Solutions to Exercises

1. The sequence (a_n) is defined by $a_0 = 0$, $a_1 = 1$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for all $n \ge 2$. Find the generating function of this sequence, and use it to solve for a_n.

   Solution. Let
   $$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots .$$
   Then
   $$\begin{aligned}(1-5x+6x^2)A(x) &= (1-5x+6x^2)(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= a_0 + (a_1 - 5a_0)x + (a_2 - 5a_1 + 6a_0)x^2 + (a_3 - 5a_2 + 6a_1)x^3 + \cdots \\ &= x,\end{aligned}$$
   so
   $$A(x) = \frac{x}{1-5x+6x^2} = \frac{x}{(1-2x)(1-3x)}.$$
   By the method of partial fractions, there exist constants C_1 and C_2 such that
   $$\frac{x}{(1-2x)(1-3x)} = \frac{C_1}{1-2x} + \frac{C_2}{1-3x}.$$
   Multiplying both sides by $(1-2x)(1-3x)$, we get
   $$x = C_1(1-3x) + C_2(1-2x) = (C_1 + C_2) - (3C_1 + 2C_2)x.$$
   Comparing the coefficients on both sides, we obtain the system of equations
   $$\begin{aligned} C_1 + C_2 &= 0, \\ 3C_1 + 2C_2 &= -1. \end{aligned}$$
   Solving this system of equations, we find $C_1 = -1$ and $C_2 = 1$, so
   $$\begin{aligned}A(x) &= \frac{1}{1-3x} - \frac{1}{1-2x} \\ &= (1 + 3x + 3^2x^2 + \cdots) - (1 + 2x + 2^2x^2 + \cdots) \\ &= (1-1) + (3-2)x + (3^2 - 2^2)x^2 + \cdots .\end{aligned}$$
   Therefore, the coefficient of x^n in $A(x)$ is given by $a_n = 3^n - 2^n$.

2. The sequences (a_n) and (b_n) are defined by $a_0 = 1$, $b_0 = 0$, and
   $$\begin{aligned} a_n &= 3a_{n-1} + 4b_{n-1}, \\ b_n &= 2a_{n-1} + 3b_{n-1} \end{aligned}$$
   for all $n \ge 1$. Find the generating functions of these sequences, and use them to solve for a_n and b_n.

Solution. Let

$$A(x) = a_0 + a_1x + a_2x^2 + \cdots,$$
$$B(x) = b_0 + b_1x + b_2x^2 + \cdots.$$

Then

$$xA(x) = a_0x + a_1x^2 + a_2x^3 + \cdots,$$
$$xB(x) = b_0x + b_1x^2 + b_2x^3 + \cdots,$$

so

$$A(x) - 3xA(x) - 4xB(x) = a_0 + (a_1 - 3a_0 - 4b_0)x + (a_2 - 3a_1 - 4b_1)x^2 + \cdots = 1,$$
$$B(x) - 2xA(x) - 3xB(x) = b_0 + (b_1 - 2a_0 - 3b_0)x + (b_2 - 3a_1 - 4b_1)x^2 + \cdots = 0.$$

We can view these equations as a system of equations in $A(x)$ and $B(x)$. Solving for $A(x)$ and $B(x)$, we find

$$A(x) = \frac{1-3x}{1-6x+x^2}, \quad B(x) = \frac{2x}{1-6x+x^2}.$$

Let

$$1 - 6x + x^2 = (1-\alpha x)(1-\beta x) = 1 - (\alpha+\beta)x + (\alpha\beta)x^2,$$

so $\alpha+\beta = 6$ and $\alpha\beta = 1$. By Vieta's Formulas, α and β are the roots of the quadratic $t^2 - 6t + 1 = 0$. By the quadratic formula, the roots of this quadratic are $3 \pm 2\sqrt{2}$, so let $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$.

By the method of partial fractions, we get

$$\begin{aligned} A(x) &= \frac{1-3x}{1-6x+x^2} \\ &= \frac{1-3x}{(1-\alpha x)(1-\beta x)} \\ &= \frac{1/2}{1-\alpha x} + \frac{1/2}{1-\beta x} \\ &= \frac{1}{2}(1 + \alpha x + \alpha^2x^2 + \cdots) + \frac{1}{2}(1 + \beta x + \beta^2 x^2 + \cdots) \\ &= 1 + \frac{\alpha+\beta}{2}x + \frac{\alpha^2+\beta^2}{2}x^2 + \cdots, \end{aligned}$$

so the coefficient of x^n in $A(x)$ is

$$a_n = \frac{\alpha^n + \beta^n}{2} = \frac{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n}{2}.$$

Similarly,

$$\begin{aligned}
B(x) &= \frac{2x}{1-6x+x^2} \\
&= \frac{2x}{(1-\alpha x)(1-\beta x)} \\
&= \frac{1/(2\sqrt{2})}{1-\alpha x} - \frac{1/(2\sqrt{2})}{1-\beta x} \\
&= \frac{1}{2\sqrt{2}}(1+\alpha x+\alpha^2 x^2+\cdots) - \frac{1}{2\sqrt{2}}(1+\beta x+\beta^2 x^2+\cdots) \\
&= \frac{\alpha-\beta}{2\sqrt{2}}x + \frac{\alpha^2-\beta^2}{2\sqrt{2}}x^2+\cdots,
\end{aligned}$$

so the coefficient of x^n in $B(x)$ is

$$b_n = \frac{\alpha^n-\beta^n}{2\sqrt{2}} = \frac{(3+2\sqrt{2})^n-(3-2\sqrt{2})^n}{2\sqrt{2}}.$$

3. Let

$$\frac{1}{1+x+x^2+x^3+x^4+x^5+x^6} = a_0+a_1x+a_2x^2+\cdots.$$

Find a_n.

Solution. We can write

$$\begin{aligned}
\frac{1}{1+x+x^2+x^3+x^4+x^5+x^6} &= \frac{1-x}{1-x^7} \\
&= (1-x)(1+x^7+x^{14}+x^{21}+\cdots) \\
&= 1-x+x^7-x^8+x^{14}-x^{15}+x^{21}-x^{22}+\cdots,
\end{aligned}$$

so

$$a_n = \begin{cases} 1 & \text{if } n\equiv 0 \pmod 7, \\ -1 & \text{if } n\equiv 1 \pmod 7, \\ 0 & \text{otherwise.} \end{cases}$$

4. For a positive integer n, let a_n denote the the number of permutations π of the numbers 1, 2, $\dots$, n, such that $|\pi(i)-i|\le 2$ for all $1\le i\le n$. The generating function of the sequence (a_n) is given by

$$\frac{1-x}{1-2x-2x^3+x^5}.$$

Using this generating function or otherwise, find a_8.

Solution. We have that

$$\frac{1-x}{1-2x-2x^3+x^5} = a_0+a_1x+a_2x^2+a_3x^3+\cdots.$$

Multiplying both sides by $1 - 2x - 2x^3 + x^5$, we get

$$\begin{aligned}1 - x &= (1 - 2x - 2x^3 + x^5)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \\ &= a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1)x^2 + (a_3 - 2a_2 - 2a_0)x^3 \\ &\quad + (a_4 - 2a_3 - 2a_1)x^4 + (a_5 - 2a_4 - 2a_2 + a_0)x^5 + (a_6 - 2a_5 - 2a_3 + a_0)x^6 + \cdots\end{aligned}$$

Comparing the coefficients on both sides, we obtain the equations

$$\begin{aligned}a_0 &= 1, \\ a_1 - 2a_0 &= -1, \\ a_2 - 2a_1 &= 0, \\ a_3 - 2a_2 - 2a_0 &= 0, \\ a_4 - 2a_3 - 2a_1 &= 0,\end{aligned}$$

and

$$a_n - 2a_{n-1} - 2a_{n-3} + a_{n-5} = 0$$

for all $n \geq 5$. Then

$$\begin{aligned}a_1 &= 2a_0 - 1 = 1, \\ a_2 &= 2a_1 = 2, \\ a_3 &= 2a_2 + 2a_0 = 6, \\ a_4 &= 2a_3 + 2a_1 = 14, \\ a_5 &= 2a_4 + 2a_2 - a_0 = 31, \\ a_6 &= 2a_5 + 2a_3 - a_2 = 73, \\ a_7 &= 2a_6 + 2a_4 - a_3 = 172, \\ a_8 &= 2a_7 + 2a_5 - a_4 = 400.\end{aligned}$$

5. Find the generating function of the sequence $a_n = n^3$.

Solution. Let

$$A(x) = \sum_{n=0}^{\infty} n^3 x^n = 0^3 + 1^3 x + 2^3 x^2 + 3^3 x^3 + \cdots.$$

Then

$$\begin{aligned}(1 - x)A(x) &= (1 - x)(0^3 + 1^3 x + 2^3 x^2 + 3^3 x^3 + \cdots + n^3 x^n + \cdots) \\ &= 0^3 + (1^3 - 0^3)x + (2^3 - 1^3)x^2 + (3^3 - 2^3)x^3 + \cdots + [n^3 - (n - 1)^3]x^n + \cdots \\ &= x + 7x^2 + 19x^3 + \cdots + (3n^2 - 3n + 1)x^n + \cdots \\ &= 3\sum_{n=1}^{\infty} n^2 x^n - 3\sum_{n=1}^{\infty} nx^n + \sum_{n=1}^{\infty} x^n.\end{aligned}$$

As derived in the handout,

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}.$$

We have that

$$\begin{aligned}(1-x)\sum_{n=1}^{\infty} nx^n &= (1-x)(x+2x^2+3x^3+\cdots) \\ &= x+(2-1)x^2+(3-2)x^3+\cdots \\ &= x+x^2+x^3+\cdots \\ &= \frac{x}{1-x},\end{aligned}$$

so

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Also,

$$\sum_{n=1}^{\infty} x^n = x+x^2+\cdots = \frac{x}{1-x}.$$

Hence,

$$\begin{aligned}(1-x)A(x) &= 3\cdot\frac{x+x^2}{(1-x)^3} - 3\cdot\frac{x}{(1-x)^2} + \frac{x}{1-x} \\ &= \frac{3x+3x^2-3x(1-x)+x(1-x)^2}{(1-x)^3} \\ &= \frac{x^3+4x^2+x}{(1-x)^3},\end{aligned}$$

so

$$A(x) = \frac{x^3+4x^2+x}{(1-x)^4}.$$

6. Show that $F_0+F_1+\cdots+F_n = F_{n+2}-1$ for all nonnegative integers n, where F_n denotes the n^{th} Fibonacci number.

Solution. We know that

$$\frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n = F_0 + F_1 x + F_2 x^2 + \cdots .$$

Dividing both sides by $1 - x$, we obtain the generating function of the partial sums:

$$\begin{aligned}\frac{x}{(1-x)(1-x-x^2)} &= \frac{F_0 + F_1 x + F_2 x^2 + \cdots}{1-x} \\ &= (F_0 + F_1 x + F_2 x^2 + \cdots)(1 + x + x^2 + \cdots) \\ &= F_0 + (F_0 + F_1)x + (F_0 + F_1 + F_2)x^2 + \cdots .\end{aligned}$$

Also, since $F_0 = 0$ and $F_1 = 1$,

$$\begin{aligned}\frac{x}{1-x-x^2} &= \sum_{n=0}^{\infty} F_n x^n \\ &= x + \sum_{n=2}^{\infty} F_n x^n \\ &= x + \sum_{n=0}^{\infty} F_{n+2} x^{n+2} \\ &= x + x^2 \sum_{n=0}^{\infty} F_{n+2} x^n.\end{aligned}$$

Then

$$x^2 \sum_{n=0}^{\infty} F_{n+2} x^n = \frac{x}{1-x-x^2} - x = \frac{x - x(1-x-x^2)}{1-x-x^2} = \frac{x^2 + x^3}{1-x-x^2},$$

so

$$\sum_{n=0}^{\infty} F_{n+2} x^n = \frac{1+x}{1-x-x^2}.$$

Subtracting

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

from both sides, we get

$$\begin{aligned}\sum_{n=0}^{\infty} (F_{n+2} - 1)x^n &= \frac{1+x}{1-x-x^2} - \frac{1}{1-x} \\ &= \frac{(1+x)(1-x) - (1-x-x^2)}{(1-x)(1-x-x^2)} \\ &= \frac{1 - x^2 - 1 + x + x^2}{(1-x)(1-x-x^2)} \\ &= \frac{x}{(1-x)(1-x-x^2)}.\end{aligned}$$

This generating function is the same as the generating function for the partial sums. Hence, $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$ for all nonnegative integers n.

7. Find the number of ways to collect \$15 from 20 people if each of the first 19 people can give a dollar or nothing, and the twentieth person can give either \$1, \$5, or nothing.

Solution. The generating function corresponding to each of the first 19 people is $1 + x$, and the generating function corresponding to the twentieth person is $1 + x + x^5$, so the generating function of the number of ways to collect money from all 20 people is given by

$$(1 + x)^{19}(1 + x + x^5).$$

We seek the coefficient of x^{15}. The only way to get a term with x^{15} is to multiply the term with x^{15} in the first factor with the term 1 in the second factor, the term with x^{14} in the first factor with the term x in the second factor, or the term with x^{10} in the first factor with the term x^5 in the second factor.

By the Binomial Theorem, the coefficient of x^k in $(1 + x)^{19}$ is $\binom{19}{k}$ for $0 \le k \le 19$. Therefore, the coefficient of x^{15} in the generating function above is

$$\binom{19}{15} + \binom{19}{14} + \binom{19}{10} = 107882.$$

8. Show that the generating function of the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = n,$$

where $0 \le x_1 \le x_2 \le x_3 \le x_4$, is

$$\frac{1}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)}.$$

Solution. Let $y_1 = x_1$, $y_2 = x_2 - x_1$, $y_3 = x_3 - x_2$, and $y_4 = x_4 - x_3$, so these variables y_i represent nonnegative integers. Then $x_1 = y_1$, $x_2 = y_2 + x_1 = y_2 + y_1$, $x_3 = y_3 + x_2 = y_3 + y_2 + y_1$, and $x_4 = y_4 + x_3 = y_4 + y_3 + y_2 + y_1$, so the given equation becomes

$$4y_1 + 3y_2 + 2y_3 + y_4 = n.$$

The generating function corresponding to the term $4y_1$ is

$$1 + x^4 + x^8 + \cdots = \frac{1}{1 - x^4}.$$

Similarly, the generating functions corresponding to the terms $3y_2$, $2y_3$, and y_4 are

$$1 + x^3 + x^6 + \cdots = \frac{1}{1 - x^3},$$
$$1 + x^2 + x^4 + \cdots = \frac{1}{1 - x^2},$$
$$1 + x + x^2 + \cdots = \frac{1}{1 - x},$$

respectively, so the generating function of the number of solutions is

$$\frac{1}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)}.$$

9. The sequence (a_n) is defined by $a_0 = 0$, $a_1 = 2$, and $a_n = 4a_{n-1} - 4a_{n-2}$ for all $n \geq 2$. Find the generating function of this sequence, and use it to solve for a_n.

   Solution. Let
$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots .$$
   Then
$$\begin{aligned}(1 - 4x + 4x^2)A(x) &= (1 - 4x + 4x^2)(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= a_0 + (a_1 - 4a_0)x + (a_2 - 4a_1 + 4a_0)x^2 + (a_3 - 4a_2 + 4a_1)x^3 + \cdots \\ &= 2x,\end{aligned}$$
   so
$$A(x) = \frac{2x}{1 - 4x + 4x^2} = \frac{2x}{(1-2x)^2}.$$
   By the method of partial fractions, there exist constants C_1 and C_2 such that
$$\frac{2x}{(1-2x)^2} = \frac{C_1}{1-2x} + \frac{C_2}{(1-2x)^2}.$$
   Multiplying both sides by $(1-2x)^2$, we get
$$2x = C_1(1-2x) + C_2 = (C_1 + C_2) - 2C_1 x.$$
   Comparing the coefficients on both sides, we obtain the system of equations
$$\begin{aligned} C_1 + C_2 &= 0, \\ -2C_1 &= 2. \end{aligned}$$
   Solving this system of equations, we find $C_1 = -1$ and $C_2 = 1$, so
$$\begin{aligned} A(x) &= \frac{1}{(1-2x)^2} - \frac{1}{1-2x} \\ &= \sum_{n=0}^{\infty} 2^n (n+1) x^n - \sum_{n=0}^{\infty} 2^n x^n \\ &= \sum_{n=0}^{\infty} n 2^n x^n. \end{aligned}$$
   Therefore, the coefficient of x^n in $A(x)$ is given by $a_n = n2^n$.

10. How many solutions in positive integers are there to the equation $y_1 + y_2 + y_3 + y_4 = 30$ such that no y_i is greater than 12?

   Solution. The generating function corresponding to each variable y_i is
$$x + x^2 + \cdots + x^{12} = x(1 + x + \cdots + x^{11}) = x \cdot \frac{1 - x^{12}}{1 - x},$$

so the generating function corresponding to $y_1 + y_2 + y_3 + y_4$ is

$$\left(x \cdot \frac{1 - x^{12}}{1 - x} \right)^4 = \frac{x^4 (1 - x^{12})^4}{(1 - x)^4}.$$

We seek the coefficient of x^{30} in this generating function, which is the same as the coefficient of x^{26} in the generating function

$$\frac{(1 - x^{12})^4}{(1 - x)^4} = (1 - 4x^{12} + 6x^{24} - 4x^{36} + x^{48}) \left[\binom{3}{3} + \binom{4}{3} x + \binom{5}{3} x^2 + \cdots \right].$$

The only way to get a term with x^{26} is to multiply the term 1 in the first factor with the term $\binom{29}{3} x^{26}$ in the second factor, the term $-4x^{12}$ in the first factor with the term $\binom{17}{3} x^{14}$ in the second factor, or the term $6x^{24}$ in the first factor with the term $\binom{5}{3} x^2$ in the second factor. Therefore, the coefficient of x^{26} is equal to

$$\binom{29}{3} - 4\binom{17}{3} + 6\binom{5}{3} = 994.$$

11. Let n be a positive integer. Show that

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots,$$

where F_n denotes the n^{th} Fibonacci number.

Hint:

$$\frac{1}{1 - x - x^2} = 1 + (x + x^2) + (x + x^2)^2 + \cdots.$$

Solution. We know that

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2},$$

which we can also write as

$$\begin{aligned} \frac{x}{1 - x - x^2} &= x[1 + (x + x^2) + (x + x^2)^2 + \cdots] \\ &= x + x(x + x^2) + x(x + x^2)^2 + \cdots. \end{aligned}$$

The coefficient of x^n in this generating function is F_n. On the other hand, the coefficient of x^n in

$$x(x + x^2)^k = x \cdot x^k (1 + x)^k = x^{k+1} (1 + x)^k$$

is the same as the coefficient of x^{n-k-1} in $(1 + x)^k$, which is

$$\binom{k}{n - k - 1}.$$

Summing over $k = n - 1, n - 2, \ldots$, we find

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots.$$

12. Expand

$$\frac{1}{(1-x)^{n+1}} = (1-x)^{-n-1}$$

using the generalized Binomial theorem.

Solution. By the generalized Binomial theorem, the coefficient of x^k in $(1-x)^{-n-1}$ is

$$\begin{aligned}(-1)^k \binom{-n-1}{k} &= (-1)^k \cdot \frac{(-n-1)(-n-2)\cdots(-n-k)}{k!} \\ &= \frac{(n+k)(n+k-1)\cdots(n+1)}{k!} \\ &= \frac{(n+k)!/n!}{k!} \\ &= \binom{n+k}{n},\end{aligned}$$

so

$$\frac{1}{(1-x)^{n+1}} = \binom{n}{n} + \binom{n+1}{n}x + \binom{n+2}{n}x^2 + \cdots.$$

13. Given $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$, express

$$a_1x + a_4x^4 + a_7x^7 + \cdots$$

in terms of $A(x)$.

Solution. Let $\omega = e^{2\pi i/3}$, so

$$\begin{aligned}A(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots, \\ A(\omega x) &= a_0 + \omega a_1x + \omega^2 a_2x^2 + a_3x^3 + \cdots, \\ A(\omega^2 x) &= a_0 + \omega^2 a_1x + \omega a_2x^2 + a_3x^3 + \cdots.\end{aligned}$$

Then

$$\begin{aligned}A(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots, \\ \omega^2 A(\omega x) &= \omega^2 a_0 + a_1x + \omega a_2x^2 + \omega^2 a_3x^3 + \cdots, \\ \omega A(\omega^2 x) &= \omega a_0 + a_1x + \omega^2 a_2x^2 + \omega a_3x^3 + \cdots.\end{aligned}$$

Adding all three equations and dividing by 3, we get

$$\frac{1}{3}[A(x) + \omega^2 A(\omega x) + \omega A(\omega^2 x)] = a_1x + a_4x^4 + a_7x^7 + \cdots.$$

14. Show that

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = 2^{n/2} \cos \frac{n\pi}{4},$$

and that

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \cdots = 2^{n/2} \sin \frac{n\pi}{4}.$$

Solution. Expanding $(1+i)^n$, we get

$$\begin{aligned}(1+i)^n &= \binom{n}{0} + \binom{n}{1} i + \binom{n}{2} i^2 + \binom{n}{3} i^3 + \cdots \\ &= \binom{n}{0} + \binom{n}{1} i - \binom{n}{2} - \binom{n}{3} i + \cdots .\end{aligned}$$

Thus, the sums we seek are the real and imaginary parts of this complex number. Expressing the complex number $1+i$ in exponential form, we get

$$1 + i = \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right) = 2^{1/2} e^{i\pi/4},$$

so

$$\begin{aligned}(1+i)^n &= (2^{1/2} e^{i\pi/4})^n \\ &= 2^{n/2} (e^{i\pi/4})^n \\ &= 2^{n/2} \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right).\end{aligned}$$

Comparing the real and imaginary parts, we find

$$\begin{aligned}\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots &= 2^{n/2} \cos \frac{n\pi}{4}, \\ \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \cdots &= 2^{n/2} \sin \frac{n\pi}{4}.\end{aligned}$$

15. Find a simple expression for

$$\prod_{k=0}^{\infty} (1 + x^{2^k}) = (1+x)(1+x^2)(1+x^4)(1+x^8) \cdots .$$

Solution. If we expand the product, then every term will be the product of some of the terms 1, x, x^2, x^4, x^8, and so on. Also, other than the term 1, each of these terms can only be used once.

By binary representation, we know that every nonnegative integer can be written uniquely as the sum of distinct powers of 2. Therefore, the product is simply

$$(1+x)(1+x^2)(1+x^4)(1+x^8) \cdots = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

16. Prove that the number of partitions of n into odd parts greater than 1 equals the number of partitions of n into distinct parts that are not powers of 2.

Solution. The generating function of the number of partitions into odd parts greater than 1 is

$$(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)(1+x^7+x^{14}+\cdots)\cdots = \frac{1}{(1-x^3)(1-x^5)(1-x^7)\cdots}.$$

As shown in the handout,

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = (1+x)(1+x^2)(1+x^3)\cdots,$$

so

$$\frac{1}{(1-x^3)(1-x^5)(1-x^7)\cdots} = (1-x)(1+x)(1+x^2)(1+x^3)\cdots.$$

By the previous problem,

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots = \frac{1}{1-x},$$

so

$$(1-x)(1+x)(1+x^2)(1+x^3)\cdots = (1+x^3)(1+x^5)(1+x^6)(1+x^7)\cdots,$$

where the product in the right-hand side is taken over all factors of the form $1+x^k$, where k is not a power of 2. This is precisely the generating function of partitions into distinct parts that are not powers of 2.

Thus, the two generating functions coincide, which means that the number of such partitions is equal.

17. Find the number of compositions of n, where each part is at least 2.

Solution. Using the same notation as in the handout, we set $f(1) = 0$ and $f(n) = 1$ for $n \geq 2$, so

$$F(x) = x^2 + x^3 + \cdots = \frac{x^2}{1-x}.$$

Then the generating function of the number of such compositions is

$$\begin{aligned}\frac{1}{1-F(x)} &= \frac{1}{1-\frac{x^2}{1-x}} \\ &= \frac{1-x}{1-x-x^2} \\ &= \frac{1-x-x^2+x^2}{1-x-x^2} \\ &= 1 + \frac{x^2}{1-x-x^2}.\end{aligned}$$

We know that

$$F_0 + F_1 x + F_2 x^2 + \cdots = \frac{x}{1 - x - x^2},$$

so

$$F_0 x + F_1 x^2 + F_2 x^3 + \cdots = \frac{x^2}{1 - x - x^2}.$$

Then

$$1 + F_0 x + F_1 x^2 + F_2 x^3 + \cdots = 1 + \frac{x^2}{1 - x - x^2}.$$

Hence, for $n \geq 1$, the number of compositions of n, where each part is at least 2, is F_{n-1}.

18. Find the number of compositions of n, where each part is odd.

Solution. Using the same notation as in the handout, we set $f(n) = 0$ if n is even and $f(n) = 1$ if n is odd, so

$$F(x) = x + x^3 + x^5 + \cdots = \frac{x}{1 - x^2}.$$

Then the generating function of the number of such compositions is

$$\begin{aligned}\frac{1}{1 - F(x)} &= \frac{1}{1 - \frac{x}{1-x^2}} \\ &= \frac{1 - x^2}{1 - x - x^2} \\ &= \frac{1 - x - x^2 + x}{1 - x - x^2} \\ &= 1 + \frac{x}{1 - x - x^2}.\end{aligned}$$

We know that

$$F_0 + F_1 x + F_2 x^2 + \cdots = \frac{x}{1 - x - x^2},$$

so

$$1 + F_1 x + F_2 x^2 + \cdots = 1 + \frac{x}{1 - x - x^2}.$$

Hence, for $n \geq 1$, the number of compositions of n, where each part is odd, is F_n.

19. Show that

$$s(n) = f(1)s(n-1) + f(2)s(n-2) + \cdots + f(n-1)s(1) + f(n)$$

for all positive integers n, where $s(n)$ is as defined in $(*)$.

Solution 1. The generating function of $s(n)$ is

$$1 + s(1)x + s(2)x^2 + \cdots = \frac{1}{1 - F(x)}.$$

Multiplying both sides by $1 - F(x)$, we get

$$\begin{aligned}1 &= [1 + s(1)x + s(2)x^2 + \cdots][1 - F(x)] \\ &= [1 + s(1)x + s(2)x^2 + \cdots][1 - f(1)x - f(2)x^2 - \cdots].\end{aligned}$$

For $n \geq 1$, the coefficient of x^n is

$$s(n) - f(1)s(n-1) - f(2)s(n-2) - \cdots - f(n-1)s(1) - f(n) = 0,$$

so

$$s(n) = f(1)s(n-1) + f(2)s(n-2) + \cdots + f(n-1)s(1) + f(n)$$

for all $n \geq 1$.

Solution 2. By definition,

$$s(n) = \sum_{a_1+a_2+\cdots+a_k=n} f(a_1)f(a_2)\cdots f(a_k),$$

where the sum is taken over all compositions $a_1 + a_2 + \cdots + a_k$ of n. Conditioning on the last part a_k, which must be between 1 and n, we can write

$$\begin{aligned}s(n) &= \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ a_k=1}} f(a_1)f(a_2)\cdots f(a_k) + \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ a_k=2}} f(a_1)f(a_2)\cdots f(a_k) \\ &\quad + \cdots + \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ a_k=n-1}} f(a_1)f(a_2)\cdots f(a_k) + \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ a_k=n}} f(a_1)f(a_2)\cdots f(a_k) \\ &= f(1) \sum_{a_1+a_2+\cdots+a_{k-1}=n-1} f(a_1)f(a_2)\cdots f(a_{k-1}) + f(2) \sum_{a_1+a_2+\cdots+a_{k-1}=n-2} f(a_1)f(a_2)\cdots f(a_{k-1}) \\ &\quad + \cdots + f(n-1) \sum_{a_1+a_2+\cdots+a_{k-1}=1} f(a_1)f(a_2)\cdots f(a_{k-1}) + f(n) \\ &= f(1)s(n-1) + f(2)s(n-2) + \cdots + f(n-1)s(1) + f(n).\end{aligned}$$

20. For an arbitrary positive integer n, consider all possible words of n letters A and B and denote by p_n the number of words containing neither $AAAA$ nor BBB. Calculate the value of

$$\frac{p_{2004} - p_{2002} - p_{1999}}{p_{2001} + p_{2000}}.$$

(Czech-Slovak, 2004)

Solution. Using the generating function

$$W(a, b) = 1 + a + b + ab + ba + aba + bab + \cdots,$$

the generating function of the words counted by p_n is

$$W(A + AA + AAA, B + BB),$$

so the generating function of the sequence (p_n) is

$$\begin{aligned} W(x + xx + xxx, x + xx) &= W(x + x^2 + x^3, x + x^2) \\ &= \frac{(1 + x + x^2 + x^3)(1 + x + x^2)}{1 - (x + x^2 + x^3)(x + x^2)} \\ &= \frac{(1 + x + x^2 + x^3)(1 + x + x^2)}{1 - x^2 - 2x^3 - 2x^4 - x^5}. \end{aligned}$$

From the denominator, we see that

$$p_n - p_{n-2} - 2p_{n-3} - 2p_{n-4} - p_{n-5} = 0$$

for all $n \geq 5$. In particular, for $n = 2004$, $p_{2004} - p_{2002} - 2p_{2001} - 2p_{2000} - p_{1999} = 0$, so

$$\frac{p_{2004} - p_{2002} - p_{1999}}{p_{2001} + p_{2000}} = 2.$$

21. Let S denote the set of strings where each letter is A or B, such that every A is next to an A and every B is next to a B. For example, the string $AAABBAA$ is allowed, but the string $AAAB$ is not.

    (a) Let $f(n, m)$ denote the number of strings in S with n As and m Bs. Find a recursive formula for $f(n, m)$.

    (b) Let $g(n)$ denote the number of strings in S with n letters. Find $g(n)$.

Solution. (a) Using the generating function

$$W(a, b) = 1 + a + b + ab + ba + aba + bab + \cdots,$$

the generating function of the words counted by $f(n, m)$ is

$$W(AA + AAA + AAAA + \cdots, BB + BB + BBBB + \cdots).$$

Hence, the generating function of $f(n, m)$ is

$$\begin{aligned} \sum_{n,m=0}^{\infty} f(n, m)x^n y^m &= W(xx + xxx + xxxx + \cdots, yy + yyy + yyyy + \cdots) \\ &= W\left(\frac{x^2}{1 - x}, \frac{y^2}{1 - y}\right) \\ &= \frac{(1 + \frac{x^2}{1-x})(1 + \frac{y^2}{1-y})}{1 - \frac{x^2}{1-x} \cdot \frac{y^2}{1-y}} \\ &= \frac{(1 - x + x^2)(1 - y + y^2)}{(1 - x)(1 - y) - x^2y^2} \\ &= \frac{(1 - x + x^2)(1 - y + y^2)}{1 - x - y + xy - x^2y^2}. \end{aligned}$$

Multiplying both sides by $1 - x - y + xy - x^2y^2$, we get

$$(1 - x + x^2)(1 - y + y^2) = (1 - x - y + xy - x^2y^2) \sum_{n,m=0}^{\infty} f(n, m)x^n y^m.$$

For n, $m \geq 2$, the coefficient of $x^n y^m$ in the right-hand side is

$$f(n, m) - f(n - 1, m) - f(n, m - 1) + f(n - 1, m - 1) - f(n - 2, m - 2),$$

and the coefficient of $x^n y^m$ in the left-hand side is 0, except that the coefficient is equal to 1 for $n = m = 2$. Therefore,

$$f(n, m) = f(n - 1, m) + f(n, m - 1) - f(n - 1, m - 1) + f(n - 2, m - 2)$$

for all n, $m \geq 2$, except for $n = m = 2$ (in which case we add 1 to the right-hand side).

(b) Setting $y = x$ in the generating function in part (a), we get

$$\begin{aligned}\sum_{n=0}^{\infty} g(n)x^n &= \frac{(1 - x + x^2)^2}{1 - 2x + x^2 - x^4} \\ &= \frac{(1 - x + x^2)^2}{(1 - x + x^2)(1 - x - x^2)} \\ &= \frac{1 - x + x^2}{1 - x - x^2} \\ &= \frac{1 - x - x^2 + 2x^2}{1 - x - x^2} \\ &= 1 + \frac{2x^2}{1 - x - x^2}.\end{aligned}$$

As seen in the solution to Exercise 17,

$$F_0 x + F_1 x^2 + F_2 x^3 + \cdots = \frac{x^2}{1 - x - x^2},$$

so

$$1 + \frac{2x^2}{1 - x - x^2} = 1 + 2F_0 x + 2F_1 x^2 + 2F_2 x^3 + \cdots.$$

Hence, $g(n) = 2F_{n-1}$ for all $n \geq 1$.

22. Show that the generating function of the words composed of the letters a, b, and c, such that no two consecutive letters are the same, is

$$W(x, y, z) = \frac{(1 + x)(1 + y)(1 + z)}{1 - xy - xz - yz - 2xyz}.$$

Generalize to n letters.

Solution. Using letters, we see that

$$W(a, b, c) = 1 + a + b + c + ab + ac + ba + bc + ca + cb + \cdots .$$

If we replace a, b, and c with $a + aa + aaa + \cdots$, $b + bb + bbb + \cdots$, and $c + cc + ccc + \cdots$, respectively, then we get

$$\begin{aligned}
&W(a + aa + aaa + \cdots , b + bb + bbb + \cdots , c + cc + ccc + \cdots)\\
&= 1 + (a + aa + aaa + \cdots) + (b + bb + bbb + \cdots) + (c + cc + ccc + \cdots)\\
&\quad + (a + aa + aaa + \cdots)(b + bb + bbb + \cdots)\\
&\quad + (a + aa + aaa + \cdots)(c + cc + ccc + \cdots)\\
&\quad + (b + bb + bbb + \cdots)(a + aa + aaa + \cdots)\\
&\quad + (b + bb + bbb + \cdots)(c + cc + ccc + \cdots)\\
&\quad + (c + cc + ccc + \cdots)(a + aa + aaa + \cdots)\\
&\quad + (c + cc + ccc + \cdots)(b + bb + bbb + \cdots) + \cdots .
\end{aligned}$$

If we expand this expression, in words, then we will find that every word composed of the letters a, b, and c appears exactly once. Hence, we can write this equation as

$$\begin{aligned}
&W(a + aa + aaa + \cdots , b + bb + bbb + \cdots , c + cc + ccc + \cdots)\\
&= 1 + (a + b + c) + (a + b + c)^2 + (a + b + c)^3 + \cdots .
\end{aligned}$$

Replacing the letters a, b, and c with the ordinary variables u, v, and w, we get

$$\begin{aligned}
&W(u + u^2 + u^3 + \cdots , v + v^2 + v^3 + \cdots , w + w^2 + w^3 + \cdots)\\
&= 1 + (u + v + w) + (u + v + w)^2 + (u + v + w)^3 + \cdots ,
\end{aligned}$$

or

$$W\left(\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-v}\right) = \frac{1}{1-(u+v+w)}.$$

Finally, let $x = \frac{u}{1-u}$, $y = \frac{v}{1-v}$, and $z = \frac{w}{1-w}$. Solving for u, v, and w, we find $u = \frac{x}{1+x}$, $v = \frac{y}{1+y}$, and $w = \frac{z}{1+z}$, so

$$W(x, y, z) = \frac{1}{1 - \frac{x}{1+x} - \frac{y}{1+y} - \frac{z}{1+z}} = \frac{(1 + x)(1 + y)(1 + z)}{1 - xy - xz - yz - 2xyz}.$$

In general, if $W(a_1, a_2, \ldots, a_n)$ is the generating function of the words composed of the letters a_1, a_2, $\ldots$, a_n, such that no two letters are consecutive, then

$$W(x_1, x_2, \ldots, x_n) = \frac{1}{1 - \frac{x_1}{1+x_1} - \frac{x_2}{1+x_2} - \cdots - \frac{x_n}{1+x_n}}.$$

23. The U.S. Social Security numbers consist of 9 digits (with initial zeros permitted). How many such numbers are there that do not contain any digit three or more times consecutively? (Crux Mathematicorum, Problem 879)

Solution. To make things easier, we think of the 10 digits as the 10 letters a_0, a_1, …, a_9. Let $W(a_0, a_1, \ldots, a_9)$ be the generating function of the words composed of the letters a_0, a_1, …, a_9, such that no two letters are consecutive, so

$$W(a_0, a_1, \ldots, a_9) = 1 + a_0 + a_1 + \cdots + a_9 + a_0a_1 + a_0a_2 + \cdots + a_9a_8 + \cdots .$$

Replacing the letter a_i with $a_i + a_ia_i$, we get

$$\begin{aligned} &W(a_0 + a_0a_0, a_1 + a_1a_1, \ldots, a_9 + a_9a_9) \\ &= 1 + (a_0 + a_0a_0) + (a_1 + a_1a_1) + \cdots + (a_9 + a_9a_9) \\ &\quad + (a_0 + a_0a_0)(a_1 + a_1a_1) + (a_0 + a_0a_0)(a_2 + a_2a_2) + \cdots + (a_9 + a_9a_9)(a_8 + a_8a_8) + \cdots . \end{aligned}$$

This is the generating function of the words we seek.

By the previous problem, if $W(a_0, a_1, \ldots, a_9)$ is the generating function of the words composed of the letters a_0, a_1, …, a_9, such that no two letters are consecutive, then

$$W(x_0, x_1, \ldots, x_9) = \frac{1}{1 - \frac{x_0}{1+x_0} - \frac{x_1}{1+x_1} - \cdots - \frac{x_9}{1+x_9}}.$$

Hence,

$$W(x_0 + x_0^2, x_1 + x_1^2, \ldots, x_9 + x_9^2) = \frac{1}{1 - \frac{x_0+x_0^2}{1+x_0+x_0^2} - \frac{x_1+x_1^2}{1+x_1+x_1^2} - \cdots - \frac{x_9+x_9^2}{1+x_9+x_9^2}}.$$

Finally, let t_n be the number of words with n letters, such that no letter appears three or more times consecutively. Then the generating function of the sequence (t_n) is obtained by replacing each variable x_i with the same variable x:

$$W(x + x^2, x + x^2, \ldots, x + x^2) = \frac{1}{1 - 10 \cdot \frac{x+x^2}{1+x+x^2}} = \frac{1 + x + x^2}{1 - 9x - 9x^2}.$$

Writing

$$\frac{1 + x + x^2}{1 - 9x - 9x^2} = t_0 + t_1x + t_2x^2 + \cdots ,$$

and multiplying both sides by $1 - 9x - 9x^2$, we get

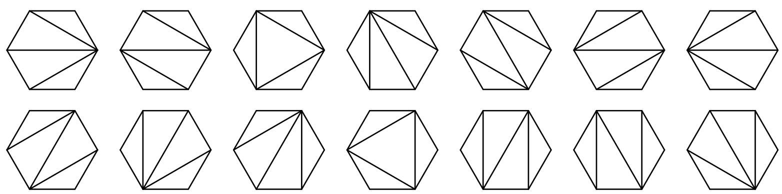
$$\begin{aligned} 1 + x + x^2 &= (1 - 9x - x^2)(t_0 + t_1x + t_2x^2 + \cdots) \\ &= t_0 + (t_0 - 9t_1)x + (t_2 - 9t_1 - 9t_0)x^2 + (t_3 - 9t_2 - 9t_1)x^3 + \cdots . \end{aligned}$$

Comparing the coefficients on both sides, we find $t_0 = 1$, $t_1 - 9t_0 = 1$, $t_2 - 9t_1 - 9t_0 = 1$, and $t_n - 9t_{n-1} - 9t_{n-2} = 0$ for all $n \geq 3$. Then

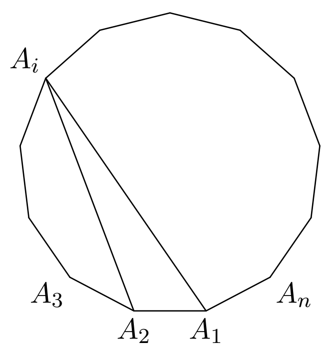
$$\begin{aligned} t_0 &= 1, \\ t_1 &= 9t_0 + 1 = 10, \\ t_2 &= 9t_1 + 9t_0 + 1 = 100, \end{aligned}$$

and $t_n = 9t_{n-1} + 9t_{n-2}$ for all $n \geq 3$. Using this recurrence, we find that $t_9 = 936845190$, the answer we seek.

24. Given a polygon, we can partition the polygon into triangles using the diagonals of the polygon, such that every vertex of a triangle is also a vertex of the polygon. Such a partition is called a *triangulation*. Show that the number of triangulations of a regular n-gon is C_{n-2}. The 14 triangulations of a hexagon are shown below.



Solution. Let T_n be the number of triangulations of a regular n-gon $A_1A_2 \cdots A_n$. Given a triangulation, let A_i be the third vertex of the triangle with edge A_1A_2, as shown below, so $3 \leq i \leq n$.



Then polygon $A_2A_3 \cdots A_i$ has $i-1$ edges, so there are T_{i-1} ways to triangulate it. Similarly, polygon $A_1A_iA_{i+1} \cdots A_n$ has $n-i+2$ edges, so there are T_{n-i+2} ways to triangulate it. Therefore, there are $T_{i-1}T_{n-i+2}$ triangulations that include triangle $A_1A_2A_i$.

Summing over $3 \leq i \leq n$, we get

$$T_n = T_2T_{n-1} + T_3T_{n-2} + \cdots + T_{n-1}T_2$$

for all $n \geq 4$. (We may set $T_2 = 1$.) Also, $T_3 = 1$.

But $C_1 = 1$, and

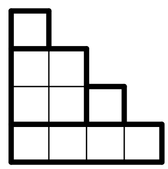
$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0$$

for all $n \geq 2$.

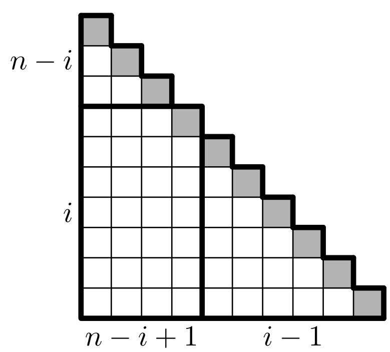
Thus, the sequences (T_n) and (C_{n-2}) have the same initial term, and satisfy the same recursion. We conclude that $T_n = C_{n-2}$ for all $n \geq 3$.

25. For a positive integer n, an n-*staircase* is a figure consisting of unit squares, with one square in the first row, two squares in the second row, and so on, up to n squares in the n^{th} row, such that all the left-most squares in each row are aligned vertically.

The n-staircase is tiled with n rectangles, such that each rectangle has integer side lengths. Show that the number of such tilings is C_n. A tiled 4-staircase is shown below.



Solution. Let T_n be the number of ways to tile an n-staircase. We say that a unit square is a *diagonal square* if it lies on the uppermost diagonal running from the upper-left square to the lower-right square, so an n-staircase has n diagonal squares. A 10-staircase is shown below, with its 10 diagonal squares shaded.



Clearly, every rectangle (in the tiling) can cover at most one diagonal square. But there are n rectangles in the tiling, and n diagonal squares, so every tile covers exactly one diagonal square.

Consider the tile that covers the bottom-left square in the n-staircase; let this tile have height i, where $1 \leq i \leq n$. This tile also covers a diagonal square (and this diagonal square must be the upper-right corner of the tile), so its width is $n - i + 1$. Furthermore, this tile divides the n-staircase into two staircases, namely an $(i-1)$-staircase and an $(n-i)$-staircase. There are T_{i-1} ways to tile an $(i-1)$-staircase, and T_{n-i} ways to tile an $(n-i)$-staircase. Therefore, there are $T_{i-1}T_{n-i}$ ways to tile an n-staircase, where the tile that covers the bottom-left square of the n-staircase has height i and width $n-i+1$. Summing over $1 \le i \le n$, we get

$$T_n = T_0 T_{n-1} + T_1 T_{n-2} + \cdots + T_{n-1} T_0$$

for all $n \ge 2$. (We may set $T_0 = 1$.) Also, $T_1 = 1$.

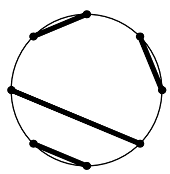
But $C_1 = 1$, and

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0$$

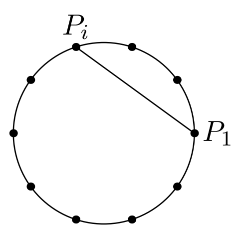
for all $n \ge 2$.

Thus, the sequences (T_n) and (C_n) have the same initial term, and satisfy the same recursion. We conclude that $T_n = C_n$ for all $n \ge 1$.

26. Given a positive integer n, take $2n$ points evenly spaced on a circle. We divide the $2n$ points into n pairs, and draw the line segment connecting the two points in the same pair. Show that the number of ways to divide the $2n$ points into n pairs, so that no two line segments intersect, is C_n. An example for $n = 4$ is shown below.



Solution. Let T_n be the number of way to divide the $2n$ points into n pairs. Label the $2n$ points P_1, P_2, $\dots$, P_{2n}, and let point P_1 be connected to point P_i, where $2 \le i \le 2n$.



Then the points P_2, P_3, $\dots$, P_{i-1} must be divided into pairs, and there are $T_{(i-2)/2}$ ways to do so. (Note that i must be even.) Similarly, the points P_{i+1}, P_{i+2}, $\dots$, P_{2n} must be divided into pairs, and there are $T_{(2n-i)/2}$ ways to do so, for a total of $T_{(i-2)/2}T_{(2n-i)/2}$ ways. Summing over all $2 \le i \le n$ where i is even, we get

$$T_n = T_0 T_{n-1} + T_1 T_{n-2} + \cdots + T_{n-1} T_0$$

for all $n \ge 2$. (We may set $T_0 = 1$.) Also, $T_1 = 1$.

But $C_1 = 1$, and

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0$$

for all $n \geq 2$.

Thus, the sequences (T_n) and (C_n) have the same initial term, and satisfy the same recursion. We conclude that $T_n = C_n$ for all $n \geq 1$.

27. Find all n such that C_n is odd.

Solution. We claim that C_n is odd if and only if $n = 2^k - 1$ for some nonnegative integer k.

Let $D_n = C_{n-1}$, so $D_1 = 1$ and

$$D_n = D_1 D_{n-1} + D_2 D_{n-2} + \cdots + D_{n-1} D_1$$

for all $n \geq 2$.

If $n = 2m + 1$, where $m \geq 1$, then

$$\begin{aligned} D_n &= D_{2m+1} \\ &= D_1 D_{2m} + D_2 D_{2m-1} + \cdots + D_{2m} D_1 \\ &= 2(D_1 D_{2m} + D_2 D_{2m-1} + \cdots + D_m D_{m+1}) \end{aligned}$$

is even.

If $n = 2m$, where $m \geq 1$, then

$$\begin{aligned} D_n &= D_{2m} \\ &= D_1 D_{2m-1} + D_2 D_{2m-2} + \cdots + D_{2m-1} D_1 \\ &= 2(D_1 D_{2m-1} + D_2 D_{2m-2} + \cdots + D_{m-1} D_{m+1}) + D_m^2. \end{aligned}$$

Hence, D_n is odd if and only if D_m is odd. In other words, if n is even, then D_n is odd if and only if $D_{n/2}$ is odd.

If we write n in the form $n = 2^k t$, where k is a nonnegative integer and t is an odd positive integer, then we see that D_n is odd if and only if D_t is odd. But from our work above, if D_t is odd and t odd, then $t = 1$.

Therefore, D_n is odd if and only if $n = 2^k$ for some nonnegative integer k, which means C_n is odd if and only if $n = 2^k - 1$ for some nonnegative integer k.

28. For $1 \leq t \leq n$, let f_t be the number of elements in S that appear in **at least** t of the sets A_1, A_2, $\ldots$, A_n, and let

$$F(x) = \sum_{t=0}^{n} f_t x^t.$$

(a) Show that

$$F(x) = \frac{xE(x) - E(1)}{x - 1}.$$

(b) Show that for $1 \le t \le n$,

$$f_t = \sum_{r=t}^{n} (-1)^{r-t} \binom{r-1}{t-1} N_r.$$

Solution. (a) Since e_t is the number of elements in S that appear in exactly t of the sets A_1, A_2, …, A_n,

$$f_t = e_t + e_{t+1} + \cdots + e_n.$$

Then

$$\begin{aligned}
\frac{xE(x) - E(1)}{x-1} &= \frac{x(e_0 + e_1 x + e_2 x^2 + \cdots + e_n x^n) - (e_0 + e_1 + e_2 + \cdots + e_n)}{x-1} \\
&= \frac{e_0(x-1) + e_1(x^2-1) + e_2(x^3-1) + \cdots + e_n(x^{n+1}-1)}{x-1} \\
&= e_0 + e_1(1+x) + e_2(1+x+x^2) + \cdots + e_n(1+x+x^2+\cdots+x^n) \\
&= (e_0 + e_1 + e_2 + \cdots + e_n) + (e_1 + e_2 + \cdots + e_n)x + (e_2 + \cdots + e_n)x^2 + \cdots + e_n x^n \\
&= f_0 + f_1 x + f_2 x^2 + \cdots + f_n x^n \\
&= F(x).
\end{aligned}$$

(b) Since $E(x) = N(x-1)$,

$$\begin{aligned}
F(x) &= \frac{xE(x) - E(1)}{x-1} \\
&= \frac{xN(x-1) - N(0)}{x-1} \\
&= \frac{x[N_0 + N_1(x-1) + N_2(x-1)^2 + \cdots + N_n(x-1)^n] - N_0}{x-1} \\
&= N_0 + N_1 x + N_2 x(x-1) + \cdots + N_n x(x-1)^{n-1} \\
&= N_0 + x \sum_{r=1}^{n} N_r (x-1)^{r-1} \\
&= N_0 + x \sum_{r=1}^{n} N_r \left(\sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{r-1}{i} x^i \right) \\
&= N_0 + \sum_{r=1}^{n} N_r \left(\sum_{i=0}^{r-1} (-1)^{r-i-1} \binom{r-1}{i} x^{i+1} \right) \\
&= N_0 + \sum_{r=1}^{n} N_r \left(\sum_{i=1}^{r} (-1)^{r-i} \binom{r-1}{i-1} x^i \right).
\end{aligned}$$

Therefore, for $1 \le t \le n$, the coefficient of x^t is

$$f_t = \sum_{r=t}^{n} (-1)^{r-t} \binom{r-1}{t-1} N_r.$$

Solutions to Miscellaneous Exercises

1. Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that $a_n^2 + a_{n+1}^2 = a_m$. (Putnam, 1999)

Solution. We have that

$$\frac{1}{1-2x-x^2} = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Multiplying both sides by $1 - 2x - x^2$, we get

$$\begin{aligned} 1 &= (1-2x-x^2)(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1 - a_0)x^2 + (a_3 - 2a_2 - a_1)x^3 + \cdots . \end{aligned}$$

Comparing the coefficients on both sides, we get $a_0 = 1$, $a_1 - 2a_0 = 0$, so $a_1 = 2$, and

$$a_n - 2a_{n-1} - a_{n-2} = 0$$

for all $n \geq 2$. We compute the first few terms:

n	a_n
0	1
1	2
2	5
3	12
4	29
5	70
6	169

We see that

$$\begin{aligned} a_0^2 + a_1^2 &= 5 = a_2, \\ a_1^2 + a_2^2 &= 29 = a_4, \\ a_2^2 + a_3^2 &= 169 = a_6. \end{aligned}$$

In general, we claim that

$$a_n^2 + a_{n+1}^2 = a_{2n+2}$$

for all $n \geq 0$, so we can take $m = 2n + 2$.

Let

$$1 - 2x - x^2 = (1-\alpha x)(1-\beta x) = 1 - (\alpha+\beta)x + (\alpha\beta)x^2,$$

so $\alpha+\beta=2$ and $\alpha\beta=-1$. By Vieta's Formulas, α and β are the roots of the quadratic $t^2-2t-1=0$. By the quadratic formula, the roots of this quadratic are $1\pm\sqrt{2}$, so let $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$.

By the method of partial fractions, we get

$$\frac{1}{1-2x-x^2}=\frac{1}{(1-\alpha x)(1-\beta x)}=\frac{C_1}{1-\alpha x}+\frac{C_2}{1-\beta x},$$

where $C_1=\frac{2+\sqrt{2}}{4}$ and $C_2=\frac{2-\sqrt{2}}{4}$. Then

$$\begin{aligned}\frac{C_1}{1-\alpha x}+\frac{C_2}{1-\beta x}&=C_1(1+\alpha x+\alpha^2x^2+\cdots)+C_2(1+\beta x+\beta^2x^2+\cdots)\\&=(C_1+C_2)+(C_1\alpha+C_2\beta)x+(C_1\alpha^2+C_2\beta^2)x^2+\cdots,\end{aligned}$$

so the coefficient of x^n is

$$a_n=C_1\alpha^n+C_2\beta^n.$$

Then

$$\begin{aligned}a_n^2+a_{n+1}^2&=(C_1\alpha^n+C_2\beta^n)^2+(C_1\alpha^{n+1}+C_2\beta^{n+1})^2\\&=(C_1\alpha^n+C_2\beta^n)^2+(\alpha C_1\alpha^n+\beta C_2\beta^n)^2\\&=C_1^2\alpha^{2n}+2C_1C_2\alpha^n\beta^n+C_2^2\beta^{2n}+\alpha^2C_1^2\alpha^{2n}+2\alpha\beta C_1C_2\alpha^n\beta^n+\beta^2C_2^2\beta^{2n}\\&=(C_1^2+\alpha^2C_1^2)\alpha^{2n}+(2C_1C_2+2\alpha\beta C_1C_2)\alpha^n\beta^n+(C_2^2+\beta^2C_2^2)\beta^{2n}\\&=\frac{10+7\sqrt{2}}{4}\alpha^{2n}+\frac{10-7\sqrt{2}}{4}\beta^{2n},\end{aligned}$$

and

$$\begin{aligned}a_{2n+2}&=C_1\alpha^{2n+2}+C_2\beta^{2n+2}\\&=\alpha^2C_1\alpha^{2n}+\beta^2C_2\beta^{2n}\\&=\frac{10+7\sqrt{2}}{4}\alpha^{2n}+\frac{10-7\sqrt{2}}{4}\beta^{2n}.\end{aligned}$$

Hence, $a_n^2+a_{n+1}^2=a_{2n+2}$ for all $n\ge 0$, as claimed.

2. If the expansion in powers of x of the function

$$\frac{1}{(1-ax)(1-bx)}$$

is given by $c_0+c_1x+c_2x^2+c_3x^3+\cdots$, prove that the expansion in powers of x of the function

$$\frac{1+abx}{(1-abx)(1-a^2x)(1-b^2x)}$$

is given by $c_0^2+c_1^2x+c_2^2x^2+c_3^2x^3+\cdots$. (Putnam, 1939)

Solution. We take the cases where $a \neq b$ and $a = b$.

If $a \neq b$, then by partial fractions,

$$\begin{aligned}\frac{1}{(1-ax)(1-bx)} &= \frac{a}{a-b} \cdot \frac{1}{1-ax} - \frac{b}{a-b} \cdot \frac{1}{1-bx} \\ &= \frac{a}{a-b}(1 + ax + a^2x^2 + \cdots) - \frac{b}{a-b}(1 + bx + b^2x^2 + \cdots) \\ &= \frac{(a-b) + (a^2-b^2)x + (a^3-b^3)x^2 + \cdots}{a-b},\end{aligned}$$

so

$$c_n = \frac{a^{n+1} - b^{n+1}}{a-b}.$$

Then

$$\begin{aligned}c_0^2 + c_1^2x + c_2^2x^2 + c_3^2x^3 + \cdots &= \sum_{n=0}^{\infty} c_n^2 x^n \\ &= \sum_{n=0}^{\infty} \left(\frac{a^{n+1} - b^{n+1}}{a-b} \right)^2 x^n \\ &= \frac{1}{(a-b)^2} \sum_{n=0}^{\infty} (a^{n+1} - b^{n+1})^2 x^n \\ &= \frac{1}{(a-b)^2} \sum_{n=0}^{\infty} (a^2 \cdot a^{2n} - 2ab \cdot a^n b^n + b^2 \cdot b^{2n}) x^n \\ &= \frac{a^2}{(a-b)^2} \sum_{n=0}^{\infty} (a^2x)^n - \frac{2ab}{(a-b)^2} \sum_{n=0}^{\infty} (abx)^n + \frac{b^2}{(a-b)^2} \sum_{n=0}^{\infty} (b^2x)^n \\ &= \frac{a^2}{(a-b)^2} \cdot \frac{1}{1-a^2x} - \frac{2ab}{(a-b)^2} \cdot \frac{1}{1-abx} + \frac{b^2}{(a-b)^2} \cdot \frac{1}{1-b^2x}.\end{aligned}$$

Putting this expression over a common denominator, we get

$$
\begin{aligned}
&\frac{a^2}{(a-b)^2}\cdot\frac{1}{1-a^2x}-\frac{2ab}{(a-b)^2}\cdot\frac{1}{1-abx}+\frac{b^2}{(a-b)^2}\cdot\frac{1}{1-b^2x}\\
&=\frac{1}{(a-b)^2}\cdot\frac{a^2(1-abx)(1-b^2x)-2ab(1-a^2x)(1-b^2x)+b^2(1-a^2x)(1-abx)}{(1-a^2x)(1-abx)(1-b^2x)}\\
&=\frac{1}{(a-b)^2}\cdot\frac{a^2-2ab+b^2+(a^3b-2a^2b^2+ab^3)x}{(1-a^2x)(1-abx)(1-b^2x)}\\
&=\frac{1}{(a-b)^2}\cdot\frac{a^2-2ab+b^2+ab(a^2-2ab+b^2)x}{(1-a^2x)(1-abx)(1-b^2x)}\\
&=\frac{1}{(a-b)^2}\cdot\frac{(a-b)^2(1+abx)}{(1-a^2x)(1-abx)(1-b^2x)}\\
&=\frac{1+abx}{(1-a^2x)(1-abx)(1-b^2x)}.
\end{aligned}
$$

Now we take the case where $a=b$. In this case,

$$\frac{1}{(1-ax)(1-bx)}=\frac{1}{(1-ax)^2}=1+2ax+3a^2x^2+\cdots,$$

so $c_n=(n+1)a^n$.
Then

$$
\begin{aligned}
c_0^2+c_1^2x+c_2^2x^2+c_3^2x^3+\cdots&=\sum_{n=0}^{\infty}c_n^2x^n\\
&=\sum_{n=0}^{\infty}(n+1)^2a^{2n}x^n\\
&=\sum_{n=0}^{\infty}(n^2+2n+1)a^{2n}x^n\\
&=\sum_{n=0}^{\infty}n^2(a^2x)^n+2\sum_{n=0}^{\infty}n(a^2x)^n+\sum_{n=0}^{\infty}(a^2x)^n.
\end{aligned}
$$

We know that

$$
\begin{aligned}
\sum_{n=0}^{\infty}n^2(a^2x)^n&=\frac{a^2x+(a^2x)^2}{(1-a^2x)^3},\\
\sum_{n=0}^{\infty}n(a^2x)^n&=\frac{a^2x}{(1-a^2x)^2},\\
\sum_{n=0}^{\infty}(a^2x)^n&=\frac{1}{1-a^2x},
\end{aligned}
$$

so

$$\begin{aligned}\sum_{n=0}^{\infty} n^2(a^2x)^n + 2\sum_{n=0}^{\infty} n(a^2x)^n + \sum_{n=0}^{\infty}(a^2x)^n &= \frac{a^2x + a^4x^2}{(1-a^2x)^3} + \frac{2a^2x}{(1-a^2x)^2} + \frac{1}{1-a^2x} \\ &= \frac{(a^2x + a^4x^2) + 2a^2x(1-a^2x) + (1-a^2x)^2}{(1-a^2x)^3} \\ &= \frac{1+a^2x}{(1-a^2x)^3}.\end{aligned}$$

3. (Vandermonde's Identity) Let m, n, and k be nonnegative integers such that $k \le m$ and $k \le n$. Prove that
$$\binom{m+n}{k} = \binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \binom{m}{2}\binom{n}{k-2} + \cdots + \binom{m}{k}\binom{n}{0}.$$

Solution. Consider the identity
$$(1+x)^{m+n} = (1+x)^m(1+x)^n.$$

The coefficient of x^k in the left-hand side is $\binom{m+n}{k}$. The only way to get a term with x^k in the right-hand side is to multiply a term with x^i in the first factor with a term with x^{k-i} in the second factor, for $0 \le i \le k$.

The coefficient of x^i in $(1+x)^m$ is $\binom{m}{i}$, and the coefficient of x^{k-i} in $(1+x)^n$ is $\binom{n}{k-i}$. Summing over $0 \le i \le k$, we find that the coefficient of x^k in $(1+x)^m(1+x)^n$ is
$$\binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \binom{m}{2}\binom{n}{k-2} + \cdots + \binom{m}{k}\binom{n}{0}.$$

Hence, the two expressions are equal.

4. Let
$$\prod_{n=1}^{1996}(1 + nx^{3^n}) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m},$$
where a_1, a_2, …, a_m are nonzero and $k_1 < k_2 < \cdots < k_m$. Find a_{1996}. (Turkey, 1996)

Solution. We see that
$$\prod_{n=1}^{1996}(1 + nx^{3^n}) = (1+x^3)(1+2x^9)(1+3x^{27})\cdots(1+1996x^{3^{1996}}).$$

If we expand the right-hand side, then every term will be the product of some of the terms 1, x^3, $2x^9$, $3x^{27}$, and so on. Also, other than the term 1, each of these terms can only be used once. Hence, the exponent k_i is the i^{th} positive integer that can be written as the sum of distinct powers of 3 (not including 3^0).

Consider the sequence of positive integers that can be written as the sum of distinct powers of 3, not including 3^0:

$$3,\ 3^2,\ 3^2+3,\ 3^3,\ 3^3+3,\ 3^3+3^2,\ 3^3+3^2+3,\ 3^4,\ \dots.$$

We can divide every term by 3, to obtain the sequence

$$1,\ 3,\ 3+1,\ 3^2,\ 3^2+1,\ 3^2+3,\ 3^2+3+1,\ 3^3,\ \dots.$$

This is the sequence of positive integers that can be written as the sum of distinct powers of 3 including 3^0. If we replace every power of 3 with a power of 2, then we obtain the sequence of positive integers that can be written as the sum of distinct powers of 2:

$$1,\ 2,\ 2+1,\ 2^2,\ 2^2+1,\ 2^2+2,\ 2^2+2+1,\ 2^3,\ \dots.$$

By binary representation, this is the sequence of positive integers.

Hence, k_{1996} can be found by writing 1996 in binary representation, replacing every power of 2 with a power of 3, and then multiplying by 3. We have that

$$1996 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^3 + 2^2,$$

so

$$k_{1996} = 3^{11} + 3^{10} + 3^9 + 3^8 + 3^7 + 3^4 + 3^3.$$

Now, the only way to get a term of $x^{k_{1996}}$ in the given product is to take the product of all terms of the form nx^{3^n}, where 3^n appears in the base-3 expansion of k_{1996}, so the coefficient of $x^{k_{1996}}$ is

$$a_{1996} = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 4 \cdot 3 = 665280.$$

5. (Hockey Stick Identity) Let n and k be nonnegative integers, with $n \ge k$. Prove that

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Solution. We know that

$$\frac{1}{(1-x)^{k+1}} = \binom{k}{k} + \binom{k+1}{k}x + \binom{k+2}{k}x^2 + \cdots.$$

Dividing both sides by $1-x$, we obtain the generating function of the partial sums:

$$\frac{1}{(1-x)^{k+2}} = \binom{k}{k} + \left[\binom{k}{k} + \binom{k+1}{k}\right]x + \left[\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k}\right]x^2 + \cdots.$$

The coefficient of x^{n-k} is

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k}.$$

But

$$\frac{1}{(1-x)^{k+2}} = \binom{k+1}{k+1} + \binom{k+2}{k+1}x + \binom{k+3}{k+1}x^2 + \cdots,$$

and the coefficient of x^{n-k} is

$$\binom{n+1}{k+1}.$$

Hence, the two expressions are equal.

6. Let

$$\frac{1}{1-x-xy} = \sum_{n,m=0}^{\infty} a_{n,m} x^n y^m.$$

Find $a_{n,m}$.

Solution 1. We can write

$$\begin{aligned}\frac{1}{1-x-xy} &= \frac{1}{1-x(1+y)} \\ &= 1 + x(1+y) + x^2(1+y)^2 + \cdots \\ &= \sum_{n=0}^{\infty} x^n (1+y)^n \\ &= \sum_{n=0}^{\infty} x^n \sum_{m=0}^{n} \binom{n}{m} y^m \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} x^n y^m.\end{aligned}$$

Therefore,

$$a_{n,m} = \binom{n}{m}$$

for $0 \le m \le n$, and $a_{n,m} = 0$ otherwise.

Solution 2. We can write

$$\begin{aligned}\frac{1}{1-x-xy} &= \frac{\frac{1}{1-x}}{1-\frac{x}{1-x}y} \\ &= \frac{1}{1-x} + \frac{x}{(1-x)^2}y + \frac{x^2}{(1-x)^3}y^2 + \cdots \\ &= \sum_{m=0}^{\infty} \frac{x^m}{(1-x)^{m+1}} y^m \\ &= \sum_{m=0}^{\infty} \left[\binom{m}{m} x^m + \binom{m+1}{m} x^{m+1} + \binom{m+2}{m} x^{m+2} + \cdots \right] y^m \\ &= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \binom{n}{m} x^n y^m.\end{aligned}$$

Therefore,

$$a_{n,m} = \binom{n}{m}$$

for $0 \le m \le n$, and $a_{n,m} = 0$ otherwise.

7. Find the sequence (a_n) if $a_0 = 1$ and

$$\sum_{k=0}^{n} a_k a_{n-k} = 1$$

for all $n \ge 0$.

Solution. Let

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Then from the given relation,

$$\begin{aligned}A(x)^2 &= a_0^2 + (a_0 a_1 + a_1 a_0)x + (a_0 a_2 + a_1^2 + a_2 a_0)x^2 + \cdots \\ &= 1 + x + x^2 + \cdots \\ &= \frac{1}{1-x},\end{aligned}$$

so

$$A(x) = \frac{1}{\sqrt{1-x}}.$$

(We take the positive square root, since the constant coefficient must be 1, i.e. $A(x)$ must satisfy $A(0) = 1$.)

By the generalized Binomial theorem,

$$\begin{aligned} A(x) &= \frac{1}{\sqrt{1-x}} \\ &= (1-x)^{-\frac{1}{2}} \\ &= 1 - \binom{-\frac{1}{2}}{1} x + \binom{-\frac{1}{2}}{2} x^2 - \cdots . \end{aligned}$$

The coefficient of x^n is $A(x)$ is then

$$\begin{aligned} a_n &= (-1)^n \binom{-\frac{1}{2}}{n} \\ &= (-1)^n \cdot \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-n+1)}{n!} \\ &= (-1)^n \cdot \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2n-1}{2})}{n!} \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} \\ &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n)}{2^n n! \cdot 2 \cdot 4 \cdots 2n} \\ &= \frac{(2n)!}{2^n n! \cdot 2^n \cdot 1 \cdot 2 \cdots n} \\ &= \frac{(2n)!}{4^n (n!)^2} \\ &= \frac{1}{4^n} \binom{2n}{n}. \end{aligned}$$

8. Let

$$(1+x+x^2+x^3+x^4)^{496} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{1984} x^{1984}.$$

Determine $\gcd(a_3, a_8, a_{13}, \ldots, a_{1983})$. (IMO Proposal, 1983)

Solution. Let $\omega = e^{2\pi i/5}$, so $\omega^5 = 1$. Furthermore, $\omega^k = 1$ if and only if k is divisible by 5. Let

$$\begin{aligned} A(x) &= x^2(1+x+x^2+x^3+x^4)^{496} \\ &= a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots \\ &= \sum_{n=0}^{1984} a_n x^{n+2}. \end{aligned}$$

Then

$$A(1) + A(\omega) + A(\omega^2) + A(\omega^3) + A(\omega^4) = \sum_{n=0}^{1984} [1 + \omega^{n+2} + \omega^{2(n+2)} + \omega^{3(n+2)} + \omega^{4(n+2)}] a_n.$$

If $n+2$ is divisible by 5, then $\omega^{n+2} = 1$, so

$$1 + \omega^{n+2} + \omega^{2(n+2)} + \omega^{3(n+2)} + \omega^{4(n+2)} = 5.$$

Otherwise, $\omega^{n+2} \neq 1$, so

$$1 + \omega^{n+2} + \omega^{2(n+2)} + \omega^{3(n+2)} + \omega^{4(n+2)} = \frac{1 - \omega^{5(n+2)}}{1 - \omega^{n+2}} = 0.$$

(It is not hard to check that $\omega^{n+2} = 1$ if and only if $n+2$ is divisible by 5, so the denominator is nonzero.)

Hence,

$$A(1) + A(\omega) + A(\omega^2) + A(\omega^3) + A(\omega^4) = 5(a_3 + a_8 + a_{13} + \cdots + a_{1983}),$$

so

$$a_3 + a_8 + a_{13} + \cdots + a_{1983} = \frac{1}{5}[A(1) + A(\omega) + A(\omega^2) + A(\omega^3) + A(\omega^4)].$$

We see that $A(1) = 5^{496}$, and for $1 \le k \le 4$, $\omega^k \neq 1$, so

$$\begin{aligned} A(\omega^k) &= \omega^{2k}(1 + \omega^k + \omega^{2k} + \omega^{3k} + \omega^{4k})^{496} \\ &= \omega^{2k}\left(\frac{1 - \omega^{5k}}{1 - \omega^k}\right)^{496} \\ &= 0. \end{aligned}$$

Therefore,

$$a_3 + a_8 + a_{13} + \cdots + a_{1983} = \frac{5^{496}}{5} = 5^{495}.$$

Let $d = \gcd(a_3, a_8, a_{13}, \dots, a_{1983})$. Then $d \mid (a_3 + a_8 + \cdots + a_{1983})$, or $d \mid 5^{495}$.

Also, $d \mid a_{1983}$. By the Binomial theorem,

$$\begin{aligned} &(1 + x + x^2 + x^3 + x^4)^{496} \\ &= [(1 + x + x^2 + x^3) + x^4]^{496} \\ &= (1 + x + x^2 + x^3)^{496} + \binom{496}{1}(1 + x + x^2 + x^3)^{495}x^4 + \binom{496}{2}(1 + x + x^2 + x^3)^{494}x^{2\cdot 4} \\ &\quad + \cdots + \binom{496}{494}(1 + x + x^2 + x^3)^2 x^{494\cdot 4} + \binom{496}{495}(1 + x + x^2 + x^3)x^{495\cdot 4} + x^{496\cdot 4} \\ &= \sum_{k=0}^{496} \binom{496}{k}(1 + x + x^2 + x^3)^{496-k}x^{4k}. \end{aligned}$$

In the general term

$$\binom{496}{k}(1 + x + x^2 + x^3)^{496-k}x^{4k},$$

the maximum degree of a term is $3(496-k)+4k=k+1488$, and $k+1488 \geq 1983$ if and only if $k \geq 495$. Therefore, a_{1983}, the coefficient of x^{1983} in $(1+x+x^2+x^3+x^4)^{496}$, is the same as the coefficient of x^{1983} in

$$\binom{496}{495}(1+x+x^2+x^3)x^{495\cdot 4}+x^{496\cdot 4}=496x^{1980}+496x^{1981}+496x^{1982}+496x^{1983}+x^{1984},$$

so $a_{1983}=496$, which means $d \mid 496$. But 5^{495} and 496 are relatively prime, so $d=1$.