AoPS FUNCTIONAL EQUATION MARATHON

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PROBLEMS

1. Find all functions $f:\mathbb{Q}_+ \to \mathbb{Q}_+$ that satisfies the following two conditions for all $x \in \mathbb{Q}_+$:
1.$f(x+1) = f(x)+1$
2.$f(x^2) = f(x)^2$

2. Determine all functions $f:\mathbb{R} \to \mathbb{R}$ such that:

$$f(x^3) - f(y^3) = (x^2 + xy + y^2)(f(x) - f(y))$$

3. Find all continuous functions $f:\mathbb{R} \to \mathbb{R}$ such that:

$$(1 + f(x)f(y))f(x+y) = f(x) + f(y)$$

4. Determine all functions $f:\mathbb{R} \to \mathbb{R}$ such that:

$$f(x^3 + y^3) = xf(x^2) + yf(y^2)$$

5. Find all functions $f:\mathbb{R}_+ \to \mathbb{R}$ satisfying

$$f(x+y) - f(y) = \frac{x}{y(x+y)}$$

6. Determine all functions $f:\mathbb{R} \to \mathbb{R}$ such that:

$$f(x + yf(x)) + f(xf(y) - y) = f(x) - f(y) + 2xy$$

7. Find least possible value of $f(1998)$ where $f:\mathbb{N} \to \mathbb{N}$ satisfies the following equation:

$$f(n^2 f(m)) = mf(n)^2$$

8. Find all functions $f:\mathbb{R}_+ \to \mathbb{R}_+$ satisfying:

$$f(x + f(y)) = f(x+y) + f(y)$$

9. Let $f:\mathbb{R} \to \mathbb{R}$ be a function such that:
$(i) f(x) + f(y) + 1 \geq f(x+y) \geq f(x) + f(y)$
(ii)For all $x \in [0,1)$, $\quad f(0) \geq f(x)$
$(iii) f(1) = -f(-1) = 1$.
Find all such functions.

10. Determine all functions $f:\mathbb{R} \to \mathbb{R}$ such that:

$$f(xy + f(x)) = xf(y) + f(x)$$

11. Find all functions $f:\mathbb{Q} \to \mathbb{Q}$ such that $f(2x) = 2f(x)$ and $f(x) + f\left(\frac{1}{x}\right) = 1$.

12. Determine all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(xf(y))+f(yf(x))=\frac{1}{2}f(2x)f(2y)$$

13. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x^5)-f(y^5)=(f(x)-f(y))(x^4+x^3y+x^2y^2+xy^3+y^4)$$

14. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(xf(x)+f(y))=y+f(x)^2$$

15. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x)^2+2yf(x)+f(y)=f(y+f(x))$$

16. Determine all polynomial functions $f:\mathbb{R}\to\mathbb{R}$, with integer coefficients, which are bijective and satisfy the relation:

$$f(x)^2=f(x^2)-2f(x)+a$$

where a is a fixed real.

17. Let k is a non-zero real constant.Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying $f(xy)=f(x)f(y)$ and $f(x+k)=f(x)+f(k)$.

18. Find all continuous and strictly-decreasing functions $f:\mathbb{R}_+\to\mathbb{R}_+$ that satisfies

$$f(x+y)+f(f(x)+f(y))=f(f(x+f(y))+f(y+f(x)))$$

19. Find all functions $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ of two variables satisfying

$$f(x,x)=x,\ f(x,y)=f(y,x),\ (x+y)f(x,y)=yf(x,x+y)$$

20. Prove that for any function $f:\mathbb{R}\to\mathbb{R}$,

$$f(x+y+xy)=f(x)+f(y)+f(xy)\iff f(x+y)=f(x)+f(y)$$

21. Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that

$$f(x)^3+f(y)^3+f(z)^3=f(x^3+y^3+z^3)$$

22. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(f(x)+y)=2x+f(f(y)-x)$$

23. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that:

$$f(f(n))+f(n+1)=n+2$$

24. Find all functions $f:\mathbb{R}_+\to\mathbb{R}_+$ such that:

$$f(x)f(yf(x))=f(x+y)$$

25. Find all functions $f:\mathbb{R}\to\mathbb{R}$ which satisfy this equation:

$$f(xf(y)+f(x))=f(yf(x))+x$$

26. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(x^2+f(y))=y+f(x)^2$$

27. If any function $f:\mathbb{R}\to\mathbb{R}$ satisfies

$$f(x^3+y^3)=(x+y)(f(x)^2-f(x)f(y)+f(y)^2)$$

then prove that $f(1996x)=1996f(x)$.

28. Find all surjective functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x-y))=f(x)-f(y)$$

29. Find all $k\in\mathbb{R}$ for which there exists a differentiable function $f:\mathbb{R}\to\mathbb{R}$ such that $f(1)\le 1$ and $f(x)^2+f'(x)^2=k$.

30. Find all $a\in\mathbb{R}$ for which there exists a non-constant function $f:(0,1]\to\mathbb{R}$ such that

$$a+f(x+y-xy)+f(x)f(y)\le f(x)+f(y)$$

31. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that

$$2n+2009\le f(f(n))+f(n)\le 2n+2011$$

32. Find all functions $f:\mathbb{R}_+\to\mathbb{R}$ satisfying $f(a)=1$ and

$$f(x)f(y)+f\left(\frac{a}{x}\right)f\left(\frac{a}{y}\right)=2f(xy)$$

33. Determine all functions $f:\mathbb{Q}\to\mathbb{C}$ such that
(i) for any rational $x_1,x_2,\dots,x_{2010}$, $f(x_1+x_2+\dots+x_{2010})=f(x_1)f(x_2)\dots f(x_{2010})$
$(ii)$for all $x\in\mathbb{Q}$, $\overline{f(2010)}f(x)=f(2010)\overline{f(x)}$.

34. Find all functions $f:\mathbb{Q}\to\mathbb{R}$ satisfying

$$f(x+y+z)=f(x)+f(y)+f(z)+3\sqrt[3]{f(x+y)f(y+z)f(z+x)}\quad \forall x,y,z\in\mathbb{Q}$$

35. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)=\max_{y\in\mathbb{R}}(2xy-f(y))$$

36.

37. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x)+y)=f(x^2-y)+4f(x)y$$

38. Find all functions $f:\mathbb{R}_+\to\mathbb{R}_+$ satisfying

$$f(x)^2+2yf(x)+f(y)=f(y+f(x))$$

39. Let $k\geq 1$ be a given integer.Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^k+f(y))=y+f(x)^k$$

40. Find all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy

$$f(xy)+f(x-y)\geq f(x+y)$$

41. Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ that satisfy $f(1)=f(-1)$ and

$$f(m)+f(n)=f(m+2mn)+f(n-2mn)$$

42. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(x+f(y+f(z)))=f(x)+f(f(y))+f(f(f(z)))$$

43. Let f be a real function defined on the positive half-axis for which $f(xy)=xf(y)+yf(x)$ and $f(x+1)\leq f(x)$ hold for every positive x. If $f\left(\frac{1}{2}\right)=\frac{1}{2}$, show that $f(x)+f(1-x)\geq -x\log_2 x-(1-x)\log_2(1-x)$ for every $x\in(0,1)$.

44. Let a be a real number and let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying $f(0)=\frac{1}{2}$ and $f(x+y)=f(x)f(a-y)+f(y)f(a-x)$ .Prove that f is a constant function.

45. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)^3=-\frac{x}{12}(x^2+7xf(x)+16f(x)^2)$$

46. Find all functions $f:\mathbb{R}\setminus\{0,1\}\to\mathbb{R}$ which satisfies

$$f(x)+f\left(\frac{1}{1-x}\right)=1+\frac{1}{x(1-x)}$$

47. Let $f:\mathbb{R}_+\to\mathbb{R}$ be a function such that

(i) If $x<y$ then $f(x)<f(y)$
(ii) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$
Show that $f(x)<0$ for some value of x.

48. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y)+f(xy)=f(x)+f(y)+f(xy+1)$$

49.A Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x)+f(y)+f(z)+f(x+y+z)=f(x+y)+f(y+z)+f(z+x)+f(0)$$

49.B Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(xf(y)+f(x))=2f(x)+xy$$

50. A function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:
$(i) f(-x)=-f(x)$
$(ii) f(x+1)=f(x)+1$
$(iii) f\left(\frac{1}{x}\right)=\frac{f(x)}{x^2}$
Prove that $f(x)=x\ \forall x \in \mathbb{R}$.

51. Find all injective functions $f\colon \mathbb{N} \to \mathbb{N}$ which satisfies

$$f(f(x)) \leq \frac{f(x)+x}{2}$$

52. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ which satisfies

$$f(f(x)+y)=f(x^2-y)+4f(x)y$$

53. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^n+f(y))=y+f(x)^n$$

where $n>1$ is a fixed natural number.

54. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x-y+f(y))=f(x)+f(y)$$

55. Find all functions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ which have the property

$$f(x)f(y)=2f(x+yf(x))$$

56. Find all functions $f\colon \mathbb{Q}_+ \to \mathbb{Q}_+$ with the property

$$f(x)+f(y)+2xyf(xy)=\frac{f(xy)}{f(x+y)}$$

57. Determine all functions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$f(x+y) - f(x-y) = 4\sqrt{f(x)f(y)}$$

58. Determine all functions $f\colon \mathbb{N}_0 \to \{1, 2, \ldots, 2000\}$ such that
(i)For $0 \le n \le 2000$, $f(n) = n$
$(ii)f(f(m) + f(n)) = f(m+n)$

59. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = y + f(x+1)$$

60. Let $n > m > 1$ be odd integers.Let $f(x) = x^m + x^n + x + 1$.Prove that $f(x)$ is irreducible over $\mathbb{Z}$.

61. A function $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfies the following equation:

$$f(m+n) + f(mn-1) = f(m)f(n) + 2$$

Find all such functions.

62. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $f\left(\sqrt{ab}\right) = \sqrt{f(a)f(b)}$ for all $a, b \in \mathbb{R}_+$ satisfying $a^2 b > 2$.Prove that the equation holds for all $a, b \in \mathbb{R}_+$

63. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$[f(m) + f(n)]f(m-n) = [f(m) - f(n)]f(m+n)$$

64. Find all polynomials which satisfy

$$P(x+1) = P(x) + 2x + 1$$

65. A rational function f (i.e. a function which is a quotient of two polynomials) has the property that $f(x) = f\left(\frac{1}{x}\right)$.Prove that f is a function in the variable $x + \frac{1}{x}$.

66. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x-y) = f(x+y)f(y)$$

67. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) = f(x) + f(y) + f(xy) - 2$$

68. Find all functions $f\colon \mathbb{R} \to \mathbb{R}_0$ such that
$(i)f(-x) = -f(x)$
$(ii)f\left(\frac{1}{x+y}\right) = f\left(\frac{1}{x}\right) + f\left(\frac{1}{y}\right) + 2(xy - 1000)$ for all $x, y \in \mathbb{R}_0$ such that

$x+y \in \mathbb{R}_0$, too.

69. Let $f(n)$ be defined on the set of positive integers by the rules: $f(1)=2$ and

$$f(n+1)=f(n)^2-f(n)+1$$

Prove that for all integers $n>1$, we have

$$1-\frac{1}{2^{2^{n-1}}}<\frac{1}{f(1)}+\frac{1}{f(2)}+\ldots+\frac{1}{f(n)}<1-\frac{1}{2^{2^n}}$$

70. Determine all functions f defined on the set of positive integers that have the property

$$f(xf(y)+y)=yf(x)+f(y)$$

and $f(p)$ is a prime for any prime p.

71. Determine all functions $f:\mathbb{R}-\{0,1\}\to\mathbb{R}$ such that

$$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)}$$

72. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y)+f(x-y)=2f(x)f(y)$$

and $|f(x)|\geq 1 \;\; \forall x\in\mathbb{R}$

73. Find all functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^3+2y)+f(x+y)=g(x+2y)$$

74. For each positive integer n let $f(n)=\dfrac{1}{\sqrt[3]{n^2+2n+1}+\sqrt[3]{n^2-1}+\sqrt[3]{n^2-2n+1}}$.
Determine the largest value of $f(1)+f(3)+\ldots+f(999997)+f(999999)$.

75. Find all strictly monotone functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x)+y)=f(x+y)+f(0)$$

76. Determine all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y)=\frac{f(x)+f(y)}{1-f(x)f(y)}$$

77. find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$xf(x)-yf(y)=(x-y)f(x+y)$$

78. For each positive integer n let $f(n)=\lfloor 2\sqrt{n}\rfloor-\lfloor\sqrt{n+1}+\sqrt{n-1}\rfloor$.Determine all values of n for which $f(n)=1$.

79. Let $f\colon \mathbb{Q} \to \mathbb{Q}$ be an injective function and $f(x) = x^n - 2x$.If $n \geq 3$, find all natural odd values of n.

80. Find all continuous, strictly increasing functions $f\colon \mathbb{R} \to \mathbb{R}$ such that
- $f(0) = 0,\ f(1) = 1$
- $\lfloor f(x+y) \rfloor = \lfloor f(x) \rfloor + \lfloor f(y) \rfloor$ for all $x, y \in \mathbb{R}$ such that $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$.

81. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(x-y)\, f\,(x+y) - (x+y)\, f\,(x-y) = 4\,x\,y\,(x^2 - y^2)$$

82. Find All Functions $f\colon \mathbb{N} \to \mathbb{N}$

$$f\,(m + f(n)) = n + f\,(m+k)$$

where k is fixed natural number.

83. Let f be a function defined for all real numbers and taking real numbers as its values. Suppose that, for all real numbers x, y the function satisfies

$$f\,(2\,x) = f\Big(\sin\Big(\frac{\pi\,x}{2} + \frac{\pi\,y}{2}\Big)\Big) + f\Big(\sin\Big(\frac{\pi\,x}{2} - \frac{\pi\,y}{2}\Big)\Big)$$

and

$$f\,(x^2 - y^2) = (x+y)\, f\,(x-y) + (x-y)\, f\,(x+y).$$

Show that these conditions uniquely determine $f\,(1990 + \sqrt[2]{1990} + \sqrt[3]{1990})$ and give its value.

84. Find all polynomials $P(x)$ Such that

$$x\,P\,(x-1) = (x-15)\,P(x)$$

85. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x)\, f\,(y\,f(x) - 1) = x^2\, f(y) - f(x)$$

86. Prove that there is no function like $f\colon \mathbb{R}_+ \to \mathbb{R}$ such that : $f\,(x+y) > y(f(x)^2)$.

87. Let f be a function de¿fined for positive integers with positive integral values satisfying the conditions:
$(i) f\,(a\,b) = f(a)\, f(b)$,
$(ii) f(a) < f(b)$ if $a < b$,
$(iii) f(3) \geq 7$
Find the minimum value for $f(3)$.

88. A function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies
$(i)\ f\,(a\,b) = f(a)\, f(b)$ whenever the gcd of a and b is 1,
$(i\,i)\ f\,(p+q) = f(p) + f(q)$ for all prime numbers p and q.
Show that $f(2) = 2$, $f(3) = 3$ and $f(1999) = 1999$.

89. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f\,(x+y)=f(x)+f(y)+f\,(x\,y)$$

90.A Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(a^3)+f(b^3)+f(c^3)=f\,(3\,a\,b\,c)$$

90.B Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(a^3)+f(b^3)+f(c^3)=a\cdot f(a^2)+b\cdot f(b^2)+c\cdot f(c^2)$$

91. Let f be a bijection from $\mathbb{N}$ into itself. Prove that one can always find three natural numbers a,b,c such that $a<b<c$ and $f(a)+f(c)=2\,f(b)$.

92. Suppose two functions $f(x)$ and $g(x)$ are defined for all x such that $2<x<4$ and satisfy $2<f(x)<4$, $2<g(x)<4$, $f(g(x))=g(f(x))=x$ and $f(x)\cdot g(x)=x^2$, for all such values of x.Prove that $f(3)=g(3)$.

93. Determine all monotone functions $f:\mathbb{R}\to\mathbb{Z}$ such that $f(x)=x,\forall x\in\mathbb{Z}$ and $f\,(x+y)\geq f(x)+f(y)$

94. Find all monotone functions $f:\mathbb{R}\to\mathbb{R}$ such that $f\,(4\,x)-f\,(3\,x)=2\,x$.

95.A Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x))=x^2-2$$

95.B Do there exist the real coefficients a,b,c such that the following functional equation $f(f(x))=a\,x^2+b\,x+c$ has at least one root?

96. Let $n\in\mathbb{N}$, such that $\sqrt{n}\notin\mathbb{N}$ and $A=\{a+b\,\sqrt{n}|a,b\in\mathbb{N},a^2-n\,b^2=1\}$. Prove that the function $f:A\to\mathbb{N}$, such that $f(x)=[x]$ is injective but not surjective.

97. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that $f\,(f(m)+f(n))=m+n$.

98. Find all functions $f:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$f\,(x^2+y^2)=f\,(x\,y)$$

99. Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that:
(i) $f(1)=f(-1)$
$(i\,i)$ $f(x)+f(y)=f\,(x+2\,x\,y)+f\,(y-2\,x\,y)$.

100. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f\,(x+y)\leq f(x)+f(y)$ and $f(x)\leq e^x-1$.

101. Let $f\colon\mathbb{R}\to\mathbb{R}$ be a function such that $f\,(x\,y)+f\,(x-y)\geq f\,(x+y)$. Prove that $f(x)\geq 0$.

102. Find all continuous functions $f\colon(0,+\infty)\to(0,+\infty)$, such that $f(x)=f\big(\sqrt{2\,x^2-2\,x+1}\big)$, for each $x>0$.

103. Determine all functions $f\colon\mathbb{N}_0\to\mathbb{N}_0$ such that $f\,(a^2-b^2)=f^2(a)-f^2(b)$, for all $a,b\in\mathbb{N}_0, a\geq b$.

104. Find all continues functions $f\colon R\longrightarrow R$ for each two real numbers x,y: $f\,(x+y)=f\,(x+f(y))$

105. Find all functions $f\colon\mathbb{R}\to\mathbb{R}$ such that

- $f\,(f(x)\,y+x)=x\,f(y)+f(x)$, for all real numbers x,y and
- the equation $f(t)=-t$ has exactly one root.

106. Find all functions $f\colon\mathbb{X}\to\mathbb{R}$ such that

$$f\,(x+y)+f\,(x\,y-1)=(f(x)+1)\,(f(y)+1)$$

for all $x,y\in\mathbb{X}$, if a) $\mathbb{X}=\mathbb{Z}$. b) $\mathbb{X}=\mathbb{Q}$.

SOLUTIONS

1. Find all functions $f\colon\mathbb{Q}_+\to\mathbb{Q}_+$ that satisfies the following two conditions for all $x\in\mathbb{Q}_+$:
1. $f(x+1)=f(x)+1$
2. $f(x^2)=f(x)^2$

Solution: From (1) we can easily find by induction that for all $n\in\mathbb{N}$,

$$f(x+n)=f(x)+n$$

Therefore by (2), we have

$$f((x+n)^2)=f(x+n)^2\Leftrightarrow f(x^2+2nx+n^2)=(f(x)+n)^2$$

$$\Longleftrightarrow f(x^2+2nx)+n^2=f(x)^2+2f(x)n+n^2\Leftrightarrow f(x^2+2nx)=f(x)^2+2nf(x)$$

Now let's put $x=\frac{p}{q}\quad p,q\in\mathbb{N}_0$ and let $n\to q$.

$$\Longrightarrow f\left(\frac{p^2}{q^2}\right)+2p=f\left(\frac{p^2}{q^2}\right)+2qf\left(\frac{p}{q}\right)$$

So $f\left(\frac{p}{q}\right)=\frac{p}{q}\quad\forall x\in\mathbb{Q}_+$ which satisfies the initial equation.

2. Determine all functions $f\colon\mathbb{R}\to\mathbb{R}$ such that:

$$f(x^3)-f(y^3)=(x^2+xy+y^2)(f(x)-f(y))$$

Solution: WLOG we may assume that $f(0)=0$.(Otherwise let $F(x)=f(x)-f(0)$.It's easy to see F also follows the given equation.)
Now putting $y=0$ we get $f(x^3)=x^2f(x)$.Substituting in the main equation we get $f(x)=xf(1)$.So all the functions are $f(x)=ax+b$ where $a,b\in\mathbb{R}$

3. Find all continuous functions $f\colon\mathbb{R}\to\mathbb{R}$ such that:

$$(1+f(x)f(y))f(x+y)=f(x)+f(y)$$

Solution: If $f(0)$ is not 0, then $P(0,0)$ gives $1+f(0)^2=2\Longrightarrow f(0)=1,-1$.
$P(0,x)$ gives $f(x)=\pm1$ each time and so by continuity we get $f(x)=1$ or $f(x)=-1$.

• **If $\boldsymbol{f(0)=0}$**

$P(x,-x)$ gives $f(-x)=-f(x)$ if $f(u)=0$ with $u\neq0$ then $f(x+u)=f(x)$
$f\left(\frac{u}{2}\right)=-f\left(-\frac{u}{2}\right)=-f\left(\frac{u}{2}\right)\Longrightarrow f\left(\frac{u}{2}\right)=0$
we also have $f(2u)=0$ (and also $f(nu)=0$ by induction)
so $f\left(\frac{n}{2^k}u\right)=0$ for every $n,k\in\mathbb{N}$ so $f(x)=0$ for every $x\in\mathbb{R}$. (Take limits and use continuity)

• **If $\boldsymbol{f(u)=0}$ only for $\boldsymbol{u=0}$**

now suppose there exist an a: $f(a) \geq 1$ so there is x_0 for which we have $f(x_0) = 1$ now let $x = y = 0.5\,x_0$ so $f\,(x_0/2) = 1$ by $[f\,(0.5\,x_0) - 1]^2 = 0$ and because of continuity $f(0) = 1$
or $f(0) = -1$ by the same argument.
So $|f(x)| < 1$ for every x now let $f(x) = \tanh\,(g(x))$ (this may be done, by the domain of tanh)
so $g\,(x + y) = g(x) + g(y)$ so $g(x) = c\,x$ so $f(x) = \tanh\,(c\,x)$.

4. Determine all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$f(x^3 + y^3) = x\,f(x^2) + y\,f(y^2)$$

Solution: Let $P(x,\,y)$ be the assertion.The following things can be proved easily:
$f(0) = 0$; $f(x^3) = x\,f(x^2)$; $f\,(x + y) = f(x) + f(y) \forall (x,y) \in \mathbb{R}^2$
$f((x + y)^3) = (x + y)\,f((x + y)^2) = (x + y)\,(f(x^2) + 2\,f\,(x\,y) + f(y^2))$
$f((x + y)^3) = f(x^3) + f(y^3) + 3\,f\,(x\,y\,(x + y))$
Comparing these two we find that

$$x\,f(y) + y\,f(x) + 2\,(x + y)\,f\,(x\,y) = 3\,f\,(x\,y\,(x + y))$$

$$\Longrightarrow f(x^2) = \frac{x\,f(1) + (2\,x - 1)\,f(x)}{2}$$

So $f(x^6) = \dfrac{x^3\,f(1) + (2\,x^3 - 1)\,x\,f(x^2)}{2}$
Also notice $f(x^6) = x^2\,f(x^4) = x^2\left(\dfrac{x^2\,f(1) + (2\,x^2 - 1)\,f(x^2)}{2}\right)$
From these two, we get

$$(x - 1)\,f(x^2) = (x - 1)\,x^2\,f(1)$$

Let's assume $x \neq 1$.So $f(x^2) = x^2\,f(1)$.The last formula also works for $x = 1$.So $f(x^3) = x\,f(x^2) = x^3\,f(1)\ \forall x \in \mathbb{R}$. So the only function satisfying $P(x,\,y)$ is $f(x) = c\,x\ \ \forall x \in \mathbb{R}$ where c is a fixed real.

5. Find all functions $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying

$$f(x + y) - f(y) = \frac{x}{y(x + y)}$$

Solution: WLOG we may assume that $f(1) = -1$.(Otherwise let $F(x) = f(x) - f(1) - 1$.It's easy to see $F(1) = -1$ and F also follows the given equation.)Now let

$$P(x,y) \Longrightarrow f(x + y) - f(y) = \frac{x}{y(x + y)}$$

$P(x,1)$ gives $f(x) = -\frac{1}{x}$.So all the functions are $f(x) = -\frac{1}{x} + c$ where $c \in \mathbb{R}$.

6. Determine all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that:

$$f(x+yf(x))+f(xf(y)-y)=f(x)-f(y)+2xy$$

Solution: Let $P(x,y)\Longrightarrow f(x+yf(x))+f(xf(y)-y)=f(x)-f(y)+2xy$.
$P(0,0)\Longrightarrow f(0)=0$
$P(0,x)\Longrightarrow f(-x)=-f(x)$
Suppose $f(a)=0$.Then $P(a,a)\Longrightarrow 0=2a^2\Longrightarrow a=0$.So $f(x)=0\Longleftrightarrow x=0$.Now let $x\neq 0$.

$$P\left(x,\frac{x+y}{f(x)}\right)+P\left(\frac{x+y}{f(x)},-x\right)\Longrightarrow f(2x+y)=2f(x)+f(y)$$

It is obviously true for $x=0$.Now make a new assertion $Q(x,y)\Longrightarrow f(2x+y)=2f(x)+f(y)$
for all $x,y\in\mathbb{R}$. $Q(x,0)\Longrightarrow f(2x)=2f(x)$ and so $f(2x+y)=f(2x)+f(y)$.Therefore $f(x+y)=f(x)+f(y)\forall x,y\in\mathbb{R}$ and the function is aditive.

$$P(y,x)\Longrightarrow f(y+xf(y))+f(yf(x)-x)=f(y)-f(x)+2xy$$

$$\Longrightarrow -f(-y+x(-f(y))-f(y(-f(x))+x)=-f(x)-(-f(y))+2xy$$

So if $f(x)$ is a solution then $-f(x)$ is also a solution.Hence wlog we may consider $f(1)\geq 0$.
Now using aditive property the original assertion becomes

$$R(x,y)\text{: } f(xf(y))+f(yf(x))=2xy$$

$R\left(x,\frac{1}{2}\right)\Longrightarrow$ f is surjective.So $\exists b$ such that $f(b)=1$.Then $R(a,a)\Longrightarrow a^2=1\Longrightarrow a=1$.
(Remember that we assumed $f(1)\geq 0$ i.e. $f(-1)\leq 0$)
$R(x,1)\Longrightarrow f(x)+f(f(x))=2x$ hence f is injective.
$R(x,x)\Longrightarrow f(xf(x))=x^2$ and so $f(x^2)=f(f(xf(x)))$.Now $R(xf(x),1)$ gives

$$f(x^2)+x^2=2xf(x)$$

So $f((x+y)^2)+(x+y)^2=2(x+y)f(x+y)\Longrightarrow f(xy)+xy=xf(y)+yf(x)$.So we have the
following properties:

$R(x,y)\Longrightarrow f(xf(y))+f(yf(x))=2xy$
$A(x,y)\Longrightarrow f(xy)=xf(y)+yf(x)-xy$
$B(x)\Longrightarrow f(f(x))=2x-f(x)$.So
$R(x,x)\Longrightarrow f(xf(x))=x^2$ ... ... (1)
$A(x,f(x))\Longrightarrow f(xf(x))=xf(f(x))+f(x)^2-xf(x)$ ... ... (2)
$B(x)\Longrightarrow f(f(x))=2x-f(x)$ ... ... (3)

So $-(1)+(2)+x(3)\Longrightarrow 0=x^2+f(x)^2-2xf(x)\Longrightarrow (f(x)-x)^2=0\Longrightarrow f(x)=x$
So all the functions are $f(x)=x\forall x\in\mathbb{R}$ and $f(x)=-x\forall x\in\mathbb{R}$.

7. Find least possible value of $f(1998)$ where $f:\mathbb{N}\rightarrow\mathbb{N}$ satisfies the following equation:

$$f(n^2 f(m)) = m f(n)^2$$

Solution: Denote $f(1) = a$, and put $m = n = 1$, therefore $f(f(k)) = a^2\, k$ and $f\,(a\,k^2) = f^2(k)$, $\forall k \in \mathbb{N}$
Thus now, we have: $f^2(x)\, f^2(y) = f^2(x)\, f\,(a\,y^2) = f\,(x^2\, f\,(f\,(a\,y^2))) = f\,(x^2\,a^3\,y^2) = f\,(a\,(a\,x\,y)^2) = f^2\,(a\,x\,y)$
$\Longleftrightarrow f\,(a\,x\,y) = f(x)\, f(y) \Rightarrow f\,(a\,x) = a\,f(x)$
$\Longleftrightarrow a\,f\,(x\,y) = f(x)\, f(y), \forall x, y \in \mathbb{N}$.
Now we can easily prove that $f(x)$ is divisible by a for each x, more likely we have that $f^k(x) = a^{k-1} \cdot f(x^k)$ is divisible by a^{k-1}.
For proving the above asertion we consider p^α and p^β the exact powers of a prime p that tivide $f(x)$ and a respectively, therefore $k\,\alpha \geq (k-1)\,\beta, \forall k \in \mathbb{N}$, therefore $\alpha \geq \beta$, so $f(x)$ is divisible by a.
Now we just consider the function $g(x) = \frac{f(x)}{a}$. Thus: $g(1) = 1$, $g\,(x\,y) = g(x)\, g(y)$, $g(g(x)) = x$. Since $g(x)$ respects the initial condition of the problem and $g(x) \leq f(x)$, we claim that it is enough to find the least value of $g(1998)$.
Since $g(1998) = g\,(2 \cdot 3^3 \cdot 37) = g(2) \cdot g^3(3) \cdot g(37)$, and $g(2)$, $g(3)$, $g(37)$ are disting prime numbers (the proof follows easily), we have that $g(1998)$, is not smaller than $2^3 \cdot 3 \cdot 5 = 120$. But g beeing a bijection, the value 120, is obtained for any g, so we have that $g(2) = 3$, $g(3) = 2$, $g(5) = 37$, $g(37) = 5$, therefore the answer is 120.

8. Find all functions $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying:

$$f(x + f(y)) = f(x + y) + f(y)$$

Solution: Find all functions $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying:

$$f\,(x + f(y)) = f\,(x + y) + f(y)$$

For any positive real numbers z, we have that

$$f(x + f(y)) + z = f(x + y) + f(y) + z$$

$$\Longleftrightarrow f\,(f(x + f(y)) + z) = f(f(x + y) + f(y) + z)$$

$$\Longleftrightarrow f\,(x + f(y) + z) + f(x + f(y) = f(x + y + f(y) + z) + f(x + y)$$

$$\Longleftrightarrow f(x + y + z) + f(y) + f(x + y) + f(y) = f(x + 2y + z) + f(y) + f(x + y)$$

$$\Longleftrightarrow f(x + y + z) + f(y) = f(x + 2y + z)$$

So $f(a) + f(b) = f(a + b)$ and by Cauchy in positive reals, then $f(x) = \alpha x$ for all $x \in (0, \infty)$. Now it's easy to see that $\alpha = 2$, then $f(x) = 2x \forall x \in \mathbb{R}_+$.

9. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function such that:
$(i)\, f(x) + f(y) + 1 \geq f(x + y) \geq f(x) + f(y)$
(ii)For all $x \in [0, 1)$, $f(0) \geq f(x)$
$(iii)\, f(1) = -f(-1) = 1$.
Find all such functions.

Solution: No complete solution was found.

10. Determine all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(xy + f(x)) = xf(y) + f(x)$$

Solution: Let $P(x, y)$ be the assertion $f(xy + f(x)) = xf(y) + f(x)$
$f(x) = 0\ \forall x$ is a solution and we'll consider from now that $\exists a$ such that $f(a) \neq 0$.
Suppose $f(0) \neq 0$. Then $P(x, 0) \implies f(f(x)) = xf(0) + f(x)$ and so $f(x_1) = f(x_2)$ $\implies x_1 = x_2$ and $f(x)$ is injective. Then $P(0, 0) \implies f(f(0)) = f(0)$ and, since $f(x)$ is injective, $f(0) = 0$, so contradiction. So $f(0) = 0$ and $P(x, 0) \implies f(f(x)) = f(x)$
$P(f(a), -1) \implies 0 = f(a)(f(-1) + 1)$ and so $f(-1) = -1$
Let $g(x) = f(x) - x$
Suppose now $\exists b$ such that $f(b) \neq b$

$$P\left(\frac{x}{f(b) - b}, b\right) \implies f\left(b\frac{x}{f(b) - b} + f\left(\frac{x}{f(b) - b}\right)\right) = \frac{x}{f(b) - b} f(b) + f\left(\frac{x}{f(b) - b}\right)$$

and so $f\left(b\frac{x}{f(b) - b} + f\left(\frac{x}{f(b) - b}\right)\right) - \left(b\frac{x}{f(b) - b} + f\left(\frac{x}{f(b) - b}\right)\right) = x$

and so $g\left(b\frac{x}{f(b) - b} + f\left(\frac{x}{f(b) - b}\right)\right) = x$ and $g(\mathbb{R}) = \mathbb{R}$

but $P(x, -1) \implies f(f(x) - x) = f(x) - x$ and so $f(x) = x\ \forall x \in g(\mathbb{R})$
And it's immediate to see that this indeed is a solution.
So we got two solutions :
$f(x) = 0\ \forall x$
$f(x) = x\ \forall x$

11. Find all functions $f: \mathbb{Q} \to \mathbb{Q}$ such that $f(2x) = 2f(x)$ and $f(x) + f\left(\frac{1}{x}\right) = 1$.

Solution: Inductively $f(2^n x) = 2^n x$ from the first equation for all integer n.Since $2f(1) = 1 \implies f(1) = \frac{1}{2}$.We get $f(2^n) = 2^{n-1}$, hence $f(2^{-n}) = 1 - 2^{n-1}$.But also $f(2^{-n}) = 2^{-n-1}$.
Then $1 - 2^{n-1} = 2^{-n-1}$, which is obviously not true for any positive integer n. Hence there is no such function.

12. Determine all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(xf(y)) + f(yf(x)) = \frac{1}{2} f(2x) f(2y)$$

Solution: In this proof, we'll show that when f is not constant, it is bijective on the separate domains $(-\infty, 0]$ and $[0, \infty)$, (not necessarily on $\mathbb{R}$) and then find all solutions on those domains. Then we get all functions f, by joining any two functions from the separate domains and checking they work. I mentioned some of the solutions in an earlier post.
Assume f is not constant and let $P(x, y) \implies f\ (x\ f(y)) + f\ (y\ f(x)) = \frac{1}{2} f(2x) f(2y)$.
$P(0, 0)$: $4f(0) = f(0)^2 \implies f(0) = 0$ or 4 ... ... (1)

Injectivity

As $f(x)=|x|$ is a solution, we cannot prove that f is injective on $\mathbb{R}$, instead we show it is injective on the domains $(-\infty,0]$ and $[0,\infty)$. So suppose there were two reals $a\neq b$ such that $f(a)=f(b)$, then we have

$\frac{1}{4}\ f\ (2\ a)^2+\frac{1}{4}\ f\ (2\ b)^2=f\ (a\ f(a))+f\ (b\ f(b))=f\ (a\ f(b))+f\ (b\ f(a))=\frac{1}{2}\,f\,(2\,a)\,f\,(2\,b)$

Which implies $\frac{1}{4}\,[f\,(2\,a)-f\,(2\,b)]^2=0\Longrightarrow f\,(2\,a)=f\,(2\,b)$

Moreover,

$$f\,(a\,f(x))+f\,(x\,f(a))=\frac{1}{2}\,f\,(2\,a)\,f\,(2\,x)$$
$$=\frac{1}{2}\,f\,(2\,b)\,f\,(2\,x)=f\,(b\,f(x))+f\,(x\,f(b))$$

This then implies $f\,(a\,f(x))=f\,(b\,f(x))$ for all $x\in\mathbb{R}$ $\quad(\star)$.

•Case 1: $\boldsymbol{f(0)=0}$

First we will show that f is injective on $[0,\infty)$. So for the sake of contradiction assume there existed $a>b>0$ such that $f(a)=f(b)$. Since $f(x)$ is continuous and not constant when $x>0$, there must be some interval $[0,c_1]$ or $[-c_1,0]$ such that f is surjective onto that interval. wlog that interval is $[0,c_1]$. So, motivated by $(\star)$ we define a strictly decreasing sequence $u_0\in[0,c_1]$, $u_{n+1}=\frac{b}{a}\,u_n$. We find that $u_n\in[0,c_1]$ for all n and therefore $f\,(a\,u_0)=f\,(b\,u_0)=f\,(a\,u_1)=\ldots=f\,(a\,u_n)$.

Now $\lim\limits_{n\to\infty}u_n\to0$, so by the continuity of f we have

$$\lim_{n\to\infty}f\,(a\,u_n)=f\left(\lim_{n\to\infty}a\,u_n\right)=f(0)=0$$

. This implies that $f\,(a\,u_0)=0$ for all $u_0\in[0,c_1]$, and therefore $f(x)=0$ when $x\in[0,a\,c_1]$.

But then for any $x\in[0,ac_1]$ we have $P(x,x)\Longrightarrow\ 0=f\,(x\,f(x))=\frac{1}{4}\,f\,(2\,x)^2$, hence $f\ (2\ x)=0$. Inductively we find that $f(x)=0$ for all $x\in\mathbb{R}^+$. Contradicting the assumption that f was not constant on that domain. Hence f is injective on the domain $[0,\infty)$.

As for the domain $(-\infty,0]$, simply alter the original assumption to $a<b<0$ such that $f(a)=f(b)$ and the same proof applies. Hence f is injective on $(-\infty,0]$ and $[0,\infty)$

•Case 2: $\boldsymbol{f(0)=4}$

Again we will consider the case $x\in[0,\infty)$. Assume there exists $a>b>0$ such that $f(a)=f(b)$.

$$P\left(\frac{x}{2},0\right)\Longrightarrow\ f\,(2\,x)+4=2\,f(x)\Longleftrightarrow f\,(2\,x)-4=2\,[f(x)-4]$$

and inductively $f\ (2^n\ x)-4=2^n\ [f(x)-4]$. So assuming there exists atleast one value such that $f(x)-4\neq0$, we will have $f(2^n)\to\pm\infty$. And since f is continuous, f will also be surjective onto at least one of: $[4,\infty)$ or $(-\infty,4]$. wlog, we will assume it $[4,\infty)$

Similar to the previous case we define the increasing sequence $u_0 \in [4, \frac{a}{b}\, 4]$ and $u_{n+1} = \frac{a}{b}\, u_n$. Again $u_n \in [4, \infty)$ and therefore $f\,(b\, u_0) = f\,(a\, u_0) = f\,(b\, u_1) = \ldots = f\,(b\, u_n)$.
Now for any $y \in [4, \infty)$ there must exists a $u_0 \in [4, \frac{a}{b}\, 4]$, such that $y = b\, u_n = b\, \frac{a^n}{b^n}\, u_0$ for some n. Hence for any value, v in the range of f, there exists some value in $x \in [4\, b, 4\, a]$ such that $f(x) = v$.
But f is continuous on the domain $[4\, b, 4\, a]$ therefore achieves a (finite) maximum. This contradicts the fact that f is surjective on $[4, \infty)$, hence our assumption is false and $f(x)$ is injective on the domain $[0, \infty)$.
We handle the negative domain $(\infty, 0]$ by changing the assumption to $a < b < 0$ and $f(a) = f(b)$. Therefore $f(x)$ is injective on both domains $x \in (-\infty, 0]$ and $[0, \infty)$. (in fact, it is bijective)
Surjectivity We already know that $f(x)$ is surjective on either $(-\infty, 4]$ or $[4, \infty)$ when $f(0) = 4$, so consider, $f(0) = 0$. We know that there exists some interval $[-c_1, 0]$ or $[0, c_1]$ such that f is surjective onto that range and f is monotonic increasing/decreasing (following from f being injective and continuous), so we consider two cases.

Case 1: f is surjective on $[0,c_1]$

Suppose f is bounded above, let $\lim_{x \to \infty} f(x) \to L_1$. Then when $f(y) > 00$ we have $P(\infty, y)$: $L_1 + f\,(L_1\, y) = \frac{L_1}{2}\, f\,(2\, y)$.
So let $y = u_0 > 0$, and $u_{n+1} = \frac{u_n}{L}$, and as we send $n \to \infty$, by the continuity of f we have: $L_1 + f(0) = \frac{L_1}{2}\, f(0) \Longrightarrow L_1 = 0$.
But this implies f is constant, and contradicts that f is surjective on $[0, c_1]$, hence f is not bounded above, and must be surjective onto $[0, \infty)$.

Case 2: f is surjective on $[-c_1,0]$

Suppose f is bounded below, let $\lim\limits_{n \to \infty} f(x) \to L_2$, then when $f(y) < 0$ we have $P(\infty, y)$: $L_2 + f\,(L_1\, y) = \frac{L_1}{2}\, f\,(2\, y)$. By a similar argument to case 1, we find $L_2 = 0$, contradicting that f is not constant. Hence $f(x)$ has no lower bound and must be surjective onto $[0, -\infty)$

Conclusion

functions when $f(0)=0$

When $f(0) = 0$, we know that there exists $2\, c \in \mathbb{R}$ such that $f\,(2\, c) = 4$, hence $f\,(c\, f(c)) = \frac{1}{4}\, f\,(2\, c)^2 = 4 = f\,(2\, c)$ So by the fact that f is injective $c\, f(c) = 2\, c \Rightarrow f(c) = 2$.
$P(x, c)$: $f\,(2\, x) + f\,(c\, f(x)) = \frac{1}{2}\, f\,(2\, c)\, f\,(2\, x) = 2\, f\,(2\, x), \Longrightarrow f\,(c\, f(x)) = f\,(2\, x) \Longrightarrow f(x) = \frac{2}{c}\, x$
Since c can be any real value, let $\frac{2}{c} = k$ we have $f(x) = k\, x$ $(\star\star)$.

functions when $f(0)=4$

When $f(0) = 4$ the above doesn't work because $c = 0$. But we do know that $f\,(2^n\, x) = 4 + 2^n\, [f(x) - 4]$. So let $f(x) = g(x) + 4$ so that $g\,(2^n\, x) = 2^n\, g(x)$ (2).
Now $P(x,\, x) \Longrightarrow f\,(x\, f(x)) = \frac{1}{4}\, f\,(2\, x)^2 = (f(x) - 2)^2 \Longleftrightarrow g\,(x\, g(x) + 4\, x) = g(x)^2 + 4\, g(x)$.
Applying (2) gives $g\,(2^n\, x\, g(x) + x) = 2^n\, g(x)^2 + g(x)$, which holds for all $n \in \mathbb{Z}, x \in \mathbb{R}^+$

Now there must exist $c\in\mathbb{R}$ such that $f(c)=1$, so, letting $x=c$ gives: $g\left(2^n c+c\right)=2^n+1$ and applying (2) gives
$f\left(2^{n+m} c+2^m\right)=2^{n+m}+2^m$ (3) which also holds for all $n,m\in\mathbb{Z}$ and $x\in\mathbb{R}$.
So now we will define a sequence that has a limit at any positive real number we choose, let that limit be $a\in\mathbb{R}^+$, and show that $g\,(a\,c)=a$, it will follow that $g\,(c\,x)=x$ for all $x\in\mathbb{R}^+$.
So pick two integers $k,\ell\in\mathbb{Z}$ such that $2^k+2^\ell<a$, and let $u_0=2^k+2^\ell$.
Now the next term in the sequence is defined by $u_{n+1}=2^{k_{n+1}}u_n^2+u_n$, where k_{n+1} is the largest possible integer such that $u_{n+1}<a$. Then the limit of this sequence as $n\to\infty$ is a.
But from (3) we have $g\,(c\,u_n)=u_n$ for all $n\in\mathbb{N}$, so by the continuity of g,
$\lim\limits_{n\to\infty} g\,(c\,u_n)=g\left(\lim\limits_{n\to\infty} c\,u_n\right)=g\,(c\,a)=a.$
This is true for all real $a\in\mathbb{R}^+$, so we have $g(x)=\frac{x}{c}$ or $f(x)=\frac{x}{c}+4$, for some $c\neq 0$.
so let $\frac{1}{c}=k$ and $f(x)=k\,x+4$ $(\star\star\star)$

All the solutions of f

$f(x)=k\,x$ $\quad k\in\mathbb{R}$
$f(x)=k\,x+4$ $\quad k\in\mathbb{R}$
And when when $k_1\leq 0,\ k_2\geq 0$, we also have

$$f(x)=\begin{cases} k_1 x & x<0\\ k_2 x & x\geq 0\end{cases}$$

$$f(x)=\begin{cases} k_1 x+4 & x<0\\ k_2 x+4 & x\geq 0\end{cases}$$

13. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x^5)-f(y^5)=(f(x)-f(y))(x^4+x^3y+x^2y^2+xy^3+y^4)$$

Solution: WLOG assume $f(0)=0$. (Otherwise let $F(x)=f(x)-f(0)$. Then you can easily see F works in equation!).
Define $P(x,y)\Longrightarrow f(x^5)-f(y^5)=(f(x)-f(y))(x^4+x^3y+x^2y^2+xy^3+y^4)$.
$P(x,0)\Longrightarrow f(x^5)=x^4 f(x)$. Now rewrite $P(x,1)$ to get

$$f(x)(x^3+x^2+x+1)=(x^3+x^2+x+1)f(1)x$$

Now suppose $x\neq -1$. Then $f(x)=x f(1)$. Now use $P(2,-1)$ to prove $f(-1)=-f(1)$.
So all the functions are $f(x)=x f(1)+f(0)$.

14. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x f(x)+f(y))=y+f(x)^2$$

Solution: Let $P(x,y)\Longrightarrow f(x f(x)+f(y))=y+f(x)^2$
$P(x,-f(x)^2)\Longrightarrow$ there exists an a such that $f(a)=0$.
$P(a,\,x)\Longrightarrow f(f(x))=x$.So the function is injective.Now comparing $P(x,\,y)$ and $P(f(x),y)$
we find $f(x)^2=x^2$.So $f(x)=x$ or $-x$ at each point.Then $f(0)=0$.Suppose $\exists a,b$ such that

$f(a)=a$ and $f(b)=-b$ and $a,b\neq 0$.$P(a,b)\Longrightarrow f(a^2-b)=b+a^2$.We know that $f(a^2-b)=a^2-b$ or $b-a^2$.But none of them is equal to $b+a^2$ for non-zero a,b.Hence such a,b can't exist.So all the functions are $f(x)=x\ \forall x\in\mathbb{R}$ and $f(x)=-x\ \forall x\in\mathbb{R}$.

15. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x)^2+2yf(x)+f(y)=f(y+f(x))$$

Solution: Let $P(x,y)$ be the assertion $f(x)^2+2\,y\,f(x)+f(y)=f\,(y+f(x))$
$f(x)=0\ \forall x$ is a solution. So we'll look from now for non all-zero solutions.
Let $f(a)\neq 0$: $P\left(a,\dfrac{u-f(a)^2}{2\,f(a)}\right)\Longrightarrow u=f(\text{something})-f(\text{something else})$ and so any real may be written as a difference $f(v)-f(w)$.
$P\,(w,-f(w))\Longrightarrow -f(w)^2+f(-f(w))=f(0)$
$P\,(v,-f(w))\Longrightarrow f(v)^2-2\,f(v)\,f(w)+f(-f(w))=f\,(f(v)-f(w))$
Subtracting the first from the second implies

$$f(v)^2-2\,f(v)\,f(w)+f(w)^2=f\,(f(v)-f(w))-f(0)$$

Therefore $f\,(f(v)-f(w))=(f(v)-f(w))^2+f(0)$
And so $f(x)=x^2+f(0)\ \forall x\in\mathbb{R}$ which indeed is a solution.
Hence the two solutions : $f(x)=0\ \forall x\ \ f(x)=x^2+a\ \forall x$

16. Determine all polynomial functions $f:\mathbb{R}\to\mathbb{R}$, with integer coefficients, which are bijective and satisfy the relation:

$$f(x)^2=f(x^2)-2f(x)+a$$

where a is a fixed real.

Solution: Let $g(x)=f(x)+1$. The equation can be written as $g(x)^2=g(x^2)+a$ and so
$g(x^2)=g(-x)^2$ and there are two cases:
• $g(x)$ is odd:
So $g(0)=0$ and so $a=0$.Thus we get $g(x)^2=g(x^2)$.It's easy to see that if $\rho e^{i\theta}$ is a root of $g(x)$,
then so is $\sqrt{\rho e^{i\theta}}$.So only roots may be 0 and 1.Since 1 does not fit, only odd polynomials matching $g(x)^2=g(x^2)$ are $g(x)=0$ and $g(x)=x^{2n+1}$.
•$g(x)$ is even:
Then,
(i)Either $g(x)=c\in\mathbb{Z}$ such that $c^2-c=a$.
(ii)Or $g(x)=h(x^2)$ and the equation becomes $h(x^2)^2=h(x^4)+a$ and so $h(x)^2=h(x^2)+a$ (remember these are polynomials)
By the same argument as before the conclution is the only solutions are $g(x)=c$ and $g(x)=x^{2n}$.
So all the solutions for $f(x)$ are:
1. If $\nexists\ c\in\mathbb{Z}$ such that $c^2-c=a$, then no solution.
2. If $\exists\ c\in\mathbb{Z}$ such that $c^2-c=a$, then $f(x)=c-1$.

3. $a=0$, then $f(x)=x^n-1$.

17. Let k is a non-zero real constant.Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying $f(xy)=f(x)f(y)$ and $f(x+k)=f(x)+f(k)$.

Solution: $f(y)\,f(x)+f(y)\,f(k)=f(y)\,f\,(x+k)$
$f\,(x\,y)+f\,(k\,y)=f\,(x\,y+y\,k)$
Now we are going to prove $f\,(x+k\,y)=f(x)+f\,(k\,y)$. If $y=0$, it's easy since $f(0)=0$. If $y\neq 0$, then we can put $\frac{x}{y}$ in x of $f\,(x\,y)+f\,(k\,y)=f\,(x\,y+y\,k)$. So $f\,(x+k\,y)=f(x)+f\,(k\,y)$. Now, since k isn't 0, we can put $\frac{y}{k}$ in y of $f\,(x+k\,y)=f(x)+f\,(k\,y)$. So $f\,(x+y)=f(x)+f(y)$. Since is an Cauchy equation, we can know that for some constant c, that $f(q)=c\,q$ when q is an rational number. But because of $f\,(x\,y)=f(x)\,f(y)$, c is 0 or 1. If $c=0$, then we can easily know that $f(x)=0$ for all real number x. If $c=1$, then $f(q)=q$. Now let's prove $f(x)=x$. Since $f\,(x\,y)=f(x)\,f(y)$, $f(x^2)=(f(x))^2$. So if $x>0$, then $f(x)>0$ since $f(x)\neq 0$. But $f(-x)=-f(x)$. So if $x<0$, then $f(x)<0$. Now let a a constant that satisfies $f(a)>a$. Then if we let $f(a)=b$, there is a rational number p that satisfies $b>p>a$. So, $f\,(p-a)+f(a)=f(p)=p$. So, $f\,(p-a)=p-f(a)=p-b<0$. But, $p-a>0$. So a contradiction! So we can know that $f(x)\leq x$. With a similar way, we can know that $f(x)\geq x$. So $f(x)=x$. We can conclude that possible functions are $f(x)=0$ and $f(x)=x$.

18. Find all continuous and strictly-decreasing functions $f:\mathbb{R}_+\to\mathbb{R}_+$ that satisfies

$$f(x+y)+f(f(x)+f(y))=f(f(x+f(y))+f(y+f(x)))$$

Solution: No complete solution was found.

19. Find all functions $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ of two variables satisfying

$$f(x,x)=x,\, f(x,y)=f(y,x),\, (x+y)f(x,y)=yf(x,x+y)$$

Solution: Substituting $f(x,y)=\dfrac{x\,y}{g(x,y)}$ we get $g(x,x)=x$, $g(x,y)=g(y,x)$, $g(x,y)=g\,(x,x+y)$. Putting $z\to x+y$, the last condition becomes $g(x,z)=g\,(x,z-x)$ for $z>x$. With $g(x,x)=x$ and symmetry, it is now obvious, by Euclidean algorithm, that $g(x,y)=\gcd\,(x,y)$, therefore $f(x,y)=\operatorname{lcm}(x,y)$.

20. Prove that for any function $f:\mathbb{R}\to\mathbb{R}$,

$$f(x+y+xy)=f(x)+f(y)+f(xy)\Longleftrightarrow f(x+y)=f(x)+f(y)$$

Solution: Let $P(x,y)$ be the assertion $f\,(x+y+x\,y)=f(x)+f(y)+f\,(x\,y)$
1) $\boldsymbol{f\,(x+y)=f(x)+f(y)\Longrightarrow P(x,y)}$
Trivial.
2) $\boldsymbol{P(x,y)\Longrightarrow f\,(x+y)=f(x)+f(y)\ \forall x,y}$
$P(x,0)\Longrightarrow f(0)=0$ $P\,(x,-1)\Longrightarrow f(-x)=-f(x)$
2.1) new assertion $\boldsymbol{R(x,y)\,:\,f\,(x+y)=f(x)+f(y)\ \forall x,y\!:x+y\neq -2}$

Let x,y such that $x+y\neq -2$

$$P\left(\frac{x+y}{2},\frac{x-y}{x+y-2}\right)\Longrightarrow f(x)=f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{x+y-2}\right)+f\left(\frac{x^2-y^2}{x+y-2}\right)$$

$$P\left(\frac{x+y}{2},\frac{y-x}{x+y-2}\right)\Longrightarrow f(y)=f\left(\frac{x+y}{2}\right)-f\left(\frac{x-y}{x+y-2}\right)-f\left(\frac{x^2-y^2}{x+y-2}\right)$$

Adding these two lines gives new assertion $Q(x,y)$: $f(x)+f(y)=2\,f(\frac{x+y}{2})$ $\forall x,y$ such that $x+y\neq -2$

$Q\,(x+y,0)\Longrightarrow f\,(x+y)=2\,f(\frac{x+y}{2})$ and so $f\,(x+y)=f(x)+f(y)$

2.2) $f\,(x+y)=f(x)+f(y)$ $\forall x,y$ such that $x+y=-2$

If $x=-2$, then $y=0$ and $f\,(x+y)=f(x)+f(y)$ If $x\neq -2$, then $(x+2)+(-2)\neq -2$ and then $R\,(x+2,-2)\Longrightarrow f(x)=f\,(x+2)+f(-2)$ and so $f(x)+f(-2-x)=f(-2)$ and so $f(x)+f(y)=f\,(x+y)$.

21. Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that

$$f(x)^3+f(y)^3+f(z)^3=f(x^3+y^3+z^3)$$

22. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(f(x)+y)=2x+f(f(y)-x)$$

23. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that:

$$f(f(n))+f(n+1)=n+2$$

24. Find all functions $f:\mathbb{R}_+\to\mathbb{R}_+$ such that:

$$f(x)f(yf(x))=f(x+y)$$

25. Find all functions $f:\mathbb{R}\to\mathbb{R}$ which satisfy this equation:

$$f(xf(y)+f(x))=f(yf(x))+x$$

26. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(x^2+f(y))=y+f(x)^2$$

27. If any function $f:\mathbb{R}\to\mathbb{R}$ satisfies

$$f(x^3+y^3)=(x+y)(f(x)^2-f(x)f(y)+f(y)^2)$$

then prove that $f(1996x)=1996f(x)$.

28. Find all surjective functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x-y))=f(x)-f(y)$$

29. Find all $k \in \mathbb{R}$ for which there exists a differentiable function $f:\mathbb{R}\to\mathbb{R}$ such that $f(1) \leq 1$ and $f(x)^2 + f'(x)^2 = k$.

30. Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f:(0,1] \to \mathbb{R}$ such that $a + f(x+y-xy) + f(x)f(y) \leq f(x) + f(y)$.

31. Find all functions $f:\mathbb{N} \to \mathbb{N}$ such that

$$2n + 2009 \leq f(f(n)) + f(n) \leq 2n + 2011$$

32. Find all functions $f:\mathbb{R}_+ \to \mathbb{R}$ satisfying $f(a) = 1$ and

$$f(x)f(y) + f\left(\frac{a}{x}\right)f\left(\frac{a}{y}\right) = 2f(xy)$$

33. Determine all functions $f:\mathbb{Q}\to\mathbb{C}$ such that
(i) for any rational $x_1, x_2, \ldots, x_{2010}$, $f(x_1 + x_2 + \ldots + x_{2010}) = f(x_1)f(x_2)\ldots f(x_{2010})$
$(ii)$for all $x \in \mathbb{Q}$, $\overline{f(2010)}f(x) = f(2010)\overline{f(x)}$.

34. Find all functions $f:\mathbb{Q} \to \mathbb{R}$ satisfying

$$f(x+y+z) = f(x) + f(y) + f(z) + 3\sqrt[3]{f(x+y)f(y+z)f(z+x)} \quad \forall x, y, z \in \mathbb{Q}$$

35. Determine all functions $f:\mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \max_{y \in \mathbb{R}} (2xy - f(y))$$

36.

37. Find all functions $f:\mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

38. Find all functions $f:\mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$f(x)^2 + 2yf(x) + f(y) = f(y + f(x))$$

39. Let $k \geq 1$ be a given integer.Find all functions $f:\mathbb{R} \to \mathbb{R}$ such that

$$f(x^k + f(y)) = y + f(x)^k$$

40. Find all functions $f:\mathbb{R} \to \mathbb{R}$ that satisfy

$$f(xy) + f(x-y) \geq f(x+y)$$

41. Find all functions $f:\mathbb{Z} \to \mathbb{Z}$ that satisfy $f(1) = f(-1)$ and

$$f(m) + f(n) = f(m + 2mn) + f(n - 2mn)$$

42. Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+f(y+f(z))) = f(x) + f(f(y)) + f(f(f(z)))$$

43. Let f be a real function defined on the positive half-axis for which $f(xy) = xf(y) + yf(x)$ and $f(x+1) \leq f(x)$ hold for every positive x. If $f\left(\frac{1}{2}\right) = \frac{1}{2}$, show that $f(x) + f(1-x) \geq -x\log_2 x - (1-x)\log_2(1-x)$ for every $x \in (0,1)$.

44. Let a be a real number and let $f: \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(0) = \frac{1}{2}$ and $f(x+y) = f(x)f(a-y) + f(y)f(a-x)$ .Prove that f is a constant function.

Solution:
Let $P(x, y)$ be the assertion $f(x+y) = f(x)f(a-y) + f(y)f(a-x)$.
$P(0,0) \Longrightarrow f(a) = \frac{1}{2}$
$P(x,0) \Longrightarrow f(x) = f(a-x)$.So $P(x, y)$ can also be written as

$$Q(x, y) \Longrightarrow f(x+y) = 2f(x)f(y)$$

$Q(a, -x) \Longrightarrow f(a-x) = f(-x)$.Hence $f(x) = f(-x)$.Then comparing $Q(x, y)$ and $Q(x, -y)$ gives $f(x+y) = f(x-y)$.Choose $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$.So $f(u) = f(v)$ and f is a constant function.

45. Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(x)^3 = -\frac{x}{12}(x^2 + 7xf(x) + 16f(x)^2)$$

46. Find all functions $f: \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$ which satisfies

$$f(x) + f\left(\frac{1}{1-x}\right) = 1 + \frac{1}{x(1-x)}$$

47. Let $f: \mathbb{R}_+ \to \mathbb{R}$ be a function such that
(i) If $x < y$ then $f(x) < f(y)$
(ii) $f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}$
Show that $f(x) < 0$ for some value of x.

48. Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(xy) = f(x) + f(y) + f(xy+1)$$

49.A Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f(y) + f(z) + f(x+y+z) = f(x+y) + f(y+z) + f(z+x) + f(0)$$

49.B Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(xf(y)+f(x)) = 2f(x)+xy$$

50. A function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:
$(i) f(-x) = -f(x)$
$(ii) f(x+1) = f(x)+1$
$(iii) f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$
Prove that $f(x) = x \ \forall x \in \mathbb{R}$.

51. Find all injective functions $f\colon \mathbb{N} \to \mathbb{N}$ which satisfies

$$f(f(x)) \le \frac{f(x)+x}{2}$$

52. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ which satisfies

$$f(f(x)+y) = f(x^2-y)+4f(x)y$$

53. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^n+f(y)) = y+f(x)^n$$

where $n > 1$ is a fixed natural number.

54. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x-y+f(y)) = f(x)+f(y)$$

55. Find all functions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ which have the property

$$f(x)f(y) = 2f(x+yf(x))$$

56. Find all functions $f\colon \mathbb{Q}_+ \to \mathbb{Q}_+$ with the property

$$f(x)+f(y)+2xyf(xy) = \frac{f(xy)}{f(x+y)}$$

57. Determine all functions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$f(x+y)-f(x-y) = 4\sqrt{f(x)f(y)}$$

58. Determine all functions $f\colon \mathbb{N}_0 \to \{1, 2, \ldots, 2000\}$ such that
(i)For $0 \le n \le 2000, \quad f(n) = n$
$(ii) f(f(m)+f(n)) = f(m+n)$

59. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x+f(y)) = y+f(x+1)$$

60. Let $n > m > 1$ be odd integers.Let $f(x) = x^m + x^n + x + 1$.Prove that $f(x)$ is irreducible over $\mathbb{Z}$.

61. A function $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfies the following equation:

$$f(m+n) + f(mn-1) = f(m)f(n) + 2$$

Find all such functions.

62. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $f\left(\sqrt{ab}\right) = \sqrt{f(a)f(b)}$ for all $a, b \in \mathbb{R}_+$ satisfying $a^2 b > 2$.Prove that the equation holds for all $a, b \in \mathbb{R}_+$

63. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$[f(m) + f(n)]f(m-n) = [f(m) - f(n)]f(m+n)$$

64. Find all polynomials which satisfy

$$P(x+1) = P(x) + 2x + 1$$

65. A rational function f (i.e. a function which is a quotient of two polynomials) has the property that $f(x) = f\left(\frac{1}{x}\right)$.Prove that f is a function in the variable $x + \frac{1}{x}$.

66. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x-y) = f(x+y)f(y)$$

67. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) = f(x) + f(y) + f(xy) - 2$$

68. Find all functions $f\colon \mathbb{R} \to \mathbb{R}_0$ such that

$(i)\, f(-x) = -f(x)$

$(ii)\, f\left(\frac{1}{x+y}\right) = f\left(\frac{1}{x}\right) + f\left(\frac{1}{y}\right) + 2(xy - 1000)$ for all $x, y \in \mathbb{R}_0$ such that $x + y \in \mathbb{R}_0$, too.

69. Let $f(n)$ be defined on the set of positive integers by the rules: $f(1) = 2$ and

$$f(n+1) = f(n)^2 - f(n) + 1$$

Prove that for all integers $n > 1$, we have

$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{f(1)} + \frac{1}{f(2)} + \ldots + \frac{1}{f(n)} < 1 - \frac{1}{2^{2^n}}$$

70. Determine all functions f defined on the set of positive integers that have the property $f(xf(y) + y) = yf(x) + f(y)$ and $f(p)$ is a prime for any prime p.

71. Determine all functions $f\colon \mathbb{R}-\{0,1\}\to\mathbb{R}$ such that

$$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)}$$

72. Find all functions $f\colon\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y)+f(x-y)=2f(x)f(y)$$

and $|f(x)|\geq 1 \quad \forall x\in\mathbb{R}$

73. Find all functions $f\colon\mathbb{R}\to\mathbb{R}$ and $g\colon\mathbb{R}\to\mathbb{R}$ such that

$$f(x^3+2y)+f(x+y)=g(x+2y)$$

74. For each positive integer n let $f(n)=\dfrac{1}{\sqrt[3]{n^2+2n+1}+\sqrt[3]{n^2-1}+\sqrt[3]{n^2-2n+1}}$.
Determine the largest value of $f(1)+f(3)+\ldots+f(999997)+f(999999)$.

75. Find all strictly monotone functions $f\colon\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x)+y)=f(x+y)+f(0)$$

76. Determine all continuous functions $f\colon\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y)=\frac{f(x)+f(y)}{1-f(x)f(y)}$$

77. find all functions $f\colon\mathbb{R}\to\mathbb{R}$ such that

$$xf(x)-yf(y)=(x-y)f(x+y)$$

78. For each positive integer n let $f(n)=\lfloor 2\sqrt{n}\rfloor-\lfloor\sqrt{n+1}+\sqrt{n-1}\rfloor$. Determine all values of n for which $f(n)=1$.

79. Let $f\colon\mathbb{Q}\to\mathbb{Q}$ be an injective function and $f(x)=x^n-2x$.If $n\geq 3$, find all natural odd values of n.

80. Find all continuous, strictly increasing functions $f\colon\mathbb{R}\to\mathbb{R}$ such that
- $f(0)=0,\ f(1)=1$
- $\lfloor f(x+y)\rfloor=\lfloor f(x)\rfloor+\lfloor f(y)\rfloor$ for all $x,y\in\mathbb{R}$ such that $\lfloor x+y\rfloor=\lfloor x\rfloor+\lfloor y\rfloor$.

81. Find all functions $f\colon\mathbb{R}\to\mathbb{R}$ such that

$$(x-y)\,f\,(x+y)-(x+y)\,f\,(x-y)=4\,x\,y\,(x^2-y^2)$$

82. Find All Functions $f\colon\mathbb{N}\to\mathbb{N}$

$$f\,(m+f(n))=n+f\,(m+k)$$

where k is fixed natural number.

83. Let f be a function defined for all real numbers and taking real numbers as its values. Suppose that, for all real numbers x, y the function satisfies

$$f\,(2\,x) = f\Big(\sin\Big(\frac{\pi\,x}{2} + \frac{\pi\,y}{2}\Big)\Big) + f\Big(\sin\Big(\frac{\pi\,x}{2} - \frac{\pi\,y}{2}\Big)\Big)$$

and

$$f\,(x^2 - y^2) = (x + y)\; f\,(x - y) + (x - y)\; f\,(x + y).$$

Show that these conditions uniquely determine $f\;(1990 + \sqrt[2]{1990} + \sqrt[3]{1990})$ and give its value.

84. Find all polynomials $P(x)$ Such that

$$x\,P\,(x - 1) = (x - 15)\,P(x)$$

85. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x)\;f\,(y\,f(x) - 1) = x^2\,f(y) - f(x)$$

86. Prove that there is no function like $f\colon \mathbb{R}_+ \to \mathbb{R}$ such that : $f\;(x + y) > y(f(x)^2)$.

87. Let f be a function de¸fined for positive integers with positive integral values satisfying the conditions:

$(i)\,f\,(a\,b) = f(a)\;f(b)$,

$(ii)\,f(a) < f(b)$ if $a < b$,

$(iii)\,f(3) \geq 7$

Find the minimum value for $f(3)$.

88. A function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies

(i) $f\,(a\,b) = f(a)\;f(b)$ whenever the gcd of a and b is 1,

$(i\,i)$ $f\,(p + q) = f(p) + f(q)$ for all prime numbers p and q.

Show that $f(2) = 2$, $f(3) = 3$ and $f(1999) = 1999$.

89. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that:

$$f\,(x + y) = f(x) + f(y) + f\,(x\,y)$$

90.A Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(a^3) + f(b^3) + f(c^3) = f\,(3\,a\,b\,c)$$

90.B Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(a^3) + f(b^3) + f(c^3) = a \cdot f(a^2) + b \cdot f(b^2) + c \cdot f(c^2)$$

91. Let f be a bijection from $\mathbb{N}$ into itself. Prove that one can always find three natural numbers a, b, c such that $a < b < c$ and $f(a) + f(c) = 2\,f(b)$.

92. Suppose two functions $f(x)$ and $g(x)$ are defined for all x such that $2<x<4$ and satisfy $2<f(x)<4$, $2<g(x)<4$, $f(g(x))=g(f(x))=x$ and $f(x)\cdot g(x)=x^2$, for all such values of x.Prove that $f(3)=g(3)$.

93. Determine all monotone functions $f\colon\mathbb{R}\to\mathbb{Z}$ such that $f(x)=x, \forall x\in\mathbb{Z}$ and $f(x+y)\geq f(x)+f(y)$

94. Find all monotone functions $f\colon\mathbb{R}\to\mathbb{R}$ such that $f(4x)-f(3x)=2x$.

95.A Does there exist a function $f\colon\mathbb{R}\to\mathbb{R}$ satisfying

$$f(f(x))=x^2-2$$

95.B Do there exist the real coefficients a, b, c such that the following functional equation $f(f(x))=ax^2+bx+c$ has at least one root?

96. Let $n\in\mathbb{N}$, such that $\sqrt{n}\notin\mathbb{N}$ and $A=\{a+b\sqrt{n}|a,b\in\mathbb{N}, a^2-nb^2=1\}$. Prove that the function $f\colon A\to\mathbb{N}$, such that $f(x)=[x]$ is injective but not surjective.

97. Find all functions $f\colon\mathbb{N}\to\mathbb{N}$ such that $f(f(m)+f(n))=m+n$.

98. Find all functions $f\colon\mathbb{R}_+\to\mathbb{R}_+$ such that

$$f(x^2+y^2)=f(xy)$$

99. Find all functions $f\colon\mathbb{Z}\to\mathbb{Z}$ such that:
(i) $f(1)=f(-1)$
(ii) $f(x)+f(y)=f(x+2xy)+f(y-2xy)$.

100. Determine all functions $f\colon\mathbb{R}\to\mathbb{R}$ such that $f(x+y)\leq f(x)+f(y)$ and $f(x)\leq e^x-1$.

101. Let $f\colon\mathbb{R}\to\mathbb{R}$ be a function such that $f(xy)+f(x-y)\geq f(x+y)$. Prove that $f(x)\geq 0$.

102. Find all continuous functions $f\colon(0,+\infty)\to(0,+\infty)$, such that $f(x)=f\left(\sqrt{2x^2-2x+1}\right)$, for each $x>0$.

103. Determine all functions $f\colon\mathbb{N}_0\to\mathbb{N}_0$ such that $f(a^2-b^2)=f^2(a)-f^2(b)$, for all $a,b\in\mathbb{N}_0, a\geq b$.

104. Find all continues functions $f\colon R\longrightarrow R$ for each two real numbers x, y: $f(x+y)=f(x+f(y))$

105. Find all functions $f\colon\mathbb{R}\to\mathbb{R}$ such that
- $f(f(x)y+x)=xf(y)+f(x)$, for all real numbers x, y and
- the equation $f(t)=-t$ has exactly one root.

106. Find all functions $f\colon\mathbb{X}\to\mathbb{R}$ such that

$$f(x+y)+f(xy-1)=(f(x)+1)(f(y)+1)$$

for all $x, y \in \mathbb{X}$, if a) $\mathbb{X} = \mathbb{Z}$. b) $\mathbb{X} = \mathbb{Q}$.