Inversion

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DGW-INVERT

§1 Reading

Chapter 8 from my geometry textbook EGMO. This is a sample chapter, so you can find it online on my site, but shouldn't you have a copy of my book by now?

§2 Lecture Notes

Problem 2.1. Let $ABCD$ be a bicentric quadrilateral with incenter I and circumcenter O. Prove that line IO passes through $\overline{AC} \cap \overline{BD}$.

Problem 2.2 (BAMO 2008/5 and 2011/4)**.** A point D lies inside triangle ABC. Let A_1, B_1, C_1 be the second intersection points of the lines AD, BD, and CD with the circumcircles of BDC, CDA, and ADB, respectively. Prove that

$$\frac{AD}{AA_1} + \frac{BD}{BB_1} + \frac{CD}{CC_1} = 1.$$

Problem 2.3. Triangle ABC has incenter I and circumcenter O. The incircle of ABC touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at points D, E, F. Show that the orthocenter of $\triangle DEF$ lies on line IO.

§3 Practice problems

Problem 3.1 (Russia 1995 et al)**.** Quadrilateral $ACDB$ is inscribed in a semicircle with diameter AB and point O is the midpoint of AB. Let K be the intersection of the circumcircles of AOC and BOD. Lines AB and CD intersect at M. Prove that $\angle OKM = 90^\circ$.

Problem 3.2 (Brazil 2009/5)**.** Let ABC be a triangle and O its circumcenter. Lines AB and AC meet the circumcircle of OBC again in $B_1 \neq B$ and $C_1 \neq C$, respectively, lines BA and BC meet the circumcircle of OAC again in $A_2 \neq A$ and $C_2 \neq C$, respectively, and lines CA and CB meet the circumcircle of OAB in $A_3 \neq A$ and $B_3 \neq B$, respectively. Prove that lines A_2A_3, B_1B_3 and C_1C_2 have a common point.

Problem 3.3 (Mixtilinear incircles)**.** Let ABC be a triangle and let T be the contact point of the A-mixtilinear incircle.

(a) (IMO 1978) Point I lies on the contact chord of the A-mixtilinear incircle.

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(b) (EGMO 2013/5) Line AT is isogonal to the A-Nagel cevian.

(c) (Iran MO 2002) Line TI passes through the midpoint of arc BAC.

Problem 3.4 (Russia 2009)**.** In triangle ABC with circumcircle Ω, the internal angle bisector of $\angle A$ intersects $\overline{BC}$ at D and Ω again at E. The circle with diameter $\overline{DE}$ meets Ω again at F. Prove that $\overline{AF}$ is a symmedian of triangle ABC.

Problem 3.5 (NIMO Winter, Aaron Lin)**.** Let ABC be a triangle and let Q be a point such that $\overline{AB} \perp \overline{QB}$ and $\overline{AC} \perp \overline{QC}$. A circle with center I is inscribed in $\triangle ABC$, and is tangent to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ at points D, E, and F, respectively. If ray QI intersects $\overline{EF}$ at P, prove that $\overline{DP} \perp \overline{EF}$.

Problem 3.6 (Cosmin Poahatza)**.** Let ABC be a triangle with circumcircle Γ and let M be an arbitrary point on Γ. Suppose the tangents from M to the incircle of ABC intersect $\overline{BC}$ at two distinct points X_1 and X_2. Prove that the circumcircle of triangle MX_1X_2 passes through the tangency point of the A-mixtilinear incircle with Γ.