My Own Problems In Number Theory

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August 11, 2015

Abstract

This is just a compilation of problems I created myself, so that I don't forget them later. Although I have forgotten most of them in the mean time. :(Anyway, they aren't sorted in any order. And I will write solutions too later if I have time.

We assume the following notations.

- *s.t.* is the short form of *such that*
- *qr* is the short form of *quadratic residue.*
- $a|b$ means b is divisible by a.
- $(a, b) = \gcd(a, b)$ is the greatest common divisor of a and b.
- $lcm(a, b) = [a, b]$ is the least common multiple of a and b.
- $a \perp b$ denotes $(a, b) = 1$ or a and b are co-prime.
- $\tau(n)$ is the number of divisors of n.
- $\sigma(n)$ is the sum of divisors of n.
- $\mathbb{P}$ denotes the set of primes.
- $p^\alpha || n$ or $\nu_p(n) = \alpha$ means α is the greatest positive integer such that $p^\alpha | n$. In other words, $p^\alpha | n$ and $p^{\alpha+1} \not| n$.
- $\omega(n)$ is the number of distinct prime factors of n.
- $\varphi(n)$ is the number of positive integers less than or equal to n and co-prime to n.
- $ord_m(a) = x$ denotes x is the order of $a \pmod m$ i.e. x is the smallest positive integer s.t. $a^x \equiv 1 \pmod m$.
- $\pi(n)$ is the number of primes less or equal to n.

1 Problems

Problem 1.1. Find all n so that

$$\sum_{i=1}^{n} \varphi(i) \geq \binom{n}{2}$$

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Problem 1.2. Find all triples of integers (a, b, c) so that their pairwise differences are prime powers.

Solution. Without loss of generality, we can assume

Problem 1.3. Find all positive integers n there are positive integers $a_1, a_2, \ldots, a_k$ strictly less than n and pair-wisely distinct satisfying

$$n \left| \left(\sum_{i=1}^{k} a_i + \prod_{i=1}^{k} \right)^2 - 1 \right.$$

for some positive integer $1 < k < n - 1$.

Problem 1.4. Find all $(m, n) \in \mathbb{N}^2$ s.t. $n^2 + 3m^2$ and $n + 3m$ both are perfect cubes.

Problem 1.5. Let c_n be the smallest positive integer s.t. $1 < c_n < n$ and $c_n \perp n$. Prove that c_n exists except for some finite n.

Problem 1.6. Find all n such that $\varphi(n)|n$.

Problem 1.7. Prove that for a prime $p > 2$, the set of complete residue class $\pmod{p}$ can be divided into two subsets of equal number of elements with sum of each group divisible by p.

Problem 1.8. A number having only one prime factor can't be a perfect number.

Problem 1.9. $\sigma(64n^2)$ is odd for any n.

Problem 1.10. Find all odd n s.t. $2013|F_n$.

Problem 1.11. Take any $2n$ integers[1] where $n > 2$. Consider all pair-wise differences we can possibly have from them. Let's denote the product of these $\binom{2n}{2}$ differences by S. Prove that S is divisible by

$$2^{n^2-n} \cdot (2n-1)(2n-3)(2n-5)$$

Problem 1.12. p is a prime of the form $3k + 2$. Prove that, there exists a set of $p - 1$ elements which forms a complete set of residue class $\pmod{p}$ and sum of elements is divisible by p^2.

Problem 1.13. If n is odd then $\tau(F_n) \geq \tau(n)$.

Problem 1.14. For all even n, $\tau(F_{2n}) \geq \tau(n)$.

Problem 1.15. If p is a prime and x is a positive integer s.t. $p > x^2 - x + 1$ then

$$\omega\left((x+1)^p - (x^p + 1)\right) \geq 4$$

Problem 1.16. For any prime $p > 2$, there are two positive integers u, v s.t. uv^{-1} is a qr of p with $u, v < \dfrac{p}{2}$.

Problem 1.17. The number of numbers less than or equal to n having odd sum of divisors is

$$\lfloor \sqrt{n} \rfloor + \left\lfloor \sqrt{\frac{n}{2}} \right\rfloor$$

[1] positive or negative whatever, including 0

Problem 1.18. Find all sequence of positive integers $\{a_i\}_{i=0}^{\infty}$ s.t.

$$[a_i, a_{i+1}] = (a_{i+1}, a_{i+2})$$

Problem 1.19. Find all primes p and $(a, b) \in \mathbb{N}^2$ s.t. $a^p + b^p$ is a perfect power of a prime.

Problem 1.20. Let $a \in \mathbb{N}$. Then, $a^{a-1} - 1$ is never square-free[2].

Problem 1.21. Show that $\forall n, 81|10^{n+1} - 9n - 10$.

Problem 1.22. Let p be a prime. Find all perfect numbers having p factors exactly.

Problem 1.23. Find all $(a, b) \in \mathbb{N}^2$ s.t. $7^a + 11^b$ is a perfect square.

Problem 1.24. Let $m, n, a_1, a_2, \ldots, a_n$ be positive integers s.t. $\forall i, a_i + m$ is a prime. Let

$$N = \prod_{i=1}^{n} p_i^{a_i}$$

and S be the number of ways to write N as a product of m positive integers. Calculate the remainder of S upon division by m^n.

Problem 1.25. Solve in positive integers:

$$\sum_{i=1}^{8} n_i^{10} = 19488391$$

Problem 1.26. Find all $n \in \mathbb{N}$ s.t. $n|2^{n!} - 1$.

Problem 1.27. Show that there exists an infinite pairs $(a, b) \in \mathbb{N}^2$ s.t. $\dfrac{a^k + b^k}{a^k b^k + 1}$ is a perfect k^{th} power.

Problem 1.28. Solve in positive integers

$$a^n + b^n = (kac)^{mn}$$

Problem 1.29. Let a, b are positive integers s.t. $a \perp b$ and $p \in \mathbb{P}$ s.t. $p|x^6 + 64$. Find all pairs (a, b) s.t.

$$2013|\frac{a^2 + b^2}{p}$$

Problem 1.30. Find all integers $n > 2$ there are positive integers $a_1, a_2, \ldots, a_k$ less than or equal to n and pair-wisely distinct s.t.

$$n|\left(\sum_{i=1}^{k} a_i + \prod_{i=1}^{k}\right)^2 - 1$$

Problem 1.31. Find all n such that the sum of number of divisors of divisors of n is n.

Problem 1.32. Say a, n, d are positive integers where $a + id \in \mathbb{P}$ for $i = 0$ to $n - 1$. Define $f(n) = 1$ if $n = a$, $f(n) = 1$ otherwise. Let $\pi(n)$ be the number of primes **strictly less than** n. Show that,

$$T = 2^{\frac{d}{2}} + 1$$

has at least $2^{2^{\pi(n)-1}-1-f(n)}$.

[2] a is square-free if it has no square factor i.e. there is no x s.t. $x^2|a$.

Problem 1.33. Let's define **General Fibonacci Number**[3] as

$$G_n = \begin{cases} a \text{ if } n = 0 \\ b \text{ if } n = 1 \\ G_{n-1} + G_{n-2} \text{ if } n > 1 \end{cases}$$

Prove that $\left|G_{n+1}G_{n-1} - G_n^2\right|$ is independent of n.

Problem 1.34. Determine true or false: G_{2n+1} has no prime factor of the form $4n+3$ for an infinite a, b. Alternatively, $\exists x, y \in \mathbb{N} : G_n = x^2 + y^2$.

Problem 1.35. Define $k(n)$ as:

$$k(n) = \sum_{d|n, d+1 \in \mathbb{P}} 1$$

and $C(n)$ is the number of positive integers x so that $x|a^n - a$ for all a. Prove that $C(n) \geq 2^{k(n)}$.

Problem 1.36. Let G be a group with $ord(G) = n$. Find all G with n elements where n is a **Carmichael number**[4].

Problem 1.37. $p \in \mathbb{P}, p > 5$.

$$X_n = \sum_{x_1+\ldots+x_n=p} \prod_{i=1}^{n} \binom{p}{x_i}$$

Find all k s.t.

$$p^3 \mid \sum_{i=1}^{k} X_i$$

Problem 1.38. Find all $n \in \mathbb{N}$ s.t.

$$\tau(n) = \varphi(n)$$

Problem 1.39. Find all n s.t. $\tau(n) = \pi(n)$.

Problem 1.40. Find all n s.t. $\varphi(n) = \pi(n)$.

Problem 1.41. For a positive real number $c > 0$, call a positive integer n, $c - good$ if for all positive integer $m < n$, $\frac{m}{n}$ can be written as

$$\frac{m}{n} = \frac{a_0}{b_0} + \ldots + \frac{a_k}{b_k}$$

for some non-negative integers $k, a_0, ..., a_k, b_0, ..., b_k$ with $k < \frac{n}{c}, 2b_i \leq n$ and $0 \leq a_i < \min(b_j), 0 \leq j \leq k$. Show that, for any positive real c there are infinite $c - good$ numbers.

Problem 1.42. Prove that $n = \varphi(n) + k$ has finite solutions for a positive integer k.

Problem 1.43. For a positive integer n, let $S_n = \{m \in \mathbb{N} : n|\tau(m)\}$. Find all n so that S_n has an infinite arithmetic problemogression as a sub-sequence.

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[3]We shall maintain this notation through this whole note.
[4]i.e. n is composite and $n|a^n - a$ for all a.

2 Solutions

Solution. Let S_n be the set of positive integers less than or equal to n. Then S_n has $\varphi(n)$ elements, where $\varphi(n)$ is the Euler function. We prove that these elements satisfy the given property for $k = \varphi(n)$. Say,

$$S_n = \{a_1, a_2, ..., a_{k-1}, a_k\}$$

$$N = \left(\sum_{i=1}^{k} a_i + \prod_{i=1}^{k}\right)^2 - 1$$

with $a_1 < a_2 < ... < a_{k-1} < a_k$. For $n > 1$, every $a_i \in S_n$ is strictly less than n. Since otherwise $\gcd(n, n) = n > 1$ would hold. F

Lemma 2.1.

$$n|\sum_{i=1}^{k} a_i$$

Proof. From Euclidean algorithm, if $\gcd(a, n) = 1$, then

$$\gcd(n, n - a) = \gcd(a, n) = 1$$

Therefore, $a_i + a_{k-i} = n$, and $\varphi(n)$ is even. So, $n|a_i + a_{k-i}$ for $0 < i < \dfrac{k}{2}$. This gives

$$n|\sum_{i=1}^{k} a_i$$

□

Lemma 2.2. *Let P_A be the product of elements of set A. Then,*

$$P_{S_n}^2 \equiv 1 \pmod{n}$$

Proof. Let $a \in S_n$. Then, all ai are distinct modulo n, otherwise we would have

$$ai \equiv aj \pmod{n}$$

implying

$$n|a(i - j)$$

with $\gcd(n, a) = 1$ and $|i - j| < n$. Take any $a \in S_n$. Then, for any $a_i \in S_n$, there is a unique j such that

$$a_i a_j \equiv a \pmod{p}$$

i.e. two of them pair up for a. Running them over S_n, we have

$$a_1 a_2 \cdots a_{k-1} a_k \equiv a^{\frac{k}{2}} \pmod{n}$$

Squaring, we have

$$P_{S_n}^2 \equiv \left(a^{\frac{k}{2}}\right)^2 \equiv a^k \equiv 1 \pmod{n}$$

□

Finally,

$$\begin{aligned} N &= \left(\sum_{i=1}^{k} a_i + \prod_{i=1}^{k}\right)^2 - 1 \\ &\equiv \left(\prod_{i=1}^{k}\right)^2 - 1 \pmod{n} \\ &\equiv 0 \pmod{n} \end{aligned}$$

So, every $n > 1$ satisfies the property.