2003 Winter Camp

More Inequality Problems

1. If a, b, c, and d are positive real numbers, prove that
$$\frac{a^3+b^3+c^3}{a+b+c}+\frac{b^3+c^3+d^3}{b+c+d}+\frac{c^3+d^3+a^3}{c+d+a}+\frac{d^3+a^3+b^3}{d+a+b}\geq a^2+b^2+c^2+d^2.$$

2. Let x, y, and z be positive real numbers such that $x+y+z=1$. Prove that
$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)\geq 64$$

3. Given that a, b, c, d, and e are real numbers such that $a+b+c+d+e=8$ and $a^2+b^2+c^2+d^2+e^2=16$, determine the maximum possible value of e.

4. The perimeter of an isosceles trapezoid is 28. If the longest side has length 13, determine the maximum possible area of the trapezoid.

5. Let $n\geq 3$ be an integer. Let $a_1,a_2,\ldots,a_n$ be real numbers, with $2\leq a_i\leq 3$ for $i=1,2,\ldots,n$. If $S=a_1+a_2+\ldots+a_n$, prove that
$$\frac{a_1^2+a_2^2-a_3^2}{a_1+a_2+a_3}+\frac{a_2^2+a_3^2-a_4^2}{a_2+a_3+a_4}+\ldots+\frac{a_n^2+a_1^2-a_2^2}{a_n+a_1+a_2}\leq 2S-2n$$
(1995 IMO Shortlist)

6. Let a, b, and c be positive real numbers such that $abc=1$. Prove that
$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\leq 1.$$
(1996 IMO Shortlist)

7. If a, b, and c are positive real numbers, prove that
$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\leq\frac{1}{abc}.$$
(1998 USAMO, Question 5)

8. Prove that for numbers a, b, and c in the interval $[0,1]$,
$$\frac{a}{b+c+1}+\frac{b}{a+c+1}+\frac{c}{a+b+1}+(1-a)(1-b)(1-c)\leq 1.$$
(1980 USAMO, Question 5)

9. Prove that for any two given positive numbers p and q with $p<q$ and real numbers a,b,c,d,e with $p<a,b,c,d,e<q$, we have:
$$(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)\leq 25+6\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2.$$
(1977 USAMO, Question 5)

10. If $x, y, z > 0$ and $x + y + z = 1$, prove that

$$x^2y + y^2z + z^2x \leq \frac{4}{27}.$$

(1999 CMO, Question 5)

11. Let $a_1, a_2, a_3, \ldots$ be real numbers such that $a_i + a_j \geq a_{i+j}$ for $i, j = 1, 2, 3, \ldots$. Prove that for all integers $n \geq 1$,

$$\frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \ldots + \frac{a_n}{n} \geq a_n.$$

(1999 APMO, Question 2)

12. Let $g(k)$ denote the greatest *odd* divisor of k. Prove that for all positive integers n,

$$0 \leq \sum_{k=1}^{n} \frac{g(k)}{k} - \frac{2n}{3} \leq \frac{2}{3}.$$

13. Prove that for any positive numbers a, b, c, and d,

$$\left(\frac{abc + abd + acd + bcd}{4}\right)^2 \leq \left(\frac{ab + ac + ad + bc + bd + cd}{6}\right)^3.$$

14. Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

(1999 USAMO, Question 2)

15. *(Hoshino's Theorem??).* Construct the incircle of $\triangle ABC$ and let its radius be r. Construct a circle that is tangent to sides AB and AC, as well as the incircle of $\triangle ABC$. Let r_1 be the radius of this circle. Construct circles with radii r_2 and r_3 similarly on the other two pairs of sides. Prove that $r_1 + r_2 + r_3 \geq r$, with equality iff $\triangle ABC$ is equilateral.

16. *(The Erdos-Mordell Inequality).* For any point P inside the triangle ABC, let X, Y, and Z be the feet of the perpendiculars from P to BC, CA, and AB, respectively. Prove that $PA+PB+PC \geq 2(PX+PY+PZ)$, with equality iff $\triangle ABC$ is equilateral and P is the centre of the triangle. *(Hint: prove that $PAsinA \geq PYsinC + PZsinB$ and play around with the Sine Law).*