Inequalities

1 Majorization

Say $x_1 \geq x_2 \geq \cdots \geq x_n$ majorizes $y_1 \geq y_2 \geq \cdots \geq y_n$ if $\sum x_i = \sum y_i$, and $\sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i$ for $k < n$.

Majorization Inequality: If f is convex, and $x_1, x_2, \ldots, x_n$ majorizes $y_1, y_2, \ldots, y_n$, then $\sum f(x_i) \geq \sum f(y_i)$.

Very general scenario: Minimize $\sum f(x_i)$ subject to the condition $\sum f(x_i)$ is constant, and possibly there is a lower and/or upper bound to all of the x_i.

f will have alternating regions of concavity/convexity (check with the second derivative).

We will now apply the "explode and collect" method (generalization of the method behind the left concave function theorem). Take some concave region containing two or more x_i, choose two of the x_i, and move them apart from each other, keeping their sum constant, until one of them hits either a bound on the x_i, or one of them hits a convex/concave boundary. Repeat this process until each concave region has at most one x_i in its interior. Now, in each convex region, take all x_i inside, and replace them with a bunch of copies of their mean (by Jensen's). We have thus reduced the problem to one in which all x_i inside each convex region are equal, there is at most one x_i strictly inside any concave region, and there may be some x_i on the upper/lower boundary.

2 Elementary Symmetric Function Fudging

This is a very general method to deal with inequalities which one can rewrite in terms of elementary symmetric polynomials. It works well where simple fudging fails, and it can be used to fudge when the condition is more complicated than $\sum x_i$ =constant.

Assume the inequality is rewritten in terms of the elementary symmetric polynomials $f(\sigma_1, \ldots, \sigma_n) \geq 0$, and the condition is rewritten as $g(\sigma_1, \sigma_2, \ldots, \sigma_n)$ =constant. The idea is that instead of fudging the individual x_i, we fudge the coefficients of the polynomial $(x - x_1)(x - x_2)\ldots(x - x_n)$ (i.e. the elementary symmetric functions), until something interesting happens. In particular, if we can continuously deform the σ_i in such a way that f is non-increasing and g remains constant **while keeping the σ_i bounded**, then we can continue doing this UNTIL either two of the x_i are equal, or one of the x_i has hit a boundary condition.

In the case of three-variable inequalities, this completely crushes the inequality, since this reduces it to a two-variable inequality in either case. Furthermore, if an x_i hits a boundary, we can reapply the method with one fewer variable.

Let's consider the ramifications of this method for homogenous inequalities: since there are no conditions placed on the x_i, we effectively just have to have f monotonic in one of the σ_i. This often happens with $\sigma_n = x_1 x_2 \ldots x_n$. In fact, one often doesn't need to expand anything out! If you had theoretically expanded everything out and found that the resulting inequality had degree $\leq 2n-1$, then it is linear in σ_n, so it will be monotonic! What's better: If you had expanded everything out and found that the resulting inequality had degree $\leq 3n-1$, then if you can show the coefficient of σ_n^2 is ≤ 0, then it is concave in σ_n, so achieves its minimum at the boundaries.

Hence we have shown that all symmetric 3-variable inequalities of expanded degree ≤ 8 are pretty doable.

Here's about as hard as it gets:

APMO 2004 $(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+ac+bc)$

When expanded out, we get $(abc)^2+2(ab+ac+bc)^2-4(a+b+c)abc+4(a+b+c)^2-17(ab+ac+bc)+8 \geq 0$. This is not decreasing in ANY of the elementary symmetric polynomials, so it appears the method is doomed to failure. However, we shouldn't give up just yet. First, $ab+ac+bc$ attains the functions minimum at $\frac{17}{4}$, so we can move it towards there until either two of a, b, c are equal, one is 0, or it attains $\frac{17}{4}$. Unfortunately, if it attains $\frac{17}{4}$, we're in a bit of a pickle. The trick is to realize that we aren't restricted to fudging one elementary symmetric polynomial at a time. If we do anything even remotely clever, we get a ridiculously fine-tuned fudge on a, b, c which with any luck destroys the problem. Indeed in this case, we have abc and $a+b+c$ intertwined in F as $(abc)^2-4(a+b+c)(abc)+4(a+b+c)^2 = (abc-2(a+b+c))^2$, so if we SIMULTANEOUSLY decrease abc twice as fast as we decrease $a+b+c$ while keeping $ab+ac+bc$ constant, we're going to force either two of a, b, c equal, or one of them 0. When one is 0, it is incredibly easy, and when $a=b$, we treat F as a quadratic in c and consider its discriminant.

MORAL Fudge the elementary symmetrics in ANY way until something breaks.

n-variable example Show that $x_1+\cdots+x_n \geq (n-1)\sqrt[n]{x_1 \ldots x_n}+\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}$

Fix $x_1+x_2+\cdots+x_n$, and consider when the R.H.S. is maximized (exists because the x_i lie in a compact set). If we consider x_i, x_j, x_k, then if we keep the product and sum fixed, the R.H.S. is decreasing in $x_ix_j+x_ix_k+x_jx_k$. Thus, we could fudge this term to increase the right hand side UNLESS two of them were equal. Hence we've shown for any three of the x_i, two are equal. Thus we can assume in the original problem WLOG the x_i take on at most two distinct values. Fixing the product to be 1, we now have basically a one variable inequality. The rest is boring.

Note If we had tried at the beginning to fix $x_1 \ldots x_n$, then it is not immediately clear that the difference of the two sides attains its maximum (since the x_i don't lie in a compact set). However, it is possible to show that if the largest of the x_i was too big, the difference couldn't attains its maximum.

2.1 Gritty details, and compact sets

How do we know that this "varying" process actually ends? A particular problem we might have is that we might vary the roots right off the real line (to infinity and beyond)!

Our main goal really is to reduce the inequality to one where the x_i only take on (almost) k distinct values. We start optimistically with $k=1$, then keep increasing it until it works (in practice, this stops at $k=2$).

To avoid all of the problems that are plaguing us, we have to do all of our reasoning inside some compact set (closed bounded set) C, which the x_i can vary in.

Here's how you do this:

1. Assume we want to show the inequality is true for a specific set of x_i. Choose a compact set C containing the x_i in n-dimensional space (possibly do this later with an ansatz).

2. If a point in C has some $k+1$ of its coordinates distinct, we try to parametrize the elementary symmetric polynomials in these $k+1$ variables in such a way that f is a non-decreasing function of the parametrization (we have no local restrictions on the parametrization except possibly having to keep the x_i inside C). This is the fudging operation.

3. Since C is compact, f attains its minimum value on C. If you want, just assume the x_i are at this global min, but in fact it turns out to be more useful sometimes to take the x_i s.t. they extremize some monovariant with respect to the fudging operation while having f evaluate to $\leq$ the original evaluation. Provided the monovariant is continuous, we can do this by intersecting C with the set of all points in space s.t. the monovariant is $\leq$ what it was for the original choice of the x_i.

4. At this specific point, none of the elementary symmetric fudges can be valid for any $k+1$-tuple of distinct coordinates. This usually implies something awesome.

Choosing C

Now, when we choose the compact set, an excellent choice if the x_i are all ≥ 0 is the set $\sigma_1 \leq r$ for some real r. The reason is that σ_1 bounded implies all of the x_i are bounded if they are non-negative. Furthermore, the only local obstruction for varying the σ_i is that σ_1 got too big, or one of the x_i hit 0.

Another possible choice is a hypercube (i.e. just bound all of the x_i individually). Unfortunately, it is not easy to fudge the x_i once they've hit the boundary of the hypercube.

2.2 Very specific 3-variable bash for people who have too much time on their hands

When doing an 3-variable elementary symmetric bash, then letting $3u = a+b+c$, $3v^2 = ab+ac+bc$, $w^3 = abc$, we have the inequalities:

$u \geq v \geq w$ (obvious)

$w^3 + 3u^3 \geq 4uv^2$ (Schur's)

$27(4(u^2-v^2)^3 - (w^3 - 3uv^2 + 2u^3)^2) = (a-b)^2(b-c)^2(c-a)^2 \geq 0$

After rewriting, this gives the absolute best bounds on w possible, and you can show that if w is between these bounds, there exist a, b, c s.t. it is actually attained.

Also, if u, v, w satisfy the crazy inequality above, then $u, v, w \geq 0 \Leftrightarrow a, b, c \geq 0$

3 Stronger Mixing Variables Lemma

Show that $f(x_1, x_2, \ldots, x_n)$ doesn't increase if we replace some two of the x_i with their mean (where the mean is chosen to match with the condition, i.e. quadratic mean if the sum of the squares is constant). In general, one only has to show this is true for the largest and the smallest of the x_i. This works wonderfully for inequalities with multiple equality cases, since in those cases it often happens that if we apply SMV to all but the largest of the x_i, or all but the smallest, we reduce it down to a two-variable inequality while preserving the equality cases.

Example: Symmetric Inequalities of degree ≤ 3 are all completely trivial (Pham Kim Hung)

If F is a symmetric (not necessarily homogenous) polynomial expression of degree ≤ 3 in n variables, then $F \geq 0$ for all non-negative $x_1, \ldots, x_n$ if and only if it is true when the x_i take on at most one other value than 0 (i.e. $\{x_1, \ldots, x_n\} \subset \{0, t\}$ for some $t \geq 0$) .

To show this, assume that there is currently an $x_i \neq x_j$. If we keep $x_i + x_j$ fixed, then F is linear in $x_i x_j$. If the coefficient of $x_i x_j$ is ≤ 0, then we can replace x_i, x_j with their average, which does not increase F. Otherwise, we can replace x_i, x_j with $0, x_i + x_j$, and we are done by induction.

4 Sum of Squares

To appreciate inequalities to their fullest, we need to adress Sum of Squares. When you have a symmetric inequality which involves rational functions with equality case $a = b = c$, try to write it in the form $S_a(b-c)^2 + S_b(a-c)^2 + S_c(a-b)^2 \geq 0$. To do this, we use David Arthur's five step method (X is used to denote the L.H.S.):

1. Group X into terms which are 0 when $a = b = c$
2. Write everything as a multiple of $a-b$, $b-c$, $c-a$.
3. Group together all terms involving $a-b$, $b-c$, $c-a$, add or subtract things from each term to ensure that they vanish when $a = b = c$
4. Write everything in terms of $(a-b)^2, (b-c)^2, (c-a)^2, (a-b)(c-b), (b-a)(c-a), (a-c)(b-c)$
5. Replace $(a-c)(b-c)$ with $\frac{1}{2}((a-c)^2 - (a-b)^2 - (b-c)^2)$

Now that you have it in SOS form, here are five criteria for when the expression is ≥ 0:

1. $S_a, S_b, S_c \geq 0$
2. $a \geq b \geq c$ and $S_b, S_b + S_c, S_b + S_a \geq 0$
3. If $a \geq b \geq c$ and $S_a, S_c, S_a + 2S_b, S_c + 2S_b \geq 0$
4. If $a \geq b \geq c$ and $S_b, S_c, a^2 S_b + b^2 S_a \geq 0$
5. If $S_a + S_b + S_c \geq 0$ and $S_a S_b + S_a S_c + S_b S_c \geq 0$

Exercise: Prove all of these.

Generally you don't need the more specialized ones, but it's nice to have them in your repertoire.

Now, David Arthur has a very nice extension of this method for cyclic inequalities (3-variable). Any cyclic inequality $\sum_{cyc} X$ can be broken up into two parts, $\frac{1}{2}\sum_{sym} X$ and $\sum_{cyc} X - \frac{1}{2}\sum_{sym} X$. The former is symmetric, and the latter is anti-symmetric. In nearly all cases, we can pull out a $(a-b)(b-c)(c-a)$ from the anti-symmetric part, so that we get the expression written as $(a-b)^2 S_c + (b-c)^2 S_a + (c-a)^2 S_b + (a-b)(b-c)(c-a)S$.

If $S_a, S_b, S_c \geq 0$, then apply AM-GM to the first three terms; it is easy to show that we have solved the inequality if $27S_aS_bS_c \geq |S^3(a-b)(b-c)(c-a)|$

Note: Sometimes when there are lots of $a-b$ type terms floating around, it is fruitful to try and fudge a, b, c by adding a constant k to each of them and seeing how the inequality varies.

5 Problems

Here's some nifty problems:

1. If $abc = 1$, show $a^2 + b^2 + c^2 + 6 \geq \frac{3}{2}(a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c})$
2. If $a, b, c \geq 0$ and $a + b + c = 3$, then $\frac{9}{10} \leq \sum_{cyc} \frac{a}{2a+bc} \leq 1$
3. $a_i \geq 0$ and $\sum a_i = n$. Show $(\sum a_i^2) + a_1a_2 \dots a_n \geq n + 1$
4. If $a^3 + b^3 + c^3 = abc(a + b + c)$, then show that $9(ab + ac + bc) + 24 \geq 17(a + b + c)$
5. If $a, b, c \geq 0$, then $2\sum_{cyc} \frac{a^3}{b+c} + (a + b + c)^2 \geq 4\sum_{cyc} a^2$
6. $4(a^3 + b^3 + c^3) + 15abc + 54 \geq 27(a + b + c)$ if $a, b, c \geq 0$
7. If $a, b, c \geq 0$ and $a + b + c = 3$, then $(1 - a + a^2)(1 - b + b^2)(1 - c + c^2) \geq 1$
8. If $a, b, c, d > 0$ are s.t. $abcd = 1$, then $\sum_{cyc}(a - 1)(a - 2) \geq 0$
9. Let x_i be non-negative numbers whose sum is 1. Show that $2\sum x_i^3 + n^2 \leq (2n + 1)\sum x_i^2$
10. If a, b, c are positive numbers whose sum is 3, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq a^2 + b^2 + c^2$.
11. Prove that for $x, y, z \geq 0$ we have $\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(x+z)^2} \geq \frac{9}{4(xy+xz+yz)}$
12. Let n be an integer ≥ 2. Prove that if the real numbers $a_1, \dots, a_n$ satisfy $a_1^2 + a_2^2 + \dots + a_n^2 = n$, then $\sum_{i<j} \frac{1}{n-a_ia_j} \leq \frac{n}{2}$. Hint: Bash. Hint2: What does the graph of $y = f(x) + c$ look like?