed in the complex plane is the modulus of their difference, proving the theorem. □

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Corollary - If $P_0P_1\dots P_{n-1}$ is a regular polygon inscribed the unit circle then we have that

$$\prod_{j,j\neq k} P_kP_j = n \tag{1}$$

There is an alternate proof of the identity in (1) that utilizes Vandermonde's Identity. The proof is as follows.

Proof. Define V to the the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \omega_1 & \omega_2 & \cdots & \omega_n \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{n-1} & \omega_2^{n-1} & \cdots & \omega_n^{n-1} \end{bmatrix},$$

where $\omega_1, \cdots, \omega_n$ are the n^{th} roots of unity. We calculate

$$\begin{aligned} & \prod_{i<j} (\omega_j - \omega_i)^2 = |V|^2 = |V| \cdot |V^T| \\ = & \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \omega_1 & \omega_2 & \cdots & \omega_n \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{n-1} & \omega_2^{n-1} & \cdots & \omega_n^{n-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & \omega_1 & \cdots & \omega_1^{n-1} \\ 1 & \omega_2 & \cdots & \omega_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n & \cdots & \omega_n^{n-1} \end{vmatrix} \\ = & \begin{vmatrix} n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & n & \cdots & 0 \end{vmatrix} \\ = & \pm n^n \end{aligned}$$

by repeated use of the identity $\omega_1^k + \cdots + \omega_n^k = n$ if $n|k$ and $\omega_1^k + \cdots + \omega_n^k = 0$ otherwise. Since $|V|^2$ is non-negative, we find that

$$\prod_{i<j} (\omega_j - \omega_i)^2 = n^n.$$

Finally, since

$$n^n = \prod_{i<j} (\omega_j - \omega_i)^2 = \prod_{i=1}^{n} \left(\prod_{j,j\neq i} (\omega_i - \omega_j) \right) = \left(\prod_{j,j\neq i} (\omega_i - \omega_j) \right)^n,$$

taking the moduli of the n^{th} root of this last equation proves the identity. $\square$

From (1) we can easily derive the identity

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}=\frac{n}{2^{n-1}},\quad n\geq 2$$

Proof. We have

$$P_0P_j^2=|\omega_n^j-1|^2=(\omega_n^j-1)(\overline{\omega_n^j-1})=2-2\cos\frac{2j\pi}{n}=4\sin^2\frac{j\pi}{n},$$

and so from (1) we obtain

$$n=\prod_{j\neq 0}P_0P_j=\prod_{j=1}^{n-1}2\sin\frac{j\pi}{n}=2^{n-1}\prod_{j=1}^{n-1}\sin\frac{j\pi}{n}.$$

Therefore, $\displaystyle\prod_{j=1}^{n-1}\sin\frac{j\pi}{n}=\frac{n}{2^{n-1}}$. □

A direct consequence of this theorem, namely that

$$\prod_{k=1}^{(p-1)/2}2\sin\frac{k\pi}{p}=\sqrt{p}$$

for an odd prime p, can be used to provide an elegant proof of the Quadratic Reciprocity Law, which the margins of this paper are too small to contain.

2 Main Results

2.1 Cyclotomic Polynomials

As the first theorem of Section 1 shows, using polynomials whose roots are the roots of unity is a powerful method capable of generating striking identities. However, in this paper it is desirable to use only those values of k that are relatively prime to n. To do this we turn to cyclotomic polynomials. We define

$$\Phi_n(x)=\prod_{\substack{1\leq k<n\\ \gcd(\mathrm{k,n})=1}}(x-\omega_n^k)$$

where ω_n, as usual, is the n^{th} root of unity $e^{2\pi i/n}$. Clearly the degree of $\Phi_n(x)$ is the Euler function $\varphi(n)$. It is well known that $\Phi_n(x)$ is a monic polynomial with integer coefficients that is irreducible over $\mathbb{Q}$. One of the most useful facts about cyclotomic polynomials, proven in a paper by Y. Gallot, is as follows:

$$\begin{aligned}&\Phi_n(1) = p \text{ when } n \text{ is a power of a prime } p\\ &\Phi_n(1) = 1 \text{ otherwise}\end{aligned}$$

Also, Gallot shows that $\Phi_q(-x) = \Phi_{2q}(x)$ when $q > 1$ is an odd integer. From this we may set $x = 1$ to find

$$\Phi_q(-1) = \Phi_{2q}(1) = 1$$

when $q > 1$ is odd, since in this case, $2q$ is obviously not a power of a prime.

2.2 The Identities

Using analogous methods as in section 1, we may now prove a variety of stunning identities, with the help of a basic lemma.

Lemma $\displaystyle\sum_{(k,n)=1} k = \frac{n\varphi(n)}{2}$.

Proof.

$$S = \sum_{(k,n)=1} k = \frac{1}{2}\left(\sum_{(k,n)=1}(k) + \sum_{(k,n)=1}(n-k)\right) = \frac{1}{2}\sum_{(k,n)=1} n = \frac{n\varphi(n)}{2}.$$

□

Theorem. If n is not a power of a prime, then

$$\prod_{(k,n)=1} \sin\frac{k\pi}{n} = \frac{1}{2^{\varphi(n)}}.$$

Proof. First,

$$\begin{aligned}1 - \omega_n^k &= 1 - \cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n} = 2\sin^2\frac{k\pi}{n} - 2i\cos\frac{k\pi}{n}\sin\frac{k\pi}{n}\\ &= 2\sin\frac{k\pi}{n}\left(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}\right)\\ &= \left(2\sin\frac{k\pi}{n}\right)\frac{1}{i}(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}).\end{aligned}$$

Since n is not a power of a prime, $n = p^a m$ for an odd prime p not dividing m. So $\varphi(n) = \varphi(p^a)\varphi(m) = p^{a-1}(p-1)\varphi(m)$. Since $p-1$ is even, $\varphi(n)$ is thus even. Now, defining P_n as the product in question, we calculate

$$\begin{aligned}
1 = \Phi_n(1) &= \prod_{(k,n)=1} (1-\omega_n^k) = \prod_{(k,n)=1} \left(2\sin\frac{k\pi}{n}\right)\frac{1}{i}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right) \\
&= \frac{2^{\varphi(n)}}{i^{\varphi(n)}} P_n \prod_{(k,n)=1} \left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right) \\
&= \frac{2^{\varphi(n)}}{i^{\varphi(n)}} P_n \cdot \left(\cos\left(\sum_{(k,n)=1} k\frac{\pi}{n}\right) + i\sin\left(\sum_{(k,n)=1} k\frac{\pi}{n}\right)\right) \\
&= 2^{\varphi(n)} P_n \cdot \frac{1}{(i^2)^{\varphi(n)/2}} \left(\cos\frac{\varphi(n)\pi}{2} + i\sin\frac{\varphi(n)\pi}{2}\right) \\
&= 2^{\varphi(n)} P_n \cdot \frac{1}{(-1)^{\varphi(n)/2}} \cdot (\cos\pi + i\sin\pi)^{\varphi(n)/2} \\
&= 2^{\varphi(n)} P_n.
\end{aligned}$$

Thus, $P_n = \frac{1}{2^{\varphi(n)}}$, as claimed. □

In an analogous manner, we obtain the following theorems.

Theorem. If $n = p^a$ for some prime p and $n > 2$, then

$$\prod_{(k,n)=1} = \frac{p}{2^{\varphi(n)}}.$$

Proof. We have $\Phi_{p^1}(1) = p$, and again $\varphi(p^a) = p^{a-1}(p-1) \equiv 0 \mod 2$, and the rest of the proof is exactly the same. □

Theorem. If $n > 1$ is odd, then the following identity holds:

$$Q_n = \prod_{(n,k)=1} \cos\frac{k\pi}{n} = \frac{(-1)^{\varphi(n)/2}}{2^{\varphi(n)}}$$

Proof. As explained above, $\varphi_n(-1) = 1$ for odd n. Also,

$$\begin{aligned}
1 + \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} &= 2\cos^2\frac{k\pi}{n} + 2\sin\frac{k\pi}{n}\cos\frac{k\pi}{n} \\
&= 2\cos\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right).
\end{aligned}$$

So we calculate

$$
\begin{aligned}
1 = \Phi_n(-1) &= \prod_{(k,n)=1} (-1-\omega_n^k) = (-1)^{\varphi(n)} \prod_{(k,n)=1} 2\cos\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right) \\
&= 2^{\varphi(n)} Q_n \prod_{(k,n)=1}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right) \\
&= 2^{\varphi(n)} Q_n \left(\cos\frac{\varphi(n)\pi}{2} + i\sin\frac{\varphi(n)\pi}{2}\right) \\
&= Q_n \cdot 2^{\varphi(n)}(-1)^{\varphi(n)/2},
\end{aligned}
$$

which implies $Q_n = \dfrac{1}{2^{\varphi(n)}(-1)^{\varphi(n)/2}} = \dfrac{(-1)^{\varphi(n)/2}}{2^{\varphi(n)}}$ as desired. □

2.3 Trigonometric Identities and Primative Polygons

These trigonometric identities have fascinating geometrical interpretations. Define the n^{th} *Primative Polygon* A_n to be the convex polygon formed by the vertices ω_n^k for k relatively prime to n in the complex plane. Although this polygon is very irregular in shape, it has some very nice properties. For example, since $(k,n) = 1 \iff (n-k,n) = 1$, the primative polygon is symmetric with respect to the x-axis. However, it is certainly not (generally) symmetric through the y-axis, as can be seen by looking at A_6 which consists of nothing but the line segment between $(1/2, \sqrt{3}/2)$ and $(1/2, -\sqrt{3}/2)$! But the trigonometric identities in the previous subsection allow us a small glimpse of its complexities.

Theorem. If $n > 1$ is odd, B_k are the vertices of A_n, and X and Y are the points 1 and -1 respectively, then

$$\prod XB_k = 1 = \prod YB_k.$$

Proof. The length from X to ω_n^k given by $|1-\omega_n^k| = |2\sin\frac{k\pi}{n}|$, and so

$$\prod XB_k = \prod_{(k,n)=1} |2\sin\frac{k\pi}{n}| = 2^{\varphi(n)}\left|\prod_{(k,n)=1}\sin\frac{k\pi}{n}\right| = 1.$$

Likewise, the distance from Y to ω_n^k is $|2\cos\frac{k\pi}{n}|$, and therefore

$$\prod YB_k = 2^{\varphi(n)}\left|\prod_{(k,n)=1}\cos\frac{k\pi}{n}\right| = 2^{\varphi(n)}\left|\frac{(-1)^{\varphi(n)/2}}{2^{\varphi(n)}}\right| = 1.$$

□

3 Bibliography

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