5 Harmonic conjugates

Harmonic conjugates (Ogilvy p. 13–14, Eves p. 82–84)

If A and B are two points on a line, any pair of points C and D on the line for which

$$\frac{AC}{CB} = \frac{AD}{DB}$$

are said to ***divide*** AB ***harmonically***. The points C and D are then said to be ***harmonic conjugates*** with respect to A and B.

Given ordinary points A and B, and given a positive number k, $k \neq 1$, there are two ordinary points C and D such that $\frac{AC}{CB} = \frac{AD}{DB} = k$. One of the points C and D is between A and B, the other is exterior to the segment AB. The midpoint C of AB satisfies $\frac{AC}{CB} = 1$, and we will adopt the convention that $\frac{AI}{IB} = 1$ (where I is the ideal point in the inversive plane). Using this convention, given two ordinary points A and B, for every positive k there are harmonic conjugates C and D for which

$$\frac{AC}{CB} = \frac{AD}{DB} = k\,.$$

Theorem 5.1. *Given four ordinary points, A, B, C, and D, if AB is divided harmonically by C and D, then CD is divided harmonically by A and B.*

The reason for the terminology is explained by the following:

Theorem 5.2. *Suppose that P, Q, R, and S are consecutive ordinary points on a line and that QS divides PR harmonically. Then the sequence of distances PQ, PR, PS forms a harmonic progression.*

Proof. The hypothesis says that

$$\frac{RQ}{QP} = \frac{RS}{SP}\,. \tag{1}$$

We want to show that $\frac{1}{PQ}$, $\frac{1}{PR}$, $\frac{1}{PS}$ are in arithmetic progression, that is, that

$$\frac{1}{PQ} - \frac{1}{PR} = \frac{1}{PR} - \frac{1}{PS}\,. \tag{2}$$

From (1) we get

$$\begin{aligned} & \frac{RQ}{QP \cdot PR} = \frac{RS}{SP \cdot PR} \\ \Longrightarrow \quad & \frac{PR - PQ}{PQ \cdot PR} = \frac{PS - PR}{PR \cdot PS} \\ \Longrightarrow \quad & \frac{1}{PQ} - \frac{1}{PR} = \frac{1}{PR} - \frac{1}{PS}\,. \end{aligned}$$

showing that (2) holds. □

The Circle of Apollonius (Ogilvy, p. 14-17)

If we are given points A and B and a positive number $k \neq 1$, we can find precisely two ordinary points X on the line AB such that $\frac{AX}{XB} = k$. There are also points X not on the line AB for which $\frac{AX}{XB} = k$.

Theorem 5.3. *Given two ordinary points A and B, and a positive number $k \neq 1$, the set of all points X in the plane for which $\frac{AX}{XB} = k$ forms a circle.*

Remark: The circle referred to in the theorem is called the ***Circle of Apollonius for A, B, and k.***

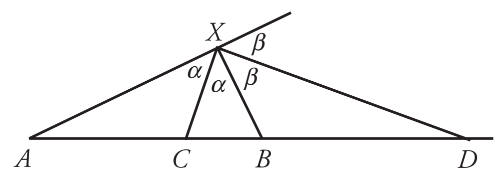
Proof. Let C and D be the two points on AB for which $\dfrac{AC}{CB} = \dfrac{AD}{DB} = k$, and let ξ be the circle with diameter CD.

There are two things to show:

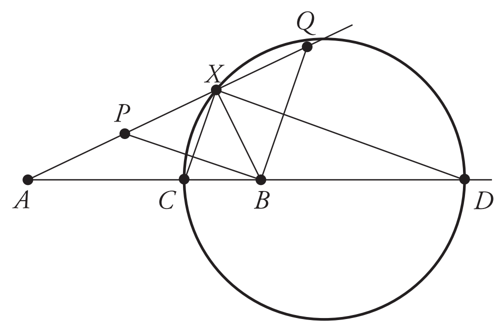
(1) Every point X for which $\frac{AX}{XB} = k$ is on ξ.

(2) Every point X on ξ satisfies $\frac{AX}{XB} = k$.

(1) Let X be a point such that $\frac{AX}{XB} = k$. Since $\frac{AC}{CB} = k$ and $\frac{AD}{DB} = k$, we know from the Angle Bisector Theorem that XC and XD are internal and external bisectors of angle AXB. Referring to the figure, we see that $\alpha + \beta = 90°$, that is, $\angle CXD$ is a right angle, so by the converse to Thales' Theorem this means that X is on the circle ξ.



(2) Let X be a point on the circle ξ. Draw $BP \parallel DX$ and $BQ \parallel CX$ as shown on the right. Since X is on the circle, then $\angle CXD = 90°$. It follows that $PBQ = 90°$. Since $\triangle APB \sim \triangle AXD$ and $\triangle AQB \sim \triangle AXC$ we also have the following:



$$\frac{AX}{XP} = \frac{AD}{DB} \qquad \text{and} \qquad \frac{AX}{XQ} = \frac{AC}{CB}.$$

Since $\dfrac{AD}{DB} = \dfrac{AC}{CB} = k$, it follows that $\dfrac{AX}{XP} = \dfrac{AX}{XQ}$, from which we get $XP = XQ$.

Since PBQ is a right angle, the point B is on the circle centred at X with radius XP (Thales' Theorem). Thus, $XB = XP$, so,

$$\frac{AX}{XB} = \frac{AX}{XP} = \frac{AD}{DB} = k\,,$$

which shows that statement (2) holds.

□

Theorem 5.4. *Let O be the centre and r the radius of the Circle of Apollonius for A, B, and k. Then:*

(i) O is on the line AB.

(ii) The points A and B are to the same side of O.

(iii) A and B are inverses with respect to the circle.

(iv) If the circle meets AB at C and D, then C and D divide AB harmonically in the ratio k.

Proof. Statements (i) and (iv) follow directly from Theorem 5.3.

(ii) We may assume that the line AB is horizontal and that A is to the left of B, that C is between A and B, and that D is not. Thus, D is located either to the left of A (as shown in figure (a)), or else to the right of B (figure (b)). We will show that statement (ii) is true for case (a)—the proof for case (b) is similar.

(a)

(b)

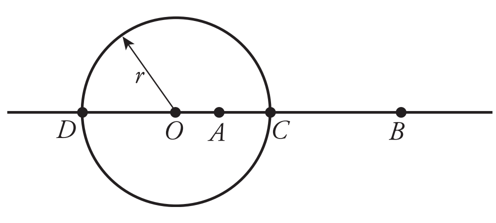
For case (a), we have $CB < DB$¿ Since C and D are on the circle of Apollonius, we also have $\frac{AC}{CB} = \frac{AD}{DB}$, and so

$$\begin{aligned} \frac{AC}{CB} \cdot CB &< \frac{AD}{DB} \cdot DB, \\ \implies \qquad AC &< AD, \end{aligned}$$

Which shows that the midpoint O of CD is to the left of A, and hence to the left of both A and B.

(iii) Assuming that O is to the left of A we have the following relationships (see the figure below):

$$AC = r - OA, \qquad AD = r + OA, \qquad CB = OB - r, \qquad DB = OB + r.$$



Since C and D are on the circle,

$$\begin{aligned} \frac{AC}{CB} &= \frac{AD}{DB} \\ \implies \qquad \frac{r - OA}{OB - r} &= \frac{r + OA}{OB + r}, \end{aligned}$$

and carrying out the Algebra will show that $OA \cdot OB = r^2$.

$\square$

Harmonic conjugates and inverses

Theorem 5.5. *A and B are harmonic conjugates with respect to C and D iff A and B are inverses with respect to the circle with diameter CD.*

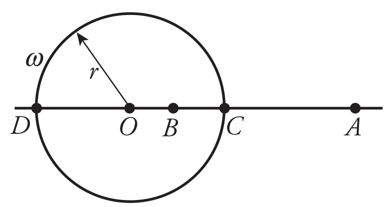
Proof. Suppose that A, B are harmonic conjugates for CD. Then C and D are harmonic conjugates for AB, that is

$$\frac{AC}{CB} = \frac{AD}{DB}.$$

Letting r be the radius of the circle with diameter CD, we want to show that $OA \cdot OB = r^2$. The proof proceeds as in the proof of statement (iii) of Theorem 5.4.

Conversely, suppose that A, B are inverses with respect to the circle ω with diameter CD. Assuming that A is outside ω as shown, to prove that A and B are harmonic conjugates for CD, it suffices to show that

$$\frac{CA/AD}{CB/BD} = 1.$$



Referring to the figure, we have

$$\begin{aligned}
\frac{CA/AD}{CB/BD} &= \frac{CA \cdot BD}{AD \cdot CB} \\
&= \frac{(OA - r) \cdot (OB + r)}{(OA + r) \cdot (r - OB)} \\
&= \frac{OA \cdot OB - r \cdot OB + r \cdot OA - r^2}{r \cdot OA + r^2 - OA \cdot OB - r \cdot OB}
\end{aligned}$$

and since $OA \cdot OB = r^2$, we get

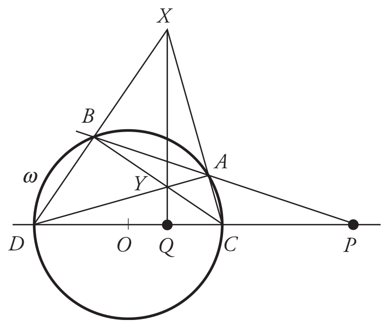
$$\begin{aligned}
\frac{CA/AD}{CB/BD} &= \frac{r \cdot OA - r \cdot OB}{r \cdot OA - r \cdot OB} \\
&= 1,
\end{aligned}$$

which finishes the proof. $\square$

The relationship between harmonic conjugates and inverses enables us to show how a straightedge alone can be used to find the inverse of a point P that is outside the circle of inversion.

Problem 5.6. *Given a point P outside a circle ω with centre O, construct the inverse of P using only a straightedge.*

Solution.



(1) Draw the line OP meeting ω at C and D.

(2) Draw a second line through P meeting ω at A and B as shown.

(3) Draw AC and BD meeting at X. Draw AD and BC meeting at Y.

(4) Draw X and Y meeting OP at Q. Then Q is the inverse of P.

Proof. Apply Ceva's Theorem to $\triangle XCD$ and cevians XQ, CB, and DA. The cevians are concurrent at Y, so

$$\frac{XA}{AC} \cdot \frac{CQ}{QD} \cdot \frac{DB}{BX} = 1 \tag{1}$$

Apply Menelaus' Theorem to $\triangle XCD$ with Menelaus points P, A, B. The points P, A, and B are collinear so

$$\frac{XA}{AC} \cdot \frac{CP}{PD} \cdot \frac{DB}{BX} = 1 \tag{2}$$

From (1) and (2) we get

$$\frac{CQ}{QD} = \frac{CP}{PD},$$

which implies that P and Q are harmonic conjugates with respect to CD. By the previous theorem, this means that P and Q are inverses with respect to ω. □

Inversion and the circle of Apollonius

We state here several theorems that are easy consequences of the previous sections.

Theorem 5.7. *If ω is the circle of Apollonius for A, B, and k, then A and B are inverses with respect to ω.*

Theorem 5.8. *The Apollonian circle for A, B, and k is the same as the Apollonian circle for B, A, and $\frac{1}{k}$.*

Remark: Note the change in order of the points A and B in the previous theorem.

Theorem 5.9. *If A and B are inverse points for a circle ω, then ω is the circle of Apollonius for A, B, and some positive number k.*

Theorem 5.10. *If α and β are orthogonal circles, then whenever either circle intersects a diameter of the other, it divides that diameter harmonically.*

The following is the converse of the previous theorem.

Theorem 5.11. *If α and β are two circles, and β divides a diameter of α harmonically, then the two circles are orthogonal.*