2005 and 1998 IMO Camp

GEOMETRY PROBLEMS FOR FUN AND PROFIT

1. Given triangles $P_1P_2P_3$ and a point P within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers P_1P/PQ_1, P_2P/PQ_2, P_3P/PQ_3 at least one is not more than 2 and at least one is not less than 2.

E 2. In ΔABC the altitudes BP and CQ fall interior to the sides AC and AB. Prove that $BC^2 = AB \cdot BQ + AC \cdot CP$.

3. Let O be a point inside ΔABC such that $\measuredangle ABO = \measuredangle OBC = \measuredangle BCO = 10^\circ$ and $\measuredangle OCA = 20^\circ$. Determine $\measuredangle BAO$.

4. In ΔABC, $AB = AC$ and $\measuredangle A = 20^\circ$. The bisector of $\measuredangle B$ meets AC at D and is extended to E so that $\measuredangle EAC = \measuredangle CAB$. Prove that $AD + DE = AB$.

5. Two circles C_1 (centre R, radius r) and C_2 (centre P, radius s) where $r > s$, touch externally at A. Their direct common tangent touches C_1 at B and C_2 at C. The line RP extended meets C_2 again at D and BC produced at E. If $BC = 6\,DE$, prove that (a) the lengths of the sides of ΔREB are in arithmetic progression, and (b) $AB = 2AC$.

6. $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are two squares for which $A_1 = A_2$ but $B_1 \neq B_2$, $C_1 \neq C_2$, $D_1 \neq D_2$. Show that the lines B_1B_2, C_1C_2, D_1D_2 are concurrent.

7. Let E and F be the midpoints of AC and AB respectively. Let D be any point on BC. Let P be the point on BF such that DP is parallel to CF, and let Q be the point on CE such that DQ is parallel to BE. Let PQ cut BE at R and CF at S. Prove that $RS = PQ/3$.

8. Determine necessary and sufficient conditions on the triangle ABC that $|GA|^2 + |GB|^2 + |GC|^2 > \frac{8}{3}R^2$ where G is the centroid and R is the circumradius.

9. Let O be a point inside triangle ABC. Let AX, BY, and CZ be segments each with O as its midpoint. Prove that the circumcircles of the triangles BCX, CAY, ABZ and XYZ are concurrent.

10. AD, BE and CF are the altitudes of an acute triangle ABC. P and Q are points on the segments DF and EF respectively. Prove that if the angles PAQ and DAC are in the same orientation and equal to each other, then AP is the bisector of the angle FPQ.

11. I is the incentre of ABC. X and Y are the points of contact of the in-centra circle with AB and BC respectively. D and E are the midpoints of BC and CA respectively. Prove that AI, XY and DE are concurrent.

12. Let H be the orthocentre of triangle ABC. Let p be any line through H. Let q, r and s be the reflections of p about BC, CA and AB respectively. Prove that the lines q, r and s meet at a point, and find the locus of this point as p rotates about H.

13. Let P and M be points on the sides DC and BC, respectively, of the square $ABCD$, such that PM is tangent to the circle with centre A and radius AB. The segments PA and MA intersect the diagonal BD in the points Q and N respectively. Show that the pentagon $PQNMC$ is concyclic.

14. Let ABC be a triangle and let a (resp. b, c) be the length of the side opposite vertex A (resp. B, C). Suppose also that the lengths of the tangents to the inscribed circle from the vertices A, B, C are respectively u, v, w. Prove that $\frac{u}{a} + \frac{v}{b} + \frac{w}{c} \geq \frac{3}{2}$.

15. H is the orthocentre of triangle ABC. D, E and F are the midpoints of BC, CA and AB respectively. A circle with centre H cuts DE at P and Q, EF at R and S, and FD at T and U. Prove that $CP = CQ = AR = AS = BT = BU$.

16. C is the midpoint of a semi-circle with AB as diameter. For any point P on the semi-circle, Q is the point on the line segment CP such that $PQ = |PA - PB| / 2$. Find the locus of Q as P moves from A to B along the semi-circle.

17. A triangle ABC has $\angle A = 90^\circ$; D is ~~the mid~~ a point on BC for which AD is perpendicular to BC. Show that the angle bisector of $\angle A$ is perpendicular to the line joining the incircles of triangle ACD and ABD.
incentres?

18. D is the midpoint of AG. On the same side of AG are erected congruent quadrilaterals ABCD and DEFG such that they have inscribed circles O and I respectively. Prove that AO, CE and GI are concurrent.

19. E is a point inside a convex quadrilateral ABCD. For each of triangles EAB, EBC and ECD, the length of each side is an integer and the perimeter is equal numerically to the area. The three areas are distinct. What is the maximum area of triangle EDA?

20. The length of each side of a convex hexagon ABCDEF is at most 1. Prove that the length of at least one of the diagonals AD, BE and CF is not greater than 2.

21. Prove that for any point P inside a triangle ABC, $PA + PB + PC$ is at least 6 times the inradius of triangle ABC.

22. Consider the three escribed circles of the triangle ABC, that is, the three distinct circles each of which touches ~~the~~ one side of triangle ABC internally and the other two externally. Each pair of escribed circles has just one common tangent which is not a side of triangle ABC, and the three such common tangents form a triangle T. O is the circumcentre of triangle ABC. Prove that OA is perpendicular to a side of T.

23. The inradius and the three exradii of a triangle are consecutive terms of a geometric progression. Determine the largest angle of the triangle.

24. On a semicircle with unit radius four consecutive chords are given: AB, BC, CD, DE with lengths a, b, c, d. Prove that $a^2 + b^2 + c^2 + d^2 + abc + bcd < 4$.

25. Prove that a convex pentagon $ABCDE$ with equal sides and for which the interior angles satisfy the condition $\angle A \geq \angle B \geq \angle C \geq \angle D \geq \angle E$ is a regular pentagon.

26. Prove that the product of two sides of a triangle is always greater than the product of the diameters of the inscribed circle and the circumscribed circle.

27. The tangents at B and C to the circumcircle of the acute-angled triangle ABC meet in X. Let M be the midpoint of BC. Prove that $\angle BAM = \angle CAX$ and $AM/AX = \cos \angle A$.

28. Given a triangle ABC and external points X, Y and Z such that $\angle BAZ = \angle CAY$, $\angle CBX = \angle ABZ$ and $\angle ACY = \angle BCX$. Prove that AX, BY and CZ are concurrent.

29. In the triangle ABC, let B_1 be on AC, E on AB, G on BC, and let EG be parallel to AC. Furthermore, EG is tangent to the inscribed circle of the triangle ABB_1 and intersects BB_1 at F. Let r, r_1 and r_2 be the inradii of the triangles ABC, ABB_1 and BFG respectively. Prove that $r = r_1 + r_2$.

30. The sphere inscribed in the tetrahedron $ABCD$ touches the sides ABD and DBC at points K and M, respectively. Prove that $\angle AKB = \angle DMC$.

31. In the plane, a circle with radius R and center w and a line l are given, and the distance of w and l is d $(d > R)$. The points M and N are chosen in such a way that the circle with diameter MN is externally tangent to the given circle. Show that there exists a point A in the plane such that all the segments MN are seen in a constant angle from A.

GEOMETRY WORKSHEET

For a triangle ABC:

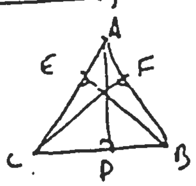
1. a) Prove that the altitudes are concurrent. (H)
   b) Prove that the medians (of a triangle, of course) are concurrent. (G)
   c) Prove that the right bisectors of the sides of a triangle are concurrent. (O)
   d) Prove that the angle bisectors are concurrent. (I)

2. a) If $O = G$ then the triangle is equilateral. Prove this.
   b) Same thing for every two of $\{O, G, H, I\}$.

3. What is the orthocentre of triangle ABH?

4. Prove the following area formulas:
   (a) $K = \frac{1}{2} bh$ (b) $K = \frac{1}{2} ab \sin C$ (c) $K = rs$
   (d) $K = \sqrt{s(s-a)(s-b)(s-c)} = \left|\begin{smallmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{smallmatrix}\right|$

5. Prove that $a = 2R \sin A$.
   Corollary The extended law of sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

6. 

   AD, BE, CF are the altitudes. Show that $EF = c \cos C$. Fill out all angles and lengths.

7. A, B, C, H are concyclic. Show that $H = A$, B or C i.e. that ABC is a right triangle.

8. Prove that $b \cos C + c \cos B = a$.

9. Find r in terms of a, b, c. Find R in terms of a, b, c.

10. Show that $\tan \frac{A}{2} = \frac{r}{s-a}$ (Hint: Find a right-angled triangle in ABC with one angle $\frac{A}{2}$ and legs r and $s-a$.)

11. Simplify $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2}$.

12. Prove that $abc = 4KR$.

13. Prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

14. Excircles. Let O_A, O_B, O_C be the excentres opposite $A, B,$ respectively. Prove that $O_A B O_C$ are collinear.

15. Prove that $r_a(s-a) = K$. Find r_a in terms of a, b, c.

16. $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = r^t$ is an identity for some t. Find t and prov the identity.

17. Prove that $c \geq (a+b)\sin\frac{C}{2}$.

18. Prove that $\cos A + \cos B + \cos C = \frac{r+R}{R}$.