2012 Mathematical Olympiad Summer Program Homework

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Chapter 1

Red and Green groups homework

1.1 Selected problems from 2012 APMO, Balkan MO, EGMO, KSA TST, and USA TST

1. [EGMO 2012] Let ABC be a triangle with circumcenter O. Points D, E, F lie in the interior of sides BC, CA, AB, respectively, such that $DE \perp CO$ and $DF \perp BO$. Let K be the circumcenter of triangle AFE. Prove that $DK \perp BC$.

2. [EGMO 2012] There are infinitely many people registered on the social network *Mugbook*. Some pairs of (different) users are registered as *friends*, but each person has only finitely many friends. Every user has at least one friend. (*Friendship is symmetric; that is, if A is a friend of B, then B is a friend of A.*)

   Each person is required to designate one of their friends as their *best friend*. If A designates B as her best friend, then (unfortunately) it does not follow that B necessarily designates A as her best friend. Someone designated as a best friend is called a *1-best friend*. More generally, if $n > 1$ is a positive integer, then a user is an *n-best friend* provided that they have been designated the best friend of someone who is an *$(n-1)$-best friend*. Someone who is a *k-best friend* for every positive integer k is called popular.

   (a) Prove that every popular person is the best friend of a popular person.

   (b) Show that if people can have infinitely many friends, then it is possible that a popular person is not the best friend of a popular person.

3. [KSA TST 2012] Let G be the centroid of triangle ABC with the side-lengths a, b, c. Prove that if $a + BG = b + AG$ and $b + CG = c + BG$, then triangle ABC is equilateral.

4. [2012 KSA TST] Let a, b, c be rational numbers such that

$$\frac{1}{a+bc} + \frac{1}{b+ac} = \frac{1}{a+b}.$$

   Prove that $\sqrt{\frac{c-3}{c+1}}$ is rational.

5. [2012 KSA TST] Consider the isosceles triangle ABC with $AB = AC$. A semicircle of diameter EF situated on the side BC, is tangent to the sides AB and AC at M and N, respectively. The line AE intersects the semicircle at P. Prove that the line PF passes through the midpoint of the chord MN.

6. [2012 KSA TST] Prove that for every positive real numbers x, y, z the following inequality holds
$$\frac{9}{x+y+z} - \frac{1}{xyz} \leq 2.$$

7. [KSA TST 2012] Show that for every positive integers $n \geq 3$ there are distinct positive integers $a_1, a_2, \ldots, a_n$ with $a_1!a_2!\ldots a_{n-1}! = a_n!$.

8. [KSA TST 2012] Find all positive integers n with the following property: there are two divisors a and b of the number n such that $a^2 + b^2 + 1$ is a multiple of n.

9. [KSA TST 2012] Let a, b, c be positive integers. Prove that if the numbers
$$\frac{a^2}{a+b}, \quad \frac{b^2}{b+c}, \quad \frac{c^2}{c+a}$$
are integers and primes, then $a = b = c$.

10. [KSA TST 2012, by Warut Suksompong] Determine all positive integers $n \geq 2$ for which the following statement is true:

    Given any n distinct points on the plane such that the distance between each pair of points is distinct, there exists a pair of points A, B for which the difference between the number of points lying on either side of the perpendicular bisector of segment AB is not greater than 1.

11. [2012 KSA TST] Let ABC be a triangle. Point D lies on side BC. Let O, O_1, and O_2 be the circumcenters of triangle ABC, ABD, and ACD, respectively. Prove that circumcircles of triangles BOO_1 and COO_2 meet on line BC.

12. [2012 KSA TST, by Dorin Andrica] Determine if there is are polynomials $p(x)$ and $q(x)$ with real coefficients such that
$$\frac{p(n)}{q(n)} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$
for every positive integer n.

13. [KSA TST 2012, by Warut Suksompong] Consider $S = \{(x, y, z) \mid x, y, z \in \{1, 2, \ldots, 2012\}\}$ as a set of 2012^3 points in three-dimensional space. For any segment joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the space, we define its *distance triplet* to be the ordered triple $(|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|)$. Alice wants to draw segments in such a way that

    (a) Each segment joins two distinct points in S;

(b) Each point in S is an endpoint of at most one segment;

(c) For any two segments, their distance triplets are different.

Find the greatest number of segments that Alice can draw.

14. [EGMO 2012] Find all functions $f : \mathbb{R} to \mathbb{R}$ such that

$$f\left(y(f(x+y)+f(x)\right) = 4x + 2yf(x+y)$$

for all real numbers x and y.

15. [EGMO 2012] Let ABC be a acute triangle with circumcircle ω and orthocenter H. Let K be a point on minor arc $\widehat{BC}$ of ω. Point L is the reflection of K in line AB, and point M is the reflection of K in line BC. Let E be the second intersection (other than B) of ω with the circumcircle of triangle BLM. Show that lines KH, EM, BC are concurrent.

16. [TST 2012] In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.

17. [RMM 2012, by Valery Senderov from Russia] Prove that there are infinitely many positive integers n such that $2^{2^n+1} + 1$ is divisible by n but $2^n + 1$ is not.

18. [TST 2012, by Ruixiang Zhang] Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for every pair of real numbers x and y,

$$f(x + y^2) = f(x) + |yf(y)|.$$

19. [KSA TST 2012, by Ruixiang Zhang] Consider (3-variable) polynomials

$$P_n(x, y, z) = (x-y)^{2n}(y-z)^{2n} + (y-z)^{2n}(z-x)^{2n} + (z-x)^{2n}(x-y)^{2n}$$

and

$$Q_n(x, y, z) = [(x-y)^{2n} + (y-z)^{2n} + (z-x)^{2n}]^{2n}.$$

Determine all positive integers n such that the quotient $Q_n(x, y, z)/P_n(x, y, z)$ is a (3-variable) polynomial with integer coefficients.

20. [RMM 2012, by Marek Cygan from Poland] Given a finite number of boys and girls, a *sociable set of boys* is a set of boys such that every girl knows at least one boy in that set; and a *sociable set of girls* is a set of girls such that every boy knows at least one girl in that set. Prove that the number of sociable sets of boys and the number of sociable sets of girls have the same parity. (Acquaintance is assumed to be mutual.)

Chapter 2

Black and blue groups homework

2.1 Selected problems from 2012 APMO, Balkan MO, EGMO, KSA TST, and USA TST

1. [Balkan] Prove that

$$\sum_{\text{cyc}} (x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx)$$

for all positive real numbers x, y, and z.

2. [KSA TST 2012, by Dorin Andrica] Triangle ABC is inscribed in circle ω. Point D is midpoint of side AC, and point M lies on segment BD with $DM = 2BM$. Ray AM meets side BC at E, and ray CM meets side BA and F. Ray FE intersects ω at N. Suppose that $AM \perp CM$. Prove that $ADEF$ is cyclic if and only if line AN bisects segment BC.

3. [KSA TST 2012] Let $ABCD$ be a convex quadrilateral such that $AB = AC = BD$. The lines AC and BD meet at point O, the circles ABC and ADO meet again at point P, and the lines AP and BC meet at point Q. Show that $\angle COQ = \angle DOQ$.

4. [KSA TST 2012] For any positive integer n denote by a_n the number of quadratic functions $f(x) = ax^2 + bx + c, a, b, c \in \{1, 2, \dots, n\}$, having only integer roots. Prove that for every $n \geq 4$, $n < a_n < n^2$.

5. [KSA TST 2012] Let $ABCDE$ be a pentagon with $\angle A = \angle B = \angle C = \angle D = 120°$. Prove that

$$4AC \cdot BD \geq 3AE \cdot ED.$$

6. [Balkan 2012] Let n be a positive integer. Let $P_n = \{2^n, 2^{n-1} \cdot 3, 2^{n-2} \cdot 3^2, \dots, 3^n\}$. For each subset X of P_n, we write S_X for the sum of all elements of X, with the convention that $S_\emptyset = 0$ where $\emptyset$ is the empty set. Suppose that y is a real number with $0 \leq yle 3^{n+1} - 2^{n+1}$. Prove that there is a subset Y of P_n such that $0 \leq y - S_Y < 2^n$.

7. [TST 2012] Determine, with proof, whether or not there exist integers $a, b, c > 2010$ satisfying the equation
$$a^3 + 2b^3 + 4c^3 = 6abc + 1.$$

8. [APMO 2012] Determine all pairs (p, n) of a prime number p and a positive integer n such that $p^n + 1$ divides $n^p + 1$.

9. [RMM 2012, by David Monk from United Kingdom] Given a non-isosceles triangle ABC, let D, E, and F denote the midpoints of the sides BC, CA, and AB respectively. The circle BCF and the line BE meet again at P, and the circle ABE and the line AD meet again at Q. Finally, the lines DP and FQ meet at R. Prove that the centroid G of the triangle ABC lies on the circle PQR.

10. [APMO 2012] Let ABC be an acute triangle inscribed in circle ω. Let M is the midpoint of side BC, and let H be the orthocenter of triangle ABC. Ray AH meets side BC at D. Ray MH meets ω at E. Ray ED meets ω again at F (other than E). Prove that $BF/CF = AB/AC$.

11. [EGMO 2012] A set A of integers is called *sum-full* if $A \subseteq A + A$; that is, each element a in A is the sum of some pair of (not necessary distinct) elements b and c in A. A set A of integers is said to be *zero-sum-free* if 0 is the only integer that cannot be expressed as the sum of elements of a finite nonempty subset of A. Does there exist a sum-full zero-sum-free set of integers?

12. [TST 2012, by Ruixiang Zhang] Consider (3-variable) polynomials
$$P_n(x, y, z) = (x - y)^{2n}(y - z)^{2n} + (y - z)^{2n}(z - x)^{2n} + (z - x)^{2n}(x - y)^{2n}$$
and
$$Q_n(x, y, z) = [(x - y)^{2n} + (y - z)^{2n} + (z - x)^{2n}]^{2n}.$$
Determine all positive integers n such that the quotient $Q_n(x, y, z)/P_n(x, y, z)$ is a (3-variable) polynomial with rational coefficients.

13. [EGMO 2012] A *word* is a finite sequence of letters from some alphabet. A word is *repetitive* if it is a concatenation of at least two identical subwords (for example, *ababab* and *abcabc* are repetitive, but *ababa* and *aabb* are not). Prove that if a word has the property that swapping any two adjacent letters makes the word repetitive, then all its letters are identical. (Note that one may swap two adjacent identical letters, leaving a word unchanged.)

14. [TST 2012] In cyclic quadrilateral $ABCD$, diagonals AC and BD intersect at P. Let E and F be the respective feet of the perpendiculars from P to lines AB and CD. Segments BF and CE meet at Q. Prove that lines PQ and EF are perpendicular to each other.

15. [TST 2012, by Ruixiang Zhang] Determine all positive integers n, $n \geq 2$, such that the following statement is true:

If $(a_1, a_2, \ldots, a_n)$ is a sequence of positive integers with $a_1 + a_2 + \cdots + a_n = 2n - 1$, then there is block of (at least two) consecutive terms in the sequence with their (arithmetic) mean being an integer.

16. [RMM 2012, by Ilya Bogdanov, Grigory Chelnokov, Dmitry Khramtsov from Russia] Given a positive integer $n \geq 3$, color each cell of an $n \times n$ square array with one of $\lfloor (n+2)^2/3 \rfloor$ colors, each color being used at least once. Prove that there is some 1×3 or 3×1 rectangular sub-array whose three cells are colored with three different colors.

17. [APMO 2012] Let n be an integer greater than or equal to 2. Prove that if the real numbers $a_1, a_2, \ldots, a_n$ satisfy $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then

$$\sum_{1 \leq i < j \leq n} \frac{1}{n - a_i a_j} \leq \frac{n}{2}.$$

18. [RMM 2012, by Ben Elliott from United Kingdom] Each positive integer is colored red or blue. A function f from the set of positive integers to itself has the following two properties:

    (a) if $x \leq y$, then $f(x) \leq f(y)$; and

    (b) if x, y and z are (not necessarily distinct) positive integers of the same color and $x + y = z$, then $f(x) + f(y) = f(z)$.

    Prove that there exists a positive number a such that $f(x) \leq ax$ for all positive integers x.

19. [RMM 2012, by Fedor Ivlev from Russia] Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A_1 distinct from A; the points B_1 and C_1 are defined similarly. Prove that the lines AA_1, BB_1 and CC_1 are concurrent at a point on the line IO.

20. [TST 2012, by Evan O'Dorney] There are 2010 students and 100 classrooms in the Olympiad High School. At the beginning, each of the students is in one of the classrooms. Each minute, as long as not everyone is in the same classroom, somebody walks from one classroom into a different classroom with at least as many students in it (prior to his move). This process will terminate in M minutes. Determine the maximum value of M.