1 Problems

1.1 Algebra

A1. Prove that for any positive numbers a, b, c the following inequality holds:

$$\sqrt{ab(a+b)} + \sqrt{bc(b+c)} + \sqrt{ca(c+a)} > \sqrt{(a+b)(b+c)(c+a)}.$$

A2. Find all pairs of positive real numbers (x, y) that satisfy both

$$\begin{aligned} 2^{x^2+y} + 2^{y^2+x} &= 8, \qquad \text{and} \\ \sqrt{x} + \sqrt{y} &= 2. \end{aligned}$$

A3. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that

$$\sum_{i,j=1}^{n} |x_i + x_j| \geq n \sum_{i=1}^{n} |x_i|.$$

A4. Find all functions f mapping rational numbers to real numbers, and satisfying

$$f(a+b+c) + f(a) + f(b) + f(c) = f(a+b) + f(b+c) + f(c+a) + f(0)$$

for rational numbers a, b, c.

1.2 Combinatorics

C1. There are eight rooks on a chessboard, no two attacking each other. Prove that some two of the pairwise distances between the rooks are equal. (The distance between two rooks is the distance between the centers of their cells.)

C2. Every vertex of the unit squares on an $n \times m$ grid is coloured either blue, green, or red, such that all the vertices on the boundary of the board are coloured red. We say that a unit square on the board is *properly coloured* if exactly one pair of adjacent vertices of the square are the same colour. Show that the number of properly coloured squares is even.

C3. $3n$ points are marked on a circle, dividing it into $3n$ arcs, n of which have length 1, n others have length 2, and n have length 3. Prove that it is possible to find two marked points diametrically opposite to each other.

C4. In a round robin chess tournament each player faces every other player exactly once. For every game, 1 point is awarded for a win, 0.5 points for a draw, and 0 points for a loss. Given a positive integer m, a tournament is $m-$special if among every set S of m players, there is one player who won all her games against the other $m-1$ players in S and one player who lost all her games against the other $m-1$ players in S. For a given integer $m \geq 4$, determine the minimum value of n (as a function of m) such that in every $m-$special round robin chess tournament with n players, the final scores of the n players are all distinct.

1.3 Geometry

G1. Let $ABCD$ be a fixed parallelogram with $AB < BC$, and let P be a variable point on side CD. Let Q be the point on side BC so that $PC = CQ$. Prove that, as P moves, the circumcircle of $\triangle APQ$ passes through a fixed point in addition to A.

G2. Let $ABCD$ be a cyclic quadrilateral. The perpendiculars to AD and BC at A and C respectively meet at M, and the perpendiculars to AD and BC at D and B meet at N. If the lines AD and BC meet at E, prove that $\angle DEN = \angle CEM$.

G3. Let ABC be an acute-angled triangle with altitudes AD, BE, CF. Let the line tangent to the circumcircle of $\triangle ABC$ at A and the line tangent to the circumcircle of $\triangle DEF$ at D intersect at a point X. Define Y and Z analogously. Prove that X, Y, and Z are collinear.

G4. In an acute triangle ABC, the angle bisector of $\angle A$ meets side BC at D. Let E and F be the feet of the perpendiculars from D to AC and AB respectively. Lines BE and CF intersect at H, and the circumcircle of $\triangle AFH$ meets BE at H and G. Show that the triangle with side lengths BG, GE, BF is right-angled.

1.4 Number Theory

N1. Can 1 be written in the form:

$$\frac{1}{3a_1 - 1} + \frac{1}{3a_2 - 1} + \ldots + \frac{1}{3a_{2011} - 1},$$

where each a_i is a positive integer?

N2. A sequence (a_i) of natural numbers has the property that for all $i \neq j$, $\gcd(a_i, a_j) = \gcd(i, j)$. Show that $a_i = i$ for all i.

N3. Let's say a natural number is *Yorky* if it can be expressed in the form $\frac{x^2-1}{y^2-1}$ for integers $x, y > 1$. Is is true that all but finitely many natural numbers are Yorky?

N4. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers such that $a_1 > 1$ and iteratively, for all $n \geq 2$, a_n is the smallest positive integer such that

$$a_n \notin \{a_1, a_2, \ldots, a_{n-1}\} \quad \text{and} \quad \gcd(a_{n-1}, a_n) > 1.$$

Prove that the sequence contains every positive integer greater than one.

2 Solutions

2.1 Algebra

A1.

$$\begin{aligned}\left(\sqrt{ab(a+b)}+\sqrt{bc(b+c)}+\sqrt{ca(c+a)}\right)^2 &= \sum_{cyc} ab(a+b) + 2\sum_{cyc}\sqrt{ab^2c(a+b)(b+c)} \\ &> \sum_{cyc} ab(a+b) + 6abc \\ &> \sum_{cyc} ab(a+b) + 2abc \\ &= (a+b)(b+c)(c+a).\end{aligned}$$

Comment: Most Olympiad inequalities are tighter approximations, and steps like replacing $6abc$ with $2abc$ will not be good enough. Expanding can help though, and the cyclic sum notation used here is a good way to write things down when you do expand.

Source: Russia 1990.

A2. Suppose $\sqrt{x}+\sqrt{y}=2$. By the power-mean inequality, we have $x+y \ge \frac{1}{2}\cdot\left(\sqrt{x}+\sqrt{y}\right)^2 \ge 2$, and $x^2+y^2 \ge \frac{1}{2}\cdot(x+y)^2 = 2$. Now, by the AM-GM inequality:

$$\begin{aligned}2^{x^2+y}+2^{y^2+x} &\ge 2\cdot\sqrt{2^{x^2+y}\cdot 2^{y^2+x}} \\ &= 2^{1+\frac{x^2+y^2}{2}+\frac{x+y}{2}} \\ &\ge 8.\end{aligned}$$

Equality holds if and only if $x=y$, so the only possible solution to the given pair of equations is $x=y=1$. Substituting this pair in, we see that it is indeed a valid solution.

Comment: Solving a system of equations like this requires doing two things: (a) proving that nothing other than $x=y=1$ works, and (b) proving that $x=y=1$ does work. It is always a good idea to check (b) separately. Your argument for (a) **might** take care of (b), but it might not, and it's better to be safe than sorry.

Source: Olymon 1997-1998.

A3. Let $a=\max_{i\in\{1,2,\ldots,n\}}|x_i|$, let S_0 denote the set of indices i such that $x_i=a$, let S_1 denote the set of indices i such that $x_i=-a$, and let S_2 denote the remaining indices. Finally, let $b=\max_{i\in S_2}|x_i|$. (If S_2 is empty, then define $b=0$.)

Consider replacing each x_i in S_0 with b and each x_i in S_1 with $-b$. This will have the following effect on each $|x_i+x_j|$ term:

- If i,j are both in S_0 or both in S_1, then $|x_i+x_j|$ will decrease from $2a$ to $2b$. There are exactly $|S_0|^2+|S_1|^2$ ordered pairs (i,j) for which this case applies.
- If one of i,j is in S_0 and the other is in S_1, then $|x_i+x_j|$ will be 0 both before and after the replacement, and therefore will be unchanged.

- If one of i, j is in $S_0 \cup S_1$ and the other is in S_2, then $|x_i + x_j|$ will decrease by exactly $a - b$. (Assume without loss of generality that $x_i = a$, and let $c = x_j$; then $|x_i + x_j|$ will decrease from $a + c$ to $b + c$. There will be no sign change since $|c| \le b \le a$.) There are exactly $2 \cdot (|S_0| + |S_1|) \cdot |S_2|$ ordered pairs (i, j) for which this case applies.
- If i, j are both in S_2, then $|x_i + x_j|$ will not change.

Therefore, $\sum_{i,j=1}^n |x_i + x_j|$ will decrease by exactly

$$\begin{array}{rl} & (a-b) \cdot \left(2 \cdot (|S_0| + |S_1|) \cdot |S_2| + 2 \cdot |S_0|^2 + 2 \cdot |S_1|^2\right) \\ \ge & (a-b) \cdot \left((|S_0| + |S_1|) \cdot |S_2| + (|S_0| + |S_1|)^2\right) \\ = & (a-b) \cdot (|S_0| + |S_1|) \cdot n. \end{array}$$

On the other hand, $n \cdot \sum_{i=1}^n |x_i|$ will decrease by exactly $(a - b) \cdot (|S_0| + |S_1|) \cdot n$. Therefore, this replacement will always decrease the left-hand side by at least as much as it will decrease the right-hand side by.

Performing this replacement repeatedly, we will eventually reach the configuration where every x_i is equal to 0, at which point it is obvious that the left-hand side is equal to the right-hand side. Therefore, we must originally have had the left-hand side being greater than or equal to the right-hand side, as required.

Solution #2: If $\{x_1, x_2, \ldots, x_n\}$ contains only non-negative numbers or only non-positive numbers, the claim is obvious.

Otherwise, assume without loss of generality that $x_1, x_2, \ldots, x_m \ge 0$ and $x_{m+1}, x_{m+2}, \ldots, x_n < 0$. Let

$$a = \frac{x_1 + \ldots + x_m}{m}, \quad b = -\frac{x_{m+1} + \ldots + x_n}{n - m}.$$

Clearly, $a, b \ge 0$. Returning to the original inequality, we can now write the left-hand side minus the right-hand side in the following form:

$$\sum_{i,j=1}^m |x_i + x_j| + \sum_{i,j=m+1}^n |x_i + x_j| + 2 \sum_{i=1}^m \sum_{j=m+1}^n |x_i + x_j| - n \sum_{i=1}^m |x_i| - n \sum_{j=m+1}^n |x_i|.$$

This simplifies further to

$$\text{LHS-RHS} = 2m^2 a + 2(n-m)^2 b + 2 \sum_{i=1}^m \sum_{j=m+1}^n |x_i + x_j| - nma - n(n-m)b.$$

Now assume without loss of generality that $a \ge b$. Since $|x + c|$ is a convex function, Jensen's inequality implies LHS-RHS is greater than or equal to

$$\begin{array}{rl} & 2m^2 a + 2(n-m)^2 b + 2m \sum_{j=m+1}^n |a + x_j| - nma - n(n-m)b \\ \ge & 2m^2 a + 2(n-m)^2 b + 2m(n-m)(a-b) - nma - n(n-m)b \\ = & a\left(2m^2 + 2m(n-m) - nm\right) + b\left(2(n-m)^2 - 2m(n-m) - n(n-m)\right) \\ = & amn + b(n^2 - 5mn + 4m^2) \\ \ge & bmn + b(n^2 - 5mn + 4m^2) \\ = & b(n^2 - 4mn + 4n^2) \\ = & b(n - 2m)^2 \ge 0. \end{array}$$

Comments: The technique of changing one or more variables, and seeing what happens to the inequality is called "smoothing" or "mixing variables". Usually, you make two variables equal, or reduce one of them to zero. On some problems, smoothing is too messy to be useful, but when the algebra is not too complicated, it is a very powerful technique.

Source: Iran 2006.

A4. Let us say a function f is "good" if it satisfies the given equation. Note that:

- If f_1 and f_2 are good, then so is $f_1 + f_2$.
- For all rational numbers A, B, C, we have $f(x) = Ax^2$ is good, $f(x) = Bx$ is good, and $f(x) = C$ is good. Therefore, all quadratical polynomials are good.

We claim this is the complete set of good functions.

Fact[1]**:** Fix real numbers a, b, c, t with $t \neq 0$. Then there exists a quadratic polynomial $P(x)$ such that $P(-t) = a, P(0) = b$, and $P(t) = c$.

Lemma: Fix a positive integer n. If f is good, then there exists a quadratic polynomial $P(x)$ such that $f\left(\frac{m}{n}\right) = P\left(\frac{m}{n}\right)$ for all integers m.

Proof of lemma: Let $P(x)$ be the quadratic polynomial that equals $f(x)$ for $x = \frac{-1}{n}, 0, \frac{1}{n}$, and let $g(x) = f(x) - P(x)$. Then g is a good function satisfying $g\left(\frac{-1}{n}\right) = g(0) = g\left(\frac{1}{n}\right) = 0$. We need to show $g\left(\frac{m}{n}\right) = 0$ for all m.

Taking $a = b = \frac{1}{n}$ and $c = \frac{-1}{n}$, we have $3g\left(\frac{1}{n}\right) + g\left(\frac{-1}{n}\right) = g\left(\frac{2}{n}\right) + 3g(0)$, and hence $g\left(\frac{2}{n}\right) = 0$. Now, for any m, take $a = b = \frac{1}{n}$ and $c = \frac{m-2}{n}$. Then

$$g\left(\frac{m}{n}\right) + 2g\left(\frac{1}{n}\right) + g\left(\frac{m-2}{n}\right) = 2g\left(\frac{m-1}{n}\right) + g\left(\frac{2}{n}\right) + g(0),$$

and hence $g\left(\frac{m}{n}\right) = 2g\left(\frac{m-1}{n}\right) - g\left(\frac{m-2}{n}\right)$. Therefore, $g\left(\frac{m}{n}\right)$ is 0 if both $g\left(\frac{m-2}{n}\right)$ and $g\left(\frac{m-1}{n}\right)$ are 0. It follows immediately that $g\left(\frac{m}{n}\right)$ is 0 for all positive integers m. Similarly, $g\left(\frac{m-2}{n}\right)$ is 0 if both $g\left(\frac{m-1}{n}\right)$ and $g\left(\frac{m}{n}\right)$ are 0, and hence $g\left(\frac{m}{n}\right)$ is also 0 for all negative integers m. $\square$

Now pick an arbitrary good function f. By the lemma, there exists a quadratic polynomial P such that $f(m) = P(m)$ for all integers m. Suppose there exists a rational number $\frac{t}{n}$ so that $f\left(\frac{t}{n}\right) \neq P\left(\frac{t}{n}\right)$. Applying the lemma again, there exists a quadratic polynomial P_2 such that $f\left(\frac{m}{n}\right) = P_2\left(\frac{m}{n}\right)$ for all m. But then $P - P_2$ is a quadratic polynomial that has roots at -1, 0, and 1, but is non-zero at $\frac{t}{n}$. However, this is impossible: a non-zero quadratic polynomial can have at most 2 roots. Therefore it must be that $f(x) \equiv P(x)$, and the problem is solved.

Comments: This problem generalizes Cauchy's equation: $f(a) + f(b) = f(a + b)$. Like Cauchy's equation, there can be all sorts of weird solutions if you let the domain of f be the set of all real numbers. It depends on whether you assume the Axiom of Choice, and is far beyond the scope of the IMO!

[1] You can quote the following more general theorem: Given any $k+1$ pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_{k+1}, y_{k+1})$ with $x_i \neq x_j$, there exists a unique degree-k polynomial P satisfying $P(x_i) = y_i$ for all i. In our case, the polynomial is $P(x) = \frac{a+c-2b}{2t^2} \cdot x^2 + \frac{c-a}{2t} \cdot x + b$.

2.2 Combinatorics

C1. Suppose that 8 rooks are placed on a chessboard with no two attacking each other. Then there must be one rook in each row and one in each column. Consider a pair of rooks that are either in adjacent rows or in adjacent columns. The distance between them is one of $\sqrt{1^2+1^2}, \sqrt{1^2+2^2}, \sqrt{1^2+3^2}, \ldots, \sqrt{1^2+7^2}$. Note that there are exactly 7 distinct distances of this form.

If all rook pairs are a different distance apart, it follows that there are at most 7 rook pairs in adjacent rows or columns. However, there are exactly 7 pairs in adjacent rows (the rook in row 1 paired with the rook in row 2, the rook in row 2 paired with the rook in row 3, etc.), and exactly 7 pairs in adjacent columns. If these combine to give only 7 pairs altogether, it must be that any time two rooks are in adjacent rows, they are also in adjacent columns. But in this case, all 7 of these rook pairs are separated by a distance of exactly $\sqrt{2}$, and we have a contradiction.

Solution #2: If there are 8 rooks on a board, this gives $\binom{8}{2} = 28$ pairs of rooks. The distance between any pair must be of the form $\sqrt{i^2+j^2}$ for $i, j \in \{1, 2, \ldots, 7\}$. There are exactly $\binom{7}{2}+7 = 28$ such distances. Therefore, if every pair of rooks is separated by a different distance, each of these 28 distances must be used.

In particular, there are two rooks separated by a distance of $\sqrt{7^2+7^2}$, meaning there are two rooks in opposite corners. But then no more rooks can be placed on the boundary without attacking one of these two. Hence, no two rooks can be separated by a distance of $\sqrt{7^2+6^2}$, and we have a contradiction.

Comments: With Pigeonhole Principle arguments like this, it is always a good idea to count the number of possible distances and the number of rook pairs. In this case, the total number of rook pairs and the total number of possible distances are both $\binom{8}{2}$, which is not enough for a contradiction. You need to tweak the argument a little bit, like we did here.

Source: Russia 2002.

C2. We will call an edge of a square good if both its endpoints are of the same colour, and bad otherwise. Any square can have either no good edges (call such squares type A), one good edge (type B), two good edges (type C), or four good edges (type D). Note that it cannot have three good edges.

Let the number of squares of type A, B, C, D be a, b, c, d respectively. For each square, count the number of good edges, and add the numbers. We must get an even number, since each edge not on the boundary is counted twice, and all $2(m+n)$ edges on the boundary are good. On the other hand, this number is $b + 2c + 4d$. Hence b is even and the result follows.

Comments: We are counting in two different ways the number of (square, good edge) pairs. This kind of argument is very important in combinatorics problems!

Source: India 2001.

C3. Assume it is not possible to find the two required points.

Starting at any marked point, divide the circumference into $6n$ arcs of length 1. Colour the endpoints of these arcs blue if they are marked, and red otherwise. By assumption, each blue point is diametrically opposite a red point. Since the total number of red points and the total number of blue points are both equal to $3n$, all red points must be used in this way, and therefore each red point is also diametrically opposite a blue point.

Let us call any arc between two adjacent blue points "elementary". Consider an elementary arc AC of length 2. Its endpoints are blue, and its midpoint B' is red. As argued above, the point B diametrically opposite to B' must be blue.

Now suppose arc AB contains a_1 elementary arcs of length 1, a_2 elementary arcs of length 2, and a_3 elementary arcs of length 3. Each length-1 elementary arc in AB is diametrically opposite a length-3 elementary arc in BC: the two blue endpoints of the length-1 arc must be opposite red points and no elementary arc can have length greater than 3. Conversely, each length-3 elementary arc in BC is diametrically opposite a length-1 elementary arc in AB. Therefore, the total number of length-3 elementary arcs in BC is exactly a_1. Since the total number of length-3 elementary arcs on the whole circle is k, we have $a_1 = k - a_3$.

Finally, arc AB is of length $3k - 1$, so we have $3k - 1 = a_1 + 2a_2 + 3a_3 = 2a_2 + 2a_2 + k$. Therefore, $2k - 1 = 2a_1 + 2a_2$, but this is impossible because the left-hand side is odd, and the right-hand side is even.

Source: Russia 1982.

C4. The answer is $2m - 3$.

For $n \leq 2m - 4$, consider a tournament with players $P_1, P_2, \ldots, P_n$ where P_i beats P_j for all $i > j$ except P_{m-1} draws against P_{m-2}. Consider any set of m players, and let i, j denote the highest and lowest index respectively of any player in this set. Clearly, $i \geq m$ and since $n \leq 2m - 4$, we also have $j \leq m - 3$. Since i, j are not $m - 1$ and $m - 2$, we know that P_i beat all other players in the set, and P_j lost to all other players in the set. Therefore, the tournament is m-special. Since players P_{m-2} and P_{m-1} have identical scores, the answer to the problem must be at least $2m - 3$.

Conversely, consider an arbitrary m-special tournament. First suppose two players P_1 and P_2 drew against each other. Let A be the set of players who beat or tied P_1, and let B be the set of players who lost to P_1. If $|A| \geq m - 2$, then consider a set of $m - 2$ players from A, along with P_1 and P_2. Each player in A beat or drew against P_1, P_1 drew against P_2, and P_2 drew against P_1. Therefore, no player lost to all other players in the set, and so the tournament cannot be m-special, which is a contradiction. Therefore, $|A| \leq m-3$. Similarly, $|B| \leq m - 3$. Since A and B together comprise all players other than P_1 and P_2, we must have $n \leq 2 + 2(m - 3) = 2m - 4$.

Next suppose there exists a "3-cycle": three players P_1, P_2, P_3 such that P_1 beat P_2, P_2 beat P_3, and P_3 beat P_1. Let A be the set of players who beat or tied P_1, and let B be the set of players who lost to P_1. Looking at size-m sets containing all of P_1, P_2, P_3 as above, we find $|A|, |B| \leq m - 4$, and hence $n \leq 2m - 5$.

Finally, consider an arbitrary tournament with no draws and with no 3-cycles. Let P be the player with the largest number of wins. Suppose some player Q is not beaten by P. Since there are no ties, it must be that Q beat P. Since there are no 3-cycles, it must also be that Q beat every player beaten by P, and hence Q has more wins than P. This contradicts the definition of P, so the only possibility is that P beat all other players. Label P as P_n and now restrict our attention to the remaining $n - 1$ players who together form a smaller tournament with no draws and with no 3-cycles. By the same argument, this tournament also has a player P_{n-1} who beats all remaining players. Proceeding recursively, we can label the players $P_1, P_2, \ldots, P_n$ so that player P_i beats P_j whenever $i > j$. Notice that no two players can have the same score for this kind of tournament.

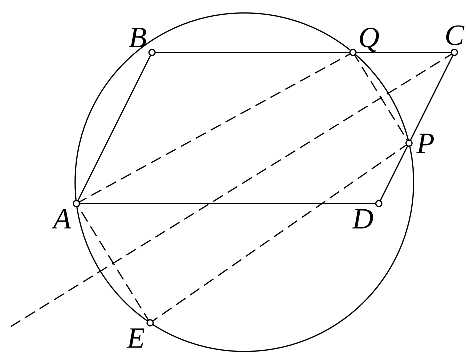
Therefore, if two players have the same score, then the tournament must have a tie or a 3-cycle. In the case of m-special tournaments, we showed that this means $n \leq 2m - 4$, and the proof is complete.

Comments: In the final case, we proved one player had beaten all other players by starting with the player who had the most wins and analyzing what properties she must have. This is called an extremal argument, and it is very useful.

Source: China 2007.

2.3 Geometry

G1. Let E be the reflection of A through the angle bisector of $\angle BCD$. Since P is the reflection of Q through the same line, it follows that PQ and AE are both perpendicular to the bisector, and that $AQ = EP$. In particular, $AQPE$ is an isosceles trapezoid, and is therefore cyclic. It follows that the circumcircle of $\triangle APQ$ passes through E no matter where P is.



Comments: For many problems where you need a show a shape always passes through a fixed point, the key is to first figure out where that point is, and then go from there. Always have a straight-edge and compass ready to help with this kind of guessing!

Source: Russia 2005.

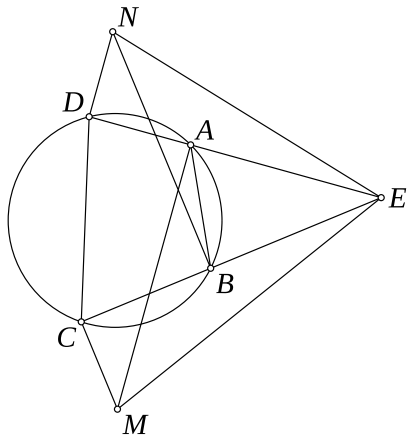
G2. Assume without loss of generality that E is closer to AB than to CD. Since $\angle EBN = \angle EDN = 90^\circ$, the points E, B, D, N lie on a circle. (We do not know what order they lie in however.) Similarly, the points E, A, C, M lie on a circle.

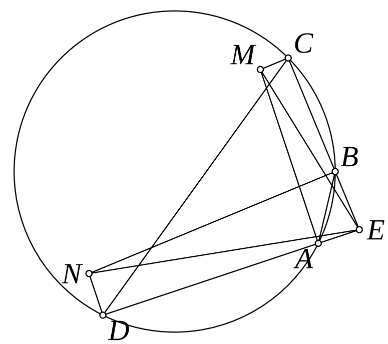
Now, suppose segments BN and DE cross. Then $\angle EBD > \angle EBN = 90^\circ$, and $\angle EAC = 180^\circ - \angle CEA - \angle ECA = 180^\circ - \angle DEB - \angle EDB = \angle EBD > 90^\circ$. It follows that segments AM and CE also cross, as shown in the left diagram.

In this configuration, we have $\angle DEN = 180^\circ - \angle BED - \angle BDN = 90^\circ - \angle BED - \angle BDA = 90^\circ - \angle AEC - \angle ACB = 180^\circ - \angle AEC - \angle ACM = \angle CEM$.

If segments AM and CE cross, then segments BN and DE also cross as argued above, and we are in the same case as before. Otherwise, it must be that AM does not cross CE, and BN does not cross DE, as shown in the right diagram.

In this configuration, we have $\angle DEN = \angle DBN = \angle DBC - 90^\circ = \angle CAD - 90^\circ = \angle CAM = \angle CEM$. Thus, $\angle DEN = \angle CEM$ in all possible configurations.





Solution #2: We know $\angle ACE = \angle ACB = \angle BDA = \angle BDE$ and $\angle CEA = \angle DEB$, so $\triangle EAC$ must be similar to $\triangle EBD$.

Also, E, A, C, M lie on a circle with diameter EM since $\angle EAM = \angle ECM = 90^\circ$. In particular, this means AM is the diameter of the circumcircle of $\triangle EAC$. Similarly, BN is the diameter of the circumcircle of $\triangle EBD$.

Now consider the similarity transformation (rotation, scaling, translation, and/or reflection) taking $\triangle EAC$ to $\triangle EBD$. This transformation will also take the point on the circumcircle of $\triangle EAC$ opposite A to the point on the circumcircle of $\triangle EBD$ opposite B; i.e., it will take M to N. Thus, the entire figure $EACM$ is similar to $EBDN$, and hence $\angle DEN = \angle CEM$.

Comments: Many students will just assume one configuration and ignore the possibility of the other one, even when the argument needs to be tweaked. This will usually cost you a point. In this problem, it would cost you multiple points. Always, always, always pay attention to all possible configurations!

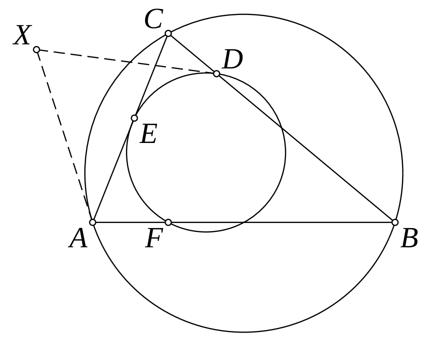
Source: Iran 2004.

G3. Let $\triangle ABC$ have angles a, b, c, and assume without loss of generality that X lies on the same side of AD as C. Then $\angle XAD = \angle XAC + \angle CAD = b + (90^\circ - c)$.

Now, $\angle AFC = \angle ADC = 90^\circ$ so $AFDC$ is a cyclic quadrilateral. Similar arguments apply for $BDEA$ and $CEFB$. Therefore,

$$\begin{aligned}\angle XDA &= \angle XDE + \angle EDA = \angle DFE + \angle EDA = \angle DFC + \angle CFE + \angle EDA \\ &= \angle DAC + \angle CBE + \angle EBA = (90^\circ - c) + (90^\circ - c) + (90^\circ - a) \\ &= 90^\circ + b - c = \angle XAD.\end{aligned}$$

Therefore, $\triangle XAD$ is isosceles and $XA = XD$. Since XA and XD are both tangents, it follows that X has equal power with respect to the circumcircles of $\triangle ABC$ and $\triangle DEF$. In particular, X is on the radical axis of these circles. The same argument can be applied to Y and Z, and so all three points lie on the one line as required.



Comments: As with G1, the trick here is to guess the line in advance. Once you know the goal is to prove X, Y, Z are on the radical axis of the two circles, it is straightforward. If you don't know what power of a point and radical axes are, you should look them up. They are important!

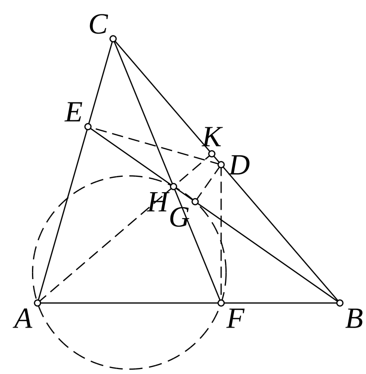
G4. First note that $\triangle AFD \cong \triangle AED$ by angle-angle-side. Now extend AH to meet BC at K. Since $AF = AE$, Ceva's theorem guarantees that:

$$\begin{aligned} \frac{BK}{CK} &= \frac{BF}{CE} \\ &= \frac{BD \cdot \cos B}{CD \cdot \cos C} \\ &= \frac{BA \cdot \cos B}{CA \cdot \cos C} \qquad \text{by the angle bisector theorem.} \end{aligned}$$

Now let K' be the base of the altitude from A to BC. Then $\frac{BK'}{CK'} = \frac{BA \cdot \cos B}{CA \cdot \cos C} = \frac{BK}{CK}$, so $K = K'$, and AK is perpendicular to BC.

Since $\angle AFD = \angle AKD = 90^\circ$, it follows that $AFDK$ is a cyclic quadrilateral. By power of a point on circles $AFDK$ and $AFGH$, we have $BK \cdot BD = BF \cdot BA = BH \cdot BG$. Therefore, $KDGH$ is also a cyclic quadrilateral. Since $\angle DKH = 90^\circ$, it has diameter DH and hence $\angle DGH = 90^\circ$ as well. In other words, DG is perpendicular to BE.

Therefore, $BG^2 - GE^2 = BD^2 - DE^2 = BD^2 - DF^2 = BF^2$ as required.



Comments: The key observations are that AK is perpendicular to BC and DG is perpendicular to BE, both of which you would probably see if you made a good straight-edge and compass diagram!

Source: China 2003.

2.4 Number Theory

N1. The answer is no. Let $X = \prod_{i=1}^{2011}(3a_i - 1)$, and let $x_i = \prod_{j \neq i}(3a_j - 1)$. Note that

$$\sum_{i=1}^{2011} \frac{1}{3a_i - 1} = \frac{1}{X} \cdot \sum_{i=1}^{2011} x_i.$$

Working modulo 3 however, we have $X \equiv 2^{2011} \equiv 2$, and $\sum_{i=1}^{2011} x_i \equiv \sum_{i=1}^{2011} 2^{2010} \equiv 2011 \cdot 1 \equiv 1$. Therefore, $X \neq \sum_{i=1}^{2011} x_i$, and hence the original sum cannot be equal to 1.

Comments: If you look at the equation mod 3, the argument is intuitively clear, but a nuisance to write up clearly. Think about how you would do it.

Source: Ukraine 2005.

N2. For any integer n, we have $n = \gcd(n, 2n) = \gcd(a_n, a_{2n})$, which divides a_n. Therefore, n divides a_n for all n.

Now suppose there exists n for which $a_n \neq n$. Then $a_n = nm$ for some $m > 1$. Since nm also divides a_{nm}, we have $nm | \gcd(a_n, a_{nm})$. However, nm does not divide $n = \gcd(n, nm)$, so this is a contradiction.

It follows that $a_n = n$ for all n.

Source: Russia 1995.

N3. Let p be an odd prime. We claim that p^2 is not Yorky, which implies there are infinitely many non-Yorky numbers.

Indeed, suppose $\frac{x^2-1}{y^2-1} = p^2$. Then:

$$\begin{aligned} & x^2 - 1 = y^2p^2 - p^2 \\ \Rightarrow \ & p^2 - 1 = y^2p^2 - x^2 \\ \Rightarrow \ & p^2 - 1 = (yp - x)(yp + x). \end{aligned}$$

If $yp - x = 1$, then $yp + x = p^2 - 1$ and $2yp = p^2$, contradicting the fact that p is odd.

Otherwise, $yp - x > 1$. Taking the equation modulo p, we have $x^2 \equiv 1 \mod p$, and hence $yp - x \equiv -x \equiv \pm 1 \mod p$. Since $yp - x$ is positive and greater than 1, it follows that $yp - x \geq p - 1$. Also, x must be greater than 1, so $yp + x > yp - x + 2 \geq p + 1$. But then $(yp - x)(yp + x) > (p - 1)(p + 1)$, and we have a contradiction.

Comments: Unless you are 100% completely sure you know whether there are infinitely many non-Yorky numbers, do NOT put all your eggs in one basket. Spend time both trying to prove there are infinitely many, and trying to prove there are not. When trying to construct non-Yorky numbers, it is natural to look at perfect squares because they let you factor the expression. The advantage of trying to construct examples is you can restrict your attention to whatever is easy to work with!

Source: Russia 2010.

N4. **Lemma 1:** Let $n > 1$ be a positive integer. If the sequence contains infinitely many multiples of n, then it contains all multiples of n.

Proof of lemma 1: Let kn be an arbitrary multiple of n. Consider the first $kn + 1$ multiples of n in the sequence. Each of these terms shares a non-trivial divisor with n, and each is followed by a distinct number. Eventually these numbers will become larger than kn, at which point kn has to be used. Therefore, kn will appear in the sequence at some point. □

Lemma 2: Let $n > 1$ be a positive integer. If the sequence contains infinitely many multiples of n, then it contains all positive integers greater than 1.

Proof of lemma 2: Let $m > 1$ be an arbitrary positive integer. By Lemma 1, the sequence contains all multiples of n, including all multiples of nm. Therefore, the sequence contains infinitely many multiples of m, which means it contains all multiples of m, including m itself. Since m was arbitrary, the proof is now complete. □

We now show that the sequence must contain infinitely many prime numbers. Suppose by way of contradiction that it contains only the primes $p_1, p_2, \ldots, p_k$. By Lemma 2, it can contain only finitely many multiples of each of these primes. Therefore, it must eventually reach a number m not divisible by any of them. Let q be the smallest prime factor of m. Then q is the smallest number sharing a non-trivial divisor with m, so m will be immediately followed by q if q has not already appeared in the sequence. Either way, q appears at some point, and we have a contradiction.

Therefore, the sequence does indeed contain infinitely many prime numbers. For any prime p, the smallest number sharing a non-trivial divisor with p is $2p$. If $2p$ has not already appeared in the sequence, it will therefore immediately follow p. Thus, having infinitely many primes in the sequence guarantees that there will be infinitely many even numbers, and we are now done by Lemma 2.

Comments: The devil here is in the details. Take a look at how the proof is made rigorous, and make sure you could do the same thing!

Source: China 2010.