AN ELEMETARY PROOF OF AN ESTIMATE FOR A NUMBER OF PRIMES LESS THAN THE PRODUCT OF THE FIRST n PRIMES

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Abstract. Let α be a real number such that $1 < \alpha < 2$ and let $x_0 = x_0(\alpha)$ be a (unique) positive solution of the equation

$$x^{\alpha-1} - \frac{\pi}{e^2\sqrt{3}}x + 1 = 0.$$

Then we prove that for each positive integer $n > x_0$ there exist at least $[n^\alpha]$ primes between the $(n+1)$th prime and the product of the first $n+1$ primes. In particular, we establish a recent Cooke's result which asserts that for each positive integer n there are at least n primes between the $(n+1)$th prime and the product of the first $n+1$ primes. Our proof is based on an elementary counting method (enumerative arguments) and the application of Stirling's formula to give upper bound for some binomial coefficients.

Ever since Euclid of Alexandria, sometimes before 300 B.C., first proved that the number of primes is infinite (see Proposition 20 in Book IX of his legendary *Elements* [8], mathematicians have amused themselves by coming up with alternate proofs. For more information about the Euclid's proof of the infinitude of primes see e.g., [6, p. 414, Ch. XVIII], and [15, Section 1]. In [15] the author of this article provided a comprehensive historical survey of different proofs of famous Euclid's theorem on the infinitude of primes which has fascinated generations of mathematicians since its first and famous demonstration given by Euclid. Quite recently, in [16] the author of this article presented a very short and elementary proof of Euclid's theorem.

Euclid's proof of the infinitude of primes is a paragon of simplicity: *given a finite list of primes $p_1, p_2, \dots p_n$, multiply them together and add one. The resulting number, say $N = p_1p_2\cdots p_n$, is not divisible by any prime on the list, so any prime factor of N is a new prime.*

A modification of the above Euclid's proof based on the factorization theorem can be found in author's survey article [15, p. 35, Section 4].

Notice that numerous proofs of the infinitude of primes yield anyone estimate for distribution of primes [15]. Applying Euclid's proof presented above with $p_1p_2\cdots p_n-1$ instead of $p_1p_2\cdots p_n+1$, we obtain that $p_{n+1} < p_1p_2\cdots p_n$

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for each $n \geq 2$, where p_k is the kth prime. In 1907 H. Bonse [2] gave an elementary proof of a stronger inequality, now called Bonse's inequality (for a simple proof based on Erdős' method [7] see [31, p. 238, Section 4.6]): if $n \geq 4$, then

$$p_{n+1}^2 < p_1 p_2 \cdots p_n. \tag{1}$$

Bonse also proved that $p_{n+1}^3 < p_1 p_2 \cdots p_n$ for all $n \geq 5$. In 2000 M. Dalezman [5, Theorem 1] gave an elementary proof of stronger inequality $p_{n+1}p_{n+2} < p_1 p_2 \cdots p_n$ with $n \geq 4$. In 1960 L. Pósa refined firstly Bonse's inequalities by proving that for every integer $k > 1$ there is an n_k such that $p_{n+1}^k < p_1 p_2 \cdots p_n$ for all $n > n_k$. Further, a syntetic proof (i.e., one not involving the limit concepts of analysis) due to S. E. Mamangakis in 1962 [14] for a theorem from which specialization lead to the following inequalities: $p_{4n} < p_1 p_2 \cdots p_n$ with $n \geq 11$ and $p_{4n}^4 < p_1 p_2 \cdots p_{4n-9}$ with $n \geq 46$. In 1971 S. Reich [25] showed that for every positive integer k there exists a positive integer $n_0 = n_0(k)$ such that $p_{n+k}^2 < p_1 p_2 \cdots p_n$ for all $n \geq n_0$. Furthermore, using a quite different approach from Bonse's, in 1988 J. Sándor [30] proved that for $n \geq 3$, $p_1 p_2 \cdots p_{n-1} + p_n + p_{p_n - 2} \leq p_1 p_2 \cdots p_n$ and that for $n \geq 24$, $p_{n+5}^2 + p_{[n/2]}^2 < p_1 p_2 \cdots p_n$ which is sharper than Bonse's inequality (1).

On the other hand, using arguments of Analytic Number Theory, many authors have been obtained stronger inequalities than those mentioned above. In 1977 H. Gupta and S. P. Khare [11] proved that $\binom{n^2}{n} < p_1 p_2 \cdots p_n$ for all $n \geq 1794$. In 1983 G. Robin [28, Théorème 4] proved that $n^n < p_1 p_2 \cdots p_n$ for each $n \geq 13$. Since by Stirling's formula easily follows that $\binom{n^2}{n} \sim \frac{e^{n-1}}{\sqrt{2\pi n}} n^n$, it follows that the mentioned inequality by Gupta and Khare is stronger than those due to Robin. Motivated by a result of Gupta and Khare, in 2011 H. Alzer and J. Sándor improved their result [1, Theorem]. Moreover, using some Rosser-Schoenfeld's [29] and Robin's estimates [28] for the prime counting function $\pi(x)$ and Chebyshev function $\theta(x)$, in 2000 L. Panaitopol [20] proved that $p_{n+1}^{n-\pi(n)} < p_1 p_2 \cdots p_n$ for every $n \geq 2$, where $\pi(x)$ is the number of primes $\leq x$. This improves Pósa's inequality in the following form: $p_{n+1}^k < p_1 p_2 \cdots p_n$ for $n \geq 2k$ with given $k \geq 1$. M. Hassani [12] refined this inequality in 2006 by proving that for $n \geq 101$ the exponent $n - \pi(n)$ can be replaced by $(1 - 1/\log n)(n - \pi(n))$. Furthermore, using Panaitopol's inequality, in 2009 S. Zhang showed [36, Corollary 1] that $2^{p_{n+1}} < p_1 p_2 \cdots p_n$ for all $n \geq 10$. This inequality yields an improvement of Pósa's inequality [24] given above and some Bonse-type inequalities. Furthermore, various Panaitopol-type inequalities and related limits are recently established by J. Sándor [33] and J. Sándor and A. Verroken [34]).

Notice that Bonse's inequality, all its refinemenets and improvements presented above does not guarantee the existence of "many primes" less than $p_1 p_2 \cdots p_n$. This is also the case with numerous known elementary proof of

Euclid's theorem on the infinitude of primes. For example, iterating the second Bonse's inequality we find that $p_{n+2} < (p_1p_2\cdots p_n)^{4/3}$, which repeating still three times gives $p_{n+4} \le (p_1p_2\cdots p_n)^{580/729}$. This shows that for $n \ge 4$ there exist at least 4 primes between the nth prime and the product of the first n primes. We see from the first Mamangakis' inequality given above that for each $n \ge 11$ there are at least $3n$ primes between the nth prime and $p_1p_2\cdots p_n$. Notice also that the first Sándor's inequality presented above and the well known estimation $p_n > n\log n$ with $n \ge 5$ (see e.g., [29, (3.10) in Theorem 3]) imply that for each $n \ge 2$ there are at least $[n\log n]-2$ primes less than $p_1p_2\cdots p_n$. Quite recently in 2011, applying two simple lemmas in the Theory of Finite Abelian Groups related to the product of some cyclic groups $\mathbb{Z}_m$, R. Cooke [4] modified Perott's proof from 1881 ([21], [26, page 10]) to establish that there are at least n primes between the $(n+1)$th prime and the product of the first $n+1$ primes. Refining the Euler's proof of the infinitude of primes presented below, in this note we improve Cooke's result by proving the following Bonse-type inequality.

Theorem 1. *Let α be a real number such that $1<\alpha<2$ and let $x_0 = x_0(\alpha)$ be a (unique) positive solution of the equation*

$$x^{\alpha-1} - \frac{\pi}{e^2\sqrt{3}}x + 1 = 0.$$

Then for each positive integer $n > x_0$ there exist at least $[n^\alpha]$ primes between the $(n+1)$th prime and the product of the first $n+1$ primes.

Moreover, for each positive integer n there are at least n primes between the $(n+1)$th prime and the product of the first $n+1$ primes.

Remark. The first assertion of Theorem 1 can be shortly written in terms of the "little o" notation as

$$p_{o(n^2)} < p_1p_2\cdots p_{n+1} \tag{2}$$

as $n\to\infty$, where p_k is the kth prime. However, our method applied for the proof of Theorem 1 cannot be applied for $\alpha = 2$; namely, this is because of the inequality (13) with $\alpha = 2$ is clearly satisfied for all $m \ge 1$.

A computation via `Mathematica 8` shows that $p_{n^2} < (p_n)^2$ for each $n \ge 5$. Namely, it is well known (see e.g., [19]) that $p_n \sim n\log n$ as $n\to\infty$, and so $(p_n)^2 \sim n^2\log^2 n$ and $p_{n^2} \sim 2n^2\log n$. This immediately implies that $2(p_n)^2 \sim p_{n^2}\log n$. Notice also that using the known estimates $\log n + \log\log n - 3/2 < p_n/n < \log n + \log\log n - 1/2$ with $n \ge 6$ (see e.g., [29, (3.10) and (3.11) in Theorem 3]) and a verification via `Mathematica 8` for $1 \le n \le 1020$, easily follows that $2(p_n)^2 > p_{n^2}\log n$ for all $n \ge 1$. Recall also that it is known that the sequence $(p_n/\log n)$ is strictly increasing (see e.g., [32, p. 106]).

A motivation for our proof given in the next section comes from a less known proof of Euclid's theorem due to Euler in 1736 (published posthumously in 1862 [9]; also see [10, Sect. 135], [6, p. 413] and citeme3) is in fact the first proof of Euclid's theorem after those of Euclid and C. Goldbach's proof presented in a letter to L. Euler in July 1730 (see [26, p. 6] and [15, Appendix C)]). As noticed in Dickson's History [6, p. 413] (see also [35, page 80]), this proof is also attributed in 1878/9 by Kummer [13] who gave essentially Euler's argument. This Euler's proof (see e.g., [3, pp. 134–135], [17] and [23, page 3]; also cf. Pinasco's proof [22]) is based on the multiplicativity of the φ-function defined as the number of positive integers not exceeding n and relatively prime to n. Namely, if $p_1, p_2, \dots, p_n$ is a list of distinct $n \geq 2$ primes with product P, then

$$\varphi(P) = (p_1 - 1)(p_2 - 1) \cdots (p_n - 1) \geq 2^{n-1} \geq 2. \tag{2}$$

The inequality (1) together with the definition of the φ-function says there exists at least an integer in the range $[2, P]$ that is relatively prime to P, but such an integer has a prime factor necessarily different from any of the p_k with $k = 1, 2, \dots, n$. This yields the infinitude of primes.

Theorem 1 may be considered as an extension of the above Euler's result in order to obtain the estimate of a number of primes less than the product of the first n primes. Proof of Theorem 1 given in the next section is combinatorial in spirit and entirely elementary. It is based on some counting arguments by using Stirling's formula to give upper bound for some binomial coefficients.

1. Proof of Theorem

Lemma 1. *Let k and N be two arbitrary fixed positive integers, and let $N(k, n)$ be the number of k-tuples $(x_1, \dots, x_k)$ of nonnegative integers $x_1, \dots, x_k$ is $\binom{n+k-1}{k}$ satisfying the inequality*

$$\sum_{i=1}^{k} x_i \leq n.$$

Then

$$N(k, n) = \binom{n+k}{n}.$$

Proof. For a fixed nonnegative integer m with $0 \leq m \leq n$, denote by $M(k, n)$ the number of k-tuples $(x_1, \dots, x_k)$ of nonnegative integers $x_1, \dots, x_k$ such that

$$\sum_{i=1}^{k} x_i = m.$$

Then by induction no $k \geq 1$ it is easy to prove the well known fact that $M(k,m) = \binom{m+k-1}{k-1}$ for each $k \geq 1$. Hence, for such a fixed k we have

$$N(k,n) = \sum_{m=0}^{n} M(k,m) = \sum_{m=0}^{n} \binom{m+k-1}{k-1}.$$

Next by induction on $n \geq 0$, using the Pascal's identity $\binom{r}{i} + \binom{r}{i+1} = \binom{r+1}{i+1}$ with $0 \leq i \leq r-1$, we immediately obtain that the sum on the right hand side of the above equality is equal to $\binom{n+k}{k}$. Therefore, we have $N(k,n) = \binom{n+k}{k} = \binom{n+k}{n}$, as desired. $\square$

Lemma 2. *Let α be a real number such that $\alpha > 1$. Then for each positive integer n*

$$\frac{1}{n!}\binom{[n^\alpha]+n}{n} < \frac{e^{2n}(n^{\alpha-1}+1)^n\sqrt{3}}{2n^{n+1}\pi} \tag{3}$$

where $[a]$ denotes the integer part of a. Furthermore, for all positive integers n we have

$$\frac{1}{n!}\binom{2n}{n} < \frac{2^{2n-1/2}e^n}{n^{n+1}\pi}. \tag{4}$$

Proof. First observe that (3) holds for $n = 1$. Stirlng's asymptotic formula $n! \approx \sqrt{2\pi n}(n/e)^n$ as $n \to \infty$ is often presented in the following refined form due to H. Robbins [27]: (also see [18])

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\gamma_n}, \quad \frac{1}{12n+1} < \gamma_n < \frac{1}{12n}, \quad n = 1, 2, \dots. \tag{5}$$

Then applying (5) to all factorials $([n^\alpha]+n)!, [n^\alpha]!, n!$, where $n^\alpha - 1 < a := [n^\alpha] \leq n^\alpha$ we find that

$$\begin{aligned} \frac{1}{n!}\binom{a+n}{n} &< \frac{a!}{a!(n!)^2}\frac{\sqrt{2\pi(a+n)}\left(\frac{a+n}{e}\right)^{a+n}}{\sqrt{2\pi a}\left(\frac{a}{e}\right)^a \cdot 2\pi n\left(\frac{n}{e}\right)^{2n}} \times \\ & e^{1/(12(a+n))-1/(12a+1)-2/(12n+1)} \\ &= \frac{e^n}{2n^{2n+1}\pi}\sqrt{1+\frac{n}{a}} \cdot \left(1+\frac{n}{a}\right)^a (a+n)^n e^{1/(12(a+n))-1/(12a+1)-2/(12n+1)}. \end{aligned} \tag{6}$$

Inserting the inequalities

$$\sqrt{1+\frac{n}{a}} < \sqrt{1+\frac{n}{n^\alpha-1}} < \sqrt{1+\frac{n}{n-1}} \leq \sqrt{3},$$

$a + n \leq n^\alpha + n$,

$$\left(1+\frac{n}{a}\right)^a = \left(\left(1+\frac{n}{a}\right)^{a/n}\right)^n < e^n,$$

and
$$\frac{1}{12(a+n)}-\frac{1}{12a+1}-\frac{2}{12n+1}<\frac{1}{12n}-\frac{2}{12n+1}<0$$
into inequality (6), we find that

$$\frac{1}{n!}\binom{a+n}{n}<\frac{e^{2n}\sqrt{3}(n^{\alpha}+n)^n}{2\pi n^{2n+1}}=\frac{e^{2n}(n^{\alpha-1}+1)^n\sqrt{3}}{2n^{n+1}\pi}. \tag{7}$$

This proves (3). Finally, taking $a=n$ into (6) we immediately obtain (4). $\square$

Lemma 3. *The real function* $f:[0,+\infty)\to\mathbf{R}$ *defined as*
$$f(x)=e^2\sqrt{3}x^{\alpha-1}-\pi x+1$$
has exactly one positive root. Moreover, if $x_0=x_0(\alpha)$ *is this root, then* f *decreases on* $[x_0,+\infty)$.

Proof. The derivative of the function f is $f'(x)=e^2\sqrt{3}(\alpha-1)x^{\alpha-2}-\pi$ whose a unique real root is $x_1=\left(e^2\sqrt{3}(\alpha-1)/\pi\right)^{1/(2-\alpha)}$. Therefore, in view of the fact that $\alpha-2<0$, we infer that f is decreasing on $[x_1,\infty)$. However, since $f(x_1)/x_1=\pi(2-\alpha)/(\alpha-1)>0$ and so $f(x_1)>0$, and $\lim_{x\to+\infty}f(x)=-\infty$, we conclude that there exists a unique positive root x_0 of the function f. As $x_0>x_1$ we see that f decreases on $[x_0,+\infty)$. $\square$

Proof of Theorem. First consider the case when $1<\alpha<2$. Then suppose that the assertion is not true. This means that there are an α with $1<\alpha<2$ and a positive integer $m>x_0=x_0(\alpha)$ for which there are less than $[m^{\alpha}]$ primes between the $(m+1)$th prime and the product of the first $m+1$ primes. For such a m, let $2=p_1<3=p_2<\cdots<p_{m+1}$ be first $m+1$ consecutive primes. Accordingly, suppose that $p_{m+2},\ldots,p_{m+1+k}$ are all the primes between p_{m+1} and the product $P:=p_1p_2\cdots p_{m+1}$ with $k\le[m^{\alpha}]-1$. Then every positive integer less than P and relatively prime to P can be factorized as $p_{m+2}^{x_1}\cdots p_{m+k+1}^{x_k}$ with nonnegative integers $x_1,\ldots,x_k$. Then obviously we have
$$p_{m+1}^{x_1+\cdots+x_k}<p_{m+2}^{x_1}\cdots p_{m+1+k}^{x_k}<P=p_1p_2\cdots p_{m+1}<p_{m+1}^{m+1},$$
whence it follows that
$$x_1+\cdots+x_k\le m. \tag{8}$$
Then by Lemma 1, a number $N(k,m)$ of k-tuples $(x_1,\ldots,x_k)$ of nonnegative integers x_i satisfying the inequality (7) is equal to $\binom{m+k}{k}$. It follows that must be
$$\binom{m+k}{m}\ge\varphi(P)=(p_1-1)(p_2-1)\cdots(p_{m+1}-1). \tag{9}$$

On the other hand, since $p_i - 1 \geq 2i$ for all $i \geq 5$, it follows that for every $n \geq 3$

$$\begin{aligned} &(p_1-1)(p_2-1)(p_3-1)(p_4-1)\cdots(p_{n+1}-1) \\ &\geq 48\cdot 2^{n-3}\cdot 5\cdot 6\cdots(n+1) = 2^{n-2}(n+1)!. \end{aligned} \tag{10}$$

Then (9) and (10) with $n = m \geq 3$ yield

$$2^{m-2}(m+1) \leq \frac{1}{m!}\binom{m+k}{m}. \tag{11}$$

Using the fact that the sequence $k \mapsto \binom{m+k}{m}$ ($k = 1, 2, \ldots$) is increasing, the previous assumption $k < [m^\alpha]$ and the inequality (3) of Lemma 2, we obtain that if $m \geq 3$, then

$$\frac{1}{m!}\binom{m+k}{m} < \frac{1}{m!}\binom{[m^\alpha]+m}{m} < \frac{e^{2m}(m^{\alpha-1}+1)^m\sqrt{3}}{2m^{m+1}\pi}. \tag{12}$$

Now from (11) and (12) it follows that if $m \geq 3$ then

$$2^{m-2}(m+1) < \frac{e^{2m}(m^{\alpha-1}+1)^m\sqrt{3}}{2m^{m+1}\pi},$$

or equivalently,

$$m^{\alpha-1}+1 > \frac{2}{e^2}\sqrt[m]{\frac{m(m+1)\pi}{2\sqrt{3}}}\cdot m. \tag{13}$$

Notice that the inequality (13) also holds for $m = 1$ and $m = 2$.

Since $\pi/(2\sqrt{3}) < 1$, it follows that

$$\sqrt[m]{\frac{m(m+1)\pi}{2\sqrt{3}}} > \sqrt[m]{\frac{\pi}{2\sqrt{3}}} \geq \frac{\pi}{2\sqrt{3}}.$$

Substituting this into (13) we have

$$m^{\alpha-1} + 1 - \pi m/(e^2\sqrt{3}) > 0. \tag{14}$$

However, using the notations of Lemma 3, since $m > x_0$ this lemma gives

$$m^{\alpha-1} + 1 - \pi m/(e^2\sqrt{3}) = f(m) < f(x_0) = 0.$$

This contradicts (14), and hence the proof when $1 < \alpha < 2$ is finished.

It remains to prove the assertion for $\alpha = 1$. Then as in the previous case, suppose that for some $m \geq 1$ there exist less than m primes between p_{m+1} and the product $p_1p_2\cdots p_{m+1}$. Then as in the previous case with the condition $k \leq m-1$ instead of $k \leq [m^\alpha]-1$, we arrive to the following inequality analogous to (11):

$$2^{m-2}(m+1) \leq \frac{1}{m!}\binom{2m-1}{m} = \frac{1}{2}\cdot\frac{1}{m!}\binom{2m}{m}, \quad m = 1, 2, \ldots. \tag{15}$$

Next from (15) and the inequality (4) of Lemma 2 for all $m \geq 1$ we get

$$2^{m-2}(m+1) < \frac{2^{2m-3/2}e^m}{m^{m+1}\pi},$$

whence it follows that if $m \geq 6$ then

$$1 < \frac{2^{m+1/2}e^m}{m^{m+1}(m+1)\pi} = \frac{\sqrt{2}}{m(m+1)\pi}\left(\frac{2e}{m}\right)^m < 1.$$

A contradiction, and hence our assertion is true if $m \geq 6$. We immediately verify that between p_{n+1} and the product $p_1p_2\cdots p_{n+1}$ there are at least n primes for all $n \in \{1,2,3,4,5\}$. This completes the proof. $\square$

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