Lecture 17 — Induction by Faith

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1 Introduction

In this paper, the phrase "Induction by Faith" is used as a personal metaphor to denote what mathematicians have historically referred to as "Proof by Induction". For many of the problems you are about to encounter would be practically impossible to solve, even with all of the world's supercomputers working in unison.

For example:

$$(37218937128931273381!)^2 > 37218937128931273381^{37218937128931273381}.$$

One can readily see this is an absurdly large number, yet a 5th grader should be able to solve this inequality, using elementary tools of induction. In competition math, pen must be especially circumspect of problems involving astronomically large numbers (as in the mathematical statement above) and consider instead to arrive at a generalization for all natural integers n (or whatever other domain is being used). A quick check by hand, for $n = 2$, $2!^2 = 2^2 = 4$ renders the equality, and any integer larger than 2 will result in a similar inequality to the above statement.

After perusing this paper, you will have little difficulty proving the inequality above and more challenging Olympiad and even Putnam style problems. Please don't hesitate to email me at Mathsolver24@gmail.com with any comments or questions, especially errors that need to be fixed.

Here is a non mathematical problem, which will give you something to think about, and appreciate how powerful Induction really is, after seeing the problems and questions that arise from this short piece. Induction can be hard on one's intuition, since it leads to conclusions that are the consequences of enormous numbers of tiny steps (from 1 to 2 to 3 to $\ldots$ n to n + 1 to ...). Each tiny step is right, so the overall conclusion must follow, but sometimes it's hard to see the intuition behind this global picture.

Once we move into situations that are not fully specied mathematically, this kind of reasoning can lead to situations that clash much more severely with intuition. In fact, the use of induction as a part of everyday, "common-sense" reasoning, as opposed to strictly as a proof technique, is a topic that has occupied philosophers and mathematicians for many years. We now discuss three examples where inductive reasoning in informal situations (i.e. situations that are not entirely mathematically precise) leads to trouble. Each of these illustrates a genre of induction problem that has been the subject of considerable philosophical debate. And all three share the property that there is no "simple explanation of how to get around the problem—they each point at something fundamental in the way inductive reasoning works.

Lifting a Cow

Farmers in Gottingébn, according to folklore, like to discuss the following conundrum. On the day a calf is born, you can pick it up and carry it quite easily. Now, calves don't grow that much in one day, so if you can lift it the day it's born, you can still lift it the next morning. And, continuing this reasoning, you can go out the next morning and still lift it, and the next morning after that, and so forth. But after two years, the calf will have grown into a full-sized, thousand-pound cow something you clearly can't lift. So where did things break down? When did you stop being able to lift it? There's clearly an inductive argument going on here: there was a basis, that you can lift it on day 1, and an induction step – if you could lift it up to day n, then it doesn't grow much at all in one day, so you can still lift it on day $n+1$. Yet the conclusion, that you can lift it on every day of its life, clearly fails, since you can't lift the thousand-pound cow it becomes.

As promised above, there's no simple "mistake." in this argument; it's genuinely puzzling. But it's clearly related to things like (a) the fact that growth is a continuous process, and we're sampling it at discrete (day-by-day) intervals, and (b) the fact that "lifting" is not entirely well-defined; maybe you just lift it less and less each day, and gradually what you're doing doesn't really count as "lifting."

The Surprise Quiz

You're take an intensive one-month summer course that meets every day during the month of June. Just before the course starts, the professor makes an announcement. "On one of the days during the course, there will be a surprise quiz. The date will be selected so that, on the day when the quiz is given, you won't be expecting it." Now, you think about this, and you realize that this means the quiz can't be on June 30: that's final day of the course, so if the quiz hasn't been given before then, it has to be given on the 30th, and so when you walk into class on the 30th you'll certainly be expecting it. But that means the quiz can't be on June 29: we just argued that the quiz can't be on the 30th, so if you walk into class on the 29th, you'll know it has to be given that day. But that means the quiz can't be on June 28: we just argued that the quiz can't be on the 29th or the 30th, so if you walk into class on the 28th, you'll know this is the last chance, and it has be given that day. But that means the quiz can't be on June 27, well, you get the idea. You can argue in this way, by induction (going down from large numbers to small), that if the quiz can't be given from day n onwards, then when you walk into class on on day $n-1$, you'll be expecting it. So it can't be given on day $n-1$ either. Yet something is wrong with all this. For example, suppose it turns out that the quiz is given on June 14. Why June 14? Exactly the point – you're surprised by the choice of June 14, so it was a surprise quiz after all.

This paradox (also called the "unexpected hanging", where the (more grisly) story is of a surprise execution) is truly vexing, and it's been claimed that over a hundred research papers have been written on the topic. Again, rather than try to identify the difficulties in the reasoning, let's simply note that any attempt to formalize what's going on will have to be very careful about what is means to be "surprised," and what is means to "know" that the quiz "must" be given on a certain date. (Source: Math Team Induction article, Cornell Univ.)

Fortunately, many of the problems that arise in the previous paradoxes won't appear here, because of the more strict nature of mathematics and all variables in math are taken into full account.

2 The Main Idea

In this section, we will discuss general mathematical induction and formalities. For sake of completion I will include a formal definition, but this is more of a heuristic process than anything else. When we are more familiar with it, we will have to pay much closer attention to the formal definition since Induction will be a much slippier slope with *IMO level* 6 type problems. Our first result is that when we divide polynomials, we can be assured to get a remainder with degree smaller than our divisor.

Theorem 2.1 (Induction, Formal Definition)**:** Let P(n) be a function or statement that depends on positive integer n (or $\mathbb{Z}^+$). If P(1) is true (or any other base case, e.g. in first sample problem it's 3) and if P(k)$\Rightarrow$ P($k+1$) for any positive integer k, then P(n) is true for all positive integers n.

Phew! What a mouthful of words, and I apologize for inserting so many parenthesis. Hopefully this one example will do a better job of explaining it.

Example 2.2 (Demonstration by Heuristics)**:** Prove that $1+3+5+\cdots+2n-1=n^2$.

Proof.

1. The base case is 1 or $1 = 1^2$,
2. So we assume it's true for some integer k
3. $$1+3+5+\cdots+(2k-1)=k^2$$
4. The next step is to get to $(k+1)$; in our case it's very easy we just have to add it to both sides, since this is a summation problem.
$$[1+3+5\ldots(2k-1)]+(2k+1)=[k^2]+(2k+1)$$
5. $$=(k+1)^2$$

□

Example 2.3: Prove that

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Proof. For our base case we see that $n=1$, and on *RHS* (Right Hand Side)

$$\frac{1\cdot 2}{2}=1$$

so now, we will try to prove it for some positive integer k which says

$$1+2+\cdots+k=\frac{k(k+1)}{2}$$

which implies

$$\begin{aligned}[1+2+\cdots+k]+(k+1) &= \left[\frac{k(k+1)}{2}\right]+(k+1)\\ &= (k+1)\left[\frac{k}{2}+1\right]\\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

hence the result is true for $n = k+1$ and by induction we are done. $\square$

3 Strengthening the Induction

Below we are going to find a new kind of inductive problem, in which none of our previous principles are going to work. We are going to have to modify it, and in fact prove an even stronger result, in which from it's obvious that the general result is true. (If I can lift 500 pounds, it follows that I can lift a 100 pounds). It seems almost counter-intuitive, that by adding restrictions we can prove something true, yet we cannot do so with the previous "flexibility".

Example 3.1: Prove that for all integers n,

$$\left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\cdots\left(1+\frac{1}{n^3}\right) < 3.$$

If we were to prove it using tools discussed previously, we'd show for our base case $n = 1$,

$$\left(1+\frac{1}{1^3}\right) = 2 < 3$$

and for our second base $n = 2$,

$$1+\left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right) = 2\left(1+\frac{1}{2^3}\right) = 2+\frac{1}{4} < 3$$

But if we went from $2 < 3$, to $2+\frac{1}{4} < 3$, which we got to by adding this inequality $\frac{1}{4} < 0$ to the first inequality, but we know that is false and induction doesn't work here. At least not the kind we're used to.

So let's restart from the beginning, we know for the base case $n = 1$ that it's true. So let $n = k$ for some integer, and let

$$A(k) = \left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\cdots\left(1+\frac{1}{k^3}\right) < 3$$

We are going to create an inductive function (below I will discuss ways to find a good function), that while making the right hand side stricter (smaller) at same time it will give us more room to work, so let

$$A(k) < 3 - \frac{1}{k}$$

For $n = k+1$, we have

$$\begin{aligned}
A(k+1) &= A(k)\left[1+\frac{1}{(k+1)^3}\right] \\
&< \left(3-\frac{1}{k}\right)\left(1+\frac{1}{(k+1)^3}\right) \\
&= 3 - \frac{1}{k} + \frac{3}{(k+1)^3} - \frac{1}{k(k+1)^3} \\
&= 3 - \frac{(k+1)^3 - 3k + 1}{k(k+1)^3}.
\end{aligned}$$

This will show the case $n = k+1$, provided

$$\frac{(k+1)^3 - 3k + 1}{k(k+1)^3} > \frac{1}{k+1},$$

which is equivalent to

$$k^3 + 3k^2 + 2 > k(k+1)^2$$

and further to

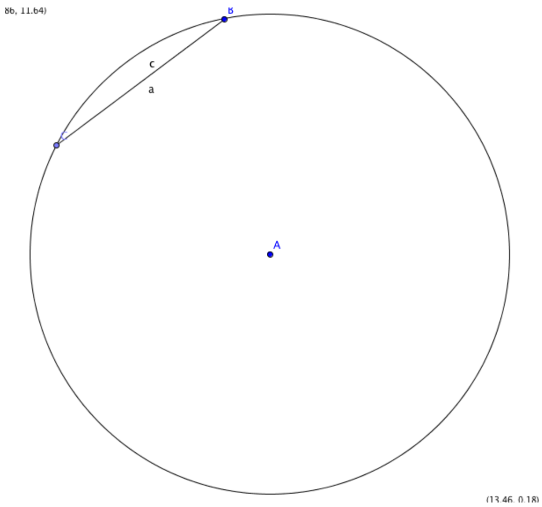
$$k^2 - k + 2 > 0$$

Since $P(x) = x^2 - x + 2$ has no real roots, and $P(0) = 0 < 2$ and $P(x) > 0$ (stated by the problem), and thus for all real numbers x, in the inequality holds, and it holds for $n = k+1$, and the induction is complete.

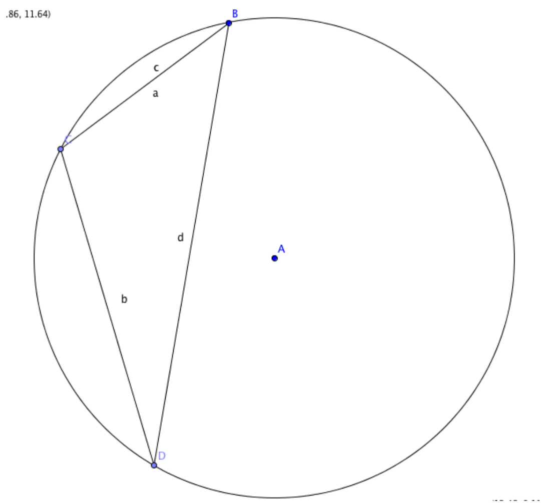
I will now show an example of a very simple but sneaky problem.

Example 3.2: Below is a circle, with one point. Every time I add a point on the circle, I will connect all the points, creating a "complete graph." I want you to find for me, the formula to finding the maximum number of regions. (So there will be no concurrent intersections or points.)

Let $f(n) = g$ where n is number of points, and g is number of regions formed. Now for two points,



Three points, four points and so on, we are seeing a pattern here, that suggest $f(n) = 2^{n-1}$ So now let's prove it.



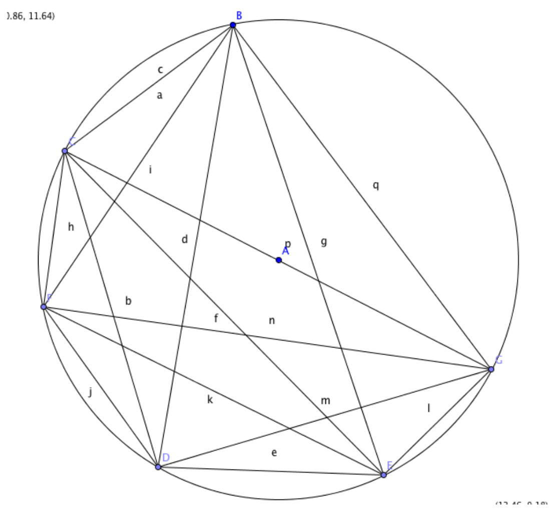
For our base case,

$$n = 1, f(1) = 2^{1-1} \to 2^0 = 1.$$

For the next case, $n = 2$, we notice that attaching another chord, and connecting them doubles every single region.

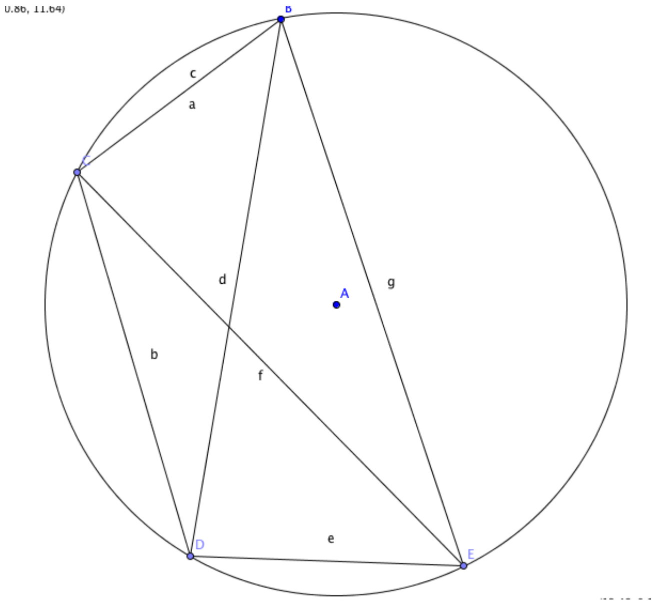
So assume it's true for some $n = k$ so that we have 2^{k-1}, and for $k+1$ we have $2 \cdot 2^{k-1} = 2^k$ regions, and we are done. Or are we? It seems like we simply stated that every additional chord doubles the amount of previously existing regions, which was nothing more than just restating our postulate. In fact we will find a counterexample for $f(6) = 32$, yet on any actual circle that you draw you will find there are always 31 regions, a problem with our proof.

Don't believe me? Draw a circle yourself, until you are satisfied that there can only be 31 regions, which puts our original conjecture in jeopardy. Below is a diagram of 6 points with 31 regions.



I will now show a correct proof, with base $n = 4$ (it's easier to show with more than one or two points, which is why we led to our erroneous solution, also we assumed that every new line would create a new another region from an old region. Nowhere did we justify any such lemmas).

In the figure above, you can see the lines connecting points 1through 4 dividing the circle into 8 total regions (i.e., $f(4) = 8$).



This figure illustrates the inductive step from $n = 4 = k$ to $n = (k + 1) = 5$ with the dashed lines. When the fifth point is added (i.e., when computing $f(5)$ using $f(4)$ recursively, this results in four $(k - 1)$ new lines being added, numbered B through E, a letter for each point that they connect to. The number of new regions introduced by the fifth point can therefore be determined by considering the number of regions added by each of the 4 lines. Set i to count the lines we are adding. Each new line can cross a number of existing lines, depending on which point it is to (the value of i). The new lines will never cross each other, except at the new point (otherwise they wouldn't be maximized). The number of lines that each new line intersects can be determined by considering the number of points on the "above" of the line and the number of points on the "below" of the line (in my case, with my drawing, but this is relative and same concept for all graphs). Since all existing points already have lines between them, the number of points on the above sec. multiplied by the number of points on the lower sec. is the number of lines that will be crossing the new line. For the line to point i, there are $n - i - 1$ points on the upper sec and $i - 1$ points on the lower section, so a total of $(n - i - 1)(i - 1)$ lines must be crossed. In this example, the lines to $i = 1$ and $i = 4$ each cross zero lines, while the lines to $i = 2$ and $i = 3$ each cross two lines (there are two points on one side and one on the other). So the recurrence can be expressed as

$$
\begin{aligned}
f(N) &= f(N-1) + \sum_{i=1}^{n-1} (1 + (n - i - 1)\, i - 1) \\
&= f(N-1) + \sum_{i=1}^{n-1} \left(2 - n + ni - i^2\right) \\
&= f(N-1) - n^2 + 3n - 2 + \sum_{i=1}^{n-1} \left(ni - i^2\right)
\end{aligned}
$$

and we could reduce it further, but we are done for now. There is a more elegant solution using combinatorics. The main lesson here that I wanted to teach is to watch out for hard problems with ridiculously easy solutions, and how to spot and prevent them from occurring.

1. Prove that for $n! > 2^n$ for all $n \geq 4$.
2. Prove that for any integer $n \geq 1$, $2^{2n} - 1$ is divisible by 3.
3. Let a and b be two distinct integers, prove that $a^n - b^n$ are divisable by $a - b$ (Hint: think of $k^n - 1$).
4. Find the maximum number R_n of regions in which the plane can be divided by n straight lines. (What special conditions would these lines have to have, in relation to each other?)

If you can solve the majority of these then you are ready to move onto the next level.

4 Intermediate Level

Theorem 4.1 (Arithmetic and Geometric Means)**:** The Arithmetic Mean-Geometric Mean Inequality (AM-GM) states that

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

In Anthony Erb's wonderful lecture on Inequalities `http://onlinemathcircle.com/wp-content/uploads/2011/01/Lecture-7.pdf`, he proved the AM-GM for base case 2, but how do we know if it's true for all other cases (that is n terms)? We are going to use what is known as a staircase induction. We will prove this induction true for all powers of two. And because all other numbers, can be stated in the form of $m < 2^r$ so this "subset" case will actually prove for all cases. This is useful, because it gives us a lot of flexibility in manipulating our number, since the prime factorization of it is very limited. We could have easily used any other prime number, p^k but for sake of simplicity we will use 2. In the case where all of the terms of

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

are equal, it's trivial that equality holds. So we will assume they are not the same terms. Then

$$\begin{aligned}
a_1 - a_2 &\neq 0 \\
(a_1 - a_2)^2 &> 0 \\
a_1^2 - 2a_1 a_2 + a_2^2 &> 0 \\
a_1^2 + 2a_1 a_2 + a_2^2 &> 4a_1 a_2 \\
(a_1 + a_2)^2 &> 4a_1 a_2 \\
\left(\frac{a_1 + a_2}{2}\right) &> a_1 a_2 \\
\frac{a_1 + a_2}{2} &> \sqrt{a_1 a_2}
\end{aligned}$$

Consider the subcase where $n = 2^k$ where k is a positive integer. We showed it's true for when $k = 1$ and now we want to show where it's also true for $k > 1$, namely $n = 2^k$. We have

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_{2^k}}{2^k} &= \frac{\frac{a_1+a_2+\cdots+a_{2^{k-1}}}{2^{k-1}} + \frac{a_{2^{k-1}+1}+a_{2^{k-1}+2}+\cdots+a_{2^k}}{2^{k-1}}}{2} \\
&\geq \frac{\sqrt[2^{k-1}]{a_1 a_2 \cdots a_{2^{k-1}}} + \sqrt[2^{k-1}]{a_{2^{k-1}+1} a_{2^{k-1}+2} \cdots a_{2^k}}}{2} \\
&\geq \sqrt{\sqrt[2^{k-1}]{a_1 a_2 \cdots a_{2^{k-1}}}\ \sqrt[2^{k-1}]{a_{2^{k-1}+1} a_{2^{k-1}+2} \cdots a_{2^k}}} \\
&= \sqrt[2^k]{a_1 a_2 \cdots a_{2^k}}
\end{aligned}$$

Where in the first inequality, the two sides are equal only if both of the following are true:

$$a_1 = a_2 = \cdots = a_{2^{k-1}} = a_{2^{k-1}+1} = a_{2^{k-1}+2} = \cdots = a_{2^k}$$

(in which case the first arithmetic mean and first geometric mean are both equal to a_1 and similarly with the second arithmetic mean and second geometric mean); and in the second inequality, the two sides are only equal if the two geometric means are equal. Since not all 2^k numbers are equal, it is not possible for both inequalities to be equalities, so we know that:

$$\frac{a_1 + a_2 + \cdots + a_{2^k}}{2^k} > \sqrt[2^k]{a_1 a_2 \cdots a_{2^k}}$$

as desired. The subcase where $n < 2^k$ If n is not a natural power of 2, then it is certainly *less* than some natural power of 2, since the sequence $2, 4, 8 ... 2^k$ is unbounded above. Therefore, without loss of generality, let m be some natural power of 2 that is greater than n. So, if we have n terms, then let us denote their arithmetic mean by α, and expand our list of terms thus:

$$a_{n+1} = a_{n+2} = \cdots = a_m = \alpha$$

We then have:

$$\begin{aligned}
\alpha &= \frac{x_1 + x_2 + \cdots + x_n}{n} \\
&= \frac{\frac{m}{n}(x_1 + x_2 + \cdots + x_n)}{m} \\
&= \frac{x_1 + x_2 + \cdots + x_n + \frac{m-n}{n}(x_1 + x_2 + \cdots + x_n)}{m} \\
&= \frac{x_1 + x_2 + \cdots + x_n + (m-n)\alpha}{m} \\
&= \frac{x_1 + x_2 + \cdots + x_n + x_{n+1} + \cdots + x_m}{m} \\
&> \sqrt[m]{x_1 x_2 \cdots x_n x_{n+1} \cdots x_m} \\
&= \sqrt[m]{x_1 x_2 \cdots x_n \alpha^{m-n}}
\end{aligned}$$

so

$$\begin{aligned}
\alpha^m &> x_1 x_2 \cdots x_n \alpha^{m-n} \\
\alpha^n &> x_1 x_2 \cdots x_n \\
\alpha &> \sqrt[n]{x_1 x_2 \cdots x_n}
\end{aligned}$$

as desired.

Theorem 4.2 (Bernoulli's Inequality)**:** The inequality states that, $(1+x)^r \geq 1+rx$ for every integer $r \geq 0$ and every real number $x \geq -1$. If the exponent r is even, then the equality is valid for ALL real numbers x.

Below we will prove the strict version, for which integer $r \geq 2$ and every real number $x \geq -1$, $x \neq 0$.

Proof. By induction, for the base case $r = 2$ we have $(1+x)^2 > 1+2x$ subtracting from both sides, we get $(1+x)^2 - (1+2x) = x^2 > 0$ so we assume this theorem is true for $n = k$ we have $(1+x)^k > 1+kx$ linking it to $k+1$ we multiply both sides by $(1+x)$ to get

$$\begin{aligned}(1+x)^{n+1} &= (1+x)(1+x)^k > 1+kx(1+x)\\ &= 1+(n+1)x+x^2 > 1+(n+1)x\end{aligned}$$

since $x^2 > 0$, and we are done. Now prove the rest of the theorem on your own (where x can be any real number, provided r is even).

Theorem 4.3 (De Moivre's Theorem.)**:** For all $n \in \mathbb{Z}$,

$$(\cos x + i\sin x)^n = \cos(nx) + i\sin(nx), n \in \mathbb{Z}.$$

De Moivre's Theorem is one of the most important and useful theorem for complex geometry on competition level (I am not qualified to assess on complex analysis or higher math, but I suspect it is useful). It is easily derived from Euler's Formula, which is $e^{\pi i} + 1 = 0$ however, de moivre's theorem gives us the freedom to map out on a compex plan specific coordinates. Generally speaking, it De Moivre's theorem doesn't hold for integer powers, because the way it derives many potential answers, and hence isn't very precise for our purposes (i.e. it may yield 30 different answers or roots of a number, which isn't very helpful to us). The truth of de Moivre's theorem can be established by mathematical induction for natural numbers, and extended to all integers from there. Consider "$S(n)$":

$$(\cos x + i\sin x)^n = \cos(nx) + i\sin(nx), n \in \mathbb{Z}$$

For $n > 0$, we proceed by mathematical induction. $S(1)$ is clearly true. For our hypothesis, we assume $S(k)$ is true for some natural k. That is, we assume

$$\left(\cos x + i\sin x\right)^k = \cos\left(kx\right) + i\sin\left(kx\right)$$

Now, considering $S(k+1)$

$$\begin{aligned}\left(\cos x + i\sin x\right)^{k+1} &= \left(\cos x + i\sin x\right)^k\left(\cos x + i\sin x\right)\\ &= \left[\cos\left(kx\right) + i\sin\left(kx\right)\right]\left(\cos x + i\sin x\right) \quad \text{by the induction hypothesis}\\ &= \cos\left(kx\right)\cos x - \sin\left(kx\right)\sin x\\ &\quad + i\left[\cos\left(kx\right)\sin x + \sin\left(kx\right)\cos x\right]\\ &= \cos\left[\left(k+1\right)x\right] + i\sin\left[\left(k+1\right)x\right] \qquad \text{by trigonometric identities}\end{aligned}$$

We deduce that $S(k)$ implies $S(k+1)$. By the principle of mathematical induction it follows that the result is true for all natural numbers. Now, $S(0)$ is clearly true since

$\cos(0x) + i\sin(0x) = 1 + i0 = 1$. Finally, for the negative integer cases, we consider an exponent of $-n$ for natural n.

$$\begin{aligned}(\cos x + i \sin x)^{-n} &= [(\cos x + i \sin x)^n]^{-1} && (1)\\ &= [\cos(nx) + i \sin(nx)]^{-1} && (2)\\ &= \cos(-nx) + i \sin(-nx). && (3)\end{aligned}$$

The equation (3) is a result of the identity $z^{-1} = \frac{\bar{z}}{|z|^2}$, for $z = \cos nx + i \sin nx$. Hence, $S(n)$ holds for all integers n.

1. A triangular array of numbers has a first row consisting of the odd integers $1, 3, 5, \ldots, 99$ in increasing order. Each row below the first has one fewer entry than the row above it, and the bottom row has a single entry. Each entry in any row after the top row equals the sum of the two entries diagonally above it in the row immediately above it. How many entries in the array are multiples of 67? (AIME I, 2008 #6)

   Solution. Let the kth number in the nth row be $a(n, k)$. Writing out some numbers, we find that $a(n, k) = 2^{n-1}(n + 2k - 2)$. We wish to find all (n, k) such that $67|a(n, k) = 2^{n-1}(n + 2k - 2)$. Since 2^{n-1} and 67 are relatively prime, it follows that $67|n+2k-2$. Since every row has one less element than the previous row, $1 \leq k \leq 51-n$ (the first row has 50 elements, the second49, and so forth; so$k$ can range from 1 to 50 in the first row, and so forth). Hence $n+2k-2 \leq n+2(51-n)-2 = 100-n \leq 100$, it follows that $67|n-2k+2$ implies that $n-2k+2 = 67$ itself. Now, note that we need n to be odd, and also that $n+2k-2 = 67 \leq 100-n \implies n \leq 33$. We can check that all rows with odd n satisfying $1 \leq n \leq 33$indeed contains one entry that is a multiple of 67, and so the answer is$\frac{33+1}{2} = \boxed{017}$.

   Proof: Indeed, note that $a(1, k) = 2^{1-1}(1 + 2k - 2) = 2k - 1$, which is the correct formula for the first row. We claim the result by induction on n. By definition of the array, $a(n, k) = a(n - 1, k) + a(n - 1, k + 1)$, and by our inductive hypothesis,

$$\begin{aligned}a(n, k) &= a(n-1, k) + a(n-1, k+1)\\ &= 2^{n-2}(n - 1 + 2k - 2) + 2^{n-2}(n - 1 + 2(k + 1) - 2)\\ &= 2^{n-2}(2n + 4k - 4)\\ &= 2^{n-1}(n + 2k - 2)\end{aligned}$$

   thereby completing our induction. The result above is fairly intuitive if we write out several rows, each time dividing the result through by 2 (as this doesn't affect divisibility by 67). The second row will be $2, 4, 6, \cdots, 98$, the third row will be $3, 5, \cdots, 97$, and so forth. Clearly, only the odd-numbered rows can have a term divisible by 67. However, with each row the row will have one less element, and the $100 - 67 = 33$rd row is the last time 67 will appear.

2. Prove for all positive prime numbers p, $a^p \equiv a \pmod p$ Hint: Prove $(x + y)^p \equiv x^p + y^p \pmod p$.

3. (Tower of Hanoi) The Tower of Hanoi or Towers of Hanoi , also called the Tower of Brahma or Towers of Brahma, is a mathematical game or puzzle. It consists of three rods, and a number of disks of different sizes which can slide onto any rod.

The puzzle starts with the disks in a neat stack in ascending order of size on one rod, the smallest at the top, thus making a conical shape. The objective of the puzzle is to move the entire stack to another rod, obeying the following rules: Only one disk may be moved at a time. Each move consists of taking the upper disk from one of the rods and sliding it onto another rod, on top of the other disks that may already be present on that rod. No disk may be placed on top of a smaller disk.

Find an explicit formula and prove it using induction. Note: This is a famous problem, and has many different solutions mostly graph/combinatorics.

4. A chessboard is an 8×8 grid (64 squares arranged in8 rows and 8 columns), but here we will call a "chessboard" any $m \times m$ square grid. We call a chessboard defective, if one of its squares is missing. Prove that any $2^n \times 2^n$ $(n \geq 1)$ *defective* chessboard can be tiled (completely covered without overlapping) with L-shaped "trominos" occupying exactly 3 squares. (a two by two square, missing a corner). Hint: For the inductive step, consider a $2^{k+1} \times 2^{k+1}$ defective chessboard and divide it into four $2^n \times 2^n$ chessboards. One of them is defective; can the other three be made defective by strategically placing an L-tromino?

5 Fibonacci Numbers

I dedicated a special section to Fibonacci Sequences, because of their prominence and unique nature in mathematics. Problems here range from Beginner to Advanced. The Fibonacci Sequence, $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$ is defined as $F_1 = F_2 = 1$ for $n \geq 3$

Before we proceed to prove "Zeckendorf's Theorem", we will need yet another form of induction called "Strong Induction", which is defined as the following: Suppose that P(n) is a statement about some positive integers and P(1) is true, and for each $k \geq 1$, if P(j) is true for all $j < k$, then P(k) is true.

Then our original function, P(n) is true for all integers $n \geq 1$.

Theorem 5.1 (Zeckendorf's Theorem)**:** (a) Every positive integer can be represented as summation of one or more distinct non consecutive Fibonacci numbers.

(b) There is only one such unique summation as described in part (b), and there exists no other summation that follow all restrictions established in part (a).

For example, 42 can only be represented as $34+8$, and while other Fibonacci numbers add up to 42 such as $34 + 5 + 3$ they are not Zeckendorf representations because $3, 5$ are consecutive Fibonacci numbers, which is illegal as described by rule (a).

Proof. We want to prove that all positive integers are in the form of $N = F_{a1} + F_{a2} + \cdots + F_{am}$ for some distinct integer a_j where $2 < a_1 < a_2 < \cdots < a_m$ To get a feel of the problem, we will take a random number, 40, where we take the largest Fibonacci number, less than or equal to 40 (because every Fib. number must be distinct and non consecutive, so we can choose largest first) which in this case is 34, leaving us to break up 6, which can only be broken up into $5 + 1$. We will now try to link this case which we'll call k into $k+1$, so adding one to both sides, we get $41 = F_2 + F_2 + F_5 + F_9$ however we repeated 1 twice, fortunately we can change that to two, but what if we had another two there already? This should convince you that the inductive hypothesis will not be as straightforward.

For $N = 1$, we have $1 = F_2$, so assume there exists a positive integer K such that all the integers from 1 to K can be represented in the given form. We want to show that $K + 1$ can also be represented in the given form.

Let a be the greatest positive integer such that $F_a \leq (K + 1)$. If $K + 1 = F_a$, then we are done. Otherwise, $Fa < K + 1$. Also, by definition of a, $K + 1 < F_{a+1}$ (kind of obvious) Then $0 < (K + 1) - F_a < F_{a+1} - Fa = F_{a-1}$

By the strong induction hypothesis, there exist positive integers $a_1, a_2, \ldots, a_m$ such that

$(K + 1) - F_a = F_{a1} + F_{a2} + \cdots + F_{am}$; where $2 < a_1 < a_2 < \cdots < a_m$ and since $(K + 1) - F_a < F_{a-1}$, none of the terms $F_{a1}, F_{a2}, \ldots, F_{am}$ are equal to F_a.

Hence, the representation

$$K + 1 = F_{a1} + F_{a2} + \cdots + F_{am} + F_a$$

satisfies the given conditions. Therefore, by strong induction, all positive integers can be represented in the given form.

QED

1. Prove that F_{3n} is even for every $n \geq 1$.
2. Prove that F_{4n} is divisible by three for every $n \geq 1$.
3. Show that
$$1 < \frac{F_n}{F_{n-1}} < 2$$
for all $n > 2$.

6 Olympiad level problems

USAMO, IMO, Putnam and other Olympiad level problems.

I will begin with my favorite and one first USAMO problems ever. I like it because it has a beautiful symmetry and represents the ideal type of problem which should not only have elegant solution, but be elegant in itself.

1. (USAMO #1, 2003) Prove that every positive integer$n$ there exists an n-digit number divisible by 5^n all of whose digits are odd.

   Before we proceed onto a formal solution, we are going to test for simple cases to get a better feel of the problem and detect any patters. For $n = 1$it's trivial. For $n = 2$ we have to find a multiple of 25 such that all of its digits are odd, that would be 75 $(25, 50$ don't work). It seems like we are just adding a new digit to the number each time, so now let's prove it.

   Assuming for some positive integer$n = k$, we want to find an odd number in the form of $10^{k-1}b_{k-1} + 10^{k-2}b_{k-2} + \cdots + 10b_1 + b_0$ which is divisible by 5^{k+1} where all the digits of b_i are odd. Factoring out 5^k from entire sequence we get

$$10^{k-1}b_{k-1} + 10^{k-2}b_{k-2} + \cdots + 10b_1 + b_0 = 10^k b_k + 5^k a$$

   where a for some positive integer a. Hence we can factor it further,

$$10^k b_k + 5^k a = 5^k(2^k b_k + a)$$

or in other words, we now have to simply show that $(2^k b_k) \equiv -a \pmod 5$ and this does work, because 2^g is relatively prime to 5 and there always will be an even and odd choice, e.g. if we get 8 we can just subtract (or other cases add) 5 (in this case to get three). What we have also shown is in fact that this n digit number is unique.

2. (USAMO 2009, #1) Let n be a positive integer and let $a_1, \dots, a_k (k \geq 2)$ be distinct integers in the set $\{1, \dots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \dots, k-1$. Prove that n doesn't divide$a_k(a_1 - 1)$. Author: Ross Atkins, Australia

   Solution. Let $n = pq$ such that $p \mid a_1$ and $q \mid a_2 - 1$. Suppose n divides $a_k(a_1 - 1)$. Note $q \mid a_2 - 1$ implies $(q, a_2) = 1$ and hence $q \mid a_3 - 1$. Similarly one has $q \mid a_i - 1$ for all i's, in particular, $p \mid a_1$ and $q \mid a_1 - 1$ force $(p, q) = 1$. Now $(p, a_1 - 1) = 1$ gives $p \mid a_k$, similarly one has $p \mid a_i$ for all i's, that is a_i's satisfy $p \mid a_i$ and$q \mid a_i - 1$, but there should be at most one such integer satisfies them within the range of $1, 2, \dots, n$ for$n = pq$ and $(p, q) = 1$. A contradiction!

3. (IMO 1997) An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \dots, 2n-1\}$ is called a silver matrix if, for each $i = 1, 2, \dots, n$ the i-throw and the i-th column together contain all elements of S. Show that

   (a) there is no silver matrix for 1997 (A marvel that they manage to get a problem with the current year on it)

   (b) silver matrices exist for infinitely many values of n
   Hint: (maybe it's not that marvelous that they managed to get a problem with 1997)

   (c) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$ where r and s are non negative integers, and no summand divides another. (For example, $23 = 9 + 8 + 6$)

4. (IMO shortlist A3) Let $x_1, x_2, \dots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1 + x_1^2} + \frac{x_2}{1 + x_1^2 + x_2^2} + \cdots + \frac{x_n}{1 + x_1^2 + \cdots + x_n^2} < \sqrt{n}.$$

Solution. We prove the following general inequality, for arbitrary positive real k:

$$\sum_{j=1}^{n} \frac{x_j}{k^2 + \sum_{i=1}^{j} x_i^2} \leq \sqrt{n}/k,$$

with equality only when $n = 0$. We proceed by induction on n. For $n = 0$, we have trivial equality. Now, suppose our inequality holds for n. Then by inductive hypothesis,

$$\sum_{j=1}^{n+1} \frac{x_j}{k^2 + \sum_{i=1}^{j} x_i^2} = \frac{x_1}{k^2 + x_1^2} + \sum_{j=2}^{n+1} \frac{x_j}{k^2 + x_1^2 + \sum_{i=2}^{j} x_i^2} \leq \frac{x_1}{k^2 + x_1^2} + \frac{\sqrt{n}}{\sqrt{k^2 + x_1^2}}.$$

If we let $t = \text{Arcsin}\left(x_1/\sqrt{x_1^2 + k^2}\right)$, then we have

$$\frac{x_1}{k^2 + x_1^2} + \frac{\sqrt{n}}{\sqrt{k^2 + x_1^2}} = (\sin t \cos t + \sqrt{n} \cos t)/k \leq (|\sin t| + \sqrt{n} \cos t)/k,$$

with equality only if $\cos t = \pm 1$. By the Cauchy-Schwarz Inequality,

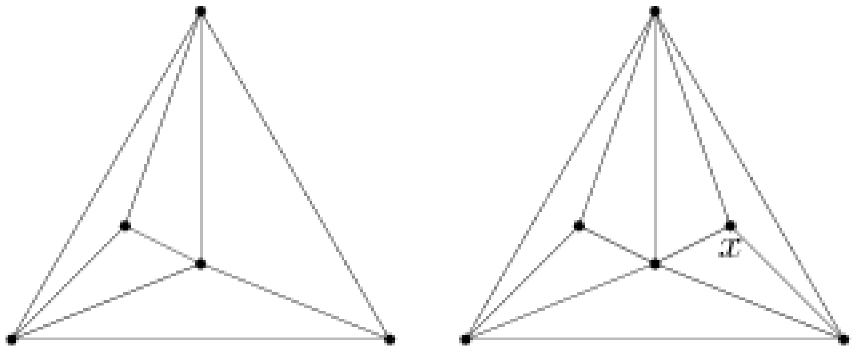
$$(|\sin t| + \sqrt{n}\cos t)/k \leq (1+n)^{1/2}(\sin^2 t + \cos^2 t)^{1/2}/k = \sqrt{n+1}/k,$$

with equality only when $(|\sin t|, \cos t) = (1/\sqrt{n^2+1}, n/\sqrt{n^2+1}$. Since $\left|n/\sqrt{n^2+1}\right| < 1$, our equality cases never coincide, so we have the desired strict inequality for $n+1$. Thus our inequality is true by induction. The problem statement therefore follows from setting $k = 1$.

5. Let k be a fixed positive integer. A company has a special method to sell sombreros. Each customer can convince two persons to buy a sombrero after he/she buys one; convincing someone already convinced does not count. Each of these new customers can convince two others and so on. If each one of the two customers convinced by someone makes at least k persons buy sombreros (directly or indirectly), then that someone wins a free instructional video. Prove that if n persons bought sombreros, then at most
$$\frac{n}{(k+2)}$$
of them got videos.
Solution. Suppose m persons receive videos. We wish to prove $n \geq m(k+2)$. We will prove the stronger bound $n \geq (m+1)(k+1)+m$ by induction on m. We say a person A is a "direct successor" of B if B directly convinced A to buy a sombrero. We say A is an "indirect successor" of B if B caused A to buy a sombrero, directly or indirectly. We will call a person who receives a video a "blossom". We will call direct successor of a blossom who is not a blossom him- or her-self a "bud". For the base case of our induction, when there is only one blossom, that blossom must have at least two buds, each of which must have at least k indirect successors. Hence $n \geq 2k+3 = (m+1)(k+1)+m$. Now, suppose there are m blossoms. Since there are only finitely many blossoms, there exists at least one blossom which has no blossoms as indirect successors. We remove this blossom, one of its buds, and all indirect successors of that bud; we then make the disconnected bud a direct successor of whatever person was a direct successor of the blossom we removed, if there was such a person. All other blossoms stay blossoms; all other buds stay buds. We have thus removed at least $2+k$ people, and we have removed one blossom. If there are n' people remaining, then the inductive hypothesis now tells us $n-(2+k) \geq n' \geq (m)(k+1)+(m-1)$, or $n \geq n'+k+2 \geq (m+1)(k+1)+m$. Therefore for all m, we have $n \geq (m+1)(k+1)+m = m(k+2)+k+1 > m(k+2)$, as desired. We note that the bound $n \geq (m+1)(k+1)+m$ is sharp, for it is possible to have blossoms $A_1, \ldots, A_m$, with A_{i+1} a direct successor of A_i, and all buds having exactly k indirect successors.

6. (1987 IMO Proposal by Yugoslavia) Prove that for every natural number k ($k \geq 2$) there exists an irrational number r such that for every natural number m, $[r^m] \equiv -1 \pmod{k}$ (Here $[x]$ denotes the greatest integer less than or equal to x).

7. Here's another tricky problem complete with a BOGUS SOLUTION, which went unreported for quite a while on AoPS WOOT course (Provided by Boy Soprano II). Tell me what's wrong with this solution and or a corrected version.



$k = 2$ $\quad$ $k = 3$

Let n be a non negative integer. Given a triangle and n points inside it, we divide the triangle into smaller smaller triangles using the $n+3$ points as vertices as shown. Show that we always end up with $2n + 1$ triangles. For the base case, when $n = 0$, there is clearly $2n + 1$ triangles. For the inductive step, assume the k points inside the triangle define $2k + 1$ triangles. If we add a point x as shown, then we lose one triangle, but create 3 more for a net of two additional triangles. Hence there are a total of $2k + 1 + 2 = 2k + 3 = 2(k + 1) + 1$ triangles, which completes the induction.

I cannot thank enough the people who directly helped me on this lecture directly, but also my revered math instructors, who I had the honor of learning with, David Hankin, Jan Siwanowicz, as well as Naoki Sato (from World Online Olympiad Training, a wonderful program with exemplary instructors) for teaching me many of the theorems and proofs mentioned here.