## Pascal's Theorem for Circles

Pascal's Theorem (discovered when he was 16 years old) applies to a hexagon inscribed in an arbitrary conic (circle, ellipse, parabola, hyperbola). An elementary approach (using Euclidean geometry) is to prove it for the circle. Then since any ellipse is a perspective image of a circle, and since the concepts "inscribed," "intersection" and "collinear" are invariant in perspective, it is also proved for an arbitrary ellipse.

The ingredients of the proof are Menelaus' Theorem, the "Secant Theorem" ("Power of a point") and the converse to Menelaus' Theorem.

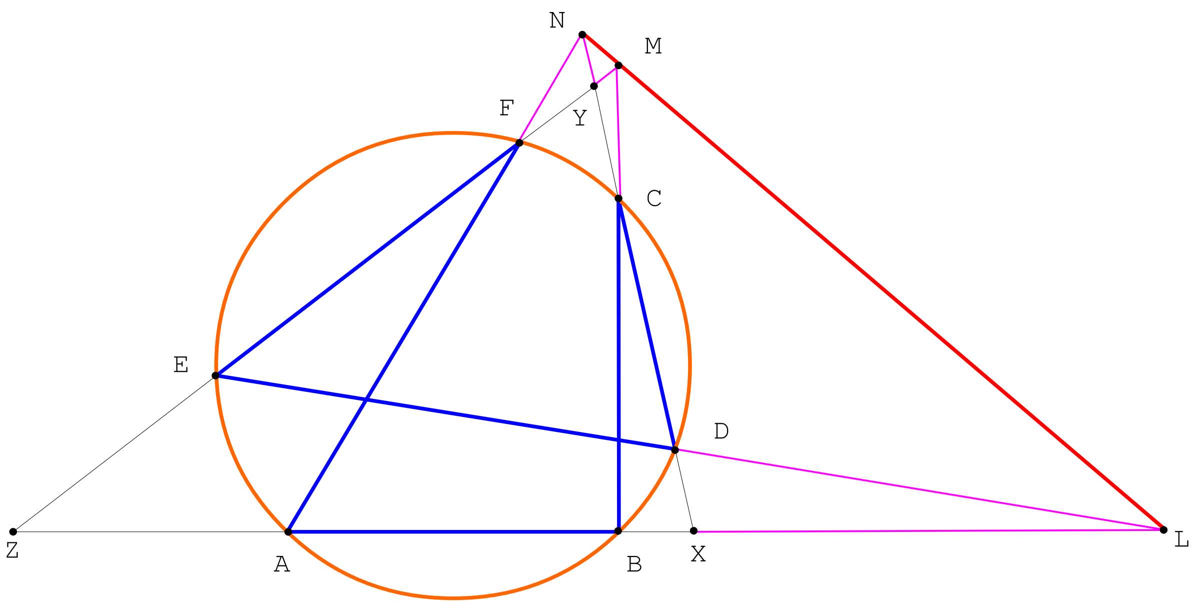
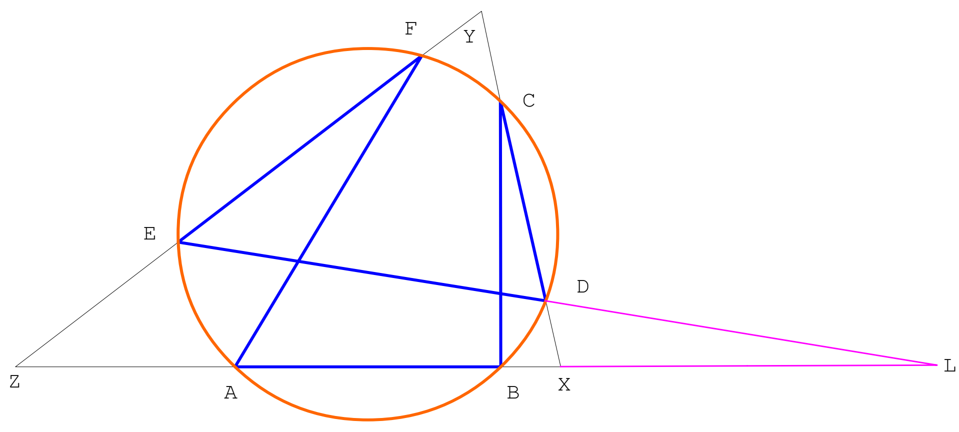



Figure 1: One possible configuration for Pascal's Theorem and its proof.

**Pascal's Theorem for circles.** If an arbirtary hexagon $ABCDEF$ is inscribed in a circle, the three intersection points of (the extensions of) opposite sides: $L = AB \cap DE$, $M = BC \cap EF$, $N = CD \cap FA$ are collinear.

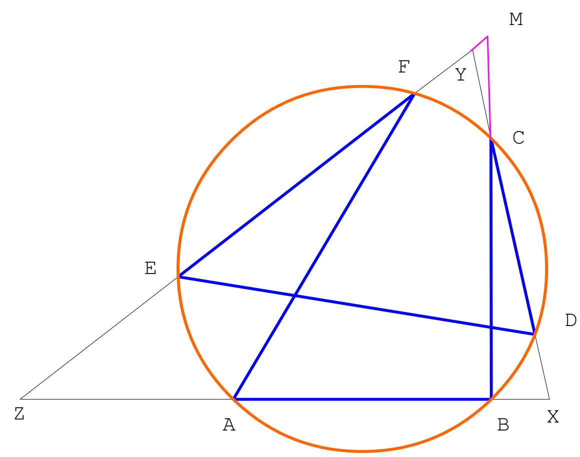
*Proof:* We will work with the triangle $XYZ$ formed by the extensions of the sides $AB, CD, EF$.

We first apply Menelaus' Theorem to the line $LDE$ and this triangle:



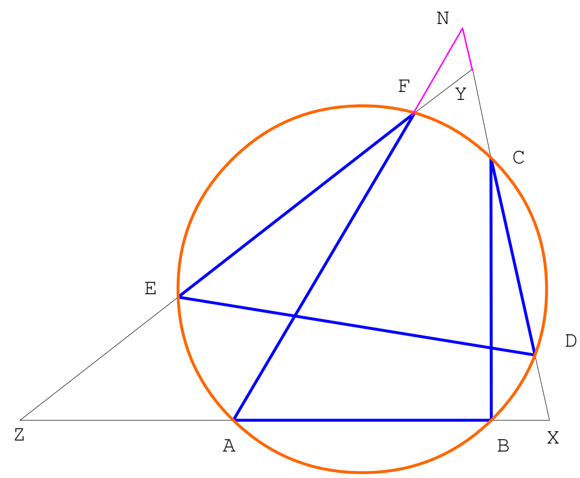

giving $\quad \dfrac{XD}{DY} \cdot \dfrac{YE}{EZ} \cdot \dfrac{ZL}{LX} = -1.$

Next we apply Menelaus' Theorem to the line $MCB$:




giving $\quad \dfrac{XC}{CY} \cdot \dfrac{YM}{MZ} \cdot \dfrac{ZB}{BX} = -1.$

And finally to line $NFA$:




giving $\quad \dfrac{XN}{NY} \cdot \dfrac{YF}{FZ} \cdot \dfrac{ZA}{AX} = -1.$

Multipying the three equations together gives

$$\frac{XD}{DY} \cdot \frac{YE}{EZ} \cdot \frac{ZL}{LX} \cdot \frac{XC}{CY} \cdot \frac{YM}{MZ} \cdot \frac{ZB}{BX} \cdot \frac{XN}{NY} \cdot \frac{YF}{FZ} \cdot \frac{ZA}{AX} = -1.$$

This equation can be simplified by applying the Secant Theorem to the points $X, Y, Z$ and our circle (figure on next page):

$$XD \cdot XC = XB \cdot XA, \quad YD \cdot YC = YF \cdot YE, \quad ZF \cdot ZE = ZB \cdot ZA.$$

(Notice that the minus signs $XA = -AX, XB = -BX$, etc. occur in pairs).

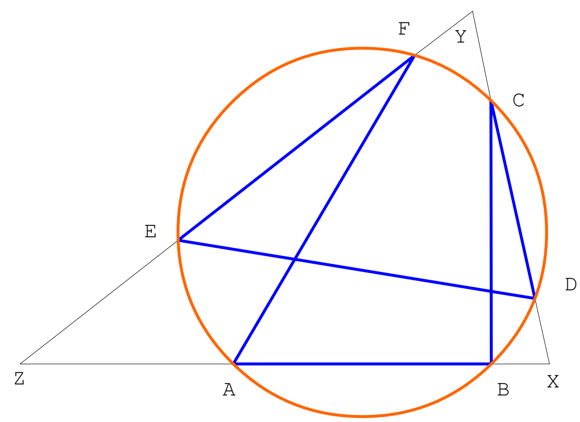
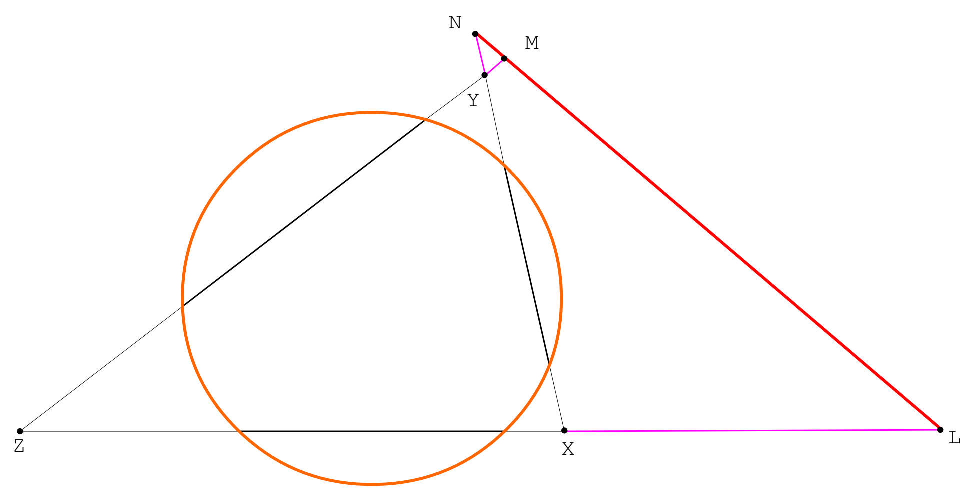



Figure 2: Three applications of the Secant Theorem.

What remains after cancellation is

$$\frac{ZL}{LX} \cdot \frac{YM}{MZ} \cdot \frac{XN}{NY} = -1;$$




By the converse to Menelaus' Theorem, applied to the triangle $XYZ$, the points $L, M, N$ are collinear. QED