COMBINATORICS PROBLEMS (2007 Winter Camp)

William
Bobby
Chen

1. A square of dimensions $(n-1)\times(n-1)$ is divided into $(n-1)^2$ unit squares in the usual manner. Each of the n^2 vertices of these squares is to be colored red or blue. Find the number of different colorings such that each unit square has exactly two red vertices. (Two coloring schemes are regarded as different if at least one vertex is colored differently in the two schemes.)

William
Bobby

2. For how many pairs of consecutive integers in the set

$$\{1000, 1001, 1002, \ldots, 2000\}$$

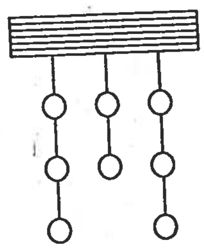
is no carrying required when the two integers are added?

William

3. Determine the number of ways to choose five numbers from the first eighteen positive integers such that any two chosen numbers differ by at least 2.

Bobby

4. In a shooting match, eight clay targets are arranged in two hanging columns of three each and one column of two, as pictured. A marksman is to break all eight targets according to the following rules: (1) The marksman first chooses a column from which a target is to be broken. (2) The marksman must then break the lowest remaining unbroken target in the chosen column. If these rules are followed, in how many different orders can the eight targets be broken? (AIME, 1990/8)



5. Let n be an odd integer greater than 1. Prove that the sequence

$$\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{\frac{n-1}{2}}$$

contains an odd number of odd numbers.

6. Twenty five of King Arthur's knights are seated at their customary round table. Three of them are chosen – all choices of three being equally likely – and are sent off to slay a troublesome dragon. Find the probability that at least two of the three had been sitting next to each other.

7. Two of the squares of a 7×7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

William X

8. Determine the smallest integer n, $n \geq 4$, for which one can choose four different numbers a, b, c, d from any n distinct integers such that $a+b-c-d$ is divisible by 20.

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9. For a finite set S of natural numbers let

$$Alt(S) = x_1 - x_2 + x_3 - \cdots,$$

where $x_1 > x_2 > x_3 > \cdots$ are the elements of S in decreasing order. Determine

$$f(n) = \sum Alt(S),$$

where the sum is extended over all ~~non-empty~~ subsets S of $\{1, 2, \ldots, n\}$. (note $Alt(\emptyset) = 0$)

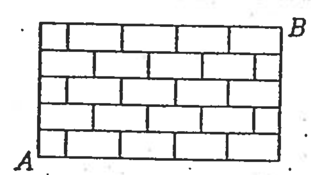
10. How many permutations $(x_1, \ldots, x_n)$ of $\{1, \ldots, n\}$ are there such that the cyclic sum $\sum_{i=1}^{n} |x_i - x_{i+1}|$ (with $x_{n+1} = x_1$) is (a) a minimum, (b) a maximum?

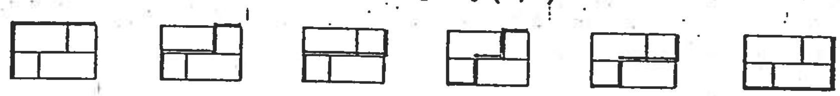
11. Let $n > 1$ be an integer. How many permutations $(a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$ are there such that

$$1 \mid a_1 - a_2, \quad 2 \mid a_2 - a_3, \quad \ldots, \quad n-1 \mid a_{n-1} - a_n?$$

12. Call a permutation π on $\{1, 2, \ldots, n\}$ an *equidistance permutation* if there is a constant $c \neq 0$ such that $|\pi(i) - i| = c$ for all $i \in \{1, 2, \ldots, n\}$. Find the number of equidistance permutations for $n = 1800$.

13. Consider an $m \times n$ "brick wall" grid of m rows and n columns, made up of 1×2 bricks with 1×1 bricks at the ends of rows where needed, and so that we always have a 1×1 brick in the lower left corner. The diagram shows the case $m = 5$, $n = 9$. Let $f(m, n)$ denote the number of walks of minimum length (using the grid lines) from A to B, so for example $f(2, 3) = 6$:





Prove that

$$f(m, n) = f(m, n-2) + f(m-1, n-1)$$

for all $m > 1$ and $n > 2$.

14. In Hilbert High School there are an infinite number of lockers, numbered by the natural numbers: $1, 2, 3, \ldots$. Each locker is occupied by exactly one student. The administration decides to rearrange the students so that the lockers are all still occupied by the same set of students, one to a locker, but in some other order. (Some students might not change lockers.) It turns out that the locker numbers of infinitely many students end up higher than before. Show that there are also infinitely many students whose locker numbers are lower than before.

Chen x

15. If $n \geq m \geq k \geq 0$ are integers such that $n + m - k + 1$ is a power of 2, prove that the sum $\binom{n}{k} + \binom{m}{k}$ is even.

chen

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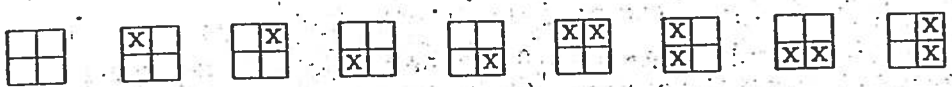
16 Find the number of unordered triples $\{a, b, c\}$ of positive integers such that $\operatorname{lcm}(a, b, c) = 1600$.

17 Find the number of ways to choose cells from a $2 \times n$ "chessboard" so that no two chosen cells are next to each other diagonally (one way is to choose no cells at all). For example, for $n = 2$ the number of ways is 9, namely

Chen



18 Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Find the number of maps f from S to S such that $f^{2080}(x) = x$ for every $x \in S$. (Here the superscript denotes iteration: $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(X))$ for all $n > 1$.)

19 Twelve friends agree to eat out once a week. Each week they will divide themselves into 3 groups of 4 each, and each of these groups will sit together at a separate table. They have agreed to meet until any two of the friends will have sat at least once at the same table at the same time. What is the minimum number of weeks this requires?

Kent

20 Pairs of numbers from the set $\{11, 12, \ldots, n\}$ are adjoined to each of the 45 different (unordered) pairs of numbers from the set $\{1, 2, \ldots, 10\}$, to obtain 45 4-element sets $A_1, A_2, \ldots, A_{45}$. Suppose that $|A_i \cap A_j| \leq 2$ for all $i \neq j$. What is the smallest n possible?

21 A 1000-element set is randomly chosen from $\{1, 2, \ldots, 2000\}$. Let p be the probability that the sum of the chosen numbers is divisible by 5. Is p greater than, smaller than, or equal to 1/5?

22 Find the least integer n so that, for every string of length n composed of the letters $a, b, c, d, e, f, g, h, i, j, k$ (repetitions allowed), one can find a non-empty block of (consecutive) letters in which no letter appears an odd number of times.

23 Imagine you are standing at a point on the edge of a half-plane extending infinitely far north, east, and west (say, on the Canada–USA border near Estevan, Saskatchewan). How many walks of n steps can you make, if each step is 1 metre either north, east, west, or south, and you never step off the half-plane? For example, there are three such walks of length 1 and ten of length 2.

24 A class of fifteen boys and fifteen girls is seated around a round table. Their teacher wishes to pair up the students and hand out fifteen tests—one test to each pair.

As the teacher is preparing to select the pairs and hand out the tests, he wonders to himself: "How many seating arrangements would allow me to match up boy/girl pairs sitting next to each other without having to ask any student to change his or her seat?" Answer the teacher's question. (Two seating arrangements are regarded as being the same if one can be obtained from the other by a rotation.)

Catherine

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Lin **25** . In a soccer tournament, each team plays another team exactly once and receives 3 points for a win, 1 point for a draw, and 0 points for a loss. After the tournament, it is observed that there is a team which has earned both the most total points and won the *fewest* games. Find the smallest number of teams in the tournament for which this is possible.

26 Twenty five boys and twenty five girls sit around a table. Prove that it is always possible to find a person both of whose neighbors are girls.

Lin Kent **27** In how many ways can the integers from 1 to n be ordered subject to the condition that, except for the first integer on the left, every integer differs by 1 from some integer to the left of it? (Putnam, 1965)

Kent **28** Let S be a 1990-element set and let $\mathcal{P}$ be a set of 100-ary sequences $(a_1, a_2, ..., a_{100})$, where a_i's are distinct elements of S. An ordered pair (x, y) of elements of S is said to *appear* in $(a_1, a_2, ..., a_{100})$ if $x = a_i$ and $y = a_j$ for some i, j with $1 \le i < j \le 100$. Assume that every ordered pair (x, y) of elements of S appears in at most one member in $\mathcal{P}$. Show that

$$|\mathcal{P}| \le 800.$$

(Proposed by the Iranian Team at the 31st IMO.)

Lin Dimitrios Frank Jonathan x Kent **29** Show that for $n \in \mathrm{N}$ with $n \ge 2$,

$$\sum_{r=1}^{n} r\sqrt{\binom{n}{r}} < \sqrt{2^{n-1}n^3}.$$

(Spanish MO, 1988)

Kent **30** Let $S = \{1, 2, ..., n\}$. Find the number of subsets A of S satisfying the following conditions:

$A = \{a, a+d, \ldots, a+kd\}$ for some positive integers a, d and k, and

$A \cup \{x\}$ is no longer an A.P. with common difference d for each $x \in S \setminus A$.

(Note that $|A| \ge 2$ and any sequence of two terms is considered as an A.P.) (Chinese Math. Competition, 1991)

31 The set $\{1, 2, \ldots, 3n\}$ is partitioned into three sets A, B, and C with each set containing n numbers. Determine with proof if it is always possible to choose one number out of each set so that one of these numbers is the sum of the other two.

32 A set T is called *even* if it has an even number of elements. Let n be a positive even integer, and let $S_1, S_2, \ldots, S_n$ be even subsets of the set $S = \{1, 2, \ldots, n\}$. Prove that there exist i and j, $1 \le i < j \le n$, such that $S_i \cap S_j$ is even.

33 We call a permutation $(x_1, \ldots, x_{2n})$ of the numbers $1, 2, \ldots, 2n$ pleasant if $|x_i - x_{i+1}| = n$ for at least one $i \in \{1, \ldots, 2n-1\}$. Prove that more than one-half of all permutations are pleasant for each positive integer n (IMO 1989).

Alex Kent x **34** Does the set $\{1, \ldots, 3000\}$ contain a subset A of 2000 elements such that $x \in A \Rightarrow 2x \notin A$ (APMO)?