CHAPTER 3

PROPERTIES OF CIRCLES

We will present in this chapter a few of the most interesting properties of circles and related problems in Mathematics Olympiads. The materials in the first section are classical and standard, their usefulness are illustrated by a series of examples. Since the techniques we learned are by no means exhaustive, some miscellaneous examples will be given in the second section.

3.1 Power of points with respect to circles

Let's recall a theorem which is certainly in Euclidean geometry: it appears in "*Elements*".

Theorem 3.1-1 (Intersecting chords theorem)

Let ω be a circle and P be a point not on ω. If L is a line through P that intersects ω at two points A and B, then the quantity $PA \times PB$ depends only on ω and P but not L. In other words, if there is another line through P which intersects ω at A' and B' then

$$PA \times PB = PA' \times PB'.$$

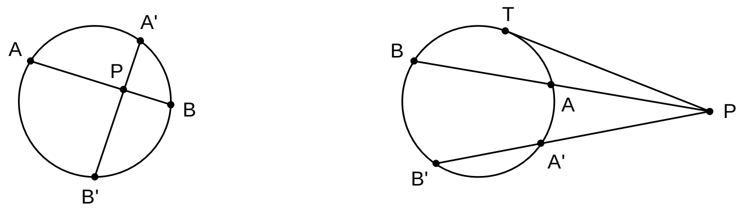

Figure 1

In the case P is outside the circle, the product $PA \times PB$ can be calculated by considering the limiting case where PT is a tangent of the circle. Then $PA \times PB = PT^2 = OP^2 - r^2$, where O and r are respectively the center and radius of the circle. This leads to the following

Definition 3.1-1 (Power of a point with respect to a circle)

Given a circle ω with center O and radius r. The **power** of a point P with respect to a circle is the number $OP^2 - r^2$.

- The power of P is negative if P lies inside ω, zero if P lies on ω, and positive if P lies outside ω. In any case $\left|OP^2 - r^2\right|$ equals to $PA \times PB$ (the product in Theorem 3.1-1).

Example 3.1-1 (St. Petersburg City Math Olympiad 1996)

Let BD be the angle bisector of angle B in triangle ABC with D on side AC. The circumcircle of triangle BDC meets AB at E, while the circumcircle of triangle ABD meets BC at F. Prove that $AE = CF$.

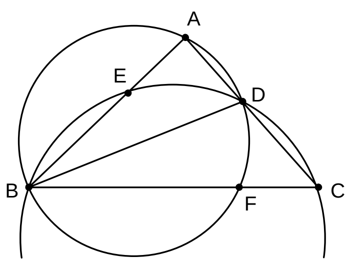

Figure 2

Solution

By intersecting chords theorem one has $AE \times AB = AD \times AC$, or

(1) $$AE = \frac{AD \times AC}{AB}.$$

Similarly, $CF \times CB = CD \times CA$ and therefore

(2) $$CF = \frac{CD \times CA}{CB}.$$

Dividing (1) by (2) gives

$$\frac{AE}{CF} = \frac{AD \times CB}{AB \times CD} = 1,$$

the last equality holds since $\dfrac{AD}{CD} = \dfrac{AB}{CB}$ by angle bisector theorem.

Q.E.D.

We mentioned in chapter 2 that the solution of Example 2.1-2 can be completed with intersecting chords theorem in hand. As usual, this problem has several elegant solutions. One of them is presented in the following example.

Example 3.1-2 (IMO 1995-1, continuation of Example 2.1-2)

Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

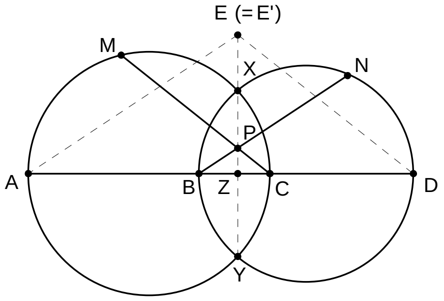

Figure 3

Solution

Draw DE parallel to CM meets XY at E, and draw AE' parallel to BN meets XY at E' (see Figure 3). We claim that $E = E'$. The reason goes as follows: note that

(3) $$\frac{ZE'}{ZE} = \frac{ZE'}{ZP} \times \frac{ZP}{ZE} = \frac{ZA}{ZB} \times \frac{ZC}{ZD}.$$

By intersecting chords theorem, $ZA \times ZC = ZX \times ZY = ZB \times ZD$. Therefore, (3) gives $ZE = ZE'$. This proved our claim. Now, AM, DN and XY are the altitudes of triangle ADE (see Figure 4), hence they are concurrent.

Q.E.D.

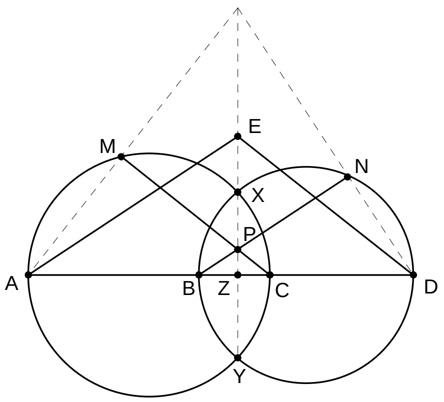

Figure 4

Before proceeding to the next example, readers are minded that the converse of intersecting chords theorem is also true. It is one of the most frequently used criteria for proving concyclic points. At least readers may agree that it is easier to apply than Ptolemy's theorem, in the sense we only need to verify an equality which looks simple. Proving concyclic points by Ptolemy's theorem is relatively difficult in general.

Theorem 3.1-2 (Converse of intersecting chords theorem)

If the lines *AB*, *CD* meet at *P* and $PA \times PB = PC \times PD$ as signed lengths, then *A*, *B*, *C*, *D* are concyclic.

Example 3.1-3.

AB is a chord of a circle, which is not a diameter. Chords A_1B_1 and A_2B_2 intersect at the midpoint *P* of *AB*. Let the tangents to the circle at A_1 and B_1 intersect at C_1. Similarly, let the tangents to the circle at A_2 and B_2 intersect at C_2. Prove that C_1C_2 is parallel to *AB*.

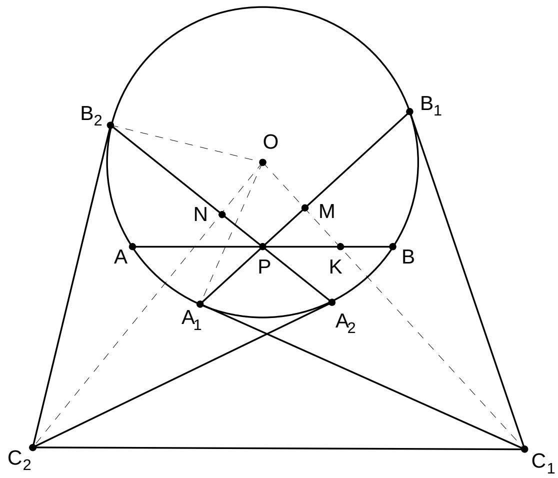

Figure 5

Solution (Due to Poon Wai Hoi)

Let *O* be the center of the circle, let OC_1 intersects A_1B_1 at *M*, let OC_2 intersects A_2B_2 at *N*, and let also OC_1 intersects *AB* at *K*. Clearly, *OM* and *ON* are respectively the perpendicular bisectors of A_1B_1 and A_2B_2. So, $\angle OMP = \angle ONP = 90°$, saying that *O*, *M*, *P*, *N* are concyclic. This implies $\angle ONM = \angle OPM = 90° - \angle MOP = \angle OKA$.

Next, we claim that *M*, C_1, C_2, *N* are concyclic. Suppose first our claim is true, it follows that

$\angle OC_1C_2 = \angle ONM = \angle OKA$ and thereby completes the proof. It remains to show M, C_1, C_2, N are concyclic. Note that ΔOA_1C_1 and ΔOB_2C_2 are right-angled triangles, therefore

$$OM \times OC_1 = OA_1^2 = OB_2^2 = ON \times OC_2 .$$

Hence, M, C_1, C_2, N are concyclic by the converse of intersecting chords theorem.

Q.E.D.

We can say something more if we don't concentrate ourselves in the case of one circle. Consider two circles with centers O_1, O_2 and radii r_1, r_2, where $O_1 \neq O_2$. It is natural to ask for the locus of points P having the same power with respect to the two circles. We leave to the reader for proving the locus is a straight line L perpendicular to the line O_1O_2. The line L is called the **radical axis**（根軸）of the two circles. In the case there are three circles with non-collinear centers, the three radical axes of the three pairs of circles intersect at a point called the **radical center**（根心）of the three circles. Figure 6 shows three circles and the three corresponding radical axes.

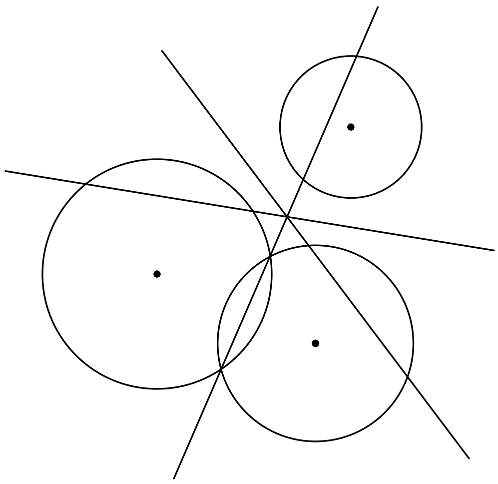

Figure 6

- If two circles intersect at points A and B, then the radical axis of the circles is simply the line AB because A and B have zero power with respect to each of the circles.

Example 3.1-4 (USAMO 1997)

Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove the lines through A, B, C, perpendicular to the lines EF, FD, DE, respectively, are concurrent.

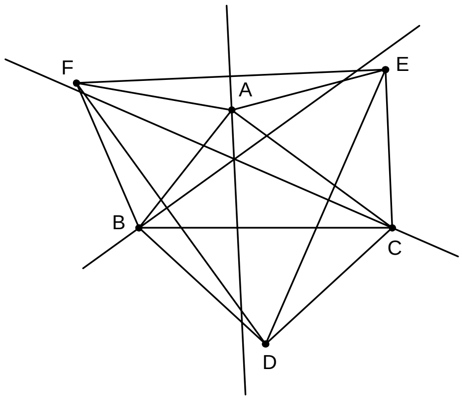

Figure 7

Solution

Let C_1 be the circle with center D and radius DB, C_2 be the circle with center E and radius EC, C_3 be the circle with center F and radius FA (see Figure 8). We claim that the three lines being proved to be concurrent are radical axes of the three pairs of circles (C_1, C_2), (C_2, C_3) and (C_3, C_1). We complete the proof by proving the claim.

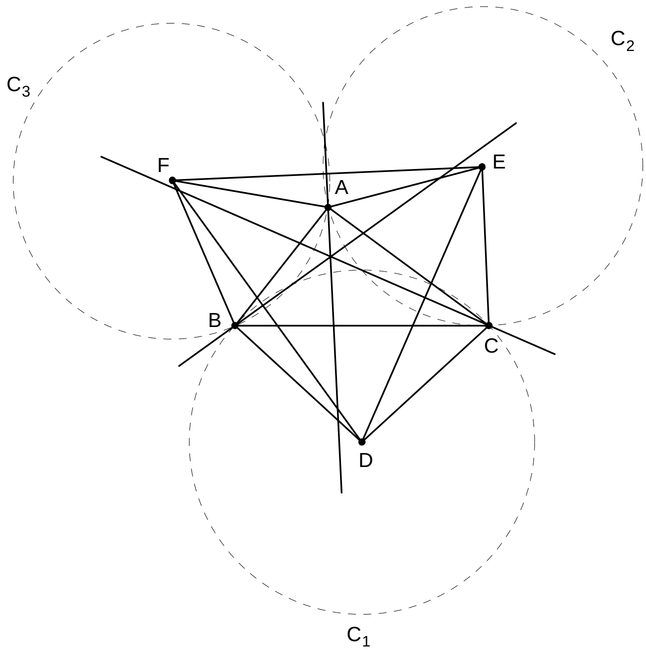

Figure 8

Since A is one of the intersections of circles C_2 and C_3, the radical axis of C_2 and C_3 is the line passing through A perpendicular to the line joining the centers E and F. Similarly, the radical axis of C_3 and C_1 is the line through B perpendicular to FD, and the radical axis of C_1 and C_2 is the line through C perpendicular to DE. These three radical axes concur at the radical center of the three circles.

Q.E.D.

Example 3.1-5 (IMO 1985-5)

A circle with centre O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangles ABC and KBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.

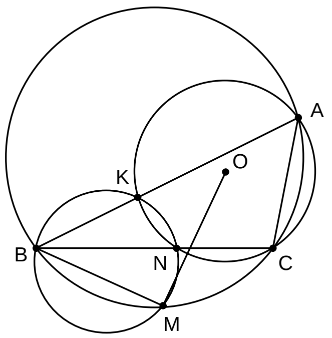

Figure 9

Analysis

The lines AC, KN, BM concur at the radical center P of the three circles involved. A possible way to show $OM \perp BP$ is proving that $OB^2 - OP^2 = MB^2 - MP^2$.

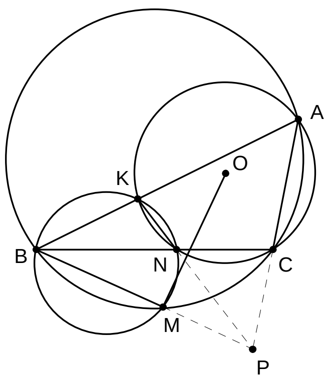

Figure 10

To prove perpendicular lines we have the following useful

Lemma 3.1-6

Let *A*, *B*, *P*, *Q* be four distinct points on a plane. Then

$$AB \perp PQ \quad \text{if and only if} \quad PA^2 - PB^2 = QA^2 - QB^2 .$$

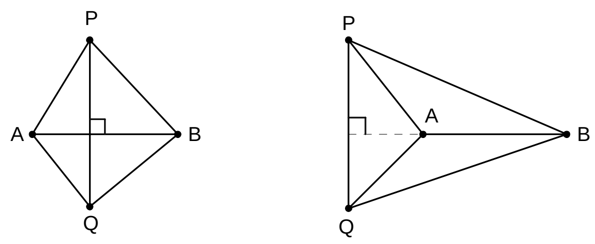

Figure 11

The "only-if-part" follows easily by Pythagoras' theorem. The "if-part" can be proved by coordinate geometry or the method of false position. (To show a point X has certain property, a common way is to construct a point X' with the desired property and then to show $X = X'$. This is called the method of false position.)

Solution to Example 3.1-5

Refer to Figure 10, the quadrilateral $PCNM$ is cyclic since $\angle PCN = \angle AKN = \angle BMN$. By intersecting chords theorem we have

(4) $$PM \times PB = PC \times PA = OP^2 - r^2 ,$$

where r is the circumradius of triangle AKC. Similarly,

(5) $$BM \times BP = BN \times BC = OB^2 - r^2 .$$

The equalities (4) and (5) together give

$$\begin{aligned} OB^2 - OP^2 &= BM \times BP - PM \times PB \\ &= BP \times (BM - PM) \\ &= (BM + PM) \times (BM - PM) \\ &= BM^2 - PM^2 \end{aligned}$$

Hence, $OM \perp BP$ by Lemma 3.1-6.

Q.E.D.

Example 3.1-6 (CMO 1997)

Let quadrilateral $ABCD$ be inscribed in a circle. Suppose lines AB and DC intersect at P and lines

AD and *BC* intersect at *Q*. From *Q*, construct the two tangents *QE* and *QF* to the circle where *E* and *F* are the points of tangency. Prove that the three points *P*, *E*, *F* are collinear.

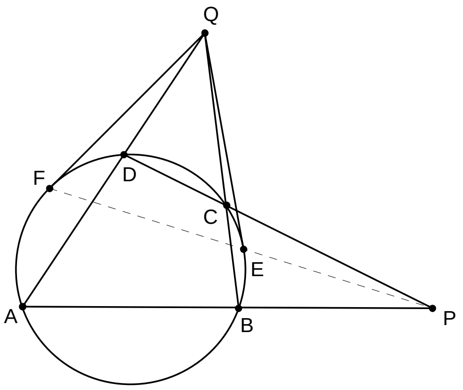

Figure 12

Solution

Let C_1 be the circumcircle of triangle *ABC* and O_1 be its center. Suppose the circumcircle C_2 of *QCD* intersects the line *PQ* at *Q* and *R*. Then the points *P*, *R*, *C*, *B* are concyclic because $\angle PRC$ = $\angle QDC = \angle ABC$. We first show $O_1R \perp PQ$ by Lemma 3.1-6 (again!).

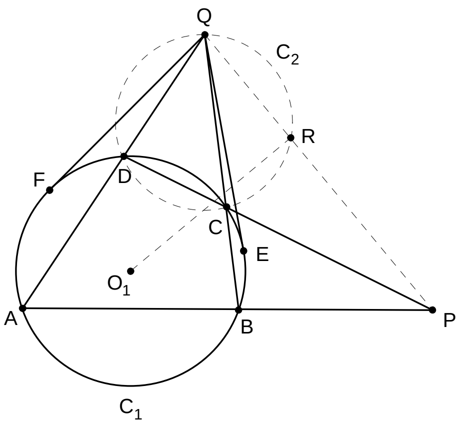

Figure 13

Similar to Example 3.1-5, we apply intersecting chords theorem:

(6) $$O_1P^2 - r_1^2 = PC \times PD = PR \times PQ$$

and also

(7) $$O_1Q^2 - r_1^2 = QC \times QB = QR \times QP\,,$$

where r_1 is the radius of C_1. Subtract (7) from (6),

$$\begin{aligned} O_1P^2 - O_1Q^2 &= PR \times PQ - QR \times QP \\ &= PQ \times (PR - QR) \\ &= (PR + QR) \times (PR - QR) \\ &= PR^2 - QR^2 \end{aligned}$$

Therefore Lemma 3.1-6 implies $O_1R \perp PQ$, i.e. the points *Q*, *F*, O_1, *E*, *R* are also concyclic. Let C_3 be the circle passes through these five points. Now, we have three circles C_1, C_2, C_3 in hand. The radical axis of C_1 and C_2 is the line *CD*, and the radical axis of C_2 and C_3 is the line *QR*. These two radical axes intersect at *P*. Hence, *P* lies on the radical axis of C_3 and C_1, namely *EF*.

Q.E.D.

- In the above solution, is it possible that *Q* coincides with *R*? If yes, is the above proof still valid? Can you modify it to make it valid? If no, why *Q* must not coincide with *R*?

- In chapter 5, we will discuss a technique called **reciprocation**（配極）. At that time a “1-line-proof” of Example 3.1-6 will be given.

Exercise

1. Prove the intersecting chords theorem and its converse.

2. Let *A*, *B* be two points and *k* be a real number. Prove that
   (a) The locus of points *P* satisfying $PA^2 - PB^2 = k$ is a straight line perpendicular to *AB*.
   (b) Hence, or otherwise, prove that the locus of points having the same power with respect to two given circles (with distinct centers) is a line perpendicular to the line joining the two centers. (This allows us to define “radical axis”.)

3. Prove Lemma 3.1-6.

4. (MOP 1995) Given triangle *ABC*, let *D*, *E* be any points on *BC*. A circle through *A* cuts the lines *AB*, *AC*, *AD*, *AE* at the points *P*, *Q*, *R*, *S*, respectively. Prove that

$$\frac{AP \times AB - AR \times AD}{AS \times AE - AQ \times AC} = \frac{BD}{CE}.$$

5. (USAMO 1998) Let ω_1 and ω_2 be concentric circles, with ω_2 in the interior of ω_1. From a point *A* on ω_1 one draws the tangent *AB* to ω_2 ($B \in \omega_2$). Let *C* be the second point of intersection of *AB* and ω_1, and let *D* be the midpoint of *AB*. A line passing through *A* intersects ω_2 at *E* and *F* in such a way that the perpendicular bisectors of *DE* and *CF* intersect at a point *M* on *AB*. Find, with proof, the ratio *AM* / *MC*.

6. (MOP 1995) Let BB', CC' be altitudes of triangles *ABC*, and assume $AB \neq AC$. Let *M* be the midpoint of *BC*, *H* the orthocenter of *ABC*, and *D* the intersection of *BC* and $B'C'$. Show that *DH* is perpendicular to *AM*.

7. (IMO 1994 proposal) A circle ω is tangent to two parallel lines l_1 and l_2. A second circle ω_1 is tangent to l_1 at *A* and to ω externally at *C*. A third circle ω_2 is tangent to l_2 at *B*, to ω externally at *D* and to ω_1 externally at *E*. Let *Q* be the intersection of *AD* and *BC*. Prove that *QC* = *QD* = *QE*.

8. (India, 1995) Let *ABC* be a triangle. A line parallel to *BC* meets sides *AB* and *AC* at *D* and *E*, respectively. Let *P* be a point inside triangle *ADE*, and let *F* and *G* be the intersection of *DE* with *BP* and *CP*, respectively. Show that *A* lies on the radical axis of the circumcircles of *PDG* and *PFE*.

3.2 Miscellaneous examples

Certainly it is not the case that every problem involving circle(s) can be solved using the concepts of power, radical axis and radical center. It is absolutely beneficial to look at different kinds of problems and solutions. Of course, the most important thing is doing more exercises.

Example 3.2-1 (Taken from "*Geometry Revisited*")

If lines *PB* and *PD*, outside a parallelogram *ABCD*, make equal angles with the sides *BC* and *DC*, respectively, as in Figure 14, then $\angle CPB = \angle DPA$. (Of course, Figure 14 a plane figure, not three dimensional!)

Analysis

The problem involves only angles, so concyclic points may play a role in the solution. Since we don't have four points being concyclic in the figure, we try to construct a set of concyclic points.

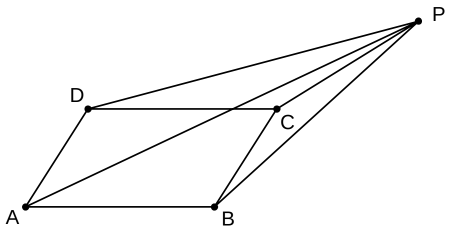

Figure 14

Solution

Complete the parallelogram *PQBC* as in Figure 15. Note that $\angle QAB = \angle PDC = \angle PBC = \angle QPB$, so the points *A*, *B*, *Q*, *P* are concyclic. Then we have $\angle APB = \angle AQB = \angle DPC$, the desired result follows.

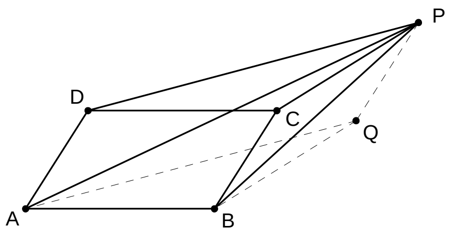

Figure 15

Example 3.2-2 (IMO 1990-1)

Chords *AB* and *CD* of a circle intersect at a point *E* inside the circle. Let *M* be an interior point of the segment *EB*. The tangent line at *E* to the circle through *D*, *E* and *M* intersects the lines *BC* and *AC* at *F* and *G*, respectively. If $\dfrac{AM}{AB} = t$, find $\dfrac{EG}{EF}$ in terms of *t*.

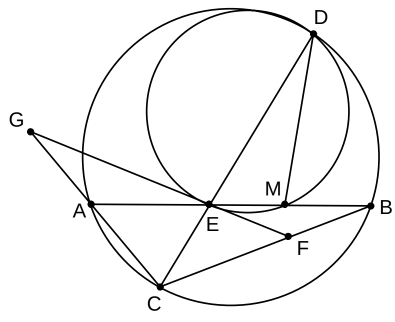

Figure 16

Analysis

To tackle a geometry problem, especially those appear in Mathematics Olympiads, we usually need to observe similar triangles and concyclic points. The crux of Example 3.2-2 is to note that $\Delta AMD \sim \Delta CEF$ and $\Delta CEG \sim \Delta BMD$.

Solution

Note that $\angle MAD = \angle ECF$ and $\angle AMD = \angle DEG = \angle CEF$, which imply $\Delta AMD \sim \Delta CEF$ and follows that $\frac{EF}{CE} = \frac{MD}{AM}$. Therefore,

(8) $$EF = \frac{MD \times CE}{AM}.$$

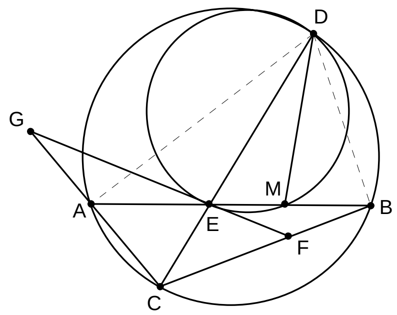

Figure 17

Similarly, $\Delta CEG \sim \Delta BMD$ because $\angle GCE = \angle DBM$ and

$$\begin{aligned} \angle CEG &= \angle CEA + \angle AEG \\ &= \angle DEM + \angle BEF \\ &= \angle DEM + \angle MDE \\ &= \angle BMD \end{aligned}$$

This pair of similar triangles gives $\frac{GE}{CE} = \frac{DM}{BM}$, or equivalently

(9) $$GE = \frac{DM \times CE}{BM}.$$

Finally, dividing (9) by (8) gives $\frac{GE}{EF} = \frac{AM}{BM} = \frac{t}{1-t}$.

Example 3.2-3 (APMO 1999)

Let Γ_1 and Γ_2 be two circles intersecting at *P* and *Q*. The common tangent, closer to *P*, of Γ_1 and Γ_2 touches Γ_1 at *A* and Γ_2 at *B*. The tangent of Γ_1 at *P* meets Γ_2 at *C*, which is different from *P* and the extension of *AP* meets *BC* at *R*. Prove that the circumcirle of triangle *PQR* is tangent to *BP* and *BR*.

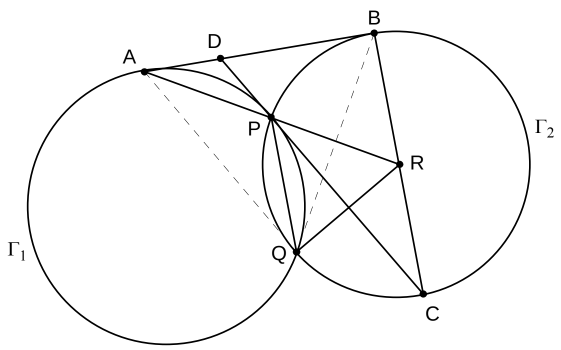

Figure 18

Analysis

To show the circumcircle of triangle *PQR* is tangent to *BP* and *BR*, it suffices to show $\angle BPR = \angle PQR = \angle BRP$. After some direct trials we found no simple reason for which the angles must be equal, so we ask ourselves: are there four concyclic points in the figure?

Solution

Note that $\angle QAR = \angle QPC = \angle QBC = \angle QBR$, so *A*, *B*, *R*, *Q* are concyclic. One has

$$\begin{aligned}\angle ARB &= \angle AQB \\ &= \angle AQP + \angle PQB \\ &= \angle BAR + \angle PQB \\ &= \angle BQR + \angle PQB \\ &= \angle PQR\end{aligned}$$

which shows *BR* is tangent to the circumcircle of triangle *PQR*. Next, we prove *BP* is also a tangent to the circumcirle of triangle *PQR*. This is no more difficult: the result follows by noting that $\angle BPR = \angle PAB + \angle PBA = \angle BQR + \angle BQP = \angle PQR$.

Q.E.D.

We have seen the importance of observing similar triangles and concyclic points. In Example 3.2-2, we are asked to find the ratio of two lengths, so there is no surprise that similar triangles play a role in the solution. In Example 3.2-3, the problem involves only angles, this explains why we looked at concyclic points instead of similar triangles. However, when the figure involves more and more points, the situation may become complicated and better insight is needed in order to find out non-trivial similar triangles or concyclic points. Let's go ahead to the next example.

Example 3.2-4

ABCDE is a convex pentagon. The sides of the pentagon intersect at P_1, P_2, P_3, P_4, P_5 as shown in the Figure 19. Construct the circumcircles of the triangles P_1AE, P_2BA, P_3CB, P_4DC and P_5ED. These circumcircles meet at five points A', B', C', D', E' which are different from *A*, *B*, *C*, *D*, *E*. Prove that the points A', B', C', D', E' are concyclic.

Analysis

As mentioned before, we don't need to consider similar triangles since there is no length involved. There are so many points in the figure, are any four of them concyclic? If there are four concyclic points, then by symmetry there should be more. So, which set of concyclic points is useful? These are the things we should keep in mind. Problem solving, not only for problems in geometry, is not a process of blind trial; it requires careful analysis on the condition(s) and conclusion.

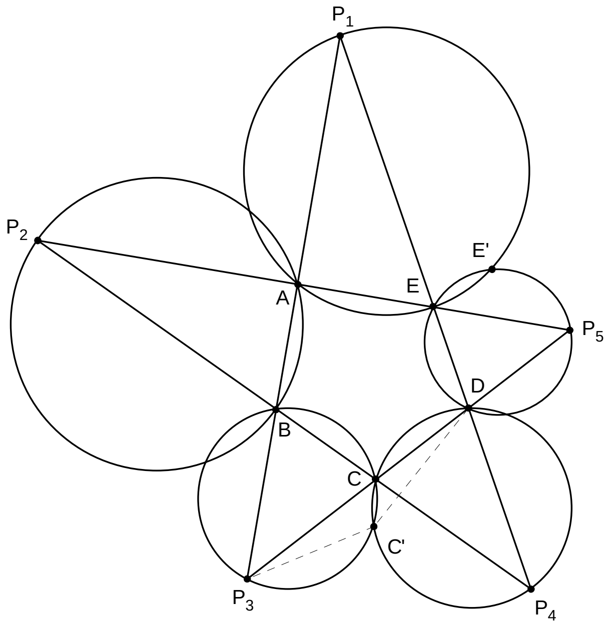

Figure 19

Back to our problem, let's ignore the concyclic points at this moment and think in another way. To show the five points A', B', C', D', E' are concyclic, it suffices to show any four of them, say A', B', C', E' are concyclic (then D' also lies on the same circle by symmetry). We will prove this happens if and only if P_1, P_3, C', E' are concyclic. Of course, the latter one may not be easier, but at least we have one more possibility.

Solution

We begin the solution by giving a sufficient (and in fact necessary) condition for which A', B', C', E' are concyclic. Note that

$$\begin{aligned}\angle A'B'C' &= \angle A'B'B + \angle BB'C' \\ &= \angle A'AP_1 + \angle BP_3C' \\ &= \angle A'E'P_1 + \angle P_1P_3C' \\ &= \angle P_1E'C' - \angle A'E'C' + \angle P_1P_3C'\end{aligned}$$

So, $\angle A'B'C' + \angle A'E'C' = \angle P_1E'C' + \angle P_1P_3C'$. That is, A', B', C', E' are concyclic if and only if P_1, P_3, C', E' are concyclic. The latter assertion holds since $\angle P_1P_3C' = \angle P_4CC' = \angle P_4DC'$, which imples P_1, P_3, C', D are concyclic and E' lies on the same circle by symmetry. We have proved that A', B', C', D', E' are concyclic.

Q.E.D.

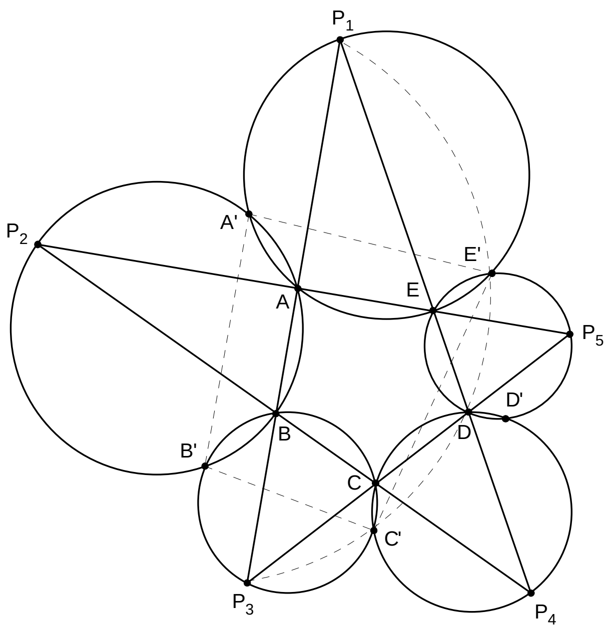

Figure 20

Example 3.2-5 (Butterfly theorem)

Let *PQ* be a chord of a circle and *M* be the midpoint of *PQ*. Through *M* two chords *AB* and *CD* of the circle are drawn. Chords *AD* and *BC* intersect *PQ* at points *X* and *Y* respectively. Prove that *M* is the midpoint of the segment *XY*.

Butterfly theorem has been around for quite a while and attracted many problem solvers. Up to now numerous proofs varying in length and difficulty has been published. The one we are going to present is certainly not the shortest one, but it is simple and elementary.

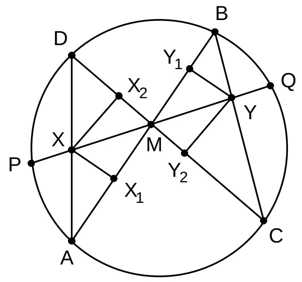

Figure 21

Solution

From X we draw perpendicular lines to AB and CD, with feet X_1 and X_2 respectively. From Y we draw perpendicular lines to AB and CD, with feet Y_1 and Y_2 respectively. For convenience, let $x = MX$, $y = MY$ and $a = PM = QM$.

By similar triangles one has

$$\frac{x}{y} = \frac{XX_1}{YY_1} = \frac{XX_2}{YY_2}, \quad \frac{XX_1}{YY_2} = \frac{AX}{CY} \quad \text{and} \quad \frac{XX_2}{YY_1} = \frac{DX}{BY}.$$

With these equalities one also has

$$\begin{aligned}
\frac{x^2}{y^2} &= \frac{XX_1}{YY_1} \times \frac{XX_2}{YY_2} \\
&= \frac{XX_1}{YY_2} \times \frac{XX_2}{YY_1} \\
&= \frac{AX}{CY} \times \frac{DX}{BY} \\
&= \frac{PX \times QX}{PY \times QY} \quad \text{(by intersecting chords theorem)} \\
&= \frac{(a+x)(a-x)}{(a+y)(a-y)} \\
&= \frac{a^2 - x^2}{a^2 - y^2}
\end{aligned}$$

which implies $\frac{x^2}{y^2} = 1$, hence $x = y$.

Q.E.D.

Example 3.2-6 (IMO 1994-2)

ABC is an isosceles triangle with $AB = AC$. Suppose that

(i) M is the mid-point of BC and O is the point on the line AM such that OB is perpendicular to AB;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F lies on the line AC such that E, Q and F are all distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

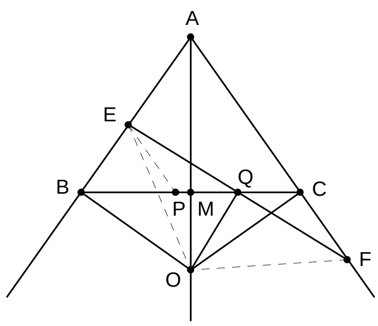

Figure 22

Solution

Suppose first $OQ \perp EF$. To prove $QE = QF$, a possible way is to show $\Delta OQE \cong \Delta OQF$. Since we already have $OQ \perp EF$, it remains to prove $\angle OEQ = \angle OFQ$. Note that $BOQE$ and $CQOF$ are cyclic because $\angle OBE$, $\angle OQE$ and $\angle OCF$ are all right angles, therefore $\angle OEQ = \angle OBQ = \angle OCQ = \angle OFQ$. This finished the "only-if-part".

Next, we suppose $QE = QF$ and try to prove $OQ \perp EF$. It suffices to show $OE = OF$. A possible reason for $OE = OF$ is that $\Delta OBE \cong \Delta OCF$. This pair of triangles are right-angled with one pair of equal sides ($OB = OC$). They are congruent to each other if and only if $BE = CF$. So, the proof can be completed by showing that $BE = CF$.

Draw a line through E parallel to AC meets BC at point P, as shown in Figure 22. It is clear that the triangles PQE and CQF are similar. Since we have the condition $QE = QF$ in hand, these two triangles are in fact congruent. It follows that $CF = PE$. Since $\angle EBP = \angle ACB = \angle EPB$, we have $BE = PE = CF$. The proof is completed.

Q.E.D.

Example 3.2-7 (IMO 1999-5)

Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points *M* and *N*, respectively. Γ_1 passes through the centre of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at *A* and *B*. The lines *MA* and *MB* meet Γ_1 at *C* and *D*, respectively. Prove that *CD* is tangent to Γ_2.

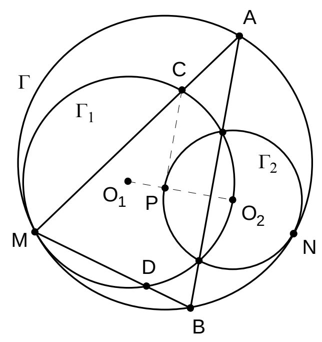

Figure 23

Solution

Let O_1 and O_2 be the centers of Γ_1 and Γ_2, respectively. The line O_1O_2 intersects Γ_2 at point *P* (see Figure 23). If we can prove $\angle CPO_2 = 90°$, then similar argument will show $\angle DPO_2 = 90°$ and therefore *CD* is a tangent of Γ_2.

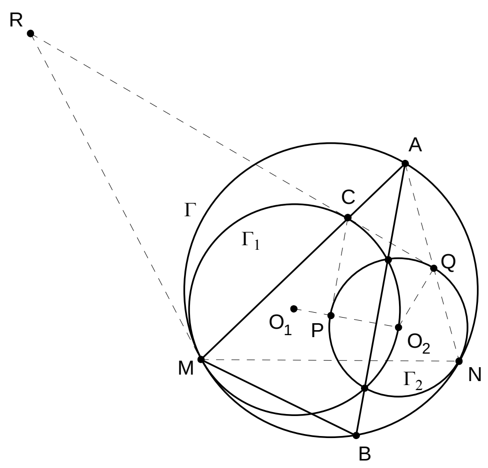

Figure 24

Join *AN* which meets Γ_2 at point *Q*, let *R* be the intersection of the line *CQ* with the tangent of Γ (and also Γ_1) at *M* (see Figure 24). We claim that *CQ* is a common tangent of Γ_1 and Γ_2. To prove our claim, we first note that *A* lies on the radical axis of Γ_1 and Γ_2, which implies

$AC \times AM = AQ \times AN$. Therefore *CMNQ* is cyclic by the converse of intersecting chords theorem.

Next, we have $\angle RCM = \angle MNQ = \angle RMC$. Recall that *RM* is a tangent to Γ_1, it forces *RC* to be another tangent of Γ_1 from *R*. We have proved *CQ* is a tangent of Γ_1 and by similar argument it is also a tangent of Γ_2 .

Finally, we prove $\angle CPO_2 = 90°$ by showing that $\Delta CPO_2 \cong \Delta CQO_2$ (see Figure 25). Since we already have $O_2P = O_2Q$, it suffices to show $\angle PO_2C = \angle QO_2C$. The argument goes as follows: note that $\frac{1}{2}\angle CO_1O_2 = \angle QCO_2$ because *CQ* is a tangent of Γ_1, so

$$\angle PO_2C = 90° - \frac{1}{2}\angle CO_1O_2 = 90° - \angle QCO_2 = \angle QO_2C ,$$

the last equality comes from the fact that *CQ* is a tangent of Γ_2 .

Q.E.D.

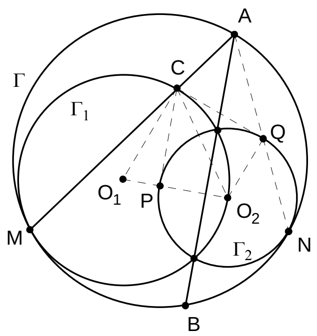

Figure 25

- An alternative solution to Example 3.2-7 using inversion will be given in chapter 5.

Exercise

1. Given two non-intersecting circles in a plane. They have two internal common tangents and two external common tangents. Show that the midpoints of these four tangents are collinear.

2. (Archimedes' "broken-chord" theorem) Point *D* is the midpoint of arc *AC* of a circle; point *B* is on minor arc *CD*; and *E* is the point on *AB* such that *DE* is perpendicular to *AB*. Prove that *AE* = *BE* + *BC*.

3. (Circle of Apollonius) Let *A*, *B* be two given points and $k \neq 1$ a positive real number. Prove that the locus of points *P* satisfying *PA* / *PB* = *k* is a circle whose center lies on *AB*.

4. (Morley's theorem) The points of intersection of the adjacent angle trisectors of any triangle form an equilateral triangle (see Figure 29). (This is one of the most surprising and beautiful theorems in elementary geometry.)

5. (Descartes's circle theorem) Let r_1, r_2, r_3, r_4 be the radii of four mutually externally tangent circles. Prove that

$$\sum_{k=1}^{4} \frac{2}{r_k^{\ 2}} = \left(\sum_{k=1}^{4} \frac{1}{r_k} \right)^2 .$$

6. (IMO 1998) In convex quadrilateral *ABCD*, the diagonals *AC* and *BD* are perpendicular and the opposite sides *AB* and *DC* are not parallel. Suppose that the point *P*, where the perpendicular bisectors of *AB* and *DC* meet, is inside *ABCD*. Prove that *ABCD* is a cyclic quadrilateral if and only if the triangles *ABP* and *CDP* have equal areas.

7. (USAMO 1993) Let *ABCD* be a convex quadrilateral with perpendicular diagonals meeting at *O*. Prove that the reflections of *O* across *AB*, *BC*, *CD*, *DA* are concyclic.

8. (IMO 1995 shortlisted problem) The incircle of triangle *ABC* touches *BC*, *CA* and *AB* at *D*, *E* and *F* respectively. *X* is a point inside triangle *ABC* such that the incircle of triangle *XBC* touches *BC* at *D* also, and touches *CX* and *XB* at *Y* and *Z* respectively. Prove that *EFZY* is a cyclic quadrilateral.

9. (IMO 1992 proposal) Circles G_1 and G_2 touch each other externally at a point *W* and are inscribed in a circle *G*. *A*, *B*, *C* are points on *G* such that *A*, G_1 and G_2 are on the same side of chord *BC*, which is also tangent to G_1 and G_2. Suppose *AW* is also tangent to G_1 and G_2. Prove that *W* is the incenter of triangle *ABC*.