

Mock 003

Time: 6:00 hours

Total marks: 28

Problem 1: Let n, k be positive integers such that $n \geq k$. n lamps are placed on a circle, which are all off. In any step we can change the state of k consecutive lamps. In the following three cases, how many states of lamps are there in all 2^n possible states that can be obtained from the initial state by a certain series of operations?

- (a) k is a prime number greater than 2;
- (b) k is odd;
- (c) k is even.

Problem: 2 Two circles O_1 and O_2 intersect at M, N . The common tangent line nearer to M of the two circles touches O_1, O_2 at A, B respectively. Let C, D be the symmetric points of A, B with respect to M respectively. The circumcircle of $\triangle DCM$ intersects circles O_1 and O_2 at points E, F respectively which are distinct from M . Prove that the circumradii of the $\triangle MEF$ and $\triangle NEF$ are equal.

Problem: 3 Find all $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies:

- (i) $f(a) = 1$ with a is a given real number
- (ii) $f(x) f(y) + f\left(\frac{a}{y}\right) f\left(\frac{a}{x}\right) = 2 f(xy)$ for all $x, y > 0$

Problem: 4 Prove that for each natural number m , there is a natural number N such that for each b that $2 \leq b \leq 1389$ sum of digits of N in base b is larger than m .

Problems collected by Fahim Ferdous (<http://www.facebook.com/fahim.neerjhor>)