HINTS: Arranging things... optimally!

David Arthur
darthur@gmail.com

A1. The answer is four. To prove it, consider only the four corner 1×1 squares. (Source: Bay Area Mathematical Olympiad 2009, #1.)

A2. Let A be the set of settlements reachable from one particular settlement, and let B be the rest. What is the minimum number of tunnels that can be missing between A and B? (Source: Bay Area Mathematical Olympiad 2004, #3.)

A3. If a colour is used at all, it must be used for at least three squares. Do you see why? (Source: Tournament of the Towns Senior O-Level, Spring 1998, #3.)

A4. The answer is 10^5. To get this many, you should use a "checksum". Start with any five digits, and then choose the sixth digit to be equal to the sum of the first five mod 10. Checksums are very important in computer science. (Source: USAMO 1990, #1.)

A5. Partition the chessboard into 32 pairs of squares, such that the squares in each pair attack each other. Then, you can put at most one knight in each pair. (Source: Tournament of the Towns, 1996-2000 sometime.)

A6. For part a, label the points from left to right as $v_1, v_2, \ldots, v_{997}$. Look at segments $\overline{v_1v_2}$, $\overline{v_1v_3}$, and $\overline{v_2v_3}$. What can you say about their midpoints?

For part b, reduce it to part a! Choose coordinate axes so that no two points have the same x coordinate. (Do you see why this is possible?) Now show that the midpoints have 1991 different x-coordinates. (Source: Part b is Asian-Pacific Math Olympiad 1991, #2)

A7. The answer is 98. To prove you can't do better, it suffices to look only at right-angled triangles with sides parallel to the sides of the board. One option is induction. Another option is to count how many rows and columns have exactly one center marked in them. (Source: Argentina Team Selection Test 2009, #1. See also USAMO 2000, #4.)

A8. Divide the triangle up into nine cells similar to how we did for Example 1. If there are nine points, then each cell must have a point. Focus on just the six central triangles and prove that is impossible.

A9. The answer is n. To find a colouring using only n colours, start by colouring the edge between vertex i and vertex j with colour $i + j$ modulo n.

So how would you think of this? Well, the colour of the edge between vertices i and j is some function of i and j modulo n that is symmetric between i and j. Addition is just the simplest option. (Source: Italy Team Selection Test 2007, #4.)

B1. Prove that every 1×5 rectangle has at least three marked squares. (Source: IberoAmerican Mathematical Olympiad 2004, #1.)

B2. The key idea for both problems is the same as the key idea for Example 2. For part b, associate each S_k with an integer between 1006 and 2010 inclusive. (Source: Iran Team Selection Test 2007, #2 for part a, and Balkan Mathematical Olympiad 2005, #4 for part b.)

B3. Suppose there are n teams and Kazakhstan has w wins and t ties. Argue that

$$3w + t > \left\lceil \frac{2 \cdot \binom{n-1}{2} + (n-1) \cdot (w+1) + t}{n-1} \right\rceil$$

and hence that $w \geq 2$. (Source: American Math Olympiad Program 1998. Original source unknown.)

B4. Let $n = 2t + 1$. Consider the $(t+1)^2$ squares in an odd-index row and an odd-index column. Each of these squares must be covered by a tromino, and no tromino can cover two of these squares. Therefore, at least $(t+1)^2$ trominos are needed. This lower bound also proves almost immediately that no tiling exists for $n < 7$; do you see why? (Source: IMO Shortlist 2002, C2)

B5. The answer is $n = 4$, achieved by a "cross" with width and height equal to 51. (Source: Russia 1998.)

B6. Suppose 39 balloons have pressure 0, and the final balloon has pressure 1. Then any pressure you get from here will be rational. What kind of denominators are possible? (Source: Russia 1999 Grade 8, #4.)

B7. Use induction to show how to delete most of the edges. For the other direction, prove the graph will always be connected and will always have an odd-length cycle. (Source: IMO Shortlist 2004, C3.)

B8. The answer is 24. Consider all columns containing a one. Altogether, suppose they contain 9 ones, b twos, c threes, and d fours. Then some column has sum at most

$$9 \cdot \frac{1 \cdot 9 + 2b + 3c + 4d}{9 + b + c + d} \leq 24.$$

When bounding the above fraction, it helps a lot to know the upper bound is 24 when you start, because then you can multiply it out right off the bat. (Source: Bosnia Herzegovina Regional Olympiad 2008 Grade 3, #4. See also All-Russian Olympiad 2004 Grade 11, #4.)

B9. For part a, the key configuration is three lottery tickets with numbers chosen from $\{1, 2, \ldots, 15\}$. For part b, prove that two tickets share a number, and then do it again. (Source: Tournament of the Towns Senior A-Level, Fall 1996, #6.)

B10. Interpret the problem as a graph. Suppose some vertex u has degree d. Show there are exactly $\binom{d}{2}$ vertices at distance 2 from u, and no vertices at distance greater than 2 from u. Then prove $d = 3$ and $d = 4$ are impossible. You can find a 16-vertex construction by looking at the $d = 5$ case. There is also a very natural construction that interprets each vertex as a 4-digit binary number. (Source: Balkan Math Olympiad 1994, #4.)

B11. The answers are 7 and 6. If you choose t cells, then each corner triangle with side length k must have at least $t-(n-k)$ cells marked in it. To actually find the optimal marking, remember to not search randomly. Use symmetry, and use patterns. (Source: Tournament of the Towns Senior A-Level, Fall 1997, #6.)

C1. For part a, bound separately the number of wins by the bottom half of participants and the top half of participants. For part b, start with n participants where each participant beats the $\frac{n}{2}$ participants that come after it in order (mod n). The problem is these participants all end up with the same score. Look for a simple way to differentiate some (but not all) of the scores. (Source: Tournament of the Towns Senior A-Level, Spring 2000, #6.)

C2. Consider only the red points first. Let r_1 be the number of points on the convex hull (of the red points), and let r_2 be the number of points in the convex hull. Prove there are at least $r_1 + 2r_2 - 4$ yellow points inside the convex hull. What does that tell you?

The construction is tricky but relies heavily on symmetry: you can get from the red points to the yellow points by rotating 60 degrees. There's a picture of the construction on Mathlinks. (Source: Japan Mathematical Olympiad Finals 2003, #5.)

C3. For the inequality, consider one vertex, and prove it can have at most $n-2$ edges out with each colour. For the equality cases, consider $n-1$ prime, and identify each vertex with a point on an $(n-1)\times(n-1)$ lattice. Does this suggest anything to you? (Source: Middle European Mathematical Olympiad 2009, #2.)

C4. Let $\ell_{v,k}$ denote the length of the longest increasing walk that ends at vertex v and that uses only some or all of the k smallest edges on the graph. Prove that $\sum_v \ell_{v,k} \geq 2k$ for all k. For part b, you need all $\ell_{v,k}$ to be equal at the end. How can you guarantee that?

C5. The answer is 89. If there is not a sequence of length 89 that has every colour, note that every sequence of length 11 has a colour that does not appear at all in the subsequent sequence of length 89. Use this argument repeatedly to either built a sequence of length 89 with every colour, or to prove a certain point is part of a sequence of 11 consecutive points that are all the same colour. Can this last case happen everywhere?

Next, you need to find a colouring where no sequence of length 88 has every colour. As always, follow a pattern. In this case, start with a sequence of 0's and 1's of length 12, and then shift it and repeat it. (Source: Iberoamerican Olympiad 2009, #6.)

C6. Suppose two vertices u and v are connected by an edge. Imagine making u and v have identical neighbours by either replacing v's neighbours with u's neighbours, or vice-versa. Prove that one of these can always be done without decreasing the number of weak quartets, and therefore, you can assume without loss of generality that the graph can be decomposed into disjoint complete graphs. Now, think of it as a straight inequality problem. (Source: IMO Shortlist 2002, C7.)