New Zealand Mathematical Olympiad Committee

2012 Squad Assignment One

Combinatorics

Due: Friday 3rd February 2012

1. Each cell of an 8×8 chess board is filled with a 0 or a 1. Prove that if we compute the sums of the numbers in each row, each column, and in each of the two diagonals, then we will get at least three sums that are equal.

2. Each of the three countries Alania, Belinga and Cartusia is inhabited by exactly n people. Each of the $3n$ inhabitants of these countries has exactly $n+1$ friends in the other two countries. Prove that one can find a group of three people, one from each country, that are mutual friends.

3. Let $n \geq 3$ be an integer. Determine the minimum number of points one has to mark inside a convex n-gon in order for the interior of any triangle with its vertices at vertices of the n-gon to contain at least one of the marked points.

4. Abby and Brian play the following game. They first choose a positive integer N, and then they take turns writing numbers on a blackboard. Abby starts by writing 1. Thereafter, when one of them has written the number n, the next player writes down either $n+1$ or $2n$, provided the number is not greater than N. The player who writes N on the blackboard wins.

   (a) Determine which player has a winning strategy if $N = 2011$.

   (b) Find the number of positive integers $N \leq 2011$ for which Brian has a winning strategy.

5. Let M and N be positive integers. Consider an $N \times N$ square array consisting of N^2 lamps that can be in two states — on or off. Initially all of the lamps are turned off.

   A *move* consists of choosing a row or column of the array and changing the state of M consecutive lamps in the chosen row or column, i.e., turning on the lamps that were off, and turning off the lamps that were on.

   Determine the necessary and sufficient condition for which it can be achieved that after a finite number of moves all of the lamps are turned on.

6. Each of 117 spies, operating in a certain country, is to assign himself to one of three missions, such that each mission has at least one spy assigned. At this point, no two spies can communicate. Headquarters will then sequentially issue a number of *passwords*, each of which allows a single pair of spies to communicate. Passwords may only be issued to a pair of spies who share the same mission, and who cannot *already* communicate directly. However, apart from these rules, headquarters may assign passwords in any fashion.

A mission is said to be *networked* if any two spies on that mission can communicate (possibly through other spies). Let n denote the number of passwords issued for which, regardless of how headquarters assigns these passwords, the spies can be certain that at least one mission will be networked. How should the spies choose their missions in order to minimise n?

7. A given list $n_1, n_2, \ldots, n_{2011}$ of positive integers has the property that $n_i n_{i+1}$ is different from $n_j n_{j+1}$ whenever i, j are distinct integers less than 2011. Find the minimum number of distinct integers that must be in any such list.

13th January 2012

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