

## Algebra

- 1 Find all functions  $f$  from the reals to the reals such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all real  $x, y$ .

- 2 Let  $a_1, a_2, \dots$  be an infinite sequence of real numbers, for which there exists a real number  $c$  with  $0 \leq a_i \leq c$  for all  $i$ , such that

$$|a_i - a_j| \geq \frac{1}{i+j} \quad \forall i, j \quad \text{with} \quad i \neq j.$$

Prove that  $c \geq 1$ .

- 3 Let  $P$  be a cubic polynomial given by  $P(x) = ax^3 + bx^2 + cx + d$ , where  $a, b, c, d$  are integers and  $a \neq 0$ . Suppose that  $xP(x) = yP(y)$  for infinitely many pairs  $x, y$  of integers with  $x \neq y$ . Prove that the equation  $P(x) = 0$  has an integer root.

- 4 Find all functions  $f$  from the reals to the reals such that

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

for all real  $x, y, z, t$ .

- 5 Let  $n$  be a positive integer that is not a perfect cube. Define real numbers  $a, b, c$  by

$$a = \sqrt[3]{n}, \quad b = \frac{1}{a - [a]}, \quad c = \frac{1}{b - [b]},$$

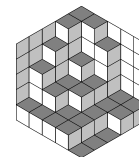
where  $[x]$  denotes the integer part of  $x$ . Prove that there are infinitely many such integers  $n$  with the property that there exist integers  $r, s, t$ , not all zero, such that  $ra + sb + tc = 0$ .

- 6 Let  $A$  be a non-empty set of positive integers. Suppose that there are positive integers  $b_1, \dots, b_n$  and  $c_1, \dots, c_n$  such that

- for each  $i$  the set  $b_i A + c_i = \{b_i a + c_i : a \in A\}$  is a subset of  $A$ , and
- the sets  $b_i A + c_i$  and  $b_j A + c_j$  are disjoint whenever  $i \neq j$

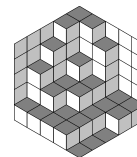
Prove that

$$\frac{1}{b_1} + \dots + \frac{1}{b_n} \leq 1.$$



## Combinatorics

- 1 Let  $n$  be a positive integer. Each point  $(x, y)$  in the plane, where  $x$  and  $y$  are non-negative integers with  $x + y < n$ , is coloured red or blue, subject to the following condition: if a point  $(x, y)$  is red, then so are all points  $(x', y')$  with  $x' \leq x$  and  $y' \leq y$ . Let  $A$  be the number of ways to choose  $n$  blue points with distinct  $x$ -coordinates, and let  $B$  be the number of ways to choose  $n$  blue points with distinct  $y$ -coordinates. Prove that  $A = B$ .
- 2 For  $n$  an odd positive integer, the unit squares of an  $n \times n$  chessboard are coloured alternately black and white, with the four corners coloured black. A tromino is an  $L$ -shape formed by three connected unit squares. For which values of  $n$  is it possible to cover all the black squares with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?
- 3 Let  $n$  be a positive integer. A sequence of  $n$  positive integers (not necessarily distinct) is called **full** if it satisfies the following condition: for each positive integer  $k \geq 2$ , if the number  $k$  appears in the sequence then so does the number  $k - 1$ , and moreover the first occurrence of  $k - 1$  comes before the last occurrence of  $k$ . For each  $n$ , how many full sequences are there?
- 4 Let  $T$  be the set of ordered triples  $(x, y, z)$ , where  $x, y, z$  are integers with  $0 \leq x, y, z \leq 9$ . Players  $A$  and  $B$  play the following guessing game. Player  $A$  chooses a triple  $(x, y, z)$  in  $T$ , and Player  $B$  has to discover  $A$ 's triple in as few moves as possible. A *move* consists of the following:  $B$  gives  $A$  a triple  $(a, b, c)$  in  $T$ , and  $A$  replies by giving  $B$  the number  $|x + y - a - b| + |y + z - b - c| + |z + x - c - a|$ . Find the minimum number of moves that  $B$  needs to be sure of determining  $A$ 's triple.
- 5 Let  $r \geq 2$  be a fixed positive integer, and let  $F$  be an infinite family of sets, each of size  $r$ , no two of which are disjoint. Prove that there exists a set of size  $r - 1$  that meets each set in  $F$ .
- 6 Let  $n$  be an even positive integer. Show that there is a permutation  $x_1, x_2, \dots, x_n$  of  $1, 2, \dots, n$  such that for every  $1 \leq i \leq n$  the number  $x_{i+1}$  is one of  $2x_i, 2x_i - 1, 2x_i - n, 2x_i - n - 1$  (where we take  $x_{n+1} = x_1$ ).
- 7 Among a group of 120 people, some pairs are friends. A *weak quartet* is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets?



## Geometry

- [1] Let  $B$  be a point on a circle  $S_1$ , and let  $A$  be a point distinct from  $B$  on the tangent at  $B$  to  $S_1$ . Let  $C$  be a point not on  $S_1$  such that the line segment  $AC$  meets  $S_1$  at two distinct points. Let  $S_2$  be the circle touching  $AC$  at  $C$  and touching  $S_1$  at a point  $D$  on the opposite side of  $AC$  from  $B$ . Prove that the circumcentre of triangle  $BCD$  lies on the circumcircle of triangle  $ABC$ .
- [2] Let  $ABC$  be a triangle for which there exists an interior point  $F$  such that  $\angle AFB = \angle BFC = \angle CFA$ . Let the lines  $BF$  and  $CF$  meet the sides  $AC$  and  $AB$  at  $D$  and  $E$  respectively. Prove that

$$AB + AC \geq 4DE.$$

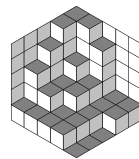
- [3] The circle  $S$  has centre  $O$ , and  $BC$  is a diameter of  $S$ . Let  $A$  be a point of  $S$  such that  $\angle AOB < 120^\circ$ . Let  $D$  be the midpoint of the arc  $AB$  which does not contain  $C$ . The line through  $O$  parallel to  $DA$  meets the line  $AC$  at  $I$ . The perpendicular bisector of  $OA$  meets  $S$  at  $E$  and at  $F$ . Prove that  $I$  is the incentre of the triangle  $CEF$ .
- [4] Circles  $S_1$  and  $S_2$  intersect at points  $P$  and  $Q$ . Distinct points  $A_1$  and  $B_1$  (not at  $P$  or  $Q$ ) are selected on  $S_1$ . The lines  $A_1P$  and  $B_1P$  meet  $S_2$  again at  $A_2$  and  $B_2$  respectively, and the lines  $A_1B_1$  and  $A_2B_2$  meet at  $C$ . Prove that, as  $A_1$  and  $B_1$  vary, the circumcentres of triangles  $A_1A_2C$  all lie on one fixed circle.
- [5] For any set  $S$  of five points in the plane, no three of which are collinear, let  $M(S)$  and  $m(S)$  denote the greatest and smallest areas, respectively, of triangles determined by three points from  $S$ . What is the minimum possible value of  $M(S)/m(S)$ ?
- [6] Let  $n \geq 3$  be a positive integer. Let  $C_1, C_2, C_3, \dots, C_n$  be unit circles in the plane, with centres  $O_1, O_2, O_3, \dots, O_n$  respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$

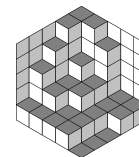
- [7] The incircle  $\Omega$  of the acute-angled triangle  $ABC$  is tangent to  $BC$  at  $K$ . Let  $AD$  be an altitude of triangle  $ABC$  and let  $M$  be the midpoint of  $AD$ . If  $N$  is the other common point of  $\Omega$  and  $KM$ , prove that  $\Omega$  and the circumcircle of triangle  $BCN$  are tangent at  $N$ .
- [8] Let  $S_1$  and  $S_2$  be circles meeting at the points  $A$  and  $B$ . A line through  $A$  meets  $S_1$  at  $C$  and  $S_2$  at  $D$ . Points  $M, N, K$  lie on the line segments  $CD, BC, BD$  respectively, with  $MN$



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parallel to  $BD$  and  $MK$  parallel to  $BC$ . Let  $E$  and  $F$  be points on those arcs  $BC$  of  $S_1$  and  $BD$  of  $S_2$  respectively that do not contain  $A$ . Given that  $EN$  is perpendicular to  $BC$  and  $FK$  is perpendicular to  $BD$  prove that  $\angle EMF = 90^\circ$ .



## Number Theory

- [1] What is the smallest positive integer  $t$  such that there exist integers  $x_1, x_2, \dots, x_t$  with

$$x_1^3 + x_2^3 + \dots + x_t^3 = 2002^{2002} ?$$

- [2] Let  $n \geq 2$  be a positive integer, with divisors  $1 = d_1 < d_2 < \dots < d_k = n$ . Prove that  $d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$  is always less than  $n^2$ , and determine when it is a divisor of  $n^2$ .
- [3] Let  $p_1, p_2, \dots, p_n$  be distinct primes greater than 3. Show that  $2^{p_1 p_2 \dots p_n} + 1$  has at least  $4^n$  divisors.

- [4] Is there a positive integer  $m$  such that the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infinitely many solutions in positive integers  $a, b, c$ ?

- [5] Let  $m, n \geq 2$  be positive integers, and let  $a_1, a_2, \dots, a_n$  be integers, none of which is a multiple of  $m^{n-1}$ . Show that there exist integers  $e_1, e_2, \dots, e_n$ , not all zero, with  $|e_i| < m$  for all  $i$ , such that  $e_1 a_1 + e_2 a_2 + \dots + e_n a_n$  is a multiple of  $m^n$ .
- [6] Find all pairs of positive integers  $m, n \geq 3$  for which there exist infinitely many positive integers  $a$  such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

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