Junior N.T. Problem Set 1

1. Prove that $\gcd(21n+4, 14n+3) = 1$ for any integer n.

2. What is the largest positive integer n for which n^3+100 is divisible by $n+10$?

3. Let m and n be positive integers such that $\operatorname{lcm}(m,n) + \gcd(m,n) = m+n$

Prove that one of the two numbers is divisible by the other.

4. Find all primes p for which p^2+20 is also a prime.

5. Find all primes p and positive integer n such that $p^3+1 = n^2$.

6. If $a \equiv b \pmod{n}$, prove that $a^n \equiv b^n \pmod{n^2}$.

7. For coprime positive integers a, b, prove that $\dfrac{(a+b-1)!}{a!\,b!}$ is also an integer.

8. Prove that for integer $a, m, n \neq 0$, $\gcd(a^m-1, a^n-1) = a^{\gcd(m,n)} - 1$

(More generally, if $a^m \equiv b^m \pmod{c}$ and $a^n \equiv b^n \pmod{c}$, then

$a^{\gcd(m,n)} \equiv b^{\gcd(m,n)} \pmod{c}$[1])

9. Show that for any positive integers a and b, the number $(36a+b)(a+36b)$ can't be a perfect power of 2.

10. Given that

$$a \equiv b \pmod{c} \quad \text{and} \quad a \equiv b \pmod{d}$$

Prove that $a \equiv b \pmod{[c,d]}$ where $[c,d]$ denotes the lcm of c, d.

11. Prove that for all integers n,

i) $n^5 - 5n^3 + 4n$ is divisible by 120.

ii) $n^2 + 3n + 5$ is not divisible 121.

12. Suppose k is an odd positive integer. Prove that for all positive integer n,

$n(n+1)$ divides $2(1^k + 2^k + \ldots + n^k)$.

13. Find the greatest positive integer x such that 23^{6+x} divides 2000!

14. Find **an** integer n such that $n^3 \equiv 2 \pmod{101}$.

(Hint: $51^{100} \equiv 1 \pmod{101}$ and $99 = 33 \times 3$)

15. Let n be a non-negative integer such that $2^n = (n+1)^2$. Find all such n's.

16. p is an odd prime. Let k be an integer between 1 and $p-1$ i.e. $1 \leqslant k \leqslant p-1$.

1. The famous exponent gcd lemma. This is very much useful. So don't forget!

Prove that,

i) $p\left|\binom{p}{k}\right.$.

ii) $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$

17. Two sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ are defined in the following way:

$x_1 = 1,\ x_2 = 3,\ x_n = 3x_{n-1} - 2x_{n-2}$ for $n \geqslant 2$. And $y_n = \dfrac{x_n^2 + 1}{x_{n-1}}$ for $n \geqslant 2$. Find all integers n for which $n|y_n$.

18. b boys and 13 girls took part in a mathematical compitition. After the result, it was seen that all the students have got $b^2 + 10b + 17$ marks in total and their average mark is an integer. Determine at most how many boys took part in that compitition.

19. Find the largest divisor of 1001001001 that does not exceed 10000.

20. Prove that $\gcd(2^{21} + 1, 2^{12} + 1) = 1$

21. Let n be a positive integer and let a_1,a_2 ...a_k ($k \geqslant 2$) be distinct integers in the set $\{1, 2, 3....n\}$ such that n divides $a_i(a_{i+1} - 1)$ for i=1, 2, 3...$k - 1$. Prove that n does not divide $a_k(a_1 - 1)$.

22.[2]A function $f: \mathbb{N} \rightarrow \mathbb{N}$ satisfies the following conditions-

i) For any two primes $p, q,\ f(p + q) = f(p) + f(q)$

ii) For any two coprime integers $a, b,\ f(ab) = f(a) \cdot f(b)$

Prove $f(3) = 3$.

23.[3] Let p be an odd prime. Suppose k is any integer such that $1 \leqslant k \leqslant p - 2$. Prove that $p|1^k + 2^k + \ldots + (p - 1)^k$.

24. Let a, b, c be three integers such that $a < b$ and $a < c$. If $\dfrac{bc}{a}$ is an integer, prove that it must be composite.[4]

25. Suppose p is a prime and $p \not\equiv 1 \pmod{8}$. Let x, y be two integers. Prove that if $p|\, x^4 + y^4$, then $p|\, x$ and $p|\, y$.

(Hint: Use Fermat's Little Theorem and problem 8)

26. Let a, b, c, d are four positive integers such that $ab = cd$. Prove that $a + b + c + d$ is not a prime.

(Hint: Think about $\gcd(a, c)$ and $\gcd(b, d)$.)

27. a, b are two non-negative integers such that $2^a \equiv 2^b$ (mod 101). Prove that $a \equiv b \pmod{100}$.

2. It is from France team selection test 2000. The original problem asks to find $f(1999)$.

3. This result is often useful. If you can solve this, try IMO shortlist 1997/12.

4. Surprisingly this problem can be solved by induction on a. For details, see mathemetical excalibur V.12 No.1 P.4.

28. The sum of two positive integers is 5432 and their lcm is 223020. Find the smaller number.

29. Find all primes p such that $17p+1$ is a perfect square.

30. Suppose that $n>1$ is an integer. Prove that $n \mid \phi(2^n-1)$.

31. Determine all pairs (x,y) of positive integers such that $xy^2+y+7 \mid x^2y+x+y$.

32. Find all pairs (k,n) of positive integers for which $7^k-3^n \mid k^4+n^2$.

All problems are collected by

Adib Hasan

Class X, Mymensingh Zilla School

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