FOREWORD

December 15, 2004

This book is intended for secondary students with some experience in school geometry. It is assumed that they have had enough elementary Euclidean geometry to cover theorems about congruences of triangles, isosceles and right triangles, basic area theorems for triangles and quadrilaterals, properties of circles and concyclic quadrilaterals. It is expected that the reader will have been introduced to the definitions of translations, rotations and reflections, but has not used them as a tool for solving geometric problems.

This is a book of basic definitions and results, along with some worked examples. However, it should not be read like a novel; the reader should pause at each definition and result to see whether it is are understood and try to look at examples of whatever is being discussed. Before reading the solutions of the problems, the reader should think about them first and try to solve them first.

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NOTATION AND TERMINOLOGY

Altitude of a triangle: The line segment that passes through a vertex of a triangle and is perpendicular to the opposite side.

Area of a figure: The area of a geometric figure is indicated by square brackets: $[\cdots]$

Centroid of a triangle: The point at which the medians intersect.

Circumcircle of a polygon: A circle that passes through the vertices of a polygon. Its radius is called the *circumradius*.

Collinear points: Points that lie on a straight line.

Concylic quadrilateral: A quadrilateral through whose four vertices passes a circle.

Congruent geometric figures: Figures either of which can be obtained from the other by a rigid transformation that preserves angles and lengths. Congruence is indicated by the notation $\equiv$.

Congruence theorems for triangles: Two triangle are congruent if any one of the folling holds: (1) Two sides and the contained angle of one are correspondingly equal to two sides and the contained angle of the second (SAS); (2) Three sides of one are correspondingly equal to three sides of the second (SSS); (3) Two angles of one and the side connecting them are correspondingly equal to two angles of the second and the side connecting them (ASA).

Distance between points: The distance between points A and B is denoted either by $|AB|$ or AB, depending on context.

Incentre of a triangle: The point at which the three angle bisectors of a triangle intersect; the centre of the incircle of the triangle.

Incircle of a polygon: A circle that is tangent to each side of a polygon. It radius is called the *inradius*.

Median of a triangle: The line segment that joins a vertex of a triangle to the midpoint of the opposite side.

Orthocentre of a triangle: The point at which the three altitudes of the triangle intersect.

Pedal point: The point at which the altitude of a triangle from a vertex intersects the opposite side.

Produced: A segment or half line is produced when the full line containing it is drawn; the full line is called the *production* of the segment or half line.

Similar geometric figures: Geometric figures of the same shape, in which one is a scaled version of the other. Figures whose linear dimensions are in a fixed proportion. Similarity is indicated by $\sim$.

Section 1

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The purpose of this book is not just to introduce a few techniques for solving geometry problems, but to encourage a different way of thinking about geometry among students who may have just been given a standard Euclidean approach. The best way to begin is with a few examples.

Problem 1.1. In the diagram, $AD = BC$ and $\angle ABD + \angle BDC = 180°$. Show that $\angle BAD = \angle BCD$.

Figure 1.1.

Solution 1. This is quite a tough problem to tackle until we get the inspiration to start moving things around. Note that we have two equal segments in the situation as well as a pair of angles that are supplementary (add up to $180°$). Let us exploit this. Turn over the triangle BCD so that the positions of B and D are interchanged and C goes to C'.

Figure 1.2.

Then $\angle DBC' + \angle ABD = \angle BDC + \angle ABD = 180°$ and $DC' = BC = AD$, so that A, B, C' are collinear and triangle ADC' is isosceles, whereupon $\angle BAD = \angle DC'B = \angle BCD$. ♠

Solution 2. Another possibility is to lay one equal side on top of the other. Suppose that we detach triangle BCD and lay point B on D and C on A, so that D falls on a position E, as in the diagram.

Figure 1.3.

Then $ABDE$ is a concyclic quadrilateral, since $\angle ABD + \angle DEA = \angle ABD + \angle BDC = 180°$. Also, angle BAD and DAE are subtended by equal chords at the circumference of the circumcircle of $ABDE$, so they are equal. Hence $\angle BAD = \angle DAE = \angle BCD$. ♠

Problem 1.2. Suppose that we have a circle of radius r, and construct as in the diagram, a rectangle whose sides OP and OQ lie along perpendicular diameters. What is the length of PQ?

Figure 1.4.

Solution. The answer is immediate once we flip the rectangle over onto itself, so that its diagonal goes from the centre O to the vertex on the circumference of the circle. ♠

Problem 1.3. The unit square is partitioned into four triangles and one quadrilateral with areas a, b, c, d and e as indicated in the diagram below.

Figure 1.12.

Each of the three partitioning lines join a vertex to the midpoint of one of the opposite sides. Determine the values of a, b, c, d, e.

Solution. It is straightforward to determine that $a = \frac{1}{4}$, $b + d = \frac{1}{2}$, $c + e = d + e = \frac{1}{4}$ and so $c = d$; this can be done for example by noting how the square can be covered by four nonoverlapping right triangles with arms of length 1 and $\frac{1}{2}$. However, this gives us only four independent equation for five variables, and it is not clear how we can find another condition to nail the values down.

However, we note that if we rotate the triangle with area e through $90°$ clockwise, it falls on part of the triangle with area d, and one can see that by expanding the linear dimensions by a factor of 2, it will exactly cover the larger triangle. (Another way of looking at it is to note that one can cover the triangle of area d by four copies of the triangle with area e.) Thus, $d = 4e$ and we find that

$$(a, b, c, d, e) = \left(\frac{1}{4}, \frac{3}{10}, \frac{1}{5}, \frac{1}{5}, \frac{1}{20}\right) . \qquad ♠$$

Problem 1.4. Suppose that a unit square is partitioned into 9 polygons by various lines joining vertices to midpoints of sides as shown in the diagram. What is the area of the quadrilateral $TUVW$ in the middle?

`Figure 1.5.`

Solution. It seems evident that $TUVW$ is in fact a square. Let us pursue this a little. Upon reflection, we realize that our intuition is fed by the symmetry of the situation. So let us try to capture this ingredient. If we rotate the square through an angle of $90°$ about its centre, then $A \to B$, $B \to C$, $C \to D$, $D \to A$, $P \to Q$, $Q \to R$, $R \to S$, $S \to P$ (we will use "$\to$" to mean "goes to"). The segment AP falls on BQ. Since the rotation is through a right angle, $AP \perp BQ$. Similarly, $BQ \perp CR$, $CR \perp DS$, so that $TUVW$ is at least a rectangle. But T, the intersection of AP and BQ falls on U the intersection of BQ and CR, and we find that $T \to U$, $U \to V$, $V \to W$ and $W \to T$; thus $TU = UV = VW = WT$.

To answer the question posed, let triangle ATQ be rotated about the point Q, so that A falls on D and T on T'. Then $T'DWT$ is a square equal to $TUVW$. Similarly, we can rotate triangle BUR, CUS and DWP to form a cross consisting of five congruent squares, one of which is $TUVW$ and all of which have combined area equal to that of the square. Accordingly, the area of $TUVW$ is one-fifth of that of $ABCD$. ♠

`Figure 1.6.`

Problem 1.5. Suppose a rectangle of dimensions 9×16 is given. With two straight cuts, partition it into three pieces that can be rearranged into a square.

Solution 1. The square, of course, will have side length 12. Begin by marking off a length 12 along one of the larger sides of the rectangle, and we will try to make this one side of the square. Slice from the mark to a vertex, as indicated in the diagram.

`Figure 1.7.`

Slide the trapezoid up so that its right side moves directly above the vertex of the lower triangle. This leaves a triangular piece protruding on the left that can be snipped and relocated to fill the triangular gap on the right. ♠

Figure 1.8.

Solution 2. There is an alternative approach. Make the first cut of length 12 from one vertex to the opposite longer side of the rectangle. Make a cut perpendicular to this to obtain the following situation. Note that the two triangles are similar.

Figure 1.9.

Now sliding two of the three pieces will give the desired rearrangement. ♠

Figure 1.10.

Exercise. One attempt at a solution is to make cuts from to ends of one of the long sides of the rectangle to the midpoint of the opposite side, and then rearrange the pieces. Does this work? Justify your answer.

Figure 1.11.

Problem 1.6. Let a rectangular piece of paper with vertices A, B, C, D be given with $AB > BC$. It is folded in such a way that the fold passes through A and a point E on BC and the vertex B falls on a point F on the side DC. Thus, the triangle ABE is folded onto triangle AFE.

Figure 1.12.

Devise a simple test to determine whether the area of triangle AEF is less than half the area of quadrilateral $AECD$, *i.e.*, $[AEF] < \frac{1}{2}[AECD]$.

Solution. In the extreme case that $AB = BC$, then the fold is along the diagonal AC, so that $E = C$ and $F = D$. The triangle covers all of the remaining (degenerate) quadrilateral. On the other hand, if AB is much longer than BC, then the point F will be close to C and the triangle will cover much less than half of the quadrilateral. Thus, the solution to the problem appears to turn on where F falls on the side CD.

We first show that, if $F = G$, the midpoint of CD, then $[AEG]$ is exactly half of $[AECD]$, or equivalently, $[AEG] = [ECG] + [AGD]$. Let H be the midpoint of AB so that $HG \| BC$; let K be the foot of the perpendicular from E to HG, and let L be the intersection of AE and HK.

Figure 1.13.

A $180°$ rotation about L carries AHL onto EKL. By noting congruent halves of rectangles, we note that

$$\begin{aligned}[AEG] &= [EKG] + [EKL] + [ALG] = [EKG] + [AHL] + [ALG] \\ &= [ECG] + [AHG] = [ECG] + [ADG] \ .\end{aligned}$$

A modification of this argument works when $F \neq G$. In this case, determine H on AB so that $HF \| BC$; let K be the foot of the perpendicular from E toHF and L the intersection of AE and

HF. Then

$$\begin{aligned}[AEF] - [ECF] - [ADF] &= [ALF] + [EKL] + [EKF] - [ECF] - [ADF] \\ &= [AHF] - [AHL] + [EKL] - [ADF] = [ADF] - [AHL] + [EKL] - [ADF] \\ &= [EKL] - [AHL] \ .\end{aligned}$$

When F lies between G and C, the triangle EKL is similar to and smaller than triangle AHL and $[AEF] < \frac{1}{2}[AECD]$. When F lies between G and D, triangle EKL is similar to and greater than $[AHL]$ and $[AEF] > \frac{1}{2}[AECD]$.

One test that can be made is to fold the side BC up to the side AD. If F lies below the fold, then $[AEF] < \frac{1}{2}[AECD]$; if above, then $[AEF] > \frac{1}{2}[AECD]$. ♠

Comment. This is not the only test that can be devised; the reader is invited to find other ways of solving the problem.

Problem 1.7. A *cycloid* is the locus of a point on the circumference of a circle that rolls without slipping along a base line. An arch of the cycloid is that arc on the curve that connects two successive positions where the moving point touches the base line. If A is the area of the rotating circle, determine the area between one arch of the cycloid and the base line along which the circle rolls.

Figure 1.14.

Solution. The required area is twice the area under that part of the arch which connects the point O on the base line to the point R at greatest distance from this line. Let ON be that segment on the base line upon which the generating circle rolls and let MR be parallel to this line, so that $ONRM$is a rectangle whose dimensions are half the circumference of the circle and the diameter of the circle. The area of rectangle $ONRM$ is twice the area of the generating circle.

Figure 1.15.

Pet P be a typical point on the cycloid. Construct the chord PQ of the generating circle at this position parallel to the base line. Let the generating circle at this position touch ON at X and MR at Y, so that $XY$is a diameter. Let $|.|$ denote the length of a line segment and $(.)$ denote the length of an arc on the circle. Then

$$\begin{aligned}|RY| &= |NX| = |ON| - |OX| \\ &= (XY) - (PX) = (PY) = (QY) \ .\end{aligned}$$

Consider the locus of the point Q. Since $|RY| = (QY)$, we can imagine a circle of the same size as the generating circle rolling along the line RM, taking the same positions as the generating circle in reverse, with Q a point on it tracing out an upside-down cycloid joining R and D. This second cycloid is congruent to the first. Indeed, a rotation of $180°$ about the centre of rectangle $ONRM$ carries one cycloid to the other.

Let B be the area between the two cycloids, *i.e.*, of the tear-shaped region $OPRQO$. At each horizontal level, the length of the chord PQ is the same as the length of the chord of the generating circle at the same level. Accordingly, the drop-shaped region and the circle have the same area. (This presumption is known as *Cavalieri's principle.*) Let C be the area between the cycloid OQR at the base line ON; C is also the area of the region $OPRYB$.

We have that $2A = B + 2C = A + 2C$ so that $2C = A$, The area under half the arch of the cycloid is $B + C = A + C$. Hence the area under the complete arch of the cycloid is $2(A + C) = 2A + 2C = 3A$, *i.e.*, thrice the area of the generating circle. ♠.

These seven problems illustrate how a dynamic approach can provide insight into the underlying structure and lead to arguments that are natural and convincing. The purpose of this monograph is to pursue this theme and give a sytematic introduction to transformations that will provide the tools for describing the procedures more precisely.

We begin with some definitions of transformations that preserve distance and angle.

Section 2

Isometries

December 15, 2007

An *isometry* is a transformation of the plane that preserves distances. That is, an isometry T is a one-one function that takes the plane onto itself for which $\mathrm{dist}(P, Q) = \mathrm{dist}(T(P), T(Q))$, where "dist" represents the (Euclidean) distance between two points and P and Q are arbitrary points in the plane. We will often use $X \to Y$ to represent $Y = T(X)$. There are a number of useful consequences of this definition; can you see why they are so?

1. The image of a line under an isometry is a line (since each point on a line through X and Y is uniquely determined by specifying its distances from X and Y respectively).

2. If, for two points P and Q, $T(P) = P$ and $T(Q) = Q$, then $T(X) = X$ for every point X on the line through P and Q. (A point X for which $T(X)$ is said to be *fixed* by T.)

3. If, for three noncollinear points, P, Q, R, we have that $T(P) = P$, $T(Q) = Q$, $T(R) = R$, then $T(X) = X$ for every point X in the plane. An isometry that fixes (carries to itself) every point of the plane is known as the *identity*.

4. If two lines l and m intersect at angle α, their images $T(l)$ and $T(m)$ are also lines that intersect at the same angle α. (This says that isometries are *conformal*; they preserve angles.)

5. If S is a plane figure with area σ, then $T(S)$ is a figure with the same area σ. (In fact, the figure $T(S)$ has the same size and shape as S, and is said to be *congruent* to S.)

Thus isometries not only preserve distances, but also angles and area, and so preserve shape. There are three main classes of isometries:

A. **Reflections.** Let a be a line, the *axis*. The *reflection* U_a with axis a is that isometry for which $U_a(A) = A$ when A lies on a and, when P does not lie on a. $U_a(P)$ is located so that the axis right bisects the line segment joining P and $U_a(P)$. Thus, $U_a(P)$ can be regarded as the mirror image of P in the axis a.

B. **Translations.** A *translation* is an isometry that moves each point a fixed distance in the same direction. It can be described in different ways. We might refer to a translation by a vector $\overrightarrow{AB}$ which takes a point P to a point Q so that $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{AB}$, or $\overrightarrow{PQ} = \overrightarrow{AB}$. Or it could be described as a mapping of the cartesian plane of the form $(x, y) \to (x + a, y + b)$ for some fixed pair (a, b).

C. **Rotations.** Let O be a point in the plane and θ an angle, measured counter-clockwise. A *rotation* $R_{O,\theta}$ maps the point O to itself and any other point P to a point P' for which the (directed) angle $\angle P'OP = \theta$.

Exercise 2.1. Suppose a line l is carried by a rotation $R_{O,\theta}$ to the line $R_{O,\theta}(l)$. Show that l and $R_{O,\theta}$ intersect at an angle θ. (Consider the perpendiculars to the line from O.)

An isometry is *direct* if given any three noncollinear points A, B, C and their images A', B', C', the vertices of both the triangle ABC and $A'B'C'$ can be read off in the same order, *i.e.*, both clockwise or counterclockwise.

Figure 2.1.

Every rotation and every translation is direct.

An isometry is *opposite* if for any three noncollinear points, A, B, C and their images A', B', C', the vertices of one triangle read in order have the opposite sense to the vertices of the other read in order, *i.e.*, if you have to read one set of vertices in the clockwise direction, the other are read in the counterclockwise cirection.

Figure 2.2.

Every reflection in an axis is opposite.

Any two isometries can be combined by following one by the other. Thus, if T_1 and T_2 are isometries, then we can define their *product* or *composite* T_2T_1 by

$$(T_2T_1)(P) = T_2(T_1(P))$$

for every point P. The identity isometry I has the property that $T = TI = IT$ for every isometry T. This operation, called *composition*, is associative: $T_3(T_2T_1) = (T_3T_2)T_1$. Do you see why?

Exercise 2.2. Give examples of pairs (U, V) of isometries for which $UV = VU$ and for which UV is distinct from VU.

Exercise 2.3. Show that the product of two direct or of two opposite isometries is direct, while the product of a direct and an opposite isometry is opposite.

Recall that the *identity* isometry is that isometry that fixes each point in the plane. Given an isometry T, we can define the *inverse* isometry, T^{-1} by

$$T^{-1}(P) = Q \qquad \text{if and only if} \qquad T(Q) = P \ .$$

Thus, T^{-1} "undoes" the effect of T by restoring each point to its original position. Symbolically, each isometry T and its inverse satisfy the relations

$$I = TT^{-1} = T^{-1}T \ .$$

Exercise 2.4. Show that $I^{-1} = I$ for the identity isometry I, that $U_a^{-1} = U_a$ for each relection and that $R_{O,\theta}^{-1} = R_{O.-\theta} = R_{O,360^\circ-\theta}$. What is the inverse of a translation?

Here are some more isometry facts:

6. An isometry is uniquely determined by its effect on three noncollinear points. This means that, if for three points P, Q, R not in a line and an isometry T, we know $T(P)$, $T(Q)$, $T(R)$, respectively, then there is only one possibility for $T(X)$ for each X in the plane. To see this, refer back to fact **3.** Suppose that T_1 and T_2 are two isometries for which $T_1(P) = T_2(P)$, $T_1(Q) = T_2(Q)$ and $T_1(R) = T_2(R)$. then $P = T_1^{-1}T_2(P)$, $Q = T_1^{-1}T_2(Q)$ and $R = T_1^{-1}T_2(R)$, so that $T_1^{-1}T_2(X) = X$ for each X in the plane. But then, applying T_1 to both sides, we find that $T_2(X) = T_1(X)$ for each X in the plane.

7. Suppose that the (nodegenerate) triangles ABC and $A'B'C'$ are congruent with $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$. Then there is a unique isometry that takes ABC to $A'B'C'$,

This isometry can be described in a number of different ways. For example, it can be effected by the product of a translation that takes A to A' and B to some point B'', say, followed by a rotation (if necessary) with centre A that takes B'' to B', and finally, if necessary, by a reflection that takes the third vertex to C'. Or it could be described as the product of two or three reflections that ensure that the three vertices of ABC eventually fall on the three vertices of $A'B'C'$ (do this).

8. The product of two reflections with distinct parallel axes is a translation in a direction perpendicular to the axes through twice the distance between the lines.

9. Suppose that a and b are two axes of reflection that intersect at the point O at an angle of θ. Then the product U_bU_a of the reflections is a rotation with centre O through an angle 2θ in the direction from a to b.

To establish this, we need specify that the product of the two reflections acts on three noncollinear points in the same way as the rotation. This is clear for the point O. Let P be any point on a and Q any point on b, both distinct from O. Then $U_bU_a(P) = P'$ where $\angle P'OP = 2\theta$ and $U_a^{-1}U_b^{-1}(Q) = Q'$ where $\angle QOQ' = 2\theta$. Since also $OP = OP'$ and $OQ = OQ'$, the three noncollinear points O. P and Q' are treated the same by the product of reflections and by the rotation.

Figure 2.3.

10. Let R_{O_1,α_1} and R_{O_2,α_2} be two rotations. If $O_1 = O_2$, then the product $R_{O_2,\alpha_2}R_{O_1,\alpha_1}$ is a rotation with centre $O_1 = O_2$ through the angle $\alpha_1 + \alpha_2$, and is in fact the identity mapping when $\alpha_1 + \alpha_2 = 360°$ or $\alpha_1 = -\alpha_2$.

In the case that $O_1 \neq O_2$ and $\alpha_1 + \alpha_2$ is equal to 0 or 360°, then $R_{O_2,\alpha_2}R_{O_1,\alpha_1}$, then the product of the two rotations is a translation. Finally, if $O_1 \neq O_2$ and $\alpha_1 + \alpha_2$ is not a multiple of 360°, then the product of the rotations is a rotation through an angle of $\alpha_1 + \alpha_2$ with a centre distinct from O_1 and O_2.

Let us see why the results in the second paragraph of **10** are so. First, let $\alpha_1 = \alpha$ and $\alpha_2 = -\alpha$ for some angle α. Consider a parallelogram PO_1QO_2 for which $O_1P = O_1O_2 = O_2Q$ and $\angle PO_1O_2 = \angle O_1O_2Q = \alpha$. Then

$$R_{O_2,-\alpha}R_{O_1,\alpha}(P) = R_{O_2,-\alpha}(O_2) = O_2$$

and

$$R_{O_2,-\alpha}R_{O_1,\alpha}(O_1) = R_{O_2,-\alpha}(O_1) = Q \ .$$

Figure 2.4.

Consider what happens to the point O_2. Let $R_{O_1,\alpha}(O_2) = S$ where $OS = O_1O_2 = O_1P$ and $\angle O_2O_1S = \angle PO_1O_2 = \alpha$. A reflection in O_1O_2 interchanges P and S, so that $PO_2 = O_2S$. Also

$$\angle O_1PO_2 = \angle O_1O_2P = \angle O_1O_2S = \angle O_1SO_2 = \frac{1}{2}(180^\circ - \alpha) \ .$$

Applying $R_{O_2,-\alpha}$, we find that $R_{O_2,-\alpha}(S) = T$, where $\angle SO_2T = \alpha$ and $O_2T = O_2S = PO_2$. Therefore $\angle PO_2T = 2((1/2)(180^\circ - \alpha) + \alpha = 180^\circ$, so that P, O_2 and T are collinear. Thus, on the noncollinear points P, O_1, O_2, $R_{O_2,-\alpha}R_{O_1,\alpha}$ behaves like a translation in the direction and distance of PO_2.

Figure 2.5.

On the other hand, the composite of two rotations with different centres O_1 and O_2 and angles respective α_1 and α_2 that do not sum to a multiple of 360° can be analyzed as follows. The diagram covers the situation that $180^\circ > \alpha_1 > \alpha_2 > 0$; the reader can check how this can be adapted to other cases.

Figure 2.6.

There is a point O not on the line O_1O_2 that is carried to itself by the composite of the two rotations. Let $\alpha = \angle O_1OO_2 = \angle O_1O'O_2$. Note that $2\alpha + \alpha_1 + \alpha_2 = 360^\circ$. Let

$$O_1'' = R_{O_2,\alpha_2}(O_1') = R_{O_2,\alpha_2}R_{O_1,\alpha_1}(O_1)$$

and

$$O_2'' = R_{O_2,\alpha_2}(O_2') = R_{O_2,\alpha_2}R_{O_1,\alpha_1}(O_2) \ .$$

Figure 2.7.

Observe that triangle $OO_1''O_2$ and OO_1O_2 are congruent and that $\angle O_1''OO_1 = 360^\circ - 2\alpha = \alpha_1 + \alpha_2$. Because of the isometries, $R_{O_2,\alpha_2}R_{O_1,\alpha_1}(O) = O$ and $R_{O_2,\alpha_2}R_{O_1,\alpha_1}(O_2) = O_2''$, we have that $OO_2 = OO_2''$. Since triangle $O_2O_1O_2'$ is isosceles, $\angle O_2'O_2O_1 = 90^\circ - (\alpha_1/2)$, so that $\angle O_2''O_2O_1 = 90^\circ - (\alpha_1/2) - \alpha_2$. Hence

$$\angle O_2''O_2O = 90^\circ - (\alpha_1/2) - \alpha_2 + (\alpha_2/2) = 90^\circ - (\alpha_1/2) - (\alpha_2/2) \ .$$

Thus,

$$\angle O_2''OO_2 = 180^\circ - 2\angle O_2''O_2O_1 = \alpha_1 + \alpha_2 \ .$$

Since the three points O, O_1 and O_2 are treated the same way by the composite as by a rotation of centre O and $\angle\alpha_1 + \alpha_2$, the desired conclusion follows.

Figure 2.8.

There is a more transparent way to obtain the result by suitably breaking each rotation into a product of reflections. As shown in the diagram, let a be the line OO_1, b be the line OO_2 and c be the line O_1O_2 produced. Then $R_{O_1,\alpha_1} = U_c \cdot U_a$, $R_{O_2,\alpha_2} = U_b \cdot U_c$, whence

$$R_{O_2,\alpha_2} R_{O_1,\alpha_1} = U_b U_c U_c U_a = U_b U_a \ ,$$

a rotation with centre O through an angle $\alpha_1 + \alpha_2$.

Figure 2.9.

Another isometry that comes into play is the *glide reflection.* This is the composite of a reflection and a translation in the direction of the axis of the axis of reflection. It does not matter in what order the reflection and translation are taken in the composition.

Exercise 2.5. Prove the various congruence theorems (SAS, ASA, SSS) for triangles.

Section 3

Using isometries to solve problems

december 15, 2007

Problem 3.1. $ABCD$ and $BEFG$ are squares inside a semi-circle $XDCFY$ whose centre O is the midpoint of side AB. If the lengths of the sides of the square $ABCD$ are each $2a$, determine the sidelength of the square $BEFG$. (Note that the smaller square is uniquely determined by the larger one.)

Figure 3.1.

Solution. Consider a reflection in the axis which makes an angle of $45°$ to the sides AB and BC of the square and passes through O. Let the axis of reflection meet BC at U and let the image of B under the reflection be V. Then V lies on the right bisector of AB and $OB = BU = VU$. Thus $OBUV$ is a square. Since $2BU = 2OB = AB = BC$, it follows that $CU = BU$. The image, W, of C under the reflection must be on the circumference of the semicircle, and $\angle CUW = 90°$, $CU = UW$. If Z is the foot of the perpendicular from W to XY, then $BZWU$ is a square, and so must be the square $BEFG$. Hence the sidelength of the smaller square is a. ♠

Figure 3.2.

Problem 3.2. AB is a diameter of a circle and C is the midpoint of one of the semicircular arcs AB. If P is any point on the opposite semicircular arc AB, and if X and Y are the respective feet of the perpendiculars from A and B to CP, then $AX + BY = CP$.

Comment. This obviously holds when P is the midpoint of the arc or P coincides with A.

Solution. A rotation of $90°$ about the centre of the circle takes $C \to A$. The chord CP falls on a chord AD that contains the point X.

Figure 3.3.

Since $P \to D$, the arc PD is one quarter of the circumference of the circle, as is BC. The reflection in the diameter right bisecting the chord CP takes the circle to itself and interchanges C and P. Since arc CB is equal to arc PD, the reflection must interchange B and D, and so must interchange lines BY and DX (which are parallel to the axis). Since CP is carried to itself by the reflection, X and Y are each carried to points on CP, from which it follows that $X \leftrightarrow Y$. Therefore, $DX = BY$, and so $AX + BY = AX + DX = AD = CP$. ♠

Comment. There are other ways of apprehending the situation. A $180°$ rotation about the centre O of the circle interchanges A and B, and interchanges the chord AD with a parallel chord through B. Thus BY and AD are equidistant from O, the centre of the circle.

Consider the reflection in the diameter d through the centre O parallel to AD and YB. This takes the chord AD to a chord on the opposite side of O through B. Thus X and Y are equidistant from d and so the reflection interchanges X and Y and interchanges B and D. The result follows from this. ♣

Problem 3.3. A page is laid on a table. A second page is laid on top of it (as indicated by a dotted line in the diagram). Does the top page cover more or less than half of the area of the first page?

Figure 3.4.

Solution. A subtle first point that has to be disposed of is to verify that the edge of the top page actually passes through the point D, so that there is nothing in the lower right corner of the bottom page that is not covered by the top page. Let the bottom page have its vertices labelled $ABCD$ and the top page $AEFG$ as indicated in the diagram. Let AH bisect the angle EAD. Consider the reflection in the axis AH. This reflection carries AE to AD, and AB to AG. Since the angle CBA is equal to the angle FGA, BC is carried onto GF. Since E lies on the side BC, its image D under the reflection must lie on GF. Observe that the top page covers the triangle AED whose area is half that of either page, so that the top page indeed covers more than half the bottom page, with respect to area. ♠

Problem 3.4. The segments BE and CF are altitudes of the acute triangle ABC, where E and F are points on the segments AC and AB, respectively. ABC is inscribed in the circle $\mathbf{Q}$ with centre O. Denote the orthocentre of ABC by H, and the midpoints of BC and AH by M and K, respectively. Let $\angle CAB = 45°$.

(a) Prove, that the quadrilateral $MEKF$ is a square.

(b) Prove that the midpoint of both diagonals of $MEKF$ is also the midpoint of the segment OH.

(c) Find the length of EF, if the radius of $\mathbf{Q}$ has length 1 unit.

Figure 3.5.

Solution 1. (a) Since AH is the hypotenuse of right triangles AFH and AHE, $KF = KH = KA = KE$. Since BC is the hypotenuse of each of the right triangles BCF and BCE, we have that $MB = MF = ME = MC$. Since $\angle BAC = 45°$, triangles ABE, HFB and ACF are isosceles right triangles, so $\angle ACF = \angle ABE = \angle FBH = \angle FHB = 45°$ and $FA = FC$, $FH = FB$.

Consider a $90°$ rotation with centre F that takes $H \to B$. Then $FA \to FC$, $FH \to FB$, so $\Delta FHA \to \Delta FBC$ and $K \to M$. Hence $FK = FM$ and $\angle KFM = 90°$.

But $FK = KE$ and $FM = ME$, so $MEKF$ is an equilateral quadrilateral with one right angle, and hence is a square.

(b) Consider a $180°$ rotation (half-turn) about the centre of the square. It takes $K \leftrightarrow M$, $F \leftrightarrow E$ and $H \leftrightarrow H'$. By part (a), $\Delta FHA \equiv \Delta FBC$ and $AH \perp BC$. Since $KH \| MH'$ (by the half-turn), $MH' \perp BC$. Since $AH = BC$, $BM = \frac{1}{2}BC = \frac{1}{2}AH = KH = MH'$, so that BMH' is a right isosceles triangle and $\angle CH'M = \angle BH'M = 45°$. Thus, $\angle BH'C = 90°$. Since $\angle BAC = 45°$, H' must be the centre of the circle through ABC. Hence $H' = O$. Since O is the image of H by a half-turn about the centre of the square, this centre is the midpoint of OH as well as of the diagonals.

(c) $|EF| = \sqrt{2}|FM| = \sqrt{2}|BM| = |OB| = 1$.

Problem 3.5. Let ABC be an acute-angled triangle, with a point H inside. Let U, V, W be respectively the reflected image of H with respect to axes BC, AC, AB. Prove that H is the orthocentre of ΔABC if and only if U, V, W lie on the circumcircle of ΔABC,

Figure 3.6.

Solution 1. Suppose that H is the orthocentre of ΔABC. Let P, Q, R be the respective feet of the altitudes from A, B, C. Since BC right bisects HU, $\Delta HBP \equiv \Delta UBP$ and so $\angle HBP = \angle UBP$. Thus

$$\begin{aligned}\angle ACB = \angle QCB &= 90° - \angle QBC = 90° - \angle HBP \\ &= 90° - \angle UBP = \angle PUB = \angle AUB\ ,\end{aligned}$$

so that $ABUC$ is concyclic and U lies on the circumcircle of ΔABC. Similarly V and W lie on the circumcircle.

Now suppose that U, V, W lie on the circumcircle. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ be the respective reflections of the circumcircle about the axes BC, CA, AB. These three circles intersect in the point H. If H' is the orthocentre of the triangle, then by the first part of the solution, the reflective image of H' about the three axes lies on the circumcircle, so that H' belongs to $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ and $H = H'$ or else HH' is a common chord of the three circles. But the latter does not hold, as the common chords AH, BH and CH of pairs of the circles intersect only in H.

Solution 2. Let H be the orthocentre, and P, Q, R the pedal points as defined in the first solution. Since $ARHQ$ is concyclic

$$\angle BAC + \angle BUC = \angle BAC + \angle BHC = \angle RAQ + \angle RHQ = 180°$$

and so $ABUC$ is concyclic. A similar argument holds for V and W.

[A. Lin] Suppose that U, V, W are on the circumcircle. From the reflection about BC, $\angle BCU = \angle BCH$. From the reflections about BA and BC, we see that $BW = BH = BU$, and so, since the equal chords BW and BU subtend equal angles at C, $\angle BCW = \angle BCU$. Hence $\angle BCW = \angle BCH$, with the result that C, H, W are collinear and CW is an altitude. Similarly, AU and BV are altitudes that contain H, and so their point H of intersection must be the orthocentre.

Problem 3.6. (*The Fermat problem.*) Given three points in the plane, determine a fourth point such that the sum of the distances from it to the three given points is minimum.

Solution. We consider first the case that no angle of the triangle exceeds $120°$. In the diagram below, A' is the image of A under a rotation of $60°$ with centre B. Let Q be an arbitrary point in the interior of the triangle and Q' be its image under the same rotation. Since $BQ = BQ'$ and $\angle QBQ' = 60°$, triangle BQQ' is equilateral. Since $A'B = AB$ and $\angle ABA' = 60°$, triangle BAA' is equilateral and $\angle A'AC = \angle A'AB + \angle BAC \leq 60° + 120° = 180°$. Hence A and B lie on opposite sides of the line $A'C$.

Figure 3.7.

We have that

$$QA + QB + QC = A'Q' + Q'Q + QC \geq A'C \ .$$

The positions of A' and C are independent of Q. Thus, the minimizing position of Q for the sum of the distances will occur when $A'Q'QC$ is a straight line. A similar observation can be made with respect to the other vertices B and C. Note that A' is a vertex of the equilateral triangle constructed externally on the side AB of triangle ABC.

Figure 3.8.

Consider the foregoing diagram: three equilateral triangles ABZ, BCX, ACY are constructed externally on the sides of triangle ABC. (We have relabeled A' as Z.) Let P be the point of intersection of the circumcircles of triangles ABZ and ACY. Then $\angle APB = \angle APC = 120°$, so that $\angle BPC = 120°$ and P also lies on the circumcircle of triangle XBC.

Since angles ZPA and ZPB are subtended by equal arcs of the circumcircle of ABZ, they are both equal to $60°$. Hence, $\angle ZPA + \angle APC = 180°$, so that Z, P, C are collinear. Similarly, P lies on AX and BY, so that P is the common point of the lines AX, BY, CZ.

P is the point that we are looking for. Since $BPAZ$ is concyclic, $\angle BZP = \angle BAP$. From this, it can be seen that the image P' of P under the rotation of $60°$ about B is on the line CPZ. Hence

$$PA + PB + PC = ZP' + P'P + PC = ZC \ .$$

Figure 3.9.

Now suppose that $\angle BAC > 120°$. Let Q be any point inside triangle ABC and construct A' and Q' as before. Now $A'C$ lies outside the triangle and $A'A$ produced penetrates the triangle and intersects QC at some point R. (Observe that AA' and QQ' intersect outside the triangle beyond A'.) Then

$$\begin{aligned} A'A + AC &\leq A'A + AR + RC = A'R + RC \\ &\leq A'Q' + Q'Q + QR + RC = AQ + BQ + CQ \ . \end{aligned}$$

In this case, the minimizing point P coincides with A. ♠

Comment. The foregoing argument is due to Hofmann in 1929. This problem has an interesting dual relationship with a second problem that was originally posed by Thos. Moss in the *Ladies' Diary, or Women's Almanack*, and again in 1810 in the *Annales de math. pures et appliqués*; the solution was given in the same journal in 1811-12. This is that, *given three points in the plane, find the largest equilateral triangle whose sides pass through the points.*

Fasbender's Theorem. (1845) Let A, B, C be three points for which no angle of the triangle ABC exceeds $120°$. Within the triangle ABC, there is a point P for which the length $PA+PB+PC$ is at the same time

• the minimum distance sum $QA + QB + QC$ for any point Q in the plane;

• the maximum altitude of an equilateral triangle circumscribed about triangle ABC; this equilateral triangle has sides perpendicular to the segments PA, PB, PC.

Proof. With the point P determined as in Problem 3.6, construct through vertices A, B, C, lines perpendicular to PA, PB, PC, respectively. These lines bound an equilateral triangle whose altitude is equal to $PA + PB + PC$ in length (why?). We show that any other circumscribed equilateral triangles has at most this altitude.

Figure 3.10.

Consider any circumscribed equilateral triangle as in the following diagram.

Figure 3.11.

For any point Q within the triangle, the altitude of the equilateral triangle is equal to $QU + QV + QW$, which does not exceed $QA+QB+QC$. In particular, this is true for the Fermat point P, so that the altitude does not exceed $PA+PB+PC$. Since this altitude is realizable, the point P leads to a solution of the equilateral triangle problem. ♠

Problem 3.7. Let PQR be a given triangle. Let points A, B, C be located outside of the triangle in such a way that $\angle PQC = \angle PRB = 45°$, $\angle QPC = \angle RPB = 30°$ and $\angle AQR = \angle ARQ = 15°$. Prove that triangle ABC is right isoceles.

Figure 3.12.

Solution. [J. Colwell] Let S be the image of R under a counterclockwise rotation about A through $90°$. Then $AR = AS = AQ$. Since $\angle QAS = 60°$, triangle QAS is equilateral and so $\angle SQR = 45°$. Since triangle ARS is right isosceles, $\angle ARS = 45°$, so that $\angle QRS = 30°$. Therefore, the three triangles PCQ, PBR and RSQ are similar, whence $CQ : QP = SQ : QR$.

Now $\angle CQS = \angle PQS + 45° = \angle PQR$. (If S lies outside of triangle PQR on the far side of PQ, then $\angle CQS = 45° - \angle PQS = \angle PQR$.) Hence triangles CQS and PQR are similar, so that

$$CS : RP = CQ : QP = BR : RP \Longrightarrow CS = BR \ .$$

Also

$$\angle CSQ = \angle PRQ \Longrightarrow \angle CSA = \angle CSQ + 60° = \angle PRQ + 45° + 15° = \angle BRA \ .$$

Since also $AS = AR$, we have that triangles CSA and ARB are congruent (SAS), so that $AC = AB$ and $\angle SAC = \angle RAB$. Hence

$$\angle BAC = \angle BAS + \angle SAC = \angle BAS + \angle RAB = \angle RAS = 90° \ .$$

Problem 3.8. Within a triangle ABC, a point P is determined so that $\angle PAC = 10°$, $\angle PCA = 20°$, $\angle PAB = 30°$ and $\angle ABC = 40°$. What is $\angle BPC$?

Figure 3.13.

Solution. Since $\angle CAB = \angle CBA = 40°$, $CA = CB$. A reflection in the right bisector of AB fixes C and interchanges A and B, as well as P and some point Q.

Figure 3.14.

Since $\angle ACB = 100°$ and $\angle PCA = \angle QCB = 20°$, it follows that $\angle PCQ = 60°$. Since $CP = CQ$, triangle CPQ is equilateral and $PQ = QC$, $\angle BQC = \angle APC = 150°$ and $\angle BQP = 360° - 60° - 150° = 150°$. A reflection in BQ produced takes the segment QP to QC while fixing B and Q. Hence triangle QCB and QPB are congruent, so that $\angle BPQ = \angle BCQ = 20°$ and $\angle BPC = 20° + 60° = 80°$. ♠

Problem 3.9. Triangle ABC is such that $\angle A = 20°$ and $\angle B = 80°$. The point D in side AB is determined so that $AD = BC$. What is $\angle ADC$?

Figure 3.15.

Solution. A combination of a translation and a rotation takes $A \to B$, $D \to C$ and $C \to E$.

Figure 3.16.

Since $\angle ABC = \angle ACB = 80°$, $AB = AC = BE$ and triangle ABE is isosceles with apex angle ABE equal to $60°$. Hence triangle ABE is equilateral and so $AE = AC$ and $\angle BAE = 60°$. Therefore, triangle ACE is isosceles with apex angle CAE equal to $40°$. Hence $\angle ACE = \angle AEC = 70°$, so that $\angle ADC = \angle BCE = 80° + 70° = 150°$. ♠

Problem 3.10. Triangle ABC is equilateral with centroid M. Points D and E are located on the respective sides CA and CB for which $CD = CE$. The point F is determined for which $DMBF$ is a parallelogram. Prove that the triangle MEF is equilateral.

Figure 3.17.

Solution. There is a translation that takes $M \to B$ and $D \to F$. Let $A \to A'$ and $C \to C'$. Since $MC = MB = CC'$, $MBC'C$ is a rhombus with $\angle CMB = 120°$. Therefore, MCC' is an equilateral triangle. Similarly, triangle MAA' is equilateral, so that $MA' = AA' = MB$. Also $\angle A'MB = \angle A'AB - \angle AMA' = 120° - 60° = 60°$.

A $60°$ rotation about M takes $C \to C'$, $B \to A'$, so that $CB \to C'A'$. Now $C'F = CD = CE$, so that $E \to F$. Thus, $ME = MF$ and $\angle EMF = 60°$. The result follows. ♠.

Problem 3.11. Determine the area between the graph of $y = \sin^2 x$ and the x−axis for $0 \le x \le \pi$.

Figure 3.18.

Solution. The required area is twice that between the two graphs for $0 \le x \le \pi/2$.

Since $\cos^2 x = 1 - \sin^2 x$, the graph of $y = \cos^2 x$ can be obtained from the graph of $y = \sin^2 x$ by a reflection in the line $y = \frac{1}{2}$ (*i.e.*, (u, v) is on the graph of one curve if and only if $(u, 1 - v)$ is on the graph of the other). Since $\cos x = \sin(\frac{\pi}{2} - x)$, the graph of $y = \cos^2 x$ can be obtained from the graph of $y = \sin^2 x$ by a reflection in the line $x = \frac{\pi}{4}$ (*i.e.*, (u, v) is on one graph if and only if $(\frac{\pi}{2} - u, v)$ is on the graph of the other).

Consider a reflection in the line $y = \frac{1}{2}$ followed by a reflection in the line $x = \frac{\pi}{4}$. The composite of these two reflections is a rotation through $180°$ about the point $(\frac{\pi}{4}, \frac{1}{2})$ that takes the graph of $y = \sin^2 x$ to itself and interchanges the $x-$axis and the line $y = 1$. Hence, on the interval defined by $0 \le x \le \frac{\pi}{2}$, the area between the graph and the $x-$axis is equal to the area between the graph and the line $y = 1$. Hence, the area under the graph is half the area of the rectangle with vertices $(0, 0)$, $(\frac{\pi}{2}, 0)$, $(\frac{\pi}{2}, 1)$, $(0, 1)$, namely $\frac{1}{2}(\frac{\pi}{2}) = \frac{\pi}{4}$. It follows that the area under the graph of $y = \sin^2 x$ for $0 \le x \le \pi$ is $\frac{\pi}{2}$. ♠

Problem 3.12. At points A and B on a circle, equal tangents AP and BQ are drawn in the same sense, as indicated in the diagram. Prove that AB produced bisects the segment PQ.

Figure 3.20.

Solution. (A. Brown) The reflection in the right bisector of AB fixes O, the centre of the circle, carries the circle to itself and interchanges A and B.

Figure 3.21.

It takes the point Q to Q', so that AQ' is tangent to the circle at A; thus Q', A, P are collinear and $Q'Q \| AB$. Consider the triangle $PQ'Q$. $Q'A = AP$ and AB produced is parallel to $Q'Q$. Therefore, AB produced must pass through the midpoint of PQ. ♠

Problem 3.13. Let POQ be a given angle. The following method is suggested for trisecting the angle using straightedge and compasses. *Select any point B on OQ and drop a perpendicular to meet OP at A. Erect an equilateral triangle ABC on the opposite side of AB to O. Join OC. Then OC trisects angle POQ, i.e., $\angle COQ = 2\angle COP$.*

Determine those angles POQ for which the method actually works.

Figure 3.22.

Solution. Suppose that the trisection succeeds. A reflection in the axis OP fixes O and A and interchanges C with some point D. Since $\angle PAC = \angle PAB - \angle CAB = 90° - 60° = 30°$, we have that $\angle DAC = 60°$ and $AD = AC$. Hence triangle DAC is equilateral.

Since $\angle DOC = 2\angle POC = \angle QOC$ and $OD = OC$, a rotation with centre O and angle DOC takes $D \to C$ and C to some point R on OQ. We have that $BC = CA = CD = CQ$. In the isosceles triangle COR, we find that

$$\angle CRO = \frac{1}{2}(180^\circ - 2\angle COQ) = 90^\circ - \angle COP\ .$$

Using an external angle for triangle OAB and the fact that CBR is isosceles, we find that

$$\begin{aligned}\angle CRO = \angle CBR &= \angle ABQ - 60^\circ \\ &= 90^\circ + \angle POQ - 60^\circ = 30^\circ + 3\angle COP\ .\end{aligned}$$

Hence $60^\circ = 4\angle COP$, so that $\angle POQ = 45^\circ$. Thus, the method works only when the angle to be trisected is 45°. ♠

Section 4

Dilations

`December 20, 2004`

Isometries are not the only transformation useful in solving geometric problems. *Dilations* (also known as *homotheties* or *central similarities*) are scale transformations determined by a fixed point, called its *centre*, and a factor that indicates the scale change.

A *dilation* $D_{C,\lambda}$ with centre C and factor $\lambda > 0$ takes the point C to itself and any other point P to a point P' located on the ray CP so that $CP' = \lambda CP$. Thus, C, P, P' are collinear and P and P' are on the same side of C. Note that, if $\lambda = 1$, we obtain the identity map; if $0 < \lambda < 1$, then P' lies between C and P and we get a contraction; if $\lambda > 1$, then P lies between C and P' and we get an expansion.

Figure 4.1.

We can define dilations with negative factors. The dilation $D_{C,-1}$, also known as a reflection in the point C, is equivalent to a rotation of $180°$ with centre C. A point P is carried to a point P' for which C is the midpoint of the segment PP'.

In general, if $\lambda < 0$, the dilation $D_{C,\lambda}$ is the composite

$$D_{C,-1} \cdot D_{C,|\lambda|} = D_{C,|\lambda|} \cdot D_{C,-1} \ .$$

The point P is taken to the point P' for which C divides the segment PP' in the ratio $1 : |\lambda|$.

Figure 4.2.

A dilation is a transformation that preserves shape but changes scale. It is direct. The image of a line is a parallel line; the image of a circle is a circle whose radius is equal to $|\lambda|$ times that of the original circle. The image of any geometric figure is a similar figure. The linear dimensions of the image are equal to $|\lambda|$ times the corresponding dimensions of the original, and the area of the image is equal to λ^2 times the area of the original.

It is straightforward to see that the composite of two dilations with the same centre is a dilation with the same centre whose factor is the product of the factors of its components. We can use analytic geometry to investigate what happens to the composite when the centres differ.

Let the dilation with centre $(0,0)$ and factor λ be followed by one with centre $(1,0)$ and factor μ. (Why is there no loss of generality in our specification of the centres?) Then the effect is given by

$$(x,y) \longrightarrow (\lambda x, \lambda y) \longrightarrow (1,0) + \mu(\lambda x - 1, \lambda y) = (1 - \mu + \lambda\mu x, \lambda\mu y) \ .$$

If $\lambda\mu = 1$, then the composite is a translation parallel to the segment joining the centres of dilation through the directed distance $1 - \mu$.

Let $\lambda\mu \neq 1$. The composite has the fixed point

$$\left(\frac{1-\mu}{1-\lambda\mu}, 0\right) .$$

We make a change of coordinates to put the fixed point at the origin. Let

$$X = x - \frac{1-\mu}{1-\lambda\mu} , \qquad Y = y .$$

Then, in the new coordinate system, the image (X', Y') of (X, Y) is given by

$$X' = 1 - \mu + \lambda\mu\left(X + \frac{1-\mu}{1-\lambda\mu}\right) - \left(\frac{1-\mu}{1-\lambda\mu}\right) = \lambda\mu X ,$$

$$Y' = \lambda\mu y = \lambda\mu Y .$$

Thus, the composite is a dilation with centre $((1-\mu)/(1-\lambda\mu), 0)$ and factor $\lambda\mu$.

Exercise. Check that this is what you would expect when either λ or μ is equal to 1.

As another exercise, let us consider the composite of a dilation and a reflection. Let the centre of a dilation of factor λ be $(0, 0)$ and let the axis of reflection be the line of equation $x = 1$. The the composite of the dilation followed by the reflection is given by

$$(x, y) \longrightarrow (\lambda x, \lambda y) \longrightarrow (2 - \lambda x, \lambda y) .$$

If $\lambda = -1$, the mapping is $(x, y) \to (-x, -y) \to (2 + x, -y)$, which is a glide reflection. When $\lambda \neq 1$, there is a fixed point $(2/(1 + \lambda), 0)$, so we define new coordinates (X, Y) that put this fixed point at the origin. Let

$$X = x - \frac{2}{1+\lambda} , \quad Y = y .$$

Then the composite, expressed in terms of the new coordinates, is given by $(X, Y) \longrightarrow (X', Y')$ where

$$X' = 2 - \lambda\left(X + \frac{2}{1+\lambda}\right) - \frac{2}{1+\lambda} = -\lambda X$$

and $Y' = \lambda Y$. Thus, the composite is a dilation followed by a reflection in an axis through the centre of the dilation.

Exercise. Examine the composite of a dilation and a reflection when the reflection comes first.

Exercise. Investigate the composite of a dilation with translations and rotations.

A *spiral similarity* is a rotation about a given point followed by a dilation about the same point. The rotation and the dilation commute, so that one gets the same result by applying the dilation first and then the rotation. A spiral similarity preserves straight lines and angles, and carries any geometric figure to a similar figure.

A *central reflection* (reflection in a point, half-turn) is a dilation with factor -1. If O is the centre of the similarity, it takes a point P to a point P' for which $PO = OP'$, *i.e.*, O is the midpoint of PP'. In space, we have to distinguish between a central reflection in a point from a half-turn ($180°$) rotation) about an axis perpendicular to a plane through the point. They have the same effect on the plane, but differ on the rest of space. The first takes points from one side of the plane to the opposite side, while the second keeps them on the same side of the plane. We also have a *dilation-reflection* on a plane, which is a reflection followed by a scaled decrease in the distance to the axis. Thus, a point P is taken to a point P' on the opposite side of an axis A for which the distance from P' to a is equal to λ times the distance from P to a for some fixed positive real number λ.

Section 5

Examples using dilations

February 28, 2005

Problem 5.1. Let $ABCD$ be a parallelogram and let E be a point on the side AB for which $AE : EB = 2 : 1$. Suppose that BD and CE intersect at F. Determine $BF : FD$.

Figure 5.1.

Solution. Observe that $DC = 3EB$, so that a dilation with centre F and factor -3 takes the point E to E' on FC and B to B' on FD with $E'B' = 3EB$. Hence it takes the segment EB to CD. In particular, BF goes to DF and $BF : FD = 1 : 3$. ♠

Problem 5.2. $ABCD$ is a trapezoid with $AB \| CD$, and M is the midpoint of AB. Suppose that P is on the side BC but not equal to B nor C. Let $X = PD \cap AB$, $Q = PM \cap AC$, $Y = DQ \cap AB$. Prove that M is the midpoint of XY.

Figure 5.2.

Solution. Let $N = CD \cap PQ$. A dilation H_Q with centre Q takes $A \to C$, $M \to N$. A central similarity H_P with centre P takes $C \to B$, $N \to M$. Then H_Q followed by H_P fixes M and takes A to B. This composite is a halfturn about M and takes $Y \to D \to X$. The result follows. ♠

Problem 5.3. Construct, using straightedge and compasses, a parallelogram $ABCD$ given A, C and the distances r and s of B and D from a given point E.

Figure 5.3.

Construction. Consider the final diagram. The central reflection in the midpoint M of the diagonals of the parallelogram interchanges A and C and interchanges B and D. Let E' be the image of E. Then $|E'D| = |EB|$, so that D is located at the intersection of the circle with centre E' and radius EB and the circle with centre E and radius ED. ♠ (*Exercise:* Complete the construction and prove that it works.)

Rider. When is the construction feasible?

Problem 5.4. Let ABC be a triangle with medians AU, BV, CW. Then there is a triangle whose sides are equal to the lengths of AU, BV and CW and whose area is equal to $3/4$ of the area of triangle ABC.

Solution. The translation in the direction of $\overrightarrow{AW}$ through the distance $\frac{1}{2}|AB|$ takes $A \to W$, $W \to B$, $V \to U$ and U goes to some point X. Since AU is equal and parallel to WX, and since WB is equal and parallel to UX, the quadrilateral $WBXU$ is a parallelogram. Let WX intersect BC in Y. The dilation with centre B and factor $\frac{1}{2}$ takes $A \to W$ and the line AU to the parallel line WX, so that $U \to Y$ and Y is the midpoint of BU as well as the midpoint of WX.

Now consider the translation in the direction $\overrightarrow{BX}$. This takes $B \to X$ and $V \to C$, so that the median BV goes to XC. Hence WXC is a triangle whose sidelengths are equal to the lengths of the medians. Furthermore

$$[WXC] = 2[WYC] = 2((3/4)[WBC]) = 2 \times (3/4) \times (1/2)[ABC] = (3/4)[ABC] \ .\spadesuit$$

Figure 5.4.

Problem 5.5. The medians of a triangle intersect in a common point which is two thirds of the way along the length of each median from its vertex.

Solution. Use the notation of Problem 5.4. Let the medians BV and CW of the triangle intersect at G. The dilation with centre A and factor $1/2$ takes $B \to W$ and $C \to V$, so that $|WV| = (1/2)|BC|$. Now consider the dilation with centre G and factor -2. This dilation takes V to a point V' on the lines GB produced and W to a point W' on GC produced, so that $|V'W'| = 2|VW|$ and $V'W' \| WV$. Since $|BC| = 2|WV|$ and $BC \| WV$, the segment BC can be none other than the segment $V'W'$. Hence $BG = 2GV$ and $CG = 2GW$. In a similar way, it can be shown that the median from A must intersect BV and CW at the point G. $\spadesuit$

Figure 5.5.

Recall that the *circumcentre* of a triangle is the intersection of the right bisectors of its sides, that the *orthocentre* is the intersection of its altitudes and the *centroid* is the intersections of its medians.

Problem 5.6. (*The Euler line.*) Let O, G and H be, respectively, the circumcentre, centroid and orthocentre of a triangle ABC. Then these three points are collinear and $HG = 2GO$.

Figure 5.6.

Solution. Let AU, BV, CW be the medians and AP, BQ, CR be the altitudes of triangle ABC. The dilation with centre G and factor $-1/2$ takes $A \to U$, $B \to V$, $C \to W$. The altitudes AP, BQ, CR of triangle ABC are carried to the altitudes of triangle UVW. In particular, AP goes to a line through U and perpendicular to WV. But $WV \| BC$, so this line is perpendicular to BC and is none other than the right bisector of BC. Similarly, BQ goes to the right bisector of AC and CR to the right bisector of AB. Hence the intersection of the altitudes of triangle UVW is the intersection of the right bisectors of the sides of triangle ABC, namely the circumcentre O. Thus the dilation takes $H \to O$, and the result follows. $\spadesuit$

Problem 5.7. (*The nine-point circle.*) Let H be the orthocentre of triangle ABC. Then the midpoints of the sides, the pedal points (feet of the altitudes) and the midpoints of AH, BH, CH all lie on a common circle.

Solution. As before, let U, V, W be the midpoints of the sides, let P, Q, R the pedal points,let X, Y, Z be the repsective midpoints of AH, BH, CH and let O be the circumcentre.

Figure 5.7.

If the circle exists, it must be the circumcircle of triangle UVW. Let S be the centre of this circumcircle. The dilation with centre G (the centroid of triangle ABC) and factor $-1/2$ must take $H \to O$ and $O \to S$, with

$$GS : GO : HG = 1 : 2 : 4$$

so that H, S, G, O are collinear and $SO : HO = 3 : 6 = 1 : 2$. Hence S is the midpoint of OH, and so must lie on the right bisector of PU (why?). Hence $PS = US$. Similarly, $QS = VS$ and $RS = WS$. Since S is the circumcentre of triangle UVW, it is equidistant from U, V, W, P, Q, R.

The dilation with centre H and factor $1/2$ takes $A \to X$, $B \to Y$ and $C \to Z$, so that the linear dimensions of triangle XYZ are half the corresponding dimensions of triangle ABC, and the sides of triangle XYZ are parallel to the respective sides of ABC. Observe that the linear dimensions of triangle UVW are half those of triangle ABC and the sides of triangle UVW are parallel to the sides of triangle ABC. Hence, triangles XYZ and UVW are congruent (SSS) and it remains to identify the isometry that links them.

The respective orthocentres of triangles XYZ and UVW are H and O. A rotation of $180°$ about S interchanges O and H. It takes the line OU to a line through H parallel to OU, namely HA. Since triangles XYZ and UVW are congruent, $XH = UO$, so that X must be the image of U under the rotation. Similarly Y and Z are the respective images of V and W. Hence $XS = US$, $YS = VS$ and $ZS = WS$ and the result follows from this. ♠

Problem 5.8. (*The Pythagorean Theorem.*) Let ABC be a right triangle with hypotenuse BC. Then the area of the square on BC is equal to the sum of the areas of the squares on AB and AC.

Figure 5.8.

Solution. Let D be the foot of the perpendicular from A to BC and let a, b, c be the respective lengths of the sides BC, AC, AB. A reflection about the bisector of angle C followed by a dilation of factor b/a takes triangle ABC to triangle DAC. Hence $[DAC] = (b/a)^2[ABC]$. Similarly, $[DBA] = (c/a)^2[ABC]$. Hence

$$[ABC] = [DAC] + [DBA] = [(b/a)^2 + (c/a)^2][ABC] = \left(\frac{b^2 + c^2}{a^2}\right)[ABC] \; .$$

Therefore $a^2 = b^2 + c^2$, as desired. ♠

Problem 5.9. Let ABC be an arbitrary triangle, let F and K be the centres of the respective squares erected outwards on the sides AB and AC, and let M be the midpoint of BC. Then $FM \perp KM$ and $FM = KM$.

Figure 5.9.

Solution. Let the squares in question be $ABDE$ and $ACGH$. Since F and M are the respective midpoints of BE and BC, a dilation with centre B and factor 2 takes $F \to E$ and $M \to C$, so that $FM \| EC$ and $2FM = EC$. Similarly, using a dilation with centre C, we find that $MK \| BH$ and $2MK = BH$. A rotation with centre A through an angle of $90°$ takes $B \to E$ and $H \to C$, so that $BH = EC$ and $BH \perp EC$. The result follows from this. ♠

Problem 5.10. Let $ABCD$ be any quadrilateral and let P, Q, R, S be the respective centres of the squares on sides AB, BC, CD, DA, respectively. Then PR is equal to and perpendicular to QS.

Figure 5.10.

Solution. Let T be the midpoint of the diagonal BD. From Problem 5.9, we see that $PT = ST$ and $PT \perp ST$, and also that $RT = QT$ and $RT \perp QT$. Consider a $90°$ rotation about the point T that takes $P \to S$ and $R \to Q$. It takes the segment PR to the segment SQ, and so the result follows. ♠

Comment. Note that, in general, T will not lie on either of the segments PR or QS. A similar argument is possible using the midpoint of the diagonal AC.

Problem 5.11. Suppose that OAB and $OA'B'$ are two equilateral triangles with the same orientation; let S be the centroid of triangle OAB, and let M and N be the respective midpoints of $A'B$ and AB'. Prove that the triangles SMB' and SNA' are similar.

Figure 5.11.

Solution. The dilation with factor 2 and centre B followed by a $60°$ rotation about O has the following effects: $S \to S' \to S$, where S' is the reflection of S in OA, and $M \to A' \to B'$. Hence $SB' = 2SM$.

The central similarity with factor 2 and centre A followed by a clockwise $60°$ rotation about O has the following effect: $S \to S$ and $N \to A'$. Hence $SA' = 2SN$. Also $\angle MSB' = 60°$ and $\angle NSA' = 60°$. The result follows. ♠

Problem 5.12. Suppose that ABC and CDE are similarly oriented equilateral triangles, each external to the other, and that P, Q, R are the respective midpoints of AE, BC, CD. Prove that triangle PQR is equilateral. (see also Problem 5.16)

Figure 5.12.

Solution. The dilation with centre A and factor $\frac{1}{2}$ takes $E \to P$ and $C \to M$, the midpoint of AC. Since

$$PM = \frac{1}{2}EC = \frac{1}{2}CD = RC$$

and $PM \| CE$, the lines PM and CD produced make an angle of $60°$.

The rotation of $60°$ with centre Q fixes Q and takes $C \to M$. The lines CR goes through a line through M that makes an angle of $60°$ with CR, so that CR must go to MP. It follows that triangle QMP is the image of triangle QCR and that $R \to P$. Thus, triangle QMP and QCR are congruent. Hence $PQ = RQ$ and $\angle PQR = 60°$, so that triangle PQR is equilateral. ♠

Problem 5.13. Suppose that $ABCDEZ$ is a regular octahedron whose pairs of opposite vertices are (A, Z), (B, D) and (C, E). The points F, G, H are chosen on the segments AB, AC, AD respectively such that $AF = AG = AH$.

(a) Show that EF and DG must intersect in a point K, and that BG and EH must intersect in a point L.

(b) Let EG meet the plane of AKL in M. Show that $AKML$ is a square.

Solution 1. (a) Since $AF : AB = AG : GC$, it follows that $FG \| BC \| ED$, while $FG < BC = ED$. Hence, $FGDE$ is a coplanar isosceles trapezoid and so EF and DG must intersect in a point K. A $90°$ rotation about the axis AZ takes $B \to C$, $F \to G$, $C \to D$, $G \to H$, $D \to E$, $E \to B$. Hence $EF \to BG$ and $DG \to EH$, so that BG and EH must intersect in a point L, which is the image of K under the rotation.

(b) KE and AB intersect in F, so that the two lines are coplanar. Also $KF : KE = FG : ED = FG : BC = AF : FB$ so that $\Delta KAF \sim \Delta EFB$ and $AK \| EB$. Hence K lies in a plane through A parallel to $BCDE$. Because the $90°$ rotation about the axis AZ (which is perpendicular to the planes $BCDE$ and AKL takes $K \to L$, $AK = AL$ and $\angle KAL = 90°$.

Consider a dilation with centre E and factor $|AB|/|FB|$. Let I be on AE with $AI = AF$. The the dilation takes $F \to K$, $H \to L$, $I \to A$ and the plane of $FGHI$ to the parallel plane AKL. The image of G under this dilation is the intersection of EG and the plane of AKL, namely M. Thus the square $FGHI$ goes to $KMLA$ which must also be a square. ♠

Solution 2. The dilation-reflection perpendicular to a plane through FG perpendicular to BE and CD with factor $|AF|/|FB|$ takes $B \to E$, $C \to D$, and fixes F and G. The lines BF and CG with intersection A gets carried to lines EF and DG which intersect in a point K for which AK is perpendicular to the plane and so parallel to BE and CD, and the distance from K to the plane is $|AF|/|FB|$ times the distance from A to the plane.

Similarly, considering a dilation-reflection to a plane through GH perpendicular to BC and DE with the same factor produces the point L with AL perpendicular to this plane and so parallel to BC and ED. Thus $\angle KAL = 90°$.

The reflection in the plane $AECZ$ fixes A, C, E, G and interchanges the points in each of the pairs (B, D) and (F, H). Hence the line pairs (EF, DG) and (EH, BG) are interchanged as is the pair K and L. Thus $AK = AL$ and KL is perpendicular to $AECZ$. The triangle AKL is in a plane through A parallel to $BCDE$. The proof that $AKML$ is a square can be completed as in

Solution 1. ♠

Problem 5.14. Let PQR be a given triangle. Let points A, B, C be located outside of the triangle in such a way that $\angle PQC = \angle PRB = 45°$, $\angle QPC = \angle RPB = 30°$ and $\angle AQR = \angle ARQ = 15°$. Prove that triangle ABC is right isoceles.

Solution. [A. Chang] Determine the point S on the same side of QR as A for which triangle QRS is equilateral. Then triangles QPC, QSA and PRB are similar. The spiral similarity with centre Q consisting of a clockwise rotation through 45° followed by a dilation with factor $|PQ|/|QC| = |QS|/|QA|$ takes $C \to P$ and $A \to S$, so that $CA \to PS$ and $|CA|/|PS| = |PQ|/|QC|$.

Figure 5.13.

The spiral similarity with centre R consisting of a rotation through 45° followed by a dilation with factor $|PR|/|RB| = |RS|/|AR|$ takes $B \to P$ and $A \to S$. Hence $|BA|/|PS| = |PR|/|BR| = |PQ|/|CQ|$.

Since $|BA|/|PS| = |CA|/|PS|$, $BA = CA$. Since a 45° rotation takes CA along PS and a 45° rotation takes BA along PS, it must be that $\angle CAB = 90°$. ♠

Comment. P. Milley had a similar approach. Note that a 45° rotation about Q carries C to C' on PQ and A to A' on QS, and that $C'A' \| PS$, Similarly, a 45° rotation counterclockwise about R takes $B \to B''$ and $A \to A''$ with $B''A'' \| PS$. ♣

Problem 5.15. Let A be a point on a circle with centre O and let B be the midpoint of OA. Let C and D be points on the circle on the same side of OA produced for which $\angle CBO = \angle DBA$. Let E be the midpoint of CD and let F be the point on EB produced for which $BF = BE$.

(a) Prove that F lies on the circle.

(b) What is the range of angle EAO?

Figure 5.14.

Solution 1. [A. Wice] We first establish a Lemma.

Lemma. Let UZ be an angle bisector of triangle UVW with Z on VW. Then

$$UZ^2 = UV \cdot UW - VZ \cdot WZ \ .$$

Figure 5.15.

Proof. By the Cosine Law,

$$UV^2 = UZ^2 + VZ^2 - 2UZ \cdot VZ \cos \angle UZW$$

and

$$UW^2 = UZ^2 + WZ^2 + 2UZ \cdot WZ \cos \angle UZW \ .$$

Eliminating the cosine term yields that

$$UV^2 \cdot WZ + UW^2 \cdot VZ = (UZ^2 + VZ \cdot WZ)(WZ + VZ) \ .$$

Now,

$$UV : VZ = UW : WZ = (UV + UW) : (VZ + WZ) \ ,$$

so that

$$UV \cdot WZ = UW \cdot VZ$$

and

$$(UV + UW) \cdot WZ = UW \cdot (WZ + VZ) \ .$$

These two equations yield that

$$\begin{aligned} UV^2 \cdot WZ + UW^2 \cdot VZ &= (UV + UW) \cdot WZ \cdot UV \\ &= (WZ + VZ) \cdot UW \cdot UV \ . \end{aligned}$$

It follows that $UW \cdot UV = UZ^2 + VZ \cdot WZ$. ♣

Let R be a point on the circle with $BR \perp OA$, S be the intersection of CD and OA produced, and D' be the reflection of D in OA. Observe that C, B, D' are collinear. (Wolog, $\angle CBO < 90°$.) Since SB is an angle bisector of triangle SCD', from the Lemma, we have that

$$BS^2 = SC \cdot SD' - CB \cdot D'B = SC \cdot SD - BR^2 = SC \cdot SD - (SR^2 - BS^2)$$

whence $SC \cdot SD = SR^2$. Using power of a point, we deduce that SR is tangent to the given circle and $OR \perp SR$.

Figure 5.16.

Now

$$\begin{aligned} 2OA \cdot AS + AS^2 &= (OA + AS)^2 - OR^2 = RS^2 = BR^2 + (AB + AS)^2 \\ &= 3AB^2 + AB^2 + 2AB \cdot AS + AS^2 \end{aligned}$$

from which $4AB \cdot AS = 4AB^2 + 2AB \cdot AS$, whence $AS = 2AB = OA$. Since $OE \perp CD$, E lies on the circle with diameter OS.

Consider the reflection in the point B (dilation in B with factor -1). It interchanges E and F, interchanges O and A, and switches the circles $ADRC$ and $OERS$. Since E lies on the latter circle, F must lie on the former circle, and the desired result (a) follows.

Ad (b), the locus of E is that part of the circle with centre A that lies within the circle with centre O. Angle EAO is maximum when E coincides with R, and minimum when D coincides with A. Since triangle ORA is equilateral, the maximum angle is $60°$ and the minimum angle is $0°$. ♠

Problem 5.16. ABC and $A'B'C$ are similarly oriented equilateral triangles intersecting only at C; P, Q, R are the respective midpoints of AB', BC, $A'C$. Prove that triangle PQR is equilateral. (see also Problem 5.12)

Figure 5.17.

Solution. Perform the following sequence of transformations: (1) a dilation with centre C and factor 2; (2) a $60°$ counterclockwise rotation about B; (3) a dilation with centre C and factor $\frac{1}{2}$. The first and third preserve the direction of any line, while the second rotates lines through an angle of $60°$; thus the three taken together have the net effect of changing the direction of any line by $60°$. The point Q is first taken to B, and then finally returned to Q, so that Q is fixed by the composite of the three transformations.

Consider what happens to R. Transformation (1) sends R to A'. Suppose that (2) sends A' to A''. Since AA'' is the image of CA' under (2), $AA'' = CA'$ and AA'' makes an angle of $60°$ with CA'. But $B'C$ makes and angle of $60°$ with CA'. Therefore $AA'' \| B'C$ and $ACB'A''$ is a parallelogram. Thus, $A''C$ bisects AB' at P. Thus the transformation (3) takes A'' to P. Therefore, the composite of the three transformations takes R to P, and so takes QR to QP. Hence, $\angle PQR = 60°$ and $QR = RP$, and the result follows. ♠

Figure 5.18.

Problem 5.17. A frustum (portion of a pyramid cut off by two parallel planes) has triangular bases. The bottom base has area A and the top area $B < A$. Inscribed in the frustum are two spheres, one touching the bottom base and the three slant faces, the other touching the top base and the three slant faces, and each touching the other. Prove that the lateral surface area (of the slant faces, but not the bases) of the frustum is equal to

$$(\sqrt[4]{A} + \sqrt[4]{B})^2(\sqrt{A} + \sqrt{B}) \ .$$

Figure 5.19.

Solution. Extend the slant edges to the apex point T to complete the tetrahedron (triangular pyramid). Consider a dilation with centre T and factor $\sqrt{B/A}$, which takes the base of area A to the base of area B. The sphere touching the base of area A and the three slant faces of the frustum is the inscribed sphere of the tetrahedron with apex T and base of area A. This gets carried by the dilation to the inscribed shphere of the tetrahedron with apex T and base of area B. Let the respective radii of these spheres by r and t.

The lower sphere of the problem has radius r; let the upper sphere touching it have radius s. Suppose that the triangular cross-section of area C is tangent to these two spheres. These two spheres are related by a dilation with centre T and factor $\sqrt{C/A}$ that takes the base of area A to the base of area C, and the base of area C to the base of area B. Thus, the composite of the dilation with itself is the dilation of factor $\sqrt{B/A}$, so that $C/A = \sqrt{B/A}$, or $C = \sqrt{AB}$. We have that

$$\frac{s}{t} = \sqrt{\frac{C}{B}} = \sqrt[4]{\frac{A}{B}} \ .$$

The tetrahedron of apex T are base of area B is the union of four tetrahedra whose common apex is the centre of the top sphere of radius t and whose bases are the slant faces and the triangle of area B. Let V be the lateral area of these three slant faces of the tetrahedron with base of area B. Then, we see that the area of the tetrahedron is $(1/3)t(V+B)$.

The same tetrahedron is also the union of three tetrahedra with the centre of the midle sphere of radius s as apex and whose faces are the three slant faces, less the tetrahedron with the same apex and base of area B. Thus, the area is also $(1/3)(V-B)$ and we deduce that $t(V+B) = s(V-B)$, or $B(s+t) = V(s-t)$.

Since the lateral area U of the tetrahedron with apex T and base of area A is equal to $(A/B)V$, by the dilation, the desired area is equal to

$$\begin{aligned} U - V &= \frac{V(A-B)}{B} = \frac{(A-B)(s+t)}{(s-t)} \\ &= \frac{(A-B)(\sqrt[4]{A} + \sqrt[4]{B})}{\sqrt[4]{A} - \sqrt[4]{B}} = \frac{(\sqrt{A} + \sqrt{B})(\sqrt{A} - \sqrt{B})(\sqrt[4]{A} - \sqrt[4]{B})}{\sqrt[4]{A} - \sqrt[4]{B}} \\ &= (\sqrt{A} + \sqrt{B})(\sqrt[4]{A} + \sqrt[4]{B})(\sqrt[4]{A} + \sqrt[4]{B}) = (\sqrt{A} + \sqrt{B})(\sqrt[4]{A} + \sqrt[4]{B})^2 \end{aligned}$$

as desired. ♠

Problem 5.18. ABC is a right triangle with $\angle A = 90°$. The point D on BC is such that $AD \perp BC$. Let U and V be the respective incentres of triangle ACD and ABD. Prove that the bisector of angle A is perpendicular to UV.

`Figure 5.20.`

Solution. The composite of a rotation and dilation with centre D takes triangle ADC to triangle BDA, and so takes DU to DV. We have that $DU : DV = AC : AB$ and $\angle UDV = \angle CAB = 90°$, so that triangles DUV and ACB are similar. Since $\angle UDA = \angle VDA = 45°$, AD bisects angle UDV.

The similarity that takes triangle DUV onto ACB takes the altitude DF of triangle DUV to the altitude AD of triangle ACB, and the bisector DA of angle UDV to the bisector AG of angle CAB. Hence $\angle FDA = \angle DAG$, so that $DF \| AG$ and $AG \perp UV$. ♠

Section 6

Inversion in a circle

February 28, 2005

In this section, we will deal with a different sort of transformation that is a kind of distorted reflection which does not preserve the exact shape of geometric objects but nevertheless does have some interesting structural properties.

Let $\mathfrak{C}$ be a circle with centre O and radius r, and let P be any point of the plane other than O. We define *inversion* in the circle $\mathfrak{C}$ as follows. Draw the ray from O that passes through P. The image P' of P is that point on the ray for which $OP \cdot OP' = r^2$. If P lies on the circle, then $P' = P$. If P lies within (outside of) the circle, then P' lies outside of (within) the circle. Finally, this transformation is an involution; this means that it has period 2, so that $P'' = (P')' = P$.

Inversion in a circle has two important properties. First, if P ranges over a circle or a straight line, then so does P'. Secondly, inversion preserves angles; if two curves intersect at an angle (defined as the angle of intersection of their tangents), then the images of these curves intersect at the same angle.

Consider the first of these properties. If m is a line through O, then as P ranges over M away from O, P' ranges over m towards O.

Suppose that m is a line that does not pass through O.

Figure 6.1.

Let the perpendicular from O to the line meet it at L and let L' be the image of L under the inversion. Suppose that P is any point on m and P' is its image. Then, since $OP \cdot OP' = OL \cdot OL'$, *i.e.* $OP : OL' = OL : OP'$ and $\angle POL = \angle L'OP'$, triangles POL and $L'OP'$ are similar. Then $\angle OP'L' = \angle OLP = 90°$, and the locus of P' is a circle with diameter OL'.

Now let P travel along a circle $\mathfrak{D}$ that passes through O. Suppose that OD is a diameter of the circle and that D' is the image of D. Let P be any point on the circle; denote its image by P'. Then it can be shown from $OP \cdot OP' = OD \cdot OD'$ that triangles $OD'P'$ and OPD are similar and $\angle OD'P' = \angle OPD = 90°$. Thus, P' travels along a line through D' perpendicular to OD.

Figure 6.2.

Finally, let P travel along a circle $\mathfrak{E}$ that does not pass through O. Let QR be the diameter of $\mathfrak{E}$ whose production passes through O, and let Q' and R' be the respective images of Q and R. There are essentially two configurations for $\mathfrak{E}$.

Case (i). O lies in the interior of $\mathfrak{E}$. Let QR be the diameter of $\mathfrak{E}$ that passes through O; this diameter is collinear with $Q'R'$, the image of QR. Since triangles OPQ and $OQ'P'$ are similar,

$\angle OPQ = \angle OQ'P'$. Similarly, $\angle OPR = \angle OR'P'$. Hence

$$\begin{aligned}\angle Q'P'R' &= 180° - (\angle OQ'P' + \angle OR'P')\\ &= 180° - (\angle OPQ + \angle OPR)\\ &= 180° - 90° = 90° \ ,\end{aligned}$$

whence we see that P' lies on the circle with diameter $Q'R'$.

Figure 6.3.

Case (ii). O lies outside of $\mathfrak{C}$. Suppose that QR is a diameter with Q lying on the segment OR. As before, $\angle OPQ = \angle OQ'P'$ and $\angle OPR = \angle OR'P'$. Therefore, $\angle Q'P'R' = \angle OR'P' - \angle OQ'P' = \angle OPR - \angle OPQ = \angle OPR = 90°$, so that P' lies on the circle with diameter $Q'R'$. ♠

Figure 6.4.

There is an alternative approach to this result. As before, $\mathfrak{C}$ is a circle that does not pass through O. For any point P on the circle, let P_1 be the second point of intersection of the circle with the line OP ($P_1 = P$ when OP is tangent to $\mathfrak{C}$). Then $OP \cdot OP_1$ is equal to a constant p. (This is true for secants passing through O when O lies outside of $\mathfrak{C}$ and for chords passing through O when O lies inside $\mathfrak{C}$.) The mapping $P \leftrightarrow P_1$ is a one-one mapping from $\mathfrak{C}$ to $\mathfrak{C}$. Consider the mapping $P_1 \rightarrow P'$. Since $OP \cdot OP' = r^2$ and $OP \cdot OP_1 = p$, it follows that $OP'/OP_1 = r^2/p$, a constant. Hence $P_1 \rightarrow P'$ is a dilation with centre O that takes the circle $\mathfrak{C}$ to a circle $\mathfrak{C}'$. Hence $P \rightarrow P'$, the composite of $P \rightarrow P_1$ and $P_1 \rightarrow P'$ takes $\mathfrak{C}$ to a circle.

We can achieve the same result using analytic geometry. Wolog, let O be at the origin, and let $\mathfrak{C}$ be the circle with equation $x^2 + y^2 = 1$. Then, if $P \sim (x, y)$,

$$P' \sim (x', y') = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) .$$

The general equation of a line or a circle is

$$d(x^2 + y^2) + ax + by + c = 0 \ ,$$

where the line corresponds to the case $d = 0$. We have that $x'^2 + y'^2 = (x^2 + y^2)^{-1}$, so that

$$\begin{aligned}0 &= d + a\left(\frac{x}{x^2 + y^2} \right) + b\left(\frac{y}{x^2 + y^2} \right) + \frac{c}{x^2 + y^2}\\ &= d + ax' + by' + c(x'^2 + y'^2) \ ,\end{aligned}$$

and (x', y') lies on a circle when $c \neq 0$ and a line when $c = 0$.

Exercise 6.1. Prove, using coordinates, that inversion is its own inverse, *i.e.*, applying it twice leads you back to where you started.

We move to the preservation of angle property. Let P be any point other than O and let l be any line that passes through P but not through O. The image of l under the inversion is a circle that passes through O and P'. Let m be the tangent to the circle at P'. For any point Q on l, $\angle OPQ$ is equal to $\angle OQ'P'$, the angle subtended at the circumference of the circle by the chord OP'. But this is equal to the angle between OP' and the tangent m. Therefore l and the image circle make the same angle with the ray OP.

If we have two lines l_1 and l_2, from the fact that each of the lines makes the same angle with each ray as their respective images, it can be deduced that the angle between l_1 and l_2 is the same as the angle between their images.

Exercise 6.2. Consider inversion in a circle with centre O and radius r. Suppose that P and Q are points whose respective distances from O are p and q and that $|PQ| = d$. Show that $|P'Q'| = (dr^2)/(pq)$. Check your answer independently in the special cases that (i) $\angle POQ = 0°$, (ii) $p = q = r$, (iii) $pq = r^2$, (iv) $p = 0$.

Soution 1. Since $|OP'| \cdot |OP| = |OQ'| \cdot |OQ| = r^2$, triangles $OP'Q'$ and OQP are similar and $P'Q' : PQ = OP' : OP$. It follows that

$$\begin{aligned} |P'Q'| &= \frac{d|OQ'|}{|OP|} \\ &= \frac{d|OQ'||OQ|}{|OP||OQ|} = \frac{dr^2}{pq} \ . \end{aligned}$$

Solution 2. Let $\angle POQ = \theta$. By the Law of Cosines,

$$d^2 = |PQ|^2 = p^2 + q^2 - 2pq \cos\theta$$

and

$$\begin{aligned} |P'Q'|^2 &= \frac{r^4}{p^2} + \frac{r^4}{q^2} - \frac{2r^4}{pq} \cos\theta \\ &= \frac{r^4}{p^2q^2}(q^2 + p^2 - 2pq\cos\theta) = \frac{r^4d^2}{p^2q^2} \ . \quad \spadesuit \end{aligned}$$

Exercise 6.3. Let $\mathfrak{D}$ be a circle in the plane. Prove that the image of $\mathfrak{D}$ is equal to $\mathfrak{D}$ if and only if $\mathfrak{D}$ intersects $\mathfrak{C}$ at right angles.

Exercise 6.4. Let $\mathfrak{D}$ be a circle whose centre coincides with the centre of the circle of inversion. Must $\mathfrak{D}'$ be a circle with the same centre?

Exercise 6.5. Given two nonintersecting circles, prove that there is an inversion that carries them to to concentric circles.

Solution. Let the two circles have centres C_1 and C_2 and respective radii r_1 and r_2. We suppose that the distance between C_1 and C_2 is the positive number d. There are two cases, according as one circle lies within the other.

Case (i). Let the circle $\mathfrak{C}_1$ with centre C_1 lie inside the circle $\mathfrak{C}_2$ with centre C_2. The strategy is to find a circle $\mathfrak{C}$ which intersects both these circles at right angles and then determine an inversion that takes $\mathfrak{C}$ and the line C_1C_2 of centres into straight lines that will intersect the images of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ at right angles. This can happen only if these images both have centres at the intersection of these lines.

The first step is to determine a point P on C_1C_2 which has tangents PT_1 and PT_2 of equal length to $\mathfrak{C}_1$ and $\mathfrak{C}_2$ respectively. Let $d = |C_1C_2|$. We select $x = |PC_1|$ to satisfy $x^2 - r_1^2 = (x+d)^2 - r_2^2$, or

$$x = \frac{1}{2d}(r_2^2 - r_1^2 - d^2) \ .$$

Let $\mathfrak{C}$ be the circle of centre P and radius $|PT_1| = |PT_2|$. This circle $\mathfrak{C}$ intersects C_1C_2 at right angles.

Figure 6.5.

Select, as circle of inversion, any circle whose centre O lies on the intersection of C_1C_2 and $\mathfrak{C}$. The inversion in this circle carries $\mathfrak{C}$ and C_1C_2 to a perpendicular pair of straight lines intersecting at some point Q. Since $\mathfrak{C}$ and C_1C_2 intersect both $\mathfrak{C}_1$ and $\mathfrak{C}_2$ at right angles, the perpendicular lines $\mathfrak{C}'$ and C_1C_2 intersect $\mathfrak{C}_1'$ and $\mathfrak{C}_2'$ at right angles. Thus, these lines must contain diameters of both circles $\mathfrak{C}_1'$ and $\mathfrak{C}_2'$, so that $\mathfrak{C}_1'$ and $\mathfrak{C}_2'$ are concentric. (Note that the positions of $\mathfrak{C}_1'$ and $\mathfrak{C}_2'$ will vary with the radius of the circle of inversion.)

Case (ii). Let the circles $\mathfrak{C}_1$ and $\mathfrak{C}_2$ lie outside of each other.

Figure 6.6.

Follow the same strategy to select P between C_1 and C_2 to make $x = |PC_1|$ satisfy $x^2 - r_1^2 = (d-x)^2 - r_2^2$ or

$$x = \frac{1}{2d}(d^2 + r_1^2 - r_2^2) \ .$$

Let $\mathfrak{C}$ be the circle with centre P and radius $|PT_1| = |PT_2|$ and invert in any circle whose centre is an intersection of $\mathfrak{C}$ and C_1C_2.

Query. What happens if the given pair of circles are already concentric?

The last exercise has an interesting consequence, which is left to the reader to establish. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be any two nonconcentric circles with $\mathfrak{C}_1$ lying inside $\mathfrak{C}_2$. We construct a chain of circles $\mathfrak{D}_i$ as follows. Let $\mathfrak{D}_1$ be any circle interior to $\mathfrak{C}_2$ and exterior to $\mathfrak{C}_1$ that touches both $\mathfrak{C}_1$ and $\mathfrak{C}_2$. For $i > 1$, let $\mathfrak{D}_i$ be a circle that rouches $\mathfrak{C}_1$. $\mathfrak{C}_2$ and $\mathfrak{D}_{i-1}$, as illustrated in the diagram.

Figure 6.7.

We say that the chain *closes in on itself*, if for some positive integer n exceeding 1, $\mathfrak{D}_n = \mathfrak{D}_1$. This means that, when we we go around the circles, we eventually get to a circle in the chain that touches the first circle $\mathfrak{D}_1$. The chain may go around the circles some whole number k of times for this to happen. Remarkably, if the chain closes in on itself for some values of n and k, it will close in on itself for the same values of n and k regardless of the position of the starting circle $\mathfrak{D}_1$. The proof is a straightforward consequence of Exercise 6.4, when we realize that the result is clearly true for a concentric pair of circles.

Section 7

Problems involving inversions

December 20, 2004

Problem 7.1. Let A be a point on a circle with centre O and let B be the midpoint of OA. Let C and D be points on the circle on the same side of OA produced for which $\angle CBO = \angle DBA$. Let E be the midpoint of CD and let F be the point on EB produced for which $BF = BE$.

(a) Prove that F lies on the circle.

(b) What is the range of angle EAO?

Solution 1. [Y. Zhao] When $\angle CBO = \angle DBA = 90°$, the result is obvious. Wolog, suppose that $\angle CBO = \angle DBA < 90°$. Suppose that the circumcircle of triangle OBD meets the given circle at G. Since $OBDG$ is concyclic and triangle OGD is isosceles,

$$\angle OBC = \angle ABD = 180° - \angle OBD = \angle OGD = \angle ODG = \angle OBG \ ,$$

so that $G = C$ and $OBDC$ is concyclic.

Let H lie on OA produced so that $OA = AH$. Since $OB \cdot OH = OA^2$, the inversion in the given circle with centre O interchanges B and H, fixes C and D, and carries the circle $OBDC$ (which passes through the centre O of inversion) to a straight line passing through H, D, C. Thus C, D, H are collinear.

This means that CD always passes through the point H on OA produced for which $OA = AH$. Since E is the midpoint of CD, a chord of the circle with centre O, $\angle OEH = \angle OED = 90°$. Hence E lies on the circle with centre A and radius OA.

Consider the reflection in the point B (the dilation with centre B and factor -1). This takes the circle with centre O and radius OA to the circle with centre A and the same radius, and also interchanges E and F. Since E is on the latter circle, F is on the given circle.

Ad (b), E lies on the arc of the circle with centre A and radius OA that joins O to the point R of intersection of this circle and the given circle. Since $RB \perp OA$, and $OA = OR = RA$, $\angle RAO = 60°$. It can be seen that $\angle EAO$ ranges from $0°$ (when CD is a diameter) to $60°$ (when $C = D = R$). ♠

Problem 7.2. Suppose that $ABCDEF$ is a convex hexagon for which $\angle A + \angle C + \angle E = 360°$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1 \ .$$

Prove that

$$\frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} = 1 \ .$$

Solution. [Y. Zhao] Consider inversion in a circle with centre F. Then

$$\begin{aligned}360^\circ &= \angle FAB + \angle BCD + \angle DEF = \angle FAB + \angle BCF + \angle FCD + \angle DEF \\ &= \angle A'B'F + \angle FB'C' + \angle C'D'F + \angle FD'E' = \angle A'B'C' + \angle C'D'E'\end{aligned}$$

whence $\angle C'B'A' = \angle C'D'E'$. (Draw a diagram and check the position and orientation of the angles.)

In the following, we use Problem 1 and suppress the factor r^2, where r is the radius of the circle of inversion. Then

$$\begin{aligned}\frac{A'B'}{B'C'} \cdot \frac{C'D'}{D'E'} &= \left(\frac{AB}{FA \cdot FB} \cdot \frac{FB \cdot FC}{BC}\right) \cdot \left(\frac{CD}{FC \cdot FD} \cdot \frac{FD \cdot FE}{DE}\right) \\ &= \frac{AB}{FA} \cdot \frac{CD}{BC} \cdot \frac{EF}{DE} = 1\end{aligned}$$

so that $A'B' : B'C' = D'E' : C'D'$. This, along with $\angle C'B'A' = \angle C'D'E'$ implies that $\Delta C'B'A' \sim \Delta C'D'E'$, so that $A'B' : A'C' = D'E' : E'C'$ or $A'B' \cdot E'C' = A'C' \cdot E'D'$.

Therefore

$$\begin{aligned}\frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} &= \left(\frac{A'B'}{FA' \cdot FB'} \cdot B'F\right) \cdot \left(\frac{1}{F'D'} \cdot \frac{FD' \cdot FE'}{D'E'}\right) \cdot \left(\frac{E'C'}{FE' \cdot FC'} \cdot \frac{FC' \cdot FA'}{C'A'}\right) \\ &= \frac{A'B'}{A'C'} \cdot \frac{E'C'}{E'D'} = 1\ ,\end{aligned}$$

as desired.

Problem 7.3. Suppose that ABC is a right triangle with $\angle B < \angle C < \angle A = 90^\circ$, and let $\mathfrak{K}$ be its circumcircle. Suppose that the tangent to $\mathfrak{K}$ at A meets BC produced at D and that E is the reflection of A in the axis BC. Let X be the foot of the perpendicular from A to BE and Y the midpoint of AX. Suppose that BY meets $\mathfrak{K}$ again in Z. Prove that BD is tangent to the circumcircle of triangle ADZ.

Solution. [Y. Zhao] Let AZ and BD intersect at M, and AE and BC intersect at P. Since PY joints the midpoints of two sides of triangle AEX, $PY \| EX$. Since $\angle APY = \angle AEB = \angle AZB = \angle AZY$, the quadrilateral $AZPY$ is concyclic. Since $\angle AYP = \angle AXE = 90^\circ$, AP is a diameter of the circumcircle of $AZPY$ and BD is a tangent to this circle.

Let O be the centre of the circle $\mathfrak{K}$. The triangles OPA and OAD are similar, whereupon $OP \cdot OD = OA^2$. The inversion in the circle $\mathfrak{K}$ interchanges P and D, carries the line BD to itself and takes the circumcircle of triangle AZP to the circumcircle of triangle AZD. As the inversion preserves tangency of circles and lines, the desired result follows.