Trigonometric and geometric Identities

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1 Trigonometry

Theorem 1.1 (Very basic theorems).

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sec^2\theta = 1 + \tan^2\theta$$
$$\csc^2\theta = 1 + \cot^2\theta$$

Theorem 1.2 (Extended Law of Sines). *If R is the circumradius of $\triangle ABC$, then $BC = 2R\sin A$.*

Theorem 1.3 (Law of Cosines). *In $\triangle ABC$,*

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ a &= b\cos C + c\cos B \end{aligned}$$

Theorem 1.4 (Addition formulae). *For any real numbers α and β,*

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\ \tan\alpha \pm \beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \\ \cot\alpha \pm \beta) &= \frac{\cot\cot\beta \mp 1}{\cot\alpha \pm \cot\beta} \end{aligned}$$

Theorem 1.5 (Product of $\sin, \cos$).

$$\begin{aligned} \sin(\alpha+\beta)\sin(\alpha-\beta) &= \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha \\ \cos(\alpha+\beta)\cos(\alpha-\beta) &= \cos^2\beta - \sin^2\alpha = \cos^2\alpha - \sin^2\beta \end{aligned}$$

Theorem 1.6 (Product to sum).

$$\begin{aligned} 2\sin a\cos b &= \sin(a+b) + \sin(a-b) \\ 2\cos a\sin b &= \sin(a+b) - \sin(a-b) \\ 2\cos a\cos b &= \cos(a+b) + \cos(a-b) \\ 2\sin a\sin b &= \cos(a-b) - \cos(a+b) \end{aligned}$$

Theorem 1.7 (Sum-to-product formulae).

$$\begin{aligned} \sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\ \sin\alpha - \sin\beta &= 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\ \cos\alpha + \cos\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\ \cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}. \end{aligned}$$

In particular, one has the double and half-angle formulae.

Theorem 1.8 (Double-angle formulae).

$$\begin{aligned} \sin 2\alpha &= 2\sin\alpha\cos\alpha \\ \cos 2\alpha &= 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \\ \tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}. \end{aligned}$$

Theorem 1.9 (Triple angle formulae).

$$\begin{aligned}
\sin 3a &= 3\sin a - 4\sin^3 a \\
\cos 3a &= 4\cos^3 a - 3\cos a \\
\tan 3a &= \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}
\end{aligned}$$

Theorem 1.10 (Relation with $\tan\frac{x}{2}$). *For any* $x \in \mathbb{R}$,

$$\sin x = \frac{2t}{1+t^2}, \ \cos x = \frac{1-t^2}{1+t^2}, \ \tan x = \frac{2t}{1-t^2}$$

$$\textit{where } t = \tan\frac{x}{2}$$

Theorem 1.11 (Half-angle formulae).

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\
\tan\frac{\alpha}{2} &= \csc\alpha - \cot\alpha.
\end{aligned}$$

Theorem 1.12 (Trigonometric Identities for Triangle).

$$\begin{aligned}
\sum_{cyc}\sin^2\frac{A}{2} &= 1 - 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\
\sum_{cyc}\sin A &= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \\
\sum_{cyc}\sin 2A &= 4\sin A\sin B\sin C \\
\sum_{cyc}\cos A &= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1 + \frac{r}{R} \\
\sum_{cyc}\cos 2A &= -1 - 4\cos A\cos B\cos C \\
\sum_{cyc}\cos^2 A &= 1 - 2\cos A\cos B\cos C \\
\sum_{cyc}\sin^2 A &= 2 + 2\cos A\cos B\cos C \\
\sum_{cyc}\tan A &= \tan A\tan B\tan C \\
\sum_{cyc}\cot\frac{A}{2} &= \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \\
\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2} &= 1 \\
\sum_{cyc}\cot A\cot B &= 1
\end{aligned}$$

2 Geometry and Identities

The half-angle formulae take a convenient form for triangles.

Theorem 2.1. *In $\triangle ABC$,*

$$\begin{aligned}
\sin\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}} = \sqrt{\frac{xy}{(y+z)(z+x)}} \\
\cos\frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}} = \sqrt{\frac{(x+y+z)z}{(y+z)(z+x)}} \\
\tan\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{xy}{(x+y+z)z}} = \frac{r}{s-c}
\end{aligned}$$

It may be helpful at times to express certain other quantities associated with a triangle in terms of the angles.

Theorem 2.2 (Identities related to R, r, s). *If $\triangle ABC$ has semiperimeter s, inradius r and circumradius R, then*

$$\begin{aligned}
r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &\iff \frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\
s = \frac{(\triangle ABC)}{r} &= 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \\
\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} &= \frac{r}{s} \\
\cos A + \cos B + \cos C &= 1 + \frac{r}{R} \\
2R\sin A\sin B\sin C &= r(\sin A + \sin B + \sin C) \\
a\cos A + b\cos B + c\cos C &= \frac{abc}{2R^2}
\end{aligned}$$

Theorem 2.3 (Area of a Triangle). $(\triangle ABC) = S$, $s = \frac{a+b+c}{2} = x+y+z$, $x = s-a, y = s-b, z = s-c$.

$$\begin{aligned}
S &= \sqrt{s(s-a)(s-b(s-c)} \\
&= \sqrt{xyz(x+y+z)} \\
&= \frac{1}{2}bc\sin A = \frac{abc}{4R} \\
&= 2R^2\sin A\sin B\sin C \\
&= sr = (s-a)r_A = (s-b)r_B = (s-c)R_C \\
&= \sqrt{rr_Ar_Br_C}
\end{aligned}$$

Theorem 2.4 (Identities and inequalities related to triangle).

$$\begin{aligned}
XG^2 &= \frac{1}{3}\sum_{cyc} XA^2 - \frac{1}{9}\sum_{cyc} BC^2 \iff 3\left(\sum_{cyc} GA^2\right) = \sum_{cyc} BC^2 \\
OH^2 &= 9R^2 - \left(\sum_{cyc} a^2\right) \iff R^2 \geq \frac{1}{9}\left(\sum_{cyc} a^2\right) \\
HG^2 &= 4R^2 - \frac{4}{9}\left(\sum_{cyc} a^2\right) \\
OG^2 &= R^2 - \frac{1}{9}\left(\sum_{cyc} a^2\right) \\
OI^2 &= R^2 - 2Rr \geq 0 \iff R \geq 2r \\
r &\leq \frac{s}{3\sqrt{3}} \leq \frac{R}{2}, \qquad \sin\frac{A}{2} \leq \frac{a}{b+c}
\end{aligned}$$

Theorem 2.5 (R, r, r_A, r_B, r_c).

$$\begin{aligned}
4R + r &= r_A + r_B + r_C,\ \frac{1}{r} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \\
r^2 &= \frac{xyz}{x+y+z},\ R^2 = \frac{(x+y)^2(y+z)^2(z+x)^2}{16xyz(x+y+z)} \\
\frac{r}{R} &= \frac{4xyz}{(x+y)(y+z)(z+x)} \\
Rr &= \frac{abc}{4s} = \frac{(x+y)(y+z)(z+x)}{4(x+y+z)}
\end{aligned}$$

Theorem 2.6 (Identities related to special points).

$$\begin{aligned}
m_a &= \frac{2b^2 + 2c^2 - a^2}{4}, \qquad m_a^2 + m_b^2 + m_c^2 = \frac{3(a^2+b^2+c^2)}{4} \\
AI &= \frac{b+c}{a+b+c}\sqrt{bc\left(1 - \left(\frac{a}{b+c}\right)^2\right)} \\
AL &= \sqrt{bc\left(1 - \left(\frac{a}{b+c}\right)^2\right)} \\
AR &= 2R\sin\left(B + \frac{A}{2}\right) \implies AR \cdot 2\cos\frac{A}{2} = 2R \cdot 2\sin\left(B + \frac{A}{2}\right)\cos\frac{A}{2}. \\
AR \cdot 2\cos\frac{A}{2} &= 2R(\sin C + \sin B) = b + c. \iff \boxed{AR = \frac{b+c}{2\cos\frac{A}{2}}}. \\
EF &= a\cos A = R\sin 2A \ (EF = \text{the side of the orthic triangle opposite to } A\ ) \\
AH &= a\cot A = 2R\cos A \\
HD &= 2R\cos B\cos C
\end{aligned}$$

Theorem 2.7 (Convex and Concave Functions). ***Convexity text.*** *Let f be twice differentiable function on $[a, b]$. Then, f is concave on $[a, b]$ iff $f''(x) \leq 0 \ \forall\, x \in [a, b]$*

Jensen's Inequality. *If f is concave in $[a, b]$, then for any $\omega_i \in [0, 1]$ with $\sum_{i=1}^{n} \omega_i = 1$ and $x_i \in [a, b]$ we have,*

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \leq f(\omega_1 x_i + \cdots + \omega_n x_n$$

$f(x) = \sin x$	$f'(x) = \cos x$	$f''(x) = -\sin x \leq 0$
$f(x) = \cos x$	$f'(x) = -\sin x$	$f''(x) = -\cos x \leq 0$
$f(x) = \tan x$	$f'(x) = \sec^2 x$	$f''(x) = 2\sec x \tan x \geq 0$
$f(x) = \ln\sin x$	$f'(x) = \frac{\cos x}{\sin x}$	$f''(x) = -\csc^x \leq 0$
$f(x) = \ln\cos x$	$f'(x) = \frac{-\sin x}{\cos x}$	$f''(x) = -\sec^2 x \leq 0$
$f(x) = \ln x$	$f'(x) = \frac{1}{x}$	$f''(x) = -\frac{1}{x^2} \leq 0$

3 References

1. *Plane Euclidean Geometry,*
2. *Problems in Plane Geometry,*
3. *Note of Hojoo Lee,* Triangle Geometry.

* This document is prepared on using LaTeX.