Problem: $f:\mathbb{R}\to\mathbb{R}, \quad f(x+f(y))=y+f(x)\,\forall x,y\in\mathbb{R}$ and $S=\left\{\dfrac{x}{f(x)}\,|\,x\in\mathbb{R}\right\}$ is finite.

Solution: Let $P(x,y)$ denote the functional equation.

$$P(0,y)\Longrightarrow f(f(y))=y+f(0)\Longrightarrow f \text{ is bijective.}$$

$$\Longrightarrow f(f(0))=f(0)\Longrightarrow f(0)=0\Longrightarrow f(f(y))=y$$

$$P(f(x),y)\Longrightarrow f(f(x)+f(y))=y+x$$

$$\Longrightarrow f(x+y)=f(f(f(x)+f(y)))=f(x)+f(y)$$

Let $|S|=k$ and $S=\{c_1,c_2,\ldots,c_k\}$. Also let $a_i=\dfrac{1}{c_i}$.

We know for every non-zero $x\in\mathbb{R}$, there exists a c_r such that

$$\frac{x}{f(x)}=c_r\Longrightarrow f(x)=\frac{1}{c_r}x=a_r x$$

Now let $Q(x,y)\Longrightarrow f(x+y)=f(x)+f(y)$

Let $U=\{a_i\,|\,\exists x>0 \text{ with } f(x)=a_i x\}$. We claim $|U|=1$.

Proof:

We proceed indirectly. Suppose $|U|>1$. So there are at least two distinct elements in U. Suppose a_m is the largest element of U and a_n is the second largest element of U. (I mean it is only smaller than a_m. If there are more than two elements in U, it is larger than all of them, only except a_m)

Consider $x,y>0$ Such that $f(x)=a_m x$ and $f(y)=a_n y$.

So there exists a_z (not necessarily equal to a_m or a_n) such that

$$f(x+y)=a_z(x+y)$$

Also notice that

$$f(x+y)=f(x)+f(y)\Longrightarrow a_z(x+y)=a_m x+a_n y$$

$$\Longrightarrow(a_m-a_z)x+(a_n-a_z)y=0 \qquad (1)$$

If $a_n\geqslant a_z$, then LHS of (1) is positive. If $a_m=a_z$, then LHS of (1) is negative. In both cases we get a contradiction. So such a_m and a_n can not exist. Therefore $|U|=1$.

So $f(x)=ax$ for some sonstant a and for all positive real x.

Obviously f is odd because $Q(x,-x)\Longrightarrow 0=f(x)+f(-x)$

So $f(-x)=-ax$ for all negaive x. Therefore $f(x)=ax\forall x\in\mathbb{R}$. Now checking shows $a\in\{1,-1\}$. So all the functions are $f(x)=x$ and $f(x)=-x$.