# Functional Equations

## 1 Introduction

In algebra, we are usually presented with a function first, and then we study its properties. However, in a *functional equation*, we are given the property first, and then we must deduce all functions that have the given property. For example, the problem might be to find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$$

for all $x, y \in \mathbb{R}$.

Here are a few definitions you should be familiar with: Let $f : A \to B$ be a function, so $A$ and $B$ are the *domain* and *codomain* of $f$, respectively. The *range* of $f$ is the set $\{f(a) : a \in A\}$, which is sometimes denoted by $f(A)$. The function $f$ is *injective* (or *1–1*) if $f(x) = f(y)$ implies $x = y$. The function $f$ is *surjective* if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. A function is *bijective* if it is both injective and surjective.

The function $f$ is *increasing* if $f(x) \geq f(y)$ for $x > y$, and *strictly increasing* if $f(x) > f(y)$ for $x > y$. Likewise, the function $f$ is *decreasing* if $f(x) \leq f(y)$ for $x > y$, and *strictly decreasing* if $f(x) > f(y)$ for $x > y$. A function is *monotone* if it is increasing or decreasing.

## 2 Substitution

The primary approach to solving any functional equation is substitution. It is usually best to start by setting the variables equal to 0 or 1, or set variables equal to each other, or any other substitution that will simplify the functional equation and help find specific values of the function.

**Problem 2.1.** Suppose that $f$ satisfies the functional equation

$$2f(x) + 3f\left(\frac{2x+29}{x-2}\right) = 100x + 80$$

for all $x \neq 2$. Find $f(3)$.

**Solution:** Since we want to find $f(3)$, we set $x = 3$ in the given functional equation, to get

$$2f(3) + 3f(35) = 380.$$

Hence, we need to find $f(35)$. Setting $x = 35$ in the given functional equation, we get

$$2f(35) + 3f(3) = 3580.$$

We now have a linear system of equations in $f(3)$ and $f(35)$. Solving for $f(3)$, we find $f(3) = 1996$.

Is it a coincidence that when we set $x = 35$ in the given equation, we obtained the original term of $f(3)$? No, because if we let

$$g(x) = \frac{2x+29}{x-2},$$

then we find

$$g(g(x)) = g\left(\frac{2x+29}{x-2}\right) = \frac{2 \cdot \frac{2x+29}{x-2} + 29}{\frac{2x+29}{x-2} - 2} = \frac{2(2x+29) + 29(x-2)}{2x+29-2(x-2)} = \frac{33x}{33} = x.$$

This tells us that 3 wasn't special; we have $g(g(x)) = x$ for all $x$ besides $x = 2$. $\square$

More generally, we say that a function $g$ is *cyclic with order* $n$ if iterating $g$ $n$ times gives back the original value; i.e.,

$$(\underbrace{g \circ g \circ \cdots \circ g}_{n \text{ times}})(x) = x,$$

so for example, the function

$$g(x) = \frac{2x+29}{x-2}$$

is cyclic with order 2. Cyclic functions can often be used to solve functional equations.

**Problem 2.2.** Let $f(n)$ be a function defined on the set of all positive integers and having its values in the same set. Suppose that

$$f(f(m) + f(n)) = m + n$$

for all positive integers $m, n$. Find all possible values for $f(1988)$. (IMO Short List, 1988)

**Solution 1:** Whenever a function is iterated in a functional equation, such as $f(f(x))$, it can be useful to make a substitution that uses the iteration itself. This has the effect of turning the iteration "inside-out," and can produce a functional equation that is easier to work with.

For example, if we substitute $m = f(a) + f(b)$ and $n = f(c) + f(d)$ in the given functional equation, we find

$$f(f(f(a) + f(b)) + f(f(c) + f(d))) = f(a) + f(b) + f(c) + f(d)$$

for all positive integers $a$, $b$, $c$, and $d$. This is a useful substitution because then we can use the fact that $f(f(a) + f(b)) = a + b$ and $f(f(c) + f(d)) = c + d$ on the left-hand side, so this equation simplifies as

$$f(a + b + c + d) = f(a) + f(b) + f(c) + f(d).$$

This equation holds for all positive integers $a$, $b$, $c$, and $d$. If we replace $a$ with $a + 1$ and $b$ with $b - 1$ (where $b \geq 2$), then we get

$$f(a + b + c + d) = f(a + 1) + f(b - 1) + f(c) + f(d).$$

Hence,

$$f(a) + f(b) = f(a + 1) + f(b - 1),$$

or

$$f(a+1) - f(a) = f(b) - f(b-1).$$

This equations tells us that if we evaluate $f$ at any two consecutive positive integer values, then the difference is always the same. Therefore, $f(n)$ is the $n^{\text{th}}$ term in an arithmetic sequence, which means $f(n) = An + B$ for some integers $A$ and $B$. Substituting into the given functional equation, we get

$$A(Am + B + An + B) + B = m + n,$$

which simplifies as

$$A^2m + A^2n + 2AB + B = m + n.$$

For this equation to hold for all positive integers $m$ and $n$, $A$ and $B$ must satisfy $A^2 = 1$ and $2AB + B = 0$. The equation $2AB + B = 0$ factors as $B(2A + 1) = 0$. Since $A$ and $B$ are integers, $B = 0$. Also, $A^2 = 1$ implies $A = \pm 1$, but the function $f$ takes positive integers to positive integers, so $A = 1$.

Therefore, $f(n) = n$ for all positive integers $n$ (and it is clear that this function satisfies the given functional equation), and in particular, $f(1988) = 1988$. □

**Solution 2:** First, we show that $f$ is injective, which means that if $f(a) = f(b)$, then $a = b$. Let $a$ and $b$ be positive integers such that $f(a) = f(b)$. Taking $m = n = a$ in given functional equation, we get

$$f(2f(a)) = 2a.$$

Similarly, $f(2f(b)) = 2b$. But $f(a) = f(b)$, so $f(2f(a)) = f(2f(b))$, which means $2a = 2b$, or $a = b$. We have shown that $f(a) = f(b)$ implies $a = b$, so $f$ is injective.

Now, let $a$ and $b$ be positive integers, where $b \geq 2$. Taking $m = a$ and $n = b$ in the given functional equation, we get

$$f(f(a) + f(b)) = a + b.$$

Taking $m = a + 1$ and $n = b - 1$ in the given functional equation, we get

$$f(f(a+1) + f(b-1)) = a + b,$$

so

$$f(f(a) + f(b)) = f(f(a+1) + f(b-1)).$$

Since $f$ is injective, we conclude that

$$f(a) + f(b) = f(a+1) + f(b-1),$$

or

$$f(a+1) - f(a) = f(b) - f(b-1).$$

We can then proceed as in Solution 1. □

# Functional Equations

**Problem 2.3.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2-y^2)$$

for all $x, y \in \mathbb{R}$. (Singapore, 1999)

**Solution:** The appearances of the expressions $x+y$ and $x-y$ (as well as the factorization $x^2-y^2 = (x+y)(x-y)$) suggest the substitution $a = x+y$ and $b = x-y$. Then $x = (a+b)/2$, $y = (a-b)/2$, and the given functional equation becomes

$$bf(a) - af(b) = (a^2-b^2)ab.$$

This holds for all $a, b \in \mathbb{R}$.

Setting $b = 0$, we get $-af(0) = 0$ for all $a \in \mathbb{R}$. Taking $a = 1$ for example, it follows that $f(0) = 0$.

Expanding the right-hand side, we get

$$bf(a) - af(b) = a^3b - ab^3,$$

so

$$bf(a) - a^3b = af(b) - ab^3.$$

Assuming both $a$ and $b$ are nonzero, we can divide both sides by $ab$, to get

$$\frac{f(a)}{a} - a^2 = \frac{f(b)}{b} - b^2.$$

This equation is significant, because each side of the equation depends only on a single variable. This is an instance of a technique known as *separating variables*. Since we can keep $a$ fixed in this equation while varying $b$, the right side must stay constant as $b$ varies. Similarly, the left side must be constant as $a$ varies. Hence, for some constant $k$,

$$\frac{f(a)}{a} - a^2 = k$$

for all nonzero $a$, which means

$$f(a) = a^3 + ka.$$

Since $f(0) = 0$, this equation also holds for $a = 0$.

Therefore, every solution is of the form $f(x) = x^3 + kx$, where $k$ is a constant. However, we are not finished, because we do not know if every solution of this form works. For example, this solution may work only for certain values of $k$. The only way to find out is to check.

Substituting into the given functional equation, the left-hand side becomes

$$\begin{aligned}
&(x-y)f(x+y)-(x+y)f(x-y)\\
&=(x-y)[(x+y)^3+k(x+y)]-(x+y)[(x-y)^3+k(x-y)]\\
&=(x-y)(x+y)^3+k(x+y)(x-y)-(x+y)(x-y)^3-k(x+y)(x-y)\\
&=(x-y)(x+y)^3-(x+y)(x-y)^3\\
&=(x-y)(x+y)[(x+y)^2-(x-y)^2]\\
&=4xy(x^2-y^2),
\end{aligned}$$

which is the right-hand side. We conclude that the solutions are of the form $f(x)=x^3+kx$, where $k$ is any constant. □

We can assemble the key pieces of our solution into the formal argument below:

**Solution:** Letting $a=x+y$ and $b=x-y$, we have $x=(a+b)/2$, $y=(a-b)/2$, so our functional equation becomes

$$bf(a)-af(b)=(a^2-b^2)ab=a^3b-ab^3$$

for all $a,b\in\mathbb{R}$. Rearranging gives

$$bf(a)-a^3b=af(b)-ab^3.$$

Letting $a=0$ gives $bf(0)=0$ for all $b$, so $f(0)=0$. For nonzero $a$ and $b$, we can divide both sides by $ab$ to get

$$\frac{f(a)}{a}-a^2=\frac{f(b)}{b}-b^2.$$

Hence, for some constant $k$, we have

$$\frac{f(a)}{a}-a^2=k$$

for all nonzero $a$, so $f(a)=a^3+ka$. Since $f(0)=0$, this equation also holds for $a=0$.

Therefore, all solutions are of the form $f(a)=a^3+ka$. For functions of this form, we have

$$\begin{aligned}
(x-y)f(x+y)-(x+y)f(x-y)&=(x-y)[(x+y)^3+k(x+y)]-(x+y)[(x-y)^3+k(x-y)]\\
&=(x-y)(x+y)^3-(x+y)(x-y)^3\\
&=4xy(x^2-y^2),
\end{aligned}$$

so the solutions of the original functional equation are of the form $f(x)=x^3+kx$, where $k$ is any constant.

Note that we had a common goal in many of the substitutions we made in this section: simplify the inputs to functions. For example, choosing a specific value of $x$ is less intimidating than dealing with $f\left(\frac{2x+29}{x-2}\right)$, the expression $f(a+b+c+d)$ is more tractable than the nested functions of $f(f(m)+f(n))$ despite the extra variables, and $b(f(a)-af(b)$ is much nicer than $(x-y)f(x+y)-(x+y)f(x-y)$.

# Functional Equations

**Exercises**

2.1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = 1 - x - y$$

for all real numbers $x$, $y$. (Slovenia, 1999)

2.2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f(x) + f(1 - x) = 2x - x^4$$

for all $x \in \mathbb{R}$.

2.3. Let $F(x)$ be a real valued function defined for all real $x$ except for $x = 0$ and $x = 1$ and satisfying the functional equation

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$

Find all functions $F(x)$ satisfying these conditions. (Putnam, 1971)

## 3 Fixed Points

Given a function $f$, a *fixed point* of $f$ is a value $x_0$ such that $f(x_0) = x_0$. Considering fixed points is another useful tool for solving functional equations.

**Problem 3.1.** The function $f$ defined by

$$f(x) = \frac{ax + b}{cx + d},$$

where $a$, $b$, $c$, and $d$ are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$, and $f(f(x)) = x$ for all values except $-d/c$. Find the unique number that is not in the range of $f$. (AIME, 1997)

**Solution:** Let

$$y = f(x) = \frac{ax + b}{cx + d}.$$

Solving for $x$, we find

$$x = -\frac{dy - b}{cy - a}.$$

Hence, there exists an $x$ such that $f(x) = y$ for all $y$ except for $y = a/c$, so $a/c$ is the unique number that is not in the range of $f$. However, we are given that $f(f(x)) = x$ for all $x$ except $x = -d/c$, which means that every number is in the range of $f$ except $-d/c$. Therefore, $a/c = -d/c$, or $d = -a$.

We are also given that $f(19) = 19$ and $f(97) = 97$, i.e. $x = 19$ and $x = 97$ are fixed points of $f$, so $x = 19$ and $x = 97$ are solutions of the equation

$$\begin{aligned} \frac{ax+b}{cx+d} &= x \\ \Leftrightarrow \quad ax+b &= x(cx+d) \\ \Leftrightarrow \quad cx^2 - (a-d)x - b &= 0. \end{aligned}$$

By Vieta's Formulas, the sum of the roots of this equation is $(a-d)/c = 2a/c$. The sum of the roots is also $19 + 97 = 116$, so $a/c = 116/2 = 58$, which is the value we seek. □

**Problem 3.2.** Find all functions $f$ defined on the set of positive real numbers that take on positive real values and satisfy the conditions:

(i) $f(xf(y)) = yf(x)$ for all positive $x$, $y$,

(ii) $f(x) \to 0$ as $x \to \infty$.

(IMO, 1983)

**Solution:** Setting $x = y$ in (i), we find

$$f(xf(x)) = xf(x)$$

for all $x > 0$. Hence, $xf(x)$ is a fixed point of $f$ for all $x > 0$. In particular, we know that $f$ has at least one fixed point.

Setting $x = 1$ in (i), we find

$$f(f(y)) = yf(1)$$

for all $y > 0$. If $y$ is a fixed point of $f$, then $f(f(y)) = f(y) = y$, so $y = yf(1)$, which means $f(1) = 1$.

Let $S$ be the set of fixed points of $f$, so in particular, $1 \in S$. Let $x_1, x_2 \in S$, so $f(x_1) = x_1$ and $f(x_2) = x_2$. Then setting $x = x_1$ and $y = x_2$ in (i), we find

$$\begin{aligned} f(x_1 f(x_2)) &= x_2 f(x_1) \\ \Rightarrow \quad f(x_1 x_2) &= x_1 x_2, \end{aligned}$$

so $x_1 x_2 \in S$. Thus, the set $S$ is closed under multiplication. Also, setting $x = 1/x_1$ and $y = x_1$ in (i), we find

$$\begin{aligned} f\left(\frac{1}{x_1} \cdot f(x_1)\right) &= x_1 f\left(\frac{1}{x_1}\right) \\ \Rightarrow \quad f(1) &= x_1 f\left(\frac{1}{x_1}\right) \\ \Rightarrow \quad f\left(\frac{1}{x_1}\right) &= \frac{1}{x_1}. \end{aligned}$$

Thus, if $x_1 \in S$, then $1/x_1 \in S$.

Let $a \in S$. Then by a straightforward induction argument, $a^n \in S$ for all positive integers $n$, i.e. $f(a^n) = a^n$ for all positive integers $n$. If $a > 1$, then this observation violates (ii). If $a < 1$, then $1/a \in S$, and again by a straightforward induction argument, $1/a^n \in S$ for all positive integers $n$, i.e.

$$f\left(\frac{1}{a^n}\right) = \frac{1}{a^n}$$

for all positive integers $n$, which again violates (ii). Therefore, the only element in $S$ can be 1, i.e. $xf(x) = 1$ for all $x > 0$, so $f(x) = 1/x$ for all $x > 0$. We easily check that this function satisfies the given conditions. $\square$

### Exercises

3.1. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$x^2(f(x) + f(y)) = (x + y)f(f(x)y)$$

holds for any positive real numbers $x$ and $y$. (Slovenia, 2005)

3.2. Let $g : S \to S$ be a function such that $g$ has exactly two fixed points, and $g \circ g$ has exactly four fixed points. Prove that there is no function $f : S \to S$ such that $g = f \circ f$.

3.3. Let $S = \{0, 1, 2, \dots\}$. Find all functions defined on $S$ taking their values in $S$ such that

$$f(m + f(n)) = f(f(m)) + f(n)$$

for all $m$ and $n$ in $S$. (IMO, 1996)

## 4 Cauchy's Functional Equation

The function $f$ is *additive* if

$$f(x + y) = f(x) + f(y),$$

for all $x$ and $y$. This functional equation is also known as *Cauchy's functional equation*. As we will see, the solutions to this functional equation depend heavily on the domain that we specify.

We see that if $f$ is of the form $f(x) = cx$, where $c$ is a constant, then $f$ is additive. We start by showing that if the domain is the set of rational numbers, then all additive functions are of the form $f(x) = cx$.

# Functional Equations

**Theorem 4.1.** If $f : \mathbb{Q} \to \mathbb{R}$ satisfies

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{Q}$, then $f$ is of the form $f(x) = cx$ for some constant $c$.

**Proof**. Our overall strategy is to first prove the result for integers, and then build to the rational numbers. Setting $x = y = 0$ in the given functional equation, we get $f(0) = 2f(0)$, so $f(0) = 0$. By a straightforward induction argument, the additive property extends to any $n$ variables; that is

$$f(x_1 + x_2 + \cdots + x_n) = f(x_1) + f(x_2) + \cdots + f(x_n)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{Q}$. If we set all the $x_i$ to the same value, say $x$, then we find

$$f(nx) = nf(x) \qquad (*)$$

for any positive integer $n$ and any $x \in \mathbb{Q}$. Setting $x = 1$, we get

$$f(n) = nf(1).$$

Let $c = f(1)$, so $f(n) = cn$ for any positive integer $n$.

Now we need to find $f(n)$ where $n$ is a negative integer. Setting $x = n$ and $y = -n$ in the given functional equation, we find $f(n) + f(-n) = f(0) = 0$, so $f(n) = -f(-n)$. Since $-n$ is a positive integer, $f(-n) = c(-n) = -cn$, so $f(n) = -(-cn) = cn$. Hence, $f(n) = cn$ for all integers $n$.

Now, let $a/b$ be an arbitrary rational number, where $a$ and $b$ are integers. We can assume that $b$ is positive. Setting $n = b$ and $x = a/b$ in $(*)$, we find

$$f(a) = bf\left(\frac{a}{b}\right),$$

so

$$f\left(\frac{a}{b}\right) = \frac{f(a)}{b}.$$

Since $a$ is an integer, $f(a) = ca$, so

$$f\left(\frac{a}{b}\right) = \frac{f(a)}{b} = c \cdot \frac{a}{b}.$$

Therefore, $f(x) = cx$ for all $x \in \mathbb{Q}$. ■

So if we have an additive function whose domain is the set of rational numbers, then the solutions are straightforward. However, when we extend the domain to the set of real numbers, something unexpected happens. We seek all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. All functions $f$ of the form $f(x) = cx$, where $c$ is a constant, are still additive. But surprisingly, there is an entire class of functions not of this form that are also additive. These functions require very technical mathematics to describe; there is no simple, explicit formula for them. These functions also exhibit some extremely bizarre behavior. (It would be impossible, for example, to graph them in any meaningful way.)

However, we can use this to our advantage. If we impose certain conditions on $f$ that force it to behave in a more normal fashion, then we can prove that the only solutions are of the form $f(x) = cx$. We draw on two important results in our proof: First, we use the fact that additive functions on the set of rational numbers are linear. Second, we use the fact that the set of rational numbers $\mathbb{Q}$ is *dense* in the set of real numbers $\mathbb{R}$, which means that there is a rational number in any nontrivial interval. We use both these facts to extend a function from the rational numbers to all real numbers.

The arguments that we present are somewhat sophisticated, but they are quite commonly used in the branch of mathematics known as *analysis*, which deals with concepts such as limits and continuity.

**Theorem 4.2.** If $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$f(x + y) = f(x) + f(y)$$
for all $x, y \in \mathbb{R}$, as well as any of the following conditions, then $f$ is of the form $f(x) = cx$ for some constant $c$.

(a) $f$ is monotone.

(b) $f(x) \geq 0$ for all $x \geq 0$.

(c) There exists a real number $C$ and a non-trivial interval $[a, b]$ such that $|f(x)| \leq C$ for all $x \in [a, b]$.

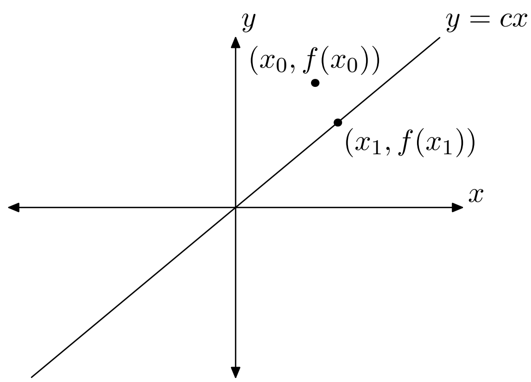
(d) There exists a real number $M$ such that $|f(x)| \leq M|x|$ for all $x \in \mathbb{R}$.

(e) $f$ is continuous.

**Proof**. By the previous theorem, we know that $f(x) = cx$ for all $x \in \mathbb{Q}$, where $c = f(1)$.

(a) If $c = 0$, then $f(x) = 0$ for all $x \in \mathbb{Q}$. Since $f$ is monotone, $f(x) = 0$ for all $x \in \mathbb{R}$. Otherwise, $c \neq 0$, so assume that $c > 0$. (The case $c < 0$ is similar.) Since $f$ is monotone, $f$ is increasing, so $f(x) \geq f(y)$ whenever $x > y$.

If we graph the function $y = f(x)$, then all points of the form $(x, f(x))$ where $x$ is rational will lie on the graph of $y = cx$. Hence, if there is any point $(x_0, f(x_0))$ that does not lie on the graph of $y = cx$, then $x_0$ is irrational. Looking at the graph, it is clear that any such point is going to break the fact that $f$ is increasing. We prove this formally.




So suppose there exists an irrational number $x_0$ such that $f(x_0) \neq cx_0$. If $f(x_0) > cx_0$, then the point $(x_0, f(x_0))$ will lie above the graph of $y = cx$, as shown. Since $f$ increasing, every point on the graph of $y = f(x)$ to the right of the point $(x_0, f(x_0))$ must also be above this point, so we can reach a contradiction by finding a point that is to the right and below this point.

Let $(x_1, f(x_1))$ be a point on the graph of $y = f(x)$. This point is to the right and below the point $(x_0, f(x_0))$ if $x_1 > x_0$ and $f(x_1) < f(x_0)$. We know that $f(x_1) = cx_1$ if $x_1$ is rational, so the inequality $f(x_1) < f(x_0)$ becomes $cx_1 < f(x_0)$, or $x_1 < \frac{f(x_0)}{c}$. Hence, it suffices to find a rational number $x_1$ such that

$$x_0 < x_1 < \frac{f(x_0)}{c}.$$

Since $x_0 < \frac{f(x_0)}{c}$, we can always find a rational number $x_1$ in this interval. Then $x_0 < x_1$ and $f(x_0) > f(x_1)$, which contradicts the fact that $f$ is increasing. The case $f(x_0) < cx_0$ is handled similarly, so $f(x_0) = cx_0$. Therefore, $f(x) = cx$ for all $x \in \mathbb{R}$.

(b) We claim that $f$ is increasing. Let $x > y$. Since $f$ is additive,

$$f(x - y) + f(y) = f((x - y) + y) = f(x),$$

so $f(x) - f(y) = f(x - y)$. But $x - y > 0$, so $f(x - y) \geq 0$, which means $f(x) - f(y) \geq 0$.

Therefore, $f(x) \geq f(y)$ for all $x > y$, which means that $f$ increasing, and the result follows from part (a).

(c) The given condition states that $f$ is bounded on the interval $[a, b]$. Let $g(x) = f(x) - cx$, where $c = f(1)$. Since $f(x) = cx$ for all $x \in \mathbb{Q}$, the function $g(x)$ has the property that $g(x) = 0$ for all $x \in \mathbb{Q}$. Furthermore, $f(x) = g(x) + cx$. Substituting into the given functional equation, we get

$$g(x + y) + c(x + y) = g(x) + cx + g(y) + cy,$$

so

$$g(x + y) = g(x) + g(y)$$

for all $x, y \in \mathbb{R}$. In other words, $g$ is additive.

First, we prove that $g$ is bounded on the interval $[a, b]$. In other words, we will show that there exists a constant $K$ such that $|g(x)| \le K$ for all $x \in [a, b]$.

We are given that $|f(x)| \le C$ for all $x \in [a, b]$, so $|g(x) + cx| \le C$ for all $x \in [a, b]$. We want to use this bound to put a bound on $|g(x)|$, so we use the Triangle Inequality:

$$|g(x)| = |(g(x) + cx) - cx| \le |g(x) + cx| + |cx|.$$

Then $|g(x) + cx| \le C$, and

$$|cx| = |c||x| \le |c| \max\{|a|, |b|\},$$

so

$$|g(x)| \le C + |c| \max\{|a|, |b|\}$$

for all $x \in [a, b]$. Hence, we can take $K = C + |c| \max\{|a|, |b|\}$, which is a constant.

We want to prove that $f(x) = cx$ for all $x \in \mathbb{R}$. Since $g(x) = f(x) - cx$, it suffices to prove that $g(x) = 0$ for all $x \in \mathbb{R}$. We already know that $g(x) = 0$ for all $x \in \mathbb{Q}$. We also know that $g$ is additive. This means if $x_0$ and $x_1$ are real numbers such that their difference $x_1 - x_0$ is a rational number, then

$$g(x_1 - x_0) + g(x_0) = g(x_1),$$

which simplifies to $g(x_0) = g(x_1)$.

But now this gives us a way to compare the value of $g$ at any point with the value of $g$ at a point in the interval $[a, b]$. More specifically, let $x_0$ be an irrational number. Then there exists a rational number $d$ such that $x_0 + d \in [a, b]$. If we set $x_1 = x_0 + d$, then by our observation above, $g(x_0) = g(x_1)$. Since $x_1 \in [a, b]$, $|g(x_1)| \le K$, so $|g(x_0)| \le K$. Therefore, $|g(x)| \le K$ for all irrational numbers $x$.

Now, since $x_0$ is an irrational number, $nx_0$ is an irrational number for all positive integers $n$, so $|g(nx_0)| \le K$. But $g$ is additive, so $g(nx_0) = ng(x_0)$. Hence, $n|g(x_0)| \le K$, or

$$|g(x_0)| \le \frac{K}{n}.$$

This inequality holds for all positive integers $n$. The only way this can happen is if $g(x_0) = 0$. Therefore, $g(x) = 0$ for all $x \in \mathbb{R}$, which means $f(x) = cx$ for all $x \in \mathbb{R}$.

(d) If $|f(x)| \le M|x|$ for all $x \in \mathbb{R}$, then $|f(x)| \le M$ for all $x \in [-1, 1]$, and the result follows from part (c).

(e) The function $f$ is continuous if for any $y \in \mathbb{R}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Taking $y = 0$ and $\epsilon = 1$, we find that there exists a $\delta > 0$ such that $|f(x)| < 1$ whenever $|x| < \delta$. In other words, $|f(x)| < 1$ for all $x \in [-\delta, \delta]$, and the result follows from part (c). ■

# Functional Equations

Several of the restrictions in Theorem 4.2 can be relaxed further. For example, notice how in our proof for (e) we only required the function to be continuous at a single point.

However, it is much more important to know and understand the ideas used in each of our five proofs above than to remember a laundry list of conditions. A good rule for working with Cauchy's functional equation is to never directly cite any facts about it in your write-ups. This will avoid the possibility of using something that is actually not true, which is a mistake that is often made by many olympiad participants. Any fact about Cauchy's functional equation worth using should be easy enough to prove yourself.

Cauchy's functional equation is important because many functional equations reduce to this equation.

**Problem 4.3.** (Jensen's Functional Equation) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$$

for all $x, y \in \mathbb{R}$.

**Solution:** It is easy to verify that all functions of the form $f(x) = ax + b$, where $a$ and $b$ are real constants, are solutions. We claim that these are the only solutions.

Let $b = f(0)$, and let $g(x) = f(x) - b$, so $f(x) = g(x) + b$. Then the given functional equation becomes

$$g\left(\frac{x+y}{2}\right) + b = \frac{g(x)+b+g(y)+b}{2},$$

or

$$g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2} \tag{$*$}$$

for all $x, y \in \mathbb{R}$. Also, $g(0) = f(0) - b = 0$.

Setting $y = 0$ in $(*)$, we find

$$g\left(\frac{x}{2}\right) = \frac{g(x)}{2}$$

for all $x \in \mathbb{R}$. Hence,

$$g\left(\frac{x+y}{2}\right) = \frac{g(x+y)}{2},$$

so equation $(*)$ becomes

$$\frac{g(x+y)}{2} = \frac{g(x)+g(y)}{2},$$

or

$$g(x+y) = g(x) + g(y)$$

for all $x, y \in \mathbb{R}$. The function $g$ is also continuous, so by Cauchy's functional equation, $g(x) = ax$ for all $x \in \mathbb{R}$, where $a = g(1)$. Then $f(x) = ax + b$ for all $x \in \mathbb{R}$, as claimed. $\square$

**Problem 4.4.** The continuous function $f : \mathbb{R} \to (0, \infty)$ satisfies

$$f(x+y) = f(x)f(y)$$

for all $x, y \in \mathbb{R}$. Show that $f(x) = [f(1)]^x$ for all $x \in \mathbb{R}$.

**Solution:** Since $f(x) > 0$ for all $x \in \mathbb{R}$, we can set $g(x) = \log f(x)$, so $f(x) = e^{g(x)}$. Then the given functional equation becomes

$$e^{g(x+y)} = e^{g(x)} \cdot e^{g(y)},$$

or $g(x+y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$. The function $g$ is also continuous, so by Cauchy's functional equation, $g(x) = g(1)x$ for all $x \in \mathbb{R}$. Then

$$f(x) = e^{g(x)} = e^{g(1)x} = [e^{g(1)}]^x = [f(1)]^x$$

for all $x \in \mathbb{R}$. □

### Exercises

4.1. A function $f$ is *multiplicative* if $f(xy) = f(x)f(y)$ for all $x$ and $y$. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that are both additive and multiplicative.

4.2. Find all continuous functions $f : (0, \infty) \to (0, \infty)$ such that

$$f(xy) = f(x)f(y)$$

for all $x, y > 0$.

4.3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y) + xy$$

for all $x, y \in \mathbb{R}$.

## 5 Tips for solving Functional Equations

- Check if there are any "trivial" solutions, such as constant functions or linear functions. It is good to get them "out of the way," and they may help guide you to other solutions, as well as preventing you from making assumptions about the solutions.

# Functional Equations

- At the same time, do not assume that the only functions you can think of are the only solutions. If your argument is not working, it may be because you have not found all the solutions.
- Seek substitutions that simplify the inputs to functions.
- Ask yourself if the function is increasing or decreasing.
- Ask yourself if the function is injective or surjective. Even if the function is not surjective, it can help to determine its range.
- Check that your solutions satisfy the given functional equation. This is not just a formality; just as solving an equation can lead to extraneous solutions, so can "solving" a functional equation. You must also show that your solutions work.
- If you derive that $f(x)^2 = 1$ for all $x$, it does not follows that $f(x) = 1$ for all $x$ or $f(x) = -1$ for all $x$. You can only conclude that $f(x) = 1$ or $f(x) = -1$ for each individual value of $x$.
- If a function $f$ satisfies $f(f(x)) = x$ for all $x$, then it is bijective.
- If a function is strictly increasing or strictly decreasing, then it is injective.
- Pay attention to the domain of the function. For example, if the domain of the function is the set of positive real numbers, then you might not be able to set any of variables in the functional equation to 0.