Geometry
An Introduction to Projective Geometry

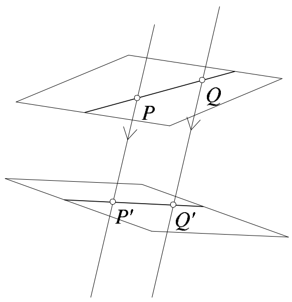
IMO Training 2006-2007

October 21, 2006 (based on the book "Geometric Transformations" by I.M.Yaglom)

1 Parallel Projection

1.1 Definition

Let Π and Π' be two distinct planes. A *parallel projection* of Π onto Π' in a direction a is a mapping of Π onto Π' that associates each point P in the plane Π the point P' in the plane Π' such that PP' is parallel to a.

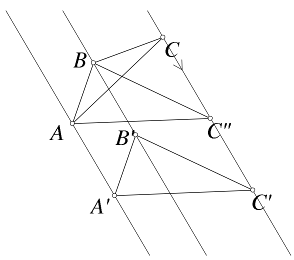


1.2 Properties

1. A parallel projection maps lines in Π onto lines in Π'
2. A parallel projection maps parallel lines into parallel lines.
3. A parallel projection preserves the ratio of the lengths of two collinear segments. It also preserves the ratio of the lengths of two segments on parallel lines.
4. A parallel projection preserves the ratio of the areas of two figures in the plane.

1.3 Fundamental Theorem

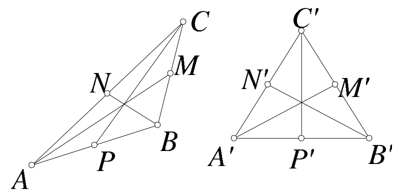
Let A, B, C be three non-collinear points in a plane Π, and let M, N, P be three non-collinear points in a plane Π'. Then the plane Π and Π' can be placed so that there exists a parallel projection of Π onto Π' which maps $\triangle ABC$ onto a $\triangle A'B'C'$ similar to $\triangle MNP$.



Basic Idea. By a parallel projection, we can transform any triangle to a specially chosen triangle so the the problem is easier to handle. Note that both concurrency, collinearity and ratio of lengths on a line are preserved under the transformation.

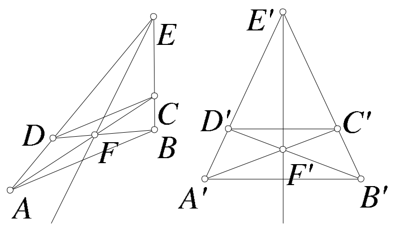
1.4 Examples

1. Prove that the three medians of a triangle are concurrent.



Proof. Let ABC be an arbitrary triangle in the plane Π. By the fundamental theorem, there exists a parallel projection of Π to a plane Π' such that the image $\triangle A'B'C'$ is an equilateral triangle. By property 3, the images of the midpoints of the sides of $\triangle ABC$ are the midpoints of the sides of $\triangle A'B'C'$. Since $\triangle A'B'C'$ is equilateral, the medians are the concurrent by symmetry. Hence the medians of the original triangle ABC are concurrent.

2. Prove that the line joining the point of intersection of the extensions of the the nonparallel sides of a trapezium to the point of intersection of its diagonals bisects the base of the trapezium.



Proof. Let $ABCD$ be a trapezium with $AB//DC$, E, F be the points of intersection of BC and AD, AC and BD respectively. Under a suitable parallel projection, the image of triangle ABE will be an isosceles triangle $A'B'E'$ with $A'E' = B'E'$; at the same time $A'B'C'D'$ is a trapezium with $A'B'//D'C'$ by property 2. Now $E'F'$ is the axis of symmetry of the trapezium $A'B'C'D'$ and bisects the base $A'B'$ and $D'C'$. Hence EF bisect the base AB and DC by property 3.

1.5 Problems

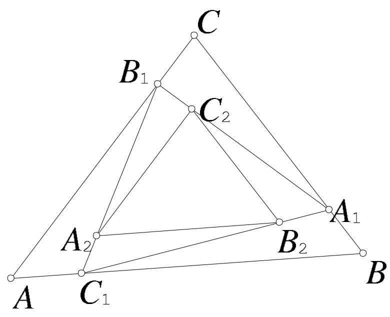
1. Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC such that

$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k.$$

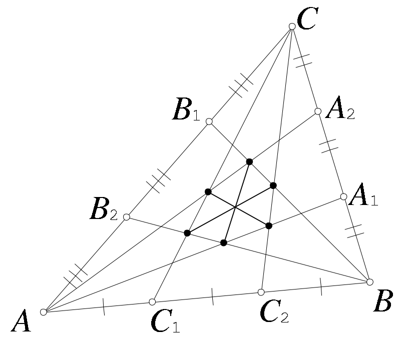
Furthermore, let A_2, B_2, C_2 be points on the sides B_1C_1, C_1A_1, A_1B_1 of a triangle $A_1B_1C_1$ such that

$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = \frac{1}{k}.$$

Show that triangle ABC and $A_2B_2C_2$ are similar.



2. Through each of the vertices of triangle ABC we draw two lines dividing the opposite side into three equal parts. These six lines determine a hexagon. Prove that the diagonals joining opposite vertices of this hexagon are concurrent.

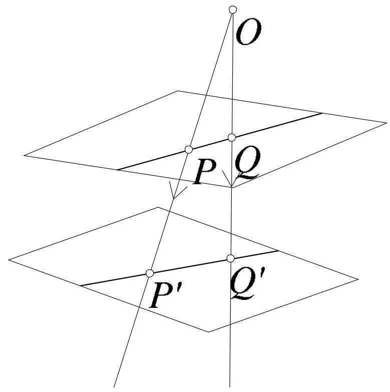


2 Central Projection

2.1 Definition

Let Π and Π' be two planes and O be a point not on either of them. The *central projection* from Π to Π' with center O is the mapping that send each point P in Π to the point P' in Π' such that OPP' is a straight line.

If Π and Π' are parallel to each other, then the central projection is simply a homothety. The case where Π and Π' are not parallel to each other is our main consideration.

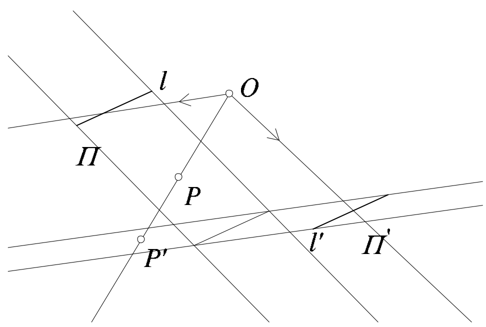


2.2 Fundamental Theorem

Let $ABCD$, $PQRS$ be quadrilaterals on Π and Π' respectively. Then there exists a central projection from Π to Π' that maps A, B, C, D onto A', B', C', D' such that the quardrilateral $A'B'C'D'$ is similar $PQRS$.

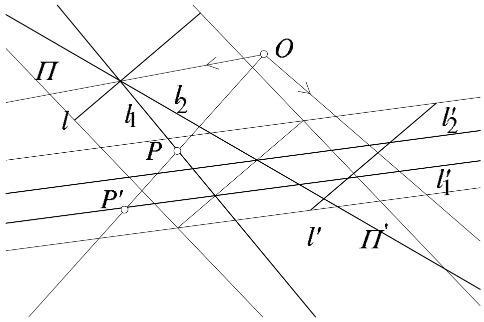
2.3 Properties

1. A central projection carries lines in Π into lines in Π' with an exception of a line l which maps to nowhere. Each line in Π' is the image of a line in Π with an exception of a line l' on Π' which no points on Π maps on it.

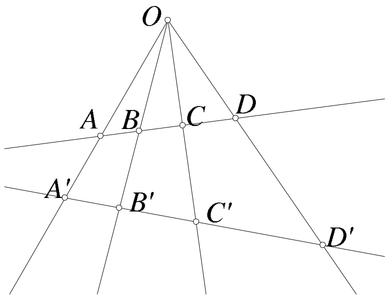


2. If l_1 and l_2 are lines on Π that intersects on a point on l, then the images l_1' and l_2' on Π' are parallel to each other. If l_1 and l_2 are parallel lines on Π, then the images l_1' and

l'_2 intersects at a point on l'. The exceptions are the case where l_1 and l_2 are parallel to l, then the images l'_1 and l'_2 are both parallel to l'.



3. Let A, B, C, D be points on a line on Π and A', B', C', D' are their images under a central projection. Then
$$\frac{AC \cdot BD}{AD \cdot BC} = \frac{A'C' \cdot B'D'}{A'D' \cdot B'C'}.$$
The fraction on the left hand side is call the cross-ratio of A, B, C, D.



4. In general, let $A_1, \ldots, A_{2n}$ be distinct points on Π and $A'_1, \ldots, A'_{2n}$ be their images under a central projection. Then
$$\frac{A_1A_2 \cdot A_3A_4 \cdots A_{2n-1}A_{2n}}{A_{2n}A_1 \cdot A_2A_3 \cdots A_{2n-3}A_{2n-1}} = \frac{A'_1A'_2 \cdot A'_3A'_4 \cdots A'_{2n-1}A'_{2n}}{A'_{2n}A'_1 \cdot A'_2A'_3 \cdots A'_{2n-3}A'_{2n-1}}.$$

2.4 Points and Line at Infinity

In order to remove the exceptions in the properties, we denote the images of points on the special line L on Π to be *points at infinity* on Π' and the image of L be the *line at infinity* on Π'. Similarly, the *points at infinity* on Π are maps onto the points on the special line L' on Π' and the *line at infinity* on Π are maps onto the special lines L'.

With this notation, two lines are parallel if and only if they intersect at a point at infinity. The first two properties of central projection can be restated as followed.

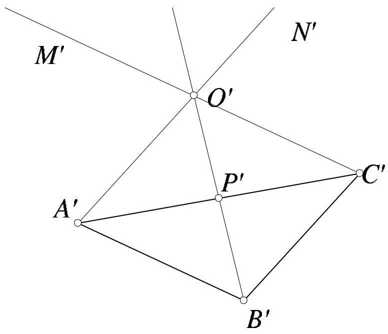
Property 1' Each line on Π are mapped onto a line on Π' under a central projection.

Property 2' A central projection preserves concurrency and collinearity.

The ordinary points together with the points at infinity form the *projective plane*. In a projective plane, every 2 distinct lines intersect at a unique point (either ordinary or at infinity).

2.5 Examples

1. (Ceva's Theorem) Three lines AN, BP and CM, where the points M, N, P lies on the sides of AB, BC, CA of $\triangle ABC$ are concurrent of parallel if and only if
$$\frac{AM}{MB} \cdot \frac{BN}{NC} \cdot \frac{CP}{PA} = 1$$
Proof. Project the plane Π of triangle ABC to a plane Π' so that MN is the special line of Π. If lines AN, BP and CM intersects in a point O or are parallel, then the figure under transformation becomes the figure below,

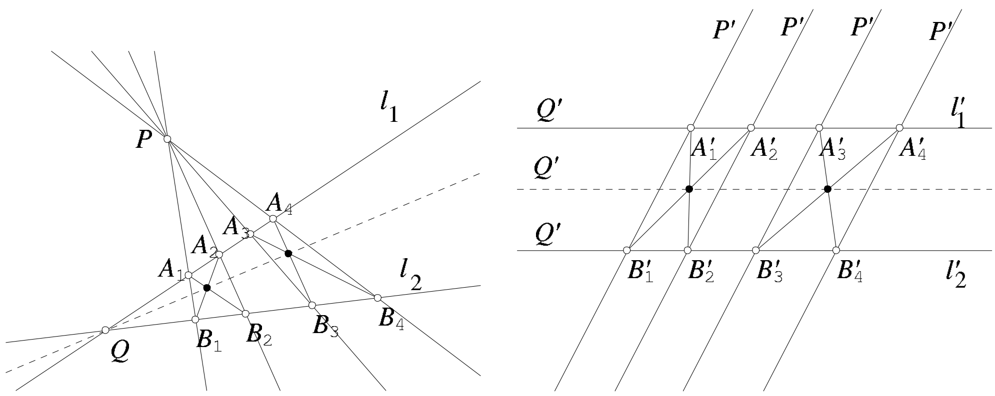


where $A'N'//B'C'$, $C'M'//B'A'$. Hence P' is the midpoint of $A'C'$ as $A'B'C'O$ is a parallelogram. We thus have
$$\frac{A'M'}{M'B'} = \frac{B'N'}{N'C'} = -1 \quad \text{and} \quad \frac{C'P'}{P'A'} = 1.$$
By property 4, we have
$$\frac{AM}{MB} \cdot \frac{BN}{NC} \cdot \frac{CP}{PA} = \frac{A'M'}{M'B'} \cdot \frac{B'N'}{N'C'} \cdot \frac{C'P'}{P'A'} = 1.$$
Conversely, assume that the equality holds. Since under our projection the point M and N are carried into points at infinity, we have
$$\frac{A'M'}{M'B'} = \frac{B'N'}{N'C'} = -1.$$
It follows that
$$\frac{C'P'}{P'A'} = 1,$$
that is P' is the midpoint of $A'C'$. Then the line $A'N'$ is parallel to $B'C'$, $C'M'$ is parallel to $B'A'$ and $B'P'$ intersect in a point O'. Hence lines AN, CM and BP are either concurrent or parallel.

2. (Menelaus' Theorem) Three points M, N, P on sides AB, BC, CA of a triangle ABC are collinear if and only if
$$\frac{AM}{MB} \cdot \frac{BN}{NC} \cdot \frac{CP}{PA} = -1$$

Proof. Project the plane Π of triangle ABC to a plane Π' so that line MN is the special line of Π. If M, N and P are colliear, then the points M', N' and P' are points at infinity on $A'B', B'C'$ and $C'A'$. Hence
$$\frac{A'M'}{M'B'} = \frac{B'N'}{N'C'} = \frac{C'P'}{P'A'} = -1.$$
By property 4
$$\frac{AM}{MB} \cdot \frac{BN}{NC} \cdot \frac{CP}{PA} = \frac{A'M'}{M'B'} \cdot \frac{B'N'}{N'C'} \cdot \frac{C'P'}{P'A'} = -1.$$
Conversely, assume that the equality holds. Since our projection carries M and N into points at infinity M' and N', it follows that
$$\frac{A'M'}{M'B'} = \frac{B'N'}{N'C'} = -1.$$
Hence
$$\frac{C'P'}{P'A'} = -1$$
and therefore P' is the point at infinity on the line $A'C'$. But then P must lie on the special line of Π. This proves the collinearity of M, N and P.

3. Let l_1 and l_2 be two lines in a plane intersecting at a point Q. Let P be a point not on either l_1 or l_2. Through P pass two lines intersecting l_1 and l_2 on A_1, B_1 and A_2, B_2 respectively. Then the locus of points of intersection of A_1B_2 and A_2B_1 is a line through Q.

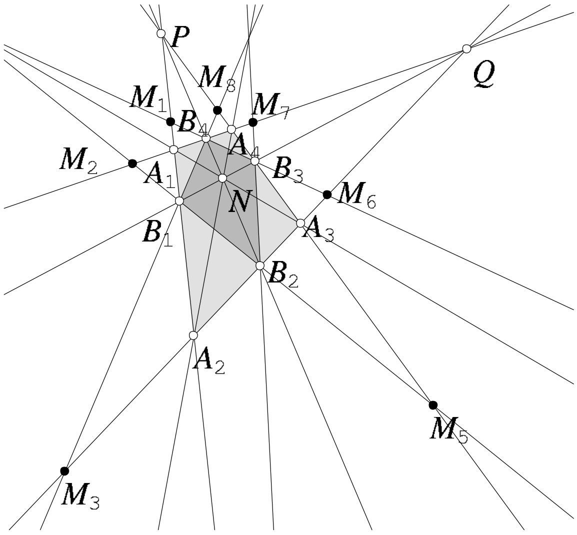


Proof. We consider a central project such that PQ is the special line L of Π. The locus of M goes into a line parallel to and equidistant from l_1' and l_2'. It follows from property 1 that the locus of M is a line. As the image of this line is parallel to l_1' and l_2', this line must pass through Q.

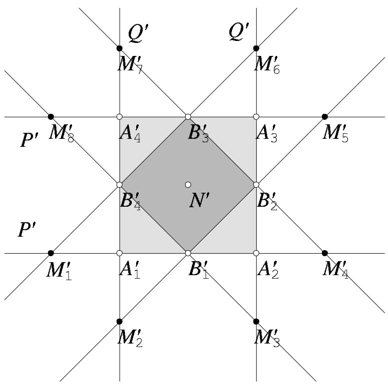
4. Let $A_1A_2A_3A_4$ be a quadrilateral whose diagonals intersects at N, and let pairs of opposite sides meet in P and Q; let B_1, B_2, B_3, B_4 be the points where the sides of the quadrilateral intersect the lines NP and NQ. Let the sides of $A_1A_2A_3A_4$ intersect the sides of the inscribed quadrilateral $B_1B_2B_3B_4$ in $M_1, \ldots, M_8$ as in the figure. Show that

   (a) Lines $M_1M_5, M_2M_6, M_3M_7, M_4M_8$ pass through N.

   (b) Lines M_2M_3 and M_6M_7 pass through P; lines M_1M_8 and M_4M_5 pass through Q.



Proof. By the fundamental theorem, there exists a central projection that maps quadrilateral $A_1A_2A_2A_4$ to a square $A'_1A'_2A'_3A'_4$. Under such a projection, P and Q are carried into points at infinity corresponding to the direction of the sides of the square; lines PN and QN are carried into the midlines of the square; points B_1, B_2, B_3, B_4 are carried into midpoints B'_1, B'_2, B'_3, B'_4 of the sides of the square; point $M_1, \ldots, M_8$ are carried into points $M'_1, \ldots, M'_8$ as shown in the figure.

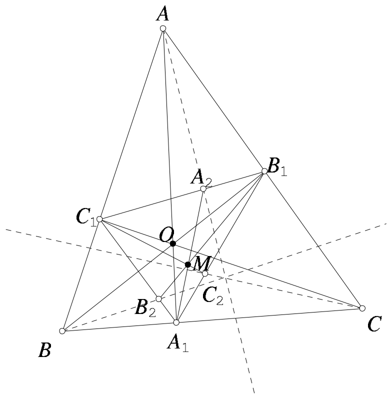


The assertions follow from the following observations:

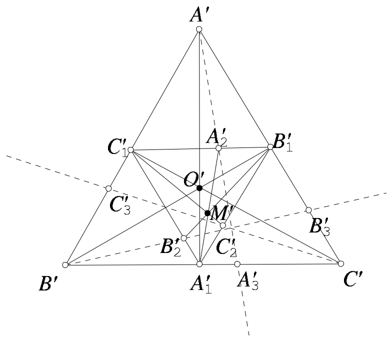
(a) Lines $M'_1M'_5, M'_2M'_6, M'_3M'_7, M'_4M'_8$ intersects at N' by symmetry.

(b) Lines $M_2'M_3'$ and $M_6'M_7'$ are parallel to $A_2'A_3'$; lines $M_1'M_8'$, $M_4'M_5'$ are parallel to $A_1'A_2'$.

5. Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC such that AA_1, BB_1, CC_1 are concurrent. If A_2, B_2, C_2 are points on the sides B_1C_1, C_1A_1, A_1B_1 of the triangle $A_1B_1C_1$ such that A_1A_2, B_1B_2, C_1C_2 are concurrent, show that AA_2, BB_2, CC_2 are also concurrent.



Proof. By the fundamental thereom, there exists a central projection mapping quadrilateral $ABCO$ to a quadrilateral $A'B'C'O'$ so that O' is the centroid of $\triangle A'B'C'$.



Under our projection $\triangle A_1'B_1'C_1'$ is the medial triangle of $\triangle A'B'C'$. Denote A_3', B_3', C_3' the points of intersection of lines $A'A_2', B'B_2'$ and $C'C_2'$ with the opposite sides of $\triangle A'B'C'$, we have

$$\frac{A'C_3'}{C_3'B'} = \frac{B_1'C_2'}{C_2'A_1'}, \quad \frac{B'A_3'}{A_3'C'} = \frac{C_1'A_2'}{A_2'B_1'}, \quad \frac{C'B_3'}{B_3'A'} = \frac{A_1'B_2'}{B_2'C_1'}.$$

Since $A_1'A_2', B_1'B_2', C_1'C_2'$ are concurrent, by Ceva's theorem, we have

$$\frac{B_1'C_2'}{C_2'A_1'} \cdot \frac{C_1'A_2'}{A_2'B_1'} \cdot \frac{A_1'B_2'}{B_2'C_1'} = 1.$$

And so

$$\frac{A'C_3'}{C_3'B'} \cdot \frac{B'A_3'}{A_3'C'} \cdot \frac{C'B_3'}{B_3'A'} = 1,$$

and hence $A'A_3', B'B_3', C'C_3'$ are concurrent by the converse of Ceva's theorem.

2.6 Problems

1. If E and F are the points intersection of opposite sides AB and DC, AD and BC of an arbitrary quadrilateral, show that

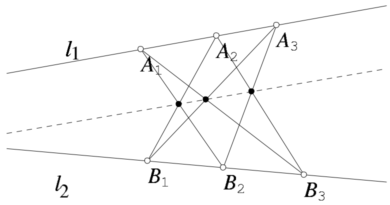
$$\frac{AE \cdot EC}{AF \cdot FC} = \frac{BE \cdot ED}{BF \cdot FD}.$$

2. Let q and two points A and B not on q be given in a plane. Let U, V be two points on q, M be the point of intersection of the lines UA and VB and N be the point of intersection of UB and VA. Then the line MN passes through a fixed point on AB as U, V vary.

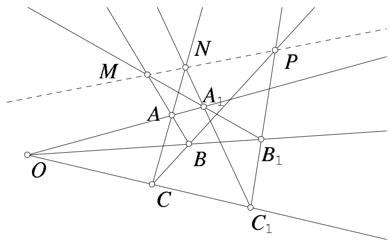
3. Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC such that AA_1, BB_1, CC_1 are concurrent at O. If A_2, B_2, C_2 are points on the sides B_1C_1, C_1A_1, A_1B_1 of the triangle $A_1B_1C_1$ such that the points of intersection of A_1A_2, B_1B_2, C_1C_2 with the opposite sides of $\triangle A_1B_1C_1$ are collinear, show that the points of intersection of AA_2, BB_2, CC_2 with the opposite sides of $\triangle ABC$ are also collinear.

3 Pappus' Theorem and Desargues' Theorem

Pappus' Theorem Let A_1, A_2, A_3 and B_1, B_2, B_3 be points on two lines l_1 and l_2 respectively. Show the intersection points of A_1B_2 and A_2B_1, A_2B_3 and A_3B_2, A_3B_1 and A_1B_3 are collinear.

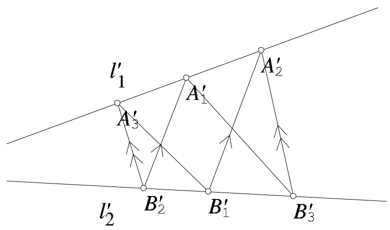


Desargues' Theorem Let ABC and $A_1B_1C_1$ be two triangles. Then the line joining the corresponding vertices of the two triangle are concurrent if and only if the point of intersection of corresponding sides of the two triangles are collinear.



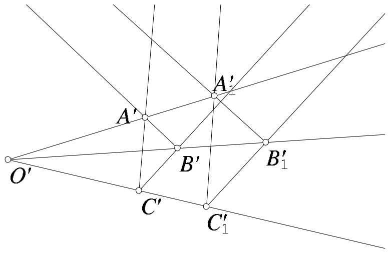
3.1 Proofs

Proof of Pappus' Theorem. Let M, N, P be the points of intersection. Consider a central projection with special line MN. The figure under transformation is shown below.



The lines $A'_1B'_2$ and $A'_2B'_1$ are parallel and the lines $A'_2B'_3$ and $A'_3B'_2$. It follows that $A'_3B'_1$ and $A'_1B'_3$ are also parallel. Hence P' is a point at infinity and lies on $M'N'$. Hence M, N, P are colliear.

Proof of Desargues' Theorem. If AA_1, BB_1, CC_1 are concurrent at O. Let M, N, P be the points of intersection of AB and A_1B_1, BC and B_1C_1, CA and C_1A_1 respectively. Consider the the central projection with MN as the special line. Then the lines $A'B'$ and $A'_1B'_1$ are parallel; the lines $B'C'$ and $B'_1C'_1$ are parallel. As $A'A'_1, B'B'_1, C'C'_1$ are concurrent at O', the triangle $A'B'C'$ are $A'_1B'_1C'_1$ are similar and O' is the center of homothety. Hence $C'A'$ and $C'_1A'_1$ are also parallel, and P' is thus a point at infinity which lies on the line at infinity $M'N'$. Hence M, N, P are collinear.



Conversely if M, N, P are collinear, using the same projection, the corresponding sides of $\triangle A'B'C'$ and $\triangle A_1'B_1'C_1'$ are parallel. Hence $A'A_1', B'B_1', C'C_1'$ are concurrent at the center of homothety.

Note that in both theorem, the exceptional cases can be included in the general case if we consider the points and line of infinity.

4 Central Projection for Circles

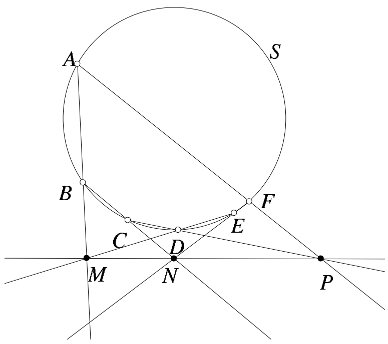
4.1 Fundamental Theorem

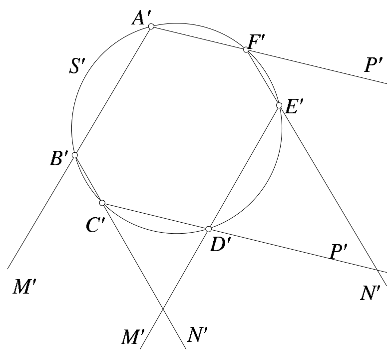
Theorem 1 Let S be a circle in a plane Π and let Q be a point in the interior of S. Then there exists a central projection from Π to a suitable plane Π' which carries S into a circle S' and Q into the center Q' of S'.

Theorem 1' Let S be a circle in a plane Π and let l be a line in Π not intersecting S. Then there exists a central projection from Π to a suitable plane Π' which carries S into a circle S' and l into the line at infintiy l'.

4.2 Pascal's Theorem and Brianchon's Theorem

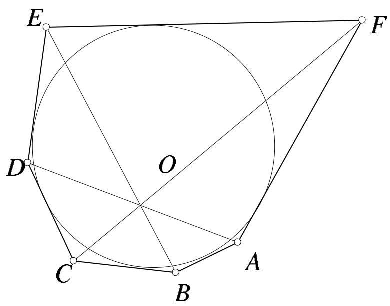
Pascal's Theorem The three points of intersections of opposite sides of a hexagon inscribed in a circle are collinear.



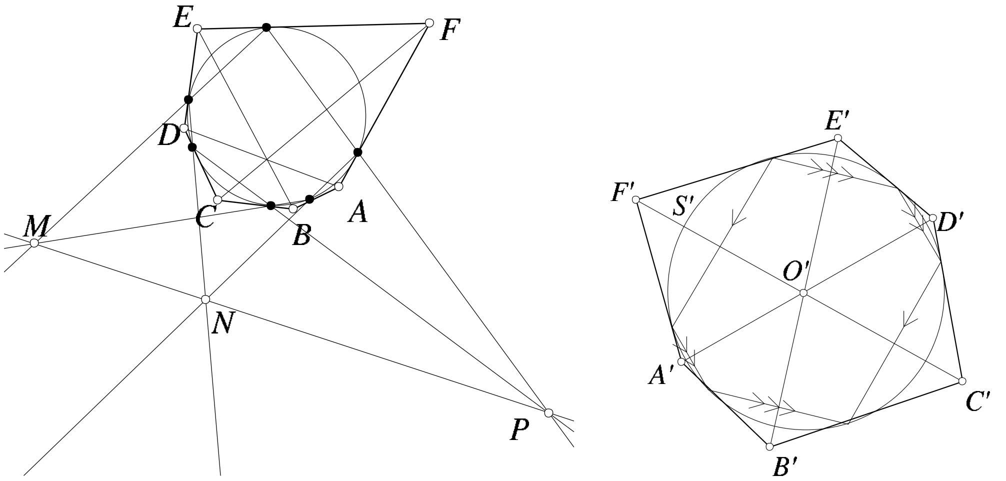


Proof. Let M, N, P be the intersections of the opposite sides. Since the tangents to S from N touch S in points of AB and DE, it follows that MN does not intersect the circle S. By fundamental theorem 1', there exists a central projection mapping S into a circle S' and MN to the line at infinity. Hence two pairs of the opposite sides of the hexagon after projection are parallel. By considering the corresponding arc length in S', it follows that the remaining pair of sides of the hexagon is also parallel. And hence P' lies on the line at infinity $M'N'$. Hence M, N, P are collinear.

Brianchon's Theorem If a circle is insribed in a hexagon, the diagonals joining the opposite vertices of the hexagon are concurrent.



Proof. Let the points of tangency of the circles form a hexagon.



By Pascal's Theorem, the intersections of the opposites sides are collinear. Similar to the proof of Pascal's theroem, the central projection mapping this line to infinity, transform the inscribed hexagon to a hexagon with opposite sides parallel. The corresponding tangent lines on opposite vertices are then symmetric about O'. Hence the corresponding diagonals are concurrent at O'. So the corresponding diagonals of the original hexagon are concurrent.

5 Polarity – Priciple of Duality

From the previous chapter, we notice that the statements of Pascal's theorem and Brianchon's theorem are closedly related. If we make correspondence between the six points in the circle of the Pascal's theorem to the six lines tangent to the circle of the Brianchon's theorem, we can further notice that the six sides of the hexagon in Pascal's theorem correspond to the six vertices of the hexagon in Brianchon's theorem; the three points of intersection of the opposites sides corresponds to the three diagonals joining the opposite vertices; the three points being collinear corresponds the three lines being concurrent. These kinds of correspondence is the basic phenomenon in projective geometry and they are called the *duality* of the projective plane.

The table below shows the correspondence in the duality of projective plane.

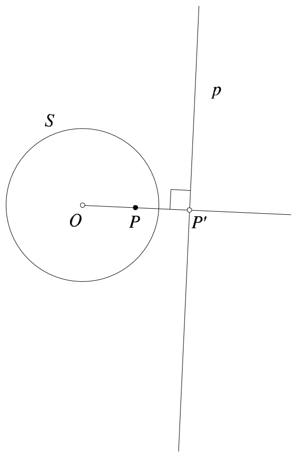
a point X	a line x
the line z passes through points X and Y	the intersecting point Z of lines x and y
points X, Y, Z are collinear	lines x, y, z are concurrent

For simplicity, we denote the point intersecting point Z of the lines x and y as $x \cdot y$.

Under this duality, we can see that the dual statement of Desargue's theorem (if XX_1, YY_1, ZZ_1 are concurrent, then the three points of intersection of lines XY and X_1Y_1, YZ and Y_1Z_1, ZX and ZX_1 are collinear) is exactly its converse statement (if the intersection of lines x and x_1, y and y_1, z and z_1 are collinear, the the lines joining the points $x \cdot y$ and $x_1 \cdot y_1$, $y \cdot z$ and $y_1 \cdot z_1$, $z \cdot x$ and $z_1 \cdot x_1$ are concurrent). Hence Desargue's theorem is self-dualed.

There are many kinds of duality in projective plane. The duality defines with respect to a fixed circle S is an important one and gives a precise duality between theorems of Pascal and Brianchon. This kind of duality is called a *polarity* with respect to S.

5.1 Definition – Pole and Polar



Let S be a circle with center O and radius r and P be a point. Let P' be the point on the line OP such that $OP \cdot OP' = r^2$. The line p through P' and perpedicular to OP is called the *pole* of P with respect to S.

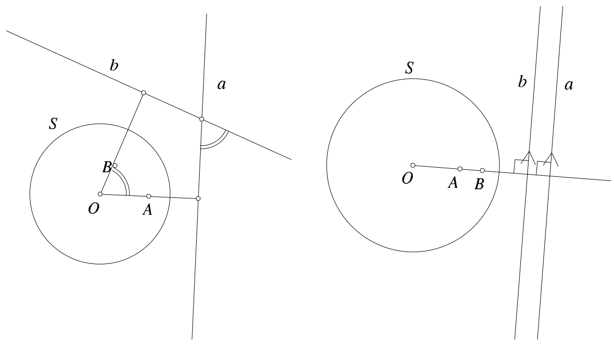
On the other hand, let q be a line and Q' be the point on q such that OQ' is perpedicular to q. Let Q be the point on OQ' such that $OQ \cdot OQ' = r^2$. The point Q is called the *polar* of q with respect to S.

By definition, p is a pole of P if and only if P is a polar of p. This duality is called the *polarity* with respect to S.

5.2 Properties

1. A point A lies on a line b if and only if the polar B lies on the pole a.

2. $\angle AOB = \theta$ if and only if the angle between the poles a, b is equal to θ. In particular, O passes through AB if and only if the poles a, b are parallel to each other.



3. P lies on S if and only if the pole p is tangent to S.

5.3 Applications

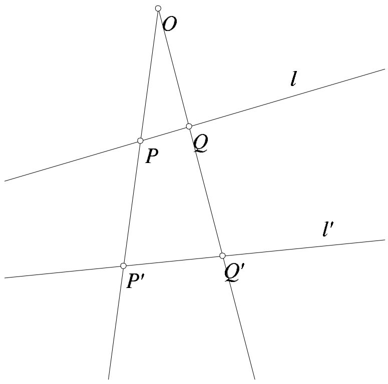
1. **(Desargue's Theorem)** Let ABC and $A_1B_1C_1$ be two triangle such that AA_1, BB_1, CC_1 are concurrent at O. Show that the points of intersection of lines AB and A_1B_1, BC and B_1C_1, CA and C_1A_1 are collinear.

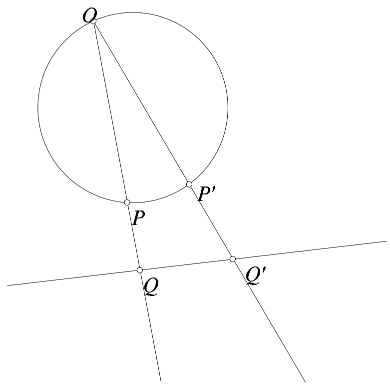
   Proof. Consider the polarity relative to a circle S with center O. By property 2, lines a and a_1, b and b_1, c and c_1 are parallel to each other. Hence the two triangles formed by a, b, c and a_1, b_1, c_1 are homothetic from a point (possibly at infinity). By property 1, the points of intersection of correspoding sides are collinear.

   The converse part can be proved by applying the Desargue's theorem by a polarity with respect to an arbitrary circle.

6 Projectivities and Cross Ratio of four points on lines and circles

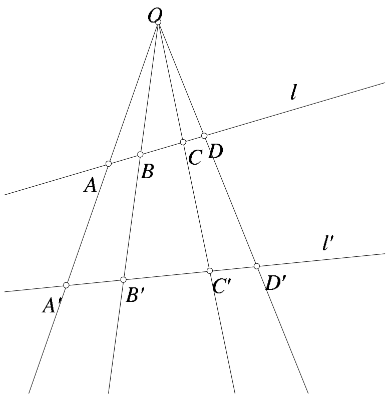
6.1 Definition

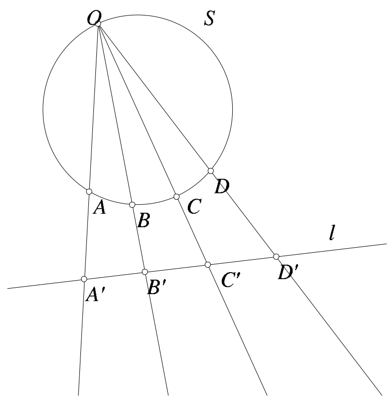




Projectivity from a point O. Let l, l' be two lines not passing through a point O. The *projectivity* from O is the central projection from O mapping points on l onto points on l'.

Projectivity of a Circle and a line. Let O be a point on a circle S and l be a line not passing through O. The *projectivity* from O is the central projection mapping points on S onto points on l (or backward).





Cross Ratio on a Line. Let A, B, C, D be points on a line, the *cross ratio* of A, B, C, D is defined by

$$\frac{AC \cdot BD}{AD \cdot BC}.$$

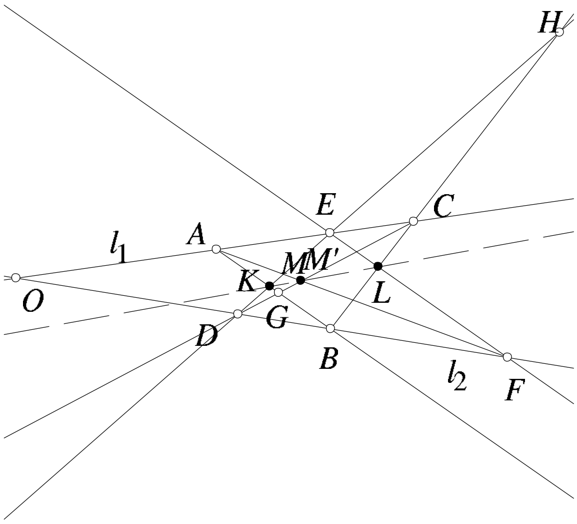
Cross Ratio on a Circle. Let A, B, C, D be four points on a circle S. Let O be a point on S, l be a line not passing through P. Let A', B', C', D' be images of A, B, C, D under the projectivity from O. The *cross ratio* of A, B, C, D is defined by the cross ratio of A', B', C', D'.

To show that the cross ratio of a circle is well-defined, we need to check that the definition is independence of the choice of O and l.

Main Theorem Projectivity preserve cross ratio of four points.

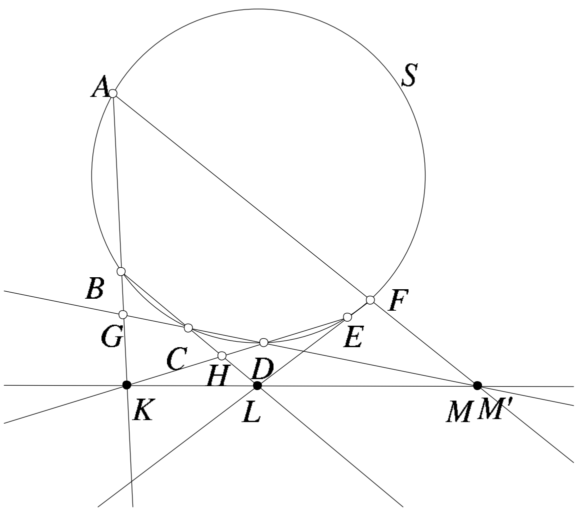
6.2 Applications

1. (Pappus' theorem) Let A, C, E and B, D, F be points on lines l_1 and l_2 respectively. Denote K, M, L the points of intersection of lines AB and ED, CD and AF, EF and CB respectively; G, H, O the points of intersection of AB and CD, CB and ED, l_1 and l_2 respectively; M' the points of intersection of KL and CD.



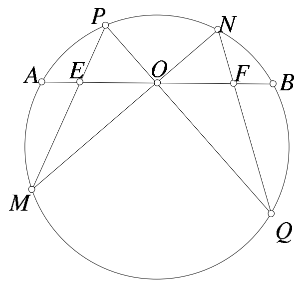
We want to show that $M = M'$. Projection of CD from A to the lines l_2 carries C, G, D, M into O, B, D, F. Projection of l_2 from E to the line CB carries O, B, D, F into C, B, H, L. Projection of CB from the K back the the line CD carries C, B, H, L into C, G, D, M'. Hence the cross ratio of C, G, D, M and C, G, D, M' are equal and hence $M = M'$.

2. (Pascal's theorem) If A, B, C, D, E, F are six points on a circle, then the points of intersection of AB and DE, BC and EF, CD and FA are collinear.



Proof. we denote the points of intersection of AB and DE, BC and EF, CD and FA, CD and KL by K, L, M, M' and the points of intersection of AB and CD, BC and DE by G and H. We want to show that $M = M'$. Projection of CD from A to the circle S carries the points G, C, D, M into points B, C, D, F. Projection of these points from E to the line BC carries them into B, C, H, L. Projection of these points from K to the line CD carries them into G, C, D, M'. Hence the cross ratio of G, C, D, M and G, C, D, M' are equal and $M = M'$.

3. (Buttlefly theorem) Let O be the midpoint of a chord AB of a circle S, and let MN and PQ be two other chords passing through O. Let E, F be the points where MP and NQ intersect AB. Show that O is the midpoint of EF.



Proof. Projection of the circle S to the line AB from M carries A, B, N, P on the circle into A, B, O, E. Projection of S to AB from Q carries A, B, N, P into A, B, F, O. It follows that the cross ratios of A, B, O, E and A, B, F, O are equal. Since $AO = OB$, we have $AF : FB = BE : EA$. Hence O is the midpoint of EF.

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