IMO Winter Camp Mock Olympiad 2010
Time: 4 Hours

1. Let $ABCD$ be a parallelogram with $AC/BD = k$. The bisectors of the angles formed by AC and BD intersect the sides of $ABCD$ at K, L, M, N. Prove that the ratio of the areas of $KLMN$ and $ABCD$ is $2k/(k+1)^2$.

2. Each edge of an $m \times n$ rectangular grid is oriented with an arrow such that

   (a) the border is oriented clockwise, and

   (b) each interior vertex has two arrows coming out of it, and two arrows going into it.

   Prove that there is at least one square whose edges are oriented clockwise.

3. Let $\mathbb{Z}^*$ denote the set of non-zero integers. A function $f : \mathbb{Z}^* \to \mathbb{Z}^{\geq 0}$ satisfies the following properties:

   (a) $f(m+n) \geq \min\{f(m), f(n)\}$ for all $m, n \in \mathbb{Z}^*, m+n \neq 0$.

   (b) $f(mn) = f(m) + f(n)$ for all $m, n \in \mathbb{Z}^*$.

   (c) $f(2010) = 1$.

   Determine the minimum and maximum possible value of $f(2010!)$.

4. If a, b, c are positive real numbers such that $a + b + c = 3$, show that

$$\frac{1}{2+a^2+b^2} + \frac{1}{2+b^2+c^2} + \frac{1}{2+c^2+a^2} \leq \frac{3}{4}$$

IMO Winter Camp Mock Olympiad 2010 Solutions

1. Let $ABCD$ be a parallelogram with $AC/BD = k$. The bisectors of the angles formed by AC and BD intersect the sides of $ABCD$ at K, L, M, N. Prove that the ratio of the areas of $KLMN$ and $ABCD$ is $2k/(k+1)^2$.

   Solution: Suppose K, L, M, N are on AB, BC, CD, DA, respectively. Let $P = AC \cap BD$. Since $ABCD$ is a parallelogram, $AP = PC$ and $BP = PD$. For any polygon $X_1X_2 \cdots X_n$, we denote its area by $[X_1X_2 \cdots X_n]$.

   By the angle bisector theorem, $AK/KB = AP/PB = AC/BD = k$. Similarly, $AN/ND = k$. Hence, $AK/AB = AN/AD = \frac{k}{k+1}$. Consequently, KN is parallel to BD. Furthermore,

$$[AKN] = \frac{k^2}{(k+1)^2}[ABD] = \frac{k^2}{2(k+1)^2}[ABCD].$$

   Similarly,

$$[BKL] = \frac{1}{2(k+1)^2}[ABCD], [CLM] = \frac{k^2}{2(k+1)^2}[ABCD], [DMN] = \frac{1}{2(k+1)^2}[ABCD].$$

   Hence,

$$\begin{aligned}[KLMN] &= [ABCD] - [AKN] - [BKL] - [CLM] - [DMN] \\ &= \left(1 - \frac{k^2}{(k+1)^2} - \frac{1}{(k+1)^2}\right)[ABCD] = \frac{2k}{(k+1)^2}[ABCD],\end{aligned}$$

   as desired. □

   Source: IMO Correspondence program, 1995-96 (from Ed Barbeau)

2. Each edge of an $m \times n$ rectangular grid is oriented with an arrow such that

   (a) the border is oriented clockwise, and

   (b) each interior vertex has two arrows coming out of it, and two arrows going into it.

   Prove that there is at least one square whose edges are oriented clockwise.

   Solution: Let a square be called half-clockwise if its top and left sides are oriented clockwise. Suppose that there is no clockwise square in the grid.

   Consider a half-clockwise square S. Since it is not a clockwise square, either the right or bottom edge must be oriented counterclockwise. If it is the right edge, let the square to the right of S be T (if there are no squares to the right of S, then the right edge of S is on the border and thus must be oriented clockwise).

Consider the top right vertex of S - it is either in the interior or on the top border of the grid. If it is an interior vertex, it has the top and right edges of S going into it, so its other edges are going out, in particular the top edge of T. If it is on the top border, then the top edge of T is clockwise. In both cases, T is a half-clockwise square.

If the bottom edge of S is oriented counterclockwise, we can similarly show that the square T below S is a half-clockwise square.

Since the top left square in the grid is half-clockwise, we can make a path of half-clockwise squares by stepping either to the right or down. This implies that the bottom right square in the grid is half-clockwise. However, the bottom and right edges of this square are clockwise-oriented, so the square is clockwise. Contradiction. $\square$

Source: IMO Training 2006, General Problems

3. Let $\mathbb{Z}^*$ denote the set of non-zero integers. A function $f : \mathbb{Z}^* \to \mathbb{Z}^{\geq 0}$ satisfies the following properties:

   (a) $f(m+n) \geq \min\{f(m), f(n)\}$ for all $m, n \in \mathbb{Z}^*, m+n \neq 0$.
   (b) $f(mn) = f(m) + f(n)$ for all $m, n \in \mathbb{Z}^*$.
   (c) $f(2010) = 1$.

   Determine the minimum and maximum possible value of $f(2010!)$.

   Solution: The minimum possible value is 30 and the maximum possible value is 2002.

   Substituting $m = n = 1$ into (b) yields $f(1) = 0$. Substituting $m = n = -1$ into (b) yields $f(-1) = 0$. Substituting $m = -1$ into (b) yields $f(n) = f(-n)$ for all $n \in \mathbb{Z}^*$.

   Let p be the smallest integer such that $f(p) = 1$. This is well defined since $f(2010) = 1 > 0$. I claim that p is a prime. Suppose $p = ab$ for some $1 < a, b < p$. Then $f(p) = f(a) + f(b)$ by (b). Hence, at least one of $f(a), f(b) = 1$. This contradicts p being the smallest integer such that $f(p) > 0$. Therefore, p is prime and $f(1) = f(2) = \cdots = f(p-1) = 0$.

   Let a be a positive integer not divisible by p. I claim that $f(a) = 0$. Write $a = qp + r$ where $0 \leq r < p$. Then $f(r) = 0$. Therefore, $0 = f(r) = f(a - qp) \geq \min\{f(a), f(-qp)\} = \min\{f(a), f(qp)\} \geq 0$. Hence, equality must hold throughout. Therefore, $\min\{f(a), f(qp)\} = 0$. Since $f(qp) = f(q) + f(p) \geq f(p) > 0$, $f(a) = 0$.

   For any positive integer m, write $m = p^r \cdot s$ where $r \in \mathbb{Z}^{\geq 0}$ and $s \nmid p$. Then $f(m) = f(p^r) + f(s) = r \cdot f(p) = r$. Hence, $f(m)$ is the number of times p divides into m for a fixed prime p. We will now verify that such a function satisfies properties (a) and (b).

Let $m = p^a \cdot b$ and $n = p^c \cdot d$ such that $a, c \in \mathbb{Z}^{\geq 0}$ and $p \nmid b, d$. Then $mn = p^{a+c} \cdot bd$ and $p \nmid bd$. Therefore, $f(mn) = a + c = f(m) + f(n)$. Hence, f satisfies (b). Finally, without loss of generality, suppose $a \leq c$. Then $m + n = p^a \cdot b + p^c \cdot d = p^a(b + p^{c-a}d)$. Hence p divides into $m + n$ at least a times. Therefore, $f(m + n) \geq a = \min\{a, c\} = \min\{f(m), f(n)\}$. Hence, f satisfies (a). Note that p can be chosen to be any prime number.

Since $f(2010) = 1$ and $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, we conclude that all of the possible values of p are $2, 3, 5, 67$. We want to find the number of times each of these values of p divide into 2010!. The maximum and minimum possible value of $f(2010!)$ is obtained by taking $p = 2$ and $p = 67$, respectively.

If $p = 2$, then the number of times 2 divides into 2010! is

$$\left\lfloor \frac{2010}{2} \right\rfloor + \left\lfloor \frac{2010}{4} \right\rfloor + \left\lfloor \frac{2010}{8} \right\rfloor + \left\lfloor \frac{2010}{16} \right\rfloor + \left\lfloor \frac{2010}{32} \right\rfloor + \left\lfloor \frac{2010}{64} \right\rfloor + \left\lfloor \frac{2010}{128} \right\rfloor + \left\lfloor \frac{2010}{256} \right\rfloor + \left\lfloor \frac{2010}{512} \right\rfloor + \left\lfloor \frac{2010}{1024} \right\rfloor,$$

$= 1005 + 502 + 251 + 125 + 62 + 31 + 15 + 7 + 3 + 1 = 2002.$

If $p = 67$, then the number of times 67 divides into 2010! is $\frac{2010}{67} = 30$. Therefore, the minimum and maximum possible value of $f(2010!)$ is 30 and 2002, respectively. $\square$

Source: Original, but this is a well-known map called a valuation map.

4. If a, b, c are positive real numbers such that $a + b + c = 3$, show that

$$\frac{1}{2 + a^2 + b^2} + \frac{1}{2 + b^2 + c^2} + \frac{1}{2 + c^2 + a^2} \leq \frac{3}{4}$$

Solution 1: Multiply both sides by 2 to get:

$$\frac{2}{2 + a^2 + b^2} + \frac{2}{2 + b^2 + c^2} + \frac{2}{2 + c^2 + a^2} \leq \frac{3}{2}$$

$\Leftrightarrow$

$$3 - \frac{2}{2 + a^2 + b^2} + \frac{2}{2 + b^2 + c^2} + \frac{2}{2 + c^2 + a^2} \geq 3 - \frac{3}{2}$$

$\Leftrightarrow$

$$\sum_{cyc} \frac{a^2 + b^2}{2 + a^2 + b^2} \geq \frac{3}{2}.$$

By Cauchy-Schwarz Inequality, we have

$$\sum_{cyc} \frac{a^2 + b^2}{2 + a^2 + b^2} \geq \frac{\left(\sum\limits_{cyc} \sqrt{a^2 + b^2}\right)^2}{\sum\limits_{cyc}(2 + a^2 + b^2)} = \frac{\sum\limits_{cyc}(a^2 + b^2) + 2\sum\limits_{cyc} \sqrt{(a^2 + b^2)(a^2 + c^2)}}{6 + 2\sum\limits_{cyc} a^2}.$$

By Cauchy-Schwarz Inequality,

$$\sqrt{(a^2 + b^2)(a^2 + c^2)} \geq a^2 + bc.$$

Hence, our expression is greater than or equal to

$$\frac{\sum_{cyc}(a^2+b^2)+2\sum_{cyc}(a^2+bc)}{6+2\sum_{cyc}a^2}=\frac{(3\sum_{cyc}a^2)+(a+b+c)^2}{6+2\sum_{cyc}a^2}=\frac{(3\sum_{cyc}a^2)+9}{6+2\sum_{cyc}a^2}=\frac{3}{2},$$

as desired. $\square$

Solution 2: (Mixing Variables - With The Cumbersome Steps Omitted) For $x,y,z>0$, let

$$f(x,y,z)=\frac{1}{2+x^2+y^2}+\frac{1}{2+y^2+z^2}+\frac{1}{2+z^2+x^2}.$$

Let $t=\frac{a+b}{2}$. We will prove that $f(a,b,c)\le f(t,t,c)\le\frac{3}{4}$ to solve the problem. Without loss of generality, suppose $a\le b\le c$.

Note that $f(t,t,c)-f(a,b,c)$

$$=\left(\frac{1}{2+2t^2}-\frac{1}{2+a^2+b^2}\right)+\left(\frac{2}{2+t^2+c^2}-\frac{1}{2+a^2+c^2}-\frac{1}{2+b^2+c^2}\right).$$

We prove each of these two terms is non-negative to prove $f(a,b,c)\le f(t,t,c)$.

$$\frac{1}{2+2t^2}-\frac{1}{2+a^2+b^2}=\frac{1}{2+2\left(\frac{a+b}{2}\right)^2}-\frac{1}{2+a^2+b^2}=\frac{(a-b)^2}{4\left(1+\left(\frac{a+b}{2}\right)^2\right)(2+a^2+b^2)}\ge 0.$$

In the second term, it suffices to show that the numerator is non-negative. The numerator is equal to

$$\begin{array}{rl} & 8+4(a^2+b^2+2c^2)+2(a^2+c^2)(b^2+c^2)-(2+t^2+c^2)(4+a^2+b^2+2c^2) \\ = & 2(a^2+b^2+a^2b^2)-t^2(a^2+b^2+4)+(a^2+b^2-2t^2)c^2 \\ = & (a-b)^2\cdot\frac{4-a^2-4ab-b^2+2c^2}{4} \end{array}.$$

It suffices to show that $a^2+4ab+b^2-2c^2\le 4$. Note that $a^2+4ab+b^2-2c^2\le(a+b)^2+\frac{(a+b)^2}{2}-2c^2$. By substituting $a+b=3-c$, the latter term being at most 4 is equivalent to $(c+19)(c-1)\ge 0$. Since $a\le b\le c$ and $a+b+c=3$, $c\ge 1$. Hence, the statement is true. Hence, we conclude that $f(a,b,c)\le f(t,t,c)$.

It remains to show that $f(t,t,c)\le\frac{3}{4}$, i.e.

$$\frac{1}{2+2t^2}+\frac{2}{2+t^2+c^2}\le\frac{3}{4}.$$

This simplifies to $3t^2c^2+3t^4+c^2-t^2-6\ge 0$. By substituting $c=3-2t$, we simplify this to proving $(t-1)^2(5t^2-2t+1)\ge 0$. Since the latter term has negative discriminant, this term is indeed non-negative and therefore the inequality is true. Hence, $f(t,t,c)\le\frac{3}{4}$, as desired. $\square$

Source: Iranian Team Selection Test 2009