Triangle geometry

In this chapter, we consider the basic properties of a generic triangle $A B C$, and how the angles and distances between points are related. We explore parametrisations of the triangle using both trigonometry and complex numbers. In the process, we develop an arsenal of identities suitable for attacking both geometrical and trigonometrical problems, noting the interchangeability between the representations.

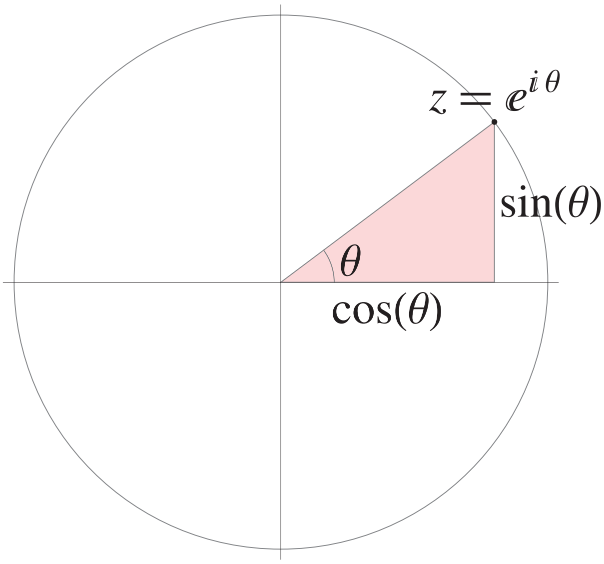
Trigonometry

The elementary trigonometric functions, namely sine and cosine, can be expressed in terms of the exponential function and *vice-versa*.

- $\sin(\theta) = \mathrm{Im}\left(e^{i\theta}\right) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. **[Definition of sine]**

- $\cos(\theta) = \mathrm{Re}\left(e^{i\theta}\right) = \frac{e^{i\theta}+e^{-i\theta}}{2}$. **[Definition of cosine]**

We can view this on the Argand plane, where we consider a point on the unit circle with Cartesian coordinates $(\cos\theta, \ \sin\theta)$ and complex representation $\cos\theta + i\sin\theta = e^{i\theta}$. From applying Pythagoras' theorem, we instantly obtain the famous identity $\sin^2\theta + \cos^2\theta = 1$.



This is arguably the most reliable approach to proving trigonometric identities, as it is a simple matter of converting each expression to its exponential counterpart and verifying that both sides of the equation are indeed equal. However, it is preferable to derive a few identities first, as working with lots of exponentials can be laborious. Perhaps the most rudimentary trigonometric identities are the *compound angle formulae*.

1. Prove that $\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$. **[Compound angle formula I]**

2. Similarly, prove that $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\phi\sin\theta$. **[Compound angle formula II]**

With these, it is no longer necessary to rely on the exponential form for proving identities. Indeed, we can now avoid using complex numbers altogether.

More sophisticated trigonometric functions can be expressed as ratios of sine and cosine.

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$; $\cot\theta = \frac{\cos\theta}{\sin\theta}$; $\sec\theta = \frac{1}{\cos\theta}$; $\mathrm{cosec}\,\theta = \frac{1}{\sin\theta}$. **[Definitions of tangent, cotangent, secant and cosecant]**

This enables us to derive a compound angle formula for the tangent function.

3. Hence prove that $\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$. **[Compound angle formula III]**

As special cases of the above, where $\theta = \phi$, we obtain the *double-angle formulae*.

- $\sin(2\theta) = 2\sin\theta\cos\theta$. **[Double-angle formula I]**
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$. **[Double-angle formula II]**

4. Prove further that $\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$. **[Extended double-angle formula]**

We can rearrange the above formula to obtain $\sin^2\theta$ and $\cos^2\theta$ in terms of $\cos 2\theta$. Hence, we can calculate the value of $\cos\frac{\pi}{6}$ from that of $\cos\frac{\pi}{3}$, for example.

5. Hence deduce that $\cos(\theta + \phi)\cos(\theta - \phi) = \frac{1}{2}(\cos 2\theta + \cos 2\phi) = \cos^2\theta - \sin^2\phi$. **[Prosthaphaeresis]**

The compound angle formulae can be used recursively to derive expressions for three angles.

6. Prove that $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$. **[Triple-angle formula]**

More generally, we can expand $\sin(\theta + \phi + \psi)$ to obtain $\sin(\theta + \phi)\cos\psi + \cos(\theta + \phi)\sin\psi$, then apply the compound angle formulae again to each term. This results in the following expression:

- $\sin(\theta + \phi + \psi) = \sin\theta\cos\phi\cos\psi + \cos\theta\sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi - \sin\theta\sin\phi\sin\psi$. **[Compound angle formula IV]**

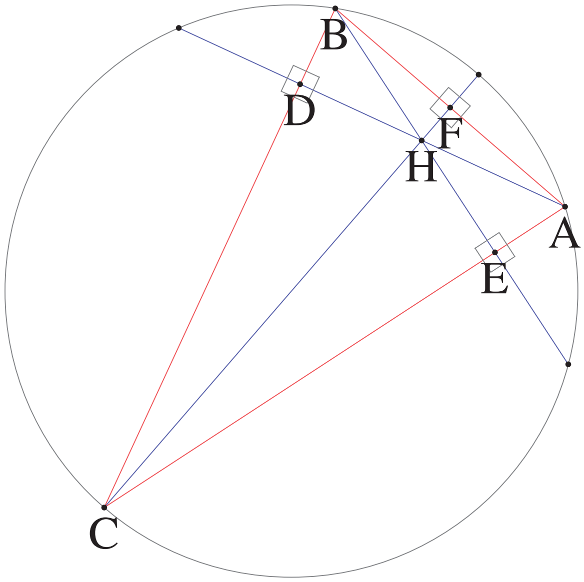
Let A, B and C be the angles of a triangle opposite sides of lengths a, b and c, respectively. R, r and s are the circumradius, inradius and semiperimeter, respectively. We can apply the trigonometric identities explored in the previous section to triangles, remembering that $A + B + C = \pi$, and thus $\sin(A + B) = \sin C$ and $\cos(A + B) = -\cos C$.

7. Prove that $\sin A\sin B\cos C + \sin A\cos B\sin C + \cos A\sin B\sin C - \cos A\cos B\cos C = 1$, and thus $\cot A + \cot B + \cot C - 1 = \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$.

8. Prove that $\tan A\tan B\tan C = \tan A + \tan B + \tan C$, and thus $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$.

Altitudes and orthocentre

Consider the triangle ABC together with its orthocentre H. The altitudes meet BC, CA and AB at D, E and F, respectively.



The orthocentric configuration exhibits a plethora of particularly interesting properties:

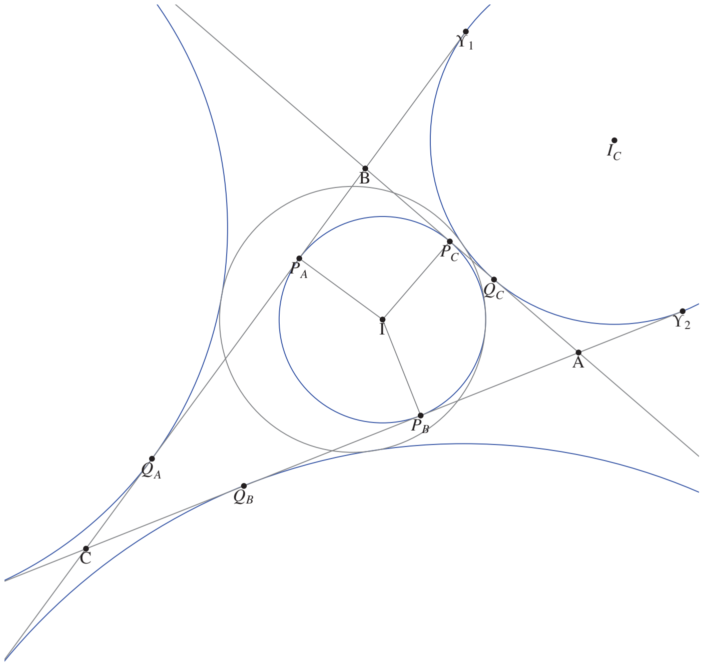
- The reflections of the orthocentre, H, in each of the sides of the triangle land on the circumcircle.
- If three points out of $\{A, B, C, D, E, F, H\}$ are collinear, the remaining four are concyclic, and *vice-versa*.
- If H is the orthocentre of ABC, then A is the orthocentre of HBC, *et cetera*. This is called an *orthocentric quadrangle* or *perpendicularogram*. All four triangles share the same nine-point circle, the centre of which is the barycentre of $\{A, B, C, H\}$.
- Due to cyclic quadrilaterals, we have $\angle DHB = \angle AHE = \angle BCA$.
- Every inter-point distance has a simple expression in terms of the circumradius and trigonometric functions of the vertex angles:

	A	B	C	D	E	F	H
A	0	$2R\sin(C)$	$2R\sin(B)$	$2R\sin(B)\sin(C)$	$2R\cos(A)\sin(C)$	$2R\cos(A)\sin(B)$	$2R\cos(A)$
B		0	$2R\sin(A)$	$2R\cos(B)\sin(C)$	$2R\sin(A)\sin(C)$	$2R\sin(A)\cos(B)$	$2R\cos(B)$
C			0	$2R\sin(B)\cos(C)$	$2R\sin(A)\cos(C)$	$2R\sin(A)\sin(B)$	$2R\cos(C)$
D				0	$2R\sin(C)\cos(C)$	$2R\sin(B)\cos(B)$	$2R\cos(B)\cos(C)$
E					0	$2R\sin(A)\cos(A)$	$2R\cos(A)\cos(C)$
F						0	$2R\cos(A)\cos(B)$
H							0

- Every rectangular hyperbola passing through A, B and C also passes through H. The centre of the hyperbola lies on the nine-point circle of triangle ABC.

Tritangential circles

We consider the four circles tangent to all three sides of a triangle, together with their centres and tangency points. One of these circles is enclosed by the triangle (the *inscribed circle*, or *incircle*, with *incentre* I), and the other three are called *escribed circles*, or *excircles*. Collectively, they are known as *tritangential circles*.



As the two tangents from a single point to a circle are equal, we have that $AP_B = AP_C$. Let l, m and n denote the distances AP_B, BP_C and CP_A, respectively. We have $a = m + n$, $b = n + l$ and $c = l + m$. This enables us to deduce that $AP_B = AP_C = l = s - a$, where $s = \frac{1}{2}(a + b + c)$ is the semiperimeter of the triangle ABC. We know that $CY_1 = CY_2$, and that $BY_1 = BQ_C$, which gives us $CB + BQ_C = Q_C A + AC = s$, from which we can deduce that $BQ_C = s - a$.

- $AP_C = BQ_C = s - a$. **[Distances to intouch and extouch points]**

This means that the line segments $P_C Q_C$ and AB share a midpoint.

Applying Pythagoras' Theorem to triangle $AP_B I$ enables the distance $AI^2 = AP_B{}^2 + P_B I^2 = (s-a)^2 + r^2$ to be determined. Similarly, we have $AI_C{}^2 = (s-b)^2 + r_C{}^2$ and $AI_A{}^2 = s^2 + r_A{}^2$.

9. Prove that $rs = r_A(s-a) = r_B(s-b) = r_C(s-c) = [ABC]$. **[Area of a triangle]**

10. Show that $\tan\frac{A}{2} = \frac{r}{s-a} = \frac{r_A}{s}$. **[Half-angle formula]**

11. Show that $AI = 4R\sin\frac{B}{2}\sin\frac{C}{2}$, and thus $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. **[Inradius formula]**

12. Prove that $s - b = r_C\tan\frac{A}{2}$. **[Complementary half-angle formula]**

With the inradius formula and half-angle formula, we obtain an expression for $s - a$; by symmetry, we also get $s - b$ and $s - c$. The complementary half-angle formula gives us r_A, r_B and r_C, and the bog-standard half-angle formula gives us s. These eight quantities are included in the table below so you can gape in awe at the elegant symmetries between the formulae.

Length	Trigonometrical expression
r	$4\,R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$
$s-a$	$4\,R\cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$
$s-b$	$4\,R\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$
$s-c$	$4\,R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$
r_A	$4\,R\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$
r_B	$4\,R\cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$
r_C	$4\,R\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$
s	$4\,R\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$

13. Prove that $[A\,B\,C]=\sqrt{r\,r_A\,r_B\,r_C}=\sqrt{s(s-a)(s-b)(s-c)}=\frac{a\,b\,c}{4\,R}$. **[Beyond Heron's formula]**

14. Hence prove that $\cot A=\frac{b^2+c^2-a^2}{4[A\,B\,C]}$.

15. Show further that $[A\,B\,C]=\frac{1}{2}\sqrt[3]{a^2\,b^2\,c^2\sin A\sin B\sin C}$. **[Gendler's formula]**

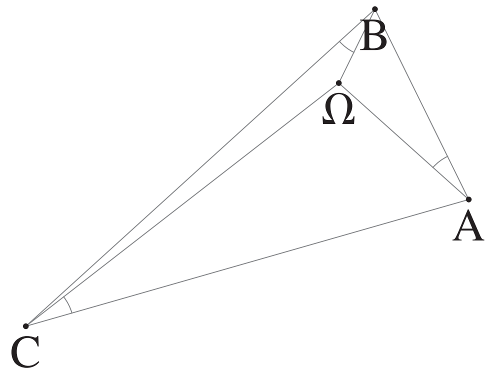
16. Prove that $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{s}{R}$.

Here are some miscellaneous properties of the tritangential circles:

- If the lines $C\,I\,I_C$ and $A\,B$ intersect at R, then $(C,\,R;\,I,\,I_C)=-1$ is a harmonic range.
- The lines $A\,P_A$, $B\,P_B$ and $C\,P_C$ are concurrent at the *Gergonne point* Ge, whilst the lines $A\,Q_A$, $B\,Q_B$ and $C\,Q_C$ are concurrent at the *Nagel point* Na. They are isotomic conjugates. The incentre, centroid, *Spieker centre* (incentre of the medial triangle) and Nagel point are collinear such that $3\,\overrightarrow{I\,G}=6\,\overrightarrow{G\,\mathrm{Sp}}=2\,\overrightarrow{\mathrm{Sp}\,\mathrm{Na}}$. This is known as the *Nagel line*, and is not unlike the Euler line.
- Feuerbach's theorem states that the incircle and three excircles are tangent to the nine-point circle.
- The excentral triangle $I_A\,I_B\,I_C$ has orthocentre I. Its nine-point circle is the circumcircle of the reference triangle.

Brocard points

The *first Brocard point* Ω is positioned such that $\angle\,\Omega\,B\,C=\angle\,\Omega\,C\,A=\angle\,\Omega\,A\,B=\omega$, where ω is known as the *Brocard angle*. The *second Brocard point* Ω' is its isogonal conjugate, where $\angle\,\Omega'\,B\,A=\angle\,\Omega'\,C\,B=\angle\,\Omega'\,A\,C=\omega$.



17. Prove that $(\cot\omega - \cot A)(\cot\omega - \cot B)(\cot\omega - \cot C) = \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$.

18. Hence show that $\cot\omega$ is a root of the cubic equation
$x^3 - (\cot A + \cot B + \cot C)\, x^2 + x - (\cot A + \cot B + \cot C) = 0$.

Let Γ_{AB} be the circle through A and B tangent to BC, and define Γ_{BC} and Γ_{CA} similarly. Ω lies on the intersection of the three circles. The other triple intersections of these three circles are the two circular points at infinity, which correspond to the imaginary roots of the cubic equation.

19. Show that the above equation has only one real root, and thus $\cot\omega = \cot A + \cot B + \cot C$. **[Brocard angle formula]**

Now that we have this expression for $\cot\omega$, we can derive further identities:

20. Prove that $\tan\omega = \frac{\sin A \sin B \sin C}{1+\cos A \cos B \cos C}$.

21. Prove that $\cot\omega = \frac{a^2+b^2+c^2}{4[ABC]}$.

22. Hence show that $\omega \le \frac{\pi}{6}$, with equality if and only if ABC is equilateral.

23. Let P be a point interior to a scalene triangle ABC. Prove that one of the angles $\angle PAB$, $\angle PBC$ and $\angle PCA$ must be less than $\frac{\pi}{6}$. **[Adapted from IMO 1991, Question 5]**

R, *r* and *s*

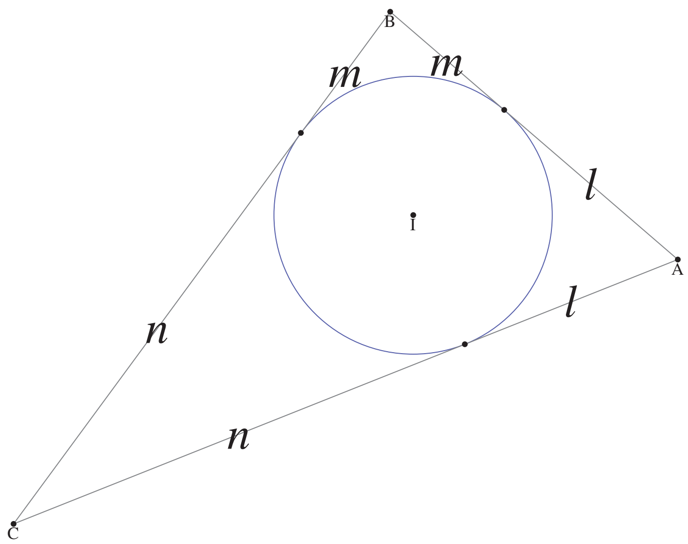
When evaluating (squared) distances between triangle centres, one often finds a symmetric polynomial in the side lengths (a, b and c). It is convenient to convert this into an expression in the circumradius R, inradius r and semiperimeter s.

24. Using the formulae $[ABC] = rs = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)}$, derive expressions for $a+b+c$, $ab+bc+ca$ and abc in terms of R, r and s.

Due to Newton's theorem of symmetric polynomials, it is possible to express any symmetric polynomial in the side lengths in terms of these elementary symmetric polynomials, and thus in terms of R, r and s.

Symmetric polynomial	R, r and s
$a+b+c$	$2\,s$
$a\,b+a\,c+b\,c$	$r^2+4\,r\,R+s^2$
$a\,b\,c$	$4\,r\,R\,s$
$a^2+b^2+c^2$	$-2\,r^2-8\,r\,R+2\,s^2$
$a^3+b^3+c^3$	$2\,s\left(-3\,r^2-6\,r\,R+s^2\right)$

When dealing with inequalities in the side lengths of a triangle, it is most convenient to convert it into an inequality in $l=s-a$, $m=s-b$ and $n=s-c$. The triangle inequality is equivalent to l, m and n being positive reals. Symmetric polynomials in l, m and n can similarly be converted into polynomials in R, r and s using this method.



Symmetric polynomial	R, r and s
$l+m+n$	s
$l\,m+l\,n+m\,n$	$r^2+4\,r\,R$
$l\,m\,n$	$r^2\,s$
$l^2+m^2+n^2$	$-2\,r^2-8\,r\,R+s^2$
$l^3+m^3+n^3$	$s^3-12\,r\,R\,s$

25. Prove that, for every triangle, $s^2 \ge 16\,R\,r-5\,r^2$. **[One half of Gerretsen's inequality]**

26. Similarly prove that $s^2 \le 4\,R^2+4\,R\,r+3\,r^2$. **[Other half of Gerretsen's inequality]**

We can go further and find the tightest bounds possible for s^2 given R and r. As l, m, n are three real roots of the equation $(x-l)\,(x-m)\,(x-n)=0$, it is a necessary and sufficient condition for the cubic $x^3-(l+m+n)\,x^2+(l\,m+m\,n+n\,l)\,x-l\,m\,n=0$ to have three (not necessarily distinct) real roots. Using the formulae for the elementary symmetric polynomials, we require the discriminant of $x^3-s\,x^2+\left(r^2+4\,R\,r\right)x-r^2\,s\,x=0$ to be non-negative. The discriminant is given by $18\,r^2\,s^2\left(r^2+4\,R\,r\right)-4\,r^2\,s^4+s^2\left(r^2+4\,R\,r\right)-4\left(r^2+4\,R\,r\right)^3-27\,r^4\,s^2$, which is a quadratic function in s^2. Solving this inequality gives us the following necessary and sufficient condition on s^2 in terms of R and r:

- $2R^2 + 10Rr - r^2 - 2\sqrt{R(R-2r)^3} \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2\sqrt{R(R-2r)^3}$.

The semiperimeter will fluctuate between these two values as the vertices of the triangle move around the circumcircle in Poncelet's porism.

27. Show that $R \geq 2r$. **[Euler's inequality]**

28. Hence prove that $\sin A + \sin B + \sin C \geq \sin 2A + \sin 2B + \sin 2C$. **[Gendler's inequality]**

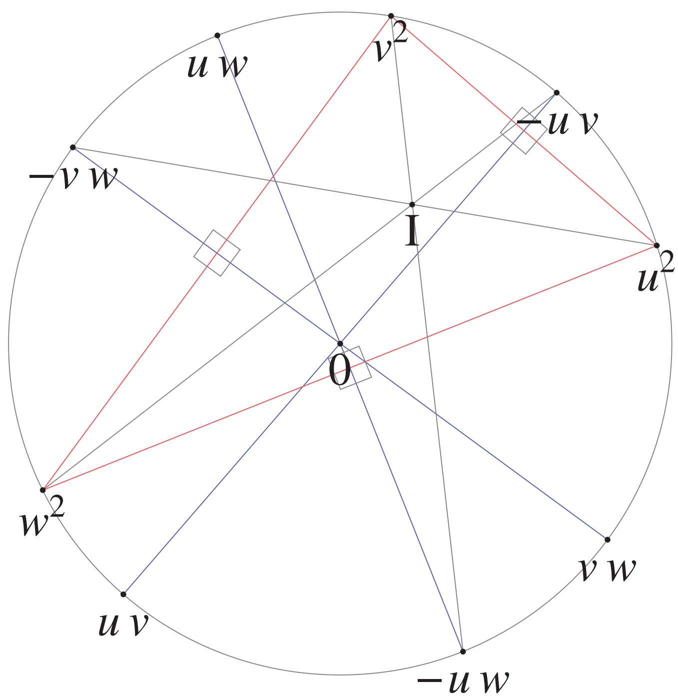
29. Express $\cot\omega$ in terms of R, r and s.

30. Prove that $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1 + \frac{r}{R}$.

31. Hence prove that $\cos A - \cos B - \cos C = 1 - 4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 1 - \frac{r_A}{R}$.

32. Show that $4R = r_A + r_B + r_C - r$.

Complex parametrisation of triangles



Consider the reference triangle ABC. If we represent the vertices with complex numbers u^2, v^2 and w^2, respectively, such that $uu^* = vv^* = ww^* = R$, and choose the signs of u, v and w such that uv lies on the arc AB containing C (and cyclic permutations thereof), then most of the useful aspects of the triangle have simple algebraic expressions:

Quantity	Symbol	Expression
Circumradius	R	$u\,u^* = v\,v^* = w\,w^*$
Inradius	r	$\frac{1}{2}\left(-v\,u^* - u\,v^* - w\,u^* - u\,w^* - w\,v^* - v\,w^*\right) - R$
Exradius opposite A	r_A	$\frac{1}{2}\left(-v\,u^* - u\,v^* - w\,u^* - u\,w^* + w\,v^* + v\,w^*\right) + R$
Semiperimeter	s	$\left(\frac{1}{2}\,i\left(v\,u^* - u\,v^* - w\,u^* + u\,w^* + w\,v^* - v\,w^*\right)\right)$
(Signed) area	[ABC]	$\left(\frac{1}{4}\,i\left(u^2\left(v^*\right)^2 - u^2\left(w^*\right)^2 - v^2\left(u^*\right)^2 + w^2\left(u^*\right)^2 + v^2\left(w^*\right)^2 - w^2\left(v^*\right)^2\right)\right)$
Side length of BC	a	$i\left(w\,v^* - v\,w^*\right)$
Angle exponential	$e^{i\,A}$	$-\frac{w\,v^*}{R}$
Sine	$\sin(A)$	$\frac{i\left(w\,v^* - v\,w^*\right)}{2\,R}$
Cosine	$\cos(A)$	$\frac{-w\,v^* - v\,w^*}{2\,R}$

There are also versions of these formulae expressed in linear factors, which can easily be multiplied and divided:

Quantity	Symbol	Expression
Inradius	r	$-\frac{1}{2}\,R\left(\frac{u}{v}+1\right)\left(\frac{w}{u}+1\right)\left(\frac{v}{w}+1\right)$
Exradius opposite A	r_A	$\frac{1}{2}\,R\left(1-\frac{u}{v}\right)\left(1-\frac{w}{u}\right)\left(\frac{v}{w}+1\right)$
Semiperimeter	s	$\frac{1}{2}\,i\,R\left(1-\frac{u}{v}\right)\left(1-\frac{w}{u}\right)\left(1-\frac{v}{w}\right)$
	$s-a$	$-\frac{1}{2}\,i\,R\left(\frac{u}{v}+1\right)\left(\frac{w}{u}+1\right)\left(1-\frac{v}{w}\right)$
(Signed) area	[ABC]	$\left(-\frac{1}{4}\,i\,R^2\left(\frac{v}{u}-\frac{u}{v}\right)\left(\frac{u}{w}-\frac{w}{u}\right)\left(\frac{w}{v}-\frac{v}{w}\right)\right)$
Side length of BC	a	$i\,R\left(\frac{w}{v}-\frac{v}{w}\right)$
Angle exponential	$e^{i\,A}$	$-\frac{w}{v}$
Sine	$\sin(A)$	$\left(\frac{1}{2}\,i\left(\frac{w}{v}-\frac{v}{w}\right)\right)$
Cosine	$\cos(A)$	$\frac{1}{2}\left(-\frac{v}{w}-\frac{w}{v}\right)$

Many triangle centres have simple quadratic expressions in u, v and w. Others, such as the Feuerbach points, are more complicated:

Point	Symbol	Expression
Vertex	A	u^2
Vertex	B	v^2
Vertex	C	w^2
Circumcentre	O	0
Centroid	G	$\left(\frac{1}{3}\left(u^2+v^2+w^2\right)\right)$
Nine−point centre	T	$\left(\frac{1}{2}\left(u^2+v^2+w^2\right)\right)$
Orthocentre	H	$u^2+v^2+w^2$
Altitude foot on BC	D	$\left(\frac{1}{2}\left(-\frac{v^2 w^2}{u^2}+u^2+v^2+w^2\right)\right)$
Incentre	I	$-u\,v-u\,w-v\,w$
Intouch point on BC	$\mathrm{P_A}$	$\frac{(u+v)\,(u+w)\,(v+w)}{2\,u}-u\,v-u\,w-v\,w$
Excentre opposite A	$\mathrm{I_A}$	$u\,v+u\,w-v\,w$
Extouch point on BC	$\mathrm{Q_A}$	$\frac{(u-v)\,(w-u)\,(v+w)}{2\,u}+u\,v+u\,w-v\,w$
Nagel point	Na	$(u+v+w)^2$
Spieker centre	Sp	$\frac{1}{2}\left(u^2+u\,v+u\,w+v^2+v\,w+w^2\right)$
Feuerbach point	F	$\frac{1}{2}\left(-\frac{R\,(u+v+w)}{u^*+v^*+w^*}+u^2+v^2+w^2\right)$
Feuerbach point on excircle $\mathrm{I_A}$	$\mathrm{F_A}$	$\frac{1}{2}\left(-\frac{R\,(u-v-w)}{u^*-v^*-w^*}+u^2+v^2+w^2\right)$

With these results, one can express any rational function of the side lengths and basic trigonometric functions as a rational function in u, v and w (and their conjugates).

Firstly, however, it is a fulfilling exercise to derive the expressions in the table above.

33. Show that $u\,v$ is the midpoint of the arc $A\,B$ containing C, and thus that $-u\,v$ is the midpoint of the arc $A\,B$ not containing C.

34. Prove that the circumcircle of $A\,B\,C$ is the nine-point circle of the *excentric triangle* $I_A\,I_B\,I_C$.

35. Hence show that $-u\,v$ is the midpoint of $I\,I_C$, and thus $u\,v$ is the midpoint of $I_A\,I_B$.

36. Hence verify the expressions for I, I_A, I_B and I_C.

With the expressions for the circumcentre and centroid, one can derive expressions for the orthocentre and nine-point centre by using the Euler line. Similarly, the Nagel line enables one to extrapolate expressions for the Spieker centre and Nagel point based on those of the incentre and centroid.

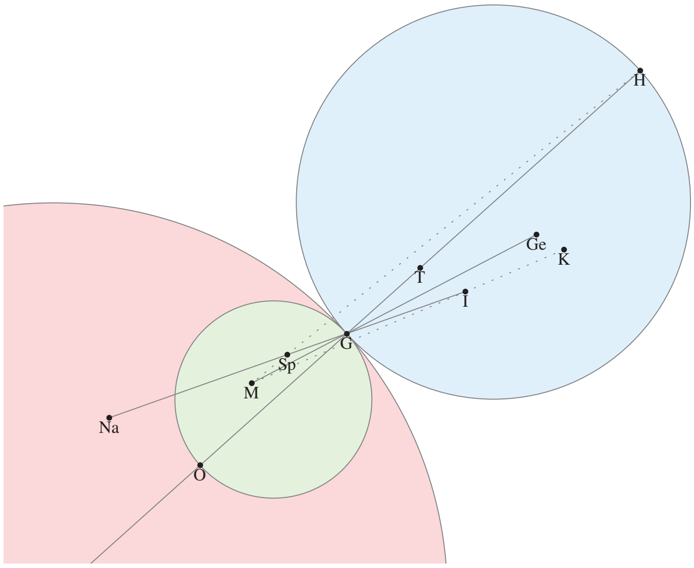
37. If J is the reflection of I in $B\,C$, show that J has representation $v^2+w^2+v\,w+(v+w)\,\frac{v\,w}{u}$, and that $\overrightarrow{I\,J}=\frac{(u+v)\,(v+w)\,(w+u)}{u}$.

38. Hence show that $2\,r=-R\left(\frac{u}{v}+1\right)\left(\frac{v}{w}+1\right)\left(\frac{w}{u}+1\right)$, and thus derive the expression for r in the table.

39. Prove that $OI^2 = R^2 - 2Rr$. **[Euler's identity]**

40. Prove that $IT = \frac{1}{2}R - r$, and thus that the nine-point circle and incircle are tangent. **[Feuerbach's theorem]**

The combination of the two above formulae results in the inequality $OI \geq 2IT$, which means that I lies in the Apollonius disc of diameter GH, known as the *Euler-Apollonius lollipop*. Geoff Smith and Christopher Bradley discovered that the symmedian point and Gergonne point also reside in this disc. As T lies on the line segment GH, it must also inhabit the Euler-Apollonius lollipop.

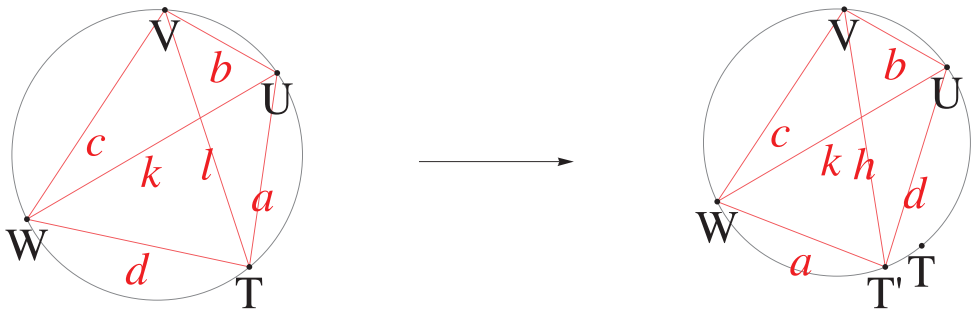


As a consequence of this, together with the Euler line and Nagel line properties, the Spieker centre lies in the disc of diameter OG and therefore outside the Euler-Apollonius lollipop. The lines IK and H Sp intersect at the symmedian point of the excentral triangle, known as the *mittenpunkt* M. M, G, Ge are collinear with ratio $MG : G\mathrm{Ge} = 1 : 2$. The mittenpunkt must therefore reside in the disc of diameter OG.

Indeed, the points shown inside the circles on the diagram above **always** remain in those circles. The red disc containing the Nagel point has diameter GL, where L is the *de Longchamps point* (reflection of H in O).

Cyclic quadrilaterals

Consider an arbitrary cyclic quadrilateral $TUVW$ (labelled anticlockwise). We denote the lengths of edges TU, UV, VW and WU with a, b, c and d, respectively. The lengths of diagonals UW and TV are denoted with k and l, respectively.



If we reflect T in the perpendicular bisector of UW to form T', we obtain a new cyclic quadrilateral $T'UVW$ with the same area and side lengths as $TUVW$, but in a different order. The diagonal UW is unaffected, and remains k. The diagonal $T'V$, however, now has length h, in general distinct from its original length l.

Hence, we can consider $TUVW$ to have *three* diagonal lengths: k and l as well as the invisible diagonal of length h obtainable by interchanging any two adjacent side lengths.

41. Show that $[TUVW] = \frac{(ab+cd)l}{4R} = \frac{hkl}{4R}$. **[Parameshvara's formula]**

Parameshvara's formula is very similar to the formula $[ABC] = \frac{abc}{4R}$. The latter can be regarded as a special case of the former, where two of the vertices of the cyclic quadrilateral are coincident. Similarly, there is a generalisation of Heron's formula applicable to cyclic quadrilaterals:

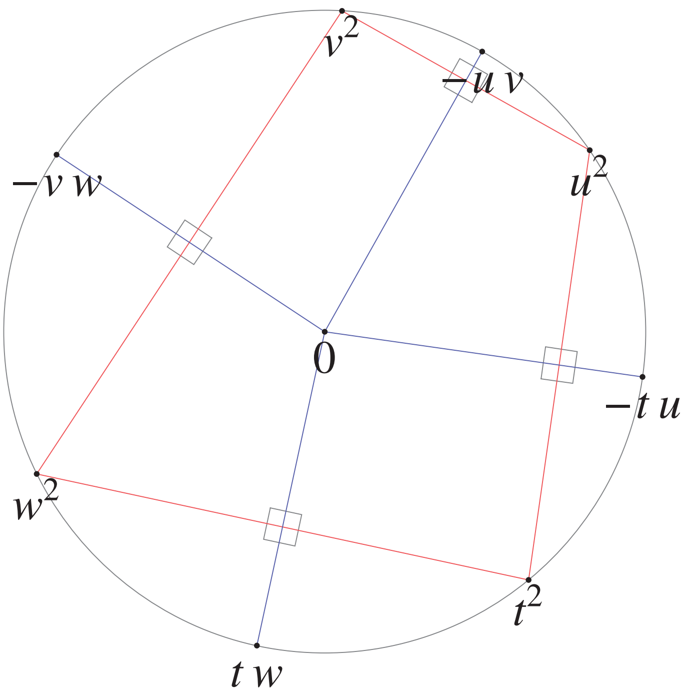
- If a cyclic quadrilateral $TUVW$ has edge lengths a, b, c, d and semiperimeter $s = \frac{1}{2}(a+b+c+d)$, then $[TUVW] = \sqrt{(s-a)(s-b)(s-c)(s-d)}$. **[Brahmagupta's formula]**

Heron's formula is a special case of Brahmagupta's formula, which is in turn a special case of Bretschneider's formula for convex quadrilaterals.

- $[TUVW] = \sqrt{\left((s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{T+V}{2}\right)\right)}$. **[Bretschneider's formula]**

The term $abcd\cos^2\left(\frac{T+V}{2}\right) = \frac{1}{4}(ac+bd+kl)(ac+bd-kl)$, where k and l are the diagonal lengths. It is easy to see, by Ptolemy's theorem, that this vanishes when the quadrilateral is cyclic.

By definition, a cyclic quadrilateral is inscribed in a circle, so we can use a related parametrisation to that used for triangles. A cyclic quadrilateral $TUVW$ is represented by complex numbers t^2, u^2, v^2 and w^2, where $tt^* = uu^* = vv^* = ww^* = R$. Unfortunately, the parametrisation is slightly less elegant for cyclic polygons with an even number of sides, as we cannot treat all vertices and edges equivalently.



Notice that the midpoints of the outer arcs of TU, UV and VW are indeed represented by $-tu$, $-uv$ and $-vw$, respectively, as one would imagine. However, due to annoying parity constraints, this forces the midpoint of the outer arc of WT to be **positive** tw. Hence, only the triangles TUV and UVW are correctly parametrised; the others have asymmetric (but equally simple) formulae associated with them. Nevertheless, we can now derive the aforementioned seven lengths and the area via Parameshvara's formula. It is thus straightforward to verify Brahmagupta's formula.

Quantity	Symbol	Expression
Side length of TU	a	$i\,R\left(\frac{u}{t}-\frac{t}{u}\right)$
Side length of UV	b	$i\,R\left(\frac{v}{u}-\frac{u}{v}\right)$
Side length of VW	c	$i\,R\left(\frac{w}{v}-\frac{v}{w}\right)$
Side length of WT	d	$-i\,R\left(\frac{t}{w}-\frac{w}{t}\right)$
Diagonal UW	k	$i\,R\left(\frac{u}{w}-\frac{w}{u}\right)$
Diagonal TV	l	$i\,R\left(\frac{t}{v}-\frac{v}{t}\right)$
Invisible diagonal	h	$i\,R\left(\frac{u}{t}+\frac{v}{w}\right)\left(\frac{t}{u}-\frac{w}{v}\right)$
	$s-a$	$-\frac{1}{2}\,i\,R\left(\frac{t}{v}+1\right)\left(1-\frac{w}{u}\right)\left(\frac{u}{t}+\frac{v}{w}\right)$
	$s-b$	$\frac{1}{2}\,i\,R\left(\frac{t}{v}+1\right)\left(\frac{w}{u}+1\right)\left(\frac{u}{t}-\frac{v}{w}\right)$
	$s-c$	$\frac{1}{2}\,i\,R\left(1-\frac{t}{v}\right)\left(\frac{w}{u}+1\right)\left(\frac{u}{t}+\frac{v}{w}\right)$
	$s-d$	$\frac{1}{2}\,i\,R\left(1-\frac{t}{v}\right)\left(1-\frac{w}{u}\right)\left(\frac{u}{t}-\frac{v}{w}\right)$
(Signed) area	[TUVW]	$-\frac{1}{4}\,i\,R^2\left(\frac{t}{v}-\frac{v}{t}\right)\left(\frac{u}{w}-\frac{w}{u}\right)\left(\frac{u}{t}+\frac{v}{w}\right)\left(\frac{t}{u}-\frac{w}{v}\right)$

42. Let the *maltitude* M_a be the line passing through the midpoint of $T\,U$ and perpendicular to $V\,W$. Define M_b, M_c and M_d similarly. Prove that the four maltitudes are concurrent at a point. **[Anticentre property]**

This concurrency point Q has representation $\frac{1}{2}\left(t^2+u^2+v^2+w^2\right)$, and is known as the *anticentre*. It is obvious from this that the centroid of the four vertices is the midpoint of $O\,Q$.

43. Let the diagonals $T\,V$ and $U\,W$ intersect at P. M and N are the midpoints of $T\,V$ and $U\,W$, respectively. Prove that the anticentre Q is the orthocentre of triangle $M\,N\,P$.

44. Let T' be the orthocentre of $U\,V\,W$, and define U', V' and W' similarly. Prove that $T'\,U'\,V'\,W'$ is congruent to $T\,U\,V\,W$.

45. Let I_T be the incentre of $U\,V\,W$, and define I_U, I_V and I_W similarly. Prove that $I_T\,I_U\,I_V\,I_W$ is a rectangle. **[Japanese theorem for cyclic quadrilaterals]**

46. Let r_T be the inradius of $U\,V\,W$, and define r_U, r_V and r_W similarly. Prove that $r_T+r_V=r_U+r_W$.

47. Suppose we have a cyclic polygon $A_1\,A_2\,\ldots\,A_n$. We draw $n-3$ non-intersecting lines between vertices to dissect the polygon into $n-2$ triangles. Let the sum of the inradii of the triangles be σ. Prove that the value of σ is independent of the choice of lines drawn. **[Japanese theorem for cyclic polygons]**

Solutions

1. We have $\sin\theta\cos\phi+\cos\theta\sin\phi=\frac{\left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\phi}+e^{-i\phi}\right)+\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{i\phi}-e^{-i\phi}\right)}{4i}=\frac{2\left(e^{i(\theta+\phi)}-e^{-i(\theta+\phi)}\right)}{4i}=\sin(\theta+\phi)$.

2. Similarly, we have $\cos\theta\cos\phi-\sin\theta\sin\phi=\frac{\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{i\phi}+e^{-i\phi}\right)+\left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\phi}-e^{-i\phi}\right)}{4}=\frac{2\left(e^{i(\theta+\phi)}+e^{-i(\theta+\phi)}\right)}{4}=\cos(\theta+\phi)$.

3. Based on the previous two theorems, $\tan(\theta+\phi)=\frac{\sin(\theta+\phi)}{\cos(\theta+\phi)}=\frac{\sin\theta\cos\phi+\cos\theta\sin\phi}{\cos\theta\cos\phi-\sin\theta\sin\phi}=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$. (The last step is where the numerator and denominator are both divided by $\cos\theta\cos\phi$.)

4. This is a consequence of the double-angle formula for cosine and the identity $\sin^2\theta+\cos^2\theta=1$.

5. $\cos(\theta+\phi)\cos(\theta-\phi)=(\cos\theta\cos\phi-\sin\theta\sin\phi)(\cos\theta\cos\phi+\sin\theta\sin\phi)=\cos^2\theta\cos^2\phi-\sin^2\theta\sin^2\phi$. Using the Pythagorean identity, we obtain
$\left(1-\sin^2\theta\right)\cos^2\phi-\left(1-\cos^2\phi\right)\sin^2\theta=\cos^2\phi-\sin^2\theta=\left(\cos^2\phi-1\right)+\left(1-\sin^2\theta\right)=\cos 2\theta+\cos 2\phi$.

6. $\sin 3\theta=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta=2\sin\theta\cos^2\theta+\cos^2\theta\sin\theta-\sin^3\theta=3\left(1-\sin^2\theta\right)\sin\theta-\sin^3\theta$. This clearly expands to $3\sin\theta-4\sin^3\theta$.

7. If we let $\theta=\frac{\pi}{2}-A$, $\phi=\frac{\pi}{2}-B$ and $\psi=\frac{\pi}{2}-C$, then the expression becomes equal to $\sin(\theta+\phi+\psi)$ by the compound angle formula. As $A+B+C=\pi$, it must be the case that $\theta+\phi+\psi=\frac{\pi}{2}$, the sine of which is 1. Dividing by $\sin A\sin B\sin C$ results in the desired equation.

8. $\tan A\tan B\tan C=\tan A\tan B\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}\tan A\tan B=\tan A+\tan B+\frac{\tan A+\tan B}{1-\tan A\tan B}=\tan A+\tan B+\tan C$. We can divide by $\tan A\tan B\tan C$ to obtain $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$.

9. The area of triangle IBC is given by $\frac{1}{2}ra$, as r is the height of the triangle when orientated such that a is the base. By symmetry, $[ABC]=\frac{1}{2}ra+\frac{1}{2}rb+\frac{1}{2}rc=rs$. The equivalence of rs to the other terms is evident from considering the similar triangles CIP_A and $CI_C\Upsilon_1$, which provides the identity $\frac{r_c}{s}=\frac{r}{s-c}$.

10. The angle $\angle IAB=\frac{A}{2}$, as I is the intersection of the three angle bisectors. We have $\tan\left(\frac{A}{2}\right)=\frac{IP}{AP}=\frac{r}{s-a}$. The second part of the formula comes from the identity $rs=r_A(s-a)$.

11. Applying the sine rule to triangle AIB gives us $\frac{AI}{\sin\frac{B}{2}}=\frac{AB}{\sin\left(\frac{\pi}{2}+\frac{C}{2}\right)}=\frac{2R\sin C}{\cos\frac{C}{2}}=4R\sin\frac{C}{2}$, with the last step utilising the double-angle formula. Rearranging results in $AI=4R\sin\frac{B}{2}\sin\frac{C}{2}$. The expression for r originates from considering the right-angled triangle AP_CI and applying basic trigonometry.

12. As the internal and external angle bisectors of A are perpendicular, we can quickly deduce that $\angle Q_CAI_C=\frac{\pi}{2}-\frac{A}{2}$ and thus $\angle AI_CQ_C=\frac{A}{2}$. We then have $\tan\frac{A}{2}=\frac{s-b}{r_C}$ from applying basic trigonometry to the right-angled triangle.

13. It is straightforward, from the expressions in the table together with the Sine Rule and double-angle formula, to verify that each term is equal to $\frac{R}{2}\sin A\sin B\sin C$.

14. A combination of the sine rule and cosine rule provides $\cot A = \frac{\cos A}{\sin A} = \frac{2R\cos A}{a} = \frac{R(b^2+c^2-a^2)}{abc}$. We also have $[ABC] = \frac{abc}{4R}$, so $\cot A = \frac{b^2+c^2-a^2}{4[ABC]}$.

15. $[ABC] = \frac{abc}{4R} = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$. The first expression is equal to the final, penultimate and antepenultimate expressions, so is trivially equal to their geometric mean.

16. $\sin A + \sin B + \sin C = \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} = \frac{s}{R} = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$, using the final expression in the table.

17. Applying the sine rule to triangle $AB\Omega$, we obtain $\frac{A\Omega}{B\Omega} = \frac{\sin(B-\omega)}{\sin\omega} = \frac{\sin B\cos\omega - \cos B\sin\omega}{\sin\omega} = (\cot\omega - \cot B)\sin B$. The cyclic product tells us that $(\cot\omega - \cot B)(\cot\omega - \cot C)(\cot\omega - \cot A)\sin A\sin B\sin C = 1$, from which we obtain the desired identity by dividing throughout by $\sin A\sin B\sin C$.

18. Expanding the previous identity results in $\cot^3\omega - (\cot A + \cot B + \cot C)\cot^2\omega + (\cot A\cot B + \cot B\cot C + \cot C\cot A)\cot\omega - \cot A\cot B\cot C = \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$. Using the identities proved in previous questions, this simplifies to $\cot^3\omega - (\cot A + \cot B + \cot C)\cot^2\omega + \cot\omega - (\cot A + \cot B + \cot C) = 0$.

19. The cubic factorises to $(x - \cot A - \cot B - \cot C)(x - i)(x + i) = 0$. As it is impossible for $\cot\omega$ to be imaginary, it must instead be $\cot A + \cot B + \cot C$.

20. $\tan\omega = \frac{1}{\cot\omega} = \frac{1}{\cot A + \cot B + \cot C} = \frac{\sin A\sin B\sin C}{\sin A\sin B\cos C + \sin A\cos B\sin C + \cos A\sin B\sin C} = \frac{\sin A\sin B\sin C}{1 + \cos A\cos B\cos C}$.

21. $\cot\omega = \cot A + \cot B + \cot C = \frac{a^2+b^2-c^2}{4[ABC]} + \frac{b^2+c^2-a^2}{4[ABC]} + \frac{c^2+a^2-b^2}{4[ABC]} = \frac{a^2+b^2+c^2}{4[ABC]}$.

22. Proving this is equivalent to showing that $\cot\omega \geq \sqrt{3}$, or $a^2 + b^2 + c^2 \geq 4\sqrt{3}\,[ABC]$. Square both sides and apply Heron's formula, giving the equivalent inequality $a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 \geq 3\left(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\right)$. Rearranging and dividing by 4 gives $a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$, which (by Muirhead's inequality) is true, with equality if and only if $a = b = c$.

23. P must lie in either triangle $A\Omega B$, $B\Omega C$ or $C\Omega A$. Without loss of generality, assume the former. Then, we have $\angle PAB \leq \angle\Omega AB < \frac{\pi}{6}$ from the previous question.

24. We clearly have $a + b + c = 2s$. As $[ABC] = rs = \frac{abc}{4R}$, we obtain $abc = 4Rrs$. By squaring Heron's formula and dividing throughout by s, we get $(s-a)(s-b)(s-c) = r^2 s$. Polynomial expansion results in $s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc = r^2 s$. The other symmetric polynomials can be replaced by the expressions in R, r and s, yielding $s^3 - 2s^3 + (ab+bc+ca)s - 4Rrs = r^2 s$. Final manipulation and division by s culminates in the expression for the third symmetric polynomial, $ab + bc + ca = s^2 + r^2 + 4Rr$.

25. Schur's inequality provides $l(l-m)(l-n) + m(m-n)(m-l) + n(n-l)(n-m) \geq 0$. We can then expand to obtain $l^3 + m^3 + n^3 + 3lmn \geq l^2 m + m^2 l + m^2 n + n^2 m + n^2 l + l^2 n$. Adding $l^3 + m^3 + n^3$ to each side of the equation yields $2\left(l^3 + m^3 + n^3\right) + 3lmn \geq (l+m+n)\left(l^2 + m^2 + n^2\right)$. Replacing each of these symmetric polynomials with their R, r and s counterparts gives $2\left(s^3 - 12Rrs\right) + 3r^2 s \geq s\left(-2r^2 - 8Rr + s^2\right)$. Rearranging, we obtain $s^3 - 16Rrs + 5r^2 s \geq 0$. Further rearrangement and division by s gives the required inequality, $s^2 + 5r^2 \geq 16Rr$. Equality occurs if and only if the triangle is equilateral.

26. We can convert the expression into an inequality in l, m, n. Firstly, we derive the expressions $r^2 = \frac{lmn}{l+m+n}$,

$4R^2 = \frac{(l+m)^2 (m+n)^2 (n+l)^2}{4 l m n (l+m+n)}$ and $4 R r = l m + m n + n l - r^2$. By multiplying the inequality by $4 l m n (l + m + n)$, we obtain the equivalent inequality $4 l m n (l + m + n)^3 \leq (l + m)^2 (m + n)^2 (n + l)^2 + 4 l m n ((l m + m n + n l) (l + m + n) + 2 l m n)$. It is now helpful to apply the $u v w$ method to express it as $w^6 + \left(2 u v^2 - 12 u^3\right) w^3 + 9 u^2 v^4 \geq 0$. This is a quadratic in w^3, so we only need to check three cases by Tejs' corollary. The third case only occurs when $F w^3 + G = 0$, or $w^3 + u v^2 - 6 u^3 = 0$. The expression is negative since $u \geq v \geq w$, so this cannot occur. Hence, we only need to consider when $l = m$ or $n = 0$. In the latter, we have a degenerate triangle comprising three collinear points, and thus $r = 0$, $2R = s$; this satisfies the inequality. In the former case, the triangle is isosceles and the inequality reduces to $4 l^2 n (2 l + n)^3 \leq 4 l^2 (l + n)^4 + 4 l^2 n \left(\left(l^2 + 2 l n\right) (2 l + n) + 2 l^2 n\right)$. Expanding out gives the equivalent inequality $4 l^6 - 8 l^5 n + 4 l^4 n^2 \geq 0$, or $4 l^4 (l - n)^2 \geq 0$. This is trivially true.

27. With Muirhead's inequality, it is trivial to verify that $(l m + m n + n l) (l + m + n) \geq 9 l m n$. Expanding each term gives $\left(r^2 + 4 R r\right) s \geq 9 r^2 s$, which simplifies to $4 R r s \geq 8 r^2 s$. Dividing throughout by $4 r s$ yields the desired inequality.

28. $[O A B] = \frac{1}{2} R^2 \sin 2C$, so we have $[A B C] = \frac{1}{2} R^2 (\sin 2A + \sin 2B + \sin 2C)$. Hence, the left-hand side of the inequality is equal to $\frac{s}{R}$, and the right-hand side is equal to $\frac{2[A B C]}{R^2} = \frac{2 r s}{R^2}$. So, we need to prove $\frac{s}{R} \geq \frac{2 r s}{R^2}$, which simplifies to Euler's inequality, $R \geq 2 r$.

29. $\cot \omega = \frac{a^2+b^2+c^2}{4[A B C]} = \frac{s^2-r^2-4 R r}{2 r s}$.

30. $\cos A + \cos B + \cos C = \sum_{\text{cyc}} \frac{b^2+c^2-a^2}{2 b c} = \sum_{\text{cyc}} \frac{a b^2+a c^2-a^3}{2 a b c} = \frac{\left(a^2+b^2+c^2\right)(a+b+c)-2\left(a^3+b^3+c^3\right)}{2 a b c}$. We can now apply the formulae to convert this expression to $\frac{4 s\left(s^2-r^2-4 R r\right)-4 s\left(s^2-3 r^2-6 R r\right)}{8 R r s}$. Some cancellation results in $\frac{2 r^2+2 R r}{2 R r}$, which further simplifies to $1 + \frac{r}{R}$. We already have $r = 4 R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$, so we are done.

31. The previous equation works for all triangles, and thus, due to the analytic continuity of sine and cosine, works for all angles such that $A + B + C = \pi$. If we use the angles $A + 2\pi$, $B - \pi$ and $C - \pi$, we obtain the equation $\cos A - \cos B - \cos C = 1 + \sin\left(\frac{A}{2} + \pi\right) \sin\left(\frac{B}{2} - \frac{\pi}{2}\right) \sin\left(\frac{C}{2} - \frac{\pi}{2}\right)$, the right-hand side of which simplifies to $1 - \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

32. Note that $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$ and $\cos A - \cos B - \cos C = 1 - \frac{r_A}{R}$ by the previous two results. We add the first equation to the cyclic sum of the second equation, yielding $0 = 4 + \frac{r}{R} - \frac{r_A}{R} - \frac{r_B}{R} - \frac{r_C}{R}$. Multiplying throughout by R and rearranging gives $4 R = r_A + r_B + r_C - r$.

33. The magnitude of $u v$ is the same as u^2 and v^2, namely R. We also have $2 \arg(u v) = \arg\left(u^2 v^2\right) = \arg\left(u^2\right) + \arg\left(v^2\right)$. Hence, it must be the bisector of one of the arcs, namely the arc containing C (as we have defined it as such).

34. The angle $\angle I A I_C$ is a right-angle. By symmetry, I must be the orthocentre of $I_A I_B I_C$, and thus $A B C$ is its orthic triangle. The circumcircle of the orthic triangle is the nine-point circle.

35. $-u v$ lies on the angle bisector $I I_C$, due to the 'equal arcs subtend equal angles' property. As it lies on the nine-point circle of $I I_A I_B I_C$, it must be the Euler point (midpoint of $I I_C$). The nine-point centre of the excentric triangle is O, the circumcentre of the reference triangle. It is also the barycentre of I, I_A, I_B and I_C, so must be the midpoint of the line joining the midpoints of $I I_C$ with $I_A I_B$. This enables us to deduce that $u v$ is indeed the midpoint of $I_A I_B$.

36. In complex coordinates, we have
$I_A = (I_A + I_B + I_C) - (I_B + I_C) = \frac{1}{2}(I_A + I_B) - \frac{1}{2}(I_B + I_C) + \frac{1}{2}(I_C + I_A) = u\,v - v\,w + w\,u$, thence it is trivial to find the representations of the other excentres and incentre.

37. Using the formula for the reflection of a point in a chord, we have that J has representation
$v^2 + w^2 + \frac{v^2\,w^2(v^*\,w^* + w^*\,u^* + u^*\,v^*)}{R^2} = v^2 + w^2 + v\,w + \frac{v^2\,w}{u} + \frac{w^2\,v}{u}$. Hence, $\overrightarrow{I\,J}$ has representation
$v^2 + w^2 - v\,w - \frac{v^2\,w}{u} - \frac{w^2\,v}{u} + u\,v + v\,w + w\,u$. Multiplying throughout by u yields the symmetric expression
$u^2\,v + v^2\,u + v^2\,w + w^2\,v + w^2\,u + u^2\,w + 2\,u\,v\,w$. This factorises to $(u+v)\,(v+w)\,(w+u)$, so the original expression is $\frac{(u+v)\,(v+w)\,(w+u)}{u}$.

38. Multiplying the expression for $\overrightarrow{I\,J}$ by its complex conjugate gives
$4\,r^2 = \frac{(u+v)\,(u^*+v^*)\,(v+w)\,(v^*+w^*)\,(w+u)\,(w^*+u^*)}{R} = R^2\left(\frac{u}{v}+1\right)\left(1+\frac{u}{v}\right)\left(\frac{v}{w}+1\right)\left(1+\frac{v}{w}\right)\left(\frac{w}{u}+1\right)\left(1+\frac{u}{w}\right)$. The square root of this is thus the distance $2\,r$.

39. We have $O\,I^2 = (u\,v + v\,w + w\,u)\,(u^*\,v^* + v^*\,w^* + w^*\,u^*) = R(3\,R + u\,w^* + w\,u^* + v\,u^* + u\,v^* + w\,v^* + v\,w^*)$.
Combined with the expression for r given in the table, this equals $R(R - 2\,r)$, as required.

40. $\overrightarrow{I\,T} = \frac{1}{2}\left(u^2 + v^2 + w^2\right) + u\,v + v\,w + w\,u = \frac{1}{2}(u+v+w)^2$. It is obvious that the modulus is
$\frac{1}{2}(u+v+w)\,(u^*+v^*+w^*) = \frac{1}{2}(3\,R + u\,w^* + w\,u^* + v\,u^* + u\,v^* + w\,v^* + v\,w^*) = \frac{1}{2}\,R - r$.

41. $[T\,U\,V\,W] = [T\,U\,V] + [V\,W\,T] = \frac{a\,b\,l}{4\,R} + \frac{c\,d\,l}{4\,R} = \frac{(a\,b + c\,d)\,l}{4\,R}$. By Ptolemy's theorem on the quadrilateral $T'\,U\,V\,W$ (where T' is the reflection of T in the perpendicular bisector of $U\,W$), we have $a\,b + c\,d = l$, giving us Parameshvara's formula.

42. The maltitude passes through $\frac{1}{2}\left(u^2 + t^2\right)$, and travels parallel to the vector $\frac{1}{2}\left(v^2 + w^2\right)$. It is clear that $\frac{1}{2}\left(t^2 + u^2 + v^2 + w^2\right)$ lies on this maltitude, and thus all four maltitudes by symmetry.

43. If we consider the other two maltitudes (from the midpoint of each diagonal perpendicular to the other diagonal), they must also pass through Q. By definition, they also pass through the orthocentre of $M\,N\,P$, so Q must be this orthocentre.

44. T' has representation $u^2 + v^2 + w^2$. This is the reflection of T in the anticentre Q. Hence, $T'\,U'\,V'\,W'$ is congruent and homothetic to the original with Q as the centre of similitude.

45. Let I_U be the incentre of $V\,W\,T$, *et cetera*. We have $I_u = -v\,w + t\,w + v\,t$, $I_T = -u\,v - v\,w - w\,u$, and $I_W = -t\,u - u\,v - v\,t$. We wish to prove that $\angle\,I_T\,I_U\,I_V = \frac{\pi}{2}$, which is equivalent to $\frac{I_V - I_U}{I_T - I_U} = \frac{(w-t)\,(u+v)}{(t+u)\,(w+v)}$ being purely imaginary. We know that t, u, v and w have equal modulus, so must be concyclic. Hence, $\frac{(w+t)\,(u+v)}{(t+u)\,(v+w)}$ is real, and we only need to prove that $\frac{w-t}{w+t} = \frac{(w-t)\,(w^*+t^*)}{(w+t)\,(w^*+t^*)}$ is imaginary. The numerator is $t^*\,w - w^*\,t$, which is equal to the negative of its conjugate and is therefore imaginary. Similarly, the denominator is equal to its conjugate and therefore real. Hence, we are done and $\angle\,I_T\,I_U\,I_V = \frac{\pi}{2}$; by symmetry, $I_T\,I_U\,I_V\,I_W$ is a rectangle.

46. For any point P in the plane of a rectangle $A\,B\,C\,D$, we have $A\,P^2 + C\,P^2 = B\,P^2 + D\,P^2$. This can be derived from assuming P is the origin and orienting the rectangle parallel to the coordinate axes and using Cartesian coordinates. Applying this to the point O (circumcentre of $T\,U\,V\,W$) and the rectangle of incentres, we obtain $O\,I_T{}^2 + O\,I_V{}^2 = O\,I_U{}^2 + O\,I_W{}^2$. We can determine each of these squared distances from Euler's formula, obtaining the equation $\left(R^2 - 2\,R\,r_T\right) + \left(R^2 - 2\,R\,r_V\right) = \left(R^2 - 2\,R\,r_U\right) + \left(R^2 - 2\,R\,r_W\right)$. Cancelling terms and dividing throughout by R yields the desired equation.

47. This is true for $n = 4$, by the previous question. We prove the general case by induction on n, assuming it is true for all $n \leq k$, and proving it for $n = k + 1$. Note that for every triangulation, there must be at least one triangle with three adjacent vertices. Suppose we have a triangulation where $A_1 A_2 A_3$ is one of the triangles. Then we can 're-triangulate' the k-gon $A_3 A_4 \ldots A_{k+1} A_1$ such that $A_1 A_3 A_4$ is also a triangle. Now, re-triangulate the cyclic quadrilateral $A_1 A_2 A_3 A_4$, so that $A_2 A_3 A_4$ is a triangle. Repeating this process, we can ensure that $A_{i-1} A_i A_{i+1}$ is a triangle for any i (with subscripts considered modulo $k + 1$). Arbitrarily re-triangulating the k-gon $A_{i+1} A_{i+2} \ldots A_{i-2} A_{i-1}$ gives any possible triangulation of the $(k + 1)$-gon. As we did not affect σ during any of the re-triangulations of the k-gons, σ has remained constant throughout the whole process.