Modular Arithmetic and Diophantine Equations

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BNW-MODS

§1 Lecture notes

- "Exponential" Diophantine equations vs. "polynomial" Diophantine equations.
- Take modulo things.
  - Squares nice mod 4.
  - Cubes nice mod 7, 9.
  - Use Fermat's theorem for general powers.
- Factor (e.g. difference of squares, $a^n \pm b^n$).

Problem 1.1. Solve $3^x = 2^y + 1$.

Problem 1.2. Solve $3^x = 2^y - 1$.

Problem 1.3 (HMMT 2016)**.** Let a and b be integers (not necessarily positive). Prove that $a^3 + 5b^3 \neq 2016$.

Problem 1.4 (Balkan MO)**.** Prove that $x^2 \neq y^5 - 4$ for any integers x and y.

§2 Practice problems

Problem 2.1 (JMO 2011)**.** Find all positive integers n such that $2^n + 12^n + 2011^n$ is a perfect square.

Problem 2.2 (MOP 2013)**.** For which primes p is $(p-1)^p + 1$ a power of p?

Problem 2.3 (Ali Gürel)**.** Solve $a^{11} + 11b^{11} + 111c^{11} = 0$ over $\mathbb{Z}$.

Problem 2.4 (JMO 2013)**.** Are there integers a and b such that $a^5b + 3$ and $ab^5 + 3$ are both perfect cubes of integers?

Problem 2.5 (Shortlist 2002)**.** What is the smallest positive integer t such that we can find integers $x_1, x_2, \ldots, x_t$ with

$$x_1^3 + x_2^3 + \cdots + x_t^3 = 2002^{2002}?$$

Problem 2.6 (AMSP 2011)**.** Find all positive integers x and y satisfying $2^x - 5 = 11^y$.

Problem 2.7 (IMO 2006/4)**.** Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$