Geometry Problems - 2

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1. In triangle ABC, $AB = AC$. Point D is the midpoint of side BC. Point E lies outside the triangle ABC such that $CE \perp AB$ and $BE = BD$. Let M be the midpoint of segment BE. Point F lies on the minor arc $\widehat{AD}$ of the circumcircle of triangle ABD such that $MF \perp BE$. Prove that $ED \perp FD$.

2. In acute triangle ABC, $AB > AC$. Let M be the midpoint of side BC. The exterior angle bisector of $\widehat{BAC}$ meet ray BC at P. Point K and F lie on line PA such that $MF \perp BC$ and $MK \perp PA$. Prove that $BC^2 = 4PF \cdot AK$.

3. Find, with proof, the point P in the interior of an acute-angled triangle ABC for which $BL^2 + CM^2 + AN^2$ is a minimum, where L, M, N are the feet of the perpendiculars from P to BC, CA, AB respectively.

4. Circles C_1 and C_2 are tangent to each other at K and are tangent to circle C at M and N. External tangent of C_1 and C_2 intersect C at A and B. AK and BK intersect with circle C at E and F respectively. If AB is diameter of C, prove that EF and MN and OK are concurrent. (O is center of circle C.)

5. A, B, C are on circle $\mathcal{C}$. I is incenter of ABC , D is midpoint of arc BAC. W is a circle that is tangent to AB and AC and tangent to $\mathcal{C}$ at P. (W is in $\mathcal{C}$) Prove that P and I and D are on a line.

6. Suppose that M is a point inside of a triangle ABC. Let A' be the point of intersection of the line AM with the circumcircle of triangle ABC (other than A). Let r be the radius of the incircle of triangle ABC. Prove that $\frac{MB \cdot MC}{MA'} \geq 2r$.

7. Let $ABCD$ be a quadrilateral, and let H_1, H_2, H_3, H_4 be the orthocenters of the triangles DAB, ABC, BCD, CDA, respectively. Prove that the area of the quadrilateral $ABCD$ is equal to the area of the quadrilateral $H_1H_2H_3H_4$.

8. Given a triangle ABC. Suppose that a circle ω passes through A and C, and intersects AB and BC in D and E. A circle S is tangent to the segments DB and EB and externally tangent to the circle ω and lies inside of triangle ABC. Suppose that the circle S is tangent to ω at M. Prove that the angle bisector of the angle $\angle AMC$ passes through the incenter of triangle ABC.

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9. Let I be the incenter of a triangle $\triangle ABC$, let (P) be a circle passing through the vertices B, C and (Q) a circle tangent to the circle (P) at a point T and to the lines AB, AC at points U, V, respectively. Prove that the points B, T, I, U are concyclic and the points C, T, I, V are also concyclic.

10. Prove the locus of the centers of ellipses that are inscribed in a quadrilateral $ABCD$, is the line connecting the midpoints of its diagonals.

11. Let $ABCD$ be a cyclic quadrilateral, and let L and N be the midpoints of its diagonals AC and BD, respectively. Suppose that the line BD bisects the angle ANC. Prove that the line AC bisects the angle BLD.

12. I and I_a are incenter and excenter opposite A of triangle ABC. Suppose II_a and BC meet at A'. Also M is midpoint of arc BC not containing A. N is midpoint of arc MBA. NI and NI_a intersect the circumcircle of ABC at S and T. Prove S, T and A' are collinear.

13. Assume A, B, C are three collinear points that $B \in [AC]$. Suppose AA' and BB' are to parrallel lines that A', B' and C are not collinear. Suppose O_1 is circumcenter of circle passing through A, A' and C. Also O_2 is circumcenter of circle passing through B, B' and C. If area of $A'CB'$ is equal to area of O_1CO_2, then find all possible values for $\angle CAA'$

14. Let H_1 be an n-sided polygon. Construct the sequence H_1, H_2, ..., H_n of polygons as follows. Having constructed the polygon H_k, H_{k+1} is obtained by reflecting each vertex of H_k through its k-th neighbor in the counterclockwise direction. Prove that if n is a prime, then the polygons H_1 and H_n are similar.

15. M is midpoint of side BC of triangle ABC, and I is incenter of triangle ABC, and T is midpoint of arc BC, that does not contain A. Prove that

$$\cos B + \cos C = 1 \iff MI = MT$$

16. In triangle ABC, if L, M, N are midpoints of AB, AC, BC. And H is orthogonal center of triangle ABC, then prove that

$$LH^2 + MH^2 + NH^2 \leq \frac{1}{4}(AB^2 + AC^2 + BC^2)$$

17. Suppose H and O are orthocenter and circumcenter of triangle ABC. ω is circumcircle of ABC. AO intersects with ω at A_1. A_1H intersects with ω at A' and A'' is the intersection point of ω and AH. We define points B', B'', C' and C'' similarly. Prove that $A'A'', B'B''$ and $C'C''$ are concurrent in a point on the Euler line of triangle ABC.

18. Assume that in traingle ABC, $\angle A = 90^\circ$. Incircle touches AB and AC at points E and F. M and N are midpoints of AB and AC respectively. MN intersects circumcircle in P and Q. Prove that E, F, P, Q lie one a circle.

19. ABC is a triangle and R, Q, P are midpoints of AB, AC, BC. Line AP intersects RQ in E and circumcircle of ABC in F. T, S are on RP, PQ such that $ES \perp PQ, ET \perp RP$. F' is on circumcircle of ABC that FF' is diameter. The point of intersection of AF' and BC is E'. S', T' are on AB, AC that $E'S' \perp AB, E'T' \perp AC$. Prove that TS and $T'S'$ are perpendicular.

20. ω is circumcirlce of triangle ABC. We draw a line parallel to BC that intersects AB, AC at E, F and intersects ω at U, V. Assume that M is midpoint of BC. Let ω' be circumcircle of UMV. We know that $R(ABC) = R(UMV)$. ME and ω' intersect at T, and FT intersects ω' at S. Prove that EF is tangent to circumcircle of MCS.

21. Let C_1, C_2 and C_3 be three circles that does not intersect and non of them is inside another. Suppose (L_1, L_2), (L_3, L_4) and (L_5, L_6) be internal common tangents of (C_1, C_2), (C_1, C_3), (C_2, C_3). Let $L_1, L_2, L_3, L_4, L_5, L_6$ be sides of polygon $AC'BA'CB'$. Prove that AA', BB', CC' are concurrent.

22. ABC is an arbitrary triangle. A', B', C' are midpoints of arcs BC, AC, AB. Sides of triangle ABC, intersect sides of triangle $A'B'C'$ at points P, Q, R, S, T, F. Prove that

$$\frac{S_{PQRSTF}}{S_{ABC}} = 1 - \frac{ab + ac + bc}{(a + b + c)^2}$$

23. Let ω be incircle of ABC. P and Q are on AB and AC, such that PQ is parallel to BC and is tangent to ω. AB, AC touch ω at F, E. Prove that if M is midpoint of PQ, and T is intersection point of EF and BC, then TM is tangent to ω.

24. In an isosceles right-angled triangle shaped billiards table , a ball starts moving from one of the vertices adjacent to hypotenuse. When it reaches to one side then it will reflect its path. Prove that if we reach to a vertex then it is not the vertex at initial position.

25. Triangle ABC is isosceles ($AB = AC$). From A, we draw a line ℓ parallel to BC. P, Q are on perpendicular bisectors of AB, AC such that $PQ \perp BC$. M, N are points on ℓ such that angles $\angle APM$ and $\angle AQN$ are $\frac{\pi}{2}$. Prove that

$$\frac{1}{AM} + \frac{1}{AN} \leq \frac{2}{AB}$$

26. Let ABC, l and P be arbitrary triangle, line and point. A', B', C' are reflections of A, B, C in point P. A'' is a point on $B'C'$ such that $AA'' \parallel l$. B'', C'' are defined similarly. Prove that A'', B'', C'' are collinear.

27. Let I be incenter of triangle ABC, M be midpoint of side BC, and T be the intersection point of IM with incircle, in such a way that I is between M and T. Prove that $\angle BIM - \angle CIM = \frac{3}{2}(\angle B - \angle C)$, if and only if $AT \perp BC$.

28. Let P_1, P_2, P_3, P_4 be points on the unit sphere. Prove that $\sum_{i \neq j} \frac{1}{|P_i - P_j|}$ takes its minimum value if and only if these four points are vertices of a regular pyramid.

29. I_a is the excenter of the triangle ABC with respect to A, and AI_a intersects the circumcircle of ABC at T. Let X be a point on TI_a such that $XI_a^2 = XA.XT$. Draw a perpendicular line from X to BC so that it intersects BC in A'. Define B' and C' in the same way. Prove that AA', BB' and CC' are concurrent.

30. In the triangle ABC, $\angle B$ is greater than $\angle C$. T is the midpoint of the arc BAC from the circumcircle of ABC and I is the incenter of ABC. E is a point such that $\angle AEI = 90°$ and $AE \parallel BC$. TE intersects the circumcircle of ABC for the second time in P. If $\angle B = \angle IPB$, find the angle $\angle A$.

31. Let $A_1A_2A_3$ be a triangle and, for $1 \leq i \leq 3$, let B_i be an interior point of edge opposite A_i. Prove that the perpendicular bisectors of A_iB_i for $1 \leq i \leq 3$ are not concurrent.

32. Let $ABCD$ be a convex quadrilateral such that $AC = BD$. Equilateral triangles are constructed on the sides of the quadrilateral. Let O_1, O_2, O_3, O_4 be the centers of the triangles constructed on AB, BC, CD, DA respectively. Show that O_1O_3 is perpendicular to O_2O_4.

33. Let $ABCD$ be a tetrahedron having each sum of opposite sides equal to 1. Prove that

$$r_A + r_B + r_C + r_D \leq \frac{\sqrt{3}}{3}$$

where r_A, r_B, r_C, r_D are the inradii of the faces, equality holding only if $ABCD$ is regular.

34. Let $ABCD$ be a non-isosceles trapezoid. Define a point $A1$ as intersection of circumcircle of triangle BCD and line AC. (Choose A_1 distinct from C). Points B_1, C_1, D_1 are de
fined in similar way. Prove that $A_1B_1C_1D_1$ is a trapezoid as well.

35. A convex quadrilateral is inscribed in a circle of radius 1. Prove that the difference between its perimeter and the sum of the lengths of its diagonals is greater than zero and less than 2.

36. On a semicircle with unit radius four consecutive chords AB, BC, CD, DE with lengths a, b, c, d, respectively, are given. Prove that

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4.$$

37. A circle C with center O on base BC of an isosceles triangle ABC is tangent to the equal sides AB, AC. If point P on AB and point Q on AC are selected such that $PB \times CQ = (\frac{BC}{2})^2$, prove that line segment PQ is tangent to circle C, and prove the converse.

38. The points D, E and F are chosen on the sides BC, AC and AB of triangle ABC, respectively. Prove that triangles ABC and DEF have the same centroid if and only if

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}$$

39. Bisectors AA_1 and BB_1 of a right triangle ABC ($\angle C = 90°$) meet at a point I. Let O be the circumcenter of triangle CA_1B_1. Prove that $OI \perp AB$.

40. A point E lies on the altitude BD of triangle ABC, and $\angle AEC = 90°$. Points O_1 and O_2 are the circumcenters of triangles AEB and CEB; points F, L are the midpoints of the segments AC and O_1O_2. Prove that the points L, E, F are collinear.

41. The line passing through the vertex B of a triangle ABC and perpendicular to its median BM intersects the altitudes dropped from A and C (or their extensions) in points K and N. Points O_1 and O_2 are the circumcenters of the triangles ABK and CBN respectively. Prove that $O_1M = O_2M$.

42. A circle touches the sides of an angle with vertex A at points B and C. A line passing through A intersects this circle in points D and E. A chord BX is parallel to DE. Prove that XC passes through the midpoint of the segment DE.

43. A quadrilateral $ABCD$ is inscribed into a circle with center O. Points P and Q are opposite to C and D respectively. Two tangents drawn to that circle at these points meet the line AB in points E and F. (A is between E and B, B is between A and F). The line EO meets AC and BC in points X and Y respectively, and the line FO meets AD and BD in points U and V respectively. Prove that $XV = YU$.

44. A given convex quadrilateral $ABCD$ is such that $\angle ABD + \angle ACD > \angle BAC + \angle BDC$. Prove that

$$S_{ABD} + S_{ACD} > S_{BAC} + S_{BDC}.$$

45. A circle centered at a point F and a parabola with focus F have two common points. Prove that there exist four points A, B, C, D on the circle such that the lines AB, BC, CD and DA touch the parabola.

46. Let B and C be arbitrary points on sides AP and PD respectively of an acute triangle APD. The diagonals of the quadrilateral $ABCD$ meet at Q, and H_1, H_2 are the orthocenters of triangles APD and BPC, respectively. Prove that if the line H_1H_2 passes through the intersection point X ($X \neq Q$) of the circumcircles of triangles ABQ and CDQ, then it also passes through the intersection point Y ($Y \neq Q$) of the circumcircles of triangles BCQ and ADQ.

47. Let ABC be an acute triangle and let ℓ be a line in the plane of triangle ABC. We've drawn the reflection of the line ℓ over the sides AB, BC and AC and they intersect in the points A', B' and C'. Prove that the incenter of the triangle $A'B'C'$ lies on the circumcircle of the triangle ABC.

48. In tetrahedron $ABCD$ let h_a, h_b, h_c and h_d be the lengths of the altitudes from each vertex to the opposite side of that vertex. Prove that

$$\frac{1}{h_a} < \frac{1}{h_b} + \frac{1}{h_c} + \frac{1}{h_d}.$$

49. Let squares be constructed on the sides BC, CA, AB of a triangle ABC, all to the outside of the triangle, and let A_1, B_1, C_1 be their centers. Starting from the triangle $A_1B_1C_1$ one analogously obtains a triangle $A_2B_2C_2$. If S, S_1, S_2 denote the areas of triangles$ABC, A_1B_1C_1, A_2B_2C_2$, respectively, prove that $S = 8S_1 - 4S_2$.

50. Through the circumcenter O of an arbitrary acute-angled triangle, chords A_1A_2, B_1B_2, C_1C_2 are drawn parallel to the sides BC, CA, AB of the triangle respectively. If R is the radius of the circumcircle, prove that

$$A_1O \cdot OA_2 + B_1O \cdot OB_2 + C_1O \cdot OC_2 = R^2.$$

51. In triangle ABC points M, N are midpoints of BC, CA respectively. Point P is inside ABC such that $\angle BAP = \angle PCA = \angle MAC$. Prove that $\angle PNA = \angle AMB$.

52. Point O is inside triangle ABC such that $\angle AOB = \angle BOC = \angle COA = 120°$. Prove that

$$\frac{AO^2}{BC} + \frac{BO^2}{CA} + \frac{CO^2}{AB} \geq \frac{AO + BO + CO}{\sqrt{3}}.$$

53. Two circles C_1 and C_2 with the respective radii r_1 and r_2 intersect in A and B. A variable line r through B meets C_1 and C_2 again at P_r and Q_r respectively. Prove that there exists a point M, depending only on C_1 and C_2, such that the perpendicular bisector of each segment P_rQ_r passes through M.

54. Two circles O, O' meet each other at points A, B. A line from A intersects the circle O at C and the circle O' at D (A is between C and D). Let M, N be the midpoints of the arcs BC, BD, respectively (not containing A), and let K be the midpoint of the segment CD. Show that $\angle KMN = 90°$.

55. Let AA', BB', CC' be three diameters of the circumcircle of an acute triangle ABC. Let P be an arbitrary point in the interior of $\triangle ABC$, and let D, E, F be the orthogonal projection of P on BC, CA, AB, respectively. Let X be the point such that D is the midpoint of $A'X$, let Y be the point such that E is the midpoint of $B'Y$, and similarly let Z be the point such that F is the midpoint of $C'Z$. Prove that triangle XYZ is similar to triangle ABC.

56. In the tetrahedron $ABCD, \angle BDC = 90^o$ and the foot of the perpendicular from D to ABC is the intersection of the altitudes of ABC. Prove that:

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

When do we have equality?

57. In a parallelogram $ABCD$, points E and F are the midpoints of AB and BC, respectively, and P is the intersection of EC and FD. Prove that the segments AP, BP, CP and DP divide the parallelogram into four triangles whose areas are in the ratio $1 : 2 : 3 : 4$.

58. Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

59. Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90°$, prove that $2\angle BDA = \angle CDE$.

60. The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of an acute-angled triangle ABC. Prove that the incenter of triangle ABC lies inside triangle XYZ.

Solutions

1. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1986970

2. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1987074

3. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1989372

4. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=16264

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6. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=18449

7. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=19329

8. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=20751

9. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=262450

10. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=99112

11. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=22914

12. http://www.artofproblemsolving.com/Forum/viewtopic.php?p=22756

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