Art of Problem Solving

WOOT 2010–11

Practice AIME 3

Instructions

- The time limit is 3 hours. These 3 hours should be continuous, e.g. do not work for 2 hours on one day, and then 1 hour on another day.
- All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor a penalty for wrong answers.
- No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted**.
- Enter your answers at the following link, before **midnight ET/9 PM PT** on **Wednesday, March 9, 2011**:

  `http://www.artofproblemsolving.com/School/WOOT/aime.php`

- Do not discuss the problems before Wednesday, March 9. The answers will be posted on Thursday, March 10.

1. Find the smallest positive integer n such that the fractions

$$\frac{19}{n+21}, \quad \frac{20}{n+22}, \quad \frac{21}{n+23}, \quad \dots, \quad \frac{91}{n+93}$$

are all irreducible.

2. In a convex polygon with 18 sides, the angles are all positive integers (when measured in degrees) and form a nonconstant arithmetic sequence. Find the measure of the smallest angle, in degrees.

3. For two sets A and B, let $f(A, B)$ denote the number of elements that are either in A or B, but not both. For example, $f(\{1, 2, 3, 4, 6\}, \{2, 3, 6, 7\}) = 3$, because of the elements 1, 4, and 7.

   For every ordered pair of sets (A, B), where A and B are subsets of $\{1, 2, 3, 4\}$, we compute $f(A, B)$. Find the sum of $f(A, B)$ taken over all such ordered pairs (A, B).

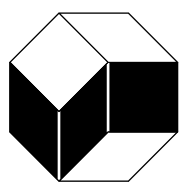
4. A floor can be covered with n identical square tiles of a certain size. If a certain smaller square tile is used, then $n + 76$ tiles are required. Find n.

5. There exist unique integers x and y such that

$$(2 - \sqrt{3})^4 x + (2 - \sqrt{3})^5 y = 1.$$

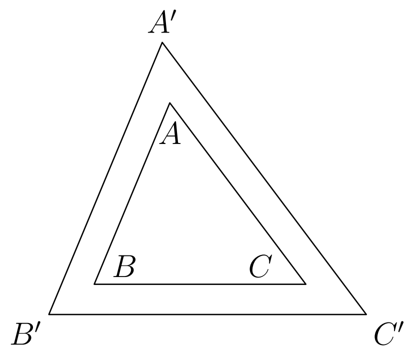
   Find x.

6. Mr. X has six tiles that are all rhombi with side length 1: one black square, one white square, two identical black rhombi with angles 45° and 135°, and two identical white rhombi with angles 45° and 135°. In how many different ways can Mr. X tile a regular octagon with side length 1 using these six tiles? An example of such a tiling is shown below.

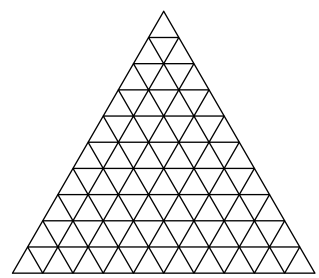


   (Even if one tiling can be obtained from another tiling by rotating or reflecting the octagon, they are still considered different.)

7. Let ABC and $A'B'C'$ be two triangles, such that triangle ABC lies inside $A'B'C'$ and the corresponding sides are parallel. The distance between each pair of corresponding sides is 2. Find the area of triangle $A'B'C'$ if the sides of triangle ABC are 13, 14, and 15.



8. An equilateral triangle is divided into 100 smaller equilateral triangles, as shown below. Find the number of equilateral triangles of any size in the diagram.



9. Let S be the set of complex numbers z such that the real part of $1/\overline{z}$ lies between $1/20$ and $1/10$. As a subset of the complex plane, the area of S can be expressed in the form $k\pi$, where k is a positive integer. Determine k.

10. Let ω be the circumcircle of square $ABCD$, and let P be a point on arc AD of ω. Let X and Y be the intersections of $\overline{PB}$ and $\overline{PC}$ with $\overline{AD}$, respectively. If $AX = 5$ and $DY = 7$, then XY can be expressed in the form $\sqrt{m} - n$, where m and n are positive integers. Find $m + n$.

11. Let
$$S = \sum_{i,j,k} \frac{1}{3^{i+j+k}},$$
where the sum is taken over all ordered triples of nonnegative integers (i, j, k), where i, j, and k are distinct, i.e. $i \neq j$, $i \neq k$, and $j \neq k$. Then S can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

12. In triangle ABC, $AB = 30$ and $AC = 41$. Let M be the midpoint of $\overline{AB}$, and let N be the point on side $\overline{AC}$ such that $AN : NC = 3 : 2$. Find BC^2 if $\overline{BN}$ is perpendicular to $\overline{CM}$.

13. Let x and y be angles such that $\sin x + \sin y = \frac{27}{25}$ and $\cos x + \cos y = \frac{39}{25}$. Then

$$\tan \frac{x}{2} + \tan \frac{y}{2}$$

can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

14. Find the number of permutations a_1, a_2, a_3, $\dots$, a_{101} of the numbers 2, 3, 4, $\dots$, 102, such that a_k is divisible by k for all $1 \le k \le 101$.

15. Let

$$p(x) = -\frac{7}{6}x^2 + \frac{11}{2}x - \frac{7}{3}.$$

The set of real numbers $\{1, 2, 4\}$ has the property that $p(1) = 2$, $p(2) = 4$, and $p(4) = 1$.

There exists another set of three distinct real numbers $\{a, b, c\}$ such that $p(a) = b$, $p(b) = c$, and $p(c) = a$. Find $a + b + c$.