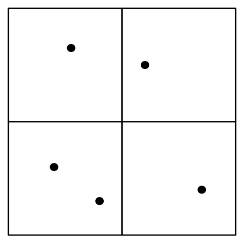
Pigeonhole Principle

Solutions to Exercises

1. Five points are chosen in a 2×2 square. Show that two points are within a distance of $\sqrt{2}$ of each other.

   Solution. Partition the 2×2 square into four 1×1 squares, as shown. By the Pigeonhole Principle, one of the 1×1 squares contains at least two points. The distance between these two points is at most $\sqrt{2}$.



2. (a) Prove that for any set of five integers, there are three integers whose sum is divisible by 3.
   (b) Prove that for any set of 17 integers, there are five integers whose sum is divisible by 5.

   Solution. (a) Consider the modulo 3 residues of the five integers. If some residue appears three times, then the sum of the corresponding three integers is divisible by 3. Otherwise, each residue appears at most twice. There are five integers, so all three residues 0, 1, and 2 must appear. Then the sum of the three integers corresponding to these residues is divisible by 3.

   (b) Consider the modulo 5 residues of the five integers. If some residue appears five times, then the sum of the corresponding five integers is divisible by 5. Otherwise, each residue appears at most four times. There are $17 = 4 \cdot 4 + 1$ integers, so all five residues 0, 1, 2, 3, and 4 must appear. Then the sum of the five integers corresponding to these residues is divisible by 5.

3. A tennis star, preparing for a tournament, decides to play at least one match a day, but no more than 20 matches altogether, over a period of two weeks. Show that during some set of consecutive days the star must play exactly 7 matches.

   Solution. For $1 \le i \le 14$, let s_i be the number of matches the tennis star played on the first i days, so
   $$1 \le s_1 < s_2 < \cdots < s_{14} \le 20.$$
   Then
   $$8 \le s_1 + 7 < s_2 + 7 < \cdots < s_{14} + 7 \le 27.$$
   Let $A = \{s_1, s_2, \dots, s_{14}\}$ and $B = \{s_1 + 7, s_2 + 7, \dots, s_{14} + 7\}$. Then all the 28 numbers in sets A and B combined lie between 1 and 27. By the Pigeonhole Principle, two of these 28 numbers are equal. In other words, the sets A and B have an element in common. Therefore, there exist indices i and j such that $s_i + 7 = s_j$.

   Then $s_j - s_i = 7$. Hence, the tennis star played exactly 7 matches between days $i + 1$ and day j, inclusive.

4. Every point in the plane is colored either red, green, or blue. Prove that there exists a rectangle in the plane such that all four of its vertices are the same color. (USAMTS, Year 18)

   Solution. Consider a 4×82 grid of lattice points in the plane, so there are 82 columns, and 4 lattice points in each column. There are $3^4 = 81$ ways to color the four lattice points in each column, so by the Pigeonhole Principle, there are two columns that have the same coloring.

   Again by the Pigeonhole Principle, two of the vertices in each column have the same color. Taking the two points in both columns gives us the desired rectangle.

5. Show that there exists a Fibonacci number F_n, $n \geq 1$, that is divisible by 10^4.

   Solution. Let S be the set of ordered pairs (x, y), where x and y are modulo 10^4 residues. Let

$$a_n = (F_n \pmod{10^4}, F_{n+1} \pmod{10^4}),$$

   so $a_0 = (0, 1)$, $a_1 = (1, 1)$, $a_2 = (1, 2)$, etc. Let $f : S \to S$ is the function defined by

$$f(x, y) = (y, x + y).$$

   Then

$$\begin{aligned} f(a_n) &= f(F_n \pmod{10^4}, F_{n+1} \pmod{10^4}) \\ &= (F_{n+1} \pmod{10^4}, F_n + F_{n+1} \pmod{10^4}) \\ &= (F_{n+1} \pmod{10^4}, F_{n+2} \pmod{10^4}) \\ &= a_{n+1}, \end{aligned}$$

   so we can generate the sequence a_0, a_1, a_2, $\ldots$ by repeatedly applying f. In other words, $a_n = f^n(a_0)$ for all n. Furthermore, the function f is invertible. To see this, let x and y be arbitrary modulo 10^4 residues. Then we want to solve

$$f(z, w) = (x, y)$$

   in z and w. We have that $f(z, w) = (w, z+w)$, so $w = x$ and $z+w = y$, which means $z = y-w = y-x$. Therefore, the unique pair (z, w) such that $f(z, w) = (x, y)$ is $(y - x, x)$. Hence, f is invertible.

   Let $M = |S|$. Consider the $M + 1$ terms

$$a_0,\ a_1,\ a_2,\ \ldots,\ a_M.$$

   By the Pigeonhole Principle, there exist indices i and j such that $a_i = a_j$, where $0 \leq i < j \leq M$. Then

$$f^i(a_0) = f^j(a_0).$$

   Since f is invertible, we can apply f^{-i} to both sides, to get

$$f^{j-i}(a_0) = a_0.$$

   In other words,

$$a_{j-i} = a_0.$$

   But $a_{j-i} = (F_{j-i} \pmod{10^4}, F_{j-i+1} \pmod{10^4})$, and $a_0 = (0, 1)$, so $F_{j-i} \equiv 0 \pmod{10^4}$.

6. Let n be a positive integer, and let S be a subset of the set $\{1, 2, \dots, 2n\}$ with $n+1$ elements.

   (a) Show that there exist two elements of S that are relatively prime.

   (b) Show that there exist two elements of S, one of which divides the other.

   Solution. (a) Partition the set $\{1, 2, \dots, 2n\}$ into the n pairs $\{1, 2\}$, $\{3, 4\}$, $\dots$, $\{2n-1, 2n\}$. By the Pigeonhole Principle, S contains two numbers in the same pair. Since these two numbers are consecutive, they are relatively prime.

   (b) Every positive integer n can be written uniquely in the form $n = 2^e m$, where e is a nonnegative integer and m is an odd positive integer. Let $f(n)$ denote this odd positive integer m.

   For $1 \le i \le 2n$, the possible values of $f(i)$ are 1, 3, 5, $\dots$, $2n-1$, for a total of n possible values. Hence, by the Pigeonhole Principle, there exist elements a and b in S such that $f(a) = f(b)$.

   Let $a = 2^e m$ and $b = 2^f m$. If $e \le f$, then a divides b, and if $e > f$, then b divides a.

7. Mathew has an unusual habit: Much to Amanda's consternation, Mathew likes to shuffle around the items in his refrigerator every day. However, at least he does it in a consistent way.

   For example, if he moves whatever is in the egg rack to the vegetable crisper on one day, then he does the same thing every day. (In other words, he applies the same permutation every day.) Prove that eventually, all the food will be returned to its original location.

   Solution. We can re-phrase the problem as follows: Let X be a finite set, and let π be a permutation on the set X. Show that there is a positive integer d such that π^d is the identity on X.

   Let $n = |X|$, so there are $n!$ permutations on X. Consider the $n!+1$ permutations

$$\pi,\ \pi^2,\ \pi^3,\ \dots,\ \pi^{n!},\ \pi^{n!+1}.$$

   By the Pigeonhole Principle, there exist i and j such that $\pi^i = \pi^j$, where $1 \le i < j \le n!+1$. As a permutation, π is invertible, so it follows that π^{j-i} is the identity, and we can take $d = j - i$.

8. Find the greatest positive integer n for which there exist n nonnegative integers x_1, x_2, $\dots$, x_n, not all zero, such that for any sequence ϵ_1, ϵ_2, $\dots$, ϵ_n of elements of $\{-1, 0, 1\}$, not all zero, n^3 does not divide $\epsilon_1 x_1 + \epsilon_2 x_2 + \dots + \epsilon_n x_n$. (Romania, 1996)

   Solution. We claim that the answer is $n = 9$.

   For $n = 9$, let $x_i = 2^{i-1}$. Then

$$\begin{aligned} |\epsilon_1 x_1 + \epsilon_2 x_2 + \dots + \epsilon_9 x_9| &= |\epsilon_1 + 2\epsilon_2 + \dots + 2^8 \epsilon_9| \\ &\le 1 + 2 + \dots + 2^8 \\ &= 2^9 - 1 \\ &= 511 \\ &< 9^3. \end{aligned}$$

Furthermore, $\epsilon_1 + 2\epsilon_2 + \cdots + 2^8\epsilon_9 = 0$ if and only if $\epsilon_i = 0$ for all i. Hence, the given condition holds for $n = 9$.

On the other hand, let $n \geq 10$. There are 2^n subsets of $\{x_1, x_2, \dots, x_n\}$. It is easy to show that $2^n > n^3$ for $n \geq 10$, so by the Pigeonhole Principle, there are two subsets A and B such that the sums of the elements in the subsets are congruent modulo n^3. (By convention, the sum for the empty set is 0.) Let $\epsilon_i = 1$ if x_i occurs in A but not in B, -1 if x_i occurs in B but not in A, and 0 otherwise. Then $\sum_{i=1}^n \epsilon_i x_i$ is divisible by n^3.