Geometry – Transformations
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Contents

Abstract

After the summer, the IMOHK trainging steped into the second phase. In the area of Geometry, after learning some basic notations and theorems, we are going to develop a deeper sense of geometric insights. We will look at the studies of the plane geometry through the ideas of transformations. Many theorems and problems will be discussed from this point of view.

1 Translations

1.1 Definition and Properties

Definition Let P and P' be two distinct points on the plane. The direction PP' defines a translation on the plane which maps a point A to another point A' such that AA' has the same length and direction of that of PP'.

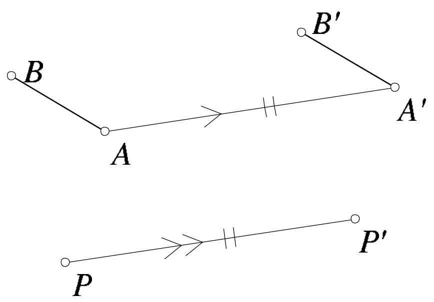

Figure 1: Translation

Properties

1. Size, shape and direction of any figure remain unchanged under translation.
2. $AB = A'B', AB//A'B'$.
3. No fixed point.
4. A line l maps to a line l' and $l//l'$.
5. A line l is fixed if and only if $PP'//l$.
6. translation + translation = translation

1.2 Examples

1. Let A', B', C' be the midpoints of sides BC, CA, AB of $\triangle ABC$ respectively. Let O_1, O_2, O_3 be the circumcenters of $\triangle AC'B', \triangle C'BA', \triangle B'A'C$ respectively; and I_1, I_2, I_3 be the incenters of $\triangle AC'B', \triangle C'BA', \triangle B'A'C$ respectively. Show that $\triangle O_1O_2O_3$ is congruent to $\triangle I_1I_2I_3$.

2. (a) At which point should a bridge MN be built across a river separating two towns A and B in order that the path $AMNB$ from town A to town B be as short as possible (the banks of the river ar assumed to be parallel straight lines, and the bridge is assumed to be perpendicular to the river)?(Figure 2)

   (b) Solve the same problem if the town A and B are separated by several rivers across which bridges must be constructed.

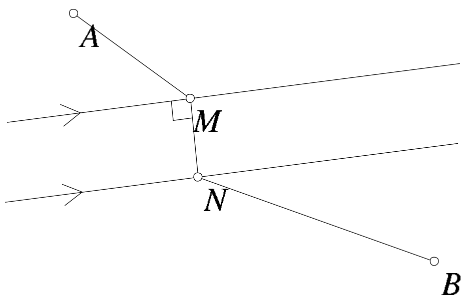

Figure 2: Shortest Path from A to B

3. Let P be a point inside a parallelogram $ABCD$ such that $\angle PAB = \angle PCB$. Show that $\angle PDC = \angle PBC$.

4. In a quadrilateral $ABCD$, M, N are the midpoints of AD, BC respectively. If $2MN = AB + CD$, show that $ABCD$ is a trapezium.

2 Rotations

2.1 Definition and Properties

Definition Let P be a point on the plane and θ be a given angle. In a rotation, the entire plane is turned about some point P through a given angle θ. (Figure 3). In particular, a point A is transformed to another point A' such that $AP = A'P$ and the directed angle $\angle A'PA = \theta$. P and θ are called the center and the angle of rotation.

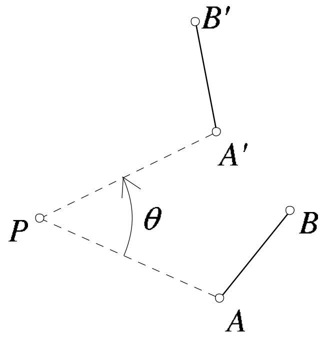

Figure 3: Rotation about P through θ

Properties

1. Size and shape of any figure remain unchanged under rotation.
2. The only point remains fixed is the center of rotation P.
3. A line l maps to a line l' and the angle between l and l' is θ.
4. There is no fixed line under rotation except when θ is a multiple of $180°$. When $\theta = 180°$, a line l is fixed if and only if P lies on l.
5. rotation + rotation = rotation or translation (?)

2.2 Examples

Pythagoras' Theorem In a triangle ABC right-angled at C, denote a, b, c the side BC, CA, AB respectively. We have $a^2 + b^2 = c^2$.

Euclid's Proof (Sketch) Let $ACIJ, CBFG, ABDE$ be squares erected externally on the side AC, CB, BA respectively. M, N are points on AB and DE such that CMN is perpendicular to AB. We need to show

1. $[ACIJ] = 2[JAB], [CBFG] = 2[ABF]$
2. $\triangle JAB \cong \triangle CAD, \triangle ABF = \triangle EBC$

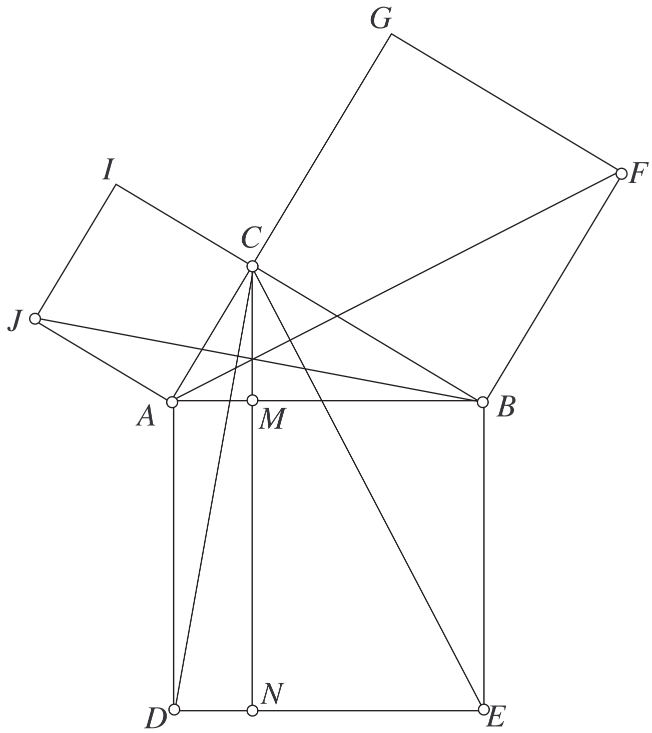

Figure 4: Pythagoras' Theorem

3. $2[CAD] = [ADNM], 2[EBC] = [MNEB]$

Hence we have $a^2 + b^2 = [ACIJ] + [CBFG] = [ADNM] + [MNBE] = [ADEB] = c^2$. $\square$

Note Do you notice that JB and CD are equal in length and perpendicular to each other? Why?

Exercise Using a similar construction to prove the Cosine Formulae

$$c^2 = a^2 + b^2 - 2ab\cos C$$

in $\triangle ABC$.

Fermat Point In a triangle ABC with $A, B, C < 120°$, there exists a unique point P inside $\triangle ABC$ such that

$$AP + BP + CP$$

is minimum. This point is called the Fermat Point. In fact, this is the unique point P in triangle ABC such that

$$\angle APB = \angle BPC = \angle CPA = 120^o.$$

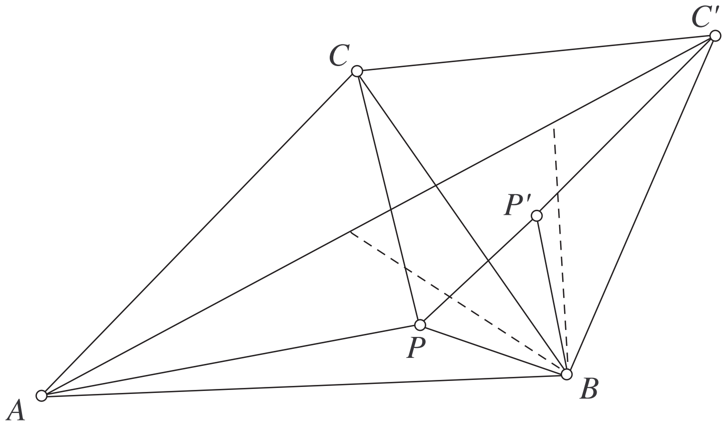

Figure 5: Fermat Point

Proof Let P be an arbitrary point inside ABC. Rotate the $\triangle CPB$ by 60^o out of the triangle with center of rotation at B into the triangle $C'P'B$. Hence we have

$$CP = C'P'.$$

Note that $\triangle PP'B$ and $\triangle CC'B$ are both equilateral, in particular, we have

$$BP = PP'.$$

Hence the sum

$$AP + BP + CP = AP + PP' + P'C',$$

which is a path of line segments from A to C'. Hence this has a minimum value AC'. This minimum is attain when

$$\angle CPB = \angle C'P'B = 180^o - 60^o = 120^o$$

and APC' is a staight line. This gives a unique construction of the Fermat point. $\square$

Exercises

1. In a triangle ABC, let CAX, ABY, BCZ be three equilateral trianglse erected externally at the sides CA, AB, BC respectively. Show that AX, BY, CZ are concurrent at the Fermat point.

2. In a square $ABCD$, if P is a point inside the square such that

$$AP = 1, \quad BP = 2, \quad \text{and} \quad CP = 3,$$

find $\angle APB$.

3. (Prelim 1993/13) Triangle ABC is equilateral. O is a point inside the triangle such that
$$AO = 3, \quad BO = 4, \quad \text{and} \quad CO = 5.$$
Find the area of $\triangle ABC$.

4. (IMO1995/5) Let $ABCDEF$ be a convex hexagon with
$$AB = BC = CD, \quad DE = EF = FA \quad \text{and} \quad \angle BCD = \angle EFA = 60^o.$$
Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120^o$. Prove that
$$AG + GB + GH + DH + HE \geq CF.$$

5. (IMO2005/5) Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

2.3 Composition of Rotations

Exploration What is the composition of two rotations? Let P_1, P_2 be the centers of the two rotations and θ_1, θ_2 be the angles of the two rotations. Hints: Consider all the cases:

1. $P_1 = P_2$
2. $P_1 \neq P_2$, $\theta_1 + \theta_2 \neq 360°$
3. $P_1 \neq P_2$, $\theta_1 + \theta_2 = 360°$.

Napoleon Triangle In a triangle ABC, let CAX, ABY, BCZ be three equilateral triangles erected externally at the sides CA, AB, BC with centroids O_1, O_2, O_3 respectively. Show that $\triangle O_1O_2O_3$ is equilateral.(Figure 6)

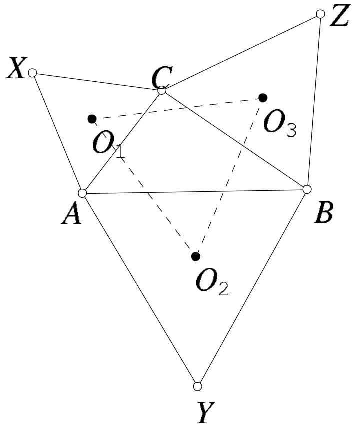

Figure 6: Napoleon Triangle

3 Reflections

3.1 Definition and Properties

Definition Let l be a line on the plane. The reflection by l maps any point A to a point A' such that l is the perpendicular bisector of AA' (Figure 7).

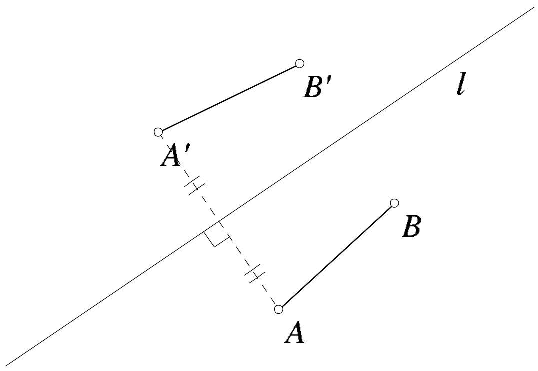

Figure 7: Reflection by l

Properties

1. Size and shape of any figure remain unchanged under reflection.
2. The points that remain fixed lies on l.
3. A line L maps to a line L'. If L is not parallel to l and meet at a point P, then L' passes through P and l is the angle bisector of the angle formed by L and L'. L is parallel to l, then L' is also parallel to l and l is equidistance from L and L'.
4. The only fixed line under reflection is l itself.
5. reflection + reflection = rotation or translation (?)

3.2 Examples

1. Let A, B be two distinct points on the same side of a line l, not lying on the line. Find the point P on l such that
$$AP + BP$$
is minimum.

   Solution. We start by reflecting B with respect to l to B'. For any point P on l, we have $BP = B'P$. Hence
$$AP + BP = AP + B'P \geq AB',$$
by triangular inequality. For the minimum value, the position of P show be on the line segment AB'. In other words, if M, N be the feet of perpendicular from A, B to l, then P is a point on the line segment MN such that
$$\frac{MP}{PN} = \frac{AM}{BN}.$$ □

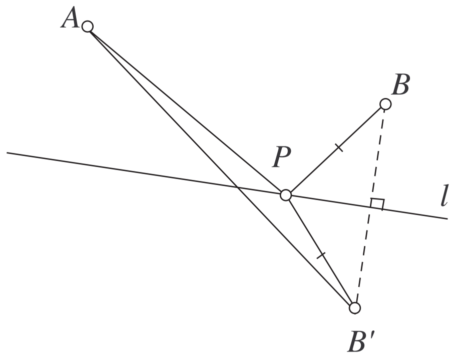

Figure 8: Shortest Path from A to B

2. (Fagnano's Problem) In an acute-angled triangle ABC, let X, Y, Z be points on the line segments CA, AB, BC respectively. Find the position of X, Y, Z such that the perimeter of $\triangle XYZ$

$$XY + YZ + ZX$$

is minimum.

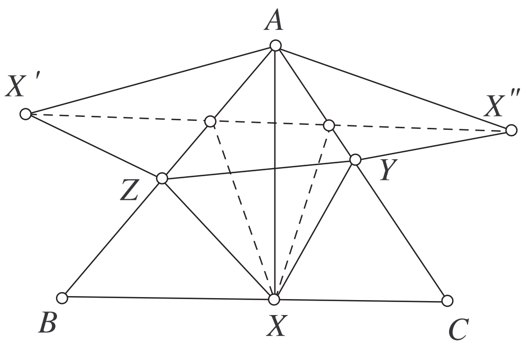

Figure 9: Fagnano's Problem

Solution. We consider the reflection X with respect to AB and AC to X' and X'' respectively. Then the perimeter

$$XY + YZ + ZX = X'Y + YZ + ZX'' \geq X'X''$$

Note that

$$\angle X'AX'' = \angle X'AX + \angle XAX'' = 2\angle BAX + 2\angle XAC = 2\angle BAC = 2A < 180^o$$

Hence $X'X'' = AX cosA$. In order for $XY + YZ + ZX$ to be minimum, we need AX to be minimum and $X'YZX''$ be a straight line. So the position of X should be at the foot of perpendicular from A to BC, i.e. AX is an altitude. For $X'YZX''$ to be a straight line, note that A is the center of the circle passing through X, X', X'', we have

$$\angle ZXX' = \angle ZX'X = \angle X''X'X = \frac{1}{2}\angle X''AX = \angle CAX = 90^o - C.$$

Also

$$\angle X'XB = 90^o - B.$$

SO

$$\angle ZXB = \angle ZXX' + \angle X'XB = (90^o - C) + (90^o - B) = 180^o - B - C = A.$$

SO $AZXC$ is concyclic and so

$$\angle AZC = \angle AXC = 90^o.$$

Thus CZ is also an altitudes of triangle ABC. Similarly, BY is an altitudes also. Hence for the perimeter to be minimum, XYZ is the orthic triangle of $\triangle ABC$. □

3. (Dutch Math Olympiad 2005) On a billiard-table in the shape of a regular hexagons $ABCDEF$ of side 4, one pushes a ball from the midpoint of the side ED. It hits subsequently the sides FA, AB and BC, before returning to the midpoint of ED. What is the length of its path?

3.3 Composition of Reflections

Exploration What is the composition of two reflections? Let l_1, l_2 be the lines of the two reflections. Hints: Consider all the cases:

1. l_1 and l_2 are parallel.
2. l_1 and l_2 are not parallel and intersect at P.

Make use of the results to determine the composition of rotations.

4 Homotheties (= Dilations)

4.1 Definition and Properties

Definition Let O be a point on the plane and r be a nonzero real number. A homothety maps a point A to another point A' such that OAA' is a straight line and

$$\frac{OA'}{OA} = r.$$

O and r are called the **center** and the **ratio** of homothety respectively (Figure 10).

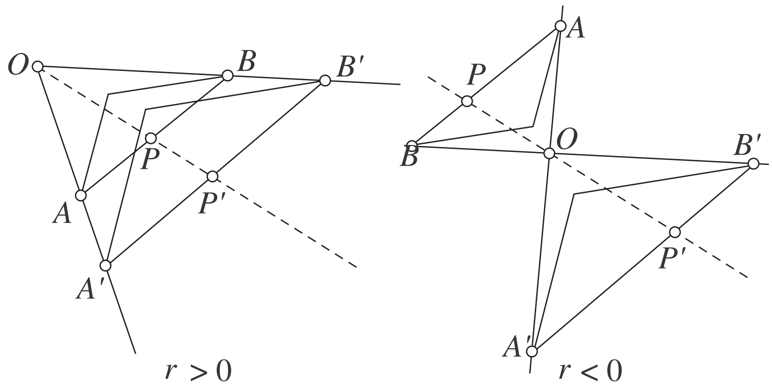

Figure 10: Homotheties with center O and ratio r

Properties

1. $\dfrac{A'B'}{AB} = r.$
2. $AB//A'B'$.

In order to check that two similar figures are homothetic to each other, it suffices to check that one pair of corresponding lines $AB, A'B'$ are parallel. Then the intersection of AA', BB' will be the center of homothety O. And for each corresponding point P, P' in the figures, we have

$$\frac{OP}{OP'} = \frac{OA}{OA'} = \frac{AB}{A'B'} = r.$$

4.2 Examples

Centroid and Medial Triangle In a triangle $A'B'C'$, denote A', B', C' the midpoints of BC, CA, AB respectively. Then $\triangle ABC$ and $\triangle A'B'C'$ are homothetic. The center of homothety is G and the ratio of homothety is $-\frac{1}{2}$. Hence we have the following results.

1. The median trisect each other, $AG : GA' = BG : GB' = CG : GC' = 2 : 1$.
2. Since the circumcenter O of $\triangle ABC$ is the orthocenter of $\triangle A'B'C'$, we have H, G, O is collinear and $HG : GO = 2 : 1$, $HA : OA' = 2 : 1$.

Nine-point Circle As before, in triangle ABC, let H be the orthocenter, O be the circumcenter, A', B', C', X, Y, Z be the midpoints on BC, CA, AB, HA, HB, HC, D, E, F be the feet of perpendicular. We need the following properties (Figure 11).

1. Extend AD, BE, CF to cut the circumcircle at D', E', F', we have
$$HD = DD', \quad HE = EE', \quad HF = FF'.$$

2. Extend HA', HB', HC' to cut the circumcircle at A'', B'', C'', we have
$$HA' = A'A'', \quad HB' = B'B'', \quad HC' = C'C''.$$

3. Of course we have
$$HX = XA, \quad HY = YB, \quad HZ = ZC.$$

Now, if we consider shrink the circumcircle with center of homothety at H and ratio $\frac{1}{2}$, the points
$$A, B, C, D', E', F', A'', B'', C''$$
on the circumcircle will shrink to the points
$$X, Y, Z, D, E, F, A', B', C'$$
respectively. Hence we have the nine-point circle and the center N of the nine-point circle is exactly the midpoint of OH, also lying on the Euler line! Consequently, the radius of the nine-point circle is $\frac{1}{2}R$.

4.3 Exercises

1. In $\triangle ABC$, let A', B', C' be the midpoints of BC, CA, AB respectively. Let S be the incenter of the medial triangle $A'B'C'$. Show that S, G, I are concurrent and $IG : GS = 2 : 1$ where G and I are the centroid and the incenter of $\triangle ABC$.

2. In $\triangle ABC$, let A', B', C' be the midpoints of BC, CA, AB respectively. Let X be a point on one of the sides of $\triangle ABC$, such that $A'X$ bisects the perimeter of $\triangle ABC$. Similarly, $B'Y$ and $C'Z$ also bisects the perimeter of $\triangle ABC$. with Y, Z lying on the sides. Show that the three lines $A'X, B'Y, C'Z$ are concurrent.

3. In $\triangle ABC$, let P, Q, R be the points where the sides BC, CA, AB are tangent to the corresponding excircles.
   (a) Show that AP, BQ, CR are concurrent.
   (b) Let J be the point of concurrency. And denote G and I the centroid and incenter of $\triangle ABC$. Show that $JG : GI = 2 : 1$.

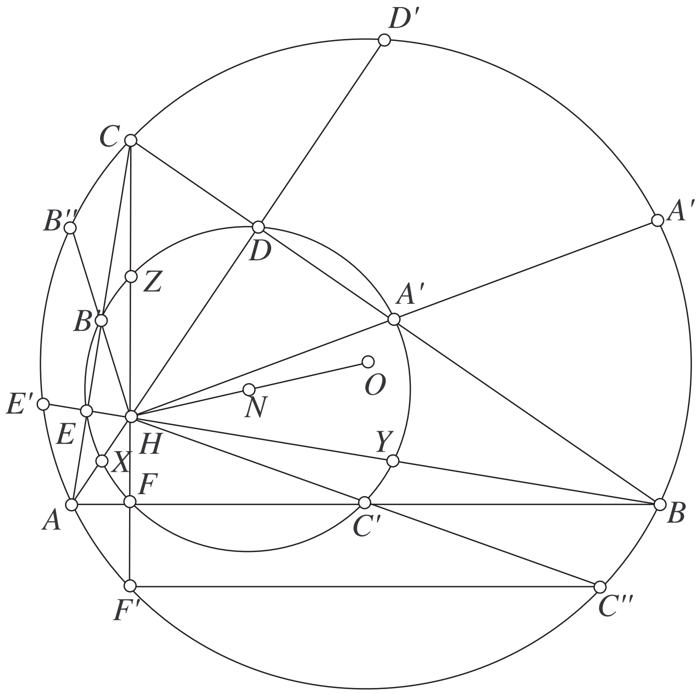

Figure 11: Nine-point Circle

5 Spiral Similarities (= Rotation + Dilation)

5.1 Definition and Properties

Definition Let O, A, A' be three distinct points on the plane and they are not lying on a line. A spiral similarity sends A to A' defined by the following composition of transformations: first rotates A to $\tilde{A}$ about O by $\angle A'OA$, and then dilates $\tilde{A}$ from O to A' (Figure 12).

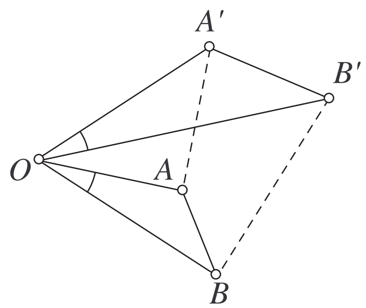

Figure 12: Spiral Similarity

Note that O,$\angle A'OA$ and OA'/OA are called the **center**, the **angle** and the **ratio** of the spiral similarity.

Properites

1. $AB : A'B' = OA : OA'$.
2. The angle bewteen AB and AB' equals $\angle A'OA$.

5.2 Examples

Ptolemy Theorem In a convex quadrilateral $ABCD$, we have

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD,$$

and equality holds if and only if $ABCD$ is concyclic.

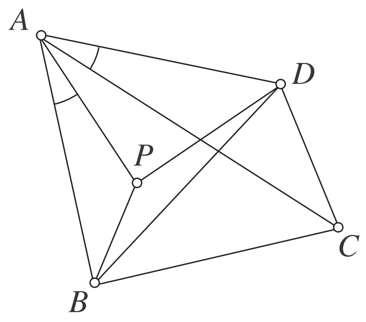

Figure 13: Ptolemy Theorem

Proof We consider a spiral similarity mapping AC to AB. We denote the the image of D by P, that is

$$\triangle ADC \sim \triangle APB.$$

By the second properties of spiral similarity, we have another pair of similar triangles

$$\triangle ABC \sim \triangle ABD.$$

From these pairs of similar triangle, we have

$$BP \cdot AC = AB \cdot CD, \quad PD \cdot AC = BC \cdot AD.$$

But the sum of BP and PD is greater than BD by triangular inequality. Hence

$$AB \cdot CD + BC \cdot DA = (BP + PD) \cdot AC \geq BD \cdot AC.$$

And equality holds if and only if P lies on BD. That is essentially

$$\angle ACD = \angle ABD,$$

which is equivalent to $ABCD$ being concyclic. $\square$

5.3 Exercises

Exploration Let $AB, A'B'$ be two line segments that are not equal in length and not parallel. Can you view this as a spiral similarity? Where is the center? What is the ratio? What is the angle?

1. Prove that the Napolean Triangle is equilateral by Spiral Similarity.

2. (IMO1994/2) ABC is an isosceles triangle with $AB = AC$. Suppose that
   (a) M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;
   (b) Q is an arbitrary point on the segment BC different from B and C;
   (c) E lies on the line AB and F lies on the line AC such that E, Q and F are collinear.
   Prove that OQ is perpendicular to EF if and only if $QE = QF$.

3. (IMO1996/2) Let P be a point inside triangle ABC such that

   $$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

   Let D, E be the incenters of triangles APB, APC respectively. Show that AP, BD and CE meet at a point.

List of Figures

References

[Yag62] I.M. Yaglom. *Geometric Transformations I,II,III.* the Mathematical Association of America, 1962.

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