The Butterfly Problem

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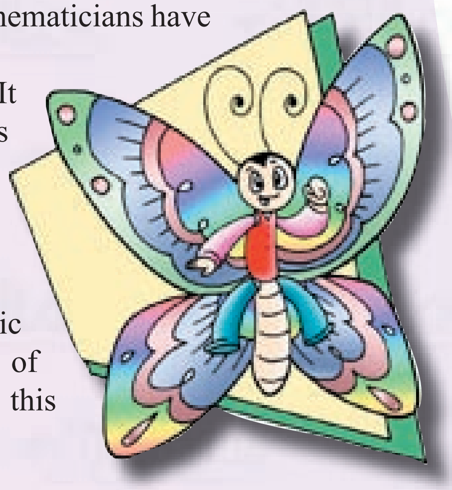
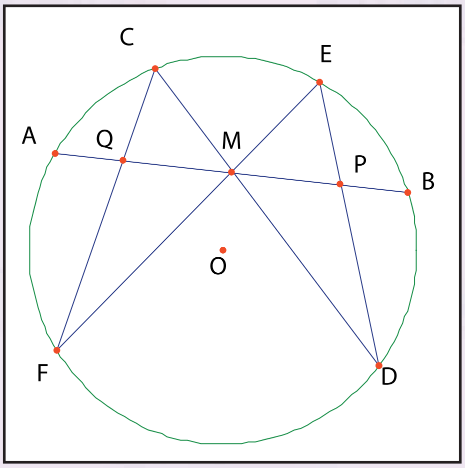
The *Butterfly problem* is one of the geometry problems that have a history dating back to 1815, when two solutions of the problem were published in *Gentleman's Diary* of 1815, a British publication that was instrumental in popularizing mathematics during the eighteenth and nineteenth centuries. Since then, a number of mathematicians have come out with different versions of the solution to the problem. Some of the solutions to the butterfly problem are very interesting. It uses high school geometry concepts and integrates several notions at once. In this article, we give one solution to the Butterfly problem—the solution presented by Horner in the *Gentleman's Diary* and Antonio Gutierrez in the *Butterfly Theorem Puzzle*. (see http://agutie.homestead.com/files/GeometryButterfly.html).



The solution uses geometric concepts, such as cyclic quadrilaterals, and the solution includes the construction of perpendiculars. We will also guide you to one more solution to this interesting problem in the worksheet.

The butterfly problem is given as follows:

Butterfly Problem. In a given circle O, M is the midpoint of chord AB. Chords, CD and EF pass through M. ED cuts AB in P and CF cuts AB in Q. Prove that $\overline{QM} = \overline{PM}$.



The problem is called the butterfly problem because the figure suggests two wings of a butterfly, formed by two pairs of chords on the circle.

We present the solution of the butterfly problem as follows:

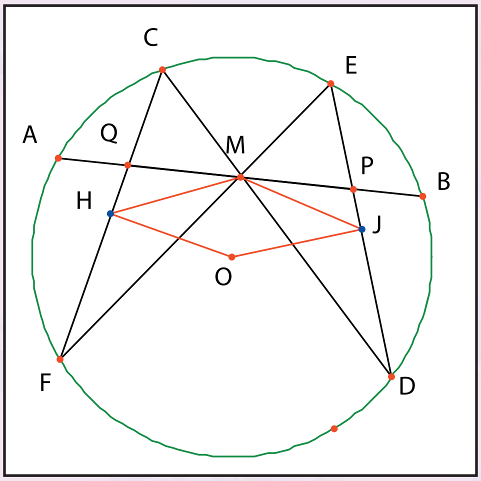
$\angle FCD \cong \angle FED$ and $\angle CFE \cong \angle EDC$ since the pairs of angles $\angle FCD$ and $\angle FED$, $\angle CFE$ and $\angle EDC$ are inscribed in the same arc and angles inscribed in the same arc are congruent.

Consider ΔFCM and ΔDEM.

Since $\angle FCM \cong \angle DEM$, $\angle CFM \cong \angle EDM$, then, by the AA Similarity result, $\Delta FCM \sim \Delta\, DEM$. Thus, their corresponding sides are proportional, that is,

$$\frac{\overline{FC}}{\overline{CM}} = \frac{\overline{DE}}{\overline{EM}}.$$

From the center of the circle O, drop a perpendicular to FC at H and DE at J.



Since OH and OJ are perpendicular to FC and DE respectively, then OH bisects FC and OJ bisects DE. (Any perpendicular from the center of a circle to a chord bisects the chord).

That is,

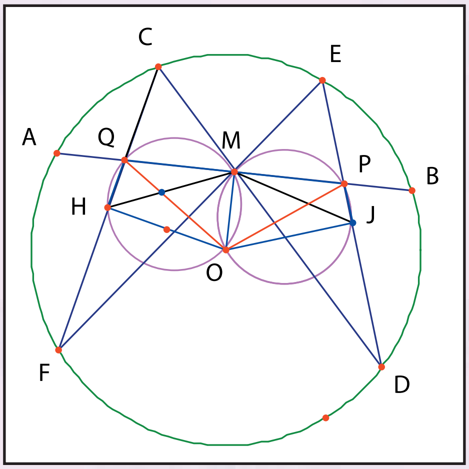
$$\overline{FC} = 2\overline{CH} \text{ and } \overline{DE} = 2\overline{EJ}.$$

Now, $\dfrac{\overline{FC}}{\overline{CM}} = \dfrac{\overline{DE}}{\overline{EM}}$ implies $\dfrac{2\overline{CH}}{\overline{CM}} = \dfrac{2\overline{EJ}}{\overline{EM}}$ or $\dfrac{\overline{CH}}{\overline{CM}} = \dfrac{\overline{EJ}}{\overline{EM}}$.

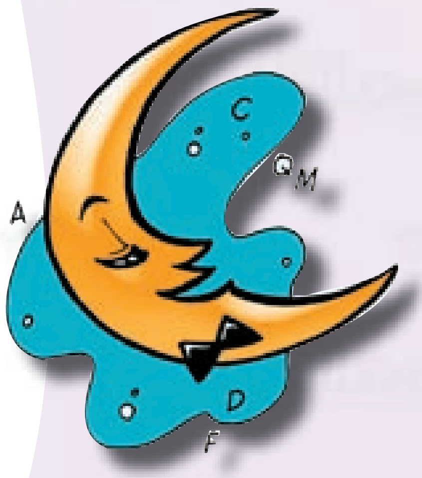
Since $\frac{\overline{CH}}{\overline{CM}} = \frac{\overline{EJ}}{\overline{EM}}$ and $\angle FCM \cong \angle DEM$, then, it follows that $\Delta HCM \sim \Delta JEM$ by the SAS Similarity theorem.

Now, connect O to M. Since O is the center of the circle and M is the midpoint of AB, then OM is perpendicular to AB. (The segment joining the center to the midpoint of a chord is perpendicular to the chord).

Consider the quadrilaterals $OMQH$ and $OMPJ$.



Since, $m\angle OMQ + m\angle OHQ = 180^0$ and $m\angle OMP + m\angle OJP = 180^0$, then the quadrilaterals $OMQH$ and $OMPJ$ are cyclic. (If opposite angles of a quadrilateral add up to 180^0 then the quadrilateral is cyclic.)



Observe that $\angle QHM$ and $\angle QOM$ intercept same arc QM, $\angle PJM$ and $\angle POM$ intercept same arc PM. Thus, $\angle QHM$ @ $\angle QOM$ and $\angle PJM \cong \angle POM$.

However, since $\Delta HCM \sim \Delta JEM$, then, $\angle QHM \cong \angle PJM$. Hence, $\angle QOM \cong \angle POM$.

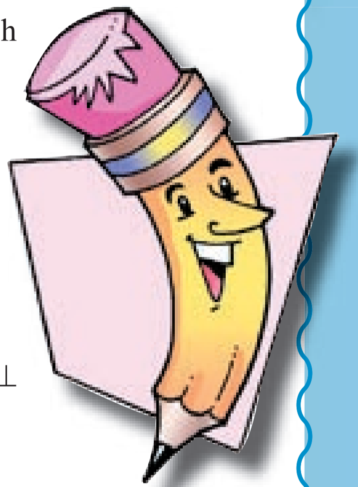
Finally, consider ΔOMQ and ΔOMP. Note that $\angle QOM \cong \angle POM$, $\angle QMO \cong \angle PMO$ with OM the common side. By the ASA Congruence theorem, $\Delta OMQ \cong \Delta OMP$.

Since corresponding sides of congruent triangles are congruent, then

$$QM \cong PM, \text{ or } \overline{QM} = \overline{PM}.$$

Activity

Worksheet

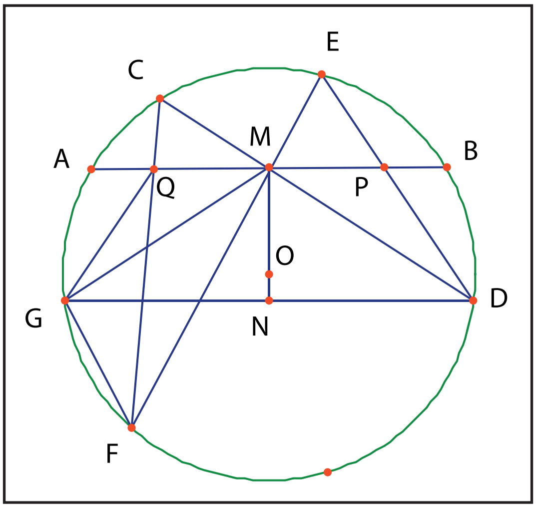


We will guide you through another solution in the worksheet. The approach for the solution is based on constructing parallel and perpendicular lines on the circle and applying notions on cyclic quadrilaterals. It is based on Posamentier's solution, given in *Teaching Secondary School Mathematics, Techniques and Enrichment Units,* 1995.

Instructions for this worksheet:

In the solution outlined below, fill in the blanks (labeled with letters **A-I)** with the appropriate geometric result that justifies the given statement.

Draw a line through D parallel to AB meeting the circle at G. Draw $MN \perp GD$. Then draw QG, MG, and FG as shown below.



$\angle GFC$ and $\angle GDC$ are angles inscribed in the same arc. Thus,

$$\angle GFC \cong \angle GDC. \qquad \textbf{(1)}$$

A. ____

Now, MG is a transversal to parallel lines AB and GD. So,

$$\angle QMG \cong \angle MGD. \qquad \textbf{(2)}$$

B. ____

Since $MN \perp GD$ and AB is parallel to GD, then $MN \perp AB$. Observe that M is the midpoint of AB.

C. ____

MN passes through the center O of the circle. Consequently, MN bisects GD.

D. ____

It follows that $GN \cong ND$.

Because $GN \cong ND$, $\angle GNM \cong \angle DNM$ with MN the common side, then by

E. ____

$\Delta GMN \cong \Delta DMN$. By the fact that corresponding parts of congruent triangles are congruent then,

$$GM \cong DM \text{ and } \angle MGD \cong \angle MDG. \qquad \textbf{(3)}$$

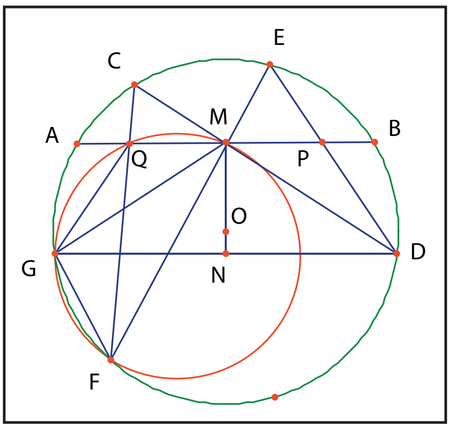
Combining (1), (2) and (3), the following are true:

$$\angle GFQ \cong \angle GDM$$
$$\angle MDG @ \angle MGD$$

$$\angle MGD \cong \angle QMG.$$

Thus,

$$\angle GFQ \cong \angle QMG. \qquad \textbf{(4)}$$



Now, consider quadrilateral $QMFG$. Since one side subtends congruent angles $\angle GFQ$ and $\angle QMG$ at the two opposite vertices, hence, quadrilateral $QMFG$ is cyclic by

F. ______________________________

With the new circle, $\angle QGM$ and $\angle QFM$ are angles inscribed in the same arc. Thus,

$$\angle QGM \cong \angle QFM. \qquad \textbf{(5)}$$

G. ______________________________

Moreover, since $\angle QFM$ and $\angle PDM$ are also angles inscribed in the same arc, then,

$$\angle QFM \cong \angle PDM. \qquad \textbf{(6)}$$

By transitivity on **(5)** and **(6)**,

$$\angle QGM \cong \angle PDM. \qquad \textbf{(7)}$$

From **(2)** and **(3)**

$$\angle QMG \cong \angle MDG. \qquad \textbf{(8)}$$

Now, with MD as transversal to parallel lines AB and GD,

$$\angle MDG \cong \angle PMD. \qquad \textbf{(9)}$$

H. ______________________________

Combining **(7)**, **(3)**, **(9)**, $\angle QGM$ @ $\angle PDM$, GM @ DM and $\angle QMG$ @$\angle PMD$ implies $\Delta QGM \cong \Delta PDM$.

I. ______________________________

Since corresponding sides of congruent triangles are congruent, then QM @ PM.

Hence, $\overline{QM} = \overline{PM}$ ..

ANSWERS

A. Angles inscribed in the same arc are congruent.
B. If two parallel lines are cut by a transversal, then any pair of alternate interior angles is congruent.
C. In the plane of a circle, the perpendicular bisector of a chord passes through the center.
D. Any perpendicular from the center of a circle to a chord bisects the chord.
E. SAS Congruence Theorem
F. If one side of the quadrilateral subtends congruent angles at the opposite vertices, then the quadrilateral is cyclic.
G. Angles inscribed in the same arc are congruent.
H. If two parallel lines are cut by a transversal, then any pair of alternate interior angles is congruent.
I. ASA Congruence Theorem