Functional Equations

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More than anything, functional equations test persistence. As the solver, it is your job to keep trying unique approaches until something finally works. There are many ideas to look for, here are some suggestions:

- **Interesting numbers.** Determine $f(0)$ and $f(1)$, and consider

  $$\{x \mid f(x) = 0\} \quad \text{and} \quad \{y \mid f(y) = 1\}.$$

  Are other numbers such as primes useful? In particular, what numbers have nice properties: $S = \{x \mid f(x) = x\}, T = \{x \mid f(x) \neq 0\}$, etc.

- **Useful range.** The easiest way to make f go away is to discover that the range of f is an interesting set, such as the domain of f, or the nonnegative numbers in the domain, etc.

- **Induction.** The typical way to get to $f(\mathbb{Q})$. Need I say more?

- **Cauchy.** Under what conditions does $f(x+y) = f(x) + f(y)$? What about $f(xy) = f(x)f(y)$?

- **Monotonicity.** The usual manner in which $f(\mathbb{Q})$ determines $f(\mathbb{R})$. It's *much* easier to conclude directly that $x > y$ implies $f(x) \geq f(y)$ than it is to conclude directly that for all x and $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x-y)| < \epsilon$ whenever $|x-y| < \delta$. Look for squeeze arguments rather than continuity based arguments.

- **Fibres.** What can be said of p and q (and f) if $f(p) = f(q)$?

- **Recursion.** Apply f to both sides of an equality and force derived results back into the original equation.

- **New function.** Introduce a suitable function in terms of the original function.

- **Symmetry.** Interchange the roles of the variables.

- **Finite candidates.** Check whether one has $f(x) \in \{f_1(x), f_2(x), \ldots, f_k(x)\}$ for all x, where $f_1, \ldots, f_k$ are known possibilities.

- **Block building.** Try to extend a property of an interval of numbers to a larger interval.

Many of these are from shortlists and should be familiar to you.

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

2. Let $f : [0,1] \to \mathbb{R}$ be a function such that

   (a) $f(1) = 1$,
   (b) $f(x) \geq 0$ for all $x \in [0,1]$,
   (c) if x, y and $x + y$ all lie in $[0,1]$, then $f(x+y) \geq f(x) + f(y)$.

   Prove that $f(x) \leq 2x$ for all $x \in [0,1]$.

*with apologies to Reid.

3. Let $n > 2$ be an integer and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for any regular n-gon $A_1 A_2 \cdots A_n$,

$$f(A_1) + f(A_2) + \cdots + f(A_n) = 0.$$

Prove that f is the zero function.

4. Find all polynomials $p(x)$ such that for all x,

$$(x - 16)p(2x) = 16(x - 1)p(x).$$

5. Find all functions $f : \mathbb{R} \to [0, \infty)$ such that for all $x, y \in \mathbb{R}$,

$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy).$$

6. For which α does there exist a nonconstant function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\alpha(x + y)) = f(x) + f(y)$ for all real numbers x, y?

7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality $f(f(x) + y) = f(x^2 - y) + 4f(x)y$ holds for all pairs of real numbers x, y.

8. Let $\mathbb{R}^+$ denote the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x)f(y) = 2f(x + yf(x))$$

for all positive real numbers x and y.

9. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x + y) + f(x)f(y) = f(xy) + 2xy + 1$$

for all real numbers x and y.

10. Find all polynomials $P(x)$ with real coefficients which satisfy the equality

$$P(a - b) + P(b - c) + P(c - a) = 2P(a + b + c)$$

for all triples a, b, c of real numbers such that $ab + bc + ca = 0$.

11. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions:

(i) $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$

(ii) $f(x) < f(y)$ for all $1 \le x < y$.

12. Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0, f(1) = 1$, and

$$f(a) + f(b) = f(a)f(b) + f(a + b - ab)$$

for all real numbers a, b with $a < 1 < b$.

13. Find all functions f from the reals to the reals such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all real x, y.

14. Find all functions f from the reals to the reals such that

$$(f(x)+f(z))(f(y)+f(t)) = f(xy-zt)+f(xt+yz)$$

for all real x, y, z, t.

15. Let T denote the set of all ordered triples (p,q,r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ such that $f(p,q,r) = 0$ if $pqr = 0$ and

$$\begin{aligned} f(p,q,r) &= 1 + \frac{1}{6}(f(p+1,q-1,r) + f(p-1,q+1,r) \\ &\quad + f(p-1,q,r+1) + f(p+1,q,r-1) + f(p,q+1,r-1) + f(p,q-1,r+1)) \end{aligned}$$

otherwise.

16. Find all functions $f : \mathbb{R} \to \mathbb{R}$, satisfying

$$f(xy)(f(x)-f(y)) = (x-y)f(x)f(y)$$

for all x, y.

17. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x-f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

18. Find all pairs of functions $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+g(y)) = xf(y) - yf(x) + g(x)$$

for all $x, y \in \mathbb{R}$.

19. (a) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \quad \text{and} \quad g(f(x)) = x^3 \quad \text{for all } x \in \mathbb{R}?$$

(b) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \quad \text{and} \quad g(f(x)) = x^4 \quad \text{for all } x \in \mathbb{R}?$$

20. Find all pairs of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} &f(x) < f(y) \text{ for } x < y; \\ &f(xy) = g(y)f(x) + f(y) \text{ for all } x, y \in \mathbb{R}. \end{aligned}$$

21. Find all functions $u : \mathbb{R} \to \mathbb{R}$ for which there exists a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x)u(y) + f(y) \quad \text{for any } x, y \in \mathbb{R}.$$

22. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all $x, y \in \mathbb{R}$,

$$f(x^3+y^3) = (x+y)(f(x)^2 - f(x)f(y) + f(y)^2).$$

Prove that for all $x \in \mathbb{R}$, $f(1996x) = 1996f(x)$.

23. Let $\mathbb{R}^+$ be the set of positive real numbers. Prove that there does not exist a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$f(x)^2 \geq f(x+y)(f(x)+y) \quad \text{for any}\, x, y \in \mathbb{R}^+.$$

24. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying

$$\begin{aligned}&\text{(a) For every}\, n \in \mathbb{N}, f(n+f(n)) = f(n).\\ &\text{(b) For some}\, n_0 \in \mathbb{N}, f(n_0) = 1.\end{aligned}$$

Show that $f(n) = 1$ for all $n \in \mathbb{N}$.

25. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ which satisfy $f(m+f(n)) = f(m) + n$ for all $m, n \in \mathbb{Z}$.

26. Let S denote the set of nonnegative integers. Find all functions $f : S \to S$ such that
$$f(m+f(n)) = f(f(m)) + f(n) \quad \text{for all}\, m, n \in S.$$

27. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$,
$$f(x+1) = f(x) + 1 \quad \text{and} \quad f(x^2) = f(x)^2.$$

28. For which integers k does there exist a function $f : \mathbb{N} \to \mathbb{Z}$ such that $f(1995) = 1996$ and
$$f(xy) = f(x) + f(y) + kf(\gcd(x, y))\, \text{for all}\, x, y \in \mathbb{N}?$$

29. Let S denote the set of nonnegative integers. Find a bijective function $f : S \to S$ such that for all $m, n \in S$,
$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$

30. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,
$$f(n) + f(n+1) = f(n+2)f(n+3) - 1996.$$

31. Determine the least possible value of $f(1998)$ where f is a function from the set of positive integers to itself such that for all positive integers m, n,
$$f(n^2 f(m)) = m(f(n))^2.$$

32. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation
$$f(x^2 + y^2 + 2f(xy)) = (f(x+y))^2$$

for all $x, y \in \mathbb{R}$.