A Nice Theorem In Multiplicative Functions

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Abstract. The theorem-to be discussed in this paper is a nice and powerful one involving multiplicative functions.

We define $f : \mathbb{N} \to \mathbb{N}$ to be a multiplicative function if $f(mn) = f(m)f(n)$ for any $m \perp n$. We also support to introduce the notation $m \perp n$ supposedly meaning that $\gcd(m, n) = 1$, in other words, m and n have no common factors other than 1. In fact, we call this *weakly multiplicative* in this case and *strongly multiplicative* if $f(mn) = f(m)f(n)$ for every $m, n \in \mathbb{N}$. In this paper, we will generally consider weak ones, unless stated. And throughout the note, let f be a multiplicative function and the prime factorization of n be,

$$n = p_1^{e_1} \cdots p_k^{e_k}$$

with $p_1, \ldots, p_k$ distinct primes or, shortly $n = \prod_{p|n} p^e$ for $p^e | n, p^{e+1} \not| n$.

1. MFT

We are talking about this particular theorem. Let's agree to call it the *MFT*, short for *Multiplicative Function Theorem*.

Theorem 1 (MFT). *Let f be a multiplicative function.*

$$\begin{aligned} \sum_{d|n} f(d) &= (1 + f(p_1) + \ldots + f(p_1^{e_1})) \cdots (1 + f(p_k) + \ldots + f(p_k^{e_k})) \\ &= \prod_{i=1}^{k} \sum_{j=0}^{e_i} f\left(p_i^j\right) \end{aligned}$$

Remark. This theorem can be re-written as: If $F(n) = \sum_{d|n} f(d)$ then

$$\sum_{d|n} f(d) = \prod_{p|n} F(p)$$

Proof. Let's assume T is the expansion of the right side of equation (1.1), and

$$S = \sum_{d|n} f(d)$$

If $d|n$, is a divisor of n, then $d = p_1^{w_i} \cdots p_k^{w_k}$ where $0 \le w_i \le e_i$ for $1 \le i \le k$. Then we have,

$$\begin{aligned} f(d) &= f\left(p_1^{w_1}\right) \cdots f\left(p_k^{w_k}\right) \\ &= f\left(p_1^{w_1}\right) \cdots f\left(p_k^{w_k}\right) \end{aligned}$$

which is a term that is present in T. Thus, we conclude that each term of S is a member of T. Now we easily find that the converse is also true. Because, after multiplying, we see that every term in T is of the form

$$f\left(p_1^{w_1}\right) \cdots f\left(p_k^{w_k}\right)$$

which can be written as

$$f\left(p_1^{w_1}\right) \cdots f\left(p_k^{w_k}\right)$$

or $f(d)$. Therefore, every term in T is also in S as well. Combining these two, $S = T$.

$\square$

Remark. Since $m \perp 1$ for all $m \in \mathbb{N}$, setting $m = 1$, $f(m \cdot 1) = f(m)f(1)$ implies $f(1) = 1$. Therefore, each term starts with 1, instead of $f(1)$ or $f(p_i^0)$.

This theorem can infer the following:

THEOREM 2. *If $F(n) = \sum_{d|n} f(d)$ with f multiplicative, then so is F.*

THEOREM 3. *If $F(d) = \sum_{d|n} f(d)$ is multiplicative, then so is $\sum_{j=0}^{e_i} f(p_i^j)$.*

2. Problems

We see some applications of this theorem by solving some problems. Note that, if we can solve the problem for $F(p)$, we are done. First we see another proof of number of divisors formula.

2.1. Find the number of divisors of n.

Solution. Set $f(d) = 1$, which is obviously multiplicative. Then,

$$\sum_{d|n} 1 = \tau(n)$$

$$\begin{aligned} F(p_i) &= f(1) + f(p_i) + \ldots + f\left(p_i^{e_i}\right) \\ &= 1 + \ldots + 1 = e_i + 1 \end{aligned}$$

Therefore, $\tau(n) = (e_1 + 1) \cdots (e_k + 1)$.

Now we prove the following generalization of a well known fact.

2.2 (Generalization Of Sum Of Divisors)**.** Let $\sigma(n)$ is the sum of divisors of n. And let's say $\sigma_r(n)$ is the sum of r^{th} power of the divisors of n, that is, if $d_1, d_2, \dots, d_{\tau(n)}$ are divisors of n,

$$\sigma_r(n) = d_1^r + \ldots + d_{\tau(n)}^r$$

Then $\sigma(n) = \sigma_1(n)$, the usual sum of divisors of n.

$$\sigma_r(n) = \frac{p_1^{(e_1+1)r} - 1}{p_1 - 1} \cdots \frac{p_k^{(e_k+1)r} - 1}{p_k - 1}$$

Solution. Set $f(n) = n^r$. This is multiplicative(in fact a strong one) since $f(mn) = (mn)^r = m^r n^r = f(m)f(n)$. Then,

$$\sigma_r(n) = \sum_{d|n} d^r = f(d)$$

$$\begin{aligned} F(p_i) &= 1 + f(p_i) + \ldots + f(p_i^{e_i}) \\ &= 1 + p_i^r + \ldots + p_i^{e_i r} \\ &= \frac{p_i^{(e_i+1)r} - 1}{p_i - 1} \end{aligned}$$

Therefore,

$$\sigma_r(n) = \frac{p_1^{(e_1+1)r} - 1}{p_1 - 1} \cdots \frac{p_k^{(e_k+1)r} - 1}{p_k - 1}$$

Note. The formula of usual sum of divisor follows if we set $r = 1$.

2.3. Prove that,

$$\sum_{d|n} \varphi(d) = n$$

where $\varphi(n)$ is the Euler function.

Solution. It is well known that φ is multiplicative[1], we can invoke MFT here.

$$\begin{aligned} F(p) &= \sum_{i=0}^{e} \varphi(p^i) \\ &= 1 + (p-1) + p(p-1) + \ldots + p^{e-1}(p-1) \\ &= 1 + (p-1)\left(1 + p + \ldots + p^{e-1}\right) \\ &= 1 + (p-1)\left(\frac{p^e - 1}{p - 1}\right) \\ &= p^e \end{aligned}$$

Hence, $\sum_{d|n} \varphi(d) = \prod_{p|n} F(p) = \prod_{p|n} p^e = n$.

[1] We don't prove it again here.

2.4. *Mobius Function* $\mu(n)$ is defined by

$$n = \begin{cases} 1 \text{ if n=1} \\ (-1)^k \text{ if } n \text{ is square-free} \\ 0 \text{ otherwise} \end{cases}$$

Prove that, $\sum_{d|n} \mu(d) = 0$ for $n > 1$.

Solution. First, note that, for a prime p, $\mu(p^a) = 0$ for $a > 1$. Therefore,

$$\begin{aligned} F(p) &= \mu(1) + \mu(p) + 0 + \ldots + 0 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

since $\mu(p) = (-1)^1 = -1$. Therefore, $\sum_{d|n} \mu(d) = \prod_{p|n} \sum_{i=0}^{e_i} \mu(p^i) = 0$.

2.5. Prove that,

$$\sum_{d|n} \mu(d)f(d) = \prod_{p|n} (1 - f(p))$$

2.6. Prove that,

$$\left| \sum_{d|n} \frac{\mu(d)\sigma(d)}{d} \right| \geq \frac{1}{n}$$

Remark. This problem is sourced from: `http://www.artofproblemsolving.com/Forum/viewtopic.php?f=57&t=487477&p=2754756&hilit=multiplicative+function#p2754756`

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