$f(x - f(y)) = f(x + y^n) + f(f(y) + y)$

Then given F.E is equivalent to $f(x) = f(x + y^n + f(y)) + f(f(y) + y)$

Let, $P(x, y) \Longrightarrow f(x) = f(x + y^n + f(y)) + f(f(y) + y)$, $n \geq 2$, $n \in \mathbb{N}$

$P(y + f(y), y) \Longrightarrow f(y + y^n + 2f(y)) = 0$

It's ugly, but not hard to check that, $f(x) = -\frac{(x + x^n)}{2}$ is not a solution.

So, there exists $u \in \mathbb{R}/\{0\}$ such that $f(u) = 0$

Define, $c = f(0)$, $S = \{x^n + f(x) \mid x \in \mathbb{R}\}$, $g(x) = f(x) - c$, $\mathcal{G} = \{g(x) \mid x \in \mathbb{R}\}$

I. $g(x + s) = g(x) + g(s)$ for all $x \in \mathbb{R}$ and $s \in S$

Proof:

$P(0, y) \Longrightarrow f(y^n + f(y)) = c - f(y + f(y))$

So, $P(x, y) \Longrightarrow f(x) = f(x + y^n + f(y)) + c - f(y^n + f(y))$

Or, $f(x + y^n + f(y)) - c = f(x) - c + f(y^n + f(y)) - c$

So, $g(x + s) = g(x) + g(s)$ for all $x \in \mathbb{R}$ and $s \in S$

II. $g\big(\sum_{1 \leq i \leq t} n_i s_i\big) = \sum n_i g(s_i)$ for all $t \in \mathbb{N}$, $n_i \in \mathbb{Z}$, $s_i \in S$

Proof:

Can easily be derived from I

III. For all $x \in \mathbb{R}$ there exists $t \in \mathbb{N} \cup \{0\}$, $s_i \in S$, $n_i \in \mathbb{Z}$, $1 \leq i \leq t$ such that, $x = \sum_{1 \leq i \leq t} n_i s_i$

Proof:

Note that we previously defined $u \neq 0$ such that, $f(u) = 0$

Now, $P(x, u) \Longrightarrow f(x) = f(x + u^n)$

$\Longrightarrow (x + u)^n - x^n = (f(x + u) + (x + u)^n) - (f(x) + x^n)$

Note that, $h(x) = (x + u)^n - x^n$ is a continious non-constant function as $u \neq 0$.

So, there must be an interval $[a, b]$, $a \neq b$ such that, $[a, b]$ is subset of co-domain of h.

Now, for any real x, $\exists t \in \mathbb{Z}$ such that $t(b - a) + b > x \geq t\,(b - a) + a$,

So, for some r, $x = t\,(b - a) + r$ where $b > r \geq a$

As, $a, b, r \in [a, b]$, $\exists i, j, k$ such that, $h(i) = a$, $h(j) = b$, $h(k) = r$

So, $x = t\,h(j) + h(k) - t\,h(i)$

As, each $h(y)$ can be represented in the form $\sum_{1 \leq i \leq t,\ n_i \in \mathbb{Z},\ s_i \in S} n_i s_i$, so is x.

IV. $g(x + y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$

Proof:

Let, $x = \sum_{1\le i\le t,\ n_i\in\mathbb{Z},\ s_i\in S} n_i s_i$ and $y = \sum_{1\le i\le t',\ n_i'\in\mathbb{Z},\ s_i'\in S} n_i' s_i'$

So, according to II,

$g(x) = \sum_{1\le i\le t,\ n_i\in\mathbb{Z},\ s_i\in S} n_i\, g\,(s_i)$ and $g(y) = \sum_{1\le i\le t',\ n_i'\in\mathbb{Z},\ s_i'\in S} n_i' g(s_i')$

So, $g(x+y)$

$= g\Big(\sum_{1\le i\le t,\ n_i\in\mathbb{Z},\ s_i\in S} n_i s_i \quad + \quad \sum_{1\le i\le t',\ n_i'\in\mathbb{Z},\ s_i'\in S} n_i' s_i' \Big)$

$= \sum_{1\le i\le t,\ n_i\in\mathbb{Z},\ s_i\in S} n_i\, g\,(s_i) \quad + \quad \sum_{1\le i\le t',\ n_i'\in\mathbb{Z},\ s_i'\in S} n_i' g(s_i')$

$= g(x) + g(y)$

V. $f(x) = 0$ for all $x \in \mathbb{R}$

Proof:

$P(x, y)$

$\Longrightarrow f(x) = f(x + y^n + f(y)) + f(f(y) + y)$

$\Longrightarrow g(x) + c = g(x + y^n + g(y) + c) + c + g(g(y) + c + y) + c$

$\Longrightarrow g(x) + c = g(x) + g(y^n) + g(g(y)) + g(c) + c + g(g(y)) + g(c) + g(y) + c$

$\Longrightarrow 2g(g(y)) + g(y) + g(y^n) = a$

Where, $a = -c - 2g(c) = -(g(2c) + c) = -f(2f(0)) = 0$, according to, $P(f(0), 0)$

So, $2g(g(y)) + g(y) + g(y^n) = 0$

Let, $Q(x) \Longrightarrow 2g(g(x)) + g(x) + g(x^n) = 0$

Then, $Q(x) + Q(y)$

$\Longrightarrow 2g(g(x)) + g(x) + g(x^n) + 2g(g(y)) + g(y) + g(y^n) = 0$

$\Longrightarrow 2g(g(x) + g(y)) + g(x+y) + g(x^n + y^n) = 0$

$\Longrightarrow 2g(g(x+y)) + g(x+y) + g(x^n + y^n) = 0$

But, $Q(x+y) \Longrightarrow 2g(g(x+y)) + g(x+y) + g((x+y)^n) = 0$

So, $g(x^n + y^n) = g(x^n) + g(y^n) = g((x+y)^n) = g(x^n) + g(y^n) + g\Big(\sum_{i=1}^{n-1} \binom{n}{i} x^i y^{n-i} \Big)$

$\Longrightarrow g\Big(\sum_{i=1}^{n-1} \binom{n}{i} x^i y^{n-i} \Big) = 0$

$y = 1 \Longrightarrow g\Big(\sum_{i=1}^{n-1} \binom{n}{i} x^i \Big) = 0$

Note that, since $n \ge 2$, $\gamma(x) = \sum_{i=1}^{n-1} \binom{n}{i} x^i$ is a non-constant continious function.

Further, $\gamma(0) = 0$ and $\lim_{x\longrightarrow\infty} \gamma(x) = \infty$

So, $(\forall x \ge 0)(\exists z \ge 0)$ such that $x = \gamma(z)$. So, $g(x) = g(\gamma(z)) = 0$ for all $x \ge 0$

But, $g(-x) + g(x) = g(0) = f(0) - c = 0$, So, $g(-x) = 0$ for all $x \ge 0$

Hence $g \equiv 0 \Longrightarrow f \equiv c$. $P(x, y) \Longrightarrow c = 2c \Longrightarrow c = 0$

So, $f(x) = 0$ for all $x \in \mathbb{R}$

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