

# Modern Projective Geometry

*by*

Claude-Alain Faure

*and*

Alfred Frölicher

*University of Geneva,  
Geneva, Switzerland*



**KLUWER ACADEMIC PUBLISHERS**

**DORDRECHT / BOSTON / LONDON**

# Table of Contents

Preface	ix
Introduction	xiii
<b>Chapter 1. Fundamental Notions of Lattice Theory</b>	<b>1</b>
1.1 Introduction to lattices . . . . .	1
1.2 Complete lattices . . . . .	5
1.3 Atomic and atomistic lattices . . . . .	7
1.4 Meet-continuous lattices . . . . .	9
1.5 Modular and semimodular lattices . . . . .	12
1.6 The maximal chain property . . . . .	15
1.7 Complemented lattices . . . . .	17
1.8 Exercises . . . . .	21
<b>Chapter 2. Projective Geometries and Projective Lattices</b>	<b>25</b>
2.1 Definition and examples of projective geometries . . . . .	26
2.2 A second system of axioms . . . . .	30
2.3 Subspaces . . . . .	34
2.4 The lattice $\mathcal{L}(G)$ of subspaces of $G$ . . . . .	36
2.5 Correspondence of projective geometries and projective lattices . . . . .	40
2.6 Quotients by subspaces and isomorphism theorems . . . . .	43
2.7 Decomposition into irreducible components . . . . .	47
2.8 Exercises . . . . .	49
<b>Chapter 3. Closure Spaces and Matroids</b>	<b>55</b>
3.1 Closure operators . . . . .	56
3.2 Examples of matroids . . . . .	59
3.3 Projective geometries as closure spaces . . . . .	63
3.4 Complete atomistic lattices . . . . .	67
3.5 Quotients by a closed subset . . . . .	70
3.6 Isomorphism theorems . . . . .	73
3.7 Exercises . . . . .	75
<b>Chapter 4. Dimension Theory</b>	<b>81</b>
4.1 Independent subsets and bases . . . . .	83

4.2	The rank of a subspace . . . . .	86
4.3	General properties of the rank . . . . .	89
4.4	The dimension theorem of degree $n$ . . . . .	92
4.5	Dimension theorems involving the corank . . . . .	97
4.6	Applications to projective geometries . . . . .	98
4.7	Matroids as sets with a rank function . . . . .	100
4.8	Exercises . . . . .	103
<b>Chapter 5. Geometries of degree <math>n</math></b>		<b>107</b>
5.1	Definition and examples . . . . .	108
5.2	Degree of submatroids and quotient geometries . . . . .	110
5.3	Affine geometries . . . . .	112
5.4	Embedding of a geometry of degree 1 . . . . .	117
5.5	Exercises . . . . .	121
<b>Chapter 6. Morphisms of Projective Geometries</b>		<b>127</b>
6.1	Partial maps . . . . .	128
6.2	Definition, properties and examples of morphisms . . . . .	133
6.3	Morphisms induced by semilinear maps . . . . .	137
6.4	The category of projective geometries . . . . .	141
6.5	Homomorphisms . . . . .	143
6.6	Examples of homomorphisms . . . . .	148
6.7	Exercises . . . . .	151
<b>Chapter 7. Embeddings and Quotient-Maps</b>		<b>157</b>
7.1	Mono-sources and initial sources . . . . .	158
7.2	Embeddings . . . . .	163
7.3	Epi-sinks and final sinks . . . . .	169
7.4	Quotient-maps . . . . .	172
7.5	Complementary subspaces . . . . .	177
7.6	Factorization of morphisms . . . . .	179
7.7	Exercises . . . . .	182
<b>Chapter 8. Endomorphisms and the Desargues Property</b>		<b>187</b>
8.1	Axis and center of an endomorphism . . . . .	188
8.2	Endomorphisms with a given axis . . . . .	191
8.3	Endomorphisms induced by a hyperplane-embedding . . . . .	195
8.4	Arguesian geometries . . . . .	197
8.5	Non-arguesian planes . . . . .	204
8.6	Exercises . . . . .	209

<b>Chapter 9. Homogeneous Coordinates</b>	<b>215</b>
9.1 The homothety fields of an arguesian geometry . . . . .	216
9.2 Coordinates and hyperplane-embeddings . . . . .	218
9.3 The fundamental theorem for homomorphisms . . . . .	221
9.4 Uniqueness of the associated fields and vector spaces . . . . .	224
9.5 Arguesian planes . . . . .	226
9.6 The Pappus property . . . . .	228
9.7 Exercises . . . . .	230
<b>Chapter 10. Morphisms and Semilinear Maps</b>	<b>235</b>
10.1 The fundamental theorem . . . . .	236
10.2 Semilinear maps and extensions of morphisms . . . . .	238
10.3 The category of arguesian geometries . . . . .	242
10.4 Points in general position . . . . .	244
10.5 Projective subgeometries of an arguesian geometry . . . . .	247
10.6 Exercises . . . . .	249
<b>Chapter 11. Duality</b>	<b>255</b>
11.1 Duality for vector spaces . . . . .	256
11.2 The dual geometry . . . . .	258
11.3 Pairings, dualities and embedding into the bidual . . . . .	261
11.4 The duality functor . . . . .	264
11.5 Pairings and sesquilinear forms . . . . .	267
11.6 Exercises . . . . .	269
<b>Chapter 12. Related Categories</b>	<b>275</b>
12.1 The category of closure spaces . . . . .	276
12.2 Galois connections and complete lattices . . . . .	278
12.3 The category of complete atomistic lattices . . . . .	281
12.4 Morphisms between affine geometries . . . . .	284
12.5 Characterization of strong morphisms . . . . .	287
12.6 Characterization of morphisms . . . . .	291
12.7 Exercises . . . . .	295
<b>Chapter 13. Lattices of Closed Subspaces</b>	<b>301</b>
13.1 Topological vector spaces . . . . .	302
13.2 Mackey geometries . . . . .	305
13.3 Continuous morphisms . . . . .	308
13.4 Dualized geometries . . . . .	310
13.5 Continuous homomorphisms . . . . .	315

13.6 Exercises . . . . .	318
<b>Chapter 14. Orthogonality</b>	<b>323</b>
14.1 Orthogeometries . . . . .	324
14.2 Ortholattices and orthosystems . . . . .	327
14.3 Orthogonal morphisms . . . . .	330
14.4 The adjunction functor . . . . .	334
14.5 Hilbertian geometries . . . . .	337
14.6 Exercises . . . . .	340
<b>List of Problems</b>	<b>345</b>
<b>Bibliography</b>	<b>347</b>
<b>List of Axioms</b>	<b>357</b>
<b>List of Symbols</b>	<b>358</b>
<b>Index</b>	<b>359</b>