Geometry Fundamentals

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1 Introduction

These notes are about geometry. But geometry is a huge subject, and I am only going to be scratching the surface here. If you are serious about doing well in Olympiads, I strongly recommend seeking out more resources on your own. Geometry Unbound by Kiran Kedlaya is a great starting point:

http://www-math.mit.edu/~kedlaya/geometryunbound/

Previous notes from Canadian winter and summer camps have been less encyclopedic, but they are exceptionally good. Take advantage of them too!

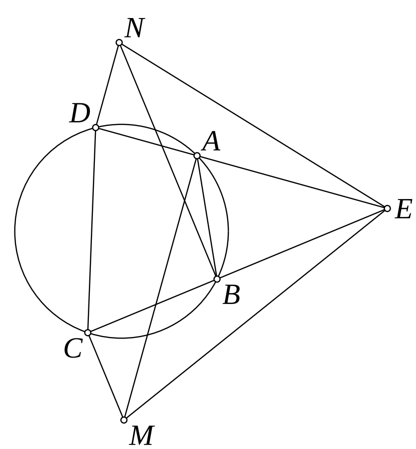
http://sites.google.com/site/imocanada/

2 Configurations and Directed Angles

All right, down to business! Instead of starting with a technique, I am going to start with a pitfall. What do the following examples have in common?

Example 1. *Let $ABCD$ be a cyclic quadrilateral. The perpendiculars to AD and BC at A and C respectively meet at M, and the perpendiculars to AD and BC at D and B meet at N. If the lines AD and BC meet at E, prove that $\angle DEN = \angle CEM$.*[1]

"Solution": Since $\angle EBN = \angle EDN = 90^\circ$, $EBDN$ is a cyclic quadrilateral. $EACM$ is also cyclic for the same reasons. Now:

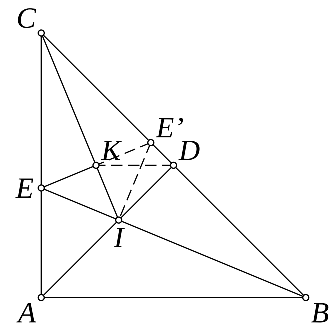


$$\begin{aligned}
\angle DEN &= 180^\circ - \angle BED - \angle BDN \\
&= 90^\circ - \angle BED - \angle BDA \\
&= 90^\circ - \angle AEC - \angle ACB \\
&= 180^\circ - \angle AEC - \angle ACM \\
&= \angle CEM,
\end{aligned}$$

and the problem is solved. □

[1] Iran 2004, and Winter Camp 2011 Warmup Problem.

Example 2. *Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.*[2]



"Solution": Let I be the incenter of $\triangle ABC$. Note that K and I both lie on the angle bisector of $\angle ACB$. Let E' be the reflection of E across line CKI. E' lies on BC – do you see why?

$\angle KE'I = \angle KEI = 45°$, and furthermore, $\angle CDA = 90°$ since $AB = AC$ and AD is an angle bisector. Because K is the incenter of $\triangle ADC$, we have $\angle KDI = \frac{1}{2} \cdot \angle CDA = 45° = \angle KE'I$. Therefore, $KE'DI$ is a cyclic quadrilateral.

Now let $\angle CAB = \theta$, so that $\angle ECB = 90 - \frac{\theta}{2}$, and $\angle EBC = 45 - \frac{\theta}{4}$. Since $\angle BEK$ is given to be $45°$ and the sum of the angles in $\triangle ECB$ is $180°$, we have $\angle CEK = \frac{3\theta}{4}$. On the other hand, since $KE'DI$ is cyclic, we know $\angle CEK = \angle CE'K = \angle CID = 90° - \angle ICD = 45 + \frac{\theta}{4}$. Equating our two expressions for θ, we are left with $\theta = 90°$. $\square$

Example 3. *Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game: Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex. Show that the player who signs first will always win by playing as well as possible.*[3]

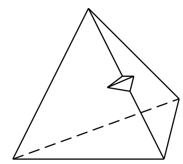
"Solution": Suppose the polyhedron has a face A with at least 4 edges. If the first player begins by signing there, then after the second player's turn, there will be 3 consecutive faces B, C, D adjacent to A, which are all unoccupied. The first player wins by signing C; after the second player's second move, at least one of B or D remains unoccupied, and either is a winning move for the first player.

It remains to show that the polyhedron has a face with at least four edges. Suppose on the contrary that each face has only three edges. Start with any face having vertices a, b, c. Let d be the third vertex that is adjacent to a. Then a, b, d and a, c, d must be faces, and then finally, b, c, d must also be a face. But we have now completed the polyhedron, and it had only 4 faces, which is a contradiction. Therefore, one face must have at least 4 edges, and the proof is complete. $\square$

So what do all these "solutions" have in common? The answer is they are all wrong!

- Actually, Example 1 is just incomplete. The proof neglects to mention the case where M is on the opposite side of CE, and also the case where N is the opposite side of DE. If M and N are both flipped, you can still solve the problem fine. If only one is flipped however, the argument fails. Good thing that case is impossible! But you really need to show that...
- In Example 2, does anything change if E' lies on the opposite side of D? What if E' is the same as D? Hint: $\theta = 90°$ is not the only possible answer! The solution here would have gotten only 4/7. It missed an answer, and it failed to verify that $\theta = 90°$ really does work.

[2] IMO 2009, #4.
[3] Putnam 2002, B2.



- In Example 3, the whole problem is wrong! Start with a tetrahedron, and consider removing a triangular chunk from one edge, as shown above. In this configuration, two faces share two different edges, and the whole proof breaks down. In fact, the first player cannot win here at all. Do you see why?

These examples illustrate the importance of making sure your argument works for *all* configurations, not just the one you drew! In particular, when you use angle-chasing, you should expect configuration issues like in Example 1. Most problems won't be as subtle as these ones were, but having two similar cases is extremely common. If you ignore some of the configurations, you are just asking to lose points.

Here is one useful (but limited) way of dealing with multiple configurations while angle chasing:

Directed Angles mod $180°$**:**

Given two lines ℓ_1 and ℓ_2, we define the *directed angle* between them: $\angle(\ell_1, \ell_2)$ to be the counter-clockwise angle that you need to rotate ℓ_1 by in order to make it parallel to ℓ_2. Note that $180°$ is the same as $0°$ in directed angles: if you rotate a line by $180°$, it is still parallel to itself.

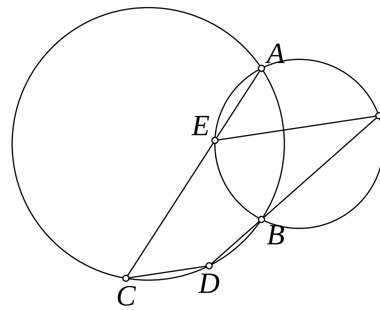
The following statements about directed angles are true, regardless of configuration:

1. $\angle(\ell_1, \ell_2) = -\angle(\ell_2, \ell_1)$
2. $\angle(\ell_1, \ell_3) = \angle(\ell_1, \ell_2) + \angle(\ell_2, \ell_3)$
3. $\angle(\ell_1, \ell_2) + \angle(\ell_2, \ell_3) + \angle(\ell_3, \ell_1) = 0$
4. A, B, C, D lie on a circle iff $\angle(AB, BC) = \angle(AD, DC)$
5. B, C, D lie on a line iff $\angle(AB, BC) = \angle(AB, BD)$
6. AB is tangent to the circle through B, C, D iff $\angle(AB, BC) = \angle(BD, DC)$
7. If A, B, C lie on a circle with center O, then $2\angle(AB, BC) = \angle(AO, OC)$

Properties 4,5,6 are what make directed angles useful. With regular angles, there are always two cases based upon the ordering of the points: the angles might be equal or supplementary. Directed angles remove the cases altogether, which makes them perfect for showing collinearity or cyclicness. They are usually **NOT** a good choice for showing two regular angles are equal however: even if two directed angles are equal, the regular angles could be supplementary instead of equal!

Example 4. *Suppose that the circles ω_1 and ω_2 intersect at distinct points A and B. Let CD be any chord on ω_1, and let E and F be the second intersections of the lines CA and BD, respectively, with ω_2. Prove EF is parallel to DC.*

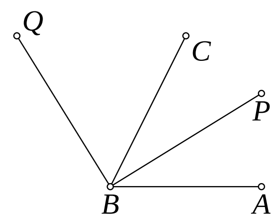
Solution: There are many, many possible configurations here, making regular angle-chasing impractical. It's a breeze with directed angles though:



$$\begin{aligned}\angle(EF, DC) &= \angle(EF, DBF) + \angle(DBF, DC)\\ &= \angle(EF, FB) + \angle(BD, DC)\\ &= \angle(EA, AB) + \angle(BA, AC)\\ &= \angle(EA, AC) = 0,\end{aligned}$$

and hence EF and DC are indeed parallel. □

Final warning: While directed angles are very useful in the right situation, there are a couple pitfalls you have to watch out for. I already mentioned that two directed angles being equal does not imply that the corresponding regular angles are equal. Here is another one: Do not ever use expressions like $\frac{1}{2} \cdot \angle(AB, BC)$, because half of a directed angle has no meaning!



For example, in the diagram to the right, $2 \cdot \angle(AB, BP) = \angle(AB, BC) = 2 \cdot \angle(AB, BQ)$, but $\angle(AB, BP) \neq \angle(AB, BQ)$. In other words, there are two possible angles that you could mean when you say $\frac{1}{2} \cdot \angle ABC$. So how do you deal with this? The answer is: never, ever look at a fraction of an angle. Modular arithmetic in number theory has the same issue.

Exercises:

1. Convince yourself that the listed properties for directed angles are true.

2. Correct the proof for Example 1.

3. Correct the proof for Example 2.

4. Fix triangle ABC, and choose points P, Q, R on lines BC, CA, and AB, respectively. Prove that the circumcircles of triangles AQR, BRP, and CPQ pass through a common point.

5. Let $\omega_1, \omega_2, \omega_3, \omega_4$ be four circles in the plane. Suppose that ω_1 and ω_2 intersect at P_1 and Q_1, ω_2 and ω_3 intersect at P_2 and Q_2, ω_3 and ω_4 intersect at P_3 and Q_3, and ω_4 and ω_1 intersect at P_4 and Q_4. Show that if P_1, P_2, P_3, and P_4 lie on a circle, then Q_1, Q_2, Q_3, and Q_4 also lie on a circle.

6. Triangle ABC is any one of the set of triangles having base BC equal to a and height from A to BC equal to h, with $h < \frac{\sqrt{3}}{2} \cdot a$. P is a point inside the triangle such that $\angle PAB = \angle PBA = \angle PCB = \alpha$. Show that the measure of α is the same for every triangle in the set.

7. Two circles intersect at points A and B. An arbitrary line through B intersects the first circle again at C and the second circle again at D. The tangents to the first circle at C and the second at D intersect at M. Through the intersection of AM and CD, there passes a line parallel to CM and intersecting AC at K. Prove that BK is tangent to the second circle.

8. Determine all finite sets S of at least three points in the plane which satisfy the following condition: For any two distinct points A and B in S, the perpendicular bisector of the segment AB is an axis of symmetry for S.

3 Power of a Point

Angle-chasing – especially to find cyclic quadrilaterals and congruent triangles – is the most important technique in all of Olympiad geometry. It doesn't always work though, and one of the simplest and most reliable alternatives to try next is power of a point.

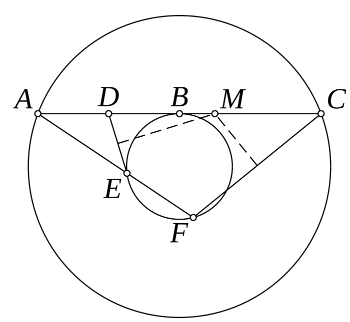
Fact 1. *Suppose lines AB and CD meet at a point P. Then A, B, C, D all lie on a circle if and only if the "directed lengths" PA, PB, PC, PD satisfy $PA \cdot PB = PC \cdot PD$.*

Fact 2. *Suppose point P lies on line AB, and C is an arbitrary point. Then, PC is tangent to the circle through A, B, C if and only if the directed lengths PA, PB, PC satisfy $PA \cdot PB = PC^2$.*

So first of all: what are directed lengths? The important thing is $PA \cdot PB$ is considered negative if P is between A and B, and positive otherwise.[4]

Hopefully, you have seen these facts before. They are already powerful enough to solve some otherwise difficult problems:

Example 5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$, one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.*[5]



Solution: If R_1 and R_2 are the radii of $\mathcal{C}_1$ and $\mathcal{C}_2$, then $AB^2 = R_1^2 - R_2^2 = BC^2$, so B is the midpoint of AC. Therefore,

$$\begin{aligned} AD \cdot AC &= \left(\frac{AB}{2}\right) \cdot (2AB) \\ &= AB^2 \\ &= AE \cdot AF \qquad \text{since } AB \text{ is tangent to } \mathcal{C}_2, \end{aligned}$$

which implies $DCFE$ is cyclic. The center of this circle lies on the perpendicular bisectors of DE and CF, which means it must be M. Therefore, $MD = MC$, and hence $MC = \frac{1}{2} \cdot DC = \frac{3}{8} \cdot AC$. Then, $AM/MC = \frac{5}{3}$. □

[4] Here is the proper definition if you are interested: Assign each direction on the plane to be positive or negative in any fashion you want, ensuring only that if two directions are opposite each other, then they have opposite signs. Then a length XY is considered positive if $\overrightarrow{XY}$ points in a positive direction, and negative otherwise. If X, Y, Z are collinear, this ensures that $XY + YZ = XZ$ no matter what order they lie on, giving directed lengths many of the same advantages as directed angles. Ceva's theorem and Menelaos's theorem are properly stated using directed lengths as well.

[5] USAMO 1998, #2.

Power of a Point:

Fix a point P and a circle ω with center O and radius r. Let ℓ be any line through P intersecting ω at A and B. By Fact 1, the product of directed lengths $PA \cdot PB$ does not depend on ℓ. This value is called the *power* of P with respect to ω.

Bonus Fact: The power of P with respect to ω is equal to $OP^2 - r^2$.

While Facts 1 and 2 are certainly handy, the main use of power of a point comes from the radical axis:

Radical Axis:

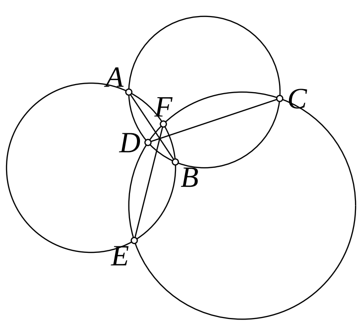
Fix two circles ω_1 and ω_2 with different centers. The set of points that have equal power with respect to ω_1 and ω_2 is a line, called the *radical axis* of ω_1 and ω_2.

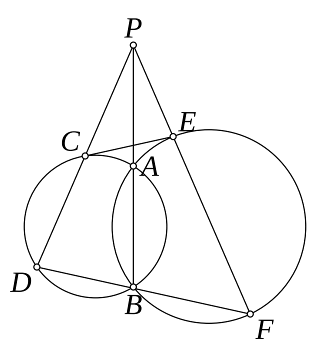
1. If ω_1 and ω_2 intersect at points A and B, then their radical axis is AB.
2. The radical axis of ω_1 and ω_2 is perpendicular to the line between their centers.

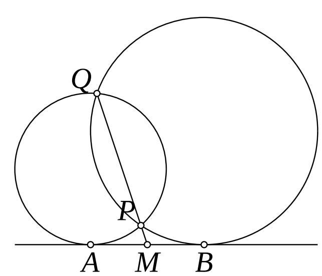
Radical Center:

Fix three circles ω_1, ω_2, and ω_3 with centers not all lying on the same line. Then the radical axes of the circles meet at a common point, which is called the *radical center* of ω_1, ω_2, and ω_3.

There are many ways to use power of a point and radical axes! A few of the most important deductions are illustrated below:



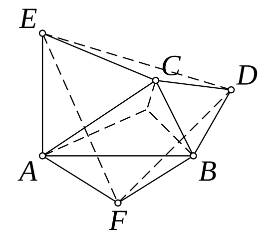




In the left figure, AB, CD, EF are radical axes, so they must all meet at a point. In the middle figure, $PC \cdot PD = PA \cdot PB = PE \cdot PF$, and so $CDFE$ is cyclic. In the right figure, $MA^2 = MP \cdot MQ = MB^2$, so M is the midpoint of AB.

And now, onto some examples from actual Olympiads!

Example 6. *Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines EF, FD, DE, respectively, are concurrent.*[6]



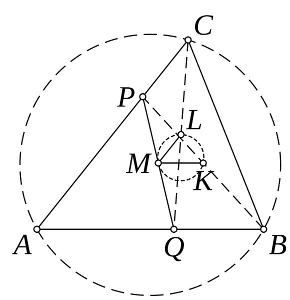
Solution: Let ω_D be the circle with center D and radius $DB = DC$. Define ω_E and ω_F analogously. The radical axis of ω_D and ω_E passes through their common point (C) and is perpendicular to the line joining their centers (DE). Thus, the problem is asking us to prove that the radical axes for three circles meet at a point, which we know to be true. □

Although it is not related to power of a point, let me also mention that Example 6 is made trivial by the following handy theorem:

Fact 3. *Fix points A, B, C, D, E, F; let ℓ_1 be the line through A perpendicular to EF, ℓ_2 be the line through B perpendicular to FD, and ℓ_3 be the line through C perpendicular to DE. Then, ℓ_1, ℓ_2, and ℓ_3 meet at a point iff $AE^2 - EC^2 + CD^2 - DB^2 + BF^2 - FA^2 = 0$.*

Now back to some real-world examples. These both come from the IMO, which means they take a couple different ideas to solve, but power of a point is key in both cases!

Example 7. *Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L, and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.*[7]



Solution: Since M and K are midpoints of PQ and BP, it must be that MK is parallel to and half the length of QB. Now,

$$\begin{aligned}\angle MLK &= \angle QMK && (QM \text{ is tangent to circle } MLK)\\ &= \angle MQA && (\text{parallel lines})\\ &= \angle AQP.\end{aligned}$$

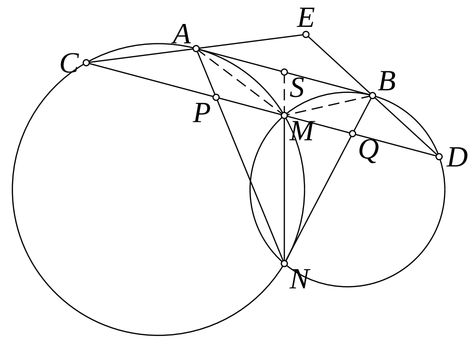
Similarly, $\angle MKL = \angle APQ$, and so $\triangle MLK$ is similar to $\triangle AQP$. In particular,

$$\begin{aligned}AQ \cdot MK = AP \cdot ML &\implies AQ \cdot (2MK) = AP \cdot (2ML)\\ &\implies AQ \cdot QB = AP \cdot PC,\end{aligned}$$

and hence P, Q have the same power with respect to the circumcircle of ABC. But this means that $OP^2 = OQ^2$, and we are done. (Recall the power of P with respect to a circle is $OP^2 - r^2$, where r is the radius of the circle.) □

Example 8. *Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.*[8]

[6]USAMO 1997, #2.
[7]IMO 2009, #2.
[8]IMO 2000, #1.



Solution: Extend NM to meet AB at S. Then $AS^2 = SM \cdot SN = SB^2$, so S is the midpoint of AB. Since AB is parallel to PQ, there exists a dilation about N taking AB to PQ. This dilation takes S to M, and hence M is the midpoint of PQ. Also:

$$\begin{aligned} \angle EAB &= \angle ACM && \text{(parallel lines)} \\ &= \angle MAB && \text{(BA is tangent to circle ACM).} \end{aligned}$$

Similarly, $\angle EBA = \angle MBA$, and hence $\triangle EAB \cong \triangle MAB$ by angle-side-angle. It follows that EM is perpendicular to AB, and is therefore perpendicular to PQ as well. Combining this with the fact that M is the midpoint of PQ gives us $\triangle EMP \cong \triangle EMQ$, and the problem is solved. □

Exercises:

1. Convince yourself that the listed properties for power of a point and radical axis are true. In particular, prove that the radical axes of three circles really must meet at a point.

2. Let BD be the angle bisector of angle B in triangle ABC with D on side AC. The circumcircle of triangle BDC meets AB at E, while the circumcircle of triangle ABD meets BC at F. Prove that $AE = CF$.

3. Draw tangents OA and OB from a point O to a given circle. Through A is drawn a chord AC parallel to OB; let E be the second intersection of OC with the circle. Prove that the line AE bisects the segment OB.

4. Two equal-radius circles ω_1 and ω_2 are centered at points O_1 and O_2. A point X is reflected through O_1 and O_2 to get points A_1 and A_2. The tangents from A_1 to ω_1 touch ω_1 at points P_1 and Q_1, and the tangents from A_2 to ω_2 touch ω_2 at points P_2 and Q_2. If P_1Q_1 and P_2Q_2 intersect at Y, prove that Y is equidistant from A_1 and A_2.

5. The altitudes through vertices A, B, C of acute triangle ABC meet the opposite sides at D, E, F, respectively. The line through D parallel to EF meets the lines AC and AB at Q and R, respectively. The line EF meets BC at P. Prove that the circumcircle of triangle PQR passes through the midpoint of BC.

6. Let P and Q be points in the plane and let ω_1, ω_2, and ω_3 be circles passing through both. If A, B, C, D, E, and F are points on a line in that order so that A and D lie on ω_1, B and E lie on ω_2, and C and F lie on ω_3, prove that $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

7. Let ω_1 and ω_2 be concentric circles with ω_2 inside ω_1. Let $ABCD$ be a parallelogram with B, C, D on ω_1 and A on ω_2. If BA intersects ω_2 again at E and CE intersects ω_2 again at P, prove that $CD = PD$.

8. Let three circles $\Omega_1, \Omega_2, \Omega_3$, which are non-overlapping and mutually external, be given in the plane. For each point P in the plane, outside the three circles, construct six points $A_1, B_1, A_2, B_2, A_3, B_3$ as follows: For each $i = 1, 2, 3$, A_i and B_i are the two points on the circle Γ_i such that the lines PA_i and PB_i are both tangents to Γ_i. Call the point P *exceptional* if the three lines A_1B_1, A_2B_2, and A_3B_3 are concurrent. Show that every exceptional point, if it exists, lies on the same circle.

4 Challenge Problems

I will end with a fairly random collection of Olympiad geometry problems for you to practice on, and challenge yourself with. The preceding sections might help you, but they also might not. Good luck – these are hard problems!

1. Two congruent circles ω_1 and ω_2 intersect at B and C. Select a point A on ω_1. Let AB and AC intersect ω_2 again at A_1 and A_2. Let X be the midpoint of BC. Let A_1X and A_2X intersect ω_1 at P_1 and P_2. Prove that $AP_1 = AP_2$.

2. The circle k is circumscribed about the isosceles triangle ABC (with $AC = BC$) and point D is the midpoint of the arc AB. Let M be a point on side AB, and let N be the second intersection of the line DM with k. Let P and Q be the intersections of the perpendicular bisector of segment MN with BC and AC, respectively. Prove that the quadrilateral $CPMQ$ is a parallelogram.

3. A convex quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC and BD intersect at P. The circumcircles of triangles ABP and CDP intersect again at Q. If O, P, and Q are three distinct points, prove that OQ is perpendicular to PQ.

4. Let D be a point on side AC of triangle ABC. Let E and F be points on the segments BD and BC respectively, such that $\angle BAE = \angle CAF$. Let P and Q be points on BC and BD respectively, such that EP and FQ are both parallel to CD. Prove that $\angle BAP = \angle CAQ$.

5. A circle with center I is inscribed in a quadrilateral $ABCD$ with $\angle BAD + \angle ADC > 180^\circ$. A line through I meets AB and CD at points X and Y, respectively. Prove that if $IX = IY$, then $AX \cdot DY = BX \cdot CY$.

6. Let ABC be an acute-angled triangle, and let L be the point where the bisector of $\angle C$ hits side AB. The point P belongs to the segment CL in such a way that $\angle APB = 180^\circ - \frac{1}{2} \cdot \angle ACB$. Let k_1 and k_2 be the circumcircles of $\triangle APC$ and $\triangle BPC$. BP meets k_1 again at Q, and AP meets k_2 again at R. The tangent to k_1 at Q and the tangent to k_2 at B meet at S. The tangent to k_2 at R and the tangent to k_1 at A meet at T. Prove that $AS = BT$.

7. Let ABC be a triangle such that $\angle A = 90^\circ$ and $\angle B < \angle C$. The tangent at A to the circumcircle ω of triangle ABC meets the line BC at D. Let E be the reflection of A in the line BC, let X be the foot of the perpendicular from A to BE, and let Y be the midpoint of the segment AX. Let the line BY intersect the circle ω again at Z. Prove that the line BD is tangent to the circumcircle of triangle ADZ.

8. Two circles Ω_1 and Ω_2 are contained inside the circle Ω, and are tangent to Ω at the distinct points M and N, respectively. Ω_1 passes through the center of Ω_2. The line passing through the two points of intersection of Ω_1 and Ω_2 meets Ω at A and B. The lines MA and MB meet Ω_1 at C and D, respectively. Prove that CD is tangent to Ω_2.

9. The point M is inside the convex quadrilateral $ABCD$, such that $MA = MC$, $\angle AMB = \angle MAD + \angle MCD$, and $\angle CMD = \angle MCB + \angle MAB$. Prove that $AD \cdot CM = BC \cdot MD$ and $BM \cdot AD = MA \cdot CD$.

10. A convex quadrilateral $ABCD$ is given. Prove that there exists a point P inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$$

if and only if the diagonals AC and BD are perpendicular.

11. A non-isosceles triangle $A_1A_2A_3$ has sides a_1, a_2, a_3 with side a_i lying opposite vertex A_i. Let M_i be the midpoint of side a_i, and let T_i be the point where the inscribed circle of triangle $A_1A_2A_3$ touches side a_i. Denote by S_i the reflection of the point T_i in the interior angle bisector of angle A_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent.

12. Let AH_1, BH_2, CH_3 be the altitudes of an acute angled triangle ABC. Its incircle touches the sides BC, CA, and AB at T_1, T_2, and T_3 respectively. Consider the symmetric images of the lines H_1H_2, H_2H_3, and H_3H_1 with respect to the lines T_1T_2, T_2T_3 and T_3T_1. Prove that these images form a triangle whose vertices lie on the incircle of ABC.

5 Hints

Exercises: Configurations and Directed Angles

2. See Warmup Solutions. Directed angles are probably not a good choice, since equal directed angles do not prove that two regular angles are equal.
3. The other possibility is $\theta = 60^\circ$. Don't forget to check that your solutions work!
4. Let Z be the intersection of circles AQR and BRP. Use directed angles to show it is on circle CPQ. Note that P does not necessarily lie on *segment* BC. (Source: Miquel's Theorem)
5. Use directed angles to calculate $\angle(Q_1Q_2, Q_2Q_3) - \angle(Q_1Q_4, Q_4Q_3)$. (Source: Geometry Unbound)
6. Use sine law. Trig Ceva works, but a cleaner method is to write PB in two ways using a, h, and α. If you are not using the condition on h, you have not solved the problem. (Source: COMC 1999, B4)
7. Let T denote the intersection of AM and CD. Use directed angles and prove $CMDA$ and $KTBA$ are cyclic. Do you see a couple possible configurations? (Source: MOP 1991)
8. You first need to prove the points are cyclic, or at least convex. Otherwise, you will never be able to make your argument precise. Can you prove all the axes of symmetry must meet at a point? (Source: IMO 1998, #1)

Exercises: Power of a Point

1. To prove the radical axis is a line, use coordinates! To prove the radical center exists, let P be the intersection of two radical axes. What is its power with respect to each circle?
2. The angle bisector theorem guarantees $\frac{AD}{AB} = \frac{DC}{BC}$. (Source: Saint Petersburg 1996)
3. Prove that BO is tangent to circle OEA. (Source: Geometry Unbound)
4. Let ω be the circle centered at X with double the radius of ω_1 and ω_2. Then Y is the radical center of ω and the zero-radius circles centered at A_1 and A_2.
5. The following are cyclic: $BFEC, DFEM, RBQC$. Show $DQ \cdot DR = PD \cdot MD$. (Source: MOP 1998)
6. Let R be the intersection of PQ and the line. Write down all lengths and use the fact that R has equal power with respect to all three circles. Use directed lengths.
7. B and C have equal power with respect to ω_2. Use this to prove triangles PCD and BEC are similar.
8. Let Q be the point where the three lines meet for some exceptional point, and let M be the midpoint of PQ. Prove M is the radical center of $\Gamma_1, \Gamma_2, \Gamma_3$ and that it is constant distance from P. (Source: APMO 2009, #3)

Challenge Problems

1. What happens when you reflect about X? Let A' be the image of A. Then you need to prove $A_1A' = A_2A'$. (Source: Po-Shen Loh)
2. Let Q' be the circumcenter of $\triangle MAN$. Prove that $\angle MAQ' = \angle MAQ$, and hence $Q' = Q$. This means $QM = QA = QN$, and the rest should follow. This technique is called working backwards, and it is very useful! (Source: MOP 1998)

3. Prove that $BQOC$ is a cyclic quadrilateral. Directed angles work nicely. (Source: China)
4. Draw triangle $A'EP$ similar to AQF, and see what happens. (Source: Crux)
5. By congruent triangles, you should get $\angle AXI$ equals $\angle DYI$ or $\angle CYI$. Only one of these two options is possible. Now chase all angles and look for similar triangles. (Source: Bulgaria 2007, #1)
6. Don't be scared! There are a lot of points, but the problem is still pure angle-chasing. Let Z be the intersection of AT and BS. Then $AZBP$ is a parallelogram, $SBCQ$ is cyclic, and S, A, C are collinear. Now look for some congruent triangles. (Source: Bulgaria 2008, #1)
7. Let AE intersect BD at P, and let AZ intersect BD at M. Prove that (a) $AZPY$ is cyclic with diameter AP, (b) M is the midpoint of PD, and then use power of a point. (Source: IMO Short List 1998, G8)
8. Let E be where AN hit Ω_2 again. Prove that CE is tangent to Ω_1 at C, then prove CO_2 bisects $\angle DCN$. (Source: IMO 1999, #5)
9. Try dissecting $ABCD$ into ABM, BCM, CDM, DAM, rescaling some of these triangles, and then putting them together to make a new quadrilateral. Try to make sure the new figure has lengths CD and $\frac{BM}{MA} \cdot AD$ appearing in it somewhere. That way, the complicated product condition simplifies to just showing two lengths are equal. (Source: IMO Short List 1999, G7)
10. A purely synthetic solution to this problem is quite involved, but other approaches work cleanly and fast. Use trig-Ceva to prove that the point exists if the diagonals are perpendicular. (You see how the point has to be constructed, right?) For the other direction, construct a quadrilateral with perpendicular diagonals, and show it is the same as what you started with. (Source: IMO Short List 2008, G6)
11. This was the hardest problem on IMO 1982, but it requires only a straightedge and compass to get the key idea! If you draw the diagram carefully, you should see the sides of $S_1S_2S_3$ are parallel to the sides of $M_1M_2M_3$, which solves the problem. Do you see why? The angle-chasing requires careful book-keeping more than anything else. (Source: IMO 1982, #2)
12. This problem is awfully hard, as befits an IMO #6, but I gave you a big hint already: the triangle formed is precisely $S_1S_2S_3$ from the previous question. To prove this, you need to show that if you take T_1, reflect it about the bisector of $\angle A$, then reflect the result about T_1T_2, you end up on H_1H_2. The composition of these reflections is a rotation centered at the intersection of these two lines. If this intersection point is X, prove $\angle BXA = 90^\circ$ and use cyclicness of $BAXH_1$ and $BIXT_1$. (Source: IMO 2000, #6)