BdMO Online Camp Exam Solutions (excluding 10,12)

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November 2 ,2011

Questions

1. Let a, b, c be positive real numbers. Prove that,

$$abc(a+b+c) \leq a^3b + b^3c + c^3a$$

2. Let a, b be real numbers. Prove that,

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

3. Let a, b, c, d be positive numbers. Prove that,

$$1 < \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} < 2$$

4. Prove that,

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$$

for all positive integers n.

5. Let a, b, c be positive real numbers. Prove that,

$$(a+b)(a+c) \geq 2\sqrt{abc(a+b+c)}$$

6. Let a, b, c be positive numbers with $abc = 1$. Prove that,

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c$$

7. Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ be positive numbers. Prove that at least one of the following must be true,

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n$$

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \geq n$$

8. Let a, b, c be real numbers. Prove that,

$$\frac{2}{b(a+b)} + \frac{2}{c(b+c)} + \frac{2}{a(c+a)} \geq \frac{27}{(a+b+c)^2}$$

9. Let $0 \leq x, y \leq 1$. Prove that,

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \leq \frac{2}{\sqrt{1+xy}}$$

[Hint : any positive real number can be written as e^u for some real number u. You can use this fact without proof. If you can solve the problem without using this hint, then you will get bonus point]

10. Let f be a function from the set of real numbers to itself such that for all real numbers x, y,

$$\frac{f(x)+f(y)}{2} - f(\frac{x+y}{2}) \geq |x-y|$$

Prove that,

$$\frac{f(x)+f(y)}{2} - f(\frac{x+y}{2}) \geq 2^n|x-y|$$

for all real numbers x, y and all non-negative integers n. Also, prove that, no such function can exist.

11. Let a, b, c, x, y, z be positive numbers such that $a \geq b \geq c$ and $x \geq y \geq z$. Prove that,

$$\frac{a^2x^2}{(by+cz)(bz+cy)} + \frac{b^2y^2}{(cz+ax)(cx+az)} + \frac{c^2z^2}{(ax+by)(ay+bx)} \geq \frac{3}{4}$$

12. Let a, b, c, d be positive numbers such that $a \leq 1$, $a+b \leq 5$, $a+b+c \leq 14$, $a+b+c+d \leq 30$. Prove that,

$$\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d} \leq 10$$

Solutions

1

By Muirhead's theorem, $[2,1,1] \leq [3,1,0]$ or, $\frac{1}{4!}(abc(a+b+c)) \leq \frac{1}{4!}(a^3b + b^3c + c^3a)$ So, $abc(a+b+c) \leq a^3b + b^3c + c^3a$[proved]

2

First we set $x = \frac{1}{a}$ and $y = \frac{1}{b}$.Then as $|mn| = |m||n|$ for all real m, n,the inequality becomes

$$\frac{|x+y|}{|xy|+|x+y|} \leq \frac{1}{1+|x|} + \frac{1}{1+|y|}$$

Or,

$$\frac{|x+y|}{|xy|+|x+y|} \leq \frac{2+|x|+|y|}{(1+|x|)(1+|y|)} = \frac{2+|x|+|y|}{1+|x|+|y|+|xy|}$$

Or,

$$(2+|x|+|y|)(|x+y|+|xy|) \geq (|x+y|)(|1+|x|+|y|+|xy|)$$

Or,

$$2|xy| + 2|x+y| + |x^2y| + |xy^2| + |x^2+xy| + |y^2+xy| \geq |x+y| +$$

$$|x^2+xy| + |y^2+xy| + |x^2y+xy^2|$$

Which is true, as

$$2|xy| + |x+y| + |x^2y| + |xy^2| + \geq |x^2y + xy^2|$$

is true because of triangle inequality and $2|xy| + |x+y|$ is always positive.

3

$$\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} > \frac{a}{a+b+c+d} + \frac{b}{a+b+c+d} + \frac{c}{a+b+c+d} + \frac{d}{a+b+c+d} = 1$$

Again,

$$\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} < \frac{a}{a+b} + \frac{b}{a+b} + \frac{c}{c+d} + \frac{d}{c+d} = 2$$

So, proved.

4

We will use induction to prove the fact. For $n = 1$, LHS=$\frac{1}{2}$=RHS Now, let the statement be true for n Then,

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$$

Now if we can prove $\frac{2n+1}{2n+2} \leq \sqrt{\frac{3n+1}{3n+4}}$, then we are done. Now,$\frac{2n+1}{2n+2} \leq \sqrt{\frac{3n+1}{3n+4}}$

$$\Leftrightarrow (\frac{2n+1}{2n+2})^2 \leq \frac{3n+1}{3n+4}$$

$$\Leftrightarrow \frac{4n^2+4n+1}{4n^2+8n+4} \leq \frac{3n+1}{3n+4}$$

$$\Leftrightarrow (4n^2+4n+1)(3n+4) \leq (4n^2+8n+4)(3n+1)$$

$$\Leftrightarrow 12n^3 + 28n^2 + 19n + 4 \leq 12n^3 + 28n^2 + 20n + 4$$

$\Leftrightarrow n \geq 0$ Which is true for all positive n. So, the statement is true for $n+1$ and thus for all $n \in N$

5

Let us set $a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}$. Then the inequality becomes,

$$(\frac{x+y}{xy})(\frac{z+x}{zx}) \geq \frac{2\sqrt{xy+yz+zx}}{xyz}$$

$$\Leftrightarrow \frac{(x+y)(x+z)}{x} \geq 2\sqrt{xy+yz+zx}$$
$$\Leftrightarrow (x+y)^2(x+z)^2 \geq 4x^2(xy+yz+zx)$$
$$\Leftrightarrow (x+y)^2(x+z)^2 \geq 4x^2\{(x+y)(x+z)-x^2\}$$
$$\Leftrightarrow (x+y)^2(x+z)^2 \geq 4(x+y)(z+x)x^2 - 4x^4$$
$$\Leftrightarrow (x+y)^2(x+z)^2 - 4(x+y)(z+x)x^2 + 4x^4 \geq 0$$
$$\Leftrightarrow \{(x+y)(x+z)-2x^2\}^2 \geq 0$$

Which is true.

6

Let's set $a = x^3, b = y^3, c = z^3$ Then the inequality becomes

$$\frac{x^3}{y^3} + \frac{y^3}{z^3} + \frac{z^3}{x^3} \geq z^3 + y^3 + x^3$$

with $xyz = 1$ Multiplying LHS with $x^3y^3z^3$ and RHS with $x^2y^2z^2$,

$$x^6z^3 + y^6x^3 + z^6y^3 \geq z^5x^2y^2 + y^5z^2x^2 + x^5y^2z^2$$

Which is true as Muirhead's inequality states $[6,3,0] \geq [5,2,2]$

7

Define $c_i = \frac{a_i}{b_i} \forall 1 \leq i \leq n$ Then, by AM-GM inequality, we get,

$$\sum_{i=1}^{n} c_i \geq n \sqrt[n]{\prod_{i=1}^{n} c_i} ...(i)$$

And

$$\sum_{i=1}^{n} \frac{1}{c_i} \geq n \sqrt[n]{\prod_{i=1}^{n} \frac{1}{c_i}} ...(ii)$$

Multiplying i and ii,

$$\sum_{i=1}^{n} c_i \cdot \sum_{i=1}^{n} \frac{1}{c_i} \geq n^2$$

Thus both of $\sum_{i=1}^{n} c_i$,$\sum_{i=1}^{n} \frac{1}{c_i}$ can't be $< n$. So one of them must be $\geq n$.

8

By AM-GM,

$$\frac{2}{b(a+b)} + \frac{2}{c(b+c)} + \frac{2}{a(c+a)} \geq 2\{\frac{3}{\sqrt[3]{abc(a+b)(b+c)(c+a)}}\}$$

$$= 2 \cdot 3\{\frac{1}{\sqrt[3]{abc}} \cdot \frac{1}{\sqrt[3]{(a+b)(b+c)(c+a)}}\}$$

By AM-GM again, $\frac{1}{\sqrt[3]{abc}} \geq \frac{1}{\frac{a+b+c}{3}}$ and $2\frac{1}{\sqrt[3]{(a+b)(b+c)(c+a)}} \geq 2\frac{1}{\frac{a+b+b+c+c+a}{3}}$ So, multiplying them we get,

$$\frac{2}{b(a+b)} + \frac{2}{c(b+c)} + \frac{2}{a(c+a)} \geq 2\{\frac{3}{\sqrt[3]{abc(a+b)(b+c)(c+a)}}\}$$

$$\geq \frac{27}{(a+b+c)^2}$$

9

Setting $a = \frac{1}{x}$ and $b = \frac{1}{y}$

$$\frac{a}{\sqrt{a^2+1}} + \frac{b}{\sqrt{b^2+1}} \leq \frac{\sqrt{ab}}{\sqrt{ab+1}}$$

$$2 \geq \sqrt{\frac{ab+1}{ab}}(\sqrt{\frac{a^2}{a^2+1}} + \sqrt{\frac{b^2}{b^2+1}})$$

$$2 \geq (\sqrt{\frac{a^3b+a^2}{a^3b+ab}} + \sqrt{\frac{ab^3+b^2}{ab^3+ab}})$$

By QM-AM

$$\sqrt{2(\frac{a^3b+a^2}{a^3b+ab} + \frac{ab^3+b^2}{ab^3+ab})}$$

$$\geq (\sqrt{\frac{a^3b+a^2}{a^3b+ab}} + \sqrt{\frac{ab^3+b^2}{ab^3+ab}})$$

So, we have to prove

$$\sqrt{\frac{1}{2}(\frac{a^3b+a^2}{a^3b+ab} + \frac{ab^3+b^2}{ab^3+ab})} \leq 1$$

$$\Leftrightarrow (ab+1)(a^2+2a^2b^2+b^2) \leq 2ab(a^2+b^2+a^2b^2+1)$$

$$\Leftrightarrow (a^2+2a^2b^2+b^2) \leq ab(2a^2+2b^2+2a^2b^2+2-(a^2+2a^2b^2+b^2))$$

$$\Leftrightarrow (a^2+2a^2b^2+b^2) \leq ab(a^2+b^2+2)$$

$$\Leftrightarrow (a^2-2ab+b^2) \leq ab(a^2+b^2-2ab)$$

But as $0 \leq x, y \leq 1$ so $a, b \geq 1$ and thus $ab \geq 1$ So

$$\Leftrightarrow (a-b)^2 \leq ab(a-b)^2$$

Proved.

11

Given that $a \geq b \geq c, x \geq y \geq z$. So $\frac{1}{by+cz} \leq \frac{1}{bz+cy}, \frac{1}{ax+cz} \leq \frac{1}{az+cx}, \frac{1}{ax+by} \leq \frac{1}{ay+bx}$ And $ax \geq by \geq cz$ And $\frac{1}{by+cz} \geq \frac{1}{ax+cz} \geq \frac{1}{ax+by}$ Thus

$$\frac{a^2x^2}{(by+cz)(bz+cy)} + \frac{b^2y^2}{(cz+ax)(cx+az)} + \frac{c^2z^2}{(ax+by)(ay+bx)}$$

$$\geq \frac{a^2x^2}{(by+cz)^2} + \frac{b^2y^2}{(cz+ax)^2} + \frac{c^2z^2}{(ax+by)^2}$$

Let us set $ax = m, by = n, cz = p$ Then we have to prove

$$\frac{m^2}{(n+p)^2} + \frac{n^2}{(m+p)^2} + \frac{p^2}{(n+m)^2} \geq \frac{3}{4}$$

But by Cauchy-Schwarz inequality ,$k\sum_{i=1}^{k} x_i^2 \geq (\sum_{i=1}^{k} x_i)^2$ So,

$$\frac{m^2}{(n+p)^2} + \frac{n^2}{(m+p)^2} + \frac{p^2}{(n+m)^2} \geq \frac{1}{3}\{\frac{m}{(n+p)} + \frac{n}{(m+p)} + \frac{p}{(n+m)}\}^2$$

But $\frac{m}{(n+p)} + \frac{n}{(m+p)} + \frac{p}{(n+m)} \geq \frac{3}{2}$[Nesbitt's inequality] So,

$$\frac{m^2}{(n+p)^2} + \frac{n^2}{(m+p)^2} + \frac{p^2}{(n+m)^2} \geq \frac{1}{3}(\frac{3}{2})^2 = \frac{3}{4}$$

And thus,

$$\frac{a^2x^2}{(by+cz)(bz+cy)} + \frac{b^2y^2}{(cz+ax)(cx+az)} + \frac{c^2z^2}{(ax+by)(ay+bx)} \geq \frac{3}{4}$$

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