NOTE: If you have any questions with any of these solutions, or would like extra problems, please don't hesitate to contact me: hoshino@mscs.dal.ca

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SOLUTIONS TO "HARD FUNCTIONAL EQUATION PROBLEMS"

1. Let $x=y=0$: $f(0)^2-f(0)=0$, so $f(0)=0$ or 1.
Let $x=y=1$: $f(1)^2-f(1)=2$, so $f(1)=2$ or -1.

Let $y=1$. Then $f(x)f(1)-f(x)=x+1 \Rightarrow f(x)\cdot[f(1)-1]=x+1 \Rightarrow f(x)=\frac{x+1}{f(1)-1}$, for all $x\in\mathbb{R}$. If $f(1)=2$, then $f(x)=x+1$, and this is easily verified to satisfy $f(x)f(y)-f(xy)=x+y \quad \forall x,y\in\mathbb{R}$ "for all"
But if $f(1)=-1$, then $f(x)=-\frac{x+1}{2}$, and so $f(0)=-\frac{1}{2}$, which is a contradiction

So we conclude that $\boxed{f(x)=x+1}$ is the unique solution.

2 Let $y=0$. Then $f(x)f(0)=f(x)+f(x) \Rightarrow f(x)\cdot[f(0)-2]=0 \quad \forall x$. Now if $f(0)\neq 2$, then $f(x)=0 \ \forall x$, and this contradicts $f(0)\neq 0$. Thus, we must have $f(0)=2$.

Let $y=1$. Then $f(x)\cdot f(1)=f(x+1)+f(x-1)$, or $2f(x+1)-5f(x)+2f(x-1)=0$, for each $x\in\mathbb{Z}$. This looks like a recurrence relation! Let $f(n)=a_n$. Then for each integer n, $2a_{n+1}-5a_n+2a_{n-1}=0$, with $a_0=2$ and $a_1=\frac{5}{2}$. The characteristic equation is $2x^2-5x+2=0$, which has roots 2 and $\frac{1}{2}$. Hence, $a_n=A\cdot 2^n+B\cdot(\frac{1}{2})^n$ for each n.

Thus, $2=a_0=A+B$ and $\frac{5}{2}=a_1=2A+\frac{1}{2}B$, and solving we get $A=B=1$. Hence, $f(n)=a_n=2^n+(\frac{1}{2})^n$. So the unique solution is $\boxed{f(x)=2^x+(\frac{1}{2})^x}$

3. Let $g(x)=2f(x)$. Then $f(xy)=2f(x)f(y) \Rightarrow \frac{g(xy)}{2}=2\cdot\frac{g(x)}{2}\cdot\frac{g(y)}{2} \Rightarrow g(xy)=g(x)g(y)$.
Letting $y=1$, we see that $g(1)=1$ (since $g(x)>0\ \forall x$, this is our only option), and let $g(2)=t$. An easy induction shows that $g(2^n)=t^n$, for all integers n. Now, $g(x^k)=(g(x))^k$ implies that for all rational numbers $\frac{a}{b}$, $g(2^{\frac{a}{b}\cdot b})=g(2^{\frac{a}{b}})^b \Rightarrow t^a=g(2^{\frac{a}{b}})^b \Rightarrow g(2^{\frac{a}{b}})=t^{\frac{a}{b}}$. Hence $g(2^n)=t^n$ holds for all rational n. Since g is continuous, we see that this implies that $g(2^n)=t^n$ holds for all real n. Hence, $g(x)=g(2^{\log_2 x})=t^{\log_2 x}=x^{\log_2 t}=x^r$, for some r.

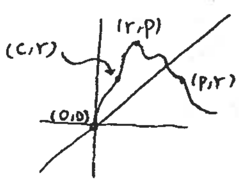
this part needs to be a bit more rigorous. see if you can add some details to make this rigorous.

Hence, the solution to $g(xy)=g(x)g(y)$ is $g(x)=x^r$, and we easily see that all possible values of r work. Therefore, our solution is $\boxed{f(x)=\frac{1}{2}x^r,\ r\in\mathbb{R}}$.

4. Let $f(0)=t$. Letting $y=0$, we have $f(x+f(0))=f(x)+0 \Rightarrow f(x+t)=f(x)$. Then $f(0+f(x+t))=f(0)+x+t$ and $f(0+f(x))=f(0)+x$. Because $f(x+t)=f(x)$, we have $f(f(x+t))=f(f(x)) \Rightarrow f(0)+x+t=f(0)+x \Rightarrow t=0$. Hence, $f(0)=0$. Letting $x=0$, $y=x$ we have $f(f(x))=x$ for all $x\in\mathbb{R}$.

Suppose $f(p)=f(q)=r$ for some p,q,r with $p\neq q$. Then $f(r)=f(f(p))=p$ and $f(r)=f(f(q))=q$, so $p=q$, a contradiction. Thus for each r, there is at most one value of p for which $f(p)=r$. And if $f(p)=r$, then we have $f(f(p))=p \Rightarrow f(r)=p$.

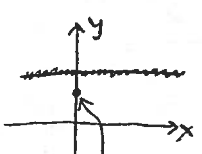
Let's prove that $p=r$. On the contrary, say $p\neq r$. WLOG, suppose $p>r$.



Consider the graph of $f(x)$, from $x=0$ to $x=r$. Since $f(0)=0$ and $f(r)=p>r$, there exists at least one number c, $0<c<r$, with $f(c)=r$, since f is continuous. Note: $c\neq p$ since $c<r$ and $r<p$. Then we have $f(c)=f(p)=r$, $c\neq p$, which is a contradiction from above.

Therefore we require $p=r$, i.e. $f(p)=p$ for each $p\in\mathbb{R}$ (note: the range of f is $\mathbb{R}$). And so we conclude that the only solution is $\boxed{f(x)=x}$.

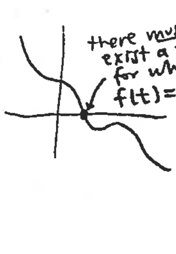
5. Let $y=1$. Then $xf(1)-f(x)=(x-1)f(x) \Rightarrow xf(1)=xf(x)$, for all x. Thus, $f(x)=f(1)$, for all $x\neq 0$. But f is continuous, so that means that $f(x)=f(1)$ for all x. So f is a constant function.



since f is continuous, we can't have $f(0)\neq f(1)$.

Let $f(x)=c$. Then $xf(y)-yf(x)=xc-yc=(x-y)c=(x-y)f(xy)$, for all $x,y\in\mathbb{R}$. Thus, the set of solutions is $\boxed{f(x)=c,\ c\in\mathbb{R}}$.

6. Let $f(a)=b$ for some a and b. Then $f(x-f(a))=1-x-a \Rightarrow f(x-b)=1-x-a$. Since a and b are finite, we can make $f(x)$ a positive number by letting x be sufficiently small, and make $f(x)$ negative by letting x be sufficiently large. Since f is continuous, there must exist a value $t\in\mathbb{R}$ for which $f(t)=0$ since f attains both positive and negative values. (see diagram).



there must exist a t for which $f(t)=0$.

Letting $y=t$, we find that $f(x)=1-x-t$. Substituting this into our functional equation, we get $1-(x-f(y))-t=1-x-y \Rightarrow 1-x+(1-y-t)-t=1-x-y \Rightarrow t=\frac{1}{2}$. Therefore, the only solution is $\boxed{f(x)=\frac{1}{2}-x}$.

7. We shall show that $f(x)=x+1$ is the only solution. First we prove this for all integers n. Letting $x=n$ and $y=1$, we get $f(n)=f(n)f(1)-f(n+1)+1 \Rightarrow f(n+1)=f(n)+1$, with $f(1)=2$. An easy induction proves that $f(x)=x+1$ for all $x\in\mathbb{Z}$. Now we prove the result for all rational numbers of the form $\frac{1}{b}$. Let $b\in\mathbb{Z}$. Letting $x=b$ and $y=\frac{1}{b}$ we get $f(b\cdot\frac{1}{b})=f(b)\cdot f(\frac{1}{b})-f(b+\frac{1}{b})+1 \Rightarrow (b+1)\cdot f(\frac{1}{b})-f(b+\frac{1}{b})=1$. Also, $f(x+1)=f(x)+1\ \forall x \Rightarrow f(b+\frac{1}{b})=f(\frac{1}{b})+b$. From these two equations we get $(b+1)\cdot f(\frac{1}{b})-f(\frac{1}{b})-b=1 \Rightarrow f(\frac{1}{b})=1+\frac{1}{b}$. So we have proven the claim for all rational numbers of the form $\frac{1}{b}$, $b\in\mathbb{Z}$.

Now suppose we have proven the claim for all rational numbers of the form $\frac{k}{b}$, where $k=1,2,\dots,t$. We shall show that the claim is true for all rational numbers of the form $\frac{t+1}{b}$, where $b\in\mathbb{Z}$. Letting $x=\frac{t}{b}$ and $y=\frac{1}{b}$, we get $f(\frac{t}{b^2})=f(\frac{t}{b})f(\frac{1}{b})-f(\frac{t+1}{b})+1 \Rightarrow f(\frac{t+1}{b})=(\frac{t}{b}+1)(\frac{1}{b}+1)+1-(\frac{t}{b^2}+1)$ by the induction hypothesis. Thus, $f(\frac{t+1}{b})=\frac{t+1}{b}+1$, as required.

Hence by induction we have proven that $f(\frac{a}{b})=\frac{a}{b}+1$ for all rational numbers $\frac{a}{b}$, and so we conclude that $\boxed{f(x)=x+1}$, for all $x\in\mathbb{Q}$.

8. Let $k=0$. Then $2f(n)=2f(0)f(n)\ \forall n\in\mathbb{Z}$. So either $f(x)=0$ for all x, or $f(0)=1$. Let $f(1)=a$. Then substituting $x=n$ and $y=1$, we get $f(n+1)-2af(n)+f(n-1)=0$. The characteristic equation is $x^2-2ax+1=0$.

Case 1: $a=1$. Then the only root of the equation is 1. Hence, $f(n)=1^n\cdot(A+Bn)$, for some constants A and B. From $f(0)=1$ and $f(1)=a=1$, we get $A=1$, $B=0$, and so $f(n)=1$ is the function. Clearly $|f(n)|\le N$ for all n, if we let $N=1$.

Case 2: $a=-1$. Then the only root of the equation is -1. Hence, $f(n)=(-1)^n(A+Bn)$ and from $f(0)=1$ and $f(1)=a=-1$, we get $A=1$ and $B=0$. So $f(n)=(-1)^n$ is the function. Checking, we see that this function satisfies the given conditions.

Case 3: $a\neq\pm1$. Then $x^2-2ax+1=0$ has two distinct roots p and q. Since $pq=1$ and p and q are distinct, either $|p|$ or $|q|$ exceeds 1. WLOG, suppose $|p|>1$. Now, $f(n)=A\cdot p^n+B\cdot q^n$ for some constants A and B. As $n\to\infty$, we have $|p|^n\to\infty$ and $|q|^n\to 0$ (since $|p|>1$ and $|q|<1$). Thus, $f(n)\overset{\text{is approx. equal to}}{\sim} A\cdot p^n$ for sufficiently large n and we can make $|f(n)|\sim|A|\cdot|p|^n$ as large as we want. So there is no integer N for which $|f(n)|\le N$ for all n. Therefore, we have no solutions.

We conclude that the only solutions are $\boxed{f(x)=1}$ and $\boxed{f(x)=(-1)^x}$, for all $x\in\mathbb{Z}$

9. $f(2)-2f(1)=0$ or 1

$\Rightarrow f(1)=\frac{f(2)}{2}$ or $\frac{f(2)-1}{2}$. Since $f(2)=0$ and $f(1)\in W$, we have $\underline{f(1)=0}$

$f(m+1)-f(m)-f(1)\geq 0 \Rightarrow f(m+1)\geq f(m)$ for each m. $(*)$

Now, $f(m+3)=f(m)+f(3)+(0 \text{ or } 1)$

$\therefore f(m+3)\geq f(m)+1$, with equality iff $f(3)=1$.

this works, but is very messy and inelegant. Can you come up with a better proof to show that $f(1982)=660$?

Now, $f(9999)=(f(9999)-f(9996))+\cdots+(f(6)-f(3))+f(3)\geq 3332+f(3)\geq 3333$

so we must have $f(3)=1$ and $f(m+3)=f(m)+1$ for each $m=3,6,\ldots,9996$

Thus, $f(3x)=x$ for $x=1,2,3,\ldots,3333$.

Since $f(1980)=660$ and $f(1983)=661$, by $(*)$, $f(1982)=660$ or 661.

Suppose $f(1982)=661$. Then $f(1985)=f(1982)+f(3)+(0 \text{ or } 1)=662$ or 663. But $f(1986)=662$, so we must have $f(1985)=662$. And also, $f(1988)=663$ by the same argument.

Thus, $f(3967)=f(1982)+f(1985)+(0 \text{ or } 1)\geq 1323$

and $f(3970)=f(1982)+f(1988)+(0 \text{ or } 1)\geq 1324$.

Then $f(7937)=f(3967)+f(3970)+(0 \text{ or } 1)\geq 2647$, but this is a contradiction since $f(7938)=2646$.

Thus, $f(1982)$ must be $\boxed{660}$. A function for which this is possible is $f(x)=\lfloor \frac{x}{3}\rfloor$. One can easily check that this function satisfies all the desired properties.

10. Suppose there is a $k\in\mathbb{R}^+$ for which $f(k)=1$. Then letting $y=k$, we have $f(xf(k))=kf(x)\Rightarrow f(x)=kf(x)$ for all $x\Rightarrow k=1$. (Clearly we can't have $f(x)=0$ for all x since $f(k)=1$).

Letting $y=\frac{1}{f(x)}$, we have $f(x\cdot f(\frac{1}{f(x)}))=1$, and so from above, we must have $x\cdot f(\frac{1}{f(x)})=1\Rightarrow f(\frac{1}{f(x)})=\frac{1}{x}$. —① So $f(1)=1$.

Also, $f(kf(y))=yf(k)\Rightarrow f(f(y))=y$ for all $y\in\mathbb{R}^+$. —②

From ① and ②, we get $f(\frac{1}{x})=f(f(\frac{1}{f(x)}))=\frac{1}{f(x)}$. So $f(x)\cdot f(\frac{1}{x})=1$.

③

Now let $f(y)=z$. Then $f(f(y))=f(z)$, but by ②, $f(f(y))=y$. Thus, $f(y)=z \Rightarrow f(z)=y$. So $f(xy)=f(x\cdot f(z))=zf(x)=f(x)f(y)$, so f is a multiplicative function.

Let $x=y$. Then $f(xf(x))=xf(x)$. So if $t=xf(x)$ for some $x\in\mathbb{R}^+$, then $f(t)=t$. Because f is multiplicative, $f(t^2)=f(t).f(t)=t^2$, and by a simple induction, $f(t^n)=[f(t)]^n=t^n$. If $t>1$, then this will contradict the given information that $\lim_{x\to\infty}f(x)=0$. So t is at most 1. Furthermore, by ③, we have $f(\frac{1}{t})=\frac{1}{f(t)}=\frac{1}{t}$, so by induction $f(\frac{1}{t^n})=\frac{1}{t^n}$. So if $t<1$, we contradict $\lim_{x\to\infty}f(x)=0$. Thus, we require $t=1$, and so the only possible value of $xf(x)$, for any $x\in\mathbb{R}^+$, is 1.

Therefore, $xf(x)=1$ for all x, and so we must have $\boxed{f(x)=\frac{1}{x}}$.

11. Sorry, I shouldn't have put this on the set: it's a lousy problem — one just pounds away at it: unfortunately the problem requires very little ingenuity.

$f(3573)=f(397)+f(9)=f(397)+f(3)+f(3)$. Since $f(x)=0$ whenever $x\equiv 3 \pmod{10}$, we get $\underline{f(397)=0}$.

Now, $0=f(10)=f(2)+f(5)$ and since $f(2)$ and $f(5)$ are non-negative, we have $f(2)=f(5)=0$. In particular, $\underline{f(5)=0}$. Thus, $f(1985)=f(5)+f(397)=\boxed{0}$

12. Let $m=n=0$. Then $f(f(0))=f(f(0))+f(0) \Rightarrow f(0)=0$. —①

Let $m=0$. Then $f(f(n))=f(f(0))+f(n)=f(0)+f(n)=f(n)$ for each $n\in\mathbb{N}$. Define a "fixed point" to be an integer x such that $f(x)=x$, i.e. x maps to itself. Then for any integer n, $f(n)$ is a fixed point since $f(f(n))=f(n)$. (In addition there may be other fixed points too). For example, 0 is a fixed point by ①.

Consider the set S of fixed points, and let k be the smallest non-zero fixed point. If no such k exists, then we must have $\underline{f(n)=0}$ for all $n\in\mathbb{N}$, and this is a trivial solution to the functional equation. (Note: we must have $f(n)=0$, for if $f(p)=q$ for some $q\neq 0$, then q is a non-zero fixed point $\rightarrow$ contradiction).

So suppose k does exist. Then $f(k)=k$. Then letting $n=k$, we have $f(m+f(k))=f(f(m))+f(k) \Rightarrow f(m+k)=f(m)+k$. By a simple induction, $f(qk)=qk$ for each integer $q\geq 0$.

Let n be an arbitrary fixed point.

Now we use the Division Algorithm: for this integer n, there exist unique integers q and r, with $0 \le r < k$, such that $n = qk + r$.

Then for this n, $f(n) = f(r+qk) = f(r + f(qk)) = f(f(r)) + f(qk) = f(r) + qk$

Since n is a fixed point, we have $f(n) = n = qk + r$. Thus, we have $f(r) + qk = qk + r \Rightarrow f(r) = r$, ie r is a fixed point. However, $0 \le r < k$ and k is the smallest non-zero fixed point. This proves that r must be 0. Hence, if n is a fixed point, then $n = qk$ for some q, ie. the fixed points of f are precisely the multiples of k.

But $f(n)$ is a fixed point for every integer n, so $k \mid f(n)$ for each n

Let $f(1) = a_1 \cdot k$, $f(2) = a_2 \cdot k$, $\ldots$, $f(k-1) = a_{k-1} \cdot k$ for some integers $a_1, a_2, \ldots, a_{k-1}$.

Then the most general function satisfying the given conditions is

$f(n) = f(qk+r) = qk + f(r) = qk + a_r \cdot k = (q + a_r)k$, where $0 \le r < k$.

(Note: $a_0 = 0$).

e.g
$f(0)=0$
$f(1)=10$
$f(2)=15$
$f(3)=0$
$f(4)=25$
$f(5)=5$
$f(6)=15$
$f(7)=20$
$f(8)=5$
$f(9)=30$
$f(10)=10$
$f(11)=20$
$f(12)=25$
$f(13)=10$
$f(14)=35$
$\vdots$
etc.

As an aside: this is extremely abstract, so let me just illustrate with an example. Say 5 is the smallest non-zero fixed point. Then the only fixed points are $0, 5, 10, 15, 20, \ldots$, ie $f(0)=0$, $f(5)=5$, $f(10)=10$, etc. We showed that $5 \mid f(n)$ for each n, so let $f(1) = 5a_1$, $f(2) = 5a_2$, $f(3) = 5a_3$, and $f(4) = 5a_4$ for any integers a_1, a_2, a_3, and a_4. Let $a_0 = 0$. Then the function $f(n) = 5(q + a_r)$ satisfies the conditions given in the question, where q and r are the unique integers for which $n = qk + r$, where $0 \le r < k$

To finish the proof, we must verify that this function satisfies the functional equation. Let $m = q_1 k + r_1$ and $n = q_2 k + r_2$, where $0 \le r_1, r_2 < k$. Then $f(m + f(n)) = f(q_1 k + r + q_2 k + a_{r_2} k) = (q_1 + q_2 + a_{r_2})k + a_{r_1} k$ $= (q_1 + q_2 + a_{r_1} + a_{r_2})k$. And $f(f(m)) + f(n) = f(q_1 k + r_1 k) + f(q_2 k + r_2) =$ $q_1 k + a_{r_1} k + q_2 k + a_{r_2} k = (q_1 + q_2 + a_{r_1} + a_{r_2})k$. Thus, $f(m + f(n)) = f(f(m)) + f(n)$.

Hence, the set of functions satisfying the functional equation is:

$f(n) = (q + a_r)k$, where k is an integer (≥ 0),

$a_0 = 0$, $a_1, a_2, \ldots, a_{k-1}$ are any non-negative integers and q and r are the unique integers for which $n = qk + r$, with $0 \le r < k$.

13. My mistake again - this is a bad problem for this set, since the solution involves one trick and then it's really straightforward. It's not something you can play with, like the other problems in this set.

Let K be the smallest number for which $|f(x)| \le K$ for each x. K is called the "least upper bound". From the given information, K is at most 1, but could be less.

Suppose on the contrary that $|g(y)|>1$, for some y. Take any x with $|f(x)|>0$ (such an x must exist because f is not identically zero). Then,

$$2K \ge |f(x+y)| + |f(x-y)| \ge |f(x+y) + f(x-y)|, \text{ by the Triangle Inequality}$$
$$= 2|f(x)||g(y)|.$$

Thus, $|f(x)| \le \frac{K}{|g(y)|} < K$, for all x. This proves that $\frac{K}{|g(y)|}$ is an upper bound for $|f(x)|$, which contradicts the fact that K is the least upper bound. Therefore, we must have $|g(y)| \le 1$ for all y.

14. The first condition implies that P is homogeneous with degree n, i.e., every term in P has degree n. See if you can convince yourself why this must be true.

when you are writing up your proof, you don't need to put this in. I just did that to illustrate one example.

Let's experiment with a special case to see what's going on: try $n=2$. Due to the homogeneity of P, $P(x,y) = ax^2+bxy+cy^2$ for some $a,b,c \in \mathbb{R}$. The second condition gives us $a[(x+y)^2+(y+z)^2+(z+x)^2] + b[x(y+z)+y(x+z)+z(x+y)] + c\cdot[x^2+y^2+z^2] = 0 \Rightarrow (x^2+y^2+z^2)(2a+c) + (xy+yz+zx)(2a+2b) = 0$. Since this equation holds for all $x,y,z \in \mathbb{R}$, we must have $2a+c = 2a+2b = 0$. The third condition gives us $1 = P(1,0) = a\cdot 1 + b\cdot 0 + c\cdot 0 \Rightarrow a = 1$. Hence, $b=-1$ and $c=-2$, so the only polynomial for $n=2$ is $P(x,y) = x^2-xy-2y^2 = (x+y)(x-2y)$

We claim that the only polynomial that satisfies the given conditions for a given n is $P(x,y) = (x+y)^{n-1}(x-2y)$.

Let's first make sure that this P satisfies the given conditions. Clearly, P is homogeneous with degree n, and $P(1,0)=1$. So conditions i) and iii) are satisfied. Let's check ii): $P(y+z,x)+P(z+x,y)+P(x+y,z) = (x+y+z)^{n-1}\cdot[(y+z-2x)+(x+z-2y)+(x+y-2z)] = 0$. So this P satisfies the given conditions. Now we have to prove that this is the only polynomial P that satisfies the given conditions.

Let $y=1-x$ and $z=0$, then $P(x,1-x) = -P(1-x,x) - 1$. $\quad$ - ①.

Let $z=1-x-y$. Then $P(1-x,x) + P(1-y,y) + P(x+y,1-x-y) = 0 \Rightarrow$

$[P(1-x,x)-1] + [P(1-y,y)-1] + [P(x+y,1-x-y)+2] = 0$

$\Rightarrow [P(1-x,x)-1] + [P(1-y,y)-1] + [-P(1-x-y,x+y)-1+2] = 0$ $\quad$ by ①

$\Rightarrow [P(1-x,x)-1] + [P(1-y,y)-1] = [P(1-x-y,x+y)-1]$.

P is a polynomial

So let $f(a) = P(1-a,a)-1$ for each $a \in \mathbb{R}$. Then we have shown that $f(x)+f(y) = f(x+y)$, for each $x,y \in \mathbb{R}$. Since P is continuous, so is f.

By a simple induction, $f(mx) = mf(x)$ for each integer m. Then for any rational number $\frac{1}{b}$, we have $f(b \cdot \frac{1}{b}) = b \cdot f(\frac{1}{b})$, so $f(\frac{1}{b}) = \frac{f(1)}{b}$, and so $f(\frac{a}{b}) = f(\frac{1}{b} \cdot a) = a \cdot f(\frac{1}{b}) = \frac{a}{b} \cdot f(1)$. Hence, $f(t) = t f(1)$ for each rational number t, and because f is continuous, that implies that $f(x) = x \cdot f(1)$ for all real x. Since $f(1) = P(0,1) - 1 = [-P(1,0)-1]-1 = -P(1,0)-2 = -3$, we have proven that $f(x) = -3x$. Therefore, $P(1-x,x) = 1-3x \ \forall x \in \mathbb{R}$.

Pick any real a and b. If $a+b \neq 0$, we have

$P(a,b) = P\left((a+b)\cdot\frac{a}{a+b}, (a+b)\cdot\frac{b}{a+b}\right) = (a+b)^n \cdot P\left(\frac{a}{a+b}, \frac{b}{a+b}\right) = (a+b)^n \cdot P\left(1-\frac{b}{a+b}, \frac{b}{a+b}\right) =$
$(a+b)^n \cdot \left(1-3\cdot\frac{b}{a+b}\right) = (a+b)^{n-1}\cdot(a-2b)$. And if $a+b=0$, we must have $P(a,b)=0$ because P is continuous.

Therefore we have proven that $P(x,y) = (x+y)^{n-1}(x-2y)$ is the only polynomial that satisfies the given conditions for a fixed positive integer n. So the desired set of polynomials is $\{(x+y)^k(x-2y) : k \text{ is a non-negative integer}\}$, and this is precisely the set of polynomials that satisfy all three given conditions.

15. First we prove that $f(1) < f(2) < f(3) < f(4) < \cdots$.

We proceed by induction: we will prove the statement $f(n) < f(m)$ whenever $n < m$ using induction on n, and this will prove that $f(1) < f(2) < f(3) < f(4) < \cdots$

Base Case: $n=1$ - suppose that $f(1)$ is not the unique minimum element of the set $\{f(1), f(2), f(3), \ldots\}$, and that $f(m)$ is a minimum element for some $m \geq 2$. Then $f(m) = f(f(m-1)+1) > f(f(m-1))$. Letting $f(m-1) = t$, we have proven that $f(m) > f(t)$ which contradicts the minimality of $f(m)$. (Note that $t = f(m-1)$ is defined because $m \geq 2$. If $m=1$, this argument doesn't work). Therefore, the minimum element of the set $\{f(1), f(2), \ldots\}$ must be $f(1)$, and so $f(1) < f(m)$ for all $1 < m$. Thus the claim is true for $n=1$.

Induction Hypothesis: suppose that $f(n) < f(m)$ whenever $n < m$ for $n = 1, 2, \ldots, k$. Essentially this means $f(1) < f(2) < \cdots < f(k)$ and $f(k) < f(m)$ whenever $k < m$. Consider the set $S = \{f(k+1), f(k+2), f(k+3), \ldots\}$. Suppose the minimum element of S is $f(k+a)$ for some $a \geq 2$. Then if we let $f(k+a-1) = \ell$, we have $f(k+a) > f(f(k+a-1)) = f(\ell)$, a contradiction (note: $\ell \geq f(k)+1 \geq k+1$, so $f(\ell)$ is in S). Thus the minimum element of S must be $f(k+1)$ and $f(k+1) < f(m)$ whenever $k+1 < m$. Thus we have proven our claim for $n = k+1$.

Therefore, we have shown that $f(1) < f(2) < f(3) < f(4) < \cdots$, i.e. the function is strictly increasing.

Since $f(1) \geq 1$, we must have $f(n) \geq n$ for each n. Suppose $f(t) > t$ for some t. Then $f(t) \geq t+1$ and because f is strictly increasing, we have $f(t) \geq t+1 \Rightarrow f(f(t)) \geq f(t+1)$. But $f(t+1) > f(f(t))$, and so $f(f(t)) > f(f(t))$ a contradiction. Thus, there is no integer t for which $f(t) > t$. Therefore, we have proven that $\boxed{f(n) = n}$, as required.

so if $p \geq 8$, then $f(p) \geq f(8)$.

16. Both f and g are strictly increasing functions. Thus, $f(1) \geq 1 \Rightarrow g(1) = f(f(1)) + 1 \geq f(1) + 1 \geq 2$. Since $g(1) \geq 2$ and 1 appears in either the set $F = \{f(1), f(2), f(3), \dots\}$ or $G = \{g(1), g(2), g(3), \dots\}$, we conclude that $1 \in F$. Since f is a strictly increasing function, $\underline{f(1) = 1}$.

Suppose there are two consecutive integers in G. Say $m, m+1 \in G$ for some m. Then $g(t) = m+1$ for some t. Then $f(f(t)) = g(t) - 1 = m$, so letting $f(t) = u$, we have $f(u) = m$, which shows that $m \in F$. However F and G are disjoint sets so m cannot be in <u>both</u> sets, so we have established a contradiction. Thus we cannot have two consecutive integers in G.

Let $f(n) = k$. Then $g(n) = f(f(n)) + 1 = f(k) + 1$. In the set $\{1, 2, 3, \dots, f(k)+1\}$, there are exactly n terms that belong to the set G because $g(1) < g(2) < \dots < g(n)$ and $g(n) = f(k) + 1$. Now let's look at the elements of $\{1, 2, 3, \dots, f(k)+1\}$ that are in F. There are exactly k such elements, because $1 = f(1) < f(2) < \dots < f(k)$ and $f(k) + 1 \notin F$, since this term is in G. Thus, the set $\{1, 2, \dots, f(k)+1\}$ has exactly n terms in G and k terms in F. But each integer from 1 to $f(k)+1$ is in exactly one of these two sets and so that proves that $f(k) + 1 = n + k \Rightarrow f(k) = n + k - 1$. We have $g(n) = n + k$ and from above, $n + k + 1 \notin G$. Thus, we must have $n + k + 1 \in F$, specifically $f(k+1) = n + k + 1$, because f is strictly increasing.

So we have proven that $f(n) = k$ implies that $\underline{f(k) = n + k - 1}$ and $\underline{f(k+1) = n + k + 1}$. The rest is just busy work. Since $f(1) = 1$, we have $f(2) = 3$ (by letting $n = k = 1$), and then we keep going:

$f(3) = 4$	(letting $n = 2, k = 3$)	$f(22) = 35$	$f(148) = 239$
$f(4) = 6$	($n = 2, k = 3$)	$f(35) = 56$	$f(240) = 148 + 239 + 1 = 388$.
$f(6) = 9$	($n = 4, k = 6$)	$f(56) = 90$	
$f(14) = 22$	($n = 6, k = 9$) etc.	$f(91) = 147$	

Therefore, $\boxed{f(240) = 388}$.

17. $f(1,y) = f(0, f(1,y-1)) = f(1,y-1)+1$. Letting $g(y) = f(1,y)$, we have $g(y) = g(y-1)+1$ and an easy induction proves that $g(n) = g(0)+n = f(1,0)+n = f(0,1)+n = n+2$. So $\underline{f(1,y) = y+2}$.

$f(2,y) = f(1, f(2,y-1)) = f(2,y-1)+2$ from above. Then solving by the same method we get $\underline{f(2,y) = 2y+3}$ since $f(2,0) = f(1,1) = 1+2 = 3$.

$f(3,y) = f(2, f(3,y-1)) = 2\cdot f(3,y-1)+3$. Let $h(y) = f(3,y)+3$. Then we have $h(y)-3 = 2\cdot[h(y-1)-3]+3 \Rightarrow h(y) = 2h(y-1)$, where $h(0) = 3+f(3,0) = 3+f(2,1) = 3+5 = 8$. So $h(y) = 2^{y+3}$ (once again, easy induction), and so $\underline{f(3,y) = 2^{y+3}-3}$.

Finally, $f(4,y) = f(3, f(4,y-1))$. Let $r(y) = f(4,y)+3$. Then we have $r(y) = 2^{r(y-1)}$, with $r(0) = f(4,0)+3 = f(3,1)+3 = 2^4-3+3 = 2^4 = 2^{2^2}$. Let $t(n)$ represent a tower of n 2's, i.e. $t(3) = 2^{2^2} = 2^4 = 16$.

Then $r(0) = t(3)$, $r(1) = 2^{r(0)} = 2^{t(3)} = t(4)$ and by induction, $r(y) = t(y+3)$. Thus, $f(4,y) = t(y+3)-3$.

$2^{2^{2^{\cdot^{\cdot^{2}}}}}$ 1984 two

We conclude that $f(4,1981)$ equals a tower of 1984 2's minus 3.

18. Let $f(1) = K$, for some integer $K \in \mathbb{N}$. Letting $m=1$, we have $f(n^2K) = (f(n))^2$ —①, and letting $n=1$, we have $f(f(m)) = mK^2$ —②.

this is one of the hardest IMO questions I have ever seen! You might want to read this solution a few times!

positive integer

By ①, we have $(f(Kx))^2 = f((Kx)^2K) = f(K^3x^2)$, for each $x \in \mathbb{N}$.

By ②, we have $f(f(Kx^2)) = Kx^2\cdot K^2 = K^3x^2$, so $f(K^3x^2) = f(f(f(Kx^2))) = f(1^2\cdot f(f(Kx^2))) = f(Kx^2)\cdot(f(1))^2 = K^2\cdot f(Kx^2) = K^2\cdot f(x^2\cdot f(1)) = K^2\cdot(f(x))^2$. Therefore, we have shown that $[f(Kx)]^2 = K^2\cdot f(x)^2 \Rightarrow \underline{f(Kx) = Kf(x)}$ —③, since $f: \mathbb{N}\to\mathbb{N}$.

Using ③, we can quickly verify that $f(K^mx) = K^mf(x)$ —④ for all positive integers m.

Thus, by ① and ③, we have $(f(x))^2 = f(x^2K) = K\cdot f(x^2)$. Let's prove that $[f(x)]^n = K^{n-1}\cdot f(x^n)$ for each $n \geq 1$. For $n=1$, the claim is trivial. We just proved it for $n=2$. Suppose the claim is true for $n = p-1$ $(p\geq 2)$. We'll prove it for $n = p+1$.

Then $[f(x)]^{p+1} = f(x)^2\cdot f(x)^{p-1} = f(x^2 f(f(x)^{p-1}))$ from the given func. equation

$= f(x^2 f(K^{p-2}f(x^{p-1})))$ by the induction hypothesis

$= f(x^2K^{p-2}\cdot f(f(x^{p-1})))$ by ④

$= f(x^2K^{p-2}\cdot x^{p-1}K^2)$ by ②

$= f(K^p x^{p+1}) = K^p f(x^{p+1})$ by ④.

Hence, if the claim holds for $n=p-1$, it holds for $n=p+1$. Since the result holds for $n=1$ and $n=2$, it holds for all positive integers n.

from here, the problem isn't that bad (well, certainly not as hard as the beginning). So on a contest, you can answer the problem for the case $K=1$, and get $f(1998) \geq 120$. It's not a complete proof because you haven't considered the case $K \geq 1$, but intuitively you'd guess that 120 is the right answer. And you'll get some part marks.

Therefore, $K^n f(x^{n+1}) = (f(x))^{n+1}$ for each integer n. Now let's show that $K \mid f(x)$. Let p be a prime divisor of K. Say p^a is the highest power of p dividing K and p^b is the highest power of p dividing $f(x)$. Then $p^{(n+1)b} \mid f(x)^{n+1} = K^n f(x^{n+1})$, so $(n+1)b \geq an \Rightarrow a \leq b(1+\frac{1}{n})$. Since this is true for all integers n, we must have $a \leq b$. This is true for all prime divisors of K, and so we conclude that $K \mid f(x)$ for each $x \in \mathbb{N}$.

Hence, we can let $g(x) = \frac{f(x)}{K}$, and then $g: \mathbb{N} \to \mathbb{N}$. Furthermore, $f(n^2 f(m)) = f(n^2 K g(m)) = K f(n^2 g(m)) = K^2 g(n^2 g(m))$, and $m(f(n))^2 = mK^2 (g(n))^2$. Since $f(n^2 f(m)) = m(f(n))^2$, we get $K^2 g(n^2 g(m)) = mK^2 (g(n))^2 \Rightarrow g(n^2 g(m)) = m \cdot (g(n))^2$. So if $K > 1$, then g is a function satisfying the given conditions, but $g(x) < f(x)$ for all $x \in \mathbb{N}$. In particular, $g(1998) < f(1998)$. So to achieve the minimum value for $f(1998)$, we want $K=1$. Thus, $\underline{f(1)=1}$. ← we did all this work to show that $f(1)=1$!

Thus, from ① and ②, we get $f(n^2) = (f(n))^2$ and $f(f(n)) = n$, for all $n \in \mathbb{N}$. So $f(xy)^2 = f(x^2y^2) = f(x^2 f(f(y^2))) = f(y^2) f(x)^2 = f(y)^2 f(x)^2 \Rightarrow \underline{f(xy) = f(x) f(y)}$ since $f(x), f(y), f(xy) > 0$. Hence, f is a multiplicative function.

Let p be a prime. Suppose $f(p) = mn$ for some $m, n \geq 2$. Then, $f(m) f(n) = f(mn) = f(f(p)) = p$, so either $f(m)=1$ or $f(n)=1$. WLOG, say $f(n)=1$. Then $n = f(f(n)) = f(1) = 1$, so $n=1$, contradiction. Thus, $f(p)$ must be prime. So $f(p) = q$ for some prime q, and $f(q) = f(f(p)) = p$.

Let $p_1, p_2, p_3, \ldots$ represent the primes and so $f(p_i) = q_i$ for some q_i (for each i). Because f is multiplicative, if $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, then we have $f(m) = f(p_1^{a_1} \cdots p_k^{a_k}) = f(p_1^{a_1}) \cdots f(p_k^{a_k}) = f(p_1)^{a_1} \cdots f(p_k)^{a_k} = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$. So the most general function that could satisfy the given conditions is a function $f: \mathbb{N} \to \mathbb{N}$ that is multiplicative, with $f(p) = q$ iff $f(q) = p$ (where p, q are prime). Let's check that such a function does indeed satisfy the given conditions: let $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and $n = q_1^{b_1} q_2^{b_2} \cdots q_k^{b_k}$ (some of the a_i's and b_i's may be 0 → we are just ordering the prime factors of m and n so they match up – see example on left). We have $f(n^2 f(m))$
$= f(q_1^{2b_1} q_2^{2b_2} \cdots q_k^{2b_k} \cdot f(p_1^{a_1} \cdots p_k^{a_k})) = f(q_1^{2b_1+a_1} \cdots q_k^{2b_k+a_k}) = p_1^{2b_1+a_1} \cdots p_k^{2b_k+a_k} = (p_1^{a_1} \cdots p_k^{a_k})[(p_1^{b_1} \cdots p_k^{b_k})]^2 = m \cdot (f(n))^2$, as required.

e.g. $m = 2^5 7^{17}$
$n = 3^{10}$
with $f(2)=11$
$f(7)=19$
$f(3)=5$.
Then we have
$m = 2^5 \cdot 7^{17} \cdot 5^0$
$n = 11^0 \cdot 19^0 \cdot 3^{10}$.

Now $f(1998) = f(2 \cdot 3^3 \cdot 37) = f(2) \cdot [f(3)]^3 \cdot f(37)$. To minimize $f(1998)$, the best we can do is $f(3) = 2$ (so $f(2)$ must be 3), and $f(37) = 5$ (so $f(5)$ must be 37). Thus, we conclude that the minimum value of $f(1998)$ is $3 \times 2^3 \times 5 = \boxed{120}$.

the next smallest prime that hasn't been taken