Practice Exam

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Time: 5 hours

[*I tried to arrange them in ascending difficulty order. But it may not be the right order for you*]

1. Find the number of subsets of $\{1, 2, ..., n\}$ that doesn't contain any two successive numbers.

2. In a tournament with n players, everybody plays with everybody else exactly once. Prove that at any state of the tournament there are always two players who have played the same number of games.

3. Find all primes p, q so that $p^2 - 2q^2 = 1$.

4. In $\triangle ABC$, AD and BE are altitudes. F, G and K are the midpoints of AH, AB and BC, respectively. Prove that, $\angle FGK$ is a right angle.

5. n people are standing with every person having one gun, where n is odd. At a signal everyone shoots, just once, the person standing closest to him.(for any person, the rest $n-1$ are at different distances) Prove that after the n shots, at least person stays alive.

6. Prove that the line segment joining the midpoints of two opposite sides of any quadrilateral bisects the line segment joining the midpoints of the diagonals.

7. $2n$ points are given in space. Altogether $n^2 + 1$ line segments are drawn between these points. Show that there is at least one set of three points which are joined pairwise by line segments.

8. Start with the positive integers $1, 2, ..., 4n - 1$. In one move you may replace any two integers by their difference. Prove that an even integer will be left after $4n - 2$ steps.

9. PT is a tangent to circle ω and A is any point on that plane such that $PT = PA$. C is any point on ω, and AC and PC intersect ω at points D and B, respectively. AB intersects ω at E. Prove that $DE \parallel AP$.

10. There are n identical cars on a circular track. Together they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from the other cars on its way around.