Art of Problem Solving

WOOT 2010–11

Practice Olympiad 4

Instructions

- You should take the test under "olympiad conditions," meaning that the test should be completed in one sitting, with handwritten solutions (just like on an actual olympiad exam). Take the test using only the resources that would be available to you on an actual olympiad, meaning you should use scrap paper, a ruler, a compass, etc., but no calculators and no reference materials.
- You should allot 3 hours to take the test.
- Completely fill out the cover sheet and make sure it is the first page of your solutions.
- On the WOOT Home Page, there is a **WOOT Practice Olympiad Answer Sheet**. Print out (or copy) several blank copies of the answer sheets, and write all of your work on these sheets. Use only black pen or very dark pencil. Make sure that the top of every answer sheet page is completely filled out. Each problem's solution should start on a new page, along with new page numbering.
- Do not discuss the problems on or before the due date of Wednesday, January 12, 2011.

How to submit your solutions

You can submit solutions by upload, email, or by fax. DO NOT MAIL YOUR SOLUTIONS!

By Upload: Scan your solutions as a single PDF file. **Check to make sure that the file is legible before submitting it!** In your "My Classes" area, follow the link that says "Submit."

By email: Scan your solutions as a single PDF file. **Check to make sure that the file is legible before emailing it!** Email to `woot@artofproblemsolving.com`. Put "WOOT Practice Olympiad 4" in the subject line, attach your solutions as a single PDF file, and write something in the message body - if you leave the message body blank, it will get blocked by our spam filters.

By fax: Fax to (619)659-8146.

Solutions are due by Wednesday, January 12, 2011

Late submissions will not be accepted except under extraordinary circumstances

WOOT Practice Olympiad Cover Sheet

Username: ____

Class ID: ____________

User ID: ________________________
(Your Class ID and User ID can be found in the "My Classes" section of the website)

Practice Olympiad Number: 4

Beginning	
Intermediate	
Advanced	

Number of pages (including cover sheet): _____

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1. Let AC and BD be two perpendicular chords of a circle with radius r, and let the two chords intersect at P. Express $PA^2 + PB^2 + PC^2 + PD^2$ in terms of r.

2. The coefficients of x^{r-1}, x^r, and x^{r+1} in the binomial expansion of $(1+x)^n$ are three consecutive terms in an arithmetic sequence, where n and r are positive integers.

   (a) Prove that $(n-2r)^2 = n+2$.

   (b) Determine all pairs of positive integers (n, r) that satisfy the equation in part (a).

3. Find all non-empty sets A of real numbers satisfying the following property: For all real numbers x and y, if $x + y \in A$, then $xy \in A$.

4. Let
$$P(x) = 2x^{2010} + x^{2009} + 2x^{2008} + x^{2007} + \cdots + 2x^2 + x + 2.$$
   Prove that if z is a complex number such that $P(z) = 0$, then $|z| = 1$.

5. Let O be the circumcenter of triangle ABC. A line through O intersects sides AB and AC at M and N, respectively. Let S and R be the midpoints of BN and CM, respectively. Prove that $\angle ROS = \angle BAC$.

6. Let $n \geq 4$ be a given integer. For every integer $m \geq 0$, let S_m denote the set $\{m+1, m+2, \ldots, m+n\}$. Let $f(n)$ be the smallest positive integer such that for every m and for every set $T \subseteq S_m$ with $|T| = f(n)$, T contains at least three (distinct) pairwise relatively prime elements. Determine $f(n)$.

7. Let a, b, c, and d be positive real numbers such that $abcd = 1$. Prove that
$$\frac{a}{\sqrt{15+a^2}} + \frac{b}{\sqrt{15+b^2}} + \frac{c}{\sqrt{15+c^2}} + \frac{d}{\sqrt{15+d^2}} \geq 1.$$

Remember . . .

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- Complete the WOOT Practice Olympiad Cover Sheet (included in this document), and make it the first page of your solutions. We will not accept your solutions without this cover sheet.
- Do not mail your solutions. Use only upload, e-mail, or fax. If you use e-mail, send your solutions to

  woot@artofproblemsolving.com,

  NOT **classes**@artofproblemsolving.com. If you use e-mail, we will only accept solutions in PDF format. In particular, solutions in JPG or Word format will not be accepted.