Art of Problem Solving

WOOT 2010–11

Practice Olympiad 2 Solutions

1. A set M containing four positive integers is called *connected* if for every element x in M, at least one of the numbers $x-1$, $x+1$ also belongs to M. For $n \geq 4$, let U_n denote the number of connected subsets of the set $\{1, 2, \ldots, n\}$.

   (a) Evaluate U_7.

   (b) Determine the least positive integer n for which $U_n \geq 2010$.

   Solution. Let M be a connected subset of $\{1, 2, \ldots, n+1\}$. Then either $n+1$ is a member of M or $n+1$ is not a member of M.

   If $n+1$ is not a member of M, then M is a connected subset of $\{1, 2, \ldots, n\}$, and there are U_n such subsets.

   If $n+1$ is a member of M, then n must also be a member of M. The other two members of M must be k and $k+1$, for some positive integer k where $1 \leq k \leq n-2$. Hence, there are $n-2$ connected subsets M where $n+1$ is a member of M.

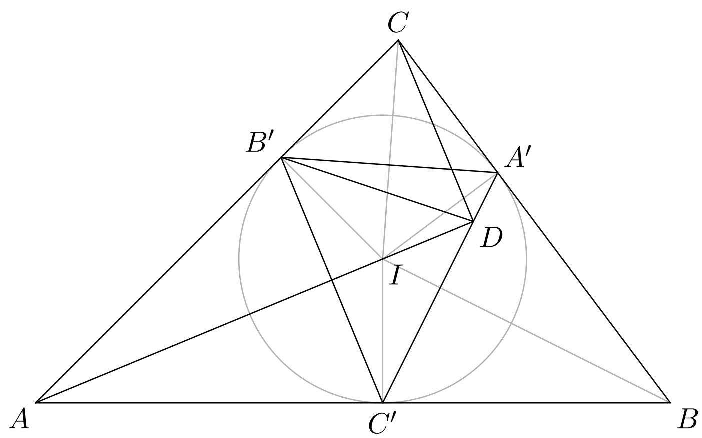
   Therefore,
$$U_{n+1} = U_n + n - 2$$
   for all $n \geq 4$.

   We see that $U_4 = 1$, and so
$$\begin{aligned} U_n &= (n-3) + U_{n-1} \\ &= (n-3) + (n-4) + U_{n-2} \\ &= \cdots \\ &= (n-3) + (n-4) + \cdots + 2 + U_4 \\ &= (n-3) + (n-4) + \cdots + 2 + 1 \\ &= \frac{(n-3)(n-2)}{2} \end{aligned}$$
   for all $n \geq 4$.

   In particular, for $n = 7$,
$$U_7 = \frac{4 \cdot 5}{2} = 10.$$
   Also,
$$U_{65} = \frac{62 \cdot 63}{2} = 1953 \quad \text{and} \quad U_{66} = \frac{63 \cdot 64}{2} = 2016,$$
   so the least positive integer n for which $U_n \geq 2010$ is $n = 66$.

2. The incircle of triangle ABC touches sides BC, AC, and AB at A', B', and C', respectively. The line $A'C'$ meets the angle bisector of $\angle A$ at D. Prove that $\angle ADC = 90°$.

   Solution. Let $\alpha = \angle A/2$, $\beta = \angle B/2$, and $\gamma = C/2$. Then $\alpha+\beta+\gamma = (A+B+C)/2 = 180°/2 = 90°$.



Let I be the incenter of triangle ABC. We claim that quadrilateral $CIDA'$ is cyclic. To prove this claim, we prove that $\angle CA'D + \angle CID = 180°$.

We see that triangle $IA'C'$ is isosceles with $IA' = IC'$. We also see that triangle $BC'I$ is a right triangle with $\angle BC'I = 90°$, so $\angle BIC' = 90° - \angle IBC' = 90° - \beta$. Similarly, $\angle BIA' = 90° - \beta$, so $\angle A'IC' = 2(90° - \beta) = 180° - 2\beta$. Then

$$\angle IA'C' = \frac{180° - \angle A'IC'}{2} = \frac{180° - (180° - 2\beta)}{2} = \beta.$$

Since $\angle CID$ is external to triangle AIC,

$$\angle CID = \angle IAC + \angle ICA = \alpha + \gamma.$$

Then

$$\begin{aligned}\angle CA'D + \angle CID &= \angle IA'D + \angle IA'C + \angle CID \\ &= \beta + 90° + \alpha + \gamma \\ &= 180°,\end{aligned}$$

so quadrilateral $CIDA'$ is cyclic.

Hence,

$$\angle ADC = \angle IDC = \angle IA'C = 90°.$$

3. The sequence x_n of integers is defined by $x_1 = 1$, $x_2 = a$, and

$$x_n = (2n+1)x_{n-1} - (n^2-1)x_{n-2}$$

for $n \geq 3$, where a is a positive integer. For which values of a does this sequence have the property that x_i divides x_j whenever $i \leq j$?

Solution. For $n = 3$,

$$x_3 = 7x_2 - 8x_1 = 7a - 8.$$

Since x_2 divides x_3,

$$\frac{x_3}{x_2} = \frac{7a-8}{a} = 7 - \frac{8}{a}$$

is an integer, which means a is a factor of 8. Therefore, a must be 1, 2, 4, or 8.

Case 1: $a = 1$.

If $a = 1$, then

$$\begin{aligned} x_3 &= 7x_2 - 8x_1 = -1, \\ x_4 &= 9x_3 - 15x_2 = -24, \\ x_5 &= 11x_4 - 24x_3 = -240, \\ x_6 &= 13x_5 - 35x_4 = -2280. \end{aligned}$$

However, -240 does not divide -2280, so a cannot be 1.

Case 2: $a = 2$.

We claim that if $a = 2$, then $x_n = n!$ for all $n \geq 1$. We prove this using strong induction.

Since $x_1 = 1$ and $x_2 = 2$, the claim is true for $n = 1$ and $n = 2$. Assume that $x_k = k!$ for all $k < n$, for some positive integer $n \geq 3$. Then

$$\begin{aligned} x_n &= (2n+1)x_{n-1} - (n^2-1)x_{n-2} \\ &= (2n+1)(n-1)! - (n^2-1)(n-2)! \\ &= n(n-1)! + (n+1)(n-1)! - (n-1)(n+1)(n-2)! \\ &= n! + (n+1)(n-1)! - (n+1)(n-1)! \\ &= n!. \end{aligned}$$

Therefore, the claim is true for $k = n$, and by strong induction, for all positive integers k. Then clearly x_i divides x_j for all $i \leq j$.

Case 3: $a = 4$.

We claim that if $a = 4$, then $x_n = (n+2)!/6$ for all $n \geq 1$. We prove this again using strong induction.

Since $x_1 = 1 = 3!/6$ and $x_2 = 4 = 4!/6$, the claim is true for $n = 1$ and $n = 2$. Assume that $x_k = (k+2)!/6$ for all $k < n$, for some positive integer $n \geq 3$. Then

$$\begin{aligned} x_n &= (2n+1)x_{n-1} - (n^2-1)x_{n-2} \\ &= \frac{1}{6}(2n+1)(n+1)! - \frac{1}{6}(n^2-1)n! \\ &= \frac{1}{6}[(2n+1)!(n+1)! - (n-1)(n+1)n!] \\ &= \frac{1}{6}[(2n+1) - (n-1)](n+1)! \\ &= \frac{1}{6}(n+2)(n+1)! \\ &= \frac{1}{6}(n+2)!. \end{aligned}$$

Therefore, the claim is true for $k = n$, and by strong induction, for all positive integers k. Then clearly x_i divides x_j for all $i \leq j$.

Case 4: $a = 8$.

If $a = 8$, then $x_3 = 48$ and $x_4 = 312$. However, 48 does not divide 312, so a cannot be 8.

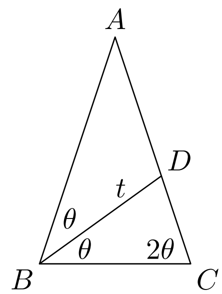
Therefore, x_i divides x_j for all $i \leq j$ if and only if $a = 2$ or $a = 4$.

4. Let ABC be a triangle with $AB = AC$. Let t be the length of the interior angle bisector from B. Find the greatest real number m and the least real number M such that

$$m < \frac{BC}{t} < M$$

for all such triangles ABC.

Solution. Let the angle bisector of $\angle B$ intersect AC at D, so $t = BD$. Let $\theta = \angle ABD = \angle CBD$, so $\angle C = 2\theta$. Then $\angle BDC = 180^\circ - 3\theta$ and $\angle A = 180^\circ - 4\theta$. Therefore, the angle θ must satisfy $0 < \theta < 45^\circ$. Conversely, θ can achieve any value within this range.



By sine law on triangle BCD,

$$\frac{BC}{t} = \frac{\sin \angle BDC}{\sin \angle BCD} = \frac{\sin(180^\circ - 3\theta)}{\sin 2\theta} = \frac{\sin 3\theta}{\sin 2\theta}.$$

By the triple angle formula for sine, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, and by the double angle formula for sine, $\sin 2\theta = 2\sin\theta\cos\theta$, so

$$\frac{BC}{t} = \frac{3\sin\theta - 4\sin^3\theta}{2\sin\theta\cos\theta} = \frac{3 - 4\sin^2\theta}{2\cos\theta} = \frac{4(1-\sin^2\theta) - 1}{2\cos\theta} = \frac{4\cos^2\theta - 1}{2\cos\theta} = 2\cos\theta - \frac{1}{2\cos\theta}.$$

On the interval $0 < \theta < 45^\circ$, $\cos\theta$ is positive and decreasing, so $-\frac{1}{2\cos\theta}$ is decreasing, which means

$$2\cos\theta - \frac{1}{2\cos\theta}$$

is decreasing. It is also continuous.

Therefore, the best bounds on BC/t are given by

$$2\cos 45^\circ - \frac{1}{2\cos 45^\circ} < \frac{BC}{t} < 2\cos 0^\circ - \frac{1}{2\cos 0^\circ},$$

or

$$\frac{\sqrt{2}}{2} < \frac{BC}{t} < \frac{3}{2}.$$

5. We say that a positive integer N is *delightful* if, when one of its digits is deleted (when written in decimal), the result is equal to $N/9$, and $N/9$ is divisible by 9. For example, the number 70875 is delightful, because if we delete the digit 0, we get 7875, which is equal to $70875/9$, and 7875 is divisible by 9.

   (a) Prove that if the positive integer N is delightful, then one of the digits of $N/9$ may be deleted, resulting in a number that is equal to $N/81$.

   (b) Find the smallest delightful positive integer.

Solution. (a) Let N be a delightful positive integer, so when one of its digits is deleted, the result is equal to $N/9$. Let

$$N = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0$$

in decimal notation, where $a_k \neq 0$, and let a_i be the deleted digit, where $0 \leq i \leq k$. Then we can write

$$\begin{aligned} N &= 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10^{i+1} a_{i+1} + 10^i a_i + 10^{i-1} a_{i-1} + \cdots + 10a_1 + a_0 \\ &= 10^{i+1} A + 10^i a_i + B, \end{aligned}$$

where $A = 10^{k-i-1} a_k + 10^{k-i-2} a_{k-1} + \cdots + a_{i+1}$ and $B = 10^{i-1} a_{i-1} + \cdots + 10a_1 + a_0$.

Deleting digit a_i in N results in $N/9$, so

$$\frac{N}{9} = 10^i A + B.$$

Since $N/9$ is divisible by 9,

$$\frac{N}{9} = 10^i A + B \equiv A + B \equiv 0 \pmod{9}.$$

But N itself is divisible by 9, so

$$N = 10^{i+1} A + 10^i a_i + B \equiv A + a_i + B \equiv 0 \pmod{9}.$$

Therefore, $a_i \equiv 0 \pmod{9}$. Since a_i is a digit, a_i is 0 or 9.

We have that

$$\frac{N}{9} = \frac{10^{i+1} A + 10^i a_i + B}{9} = 10^i A + B,$$

so

$$10^{i+1} A + 10^i a_i + B = 9 \cdot 10^i A + 9B.$$

This simplifies to

$$10^i A + 10^i a_i = 8B.$$

We see that $B = 10^{i-1} a_{i-1} + \cdots + 10a_1 + a_0 < 10^i$, so

$$10^i A + 10^i a_i = 8B < 8 \cdot 10^i,$$

which implies $A + a_i < 8$. Hence, the deleted digit a_i must be 0, and A must be a one-digit number, i.e. $A = a_{i+1}$ and $k = i + 1$.

Then $10^i a_{i+1} = 8B$, and

$$\frac{N}{9} = 10^i a_{i+1} + B = 9B,$$

so

$$\frac{N}{81} = B.$$

Thus, deleting the digit a_{i+1} in $N/9$ results in $B = N/81$.

(b) A delightful positive integer N must contain at least three digits, so we first look for a positive integer of the form $N = 100A + B$, where A and B are digits. Furthermore, by part (a), N must be a multiple of 81.

The smallest multiple of 81 of the form $100A + B$ is 405. The positive integer 405 is delightful, so it is the smallest delightful positive integer.

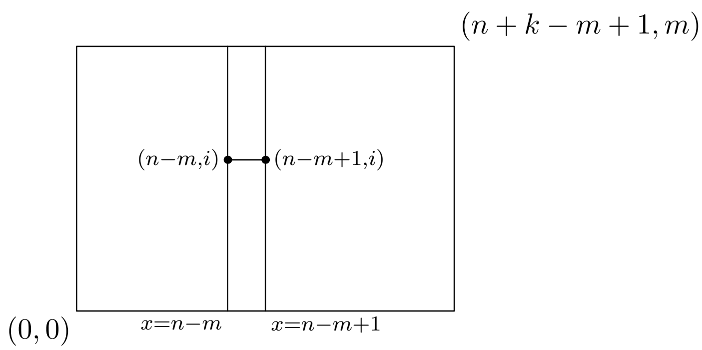
6. Let m, n, and k be positive integers such that $n \geq m \geq 2$. Prove that

$$\binom{n}{m}\binom{k}{0} + \binom{n-1}{m-1}\binom{k+1}{1} + \binom{n-2}{m-2}\binom{k+2}{2} + \dots + \binom{n-m}{0}\binom{k+m}{m} = \binom{n+k+1}{m}.$$

Solution. Consider a path in the coordinate plane from $(0,0)$ to $(n+k-m+1, m)$, consisting of steps to the right and upwards. There are always $n+k-m+1$ steps to the right and m steps upwards, and we can order these $(n+k-m+1)+m = n+k+1$ steps in any way, so there are a total of

$$\binom{n+k+1}{m}$$

such paths.



At some point, the path must take a step right from the line $x = n-m$ to the line $x = n-m+1$. Suppose that the path goes from $(n-m, i)$ to $(n-m+1, i)$, where $0 \leq i \leq m$. There are $\binom{n-m+i}{i}$ paths from $(0,0)$ to $(n-m, i)$, and $\binom{k+m-i}{m-i}$ paths from $(n-m+1, i)$ to $(n+k-m+1, m)$. Summing over i, we find that the total number of paths from $(0,0)$ to $(n+k-m+1, m)$ is

$$\begin{aligned}&\sum_{i=0}^{m}\binom{n-m+i}{i}\binom{k+m-i}{m-i}\\ &= \binom{n-m}{0}\binom{k+m}{m} + \binom{n-m+1}{1}\binom{k+m-1}{m-1} + \dots + \binom{n}{m}\binom{k}{0}.\end{aligned}$$

Hence, the two expressions are equal.

7. There exist 5 points in the plane, such that the area of any triangle formed by three of these points is at least 2. Prove that one of these triangles has area at least 3.

Solution. Let the five points be A, B, C, D, and E.

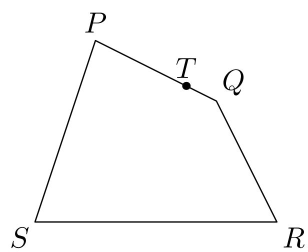
First, we take the case where the five points do not form a convex pentagon. In this case, one of the points lies inside the convex hull of the five points, which means it lies inside the triangle formed by three of the other points. Without loss of generality, assume that D lies inside triangle ABC. Then

$$[ABC] = [DBC] + [ADC] + [ABD] \geq 2 + 2 + 2 = 6.$$

Now, assume that $ABCDE$ is a convex pentagon. We prove the following lemma.

Lemma. Let $PQRS$ be a convex quadrilateral and T a point on side PQ. Then

$$[TSR] \geq \min([PSR], [QSR]).$$

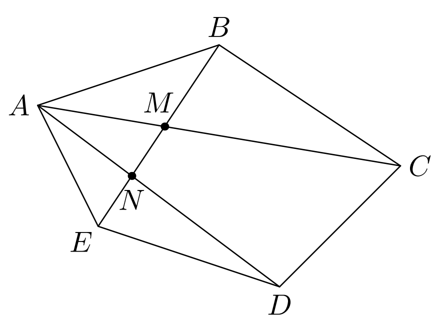


Proof. For a point X and a line ℓ, let $d(X, \ell)$ denote the distance from X to ℓ. Since T lies on PQ, either $d(T, SR) \geq d(P, SR)$ or $d(T, SR) \geq d(Q, SR)$, so

$$d(T, SR) \geq \min(d(P, SR), d(Q, SR)).$$

Then $[TSR] = \frac{1}{2} SR \cdot d(T, SR)$, etc., and the result follows. ■

Let M and N be the intersections of BE with AC and AD, respectively.



Suppose that $BM \geq BE/3$, which means $BM \geq ME/2$. Then

$$[BDE] = [BDM] + [MDE] \geq \frac{1}{2}[MDE] + [MDE] = \frac{3}{2}[MDE].$$

By the lemma,

$$[MDE] \geq \min([CDE], [ADE]),$$

so

$$[BDE] \geq \frac{3}{2}\min([CDE], [ADE]) \geq 3.$$

The case when $NE \geq BE/3$ is similar, so the only case remaining is $MN \geq BE/3$. Then

$$[AMN] \geq \frac{1}{3}[ABE] \geq \frac{2}{3},$$
$$[MND] \geq \frac{1}{3}[BED] \geq \frac{2}{3}.$$

By the lemma,

$$[MCD] \geq \min([BCD], [ECD]) \geq 2.$$

Therefore,

$$[ACD] = [AMN] + [MND] + [MCD] \geq \frac{2}{3} + \frac{2}{3} + 2 = \frac{10}{3} > 3.$$