New Zealand Mathematical Olympiad Committee

2012 Squad Assignment One

Combinatorics

Due: Friday 3rd February 2012

1. *Each cell of an 8×8 chess board is filled with a 0 or a 1. Prove that if we compute the sums of the numbers in each row, each column, and in each of the two diagonals, then we will get at least three sums that are equal.*

   Solution: There are 18 sums, and they can be any of the nine numbers 0 to 8. If there are no three equal sums, each possible sum from 0 to 8 must occur exactly twice. Suppose without loss of generality that a total of 8 appears in a row (it cannot appear on a diagonal, as then there is only one possible way to get a sum of 0, namely the other diagonal). Then the sum of the numbers in any column or diagonal is at least 1, so the two sums of 0 must be in the rows as well, and consequently the other sum of 8 must appear in a row too.

   Since there are two rows with just 0s in them, no column or diagonal can add up to 7, and similarly no column or diagonal can add up to 1. Hence these sums must occur in the rows as well, and all the other sums must occur in the columns or diagonals.

   Now there are four rows which together contain exactly two 1s (the rows summing to 0, and the rows summing to 1), and similarly four rows that together contain exactly two zeros. Therefore there must be at least four columns that contain four 0s and four 1s, giving a sum of 4 each. This completes the proof. □

2. *Each of the three countries Alania, Belinga and Cartusia is inhabited by exactly n people. Each of the $3n$ inhabitants of these countries has exactly $n + 1$ friends in the other two countries. Prove that one can find a group of three people, one from each country, that are mutual friends.*

   Solution: Consider a person with the smallest number of friends in a single country. Let this number be k. By assumption $k \geq 1$. Let us further assume (without loss of generality) that this person lives in Alania, has k friends in Belinga and $n+1-k$ friends in Cartusia. Now consider one of the k friends in Belinga. This friend must have at least k friends in Cartusia (by minimality of k). Since $k+(n+1-k) = n+1 > n$, one of these friends must be friends with the original Alanian we started with, so we are done. □

3. *Let $n \geq 3$ be an integer. Determine the minimum number of points one has to mark inside a convex n-gon in order for the interior of any triangle with its vertices at vertices of the n-gon to contain at least one of the marked points.*

   Solution: Since the diagonals from one vertex divide an n-gon into $n - 2$ disjoint triangles, at least $n - 2$ points are necessary. We claim that it is in fact possible to mark $n - 2$ points so that the given condition is satisfied.

Denote the vertices of the given n-gon by $A_1, A_2, \ldots, A_n$. Draw all of the diagonals of the given n-gon, and colour the areas bounded by the diagonals A_1A_k, A_kA_n and $A_{k-1}A_{k+1}$ for each $k \in \{2, 3, \ldots, n-1\}$.

If we mark one point in each coloured area then every triangle with vertices at vertices of the n-gon must contain at least one of the marked points. Indeed, if $1 \le l < k < m \le n$ then triangle $A_lA_kA_m$ contains the whole coloured region bounded by the diagonals A_1A_k, A_kA_n, $A_{k-1}A_{k+1}$, and so contains the corresponding marked point. □

4. *Abby and Brian play the following game. They first choose a positive integer N, and then they take turns writing numbers on a blackboard. Abby starts by writing 1. Thereafter, when one of them has written the number n, the next player writes down either $n+1$ or $2n$, provided the number is not greater than N. The player who writes N on the blackboard wins.*

   (a) *Determine which player has a winning strategy if $N = 2011$.*

   (b) *Find the number of positive integers $N \le 2011$ for which Brian has a winning strategy.*

   Solution:

   (a) Abby has a winning strategy for odd N, and so wins when $N = 2011$. Observe that, whenever a player writes down an odd number, the next is forced to write down an even number. By adding 1 to that number, the first player can write down another odd number. Since Abby starts the game by writing down an odd number, she can force Brian to write down even numbers only. Since N is odd, Abby will win the game, and in particular, she wins for $N = 2011$.

   (b) For even N we consider two cases, according to the value of $N \bmod 4$.

   - Let $N = 4k$. If any player is forced to write down a number $m \in \{k+1, k+2, \ldots, 2k\}$, the other player wins by writing down $2m \in \{2k+2, 2k+4, \ldots, 4k\}$, for the players will then have to write down the remaining numbers one after the other. Since there is an even number of numbers remaining, the latter player wins. This implies that the player who can write down k (that is, has a winning strategy for $N = k$)), wins the game for $N = 4k$.
   - Similarly, let $N = 4k+2$. If any player is forced to write down a number $m \in \{k+1, k+2, \ldots, 2k+1\}$, the other player wins the game by writing down $2m \in \{2k+2, 2k+4, \ldots, 4k+2\}$, as in the previous case. Analogously, this implies that the player who has a winning strategy for $N = k$ wins the game for $N = 4k+2$.

   Since Abby wins the game for $N = 1, 3$, while Brian wins the game for $N = 2$, Brian wins the game for $N = 8, 10$ as well, and thus for $N = 32, 34, 40, 42$ too. Then Brian wins the game for a further 8 values of N between 128 and 170, and thence a further 16 values between 512 and 682, and for no other values with $N \le 2011$. Hence Brian has a winning strategy for precisely 31 values of N with $N \le 2011$.

□

0	1	2	3	0	1	2	3	0	1	2
1	2	3	0	1	2	3	0	1	2	3
2	3	0	1	2	3	0	1	2	3	0
3	0	1	2	3	0	1	2	3	0	1
0	1	2	3	0	1	2	3	0	1	2
1	2	3	0	1	2	3	0	1	2	3
2	3	0	1	2	3	0	1	2	3	0
3	0	1	2	3	0	1	2	3	0	1
0	1	2	3	0	1	2	3	0	1	2
1	2	3	0	1	2	3	0	1	2	3
2	3	0	1	2	3	0	1	2	3	0

Figure 1: An example of the colouring, and the division of the array into subarrays, for $N = 11$ and $M = 4$. In this case we have $N = 2M + 3$, and the double lines divide the array into subarrays of dimensions $2M \times 2M$, $2M \times 3$, $3 \times 2M$ and 3×3. In the 3×3 square we see that colour 2 appears one more time than colour 3; since these two colours appear an equal number of times in each of the other subarrays, this implies that they appear in the whole array with different parities.

5. *Let M and N be positive integers. Consider an $N \times N$ square array consisting of N^2 lamps that can be in two states — on or off. Initially all of the lamps are turned off.*

   A move *consists of choosing a row or column of the array and changing the state of M consecutive lamps in the chosen row or column, i.e., turning on the lamps that were off, and turning off the lamps that were on.*

   Determine the necessary and sufficient condition for which it can be achieved that after a finite number of moves all of the lamps are turned on.

   Solution: The sought condition is that M divides N.

   It is easy to see that if M divides N we can choose a sequence of moves after which all lamps are turned on. In this case each row may be divided into $d = N/M$ disjoint blocks of size $1 \times M$, and then in a sequence of dN moves we may turn on all the lamps in each block in turn.

   To prove necessity, colour the lamps in M colours (named $0, 1, 2, \ldots, M-1$) in the way presented in Figure 1, i.e. colour the lamp in the ith row and jth column colour $i + j - 2 \bmod M$.

   In every move we change the state of exactly one lamp of each colour. In the beginning all the lamps are turned off, so after each step the parity of the number of lamps of each colour that are turned on must be the same. Thus, if it is possible to achieve that after some move all of the lamps are turned on, then the parity of the number of lamps of each colour must be the same.

   Assume on the contrary that M does not divide N and let $N = Mk + r$, where $1 \le r \le M - 1$. Divide the $N \times N$ array into four subarrays of dimensions $Mk \times Mk$, $Mk \times r$, $r \times Mk$ and $r \times r$, as seen in Figure 1.

Since each of the subarrays of dimensions $Mk \times Mk$, $Mk \times r$ and $r \times Mk$ are a disjoint union of blocks of dimensions $1 \times M$ or $M \times 1$, we see that the number of lamps of each colour in their union is the same (it is $Mk^2 + 2kr$).

Consider the remaining $r \times r$ subarray. In the figure we see that the number of lamps of colour $r-1$ equals r, but the number of lamps of colour r equals $r-1$. Indeed, the lamps of colour $r-1$ appear in each row of the subarray exactly once, while the lamps of colour r appear in all but the first row. Also, there is no row with two or more lamps of colour r, because r is strictly less than M, so all the lamps in a given row are different colours.

Hence in the whole array there is one fewer lamp of colour r then there is of colour $r-1$, so the numbers of lamps of these colours have different parities. Consequently, we can never achieve the state in which all of the lamps are turned on.

Alternate solution (Malcolm Granville). We work modulo 2. Assign the monomial x^{i+j} to the (i, j) square, where i, j run from 0 to $N-1$. We will consider in two ways the sum over all strips of M lamps changed the sum of the monomials in the strip. Firstly, it is clearly divisible by $P = 1+\cdots+x^{M-1}$. Secondly, since all lamps are on at the end, each is flipped an odd (=1 mod 2) number of times, so the sum is also $Q = (1+\cdots+x^{N-1})^2$. Hence $P|Q$, or $(x-1)(x^M-1)|(x^N-1)^2 = x^{2N}-1$. It then follows from $\gcd(x^a-1, x^b-1) = x^{\gcd(a,b)}-1$ that we need $M|2N$.

Suppose M doesn't divide N, and set $y = x^M$. Then $2N/M = k$ is odd, so the polynomial

$$\frac{x^{2N}-1}{x^M-1} = \frac{y^k-1}{y-1} = y^{k-1} + \cdots + 1$$

doesn't have 1 as a root, and therefore $x-1$ can't divide $x^{2N}-1$ also, a contradiction. Thus $M|N$, and clearly this works. $\square$

6. *Each of 117 spies, operating in a certain country, is to assign himself to one of three missions, such that each mission has at least one spy assigned. At this point, no two spies can communicate. Headquarters will then sequentially issue a number of* passwords, *each of which allows a single pair of spies to communicate. Passwords may only be issued to a pair of spies who share the same mission, and who cannot* already *communicate directly. However, apart from these rules, headquarters may assign passwords in any fashion.*

   A mission is said to be networked *if any two spies on that mission can communicate (possibly through other spies). Let n denote the number of passwords issued for which, regardless of how headquarters assigns these passwords, the spies can be certain that at least one mission will be networked. How should the spies choose their missions in order to minimise n?*

   Solution: We represent the given information by a graph, with a vertex for each spy, and an edge between two spies if they share a password which allows them to communicate directly. We need to ensure that at least one of the subgraphs formed by spies sharing the same mission is connected.

   To find the number of edges needed to ensure that a graph with $a \geq 2$ vertices is connected, no matter how the edges are placed, we first observe that the complete graph with $a-1$

vertices has $\frac{(a-1)(a-2)}{2}$ edges. We next prove that it is sufficient to have

$$\frac{(a-1)(a-2)}{2}+1=\frac{a^2-3a+4}{2}$$

edges. To see this, assume that the graph is not connected. Then there exist two subgraphs, not connected to each other, one with $x \geq 1$ and the other with $a-x \geq 1$ vertices. The number of edges of such a graph is at most

$$\begin{aligned}\frac{(a-x)(a-x-1)}{2}+\frac{x(x-1)}{2} &= \frac{2x^2-2ax+a^2-a}{2}\\ &= \frac{a^2-3a+2}{2}-(a-x-1)(x-1)\\ &\leq \frac{a^2-3a+2}{2}=\frac{(a-1)(a-2)}{2}.\end{aligned}$$

If the sizes of the three missions are a, b and c, the above shows that issuing

$$n=\frac{a^2-3a+2}{2}+\frac{b^2-3b+2}{2}+\frac{c^2-3c+2}{2}+1$$

passwords ensures that at least one mission is networked.

Finally, we need to minimise

$$2n=a^2+b^2+c^2-3(a+b+c)+8$$

subject to the condition that $a+b+c=117$. Because

$$\begin{aligned}3(a^2+b^2+c^2)-(a+b+c)^2 &= 2(a^2+b^2+c^2-ab-bc-ca)\\ &= (a-b)^2+(b-c)^2+(c-a)^2 \geq 0,\end{aligned}$$

we obtain

$$2n \geq \frac{(a+b+c)^2}{3}-3(a+b+c)+8=\frac{117^2}{3}-3\cdot 117+8=4220,$$

with equality iff $a=b=c=39$. This shows that the minimum number of passwords needed to ensure that at least one mission is networked is $n=2110$. This number is sufficient iff the three missions are of equal size. □

7. *A given list $n_1, n_2, \ldots, n_{2011}$ of positive integers has the property that $n_i n_{i+1}$ is different from $n_j n_{j+1}$ whenever i, j are distinct integers less than 2011. Find the minimum number of distinct integers that must be in any such list.*

Solution: Suppose a list $L=(n_i)_{i=1}^N$ is made up only of numbers selected from the (possibly much shorter) list of distinct numbers $(m_i)_{i=1}^k$, and that the products $n_i n_{i+1}$ are all distinct. We write down the indices of the numbers m_i occurring in L as another list M, also of length N (so if L starts $m_3, m_7, m_2, m_3, m_9, \ldots$ then M begins 3,7,2,3,9, ...).

If $n_i=m_a$ and $n_{i+1}=m_b$, then the product $n_i n_{i+1}$ is determined by the set $\{a,b\}$. Hence, any two (not necessarily distinct) numbers a, b from $\{1,2,\ldots,k\}$ can appear as neighbours in the list M at most once. Let G be the graph whose vertices are the numbers $\{1, 2, \ldots, k\}$, with an edge from each vertex to every other vertex (including a loop from each vertex to itself). Then M can be thought of as a walk in G, and the condition that any two numbers a, b can appear as neighbours at most once means that M traverses each edge at most once. Thus G must have at least 2010 edges, since M visits 2011 vertices. Now since G has k vertices each of degree $k + 1$ it has a total of $k(k + 1)/2$ edges, and so we must have $k(k + 1)/2 \geq 2010$, i.e. $k \geq 63$.

Moreover, for k odd (and in particular, for $k = 63$), every vertex of G has even degree, and so G has an Eulerian circuit. Following this circuit until we have visited 2011 vertices gives a list M as required. Let $M = (\mu_i)_{i=1}^{2011}$ be this list, and let m_i be the ith prime number. Defining $n_i = m_{\mu_i}$ we obtain a list $L = (n_i)_{i=1}^{2011}$ which satisfies the conditions of the problem. $\square$

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