GENERATING FUNCTIONS

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1. Introduction

In this session you will develop the basic tools of "generating functions". This is a powerful tool for solving combinatorial problems, particularly problems of a recursive nature.

The basic idea is to define (i.e. *generate*) a polynomial (i.e. *function*) whose coefficients are the answers to combinatorial questions you might have. We will then use tools from algebra, number theory, calculus and complex numbers to help us answer our combinatorics question.

Theorem 1.1 (**Main Idea**). *Rewrite a combinatorics problem in the language of polynomials. Use your polynomial tools to manipulate it. Translate your manipulated polynomial back into the language of combinatorics to solve your original problem.*

2. Example 0 - Coin Flipping

The next question is easy, but illustrates the main ideas. Pay attention to the ideas, not the actual problem.

Question 2.1. *You flip 3 coins. How many ways are there to get n total heads?*

Let's brute force this question and just write them down:

$$\text{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH}$$

We can keep track of this information in this table:

n	0	1	2	3	4	5
# of ways	1	3	3	1	0	0

Tables are boring though (which should really be a theorem). Let's code the exact same information in a polynomial:

n	x^0	x^1	x^2	x^3	x^4	x^5
coefficient	1	3	3	1	0	0

Oops! That's still a table. Silly me. Let's put that into a polynomial:

$$\begin{aligned} p(x) &= 1x^0 + 3x^1 + 3x^2 + 1x^3 + 0x^4 + 0x^5 + \dots \\ &= 1 + 3x + 3x^2 + x^3 \end{aligned}$$

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Here the generating function

$p(x) = (\text{ways to get 0 heads})x^0 + (\text{ways to get 1 heads})x^1 + (\text{ways to get 2 heads})x^2 + \dots$.

Let's take that polynomial and do some polynomial-stuff with it.

(1) **Plug in some numbers**. There are many different numbers we can plug in, but $p(1)$ gives something interesting. It gives the sum of the coefficients, which *in the language of combinatorics*, is "the total number of ways to flip 3 coins".
(2) **Factor it**. In this case we get $p(x) = (1+x)(1+x)(1+x)$. This is interesting. If we want to figure out what the coefficient in front of x^2 is in $p(x)$, we see that each factor contributes either 1 OR x. Since we want exactly two contributions of x, we see that there are three ways to do this, (just figure out which factor is contributing just its "1").

This might seem obvious, and weird to contort ourselves with generating functions to answer such an easy question, but it will be useful for more complicated combinatorics questions.

3. Example 1 - The Sicherman Dice

Have you ever played Settlers of Catan? What about Monopoly? In these games you roll two six-sided dice, then *add up their values* to determine how much your piece moves/whether you get out of jail/whether you get to move the thief. The only thing you care about is the *sum* of the values on the two dice.

Question 3.1. *There is an alternate labeling on a pair of six-sided dice whose sums have the same probability distribution as the regular dice. i.e. You can play Settlers of Catan with these weird dice, and it doesn't change the game. Dice labeled this way are called the Sicherman Dice. Prove that the regular labeling and the Sicherman labeling are the only two labelings with this property.*

Note that a valid labeling must use only positive integers.

We're going to go through this question in parts.

(1) **Exercise 3.1**. Find a generating function (i.e. a polynomial) that represents all of the combinatorial information about the sums of values of the dice. Call this $p(x)$. Write it down explicitly.
(2) **Exercise 3.2**. Find the "square root" of $p(x)$. That is, find a polynomial $d(x)$ such that $p(x) = d(x)d(x)$. In this problem $d(x)$ is also a generating function. What does it represent?
(3) **Exercise 3.3**. You are a hero. Factor $d(x) = f_1(x) \dots f_N(x)$ into its irreducible factors.
(4) **Exercise 3.4**. Keep track of $f_1(1), f_2(1), \dots, f_N(1)$; they will be useful.
(5) **Exercise 3.5**. In terms of the factors of $p(x)$, what would another labeling of two six-sided dice represent? i.e. In the language of factors of $p(x)$, what is the question asking you to find?
(6) **Exercise 3.6**. What does the condition of "the dice are six-sided" correspond to?
(7) **Exercise 3.7**. What other conditions are placed on the labels?
(8) **Exercise 3.8**. Find the Sicherman labeling. See that it is the only option other than the regular labeling that satisfies the constraints of the question.

Exercise You can't actually play Monopoly with the Sicherman dice. Why?

4. Partial Fractions Review

We're going to move on to some more powerful applications of generating functions, but to do that we will need to remember some techniques from algebra: partial fractions and power series. **Skip these sections if you feel comfortable with these two techniques**, or just use this as a reference if you get stuck in the later examples. Okay, now's your last chance to get off... I'm going to move on now... Ready?

Partial fractions is the method of simplifying rational functions like this:

$$\frac{1}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x}$$

The idea is (by multiplying through), solve a related equality and use the fact that **two polynomials are equal if and only if all of their coefficients are same** (on the appropriate powers of x).

$$0x + 1 = A(1-x) + B(1+x) = (B-A)x + (A+B)$$

In this case we see that $A = B = \frac{1}{2}$. (If you are really bad, and the idea of dividing by 0 doesn't make you sick, you can plug $x = -1$ and $x = 1$ into the above equation to see what A and B should be.)

4.1. Irreducible Quadratic Factors.

This method also works for quadratic factors, but instead of a constant, you pick up a linear term. For example, let's say we want to simplify $\frac{x^2+x+1}{(1+x^2)(x)}$. Then we write:

$$\frac{x^2+x+1}{(1+x^2)(x)} = \frac{Ax+B}{1+x^2} + \frac{C}{x}$$

and solve

$$x^2 + x - 1 = (Ax+B)x + C(1+x^2) = (A+C)x^2 + (B)x + C$$

So we see that $C = -1, B = 1$ and $A + C = 1$, so $A = 2$.

4.2. Repeated Factors.

In the case of repeated factors we need to do a little more:

$$\text{Every factor } \frac{1}{(x-a)^k} \text{ contributes } \frac{1}{x-a} + \frac{1}{(x-a)^2} + \ldots + \frac{1}{(x-a)^k}.$$

Something similar happens for repeated quadratic factors:

$$\text{Every factor } \frac{1}{(x^2+bx+c)^k} \text{ contributes } \frac{1}{x^2+bx+c} + \frac{1}{(x^2+bx+c)^2} + \ldots + \frac{1}{(x^2+bx+c)^k}.$$

4.3. Repeated Factors Example.

Here's an example for you to look at:

$$\begin{aligned}\frac{x+1}{(1+x^2)^2(x-4)^3} &= \frac{Ax+B}{1+x^2} + \frac{Cx+D}{(1+x^2)^2} \\ &\quad + \frac{E}{x-4} + \frac{F}{(x-4)^2} + \frac{G}{(x-4)^3}\end{aligned}$$

and solve

$$\begin{aligned}x + 1 &= (Ax+B)(1+x^2)(x-4)^3 + (Cx+D)(x-4)^3 \\ &\quad + E(1+x^2)^2(x-4)^2 + F(1+x^2)^2(x-4) + G(1+x^2)^2\end{aligned}$$

At this stage you would group like terms, to set up a list of 7 equations with 7 unknowns. (Do you see why the *repeated* factors required special attention?)

You aren't likely to see these sorts of beasts in the wild, so don't worry about trying to solve it. The point is just to give you the idea of how you *would* solve something like this by hand (if civilization collapsed). Seriously, only a maniac would try to solve this. A maniac! You're not going to solve it are you? Don't do it. I didn't even check to make sure the numbers are nice. You might end up with a $\frac{981}{\sqrt{76}}$. Actually, maybe you can't get square roots here? Huh. (**Exercise**: Show you can't get (irrational) square roots showing up as the constants.)

5. Power Series Review

Next we will discuss power series. This will be a very pratical discussion designed to get you to the point where we can manipulate them.

Intuition: A power series is just a (possibly infinite) polynomial in some variable x. The powers of x are just placeholders, and the coefficients are real numbers which for us will usually answer some combinatorial problem.

For example:

- $g(x) = 1 + x + x^2 + x^3 + x^4 + \ldots$ is a power series where all the coefficients are 1. This is called the **Geometric series**.
- $F(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \ldots$ is a power series where the nth coefficient is (**Exercise**).
- $p(x) = 1 + 0x + x^2 + 0x^3 + 0x^4 + \ldots$ is a (finite) power series because only finitely many of the coefficients are non-zero.

5.1. Notation.

Because it's impossible to write infinitely many things down, (**Exercise**: Try to write down infinitely many things), we use **Sigma Notation** to denote these power series. We write a general power series as $\sum_{n\geq 0} a_n x^n$, and it is implicit that we only let n range over (non-negative) integers.

For example:

- $g(x) = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n\geq 0} x^n$.
- $F(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \ldots = \sum_{n\geq 0} F_n x^n$.
- $p(x) = 1 + 0x + x^2 + 0x^3 + 0x^4 + \ldots = 1 + x^2 + \sum_{n\geq 3} 0x^n$.

Recall the following identity that you know for the finite geometric series:

$$1 + x + \ldots + x^N = \frac{1 - x^{N+1}}{1 - x}$$

The remarkable thing is that there is an infinite version of this. If x is small (say $-1 < x < 1$), as N gets large it *crushes* x, and x^{N+1} is basically nothing. So we get the following equality which is "valid" for small values of x.

$$\sum_{n\geq 0} x^n = \frac{1}{1 - x}$$

Question 5.1. *Are you allowed to plug numbers into power series?*

Yes, and no. Definitely for finite power series we can plug in values. However, what does an infinite amount of additions look like? It's weird, and we aren't going to talk about it. **Exercise**. Plug $x = 2$ into the infinite geometric series equality and behold the weirdness.

Question 5.2. *Should I be worried about this?*

No, not really. You'll see that we are only going to be doing *formal*, algebraic manipulations of this sort, and we're never going to claim something about infinite additions. Remember, the power series (**Exercise**: What's the plural of series? Kevin would know if he didn't fall asleep in his Latin class. *si dormiveris in schola linguae latinae, vinceris*) are just placeholders for combinatorial information; we're going to ask questions like "What's the nth Fibonacci number?" or "How many ways are there to stack 2015 coins in a 'fountain' pattern?". These questions are not about infinity, so don't worry about the infinity stuff.

5.2. **The Important Power Series.** There are many power series worth thinking about, but we will only mention a couple that have nice closed-forms.

$$\sum_{n=0}^{N} x^n = 1 + x + x^2 + \ldots + x^N = \frac{1 - x^{N+1}}{1 - x} \tag{1}$$

$$\sum_{k=0}^{N} \binom{N}{k} x^k = 1 + \binom{N}{1} x + \ldots + \binom{N}{N} x^N = (1+x)^N \tag{2}$$

$$\sum_{n\geq 0} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x} \tag{3}$$

Question 5.3. *What sort of algebraic manipulations can I do to power series? Can I add them together?*

The short answer is that adding power series and multiplying by a constant works the way you think it should. Your intuition from polynomials should be correct. For example:

$$\sum_{n\geq 0} 2x^n = 2\sum_{n\geq 0} x^n = \left(\sum_{n\geq 0} x^n\right) + \left(\sum_{n\geq 0} x^n\right) = \frac{2}{1-x}$$

and addition of two power series is done by adding up the coefficients in front of like terms, e.g.

$$(2+3x+4x^2+5x^3+\ldots)+(1+3x+5x^2+7x^3+\ldots) = 3+6x+9x^2+12x^3+\ldots = \frac{3}{1-x}$$

5.3. **New Power Series from Old.** We can get new power series from old power series. In particular, we can manipulate the geometric series as follows:

$$\frac{1}{1-2x} = \sum_{n\geq 0} (2x)^n = 1 + 2x + 4x^2 + 8x^3 + \ldots$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n\geq 0} (-x)^n = 1 - x + x^2 - x^3 + \ldots$$

This versatile technique allows us to find power series like:

$$\begin{aligned}\frac{1}{1-x^2} &= \frac{1}{(1-x)(1+x)}\\ &= \frac{1}{2}\frac{1}{1-x}+\frac{1}{2}\frac{1}{1+x}\\ &= \frac{1}{2}\sum_{n\geq 0}x^n+\frac{1}{2}\sum_{n\geq 0}(-x)^n\\ &= 1+x^2+x^4+x^6+\dots\end{aligned}$$

Exercise Find a faster/cleaner way to find a power series for $\frac{1}{1-x^2}$.

5.4. **Power Series to Gaze upon.** The following ones are nice to look at, but we won't be using them.

$$\begin{aligned}\sum_{n\geq 0}\frac{(-1)^n}{n+1}x^{n+1} &= x-\frac{x^2}{2}+\frac{x^3}{3}+\dots=\ln(1+x)\\ \sum_{n\geq 0}\frac{1}{n!}x^n &= 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots=e^x\\ \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)!}x^{2n+1} &= x-\frac{x^3}{6}+\frac{x^5}{120}+\dots=\sin(x)\\ \sum_{n\geq 0}\frac{(-1)^n}{(2n)!}x^{2n} &= 1-\frac{x^2}{2}+\frac{x^4}{24}+\dots=\cos(x)\\ \sum_{n\geq 0}\frac{(-1)^n}{2n+1}x^{2n+1} &= x-\frac{x^3}{3}+\frac{x^5}{5}+\dots=\arctan(x)\end{aligned}$$

This is more than just something cool to look at; this is how your calculator computes $\sin(x)$.

6. Example 2 - First Recursion Example

Now it's time to see the power of generating functions. Well, sort of. Generating functions can be a bit complicated, so let's ease into it. Let's start with a nice example that is also the first example in Wilf's "generatingfunctionology".

Question 6.1. *Consider the following sequence: $a_{n+1}=2a_n+1$ for $n\geq 0$ and $a_0=0$. Using generating functions, can you find a closed form for this sequence?*

Let me first point out that you can easily guess what this sequence is, then prove it by induction. That's not the point. We want to see how the machinery of generating functions can help answer some recursion questions.

Question 6.2. *What generating function should we use?*

Our generating function should contain the relevant information in the coefficients, and since we are looking for the a_n, we'll make those our coefficients; simple. So let

$$P(x)=\sum_{n\geq 0}a_nx^n$$

Now the general strategy will be to use the recursion to form two different (but necessarily equal!) power series. Let's do it!

Take the recursion $a_{n+1} = 2a_n + 1$, multiply through by x^n,

$$a_{n+1}x^n = 2a_nx^n + x^n$$

then sum both sides to get

$$\sum_{n\geq 0} a_{n+1}x^n = \sum_{n\geq 0} 2a_nx^n + \sum_{n\geq 0} x^n$$

We can simplify the RHS as $2P(x) + \frac{1}{1-x}$. Now to deal with the LHS:

$$\begin{aligned}\sum_{n\geq 0} a_{n+1}x^n &= a_1 + a_2x + a_3x^2 + \dots \\ &= \frac{a_1x + a_2x^2 + a_3x^3 + \dots}{x} \\ &= \frac{(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) - a_0}{x} \\ &= \frac{P(x)}{x}\end{aligned}$$

where we used the fact that $a_0 = 0$. Combining the two we get:

$$\frac{P(x)}{x} = 2P(x) + \frac{1}{1-x}$$

Solving for $P(x)$ we get

$$P(x) = \frac{x}{(1-x)(1-2x)}$$

Exercise. Write this fraction as a power series, then read off its coefficients.

7. Formal Derivatives

At this stage you are ready to start attempting the questions in the next section. You might need the following nice power series:

$$\sum_{n\geq 1} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \dots = \frac{1}{(1-x)^2}$$

How are these related? Well the terms of this (nx^{n-1}) are the *derivatives* of the terms of the geometric series (x^n), and $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$.

If you know a little bit of calculus, then you can get a whole bunch of new power series. This shows the power of translating a question into other language; now when we're solving combinatorics problems, we get to use calculus!

8. Other questions

These questions are roughly in order of the way the material was presented.

(1) Find the smallest n such that two regular dice labeled $1, 2, \dots, n$ admit alternative labelings, but don't affect the probability distribution of their sums.
(2) Find the coefficient of x^{21} in $(x^2 + x^3 + x^4 + x^5 + x^6)^8$.
(3) Suppose that you want to play Settlers of Catan, but since you're really cool you want to do it with as many dice as possible (but you don't care about how many sides they have). Your friends don't like negative numbers, but they are okay with zeros (so long as each die has a nonzero label on it). What is the most number of dice can you use without changing the probability distribution of their sum?
(4) Using generating functions, find a closed form for a_n, where $a_{n+1} = 2a_n + n$ for $n \geq 0$ and $a_0 = 1$.
(5) Consider the k-variable generating function $F_k(x_1, \dots, x_k) = (1 + x_1)(1 + x_2)\dots(1 + x_k)$. Expand F_3 and compare it with $F_3(1, 1, 1)$. What have you just proved?
(6) How many subsets of $\{1, 2, \dots, 100\}$ are there that do not contain adjacent numbers? (This question is tricky. There is a hint at the end.)
(7) Use generating functions to find a closed form for the Fibonacci numbers. Recall $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$ and $F_0 = 0$, $F_1 = 1$.
(8) Let $f(n)$ be defined for all $n \geq 1$ such that:
   - $f(1) = 1$;
   - $f(2n) = f(n)$;
   - $f(2n + 1) = f(2n)$.

   and define its generating function as $F(x) = \sum_{n\geq 1} f(n)x^n$. Show that $F(x) = (1 + x + x^2)F(x^2)$. Use this to write $F(x)$ as an infinite product.
(9) Consider all possible sums of the form

$$x_1 + x_2 + 2x_3 + 5x_4 + 10x_5 + 10x_6 + 20x_7 + 50x_8$$

   where $x_i = 0, 1$. Let $C_n :=$ the number of ways of getting n as this type of sum, and write the generating function $C(x) = \sum_{n=0}^{99} C_n x^n$.
   (a) Write $C(x)$ as a product.
   (b) Consider the corresponding sums where this time $x_i = -1, 0, 1$. Let $D_n :=$ the number of ways of getting n as this type of sum. Show that there is a $D_n \geq 33$.
   (c) Generalize the first two parts of this question.
(10) 100 is your favourite number. Let $A_k = 1^k + 2^k + 3^k + \ldots + 100^k$. Use generating functions to come up with a closed form for A_1 and A_2.
(11) Let $A(x)$ be a power series.
   (a) Find a power series $B(x)$ (called the bisection of $A(x)$) which has the same coefficients as $A(x)$ for even powers of x and has 0 as its odd coefficients.
   (b) Find a power series $C(x)$ which has the same coefficients as $A(x)$ for odd powers of x and has 0 as its even coefficients.
(12) Prove Euler's Formula $e^{i\pi} + 1 = 0$, where $i^2 = -1$. (Ignore issues of convergence.)

Questions [4,8,9] were taken from Wilf's "Generatingfunctionology". Questions [2,5,11] were taken from Bender and Williamson's "Foundations of Combinatorics with Applications".

9. Hints

(Question 6) In the language of strings of 0s and 1s, this asks for the number of such strings of length 100 that do not have consecutive 1s. Note that these strings are exactly the strings of the form:

$$\text{(All zeros)OR((0s)1(any number of: (0s, 0, 1))(0s))}$$