2010 IMO Summer IMO Training: Functional Equations

Adrian Tang
tang @ math.ucalgary.ca

1. A Problem Walkthrough

We will go through in detail the following problem that appeared on your mock olympiad.

Problem 1: (IMO 2009 Shortlist) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x+y)) = f(yf(x)) + x^2,$$

for all $x, y \in \mathbb{R}$.

Functional Equation Strategies:

1. **(Find Solutions:)** Find a class of solutions by inspection, i.e. $f(x) = c$, a constant, $f(x) = ax + b$ for some $a, b \in \mathbb{R}$, $f(x) = x^2, a^x$, etc. If the functional equation contains trignometric terms, you can consider terms of the form $f(x) = a \sin x + b \cos x$, etc. Your class of solutions often dictates your strategies. At this point forward, make sure that you follow the following rule:

   Make sure everything you do from this point onwards is consistent with these equations.

2. **(Basic Substitutions:)** Apply good substitutions, often to either cancel things out or make terms equal to zero.

   (a) In this exercise, substituting $x = 0, y = 0$ or both are good starts.

   (b) You can consider whether you can make $xf(x+y) = yf(x)$, so that you can cancel the terms $f(xf(x+y)) = f(yf(x))$. It turns out that in this case you cannot do so easily. However, you should still consider doing it.

3. Try to find $f(k)$ for some fixed k, usually $k = 0$ or $k = \pm 1$. In our case, try to prove that $f(0) = 0$.

4. Is f an odd or even function? i.e. $f(x) = f(-x), \forall x \in \mathbb{R}$? $f(x) = -f(-x), \forall x \in \mathbb{R}$?

5. Is f injective or surjective? If so, can you prove it? We will show how powerful it is to know that f is injective, or surjective, or both.

6. Is f strictly (or monotone) increasing (or decreasing)?

7. **Fixed Points:** Is there some thing special about fixed points of f? i.e. all $w \in \mathbb{R}$ such that $f(w) = w$. See Section 2.

8. Something more advanced; if $z_1, z_2 \in \text{Range}(f)$, can every $x \in \mathbb{R}$ be written in the form $x = z_1 - z_2$? Or something similar? Then if you are trying to prove, say, $f(x) = x^2, \forall x \in \mathbb{R}$, then you can try to prove $f(z_1 - z_2) = (z_1 - z_2)^2, \forall z_1, z_2 \in \text{Range}(f)$.[1] See Section 3.

9. Is the function f periodic? i.e. is there a positive d such that $f(x+d) = f(x)$ for some or all values of x?

10. Can you reduce the problem to a Cauchy equation? i.e. $f(x+y) = f(x) + f(y), \forall x, y \in \mathbb{R}$? We know that if f satisfies any of the following conditions, then $f(x) = cx$ for some constant c.

    (a) f is defined over the rationals

    (b) f is bounded on any closed interval of $\mathbb{R}$, i.e. $[a, b] = \{x | a \le x \le b\}$, where $a, b \in \mathbb{R}, a < b$.

    (c) f is monotone increasing, i.e. $f(x) \le f(y)$ whenever $x \le y$ or monotone decreasing, i.e. $f(x) \ge f(y)$ whenever $x \le y$.

11. At the end, remember to check that your solutions work in your solution write-up!

These are the main things you should look for. It is impossible to lay out every method possible to solve a functional equation. A lot of it does come down to a jigsaw puzzle approach, i.e. piecing together results to obtain bigger results. For this reason, **it is important that you highlight your main observations clearly**, i.e. not an obscure line in a 42-line paragraph in small hand-writing.

Let's return to our problem.

We inspect that $f(x) = x$ and $f(x) = -x, \forall x \in \mathbb{R}$ are solutions of the functional equations. There are possibly other solutions. But these are the ones you have found for now, so you will assume that these are the only solutions until shown otherwise.

Substituting $x = 0$ into the original equation yields $f(0) = f(yf(0)), \forall y \in \mathbb{R}$. Suppose $f(0) \neq 0$. Then $yf(0)$ ranges over all reals as y varies over all reals. (Remember this technique!)

Hence, $f(0) = f(z)$ for all $z \in \mathbb{R}$. Consequently, f is a constant. It is easy to verify that f is not a constant. Therefore, $f(0) = 0$.

Suppose $f(z) = 0$. We will prove that $z = 0$. Substituting $x = z, y = -z$ into the original equation yields $f(zf(0)) = f(-zf(z)) + z^2 = z^2$. Since $f(0) = f(z) = 0$, $0 = 0 + z^2$. Therefore, $z = 0$. We have

$$f(z) = 0 \Leftrightarrow z = 0, \forall z \in \mathbb{R} \tag{1}$$

Now that we know (1), it is good idea to make a lot of things zero in the functional equation to see what happens. Some things turn out to be useful, and some may not.

[1] Of course, $z_1 - z_2$ is only an example. The term in equation can be some other combination of z_1, z_2. We will see an example where this is a very powerful technique.

Since $f(0) = 0$, substituting $y = 0$ into the original equation yields

$$f(xf(x)) = x^2, \forall x \in \mathbb{R} \tag{2}$$

Substituting $y = -x$ yields

$$f(-xf(x)) = -x^2, \forall x \in \mathbb{R} \tag{3}$$

Not only have we established two interesting identities, we just showed that f is surjective. (Do you see how?) *Always be on the lookout for terms written in the form $f(\cdot) =$ an expression not dependent on the function f. If the right hand side ranges over all reals, then f is surjective. (Even showing that f ranges over non-negative reals is a nice result.)*

We now prove that f is injective. Suppose $f(z) = f(z + r)$ for some $z, r \in \mathbb{R}$. Substituting $x = z, y = r$ into the original equation yields $f(zf(z + r)) = f(rf(z)) + z^2$. Since $f(z) = f(z + r)$, $f(zf(z)) = f(rf(z)) + z^2$. By (2), $f(zf(z)) = z^2$. Therefore, $f(rf(z)) = 0$. By (1), $rf(z) = 0$. Therefore, $f(z) = 0$ or $r = 0$. If $f(z) = 0$, then $z = 0$, Hence, $f(0) = f(0 + r)$. By (1), $r = 0$. Hence, in either case, $r = 0$. Therefore, f is injective.

Now, we abuse the surjectivity property. Can we make the right-hand side of the original equation zero.

We fix $x = z \neq 0$. Then $f(z) \neq 0$. Therefore, $yf(z)$ ranges over all reals. Since f is surjective, $f(yf(z))$ ranges over all reals. Remember we want to make things zero? Choose y such that $f(yf(z)) = -z^2$. Then $f(z(z + y)) = 0$. Since $z \neq 0$, $y = -z$. Then $f(-zf(z)) = -z^2$. By (3), we already know this. Well, that yielded nothing new. Let's try $x = -z \neq 0$ instead. Then $f(-z(-z+y)) = f(yf(-z))+z^2$. Again, there exist y such that $f(yf(-z)) = -z^2$. Then $f(-z(-z + y)) = 0$. Since $z \neq 0$, $y = z$. Then $f(zf(-z)) = -z^2$. But $f(-zf(z)) = -z^2$ as well. Time to abuse injectivity. We then have $zf(-z) = -zf(z)$. Since $z \neq 0$, $f(-z) = -f(z)$. If $z = 0$, then clearly $f(-z) = -f(z)$. This is an important result! Write it down!

$$f(-x) = -f(x), \forall x \in \mathbb{R}. \tag{4}$$

Given this fact. Substituting $x, y = \pm 1$ is a great idea. Since f is surjective, another alternative is to let $z \in \mathbb{R}$ such that $f(z) = 1$. You should try all of these things. It turns out that it is the latter that gives us more information.

Since f is surjective, there exist $z \in \mathbb{R}$ such that $f(z) = 1$. Then substituting $x = z$ into (2) yields $f(z) = z^2$. Since $f(z) = 1$, $z = \pm 1$. Therefore, by (4) $(f(1), f(-1)) = (1, -1)$ or $(f(-1), f(1)) = (-1, 1)$.

If $f(1) = 1$ and $f(-1) = -1$, then substituting $x = 1$ yields

$$f(f(y + 1)) = f(y) + 1, \forall y \in \mathbb{R} \tag{5}$$

and substituting $x = -1$ yields $f(-f(y - 1)) = f(-y) + 1$, or by Lemma 3,

$$-f(f(y-1)) = -f(y)+1, \forall y \in \mathbb{R}. \tag{6}$$

Therefore, $-f(f(y+1)) = -f(y+2)+1, \forall y \in \mathbb{R}$. Adding this equation with (5) yields $0 = f(y) - f(y+2)+2$. Hence, $f(y+2)-f(y) = 2$. Similarly, $f(y+4)-f(y+2) = 2$. Therefore,

$$f(y+4)-f(y) = 4, \forall y \in \mathbb{R}. \tag{7}$$

Now why did I do this? First of all, I would have preferred to have proven $f(y+1)-f(y) = 1, \forall y \in \mathbb{R}$. Then I can use (5) to yield $f(f(y+1)) = f(y+1)$. Since f is injective, $f(y+1) = y+1$. We then conclude that $f(x) = x$ for all $x \in \mathbb{R}$. Awesome. But instead I have $f(y+2)-f(y) = 2$. I want to substitute $x = 2$ to make the 4 appear in the original equation. Hence, I created (6). I suppose I could instead substitue $x = \sqrt{2}$, but that would be too irrational.

Substituting $x = 2$ into the original equation fields $f(2f(y+2)) = f(yf(2))+4$. By (7), we have $f(yf(2)+4) = f(yf(2))+4$. Therefore, $f(2f(y+2)) = f(yf(2)+4)$. Since f is injective, $2f(y+2) = yf(2)+4$. Hence, $f(y) = \frac{f(2)}{2}(y-2)+2$. Let $c = \frac{f(2)}{2}$. Then $f(y) = cy+(2-2c)$. Since $y = 0$ and $f(0) = 0$, $c = 1$. Then $f(y) = y, \forall y \in \mathbb{R}$. This is a solution to the functional equation since $f(xf(x+y)) = x(x+y)$ and $f(y(f(x)))+x^2 = xy+x^2 = x(x+y)$.

If $f(1) = -1$ and $f(-1) = 1$, then I leave to the reader to finish the problem similarly to the previous case.

2. Fixed Points

Given a function f, a *fixed point* of f is defined to be $w \in \mathbb{R}$ such that $f(w) = w$. It is often good to observe fixed points of a function.

Problem 2: (IMO 1993) Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(xf(y)) = yf(x)$$

for all $x, y \in \mathbb{R}^+$ and as $x \to \infty$, $f(x) \to 0$.

Solution: Let's inspect some functions first. So $f(x)$ approaches 0 as x gets large. So forget $f(x) = x$ or constants. It's probably something like $f(x) = \frac{1}{x}$. Well, it works. So let's keep this in mind.

Maybe this function is injective. Suppose $f(z_1) = f(z_2)$. Substitute $x = 1$. Then $z_1 f(x) = f(xf(z_1)) = f(xf(z_2)) = z_2 f(x)$. Since $f(x) \neq 0$ for any x, $z_1 = z_2$. Hence, f is injective. So far so good.

Let's find $xf(y) = x$ to try to clear the terms $f(xf(y)) = f(x)$. Let $x = y = 1$. Then $f(f(1)) = f(1)$. Since f is injective, $f(1) = 1$. So 1 is a fixed point.

So maybe I can substitute $y = 1$. I get $f(x) = f(x)$. Well, thanks for nothing. Maybe I should try $x = 1$ instead. Then $f(f(y)) = yf(1) = y$. This sounds good. I should write this down.

$$f(f(y)) = y, \forall y \in \mathbb{R}.^2 \tag{8}$$

Substituting $x = y$ yields $f(xf(x)) = xf(x)$. Therefore, $xf(x)$ is a fixed point for all $x \in \mathbb{R}$. The funny thing is, since $f(x) \to 0$ as $x \to \infty$. Hence, the set of fixed points of f is bounded above. I'm also trying to "prove" that $f(x) = \frac{1}{x}$, i.e. 1 is the only fixed point. Let w be a fixed point of f. Suppose $w > 1$ is a fixed point of f, i.e. $f(w) = w$. Then substituting $x = y = w$ into the original equation yields $f(w^2) = w^2$. Therefore, w^2 is also a fixed point. Inductively, w^{2^n} is a fixed point for any $n \in \mathbb{N}$. Since $w > 1$, $w^{2^n} \to \infty$. But the set of fixed points of f is bounded. Hence, $w \leq 1$. If $w < 1$, then substituting $x = \frac{1}{w}$ and $y = w$ yields $f(1) = wf(\frac{1}{w}$. Therefore, $f(1/w) = 1/w$. Hence, $1/w > 1$ is a fixed point. This is again a contradiction.

Therefore, 1 is the only fixed point of f. Hence, since $xf(x)$ is a fixed point for all $x \in \mathbb{R}$, $xf(x) = 1$ for all $x \in \mathbb{R}$. Therefore, $f(x) = 1/x$. This is indeed a solution to the equation since $f(xf(y)) = f(x/y) = y/x$ and $yf(x) = y/x$. $\square$

3. An Advanced Technique

Consider the following problem.

Problem 3: (IMO 1999) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

Solution: It appear difficult to find a solution to this functional equation. Maybe we should move on and try to find some properties of f first, before conjecturing what f is.

Let's set $x = y = 0$. Then $f(-f(y)) = f(f(y)) + f(0) - 1$. Then substituting $x = f(y)$ yields $f(0) = f(f(y)) + f(y)^2 + f(f(y)) - 1$. Hence, $z^2 + 2f(z) = f(0)$ for all $z \in \text{Range}(f)$, by substituting $z = f(y)$. This would be awesome if $\text{Range}(f) = \mathbb{R}$. Because then $f(z) = f(0) - \frac{z^2}{2}$ for all $z \in \mathbb{R}$. But we do have this statement for all $z \in \text{Range}(f)$.

Note that $f(z) - f(0) = -\frac{z^2}{2}$, which can be any non-positive number and $f(0) - f(z) = \frac{z^2}{2}$, which can be any non-negative number. Therefore, $\{z_1 - z_2 | z_1, z_2 \in \text{Range}(f)\} = \mathbb{R}$. Hence, it suffices to show that

$$f(z_1 - z_2) = f(0) - \frac{(z_1 - z_2)^2}{2},$$

[2] We of course conclude that f is surjective on the positive reals. You should certainly note that. But it turns out that this is not necessary for this proof.

for all $z_1, z_2 \in \text{Range}(f)$. Then substituting $x = z_1, f(y) = z_2$ (yes we can do this! :D) yields

$$f(z_1-z_2) = f(z_2)+z_1z_2+f(z_1)-1 = 2f(0)-\frac{z_1^2+z_2^2}{2}+z_1z_2-1 = (2f(0)-1)-\frac{(z_1-z_2)^2}{2} = c-\frac{(z_1-z_2)^2}{,}$$

for some constant c. Now substitute this back into the original equation to obtain $c = 1$. $\square$

Exercises:

1. Consider the functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(m+n) \geq f(m) + f(f(n)) - 1$$
for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x + f(y)) = f(x + y) + f(y).$$

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(xf(y) + f(x)) = 2f(x) + xy$$
for all $x, y \in \mathbb{R}$.

4. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that
$$f(m + f(n)) = f(f(m)) + f(n),$$
for all $m, n \in \mathbb{N}_0$, where $\mathbb{N}_0$ denotes the set of non-negative integers.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x)^2 + 2yf(x) + f(y) = f(y + f(x)).$$