Winter Camp 2009
Mock Olympiad

1. At a party, one or more pairs of people shook hands with each other. We say two people were "close" if either they shook hands with each other or if there was a third person they both shook hands with. Show there were two people who (a) were close, and (b) shook hands with the exact same number of people.

2. Let $ABCD$ be a cyclic quadrilateral such that $AD + BC = AB$. Prove that the bisectors of the angles ADC and BCD meet on the line AB.

3. Find all integer-valued functions, f and g, defined on the integers, for which g is one-to-one and

$$f(g(x) + y) = g(f(y) + x)$$

for all integers x, y.

4. Prove that any set of 10 positive integers, ranging between 1 and 2009, contains three distinct elements a, b, c such that $\gcd(a, b)$ divides c.

Solutions

1. For each person p, let n_p denote the number of people that p shook hands with. Let $M = \max n_p$, let x_0 be a person for which $n_p = M$, and let $\{x_1, x_2, \ldots, x_M\}$ be the people that x_0 shook hands with.

   For each i, we have $n_{x_i} \geq 1$ since each person x_i shook hands with x_0, and $n_{x_i} \leq M$ by definition. Therefore, the Pigeonhole Principle implies that two of the values $\{n_{x_0}, n_{x_1}, \ldots, n_{x_M}\}$ are equal. But if $n_{x_p} = n_{x_q}$, then x_p and x_q have the property we are looking for.

2. Choose E on AB so that $AE = AD$ and hence $BE = BC$. Also let F denote the second intersection of AB and the circumcircle of $\triangle CDE$. Note that F lies between A and B.[1] Also note that regardless of configuration, $\angle AFD = \angle ECD$.

   Therefore, $\angle ADC = 180^\circ - \angle EBC = 2\angle BCE = 2\angle BCD - 2\angle ECD = 360^\circ - 2\angle FAD - 2\angle AFD = 2\angle ADF$. Similarly, $\angle BCD = 2\angle BCF$, so F is the intersection of the bisectors of $\angle BCD$ and $\angle ADC$.

   We know that F lies on AB, however, so the result follows.

   Source: American Math Olympiad Program 1999

3. Let $m = f(0)$ and $n = g(0)$. Setting $x = 0$ in the given equation yields $f(n + y) = g(f(y))$ (1). Setting $y = 0$ in the given equation yields $g(m + x) = f(g(x))$ (2).

   Setting $y = g(x)$ in equation (1) gives us $g(f(g(x))) = f(n + g(x))$, which we know from the original equation is equal to $g(f(n) + x)$. Since g is one-to-one, it follows that $f(g(x)) = f(n) + x$. Combining this with (2), we have $g(m + x) = f(n) + x$. In particular, there exists a constant C so that $g(x) = x + C$.

   Substituting this into (2) yields $m + x + C = f(x + C)$, which implies there exists a constant B so that $f(x) = x + B$.

   Conversely, it is easy to check that $f(x) = x + B$ and $g(x) = x + C$ satisfies the original functional equation.

   Source: Latvia

4. Consider a set $\{x_1, x_2, \ldots, x_{10}\}$ of positive integers for which $\gcd(x_i, x_j)$ does not divide x_k for all distinct i, j, k. We claim $\max(x_i) > 2009$. If each x_i is even, we can divide through by 2 to achieve a set with smaller elements. Therefore, we may assume without loss of generality that the set contains an odd element, say x_1.

   For $i > 1$, let $g_i = \gcd(x_1, x_i)$. If $g_i | g_j$ for any $i \neq j$, then $\gcd(x_1, x_i) | x_j$, which is impossible. Therefore, x_1 has 9 factors (g_2, g_3, …, g_9), none of which divide each other.

   Now, let $p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$ be the prime factorization of x_1, ordered so that $e_1 \geq e_2 \geq \ldots \geq e_k$. Each g_i can be written in the form $p_1^{e_{i,1}} \cdot p_2^{e_{i,2}} \cdot \ldots \cdot p_k^{e_{i,k}}$. If $e_{i,t} = e_{j,t}$ for all $t > 1$, then one of g_i or g_j must divide the other, which is impossible. However, there are only $\prod_{j>1}(e_j + 1)$ possible values for $\{e_{i,2}, e_{i,3}, \ldots, e_{i,t}\}$, so if these are all distinct, we must have $\prod_{j>1}(e_j + 1) \geq 9$.

[1] For example, if B, E, A, F occur on line BA in that order, then by continuity, there exists X between E and A for which $BX \cdot XA = EX \cdot XF$. This point X must lie on the radical axis of circle $ABCD$ and circle CDE, which is CD. Therefore, CD intersects AB between A and B, which is impossible.

Now, if $k = 2$, then $e_2 \geq 8$ and $x_1 \geq 3^8 \cdot 5^8$. If $k = 3$, then $e_2, e_3 \geq 2$ and $x_1 \geq 3^2 \cdot 5^2 \cdot 7^2$, *or* $e_2 \geq 4$ and $x_1 \geq 3^4 \cdot 5^4 \cdot 7$. If $k = 4$, then $a_2 \geq 2$ and $x_1 \geq 3^2 \cdot 5^2 \cdot 7 \cdot 11$. If $k = 5$, then $x_1 \geq 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. Regardless, $x_1 > 2009$, and the result is proven.

Source: Romanian Math Stars Competition 2007, #8, except they consider 27 integers instead of 10. They also conjecture the result holds even for 6 integers.