Problem Set (Graph Theory)

October 2018

Turan's Theorem

1) For a pair $A = (x_1, y_1)$ and $B = (x_2, y_2)$ of points on the coordinate plane, let $d(A, B) = |x_1 - x_2| + |y_1 - y_2|$. We call a pair (A, B) of (unordered) points harmonic if $1 < d(A, B) \leq 2$. Determine the maximum number of harmonic pairs among 100 points in the plane.

2) 155 birds $P_1, \ldots, P_{155}$ are sitting down on the boundary of a circle C. Two birds P_i, P_j are mutually visible if the angle at centre $m(\cdot)$ of their positions $m(P_iP_j) \leq 10°$. Find the smallest number of mutually visible pairs of birds, i.e. minimal set of pairs $\{x, y\}$ of mutually visible pairs of birds with $x, y \in \{P_1, \ldots, P_{155}\}$. One assumes that a position (point) on C can be occupied simultaneously by several birds, e.g. all possible birds.

Extremal Graph Theory

3) Among a group of 120 people, some pairs are friends. A weak quartet is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets?

4) Let $n \geq 1$ be an integer and let X be a set of $n^2 + 1$ positive integers such that in any subset of X with $n+1$ elements there exist two elements $x \neq y$ such that $x \mid y$. Prove that there exists a subset $\{x_1, x_2, \ldots, x_{n+1}\} \in X$ such that $x_i \mid x_{i+1}$ for all $i = 1, 2, \ldots, n$.

Ramsey's Theorem

5) An international society has its members from six different countries. The list of members contain 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

6) Six points are joined pairwise by red or blue segments. Must there exist a closed path consisting of four of the segments, all of the same color?

Matching

7) Let A be a finite set with subsets $A_1, ..., A_n$, and let $d_1, ..., d_n \in N$. Show that there are disjoint subsets $D_k \subset A_k$ with $D_k = d_k$ for all k, if and only if

$$\bigcup_{i \in I} A_i \geq \sum_{i \in I} d_i$$

for all $I \subset 1, . . . ,$ n.

8) A holey triangle is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A diamond is a $60° - 120°$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \leq k \leq n$.

Unexpected Application

9) Let n be a positive integer and let S be a set of $2^n + 1$ elements. Let f be a function from the set of two-element subsets of S to $\{0, \ldots, 2^{n-1} - 1\}$. Assume that for any elements x, y, z of S, one of $f(\{x, y\}), f(\{y, z\}), f(\{z, x\})$ is equal to the sum of the other two. Show that there exist a, b, c in S such that $f(\{a, b\}), f(\{b, c\}), f(\{c, a\})$ are all equal to 0.