Harmonic Bundles Continued

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DGX-HARMONIC

§1 Reading

Either of the following:

- §1 and §2 of Cross Ratios (MOP 2016, available at https://www.dropbox.com/s/5ab1mhanp81n5jo/CrossRatios.pdf?dl=0).
- Or, EGMO §9.2, §9.3.

Quick additional fact not mentioned, but occasionally useful:

Fact 1.1. If $ABXYA'B'X'Y'$ lie on a circle with $\overline{AA'}$, $\overline{BB'}$, $\overline{XX'}$, $\overline{YY"}$ concurrent at a point P, then $(AB;XY) = (A'B';X'Y')$.

§2 Lecture notes

§2.1 Review

Problem 2.1 (JMO 2011/5)**.** Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.

Problem 2.2 (Brazil 2011/5)**.** Let ABC be an acute triangle with orthocenter H and altitudes $\overline{BD}$, $\overline{CE}$. The circumcircle of ADE cuts the circumcircle of ABC at $F \neq A$. Prove that the angle bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on $\overline{BC}$.

Problem 2.3 (Shortlist 2015 G3)**.** Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that $\overline{CH}$ bisects $\overline{AD}$. Let P be the intersection point of the lines $\overline{BD}$ and $\overline{CH}$. Let ω be the semicircle with diameter $\overline{BD}$ that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines $\overline{CQ}$ and $\overline{AD}$ meet on ω.

Problem 2.4 (Taiwan TST 2014/1J/3)**.** In $\triangle ABC$ with incenter I, the incircle is tangent to $\overline{CA}$, $\overline{AB}$ at E, F. The reflections of E, F across I are G, H. Let Q be the intersection of $\overline{GH}$ and $\overline{BC}$, and let M be the midpoint of $\overline{BC}$. Prove that $\overline{IQ}$ and $\overline{IM}$ are perpendicular.

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§2.2 More examples

Problem 2.5 (TSTST 2015/2)**.** Let ABC be a scalene triangle. Let K_a, L_a, and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at a point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.

Problem 2.6 (USAMO 2008)**.** Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside triangle ABC. Prove that points A, N, F, and P all lie on one circle.

Problem 2.7 (Shortlist 2016, by me)**.** Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

§3 Practice problems

Problem 3.1 (China TST 2002)**.** Let $ABCD$ be a quadrilateral. Point E is the intersection of lines AB and CD while point F is the intersection of lines BC and DA. The diagonals of the quadrilateral meet at P, and point O is the foot from P to $\overline{EF}$. Prove that $\angle BOC = \angle AOD$.

Problem 3.2. Let ABC be a triangle whose incircle γ touches the sides BC, CA, AB at D, E, F. Line AD meets γ at $T \neq D$, and the tangent to γ at T meets line EF at P. Prove that if H lies on $\overline{AT}$ with $\overline{HP} \parallel \overline{AB}$ then $\angle HEF = 90^\circ$.

Problem 3.3 (JMO 2015)**.** Let $ABCD$ be a cyclic quadrilateral. Prove that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if there exists a point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$ and $\angle CDB = \angle YDA$.

Problem 3.4 (APMO 2013)**.** Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of $\overline{AC}$ such that $\overline{PB}$ and $\overline{PD}$ are tangent to ω. The tangent at C intersects $\overline{PD}$ at Q and the line AD at R. Let E be the second point of intersection between $\overline{AQ}$ and ω. Prove that B, E, R are collinear.

Problem 3.5 (Shortlist 2005 G6)**.** Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to $\overline{BC}$, intersect the incircle γ again in two points X and Y. Let the lines AX and AY intersect BC again at the points P and Q. Prove that $BP = CQ$.

Problem 3.6 (HMMT 2017, Sam Korsky)**.** Let LBC be a fixed triangle with $LB = LC$, and let A be a variable point on arc LB of its circumcircle. Let I be the incenter of $\triangle ABC$ and $\overline{AK}$ the altitude from A. The circumcircle of $\triangle IKL$ intersects lines KA and BC again at $U \neq K$ and $V \neq K$. Finally, let T be the projection of I onto line UV. Prove that the line through T and the midpoint of $\overline{IK}$ passes through a fixed point as A varies.

Problem 3.7 (Shortlist 2009 G4)**.** Given a cyclic quadrilateral $ABCD$, let $E = \overline{AC} \cap \overline{BD}$, $F = \overline{AD} \cap \overline{BC}$. The midpoints of $\overline{AB}$ and $\overline{CD}$ are G and H, respectively. Show that $\overline{EF}$ is tangent at E to the circle through the points E, G, and H.

Problem 3.8 (ELMO 2016, James Lin)**.** Elmo is now learning olympiad geometry. In a triangle ABC with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\angle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\angle XSY = \angle XTY = 90^\circ$. Finally, let γ be the circumcircle of $\triangle AST$.

(a) Help Elmo show that γ is tangent to the circumcircle of $\triangle ABC$.

(b) Help Elmo show that γ is also tangent to the incircle of $\triangle ABC$.