IMO Training - July 9th, 1999
& 2000

Really Tough Problems on Inequalities

1. Given that a, b, c, d, and e are real numbers such that $a+b+c+d+e=8$ and $a^2+b^2+c^2+d^2+e^2=16$, determine the maximum possible value of e.

2. The perimeter of an isosceles trapezoid is 28. If the longest side has length 13, determine the maximum possible area of the trapezoid.

3. Prove that for any positive numbers a, b, c, and d with $a \leq b \leq c \leq d$, we have $a^b b^c c^d d^a \geq b^a c^b d^c a^d$.

4. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

5. Suppose $\sqrt{7} > \frac{m}{n}$, where m and n are integers. Prove that $\sqrt{7} - \frac{m}{n} > \frac{1}{mn}$.

6. Let $g(k)$ denote the greatest *odd* divisor of k. Prove that for all positive integers n,
$$0 \leq \sum_{k=1}^{n} \frac{g(k)}{k} - \frac{2n}{3} \leq \frac{2}{3}.$$

7. Let S_n be the set of permutations of $\{1,2,3,\ldots,n\}$. Over all such permutations p, determine the minimum and maximum value of
$$|p_1-p_2|+|p_2-p_3|+|p_3-p_4|+\ldots+|p_{n-1}-p_n|+|p_n-p_1|.$$

8. Let $x_1, x_2, \ldots, x_n$ be real numbers with $x_1+x_2+\ldots+x_n=0$ and $x_1^2+x_2^2+\ldots+x_n^2=1$. Show that $x_ix_j \leq -\frac{1}{n}$ for some i and j.

9. Suppose that the coefficients $a_1, a_2, \ldots a_{n-1}$ of the polynomial $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-2}x^2 + a_{n-1}x + 1$ are all non-negative real numbers, and that $f(x)$ has n real roots. Prove that $f(2) \geq 3^n$. If $f(2) = 3^n$, what can you say about $f(x)$?

10. Let a, b, and c be the sides of a triangle. Prove that
$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \leq 3abc$$
Determine when equality occurs. (1964 IMO).

11. Let a, b, and c be the sides of a triangle. Let T be its area. Show that $a^2+b^2+c^2 \geq 4T\sqrt{3}$. When does equality hold? (1961 IMO).

12. Let a, b, and c be the sides of a triangle. Prove that $a^2b(a-b)+b^2c(b-c)+c^2a(c-a) \geq 0$. Determine when equality occurs. (1983 IMO, Question 6)

13. Prove that for any positive numbers a, b, c, and d,

$$\left(\frac{abc + abd + acd + bcd}{4}\right)^2 \leq \left(\frac{ab + ac + ad + bc + bd + cd}{6}\right)^3.$$

14. Prove that for any two given positive numbers p and q with $p < q$ and real numbers a, b, c, d, e with $p < a, b, c, d, e < q$, we have:

$$(a + b + c + d + e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \leq 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2.$$