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Applications of the
Pigeonhole Principle

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§1. Introduction

We begin our discussion with a common daily embarrassing moment. Suppose that in one's dresser drawer, he has socks of three different colours (all placed in *messy* order). Having to get up early in the morning while it is still dark, how does he ensure that he gets a matching pair of same coloured socks in the most convenient way without disturbing his partner? While, the answer is simple! He just has to take 4 socks from the drawer! The answer behind this is of course, the **Pigeonhole Principle** which we will be exploring in this Maths Project.

What is the Pigeonhole Principle then? Let me give you an example to illustrate this principle. For instance, there are 3 pigeonholes around. A pigeon is delivering 4 mails and has to place all its mails into the available pigeonholes. With only 3 pigeonholes around, there bound to be 1 pigeonhole with at least 2 mails!.

Thus, the general rule states **when there are k pigeonholes and there are k+1 mails, then they will be 1 pigeonhole with at least 2 mails**. A more advanced version of the principle will be the following: **If mn + 1 pigeons are placed in n pigeonholes, then there will be at least one pigeonhole with m + 1 or more pigeons in it.**

The Pigeonhole Principle sounds trivial but its uses are deceiving astonishing! Thus, in our project, **we aim to learn and explore more about the Pigeonhole Principle and illustrate its numerous interesting applications in our daily life.**

We begin with the following simple example:

§2. Pigeonhole Principle and the Birthday problem

We have always heard of people saying that in a large group of people, it is not difficult to find two persons with their birthday on the same month. For instance, 13 people are involved in a survey to determine the month of their birthday. As we all know, there are 12 months in a year, thus, even if the first 12 people have their birthday from the month of January to the month of December, the 13^{th} person has to have his birthday in any of the month of January to December as well. Thus, we are right to say that there are at least 2 people who have their birthday falling in the same month.

In fact, we can view the problem as there are 12 pigeonholes (months of the year) with 13 pigeons (the 13 persons). Of course, by the Pigeonhole Principle, there will be at least one pigeonhole with 2 or more pigeons*!*

Here's another example of the application of Pigeonhole Principle with people's relationship:

§3. Pigeonhole Principle and problems on relations

Assume that the relation `*to be acquainted with*' is symmetric: if Peter is acquainted with Paul, then Paul is acquainted with Peter.

Suppose that there are 50 people in the room. Some of them are acquainted with each other, while some not. Then we can show that there are two persons in the room who have equal numbers of acquaintances.

Let's assume that there is one person in the room that has ***no acquaintance*** at all, then the others in the room will have either 1, 2, 3, 4, …, 48 acquaintance, or do not have acquaintance at all. Therefore we have 49 "pigeonholes" numbered 0, 1, 2, 3, ………, 48 and we have to distribute between them 50 "pigeons". So, there are *at least two* persons that have the same number of acquaintance with the others.

Next, if everyone in the room has **at least one acquaintance**, we will still have 49 "pigeonholes" numbered 1, 2, 3, ………, 48, 49 and we have to be distribute between them 50 "pigeons"!

Also, we can apply the Pigeonhole Principle in the proving of numerical properties. The following are two of such examples:

§4. Pigeonhole Principle and divisibility

Consider the following random list of 12 numbers say, 2, 4, 6, 8, 11, 15, 23, 34, 55, 67, 78 and 83. Is it possible to choose two of them such that their difference is divisible by 11? Can we provide an answer to the problem by applying the Pigeonhole Principle?

There are 11 possible remainders when a number is divided by 11:

0, 1, 2, 3, …………….., 9, 10.

But we have 12 numbers. If we take the remainders for "pigeonholes" and the numbers for "pigeons", then by the Pigeon-Hole Principle, there are at least two pigeons sharing the same hole, ie two numbers with the same remainder. The difference of these two numbers is thus divisible by 11!

In fact, in our example, there are several answers as the two numbers whose difference is divisible by 11 could be 4 & 15; 34 & 67 or 6 & 83.

§5. Pigeonhole Principle and numerical property

We can also apply the Pigeonhole Principle in determining useful numerical properties. Consider a sequence of any 7 distinct real numbers. Is it possible to select two of them say x and y, which satisfy the inequality that $\mathbf{0 < \frac{x - y}{1 + xy} < \frac{1}{\sqrt{3}}}$?

The problem sounds difficult as we may need to consider more advanced calculus and trigonometrical methods in the determination of the result. Well, to answer the above problem, one will be surprised to know that we just need a simple trigonometrical identity and apply the Pigeonhole Principle!

Before proceeding to answer the problem, we first note that given any real number x, we can always find a real number α where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ such that $\tan \alpha = x$. For example, $\sqrt{3} = \tan\frac{\pi}{3}$, $-1 = \tan(-\frac{\pi}{4})$. So, with given 7 distinct numbers n_1, n_2, ……..and n_7, we can find 7 distinct numbers α_1, α_2, …….α_7 in the above stated range such that :

$$n_1 = \tan \alpha_1,\ n_2 = \tan \alpha_2,\ \ldots\ldots,\ n_7 = \tan \alpha_7$$

Now, if we were to divide the interval $(-\pi, \pi)$ into 6 equal intervals, we obtain the following sub-intervals:

$$(-\frac{1}{2}\pi,\ -\frac{1}{3}\pi),\ [-\frac{1}{3}\pi, -\frac{1}{6}\pi),\ [-\frac{1}{6}\pi,\ 0),\ [0, \frac{1}{6}\pi),\ [\frac{1}{6}\pi, \frac{1}{3}\pi) \text{ and } [\frac{1}{3}\pi,\ \frac{1}{2}\pi).$$

For the 7 distinct numbers α_1, α_2, …….α_7, by the Pigeonhole Principle, there should be two values say, α_i and α_j such that $\boldsymbol{\alpha_I > \alpha_j}$ and ***α_i & α_j are in the same interval!*** For these two values α_i and α_j, we should have $\mathbf{0 <} \alpha_I - \alpha_j \mathbf{<} \frac{1}{6}\pi$**.**

We may recall an important trigonometrical identity:

$$\mathbf{\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}.$$

Thus, if $n_i = \alpha_i$ and $n_j = \alpha_j$, then $\mathbf{\frac{n_i - n_j}{1 + n_i n_j} = \frac{\tan\alpha_i - \tan\alpha_j}{1 + \tan\alpha_i \tan\alpha_j}}$

$$\mathbf{= \tan(\alpha_i - \alpha_j)}$$

As $\mathbf{0 <} \alpha_I - \alpha_j \mathbf{<} \frac{1}{6}\pi$, $\quad \mathbf{\tan 0 < \tan(\alpha_i - \alpha_j) < \tan \frac{1}{6}\pi}$

$$\mathbf{0 < \tan(\alpha_i - \alpha_j) < \frac{1}{\sqrt{3}}}$$

and so, $\quad \mathbf{0 < \frac{n_i - n_j}{1 + n_i n_j} < \frac{1}{\sqrt{3}}},$

which is the result we are seeking!

We may also apply the Pigeonhole Principle in the proving of useful daily geometrical results.. The following examples illustrate such usages:

§6. Pigeonhole Principle and Geometry

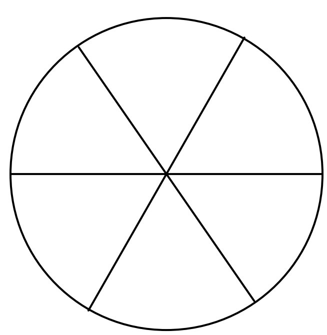
a. Dartboard applications

Another common type of problem requiring the pigeonhole principle to solve are those which involve the dartboard. In such questions, a given number of darts are thrown onto a dartboard, the general shape and size of which are known. Possible maximum distance between two certain darts is then to be determined. As with most questions involving the pigeonhole principle, the hardest part is to identify the pigeons and pigeonholes.

Example 1:

Seven darts are thrown onto a circular dartboard of radius 10 units. Can we show that there will always be two darts which are at most 10 units apart?

To prove that the final statement is always true, we first divide the circle into six equal sectors as shown;



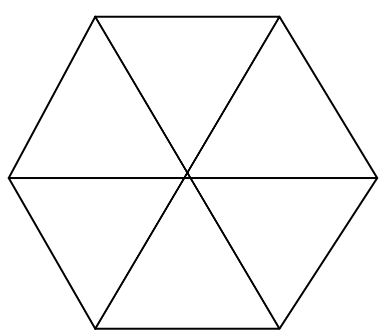
Allowing each sector to be a pigeonhole and each dart to be a pigeon, we have seven pigeons to go into six pigeonholes. By pigeonhole principle, there is at least one sector containing a minimum of two darts. Since the greatest distance between two points lying in a sector is 10 units, the statement is proven to be true in any case.

In fact, it is also possible to prove the scenario with only six darts. In such a case, the circle is this time divided into five sectors and all else follows. However, take note that this is not always true anymore with only five darts or less.

Example 2:

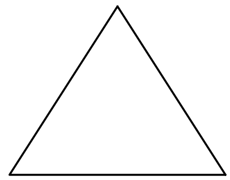
Nineteen darts are thrown onto a dartboard which is shaped as a regular hexagon with side length of 1 unit. Can we prove that there are two darts within $\frac{\sqrt{3}}{3}$ units of each other ?

Again, we identify our pigeonholes by dividing the hexagon into six equilateral triangles as shown below.



With the six triangles as our pigeonholes and the 19 darts as pigeons, we find that there must be at least one triangle with a minimum of 4 darts in it.

Now, considering the best case scenario, we will have to try an equilateral triangle of side 1 unit with 4 points inside.



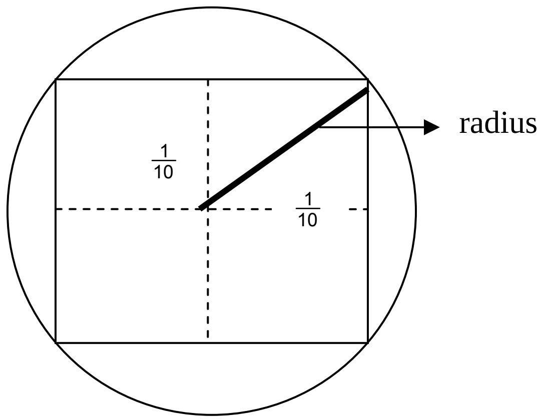
If we try to put the points as far apart from each other as possible, we will end up assigning each of the first three points to the vertices of the triangle. The last point will then be at the exact centre of the triangle. As we know that the distance from the centre of the triangle to each vertex is two-third of the altitude of this triangle, that is, $\frac{2}{3}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}$ units, we can see that it is definitely possible to find two darts which are $\frac{\sqrt{3}}{3}$ units apart within the equilateral triangle!

b. Encompassing problems

Consider the following problem:

51 points are placed, in a random way, into a square of side 1 unit. Can we prove that 3 of these points can be covered by a circle of radius $\frac{1}{7}$ units ?

To prove the result, we may divide the square into 25 equal smaller squares of side $\frac{1}{5}$ units each. Then by the Pigeonhole Principle, at least one of these small squares (so call “pigeonholes”) should contain at least 3 points (ie the “pigeons”). Otherwise, each of the small squares will contain 2 or less points which will then mean that the total number of points will be less than 50 , which is a contradiction to the fact that we have 51 points in the first case !



Now the circle circumvented around the particular square with the three points inside should have

$$\text{radius} = \sqrt{(\frac{1}{10})^2 + (\frac{1}{10})^2} = \sqrt{\frac{2}{100}} = \sqrt{\frac{1}{50}} < \sqrt{\frac{1}{49}} = \mathbf{\frac{1}{7}} \textbf{ units !}$$

It will be worthwhile to note the above technique can be useful in analyzing accuracy of weapons in shooting practices and tests.

Next, we will like to proceed to a more creative aspect of the application of Pigeonhole Principle by showing how it can be used to design interesting games:

§7. Application of pigeonhole principle in card games

We like to introduce the application of pigeonhole principle in two exciting card tricks:

a. Combinatorial Card Trick :

Here's the trick:

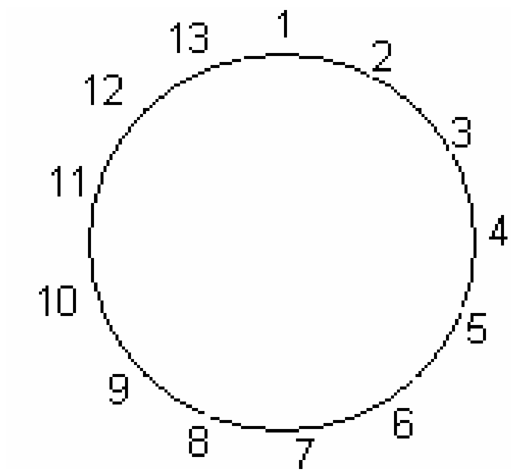
A magician asks an unsuspecting observer to randomly choose five cards from a standard deck of playing cards. The participant does not show these cards to the magician, but does show them to the magician's accomplice. The accomplice looks at the five cards, chooses four of them, and shows these four to the magician in a certain ordered manner. The magician immediately identifies the fifth hidden card.

How does the trick work?

The following is an explanation of our working strategy:

(1) First of all, **notice that in any hand of five cards there must be two cards of the same suit (an application of Pigeonhole Principle).** ***The <u>first</u> card that the accomplice shows to the magician is one of these two cards.*** The other card of the same suit is never shown – it is the mystery card, the card which the magician must discover. Thus, the accomplice can easily communicate the suit of the hidden card: the hidden card has same suit as the first card shown to the magician. Specifying the rank of the mystery card (ie its value) is a little trickier but can be accomplished with a little "circular counting" manner which we will explained below

Number the cards in a suit circularly from 1(ace) to 11 (jack), 12 (queen) and 13 (king) so that 1 follows 13 i.e. the list is ordered in a clockwise direction.



Now, given any two cards A and B, define ***distance (A,B)*** **as the clockwise distance from A to B.** It is easy to see that for any two cards A and B either distance(A,B) or distance(B,A) must always be less than or equal to 6. **Again as an application of the Pigeonhole Principle, we note that if they are both 7 or more, then there will be at least 2 x 7 = 14 cards in a standard suit of cards!!**

Example

Cards: 3 and Jack (11)

distance(Jack, 3) = 5*; distance (3, Jack) = 8*

Cards: Ace(1) and 7

distance (Ace, 7) = 6*; distance (7,Ace) = 7*

(2) Our working strategy thus proceeds as follows.:

From those two cards of the same suit, A and B, the accomplice shows the magician card A such that distance(A, B) is 6 or less.

For example, given the choice between the three of clubs and the Jack of clubs, the accomplice reveals the Jack (since distance (Jack ,3) = 5 and distance(3, Jack)= 8). The three of clubs remains hidden.

If the two same-suit cards are the five of hearts and the six of hearts, the accomplice chooses the five (since distance (5,6) = 1 but distance (6,5) = 12) leaving the six of hearts as the mystery card.

(3) Finally, the accomplice arranges the last three cards to encode a number from 1 to 6 – the distance from the value of first card to that of the hidden card. A quick calculation allows the magician to discover the value of the mystery card. Notice that although the magician must decode only one of 6 possibilities, it should not present a problem, even to the slowest of magicians.

To facilitate the explanation for the last step involved, we may assign each card a number from 1 to 52 for ranking purpose. For example,

> the ace of spade can be numbered 1 (the highest ranking card),
>
> ace of heart numbered 2,
>
> ace of club numbered 3,
>
> ace of diamond numbered 4,
>
> king of spade numbered 5,

……………………………..,

queen of spade numbered 9,

……………………………….,

jack of spade numbered 13,

……………………………….,

10 of spade numbered 17,

…………………………… ,

…………………………… ,

2 of diamond numbered 52 (the lowest ranking card).

We will now proceed to explain the last step using the following example:

***Example**:*

Suppose the five cards chosen are the following:

(i) <u>3 of Hearts (numbered 46)</u>

(ii) 5 of Spades (numbered 37)

(iii) 6 of Clubs (numbered 35)

(iv) <u>7 of Hearts (numbered 30)</u>

(v) 2 of Diamonds (numbered 52)

The accomplice notices that the 3 and the 7 have the same suit-- hearts. Since the distance(3 ,7) = 4 and distance(7, 3) = 9, the accomplice chooses the 3 as the first card to show the magician, leaving the 7 of hearts as the hidden card. The magician now knows that the suit of the mystery card is ***hearts.*** The accomplice's next task is thus to let the magician know that he must **add the value 4 to the number 3 to obtain the final value of 7 for the hidden card!**

How can he achieve this? Basically, he can arrange the other three cards in 3! = 6 ways. Based on the numbering method explained earlier, the 3 remaining cards can be ranked 1^{st}, 2^{nd} and 3^{rd}. In our example, the 6 of Clubs will be ranked 1, the 5 of Spades will be ranked 2 and the 2 of Diamonds will be ranked 3. The accomplice may agree with the magician earlier that the arrangement of these 3 cards represent specific numbers as shown below:

Order in which 3 remaining cards are shown	Number represented by the arrangement
1, 2, 3	1
1, 3, 2	2
2, 1, 3	3
2, 3, 1	**4**
3, 1, 2	5
3, 2, 1	6

Thus in our example, the accomplice should display the cards in the following manner: ***firstly, the 5 of Spades, then the 2 of Diamonds and lastly, the 6 of Clubs !***

b. Permutation Card Trick:

Here's the trick:

A magician asks an unsuspecting observer to randomly arrange 10 cards which are labelled 1 to 10 in a hidden "face down" manner. The participant does not show the arrangement of these cards to the magician, but does show them to the magician's accomplice. The accomplice looks at the ten cards and flips over six of the cards in a certain ordered manner to reveal their values to the magician. The magician immediately identifies the values of the four remaining unknown cards.

How does the trick work?

We first note that by applying the Pigeonhole Principle, we can show that ***in any permutation of 10 distinct numbers there exists an increasing subsequence of at least 4 numbers or a decreasing subsequence of at least 4 numbers.*** *(refer next section of our discussion).* These are the numbers that remain hidden in our trick. The magician will know that the sequence is increasing if the accomplice flips over the other six cards from the left to right and it is decreasing if the other six cards are flipped over from the right to the left.

We will now proceed to explain the trick behind the game:

The trick behind the game:

Given any sequence of mn+1 real numbers, some subsequence of (m+1) numbers is increasing or some subsequence of (n+1) numbers is decreasing.

We shall prove the result by 'Contradiction' method.

Assume that the result is false. For each number *x* in the sequence, we have the ordered pair (i, j), where i is the length of the longest increasing subsequence beginning with x, and j is the length of the longest decreasing subsequence ending with x. Then, since the result is false, $1 \le i \le m$ and $1 \le j \le n$. Thus we have mn+1 ordered pairs, of which at most mn are distinct. Hence ***by the Pigeonhole Principle,*** two members of the sequence, say a and b, are associated with the same ordered pair (s, t). Without loss of generality, we may assume that a precedes b in the sequence.

If a<b, then a, together with the longest increasing subsequence beginning with b, is an increasing subsequence of length (s+1), contradicting the fact that s is the length of the longest increasing subsequence beginning with a. Hence $a \ge b$. But then, b, together with the longest decreasing subsequence ending with a, is a subsequence of length (t+1), contradicting that the longest decreasing subsequence ending with b is of length t. This is clearly a contradiction to our assumption and so the result must be true.

Thus, in our trick, we should have an increasing subsequence of at least (3+1) numbers or a decreasing subsequence of at least (3+ 1) numbers in a permutation of (3x3+ 1) distinct numbers!

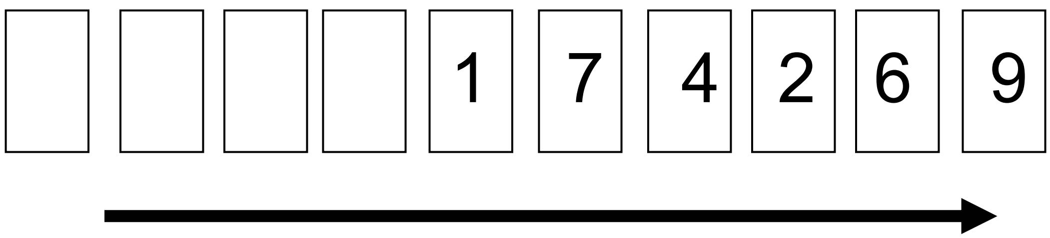
Here is an example of how the trick can be performed:

Example

Suppose the participant arranges the 10 cards in the following manner (value faced down from left to right): 3, 5, 8, 10, 1, 7, 4, 2, 6, 9.

Upon careful inspection, the accomplice notices that an increasing subsequence can be 3, 5, 8, 10 while a decreasing subsequence can be 10, 7, 4, 2.

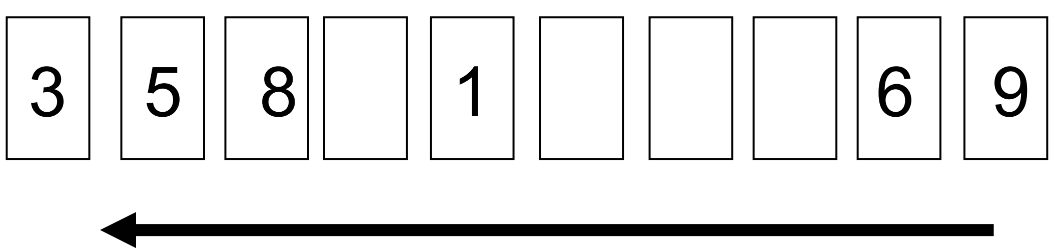
If he decides to use the increasing subsequence, he should leave the first four cards untouched and flips the other six cards over in a leftward manner as shown:



Direction of flip

The magician on realising that the four missing numbers are 3, 5, 8 and 10 and the leftward direction of flip, will thus proclaim the 4 hidden numbers to be 3, 5, 8, and 10 respectively!

If the accomplice decides to use the decreasing subsequence, he should leave the cards bearing the numbers 10, 7, 4, 2 untouched and flips the other six cards over in a rightward manner as shown:



Direction of flip

The magician on realising that the four missing numbers are 2, 4, 7 and 10 and the rightward direction of flip, will thus proclaim the 4 hidden numbers (from left to right) to be 10, 7, 4, 2 respectively!

§8. Conclusion

Although the Pigeonhole Principle seems simple and trivial, it is extremely useful in helping one to formulate and facilitate calculation and proving steps for numerous important Mathematical results and applications. We have included just a substantial amount of its applications in our project discussion. More importantly, we will like to show that a simple Mathematical concept like the Pigeonhole Principle does have numerous interesting and beneficial application in our daily life!

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