Synthetic Geometry

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1 Introduction

This handout includes a rough outline of the solutions presented in the talk and a list of some common theorems and results useful in many geometry problems. Many of the theorems and lemmas listed have interesting proofs and it is recommended that you try to prove some of them. The solutions to the examples are outlines which are intended to illustrate certain techniques. In some cases, calculations and dealing with special diagram cases are omitted.

2 Warmup

Here are two problems to start, both of which have short clean solutions. The first warmup applies a transformation after which the result becomes much clearer. The solution to the second warmup shifts the focus to a new triangle after which the problem becomes tractable.

Warmup 1. *(JBMO 2002) An isosceles triangle ABC satisfies that $CA = CB$. A point P is on the circumcircle between A and B and on the opposite side of the line AB to C. If D is the foot of the perpendicular from C to PB, show that $PA + PB = 2 \cdot PD$.*

Solution: Let the point Q be such that triangles QCB and PCA are congruent. Since $PACB$ is cyclic,

$$\angle CBQ = \angle CAP = 180^\circ - \angle CBP$$

which implies that P, B and Q are collinear. Since QCB and PCA are congruent, CPQ is isosceles and thus D is the midpoint of PQ. Therefore

$$PA + PB = PQ = 2 \cdot PD$$

Warmup 2. *(Russia 2005) In an acute-angled triangle ABC, AM and BN are altitudes. A point D is chosen on arc ACB of the circumcircle of the triangle. Let the lines AM and BD meet at P and the lines BN and AD meet at Q. Prove that MN bisects segment PQ.*

Solution: Assume without the loss of generality that D is on arc AC not including B. Let H be the orthocenter of ABC. Since $ADCB$ is cyclic,

$$\angle PAN = \angle DAC = \angle DBC = \angle QBM.$$

Also, it follows that

$$\angle NAH = 90^\circ - \angle ACB = \angle MBH.$$

Since $HP \perp AN$ and $HQ \perp BM$, PAN is similar to QBM and NAH is similar to MBH. Therefore

$$\frac{PM}{MH} = \frac{PM/BM}{MH/BM} = \frac{QN/AN}{NH/AN} = \frac{QN}{NH}$$

If X denotes the midpoint of PQ, then

$$\frac{PM}{MH} \cdot \frac{NH}{QN} \cdot \frac{QX}{XP} = 1$$

and by Menelaus' Theorem applied to triangle HPQ, points X, M and N are collinear.

3 Redefining Points to be Easier

Often points in geometry problems are defined in ways that are difficult to deal with. For any points that seem difficult to understand or work with, it is often best to redefine them in a useful way. Specifically, if P is a point in the diagram that is difficult to deal with, it is often best to define P' in some other way using a property we think is true of P and which can be used to define P, and then prove that $P' = P$. One thing to note is that this method requires that we have a property of P in mind. Finding out what is true of P is usually the most difficult part of problems that can be solved using this method. There is no best way to look for properties of P. However, it is often useful to think about what properties of P would solve or yield significant progress on the problem and what properties seem as though they may be true based on the diagram.

Often the best conjectures are simple, such as P lies on a line in the diagram, P lies on a circle in the diagram or is concyclic with other points in the diagram, that two lines are parallel or perpendicular, or that two triangles are similar or congruent. It is important though to make sure that you are not only trying to conjecture about the diagram and also trying to make direct progress on the problem. The examples in this section are intended to illustrate how this method of redefining points can be applied to problems.

Example 1. *An acute-angled triangle ABC is inscribed in a circle ω. A point P is chosen inside the triangle. Line AP intersects ω at the point A_1. Line BP intersects ω at the point B_1. A line ℓ is drawn through P and intersects BC and AC at the points A_2 and B_2. Prove that the circumcircles of triangles A_1A_2C and B_1B_2C intersect again on line ℓ.*

We want to analyze the second intersection of the circumcircles of triangles A_1A_2C and B_1B_2C. How much we can prove about this intersection Q varies greatly with how we define Q. Consider the two different methods below:

Method #1: First let's try defining Q directly as the intersection of the circumcircles of triangles A_1A_2C and B_1B_2C. From this, we know that $\angle CQB_2 = 180° - \angle CB_1B_2$ and $\angle CQA_2 = 180° - \angle CA_1A_2$. What we want is to show that $\angle CQB_2 + \angle CQA_2 = 180°$ which now is equivalent to $\angle CB_1B_2 + \angle CA_1A_2 = 180°$. However, this is not immediately true given the conditions in the problem. Now consider a second method.

Method #2: From the diagram, it looks like B_1PQA_1 is cyclic. From this information, consider defining Q' as the intersection of the circumcircle of B_1PA_1 and ℓ. From cyclic quadrilaterals, we have

$$\angle B_1Q'P = \angle B_1A_1P = \angle B_1CB_2$$

which implies that Q' is on the circumcircle of B_1B_2C. By a similar argument, we have that Q' is on the circumcircle of A_1A_2C. Together these imply that $Q = Q'$. Thus Q lies on ℓ.

A solution can also be obtained by defining Q' as the intersection of the circumcircle of B_1B_2C and ℓ. The way we define Q' above can be motivated by more than a conjecture based off of a conjecture from a diagram. We want to define Q' in some way and then use this way to show it lies on circles. The cleanest way to do this is to show the angle conditions for a cyclic quadrilateral. In order to get these angle conditions, one promising approach is to define Q' as the intersection of a circle with something, which in this case is ℓ.

One note for completeness is that the condition $\angle CB_1B_2 + \angle CA_1A_2 = 180°$ in Method #1 is a direct implication of Pascal's Theorem. In this case, Method #2 has saved us having to cite a deep theorem such as Pascal's Theorem. In other cases, redefining points can avoid much more complicated applications of advanced theorems.

These next examples illustrate this same method applied in more situations. Particularly in Example 3, it is hard to find a clean solution without the observations used to define P'.

Example 2. *(China 2012) In the triangle ABC, $\angle A$ is biggest. On the circumcircle of ABC, let D be the midpoint of arc ABC and E be the midpoint of arc ACB. The circle c_1 passes through A, B and is tangent to AC at A, the circle c_2 passes through A, E and is tangent AD at A. Circles c_1 and c_2 intersect at A and P. Prove that AP bisects $\angle BAC$.*

If the result is true, then by the tangency conditions $\angle APB = 180° - \angle BAC$ and $\angle PBA = 180° - \angle APB - \angle PAB = \frac{1}{2}\angle BAC = \angle PAB$. Therefore if the problem is true, then P lies on the perpendicular bisector of AB. This gives us the hint to try defining P based on this. The method below defines P' as the intersection of c_1 and the perpendicular bisector of AB.

Solution: Let the center of c_1 be O_1 and let the center of c_2 be O_2. Since c_1 is tangent to AC, it follows that $\angle BO_1A = 2\angle BAC$. Since O_1 and E both lie on the perpendicular bisector of AB, it follows that O_1E bisects angle $\angle BO_1A$ which implies that $\angle BO_1A = \angle BAC$ and hence that $\angle BP'E = 90° + \frac{1}{2}\angle BAC$. However, since P' lies on the perpencular bisector EO_1 of AB, A is the reflection of B about EO_1 and $\angle AP'E = \angle BP'E = 90° + \angle BAC$. Since c_2

is tangent to AD and passes through E, it follows that $\angle AO_2E = 2\angle DAE = 180^\circ - \angle BAC$. Combining this with the angle relation above yields that P' lies on c_2. Hence P' lies on both c_1 and c_2 and $P = P'$. Therefore $\angle BAP = \frac{1}{2}\angle BO_1P = \frac{1}{2}\angle BAC$ which implies the result.

Example 3. *(IMO 2011) Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.*

Experimenting with what we can prove yields that we can get almost no information that seems to lead to proving the desired result through standard techniques. The main issue is that we know almost nothing about the point of tangency if the problem is true. The key to the simplest solution to this problem is to find a way to define this supposed point of tangency. We try intersecting circumcircles in order to obtain angle information to prove that the point of intersection lies on Γ, the circumcircle of the triangle determined by the three lines and prove that the circles are tangent at this point.

Solution Outline: Let A', B' and C' be the intersections of ℓ_b and ℓ_c, ℓ_a and ℓ_c, and ℓ_a and ℓ_b, respectively. Let P be the point of tangency between Γ and ℓ and let Q be the reflection of P through BC. Now let T be the second intersection of the circumcircles of $BB'Q$ and $CC'Q$. It can be shown that T lies on Γ and the circumcircle of $A'B'C'$ by angle chasing. Similarly, T can be shown to be a point of tangency between the circles by angle chasing. The angle chasing is made easier by first showing that AA', BB' and CC' meet at the incenter I of $A'B'C'$.

Example 4. *(CMO 2013) Let O denote the circumcenter of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.*

Here, we use the method above to define the reflection R of the point of tangency in line PQ as the intersection of triangle OBP with side BC. This construction can be motivated either by noticing this pattern in the diagram, noting that this method of intersecting circles obtains angles in exactly the way needed to prove the result, or by trying to complete the Miquel configuration. The Miquel configuration is described in greater detail in Section 5.

Solution: Let the circumcircle of triangle OBP intersect side BC at the points R and B and let $\angle A$, $\angle B$ and $\angle C$ denote the angles at vertices A, B and C, respectively. Now note that since $\angle BOP = \angle B$ and $\angle COQ = \angle C$, it follows that

$$\angle POQ = 360^\circ - \angle BOP - \angle COQ - \angle BOC = 360^\circ - (180 - \angle A) - 2\angle A = 180^\circ - \angle A.$$

This implies that $APOQ$ is a cyclic quadrilateral. Since $BPOR$ is cyclic,

$$\angle QOR = 360^\circ - \angle POQ - \angle POR = 360^\circ - (180^\circ - \angle A) - (180^\circ - \angle B) = 180^\circ - \angle C.$$

This implies that $CQOR$ is a cyclic quadrilateral. Since $APOQ$ and $BPOR$ are cyclic,

$$\angle QPR = \angle QPO + \angle OPR = \angle OAQ + \angle OBR = (90^\circ - \angle B) + (90^\circ - \angle A) = \angle C.$$

Since $CQOR$ is cyclic, $\angle QRC = \angle COQ = \angle C = \angle QPR$ which implies that the circumcircle of triangle PQR is tangent to BC. Further, since $\angle PRB = \angle BOP = \angle B$,

$$\angle PRQ = 180^\circ - \angle PRB - \angle QRC = 180^\circ - \angle B - \angle C = \angle A = \angle PAQ.$$

This implies that the circumcircle of PQR is the reflection of Γ in line PQ. By symmetry in line PQ, this implies that the reflection of BC in line PQ is tangent to Γ.

4 Spiral Similarity and Applying Transformations

One of the most useful techniques in synthetic geometry problems is applying transformations to a diagram. Often solutions to difficult problems introduce a point to the diagram which allows for a clean quick solution. These points can often be viewed as completions of transformations already present in a diagram. For example, a diagram may contain a parallelogram $ABCD$ in which cases there is a translation mapping AB to DC. A diagram may contain a trapezoid $ABCD$ with $AB \| CD$ in which case there is a homothety mapping AB to CD. The transformations that most commonly appear are spiral similarities, rotations, homotheties and translations. The first few examples illustrate different ways to apply spiral similarities. The first example is one direction of Ptolemy's Theorem.

Example 5. *(Ptolemy) If $ABCD$ is a cyclic quadrilateral, then*

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

Here we construct similar triangles by applying a spiral similarity with center A mapping the C to D. We let the point B be mapped to P under this transformation.

Solution: Let P be the point on BD such that $\angle APD = \angle ABC$. Note that since $\angle ADP = \angle ACB$ which implies that triangles ABC and APD are similar. This implies that triangles ADC and APB are similar. Therefore $\frac{AD}{AC} = \frac{PD}{BC}$ and $\frac{AB}{AC} = \frac{BP}{CD}$. Therefore

$$BD = BP + PD = \frac{AB \cdot CD}{AC} + \frac{AD \cdot BC}{AC}$$

which implies on multiplying up that $AB \cdot CD + AD \cdot BC = AC \cdot BD$.

Example 6. *(IMO Shortlist 2000) Let $ABCD$ be a convex quadrilateral. The perpendicular bisectors of its sides AB and CD meet at Y. Denote by X a point inside the quadrilateral*

$ABCD$ such that $\measuredangle ADX = \measuredangle BCX < 90^\circ$ and $\measuredangle DAX = \measuredangle CBX < 90^\circ$. Show that $\measuredangle AYB = 2 \cdot \measuredangle ADX$.

In this example we consider the spiral similarity with center B mapping line CX to the perpendicular bisector of AB in order to obtain the angle we want Y to have at the image Y' of C. We then show that $Y = Y'$ in the same way as in the previous section.

Solution: Let X' and Y' be such that $AX' = BX'$, $AY' = BY'$, $\measuredangle AX'B = 2 \cdot \measuredangle BXC$ and $\measuredangle AY'B = 2 \cdot \measuredangle BCX$. We have that $AX'Y'$ and AXD are similar, and that $BX'Y'$ and BXC are similar. These similarities imply that triangles AXX' and ADY' are similar and that triangles BXX' and BCY' are similar. The ratios of similarity give that

$$DY' = \frac{AY' \cdot XX'}{AX'} = \frac{BY' \cdot XX'}{BX'} = CY'$$

Thus Y' lies on the perpendicular bisector of CD and $Y' = Y$. Therefore $\measuredangle AYB = 2 \cdot \measuredangle ADX$.

Example 7. *(IMO 1996) Let P be a point inside a triangle ABC such that*

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE meet at a point.

Solution: Here we use spiral similarity to construct exactly the given angle condition. By the angle bisector theorem, it suffices to show that $\frac{AB}{BP} = \frac{AC}{CP}$. Let Q be such that triangles APB and ACQ are similar. It follows that APC and ABQ are similar. It follows that

$$\angle CBQ = \angle APC - \angle ABC = \angle APB - \angle ACB = \angle BCQ$$

and thus $BQ = CQ$. Ratios of similarity finish the problem since

$$\frac{AB}{BP} = \frac{AQ}{CQ} = \frac{AQ}{BQ} = \frac{AC}{CP}$$

The next problem illustrates an often useful transformation when there is a midpoint of the side of a triangle. It is often useful to perform a 180° rotation about the midpoint to produce a parallelogram as in the example below which is from Challenging Problems in Geometry.

Example 8. *Let ABC be a given triangle and M be the midpoint of BC. If $\angle CAM = 2 \cdot \angle BAM$ and D is a point on line AM such that $\angle DBA = 90^\circ$, prove that $AD = 2 \cdot AC$.*

Solution: There is a very short trigonometric solution to this problem, but we present a synthetic one to illustrate the transformation mentioned above. Let D be such that $ABDC$ is a parallelogram. If N is the midpoint of AD, then M is the midpoint of AD. Now note

$$\angle BND = 2 \cdot \angle BAM = \angle CAM = \angle NDB$$

and thus $BD = BN$. This implies that $AC = BD = BN = \frac{1}{2}AD$.

The next example illustrates applying translations, which are particularly useful when there is a parallelogram in the diagram.

Example 9. *(2013 British MO) The point P lies inside triangle ABC so that $\angle ABP = \angle PCA$. The point Q is such that $PBQC$ is a parallelogram. Prove that $\angle QAB = \angle CAP$.*

Solution: Let R be such that $RACP$ is a parallelogram. It follows that $\angle ARP = \angle PCA = \angle ABP$ which implies that $RAPB$ is cyclic. It follows that BRP and QAC are congruent and thus $\angle QAC = \angle BRP = \angle BAP$. This implies that $\angle QAB = \angle CAP$.

5 Geometry Facts

The theorems and facts below are many that I find useful. This list is a work in progress so there are many useful ideas that I have omitted. I plan to add sections on collinearity, concurrency, and miscellaneous useful facts. If any come to mind, please let me know. For now, I have not included ideas from inversive and projective geometry. The theorems below all apply to points in a plane. Quadrilaterals are named $ABCD$ such that the sides of the quadrilateral are AB, BC, CD and DA. The directed angle $\measuredangle ABC$ is the counter-clockwise angle between 0 and 180° needed to rotate line AB to line BC. A triangle ABC is taken to have angles a, b and c.

Cyclic Quadrilaterals:

1. A convex quadrilateral $ABCD$ is cyclic if and only if either:
   (a) $\angle ADB = \angle ACB$
   (b) $\angle DAB + \angle BCD = 180°$
2. The above two conditions can be restated as a single condition in terms of directed angles: Four points A, B, C and D are concyclic if and only if $\measuredangle ABC = \measuredangle ADC$.
3. (Power of a Point) Let $ABCD$ be a convex quadrilateral such that AB and CD intersect at P and diagonals AC and BD intersect at Q. $ABCD$ is cyclic if and only if either:
   (a) $AQ \cdot QC = BQ \cdot QD$ or equivalently QAD and QBC are similar
   (b) $PA \cdot PB = PC \cdot PD$ or equivalently PAD and PCB are similar
4. Given a triangle ABC, the intersections of the internal and external bisectors of angle $\angle BAC$ with the perpendicular bisector of BC both lie on the circumcircle of ABC.

5. (Ptolemy's Theorem) A quadrilateral $ABCD$ is cyclic if and only if

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

6. Let $ABCD$ be a cyclic quadrilateral such that AB and CD intersect at P and diagonals AC and BD intersect at Q. Then:

$$\frac{BQ}{QD} = \frac{AB \cdot BC}{AD \cdot DC} \quad \text{and} \quad \frac{PB}{PA} = \frac{BC \cdot BD}{AC \cdot AD}$$

7. (Polars) Let $ABCD$ be a cyclic quadrilateral inscribed in circle Γ such that AB and CD intersect at P and diagonals AC and BD intersect at Q. If the tangents drawn from P to Γ touch Γ at R and S, then R, Q and S are collinear.

Circles:

1. (Power of a Point) Given a circle Γ with center O and a point P then for any line ℓ through P that intersects Γ at A and B, the value $PA \cdot PB$ is constant as ℓ varies and is equal to the power of the point P with respect to Γ.
   (a) The power of P is equal to $r^2 - PO^2$ if P is inside Γ and $PO^2 - r^2$ otherwise.
   (b) If PA is tangent to Γ, then the power of P is equal to PA^2.

2. (Radical Axis) Given two circles Γ_1 and Γ_2, the set of all points P with equal powers with respect to Γ_1 and Γ_2 is a line which is the radical axis of the two circles.
   (a) The radical axis is perpendicular to the line through the centers of Γ_1 and Γ_2.
   (b) If Γ_1 and Γ_2 intersect at A and B, then the radical axis passes through A and B.
   (c) If AB is a common tangent with A on Γ_1 and B on Γ_2, then the radical axis passes through the midpoint of AB.

3. (Radical Center) Given three circles Γ_1, Γ_2 and Γ_3, the three radical axes between pairs of the three circles meet at a common point P which is the radical center of the circles.

4. A point P is a circle of radius zero and the radical axis of P and a circle Γ is the line through the midpoints of PA and PB where A and B are points on Γ such that PA and PB are tangent to Γ.

5. (Monge's Theorem) Given three circles Γ_1, Γ_2 and Γ_3. If P, Q and R are the external centers of homothety between pairs of the three circles, then P, Q and R are collinear. If P and Q are internal centers of homothety, then P, Q and R are also collinear.

6. Two circles Γ_1 and Γ_2 intersect at R and have centers O_1 and O_2. If P and Q are the internal and external centers of homothety between the two circles, then $\angle PRQ = 90^\circ$. The lines RP and RQ are the internal and external bisectors of $\angle O_1RO_2$.

Triangle Geometry:

1. (Angle Bisector Theorem) Let ABC be a given triangle and let P and Q be the intersections of the internal and external bisectors of angle $\angle ABC$ with line AC. Then
$$\frac{AB}{BC} = \frac{AP}{PC} = \frac{AQ}{QC}$$

2. Angles around the centers of a triangle ABC:
   (a) If I is the incenter of ABC then $\angle BIC = 90^\circ + \frac{a}{2}$, $\angle IBC = \frac{b}{2}$ and $\angle ICB = \frac{c}{2}$.
   (b) If H is the orthocenter of ABC then $\angle BHC = 180^\circ - a$, $\angle HBC = 90^\circ - c$ and $\angle HCB = 90^\circ - b$.
   (c) If O is the circumcenter of ABC then $\angle BOC = 2a$ and $\angle OBC = \angle OCB = 90^\circ - a$.
   (d) If I_a is the A-excenter of ABC then $\angle AI_aB = \frac{c}{2}$, $\angle AI_aC = \frac{b}{2}$ and $\angle BI_aC = 90^\circ - \frac{a}{2}$.

3. Pedal triangles of the centers of a triangle ABC:
   (a) If DEF is the triangle formed by projecting the incenter I onto sides BC, AC and AB, then I is the circumcenter of DEF and $\angle EDF = 90^\circ - \frac{a}{2}$.
   (b) If DEF is the triangle formed by projecting the orthocenter H onto sides BC, AC and AB, then H is the incenter of DEF and $\angle EDF = 180^\circ - 2a$.
   (c) The medial triangle of ABC is the pedal triangle of the circumcenter O of ABC and O is its orthocenter.

4. Alternate methods of defining the orthocenter and circumcenter:
   (a) O is the circumcenter of ABC if and only if $\measuredangle AOB = 2\measuredangle ACB$ and $OA = OB$.
   (b) H is the orthocenter of ABC if and only if H lies on the altitude from A and satisfies that $\measuredangle BHC = 180^\circ - \measuredangle BAC$.

5. Facts related to the orthocenter H of a triangle ABC with circumcircle Γ:
   (a) If O is the circumcenter of ABC, then $\angle BAH = \angle CAO$.
   (b) If D is the point diametrically opposite to A on Γ and M is the midpoint of BC, then M is also the midpoint of HD.
   (c) If AH, BH and CH intersect Γ again at D, E and F, then there is a homothety centered at H sending the pedal triangle of H to DEF with ratio 2.
   (d) If D and E are the intersections of AH with BC and Γ, respectively, then D is the midpoint of HE.
   (e) H lies on the three circles formed by reflecting Γ about AB, BC and AC.

(f) If M is the midpoint of BC then $AH = 2 \cdot OM$.

(g) If BH and CH intersect AC and AB at D and E, and M is the midpoint of BC, then M is the center of the circle through B, D, E and C, and MD and ME are tangent to the circumcircle of ADE.

6. Facts related to the incenter I and excenters I_a, I_b, I_c of ABC with circumcircle Γ:

   (a) If the incircle of ABC is tangent to AB and AC at points D and E and s is the semiperimeter of ABC then

$$AD = AE = \frac{AB + AC - BC}{2} = s - BC$$

   (b) If AI intersects Γ at D then $DB = DI = DC$, D is the midpoint of II_a, and II_a is a diameter of the circle with center D which passes through B and C.

   (c) If AI, BI and CI intersect Γ at D, E and F, then $I_aI_bI_c$, DEF and the pedal triangle of I are similar and have parallel sides.

   (d) I is the orthocenter of $I_aI_bI_c$ and Γ is the nine-point circle of $I_aI_bI_c$.

   (e) If BI and CI intersect Γ again at D and E, then I is the reflection of A in line DE and if M is the intersection of the external bisector of $\angle BAC$ with Γ, then $DMEI$ is a parallelogram.

   (f) If the incircle and A-excircle of ABC are tangent to BC at D and E, $BD = CE$.

   (g) If the A-excircle of ABC is tangent to AB, AC and BC at D, E and F then $AB + BF = AC + CF = AD = AE = s$ where s is the semi-perimeter of ABC.

   (h) If M is the midpoint of arc BAC of Γ, then M is the midpoint of I_bI_c and the center of the circle through I_b, I_c, B and C.

7. (Nine-Point Circle) Given a triangle ABC, let Γ denote the circle passing through the midpoints of the sides of ABC. If H is the orthocenter of ABC, then Γ passes through the midpoints of AH, BH and CH and the projections of H onto the sides of ABC.

8. (Euler Line) If O, H and G are the circumcenter, orthocenter and centroid of a triangle ABC, then G lies on segment OH with $HG = 2 \cdot OG$.

9. (Symmedian) Given a triangle ABC such that M is the midpoint of BC, the symmedian from A is the line that is the reflection of AM in the bisector of angle $\angle BAC$.

   (a) If the tangents to the circumcircle Γ of ABC at B and C intersect at N, then N lies on the symmedian from A and $\angle BAM = \angle CAN$.

   (b) If the symmedian from A intersects Γ at D, then $AB/BD = AC/CD$.

10. If the median from A in a triangle ABC intersects the circumcircle Γ of ABC at D, then $AB \cdot BD = AC \cdot CD$.

11. (Euler's Formula) Let O, I and I_a be the circumcenter, incenter and A-excenter of a triangle ABC with circumradius R, inradius r and A-exradius r_a. Then:
    (a) $OI = \sqrt{R(R-2r)}$.
    (b) $OI_a = \sqrt{R(R-2r_a)}$.

12. (Poncelet's Porism) Let Γ and ω be two circles with centers O and I and radii R and r, respectively, such that $OI = \sqrt{R(R-2r)}$. Let A, B and C be any three points on Γ such that lines AB and AC are tangent to ω. Then line BC is also tangent to ω.

13. (Apollonius Circle) Let ABC be a given triangle and let P be a point such that $AB/BC = AP/PC$. If the internal and external bisectors of angle $\angle ABC$ meet line AC at Q and R, then P lies on the circle with diameter QR.

Trigonometry:

1. (Sine Law) Given a triangle ABC with circumradius R
$$\frac{BC}{\sin\angle A} = \frac{AC}{\sin\angle B} = \frac{AB}{\sin\angle C} = 2R$$

2. (Cosine Law) Given a triangle ABC
$$BC^2 = AB^2 + AC^2 - 2\cdot AB\cdot AC\cdot\cos\angle A$$

3. (Pythagorean Theorem) If ABC is a triangle, then $\angle ABC = 90^\circ$ if and only if
$$AB^2 + BC^2 = AC^2$$

4. Given a triangle ABC and a point D on line BC, then
$$\frac{\sin\angle BAD}{\sin\angle CAD} = \frac{BD\cdot AC}{CD\cdot AB}$$

Miscellaneous Synthetic Facts:

1. (Spiral Similarity) Let OAB and OCD be directly similar triangles. Then OAC and OBD are also directly similar triangles.

2. The unique center of spiral similarity sending AB to CD is the second intersection of the circumcircles of QAB and QCD where AC and BD intersect at Q.

3. Lines AB and CD are perpendicular if and only if $AC^2 - AD^2 = BC^2 - BD^2$.

4. (Apollonius Circle) Given two points A and B and a fixed $r > 0$, then the locus of points Q such that $AQ/BQ = r$ is a circle Γ with center at the midpoint of Q_1Q_2 where Q_1 and Q_2 are the two points on line AB satisfying $AQ_i/BQ_i = r$ for $i = 1, 2$.

5. Let $ABCD$ be a convex quadrilateral. The four interior angle bisectors of $ABCD$ are concurrent and there exists a circle Γ tangent to the four sides of $ABCD$ if and only if $AB + CD = AD + BC$.

6 Problems

The problems below have been arranged roughly in order of difficulty. I divided the problems into three difficulty classes: A, B and C.

A1. (CMO 1997) The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180°$. Prove that $\angle OBC = \angle ODC$.

A2. (APMO 2007) Let ABC be an acute angled triangle with $\angle BAC = 60°$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

A3. (IMO 2006) Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

A4. (IMO 2008) Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2. Prove that six points A_1 , A_2, B_1, B_2, C_1 and C_2 are concyclic.

A5. (Russia 2012) The points A_1, B_1 and C_1 lie on the sides BC, CA and AB of the triangle ABC, respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let O_A, O_B and O_C be the circumcenters of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the incenter of triangle $O_AO_BO_C$ is the incenter of triangle ABC.

A6. (IMO Shortlist 2006) Let ABC be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $\frac{AK}{KB} = \frac{DL}{LC}$. Suppose that there are points P and Q on the line segment KL satisfying $\angle APB = \angle BCD$ and $\angle CQD = \angle ABC$. Prove that the points P, Q, B and C are concyclic.

A7. (IMO Shortlist 2008) Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

B1. (IMO Shortlist 2000) Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that $OD + DH = OE + EH = OF + FH$ and the lines AD, BE, and CF are concurrent.

B2. (IMO Shortlist 2012) In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

B3. (IMO Shortlist 2005) Let $ABCD$ be a parallelogram. A variable line g through the vertex A intersects the rays BC and DC at the points X and Y, respectively. Let K and L be the A-excenters of the triangles ABX and ADY. Show that the angle $\measuredangle KCL$ is independent of the line g.

B4. (Tuymaada MO 2012) Point P is taken in the interior of the triangle ABC, so that

$$\angle PAB = \angle PCB = \frac{1}{4}(\angle A + \angle C).$$

Let L be the foot of the angle bisector of $\angle B$. The line PL meets the circumcircle of $\triangle APC$ at point Q. Prove that QB is the angle bisector of $\angle AQC$.

B5. (Japan MO 2009) Let Γ be the circumcircle of a triangle ABC. A circle with center O touches to line segment BC at P and touches the arc BC of Γ which doesn't have A at Q. If $\angle BAO = \angle CAO$, then prove that $\angle PAO = \angle QAO$.

B6. (Bulgarian TST 2004) The points P and Q lie on the diagonals AC and BD, respectively, of a quadrilateral $ABCD$ such that $\frac{AP}{AC} + \frac{BQ}{BD} = 1$. The line PQ meets the sides AD and BC at points M and N. Prove that the circumcircles of the triangles AMP, BNQ, DMQ, and CNP are concurrent.

B7. (Chinese TST 2002) Circles ω_1 and ω_2 intersect at points A and B. Points C and D are on circles ω_1 and ω_2, respectively, such that lines AC and AD are tangent to circles ω_2 and ω_1, respectively. Let I_1 and I_2 be the incenters of triangles ABC and ABD, respectively. Segments I_1I_2 and AB intersect at E. Prove that $\frac{1}{AE} = \frac{1}{AC} + \frac{1}{AD}$.

C1. (Russia 2012) The point E is the midpoint of the segment connecting the orthocenter of the scalene triangle ABC and the point A. The incircle of triangle ABC incircle is tangent to AB and AC at points C' and B', respectively. Prove that point F, the point symmetric to point E with respect to line $B'C'$, lies on the line that passes through both the circumcenter and the incenter of triangle ABC.

C2. (Chinese TST 2005) Let ω be the circumcircle of $\triangle ABC$. P is an interior point of $\triangle ABC$. A_1, B_1, C_1 are the intersections of AP, BP, CP respectively and A_2, B_2, C_2 are the symmetrical points of A_1, B_1, C_1 with respect to the midpoints of side BC, CA, AB. Show that the circumcircle of $\triangle A_2B_2C_2$ passes through the orthocentre of $\triangle ABC$.

C3. (USA TST 2005) Let ABC be an acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.

C4. (USA TST 2006) Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.

C5. (RMM 2012) Let ABC be a triangle and let I and O denote its incentre and circumcentre respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.

C6. (RMM 2011) A triangle ABC is inscribed in a circle ω. A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω. Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.