1998 IMO Camp

Problem Set - Graphs and Algorithms

1. A chain is a sequence of edges in a graph so that consecutive edges have a vertex in common. Given a graph with n vertices and q edges numbered $1, 2, 3, \ldots, q$, Show that there exists a chain of at least $\lceil 2q/n \rceil$ edges that is monotonic with respect to the edge numbering.

~~[illegible]~~

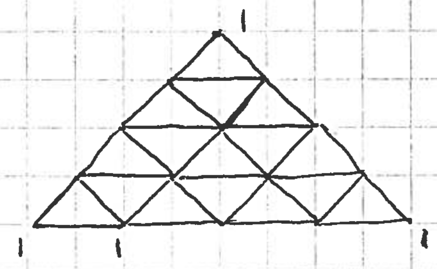
2. Show that the edges of any graph with n vertices can be covered using $\lfloor n^2/4 \rfloor$ or fewer edges (K_2's) and triangles (K_3's).

3. Show that the edges of any graph with n vertices can be partitioned using $\lfloor n^2/4 \rfloor$ or fewer edges and triangles

4. Place a beetle at each of the vertices of an equilateral triangle of side 6. The beetles all move at the same speed. When they reach a vertex they must turn, reversing is not allowed.
   a. show that at some time there must be two beetles at the same vertex.
   b. is this true for a 5-sided equilateral triangle?

5. An equilateral triangle is made up of rows of smaller equilateral triangles: 1 in the first, 3 in the second, 5 in the third, ..., $2n-1$ triangles in the n^{th} row. At the start each edge is blue. There are three paint machines A, B and C. They start in the three corners of the big triangle. They then move in turn, first A then B then C. They move along the edge of a small triangle, but only if the edge is blue, they are not allowed to move along a red edge. They paint the edge red as they move along it. More than one machine is allowed to be at the same vertex. Show, by induction that for every n all edges of the triangle can be coloured red.

6. Take an equilateral triangle of side n which is divided up into smaller unit equilateral triangles.



Four 1's are placed, one at each of the three corners of the large triangle and the fourth next to the bottom left corner as in the figure, and 0's at all the other vertices.
The vertices of a small rombus (formed by two of the unit equilateral triangles) can all be increased or decreased by 1. Is it possible to change all numbers to 0's? Explain! (If yes, give an algorithm, if no: prove!)

7. In town A, there are n girls and n boys, and each girl knows each boy. In town B, there are n girls $g_1, g_2, \dots, g_n$ and $2n-1$ boys $b_1, b_2, \dots, b_{2n-1}$.
The girl g_i, $i = 1, 2, \dots, n$, knows the boys $b_1, b_2, \dots, b_{2i-1}$, and no others. For all $r = 1, 2, \dots, n$, denote by $A(r), B(r)$ the number of different ways in which r girls from town A, respectively town B, can dance with r boys from their own town, forming r pairs, each girl with a boy she knows. Prove that $A(r) = B(r)$ for each $r = 1, 2, \dots, n$.

8. Let G be a graph, with a function defined on the vertices, taking on non-negative integer values such that:
   a) if v and w are adjacent, then $|f(v) - f(w)| \leq 1$;
   b) if $f(v) > 0$ then v is adjacent to at least one vertex w such that $f(w) < f(v)$; and
   c) there is exactly one vertex v_0 such that $f(v_0) = 0$.

Show that $f(v)$ is the distance from v to v_0.

9. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, 3, \dots, k$ in such a way that at each vertex which belongs to two or more edges the greatest common divisor of the integers labelling those edges is equal to 1.

10. Prove that for $n \geq 5$, every graph with n vertices and $\lfloor n^2/4 \rfloor + 2$ edges contains two triangles with exactly one vertex in common (a bowtie).

11. On an infinite chess board, a game is played as follows. At the start, n^2 pieces are arranged on the chess board in an $n \times n$ block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece that has been jumped over is then removed. Find the values of n for which the game can end with only one piece remaining on the board.

12. Let K_9 be the complete graph on 9 vertices. Begin to colour the edges of this graph, one at a time, using either red or blue. What is the smallest number of edges you would have to colour this way to be sure that among the coloured edges there will always be a monochromatic triangle?

13. There are 1001 steps going up a hill. Some steps have rocks on them (no more than one rock per step). Christopher may pick up any rock and raise it one or more steps up to the next empty step. His opponent, Georg, rolls some rock (with an empty step directly below it) down one step. To start, there are 500 rocks on the first 500 steps. Christopher and Georg move rocks in turn with Christopher making the first move. Christopher wins if he gets a rock onto the top step. Can he win?

14. Take k crosses each with 4 arms. A move is to join one arm with another by a curve which does not cross any of the already drawn curves. After a curve is drawn, put a new cross arm at some point on the curve. Assume that this is a game for two players with the players making alternate moves and the last player to move wins. Assume that the game finishes, which player wins?

15. Two players play on an 8×8 chess-board. The first player places a knight on the chess-board. They then take turns in moving the knight starting with the second player. The knight is not allowed to revisit a square. Who wins?