Linearly Recurrent Sequences

Solutions to Exercises

1. Let x, y, and z be real numbers such that $x+y+z=0$. Prove that
$$\frac{x^2+y^2+z^2}{2}\cdot\frac{x^5+y^5+z^5}{5}=\frac{x^7+y^7+z^7}{7}.$$

Solution. Using the same notation as in the handout,
$$S_6=-AS_4+BS_3=-2A^3+3B^2,$$
$$S_7=-AS_5+BS_4=7A^2B,$$
so
$$\frac{S_2}{2}\cdot\frac{S_5}{5}=A^2B=\frac{S_7}{7}.$$

Note: Problem 2 on the 1982 USAMO generalizes this result as follows: Let $S_r=x^r+y^r+z^r$, with x, y, z real. It is known that if $S_1=0$,
$$\frac{S_{m+n}}{m+n}=\frac{S_m}{m}\cdot\frac{S_n}{n} \qquad (*)$$
for $(m,n)=(2,3)$, $(3,2)$, $(2,5)$, or $(5,2)$. Determine *all* other pairs of integers (m,n) if any, so that $(*)$ holds for all real numbers x, y, z such that $x+y+z=0$.

2. Find ax^5+by^5 if the real numbers a, b, x, and y satisfy the equations
$$ax+by=3,$$
$$ax^2+by^2=7,$$
$$ax^3+by^3=16,$$
$$ax^4+by^4=42.$$

(AIME, 1990)

Solution. Let $S_n=ax^n+by^n$. Then the sequence (S_n) satisfies a linear recurrence, whose characteristic polynomial is
$$(t-x)(t-y)=t^2-(x+y)t+xy.$$
Let $A=x+y$ and $B=xy$, so the characteristic polynomial can also be written as t^2-At+B. Then
$$S_n=AS_{n-1}-BS_{n-2}$$
for all $n\geq 3$. Setting $n=3$ and $n=4$, we obtain the system of equations
$$7A-3B=16,$$
$$16A-7B=42.$$
Solving for A and B, we find $A=-14$ and $B=-38$. Therefore,
$$ax^5+by^5=S_5=AS_4-BS_3=(-14)(42)-(-38)(16)=20.$$

3. Let (x_n) be a sequence such that $x_0 = x_1 = 5$ and

$$x_n = \frac{x_{n-1} + x_{n+1}}{98}$$

for all positive integers n. Prove that $(x_n + 1)/6$ is a perfect square for all n.

Solution. From the given equation,

$$x_{n+1} - 98x_n + x_{n-1} = 0$$

for all $n \geq 1$. At this point, we could solve for x_n, but we take another approach. We compute the first few terms of the sequence:

n	x_n	$(x_n + 1)/6$
0	5	1
1	5	1
2	485	81
3	47525	7921
4	4656965	776161

We see that the few terms of the sequence $(x_n + 1)/6$ are perfect squares, and that their square roots are 1, 1, 9, 89, 881. Since the sequence (x_n) satisfies a linear recurrence, we suspect that these square roots may satisfy a linear recurrence as well.

First, we try a linear recurrence where each term depends on the previous two terms. If the coefficients of these two terms in the linear recurrence are A and B, then

$$A + B = 9,$$
$$9A + B = 89.$$

Solving this system of equations, we find $A = 10$ and $B = -1$. These coefficients produce a linear recurrence that is consistent with the other square roots that we have computed.

Hence, we define the sequence (y_n) by $y_0 = y_1 = 1$ and $y_n = 10y_{n-1} - y_{n-2}$ for all $n \geq 2$. Clearly, y_n is an integer for all $n \geq 0$. The characteristic polynomial for this linear recurrence is $t^2 - 10t + 1$. Let the roots of this quadratic be α and β, so by Vieta's Formulas, $\alpha + \beta = 10$ and $\alpha\beta = 1$. Also,

$$y_n = c_1\alpha^n + c_2\beta^n$$

for some constants c_1 and c_2. Now, let

$$z_n = 6y_n^2 - 1.$$

We want to show that $z_n = x_n$ for all n.

We have that

$$\begin{aligned} z_n &= 6y_n^2 - 1 \\ &= 6(c_1\alpha^n + c_2\beta^n)^2 - 1 \\ &= 6c_1^2\alpha^{2n} + 12c_1c_2\alpha^n\beta^n + 6c_2^2\beta^{2n} - 1 \\ &= 12c_1c_2 - 1 + 6c_1^2(\alpha^2)^n + 6c_2^2(\beta^2)^n. \end{aligned}$$

We see that the sequence (z_n) satisfies a linear recurrence, whose characteristic polynomial is

$$(t-1)(t-\alpha^2)(t-\beta^2) = (t-1)[t^2 - (\alpha^2+\beta^2) + \alpha^2\beta^2].$$

Squaring $\alpha+\beta = 10$, we get $\alpha^2 + 2\alpha\beta + \beta^2 = 100$, so $\alpha^2+\beta^2 = 100 - 2\alpha\beta = 98$. Squaring $\alpha\beta = 1$, we get $\alpha^2\beta^2 = 1$. Thus, the characteristic polynomial is

$$(t-1)(t^2 - 98t + 1) = t^3 - 99t^2 + 99t - 1,$$

which means

$$z_n - 99z_{n-1} + 99z_{n-2} + z_{n-3} = 0$$

for all $n \ge 3$.

The given sequence (x_n) satisfies $x_n - 98x_{n-1} + x_{n-2} = 0$ for all $n \ge 2$. The characteristic polynomial for this linear recurrence is $t^2 - 98t + 1$. We found that the roots of this quadratic are α^2 and β^2, so

$$x_n = d_1\alpha^{2n} + d_2\beta^{2n}$$

for some constants d_1 and d_2. We can also write

$$x_n = 0 \cdot 1^n + d_1\alpha^{2n} + d_2\beta^{2n}.$$

Hence, the sequence (x_n) also satisfies the linear recurrence whose characteristic polynomial is

$$(t-1)(t^2 - 98t + 1) = t^3 - 99t^2 + 99t - 1.$$

(More generally, if a sequence (x_n) satisfies a linear recurrence whose characteristic polynomial is $p(x)$, then the sequence (x_n) satisfies the linear recurrence whose characteristic polynomial is any multiple of $p(x)$.) Therefore,

$$x_n - 99x_{n-1} + 99x_{n-2} + x_{n-3} = 0$$

for all $n \ge 3$. Furthermore, $x_0 = z_0 = 5$, $x_1 = z_1 = 5$, and $x_2 = z_2 = 485$. We conclude that $x_n = z_n$ for all n, which means $(x_n+1)/6 = (z_n+1)/6 = y_n^2$ is a perfect square for all n.

4. Let a, b, and c be the roots of the equation $x^3 - x^2 - x - 1 = 0$. Show that a, b, and c are distinct, and that

$$\frac{a^{1982} - b^{1982}}{a-b} + \frac{b^{1982} - c^{1982}}{b-c} + \frac{c^{1982} - a^{1982}}{c-a}$$

is an integer. (Canada, 1982)

Solution. By Vieta's Formulas, $a+b+c = 1$, $ab+ac+bc = -1$, and $abc = 1$.

Suppose that two of the roots are equal. Without loss of generality, assume that $b = c$. Then from the equations above, $a + 2b = 1$, $2ab + b^2 = -1$, and $ab^2 = 1$. From the first equation, $a = 1 - 2b$. Substituting this expression for a into the equation $2ab + b^2 = -1$, we get $2(1-2b)b + b^2 = -1$, which simplifies as

$$3b^2 - 2b - 1 = (b-1)(3b+1) = 0,$$

so $b = 1$ or $b = -1/3$. But neither of these values satisfy the cubic $x^3 - x^2 - x - 1 = 0$, contradiction. Therefore, the roots a, b, and c are distinct.

Now, let

$$\begin{aligned} S_n &= \frac{a^n - b^n}{a - b} + \frac{b^n - c^n}{b - c} + \frac{c^n - a^n}{c - a} \\ &= \left(\frac{1}{a - b} - \frac{1}{c - a} \right) a^n + \left(\frac{1}{b - c} - \frac{1}{a - b} \right) b^n + \left(\frac{1}{c - a} - \frac{1}{b - c} \right) c^n. \end{aligned}$$

Then the sequence (S_n) satisfies a linear recurrence, whose characteristic polynomial is

$$(x - a)(x - b)(x - c) = x^3 - x^2 - x - 1.$$

Hence,

$$S_n - S_{n-1} - S_{n-2} - S_{n-3} = 0$$

for all $n \geq 3$.

Furthermore, the first few terms of the sequence are

$$\begin{aligned} S_0 &= \frac{a^0 - b^0}{a - b} + \frac{b^0 - c^0}{b - c} + \frac{c^0 - a^0}{c - a} = 0, \\ S_1 &= \frac{a - b}{a - b} + \frac{b - c}{b - c} + \frac{c - a}{c - a} = 3, \\ S_2 &= \frac{a^2 - b^2}{a - b} + \frac{b^2 - c^2}{b - c} + \frac{c^2 - a^2}{c - a} = (a + b) + (b + c) + (c + a) = 2a + 2b + 2c = 2. \end{aligned}$$

It follows that S_n is an integer for all $n \geq 0$. In particular,

$$S_{1982} = \frac{a^{1982} - b^{1982}}{a - b} + \frac{b^{1982} - c^{1982}}{b - c} + \frac{c^{1982} - a^{1982}}{c - a}$$

is an integer.

5. For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Putnam, 1980)

   Solution. The first few terms of the sequence are

$$\begin{aligned} u_1 &= 2u_0 - 0 = 2a, \\ u_2 &= 2u_1 - 1 = 4a - 1, \\ u_3 &= 2u_2 - 4 = 8a - 6. \end{aligned}$$

   From the given recurrence,

$$u_{n+1} - 2u_n = n^2$$

for all $n \geq 0$. Shifting the index n by 1, we get

$$u_{n+1} - 2u_n = n^2,$$
$$u_n - 2u_{n-1} = (n-1)^2.$$

Subtracting these equations, we get

$$u_{n+1} - 3u_n + 2u_{n-1} = n^2 - (n-1)^2 = 2n - 1$$

for all $n \geq 1$.

Shifting the index n by 1 again, we get

$$u_{n+1} - 3u_n + 2u_{n-1} = 2n - 1,$$
$$u_n - 3u_{n-1} + 2u_{n-2} = 2(n-1) - 1.$$

Subtracting these equations, we get

$$u_{n+1} - 4u_n + 5u_{n-1} - 2u_{n-2} = 2n - 1 - [2(n-1) - 1] = 2$$

for all $n \geq 2$.

Shifting the index n by 1 again, we get

$$u_{n+1} - 4u_n + 5u_{n-1} - 2u_{n-2} = 2,$$
$$u_n - 4u_{n-1} + 5u_{n-2} - 2u_{n-3} = 2.$$

Subtracting these equations, we get

$$u_{n+1} - 5u_n + 9u_{n-1} - 7u_{n-2} + 2u_{n-3} = 0$$

for all $n \geq 3$.

Hence, the sequence (u_n) satisfies a linear recurrence, whose characteristic polynomial is

$$x^4 - 5x^3 + 9x^2 - 7x + 2 = (x-1)^3(x-2).$$

Therefore,

$$u_n = c_1 + c_2 n + c_3 n^2 + c_4 2^n$$

for some constants c_1, c_2, c_3, and c_4. Setting $n = 0$, 1, 2, and 3, we obtain the system of equations

$$c_1 + c_4 = a,$$
$$c_1 + c_2 + c_3 + 2c_4 = 2a,$$
$$c_1 + 2c_2 + 4c_3 + 4c_4 = 4a - 1,$$
$$c_1 + 3c_2 + 9c_3 + 8c_4 = 8a - 6.$$

Solving this system of equations, we find $c_1 = 3$, $c_2 = 2$, $c_3 = 1$, and $c_4 = a - 3$, so

$$u_n = 3 + 2n + n^2 + (a-3)2^n$$

for all $n \geq 0$. In this expression, for large n, the term 2^n grows the fastest. (In other words, 2^n dominates 1, n, and n^2 for large n.) The coefficient of 2^n is $a - 3$, so $u_n > 0$ for all $n \geq 0$ if and only if $a \geq 3$.

6. An integer sequence is defined by $a_0 = 0$, $a_1 = 1$, and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$. Prove that 2^k divides a_n if and only if 2^k divides n. (IMO Short List, 1988)

Solution. The characteristic polynomial of the sequence (a_n) is $x^2 - 2x - 1$. By the quadratic formula, the roots of this quadratic are $1 \pm \sqrt{2}$, so let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Then

$$a_n = c_1\alpha^n + c_2\beta_n$$

for some constants c_1 and c_2. Setting $n = 0$ and $n = 1$, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 0, \\ \alpha c_1 + \beta c_2 &= 1. \end{aligned}$$

Solving this system of equations, we find $c_1 = \frac{1}{\alpha-\beta} = \frac{1}{2\sqrt{2}}$ and $c_2 = -\frac{1}{\alpha-\beta} = -\frac{1}{2\sqrt{2}}$, so

$$a_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$$

for all $n \geq 0$.

Since $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$,

$$a_n \equiv a_{n-2} \pmod{2}$$

for all $n \geq 2$. Since $a_1 = 1$, a_n is odd for all odd n.

Now, let

$$b_n = \frac{a_{2n}}{2a_n} = \frac{(\alpha^{2n} - \beta^{2n})/(2\sqrt{2})}{2(\alpha^n - \beta^n)/(2\sqrt{2})} = \frac{\alpha^{2n} - \beta^{2n}}{2(\alpha^n - \beta^n)} = \frac{\alpha^n + \beta^n}{2}$$

for $n \geq 1$. Then the sequence (b_n) satisfies a linear recurrence, whose characteristic polynomial is $(x - \alpha)(x - \beta) = x^2 - 2x - 1$, so

$$b_n - 2b_{n-1} - b_{n-2} = 0$$

for all $n \geq 3$. Also, $b_1 = a_2/(2a_1) = 1$ and $b_2 = a_4/(2a_2) = 3$. It follows that b_n is an integer for all $n \geq 1$. Furthermore,

$$b_n = 2b_{n-1} + b_{n-2} \equiv b_{n-2} \pmod{2}$$

for all $n \geq 2$. Since $b_1 = 1$ and $b_2 = 3$ are odd, b_n is odd for all $n \geq 1$.

Given a positive integer n, we can write n uniquely in the form $n = 2^e \cdot t$, where e is a nonnegative integer and t is an odd positive integer. Then

$$\begin{aligned} a_n &= a_{2^e \cdot t} \\ &= 2^e \cdot \frac{a_{2^e \cdot t}}{2a_{2^{e-1} \cdot t}} \cdot \frac{a_{2^{e-1} \cdot t}}{2a_{2^{e-2} \cdot t}} \cdots \frac{a_{2t}}{2a_t} \cdot a_t \\ &= 2^e b_{2^{e-1} \cdot t} b_{2^{e-2} \cdot t} \cdots b_t a_t. \end{aligned}$$

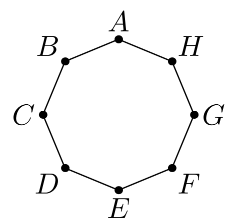
All the factors b_i are odd, and a_t is odd since t is odd, so a_n has exactly e factors of 2. In other words, n and a_n always have the same number of factors of 2. It follows that 2^k divides a_n if and only if 2^k divides n.

7. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$ and

$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1})$$

for all $n = 1, 2, 3, \ldots$, where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$. (IMO, 1979)

Solution. Label the vertices of the octagon as shown.



Clearly, the frog can be only at one of the vertices A, C, E, or G after an even number of jumps, so $a_{2n-1} = 0$ for all $n \geq 1$.

For all $n \geq 1$, let b_n be the number of distinct paths of exactly n jumps starting at A and ending at C, and let c_n be the number of distinct paths of exactly n jumps starting at A and ending at A. By symmetry, b_n is also the number of distinct paths of exactly n jumps starting at A and ending at G.

If the frog is at vertex E after n jumps, then it must have been either at vertex C or G two jumps before (but not at vertex E, because once the frog reaches E, it stays there), so

$$a_n = 2b_{n-2}$$

for all $n \geq 2$.

If the frog is at vertex C after n jumps, then it must have been either at vertex A or C two jumps before (but not at vertex E), and there are two ways to go from C back to C after two jumps, so

$$b_n = 2b_{n-2} + c_{n-2}$$

for all $n \geq 2$.

Finally, if the frog is at vertex A after n jumps, then it must have been either at vertex A, C, or G two jumps before, and there are two ways to go from A back to A after two jumps, so

$$c_n = 2b_{n-2} + 2c_{n-2}$$

for all $n \geq 2$. Thus, we have the linear recurrences

$$a_n = 2b_{n-2}, \tag{1}$$
$$b_n = 2b_{n-2} + c_{n-2}, \tag{2}$$
$$c_n = 2b_{n-2} + 2c_{n-2} \tag{3}$$

for all $n \geq 2$. The first few values of a_n, b_n, and c_n are as follows:

n	a_n	b_n	c_n
0	0	0	1
2	0	1	2
4	2	4	6
6	8	14	20
8	28	48	68

From equation (2),

$$c_{n-2} = b_n - 2b_{n-2}$$

for all $n \geq 2$. Then $c_n = b_{n+2} - 2b_n$ for all $n \geq 0$. Substituting these expressions into equation (3), we get

$$b_{n+2} - 2b_n = 2b_{n-2} + 2(b_n - 2b_{n-2}),$$

or

$$b_{n+2} - 4b_n + 2b_{n-2} = 0$$

for all $n \geq 2$. Then

$$2b_{n+2} - 8b_n + 4b_{n-2} = 0$$

for all $n \geq 2$, so from equation (1),

$$a_{n+4} - 4a_{n+2} + 2a_n = 0$$

for all $n \geq 2$.

Let $d_n = a_{n/2}$ for all even integers $n \geq 1$, so

$$d_{n+2} - 4d_{n+1} + 2d_n = 0$$

for all $n \geq 1$. Hence, the sequence (d_n) satisfies a linear recurrence, whose characteristic polynomial is $t^2 - 4t + 2$. By the quadratic formula, the roots of this quadratic are $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$. Then

$$d_n = Ax^n + By^n$$

for some constants A and B. Setting $n = 1$ and $n = 2$, we obtain the system of equations

$$\begin{aligned} xA + yB &= 0, \\ x^2A + y^2B &= 2. \end{aligned}$$

Solving this system of equations, we find

$$\begin{aligned} A &= \frac{2}{x(x-y)} = \frac{2}{2\sqrt{2}(2+\sqrt{2})} = \frac{-1+\sqrt{2}}{2}, \\ B &= \frac{2}{y(y-x)} = -\frac{2}{2\sqrt{2}(2-\sqrt{2})} = \frac{-1-\sqrt{2}}{2}, \end{aligned}$$

so

$$\begin{aligned}
a_{2n} &= d_n \\
&= \frac{-1+\sqrt{2}}{2}(2+\sqrt{2})^n - \frac{1+\sqrt{2}}{2}(2-\sqrt{2})^n \\
&= \frac{(-1+\sqrt{2})(2+\sqrt{2})}{2}(2+\sqrt{2})^{n-1} - \frac{(1+\sqrt{2})(2-\sqrt{2})}{2}(2-\sqrt{2})^{n-1} \\
&= \frac{\sqrt{2}}{2}(2+\sqrt{2})^{n-1} - \frac{\sqrt{2}}{2}(2-\sqrt{2})^{n-1} \\
&= \frac{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}}{\sqrt{2}}
\end{aligned}$$

for all $n \geq 1$.

8. A sequence (a_n) is defined by $a_0 = a_1 = 0$, $a_2 = 1$, and $a_{n+3} = a_{n+1} + 1998a_n$ for all $n \geq 0$. Prove that $a_{2n-1} = 2a_n a_{n+1} + 1998a_{n-1}^2$ for every positive integer n. (Komal)

Solution 1. More generally, let

$$a_{n+3} = a_{n+1} + ka_n$$

for all $n \geq 0$. We claim that

$$a_{2n-1} = 2a_n a_{n+1} + ka_{n-1}^2$$

for all $n \geq 1$. The sequence (a_n) satisfies a linear recurrence, whose characteristic polynomial is $x^3 - x - k$. Let α, β, and γ be the roots of this cubic, so

$$a_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n$$

for some constants c_1, c_2, and c_3. Let

$$\begin{aligned}
b_n &= a_{2n-1} - 2a_n a_{n+1} - ka_{n-1}^2 \\
&= c_1\alpha^{2n-1} + c_2\beta^{2n-1} + c_3\gamma^{2n-1} - 2(c_1\alpha^n + c_2\beta^n + c_3\gamma^n)(c_1\alpha^{n+1} + c_2\beta^{n+1} + c_3\gamma^{n+1}) \\
&\quad - k(c_1\alpha^{n-1} + c_2\beta^{n-1} + c_3\gamma^{n-1})^2.
\end{aligned}$$

Expanding this expression, we find that

$$b_n = d_1\alpha^{2n} + d_2\beta^{2n} + d_3\gamma^{2n} + d_4\alpha^n\beta^n + d_5\alpha^n\gamma^n + d_6\beta^n\gamma^n$$

for some constants d_1, d_2, d_3, d_4, d_5, and d_6. Hence, the sequence (b_n) satisfies a linear recurrence, whose characteristic polynomial is

$$(x-\alpha^2)(x-\beta^2)(x-\gamma^2)(x-\alpha\beta)(x-\alpha\gamma)(x-\beta\gamma).$$

This is a sixth degree polynomial, so the sequence (b_n) satisfies a linear recurrence, where each term depends on the previous six terms.

We compute the first few terms of the sequence (a_n):

n	a_n
0	0
1	0
2	1
3	0
4	1
5	k
6	1
7	$2k$
8	k^2+1
9	$3k$
10	$3k^2+1$
11	k^3+4k

We can then compute the first few terms of the sequence (b_n):

$$\begin{aligned}
b_1 &= a_1 - 2a_1a_2 - ka_0^2 = 0, \\
b_2 &= a_3 - 2a_2a_3 - ka_1^2 = 0, \\
b_3 &= a_5 - 2a_3a_4 - ka_2^2 = k - k = 0, \\
b_4 &= a_7 - 2a_4a_5 - ka_3^2 = 2k - 2k = 0, \\
b_5 &= a_9 - 2a_5a_6 - ka_4^2 = 3k - 2k - k = 0, \\
b_6 &= a_{11} - 2a_6a_7 - ka_5^2 = k^3 + 4k - 4k - k^3 = 0.
\end{aligned}$$

The first six terms of the sequence (b_n) are all 0, so $b_n = 0$ for all $n \ge 1$. Hence, $a_{2n-1} = 2a_na_{n+1} + ka_{n-1}^2$ for all $n \ge 1$.

Solution 2. Consider an infinite sequence of cities C_0, C_1, C_2, $\ldots$, where there is one road from C_i to C_{i+2}, and 1998 roads from C_i to C_{i+3}, for all $i \ge 0$. Let b_n be the number of possible paths from C_0 to C_n. Then $b_0 = 1$, $b_1 = 0$, and $b_2 = 1$.

Let $n \ge 0$. The only way to reach city C_{n+3} is to either go through city C_{n+1} or city C_n. There is one way from C_{n+1} to C_{n+3}, and 1998 ways from C_n to C_{n+3}, so

$$b_{n+3} = b_{n+2} + 1998b_n.$$

Furthermore, $b_0 = a_2 = 1$, $b_1 = a_3 = 0$, and $b_2 = a_4 = 1$. It follows that $b_n = a_{n+2}$ for all $n \ge 0$. Hence, the problem has become showing that

$$b_{2n-3} = 2b_{n-2}b_{n-1} + 1998b_{n-3}^2$$

for all $n \ge 3$. (It is easy to verify that $a_{2n-1} = 2a_na_{n+1} + 1998a_{n-1}^2$ for $n = 1$ and 2.)

Every path from C_0 to C_{2n-3} satisfies exactly one of the following conditions:

(a) The path passes through C_{n-1}.

(b) The path passes through C_{n-2}.

(c) The path passes through neither C_{n-1} nor C_{n-2}.

In case (a), there are b_{n-1} paths from C_0 to C_{n-1}, and b_{n-2} paths from C_{n-1} to C_{2n-3}, so there are $b_{n-2}b_{n-1}$ such paths.

In case (b), there are b_{n-2} paths from C_0 to C_{n-2}, and b_{n-1} paths from C_{n-2} to C_{2n-3}, so there are again $b_{n-2}b_{n-1}$ such paths.

In case (c), the path must pass through C_{n-3}, then go to C_n. There are b_{n-3} paths from C_0 to C_{n-3}, 1998 paths from C_{n-3} to C_n, and b_{n-3} paths from C_n to C_{2n-3}, so there are $1998b_{n-3}^2$ such paths.

Hence, the total number of paths from C_0 to C_{2n-3} is equal to

$$b_{2n-3} = 2b_{n-1}b_{n-2} + 1998b_{n-3}^2.$$