THURSDAY PROBLEM 1999 Winter Camp

1. Find a positive integer, the first digit of which is 1 and which has the property that, if this digit is transferred to the end of the number, the number is tripled.

2. Show that the number x is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence
$$x,\ x+1,\ x+2,\ x+3,\ \ldots$$

3. Find all integers a, b and c for which
$$(x-a)(x-10)+1=(x+b)(x+c)$$
for all x.

4. A number of schools took part in a tennis tournament. No two players from the same school played against each other. Every two players from different schools played exactly one match against each other. A match between two boys or between two girls was called a single and that between a boy and a girl was called a mixed single. The total number of boys differed from the total number of girls by at most 1. The total number of singles differed from the total number of mixed singles by at most 1. At most how many schools were represented by an odd number of players?

5. Let $y_1, y_2, y_3, \ldots$ be a sequence such that $y_1=1$ and, for $k>0$, is defined by the relationship
$$y_{2k}=\begin{cases}2y_k & \text{if } k \text{ is even}\\ 2y_k+1 & \text{if } k \text{ is odd}\end{cases}$$
$$y_{2k+1}=\begin{cases}2y_k & \text{if } k \text{ is odd}\\ 2y_k+1 & \text{if } k \text{ is even}\end{cases}$$
Show that [illegible]

T. P.

6. Let f be a function mapping positive integers into positive integers. Suppose that

$$f(n+1) > f(n) \text{ and } f(f(n)) = 3n$$

for all positive integers.

Determine $f(1999)$.

7. Does there exist a number such that when written to base k contains each of the digits $0, 1, 2, \ldots, k-1$ at least once for each $k = 2, 3, 4, 5, \ldots, 1999$?

8. Find four positive integers the product of which is divisible by the sum of every pair of them. Can you find a set of five or more numbers with the same property?

9. Find a pair of integers r, s such that $0 < s < 200$ and

$$\frac{45}{61} < \frac{r}{s} < \frac{59}{80}.$$

Prove that there is exactly one such pair (r, s).

10. Prove that if x and y are rational number satisfying the equation

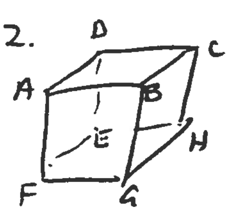
$$x^5 + y^5 = 2x^2y^2$$

then $1 - xy$ is the square of a rational number.

11. Find, showing your method, a six-digit integer n with the following properties:

(a) the number formed by the last three digits of n is exactly one greater than the number formed by the first three digits of n (so it might look like 123124);

(b) n is a perfect square.

12. 

ABCDEFG is a cube of side 2.
M is the midpoint of BC; N is the midpoint of EF
P is the midpoint of AB; Q is the midpoint of HE
AM meets CP at X; HN meets FQ at Y.

(a) Find the area of quadrilateral AMHN.

(b) Find the length of XY.