2003 Winter Camp

Notes on Inequalities

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Inequalities are a popular topic on math contests. In this handout, we describe the major ideas and tools needed to solve difficult Olympiad level inequality problems. There are four big tools, which we describe below. The first two (AM-GM and Cauchy-Schwarz) are particularly important. I would recommend you read this handout as follows: read the descriptions of the first two inequalities, then read the solutions to the first few problems to see how these inequalities are used. Once you get a feel for what is going on, then come back and read the descriptions of the Power Mean inequality and Jensen's Inequality.

1. **AM-GM** (Arithmetic Mean – Geometric Mean Inequality)

   If $a_1, a_2, \ldots, a_n$ are non-negative real numbers, then

$$\underset{\substack{\Uparrow \\ \text{“arithmetic”} \\ \text{mean}}}{\frac{a_1+a_2+\ldots+a_n}{n}} \geq \underset{\substack{\Uparrow \\ \text{“geometric”} \\ \text{mean}}}{\sqrt[n]{a_1 \cdot a_2 \cdot \cdots \cdot c_n}},$$

   with equality occurring $\underbrace{\text{if and only if}}_{\text{“iff”}}$ $a_1 = a_2 = \ldots = a_n$.

   For example, let us demonstrate the inequality for the case $n = 2$. The AM-GM inequality is saying that

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2},$$

   for any two non-negative numbers a_1 and a_2.

   You can prove this as follows: we start off with a true statement, and manipulate the terms until we get the inequality we want.

$$\begin{aligned} (a_1 - a_2)^2 &\geq 0 \\ a_1^2 - 2a_1a_2 + a_2^2 &\geq 0 \\ a_1^2 + 2a_1a_2 + a_2^2 &\geq 4a_1a_2 \\ (a_1 + a_2)^2 &\geq 4a_1a_2 \\ \left(\frac{a_1 + a_2}{2}\right)^2 &\geq a_1a_2 \\ \frac{a_1 + a_2}{2} &\geq \sqrt{a_1a_2} \end{aligned}$$

   Here, we see that equality occurs, i.e., $\dfrac{a_1 + a_2}{2} = \sqrt{a_1a_2}$ precisely when $a_1 = a_2$.

   It is much more difficult to prove the AM-GM inequality for $n \geq 3$, so we omit the proof. One can prove the inequality by a complicated technique known as *backwards induction.*

2. **Cauchy-Schwarz**

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are non-negative real numbers, then

$$(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) \geq (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

with equality occurring iff $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \ldots = \dfrac{a_n}{b_n}$.

I'd like you to verify that this inequality holds for $n = 2$.

This inequality is saying that $(a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1b_1 + a_2b_2)^2$. See if you can prove this.

3. **Power Mean**

Let $f(x) = \left(\dfrac{a_1^x + a_2^x + \ldots + a_n^x}{n}\right)^{1/x}$, where $a_1, a_2, \ldots, a_n$ are non-negative real numbers, and $n \geq 1$. Suppose x and y are integers with $x \geq y$. Then, $f(x) \geq f(y)$, with equality occurring iff $a_1 = a_2 = \ldots = a_n$.

For example, the QM-AM-GM-HM inequality is derived from the Power Mean, since $f(2) \geq f(1) \geq f(0) \geq f(-1)$.

$$\underbrace{\sqrt{\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}}}_{\text{“quadratic mean”}} \geq \frac{a_1+a_2+\ldots+a_n}{n} \geq \sqrt[n]{a_1a_2\ldots a_n} \geq \underbrace{\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}}_{\text{“harmonic mean”}}$$

Actually, $f(0)$ doesn't exist, but as $x \to 0$, $f(x)$ approaches $\sqrt[n]{a_1a_2\ldots a_n}$.

4. **Jensen's Inequality**

This is extremely neat but rather complicated. Suppose that $f(x)$ is a real continuous function that is convex (i.e. concave up) on an interval. You test for convexity by showing that $f''(x) \geq 0$ for all x in that interval $\to$ the double or second derivative if you've ever taken calculus. An easier way to see it is if you pick any two points on the curve, and join them. If the line lies above (or on) the curve, it is convex. If the line lies below, then it is concave, i.e. concave down (see diagram).

Let $a_1, a_2, \ldots, a_n$ be n real numbers in an interval S where $f(x)$ is convex for all x in S. Then, Jensen's Inequality states that

$$\frac{f(a_1) + f(a_2) + \ldots + f(a_n)}{n} \geq f\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)$$

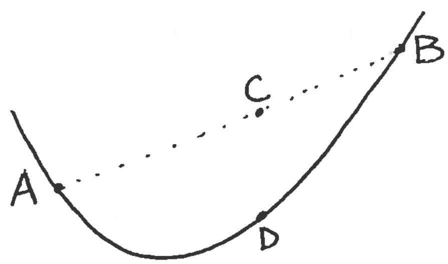
with equality occurring if and only if $a_1 = a_2 = \ldots = a_n$.

If $f(x)$ is concave for all x in S, then Jensen's Inequality states that

$$\frac{f(a_1)+f(a_2)+\ldots+f(a_n)}{n} \leq f\left(\frac{a_1+a_2+\ldots+a_n}{n}\right).$$

Same thing, except now the sign is reversed. Same conditions for equality as well (i.e., $a_1 = a_2 = \ldots = a_n$).

Let's prove Jensen's Inequality for the case $n = 2$, so you can get a better understanding of the idea behind this inequality.



Let $A(a_1, f(a_1))$, and $B(a_2, f(a_2))$ be any two points on the *convex* function $f(x)$.

Let C be the midpoint of the line AB, and D be the point indicated. The x-coordinates of C and D are the same, but C is above D, i.e. equality occurs iff $a_1 = a_2$, i.e. A and B are the same point.

$$\frac{f(a_1)+f(a_2)}{2} \geq f\left(\frac{a_1+a_2}{2}\right).$$

The inequality will go the other direction if $f(x)$ is a *concave* function.

Now let us solve some problems. Here are ten problems of varying levels of difficulty. Some are from recent math olympiads.

1. **What is the minimum possible positive value of $x + \dfrac{9}{x}$?**

Clearly, we don't want x to be negative. If $x > 0$, then both x and $\dfrac{9}{x}$ are positive, so by the AM-GM inequality,

$$\frac{x + \frac{9}{x}}{2} \geq \sqrt{x \cdot \frac{9}{x}} = \sqrt{9} = 3.$$

Thus, $x + \dfrac{9}{x} \geq 6$, with equality occurring iff $x = \dfrac{9}{x}$, i.e., $x^2 = 9$ or $x = 3$. (Note: $x \neq -3$, since x is positive.) We see that, indeed, when $x = 3$, we have $x + \dfrac{9}{x} = 3 + 3 = 6$, thus the minimum possible value of $x + \dfrac{9}{x}$ is 6.

2. **Prove that $(a+b+c)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) \geq 9$ if $a, b, c > 0$.**

<u>Solution 1</u>: Expanding, we have

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 3 + \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b}.$$

By AM-GM, since $a, b, c > 0$, we have

$$\frac{\frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b}}{6} \geq \sqrt[6]{\frac{a}{b} \cdot \frac{a}{c} \cdot \frac{b}{a} \cdot \frac{b}{c} \cdot \frac{c}{a} \cdot \frac{c}{b}} = 1.$$

Thus, $\dfrac{a}{b} + \dfrac{a}{c} + \dfrac{b}{a} + \dfrac{b}{c} + \dfrac{c}{a} + \dfrac{c}{b} \geq 6$, and so $(a+b+c)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) \geq 3 + 6 = 9$, as required.

<u>Solution 2</u>: By Cauchy-Schwarz, let $a_1 = \sqrt{a}$, $a_2 = \sqrt{b}$, $a_3 = \sqrt{c}$, $b_1 = \frac{1}{\sqrt{a}}$, $b_2 = \frac{1}{\sqrt{b}}$, and $b_3 = \frac{1}{\sqrt{c}}$. Thus,

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq (1+1+1)^2 = 3^2 = 9,$$

and we are done. Equality occurs iff $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$, i.e. iff $a = b = c$.

3. **Find the maximum value of $x^3(4-x)$, where $0 < x < 4$.**

Yuck. How can we use our knowledge of inequalities here? No really obvious way. That's why a little ingenuity is needed:

Really nice trick here.
See how the left side
simplifies nicely?
↓

Since $0 < x < 4$, we have by AM-GM, $\dfrac{\frac{x}{3} + \frac{x}{3} + \frac{x}{3} + (4-x)}{4} \geq \sqrt[4]{\left(\dfrac{x}{3}\right)^3 \cdot (4-x)}$.

$$\therefore \frac{x + (4-x)}{4} = 1 \geq \sqrt[4]{\frac{x^3(4-x)}{27}} \Rightarrow x^3(4-x) \leq 27. \quad \text{Maximum value is 27.}$$

Equality occrs iff $\dfrac{x}{3} = \dfrac{x}{3} = \dfrac{x}{3} = 4 - x$, i.e. if $\dfrac{4x}{3} = 4$, or $x = 3$.
Checking, we see that if $x = 3$, the maximum value of 27 is indeed attained.

4. **If $a+b+c=1$, show that $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2 \geq \frac{100}{3}$, where $a,b,c>0$.**

By the Cauchy-Schwarz Inequality,

$$\begin{aligned}
\left[\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2\right]\left[1^2+1^2+1^2\right] &\geq \left[\left(a+\frac{1}{a}\right)+\left(b+\frac{1}{b}\right)+\left(c+\frac{1}{c}\right)\right]^2 \\
&= \left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \\
&= \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2
\end{aligned}$$

Using Cauchy again, we have $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b+c) \geq (1+1+1)^2 = 9$. (See how we used Cauchy here? Recall what we did in Question 2)

Since $a+b+c=1$, that means $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$.

Hence, $\left[\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2\right] \times 3 \geq \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \geq (1+9)^2 = 100$.

Thus, $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2 \geq \frac{100}{3}$, as required. Equality occurs iff $a=b=c=\frac{1}{3}$.

5. **If the roots of the polynomial $x^6-6x^5+ax^4+bx^3+cx^2+dx+1$ are all positive, find a,b,c, and d.**

There are two things you should be thinking: (i) how is this an inequality problem? and (ii) surely there isn't enough information to figure this out! Check this out:

Let the roots of the polynomial be p_1,p_2,p_3,p_4,p_5 and p_6. We are given that $p_1,p_2,\ldots,p_6>0$. Also, by the relationship between the roots of a polynomial and its coefficients, we have:

$$\begin{cases} p_1+p_2+p_3+p_4+p_5+p_6=6 \\ p_1p_2p_3p_4p_5p_6=1 \end{cases}$$

By the AM-GM inequality, $\frac{p_1+p_2+p_3+p_4+p_5+p_6}{6} \geq \sqrt[6]{p_1p_2p_3p_4p_5p_6}$, which simplifies to $1=1$. Note that we can use the AM-GM inequality only because all the terms are non-negative.

Remember, equality occurs if and only if $p_1=p_2=p_3=p_4=p_5=p_6$. And we do have equality, because this expression simplifies to $1=1$. Thus, all six roots *must* be equal.

Since $p_1+p_2+\ldots+p_6=6$, that tells us that each term is equal to 1.

Hence all the roots of the polynomial are 1, so:

$$\begin{aligned}
x^6-6x^5+ax^4+bx^3+cx^2+dx+1 &= (x-1)^6 \\
&= x^6-6x^5+15x^4-20x^3+15x^2-6x+1
\end{aligned}$$

and matching up coefficients, we get $a=15$, $b=-20$, $c=15$ and $d=-6$.

Isn't that a great question?

6. **Let A, B, and C be the angles of a triangle. Show that $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$.**

Here is where Jensen's Inequality comes in handy. Let $f(x) = \sin x$. Then $f'(x) = \cos x$, and $f''(x) = -\sin x$. Since A, B, and C are angles of a triangle, $0° < A, B, C < 180°$. (Also, $A+B+C = 180°$, and we'll need that later). For all x from $0°$ to $180°$, $f''(x) = -\sin x < 0$, and thus is concave in that interval.

Since A, B, and C lie in this interval, by Jensen's Inequality,

$$\begin{aligned} \frac{f(A)+f(B)+f(C)}{3} &\leq f\left(\frac{A+B+C}{3}\right) \\ \therefore \quad \frac{\sin A + \sin B + \sin C}{3} &\leq \sin\left(\frac{180°}{3}\right) = \frac{\sqrt{3}}{2}, \quad \text{since } A+B+C = 180°. \end{aligned}$$

Multiplying both sides of the inequality by 3, we arrive at the desired result.

7. **Let a, b, and c be positive real numbers. Show that $a^a b^b c^c \geq (abc)^{a+b+c/3}$.**
(1995 CMO, Question 2)

There are several ways to do this, but this one is really ingenious.

Let $f(x) = (\ln x) \cdot x$. Then $f'(x) = \ln x + \frac{1}{x} \cdot (x) = \ln x + 1$, and $f''(x) = \frac{1}{x}$ and $f''(x) > 0$ for all positive values of x. This, by Jensen's Inequality, for positive a, b, and c, we have:

$$\begin{aligned} \frac{f(a)+f(b)+f(c)}{3} &\geq \left(\frac{a+b+c}{3}\right) \\ \therefore \quad \frac{a\ln a + b\ln b + c\ln c}{3} &\geq \frac{a+b+c}{3} \cdot \ln\left(\frac{a+b+c}{3}\right) \\ \Leftrightarrow \quad \frac{1}{3}\cdot(\ln a^a + \ln b^b + \ln c^c) &\geq \frac{1}{3} \cdot \ln\left(\frac{a+b+c}{3}\right)^{a+b+c} \\ \Leftrightarrow \quad \ln a^a b^b c^c &\geq \ln\left(\frac{a+b+c}{3}\right)^{a+b+c} \\ \Leftrightarrow \quad a^a b^b c^c &\geq \left(\frac{a+b+c}{3}\right)^{a+b+c}. \end{aligned}$$

By AM-GM, $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

Therefore, $a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c} \geq (\sqrt[3]{abc})^{a+b+c} = (abc)^{\frac{a+b+c}{3}}$.

Thus, we have proven the desired inequality. (Note: equality occurs iff $a = b = c$.)

8. **Suppose a, b, and c are the sides of a triangle.**
Prove that $\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$.
(1996 APMO, Question 5)

Since a, b, and c are the sides of a triangle $\begin{cases} a+b-c > 0 \\ a+c-b > 0 \\ b+c-a > 0 \end{cases}$.

Since all three of these expressions are positive, let $x = a+b-c$, $y = a+c=b$ and $z = b+c-a$. This is often a very useful substitution to make if you are given that a, b, and c are sides of a triangle.

Then $x, y, z > 0$ and we can express a, b, and c in terms of x, y, and z.

Our inequality then becomes, (i.e., is equivalent to):

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \leq \sqrt{\frac{x+y}{2}}+\sqrt{\frac{x+z}{2}}+\sqrt{\frac{y+z}{2}}.$$

Since $x, y, z > 0$, and thus, $\sqrt{x}, \sqrt{y}, \sqrt{z} > 0$. So let's make another substitution: $x = p^2, y = q^2$, and $z = r^2$, where $p, q, r > 0$. Then we need to prove that:

$$p+q+r \leq \sqrt{\frac{p^2+q^2}{2}}+\sqrt{\frac{q^2+r^2}{2}}+\sqrt{\frac{q^2+r^2}{2}}.$$

But by QM-AM, we have $\sqrt{\frac{p^2+q^2}{2}} \geq \frac{p+q}{2}$, $\sqrt{\frac{p^2+r^2}{2}} \geq \frac{p+r}{2}$, and $\sqrt{\frac{q^2+r^2}{2}} \geq \frac{q+r}{2}$. (Really neat use of symmetry here. Always try to exploit symmetry.)

Adding up these three inequalities, we get

$$\sqrt{\frac{p^2+q^2}{2}}+\sqrt{\frac{p^2+r^2}{2}}+\sqrt{\frac{q^2+r^2}{2}} \geq \frac{p+q}{2}+\frac{p+r}{2}+\frac{q+r}{2}=p+q+r$$

as desired. Thus, we are done.

9. **Suppose a, b, and c are all positive. Prove that $(a^3+b^3+abc)^{-1}+(b^3+c^3+abc)^{-1}+(c^3+a^3+abc)^{-1} \leq (abc)^{-1}$. (1998 USAMO, Question 2)**

Here's a really nice technique to remember, called the *principle of homogeneity*. If we replace a, b, and c by ka, kb, and kc, all the terms with k's will cancel out, and we'll get back to the original inequality $\rightarrow$ try it, you'll see that all the k's will disappear. Thus, we can assume without loss of generality that $abc = 1$!!! It makes things so much easier! Even if a, b and c aren't numbers that multiply to 1, we can multiply all of them by a constant k so that the relation holds, so that's why we can do that. When a given inequality is *homogeneous*, i.e. every term has the same degree, we can use that to our advantage and assume any one thing we want. In our case, we assumed that $abc = 1$.

Furthermore, we can let $x = a^3$, $y = b^3$, and $z = c^3$, since that will make the simplification easier. Since $abc = 1$, we have $xyz = a^3b^3c^3 = 1$. So now our inequality then becomes: $(x+y+1)^{-1}+(y+z+1)^{-1}+(z+x+1)^{-1} \leq 1$, where $x, y, z > 0$ and $xyz = 1$.

Much easier, isn't it? Well, after simplification, we get:

$$2(x+y+z) \leq x^2y+xy^2+xz^2+x^2z+yz^2+y^2z=(x+y+z)(xy+yz+zx)-3xyz$$

$\therefore$ $(x+y+z)(xy+yz+zx-2) \geq 3xyz = 3$. That is the inequality that we need to prove.

And because $\frac{x+y+z}{3} \geq \sqrt[3]{xyz} = 1$ and $\frac{xy+yz+zx}{3} \geq \sqrt[3]{x^2y^2z^2} = 1$, by AM-GM, we have $(x+y+z)(xy+yz+zx-2) \geq 3 \cdot (3-2) = 3$, and so we are done.

10. **Suppose that a, b, and c are positive real numbers such that $abc = 1$. Prove that** $\dfrac{1}{a^3(b+c)} + \dfrac{1}{b^3(a+c)} + \dfrac{1}{c^3(a+b)} \geq \dfrac{3}{2}$.
(1995 IMO, Question 2)

The key insight is to make the substitution $a = \dfrac{1}{x}$, $b = \dfrac{1}{y}$, and $c = \dfrac{1}{z}$. If you don't do this, the problem is virtually impossible to solve. See how that trick is useful? Try to recognize substitutions like these that will give you an inequality that is easier to prove. Remember, when you see an inequality problem, be clever and try to use some of the ideas detailed in these solutions. Who knows, that might be the way to do it! With problems like these, perseverance and tenacity is what you need – you might have to try 5 to 10 (or more!) different methods before you finally get it!

Thus, we have

$$\frac{1}{\frac{1}{x^3}\left(\frac{1}{y}+\frac{1}{z}\right)} + \frac{1}{\frac{1}{y^3}\left(\frac{1}{x}+\frac{1}{z}\right)} + \frac{1}{\frac{1}{z^3}\left(\frac{1}{y}+\frac{1}{x}\right)} \geq \frac{3}{2}$$
$$\Leftrightarrow \frac{x^3yz}{y+z} + \frac{y^3xz}{x+z} + \frac{z^3xy}{x+y} \geq \frac{3}{2}$$
$$\Leftrightarrow \frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq \frac{3}{2} \quad \text{(since } xyz = 1 \text{ because } xyz = \frac{1}{abc} = 1\text{)}.$$

Hence, if we can prove this inequality, we will be done. There are now a couple of ways to proceed.

<u>**Method 1:**</u> (Cauchy-Schwarz).

Since $x, y, z \geq 0$, we have by Cauchy-Schwarz:

$$\left(\frac{x^2}{y+z} + \frac{y^2}{x+x} + \frac{z^2}{x+y}\right)((y+z)+(z+z)+(x+y)) \geq (x+y+z)^2$$
$$\Leftrightarrow \left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right)\cdot(2x+2y+2z) \geq (x+y+z)^2$$
$$\Leftrightarrow \frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq \frac{x+y+z}{2}$$

By AM-GM, we have $\frac{x+y+z}{3} \geq \sqrt[3]{xyz} = 1$, so $x+y+z \geq 3$. Hence, $\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq \frac{3}{2}$, and we are done.

<u>**Method 2:**</u> (Jensen).

Let $f(p) = \dfrac{p^2}{x+y+z-p}$.

Then one can show that $f''(p) = \dfrac{2}{x+y+z-p} + \dfrac{2p(2x+2y+2z-p)}{(x+y+z-p)^3}$, and this is clearly positive if $0 < p < x+y+z$. So $f(p)$ is convex in the interval $[0, x+y+z]$.

Since each of x, y, and z are betewen 0 and $x+y+z$, by Jensen's Inequality, we have $\dfrac{f(x)+f(y)+f(z)}{3} \geq f\left(\dfrac{x+y+z}{3}\right)$. This gives us

$$\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq 3\cdot\frac{\left(\frac{x+y+z}{3}\right)^2}{(x+y+z)-\frac{x+y+z}{3}} = \frac{x+y+z}{2}.$$

By AM-GM, $x+y+z \geq 3$ (same as before), so $\dfrac{x^2}{y+z} + \dfrac{y^2}{x+z} + \dfrac{z^2}{x+y} \geq \dfrac{3}{2}$, as required.

More Problems on Inequalities

1. Show that if $a, b, c > 0$, then $a^3 + b^3 + c^3 \geq 3abc$.

2. Show that if $a, b, c > 0$, then $\frac{ab}{c^2} + \frac{ac}{b^2} + \frac{bc}{a^2} \geq 3$.

3. Show that if $a, b, c > 0$, then $\frac{a}{2} + \frac{b}{3} + \frac{c}{6} \geq a^{1/2}b^{1/3}c^{1/6}$.

4. Show that if $a, b, c > 0$, then $(a+b)(a+c)(b+c) \geq 8abc$.

5. Show that if $a, b, c > 0$, then $a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3}$.

6. Find the greatest value of x^2y^3z given that $x, y, z > 0$ and $x + y + z = 6$.

7. Show that if $a, b, c > 0$, then $(a+b+c)\left(\dfrac{1}{a+b} + \dfrac{1}{a+c} + \dfrac{1}{b+c}\right) \geq \dfrac{9}{2}$. When does equality occur?

8. If $a, b > 0$, prove that $\dfrac{1+nb}{n+1} \geq \sqrt[n+1]{ab^n}$, where n is a positive integer. Using this, can you show that the sequence $\left(1 + \dfrac{1}{n}\right)^n$ is increasing?

9. Show that for all $n > 1$, we have $\left(\dfrac{n+1}{2}\right)^n > n!$ (Note: We <u>never</u> have equality.)

10. If a, b, c, x, y, and z are real and $a^2 + b^2 + c^2 = 25$, $x^2 + y^2 + z^2 = 36$, and $ax + by + cz = 30$, compute the value of $\frac{a+b+c}{x+y+z}$.

11. (a) Prove that among all rectangles with a fixed perimeter p, the square has the greatest area.

    (b) Prove that among all rectangles with a fixed area A, the square has the least perimeter.

12. Prove that among all triangles with a fixed perimeter p, the equilateral triangle has the greatest area.

13. Prove that $(x+y)(x+z)(y+z) \geq 8(x+y-z)(y+z-x)(z+x-y)$ where x, y, and z are positive real numbers.

14. Solve the system:

$$\begin{cases} 2^{x+y+z} &= 64 \\ xyz &= 8 \end{cases},$$

where x, y, z are positive real numbers.

15. If A, B and C are the angles of a triangle, show that $\cos A + \cos B + \cos C \leq \frac{3}{2}$.

16. Suppose that $a_1, a_2, \ldots, a_n > 0$. Show that

$$\frac{a_1^2}{a_1 + a_2} + \frac{a_2^2}{a_2 + a_3} + \ldots + \frac{a_n^2}{a_n + a_1} \geq \frac{a_1 + a_2 + \ldots + a_n}{2}$$

17. Suppose $a_1, a_2, \ldots, a_{1999}$ are 1999 positive real numbers, and let $b_1, b_2, \ldots, b_{1999}$ be some rearrangement of these numbers. What is the minimum value of $\frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_{1999}}{b_{1999}}$?

18. Prove the AM-GM inequality using Jensen's Inequality. (Hint: Let $f(x) = \ln x$.)

19. Suppose that $a_1 \geq a_2 \geq \ldots \geq a_n > 0$, and that $b_1 \geq a_1$, $b_1 b_2 \geq a_1 a_2$, $b_1 b_2 b_3 \geq a_1 a_2 a_3$, ..., $b_1 b_2 \ldots b_n \geq a_1 a_2 \ldots a_n$.

    Prove that $b_1 + b_2 + \ldots + b_n \geq a_1 + a_2 + \ldots + a_n$.

    (Hint: let $c_i = \frac{b_1}{a_1} \cdot \frac{b_2}{a_2} \cdots \frac{b_i}{a_i}$).

20. Let n be an integer, $n \geq 3$. Let $a_1, a_2, \ldots, a_n$ be real numbers, where $2 \leq a_i \leq 3$ for $i = 1, 2, \ldots, n$. If $s = a_1 + a_2 + \ldots + a_n$, prove that:

$$\frac{a_1^2 + a_2^2 - a_3^2}{a_1 + a_2 - a_3} + \frac{a_2^2 + a_3^2 - a_4^2}{a_2 + a_3 - a_4} + \ldots + \frac{a_n^2 + a_1^2 - a_2^2}{a_n + a_1 - a_2} \leq 2s - 2n.$$

    (1995 IMO Shortlist Problem)