

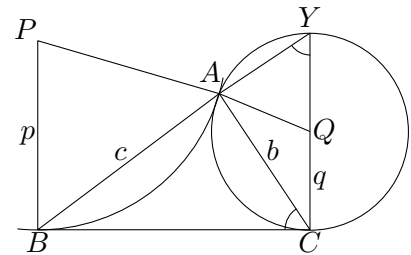
Plane Geometry : Ceva's Theorem Problems with Solutions

Problems.

- For $\triangle ABC$, let p and q be the radii of two circles through A , touching BC at B and C , respectively. Prove $pq = R^2$.

Solution. Let P be the centre of the circle of radius p through A , touching BC at B , and let Q be the centre of the circle of radius q through A , touching BC at C . Produce CQ to meet the circle centred at Q again at Y , so that CY is a diameter. Then

$$\begin{aligned}\angle C &= \angle ACB = \angle AYB, && \text{by Theorem 29} \\ \angle CAY &= 90^\circ, && \text{by Theorem 20}\end{aligned}$$



$$\therefore \sin C = \sin(\angle AYB) = \frac{b}{2q}$$

$$\therefore 2q = \frac{b}{\sin C} \tag{1}$$

$$\text{Similarly, } 2p = \frac{c}{\sin B} \tag{2}$$

Multiplying, (2) and (1) we have

$$\begin{aligned}\therefore 2p \cdot 2q &= \frac{b}{\sin C} \cdot \frac{c}{\sin B} \\ &= \frac{b}{\sin B} \cdot \frac{c}{\sin C} \\ &= 2R \cdot 2R,\end{aligned}$$

by the Sine Rule

$$\therefore pq = R^2.$$

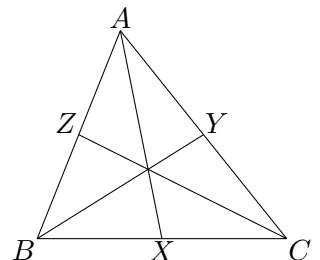
- If X, Y, Z are the midpoints of sides BC, CA, AB , respectively, of $\triangle ABC$, prove the cevians AX, BY, CZ are concurrent.

The cevians AX, BY, CZ here, are the *medians* of $\triangle ABC$ and the point at which they concur is the *centroid* or *centre of gravity* of $\triangle ABC$.

Solution. If the cevians are medians we have:

$$\begin{aligned}BX &= XC, \quad CY = YA, \quad AZ = ZB \\ \therefore \frac{BX}{XC} \frac{CY}{YA} \frac{AZ}{ZB} &= 1.\end{aligned}$$

Hence by Ceva's Theorem, AX, BY, CZ concur. The point of concurrence is often labelled G (memonical for centre of Gravity).



3. Prove cevians perpendicular to the opposite sides are concurrent.

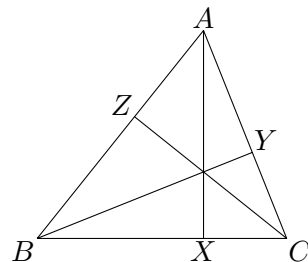
Such cevians of a triangle are its *altitudes* and the point at which they concur is the *orthocentre*.

Solution. Suppose cevians AX , BY , CZ are altitudes of $\triangle ABC$. Then

$$\begin{aligned}\angle AXC &= \angle BYC = 90^\circ \\ \angle XCA &= \angle YCB, & \text{(same angle)} \\ \therefore \triangle AXC &\sim \triangle BYC, & \text{by the AA Rule} \quad (3)\end{aligned}$$

Similarly,

$$\begin{aligned}\triangle CZA &\sim \triangle BYA & (4) \\ \text{and } \triangle AXB &\sim \triangle CZB & (5) \\ \therefore \frac{CY}{XC} &= \frac{YC}{XC} = \frac{BY}{AX}, & \text{by (3)} \\ \frac{AZ}{YA} &= \frac{ZA}{YA} = \frac{CZ}{BY}, & \text{by (4)} \\ \frac{BX}{ZB} &= \frac{XB}{ZB} = \frac{AX}{CZ}, & \text{by (5)} \\ \therefore \frac{BX}{XC} \frac{CY}{YA} \frac{AZ}{ZB} &= \frac{CY}{XC} \frac{AZ}{YA} \frac{BX}{ZB} \\ &= \frac{BY}{AX} \frac{CZ}{BY} \frac{AX}{CZ} \\ &= 1\end{aligned}$$



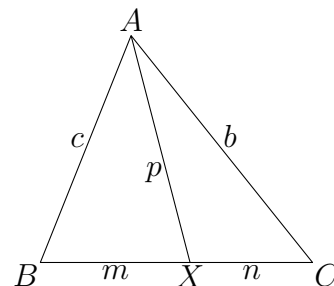
Hence by Ceva's Theorem, AX , BY , CZ concur. The point of concurrence is often labelled H (perhaps memonical for *ortHocentre*).

4. Let AX be a cevian of $\triangle ABC$ of length p dividing BC into segments $BX = m$ and $XC = n$. Prove

$$a(p^2 + mn) = b^2m + c^2n.$$

This result is known as *Stewart's Theorem*.

Hint. Use the Cosine Rule on each of $\triangle ABX$ and $\triangle AXC$, in each case taking the cosine of the angle at X . What relationship do the cosines of supplementary angles have?



Solution. Applying the Cosine Rule to $\triangle ABX$ and $\triangle AXC$, we have

$$c^2 = p^2 + m^2 - 2pm \cos(\angle AXB) \quad (6)$$

$$b^2 = p^2 + n^2 - 2pn \cos(\angle AXC) \quad (7)$$

Now $\angle AXB$ and $\angle AXC$ are supplementary; so their cosines are the negatives of one another. So we multiply each of (6) and (7) by appropriate constants so that the cosine expressions will cancel when the resulting equations are added:

$$n \times (6) : nc^2 = np^2 + nm^2 - 2pmn \cos(\angle AXB) \quad (8)$$

$$m \times (7) : mb^2 = mp^2 + mn^2 - 2pmn \cos(\angle AXC) \quad (9)$$

$$\therefore nc^2 + mb^2 = (n + m)p^2 + (n + m)mn, \quad (8) + (9)$$

$$\therefore nc^2 + mb^2 = ap^2 + amn, \quad \text{since } m + n = BC = a$$

$$\therefore a(p^2 + mn) = b^2m + c^2n$$

5. Prove that the medians of a triangle dissect the triangle into six smaller triangles of equal area.

Solution. Suppose $\triangle ABC$ has medians AX, BY, CZ as per the diagram.
First, since AX is a median,

$$BX = XC.$$

Regarding these as the bases of $\triangle GBX$ and $\triangle GXC$, respectively, which then have a common altitude to G from BC , it follows that the areas of these triangles are equal, i.e.

$$(GBX) = (GXC) = x \text{ say.}$$

Similarly, since BY is a median, $CY = YA$ which are bases for $\triangle GCY$ and $\triangle GYA$ that have a common altitude to G from CA , so that

$$(GCY) = (GYA) = y \text{ say,}$$

and since CZ is a median, $AX = ZB$ which are bases for $\triangle GAZ$ and $\triangle GZB$ that have a common altitude to G from AB , so that

$$(GAZ) = (GZB) = z \text{ say.}$$

Updating our diagram with x, y, z we have the diagram, at right.
Since, $\triangle ABX$ and $\triangle AXC$ have equal bases BX and XC , with a common altitude to A from BC . So,

$$\begin{aligned} 2z + x &= (ABX) = (AXC) = 2y + x \\ \therefore 2z &= 2y \\ z &= y \end{aligned}$$

Similarly, $\triangle CAZ$ and $\triangle CZB$ have equal bases AZ and ZB , with a common altitude to C from AB . So,

$$\begin{aligned} 2y + z &= (CAZ) = (CZB) = 2x + z \\ \therefore 2y &= 2x \\ y &= x \\ \therefore x &= y = z. \end{aligned}$$

Thus the six smaller triangles all have equal area, and so the medians of a triangle dissect $\triangle ABC$ into six smaller triangles of equal area.

6. Prove the medians of a triangle divide each other in the ratio 2 : 1, i.e. the medians of a triangle *trisect* one another.

Solution. From the previous problem we have that the medians of a triangle dissect the triangle into six triangles of a common area x , say.

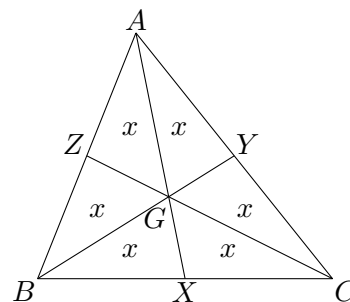
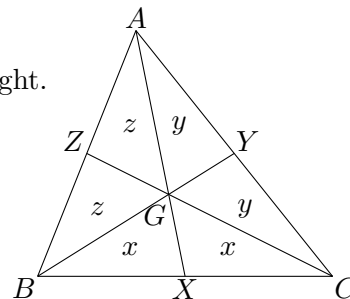
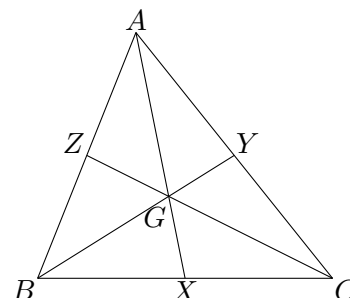
Now, $\triangle CAG$ and $\triangle CGX$ can be taken to have bases AG and GX , respectively, with common altitude to C from AX . So the ratio of their bases is equal to the ratio of their areas, i.e.

$$\begin{aligned} AG : GX &= (CAG) : (CGX) \\ &= 2x : x \\ &= 2 : 1. \end{aligned}$$

Similarly,

$$BG : GY = CG : GZ = 2 : 1.$$

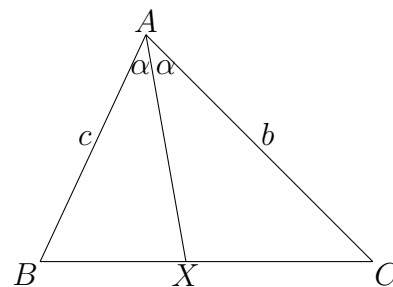
Hence the medians trisect one another.



7. Prove that each (internal) angle bisector of a triangle divides the opposite side into segments proportional in length to the adjacent sides, e.g. if AX is the cevian of $\triangle ABC$ that bisects the angle at A internally, then $BX : XC = c : b$.

Hint. Use the Sine Rule on each of $\triangle ABX$ and $\triangle AXC$, in each case taking the sine of the angle at X .

What relationship do the sines of supplementary angles have?



Solution. Let us abbreviate the size of $\angle AXB$ and $\angle AXC$ by X_{\bullet} and X_{\circ} , respectively. Then, since X_{\bullet} and X_{\circ} are supplementary, i.e. $X_{\bullet} + X_{\circ} = 180^\circ$, we have

$$\sin X_{\bullet} = \sin X_{\circ}.$$

Applying, the Sine Rule in $\triangle ABX$, we have

$$\frac{BX}{\sin \alpha} = \frac{c}{\sin X_{\bullet}}.$$

Similarly, in $\triangle AXC$ we have

$$\frac{XC}{\sin \alpha} = \frac{b}{\sin X_{\circ}}.$$

So, we have

$$\frac{BX}{XC} = \frac{BX/\sin \alpha}{XC/\sin \alpha} = \frac{c/\sin X_{\bullet}}{b/\sin X_{\circ}} = \frac{c}{b},$$

i.e. $BX : XC = c : b$.

8. The angle bisector of the angle between two sides is the locus of points that are equidistant from the sides making the angle. One consequence of this is that any pair of internal angle bisectors of a triangle meet at a point that is equidistant from all three sides of the triangle, and hence in fact the three internal angle bisectors are concurrent.

The point at which the angle bisectors of a triangle concur is the *incentre* I , the common (perpendicular) distance from I to the three sides is the *inradius* r , and the circle with centre I and radius r thus touches each side tangentially and is called the *incircle* of the triangle.

Find an alternative proof that the (internal) angle bisectors of a triangle concur, using Ceva's Theorem and the result of the previous problem.

Solution. Let AX, BY, CZ be angle bisectors of $\triangle ABC$. From the previous problem, since AX is an angle bisector

$$\frac{BX}{XC} = \frac{c}{b}.$$

Likewise, since BY is an angle bisector,

$$\frac{CY}{YA} = \frac{a}{c},$$

and, since CZ is an angle bisector,

$$\frac{AZ}{ZB} = \frac{b}{a}.$$

Hence,

$$\begin{aligned} \frac{BX}{XC} \frac{CY}{YA} \frac{AZ}{ZB} &= \frac{c}{b} \frac{a}{c} \frac{b}{a} \\ &= 1, \end{aligned}$$

and so by Ceva's Theorem, cevians AX, BY, CZ are concurrent.