

PARTIALLY ORDERED SETS

A *partially ordered set*, or *poset* for short, is a collection of elements, drawn as small circles, linked together by straight lines, where if an element a is connected to an element b by a straight line, and a is lower on the page than b is, then a is thought of as *less* than b in some way. There are many examples in mathematics (or real life) when objects are ordered in some way, and they are examples of posets.

- Take some numbers, say 1, 2, 3 and 8. We can order them by the usual $<$ and get the poset



Here, the line from 3 to 8 indicates that 3 is less than 8 (under the ordering $<$), because 3 is lower on the page than 8 is. It is not necessary that the line be vertical, only that it end at a higher point on the page than where it started. Also, the length of the line is unimportant. Note that 2 is less than 8, but we have not drawn a line from 2 to 8; this is because there is an element 3 in between, so from the facts that $2 < 3$ and $3 < 8$, illustrated by the lines between 2 and 3 and between 3 and 8, we can deduce that 2 is less than 8, so we don't need a line to tell us that. (Or, in other words, the *upward path* from 2 through 3 to 8 tells us that $2 < 8$.)

- Next, use the same numbers, but order them by “divides into” rather than $<$. That is, 1 divides into 2, but 2 does not divide into 3, and so on. This time we get the poset



Notice that, since 2 does not divide into 3, there is no line from 2 to 3 which goes up the page. It is unimportant whether the element 3 is higher or lower on the page than 2 is.

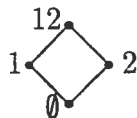
- Suppose we take a set, say $\{1, 2\}$, and list all its subsets. We get

$$\{\emptyset, \{1\}, \{2\}, \{1, 2\}\},$$

where \emptyset is the symbol for the *empty set*, by which we mean the set with no elements in it. Usually we will save ourselves some notation by dropping the parentheses and commas in each set, so we will write the above sets as

$$\{\emptyset, 1, 2, 12\}.$$

Now suppose we order these sets by \subset (containment), so that $1 \subset 12$ for example (and of course the empty set is contained in every set). Then we get the poset



Notice that once again we have not drawn a line from \emptyset to 12, even though $\emptyset \subset 12$ is true; this is because either of the two paths from \emptyset to 12 already in the picture will be enough to show this. We always omit all such unnecessary lines from posets. Suppose we use three or four elements in our set instead of two, and still list all subsets and order them by \subset : what do the posets look like then?

Maybe the simplest question we could ask about posets is: how many are there of a certain size? Here we don't care what the elements are called, only what the pictures look like. And things like length and slope of lines don't matter either. For example, there is only one poset with one element (a dot), and just two with two elements, namely

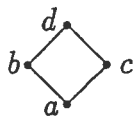


There are 5 posets with three elements and 16 posets with four elements: find them.

Problem 1. How many posets with five elements are there?

Chains and Antichains.

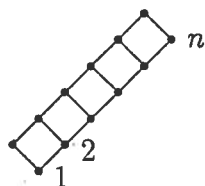
A poset in which every two elements are ordered one way or the other is called a *chain*. A chain with n elements in it is denoted by n . In fact, a subset of the elements of a poset, which form a chain by themselves, is also called a chain. For example, the poset



is not a chain, because b is not less than c or bigger than c . It has lots of chains in it, though; each single element is a chain, so is $\{a, b\}$ and most other two-element subsets, even $\{a, d\}$ is a chain because $a < d$ is true, but $\{b, c\}$ is not a chain. Also, $\{a, b, d\}$ and $\{a, c, d\}$ are chains. Finally, the empty set \emptyset is also considered a chain in any poset.

How many chains does the poset n have?

Problem 2. How many chains does the poset



have?

An *antichain* is a poset, or a subset of a poset, in which no two elements are ordered. A poset with n elements which is an antichain is denoted n .

How many antichains does n have? How many does n have?

Problem 3. How many antichains does the poset

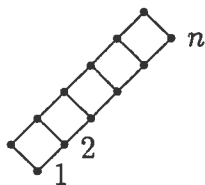


have?

Suppose that the largest chain of a poset has n elements. Prove that the poset can be partitioned into n antichains, but no fewer.

Problem 4. Suppose that the largest antichain of a poset has n elements. Prove that the poset can be partitioned into n chains, but no fewer. (This is called Dilworth's Theorem, and is harder than the previous question.)

The maximal chains of a poset do not have to be all the same size, but if they are, we say that the poset is *graded*, and each element of the poset has a height. The elements at the same height form a *level* of the poset. For instance, the poset



is graded, because all maximal chains have $n + 1$ elements. The bottom element has height 0 and the top element height n . There are $n + 1$ levels. Note that every level is an antichain, but not every antichain is in only one level.

Find a graded poset which has only one antichain of largest size, and this antichain is not a level.

Now we'll look again at the poset of all subsets of a set, ordered by \subset . The poset of all subsets of the set $\{1, 2, \dots, n\}$ is denoted B_n .

What does B_n look like? Show it is graded. How many levels does it have? What are the sizes of the levels? Show that the biggest level is the middle one(s).

It is true that in B_n , no antichain is larger than the largest level. This result is called *Sperner's Theorem*.

Problem 5. Prove Sperner's Theorem three ways:

Proof 1: Suppose \mathcal{A} is an antichain in B_n . Let p_k be the number of members of \mathcal{A} of size k , for each integer k . For each member A of \mathcal{A} which has size k , count the number of permutations of the set $\{1, 2, \dots, n\}$ which begin with the elements of A in some order. Then show that

$$\sum_{k=0}^n k!(n-k)!p_k \leq n! \quad \text{and thus} \quad \sum_{k=0}^n \frac{p_k}{\binom{n}{k}} \leq 1.$$

Now note that $|\mathcal{A}| = \sum_{k=0}^n p_k$ and hence show that $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof 2: The idea of this proof is that if \mathcal{A} is an antichain in B_n which contains sets of size different from $(n-1)/2$ (if n is odd) or $n/2$ (if n is even), then such sets can be replaced by just as many other sets closer in size to $\lfloor n/2 \rfloor$ while still getting an antichain. In this way all sets can be replaced by sets all of the same size. To make this proof work we need to prove the following. Suppose that \mathcal{B} is any collection of k -element subsets of

$\{1, 2, \dots, n\}$, where $k < n$. Let $\nabla\mathcal{B}$ denote the collection of all $(k+1)$ -element subsets of $\{1, 2, \dots, n\}$ which contain some member of \mathcal{B} . We now count the number of ordered pairs (B, D) where $B \in \mathcal{B}$, $D \in \nabla\mathcal{B}$ and $B \subset D$. Show that this number is exactly $(n-k)|\mathcal{B}|$ and is at most $(k+1)|\nabla\mathcal{B}|$, and thus prove that

$$|\nabla\mathcal{B}| \geq \frac{n-k}{k+1}|\mathcal{B}|.$$

Conclude that if $k \leq (n-1)/2$ then $|\nabla\mathcal{B}| \geq |\mathcal{B}|$. Using a symmetrical result for $k \geq (n+1)/2$, get Sperner's Theorem.

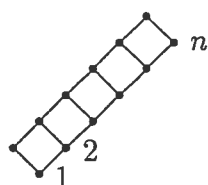
Proof 3: First we define a *symmetric chain* to be a chain in B_n that contains a set from every level from s to $n-s$, for some $s \leq n/2$. The idea of this proof is to show that B_n can be partitioned into disjoint symmetric chains, from which the proof will follow quickly. The proof that B_n is a disjoint union of symmetric chains can be done by induction on n , using that B_n is the disjoint union of B_{n-1} and the set of all subsets of B_n containing the element n . The trick is to figure out how to turn the symmetric chains in B_{n-1} (which exist by induction hypothesis) into symmetric chains in B_n .

Linear Extensions.

Here's another example of a poset. Suppose there are a number of tennis players, and they play a number of matches. Each two players play each other at most once, and we assume that there are no draws. We also assume that the players are naturally ordered by ability, and that the better player always beats the worse player. (Of course, this ordering is not known in advance.) Then the result of a number of matches can be illustrated by a poset, where the elements are the players, and for any players a and b , a line from a up to b in the poset means that b has beaten a (so we know that b is better than a). Before any matches have been played, the poset will be an antichain. When everyone has played everyone (or probably earlier than that), the poset will be a chain.

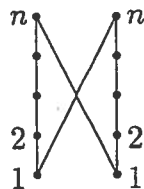
Now suppose only some matches have been played. Suppose we'd like to guess at what the final ordering of the players, from worst to best, might be. We would need to form a chain using all the players which preserves all the ordering found so far. Such a chain is called a *linear extension* of the poset.

How many linear extensions does n have? How many does n have? How many does the poset



have? (You obtain a sequence of numbers known as the Catalan numbers.)

Problem 6. How many linear extensions does the poset



have?

The next problem shows that the posets B_n contain every poset inside them.

Problem 7. Show that every (finite) poset is a subset of some B_n .

The *smallest* positive integer n so that poset P is a subset of B_n is called the *dimension* of P . This is also the smallest number of linear extensions of P whose “intersection” is P . Find the dimensions of the various posets considered above.

Two Open Questions.

To finish, here are two questions which are not hard to understand but which are still unsolved. So don’t spend too much time on them!

1) Look again at the tennis players poset. Suppose some games have been played, so we have a poset P , but that we still don’t know the complete ordering of the players (so P is not a chain). It seems reasonable to define the *probability* that player a is worse than player b to be

$$p(a < b) = \frac{\text{number of linear extensions of } P \text{ in which } a < b}{\text{total number of linear extensions of } P}.$$

Suppose we wanted to find two players a and b for which $p(a < b)$ is $1/2$. Of course exactly $1/2$ isn’t always possible, but how close can we always get? There is a small poset in which the best we can do is find a and b so that $p(a < b) = 1/3$ (or $2/3$). So here is the unsolved problem:

For every finite poset P which is not a chain, are there always two elements a and b so that $p(a < b)$ is between $1/3$ and $2/3$??

It has been proved that this statement is correct for the numbers $3/11$ and $8/11$!

2) For each positive integer n ,

- find a poset containing exactly n antichains. (Easy.)
- show that there is a poset whose largest chain has only two elements and which contains exactly n antichains. (Solved by V. Linek when he was an undergrad student at the University of Calgary.)
- find a poset containing exactly n chains. (Easy.)

And finally, the unsolved problem:

- show that there is a poset whose largest antichain has only two elements and which contains exactly n chains. (Unknown if this is true, but it probably is.)

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