even
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IMO Winter Camp 2006

1. Let n be a positive integer. Find
$$\binom{n}{1} - \left(1+\frac{1}{2}\right)\binom{n}{2} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\binom{n}{3} - \cdots + (-1)^{n+1}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\binom{n}{n}.$$

2. Prove that if x and y are rational numbers satisfying the equation $x^5+y^5 = 2x^2y^2$, then $1-xy$ is the square of a rational number.

3. Determine all real values of the parameter a, for which the equation
$$16x^4 - ax^3 + (2a+17)x^2 - ax + 16 = 0$$
has exactly four distinct real roots which form a geometric progression.

4. You are given the sequence $u_1, u_2, \ldots, u_{2^n}$, where $u_i \in \{-1, 1\}$ for all i. Construct the new sequence $u_1u_2, u_2u_3, \ldots, u_{2^n}u_1$. Use the same rule to successively construct new sequences. Prove that after at most 2^n steps, the resulting sequence will consist entirely of 1s. (IMO Proposal, 1977)

5. Let K_n denote the complete graph on n vertices (where every pair of vertices is joined by an edge). Prove that the graph K_n can be decomposed into $n-1$ disjoint paths of length $1, 2, \ldots, n-1$.

6. Let S be a set of binary strings, such as $\{101, 1001, 010100, 00000\}$. We say the set is *prefix-free* if there are no two strings x and y such that x is exactly the beginning of y. (For example, a prefix-free set with 101 in it can't include 1010 or 101000 or 1010101 or 10. (In the last case, 10 is a prefix of 101, so we can't include it).)

   Let $|x|$ denote the length of string x. Show that for any prefix-free set S,
$$\sum_{x\in S} \frac{1}{2^{|x|}} \le 1.$$

7. Let b_n be the number of partitions of n into non-negative powers of 2. For example, $b_4 = 4$: $1+1+1+1$, $1+1+2$, $2+2$, 4. Let c_n be the number of partitions of n which include at least one power of 2 from 1 up to the highest in the partition. For example, $c_4 = 2$: $1+1+1+1$, $1+1+2$. Show that $b_{n+1} = 2c_n$. (BMO, 1984)

8. Consider the "half-Pascal's triangle," the first seven rows of which appear as follows:
$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 0 & & & & & \\
 & & & & 1 & & 1 & & 0 & & & & \\
 & & & 1 & & 2 & & 0 & & 0 & & & \\
 & & 1 & & 3 & & 2 & & 0 & & 0 & & \\
 & 1 & & 4 & & 5 & & 0 & & 0 & & 0 & \\
1 & & 5 & & 9 & & 5 & & 0 & & 0 & & 0
\end{array}$$
   The array x of integers is defined as follows:
$$x_{i,j} = \begin{cases} 1 & \text{if } j = 0, \\ x_{i-1,j-1} + x_{i-1,j} & \text{if } 1 \le i \text{ and } 1 \le j \le \lfloor i/2 \rfloor, \\ 0 & \text{if } 1 \le i \text{ and } \lfloor i/2 \rfloor < j \le i. \end{cases}$$
   Show that the sum of the entries in the nth row is
$$\binom{n}{\lfloor n/2 \rfloor}.$$

9. A ski course, which begins and ends at the same point, intersects itself several times without ever traversing the same stretch twice or passing through the same point three times. A skier goes along the course, and plants flags numbered 1, 2, ... in order at the points of self-intersection. Prove that an odd and an even number appear at each intersection.

10. Two players A and B take stones one after the other from a heap with $n \geq 2$ stones. A begins the game and takes at least 1 stone but no more than $n - 1$ stones. Each player on his turn must take at least 1 stone but no more than the other player has taken before him. The player who takes the last stone is the winner. Find who of the players has a winning strategy.

11. n passengers are lined up to board a plane, and the kth person in line has a ticket for the kth seat. However, the first passenger goes crazy (possibly from trying to solve too many math problems), and decides to sit in a random seat.

    Every passenger afterwards chooses his seat the following way: If his assigned seat is free, then he sits in it; otherwise, he chooses a random seat. What is the probability that the last passenger sits in his assigned seat?

12. Let n be a positive integer, and let

$$\frac{a}{b} = \frac{2 \cdot 5 \cdot 8 \ldots (3n-1)}{n!},$$

    where a and b are relatively prime. Prove that $b = 3^m$ for some non-negative integer m.

13. Prove that for each $n \geq 3$, the number $n!$ can be represented as the sum of n distinct divisors of itself.

14. Let n be an integer. Prove that if $2 + 2\sqrt{28n^2 + 1}$ is an integer, then it is a perfect square.

15. Find the greatest common divisor of the set of integers $\{16^n + 10n - 1 | n = 1, 2, 3, \ldots\}$.

16. Prove that for every prime p,

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k^3}{p} \right\rfloor = \frac{(p-2)(p-1)(p+1)}{4}.$$

17. In triangle ABC, $AB = AC$ and $\angle BAC = 20°$. Let D be a point on side AB such that $\angle DCA = 10°$. Prove that $AD = BC$.

18. In triangle ABC, $\tan A = 3$ and $\tan B = 2$. Let A_1 be the projection of A onto side BC. Prove that the orthocentre of triangle ABC is the mid-point of AA_1.

19. Let us choose arbitrarily n vertices of a regular $2n$-gon and colour them red. The remaining vertices are coloured blue. We arrange all red-red distances into a non-decreasing sequence and do the same with the blue-blue distances. Prove that the sequences are equal.

20. In acute-angled triangle ABC, the orthocentre H divides altitude BD in the ratio $BH : HD = 3 : 1$. Let K be mid-point of ~~triangle~~ line BD. Prove that $\angle AKC = 90°$.

21. Two circles C_1 and C_2 of radii r_1 and r_2 touch their common external tangent at A_1 and A_2. The circles intersect at points M, N. Prove that the circumradius of the triangle A_1MA_2 does not depend on the length of A_1A_2 and is equal to $\sqrt{r_1r_2}$.

22. In triangle ABC, $\angle A = 60°$. Prove that the Euler line intersects sides AB and AC at 60°.

23. Show that

$$1 < \frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{3001} < \frac{4}{3}.$$

24. Prove that for all positive integers $n > 1$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2}.$$

25. Let $\sigma(1)$, $\sigma(2)$, $\ldots$, $\sigma(100)$ be a permutation of 1, 2, $\ldots$, 100. Find the minimum and maximum values of

$$S = |\sigma(1) - \sigma(2)| + |\sigma(2) - \sigma(3)| + |\sigma(99) - \sigma(100)| + \cdots + |\sigma(100) - \sigma(1)|.$$

26. Let x_1, x_2, $\ldots$, x_n be real numbers, such that $|x_i| \leq 2$ for all i, and $x_1 + x_2 + \cdots + x_n = 0$. Show that $|x_1^3 + x_2^3 + \cdots + x_n^3| \leq 2n$.

27. Prove that if $a_1 > a_2 > a_3 > a_4 > 0$ and $a_1^2 + a_4^2 = a_2^2 + a_3^2$, then $a_1^3 + a_4^3 > a_2^3 + a_3^3$.

28. Let $0 < b < a$. Show that

$$\frac{(a-b)^2}{8a} < \frac{a+b}{2} - \sqrt{ab} < \frac{(a-b)^2}{8b}.$$