

Problems in Elementary Number Theory

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God does arithmetic. C. F. Gauss

Chapter 1

Introduction

The heart of Mathematics is its problems.

Paul Halmos

Number Theory is a beautiful branch of Mathematics. The purpose of this book is to present a collection of interesting problems in elementary Number Theory. Many of the problems are mathematical competition problems from all over the world like IMO, APMO, APMC, Putnam and many others. The book has a supporting website at

<http://www.problem-solving.be/pen/>

which has some extras to offer, including problem discussion and (where available) solutions, as well as some history on the book. If you like the book, you'll probably like the website.

I would like to stress that this book is **unfinished**. Any and all feedback, especially about errors in the book (even minor typos), is appreciated. I also appreciate it if you tell me about any challenging, interesting, beautiful or historical problems in elementary number theory (by email or via the website) that you think might belong in the book. On the website you can also help me collecting solutions for the problems in the book (all available solutions will be on the website only). You can send all comments to both authors at

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or (preferred) through the website.

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Lastly, note that I will use the following notations in the book:

\mathbb{Z} the set of integers,
 \mathbb{N} the set of (strictly) positive integers,
 \mathbb{N}_0 the set of nonnegative integers.

Enjoy your journey!

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Chapter 2

Divisibility Theory

Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is.

Paul Erdős

A 1. Show that if x, y, z are positive integers, then $(xy + 1)(yz + 1)(zx + 1)$ is a perfect square if and only if $xy + 1$, $yz + 1$, $zx + 1$ are all perfect squares.

Kiran S. Kedlaya

A 2. Find infinitely many triples (a, b, c) of positive integers such that a, b, c are in arithmetic progression and such that $ab + 1$, $bc + 1$, and $ca + 1$ are perfect squares.

AMM, Problem 10622, M. N. Deshpande

A 3. Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer.

IMO 1988/6

A 4. If a, b, c are positive integers such that

$$0 < a^2 + b^2 - abc \leq c,$$

show that $a^2 + b^2 - abc$ is a perfect square.¹

CRUX, Problem 1420, Shailesh Shirali

A 5. Let x and y be positive integers such that xy divides $x^2 + y^2 + 1$. Show that

$$\frac{x^2 + y^2 + 1}{xy} = 3.$$

¹This is a generalization of **A3**! Indeed, $a^2 + b^2 - abc = c$ implies that $\frac{a^2 + b^2}{ab + 1} = c \in \mathbb{N}$.

A 6.

- (a) Find infinitely many pairs of integers a and b with $1 < a < b$, so that ab exactly divides $a^2 + b^2 - 1$.
- (b) With a and b as above, what are the possible values of

$$\frac{a^2 + b^2 - 1}{ab}?$$

CRUX, Problem 1746, K. Guy and Richard J. Nowakowski

A 7. Let n be a positive integer such that $2 + 2\sqrt{28n^2 + 1}$ is an integer. Show that $2 + 2\sqrt{28n^2 + 1}$ is the square of an integer.

1969 Eötvös-Kürschák Mathematics Competition

A 8. The integers a and b have the property that for every nonnegative integer n the number of $2^n a + b$ is the square of an integer. Show that $a = 0$.

Poland 2001

A 9. Prove that among any ten consecutive positive integers at least one is relatively prime to the product of the others.

[IHH, pp. 211]

A 10. Let n be a positive integer with $n \geq 3$. Show that

$$n^{n^{n^n}} - n^{n^n}$$

is divisible by 1989.

[UmDz pp.13] Unused Problem for the Balkan MO

A 11. Let a, b, c, d be integers. Show that the product

$$(a - b)(a - c)(a - d)(b - c)(b - d)(c - d)$$

is divisible by 12.

Slovenia 1995

A 12. Let k, m , and n be natural numbers such that $m + k + 1$ is a prime greater than $n + 1$. Let $c_s = s(s + 1)$. Prove that the product

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \cdots (c_{m+n} - c_k)$$

is divisible by the product $c_1 c_2 \cdots c_n$.

Putnam 1972

A 13. Show that for all prime numbers p ,

$$Q(p) = \prod_{k=1}^{p-1} k^{2k-p-1}$$

is an integer.

AMM, Problem E2510, Saul Singer

A 14. Let n be an integer with $n \geq 2$. Show that n does not divide $2^n - 1$.

A 15. Suppose that $k \geq 2$ and $n_1, n_2, \dots, n_k \geq 1$ be natural numbers having the property

$$n_2 \mid 2^{n_1} - 1, n_3 \mid 2^{n_2} - 1, \dots, n_k \mid 2^{n_{k-1}} - 1, n_1 \mid 2^{n_k} - 1.$$

Show that $n_1 = n_2 = \dots = n_k = 1$.

IMO Long List 1985 P (RO2)

A 16. Determine if there exists a positive integer n such that n has exactly 2000 prime divisors and $2^n + 1$ is divisible by n .

IMO 2000/5

A 17. Let m and n be natural numbers such that

$$A = \frac{(m+3)^n + 1}{3m}$$

is an integer. Prove that A is odd.

Bulgaria 1998

A 18. Let m and n be natural numbers and let $mn + 1$ be divisible by 24. Show that $m + n$ is divisible by 24.

Slovenia 1994

A 19. Let $f(x) = x^3 + 17$. Prove that for each natural number $n \geq 2$, there is a natural number x for which $f(x)$ is divisible by 3^n but not 3^{n+1} .

Japan 1999

A 20. Determine all positive integers n for which there exists an integer m such that $2^n - 1$ divides $m^2 + 9$.

IMO Short List 1998

A 21. Let n be a positive integer. Show that the product of n consecutive integers is divisible by $n!$

A 22. Prove that the number

$$\sum_{k=0}^n \binom{2n+1}{2k+1} 2^{3k}$$

is not divisible by 5 for any integer $n \geq 0$.

IMO 1974/3

A 23. (Wolstenholme's Theorem) Prove that if

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p-1}$$

is expressed as a fraction, where $p \geq 5$ is a prime, then p^2 divides the numerator.

[GhEw pp.104]

A 24. Let $p > 3$ is a prime number and $k = \lfloor \frac{2p}{3} \rfloor$. Prove that

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

is divisible by p^2 .

Putnam 1996

A 25. Show that $\binom{2n}{n} \mid \text{lcm}(1, 2, \dots, 2n)$ for all positive integers n .

A 26. Let m and n be arbitrary non-negative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.²

IMO 1972/3

A 27. Show that the coefficients of a binomial expansion $(a+b)^n$ where n is a positive integer, are all odd, if and only if n is of the form $2^k - 1$ for some positive integer k .

A 28. Prove that the expression

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of positive integers (m, n) with $n \geq m \geq 1$.

Putnam 2000

A 29. For which positive integers k , is it true that there are infinitely many pairs of positive integers (m, n) such that

$$\frac{(m+n-k)!}{m!n!}$$

is an integer?

AMM Problem E2623, Ivan Niven

A 30. Show that if $n \geq 6$ is composite, then n divides $(n-1)!$.

A 31. Show that there exist infinitely many positive integers n such that $n^2 + 1$ divides $n!$.

Kazakhstan 1998

A 32. Let p and q be natural numbers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that p is divisible by 1979.

²Note that $0! = 1$.

IMO 1979/1

A 33. Let $a, b, x \in \mathbb{N}$ with $b > 1$ and such that $b^n - 1$ divides a . Show that in base b , the number a has at least n non-zero digits.

IMO Short List 1996

A 34. Let p_1, p_2, \dots, p_n be distinct primes greater than 3. Show that

$$2^{p_1 p_2 \cdots p_n} + 1$$

has at least 4^n divisors.

IMO Short List 2002 N3

A 35. Let $p \geq 5$ be a prime number. Prove that there exists an integer a with $1 \leq a \leq p - 2$ such that neither $a^{p-1} - 1$ nor $(a + 1)^{p-1} - 1$ is divisible by p^2 .

IMO Short List 2001 N4

A 36. Let n and q be integers with $n \geq 5$, $2 \leq q \leq n$. Prove that $q - 1$ divides $\left\lfloor \frac{(n-1)!}{q} \right\rfloor$.

Australia 2002

A 37. If n is a natural number, prove that the number $(n + 1)(n + 2) \cdots (n + 10)$ is not a perfect square.

Bosnia and Herzegovina 2002

A 38. Let p be a prime with $p > 5$, and let $S = \{p - n^2 | n \in \mathbb{N}, n^2 < p\}$. Prove that S contains two elements a and b such that $a|b$ and $1 < a < b$.

MM, Problem 1438, David M. Bloom

A 39. Let n be a positive integer. Prove that the following two statements are equivalent.

- n is not divisible by 4
- There exist $a, b \in \mathbb{Z}$ such that $a^2 + b^2 + 1$ is divisible by n .

A 40. Determine the greatest common divisor of the elements of the set

$$\{n^{13} - n \mid n \in \mathbb{Z}\}.$$

[PJ pp.110] UC Berkeley Preliminary Exam 1990

A 41. Show that there are infinitely many composite numbers n such that $3^{n-1} - 2^{n-1}$ is divisible by n .

[Ae pp.137]

A 42. Suppose that $2^n + 1$ is an odd prime for some positive integer n . Show that n must be a power of 2.

A 43. Suppose that p is a prime number and is greater than 3. Prove that $7^p - 6^p - 1$ is divisible by 43.

Iran 1994

A 44. Suppose that $4^n + 2^n + 1$ is prime for some positive integer n . Show that n must be a power of 3.

Germany 1982

A 45. Let $b, m, n \in \mathbb{N}$ with $b > 1$ and $m \neq n$. Suppose that $b^m - 1$ and $b^n - 1$ have the same set of prime divisors. Show that $b + 1$ must be a power of 2.

IMO Short List 1997

A 46. Let a and b be integers. Show that a and b have the same parity if and only if there exist integers c and d such that $a^2 + b^2 + c^2 + 1 = d^2$.

Romania 1995, I. Cucurezeanu

A 47. Let n be a positive integer with $n > 1$. Prove that

$$\frac{1}{2} + \cdots + \frac{1}{n}$$

is not an integer.

[Imv, pp. 15]

A 48. Let n be a positive integer. Prove that

$$\frac{1}{3} + \cdots + \frac{1}{2n+1}$$

is not an integer.

[Imv, pp. 15]

A 49. Prove that there is no positive integer n such that, for $k = 1, 2, \dots, 9$, the leftmost digit ³ of $(n + k)!$ equals k .

IMO Short List 2001 N1

A 50. Show that every integer $k > 1$ has a multiple less than k^4 whose decimal expansion has at most four distinct digits. ⁴

Germany 2000

A 51. Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m , then $a = 1$.

IMO 1984/6

A 52. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a and b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

IMO 1986/1

³Base 10.

⁴Base 10.

A 53. Suppose that x, y , and z are positive integers with $xy = z^2 + 1$. Prove that there exist integers a, b, c , and d such that $x = a^2 + b^2$, $y = c^2 + d^2$, and $z = ac + bd$.

Iran 2001

A 54. A natural number n is said to have the property P , if whenever n divides $a^n - 1$ for some integer a , n^2 also necessarily divides $a^n - 1$.

- (a) Show that every prime number n has the property P .
- (b) Show that there are infinitely many composite numbers n that possess the property P .

IMO ShortList 1993 IND5

A 55. Show that for every natural number n the product

$$\left(4 - \frac{2}{1}\right) \left(4 - \frac{2}{2}\right) \left(4 - \frac{2}{3}\right) \cdots \left(4 - \frac{2}{n}\right)$$

is an integer.

Czech and Slovak Mathematical Olympiad 1999

A 56. Let a, b , and c be integers such that $a + b + c$ divides $a^2 + b^2 + c^2$. Prove that there are infinitely many positive integers n such that $a + b + c$ divides $a^n + b^n + c^n$.

Romania 1987, L. Panaitopol

A 57. Prove that for every $n \in \mathbb{N}$ the following proposition holds: $7|3^n + n^3$ if and only if $7|3^n n^3 + 1$.

Bulgaria 1995

A 58. Let $k \geq 14$ be an integer, and let p_k be the largest prime number which is strictly less than k . You may assume that $p_k \geq \frac{3k}{4}$. Let n be a composite integer. Prove that

- (a) if $n = 2p_k$, then n does not divide $(n - k)!$,
- (b) if $n > 2p_k$, then n divides $(n - k)!$.

APMO 2003/3

A 59. Suppose that n has (at least) two essentially distinct representations as a sum of two squares. Specifically, let $n = s^2 + t^2 = u^2 + v^2$, where $s \geq t \geq 0$, $u \geq v \geq 0$, and $s > u$. Show that $\gcd(su - tv, n)$ is a proper divisor of n .

[AaJc, pp. 250]

A 60. Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t , the number $at + b$ is a triangular number if and only if t is a triangular number⁵.

Putnam 1988/B6

⁵The triangular numbers are the $t_n = n(n + 1)/2$ with $n \in \{0, 1, 2, \dots\}$.

A 61. For any positive integer $n > 1$, let $p(n)$ be the greatest prime divisor of n . Prove that there are infinitely many positive integers n with

$$p(n) < p(n+1) < p(n+2).$$

Bulgaria 1995

A 62. Let $p(n)$ be the greatest odd divisor of n . Prove that

$$\frac{1}{2^n} \sum_{k=1}^{2^n} \frac{p(k)}{k} > \frac{2}{3}.$$

Germany 1997

A 63. There is a large pile of cards. On each card one of the numbers $1, 2, \dots, n$ is written. It is known that the sum of all numbers of all the cards is equal to $k \cdot n!$ for some integer k . Prove that it is possible to arrange cards into k stacks so that the sum of numbers written on the cards in each stack is equal to $n!$.

[Tt] Tournament of the Towns 2002 Fall/A-Level

A 64. The last digit⁶ of the number $x^2 + xy + y^2$ is zero (where x and y are positive integers). Prove that two last digits of this numbers are zeros.

[Tt] Tournament of the Towns 2002 Spring/O-Level

A 65. Clara computed the product of the first n positive integers and Valerid computed the product of the first m even positive integers, where $m \geq 2$. They got the same answer. Prove that one of them had made a mistake.

[Tt] Tournament of the Towns 2001 Fall/O-Level

A 66. (Four Number Theorem) Let a, b, c , and d be positive integers such that $ab = cd$. Show that there exists positive integers p, q, r, s such that

$$a = pq, \quad b = rs, \quad c = pt, \quad d = su.$$

[PeJs, pp. 5]

A 67. Suppose that $S = \{a_1, \dots, a_r\}$ is a set of positive integers, and let S_k denote the set of subsets of S with k elements. Show that

$$\text{lcm}(a_1, \dots, a_r) = \prod_{i=1}^r \prod_{s \in S_i} \gcd(s)^{((-1)^i)}.$$

[Her, pp. 14]

A 69. Prove that if the odd prime p divides $a^b - 1$, where a and b are positive integers, then p appears to the same power in the prime factorization of $b(a^d - 1)$, where $d = \gcd(b, p-1)$.

MM, June 1986, Problem 1220, Gregg Partuno

⁶Base 10.

A 70. Suppose that $m = nq$, where n and q are positive integers. Prove that the sum of binomial coefficients

$$\sum_{k=0}^{n-1} \binom{\gcd(n,k)q}{\gcd(n,k)}$$

is divisible by m .

MM, Sep. 1984, Problem 1175

A 71. Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

IMO 1990/3 (ROM5)

A 72. Determine all pairs (n, p) of nonnegative integers such that

- p is a prime,
- $n < 2p$,
- $(p-1)^n + 1$ is divisible by n^{p-1} .

IMO 1999/4

A 73. Determine all pairs (n, p) of positive integers such that

- p is a prime, $n > 1$,
- $(p-1)^n + 1$ is divisible by n^{p-1} .⁷

A 74. Find an integer n , where $100 \leq n \leq 1997$, such that

$$\frac{2^n + 2}{n}$$

is also an integer.

APMO 1997/2

A 75. Find all triples (a, b, c) of positive integers such that $2^c - 1$ divides $2^a + 2^b + 1$.

APMC 2002

A 76. Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1) \quad \text{is a divisor of} \quad abc - 1.$$

IMO 1992/1

⁷The answer is $(n, p) = (2, 2), (3, 3)$. Note that this problem is a very nice generalization of the above two IMO problems A 71 and A 72 !

A 77. Find all positive integers, representable uniquely as

$$\frac{x^2 + y}{xy + 1},$$

where x and y are positive integers.

Russia 2001

A 78. Determine all ordered pairs (m, n) of positive integers such that

$$\frac{n^3 + 1}{mn - 1}$$

is an integer.

IMO 1994/4

A 79. Determine all pairs of integers (a, b) such that

$$\frac{a^2}{2a^2b - b^3 + 1}$$

is a positive integer.

IMO 2003/2

A 80. Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

IMO 2002/3

A 81. Determine all triples of positive integers (a, m, n) such that $a^m + 1$ divides $(a + 1)^n$.

IMO Short List 2000 N4

A 82. Which integers can be represented as

$$\frac{(x + y + z)^2}{xyz}$$

where x , y , and z are positive integers?

AMM, Problem 10382, Richard K. Guy

A 83. Find all $n \in \mathbb{N}$ such that $\lfloor \sqrt{n} \rfloor$ divides n .

[Tma pp. 73]

A 84. Determine all $n \in \mathbb{N}$ for which

- n is not the square of any integer,
- $\lfloor \sqrt{n} \rfloor^3$ divides n^2 .

India 1989

A 85. Find all $n \in \mathbb{N}$ such that 2^{n-1} divides $n!$.

[ElCr pp. 11]

A 86. Find all positive integers (x, n) such that $x^n + 2^n + 1$ divides $x^{n+1} + 2^{n+1} + 1$.

Romania 1998

A 87. Find all positive integers n such that $3^n - 1$ is divisible by 2^n .

A 88. Find all positive integers n such that $9^n - 1$ is divisible by 7^n .

A 89. Determine all pairs (a, b) of integers for which $a^2 + b^2 + 3$ is divisible by ab .

Turkey 1994

A 90. Determine all pairs (x, y) of positive integers with $y|x^2 + 1$ and $x|y^3 + 1$.

Mediterranean Mathematics Competition 2002

A 91. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

IMO 1998/4

A 92. Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r . Find all pairs (a, b) such that $q^2 + r = 1977$.

IMO 1977/5

A 93. Find the largest positive integer n such that n is divisible by all the positive integers less than $\sqrt[3]{n}$.

APMO 1998

A 94. Find all $n \in \mathbb{N}$ such that $3^n - n$ is divisible by 17.

A 95. Suppose that a and b are natural numbers such that

$$p = \frac{b}{4} \sqrt{\frac{2a-b}{2a+b}}$$

is a prime number. What is the maximum possible value of p ?

Iran 1998

A 96. Find all positive integers n that have exactly 16 positive integral divisors d_1, d_2, \dots, d_{16} such that $1 = d_1 < d_2 < \dots < d_{16} = n$, $d_6 = 18$, and $d_9 - d_8 = 17$.

Ireland 1998

A 97. Suppose that n is a positive integer and let

$$d_1 < d_2 < d_3 < d_4$$

be the four smallest positive integer divisors of n . Find all integers n such that

$$n = d_1^2 + d_2^2 + d_3^2 + d_4^2.$$

Iran 1999

A 98. Let n be a positive integer with $k \geq 22$ divisors $1 = d_1 < d_2 < \cdots < d_k = n$, all different. Determine all n such that

$$d_7^2 + d_{10}^2 = \left(\frac{n}{d_{22}}\right)^2.$$

Belarus 1999, I. Voronovich

A 99. Let $n \geq 2$ be a positive integer, with divisors

$$1 = d_1 < d_2 < \cdots < d_k = n.$$

Prove that

$$d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$$

is always less than n^2 , and determine when it divides n^2 .

IMO 2002/4

A 100. Find all positive integers n such that n has exactly 6 positive divisors $1 < d_1 < d_2 < d_3 < d_4 < n$ and $1 + n = 5(d_1 + d_2 + d_3 + d_4)$.

Singapore 1997

A 101. Find all composite numbers n having the property that each proper divisor d of n has $n - 20 \leq d \leq n - 12$.

Belarus 1998, E. Barabanov, I. Voronovich

A 102. Determine all three-digit numbers N having the property that N is divisible by 11, and $\frac{N}{11}$ is equal to the sum of the squares of the digits of N .

IMO 1960/1

A 103. When 4444^{4444} is written in decimal notation, the sum of its digits is A . Let B be the sum of the digits of A . Find the sum of the digits of B . (A and B are written in decimal notation.)

IMO 1975/4

A 104. A wobbly number is a positive integer whose *digits* in base 10 are alternatively non-zero and zero the units digit being non-zero. Determine all positive integers which do not divide any wobbly number.

IMO Short List 1994 N7

A 105. Find the smallest positive integer n such that

- n has exactly 144 distinct positive divisors,
- there are ten consecutive integers among the positive divisors of n .

IMO Long List 1985 (TR5)

A 106. Determine the least possible value of the natural number n such that $n!$ ends in exactly 1987 zeros.

IMO Long List 1987

A 107. Find four positive integers, each not exceeding 70000 and each having more than 100 divisors.

IMO Short List 1986 P10 (NL1)

A 108. For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n . Determine all triples (x, y, z) of distinct positive integers satisfying

- x, y, z are in arithmetic progression,
- $p(xyz) \leq 3$.

British Mathematical Olympiad 2003, 2-1

A 109. Find all positive integers a and b such that

$$\frac{a^2 + b}{b^2 - a} \text{ and } \frac{b^2 + a}{a^2 - b}$$

are both integers.

APMO 2002/2

A 110. For each positive integer n , write the sum $\sum_{m=1}^n 1/m$ in the form p_n/q_n , where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n .

Putnam 1997/B3

A 111. Find all natural numbers n such that the number $n(n+1)(n+2)(n+3)$ has exactly three different prime divisors.

Spain 1993

A 112. Prove that there exist infinitely many pairs (a, b) of relatively prime positive integers such that

$$\frac{a^2 - 5}{b} \text{ and } \frac{b^2 - 5}{a}$$

are both positive integers.

Germany 2003

A 113. Find all triples (l, m, n) of distinct positive integers satisfying

$$\gcd(l, m)^2 = l + m, \gcd(m, n)^2 = m + n, \text{ and } \gcd(n, l)^2 = n + l.$$

Russia 1997

A 114. What is the greatest common divisor of the set of numbers

$$\{16^n + 10n - 1 \mid n = 1, 2, \dots\}?$$

[EbMk, pp. 16]

A 115. Does there exist a 4-digit integer (in decimal form) such that no replacement of three of its digits by any other three gives a multiple of 1992?

[Ams, pp. 102], I. Selishev

A 116. What is the smallest positive integer that consists base 10 of each of the ten digits, each used exactly once, and is divisible by each of the digits 2 through 9?

[JDS, pp. 27]

A 117. Find the smallest positive integer n such that

$$2^{1989} \mid m^n - 1$$

for all odd positive integers $m > 1$.

[Rh2, pp. 98]

A 118. Determine the highest power of 1980 which divides

$$\frac{(1980n)!}{(n!)^{1980}}.$$

MM, Jan. 1981, Problem 1089, M. S. Klamkin

Chapter 3

Arithmetic in Z_n

Mathematics is the queen of the sciences and number theory is the queen of Mathematics.
Johann Carl Friedrich Gauss

3.1 Primitive Roots

B 1. Let n be a positive integer. Show that there are infinitely many primes p such that the smallest positive primitive root¹ of p is greater than n .

B 2. Let p be a prime with $p > 4 \left(\frac{p-1}{\phi(p-1)} \right)^2 2^{2k}$, where k denotes the number of distinct prime divisors of $p-1$, and let M be an integer. Prove that the set of integers $\{M+1, M+2, \dots, M+2 \left\lfloor \frac{p-1}{\phi(p-1)} 2^k \sqrt{p} \right\rfloor - 1\}$ contains a primitive root modulo p .

[Imv, pp. 90]

B 3. Show that for each odd prime p , there is an integer g such that $1 < g < p$ and g is a primitive root modulo p^n for every positive integer n .

MM, Problem 1419, William P. Wardlaw

B 4. Let g be a Fibonacci primitive root $(\text{mod } p)$. i.e. g is a primitive root $(\text{mod } p)$ satisfying $g^2 \equiv g+1 \pmod{p}$. Prove that

(a) $g-1$ is also a primitive root $(\text{mod } p)$.

(b) if $p = 4k+3$ then $(g-1)^{2k+3} \equiv g-2 \pmod{p}$, and deduce that $g-2$ is also a primitive root $(\text{mod } p)$.

[Km, Problems Sheet 3-9]

¹For a definition of primitive roots, see <http://mathworld.wolfram.com/PrimitiveRoot.html>.

B 5. Let p be an odd prime. If $g_1, \dots, g_{\phi(p-1)}$ are the primitive roots $(\text{mod } p)$ in the range $1 < g \leq p-1$, prove that

$$\sum_{i=1}^{\phi(p-1)} g_i \equiv \mu(p-1) \pmod{p}.$$

[Km, Problems Sheet 3-11]

B 6. Suppose that m does not have a primitive root. Show that

$$a^{\frac{\phi(m)}{2}} \equiv 1 \pmod{m}$$

for every a relatively prime m .

[AaJc, pp. 178]

B 7. Suppose that $p > 3$ is prime. Prove that the products of the primitive roots of p between 1 and $p-1$ is congruent to 1 modulo p .

[AaJc, pp. 181]

B 8. Let p be a prime. Let g be a primitive root of modulo p . Prove that there is no k such that $g^{k+2} \equiv g^{k+1} + 1 \equiv g^k + 2 \pmod{p}$.

[Her, pp. 99]

3.2 Quadratic Residues

C 1. Find all positive integers n that are quadratic residues modulo all primes greater than n .

CRUX, Problem 2344, Murali Vajapeyam

C 2. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

IMO 1996/4

C 3. Let p be an odd prime number. Show that the smallest positive quadratic nonresidue of p is smaller than $\sqrt{p} + 1$.

[IHH pp.147]

C 4. Let M be an integer, and let p be a prime with $p > 25$. Show that the set $\{M, M+1, \dots, M+3\lfloor\sqrt{p}\rfloor-1\}$ contains a quadratic non-residue to modulus p .

[Imv, pp. 72]

C 5. Let p be an odd prime and let Z_p denote (the field of) integers modulo p . How many elements are in the set

$$\{x^2 : x \in Z_p\} \cap \{y^2 + 1 : y \in Z_p\}?$$

Putnam 1991/B5

C 6. Let a, b, c be integers and let p be an odd prime with

$$p \nmid a \text{ and } p \nmid b^2 - 4ac.$$

Show that

$$\sum_{k=1}^p \left(\frac{ak^2 + bk + c}{p} \right) = - \left(\frac{a}{p} \right).$$

[Ab, pp. 34]

3.3 Congruences

D 1. If p is an odd prime, prove that

$$\binom{k}{p} \equiv \left\lfloor \frac{k}{p} \right\rfloor \pmod{p}.$$

[Tma, pp. 127]

D 2. Suppose that p is an odd prime. Prove that

$$\sum_{j=0}^p \binom{p}{j} \binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

Putnam 1991/B4

D 3. Show that

$$(-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \equiv 4^{p-1} \pmod{p^3}$$

for all prime numbers p with $p \geq 5$.

Morley

D 4. Let n be a positive integer. Prove that n is prime if and only if

$$\binom{n-1}{k} \equiv (-1)^k \pmod{n}$$

for all $k \in \{0, 1, \dots, n-1\}$.

MM, Problem 1494, Emeric Deutsch and Ira M. Gessel

D 5. Prove that for $n \geq 2$,

$$\underbrace{2^{2^{\dots^2}}}_{n \text{ terms}} \equiv \underbrace{2^{2^{\dots^2}}}_{n-1 \text{ terms}} \pmod{n}.$$

Putnam 1997/B5

D 6. Show that, for any fixed integer $n \geq 1$, the sequence

$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \dots \pmod{n}$$

is eventually constant.

USA 1991

D 7. Somebody incorrectly remembered Fermat's little theorem as saying that the congruence $a^{n+1} \equiv a \pmod{n}$ holds for all a if n is prime. Describe the set of integers n for which this property is in fact true.

[DZ] posed by Don Zagier at the St Andrews Colloquium 1996

D 8. Characterize the set of positive integers n such that, for all integers a , the sequence a, a^2, a^3, \dots is periodic modulo n .

D 9. Show that there exists a composite number n such that $a^n \equiv a \pmod{n}$ for all $a \in \mathbb{Z}$.

D 10. Let p be a prime number of the form $4k + 1$. Suppose that $2p + 1$ is prime. Show that there is no $k \in \mathbb{N}$ with $k < 2p$ and $2^k \equiv 1 \pmod{2p + 1}$.

D 11. During a break, n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule. He selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of n for which eventually, perhaps after many rounds, all children will have at least one candy each.

APMO 1991/4

D 12. Suppose that $m > 2$, and let P be the product of the positive integers less than m that are relatively prime to m . Show that $P \equiv -1 \pmod{m}$ if $m = 4, p^n$, or $2p^n$, where p is an odd prime, and $P \equiv 1 \pmod{m}$ otherwise.

[AaJc, pp. 139]

D 13. Let Γ consist of all polynomials in x with integer coefficients. For f and g in Γ and m a positive integer, let $f \equiv g \pmod{m}$ mean that every coefficient of $f - g$ is an integral multiple of m . Let n and p be positive integers with p prime. Given that f, g, h, r and s are in Γ with $rf + sg \equiv 1 \pmod{p}$ and $fg \equiv h \pmod{p}$, prove that there exist F and G in Γ with $F \equiv f \pmod{p}$, $G \equiv g \pmod{p}$, and $FG \equiv h \pmod{p^n}$.

Putnam 1986/B3

D 14. Determine the number of integers $n \geq 2$ for which the congruence

$$x^{25} \equiv x \pmod{n}$$

is true for all integers x .

Purdue POW, Spring 2003 Series/5

D 15. Let n_1, \dots, n_k and a be positive integers which satisfy the following conditions:

- for any $i \neq j$, $(n_i, n_j) = 1$,
- for any i , $a^{n_i} \equiv 1 \pmod{n_i}$,
- for any i , n_i does not divide $a - 1$.

Show that there exist at least $2^{k+1} - 2$ integers $x > 1$ with $a^x \equiv 1 \pmod{x}$.

Turkey 1993

D 16. Determine all positive integers $n \geq 2$ that satisfy the following condition; For all integers a, b relatively prime to n ,

$$a \equiv b \pmod{n} \iff ab \equiv 1 \pmod{n}.$$

D 17. Determine all positive integers n such that $xy + 1 \equiv 0 \pmod{n}$ implies that $x + y \equiv 0 \pmod{n}$.

AMM, Problem???, M. S. Klamkin and A. Liu

D 18. Let p be a prime number. Determine the maximal degree of a polynomial $T(x)$ whose coefficients belong to $\{0, 1, \dots, p-1\}$, whose degree is less than p , and which satisfies

$$T(n) = T(m) \pmod{p} \implies n = m \pmod{p}$$

for all integers n, m .

Turkey 2000

D 19. Let a_1, \dots, a_k and m_1, \dots, m_k be integers with $2 \leq m_1$ and $2m_i \leq m_{i+1}$ for $1 \leq i \leq k-1$. Show that there are infinitely many integers x which do not satisfy any of congruences

$$x \equiv a_1 \pmod{m_1}, x \equiv a_2 \pmod{m_2}, \dots, x \equiv a_k \pmod{m_k}.$$

Turkey 1995

D 20. Show that 1994 divides $10^{900} - 2^{1000}$.

Belarus 1994

D 21. Determine the last three digits of

$$2003^{2002^{2001}}.$$

Canada 2003

D 22. Prove that $1980^{1981^{1982}} + 1982^{1981^{1980}}$ is divisible by 1981^{1981} .

China 1981

D 23. Let p be an odd prime of the form $p = 4n + 1$.

(a) Show that n is a quadratic residue \pmod{p} .

(b) Calculate the value $n^n \pmod{p}$.

Chapter 4

Primes and Composite Numbers

Wherever there is number, there is beauty.

Proclus Diadochus

E 1. Prove that the number $512^3 + 675^3 + 720^3$ is composite.¹

[DfAk, pp. 50] Leningrad Mathematical Olympiad 1991

E 2. Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that $ac + bd = (b + d + a - c)(b + d - a + c)$. Prove that $ab + cd$ is not prime.

IMO 2001/6

E 3. Find the sum of all distinct positive divisors of the number 104060401.

MM, Problem Q614, Rod Cooper

E 4. Prove that 1280000401 is composite.

E 5. Prove that $\frac{5^{125}-1}{5^{25}-1}$ is a composite number.

IMO Short List 1992 P16

E 6. Find a factor of $2^{33} - 2^{19} - 2^{17} - 1$ that lies between 1000 and 5000.

MM, Problem Q684, Noam Elkies

E 7. Show that there exists a positive integer k such that $k \cdot 2^n + 1$ is composite for all $n \in \mathbb{N}_0$.

USA 1982

E 8. Show that for all integer $k > 1$, there are infinitely many natural numbers n such that $k \cdot 2^{2^n} + 1$ is composite.

[VsAs]

E 9. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers a, b, c, d are replaced by $a - b, b - c, c - d, d - a$). Is it possible after 1996 such steps to have numbers a, b, c and d such that the numbers $|bc - ad|$, $|ac - bd|$ and $|ab - cd|$ are primes?

¹Hint: $2z^2 = 3xy \Rightarrow x^3 + y^3 + z^3 = x^3 + y^3 + (-z)^3 - 3xy(-z)$.

E 10. Represent the number $989 \cdot 1001 \cdot 1007 + 320$ as a product of primes.

[DfAk, pp. 9] *Leningrad Mathematical Olympiad 1987*

E 11. In 1772 Euler discovered the curious fact that $n^2 + n + 41$ is prime when n is any of $0, 1, 2, \dots, 39$. Show that there exist 40 consecutive integer values of n for which this polynomial is *not* prime.

[JDS, pp. 26]

E 12. Show that there are infinitely many primes.

E 13. Find all natural numbers n for which every natural number whose decimal representation has $n - 1$ digits 1 and one digit 7 is prime.

IMO Short List 1990 USS1

E 14. Prove² that there do not exist polynomials P and Q such that

$$\pi(x) = \frac{P(x)}{Q(x)}$$

for all $x \in \mathbb{N}$.

[Tma, pp. 101]

E 15. Show that there exist two consecutive squares such that there are at least 1000 primes between them.

MM, Problem Q789, Norman Schaumberger

E 16. Prove that for any prime p in the interval³ $]n, \frac{4n}{3}]$, p divides

$$\sum_{j=0}^n \binom{n}{j}^4.$$

MM, Problem 1392, George Andrews

E 17. Let a , b , and n be positive integers with $\gcd(a, b) = 1$. Without using Dirichlet's theorem⁴, show that there are infinitely many $k \in \mathbb{N}$ such that $\gcd(ak + b, n) = 1$.

[AaJc pp.212]

E 18. Without using Dirichlet's theorem, show that there are infinitely many primes ending in the digit 9.

E 19. Let p be an odd prime. Without using Dirichlet's theorem, show that there are infinitely many primes of the form $2pk + 1$.

²Denote $\pi(x)$ the number of primes p with $p \leq x$.

³Here, $]a, b]$ denotes the same as $(a, b]$. It seems both notations are used.

⁴For any $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$, there are infinitely many primes of the form $ak + b$.

[AaJc pp.176]

E 20. Verify that, for each $r \geq 1$, there are infinitely many primes p with $p \equiv 1 \pmod{2^r}$.

[GjJj pp.140]

E 21. Prove that if p is a prime, then $p^p - 1$ has a prime factor that is congruent to 1 modulo p .

[Ns pp.176]

E 22. Let p be a prime number. Prove that there exists a prime number q such that for every integer n , $n^p - p$ is not divisible by q .

IMO 2003/6

E 23. Let $p_1 = 2, p_2 = 3, p_3 = 5, \dots, p_n$ be the first n prime numbers, where $n \geq 3$. Prove that

$$\frac{1}{p_1^2} + \frac{1}{p_2^2} + \dots + \frac{1}{p_n^2} + \frac{1}{p_1 p_2 \dots p_n} < \frac{1}{2}.$$

Yugoslavia 2001

E 24. Let p_n again denote the n th prime number. Show that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges.

E 25. Prove that $\ln n \geq k \ln 2$, where n is a natural number and k is the number of distinct primes that divide n .

[Er pp.10] Eötvös Competition 1896

E 26. Find the smallest prime which is not the difference (in some order) of a power of 2 and a power of 3.

MM, Problem 1404, H. Gauchmen and I. Rosenholtz

E 27. Prove that for each positive integer n , there exist n consecutive positive integers none of which is an integral power of a prime number.

IMO 1989/5

E 28. Show that $n^{\pi(2n) - \pi(n)} < 4^n$ for all positive integer n .

[GjJj pp.36]

E 29. Let s_n denote the sum of the first n primes. Prove that for each n there exists an integer whose square lies between s_n and s_{n+1} .

[Tma, pp. 102]

E 30. Given an odd integer $n > 3$, let k and t be the smallest positive integers such that both $kn + 1$ and tn are squares. Prove that n is prime if and only if both k and t are greater than $\frac{n}{4}$

[Tma, pp. 128]

E 31. Suppose n and r are nonnegative integers such that no number of the form $n^2 + r - k(k+1)$ ($k \in \mathbb{N}$) equals to -1 or a positive composite number. Show that $4n^2 + 4r + 1$ is 1, 9 or prime.

CRUX, Problem 1608, Seung-Jin Bang

E 32. Let $n \geq 5$ be an integer. Show that n is prime if and only if $n_i n_j \neq n_p n_q$ for every partition of n into 4 integers, $n = n_1 + n_2 + n_3 + n_4$, and for each permutation (i, j, p, q) of $(1, 2, 3, 4)$.

Singapore 1989

E 33. Prove that there are no positive integers a and b such that for all different primes p and q greater than 1000, the number $ap + bq$ is also prime.

Russia 1996

E 34. Let p_n denote the n th prime number. For all $n \geq 6$, prove that

$$\pi(\sqrt{p_1 p_2 \cdots p_n}) > 2n.$$

[Rh, pp. 43]

E 35. There exists a block of 1000 consecutive positive integers containing no prime numbers, namely, $1001! + 2, 1001! + 3, \dots, 1001! + 1001$. Does there exist a block of 1000 consecutive positive integers containing exactly five prime numbers?

[Tt] Tournament of the Towns 2001 Fall/O-Level

E 36. Prove that there are infinitely many twin primes if and only if there are infinitely many integers that cannot be written in any of the following forms:

$$6uv + u + v, \quad 6uv + u - v, \quad 6uv - u + v, \quad 6uv - u - v,$$

for some positive integers u and v .

[PeJs, pp. 160], S. Golomb

E 37. It's known that there is always a prime between n and $2n - 7$ for all $n \geq 10$. Prove that, with the exception of 1, 4, and 6, every natural number can be written as the sum of distinct primes.

[PeJs, pp. 174]

E 38. Prove that if $c > \frac{8}{3}$, then there exists a real numbers θ such that $\lfloor \theta^{c^n} \rfloor$ is prime for every positive integer n .

[PbAw, pp. 1]

E 39. Let c be a nonzero real numbers. Suppose that $g(x) = c_0 x^r + c_1 x^{r-1} + \cdots + c_{r-1} x + c_r$ is a polynomial with integer coefficients. Suppose that the roots of $g(x)$ are b_1, \dots, b_r . Let k be a given positive integer. Show that there is a prime p such that $p > \max(k, |c|, |c_r|)$, and moreover if t is a real between 0 and 1, and j is one of $1, \dots, r$, then

$$|(c^r b_j g(tb_j))^{p e^{(1-t)b_j}}| < \frac{(p-1)!}{2r}.$$

Furthermore, if

$$f(x) = \frac{e^{rp-1}x^{p-1}(g(x))^p}{(p-1)!}$$

then

$$\left| \sum_{j=1}^r \int_0^1 e^{(1-t)b_j} f(tb_j) dt \right| \leq \frac{1}{2}.$$

[Wsa, pp. 217]

E 40. Prove that there do not exist eleven primes, all less than 20000, which form an arithmetic progression.

[DNI, 19]

E 41. Let n be a positive integer. Show that n is prime if and only if

$$\lim_{r \rightarrow \infty} \lim_{s \rightarrow \infty} \lim_{t \rightarrow \infty} \sum_{u=0}^s \left(1 - \left(\cos \frac{(u!)^r \pi}{n} \right)^2 t \right) = n.$$

[Her, pp. 11], G. H. Hardy

Chapter 5

Rational and Irrational Numbers

God made the integers, all else is the work of man.

Leopold Kronecker

5.1 Rational Numbers

F 1. Suppose that a rectangle with sides a and b is arbitrarily cut into squares with sides x_1, \dots, x_n . Show that $\frac{x_i}{a} \in \mathbb{Q}$ and $\frac{x_i}{b} \in \mathbb{Q}$ for all $i \in \{1, \dots, n\}$.

[Vvp, pp. 40]

F 2. Find all x and y which are rational multiples of π with $0 < x < y < \frac{\pi}{2}$ and $\tan x + \tan y = 2$.

CRUX, Problem 1632, Stanley Rabinowitz

F 3. Let α be a rational number with $0 < \alpha < 1$ and $\cos(3\pi\alpha) + 2\cos(2\pi\alpha) = 0$. Prove that $\alpha = \frac{2}{3}$.

IMO ShortList 1991 P19 (IRE 5)

F 4. Suppose that $\tan \alpha = \frac{p}{q}$, where p and q are integers and $q \neq 0$. Prove the number $\tan \beta$ for which $\tan 2\beta = \tan 3\alpha$ is rational only when $p^2 + q^2$ is the square of an integer.

IMO Long List 1967 P20 (DDR)

F 5. Prove that there is no positive rational number x such that

$$x^{\lfloor x \rfloor} = \frac{9}{2}.$$

Austria 2002

F 6. Let x, y, z non-zero real numbers such that xy, yz, zx are rational.

(a) Show that the number $x^2 + y^2 + z^2$ is rational.

(b) If the number $x^3 + y^3 + z^3$ is also rational, show that x, y, z are rational.

Romania 2001, Marius Ghergu

F 7. If x is a positive rational number, show that x can be uniquely expressed in the form

$$x = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots,$$

where a_1, a_2, \dots are integers, $0 \leq a_n \leq n-1$ for $n > 1$, and the series terminates. Show also that x can be expressed as the sum of reciprocals of different integers, each of which is greater than 10^6 .

IMO Long List 1967 (GB)

F 8. Find all polynomials W with real coefficients possessing the following property: if $x + y$ is a rational number, then $W(x) + W(y)$ is rational.

Poland 2002

F 9. Prove that every positive rational number can be represented in the form

$$\frac{a^3 + b^3}{c^3 + d^3}$$

for some positive integers a, b, c , and d .

IMO Short List 1999

F 10. The set S is a finite subset of $[0, 1]$ with the following property: for all $s \in S$, there exist $a, b \in S \cup \{0, 1\}$ with $a, b \neq s$ such that $s = \frac{a+b}{2}$. Prove that all the numbers in S are rational.

Berkeley Math Circle Monthly Contest 1999-2000

F 11. Let $S = \{x_0, x_1, \dots, x_n\} \subset [0, 1]$ be a finite set of real numbers with $x_0 = 0$ and $x_1 = 1$, such that every distance between pairs of elements occurs at least twice, except for the distance 1. Prove that all of the x_i are rational.

Iran 1998

F 12. Does there exist a circle and an infinite set of points on it such that the distance between any two points of the set is rational?

[Zh, PP. 40] Mediterranean MC 1999 (Proposed by Ukraine)

F 13. Prove that numbers of the form

$$\frac{a_1}{1!} + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots,$$

where $0 \leq a_i \leq i-1$ ($i = 2, 3, 4, \dots$) are rational if and only if starting from some i on all the a_i 's are either equal to 0 (in which case the sum is finite) or all are equal to $i-1$.

[PeJs, pp. 99]

F 14. Let k and m be positive integers. Show that

$$S(m, k) = \sum_{n=1}^{\infty} \frac{1}{n(mn + k)}$$

is rational if and only if m divides k .

[PbAw, pp. 2]

F 15. Find all rational numbers k such that $0 \leq k \leq \frac{1}{2}$ and $\cos k\pi$ is rational.

[EbMk, pp. 15]

F 16. Prove that for any distinct rational numbers a, b, c , the number

$$\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2}$$

is the square of some rational number.

[EbMk, pp. 23]

5.2 Irrational Numbers

G 1. Find the smallest positive integer n such that

$$0 < \sqrt[4]{n} - \lfloor \sqrt[4]{n} \rfloor < 0.00001.$$

[GML, pp. 167]

G 2. Prove that for any positive integers a and b

$$\left| a\sqrt{2} - b \right| > \frac{1}{2(a+b)}.$$

[GML, pp. 183]

G 3. Prove that there exist positive integers m and n such that

$$\left| \frac{m^2}{n^3} - \sqrt{2001} \right| < \frac{1}{10^8}.$$

The Grosman Memorial Mathematical Olympiad 1999

G 4. Let a, b, c be integers, not all zero and each of absolute value less than one million. Prove that

$$\left| a + b\sqrt{2} + c\sqrt{3} \right| > \frac{1}{10^{21}}.$$

Belarus 2002

G 5. Let a, b, c be integers, not all equal to 0. Show that

$$\frac{1}{4a^2 + 3b^2 + 2c^2} \leq |\sqrt[3]{4a} + \sqrt[3]{2b} + c|.$$

Belarus 2001

G 6. Prove that for any irrational number ξ , there are infinitely many rational numbers $\frac{m}{n}$ ($(m, n) \in \mathbb{Z} \times \mathbb{N}$) such that

$$\left| \xi - \frac{n}{m} \right| < \frac{1}{\sqrt{5}m^2}.$$

Putnam 1980, Hurwitz

G 7. Show that π is irrational.

CRUX, Problem A240, Mohammed Aassila

G 8. Show that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is irrational.

G 9. Show that $\cos \frac{\pi}{7}$ is irrational.

G 10. Show that $\frac{1}{\pi} \arccos \left(\frac{1}{\sqrt{2003}} \right)$ is irrational.

G 11. Show that $\cos 1^\circ$ is irrational.

G 12. An integer-sided triangle has angles $p\theta$ and $q\theta$, where p and q are relatively prime integers. Prove that $\cos \theta$ is irrational.

G 13. It is possible to show that $\csc \frac{3\pi}{29} - \csc \frac{10\pi}{29} = 1.999989433\dots$. Prove that there are no integers j, k, n with odd n satisfying $\csc \frac{j\pi}{n} - \csc \frac{k\pi}{n} = 2$.

G 14. For which angles θ , with θ a rational number of degrees, is $\tan^2 \theta + \tan^2 2\theta$ irrational?

CRUX, Problem 2305, Richard I. Hess

G 15. Prove that for any $p, q \in \mathbb{N}$ with $q > 1$ the following inequality holds:¹

$$\left| \pi - \frac{p}{q} \right| \geq q^{-42}.$$

(K. Mahler, 1953) AMM, Problem 10630, Richard Strong

G 16. For each integer $n \geq 1$, prove that there is a polynomial $P_n(x)$ with rational coefficients such that $x^{4n}(1-x)^{4n} = (1+x)^2 P_n(x) + (-1)^n 4^n$. Define the rational number a_n by

$$a_n = \frac{(-1)^{n-1}}{4^{n-1}} \int_0^1 P_n(x) dx, \quad n = 1, 2, \dots$$

Prove that a_n satisfies the inequality

$$|\pi - a_n| < \frac{1}{4^{5n-1}}, \quad n = 1, 2, \dots$$

MM, Problem 1372, Nick Lord

¹This is a deep theorem in transcendental number theory. Note that it follows from this result that π is irrational! In fact, it's known that for sufficiently large q , the exponent 42 can be replaced by 30. Here is a similar result due to A. Baker: for any rationals $\frac{p}{q}$, one has $|\ln 2 - \frac{p}{q}| \geq 10^{-100000} q^{-12.5}$. [AI, pp. 106]

G 17. Suppose that $p, q \in \mathbb{N}$ satisfy the inequality

$$\exp(1) \cdot (\sqrt{p+q} - \sqrt{q})^2 < 1.$$

Show that $\ln\left(1 + \frac{p}{q}\right)$ is irrational.

(*K. Alladi, M. Robinson, 1979*) [*AI*, pp. 106] For a proof, See [*Kmh*].

G 18. Show that the cube roots of three distinct primes cannot be terms in an arithmetic progression.

[*KhKw*, pp. 11]

G 19. Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with a rational area.

[*AI*, pp. 116] For a proof, See [*KaMr*].

G 20. You are given three lists A, B, and C. List A contains the numbers of the form 10^k in base 10, with k any integer greater than or equal to 1. Lists B and C contain the same numbers translated into base 2 and 5 respectively:

<i>A</i>	<i>B</i>	<i>C</i>
10	1010	20
100	1100100	400
1000	1111101000	13000
\vdots	\vdots	\vdots

Prove that for every integer $n > 1$, there is exactly one number in exactly one of the lists B or C that has exactly n digits.

APMO 1994/5

G 21. Prove that if α and β are positive irrational numbers satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then the sequences

$$\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \dots$$

and

$$\lfloor \beta \rfloor, \lfloor 2\beta \rfloor, \lfloor 3\beta \rfloor, \dots$$

together include every positive integer exactly once.

(*Beatty*) *IMO 1987/5*

G 22. For a positive real number α , define

$$S(\alpha) = \{\lfloor n\alpha \rfloor \mid n = 1, 2, 3, \dots\}.$$

Prove that \mathbb{N} cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

Putnam 1995/B6

G 23. Let $f(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x}{2^n}\right)$. Show that at the point $x = 1$, $f(x)$ and all its derivatives are irrational.

G 24. Let $\{a_n\}_{n \geq 1}$ be a sequence of positive numbers such that

$$a_{n+1}^2 = a_n + 1, \quad n \in \mathbb{N}.$$

Show that the sequence contains an irrational number.

Putnam 1995

G 25. Show that $\tan\left(\frac{\pi}{m}\right)$ is irrational for all positive integers $m \geq 5$.

MM, Problem 1385, Howard Morris

G 26. Prove that if $g \geq 2$ is an integer, then two series

$$\sum_{n=0}^{\infty} \frac{1}{g^{n^2}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{g^{n!}}$$

both converge to irrational numbers.

[Ae, pp. 226]

G 27. Let $1 < a_1 < a_2 < \cdots$ be a sequence of positive integers. Show that

$$\frac{2^{a_1}}{a_1!} + \frac{2^{a_2}}{a_2!} + \frac{2^{a_3}}{a_3!} + \cdots$$

is irrational.

[PeJs, pp. 95]

G 28. Do there exist real numbers a and b such that

(a) $a + b$ is rational and $a^n + b^n$ is irrational for all $n \in \mathbb{N}$ with $n \geq 2$?

(b) $a + b$ is irrational and $a^n + b^n$ is rational for all $n \in \mathbb{N}$ with $n \geq 2$?

(N. Agahanov) [PeJs, pp. 99]

G 29. Let $p(x) = x^3 + a_1x^2 + a_2x + a_3$ have rational coefficients and have roots r_1, r_2 , and r_3 . If $r_1 - r_2$ is rational, must r_1, r_2 , and r_3 be rational?

[PbAw, pp. 2]

G 30. Let $\alpha = 0.d_1d_2d_3 \cdots$ be a decimal representation of a real number between 0 and 1. Let r be a real number with $|r| < 1$.

(a) If α and r are rational, must $\sum_{i=1}^{\infty} d_i r^i$ be rational?

(b) If $\sum_{i=1}^{\infty} d_i r^i$ and r are rational, α must be rational?

[Ams, pp. 14]

Chapter 6

Diophantine Equations

To divide a cube into two other cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it. Pierre de Fermat, in the margin of his copy of Diophantus' *Arithmetica*

H 1. One of Euler's conjectures¹ was disproved in the 1980s by three American Mathematicians² when they showed that there is a positive integer n such that

$$n^5 = 133^5 + 110^5 + 84^5 + 27^5.$$

Find the value of n .

AIME 1989/9

H 2. The number 21982145917308330487013369 is the thirteenth power of a positive integer. Which positive integer?

UC Berkeley Preliminary Exam 1983

H 3. Does there exist a solution to the equation

$$x^2 + y^2 + z^2 + u^2 + v^2 = xyzuv - 65$$

in integers with x, y, z, u, v greater than 1998?

Taiwan 1998

H 4. Find all pairs (x, y) of positive rational numbers such that $x^2 + 3y^2 = 1$.

H 5. Find all pairs (x, y) of rational numbers such that $y^2 = x^3 - 3x + 2$.

¹In 1769, Euler, by generalizing Fermat's Last Theorem, conjectured that 'it is impossible to exhibit three fourth powers whose sum is a fourth power, four fifth powers whose sum is a fifth power, and similarly for higher powers' [Rs]

²L. J. Lander, T. R. Parkin, and J. L. Selfridge

H 6. Show that there are infinitely many pairs (x, y) of rational numbers such that $x^3 + y^3 = 9$.

H 7. Determine all pairs (x, y) of positive integers satisfying the equation

$$(x + y)^2 - 2(xy)^2 = 1.$$

Poland 2002

H 8. Show that the equation

$$x^3 + y^3 + z^3 + t^3 = 1999$$

has infinitely many integral solutions.³

Bulgaria 1999

H 9. Determine all integers a for which the equation

$$x^2 + axy + y^2 = 1$$

has infinitely many distinct integer solutions x, y .

Ireland 1995

H 10. Prove that there are unique positive integers a and n such that

$$a^{n+1} - (a + 1)^n = 2001.$$

Putnam 2001

H 11. Find all $(x, y, n) \in \mathbb{N}^3$ such that $\gcd(x, n + 1) = 1$ and $x^n + 1 = y^{n+1}$.

India 1998

H 12. Find all $(x, y, z) \in \mathbb{N}^3$ such that $x^4 - y^4 = z^2$.

H 13. Find all pairs (x, y) of positive integers that satisfy the equation ⁴

$$y^2 = x^3 + 16.$$

Italy 1994

H 14. Show that the equation $x^2 + y^5 = z^3$ has infinitely many solutions in integers x, y, z for which $xyz \neq 0$.

Canada 1991

³More generally, the following result is known: let n be an integer, then the equation $x^3 + y^3 + z^3 + w^3 = n$ has infinitely many integral solutions (x, y, z, w) if there can be found one solution $(x, y, z, w) = (a, b, c, d)$ with $(a + b)(c + d)$ negative and with either $a \neq b$ and $c \neq d$. [Eb2, pp.90]

⁴It's known that there are (infinitely) many integers k so that the equation $y^2 = x^3 + k$ has no integral solutions. For example, if k has the form $k = (4n - 1)^3 - 4m^2$, where m and n are integers such that no prime $p \equiv -1 \pmod{4}$ divides m , then the equation $y^2 = x^3 + k$ has no integral solutions. For a proof, see [Tma, pp. 191].

H 15. Prove that there are no integers x and y satisfying $x^2 = y^5 - 4$.

Balkan Mathematical Olympiad 1998

H 16. Find all pairs (a, b) of different positive integers that satisfy the equation $W(a) = W(b)$, where $W(x) = x^4 - 3x^3 + 5x^2 - 9x$.

Poland 2003

H 17. Find all positive integers n for which the equation

$$a + b + c + d = n\sqrt{abcd}$$

has a solution in positive integers.

Vietnam 2002

H 18. Determine all positive integer solutions (x, y, z, t) of the equation

$$(x + y)(y + z)(z + x) = xyzt$$

for which $\gcd(x, y) = \gcd(y, z) = \gcd(z, x) = 1$.

Romania 1995, M. Becheanu

H 19. Find all $(x, y, z, n) \in \mathbb{N}^4$ such that $x^3 + y^3 + z^3 = nx^2y^2z^2$.

[UmDz pp.14] Unused Problem for the Balkan MO

H 20. Determine all positive integers n for which the equation

$$x^n + (2 + x)^n + (2 - x)^n = 0$$

has an integer as a solution.

APMO 1993/4

H 21. Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

APMO 1989/2

H 22. Find all integers a, b, c, x, y, z such that

$$a + b + c = xyz, \quad x + y + z = abc, \quad a \geq b \geq c \geq 1, \quad x \geq y \geq z \geq 1.$$

Poland 1998

H 23. Find all $(x, y, z) \in \mathbb{Z}^3$ such that $x^3 + y^3 + z^3 = x + y + z = 3$.

H 24. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n.$$

has a solution in integers (x, y) , then it has at least three such solutions. Show that the equation has no solutions in integers when $n = 2891$.

IMO 1982/4

H 25. What is the smallest positive integer t such that there exist integers x_1, x_2, \dots, x_t with

$$x_1^3 + x_2^3 + \dots + x_t^3 = 2002^{2002} \quad ?$$

IMO Short List 2002 N1

H 26. Solve in integers the following equation

$$n^{2002} = m(m+n)(m+2n) \cdots (m+2001n).$$

Ukraine 2002

H 27. Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semi-perimeter and the inradius of a triangle with integer side lengths.

IMO Short List 2000 N5

H 28. Let a, b, c be positive integers such that a and b are relatively prime and c is relatively prime either to a or b . Prove that there exist infinitely many triples (x, y, z) of distinct positive integers such that

$$x^a + y^b = z^c.$$

IMO Short List 1997 N6

H 29. Find all pairs of integers (x, y) satisfying the equality

$$y(x^2 + 36) + x(y^2 - 36) + y^2(y - 12) = 0.$$

Belarus 2000

H 30. Let a, b, c be given integers, $a > 0$, $ac - b^2 = p$ a squarefree positive integer. Let $M(n)$ denote the number of pairs of integers (x, y) for which $ax^2 + bxy + cy^2 = n$. Prove that $M(n)$ is finite and $M(n) = M(p^k \cdot n)$ for every integer $k \geq 0$.

IMO Short List 1993 GEO3

H 31. Determine all integer solutions of the system

$$2uv - xy = 16,$$

$$xv - yu = 12.$$

[Eb1, pp. 19] AMM 61(1954), 126; 62(1955), 263

H 32. Let n be a natural number. Solve in whole numbers the equation

$$x^n + y^n = (x - y)^{n+1}.$$

IMO Long List 1987 (Romania)

H 33. Does there exist an integer such that its cube is equal to $3n^2 + 3n + 7$, where n is integer?

IMO Long List 1967 P (PL)

H 34. Are there integers m and n such that $5m^2 - 6mn + 7n^2 = 1985$?

IMO Long List 1985 (SE1)

H 35. Find all cubic polynomials $x^3 + ax^2 + bx + c$ admitting the rational numbers a , b and c as roots.

IMO Long List 1985 (TR3)

H 36. Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions (a, b, c) .

Italy 1996

H 37. Prove that for each positive integer n there exist odd positive integers x_n and y_n such that $x_n^2 + 7y_n^2 = 2^n$.

Bulgaria 1996

H 38. Suppose that p is an odd prime such that $2p + 1$ is also prime. Show that the equation $x^p + 2y^p + 5z^p = 0$ has no solutions in integers other than $(0, 0, 0)$.

[JeMm, pp. 10]

H 39. Let A, B, C, D, E be integers, $B \neq 0$ and $F = AD^2 - BCD + B^2E \neq 0$. Prove that the number N of pairs of integers (x, y) such that

$$Ax^2 + Bxy + Cx + Dy + E = 0,$$

satisfies $N \leq 2d(|F|)$, where $d(n)$ denotes the number of positive divisors of positive integer n .

[KhKw, pp. 9]

H 40. Determine all pairs of rational numbers (x, y) such that

$$x^3 + y^3 = x^2 + y^2.$$

[EbMk, pp. 44]

H 41. Suppose that $A = 1, 2$, or 3 . Let a and b be relatively prime integers such that $a^2 + Ab^2 = s^3$ for some integer s . Then, there are integers u and v such that $s = u^2 + Av^2$, $a = u^3 - 3Avu^2$, and $b = 3u^2v - Av^3$.

[Wsa, pp. 39]

H 42. Find all integers a for which $x^3 - x + a$ has three integer roots.

[GML, pp. 2]

H 43. Find all solutions in integers of $x^3 + 2y^3 = 4z^3$.

[GML, pp. 33]

H 44. For all $n \in \mathbb{N}$, show that the number of integral solutions (x, y) of

$$x^2 + xy + y^2 = n$$

is finite and a multiple of 6.

[GML, pp. 192]

H 45. Show that there cannot be four squares in arithmetical progression.

(Fermat) [Ljm, pp. 21]

H 46. Let a, b, c, d, e, f be integers such that $b^2 - 4ac > 0$ is not a perfect square and $4acf + bde - ae^2 - cd^2 - fb^2 \neq 0$. Let

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$$

Suppose that $f(x, y) = 0$ has an integral solution. Show that $f(x, y) = 0$ has infinitely many integral solutions.

(Gauss) [Ljm, pp. 57]

H 47. Show that the equation $x^4 + y^4 + 4z^4 = 1$ has infinitely many rational solutions.

[Ljm, pp. 94]

H 48. Solve the equation $x^2 + 7 = 2^n$ in integers.

[Ljm, pp. 205]

H 49. Show that the only solutions of the equation $x^3 - 3xy^2 - y^3 = 1$ are given by $(x, y) = (1, 0), (0, -1), (-1, 1), (1, -3), (-3, 2), (2, 1)$.

[Ljm, pp. 208]

H 50. Show that the equation $y^2 = x^3 + 2a^3 - 3b^2$ has no solution in integers if $ab \neq 0$, $a \not\equiv 1 \pmod{3}$, 3 does not divide b , a is odd if b is even, and $p = t^2 + 27u^2$ has a solution in integers t, u if $p|a$ and $p \equiv 1 \pmod{3}$.

[Her, pp. 287]

H 51. Prove that the product of five consecutive positive integers is never a perfect square.

[Rh3, pp. 207]

H 52. Do there exist two right-angled triangles with integer length sides that have the lengths of exactly two sides in common?

H 53. Suppose that a, b , and p are integers such that $b \equiv 1 \pmod{4}$, $p \equiv 3 \pmod{4}$, p is prime, and if q is any prime divisor of a such that $q \equiv 3 \pmod{4}$, then $q^p|a^2$ and p does not divide $q - 1$ (if $q = p$, then also $q|b$). Show that the equation

$$x^2 + 4a^2 = y^p - b^p$$

has no solutions in integers.

MM, Jan. 1985, Lorraine L. Foster, pp. 50-51

H 54. Show that the number of integral-sided right triangles whose ratio of area to semi-perimeter is p^m , where p is a prime and m is an integer, is $m + 1$ if $p = 2$ and $2m + 1$ if $p \neq 2$.

MM, Sep. 1980, Problem 1077, Henry Klostergaard

H 55. Given that

$$34! = 95232799cd96041408476186096435ab000000_{(10)},$$

determine the digits a, b, c , and d .

British Mathematical Olympiad 2002/2003, 1-1

H 56. Prove that the equation⁵ $\prod_{cyc}(x_1 - x_2) = \prod_{cyc}(x_1 - x_3)$ has a solution in natural numbers where all x_i are different.

Latvia 1995

H 57. Show that the equation $\binom{n}{k} = m^l$ has no integral solution with $l \geq 2$ and $4 \leq k \leq n-4$.

(P. Erdős) [MaGz pp.13-16]

H 58. Solve in positive integers the equation $10^a + 2^b - 3^c = 1997$.

Belarus 1999, S. Shikh

H 59. Solve the equation $28^x = 19^y + 87^z$, where x, y, z are integers.

IMO Long List 1987 (Greece)

H 60. Show that the equation $x^7 + y^7 = 1998^z$ has no solution in positive integers.

[VsAs]

H 61. Solve the equation $2^x - 5 = 11^y$ in positive integers.

CRUX, Problem 1797, Marcin E. Kuczma

H 62. Solve the equation $7^x - 3^y = 4$ in positive integers.

India 1995

H 63. Show that $|12^m - 5^n| \geq 7$ for all $m, n \in \mathbb{N}$.

H 64. Show that there is no positive integer k for which the equation

$$(n-1)! + 1 = n^k$$

is true when n is greater than 5.

[Rdc pp.51]

H 65. Determine all pairs (x, y) of integers such that

$$(19a + b)^{18} + (a + b)^{18} + (19b + a)^{18}$$

is a nonzero perfect square.

Austria 2002

⁵There stands $(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 - x_5)(x_5 - x_6)(x_6 - x_7)(x_7 - x_1) = (x_1 - x_3)(x_2 - x_4)(x_3 - x_5)(x_4 - x_6)(x_5 - x_7)(x_6 - x_1)(x_7 - x_2)$.

H 66. Let b be a positive integer. Determine all 2002-tuples of non-negative integers $(a_1, a_2, \dots, a_{2002})$ satisfying

$$\sum_{j=1}^{2002} a_j^{a_j} = 2002b^b.$$

Austria 2002

H 67. Is there a positive integer m such that the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infinitely many solutions in positive integers a, b, c ?

IMO Short List 2002 N4

H 68. Consider the system

$$x + y = z + u,$$

$$2xy = zu.$$

Find the greatest value of the real constant m such that $m \leq \frac{x}{y}$ for any positive integer solution (x, y, z, u) of the system, with $x \geq y$.

IMO Short List 2001 N2

H 69. Determine all positive rational numbers $r \neq 1$ such that $r^{-1}\sqrt[r]{r}$ is rational.

Hong Kong 2000

H 70. Show that the equation⁶ $\{x^3\} + \{y^3\} = \{z^3\}$ has infinitely many rational non-integer solutions.

Belarus 1999

H 71. Let n be a positive integer. Prove that the equation

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

does not have solutions in positive rational numbers.

Baltic Way 2002

H 72. Find all pairs (x, y) of positive rational numbers such that $x^y = y^x$.

H 73. Find all pairs (a, b) of positive integers that satisfy the equation

$$a^{b^2} = b^a.$$

IMO 1997/5

H 74. Find all pairs (a, b) of positive integers that satisfy the equation

$$a^{a^a} = b^b.$$

⁶Here, $\{x\}$ denotes $x - \lfloor x \rfloor$.

Belarus 2000

H 75. Let a, b , and x be positive integers such that $x^{a+b} = a^b b$. Prove that $a = x$ and $b = x^x$.

Iran 1998

H 76. Find all pairs (m, n) of integers that satisfy the equation

$$(m - n)^2 = \frac{4mn}{m + n - 1}.$$

Belarus 1996

H 77. Find all pairwise relatively prime positive integers l, m, n such that

$$(l + m + n) \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \right)$$

is an integer.

Korea 1998

H 78. Let x, y , and z be integers with $z > 1$. Show that

$$(x + 1)^2 + (x + 2)^2 + \cdots + (x + 99)^2 \neq y^z.$$

Hungary 1998

H 79. Find all positive integers m and n for which

$$1! + 2! + 3! + \cdots + n! = m^2.$$

[Eb2, pp. 20] Q657, MM 52(1979), 47, 55

H 80. Prove that if a, b, c, d are integers such that $d = (a + \sqrt[3]{2}b + \sqrt[3]{4}c)^2$ then d is a perfect square.

IMO Short List 1980 (GB)

H 81. Find a pair of relatively prime four digit natural numbers A and B such that for all natural numbers m and n , $|A^m - B^n| \geq 400$.

[DfAk, pp. 18] Leningrad Mathematical Olympiad 1988

H 82. Find all triples (a, b, c) of positive integers to the equation

$$a!b! = a! + b! + c!.$$

British Mathematical Olympiad 2002/2003, 1-5

H 83. Find all pairs (a, b) of positive integers such that

$$(\sqrt[3]{a} + \sqrt[3]{b} - 1)^2 = 49 + 20\sqrt[3]{6}.$$

British Mathematical Olympiad 2000, 2-3

H 84. For what positive numbers a is

$$\sqrt[3]{2 + \sqrt{a}} + \sqrt[3]{2 - \sqrt{a}}$$

an integer?

MM, Problem 1529, David C. Kay

H 85. Find all integer solutions to $2(x^5 + y^5 + 1) = 5xy(x^2 + y^2 + 1)$.

MM, Problem 1538, Murray S. Klamkin and George T. Gilbert.

H 86. A triangle with integer sides is called Heronian if its area is an integer. Does there exist a Heronian triangle whose sides are the arithmetic, geometric and harmonic means of two positive integers?

CRUX, Problem 2351, Paul Yiu

H 87. What is the smallest perfect square that ends in 9009?

[EbMk, pp. 22]

H 88. (Leo Moser) Show that the Diophantine equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \cdots x_n} = 1$$

has at least one solution for every positive integers n .

[EbMk, pp. 46]

H 89. Prove that the number $99999 + 111111\sqrt{3}$ cannot be written in the form $(A + B\sqrt{3})^2$, where A and B are integers.

[DNI, 42]

H 90. Find all triples of positive integers (x, y, z) such that

$$(x + y)(1 + xy) = 2^z.$$

Vietnam 2004

H 91. If R and S are two rectangles with integer sides such that the perimeter of R equals the area of S and the perimeter of S equals the area of R , then we call R and S a friendly pair of rectangles. Find all friendly pairs of rectangles.

[JDS, pp. 29]

Chapter 7

Functions in Number Theory

Number theorists are like lotus-eaters - having tasted this food they can never give it up.
Leopold Kronecker

7.1 Floor Function and Fractional Part Function

I 1. Let α be the positive root of the equation $x^2 = 1991x + 1$. For natural numbers m and n define

$$m * n = mn + \lfloor \alpha m \rfloor \lfloor \alpha n \rfloor.$$

Prove that for all natural numbers p , q , and r ,

$$(p * q) * r = p * (q * r).$$

IMO ShortList 1991 P20 (IRE 3)

I 2. Prove that for any positive integer n ,

$$\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor.$$

[EbMk, pp. 5]

I 3. Prove that for any positive integer n ,

$$\left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n+4}{8} \right\rfloor + \left\lfloor \frac{n+8}{16} \right\rfloor + \cdots = n.$$

[EbMk, pp. 7]

I 4. Show that for all positive integers n ,

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor = \lfloor \sqrt{4n+3} \rfloor.$$

Canada 1987

I 5. Find all real numbers α for which the equality

$$\lfloor \sqrt{n} + \sqrt{n + \alpha} \rfloor = \lfloor \sqrt{4n + 1} \rfloor$$

holds for all positive integers n .

CRUX, Problem 1650, Iliya Bluskov

I 6. Prove that for all positive integers n ,

$$\lfloor \sqrt{n} + \sqrt{n + 1} + \sqrt{n + 2} \rfloor = \lfloor \sqrt{9n + 8} \rfloor.$$

Iran 1996

I 7. Prove that for all positive integers n ,

$$\lfloor \sqrt[3]{n} + \sqrt[3]{n + 1} \rfloor = \lfloor \sqrt[3]{8n + 3} \rfloor.$$

MM, Problem 1410, Seung-Jin Bang

I 8. Prove that $\lfloor \sqrt[3]{n} + \sqrt[3]{n + 1} + \sqrt[3]{n + 2} \rfloor = \lfloor \sqrt[3]{27n + 26} \rfloor$ for all positive integers n .

Can. Math. Soc. Notes, Problem P11, Mihály Bencze

I 9. Show that for all positive integers m and n ,

$$\gcd(m, n) = m + n - mn + 2 \sum_{k=0}^{m-1} \left\lfloor \frac{kn}{m} \right\rfloor.$$

Taiwan 1998

I 10. Show that for all primes p ,

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k^3}{p} \right\rfloor = \frac{(p+1)(p-1)(p-2)}{4}.$$

AMM, Problem 10346, David Doster

I 11. Let p be a prime number of the form $4k + 1$. Show that

$$\sum_{i=1}^{p-1} \left(\left\lfloor \frac{2i^2}{p} \right\rfloor - 2 \left\lfloor \frac{i^2}{p} \right\rfloor \right) = \frac{p-1}{2}.$$

Korea 2000

I 12. Let $p = 4k + 1$ be a prime. Show that

$$\sum_{i=1}^k \left\lfloor \sqrt{ip} \right\rfloor = \frac{p^2 - 1}{12}.$$

[IHH pp.142]

I 13. Suppose that $n \geq 2$. Prove that

$$\sum_{k=2}^n \left\lfloor \frac{n^2}{k} \right\rfloor = \sum_{k=n+1}^{n^2} \left\lfloor \frac{n^2}{k} \right\rfloor.$$

I 14. Let a, b, n be positive integers with $\gcd(a, b) = 1$. Prove¹ that

$$\sum_k \left\{ \frac{ak + b}{n} \right\} = \frac{n-1}{2},$$

where k runs through a complete system of residues modulo n .

I 15. Find the total number of different integer values the function

$$f(x) = [x] + [2x] + \left\lfloor \frac{5x}{3} \right\rfloor + [3x] + [4x]$$

takes for real numbers x with $0 \leq x \leq 100$.

APMO 1993/2

I 16. Prove or disprove that there exists a positive real number u such that $[u^n] - n$ is an even integer for all positive integer n .

I 17. Determine all real numbers a such that

$$4[an] = n + [a[an]] \text{ for all } n \in \mathbb{N}.$$

Bulgaria 2003

I 18. Do there exist irrational numbers $a, b > 1$ and $[a^m] \neq [b^n]$ for any positive integers m and n ?

[Tt] Tournament of the Towns 2002 Spring/A-Level

I 19. Let a, b, c , and d be real numbers. Suppose that $[na] + [nb] = [nc] + [nd]$ for all positive integers n . Show that at least one of $a + b$, $a - c$, $a - d$ is an integer.

[PbAw, pp. 5]

I 20. Find all integer solutions of the equation

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \cdots + \left\lfloor \frac{x}{10!} \right\rfloor = 1001.$$

(S. Reznichenko) [Ams, pp. 45]

7.2 Divisor Functions

J 1. Let n be an integer with $n \geq 2$. Show that $\phi(2^n - 1)$ is divisible by n .

¹We will denote $\{x\} = x - [x]$.

J 2. Show that for all $n \in \mathbb{N}$,

$$n = \sum_{d|n} \phi(d).$$

Gauss

J 3. If p is a prime and n an integer such that $1 < n \leq p$, then

$$\phi\left(\sum_{k=0}^{p-1} n^k\right) \equiv 0 \pmod{p}.$$

MM, Problem 1376, Eric Canning

J 4. Let m, n be positive integers. Prove that, for some positive integer a , each of $\phi(a), \phi(a+1), \dots, \phi(a+n)$ is a multiple of m .

AMM, Problem 10837, Hojoo Lee

J 5. If n is composite, prove that $\phi(n) \leq n - \sqrt{n}$.

[Km, Problems Sheet 1-11]

J 6. Show that if m and n are relatively prime positive integers, then $\phi(5^m - 1) \neq 5^n - 1$.

AMM, Problem 10626, Florian Luca

J 7. Show that if the equation $\phi(x) = n$ has one solution, it always has a second solution, n being given and x being the unknown.

[Rdc pp.36]

J 8. Prove that for any δ greater than 1 and any positive number ϵ , there is an n such that $\left|\frac{\phi(n)}{n} - \delta\right| < \epsilon$.

[PeJs, pp. 237]

J 9. Show that the set of all numbers $\frac{\phi(n+1)}{\phi(n)}$ is dense² in the set of all positive reals.

(Schinzel, Sierpiński) [Pr, pp. 38]

J 10. Show that

(a) if $n > 49$, then there are positive integers $a > 1$ and $b > 1$ such that $a + b = n$ and $\frac{\phi(a)}{a} + \frac{\phi(b)}{b} < 1$.

(b) if $n > 4$, then there are $a > 1$ and $b > 1$ such that $a + b = n$ and $\frac{\phi(a)}{a} + \frac{\phi(b)}{b} > 1$.

MM, Sep. 1983, Problem 1153, Charles R. Wall

J 11. Prove³ that $\{d((n^2 + 1)^2)\}_{n \geq 1}$ does not become monotonic from any given point onwards.

Russia 1998

²A set A is called *dense* in a set B if for any point $b \in B$, any neighborhood of b contains at least one point from A .

³We will denote $\sigma(n)$ the sum of positive divisors of n and $d(n)$ the number of positive divisors of n .

J 12. Determine all positive integers n such that $n = d(n)^2$.

Canada 1999

J 13. Determine all positive integers k such that

$$\frac{d(n^2)}{d(n)} = k$$

for some $n \in \mathbb{N}$.

IMO 1998/3

J 14. Find all positive integers n such that $d(n)^3 = 4n$.

IMO Short List 2000 N2

J 15. Determine all positive integers for which $d(n) = \frac{n}{3}$ holds.

Canada 1992

J 16. We say that an integer $m \geq 1$ is super-abundant if

$$\frac{\sigma(m)}{m} > \frac{\sigma(k)}{k}$$

for all $k \in \{1, 2, \dots, m-1\}$. Prove that there exists an infinite number of super-abundant numbers.

IMO Short List 1983 (Belgium)

J 17. Show that $\phi(n) + \sigma(n) \geq 2n$ for all positive integers n .

[Rh pp.104] Quantum, Problem M59, B. Martynov

J 18. Prove that for any δ greater than 1 and any positive number ϵ , there is an n such that $\left| \frac{\sigma(n)}{n} - \delta \right| < \epsilon$.

[PeJs, pp. 237]

J 19. Prove that $\sigma(n)\phi(n) < n^2$, but that there is a positive constant c such that $\sigma(n)\phi(n) \geq cn^2$ holds for all positive integers n .

[PeJs, pp. 237]

J 20. Show that $\sigma(n) - d(m)$ is even for all positive integers m and n where m is the largest odd divisor of n .

[Jjt, pp. 95]

J 21. Show that for any positive integer n ,

$$\frac{\sigma(n!)}{n!} \geq \sum_{k=1}^n \frac{1}{k}.$$

[Dmb, pp. 108]

J 22. Let n be an odd positive integer. Prove that $\sigma(n)^3 < n^4$.

Belarus 1999, D. Bazylev

7.3 Functional Equations

K 1. Prove that there is a function f from the set of all natural numbers into itself such that $f(f(n)) = n^2$ for all $n \in \mathbb{N}$.

Singapore 1996

K 2. Find all surjective functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $m, n \in \mathbb{N}$:

$$m|n \iff f(m)|f(n).$$

Turkey 1995

K 3. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(n+1) > f(f(n)).$$

IMO 1977/6

K 4. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(f(n))) + f(f(n)) + f(n) = 3n.$$

K 5. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(m) + f(n)) = m + n.$$

K 6. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f^{(19)}(n) + 97f(n) = 98n + 232.$$

IMO unused 1997

K 7. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(n)) + f(n) = 2n + 2001 \text{ or } 2n + 2002.$$

Balkan 2002

K 8. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(f(n))) + 6f(n) = 3f(f(n)) + 4n + 2001.$$

USAMO Summer Program 2001

K 9. Find all functions $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$:

$$f(f(n)) + f(n) = 2n + 6.$$

Austria 1989

K 10. Find all functions $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$:

$$f(m + f(n)) = f(f(m)) + f(n).$$

IMO 1996/3

K 11. Find all functions $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that for all $m, n \in \mathbb{N}_0$:

$$mf(n) + nf(m) = (m+n)f(m^2 + n^2).$$

Canada 2002

K 12. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $m, n \in \mathbb{N}$:

- $f(2) = 2$,
- $f(mn) = f(m)f(n)$,
- $f(n+1) > f(n)$.

Canada 1969

K 13. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that for all $m \in \mathbb{Z}$:

$$f(f(m)) = m + 1.$$

Slovenia 1997

K 14. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that for all $m \in \mathbb{Z}$:

- $f(m+8) \leq f(m) + 8$,
- $f(m+11) \geq f(m) + 11$.

K 15. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that for all $m, n \in \mathbb{Z}$:

$$f(m + f(n)) = f(m) - n.$$

APMC 1997

K 16. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that for all $m, n \in \mathbb{Z}$:

$$f(m + f(n)) = f(m) + n.$$

South Africa 1997

K 17. Find all functions $h : \mathbb{Z} \rightarrow \mathbb{Z}$ such that for all $x, y \in \mathbb{Z}$:

$$h(x+y) + h(xy) = h(x)h(y) + 1.$$

Belarus 1999

K 18. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

$$f(xy) = f(x)f(y) - f(x+y) + 1.$$

APMC 1984

K 19. Find all functions $f : \mathbb{Q}^+ \rightarrow \mathbb{Q}^+$ such that for all $x, y \in \mathbb{Q}$:

$$f\left(x + \frac{y}{x}\right) = f(x) + \frac{f(y)}{f(x)} + 2y, \quad x, y \in \mathbb{Q}^+.$$

K 20. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{Q}$ such that for all $x, y \in \mathbb{Q}$:

$$f(x + y) + f(x - y) = 2(f(x) + f(y)).$$

Nordic Mathematics Contest 1998

K 21. Find all functions $f, g, h : \mathbb{Q} \rightarrow \mathbb{Q}$ such that for all $x, y \in \mathbb{Q}$:

$$f(x + g(y)) = g(h(f(x))) + y.$$

KMO Winter Program Test 2001

K 22. Find all functions $f : \mathbb{Q}^+ \rightarrow \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$:

- $f(x + 1) = f(x) + 1$,
- $f(x^2) = f(x)^2$.

Ukraine 1997

K 23. Let \mathbb{Q}^+ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \rightarrow \mathbb{Q}^+$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all $x, y \in \mathbb{Q}^+$.

IMO 1990/4

K 24. A function f is defined on the positive integers by

$$\begin{cases} f(1) &= 1, \\ f(3) &= 3, \\ f(2n) &= f(n), \\ f(4n+1) &= 2f(2n+1) - f(n), \\ f(4n+3) &= 3f(2n+1) - 2f(n), \end{cases}$$

for all positive integers n . Determine the number of positive integers n , less than or equal to 1988, for which $f(n) = n$.

IMO 1988/3

K 25. Consider all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying $f(t^2 f(s)) = s(f(t))^2$ for all s and t in \mathbb{N} . Determine the least possible value of $f(1998)$.

IMO 1998/6

K 26. The function $f : \mathbb{N} \rightarrow \mathbb{N}_0$ satisfies for all $m, n \in \mathbb{N}$:

$$f(m + n) - f(m) - f(n) = 0 \text{ or } 1, \quad f(2) = 0, \quad f(3) > 0, \quad \text{and } f(9999) = 3333.$$

Determine $f(1982)$.

IMO 1982/1

K 27. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $m, n \in \mathbb{N}$:

$$f(f(m) + f(n)) = m + n.$$

IMO Short List 1988

K 28. Find all surjective functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(n) \geq n + (-1)^n.$$

Romania 1986

K 29. Find all functions $f : \mathbb{Z} \setminus \{0\} \rightarrow \mathbb{Q}$ such that for all $x, y \in \mathbb{Z} \setminus \{0\}$:

$$f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y)}{2}, \quad x, y \in \mathbb{Z} - \{0\}$$

Iran 1995

K 30. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$f(f(f(n))) + f(f(n)) + f(n) = 3n.$$

K 31. Find all strictly increasing functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$f(f(n)) = 3n.$$

K 32. Find all functions $f : \mathbb{Z}^2 \rightarrow \mathbb{R}^+$ such that for all $i, j \in \mathbb{Z}$:

$$f(i, j) = \frac{f(i+1, j) + f(i, j+1) + f(i-1, j) + f(i, j-1)}{4}.$$

K 33. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{Q}$ such that for all $x, y, z \in \mathbb{Q}$:

$$f(x+y+z) + f(x-y) + f(y-z) + f(z-x) = 3f(x) + 3f(y) + 3f(z).$$

K 34. Show that there exists a bijective function $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that for all $m, n \in \mathbb{N}_0$:

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$

K 35. Show that there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$\underbrace{f(f(\dots f(1996)\dots))}_{1996 \text{ times}} = 2n, \quad n \in \mathbb{N}.$$

Chapter 8

Sequences of Integers

A peculiarity of the higher arithmetic is the great difficulty which has often been experienced in proving simple general theorems which had been suggested quite naturally by numerical evidence.

Harold Davenport

8.1 Linear Recurrences

L 1. An integer sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_0 = 0, a_1 = 1, a_{n+2} = 2a_{n+1} + a_n$$

Show that 2^k divides a_n if and only if 2^k divides n .

IMO Short List 1988

L 2. The Fibonacci sequence $\{F_n\}$ is defined by

$$F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n.$$

Show that $\gcd(F_m, F_n) = F_{\gcd(m, n)}$ for all $m, n \in \mathbb{N}$.

[Nv pp.58]

L 3. The Fibonacci sequence $\{F_n\}$ is defined by

$$F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{mn-1} - F_{n-1}^m$ is divisible by F_n^2 for all $m \geq 1$ and $n > 1$.

[Nv pp.74]

L 4. The Fibonacci sequence $\{F_n\}$ is defined by

$$F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{mn} - F_{n+1}^m + F_{n-1}^m$ is divisible by F_n^3 for all $m \geq 1$ and $n > 1$.

[Nv pp.75]

L 5. The Fibonacci sequence $\{F_n\}$ is defined by

$$F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n.$$

Show¹ that $F_{2n-1}^2 + F_{2n+1}^2 + 1 = 3F_{2n-1}F_{2n+1}$ for all $n \geq 1$.

[Eb1 pp.21]

L 6. Prove that no Fibonacci number can be factored into a product of two smaller Fibonacci numbers, each greater than 1.

MM, Problem 1390, J. F. Stephany

L 7. Let m be a positive integer. Define the sequence $\{a_n\}_{n \geq 0}$ by

$$a_0 = 0, a_1 = m, a_{n+1} = m^2 a_n - a_{n-1}.$$

Prove² that an ordered pair (a, b) of non-negative integers, with $a \leq b$, gives a solution to the equation

$$\frac{a^2 + b^2}{ab + 1} = m^2$$

if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \geq 0$.

Canada 1998

L 8. Let $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ be two sequences defined recursively as follows

$$x_0 = 1, x_1 = 4, x_{n+2} = 3x_{n+1} - x_n,$$

$$y_0 = 1, y_1 = 2, y_{n+2} = 3y_{n+1} - y_n.$$

(a) Prove that $x_n^2 - 5y_n^2 + 4 = 0$ for all non-negative integers.

(b) Suppose that a, b are two positive integers such that $a^2 - 5b^2 + 4 = 0$. Prove that there exists a non-negative integer k such that $a = x_k$ and $b = y_k$.

Vietnam 1999

L 9. Let $\{u_n\}_{n \geq 0}$ be a sequence of positive integers defined by

$$u_0 = 1, u_{n+1} = au_n + b,$$

where $a, b \in \mathbb{N}$. Prove that for any choice of a and b , the sequence $\{u_n\}_{n \geq 0}$ contains infinitely many composite numbers.

Germany 1995

L 10. The sequence $\{y_n\}_{n \geq 1}$ is defined by

$$y_1 = y_2 = 1, y_{n+2} = (4k - 5)y_{n+1} - y_n + 4 - 2k.$$

Determine all integers k such that each term of this sequence is a perfect square.

¹See **A5**

²See **A3**

L 11. Let the sequence $\{K_n\}_{n \geq 1}$ be defined by

$$K_1 = 2, K_2 = 8, K_{n+2} = 3K_{n+1} - K_n + 5(-1)^n.$$

Prove that if K_n is prime, then n must be a power of 3.

MM, Problem 1558, Mansur Boase

L 12. The sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = 1, a_2 = 12, a_3 = 20, a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n.$$

Prove that $1 + 4a_n a_{n+1}$ is a square for all $n \in \mathbb{N}$.

[Ae, pp. 226]

L 13. The sequence $\{x_n\}_{n \geq 1}$ is defined by

$$x_1 = x_2 = 1, x_{n+2} = 14x_{n+1} - x_n - 4.$$

Prove that x_n is always a perfect square.

[Rh2, pp. 197]

8.2 Recursive Sequences

M 1. Let $P(x)$ be a nonzero polynomial with integer coefficients. Let $a_0 = 0$ and for $i \geq 0$ define $a_{i+1} = P(a_i)$. Show that $\gcd(a_m, a_n) = a_{\gcd(m,n)}$ for all $m, n \in \mathbb{N}$.

M 2. An integer sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = 1, a_{n+1} = a_n + \lfloor \sqrt{a_n} \rfloor.$$

Show that a_n is a square if and only if $n = 2^k + k - 2$ for some $k \in \mathbb{N}$.

AMM, Problem E2619, Thomas C. Brown

M 3. Let $f(n) = n + \lfloor \sqrt{n} \rfloor$. Prove that, for every positive integer m , the sequence

$$m, f(m), f(f(m)), f(f(f(m))), \dots$$

contains at least one square of an integer.

Putnam 1983

M 4. The sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = 1, a_2 = 2, a_3 = 24, a_{n+2} = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1} a_{n-2}^2}{a_{n-2} a_{n-3}}.$$

Show that a_n is an integer for all n .

Putnam 1999

M 5. Show that there is a unique sequence of integers $\{a_n\}_{n \geq 1}$ with

$$a_1 = 1, a_2 = 2, a_4 = 12, a_{n+1}a_{n-1} = a_n^2 \pm 1 \quad (n \geq 2).$$

United Kingdom 1998

M 6. The sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = 1, a_{n+1} = 2a_n + \sqrt{3a_n^2 + 1}.$$

Show that a_n is an integer for every n .

Serbia 1998

M 7. Prove that the sequence $\{y_n\}_{n \geq 1}$ defined by

$$y_0 = 1, y_{n+1} = \frac{1}{2} \left(3y_n + \sqrt{5a_n^2 - 4} \right)$$

consists only of integers.

United Kingdom 2002

M 8. The Bernoulli sequence³ $\{B_n\}_{n \geq 0}$ is defined by

$$B_0 = 1, B_n = -\frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} B_k \quad (n \geq 1)$$

Show that for all $n \in \mathbb{N}$,

$$(-1)^n B_n - \sum \frac{1}{p},$$

is an integer where the summation is done over all primes p such that $p|2k-1$.

C. von Staudt, [KiMr pp. 233]

M 9. An integer sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = 2, a_{n+1} = \left\lfloor \frac{3}{2}a_n \right\rfloor.$$

Show that it has infinitely many even and infinitely many odd integers.

Putnam 1983

M 10. An integer sequence satisfies $a_{n+1} = a_n^3 + 1999$. Show that it contains at most one square.

APMC 1999

M 11. Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and

$$a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, n \geq 4.$$

Determine $a_{14^{14}}$.

³ $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, \dots$

M 12. Let k be a fixed positive integer. The sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_1 = k + 1, a_{n+1} = a_n^2 - ka_n + k.$$

Show that if $m \neq n$, then the numbers a_m and a_n are relatively prime.

Poland 2002

M 13. The sequence $\{x_n\}$ is defined by

$$x_0 \in [0, 1], x_{n+1} = 1 - |1 - 2x_n|.$$

Prove that the sequence is periodic if and only if x_0 is irrational.

[Ae pp.228]

M 14. Let x_1 and x_2 be relatively prime positive integers. For $n \geq 2$, define $x_{n+1} = x_n x_{n-1} + 1$.

(a) Prove that for every $i > 1$, there exists $j > i$ such that x_i^i divides x_j^j .

(b) Is it true that x_1 must divide x_j^j for some $j > 1$?

IMO Short List 1994 N6

M 15. For a given positive integer k denote the square of the sum of its digits by $f_1(k)$ and let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{1991}(2^{1990})$.

IMO Short List 1990 HUN1

M 16. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i ?

Putnam 1985/A4

M 17. A sequence of integers, $\{a_n\}_{n \geq 1}$ with $a_1 > 0$, is defined by

$$\begin{aligned} a_{n+1} &= \frac{a_n}{2} & \text{if } n \equiv 0 \pmod{4}, \\ a_{n+1} &= 3a_n + 1 & \text{if } n \equiv 1 \pmod{4}, \\ a_{n+1} &= 2a_n - 1 & \text{if } n \equiv 2 \pmod{4}, \\ a_{n+1} &= \frac{a_n + 1}{4} & \text{if } n \equiv 3 \pmod{4}. \end{aligned}$$

Prove that there is an integer m such that $a_m = 1$.

CRUX, Problem 2446, Catherine Shevlin

M 18. Given is an integer sequence $\{a_n\}_{n \geq 0}$ such that $a_0 = 2$, $a_1 = 3$ and, for all positive integers $n \geq 1$, $a_{n+1} = 2a_{n-1}$ or $a_{n+1} = 3a_n - 2a_{n-1}$. Does there exist a positive integer k such that $1600 < a_k < 2000$?

Netherlands 1994

M 19. A sequence with first two terms equal 1 and 24 respectively is defined by the following rule: each subsequent term is equal to the smallest positive integer which has not yet occurred in the sequence and is not coprime with the previous term. Prove that all positive integers occur in this sequence.

[Tt] *Tournament of the Towns 2002 Fall/A-Level*

M 20. Each term of a sequence of natural numbers is obtained from the previous term by adding to it its largest digit. What is the maximal number of successive odd terms in such a sequence?

[Tt] *Tournament of the Towns 2003 Spring/O-Level*

M 21. In the sequence $1, 0, 1, 0, 1, 0, 3, 5, \dots$, each member after the sixth one is equal to the last digit of the sum of the six members just preceding it. Prove that in this sequence one cannot find the following group of six consecutive members:

$$0, 1, 0, 1, 0, 1$$

[JtPt, pp. 93] *Russia 1984*

M 22. Let a , and b be odd positive integers. Define the sequence $\{f_n\}_{n \geq 1}$ by putting $f_1 = a$, $f_2 = b$, and by letting f_n for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for sufficiently large n and determine the eventual value as a function of a and b .

USA 1993

M 23. Define

$$\begin{cases} d(n, 0) = d(n, n) = 1 & (n \geq 0), \\ md(n, m) = md(n-1, m) + (2n-m)d(n-1, m-1) & (0 < m < n). \end{cases}$$

Prove that $d(n, m)$ are integers for all $m, n \in \mathbb{N}$.

IMO Long List 1987 (GB)

M 24. Let k be a given positive integer. The sequence x_n is defined as follows: $x_1 = 1$ and x_{n+1} is the least positive integer which is not in $\{x_1, x_2, \dots, x_n, x_1 + k, x_2 + 2k, \dots, x_n + nk\}$. Show that there exist real number a such that $x_n = \lfloor an \rfloor$ for all positive integer n .

Vietnam 2000

M 25. Let $\{a_n\}_{n \geq 1}$ be a sequence of positive integers such that

$$0 < a_{n+1} - a_n \leq 2001 \quad \text{for all } n \in \mathbb{N}.$$

Show that there are infinitely many pairs (p, q) of positive integers such that $p > q$ and $a_q \mid a_p$.

Vietnam 1999

M 26. Let p be an odd prime p such that $2h \not\equiv 1 \pmod{p}$ for all $h \in \mathbb{N}$ with $h < p-1$, and let a be an even integer with $a \in [\frac{p}{2}, p[$. The sequence $\{a_n\}_{n \geq 0}$ is defined by $a_0 = a$, $a_{n+1} = p - b_n$ ($n \geq 0$), where b_n is the greatest odd divisor of a_n . Show that the sequence $\{a_n\}_{n \geq 0}$ is periodic and find its minimal (positive) period.

Poland 1995

M 27. Let $p \geq 3$ be a prime number. The sequence $\{a_n\}_{n \geq 1}$ is defined by $a_n = n$ for all $0 \leq n \leq p-1$, and $a_n = a_{n-1} + a_{n-p}$ for all $n \geq p$. Compute $a_{p^3} \pmod{p}$.

Canada 1986

M 28. Let $\{u_n\}_{n \geq 0}$ be a sequence of integers satisfying the recurrence relation $u_{n+2} = u_{n+1}^2 - u_n$ ($n \in \mathbb{N}$). Suppose that $u_0 = 39$ and $u_1 = 45$. Prove that 1986 divides infinitely many terms of this sequence.

China 1991

M 29. The sequence $\{a_n\}_{n \geq 1}$ is defined by $a_1 = 1$ and

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{4a_n} \quad (n \in \mathbb{N}).$$

Prove that $\sqrt{\frac{2}{2a_n^2 - 1}}$ is a positive integer for $n > 1$.

MM, Problem 1545, Erwin Just

M 30. Let k be a positive integer. Prove that there exists an infinite monotone increasing sequence of integers $\{a_n\}_{n \geq 1}$ such that

$$a_n \text{ divides } a_{n+1}^2 + k \text{ and } a_{n+1} \text{ divides } a_n^2 + k$$

for all $n \in \mathbb{N}$.

[Rh, pp. 276]

M 31. Each term of an infinite sequence of natural numbers is obtained from the previous term by adding to it one of its nonzero digits. Prove that this sequence contains an even number.

[Tt] Tournament of the Towns 2002 Fall/O-Level

M 32. In an increasing infinite sequence of positive integers, every term starting from the 2002-th term divides the sum of all preceding terms. Prove that every term starting from some term is equal to the sum of all preceding terms.

[Tt] Tournament of the Towns 2002 Spring/A-Level

M 33. The sequence $\{x_n\}_{n \geq 1}$ is defined by

$$x_1 = 2, x_{n+1} = \frac{2 + x_n}{1 - 2x_n} \quad (n \in \mathbb{N}).$$

Prove that

- (a) $x_n \neq 0$ for all $n \in \mathbb{N}$,
- (b) $\{x_n\}_{n \geq 1}$ is not periodic.

[Ae, pp. 227]

M 34. The sequence of integers $\{x_n\}_{n \geq 1}$ is defined as follows:

$$x_1 = 1, \quad x_{n+1} = 1 + x_1^2 + \cdots + x_n^2 \quad (n = 1, 2, 3, \dots).$$

Prove that there are no squares of natural numbers in this sequence except x_1 .

(A. Perlin) [Ams, pp. 104]

M 35. The first four terms of an infinite sequence S of decimal digits are 1, 9, 8, 2, and succeeding terms are given by the final digit in the sum of the four immediately preceding terms. Thus S begins 1, 9, 8, 2, 0, 9, 9, 0, 8, 6, 3, 7, 4, \dots . Do the digits 3, 0, 4, 4 ever come up consecutively in S ?

[Rh3, pp. 103]

8.3 More Sequences

N 1. Show that the sequence $\{a_n\}_{n \geq 1}$ defined by $a_n = \lfloor n\sqrt{2} \rfloor$ contains an infinite number of integer powers of 2.

IMO Long List 1985 (RO3)

N 2. Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence a_1, a_2, a_3, \dots become periodic after a finite number of terms?

IMO Short List 1991 P14 (USS 2)

N 3. Let $n > 6$ be an integer and a_1, a_2, \dots, a_k be all the natural numbers less than n and relatively prime to n . If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

IMO 1991/2

N 4. Show that if an infinite arithmetic progression of positive integers contains a square and a cube, it must contain a sixth power.

IMO Short List 1993

N 5. Prove that there exist two strictly increasing sequences a_n and b_n such that $a_n(a_n + 1)$ divides $b_n^2 + 1$ for every natural n .

IMO Short List 1999 N3

N 6. Let $\{a_n\}$ be a strictly increasing positive integers sequence such that $\gcd(a_i, a_j) = 1$ and $a_{i+2} - a_{i+1} > a_{i+1} - a_i$. Show that the infinite series

$$\sum_{i=1}^{\infty} \frac{1}{a_i}$$

converges.

Pi Mu Epsilon Journal, Problem 339, Paul Erdős

N 7. Let $\{n_k\}_{k \geq 1}$ be a sequence of natural numbers such that for $i < j$, the decimal representation of n_i does not occur as the leftmost digits of the decimal representation of n_j . Prove that

$$\sum_{k=1}^{\infty} \frac{1}{n_k} \leq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{9}.$$

Iran 1998

N 8. An integer sequence $\{a_n\}_{n \geq 1}$ is given such that

$$2^n = \sum_{d|n} a_d$$

for all $n \in \mathbb{N}$. Show that a_n is divisible by n for all $n \in \mathbb{N}$.

IMO Short List 1989

N 9. Let q_0, q_1, \dots be a sequence of integers such that

- for any $m > n$, $m - n$ is a factor of $q_m - q_n$, item $|q_n| \leq n^{10}$ for all integers $n \geq 0$.

Show that there exists a polynomial $Q(x)$ satisfying $q_n = Q(n)$ for all n .

Taiwan 1996

N 10. Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence n_1, n_2, \dots, n_k of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i ($1 \leq i < k$).

USA 2002

N 11. The infinite sequence of 2's and 3's

$$\begin{aligned} &2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, \\ &3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \dots \end{aligned}$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n , the n th term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m .

Putnam 1993/A6

N 12. The sequence $\{a_n\}_{n \geq 1}$ is defined by

$$a_n = 1 + 2^2 + 3^3 + \cdots + n^n.$$

Prove that there are infinitely many n such that a_n is composite.

Vietnam 2001

N 13. One member of an infinite arithmetic sequence in the set of natural numbers is a perfect square. Show that there are infinitely many members of this sequence having this property.

Croatia 1994

N 14. In the sequence 00, 01, 02, 03, \dots , 99 the terms are rearranged so that each term is obtained from the previous one by increasing or decreasing one of its digits by 1 (for example, 29 can be followed by 19, 39, or 28, but not by 30 or 20). What is the maximal number of terms that could remain on their places?

[Tt] Tournament of the Towns 2003 Spring/O-Level

N 15. Does there exist positive integers $a_1 < a_2 < \dots < a_{100}$ such that for $2 \leq k \leq 100$, the least common multiple of a_{k-1} and a_k is greater than the least common multiple of a_k and a_{k+1} ?

[Tt] Tournament of the Towns 2001 Fall/A-Level

N 16. Does there exist positive integers $a_1 < a_2 < \dots < a_{100}$ such that for $2 \leq k \leq 100$, the greatest common divisor of a_{k-1} and a_k is greater than the greatest common divisor of a_k and a_{k+1} ?

[Tt] Tournament of the Towns 2001 Fall/A-Level

N 17. Suppose that a and b are distinct real numbers such that

$$a - b, a^2 - b^2, \dots, a^k - b^k, \dots$$

are all integers. Show that a and b are integers.

[GML, pp. 173]

Chapter 9

Combinatorial Number Theory

In great mathematics there is a very high degree of unexpectedness, combined with inevitability and economy.

Godfrey Harold Hardy

O 1. Suppose all the pairs of a positive integers from a finite collection

$$A = \{a_1, a_2, \dots\}$$

are added together to form a new collection

$$A^* = \{a_i + a_j \mid 1 \leq i < j \leq n\}.$$

For example, $A = \{2, 3, 4, 7\}$ would yield $A^* = \{5, 6, 7, 9, 10, 11\}$ and $B = \{1, 4, 5, 6\}$ would give $B^* = \{5, 6, 7, 9, 10, 11\}$. These examples show that it's possible for different collections A and B to generate the same collections A^* and B^* . Show that if $A^* = B^*$ for different sets A and B , then $|A| = |B|$ and $|A| = |B|$ must be a power of 2.

(Erdős) [Rh2, pp. 243]

O 2. Let p be a prime. Find all positive integers k such that the set $\{1, 2, \dots, k\}$ can be partitioned into p subsets with equal sum of elements.

IMO Long List 1985 (PL2)

O 3. Prove that the set of integers of the form $2^k - 3$ ($k = 2, 3, \dots$) contains an infinite subset in which every two members are relatively prime.

IMO 1971/3

O 4. The set of positive integers is partitioned into finitely many subsets. Show that some subset S has the following property: for every positive integer n , S contains infinitely many multiples of n .

Berkeley Math Circle Monthly Contest 1999-2000

O 5. Let M be a positive integer and consider the set

$$S = \{n \in \mathbb{N} \mid M^2 \leq n < (M+1)^2\}.$$

Prove that the products of the form ab with $a, b \in S$ are distinct.

India 1998

O 6. Let S be a set of integers such that

- there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a - 2, b - 2) = 1$,
- if $x, y \in S$, then $x^2 - y \in S$.

Prove that $S = \mathbb{Z}$.

USA 2001

O 7. Show that for each $n \geq 2$, there is a set S of n integers such that $(a - b)^2$ divides ab for every distinct $a, b \in S$.

USA 1998

O 8. Let a and b be positive integers greater than 2. Prove that there exists a positive integer k and a finite sequence n_1, \dots, n_k of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i ($1 \leq i \leq k$).

Romania 1998

O 9. Let n be an integer, and let X be a set of $n + 2$ integers each of absolute value at most n . Show that there exist three distinct numbers $a, b, c \in X$ such that $c = a + b$.

India 1998

O 10. Let $m \geq 2$ be an integer. Find the smallest integer $n > m$ such that for any partition of the set $\{m, m + 1, \dots, n\}$ into two subsets, at least one subset contains three numbers a, b, c such that $c = a^b$.

Romania 1998

O 11. Let $S = \{1, 2, 3, \dots, 280\}$. Find the smallest integer n such that each n -element subset of S contains five numbers which are pairwise relatively prime.

IMO 1991/3

O 12. Let m and n be positive integers. If x_1, x_2, \dots, x_m are positive integers whose arithmetic mean is less than $n + 1$ and if y_1, y_2, \dots, y_n are positive integers whose arithmetic mean is less than $m + 1$, prove that some sum of one or more x 's equals some sum of one or more y 's.

MM, Problem 1466, David M. Bloom

O 13. Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set $M = \{1, 2, \dots, n - 1\}$ is colored either blue or white. It is given that

- for each $i \in M$, both i and $n - i$ have the same color,
- for each $i \in M, i \neq k$, both i and $|i - k|$ have the same color.

Prove that all numbers in M have the same color.

IMO 1985/2

O 14. Let p be a prime number, $p \geq 5$, and k be a digit in the p -adic representation of positive integers. Find the maximal length of a non constant arithmetic progression whose terms do not contain the digit k in their p -adic representation.

Romania 1997, Marian Andronache and Ion Savu

O 15. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5 , no three of which are consecutive terms of an arithmetic progression?

IMO 1983/5

O 16. Is it possible to find 100 positive integers not exceeding 25000 such that all pairwise sums of them are different?

IMO Short List 2001

O 17. Find the maximum number of pairwise disjoint sets of the form

$$S_{a,b} = \{n^2 + an + b \mid n \in \mathbb{Z}\},$$

with $a, b \in \mathbb{Z}$.

Turkey 1996

O 18. Let p be an odd prime number. How many p -element subsets A of $\{1, 2, \dots, 2p\}$ are there, the sum of whose elements is divisible by p ?

IMO 1995/6

O 19. Let $m, n \geq 2$ be positive integers, and let a_1, a_2, \dots, a_n be integers, none of which is a multiple of m^{n-1} . Show that there exist integers e_1, e_2, \dots, e_n , not all zero, with $|e_i| < m$ for all i , such that $e_1 a_1 + e_2 a_2 + \dots + e_n a_n$ is a multiple of m^n .

IMO Short List 2002 N5

O 20. Determine the smallest integer $n \geq 4$ for which one can choose four different numbers a, b, c , and d from any n distinct integers such that $a + b - c - d$ is divisible by 20.

IMO Short List 1998 P16

O 21. A sequence of integers a_1, a_2, a_3, \dots is defined as follows: $a_1 = 1$, and for $n \geq 1$, a_{n+1} is the smallest integer greater than a_n such that $a_i + a_j \neq 3a_k$ for any i, j , and k in $\{1, 2, 3, \dots, n+1\}$, not necessarily distinct. Determine a_{1998} .

IMO Short List 1998 P17

O 22. Prove that for each positive integer n , there exists a positive integer with the following properties:

- it has exactly n digits,
- none of the digits is 0,
- it is divisible by the sum of its digits.

IMO ShortList 1998 P20

O 23. Let k, m, n be integers such that $1 < n \leq m-1 \leq k$. Determine the maximum size of a subset S of the set $\{1, 2, \dots, k\}$ such that no n distinct elements of S add up to m .

IMO Short List 1996

O 24. Find the number of subsets of $\{1, 2, \dots, 2000\}$, the sum of whose elements is divisible by 5.

[TaZf pp.10] *High-School Mathematics (China) 1994/1*

O 25. Let A be a non-empty set of positive integers. Suppose that there are positive integers b_1, \dots, b_n and c_1, \dots, c_n such that

- for each i the set $b_i A + c_i = \{b_i a + c_i | a \in A\}$ is a subset of A ,
- the sets $b_i A + c_i$ and $b_j A + c_j$ are disjoint whenever $i \neq j$.

Prove that

$$\frac{1}{b_1} + \dots + \frac{1}{b_n} \leq 1.$$

IMO Short List 2002 A6

O 26. A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called historic if $\{z - y, y - x\} = \{1776, 2001\}$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.

IMO Short List 2001 C4

O 27. Let p and q be relatively prime positive integers. A subset $S \subseteq \mathbb{N}_0$ is called ideal if $0 \in S$ and, for each element $n \in S$, the integers $n + p$ and $n + q$ belong to S . Determine the number of ideal subsets of \mathbb{N}_0 .

IMO Short List 2000 C6

O 28. Prove that the set of positive integers cannot be partitioned into three nonempty subsets such that, for any two integers x, y taken from two different subsets, the number $x^2 - xy + y^2$ belongs to the third subset.

IMO Short List 1999 A4

O 29. Let A be a set of N residues $(\text{mod } N^2)$. Prove that there exists a set B of N residues $(\text{mod } N^2)$ such that the set $A + B = \{a + b | a \in A, b \in B\}$ contains at least half of all the residues $(\text{mod } N^2)$.

IMO Short List 1999 C4

O 30. Determine the largest positive integer n for which there exists a set S with exactly n numbers such that

- each member in S is a positive integer not exceeding 2002,
- if $a, b \in S$ (not necessarily different), then $ab \notin S$.

Australia 2002

O 31. Prove that, for any integer $a_1 > 1$, there exist an increasing sequence of positive integers a_1, a_2, a_3, \dots such that

$$a_1 + a_2 + \dots + a_n \mid a_1^2 + a_2^2 + \dots + a_n^2$$

for all $n \in \mathbb{N}$.

[Ae pp.228]

O 32. An odd integer $n \geq 3$ is said to be nice if and only if there is at least one permutation a_1, \dots, a_n of $1, \dots, n$ such that the n sums $a_1 - a_2 + a_3 - \dots - a_{n-1} + a_n, a_2 - a_3 + a_4 - \dots - a_n + a_1, a_3 - a_4 + a_5 - \dots - a_1 + a_2, \dots, a_n - a_1 + a_2 - \dots - a_{n-2} + a_{n-1}$ are all positive. Determine the set of all ‘nice’ integers.

IMO ShortList 1991 P24 (IND 2)

O 33. Assume that the set of all positive integers is decomposed into r disjoint subsets A_1, A_2, \dots, A_r $A_1 \cup A_2 \cup \dots \cup A_r = \mathbb{N}$. Prove that one of them, say A_i , has the following property: There exist a positive integer m such that for any k one can find numbers a_1, \dots, a_k in A_i with $0 < a_{j+1} - a_j \leq m$ ($1 \leq j \leq k-1$).

IMO Short List 1990 CZE3

O 34. Determine for which positive integers k , the set

$$X = \{1990, 1990 + 1, 1990 + 2, \dots, 1990 + k\}$$

can be partitioned into two disjoint subsets A and B such that the sum of the elements of A is equal to the sum of the elements of B .

IMO Short List 1990 MEX2

O 35. Let $n \geq 3$ be a prime number and $a_1 < a_2 < \dots < a_n$ be integers. Prove that a_1, \dots, a_n is an arithmetic progression if and only if there exists a partition of $\{0, 1, 2, \dots\}$ into sets A_1, A_2, \dots, A_n such that

$$a_1 + A_1 = a_2 + A_2 = \dots = a_n + A_n,$$

where $x + A$ denotes the set $\{x + a | a \in A\}$.

USA 2002

O 36. Let a and b be non-negative integers such that $ab \geq c^2$ where c is an integer. Prove that there is a positive integer n and integers $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ such that

$$x_1^2 + \dots + x_n^2 = a, y_1^2 + \dots + y_n^2 = b, x_1 y_1 + \dots + x_n y_n = c$$

IMO Short List 1995

O 37. Let n, k be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in the decimal representation is k .

IMO Short List 1999

O 38. Prove that for every real number M there exists an infinite arithmetical progression of positive integers such that

- the common difference is not divisible by 10,
- the sum of digits of each term exceeds M .

IMO Short List 1999

O 39. Find the smallest positive integer n for which there exist n different positive integers a_1, a_2, \dots, a_n satisfying

- $\text{lcm}(a_1, a_2, \dots, a_n) = 1985$,
- for each $i, j \in \{1, 2, \dots, n\}$, $\gcd(a_i, a_j) \neq 1$,
- the product $a_1 a_2 \cdots a_n$ is a perfect square and is divisible by 243,

and find all such n -tuples (a_1, \dots, a_n) .

Romania 1995

O 40. Let X be a non-empty set of positive integers which satisfies the following:

- if $x \in X$, then $4x \in X$,
- if $x \in X$, then $\lfloor \sqrt{x} \rfloor \in X$.

Prove that $X = \mathbb{N}$.

Japan 1990

O 41. Prove that for every positive integer n there exists an n -digit number divisible by 5^n all of whose digits are odd.

USA 2003

O 42. Let N_n denote the number of ordered n -tuples of positive integers (a_1, a_2, \dots, a_n) such that

$$1/a_1 + 1/a_2 + \dots + 1/a_n = 1.$$

Determine whether N_{10} is even or odd.

Putnam 1997/A5

O 43. Is it possible to find a set A of eleven positive integers such that no six elements of A have a sum which is divisible by 6?

British Mathematical Olympiad 2000

O 44. A set C of positive integers is called good if for every integer k there exist distinct $a, b \in C$ such that the numbers $a + k$ and $b + k$ are not relatively prime. Prove that if the sum of the elements of a good set C equals 2003, then there exists $c \in C$ such that the set $C - \{c\}$ is good.

Bulgaria 2003

O 45. Find all positive integers n with the property that the set

$$\{n, n+1, n+2, n+3, n+4, n+5\}$$

can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

IMO 1970/4

O 46. Suppose p is a prime with $p \equiv 3 \pmod{4}$. Show that for any set of $p - 1$ consecutive integers, the set cannot be divided into two subsets so that the product of the members of the one set is equal to the product of the members of the other set.

CRUX, Problem A233, Mohammed Aassila

O 47. Let S be the set of all composite positive odd integers less than 79.

(a) Show that S may be written as the union of three (not necessarily disjoint) arithmetic progressions.

(b) Show that S cannot be written as the union of two arithmetic progressions.

[KhKw, pp. 12]

O 48. Let a_1, \dots, a_{44} be natural numbers such that

$$0 < a_1 < a_2 < \dots < a_{44} < 125.$$

Prove that at least one of the 43 differences $d_j = a_{j+1} - a_j$ occurs at least 10 times.

[KhKw, pp. 13]

O 49. Consider the set of all five-digit numbers whose decimal representation is a permutation of the digits 1, 2, 3, 4, 5. Prove that this set can be divided into two groups, in such a way that the sum of the squares of the numbers in each group is the same.

(D. Fomin) [Ams, pp. 12]

O 50. What's the largest number of elements that a set of positive integers between 1 and 100 inclusive can have if it has the property that none of them is divisible by another?

[Prh, pp. 29]

O 51. Prove that among 16 consecutive integers it is always possible to find one which is relatively prime to all the rest.

[DNI, 19]

O 52. Is there a set S of positive integers such that a number is in S if and only if it is the sum of two distinct members of S or a sum of two distinct positive integers not in S ?

[JDS, pp. 31]

O 53. Suppose that the set $M = \{1, 2, \dots, n\}$ is split into t disjoint subsets M_1, \dots, M_t where the cardinality of M_i is m_i , and $m_i \geq m_{i+1}$, for $i = 1, \dots, t - 1$. Show that if $n > t! \cdot e$ then at least one class M_z contains three elements x_i, x_j, x_k with the property that $x_i - x_j = x_k$.

[Her, pp. 16]

O 54. Let S be a subset of $\{1, 2, 3, \dots, 1989\}$ in which no two members differ by exactly 4 or by exactly 7. What is the largest number of elements S can have?

[Rh2, pp. 89]

O 55. The set M consists of integers, the smallest of which is 1 and the greatest 100. Each member of M , except 1, is the sum of two (possibly identical) numbers in M . Of all such sets, find one with the smallest possible number of elements.

[Rh2, pp. 125]

O 56. Show that it is possible to color the set of integers

$$M = \{1, 2, 3, \dots, 1987\},$$

using four colors, so that no arithmetic progression with 10 terms has all its members the same color.

[Rh2, pp. 145]

O 57. Prove that every selection of 1325 integers from $M = \{1, 2, \dots, 1987\}$ must contain some three numbers $\{a, b, c\}$ which are pairwise relatively prime, but that it can be avoided if only 1324 integers are selected.

[Rh2, pp. 202]

O 58. Prove that every infinite sequence S of distinct positive integers contains either an infinite subsequence such that for every pair of terms, neither term ever divides the other, or an infinite subsequence such that in every pair of terms, one always divides the other.

[Rh3, pp. 213]

O 59. Let $a_1 < a_2 < a_3 < \dots$ be an infinite increasing sequence of positive integers in which the number of prime factors of each term, counting repeated factors, is never more than 1987. Prove that it is always possible to extract from A an infinite subsequence $b_1 < b_2 < b_3 < \dots$ such that the greatest common divisor (b_i, b_j) is the same number for every pair of its terms.

[Rh3, pp. 51]

Chapter 10

Additive Number Theory

I remember once going to see him when he was lying ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.' G. H. Hardy, on Ramanujan

P 1. Show that any integer can be expressed as a sum of two squares and a cube.

AMM, Problem 10426, Noam Elkies and Irving Kaplanky

P 2. Show that each integer n can be written as the sum of five perfect cubes (not necessarily positive).

Netherlands 1994

P 3. Prove that infinitely many positive integers cannot be written in the form

$$x_1^3 + x_2^5 + x_3^7 + x_4^9 + x_5^{11},$$

where $x_1, x_2, x_3, x_4, x_5 \in \mathbb{N}$.

Belarus 2002, V. Bernik

P 4. Determine all positive integers that are expressible in the form

$$a^2 + b^2 + c^2 + c,$$

where a, b, c are integers.

MM, Problem Q817, Robert B. McNeill

P 5. Show that any positive rational number can be represented as the sum of three positive rational cubes.

P 6. Show that every integer greater than 1 can be written as a sum of two square-free¹ integers.

¹A positive integer n is square-free if there is no prime p such that $p^2 \mid n$.

[IHH, pp. 474]

P 7. Prove that every integer $n \geq 12$ is the sum of two composite numbers.

[Tma, pp. 22]

P 8. Prove that any positive integer can be represented as an aggregate of different powers of 3, the terms in the aggregate being combined by the signs + and – appropriately chosen.

[Rdc pp.24]

P 9. The integer 9 can be written as a sum of two consecutive integers: $9=4+5$. Moreover it can be written as a sum of (more than one) consecutive positive integers in exactly two ways, namely $9=4+5=2+3+4$. Is there an integer which can be written as a sum of 1990 consecutive integers and which can be written as a sum of (more than one) consecutive positive integers in exactly 1990 ways?

IMO Short List 1990 AUS3

P 10. For each positive integer n , $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.

(a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.

(b) Find an integer n such that $S(n) = n^2 - 14$.

(c) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

IMO 1992/6

P 11. For each positive integer n , let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:

$$4, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1.$$

Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

IMO 1997/6

P 12. The positive function $p(n)$ is defined as the number of ways that the positive integer n can be written as a sum of positive integers.² Show that, for all positive integers $n \geq 2$,

$$2^{\lfloor \sqrt{n} \rfloor} < p(n) < n^{3\lfloor \sqrt{n} \rfloor}.$$

[Hua pp.199]

P 13. Let $a_1 = 1, a_2 = 2, a_3, a_4, \dots$ be the sequence of positive integers of the form $2^\alpha 3^\beta$, where α and β are nonnegative integers. Prove that every positive integer is expressible in the form

$$a_{i_1} + a_{i_2} + \dots + a_{i_n},$$

where no summand is a multiple of any other.

²For example, $5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$ gives us $p(5) = 7$.

MM, Problem Q814, Paul Erdős

P 14. Let n be a non-negative integer. Find all non-negative integers a, b, c, d such that

$$a^2 + b^2 + c^2 + d^2 = 7 \cdot 4^n.$$

Romania 2001, Laurentiu Panaitopol

P 15. Find all integers $m > 1$ such that m^3 is a sum of m squares of consecutive integers.

AMM, Problem E3064, Ion Cucurezeanu

P 16. Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers.

Putnam 2000

P 17. Let p be a prime number of the form $4k+1$. Suppose that r is a quadratic residue of p and that s is a quadratic nonresidue of p . Show that $p = a^2 + b^2$, where

$$a = \frac{1}{2} \sum_{i=1}^{p-1} \left(\frac{i(i^2 - r)}{p} \right), b = \frac{1}{2} \sum_{i=1}^{p-1} \left(\frac{i(i^2 - s)}{p} \right).$$

Here, $\left(\frac{k}{p} \right)$ denotes the Legendre Symbol.

Jacobsthal

P 18. Let p be a prime with $p \equiv 1 \pmod{4}$. Let a be the unique integer such that

$$p = a^2 + b^2, \quad a \equiv -1 \pmod{4}, \quad b \equiv 0 \pmod{2}$$

Prove that

$$\sum_{i=0}^{p-1} \left(\frac{i^3 + 6i^2 + i}{p} \right) = 2 \left(\frac{2}{p} \right),$$

where $\left(\frac{k}{p} \right)$ denotes the Legendre Symbol.

AMM, Problem 2760, Kenneth S. Williams

P 19. Let n be an integer of the form $a^2 + b^2$, where a and b are relatively prime integers and such that if p is a prime, $p \leq \sqrt{n}$, then p divides ab . Determine all such n .

APMO 1994/3

P 20. If an integer n is such that $7n$ is the form $a^2 + 3b^2$, prove that n is also of that form.

India 1998

P 21. Let A be the set of positive integers of the form $a^2 + 2b^2$, where a and b are integers and $b \neq 0$. Show that if p is a prime number and $p^2 \in A$, then $p \in A$.

Romania 1997, Marcel Tena

P 22. Show that an integer can be expressed as the difference of two squares if and only if it is not of the form $4k+2$ ($k \in \mathbb{Z}$).

P 23. Show that there are infinitely many positive integers which cannot be expressed as the sum of squares.

P 24. Show that any integer can be expressed as the form $a^2 + b^2 - c^2$, where $a, b, c \in \mathbb{Z}$.

P 25. Let a and b be positive integers with $\gcd(a, b) = 1$. Show that every integer greater than $ab - a - b$ can be expressed in the form $ax + by$, where $x, y \in \mathbb{N}_0$.

P 26. Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where $x, y, z \in \mathbb{N}_0$

IMO 1983/3

P 27. Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

IMO 1976/4

P 28. Prove that any positive integer can be represented as a sum of Fibonacci numbers, no two of which are consecutive.

Zeckendorf

P 29. Show that the set of positive integers which cannot be represented as a sum of distinct perfect squares is finite.

IMO Short List 2000 N6

P 30. Let a_1, a_2, a_3, \dots be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j , and k are not necessarily distinct. Determine a_{1998} .

IMO Short List 1998 P21

P 31. A finite sequence of integers a_0, a_1, \dots, a_n is called *quadratic* if for each $i \in \{1, 2, \dots, n\}$ we have the equality $|a_i - a_{i-1}| = i^2$.

- (a) Prove that for any two integers b and c , there exists a natural number n and a quadratic sequence with $a_0 = b$ and $a_n = c$.
- (b) Find the smallest natural number n for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.

IMO Short List 1996 N3

P 32. A composite positive integer is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite positive integer is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

Putnam 1988/B1

P 33. Let a_1, a_2, \dots, a_k be relatively prime positive integers. Determine the largest integer which cannot be expressed in the form

$$x_1 a_2 a_3 \cdots a_k + x_2 a_1 a_3 \cdots a_k + \cdots + x_k a_1 a_2 \cdots a_{k-1}$$

for some nonnegative integers x_1, x_2, \dots, x_k .

MM, Problem 1561, Emre Alkan

P 34. If n is a positive integer which can be expressed in the form $n = a^2 + b^2 + c^2$, where a, b, c are positive integers, prove that for each positive integer k , n^{2k} can be expressed in the form $A^2 + B^2 + C^2$, where A, B, C are positive integers.

[KhKw, pp. 21]

P 35. Prove that every positive integer which is not a member of the infinite set below is equal to the sum of two or more distinct numbers of the set

$$\{3, -2, 2^2 3, -2^3, \dots, 2^{2k} 3, -2^{2k+1}, \dots\} = \{3, -2, 12, -8, 48, -32, 192, \dots\}.$$

[EbMk, pp. 46]

P 36. Let k and s be odd positive integers such that

$$\sqrt{3k-2} - 1 \leq s \leq \sqrt{4k}.$$

Show that there are nonnegative integers t, u, v , and w such that

$$k = t^2 + u^2 + v^2 + w^2, \text{ and } s = t + u + v + w.$$

[Wsa, pp. 271]

P 37. Let $S_n = \{1, n, n^2, n^3, \dots\}$, where n is an integer greater than 1. Find the smallest number $k = k(n)$ such that there is a number which may be expressed as a sum of k (possibly repeated) elements in S_n in more than one way. (Rearrangements are considered the same.)

[GML, pp. 37]

P 38. Find the smallest possible n for which there exist integers x_1, x_2, \dots, x_n such that each integer between 1000 and 2000 (inclusive) can be written as the sum (without repetition), of one or more of the integers x_1, x_2, \dots, x_n .

[GML, pp. 144]

P 39. In how many ways can 2^n be expressed as the sum of four squares of natural numbers?

[DNI, 28]

P 40. Show that

- (a) infinitely many perfect squares are a sum of a perfect square and a prime number,
- (b) infinitely many perfect squares are not a sum of a perfect square and a prime number.

[JDS, pp. 25]

P 41. The famous conjecture of Goldbach is the assertion that every even integer greater than 2 is the sum of two primes. Except 2, 4, and 6, every even integer is a sum of two positive composite integers: $n = 4 + (n - 4)$. What is the largest positive even integer that is not a sum of two odd composite integers?

[JDS, pp. 25]

P 42. Prove that for each positive integer K there exist infinitely many even positive integers which can be written in more than K ways as the sum of two odd primes.

MM, Feb. 1986, Problem 1207, Barry Powell

P 43. A positive integer n is abundant if the sum of its proper divisors exceeds n . Show that every integer greater than 89×315 is the sum of two abundant numbers.

MM, Nov. 1982, Problem 1130, J. L. Selfridge

Chapter 11

Various Problems

The only way to learn Mathematics is to do Mathematics.

Paul Halmos

11.1 Polynomials

Q 1. Suppose $p(x) \in \mathbb{Z}[x]$ and $P(a)P(b) = -(a - b)^2$ for some distinct $a, b \in \mathbb{Z}$. Prove that $P(a) + P(b) = 0$.

MM, Problem Q800, Bjorn Poonen

Q 2. Prove that there is no nonconstant polynomial $f(x)$ with integral coefficients such that $f(n)$ is prime for all $n \in \mathbb{N}$.

Q 3. Let $n \geq 2$ be an integer. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.

IMO 1987/6

Q 4. A prime p has decimal digits $p_n p_{n-1} \cdots p_0$ with $p_n > 1$. Show that the polynomial $p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0$ cannot be represented as a product of two nonconstant polynomials with integer coefficients

Balkan Mathematical Olympiad 1989

Q 5. (Eisenstein's Criterion) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a nonconstant polynomial with integer coefficients. If there is a prime p such that p divides each of a_0, a_1, \dots, a_{n-1} but p does not divide a_n and p^2 does not divide a_0 , then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

[Twh, pp. 111]

Q 6. Prove that for a prime p , $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbb{Q}[x]$.

[Twh, pp. 114]

Q 7. Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

Q 8. Show that a polynomial of odd degree $2m + 1$ over \mathbb{Z} ,

$$f(x) = c_{2m+1}x^{2m+1} + \cdots + c_1x + c_0,$$

is irreducible if there exists a prime p such that

$$p \nmid c_{2m+1}, p \mid c_{m+1}, c_{m+2}, \dots, c_{2m}, p^2 \mid c_0, c_1, \dots, c_m, \text{ and } p^3 \nmid c_0.$$

(Eugen Netto) [Ac, pp. 87] For a proof, see [En].

Q 9. For non-negative integers n and k , let $P_{n,k}(x)$ denote the rational function

$$\frac{(x^n - 1)(x^n - x) \cdots (x^n - x^{k-1})}{(x^k - 1)(x^k - x) \cdots (x^k - x^{k-1})}.$$

Show that $P_{n,k}(x)$ is actually a polynomial for all $n, k \in \mathbb{N}$.

CRUX, Problem A230, Naoki Sato

Q 10. Suppose that the integers a_1, a_2, \dots, a_n are distinct. Show that

$$(x - a_1)(x - a_2) \cdots (x - a_n) - 1$$

cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

[Ae, pp. 257]

Q 11. Show that the polynomial $x^8 + 98x^4 + 1$ can be expressed as the product of two nonconstant polynomials with integer coefficients.

[Ae, pp. 258]

Q 12. Prove that if the integers a_1, a_2, \dots, a_n are all distinct, then the polynomial

$$(x - a_1)^2(x - a_2)^2 \cdots (x - a_n)^2 + 1$$

cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

[DNI, 47]

Q 13. On Christmas Eve, 1983, Dean Jixon, the famous seer who had made startling predictions of the events of the preceding year that the volcanic and seismic activities of 1980 and 1981 were connected with mathematics. The diminishing of this geological activity depended upon the existence of an elementary proof of the irreducibility of the polynomial

$$P(x) = x^{1981} + x^{1980} + 12x^2 + 24x + 1983.$$

Is there such a proof?

MM, Jan. 1982, Problem 1113, William H. Gustafson

11.2 The Geometry of Numbers

R 1. Does there exist a convex pentagon, all of whose vertices are lattice¹ points in the plane, with no lattice point in the interior?

MM, Problem 1409, Gerald A. Heur

R 2. Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

Putnam 1993/B5

R 3. Prove no three lattice points in the plane form an equilateral triangle.

R 4. The sidelengths of a polygon with 1994 sides are $a_i = \sqrt{i^2 + 4}$ ($i = 1, 2, \dots, 1994$). Prove that its vertices are not all on lattice points.

Israel 1994

R 5. A triangle has lattice points as vertices and contains no other lattice points. Prove that its area is $\frac{1}{2}$.

R 6. Let R be a convex region symmetrical about the origin with area greater than 4. Show that R must contain a lattice point different from the origin.

[Hua pp.535]

R 7. Show that the number $r(n)$ of representations of n as a sum of two squares has π as arithmetic mean, that is

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n r(m) = \pi.$$

[GJ.Jj pp.215]

R 8. Prove that on a coordinate plane it is impossible to draw a closed broken line such that

- coordinates of each vertex are rational,
- the length of its every edge is equal to 1,
- the line has an odd number of vertices.

IMO Short List 1990 USS3

R 9. Prove that if a lattice parallelogram contains an odd number of lattice points, then its centroid.

[PeJs, pp. 125]

R 10. Prove that if a lattice triangle has no lattice points on its boundary in addition to its vertices, and one point in its interior, then this interior point is its center of gravity.

[PeJs, pp. 125]

R 11. Prove that if a lattice parallelogram contains at most three lattice points in addition to its vertices, then those are on one of the diagonals.

[PeJs, pp. 125]

R 12. Find coordinates of a set of eight non-collinear planar points so that each has an integral distance from others.

[Jjt, pp. 75]

¹A point with integral coordinates

11.3 Miscellaneous problems

Mathematics is not yet ready for such problems.

Paul Erdős

S 1. Two positive integers are chosen. The sum is revealed to logician A , and the sum of squares is revealed to logician B . Both A and B are given this information and the information contained in this sentence. The conversation between A and B goes as follows: B starts

B: 'I can't tell what they are.'
A: 'I can't tell what they are.'
B: 'I can't tell what they are.'
A: 'I can't tell what they are.'
B: 'I can't tell what they are.'
A: 'I can't tell what they are.'
B: 'Now I can tell what they are.'

- (a) What are the two numbers?
- (b) When B first says that he cannot tell what the two numbers are, A receives a large amount of information. But when A first says that he cannot tell what the two numbers are, B already knows that A cannot tell what the two numbers are. What good does it do B to listen to A ?

MM, May 1984, Problem 1173, Thomas S. Ferguson

S 2. It is given that 2^{333} is a 101-digit number whose first digit is 1. How many of the numbers 2^k , $1 \leq k \leq 332$, have first digit 4?

[Tt] Tournament of the Towns 2001 Fall/A-Level

S 3. Is there a power of 2 such that it is possible to rearrange the digits giving another power of 2?

[Pt] Tournament of the Towns

S 4. If x is a real number such that $x^2 - x$ is an integer, and for some $n \geq 3$, $x^n - x$ is also an integer, prove that x is an integer.

Ireland 1998

S 5. Suppose that both $x^3 - x$ and $x^4 - x$ are integers for some real number x . Show that x is an integer.

Vietnam 2003 (Tran Nam Dung)

S 6. Suppose that x and y are complex numbers such that

$$\frac{x^n - y^n}{x - y}$$

are integers for some four consecutive positive integers n . Prove that it is an integer for all positive integers n .

S 7. Let n be a positive integer. Show that

$$\sum_{k=1}^n \tan^2 \frac{k\pi}{2n+1}$$

is an odd integer.

S 8. The set $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ contains arithmetic progressions of various lengths. For instance, $\frac{1}{20}, \frac{1}{8}, \frac{1}{5}$ is such a progression of length 3 and common difference $\frac{3}{40}$. Moreover, this is a maximal progression in S since it cannot be extended to the left or the right within S ($\frac{11}{40}$ and $\frac{-1}{40}$ not being members of S). Prove that for all $n \in \mathbb{N}$, there exists a maximal arithmetic progression of length n in S .

British Mathematical Olympiad 1997

S 9. Suppose that

$$\prod_{n=1}^{1996} (1 + nx^{3^n}) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m}$$

where a_1, a_2, \dots, a_m are nonzero and $k_1 < k_2 < \cdots < k_m$. Find a_{1996} .

Turkey 1996

S 10. Let p be an odd prime. Show that there is at most one non-degenerate integer triangle with perimeter $4p$ and integer area. Characterize those primes for which such triangle exist.

CRUX, Problem 2331, Paul Yiu

S 11. For each positive integer n , prove that there are two consecutive positive integers each of which is the product of n positive integers greater than 1.

[Rh, pp. 165] Unused problems for 1985 CanMO

S 12. Let

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n) .

Putnam 1985/B3

S 13. The sum of the digits of a natural number n is denoted by $S(n)$. Prove that $S(8n) \geq \frac{1}{8}S(n)$ for each n .

Latvia 1995

S 14. Let p be an odd prime. Determine positive integers x and y for which $x \leq y$ and $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is nonnegative and as small as possible.

IMO Short List 1992 P17

S 15. Let $\alpha(n)$ be the number of digits equal to one in the dyadic representation of a positive integer n . Prove that

- (a) the inequality $\alpha(n^2) \leq \frac{1}{2}\alpha(n)(1 + \alpha(n))$ holds,
- (b) equality is attained for infinitely $n \in \mathbb{N}$,
- (c) there exists a sequence $\{n_i\}$ such that $\lim_{i \rightarrow \infty} \frac{\alpha(n_i^2)}{\alpha(n_i)} = 0$.

S 16. Show that if a and b are positive integers, then

$$\left(a + \frac{1}{2}\right)^n + \left(b + \frac{1}{2}\right)^n$$

is an integer for only finitely many positive integer n .

[Ns pp.4]

S 17. Determine the maximum value of $m^2 + n^2$, where m and n are integers satisfying $m, n \in \{1, 2, \dots, 1981\}$ and $(n^2 - mn - m^2)^2 = 1$.

IMO 1981/3

S 18. Denote by S the set of all primes p such that the decimal representation of $\frac{1}{p}$ has the fundamental period of divisible by 3. For every $p \in S$ such that $\frac{1}{p}$ has the fundamental period $3r$ one may write

$$\frac{1}{p} = 0.a_1a_2 \cdots a_{3r}a_1a_2 \cdots a_{3r} \cdots,$$

where $r = r(p)$. For every $p \in S$ and every integer $k \geq 1$ define

$$f(k, p) = a_k + a_{k+r(p)} + a_{k+2r(p)}.$$

- (a) Prove that S is finite.
- (b) Find the highest value of $f(k, p)$ for $k \geq 1$ and $p \in S$.

IMO Short List 1999 N4

S 19. Determine all pairs (a, b) of real numbers such that $a[b n] = b[a n]$ for all positive integer n .

IMO Short List 1998 P15

S 20. Let n be a positive integer that is not a perfect cube. Define real numbers a, b, c by

$$a = \sqrt[3]{n}, \quad b = \frac{1}{a - [a]}, \quad c = \frac{1}{b - [b]}.$$

Prove that there are infinitely many such integers n with the property that there exist integers r, s, t , not all zero, such that $ra + sb + tc = 0$.

S 21. Find, with proof, the number of positive integers whose base- n representation consists of distinct digits with the property that, except for the leftmost digit, every digit differs by ± 1 from some digit further to the left.

USA 1990

S 22. The decimal expression of the natural number a consists of n digits, while that of a^3 consists of m digits. Can $n + m$ be equal to 2001?

[Tt] Tournament of the Towns 2001 Spring/O-Level

S 23. Observe that

$$\begin{aligned}\frac{1}{1} + \frac{1}{3} &= \frac{4}{3}, & 4^2 + 3^2 &= 5^2, \\ \frac{1}{3} + \frac{1}{5} &= \frac{8}{15}, & 8^2 + 15^2 &= 17^2, \\ \frac{1}{5} + \frac{1}{7} &= \frac{12}{35}, & 12^2 + 35^2 &= 37^2.\end{aligned}$$

State and prove a generalization suggested by these examples.

[EbMk, pp. 10]

S 24. A number n is called a Niven number, named for Ivan Niven, if it is divisible by the sum of its digits. For example, 24 is a Niven number. Show that it is not possible to have more than 20 consecutive Niven numbers.

(C. Cooper, R. E. Kennedy) [Jjt, pp. 58]

S 25. Prove that if the number α is given by decimal $0.9999\dots$, where there are at least 100 nines, then $\sqrt{\alpha}$ also has 100 nines at the beginning.

[DNI, 20]

S 26. Prove that there does not exist a natural number which, upon transfer of its initial digit to the end, is increased five, six or eight times.

[DNI, 12]

S 27. Which integers have the following property? If the final digit is deleted, the integer is divisible by the new number.

[DNI, 11]

S 28. Let A be the set of the 16 first positive integers. Find the least positive integer k satisfying the condition: In every k -subset of A , there exist two distinct $a, b \in A$ such that $a^2 + b^2$ is prime.

Vietnam 2004

S 29. What is the rightmost nonzero digit of $1000000!$?

[JDS, pp. 28]

S 30. For how many positive integers n is

$$\left(1999 + \frac{1}{2}\right)^n + \left(2000 + \frac{1}{2}\right)^n$$

an integer?

[JDS, pp. 30]

S 31. Is there a 3×3 magic² square consisting of distinct Fibonacci numbers (both f_1 and f_2 may be used; thus two 1s are allowed)?

[JDS, pp. 31]

S 32. Alice and Bob play the following number-guessing game. Alice writes down a list of positive integers x_1, \dots, x_n , but does not reveal them to Bob, who will try to determine the numbers by asking Alice questions. Bob chooses a list of positive integers a_1, \dots, a_n and asks Alice to tell him the value of $a_1x_1 + \dots + a_nx_n$. Then Bob chooses another list of positive integers b_1, \dots, b_n and asks Alice for $b_1x_1 + \dots + b_nx_n$. Play continues in this way until Bob is able to determine Alice's numbers. How many rounds will Bob need in order to determine Alice's numbers?

[JDS, pp. 57]

S 33. Four consecutive even numbers are removed from the set

$$A = \{1, 2, 3, \dots, n\}.$$

If the arithmetic mean of the remaining numbers is 51.5625, which four numbers were removed?

[Rh2, pp. 78]

S 34. Let S_n be the sum of the digits of 2^n . Prove or disprove that $S_{n+1} = S_n$ for some positive integer n .

MM Nov. 1982, Q679, M. S. Klamkin and M. R. Spiegel

S 35. Counting from the right end, what is the 2500th digit of $10000!$?

MM Sep. 1980, Problem 1075, Phillip M. Dunson

S 36. For every natural number n , denote $Q(n)$ the sum of the digits in the decimal representation of n . Prove that there are infinitely many natural numbers k with $Q(3^k) > Q(3^{k+1})$.

Germany 1996

S 37. Let n and k are integers with $n > 0$. Prove that

$$-\frac{1}{2n} \sum_{m=1}^{n-1} \cot \frac{\pi m}{n} \sin \frac{2\pi km}{n} = \begin{cases} \frac{k}{n} - \lfloor \frac{k}{n} \rfloor - \frac{1}{2} & \text{if } k \nmid n \\ 0 & \text{otherwise} \end{cases}.$$

[Tma, pp.175]

S 38. The function $\mu : \mathbb{N} \rightarrow \mathbb{C}$ is defined by

$$\mu(n) = \sum_{k \in R_n} \left(\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \right),$$

where $R_n = \{k \in \mathbb{N} | 1 \leq k \leq n, \gcd(k, n) = 1\}$. Show that $\mu(n)$ is an integer for all positive integer n .

²A magic square is an $n \times n$ matrix, containing the numbers $1, 2, \dots, n^2$ exactly once each, such that the sum of the elements in each row, each column, and each of both main diagonals is equal.

Chapter 12

References

Abbreviations used in the book

AIME	American Invitational Mathematics Examination
APMO	Asian Pacific Mathematics Olympiads
IMO	International Mathematical Olympiads
CRUX	Crux Mathematicorum (with Mathematical Mayhem)
MM	Math. Magazine
AMM	American Math. Monthly

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¹ Australian Mathematics Trust

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