# Arranging things... optimally!

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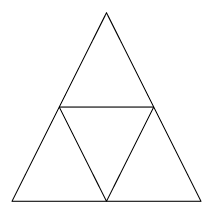
Sometimes you will run into problems asking you to choose or place or label a bunch of objects in the “best” way possible. Here’s one example that can be solved with a trick you might have seen before:

**Example 1.** *$n$ points are marked inside an equilateral triangle with side length 1 in such a way that no two marked points are within distance $0.5$ of each other. What is the largest possible value for $n$?*

*Solution.* For $n = 4$, we can mark one point at each vertex of the triangle and one at the center. The minimum distance between any two of these points is $\frac{1}{\sqrt{3}} > 0.5$. Now suppose $n \geq 5$. Partition the triangle into four regions as shown:



By the Pigeonhole Principle, two points must be in the same region, and the distance between those two points has to be at most 0.5, giving a contradiction. Therefore, the answer is $n = 4$. $\square$

As you can see, there are two completely separate parts to this problem. First, we need to find a configuration that works for $n = 4$. Then we need to show there are no valid configurations for $n > 4$. In fact, almost all optimization problems break down in this way, and either part can be quite challenging.

## 1 Showing a configuration is optimal

A lot of the time, it is actually pretty easy to guess what the optimal configuration should be. After that, you are just left with trying to *prove* it is optimal. There is one principle above else that you should keep in mind while doing this:

**KISS (Keep It Simple, Stupid):**

Optimization problems, like other Olympiad problems, have *clean* solutions 9 times out of 10. This means that you should be looking for *simple* things to analyze, and you should be staying away from messy calculations, intuition-based hand-waving, and heavy case analysis.

Example 1 shows KISS in action. You *could* write down equations for the distances between all points and then try manipulating them, but can you see how to complete a proof from there? No? Me neither. But once you divide the triangle up and use the Pigeonhole Principle, the problem becomes easy.

The only question is: why would you think to divide the triangle up like that? Well, the good news is there are a few specific ideas that come up again and again, and this is one of them. Here's a handy list of some of these tricks. It will be a little confusing at first, but try a couple problems and refer to the list as you go, and hopefully you will start to see some patterns.

1. Suppose you want to choose as many points as possible from some space in such a way that no two points are close together. Then, divide the space into regions so that it is "obvious" that you cannot choose more than one point (or two points or three) from any one region. This is exactly what we did in Example 1.

2. More generally, suppose you have a list of "things", and you want to pick as many things as possible subject to the constraint that certain combinations of things are illegal. Then divide the set of all things into groups so that it is "obvious" that you cannot choose more than one thing (or two or three) from any group. See Example 2 and Problem A5.

3. Suppose you want to cover a space with a small number of circles, squares, etc. Then focus on only a few specific points such that no one object can cover too many of these points. Often, you can completely ignore everything else during your proof. See Problem B4.

4. More generally, suppose you have a list of "things", and you want to pick as few things as possible subject to a bunch of constraints of the following form:

   (a) You must choose at least one thing with some property $A$.

   (b) You must choose at least one thing with some property $B$.

   (c) etc.

   Once again, focus on only a few specific constraints such that no one object can cover too many of these constraints. See Example 3.

5. For graph problems, focus on one vertex and its neighbours. For colouring problems, focus on one single object and other objects that obviously cannot be the same colour as it. Hopefully, a large percentage of these must be different colours from each other as well. See Example 4. (That example is not a graph problem, but it *is* essentially a colouring problem and the principle applies.)

6. Smoothing (a more advanced technique): look at some very small piece of an optimal solution. Prove that you can find a better solution unless it is configured in some particular way. Now ask yourself: what solutions *are* configured in that particular way? If you're lucky, these solutions might be easier to analyze than completely general solutions. See Problem C6. This approach is also the key idea to CMO 2009, #5, even though that problem is not framed as an optimization problem.

7. Invariants: is there some *simple* quantity or concept that behaves in a very predictable way as you build your solution? Can you deduce something as a result? See Problems B6 and B7.

It's completely okay if this list seems a little too complicated or a little too abstract right now. Just read it once, try a few problems, and then try to see how the solutions fit into a common pattern. But notice how the key to all of these methods is to focus on something small and easy to understand. If you don't focus on something much smaller than the whole, then optimizations problems are usually just too complicated to deal with[1].

To get you started, here are a couple more worked examples:

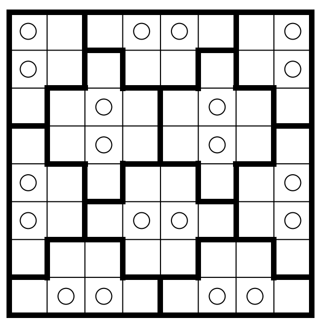
**Example 2.** *Let $A$ be the largest subset of $\{1, 2, \ldots, 100\}$ such that $A$ does not contain two elements with one dividing the other. What is $|A|$?*

*Solution.* For each odd number $k$, let $S_k$ denote the set $\{k, 2k, 4k, 8k, \ldots\}$. Any two elements of $S_k$ divide each other, so $A$ can have at most one element from $S_k$ for all $k$. On the other hand, every number between 1 and 100 belongs to exactly one $S_k$ for some odd $k$ between 1 and 100. There are exactly 50 such $k$, so it follows that $|A| \leq 50$.

Conversely, the set $\{51, 52, \ldots, 100\}$ satisfies the given property since $2 \cdot 51 > 100$, and hence $|A| \geq 50$. Therefore, $|A| = 50$. □

**Example 3.** *We say that two different cells of an $8 \times 8$ board are* neighbouring *if they have a common side. Find the minimal number of cells on the $8 \times 8$ board that must be marked so that every cell (marked or not marked) has a marked neighbouring cell.*

*Solution.* Divide the board into 10 regions as shown below:



Note that within each region, every square neighbours one of the two "central" squares. Therefore, if we mark each of the 20 central squares as shown, we have a valid solution. Conversely, the two central squares within each region *only* neighbour other squares in that region, and they do not share any neighbours with each other. Therefore, we must mark at least two squares within each region, and hence 20 marked squares is the smallest possible. □

This last example is pretty evil; it is actually based on IMO 1999 #3[2]. So if you are wondering how anyone would think of this tiling, don't worry about it too much. An IMO #3 is always going to be hard! But *nonetheless*, notice that as hard as this problem is, the basic approach is really just the exact same as in Example 1. Divide the thing up into regions such that each region is very easy to understand. Even for the hardest optimization problems, ideas like this can go a long way.

[1] Like all good rules of thumb, there are exceptions to this. Often, these exceptions rely on interpreting the problem as a special case of some deeper mathematical structure, like linearly independent vectors or partially ordered sets. Yufei wrote some notes on these topics if you are interested: http://web.mit.edu/yufeiz/www/imo2008/zhao-combinatorics.pdf

[2] IMO 1999 #3 is identical to this problem except you have to do $n \times n$ boards for all even $n$, not just $8 \times 8$. The tiling I showed here can be extended to a tiling for all such boards and thus solve the whole IMO problem, although you might have to play around with it a little bit to see how.

## 2 Finding the optimal configuration

Occasionally, you will run into problems where finding the optimal configuration is as hard or harder than any proofs. Here is one example that some of you might remember from last year's buffet contest:

**Example 4.** *A mathematics competition has $n$ contestants and five problems. On each problem, each contestant is assigned a positive integer score which is at most seven. It turns out no pair of contestants got the same score on two different problems. Find the maximum possible value for $n$.*

*Solution.* There are seven possible scores on each question. If $n \geq 50$, then at least $\lceil \frac{50}{7} \rceil = 8$ contestants got the same score on problem 1. But then two of those contestants must have gotten the same score on problem 2, which is impossible.

Now, for $1 \leq i, j \leq 7$, let $x_{i,j,k}$ denote the value in $\{1, 2, \ldots, 7\}$ that is congruent to $i + jk \pmod 7$. Consider 49 contestants $C_{i,j}$ where contestant $C_{i,j}$ receives score $x_{i,j,k}$ on problem $k$. Suppose that two contestants $C_{i_1,j_1}$ and $C_{i_2,j_2}$ got the same scores on questions $k_1$ and $k_2$. Then $i_1 - i_2 + (j_1 - j_2)k_1 \equiv i_1 - i_2 + (j_1 - j_2)k_2 \equiv 0 \pmod 7$. Subtracting, we have $(j_1 - j_2)(k_1 - k_2) \equiv 0 \pmod 7 \implies j_1 \equiv j_2 \pmod 7 \implies j_1 = j_2$. But then we must also have $i_1 = i_2$, which is a contradiction.

Therefore, it is possible to satisfy the required condition with 49 contestants, and hence 49 is the maximum possible value for $n$. □

Where did this construction come from? Well, we can think of each student's scores being given by a function from $[1, 5]$ to $[1, 7]$. So we need to come up with a large family of functions for which any two functions have the same value at only one point. But we know that any two line functions have this property! So we assign each student scores according to an equation like this: $Score = A \cdot ProblemNumber + B$. The rest is just easy algebra[3]! Now, what if we modified the problem so that no pair of contestants got the same score on *three* different problems? Can you see how to assign scores for $n = 343$?

There are plenty of problems where you can just guess randomly and eventually come up with an optimal configuration, but as with this example, there are also times where that is impractical. Here are some things to keep in mind when you run into one of those situations:

1. Always look for solutions that are symmetrical or that follow some simple pattern. If the optimal configuration is just random garbage, it would be a nightmare to prove it has the properties you want (unless it is very small). Since Olympiad problems don't have nightmare solutions, look for something less random. Bear this in mind if you try Problems C2 and C5!

2. When assigning values between 1 and $n$, think modulo $n$. The ability to add, subtract, and multiply (and also divide if $n$ is prime), can lead to constructions you would not have thought of otherwise. This was key to our solution for Example 4.

3. If you are trying to prove a configuration exists and it seems impractical to write the configuration down (because it is too big, or because you need to prove a configuration exists in many different situations at once), use induction.

[3] Implicit in this approach is the idea that integers mod 7 behave in a very similar way to real numbers, so our intuition about two lines intersecting only once is still true. This does work for 7, and it also works for any prime. It does *not* work for non-primes though. Try doing this construction in the case where there are only six possible scores on each problem, and see for yourself where it breaks down.

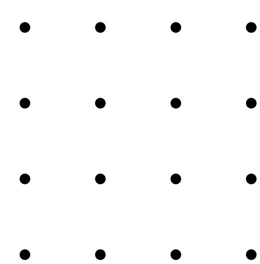
4. A more advanced technique: If you are trying to prove there exists a configuration or object with some value at least equal to $C$, calculate the average of this value over all choices of configurations/objects. If the *average* is at least $C$, then you're done! See Problem B8.

## 3 Problems

For almost all of the following problems, you will have to both find an optimal configuration and then independently prove it is optimal. I strongly recommend thinking about both parts of the problem as you go. Partial progress on the proof can help guide your search for a good configuration, and time spent looking for a good configuration can sometimes give you intuition for the proof. Plus, you don't want to spend a long time trying to prove a configuration is optimal only to find out there is an even better configuration that you missed!

The questions are arranged roughly in order, with A-level problems being pre-IMO difficulty, B-level problems being "easy" and medium IMO difficulty, and C-level problems being hard IMO difficulty. Good luck!

A1. A square grid of 16 dots (see the figure) contains the corners of nine $1 \times 1$ squares, four $2 \times 2$ squares, and one $3 \times 3$ square, for a total of 14 squares whose sides are parallel to the sides of the grid. What is the smallest possible number of dots you can remove so that, after removing those dots, each of the 14 squares is missing at least one corner?



A2. NASA has proposed populating Mars with 2010 settlements. The only way to get from one settlement to another will be by a connecting tunnel. A bored bureaucrat draws on a map of Mars, randomly placing $N$ tunnels connecting the settlements in such a way that no two settlements have more than one tunnel connecting them. What is the smallest value of $N$ that guarantees that, no matter how the tunnels are drawn, it will be possible to travel between any two settlements?

A3. What is the maximum number of colours that can be used to paint an $8 \times 8$ chessboard such that every square is painted in a single colour, and is adjacent, horizontally, vertically, or diagonally, to at least two other squares of its own colour?

A4. A certain province issues license plates consisting of six digits (from 0 to 9). The province requires that any two license plates differ in at least two places. (For instance, the numbers 027592 and 020592 cannot both be used.) Determine, with proof, the maximum number of license plates that the province can use.

A5. What is the largest number of knights that can be placed on an $8 \times 8$ chessboard so that no two knights are attacking each other?

A6. (a) Suppose there are 997 points given on a line. If every two points are joined by a line segment with its midpoint coloured in red, show that there are at least 1991 red points on the line. Can you find a special case with exactly 1991 red points?

(b) Do the same problem on the plane, instead of on a line.

A7. On a $50 \times 50$ board, the centers of several unit squares are coloured black. Find the maximum number of centers that can be coloured black such that no three black points form a right-angled triangle.

A8. $n$ points are marked inside an equilateral triangle with side length 1 in such a way that no two marked points are within distance $\frac{1}{3}$ of each other. What is the largest possible value for $n$?

A9. McNutty colours the vertices and edges of a complete graph on $n$ vertices such that:

(a) If two edges touch the same vertex, then those two edges are coloured differently, and

(b) If an edge touches a vertex, then the edge and vertex are coloured differently.

What is the minimum number of colours that McNutty could have used?

B1. On a $1001 \times 1001$ board, $m$ squares are marked so that:

(a) In every pair of adjacent squares, at least one square is marked, and

(b) In every set of six consecutive squares within a row or column, there are at least two adjacent squares that are marked.

Find the smallest possible value for $m$.

B2. (a) Let $A$ be the largest subset of $\{1, 2, \ldots, n\}$ such that for each $x \in A$, $x$ divides at most one other element of $A$. Prove that

$$\frac{2n}{3} \leq |A| \leq \left\lceil \frac{3n}{4} \right\rceil.$$

(b) Let $B$ be the largest subset of $\{1, 2, \ldots, 2010\}$ such that $B$ neither contains two elements one of which divides the other, nor contains two elements which are relatively prime. What is $|B|$?

B3. In an international soccer tournament, each team plays each other team exactly once and receives three points for a win, one point for a draw, and zero points for a loss. After the tournament, it is observed that the team from Glorious Nation of Kazakhstan won fewer games than any other team but also earned more points than any other team. Find the smallest number of teams that could have competed in the tournament.

B4. For $n$ an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. A *tromino* is an $L$-shape formed by three consecutive unit squares. For which values of $n$ is it possible to cover all the black squares with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?

B5. We have a $102 \times 102$ sheet of graph paper and a connected figure of unknown shape consisting of 101 squares. What is the smallest number of copies of the figure which can be cut out of the square (assuming the cutting is done optimally)?

B6. We are given 40 balloons, the air pressure inside each of which is unknown and may differ from balloon to balloon. It is permitted to choose up to $k$ of the balloons and equalize the pressure in them (to the arithmetic mean of their respective original pressures). What is the smallest $k$ for which it is always possible to equalize the pressure in all of the balloons?

B7. The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on $n$ vertices.

B8. A rectangular table with 9 rows and 2008 columns is filled with the numbers $1, 2, \ldots, 2008$ in such a way that each number appears exactly 9 times and the difference between any two numbers from the same column is at most 3. Let $S$ denote the sum of the entries in the column for which this is minimal. What is the largest possible value for $S$?

B9. When you buy a "mathlotto" ticket, you choose 10 of the first 100 positive integers. Then 10 of the integers from 1 to 100 are drawn, and a winning ticket is one which does not contain any of them. Prove that

(a) If you buy 13 tickets, you can choose your numbers so that regardless of which numbers are drawn, you are guaranteed to have at least one winning ticket;

(b) If you buy only 12 tickets, it is possible for you not to have any winning tickets, regardless of how you choose your numbers.

B10. Find the smallest integer $n \geq 5$ for which there exists a set of $n$ people, such that any two people who are acquainted have no common acquaintances, and any two people who are not acquainted have exactly two common acquaintances.

B11. Points on each side of an equilateral triangle divide the side into $n$ equal segments. Lines parallel to the sides of the triangle are drawn through all of these points, dividing the original triangle into $n^2$ small triangles or "cells". The cells between any two adjacent parallel lines from a "stripe". What is the maximum number of cells that can be chosen such that no two of them belong to a single stripe in any of the three orientations, if

(a) $n = 10$;

(b) $n = 9$?

C1. In a chess tournament, every pair of participants play against each other once. A win is worth one point, a draw is worth half a point, and a loss is worth zero points. Let us call a game an "upset" if at the end of the tournament, the player who won the game ends up with fewer points than the player who lost.

(a) Prove that no matter what the result of the tournament is, strictly less than $\frac{3}{4}$ of the games were upsets.

(b) Prove that one cannot replace the number $\frac{3}{4}$ in (a) by a smaller number.

C2. Find the largest integer $n$ for which it is possible to place $n$ points in the plane with no three collinear and then to colour each of them red, green, or yellow so that:

- There is a green point inside each triangle formed by three red points.
- There is a yellow point inside each triangle formed by three green points.
- There is a red point inside each triangle formed by three yellow points.

C3. Suppose that we have $n \geq 3$ distinct colours. Let $f(n)$ be the greatest integer with the property that every side and every diagonal of a convex polygon with $f(n)$ vertices can be coloured with one of the $n$ colours in the following way:

(a) At least two colours are used,

(b) Any three vertices of the polygon determine either three segments of the same colour or three of different colours.

Show that $f(n) \leq (n-1)^2$ with equality for infinitely many values of $n$.

C4. Each edge of a graph is labeled with a distinct real number. We say it has an "increasing walk of length $n$" if there is a sequence of $n+1$ vertices $v_0, v_1, \ldots, v_n$ such that each $v_i$ and $v_{i+1}$ are connected by an edge and such that the labels on these edges are increasing with $i$.

(a) Let $n$ be a positive integer. Prove that if the graph has average degree more than $n$, then it has an increasing walk of length $n+1$.

(b) For each positive integer $n$, prove there exists a graph with average degree equal to $n$, and with no increasing walk of length $n+1$.

C5. 6000 points are marked on the boundary of a circle, and they are coloured with 10 colours in such a way that within every group of 100 consecutive points, all the colours are used. Determine the least positive integer $k$ with the following property: In every colouring satisfying the condition above, it is possible to find a group of $k$ consecutive points in which all the colours are used.

C6. Among a group of 120 people, some pairs are friends. A *weak quartet* is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets?