Junior problems

J223. Let a and b be real numbers such that $\sin^3 a - \frac{4}{3}\cos^3 a \le b - \frac{1}{4}$. Prove that

$$\frac{3}{4}\sin a - \cos a \le b + \frac{1}{6}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J224. Consider 25 points inside the unit circle. Prove that among them there are two at most $\frac{1}{2}$ apart.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J225. Let a, b, c be nonnegative real numbers such that $a + b + c = 1$. Prove that

$$ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) + abc \le \frac{1}{8}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J226. We are given $n \ge 4$ points in the plane, no three collinear. Denote by T_p the set of triangles with vertices in these points whose interior contains at least one of the other points. Prove that if $|T_p| \le n - 4$, then $|T_p| = 0$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J227. For a positive integer N let $r(N)$ be the number obtained by reversing the digits of N. Find all 3-digit numbers N such that $r^2(N) - N^2$ is the cube of a positive integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J228. Prove that a square of side 1 and a square of side 2 cannot fit inside a square of side less than 3 without overlapping.

Proposed by Roberto Bosch Cabrera, Texas, USA

Senior problems

S223. We define *magic numbers* as follows:

(i) all numbers from 0 to 9 are magic;

(ii) a number greater than 9 is magic if it is divisible by the number of its digits and the number obtained by deleting its final digit is also magic.

Find the greatest magic number.

Proposed by Roberto Bosch Cabrera, Texas, USA

S224. Let a, b, c be real numbers greater than 2 such that

$$\frac{7-2a}{3a-6}+\frac{7-2b}{3b-6}+\frac{7-2c}{3c-6}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S225. Let ABC be a triangle. Determine the points X on the median from vertex A for which the ratio $\frac{BX}{CX}$ is minimal or maximal.

Proposed by Roberto Bosch Cabrera, USA and Francisco Javier García Capitán, Spain

S226. Let x, y, z be pairwise distinct positive real numbers. Prove that

$$\frac{x+y}{(x-y)^2}+\frac{y+z}{(y-z)^2}+\frac{z+x}{(z-x)^2}\geq \frac{9}{x+y+z}.$$

Proposed by Mircea Lascu and Marius Stanean, Zalau, Romania

S227. Let $\mathbb{N}^*$ be the set of positive integers. Find all functions $f:\mathbb{N}^*\to\mathbb{N}^*$ such that

$$f(n+1)>\frac{f(n)+f(f(n))}{2}$$

for all n.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

S228. Given triangle ABC, let M, N, P be the midpoints of BC, CA, AB, respectively. Let D, E, F be the tangency points of the incircle $\omega(I,r)$ with sides BC, CA, AB, respectively. Let X be a point on the line AI such that $\frac{AI}{IX}=2$, with I lying on the segment AX. Similarly, define Y and Z. Prove that

a) Lines MX, NY, PZ are parallel.

b) Lines DX, DY, DZ are concurrent on ω.

Proposed by Luiz Gonzalez, Maracaibo, Venezuela and Cosmin Pohoata, Princeton University, USA

Undergraduate problems

U223. Let $(x_k)_{k\geq 1}$ be the positive roots of the equation $\tan x = x$. Prove that

$$\sum_{k=1}^{\infty} \frac{1}{x_k^2} = \frac{1}{10}.$$

Proposed by Roberto Bosch Cabrera, Texas, USA

U224. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers satisfying $a_1 = a \neq 0$ and

$$a_{n+1} = \sqrt[3]{\frac{a_1^2}{1} + \frac{a_2^2}{2} + \cdots + \frac{a_n^2}{n}},\ n \geq 1.$$

Find $\lim\limits_{n\to\infty} (3a_n - \log n)$.

Proposed by Cezar Lupu, University of Pittsburgh, USA and Tudorel Lupu, Decebal High School, Constanta, Romania

U225. Find the maximal number of edges of the n-dimensional unit cube that are cut by a hyperplane.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

U226. Let $\mathbf{x}$ and $\mathbf{y}$ be points on $\mathbb{S}^n$ which are randomly chosen from the uniform distribution on the unit n-sphere. Evaluate $\mathrm{E}\big[||\mathbf{x} - \mathbf{y}||^2\big]$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U227. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a-b) + f(b-c) + f(c-a) = 2f(a+b+c)$$

whenever a, b, c are real numbers such that $ab + bc + ca = 0$.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

U228. Let $L \diagup K$ be a separable algebraic extension of fields and let V, W and U be L-vector spaces. Furthermore, let $h : V \times W \mapsto U$ be a K-bilinear map satisfying

$$h\,(xa, xb) = x^2 h\,(a, b) \text{ for every } x \in L,\ a \in V, \text{ and } b \in W.$$

Prove that h is L-bilinear.

Proposed by Darij Grinberg, Massachusetts Institute of Technology, USA

Olympiad problems

O223. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(y + f(x)) = f(x)f(y) + f(f(x)) + f(y) - xy$$

for all $x, y \in \mathbb{R}$.

Proposed by Preudtanan Sriwongleang, Ramkamhaeng University, Thailand

O224. Let a, b, c be positive real numbers. Prove that

$$\frac{3(a^3 + b^3 + c^3)}{2(a + b + c)(a^2 + b^2 + c^2)} \leq \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} - 1 \leq \frac{3(a^3 + b^3 + c^3)}{2(a + b + c)(ab + bc + ca)}.$$

Proposed by Cezar Lupu, University of Pittsburgh, USA and Duc Huu Pham, Ballajura, Australia

O225. For any prime $p > 3$, prove that

$$p \sum_{j=0}^{p-1} \frac{(-3)^j}{2j + 1} \equiv \left(\frac{p}{3}\right) \pmod{p^2},$$

where $\left(\frac{p}{3}\right)$ is the Legendre symbol.

Proposed by Cosmin Pohoata, Princeton University, USA

O226. Let $n > 1$ be an odd integer and let $A_1 \ldots A_n$ be a regular polygon. Find the number of triangles $A_iA_jA_k$ up to a permutation that contain the center of the n-gon.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

O227. Decide whether there is a polynomial $f \in \mathbb{Q}[X]$ such that $f(\mathbb{Z}) \subset \mathbb{Z}$ and

a) there is no $g \in \mathbb{Z}[X]$ such that $f(\mathbb{Z}) = g(\mathbb{Z})$.

b) there is $h \in \mathbb{Z}[X, Y]$ such that $f(\mathbb{Z}) = h(\mathbb{Z} \times \mathbb{Z})$.

Proposed by Gabriel Dospinescu, Ecole Polytechnique, France

O228. Let Γ be an arbitrary circle in the plane of a given triangle ABC. Let $\mathcal{K}'_A$ and $\mathcal{K}''_A$ be the circles through B and C which are tangent to Γ at X' and X'', respectively. Similarly define $\mathcal{K}'_B$, $\mathcal{K}''_B$, $\mathcal{K}'_C$, $\mathcal{K}''_C$ and their tangency points with Γ, Y', Y'', Z', and Z'', respectively. Prove that the circumcircles of triangles $AX'X''$, $BY'Y''$, and $CZ'Z''$ are coaxal.

Proposed by Cosmin Pohoata, Princeton University, USA and Paul Yiu, Florida Atlantic University, USA