Combinatorial Number Theory

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Most of these problems were shamelessly stolen from Gabriel Carroll's BMC presentation on combinatorial number theory, available at `http://www.people.fas.harvard.edu/~gcarroll/math/cnt.tex`.

1. Prove that if one chooses more than n numbers from the set $\{1, 2, 3, \ldots, 2n\}$, then two of them are relatively prime.
2. Prove that if one chooses more than n numbers from the set $\{1, 2, 3, \ldots, 2n\}$, then one number is a multiple of another. Can this be avoided with exactly n numbers? (Paul Erdős)
3. Does there exist an infinite sequence of positive integers, containing every positive integer exactly once, such that the sum of the first n terms is divisible by n for every n?
4. Prove that, for each integer $n \geq 2$, there is a set S of n integers such that ab is divisible by $(a-b)^2$ for all distinct $a, b \in S$. (USA, 1998)
5. Given 81 positive integers all of whose prime factors are in the set $\{2, 3, 5\}$, prove that there are 4 numbers whose product is the fourth power of an integer. (Greece, 1996)

- Schur's Theorem: For every positive integer k, there is some n such that when the integers from 1 through n are partitioned into k subsets, some subset contains three distinct numbers x, y, z such that $x + y = z$.
- Van der Waerden's Theorem: If the set of all positive integers is partitioned into finitely many subsets, one of the subsets contains arbitrarily long arithmetic progressions.

6. Is it possible for the numbers $1, 2, \ldots, 100$ to be the terms of 12 geometric progressions? (Russia, 1995)
7. You want to color the integers from 1 to 100 so that no number is divisible by a different number of the same color. What is the smallest possible number of colors you must have?
8. A set of S positive integers is called a *finite basis* if there exists some n such that every sufficiently large positive integer can be written as a sum of at most n elements of S. If the set of positive integers is divided into finitely many subsets, must one of them necessarily be a finite basis?
9. Suppose that the positive integers have been colored in four colors—red, green, blue, and yellow. Let x and y be odd integers of different absolute values. Show that there exist two numbers of the same color whose difference has one of these values: $x, y, x - y$, or $x + y$. (IMO Proposal, 1999)
10. The set of all integers is partitioned into arithmetic progressions. Prove that some two of them have the same common difference.
11. Prove that any set of n integers has a nonempty subset whose sum is divisible by n.
12. Fifty numbers are chosen from the set $\{1, \ldots, 99\}$, no two of which sum to 99 or 100. Prove that the chosen numbers must be 50, 51, 52, …, 99. (St. Petersburg, 1997)
13. Prove that from a set of ten distinct two-digit numbers, it is possible to select two disjoint nonempty subsets whose members have the same sum. (IMO, 1972)
14. Let $x_1, x_2, \ldots, x_{19}$ be positive integers less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i equal to a sum of some y_j. (Putnam, 1993)
15. Let p be an odd prime. Determine the number of p-element subsets of $\{1, 2, \ldots, 2p\}$ such that the sum of the elements is divisible by p. (IMO, 1995)
16. Given $2n - 1$ integers, prove that one can choose exactly n of them whose sum is divisible by n. (Paul Erdős)