IMO Shortlist2010

Algebra

$\boxed{1}$ Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

Proposed by Pierre Bornsztein, France

$\boxed{2}$ Let the real numbers a, b, c, d satisfy the relations $a + b + c + d = 6$ and $a^2 + b^2 + c^2 + d^2 = 12$. Prove that
$$36 \le 4\left(a^3 + b^3 + c^3 + d^3\right) - \left(a^4 + b^4 + c^4 + d^4\right) \le 48.$$

Proposed by Ukraine

$\boxed{3}$ Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \le 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum $S = \sum_{i=1}^{100} x_i x_{i+2}$.

Proposed by Russia

$\boxed{4}$ A sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k, x_{2k-1} = (-1)^{k+1} x_k$ for all $k \ge 1$. Prove that $\forall n \ge 1 \ x_1 + x_2 + \ldots + x_n \ge 0$.

Proposed by Austria

$\boxed{5}$ Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f\left(f(x)^2 y\right) = x^3 f(xy).$$

Proposed by Switzerland

$\boxed{6}$ Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n.

Proposed by Germany

$\boxed{7}$ Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers, and s be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \le k \le n-1\} \ \text{ for all } \ n > s.$$

Prove there exist positive integers $\ell \leq s$ and N, such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \geq N.$$

Proposed by Morteza Saghafiyan, Iran

8 Given six positive numbers a, b, c, d, e, f such that $a < b < c < d < e < f$. Let $a + c + e = S$ and $b + d + f = T$. Prove that

$$2ST > \sqrt{3(S+T)\left(S(bd + bf + df) + T(ac + ae + ce)\right)}.$$

Proposed by South Korea

Combinatorics

1 In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

Proposed by Austria

2 On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

Proposed by Croatia

3 2500 chess kings have to be placed on a 100×100 chessboard so that

(i) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex); **(ii)** each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

Proposed by Russia

4 Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

Type 1) Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

Type 2) Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

Proposed by Netherlands

5 $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and l_i be respectively the number of wins and losses of the ith player. Prove that

$$\sum_{i=1}^{n} (w_i - l_i)^3 \geq 0.$$

<div align="right">Proposed by South Korea</div>

6 Given a positive integer k and other two integers $b > w > 1$. There are two strings of pearls, a string of b black pearls and a string of w white pearls. The length of a string is the number of pearls on it. One cuts these strings in some steps by the following rules. In each step:

(i) The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then k first ones (if they consist of more than one pearl) are chosen; if there are less than k strings longer than 1, then one chooses all of them. **(ii)** Next, one cuts each chosen string into two parts differing in length by at most one. (For instance, if there are strings of $5, 4, 4, 2$ black pearls, strings of $8, 4, 3$ white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts $(4, 4), (3, 2), (2, 2)$ and $(2, 2)$ respectively.) The process stops immediately after the step when a first isolated white pearl appears.

Prove that at this stage, there will still exist a string of at least two black pearls.

<div align="right">Proposed by Canada</div>

7 Let $P_1, \ldots, P_s$ be arithmetic progressions of integers, the following conditions being satisfied:

(i) each integer belongs to at least one of them; **(ii)** each progression contains a number which does not belong to other progressions.

Denote by n the least common multiple of the ratios of these progressions; let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ its prime factorization.

Prove that

$$s \geq 1 + \sum_{i=1}^{k} \alpha_i (p_i - 1).$$

<div align="right">Proposed by Germany</div>

Geometry

$\boxed{1}$ Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

Proposed by United Kingdom

$\boxed{2}$ Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

Proposed by Poland

$\boxed{3}$ Let $A_1 A_2 \ldots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, \ldots, P_n$ onto lines $A_1 A_2, \ldots, A_n A_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \ldots, X_n$ on sides $A_1 A_2, \ldots, A_n A_1$ respectively,

$$\max \left\{ \frac{X_1 X_2}{P_1 P_1}, \ldots, \frac{X_n X_1}{P_n P_1} \right\} \geq 1.$$

Proposed by Armenia

$\boxed{4}$ Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

Proposed by Tai Wai Ming and Wang Chongli, Hong Kong

$\boxed{5}$ Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90°$, prove that $2\angle BDA = \angle CDE$.

Proposed by Ukraine

$\boxed{6}$ The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of an acute-angled triangle ABC. Prove that the incenter of triangle ABC lies inside triangle XYZ.

Proposed by Bulgaria

7 | Three circular arcs γ_1, γ_2, and γ_3 connect the points A and C. These arcs lie in the same half-plane defined by line AC in such a way that arc γ_2 lies between the arcs γ_1 and γ_3. Point B lies on the segment AC. Let h_1, h_2, and h_3 be three rays starting at B, lying in the same half-plane, h_2 being between h_1 and h_3. For $i, j = 1, 2, 3$, denote by V_{ij} the point of intersection of h_i and γ_j (see the Figure below). Denote by $\widehat{V_{ij}V_{kj}}\widehat{V_{kl}V_{il}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{il}$, $V_{kj}V_{kl}$ and arcs $V_{ij}V_{kj}$ and $V_{il}V_{kl}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs. Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}, \widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}, \widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed, too.

Proposed by Hungary

Number Theory

$\boxed{1}$ Find the least positive integer n for which there exists a set $\{s_1, s_2, \ldots, s_n\}$ consisting of n distinct positive integers such that

$$\left(1 - \frac{1}{s_1}\right)\left(1 - \frac{1}{s_2}\right) \cdots \left(1 - \frac{1}{s_n}\right) = \frac{51}{2010}.$$

$\boxed{2}$ Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m\left(2^{n+1} - 1\right).$$

Proposed by Australia

$\boxed{3}$ Find the smallest number n such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \ldots + f_n(x)^2.$$

Proposed by Poland

$\boxed{4}$ Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is n-good if $n | P(m) - P(k)$ implies $n | m - k$ for all integers m, k. We say that (a, b) is *very good* if (a, b) is n-good for infinitely many positive integers n. **(a)** Find a pair (a, b) which is 51-good, but not very good.[/*:m] **(b)** Show that all 2010-good pairs are very good.[/*:m]

Proposed by Turkey

$\boxed{5}$ Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m) + n)(g(n) + m)$$

is perfect square for all $m, n \in \mathbb{N}$.

Proposed by Gabriel Carroll, USA

$\boxed{6}$ The rows and columns of a $2^n \times 2^n$ table are numbered from 0 to $2^n - 1$. The cells of the table have been coloured with the following property being satisfied: for each $0 \leq i, j \leq 2^n - 1$, the j-th cell in the i-th row and the $(i + j)$-th cell in the j-th row have the same colour. (The indices of the cells in a row are considered modulo 2^n.) Prove that the maximal possible number of colours is 2^n.

Proposed by Iran