Combinatorial Geometry In Olympiad

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1 Introduction

Recently, combinatorial geometry has become popular in international olympiads like IMO.These problems are really nice and it's a lot of fun just passing some time thinking over them, and so it doesn't matter if one can solve them or not. It's because they hardly require much knowledge to begin with. One needs pure thinking instead of brute force approach or great deal of knowledge. That's where the beauty of these problems lie. However, in combinatorial geometrical problems, contestants are hardly asked to do much geometry. Rather it generally involves some constructions or algorithms or rather arguments such as extreme principle, invariant principle or just plain arguments. This note should work as an introduction to combinatorial geometry problems. But first we must discuss some common techniques, since I intend to keep this note self-contained. For that purpose, we shall discuss problems regarding elementary arguments, construction or algorithms only, since combinatorial geometry spans a wider area than that. Throughout the article, if not stated otherwise, the points will all be in the same plane.

The following problem appeared recently at the IMO 2015.

Problem 1.1 (IMO 2015, Problem 1)**.** We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

1. Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.
2. Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of points.

Proposed by Netherlands

Next up is a problem posed at APMO 2015.

Problem 1.2 (APMO 2015, Problem 4)**.** In a plane, there are $2n$ distinct lines where n is a positive integer. Among them, n lines are colored blue and n lines are colored red and no two lines are parallel. Let $\mathcal{B}(\mathcal{R})$ be the set of all points that lie on at least one blue(red) line. Prove that, there exists a circle that intersects $\mathcal{B}$ and $\mathcal{R}$ in exactly $2n-1$ points.

Proposed by Pakawut Jiradilok and Warut Suksompong, Thailand

We will solve these problems along with some other nice and comparatively easy problems in order to understand the motivation behind the key idea of a solution.

Definition 1.1 (Lattice point)**.** A point with integer coordinates is called a *lattice point*. Lattice points are often of interest in olympiad problems.

Definition 1.2 (Convexity)**.** We call a geometrical shape *convex* if the line connecting any two points inside it is inside the figure as well.

Definition 1.3 (Polygons)**.** A *polygon* is a set of vertices $P_1 ... P_n$ where $P_i P_{i+1}$ is an *edge* of the polygon for $i = 1, ..., n-1$ and P_n is connected to P_1. A polygon can be *simple* or *complex*. In a complex polygon there is an edge which intersects the polygon itself i.e. has a self-intersecting edge. Simple polygons can be *convex* or *concave*. In a convex polygon, every angle of the polygon is less than 180, whereas a concave polygon has at least one angle more than 180. A regular polygon is of interest too, which has n sides equal in length and therefore all the angles are equal too.

Problem 1.3. Prove that, if a polygon is convex, then all its interior angles are less than 180.

Definition 1.4 (Convex Hull)**.** The smallest convex polygon that contains a set of points is the convex hull for that set(some points may be collinear). An example is shown in the figure below. It can also be thought as

- the intersection of all convex polygons that contain those points.
- the convex polygon with smallest perimeter.
- a rubber band held by some sticks, where a stick is planted in every point of the set.

Make sense of the fact that the minimum polygon containing a set of points must be convex. Otherwise, it will have three consecutive points A, B, C on the polygon so that $\angle ABC$ is greater than 180. But then we can just join A and C, which produces a smaller length $|AC|$ than $|AB| + |BC|$ according to triangle inequality and keeps all the points inside. Now, try to answer the following question. Does every closed polygon(not necessarily convex) have a convex angle?

Definition 1.5 (Minimum Enclosing Circle)**.** As the name suggests, the circle with the smallest radius that contains the given set of points is the minimum enclosing circle for that set.

Extremal Principle We can take a *maximal* or *minimal* element under circumstances(a greedy approach). For example, every set of positive integers have a minimum element, which is the **well-ordering principle**.

As an exercise, we see the following problem.

Problem 1.4. For a set of points S, if two points A and B is in S, then so is their midpoint. Prove that S is infinite.

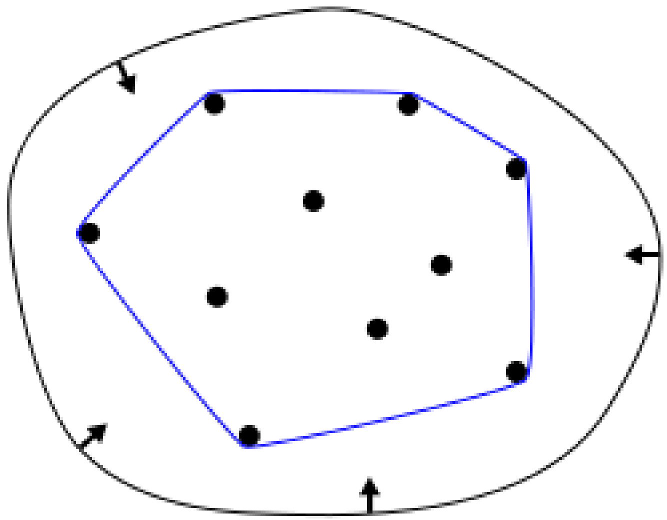

Figure 1: Rubber analogy of Convex Hull(taken from Wikipedia)

Solution. For the sake of contradiction, let's assume that S is finite. Then there exist two points A, B so that the distance between them $|AB|$ is minimum. Since their midpoint M is also in the set, $|AM| < |AB|$ contradicts the minimality of $|AB|$. Therefore, S must be infinite.

Sorting Sometimes we have to sort distances or angles depending on the problem. This type of idea isn't easy to pop up in head unless one has some experience. A particular technique is **angular sorting**. We see the following problem for an implementation of angular sorting.

Problem 1.5. n points of a polygon are given in some order. Find a permutation of the points so that they create a valid simple polygon.

Solution. In angular sorting we fix a point(usually an extremal point). In this case, we take the left-most point which is also the lowest i.e. the point with minimum coordinates(imagine X axis and Y axis mentally, though they don't have much to do) and call it P. So we are sure that there is no point left to it. To create an angle, we need one more point but we don't want to fix another point too since that will make things complicated. Instead we take a line perpendicular to the X axis which goes through P, and call this line l and the intersection of l and X axis K. Now join all other vertices with P and we get the angles such as $\angle KPB$ or $\angle KPG$. A natural question is why did we take a perpendicular line? What if we took a parallel one? The answer is, depending on construction, a parallel line will work too. We did this so that all angles will be between 0 and 180. See the figure for a better understanding. The advantage is that if two points aren't collinear with P then they will be at different angles. Therefore, we can sort them according to angles. If that's not the case, such as J and C are collinear with P, the angle created is the same but the distance isn't. So the closer one comes first in the permutation(depending on our orientation of-course, here we will go anti-clockwise). Thus, after doing an angular sorting, we have a valid permutation for the polygon which is $PBEFJCIGDA$.

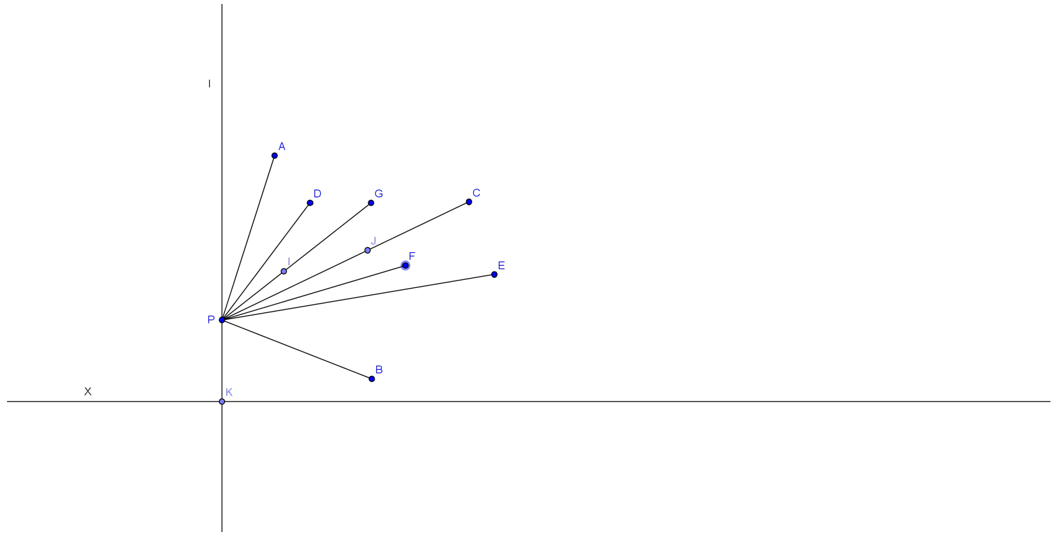

Figure 2: Angular Sorting

Question is, how do we prove that this construction leads to a valid polygon? We leave that as an exercise. Also, notice that we had to bring in some auxiliary objects such as the line l. One can't do that without having understood what the problem requires and how to handle it. That's what makes combinatorial geometry problems more interesting. Also, be aware that angular sorting doesn't guarantee an output of convex polygon. It only provides a valid polygon, not necessarily convex. The point is, while it's interesting to solve these problems, it's also easy to make mistakes or assume something that's not true.

Problem 1.6. Design an algorithm to find the convex hull of a given set of points.

There are many well known algorithms for constructing convex hull of some points quite efficiently. But in olympiad problems, efficiency is not an issue, solving the problem is. We provide two ways here. They are not so common or efficient but quite intuitive.

Solution (First algorithm). if the number of points is less than 3, we have nothing to do. Now, see figure (3) first. Let's find the leftmost point P first. Now take an arbitrary line ℓ_1 that's outside the polygon(for example, just drop a perpendicular on X-axis). Now we will rotate it(either clockwise or anti-clockwise but not both) using P as a pivot until it intersects a point. In the figure, we rotated anti-clockwise. The first point that this line intersects is G and this is the next pivot. Then we again rotate in the same direction and again we find E is the pivot. Continuing this process, D, C, A are pivots and the process terminates when it hits P again. Therefore the convex hull is $PGEDCAP$.

Solution (Second Algorithm). First take the leftmost point P like before. Drop a perpendicular on X-axis from this point. Consider the angles all other points create with P and take the point which is the minimum(in the figure G) and this is new pivot. Repeat this until we get to point P.

Note. In both algorithms, if two or more points are collinear with the pivot, then we take the farthest point from the current pivot that's collinear as the new pivot. In the second algorithm an angle is in the range $0 < \theta < 360$. And we can't change the direction of rotation once we have started.

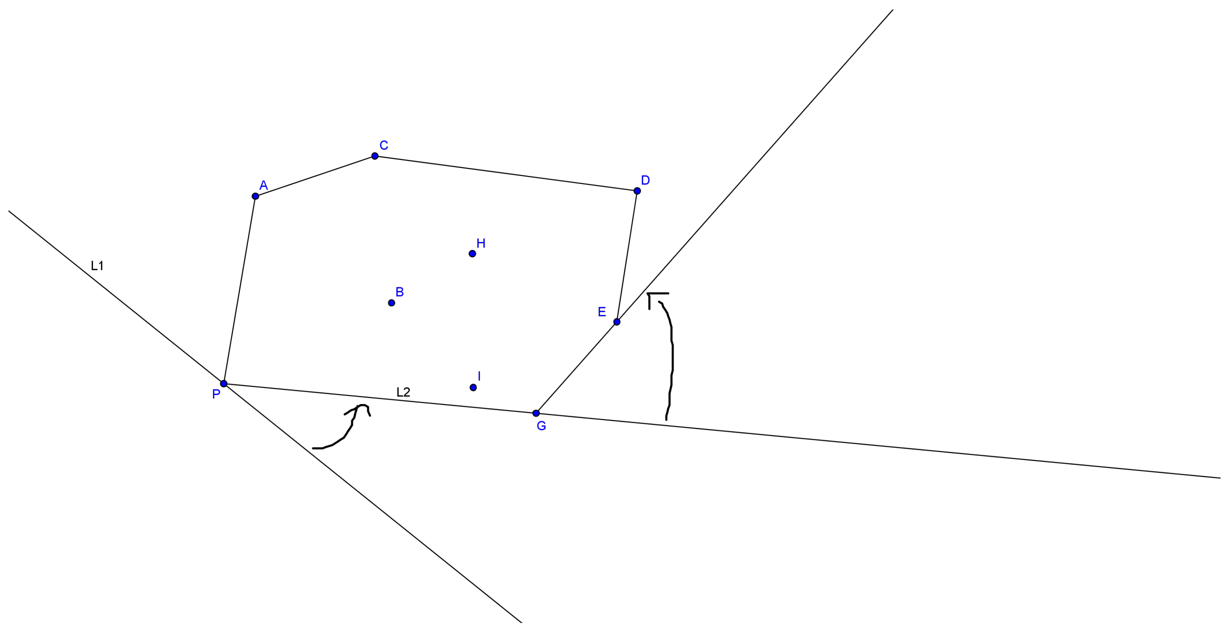

Figure 3: Convex hull using rotation

Pigeonhole Principle If there are n pigeons and k pigeonholes, there will be at least one pigeonhole containing at least $\lceil \frac{n}{k} \rceil$ pigeons. In other words, if there are $nk + 1$ objects are distributed among k sets then at least one set will have at least $n + 1$ objects.

A trivial problem.

Problem 1.7. A line is on the same plane as ΔABC. Prove that, it can't cut all three edges of ΔABC.

2 Problems

Problem 2.1. Prove that, the minimum enclosing circle of a set of points has at least two points on its border.

Solution. Let's make some observations. An obvious but crucial fact to solve this one is: We are trying to create a circle with radius as little as possible. Let the points be $P_1, ..., P_n$.

Observation. If the distance of a point from the center of a circle is less than or equal to its radius then that point is inside or on the circle.

It's straightforward from the definition of a circle.

Observation. We can enclose all the points in a circle with an arbitrary center and a radius large enough.

Let's say the center O is chosen arbitrarily. Then if the radius R centering O is large enough, it will contain all of the points inside.

Observation. A good circle has at least one point on it's border.

This is obvious and convenient since we have to enclose all the points first. So first we enclose them and then we try to minimize the radius as much as possible. If no point is on the border, we can just shrink it until it touches a point and it'll still be a good one since all the points are inside. Eventually, it touches a point, let's call it P. We could state this another way. Taking O as a center, consider all the distances $OP_1, ..., OP_n$. Then the maximum of these distances is our desired R since from our first observation, all other points will be on or inside this circle. This observation also says that the more points we can add to the boundary, the less the radius becomes. And it can improved to two points too.

Now we prove the statement. If two points are not on border, we minimize the radius while the first point on boundary is fixed. In other words, we keep P fixed. Now, we translate O towards P and keep decreasing R until it touches another point. When it does, we stop and the radius now is the new R. And now we have two points on the circle.

Question. Can we guarantee that three points must be on the circle as well? If so, why? If not, what can we say about it?

Problem 2.2. n points are given in a plane. Devise a way(an algorithm) to determine if they can be enclosed in a circle of radius r.

Solution. Let's say, the circle has center O. And we call a circle *good* if it has a radius less than or equal to r and contains all the points. It's a good thing we are given the radius. Thus, if we can know somehow where the center is, we can check if the circle contains all points. From the previous example, we know that two points must be on this circle. Therefore, let's take two arbitrary points A and B from the set. Now, if A and B are the points on the circle, the center O which must be equidistant from A and B i.e $OA = OB$. Therefore, O must lie on the perpendicular bisector of AB. Let the perpendicular bisector of AB be l. Draw a circle of radius r centering A which intersects l and call the intersection point O. This O is our desired center(why?). So if we draw a circle with center O and radius r and it contains all the points, they are enclosed, otherwise not. But since we can't choose A and B freely, we have to check for all possible $\binom{n}{2}$ pairs of points. If the above construction provides a good circle for one of those pairs then it is possible, otherwise not.

Problem 2.3. Five lattice points are chosen in the plane lattice. Prove that you can always choose two of these points such that the segment joining these points passes through another lattice point.

Solution. Since midpoint of two points A and B is simply $\dfrac{A+B}{2}$, we just need to check the parity of the points so that the midpoint is a lattice point too. There are only four possible combinations of the parity $(0,0), (0,1), (1,0), (1,1)$ where 1 means odd and 0 means even coordinate. But there are five points, so at least one parity configuration is sure to repeat. If those two points are $A(a, b)$ and $B(c, d)$, then their midpoint is $M\left(\dfrac{a+c}{2}, \dfrac{b+d}{2}\right)$. Both a, c and b, d are of the same parity. Therefore, their average is an integer and so M is a lattice point.

Problem 2.4 (Used at national camp 2015, Bangladesh)**.** 2015 points are given in a plane. No three are collinear. Prove that, there is a circle which has exactly 1007 points outside and 1007 points inside.

Solution. How do we proceed to solve this one? Instead of going directly for the solution, let's try to understand what the problem really requires. Let's say we have already found the circle and it has center O and radius R. 1007 points are strictly outside and 1007 are inside, this means the other point must be on the boundary. This is quite useful, which tells us to consider the distances of the points from the center. Call the points $P_1, ...P_n$ where $n = 2015$. Without loss of generality, we can assume that P_1008 lies on the boundary and the points $OP_1, ..., OP_{1007}$ are inside the circle of radius OP_{1008}. Then $OP_{1009}, ..., OP_{2015}$ are outside the circle. If OP_i is inside the circle then we must have $OP_i < OP_{1008}$, otherwise $OP_i > OP_{1008}$. This should tell you to sort the distances somehow. In other words, we need a construction for the center O so that the distances of P_i are sorted. We are done if we can find O so that all the distances are distinct. In order to find such a construction, we can think the opposite. When will two distances be equal? $OP_i = OP_j$ is possible only if O lies on the perpendicular bisector of P_iP_j. Since we want all the distances distinct, we need to take O so that it doesn't lie on any perpendicular bisector of P_iP_j for all i, j. And obviously there are infinite such points. Now, we can sort the points according to distances i.e. $OP_1 < OP_2 < ... < OP_{2015}$. Therefore, we make O center and draw a circle with radius OP_{1008} and we are done. For a better understanding, see the figure with 5 points. The lines drawn are the perpendicular bisectors of all 10 lines(the lines aren't shown).

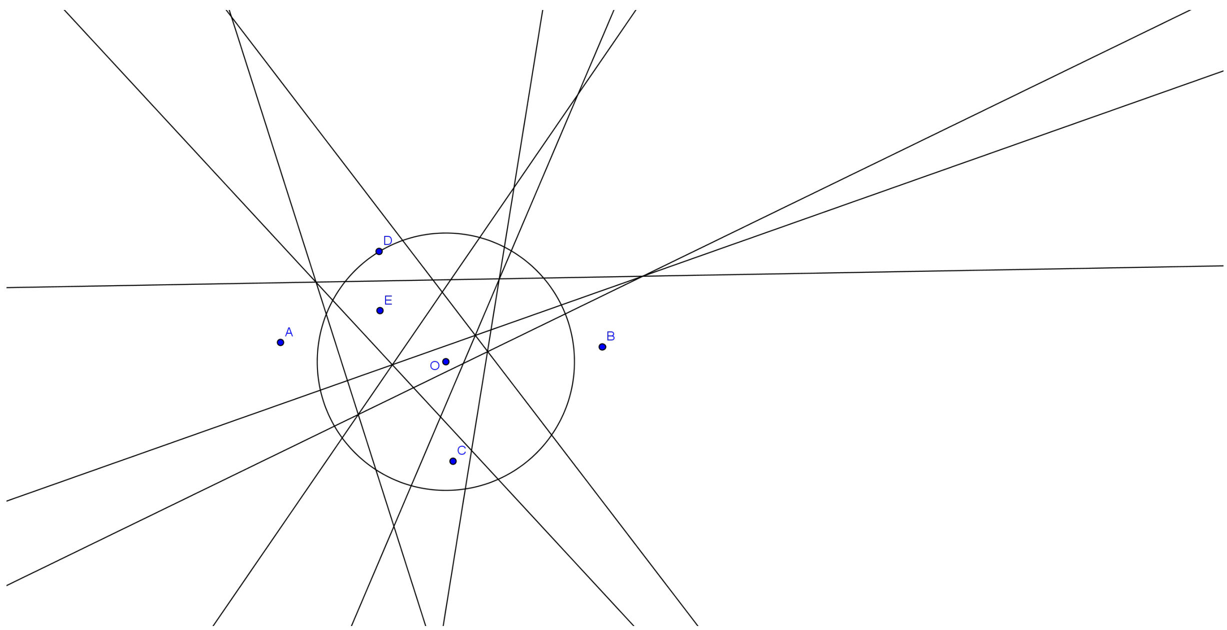

Figure 4: Construction with 5 points

There are other ways to solve this problem too. We provide with an alternative here.

Solution (Alternative). We can make the following claim.

Lemma 2.1. *There exists a line which has exactly two points of the set on this line,* 1007 *points on one side and rest of the* 1006 *points on the other side of the line.*

Proof. Consider a left-most point P and the angular sorting taking P as pivot. We make use of the fact that no three points are collinear. Because of that, we can guarantee each point creates a different angle. Except the pivot element, there are 2014 points. Now, consider each line connecting P with other points and label them $P_1, ..., P_{2014}$ according to sorted order. Then the line joining P and P_{1007} is our desired line(line PP_2 in the figure). $\square$

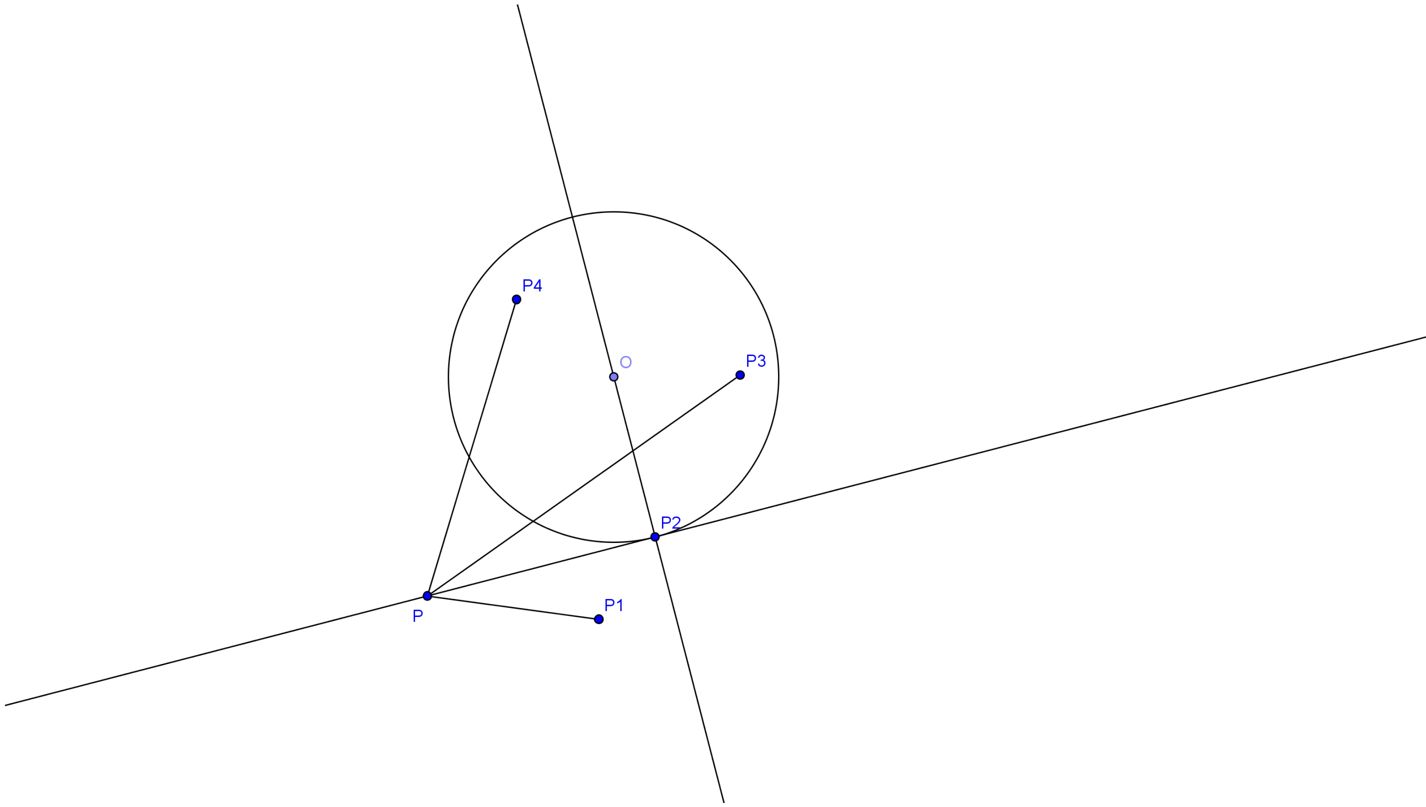

Figure 5: Alternative construction with 5 points

Now, we construct the circle. Note that we have exactly 1007 points on one side and 1006 on the other. If we draw a circle that touches PP_{1007} at P_{1007}(P_2 in the figure) with an appropriate radius, then we are done. But how do we find this appropriate radius? First we draw a circle with an arbitrary radius. Then on the perpendicular line of PP_{1007}, we take a point O which will act as a center. We can move O as we wish to get all those 1007 points inside the circle. Note that, since the circle is tangent at P_{1007}, it will never go through P. Therefore, the 1006 points outside this circle, along with P, total 1007 points will be outside the circle, as desired.

Question. Can we find a solution using convex hull or minimum enclosing circle? Think about it.

The following problem is a very nice one.

Problem 2.5 (APMO 2015, Problem 4)**.** In a plane, there are $2n$ distinct lines where n is a positive integer. Among them, n lines are colored blue and n lines are colored red and no two lines are parallel. Let $\mathcal{B}(\mathcal{R})$ be the set of all points that lie on at least one blue(red) line. Prove that, there exists a circle that intersects $\mathcal{B}$ and $\mathcal{R}$ in exactly $2n-1$ points.

Solution. To reword the problem, we need to find a circle which intersects both red and blue lines at $2n-1$ points(can touch as well of-course). We know that a circle can intersect a line at two points or touch at one point or doesn't intersect at all. So, when you look at the number $2n-1$, it should strike you as a hint since $2n-1$ is odd. There are n lines of each color. Let x be the number of lines the circle intersect, then it touches the rest $n-x$ lines. The number of intersection(consider touching as a single intersection) is $2x+n-x=2n-1$ which gives $x=n-1$. Therefore, we have to find a circle that intersects exactly $n-1$ lines of each color and touches the rest one.

The crucial fact: The circle has to touch exactly one line of each color. It may happen if the center O is equidistant from the two lines. In other words, O lies on the angle bisector of a red line and a blue line. Therefore, we have to consider all pairs of red and blue lines for their angle bisector. Since O has to touch exactly a red line and a blue line, O can't be on any other angle bisector. So we choose O so that it lies on only one of the angular bisectors and not on any other. There are infinite such points. But will it suffice to take such a point arbitrarily? If you think clearly, you can easily understand that's not the case. See the figure above where $\angle AIF$ is the maximum angle among all the pairs of red and blue lines and O is on the angle bisector of $\angle AIF$, as constructed.

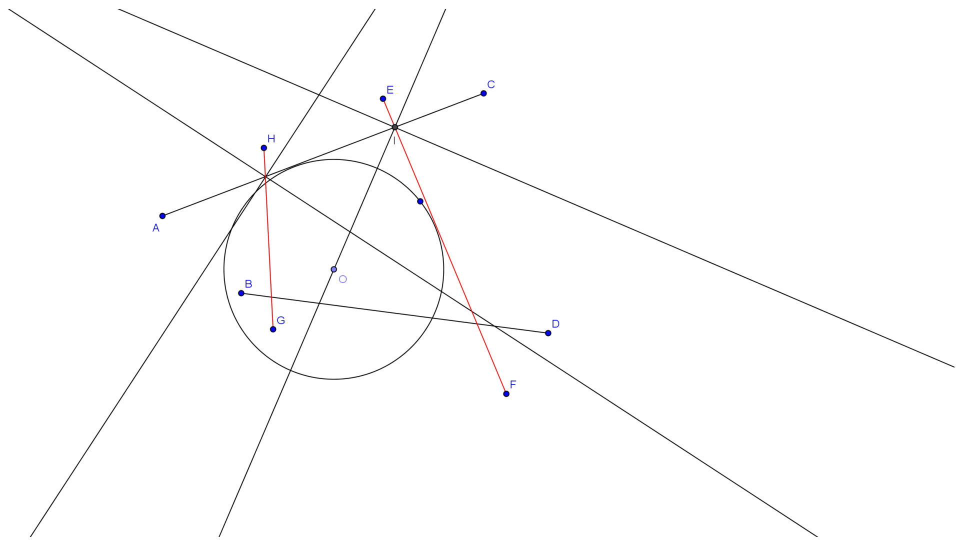

Figure 6: Construction with $n=2$

Why was the maximum angle part necessary? After we have found a possible construction for O, we only need to make sure that the circle surely intersects rest of the $n-1$ lines. Let the lines be X_1X_2 and Y_1Y_2 which intersect at Z. There are four rays, ZX_1, $ZX2$, $ZY1$, and $ZY2$. Now, if the angle between X_1X_2 and Y_1Y_2 (say, $\theta = \angle X_1ZY_1 = \angle X_2ZY_2$) is maximum among all angles created by intersection of lines, any line that intersect both ZX_1 and ZY_1 or ZX_2 and ZY_2 will create an angle greater than θ; which is a contradiction. So, we have that every line intersects either ZX_1 (and ZY_2) or ZY_1(and ZX_2). If we now take O far enough from Z and X, Y its perspective on the lines, every line will intersect either segment ZX or ZY, and from problem (1.7), the lines will cut exactly two of the edges of $\triangle XYZ$. This implies every line cuts XY, and thus also the circle centered at O and with radius $OX = OY$.

Solution (Using convex hull). We could handle the latter part of the solution using convex hull. Take all the intersection points of red and blue lines and consider their convex hull. Now any angle of the hull has one red line and one blue line. That part is taken care of. But the advantage is that, now we don't need to take a maximum angle anymore. Since it is a convex polygon, any angle will be less than 180. Therefore, any circle we draw touching that red and blue line, is bound to intersect all other lines(why?). Then we just need to draw the angle bisector and take O as we did before.

Question. Where did we use the fact that no two lines are parallel?

Problem 2.6 (IMO 2015, Problem 1)**.** We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

1. Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.
2. Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of points.

Proposed by Netherlands

Solution. For part (a), since we have the condition that $AC = BC$, one should instantly have a regular polygon in mind. But do they take care of all n? The answer is no. But experimenting by hand, one can see it works for all odd n. What if n is even? A regular polygon doesn't do. As a base case, we first need to construct a solution for $n = 4$. Let $ABCD$ be a balanced quadrilateral. Then $AC = BC, AD = BD, AC = AD, AC = CD, BC = CD$ which gives $AC = BC = AC$ and $AC = AD = CD$ i.e. ΔABC is equilateral and similarly ΔACD are equilateral. Therefore, we construct $ABCD$ this way. Take AB and draw an equilateral triangle ΔABC. Then on the face of ΔABC, draw another equilateral triangle of the same side ΔACD. Then we have a quadrilateral. This easily leads to the solution. With the same length, we just have to add an equilateral triangle with A fixed. Then we get balanced set for $6, 8, ...$ points.

For part (b), the construction above is true for odd n again. But this time, there is no such construction for even $n = 2k$. First note that, for any pair of points A, B there is a point P with $PA = PB$. The question is- for how many pairs can a fixed point P hold the same relation? There are $k(2k-1)$ pairs of points but a point can hold this true for at most $k - 1$ pairs. Otherwise the relation $PA = PB = PC$ will break since two pairs will have a point in common(remember pigeonhole principle?). Thus the number of such P is at most $n * (k-1) = 2k(k-1)$, if each point can do the same. But $2k(k-1) < k(2k-1)$ for all positive integer k. Therefore, for even k, it is not possible.

The following problem tells you to always keep **rotation** and **translation** in mind.

Problem 2.7 (BdMO 2014, Problem 10)**.** There are n points in space. Prove that you can divide these n points with $n - 1$ planes each parallel to each other.

Solution. The problems we have encountered so far only involved points in a plane. But there are points in the space now. The obvious question is: Is there a way we can make this problem into an equivalent one in a plane? And fortunately, there is. But think on it for yourself first. We are not gonna discuss the motivation behind such a mapping, we will only discuss the strategy. **Take a plane that's not parallel or perpendicular to any of the $X - Y$, $Y - Z$ or $Z - X$ plane**. And drop perpendiculars on this plane from all of those points. Now we are sure the points mapped to that plane are mutually distinct(so this can be a powerful tool as well). Then we are done if we can show the $2d$ version of the problem.

Lemma 2.2. *Given n distinct points on a plane. Then there exists $n - 1$ mutually parallel lines on the same plane which divide those n points.*

In fact this lemma can be treated as a corollary of another lemma.

Lemma 2.3. *Given three distinct points and two lines separating them. Then we can rotate the lines in a way so that they are parallel.*

We leave the proof as an exercise. And we believe the reader can put the parts together to find the solution.

3 Practice Problems

Problem 3.1. Design an algorithm to construct the minimum enclosing circle of a set of points.

Problem 3.2. Is the following algorithm to find the minimum enclosing circle correct? Prove.

Take the convex hull of all the points. Then draw a circle taking the segment that connects the two points of the hull with the largest distance as a diameter.

Problem 3.3. Prove that the two algorithms of finding convex hull are correct.

Problem 3.4. Find yet another algorithm for constructing convex hull.

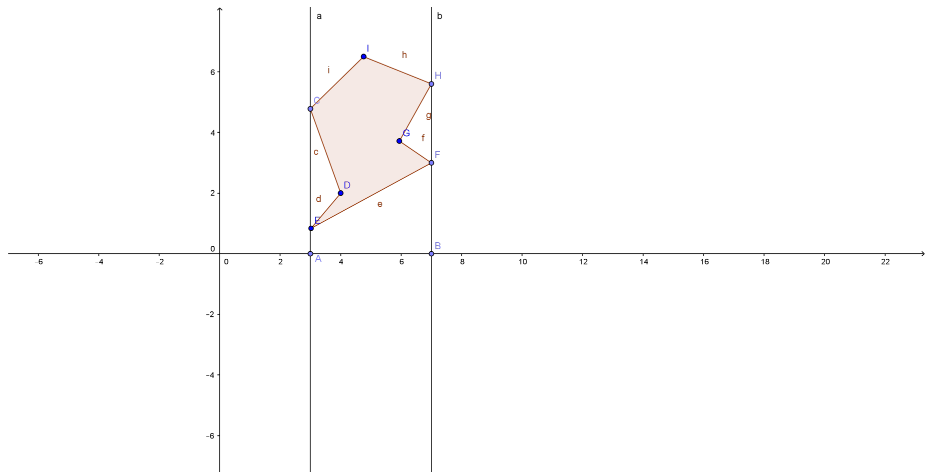

Figure 7: Polygon passes and $AB = \ell(\mathcal{P})$

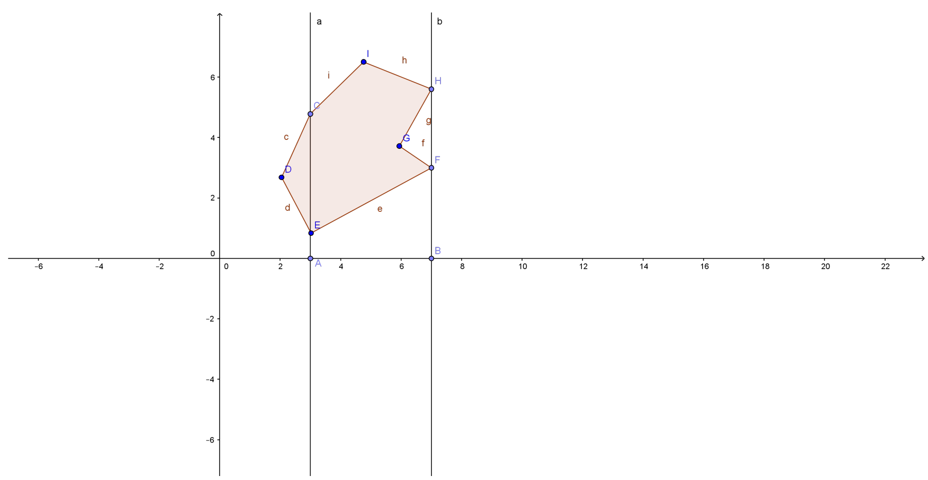

Figure 8: Polygon doesn't pass

Problem 3.5. Given a polygon $\mathcal{P}$(not necessarily convex and never assume so unless stated in the problem). Consider two points $(x_1, 0)$ and $(x_2, 0)$ on the X-axis and draw perpendiculars from those points. We call the length $\ell = |x_1 - x_2|$ good for polygon $\mathcal{P}$ if $\mathcal{P}$ can be passed through these two parallel lines $\ell(\mathcal{P})$ length apart(see the figures for clarification) only by rotating and translating the whole polygon. Design an algorithm to find the minimum value of $\ell(\mathcal{P})$.

Problem 3.6. Can APMO problem (2.5) be generalized for any given $k \leq 2n$ i.e.

Does there exist a circle which intersect in exactly k points where $k \leq 2n$?

Problem 3.7 (IMO 2013, Problem 2)**.** In the plane, 2013 red points and 2014 blue points are marked so that no three of the marked points are collinear. One needs to draw k lines not passing through the marked points and dividing the plane into several regions. The goal is to do it in such a way that no region contains points of both colors.

Find the minimal value of k such that the goal is attainable for every possible configuration of 4027 points.

Problem 3.8. 6 circles have an interior point in common. Prove that there exists a circle among them that contains the center of another of those circles in its interior.

Problem 3.9. 2015 points on the plane. No 4 lie on one circle. What is the maximum number of intersection points of described circles of triangles with vertices in that points?

Problem 3.10. Choose arbitrarily n vertices of a regular $2n-$gon and colour them red. The remaining vertices are coloured blue. We arrange all red-red distances into a nondecreasing sequence and do the same with the blue-blue distances. Prove that the two sequences thus obtained are identical.