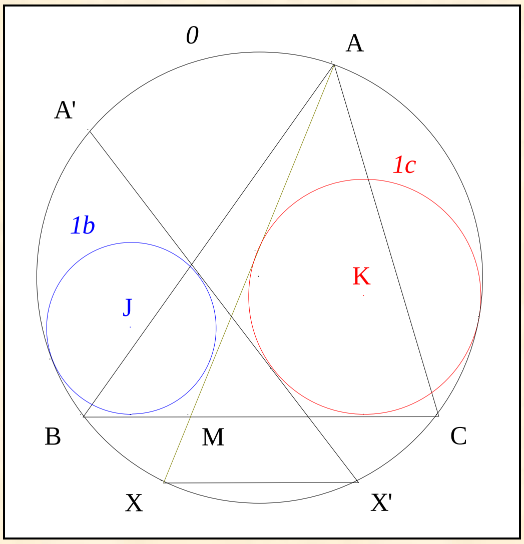
TWO PARALLEL TANGENT THEOREMS

†

Jean - Louis AYME



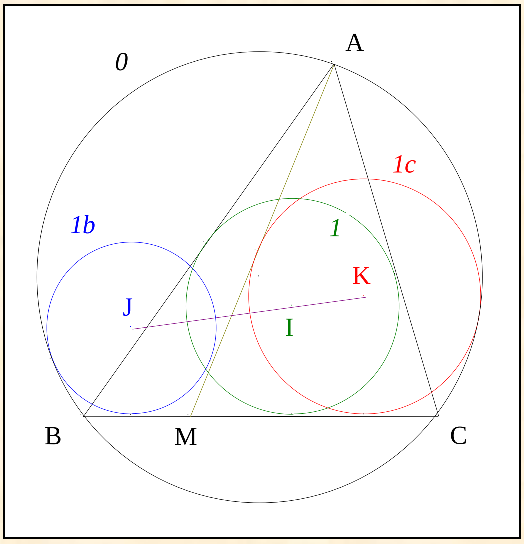
Abstract. The author presents a synthetic proof of the parallel tangent theorem with two applications. Another parallel tangent theorem is also presented.
The Appendix recalls the rediscovery of Michail Thiomkin.
The figures are all in general position and all the theorems quoted can be proved synthetically.

Summary

A. THÉBAULT's THEOREM

VISION

Figure :

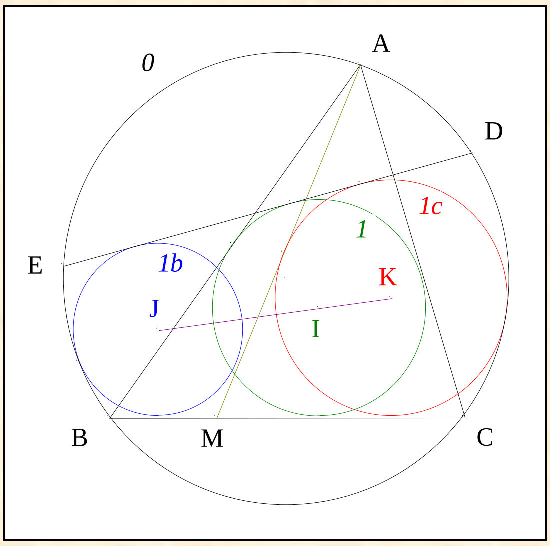


Features : ABC a triangle,
0 the circumcircle of ABC,
M a point on the segment BC,
1 the incircle of ABC,
I the center of *1*,
1b, *1c* the B, C-Thébault's circles of ABC wrt M
and J, K the centers of *1b*, *1c*.

Given : I, J and K are collinear [1].

Remarks : **(1)** the second external tangent

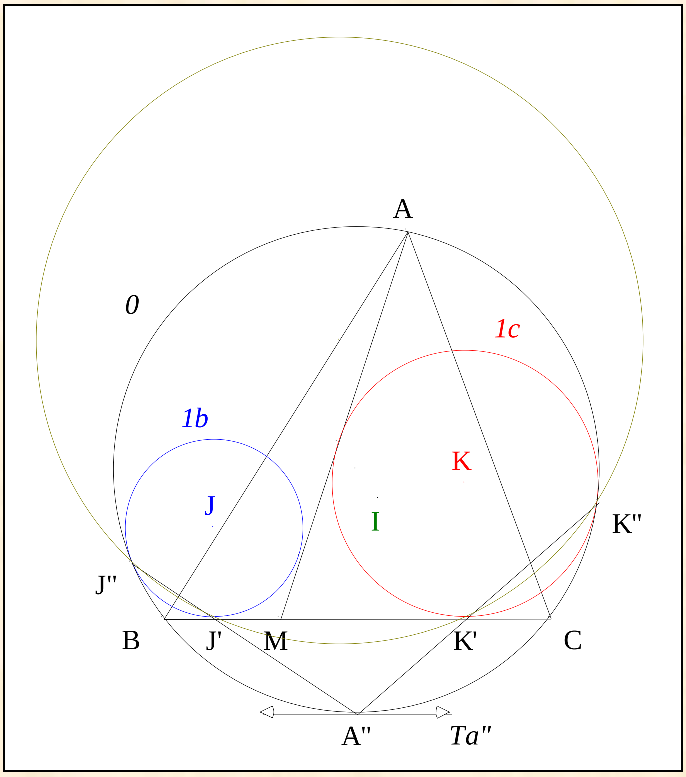
[1] Sawayama Y., *American Mathematical Monthly* vol. **12** (1905) 222-224.
Thébault V., Problem 3887, Three circles with collinear centers, *Amer. Math. Monthly* **45** (1938) 482-483.
Ayme J.-L., Sawayama and Thébault's theorem, G.G.G. vol. **10** ; http://perso.orange.fr/jl.ayme.
Ayme J.-L., Sawayama and Thébault's theorem, *Forum Geometricorum* (2003) 225-229 ; http://forumgeom.fau.edu/.



- Note D, E the points of intersection of the second common external tangent to *1b*, *1c* with *0*.

- **Conclusion :** by symmetry wrt IJK, DE is tangent to *1*.

(2) Four concyclic points



- Note J", K" the points of contact of *1b*, *1c* with *0*,
  A" the midpoint of the arc BC which doesn't contain A
  and *Ta"* the tangent to *0* at A".

- **Remarks :** **(1)** J", J' and A" are collinear
  (2) K", K' and A" are collinear
  (3) *Ta"* // J'K'.

- **Conclusion :** the circle *0*, the basic points J" and K", the borning monians A"J"J' and A"K"K',

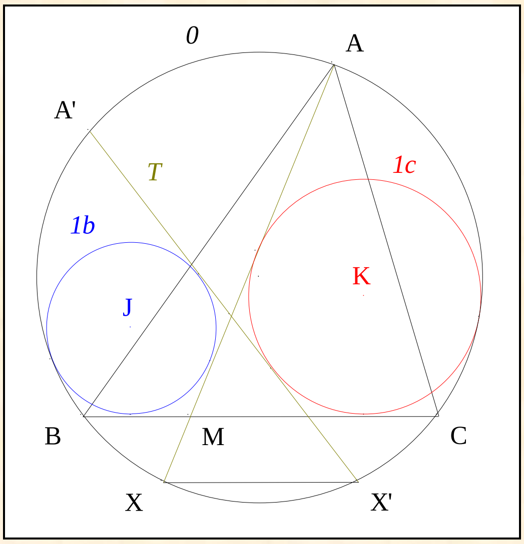
the parallels *Ta"* and J'K', lead to Reim's theorem **1"** ;
consequently, J", K", J', K' are concyclic.

B. THE PARALLEL TANGENT THEOREM

1. Theorem

VISION

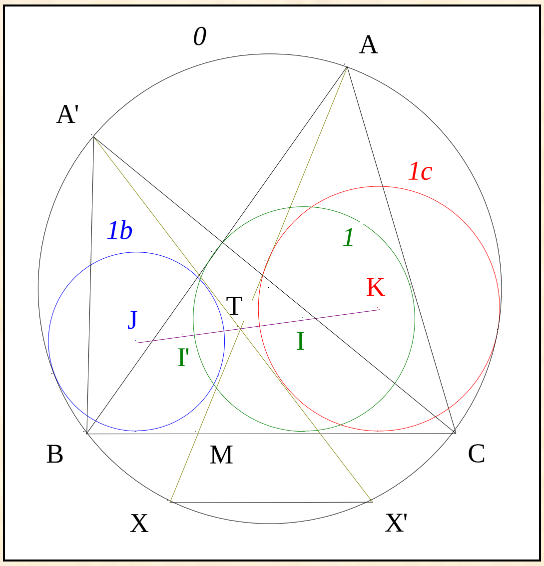
Figure :



Features :	ABC	a triangle,
	0	the circumcircle of ABC,
	M	a point on the segment BC,
	1b, *1c*	the B, C-Thébault's circles of ABC wrt M,
	J, K	the centers of *1b*, *1c*,
	X	the second point of intersection of AM with *0*,
	T	the second common internal tangent of *1b* and *1c*,
and	A', X'	the point of intersection of *T* with *0* as shown in the figure.

Given : XX' and BC are parallel.

VISUALIZATION

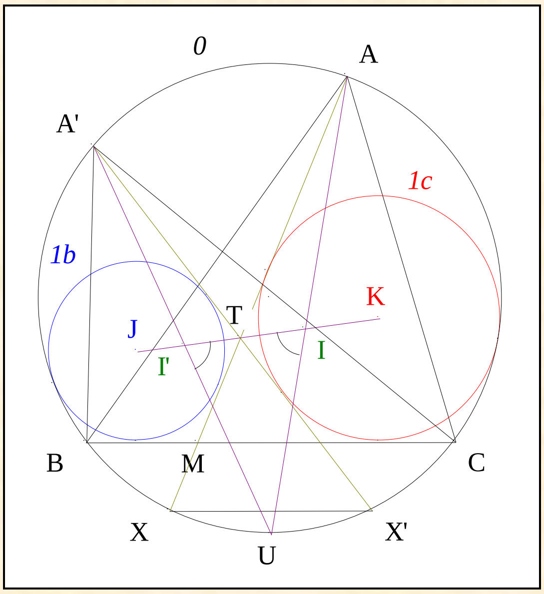


- Note *1* the incircle of ABC,
  I the center of *1*,
  I' the center of the incircle of the triangle A'BC
  and T le point of intersection of AX and A'X'.

- **Remarks :** **(1)** J, K and T are collinear
  (2) AX and A'X' are symmetric wrt JKT.

- According to **A.** Thébault's theorem applied to ABC wrt M, **(1)** I, J and K are collinear
  (2) I', J and K are collinear.

- **Partial conclusion :** I, I', J, K and T are collinear.



- Note U the midpoint of the arc BC which doesn't contain A.

- **Remarks :** **(1)** A, I and U are collinear
  (2) A', I' and U are collinear.

- According to "A Mention's circle" (Cf. Annex 1), the triangle UII' is U-isosceles.

- An geometric angle chasing :

UII' being U-isosceles, <UI'I = <I'IU ;

by symmetry, we have : <ATI = <A'TI' ;

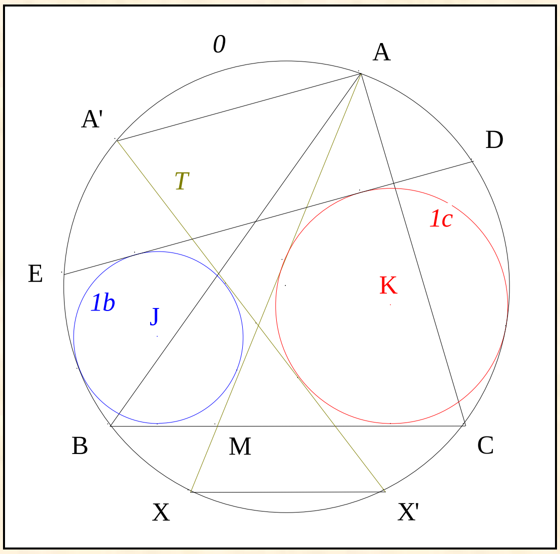
according to "Sum of the angles of a triangle" applied to the triangle ATI and A'TI', <IAT = <I'A'T or <UAX = <UAX'.

- **Partial conclusion :** U is the midpoint of the arc XX' which doesn't contain A.

- **Conclusion :** XX' and BC are parallel.

Historic note : This result was reported in 2002 by Shay Gueron[2] with his "theorem 2" and rediscovered by the beloved Juan Carlos Salazar[3] on *Mathlinks* where metric and trigonometric proofs are presented.
The above proof is inspired from that of "Huyền Vũ".

Remarks : **(1)** with the other external tangent

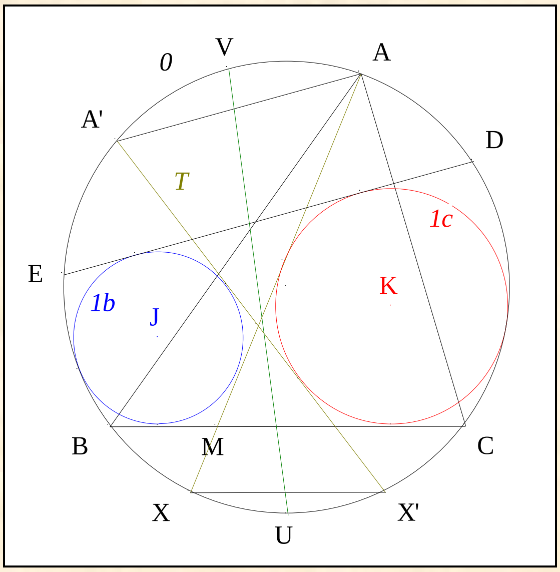


- Note D, E the points of intersection of the second common external tangent to *1b* and *1c* with *0.*

- **Conclusion :** mutatis mutandis, we would prove that AA' and DE are parallel.

(2) Radical axis of *1b* and *1c*

[2] Shay Gueron, Two Applications of the Generalized Ptolemy Theorem, *Amer. Math. Monthly* **109** (2002) 362-370.
[3] Salazar J. C., Parallel tangent, *Mathlinks* of 08/26/2004 ; http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=15945.
Parallel problem # 20, *Mathlinks* du 07/28/2010 ; http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=35929 ;
Circle, *Mathlinks* of 02/23/2011 ; http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=393179.



- Note U the midpoint of the arc BC which doesn't contain A
  and V the midpoint of the arc DE which doesn't contain X.

- **Conclusion :** UV is the radical axis of *1b* and *1c*.

Historic note : This result was reported in 2004 by "Grobber"[4] and rediscovered by Jean-Pierre Ehrmann[5] on *Hyacinthos.* A synthetic proof can be seen on *Geometry * Géométrie * Geometria* [6].

2. A short note about Shay Gueron

After his Ph.D. in 1991 at Technion, Israel Institute of Technology, Shay Gueron

4 Parallel tangent, *Mathlinks* of 08/26/2004 ; http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=15945.

5 Ehrmann J.-P., on the Thebault circles of a cevian (b), Message *Hyacinthos* # 15979 of 01/05/2008 ; http://tech.groups.yahoo.com/group/Hyacinthos/message/15979.

6 Ayme J.-L., A new mixtilinear incircle adventure III, G.G.G. vol. 4 p. 42-43 ; http://perso.orange.fr/jl.ayme.

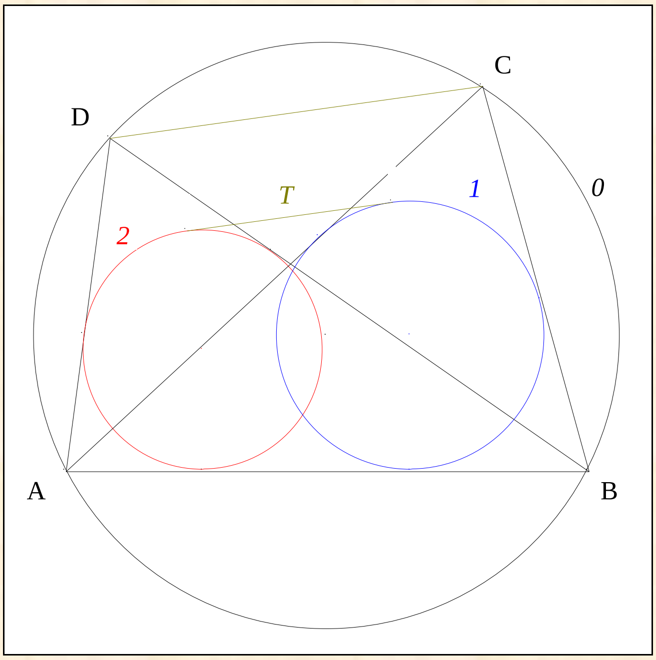
is nowadays professor of mathematics at the University of Haifa (Israel).

C. APPLICATIONS

1. With two incircles

VISION

Figure :

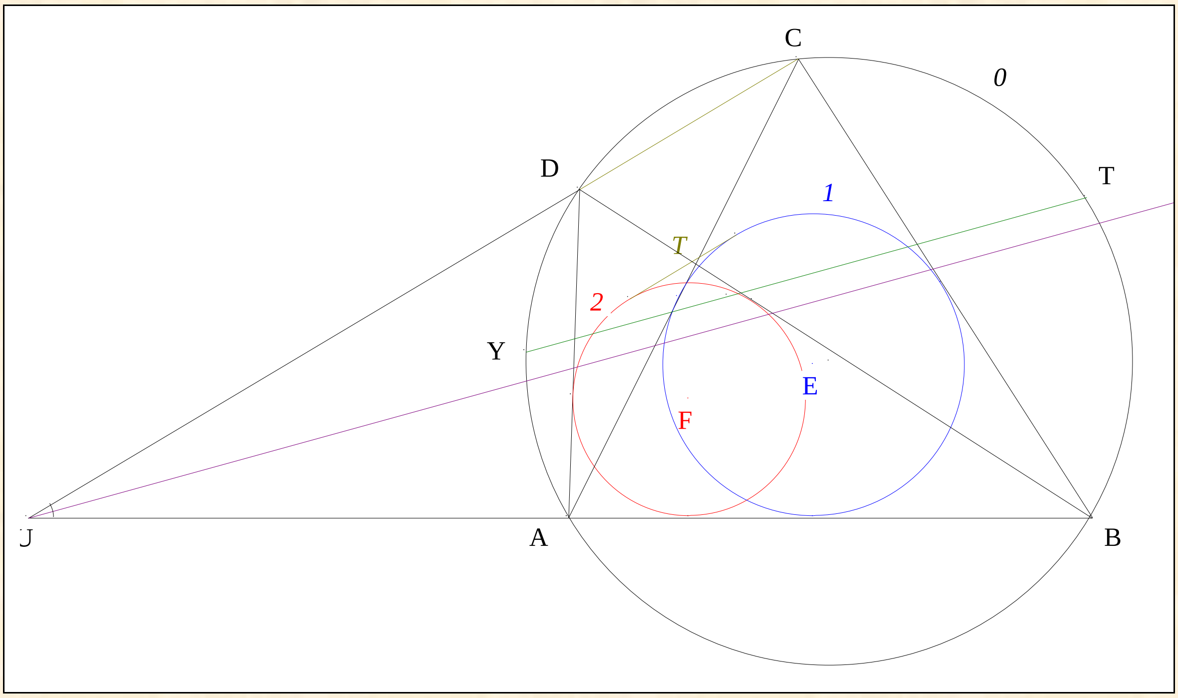


Features : ABCD a cyclic quadrilateral,
0 the circumcircle of ABCD,
1, *2* the incircles of the resp. triangles CAB, DAB,
T the second common external tangent of *1* and *2*,

Given : *T* is parallel to CD.[7]

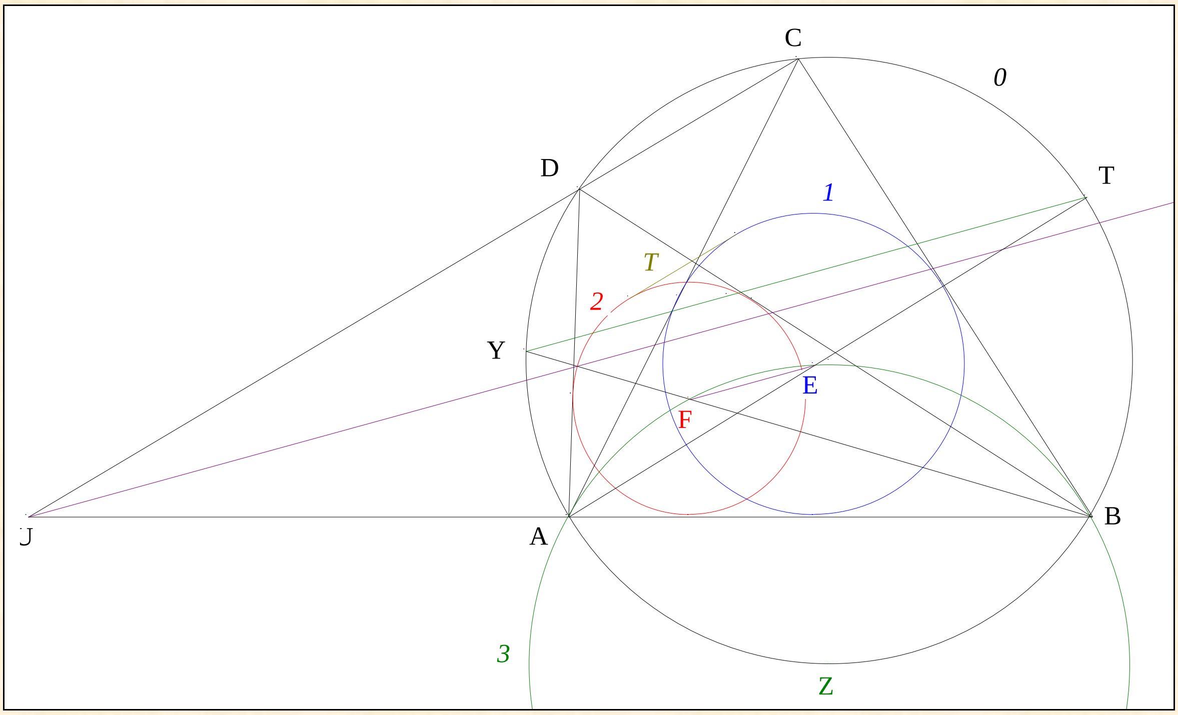
VISUALIZATION

[7] Ayme J.-L., A variant of the parallel tangent theorem, *Mathlinks* /03/2011 ; http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=394862.



- Note Y the midpoint of the arc DA which doesn't contain B,
  T the midpoint of the arc BC which doesn't contain D,
  E, F the centers resp. of *1*, *2*
  and U the point of intersection of AB and CD.

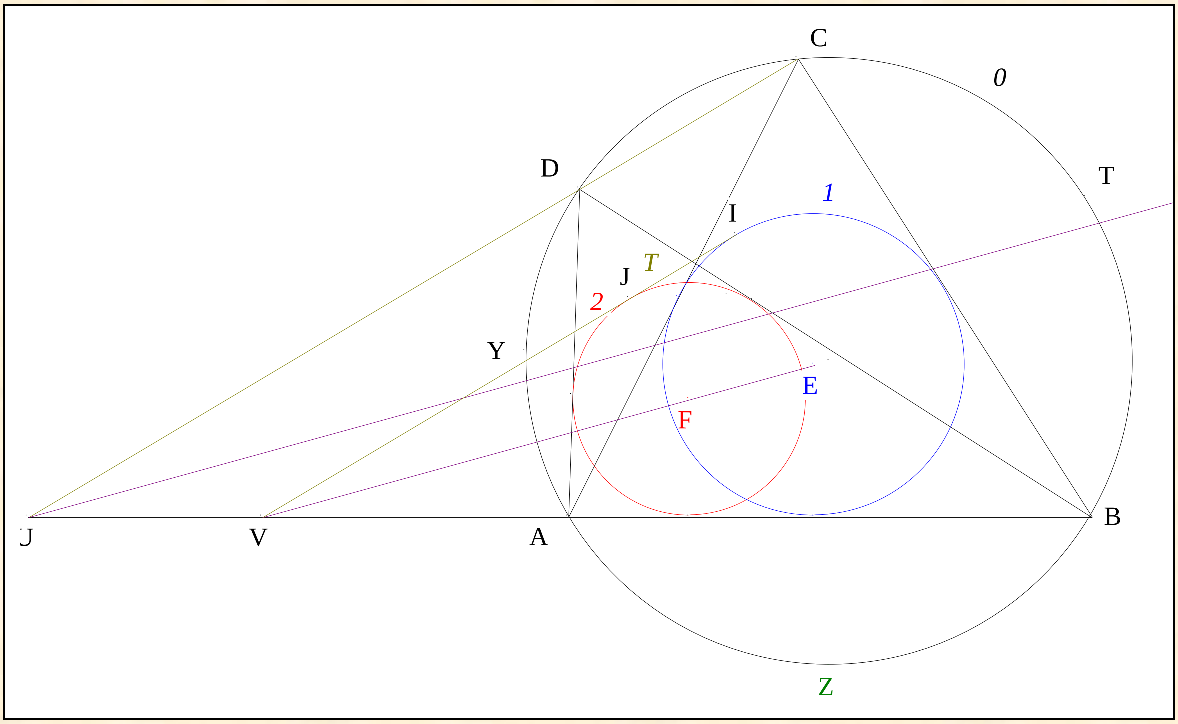
- According to Steiner "Bisectors and cyclic quadrilateral" (Cf. Annex 2), TY is parallel to the internal bisector of <BUC.



- Note Z the midpoint of the arc AB which doesn't contain C,
  and *3* the A-Mention circle of the triangle ABC ;
  its center is Z and it goes through A, E, F and B.

- The circles *0* and *3*, the basic points A and B, the monians TAE and YBF, lead to the Reim's theorem **0** ; consequently TY // EF.

- **Partial conclusion :** VEF is parallel to the bisector of <BUC.



- Note V the point of intersection of AB and EF,
  and I, J the points of contact of *T* resp. with *1* and *2*.

- **Remark :** *T* being the symmetric of AB wrt VEF, *T* goes through V.

- <BVI and <BUC having the side-line AB in common and their internal bisectors parallel, VIJ is parallel to UCD.

- **Conclusion :** *T* is parallel to CD.

2. Incircles of HBC triangles

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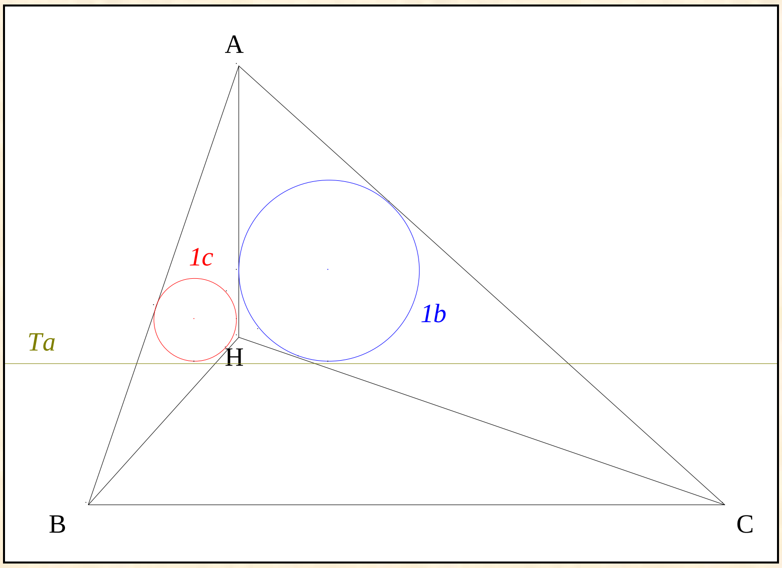
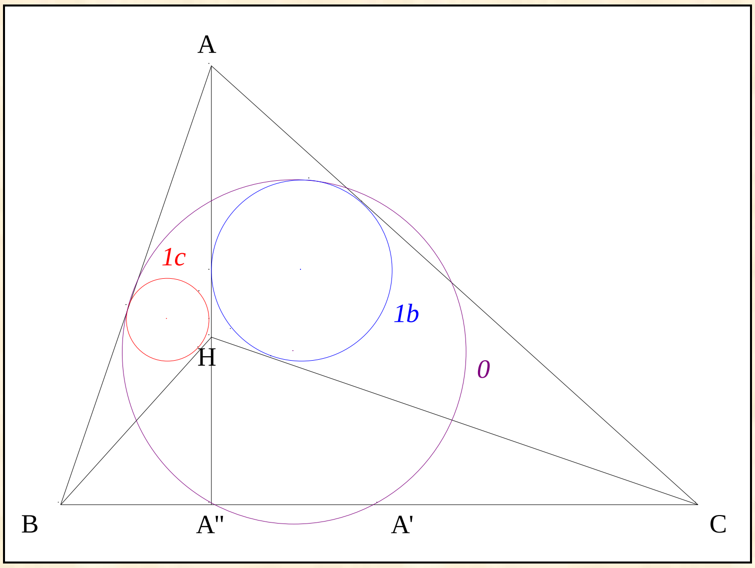
Figure :



Features : ABC an acute triangle,
H the orthocenter of ABC,
1b, *1c* the incircles of the resp. triangles HCA, HAB
and *Ta* the external common tangent to *1c* and *1''c* which is near H.

Given : *Ta* is parallel to BC.[8]

VISUALIZATION



- Note A' the midpoint of the segment BC,
  A'' the foot of the A-altitude of ABC
  and *0* the Euler's circle of ABC.

- **Remark :** *0* is the Euler's circle of the resp. triangles HBC, HCA et HAB.

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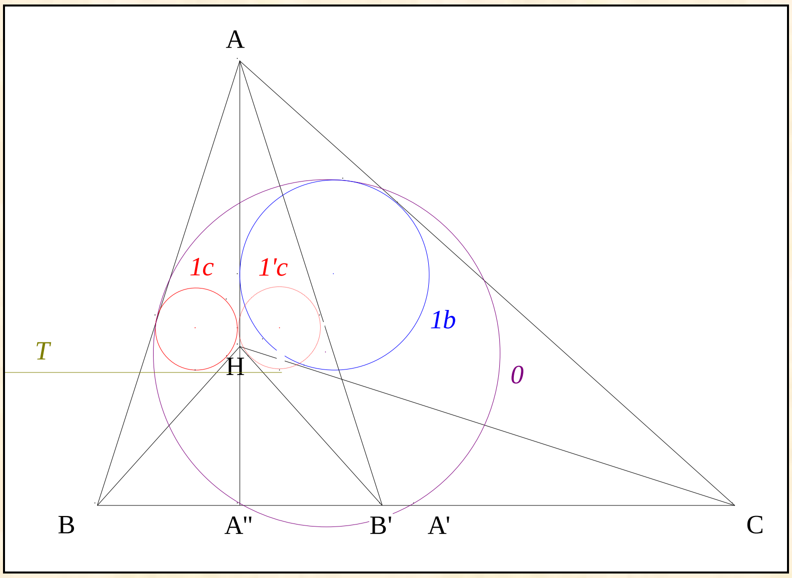
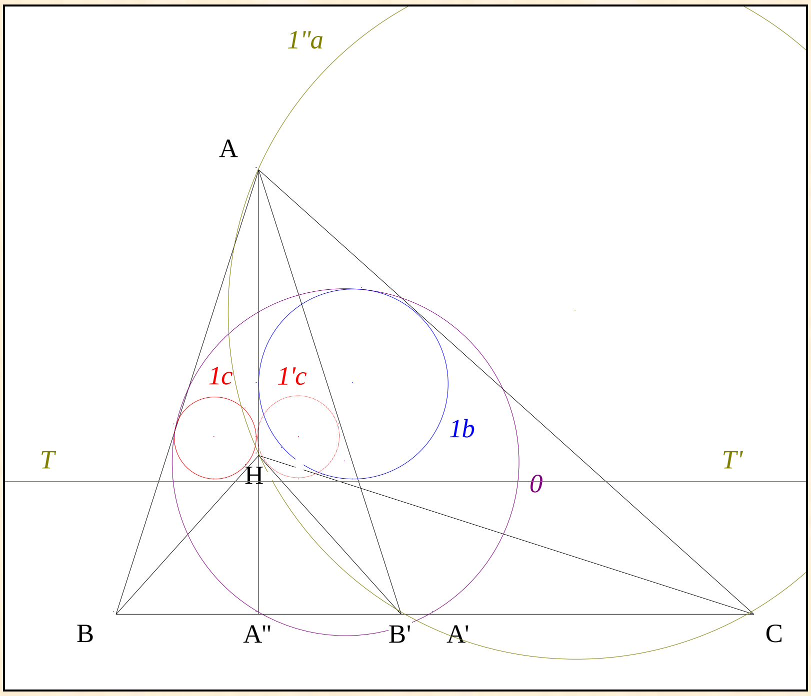
[8] Ayme J.-L., Another variant of the parallel tangent theorem, *Mathlinks* (03/05/2011) ;
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=395020 ;
Salazar J. C., Parallel tangent, , *Mathlinks* (08/26/2004)
http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=15945 ;
Orthocenter and circles, Message *Hyacinthos* # 9127 (01/24/2004) ;
http://tech.groups.yahoo.com/group/Hyacinthos/.

- According to "The Feuerbach's point", *0* is tangent to *1b* and *1c*.



- Note B' the symmetric of B wrt A",
  1"c the incircle of the triangle AHB'
  and *T* the external common tangent to *1c* and *1"c* which is between H and A".

- **Partial conclusion :** *1"c* being the symmetric of *1c* wrt AH, *T* is parallel to BC.



- An angle chasing modulo Π :
  according to "The perpendicular sides angles theorem" <ACH = <HBA ;
  by symmetry wrt AH, <HBA = <AA'H ;
  by transitivity of the relation =, <ACH = <AA'H ;
  according to "The chordal angle theorem", A, H, B' and C are concyclic.

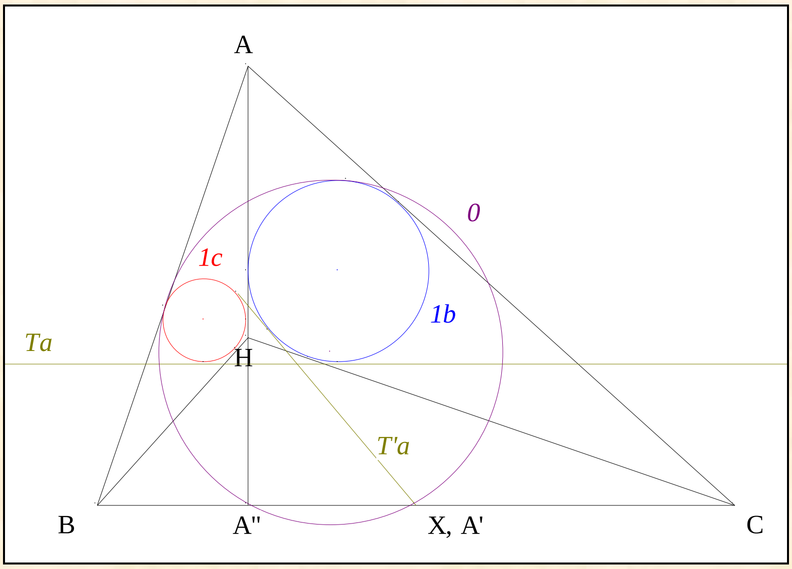
- Note *1"a* this circle
  and *T'* the external common tangent to *1'c* and *1b* which is between H and A".

- According to **C. 1.** With two incircles applied to *1'c* and *1b*, *T'* is parallel to BC ;
  consequently, *T'* = *T* (=*Ta*).

- **Conclusion :** *Ta* is parallel to BC.

Historic note : the solution to this problem is based on that of South Korean Shin Han-sol best known under the pseudonym of "Leonhard Euler" on *Mathlinks*.

Remark : Paul Yiu's result [9]



- Note *T'a* the second internal tangent of *1b* and *1c*.
  and X the point of intersection of *Ta* and *0* as shown in the figure.

- According to **B. 1.** The parallel tangent theorem, A"X // *Ta* ;
  we have, *Ta* // A"A' ;
  by transitivity of the relation //, A"X // A"A' ;
  according to the Euclide's postulate, A"X = A"A' ;
  consequently, A", X and A' are collinear
  and A' and X are identic.

- **Conclusion :** *Ta* goes through A'.

Historic note : this problem has attracted a dozen mail *Hyacinthos* group without this provide a solution.
A development of this problem can be the cause of a theorem of Quidde-Mannheim[10] dating from 1864.

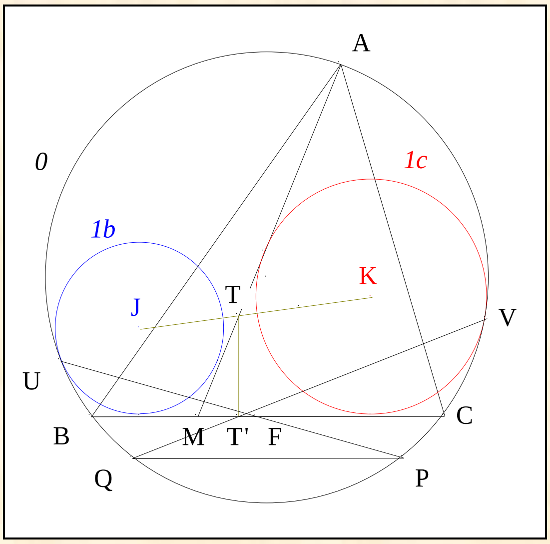
9 Yiu P., One more triad of circles, Message *Hyacinthos* # 888 (16/04/2003) ; http://tech.groups.yahoo.com/group/Hyacinthos/.

10 Mannheim A., F.G.M., Exercices de Géométrie, 6th ed. (1920). Gabay reprint, Paris (1991) Théorème #164 p. 326.

D. ANOTHER PARALLEL TANGENT THEOREM

VISION

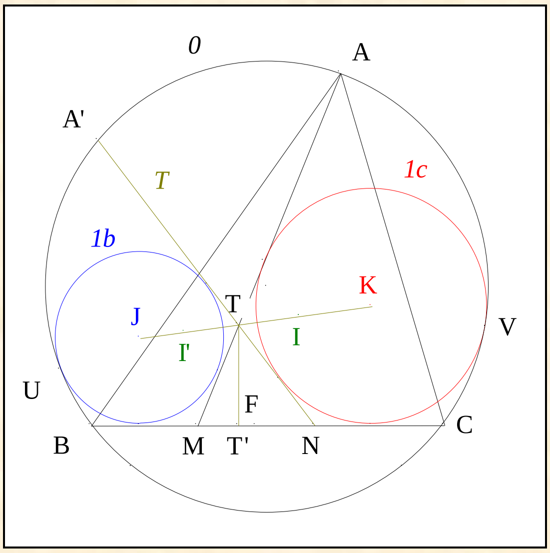
Figure :



Features :	ABC	a triangle,
	0	the circumcircle of ABC,
	M	a point on the segment BC,
	1b, *1c*	the B, C-Thébault's circles of ABC wrt M,
	J, K	the centers of *1b*, *1c*,
	U, V	the points of contact of *1b*, *1c* with *0*,
	T	the point of intersection of JK and AM,
	T'	the foot of the perpendicular to BC through T,
	F	the midpoint of the segment DE
and	P, Q	the second points of intersection of UF, VT' with *0*.

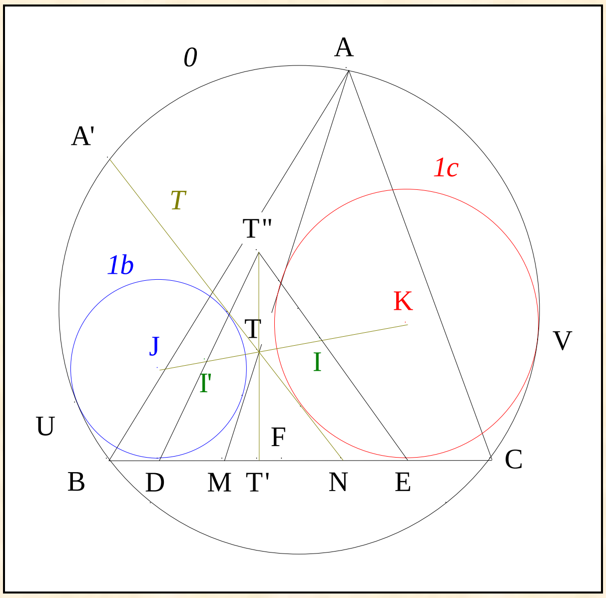
Given : PQ and BC are parallel.

VISUALIZATION



- Note   *T*   the second common internal tangent of *1b* and *1c* ; it goes through T ;
  N   the point of intersection of *T* and BC,
  A'   the point of intersection of *T* with *0* as shown in the figure,
  I   the incenter center of ABC
  and   I'   the incenter of the triangle A'BC

- According to **B.** The parallel tangent theorem,   I, I', J, K and T are collinear.

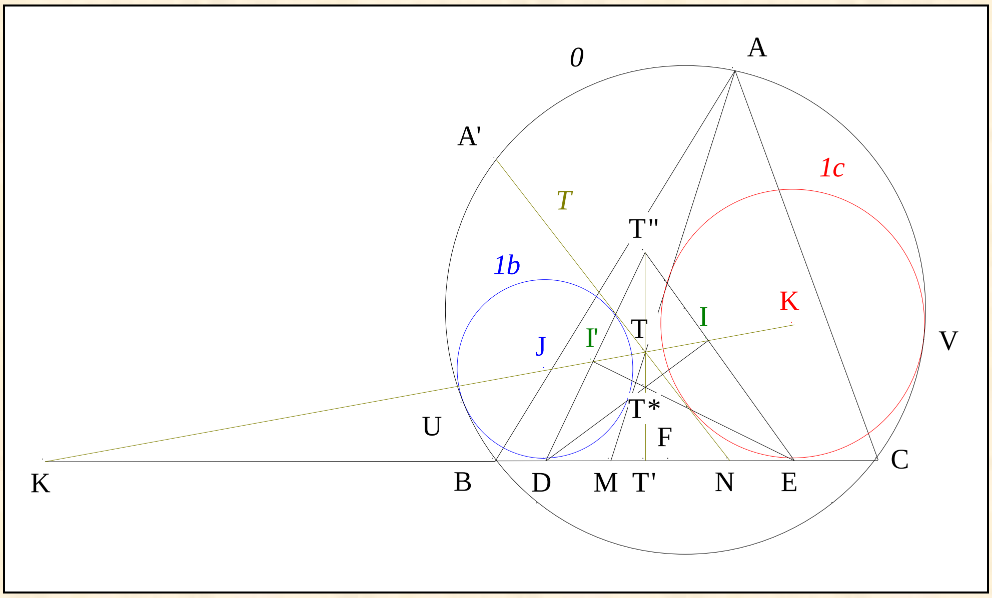


- Note   D, E   the points of contact of *1b*, *1c* with BC.

- According to Hadamard[11]   T"T is the T"-altitude of the triangle T"DE ;

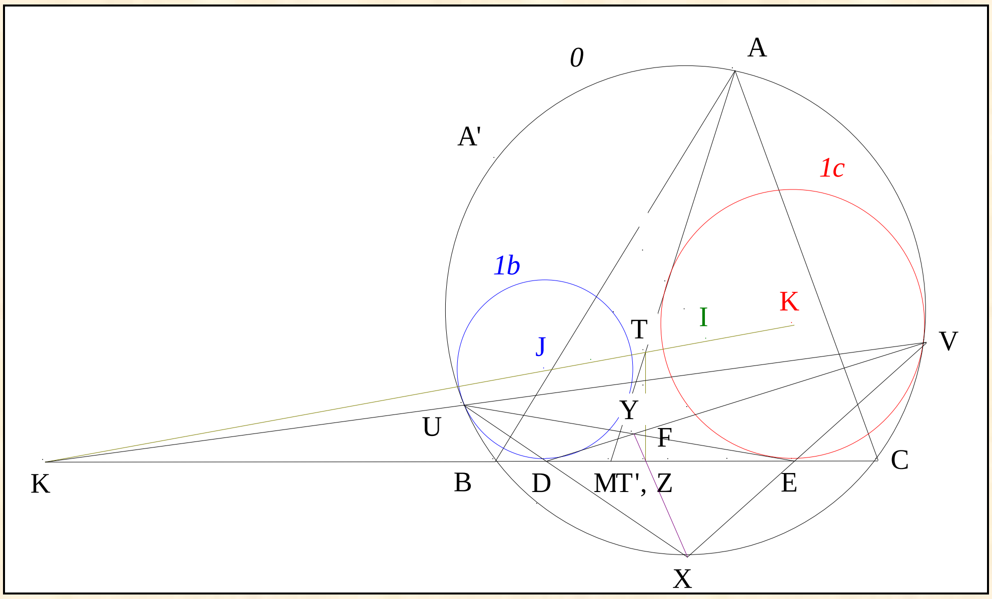
- We have :   TT" ⊥ BC   and   BC ⊥ TT'
  according to the perpendicularity axiom IVa,   TT" // TT'
  according to the Euclide's postulate,   TT" = TT'
  consequently,   T", T and T' are collinear.

[11] Ayme J.-L., From Sharygin to Hadamard, G.G.G. vol. **10** ; http://perso.orange.fr/jl.ayme.



- Note T* the orthocentre of T"DE
  and K the point of intersection of JK and BC.

- According to Pappus "Diagonals of a quadrilateral" (Cf. Annex 3)
  applied to DEII', the quaterne (D, E, T', K) is harmonic.



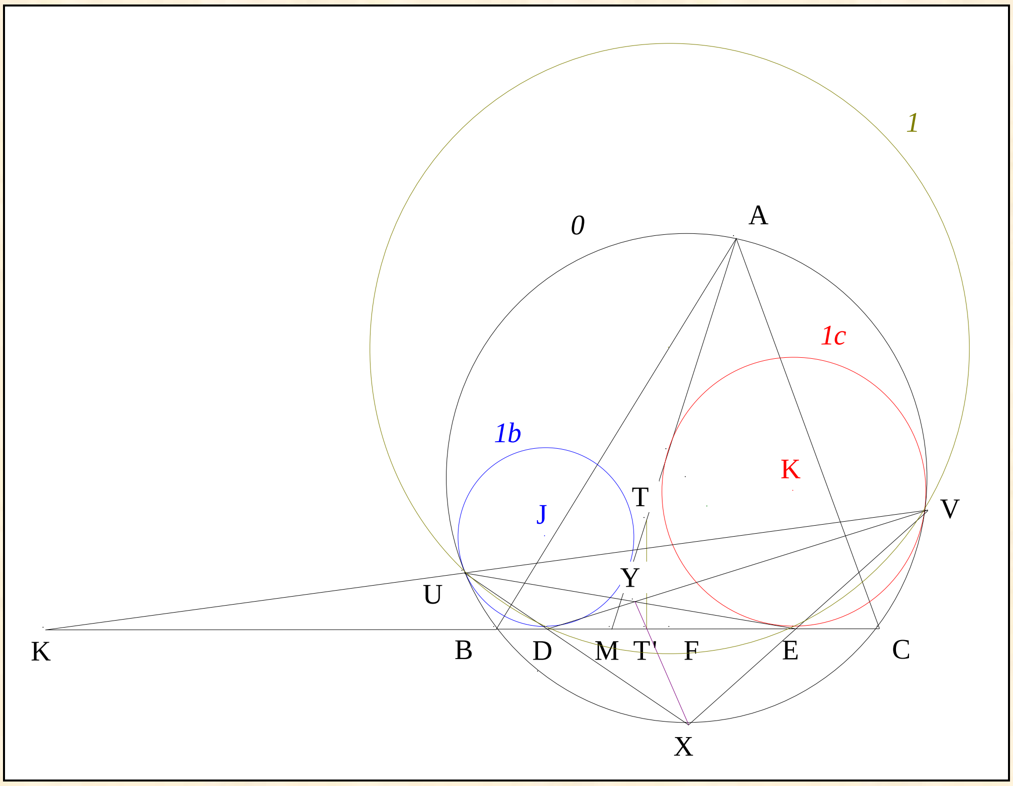
- Note X the midpoint of the arc BC which doesn't contain A,
  Y the point of intersection of UE and VD,
  and Z the point of intersection of XY and BC.

- **Remarks :** **(1)** U, D and X are collinear
  (2) V, E and X are collinear.

- According to "d'Alembert's theorem", UV goes through K.

- According to Pappus "Diagonals of a quadrilateral" (Cf. Annex 3)
  applied to DEVU, the quaterne (D, E, Z, K) is harmonic.
  consequently, Z and T' are identic.

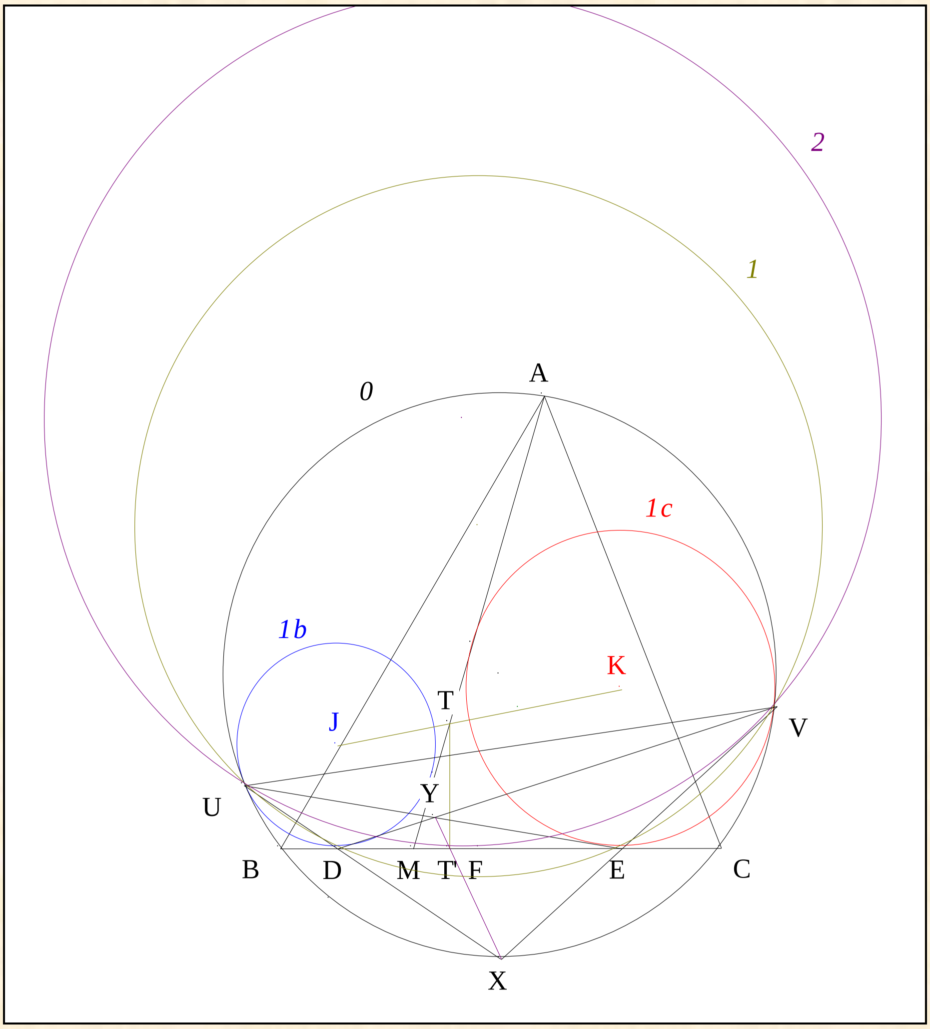
- **Partial conclusion :** X, Y and T' are collinear.[12]



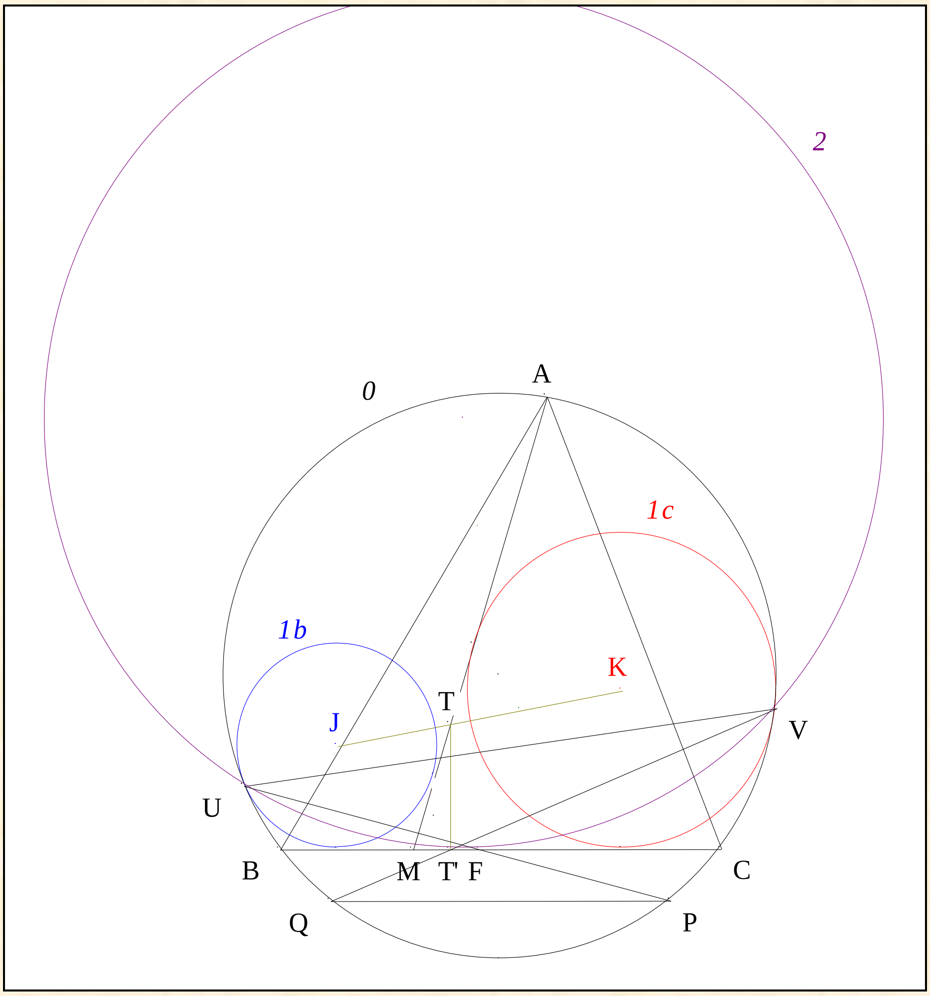
- According to **A.** Thébault's theorem, Remark 2, U, D, E and V are concyclic.

- Note *1* this circle.

12 Ayme J.-L., With the Thébault's circles (own), *Mathlinks* /02/2011 ; http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=393816.



- According to Thyomkyn' rediscovery (Cf. Appendix 1), U, T', F and V are concyclic.
- Note *2* this circle.



- The circles *2* and *0*, the basic points U and V, the monians PUF and QVT', lead to Reim's theorem **0** ; consequently, PQ // FT'.

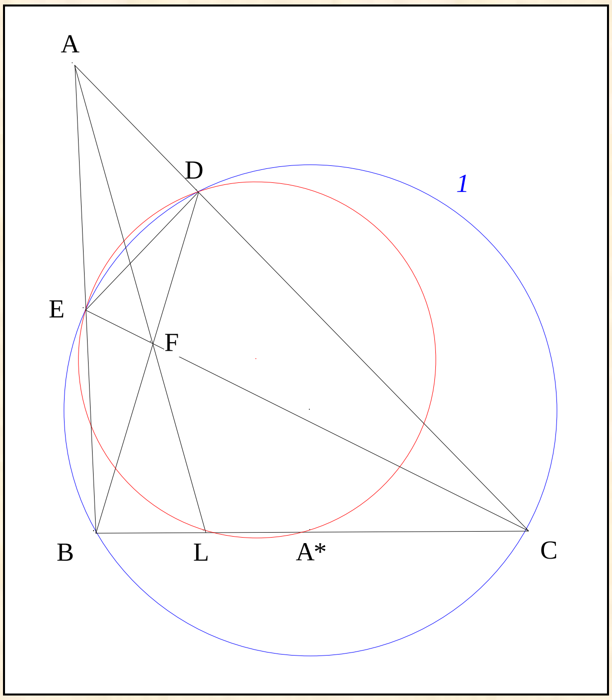
- **Conclusion :** PQ and BC are parallel.

E. APPENDIX

1. The rediscovery of Michail Tyomkyn

VISION

Figure:

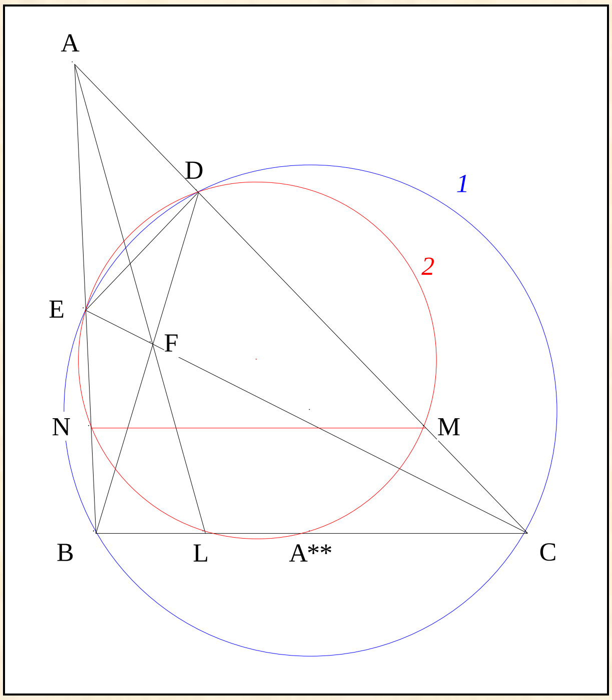


Features : ABC a triangle,
1 a circle going through B and C,
D, E the second points of intersection of AC, AB with *1*,
F the point of intersection of AD and CE,
L the point of intersection of AF and BC,
and A* the midpoint of the segment BC.

Given : D, E, F and A* are concyclic. [13]

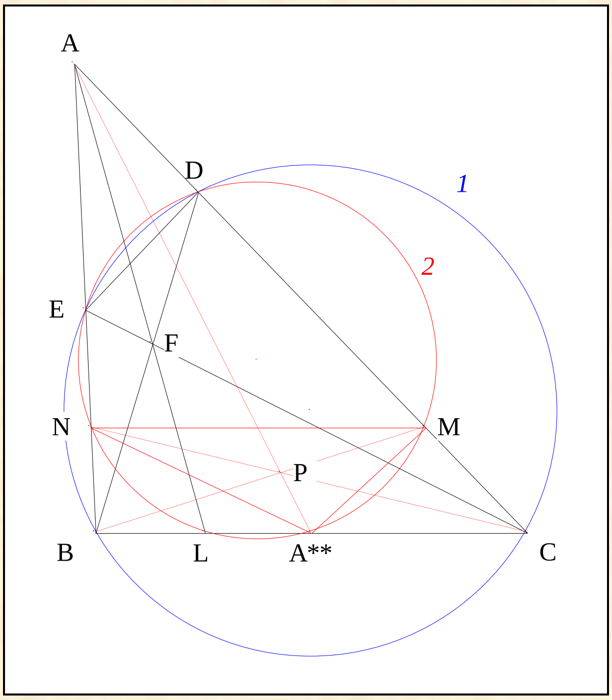
VISUALIZATION

[13] Altshiller-Court N., *College Geometry*, Richmond (1923) 247, exercice 5.



- Note 2 the circle going through D, E, L,
  A** the second point of intersection of BC with *2*
  and M, N the second points of intersections of AC, AB with *2*.

- the circles *1* and *2*, the basic points D and E, the monians CDM and BEN, lead to Reim's theorem **0** ;
  consequently CB // MN.



- According to "The Terquem's circle" [14],
  the triangle A**MN is cevian ; consequently, AA**, BM and CN are concurrent.

- Note P this point of concurs.

[14] Ayme J.-L., A new point on Euler line, G.G.G. vol. **5**, p. 3 ; http://perso.orange.fr/jl.ayme.

- According to "The complete trapeze" (Cf. Annex 4), AP goes through A* ;
  consequently, A** are A* identic.

- **Conclusion :** D, E, F and A* are concyclic.

Theorem : given a triangle ABC
and some point P with cevian triangle A'B'C' such that A'B' is antiparallel to AB
(i. e., the points B', C', B and C lie on one circle).
Then the midpoint A* of BC lies on the circle through the points A', B' and C'.

Remark : *if,* E is the orthocenter of ABC *then,* *2* is the Euler's circle of ABC.

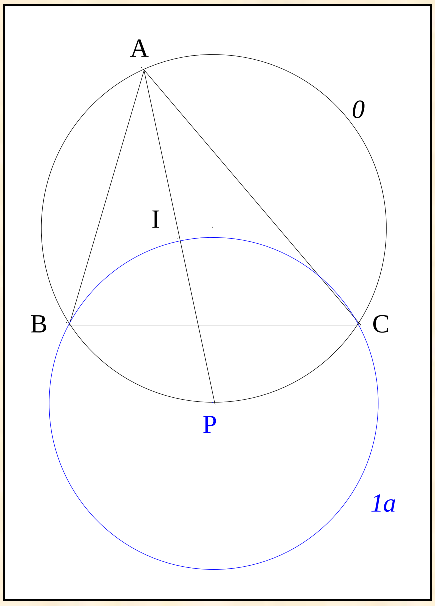
Historic note : This nice result of Nathan Altshiller-Court given as exercise in 1923 was rediscovered by Michail Tyomkyn, member of the German team for the O.I.M. of 2003 which took place in Tokyo (Japan).
This result appears as a generalization of the Euler's circle.

IMO-Team 2003

v.l.n.r.:
Prof. Dr. Hans-Dietrich Gronau, Peter Eberhard, Friedrich Feuerstein,
Alex Schreiber, Christian Reiher, Richard Bamler, Michael Tyomkyn, Arend Bayer

F. ANNEX

1. A Mention's circle



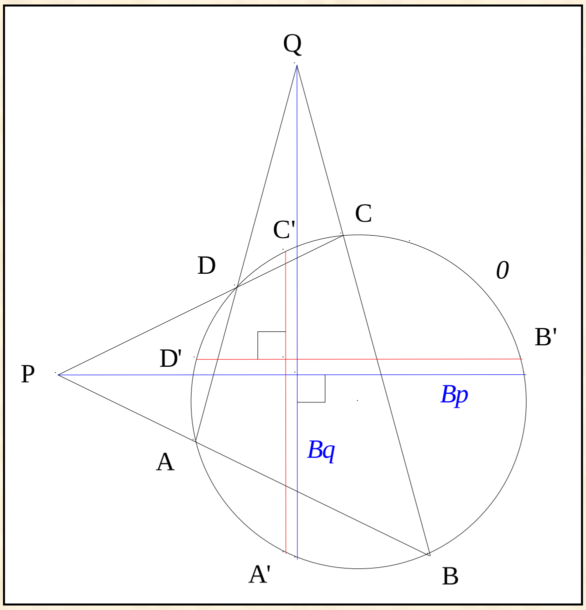
Features :	ABC	a triangle,
	0	the circumcircle of ABC,
	I	the incenter of ABC,
	P	the second point of intersection of AI with *0*
and	*1a*	the circle centered at P through B and C.

Given : *1a* goes through I.

Definition : *1a* is "the A-Mention's circle of ABC".

2. Bisectors and cyclic quadrilateral [15]

[15] Steiner J..

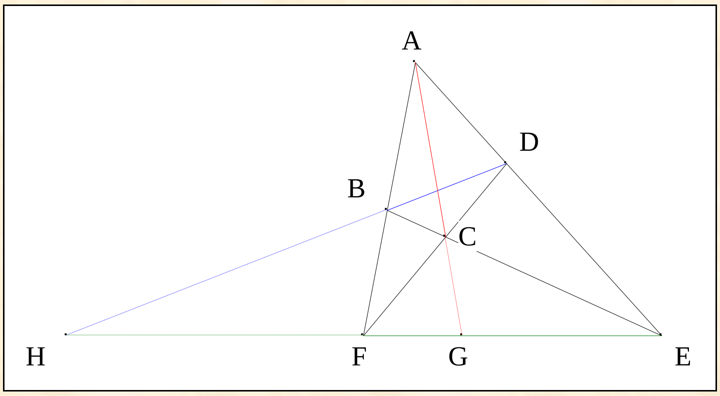


Features :	ABCD	a cyclic quadrilateral,
	0	the circumcircle of ABCD,
	P	the point of intersection of AB and CD,
	Q	the point of intersection of AD and BC,
	A', B', C', D'	the midpoint of the resp. arc AB, BC, CD, DA as shown in the figure
and	*Bp* , *Bq*	the P, Q-internal bisectors of the resp. triangles PAD, QDC.

Given : *Bp* (resp. *Bq*) is parallel to B'D' (resp. A'C').

Theorem : in any cyclic quadrilateral,
the bisectors of the angles formed by the opposite sides are parallel to the bisectors of the angles formed by the diagonals.

3. Diagonals of a quadrilateral [16]

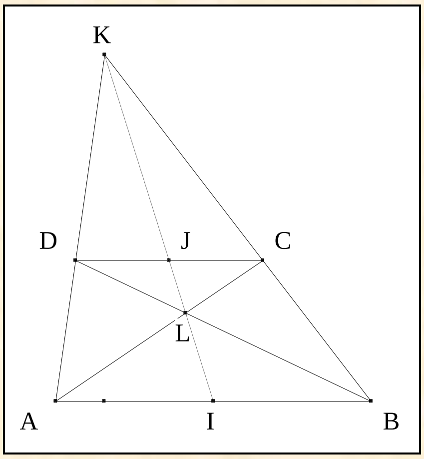


Features :	ABCD	a quadrilateral,
	E, F	the points of intersection resp. of AD and BC, AB and CD,
and	G, H	the points of intersection resp. of AC and EF, BD and EF.

Given : the quaterne (E, F, G, H) is harmonic.

[16] Pappus, *Collections*, Livre **7**, proposition 131.

4. The complete trapeze



Features :	ABCD	a quadrilateral,
	I	the midpoint of the segment AB,
	J	the midpoint of the segment CD,
	K	the point of intersection of AD and BC,
and	L	the point of intersection of AC and BD.

Given : ABCD is a trapeze with basis AB and CD *if, and only if,* I, J, K and L are collinear.