Art of Problem Solving

WOOT 2010–11

Induction FAIL

Here's a problem that has already tormented many WOOTers. In this article, we see how it tormented me. I thought I had found a slick, quick solution, but my solution nagged at me and nagged at me. It seemed just a little bit too easy for a USAMO #3. So I was bothered by it. Only when I wrote up my solution for the WOOT Induction Handout did I stumble on the flaw in my reasoning. OK, it was well after I wrote it up, but I finally saw the error. Can you find it?

Problem. The 2010 positive numbers $a_1, a_2, \ldots, a_{2010}$ satisfy the inequality $a_ia_j \le i + j$ for all distinct indices i, j. Determine, with proof, the largest possible value of the product $a_1a_2 \cdots a_{2010}$. (USAMO, 2010)

Solution. Right away, we at least have an upper bound for the maximum product. We can take inequalities corresponding to 1005 disjoint pairs of the a_i and multiply those to get an upper bound. For example, multiplying

$$\begin{aligned} a_1a_2 &\le 3,\\ a_3a_4 &\le 7,\\ &\vdots\\ a_{2009}a_{2010} &\le 4019, \end{aligned}$$

gives

$$a_1a_2a_3a_4 \cdots a_{2010} \le (3)(7)(11)(15)\cdots(4019).$$

However, this isn't the only way we can combine 1005 pairs to get an upper bound. For example, we could replace the first two inequalities above with $a_1a_3 \le 4$ and $a_2a_4 \le 6$. But the product of these gives $a_1a_2a_3a_4 \le 24$, which is a less tight bound that $a_1a_2a_3a_4 \le 21$. So, we first have to show that our 1005 products shown above give the least possible upper bound from among those we can form by multiplying 1005 of the inequalities.

To prove that this group of 1005 inequalities offers the least upper bound, we must show that if $p < q < r < s$, then the least upper bound of $a_pa_qa_ra_s$ comes from multiplying $a_pa_q \le p + q$ and $a_ra_s \le r + s$. The three possible pairings of these four a_i are

$$\begin{aligned} a_pa_q \le (p+q), a_ra_s \le (r+s) \quad &\Longrightarrow \quad a_pa_qa_ra_s \le (p+q)(r+s),\\ a_pa_r \le (p+r), a_qa_s \le (q+s) \quad &\Longrightarrow \quad a_pa_qa_ra_s \le (p+r)(q+s),\\ a_pa_s \le (p+s), a_ra_q \le (r+q) \quad &\Longrightarrow \quad a_pa_qa_ra_s \le (p+s)(r+q). \end{aligned}$$

To see that the first gives the least upper bound, we note that

$$\begin{aligned} (p+r)(q+s) - (p+q)(r+s) &= pq + rs - pr - qs = (p-r)(q-s) \ge 0,\\ (p+s)(r+q) - (p+q)(r+s) &= pq + rs - ps - qr = (p-s)(q-r) \ge 0, \end{aligned}$$

where the final step in each chain comes from the fact that $p, q < r, s$. The only 1005 disjoint pairs of the a_i for which every two pairs a_pa_q and a_ra_s satisfy $p, q < r, s$ are the pairs shown above. We have therefore shown that

$$a_1a_2a_3a_4 \cdots a_{2010} \le (3)(7)(11)(15)\cdots(4019),$$

and we don't have any easy way to tighten the upper bound. We at least now have a target. If we can show that this bound can be achieved, then we have our desired maximum.

We'll try induction. First, we tackle a base case, and thereby make sure we aren't barking up the wrong tree with trying to prove that this particular bound can be achieved. Let's tackle the problem with just a_1, a_2, a_3, a_4. We seek values such that $a_1a_2 = 3, a_3a_4 = 7$, and $a_ia_j \le i + j$ for the other four possible pairs. Sure enough, we can find lots of values that work, such as $(a_1, a_2, a_3, a_4) = (\frac{3}{2}, 2, \frac{7}{3}, 3)$. Now that we have a base case, on to the inductive step.

We assume that there exist $a_1, a_2, \ldots, a_{2n}$ such that $a_{2k-1}a_{2k} = 4k - 1$ for $1 \le k \le n$, and that $a_ia_j \le i + j$ for all $1 \le i, j \le 2n$.

We wish to show that this assumption implies that there exist $a_1, a_2, \ldots, a_{2n+2}$ such that $a_{2k-1}a_{2k} = 4k - 1$ for $1 \le k \le n + 1$, and that $a_ia_j \le i + j$ for all $i, j \le 2n + 2$. A natural guess is to start with the values of $a_1, a_2, \ldots, a_{2n}$ in our inductive hypothesis, and prove that we can tack on a_{2n+1} and a_{2n+2} such that

$$\begin{aligned} a_{2n+1}a_{2n+2} &= (2n + 1) + (2n + 2), \\ a_{2n+1}a_i &\le 2n + 1 + i \text{ for all } i \le 2n, \\ a_{2n+2}a_j &\le 2n + 2 + j \text{ for all } j \le 2n. \end{aligned}$$

Our inductive hypothesis assures us that $a_{2k-1}a_{2k} = 4k - 1$ for $1 \le k \le n$, and that $a_ia_j \le i + j$ for all $i, j \le 2n$, so we only have to worry about satisfying the inequalities for a_{2n+1} and a_{2n+2} above. Since $a_{2n+1}a_{2n+2} = (2n + 1) + (2n + 2)$, we have

$$a_{2n+2} = \frac{(2n + 1) + (2n + 2)}{a_{2n+1}}.$$

So, we have

$$\begin{aligned} & a_{2n+2}a_j \le 2n + 2 + j \\ \Leftrightarrow \quad & \frac{(2n + 1) + (2n + 2)}{a_{2n+1}} \cdot a_j \le 2n + 2 + j \\ \Leftrightarrow \quad & \frac{((2n + 1) + (2n + 2))a_j}{2n + 2 + j} \le a_{2n+1} \end{aligned} \tag{1}$$

We also have $a_{2n+1}a_i \le 2n + 1 + i \Leftrightarrow a_{2n+1} \le \frac{2n+1+i}{a_i}$. Combining this with 1 above, we must now show that there exists an a_{2n+1} such that

$$\frac{((2n + 1) + (2n + 2))a_j}{2n + 2 + j} \le a_{2n+1} \le \frac{2n + 1 + i}{a_i}$$

for all $1 \le i, j, \le 2n$. Such an a_{2n+1} exists if and only if each of these lower bounds is below each of the upper bounds, or

$$\frac{2n + 1 + i}{a_i} - \frac{((2n + 1) + (2n + 2))a_j}{2n + 2 + j} \ge 0.$$

Now, we just crank through the algebra:

$$\begin{aligned} & \frac{2n+1+i}{a_i} - \frac{((2n+1)+(2n+2))a_j}{2n+2+j} \geq 0 \\ \Leftrightarrow \quad & (2n+2+j)(2n+1+i) - ((2n+1)+(2n+2))a_ia_j \geq 0. \end{aligned}$$

By our inductive hypothesis, we have $a_ia_j \leq i+j$, so

$$\begin{aligned} & (2n+2+j)(2n+1+i) - ((2n+1)+(2n+2))a_ia_j \\ & \geq ((2n+2)+j)((2n+1)+i) - ((2n+1)+(2n+2))(i+j) \\ & = (2n+2)(2n+1) - (2n+1)i - (2n+2)j + ij \\ & = ((2n+1)-j)((2n+2)-i) \\ & \geq 0, \end{aligned}$$

where the final step follows from the fact that $i, j \leq 2n$.

This completes the inductive step and shows that the bound $3 \cdot 7 \cdot 11 \cdots 4019$ can indeed be achieved.

Or does it? Where's the flaw?

Moral of the story: if a problem seems too easy for its placement on a particular contest, then you should be extra skeptical about your solution. After you find the flaw, think about how you could avoid making a similar mistake when performing an induction.