Trigonometry Enrichment Program

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 March 2012 version adds the Cotangent rule and a new identity proof

How do you solve trig identities and trig equations? Each one can seem like a completely new puzzle, on which the methods you used on the last one no longer apply. But there are some basics. Your tools are the following identities *which you should memorize.*

Symmetry Identities

$$\cos(-x) = \cos(x) \qquad \sin(-x) = -\sin(x) \qquad \tan(-x) = -\tan(x)$$

$$\sec(-x) = \sec(x) \qquad \csc(-x) = -\csc(x) \qquad \cot(-x) = -\cot(x)$$

The Pythagorean Identity

$\sin^2(x)+\cos^2(x) = 1$ hence $\cos^2(x) = 1-\sin^2(x)$ and $\sin^2(x) = 1-\cos^2(x)$

Cosine of sums and differences Identities

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

Sine of sums and differences Identities

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

Definitions of tangent, cotangent, secant, and cosecant

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$$

$$\sec(x) = \frac{1}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

From the above you can easily derive these:

$$\sec^2(x) = \frac{1}{\cos^2(x)} = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} = \tan^2(x) + 1$$

Derivation of a similar identity involving $\csc^2$ and $\cot^2$ is left as an exercise to the reader.

Double angle formulas:

$$\begin{aligned}
\sin(2x) &= \sin(x+x) = \sin(x)\cos(x) + \sin(x)\cos(x) = 2\sin(x)\cos(x) \\
\cos(2x) &= \cos(x+x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x)
\end{aligned}$$

If you replace $\sin^2(x)$ with $1 - \cos^2(x)$ in the latter, you get

$$\cos(2x) = 2\cos^2(x) - 1$$

Likewise if you replace $\cos^2(x)$ with $1 - \sin^2(x)$ in the same equation, you get

$$\cos(2x) = 1 - 2\sin^2(x)$$

For tangent of the double angle:

$$\tan(2x) = \frac{\sin(2x)}{\cos(2x)} = \frac{2\sin(x)\cos(x)}{\cos^2(x) - \sin^2(x)} = \frac{2\frac{\sin(x)}{\cos(x)}}{\frac{\cos^2(x)}{\cos^2(x)} - \frac{\sin^2(x)}{\cos^2(x)}} = \frac{2\tan(x)}{1 - \tan^2(x)}$$

You can use a similar approach to show that for any x and y:

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

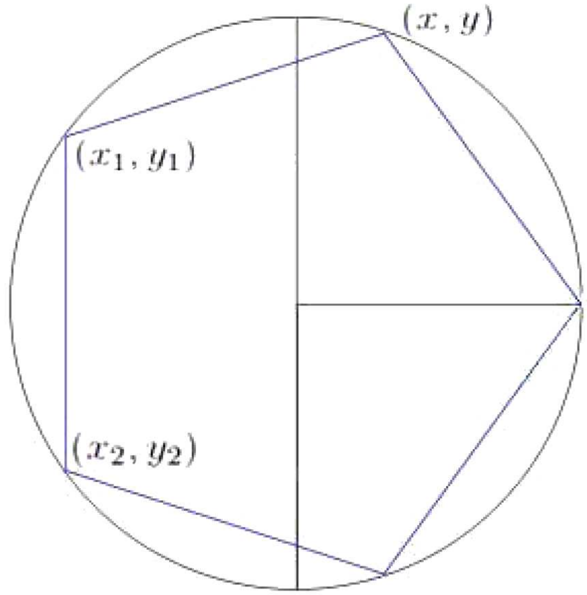
So much for the ones you need to memorize. From these and the application of algebra you can prove all the other identities and solve all the trig equations that you might be assigned to do as homework problems.

Triple angle formulas:

$$\begin{aligned}
\sin(3x) &= \sin(x+2x) = \sin(x)\cos(2x) + \sin(2x)\cos(x) \\
&= \sin(x)\left(\cos^2(x) - \sin^2(x)\right) + 2\sin(x)\cos^2(x) \\
&= \sin(x)\left(1 - 2\sin^2(x)\right) + 2\sin(x)\left(1 - \sin^2(x)\right) \\
&= \sin(x) - 2\sin^3(x) + 2\sin(x) - 2\sin^3(x) \\
&= 3\sin(x) - 4\sin^3(x)
\end{aligned}$$

$$\begin{aligned}
\cos(3x) &= \cos(x+2x) = \cos(x)\cos(2x) - \sin(x)\sin(2x) \\
&= \cos(x)\left(\cos^2(x) - \sin^2(x)\right) - \sin(x)(2\sin(x)\cos(x)) \\
&= \cos(x)(2\cos^2(x) - 1) - 2\sin^2(x)\cos(x) \\
&= 2\cos^3(x) - \cos(x) - 2(1 - \cos^2(x))\cos(x) \\
&= 4\cos^3(x) - 3\cos(x)
\end{aligned}$$

Problem: Find the coordinates of the vertices of a pentagon. Assume the circle has radius of 1. If θ is the angle from one vertex to the next then the angle from the x-axis going counterclockwise to (x_1, y_1) is 2θ. The angle from the x-axis going counterclockwise to (x_2, y_2) is 3θ. Observe that $x_1 = x_2$. Since $x_1 = \cos(2x)$ and $x_2 = \cos(3\theta)$, it follows that $\cos(2\theta) = \cos(3\theta)$. We apply the double and triple angle formulas to form an equation that we can solve for $\cos(\theta)$.



$$\begin{aligned}
\cos(2\theta) &= \cos(3\theta) \\
2\cos^2(\theta) - 1 &= 4\cos^3(\theta) - 3\cos(\theta) \\
0 &= 4\cos^3(\theta) - 2\cos^2(\theta) - 3\cos(\theta) + 1
\end{aligned}$$

If you replace $\cos(\theta)$ with x, you have the cubic polynomial, $0 = 4x^3 - 2x^2 - 3x + 1$. Observe that $\theta = 0$ must be a solution because twice zero is equal to three times zero. Since $\cos(0) = 1$, it must be that $x = 1$ is a solution. So we can divide it out using polynomial long division.

$$\begin{array}{r l}
 & 4x^2 + 2x - 1 \\ \cline{2-2}
x - 1 \big) & 4x^3 - 2x^2 - 3x + 1 \\
 & -4x^3 + 4x^2 \\ \cline{2-2}
 & \qquad 2x^2 - 3x \\
 & \qquad -2x^2 + 2x \\ \cline{2-2}
 & \qquad\qquad -x + 1 \\
 & \qquad\qquad\ \ x - 1 \\ \cline{2-2}
 & \qquad\qquad\qquad 0
\end{array}$$

It's easy enough to apply the quadratic formula to the resuling quotient:

$$x = \frac{-2 \pm \sqrt{4 + 16}}{8} = \frac{\pm\sqrt{5} - 1}{4}$$

So why two solutions? We expected that solving this would give us x in the (x, y) shown in the figure. That would be the positive solution. The other solution is x_1, which we already determined is the same as x_2. Our original setup was to solve for the cosine of the angle whose double and triple angles have the same cosine. Notice if you take twice the angle from the x-axis to (x_1, y_1) (which takes you nearly around the circle to the lower-right vertex), and three times that same angle (which takes you fully around to (x, y)), both of those destinations have the same x coordinate and hence the same cosine. So the negative solution is legitimate also.

To find the y coordinate of (x, y), we use the Pythagorean identity, $\sin^2(\theta) = 1 - \cos^2(\theta)$. Hence

$$y^2 = 1 - x^2 = 1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2 = \frac{5 - \sqrt{5}}{8}$$

By taking the the square root of the above result you arrive at y.

From the diagram you can see that $y_2 = -y_1$. So we could have solved for θ for which $\sin(2\theta) = -\sin(3\theta)$. By applying the double and triple angle formulas for sine:

$$2\sin(\theta)\cos(\theta) = 4\sin^3(\theta) - 3\sin(\theta)$$

Clearly again $\theta = 0$ is a solution because $\sin(0) = 0$. But we eliminate that solution by dividing the common factor of $\sin(\theta)$ out of the above.

$$2\cos(\theta) = 4\sin^2(\theta) - 3$$

Now replace $\sin^2(\theta)$ with $1 - \cos^2(\theta)$

$$2\cos(\theta) = 4 - 4\cos^2(\theta) - 3 = 1 - 4\cos^2(\theta) \quad \text{hence} \quad 4\cos^2(\theta) + 2\cos(\theta) - 1 = 0$$

If you again replace $\cos(\theta)$ with x, you get the same quadratic we got doing it the first way, and therefore the same solutions.

Quad-angle formulas:

$$\begin{aligned}
\cos(4x) &= \cos(2x+2x) \\
&= 2\cos^2(2x) - 1 \\
&= 2\left(2\cos^2(x) - 1\right)^2 - 1 \\
&= 8\cos^4(x) - 8\cos^2(x) + 2 - 1 \\
&= 8\cos^4(x) - 8\cos^2(x) + 1
\end{aligned}$$

$$\begin{aligned}
\sin(4x) &= \sin(2x+2x) \\
&= 2\sin(2x)\cos(2x) \\
&= 4\sin(x)\cos(x)\left(2\cos^2(x) - 1\right) \\
&= 8\sin(x)\cos^3(x) - 4\sin(x)\cos(x) \\
\text{or} & \\
&= 4\sin(x)\cos(x)\left(1 - 2\sin^2(x)\right) \\
&= 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x)
\end{aligned}$$

We can use quad-angle formulas to solve the pentagon problem also. If θ is the angle from the x-axis to (x, y), then 4θ takes you from the x-axis to the lower right vertex, which has the same x value. So by solving for θ where $\cos(\theta) = \cos(4\theta)$, we should arrive at the same solutions as before. Applying the quad-angle formula for cosine:

$$\begin{aligned}
\cos(\theta) &= 8\cos^4(\theta) - 8\cos^2(\theta) + 1 \\
0 &= 8\cos^4(\theta) - 8\cos^2(\theta) - \cos(\theta) + 1
\end{aligned}$$

This time the polynomial to solve is $8x^4 - 8x^2 - x + 1 = 0$. As before, $\theta = 0$ is a trivial solution that corresponds to $x = \cos(0) = 1$. Dividing out that root,

$$\begin{array}{r|rrrrr}
 & 8x^3 & +8x^2 & & -1 & \\
\hline
x-1 & 8x^4 & & -8x^2 & -x & +1 \\
 & -8x^4 & +8x^3 & & & \\
\hline
 & & 8x^3 & -8x^2 & & \\
 & & -8x^3 & +8x^2 & & \\
\hline
 & & & & -x & +1 \\
 & & & & x & -1 \\
\hline
 & & & & & 0
\end{array}$$

We see that the quotient is a cubic. We expect that we should get the same two solutions solving the pentagon problem this way as we did solving it using the double and triple angle formulas. Yet a cubic must have either one or three real roots – never two. So is there a solution to $\cos(\theta) = \cos(4\theta)$

that we haven't accounted for? You have to think about it for a few minutes. The unaccounted for solution, it turns out, has nothing to do with pentagons. It arises from the fact that going around the circle $\frac{1}{3}$ times ($\frac{2\pi}{3}$ radians or $120°$) is the same as going around $\frac{4}{3}$ times ($\frac{8\pi}{3}$ radians or $480°$). It means that $\theta = \frac{2\pi}{3}$ is a solution. Since $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, it must be the case that $2x+1$ evenly divides $8x^3+8x^2-1$.

$$
\begin{array}{r|l}
\multicolumn{2}{r}{4x^2 + 2x - 1} \\
\cline{2-2}
2x+1 & 8x^3 + 8x^2 \qquad - 1 \\
 & -8x^3 - 4x^2 \\
\cline{2-2}
 & \qquad 4x^2 \\
 & \quad -4x^2 - 2x \\
\cline{2-2}
 & \qquad\qquad -2x - 1 \\
 & \qquad\qquad\ \ 2x + 1 \\
\cline{2-2}
 & \qquad\qquad\qquad 0
\end{array}
$$

This, as you can see, results in the same quadratic as we got solving the pentagon problem twice before.

Application to compass and straight-edge construction: Algebra and geometry are more intertwined than perhaps you had imagined. The quadratic solution we arrived at for the vertices of a pentagon implies that a regular pentagon can be constructed using a compass and straight-edge. The reason has to do with constructable lengths. If you choose some unit length, then any integer multiple of that length is constructable simply by walking the compass down the straight-edge that number of times. Likewise the sum or difference of any two constructable lengths is also constructable. In addition the square root of any constructable length is constructable. This is because if x is a constructable length then so are $x+1$ and $|x-1|$. By constructing two perpendicular lines then measuring $|x-1|$ from their interesection along one of them, it is possible to construct a right triangle whose one side is $|x-1|$ and whose hypotenuse is $x+1$. By Pythagoras, the remaining side must be $2\sqrt{x}$. You find $\sqrt{x}$ by bisecting that remaining side. Indeed if you have any constructable length, that length divided by any power of 2 is also constructable by repeated bisection (note that it is also possible to n-sect a line segment for any positive integer n, so any rational fraction of a constructable length is also constructable).

The consequence is that the solutions we found to the quadratic, $4x^2+2x-1$, are constructable using compass and straight-edge. This is because $1 \pm \sqrt{5}$ is constructable, as is that value divided by 4 by virtue of 4 being an integer divisor. It follows that a regular pentagon is also constructable.

But suppose we were to try the same approach to finding the vertices of a regular heptagon (a figure with 7 equal sides and angles). We would use the triple angle formula together with the quad-angle formula to find θ where $\cos(3\theta) = \cos(4\theta)$. Applying the formulas:

$$4\cos^3(\theta) - 3\cos(\theta) = 8\cos^4(x) - 8\cos^2(\theta) + 1$$

This results in having to solve the polynomial, $8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0$. Even when you divide out the trivial root, $x - 1$, you still have the cubic, $8x^3 + 4x^2 - 4x - 1 = 0$. This cubic has three real roots that can be found only by taking cube roots of values derived from the cubic's coefficients. Compass and straight-edge constructions can make constructable lengths of square roots of other constructable lengths, but no combination of summing and taking square roots can ever find the cube root of any value other than zero and one. It follows that no matter how clever you are, you will never be able to construct a perfect regular heptagon using only compass and straight-edge.

Half-angle formulas: Because we know that $\cos(2\theta) = 2\cos^2(\theta) - 1$, we can derive a half-angle formula:

$$\begin{aligned}
\cos(\theta) &= \cos\left(\frac{\theta}{2} + \frac{\theta}{2}\right) \\
&= 2\cos^2\left(\frac{\theta}{2}\right) - 1 \\
\frac{1}{2}(\cos(\theta) + 1) &= \cos^2\left(\frac{\theta}{2}\right) \\
\sqrt{\frac{1}{2}(\cos(\theta) + 1)} &= \cos\left(\frac{\theta}{2}\right)
\end{aligned}$$

Likewise using $\cos(2\theta) = 1 - 2\sin^2(\theta)$ we can derive,

$$\sqrt{\frac{1}{2}(1 - \cos(\theta))} = \sin\left(\frac{\theta}{2}\right)$$

The impossibility of trisecting angles. We now extend our discussion of compass and straight-edge constructions. You can see from the cosine half-angle formula that if $\cos(\theta)$ is a constructable length, then $\cos\left(\frac{\theta}{2}\right)$ is also a constructable length. Why? Because it can be calculated using a sum of constructable lengths, division by an integer, and taking a square root. This implies that it is possible to bisect any angle using compass and straight-edge alone. But suppose we used the triple angle formula in the same way to derive a formula for cosine of $\frac{1}{3}$ of a known angle, θ?

$$\begin{aligned}\cos(\theta) &= \cos\left(3\,\frac{\theta}{3}\right) \\ &= 4\cos^3\left(\frac{\theta}{3}\right) - 3\cos\left(\frac{\theta}{3}\right)\end{aligned}$$

Suppose a constructable length, ℓ, is known to be equal to $\cos(\theta)$ and we want to find $x = \cos\left(\frac{\theta}{3}\right)$. Then we would have to solve the polynomial,

$$4x^3 - 3x - \ell = 0$$

Except for special values of ℓ such as $\ell = 0$, solving this cubic for x will involve extracting cube roots. Again no amount of summing, dividing by integers, and taking square roots can ever result in the cube root of any value except zero and one. This means that, except in special cases, x is an unconstructable length. But if angle $\frac{\theta}{3}$ were constructable, its cosine would also be constructable. It follows that it is impossible to trisect an arbitrary angle using only compass and straight-edge.

Challenge problem: Apply the double angle and triple angle formulas for sine and cosine to $\cos(5\theta) = \cos(3\theta + 2\theta)$ to arrive at a quint-angle formula for cosine that is strictly in terms of powers of $\cos(\theta)$. When you have done the algebra, see ***http://mathworld.wolfram.com/Multiple-AngleFormulas.html*** to compare your result with the correct answer. Use this result together with the quad-angle formula for cosine to arrive at a polynomial that must be solved to determine the vertices of a regular nonagon (the 9-sided figure with equal sides and angles). Do this by expanding $\cos(5\theta) = \cos(4\theta)$.

Product formulas for Sine and Cosine and their applications. From the double angle formula for cosine, we can easily derive another formula for the square of the cosine in terms of the double angle.

$$\begin{aligned}\cos(2\theta) &= 2\cos^2(\theta) - 1 \\ \cos^2(\theta) &= \frac{1}{2}\left(1 + \cos(2\theta)\right)\end{aligned}$$

If you replace 1 with $\cos(0)$, you have

$$\cos^2(\theta) = \frac{1}{2}(\cos(0) + \cos(2\theta))$$

Look at this result carefully. We are taking the product, $\cos(\theta) \times \cos(\theta)$, and ending up with a sum involving cosine of $\theta - \theta$ and cosine of $\theta + \theta$. That is the product of the cosines of two angles (which in this case are identical) yields a sum involving cosines of the sum of the two angles and the difference of two angles. Of course so far we have seen this to be the case only when those two angles are equal. But this pattern is generally true. Here's the proof based upon the cosine-of-a-sum formula.

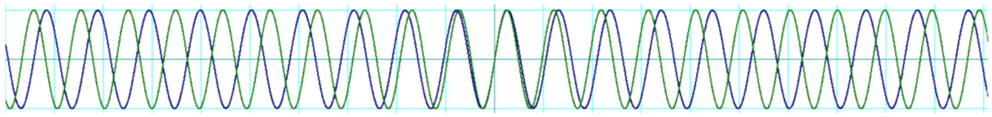
$$\begin{array}{rcl}
\cos(\theta - \phi) & = & \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) \\
\cos(\theta + \phi) & = & \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \\
\hline
\cos(\theta - \phi) + \cos(\theta + \phi) & = & 2\cos(\theta)\cos(\phi)
\end{array}$$

If you take the difference of the above equations you get the formula for the product of sines. To get the cross-product of sine with cosine, use the sine-of-a-sum formula:

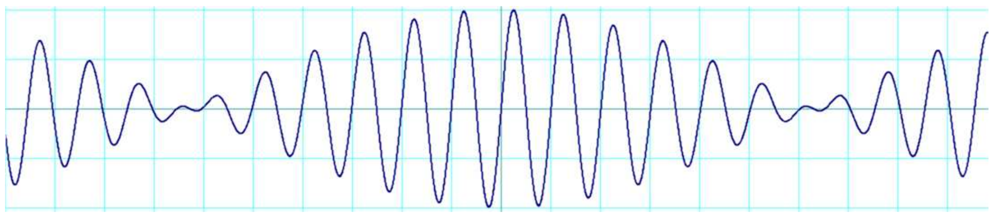
$$\begin{array}{rcl}
\sin(\theta + \phi) & = & \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) \\
\sin(\theta - \phi) & = & \sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi) \\
\hline
\sin(\theta + \phi) + \sin(\theta - \phi) & = & 2\sin(\theta)\cos(\phi)
\end{array}$$

In every case you find that the product formula is half the sum (or difference) of sines or cosines of the sum and difference of θ and ϕ.

In engineering it is common for a signal to be a sinusoidal function of time. The typical formula is $y(t) = A\sin(\omega t + \phi)$, where A is called the *amplitude*, ω the *radian frequency*, and ϕ the phase. An experiment you can do if you can find two identical tuning forks is this. Wrap rubber bands tightly around each of the tines of one of the tuning forks. This will lower its frequency slightly. Strike both tuning forks simultaneously, touch their handles to the same surface, and listen. You will hear them *beating*. That is you will hear the tone grow louder and softer at a fixed rate. If you can't find tuning forks, you can also do this experiment with musical instruments – two trumpets for example. Tune one of them so it is slightly out of tune with the other. Then have two trumpet players each blow a middle C. You will hear them beating against each other. On a standard B-flat trumpet, middle C is around 465 Hz. Suppose one trumpet is tuned for middle C at 463 Hz. and the other tuned to 467 Hz. The graph below shows what happens when both trumpets are blown at once.

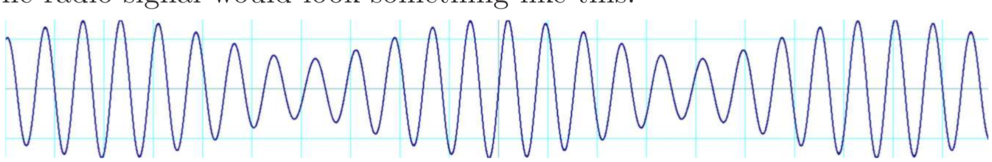


You can see that near the center of the graph, the signals from the two trumpets are aligned. They re-enforce each other. On each side near the outside of the graph, the two trumpets fight each other. When you add the two signals you get a graph that looks like this:



To get the radian frequency of a signal, you multiply the Hz. frequency by 2π. You can imagine the signals from the two trumpets being $A\sin\left(2\pi\left(465+2\right)t\right)$ and $A\sin\left(2\pi\left(465-2\right)t\right)$. From the formulas we developed, we expect this sum to be equivalent to $2A\sin\left(\left(2\pi\times 465\right)t\right)\cos\left(\left(2\pi\times 2\right)t\right)$. Can you see how the latter function is expressed in the graph? Near the middle (where $t=0$), the cosine function is close to unity. So at that time you hear the trumpets loudly. Near the outside of the graph, the cosine function is close to zero. At that time the two trumpets cancel each other. The frequency of the beats is the frequency with which $\cos\left(2\pi\times 2t\right)$ crosses zero – that is 4 beats per second.

AM radio broadcasts a signal whose frequency is in the range of 520 kHz to 1610 kHz. When you tune an AM radio to a station, you are selecting the frequency in that range on which that station transmits. The program – say the announcer's voice – is encoded on the radio transmission by making the signal stronger and weaker in time with the vibrations of the announcer's voice. Suppose an AM radio station were to transmit a signal that would cause your radio receiver to produce a sinusoidal sound from the speaker. The radio signal would look something like this:



We can describe this signal mathematically using the expression, $A\sin\left(\omega_1 t\right)\left(1+h\cos\left(\omega_2 t\right)\right)$, where ω_1 is the station's broadcast radian frequency and ω_2 is the radian frequency of the sound the speaker is producing. Note that ω_1 is much greater than ω_2. h, which must be between 0 and 1, is called the *modulation index* (in the graph, $h=0.4$). When you multiply this function out you get a component, $A\sin(\omega_1 t)$, which radio engineers call the *carrier*, and another component, $Ah\sin(\omega_1 t)\cos(\omega_2 t)$. Using the product formula we see that the latter is equivalent to $\frac{1}{2}Ah\sin((\omega_1+\omega_2)t)+\frac{1}{2}Ah\sin((\omega_1-\omega_2)t)$. Radio engineers call these two terms the *sidebands* because one component is ω_2 higher than the carrier, the other ω_2 below the carrier – that is they are on either side of the carrier. The sidebands are not just theoretical. Their frequencies are broadcast over the air along with the carrier. In the United States, regulations restrict each AM broadcast station to 20 kHz. of the radio spectrum. This means that the highest frequency of sound that an AM radio station can send to the speaker of a radio receiver is 10 kHz. Attempting to broadcast higher frequency sounds would result in sidebands that fall outside of the station's designated channel.

If you paid a lot of money for a home audio system, you would expect that it faithfully reproduce the sounds played through it. This means that the output signal (what comes out of the speakers) should be strictly a constant multiple of the input signal (e.g., what's on your CD or the program coming from a radio station). That is, we want input, $G(t)$, and output, $H(t)$, to have this relationship:

$$H(t)=kG(t)$$

where k is the amplification factor. Another way of putting this is that we want the output to be a linear polynomial of the input. The degree to which the output deviates from being a linear polynomial of the input is called the distortion. So how might an audio engineer measure the distortion of a sound system? We can employ the product formulas developed in the previous paragraphs to see how. Supposing we were to allow the input to be the sum of two sinusoidal signals, one at 2000 Hz., the other at 2200 Hz.

$$G(t)=A(\sin(2\pi\times 2000t)+\sin(2\pi\times 2200t))$$

If the system were perfect, we expect that the output should contain these two frequencies and nothing else.

$$\begin{aligned} H(t)=kG(t) &= kA(\sin(2\pi\times 2000t)+ \\ &\qquad \sin(2\pi\times 2200t)) \end{aligned}$$

But if the system were not perfect, then the transfer function – that is the function that gives you the output in terms of the input – would be a higher degree polynomial. Suppose we add a second degree term to the transfer function:

$$H(t) = kG(t) + \xi (G(t))^2$$

Even if ξ is very small compared to k, there is a strategy by which we can measure it. Expanding our sum of 2000 Hz. and 2200 Hz. using the transfer function we get

$$\begin{aligned} H(t) &= kA(\sin(2\pi \times 2000t) + \sin(2\pi \times 2200t)) + \\ &\quad \xi A^2 (\sin(2\pi \times 2000t) + \sin(2\pi \times 2200t))^2 \end{aligned}$$

When you multiply out the square term you find that you have sine squared of both the 2000 Hz. and the 2200 Hz. terms present (which produce components at 4000 Hz. and 4400 Hz. respectively). But you also find that you have a term, $2\xi A^2 \sin(2\pi \times 2000t) \sin(2\pi \times 2200t)$. Our formula tells us that this is the same as $\xi A^2 (\cos(2\pi \times 200t) - \cos(2\pi \times 4200t))$. Using electronic filters it is possible to eliminate all of the higher frequencies and single out the 200 Hz. signal only. If the system were perfect (that is, if $\xi = 0$), there should be no 200 Hz. component in the output whatsoever. The strength of whatever 200 Hz. signal is present in the output will be in proportion to magnitude of ξ. And ξ is a measurement of the distortion produced by the system. This type of test of an amplifier is called an *intermodulation test*.

Challenge: Suppose you were a technician evaluating an audio system. You have already determined that $\xi (G(t))^2 = 0$ to within the sensitivity of your measurement. You suspect though that there is a higher order term, $\zeta (G(t))^3$, present in the system's output that is measurably nonzero. You need to set up an intermodulation test to measure this. What pair of frequencies might you use as input in order to measure ζ by singling out and measuring the amplitude of 200 Hz in the output?

Many Worked Trig Identities

What follows is a sampling of the many trig identity problems that people have emailed me over time along with my worked solutions. Each solution is on the page following the problem. Try to work each of them on your own before looking at its solution. In each case where there are tangents, cotangents, secants, and/or cosecants, I begin with the method of reconstructing the problem in terms of sines and cosines only. This is not always the shortest path to a result, but it always works, and so it is what I recommend to use always as a first step to students who need a consistent methodology for solving trig identity problems. Students who are able on their own to spot the shorter method, where it can be found, are not likely to be in need of the review provided here.

Building a Bridge Backward. The objective when you prove a trig identity is to build a bridge from some pair of expressions whose equality is obvious, step by step, to the equation whose equality you are wanting to prove. But at the outset, you do not know what that pair of equal expressions is. That is, you don't know where the bridge begins. You find it by building the bridge backward. That is you start with the equation whose equality you are trying to prove, and step by step, work toward a pair of expressions whose equality is obvious. But always keep in mind that you have to be able to travel the bridge you build in the reverse direction from the way you build it. That means that each step you take must be reversible. Here is an extreme example of why that is. If you allow irreversible operations, you can prove that $1 = 2$ simply by multiplying both sides of that equation by zero. That step is not reversible because you can never divide by zero. But multiplying both sides of an equation by, say, 2, is allowable because the reverse, dividing by 2, is always possible. Multiplying both sides by, say, $\sin(\theta)$, is also allowable, but only with the understanding that it does not apply to values of θ for which $\sin(\theta) = 0$, that is for values of θ that are integer multiples of π. That understanding includes the requirement that the identity be proved by other means for those θ's. As you observe the steps that prove the identities that follow, see if you can identify the discrete points for which the proof might not apply – that is those values of θ for which some step of the proof is irreversible. Challenge: What can you determine about the reversibility of squaring both sides of an equation as part of a proof? That is, under what conditions should that step be allowed?

1) Prove that

$$\frac{1}{\sin(\theta)+1} = \sec^2(\theta) - \tan(\theta)\sec(\theta)$$

Replace $\sec(\theta)$ wherever it occurs with $\dfrac{1}{\cos(\theta)}$. Replace $\tan(\theta)$ wherever it occurs with $\dfrac{\sin(\theta)}{\cos(\theta)}$.

$$\frac{1}{\sin(\theta)+1} = \frac{1}{\cos^2(\theta)} - \frac{\sin(\theta)}{\cos^2(\theta)}$$

Multiply top and bottom of the left side by $1-\sin(\theta)$. Use the difference of squares on the bottom.

$$\frac{1-\sin(\theta)}{1-\sin^2(\theta)} = \frac{1}{\cos^2(\theta)} - \frac{\sin(\theta)}{\cos^2(\theta)}$$

Use the Pythagorean identity on the bottom of the left side.

$$\frac{1-\sin(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} - \frac{\sin(\theta)}{\cos^2(\theta)}$$

Now simply observe the common denominator among the two terms to the right of the equal to complete the proof.

2) Prove that

$$\frac{\sin(\theta)}{\sin(\theta)-\cos(\theta)} = \frac{1}{1-\cot(\theta)}$$

Replace $\cot(\theta)$ with $\dfrac{\cos(\theta)}{\sin(\theta)}$

$$\frac{\sin(\theta)}{\sin(\theta)-\cos(\theta)}=\frac{1}{1-\frac{\cos(\theta)}{\sin(\theta)}}$$

Multiply top and bottom of the right-hand side by $\sin(\theta)$

$$\frac{\sin(\theta)}{\sin(\theta)-\cos(\theta)}=\frac{\sin(\theta)}{\sin(\theta)-\cos(\theta)}$$

Done.

3) Prove that

$$\frac{\csc(\theta)+\cot(\theta)}{\tan(\theta)+\sin(\theta)}=\cot(\theta)\csc(\theta)$$

Replace $\tan(\theta)$ with $\frac{\sin(\theta)}{\cos(\theta)}$. Replace $\cot(\theta)$ with $\frac{\cos(\theta)}{\sin(\theta)}$. Replace $\csc(\theta)$ with $\frac{1}{\sin(\theta)}$.

$$\frac{\frac{1}{\sin(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}}{\frac{\sin(\theta)}{\cos(\theta)} + \sin(\theta)} = \frac{\cos(\theta)}{\sin(\theta)} \frac{1}{\sin(\theta)}$$

Multiply top and bottom of the left side by $\sin(\theta)\cos(\theta)$.

$$\frac{\cos(\theta) + \cos^2(\theta)}{\sin^2(\theta) + \sin^2(\theta)\cos(\theta)} = \frac{\cos(\theta)}{\sin(\theta)} \frac{1}{\sin(\theta)}$$

Find the common factors in the left side. On the top you can factor out $\cos(\theta)$. On the bottom you can factor out $\sin^2(\theta)$.

$$\frac{\cos(\theta)(1 + \cos(\theta))}{\sin^2(\theta)(1 + \cos(\theta))} = \frac{\cos(\theta)}{\sin(\theta)} \frac{1}{\sin(\theta)}$$

Cancel the common factor on top and bottom of $1 + \cos(\theta)$. Multiply the two terms of right hand side, and you're done.

$$\frac{\cos(\theta)\cancel{(1 + \cos(\theta))}}{\sin^2(\theta)\cancel{(1 + \cos(\theta))}} = \frac{\cos(\theta)}{\sin^2(\theta)}$$

Observe in the previous two problems the technique of multiplying the top and bottom of a nasty fractions-inside-a-fraction in order to simplify it. Of the denominators of the fractions inside the fraction, find the least common multiple of them (often by multiplying all of those denominators all together). That product will be what you multiply top and bottom of the big fraction by.

4) Prove that

$$(\sin(\theta) - \cos(\theta))^2 = 1 - 2\sin(\theta)\cos(\theta)$$

Square out the left hand side:

$$\sin^2(\theta) - 2\sin(\theta)\cos(\theta) + \cos^2(\theta) = 1 - 2\sin(\theta)\cos(\theta)$$

Rearrange the terms on the left and then apply the Pythagorean identity, and you're done.

$$\sin^2(\theta) + \cos^2(\theta) - 2\sin(\theta)\cos(\theta) = 1 - 2\sin(\theta)\cos(\theta)$$

Observe that if you apply the double-angle formula for sine to the right, you have the further identity:

$$(\sin(\theta) - \cos(\theta))^2 = 1 - \sin(2\theta)$$

5) Prove that

$$\frac{1 + \cos(2\theta)}{2\cos(\theta)} = \cos(\theta)$$

Apply the double-angle formula for cosine. You may have observed it has three different forms. When you try each one, you find that the one that is useful is $\cos(2\theta) = 2\cos^2(\theta) - 1$.

$$\frac{1 + 2\cos^2(\theta) - 1}{2\cos(\theta)} = \cos(\theta)$$

Take the cancellation of the plus 1 with the minus 1, then cancel the common factor of $2\cos(\theta)$ from top and bottom, and you're done.

$$\frac{\cancel{1} + 2\cos^2(\theta) \cancel{-1}}{2\cos(\theta)} = \cos(\theta)$$

$$\frac{\cancel{2}\cos^{\cancel{2}}(\theta)}{\cancel{2}\cancel{\cos(\theta)}} = \cos(\theta)$$

6) Prove that

$$\frac{1 - \tan^2(\theta)}{1 - \cot^2(\theta)} = 1 - \sec^2(\theta)$$

Replace $\tan(\theta)$ with $\dfrac{\sin(\theta)}{\cos(\theta)}$. Replace $\cot(\theta)$ with $\dfrac{\cos(\theta)}{\sin(\theta)}$. Replace $\sec(\theta)$ with $\dfrac{1}{\cos(\theta)}$.

$$\frac{1-\frac{\sin^2(\theta)}{\cos^2(\theta)}}{1-\frac{\cos^2(\theta)}{\sin^2(\theta)}} = 1 - \frac{1}{\cos^2(\theta)}$$

Multiply both sides by $\cos^2(\theta)$. This will get rid of the fraction on the right. It will also get rid of the fraction in the numerator of the left hand side.

$$\frac{\cos^2(\theta)-\sin^2(\theta)}{1-\frac{\cos^2(\theta)}{\sin^2(\theta)}} = \cos^2(\theta) - 1$$

Multiplying the denominator of the left side by $\sin^2(\theta)$ is the same as dividing that entire left side by the same. So we do that to the left as well as the right.

$$\frac{\cos^2(\theta)-\sin^2(\theta)}{\sin^2(\theta)-\cos^2(\theta)} = \frac{\cos^2(\theta)}{\sin^2(\theta)} - \frac{1}{\sin^2(\theta)}$$

The left hand side is clearly equal to -1. The right has two terms over a common denominator. So combine them. Note that $\cos^2(\theta)-1 = -(1-\cos^2(\theta))$. Apply the Pythagorean identity to finish it.

7) Prove that

$$\tan(\theta)\sin(2\theta) = 1 - \cos(2\theta)$$

Replace $\tan(\theta)$ with $\dfrac{\sin(\theta)}{\cos(\theta)}$.

$$\frac{\sin(\theta)}{\cos(\theta)}\sin(2\theta) = 1 - \cos(2\theta)$$

Apply the double angle formula for sine.

$$\frac{\sin(\theta)}{\cos(\theta)}\left(2\sin(\theta)\cos(\theta)\right) = 1 - \cos(2\theta)$$

Combine terms to simplify the left hand side.

$$2\sin^2(\theta) = 1 - \cos(2\theta)$$

Apply the double angle formula for cosine to the right. Again there are three forms of this identity. The one that is the most useful in this problem is $\cos(2\theta) = 2\cos^2(\theta) - 1$.

$$2\sin^2(\theta) = 1 - 2\cos^2(\theta) + 1 = 2 - 2\cos^2(\theta)$$

Factor the 2 out of the right hand side. Apply the Pythagorean identity to the result, and you're done.

8) Prove that

$$\frac{1-\tan^2(\theta)}{1-\cot^2(\theta)} = 1 - \sec^2(\theta)$$

Replace $\tan(\theta)$ with $\dfrac{\sin(\theta)}{\cos(\theta)}$. Replace $\cot(\theta)$ with $\dfrac{\cos(\theta)}{\sin(\theta)}$. Replace $\sec(\theta)$ with $\dfrac{1}{\cos(\theta)}$.

$$\frac{1-\frac{\sin^2(\theta)}{\cos^2(\theta)}}{1-\frac{\cos^2(\theta)}{\sin^2(\theta)}} = 1 - \frac{1}{\cos^2(\theta)}$$

Multiply both sides by $\cos^2(\theta)$ to remove the cosines from both the denonminators in which they occur.

$$\frac{\cos^2(\theta)-\sin^2(\theta)}{1-\frac{\cos^2(\theta)}{\sin^2(\theta)}} = \cos^2(\theta) - 1$$

Multiply top and bottom of the left side by $\sin^2(\theta)$ to remove the sine from the denominator in which it occurs.

$$\frac{\sin^2(\theta)\left(\cos^2(\theta)-\sin^2(\theta)\right)}{\sin^2(\theta)-\cos^2(\theta)} = \cos^2(\theta) - 1$$

Observe that $\cos^2(\theta)-\sin^2(\theta) = -\left(\sin^2(\theta)-\cos^2(\theta)\right)$. Applying this to the left gives a common factor of $\sin^2(\theta)-\cos^2(\theta)$ on top and bottom. Cancel them.

$$\frac{-\sin^2(\theta)\cancel{\left(\sin^2(\theta)-\cos^2(\theta)\right)}}{\cancel{\sin^2(\theta)-\cos^2(\theta)}} = -\sin^2(\theta) = \cos^2(\theta) - 1$$

Observe that $\cos^2(\theta)-1 = -\left(1-\cos^2(\theta)\right)$. Apply the Pythagorean identity and you're done.

Alternatively, if you remember your identities among tangent, cotangent, and secant – in particular that $\tan(\theta)\cot(\theta) = 1$ and that $1+\tan^2(\theta) = \sec^2(\theta)$ – you can multiply top and bottom of the left by $\tan^2(\theta)$.

$$\frac{\tan^2(\theta)\left(1-\tan^2(\theta)\right)}{\tan^2(\theta)-1} = 1 - \sec^2(\theta)$$

You get a cancellation on the left of the nasty stuff, leaving $-\tan^2(\theta) = 1-\sec^2(\theta)$. Add $\tan^2(\theta)+\sec^2(\theta)$ to both sides, apply the identity, and you're done.

9) Prove that

$$\csc(\theta) - \tan\left(\frac{\theta}{2}\right) = \cot(\theta)$$

Replace $\tan\left(\dfrac{\theta}{2}\right)$ with $\dfrac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$. Replace $\cot(\theta)$ with $\dfrac{\cos(\theta)}{\sin(\theta)}$. Replace $\csc(\theta)$ with $\dfrac{1}{\sin(\theta)}$.

$$\frac{1}{\sin(\theta)} - \frac{1}{2}\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \frac{\cos(\theta)}{\sin(\theta)}$$

When there is a mix of half-angles and full-angles, the best next step is to make the substitution, $2u = \theta$ or equivalently $u = \dfrac{\theta}{2}$.

$$\frac{1}{\sin(2u)} - \frac{\sin(u)}{\cos(u)} = \frac{\cos(2u)}{\sin(2u)}$$

Now apply the double angle formulas for sine and cosine. Again the best version of the cosine formula is $\cos^2(2u) = 2\cos(u) - 1$.

$$\frac{1}{2\sin(u)\cos(u)} - \frac{\sin(u)}{\cos(u)} = \frac{2\cos^2(u) - 1}{2\sin(u)\cos(u)}$$

Add $\dfrac{1}{2\sin(u)\cos(u)}$ to both sides.

$$\frac{1}{\sin(u)\cos(u)} - \frac{\sin(u)}{\cos(u)} = \frac{\cancel{2}\cos^{\cancel{2}}(u)}{\cancel{2}\sin(u)\cancel{\cos(u)}}$$

Observe the cancellations you get subsequent to that last step. Now add $\dfrac{\sin(u)}{\cos(u)}$ to both sides.

$$\frac{1}{\sin(u)\cos(u)} = \frac{\cos(u)}{\sin(u)} + \frac{\sin(u)}{\cos(u)}$$

Put the two terms on the right over the common denominator of $\sin(u)\cos(u)$.

$$\frac{1}{\sin(u)\cos(u)} = \frac{\cos^2(u) + \sin^2(u)}{\sin(u)\cos(u)}$$

Finally apply the Pythagorean identity to the numerator of the right side, and you're done.

10) Prove that

$$4\sin(\theta)\cos(\theta)\cos(2\theta) = \sin(4\theta)$$

Apply the double angle formula for cosine to the left. This time the best version is $\cos(2\theta) = 1 - 2\sin^2(\theta)$.

$$4\sin(\theta)\cos(\theta)\left(1 - 2\sin^2(\theta)\right) = \sin(4\theta)$$

Apply the quad-angle formula for sine to the right.

$$4\sin(\theta)\cos(\theta)\left(1 - 2\sin^2(\theta)\right) = 4\sin(\theta)\cos(\theta) - 8\sin^3(\theta)\cos(\theta)$$

Factor $\sin(\theta)\cos(\theta)$ out of the two terms of the right-hand side.

$$4\cancel{\sin(\theta)\cos(\theta)}\left(1 - 2\sin^2(\theta)\right) = \left(4 - 8\sin^2(\theta)\right)\cancel{\sin(\theta)\cos(\theta)}$$

Observe the cancellation of the common factor from both sides. All that's left is to factor a 4 from the right-hand side, and you're done.

11) Prove that

$$2\csc(2\theta) = \tan(\theta) + \cot(\theta)$$

Replace $\csc(2\theta)$ with $\dfrac{1}{\sin(2\theta)}$. Replace $\tan(\theta)$ with $\dfrac{\sin(\theta)}{\cos(\theta)}$. Replace $\cot(\theta)$ with $\dfrac{\cos(\theta)}{\sin(\theta)}$.

$$\frac{2}{\sin(2\theta)} = \frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}$$

Apply the double angle formula for sine to the left side.

$$\frac{\cancel{2}}{\cancel{2}\sin(\theta)\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}$$

Put the two terms of the right side over the common denominator, $\sin(\theta)\cos(\theta)$.

$$\frac{1}{\sin(\theta)\cos(\theta)} = \frac{\sin^2(\theta) + \cos^2(\theta)}{\sin(\theta)\cos(\theta)}$$

Now apply the Pythagorean identity, and you're done.

12) Simplify

$$2\csc^2(\theta) - \csc^4(\theta) + \cot^4(\theta)$$

Replace $\csc(\theta)$ with $\dfrac{1}{\sin(\theta)}$. Replace $\cot(\theta)$ with $\dfrac{\cos(\theta)}{\sin(\theta)}$.

$$\frac{2}{\sin^2(\theta)} - \frac{1}{\sin^4(\theta)} + \frac{\cos^4(\theta)}{\sin^4(\theta)}$$

Observe that the common denominator here is $\sin^4(\theta)$.

$$\frac{2\sin^2(\theta) - 1 + \cos^4(\theta)}{\sin^4(\theta)}$$

Replace $\sin^2(\theta)$ with $1 - \cos^2(\theta)$.

$$\frac{1 - 2\cos^2(\theta) + \cos^4(\theta)}{\sin^4(\theta)}$$

Observe that the numerator is a perfect square. Factor it.

$$\frac{\left(1 - \cos^2(\theta)\right)^2}{\sin^4(\theta)}$$

Replace $1 - \cos^2(\theta)$ with $\sin^2(\theta)$, then take the cancellation. Replace $\dfrac{1}{\sin(\theta)}$ with $\csc(\theta)$ to get your final answer.

13) Find the solutions to

$$\sin(2\theta) + \sin(4\theta) = 0$$

Clearly $\theta = 0$ is a solution, as is any value of θ for which $\sin(2\theta) = 0$. This happens when

$$2\theta = k\pi \qquad \text{where } k \text{ is any integer}$$

Or in other words, whenever θ is an integer multiple or half-integer multiple of π. But there are more solutions. Use the double angle formula on $\sin(4\theta)$:

$$\sin(2\theta) + 2\sin(2\theta)\cos(2\theta) = 0$$

Extract the common factor.

$$\sin(2\theta)\left(1 + 2\cos(2\theta)\right) = 0$$

Since we have already accounted for the solutions arising when $\sin(2\theta) = 0$, divide that out and you are left with

$$1 + 2\cos(2\theta) = 0$$

This is true for all values of θ for which $\cos(2\theta) = -\frac{1}{2}$. So which are those? You need to look at the graph of the cosine function and recall your facts about cosine of multiples of $\frac{\pi}{3}$. When

$$2\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \ldots$$

you find that it solves the last equation. See the pattern? Whenever the integer multiplier in the numerator is even but not a multiple of 6. If you divide the whole thing by 2 to solve for θ, you have

$$\theta = \frac{k\pi}{3} \qquad \text{for integer } k \text{ that is not a multiple of } 3$$

Combine this solution set with the first solution set we found and you have the whole solutions set.

14) Prove that

$$\frac{\sec(\theta)}{1 + \sec(\theta)} - \frac{\sec(\theta)}{1 - \sec(\theta)} = 2\csc^2(\theta)$$

Replace $\sec(\theta)$ with $\dfrac{1}{\cos(\theta)}$. Replace $\csc(\theta)$ with $\dfrac{1}{\sin(\theta)}$.

$$\frac{\frac{1}{\cos(\theta)}}{1+\frac{1}{\cos(\theta)}} - \frac{\frac{1}{\cos(\theta)}}{1-\frac{1}{\cos(\theta)}} = \frac{2}{\sin^2(\theta)}$$

On the left, multiply top and bottom of both fractions by $\cos(\theta)$.

$$\frac{1}{\cos(\theta)+1} - \frac{1}{\cos(\theta)-1} = \frac{2}{\sin^2(\theta)}$$

Put both of the fractions on the left over a common denominator. That common denominator is $(\cos(\theta)+1)(\cos(\theta)-1) = \cos^2(\theta)-1$.

$$\frac{\cos(\theta)-1-(\cos(\theta)+1)}{\cos^2(\theta)-1} = \frac{2}{\sin^2(\theta)}$$

Take the sum on the left. Also apply the Pythagorean identity to the denominator on the left, and you're done.

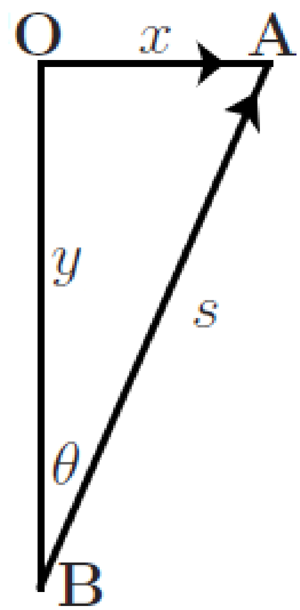
15) A bird flies horizontally out of a tree 5.6 meters directly above the hunter's gun and flies at a constant velocity of 28 meters/second. If the velocity of the bullet is 35 meters/second, and the gun is fired at the moment the bird leaves the tree:

a) What is the angle (with respect to the vertical) at which the gun must be pointed to hit the bird? and

b) What is the flight time of the bullet?

Let the angle be θ. Let the horizontal distance the bullet travels be x and the vertical distance it travels (which is given as 5.6 meters) be y. Then, according to Pythagoras, the total distance, s, that the bullet travels is given by

$$s^2 = x^2 + y^2$$



And from what we know about right triangles,

$$x = s\sin(\theta) \quad \text{and} \quad y = s\cos(\theta)$$

If t is the time of flight, v_A the velocity of the bird, and v_B the velocity of the bullet, then

$$x = v_A t \quad \text{and} \quad s = v_B t$$

Both v_A and v_B are given in the problem. Substituting, $x = v_B t\sin(\theta)$ and consequently $v_A = v_B \sin(\theta)$. From this we know the sine of the angle. Since

$$\tan(\theta) = \frac{\sin(\theta)}{\sqrt{1-\sin^2(\theta)}}$$

we have, by doing a little algebra,

$$\tan(\theta) = \frac{v_A}{\sqrt{v_B^2 - v_A^2}}$$

Putting the numbers in you get $\theta = \arctan\left(\frac{4}{3}\right)$ or about $53.13°$. To determine the flight time, we take advantage of y being given. Again substituting from prior equations, $y = s\cos(\theta) = v_B t\cos(\theta)$. Since y, v_B, and θ are all now known,

$$t = \frac{y}{v_B\cos(\theta)} = \frac{5.6 \text{ meters}}{21 \text{ meters/sec}} = \frac{4}{15} \text{ seconds} = 0.2667 \text{ seconds}$$

16) Prove that

$$1 - \csc(\theta)\sin(3\theta) = 2\cos(2\theta)$$

Replace $\csc(\theta)$ with $\dfrac{1}{\sin(\theta)}$:

$$1 - \frac{\sin(3\theta)}{\sin(\theta)} = 2\cos(2\theta)$$

Apply the triple-angle formula for sine from page 2 of this booklet to the numerator:

$$1 - \frac{3\sin(\theta) - 4\sin^3(\theta)}{\sin(\theta)} = 2\cos(2\theta)$$

Cancel the common factor of $\sin(\theta)$ from top and bottom of the fraction:

$$1 - \left(3 - 4\sin^2(\theta)\right) = 2\cos(2\theta)$$

Simplify the left side by combining the constant terms and factoring out the common factor of 2:

$$2\left(-1 + 2\sin^2(\theta)\right) = 2\cos(2\theta)$$

Apply the appropriate double-angle formula for cosine from page 2 to the right to complete the proof.

Challenge problem: Prove that

$$\frac{\cos(3\theta)-\cos(5\theta)}{\sin(3\theta)+\sin(5\theta)}=\tan(\theta)$$

Clearly you must apply the triple angle and quint-angle formulas for both sine and cosine. In a previous challenge problem you were asked to derive the quint-angle formula for cosine. You are encouraged to do the same for sine. But you can also look both quint-angle formulas up at ***http://mathworld.wolfram.com/Multiple-AngleFormulas.html***. There are several forms of each quint-angle formula. I have applied the most useful for this problem below. But first a rearrangement of the triple angle for sine. Earlier I listed it as $\sin(3\theta)=3\sin(\theta)-4\sin^3(\theta)$. But

$$\begin{aligned} 3\sin(\theta)-4\sin^3(\theta) &= \sin(\theta)\left(3-4\sin^2(\theta)\right) \\ &= \sin(\theta)\left(3-4\left(1-\cos^2(\theta)\right)\right) \\ &= \sin(\theta)\left(4\cos^2(\theta)-1\right) \end{aligned}$$

Substituting all the multiple angle formulas into the original equation:

$$\frac{\left(4\cos^3(\theta)-3\cos(\theta)\right)-\left(16\cos^5(\theta)-20\cos^3(\theta)+5\cos(\theta)\right)}{\sin(\theta)\left(4\cos^2(\theta)-\cancel{1}+16\cos^4(\theta)-12\cos^2(\theta)+\cancel{1}\right)}=\tan(\theta)$$

Now gather terms.

$$\frac{-16\cos^5(\theta)+24\cos^3(\theta)-8\cos(\theta)}{\sin(\theta)\left(16\cos^4(\theta)-8\cos^2(\theta)\right)}=\tan(\theta)$$

Ok. Now let's do some factoring. There are plenty of common factors to pull out. Factor $-8\cos(\theta)$ from the top and $8\cos^2(\theta)$ from the bottom.

$$\frac{-\cancel{8}\cos(\theta)\left(2\cos^4(\theta)-3\cos^2(\theta)+1\right)}{\cancel{8}\sin(\theta)\cos^2(\theta)\left(2\cos^2(\theta)-1\right)}=\tan(\theta)$$

Observe that $\cos^4(\theta)-3\cos^2(\theta)+1$ factors into $\left(2\cos^2(\theta)-1\right)\left(\cos^2(\theta)-1\right)$. That results in a common factor of $2\cos^2(\theta)-1$ on the top and bottom. So cancel that out and you get

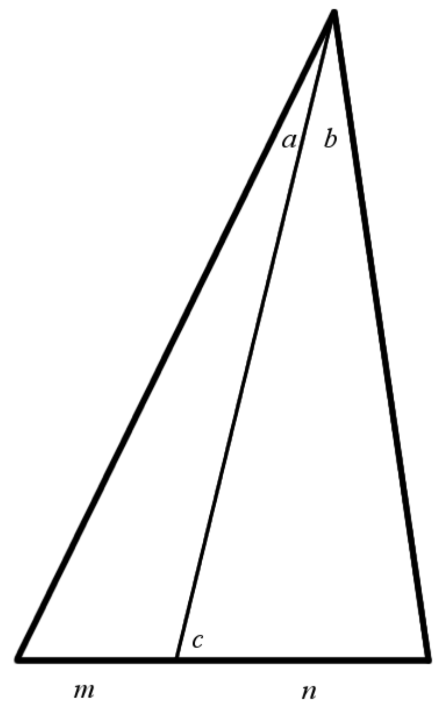
$$\frac{-\cos(\theta)\left(\cos^2(\theta)-1\right)}{\sin(\theta)\cos^2(\theta)}=\tan(\theta)$$

But $-\left(\cos^2(\theta)-1\right)=1-\cos^2(\theta)=\sin^2(\theta)$. So you have

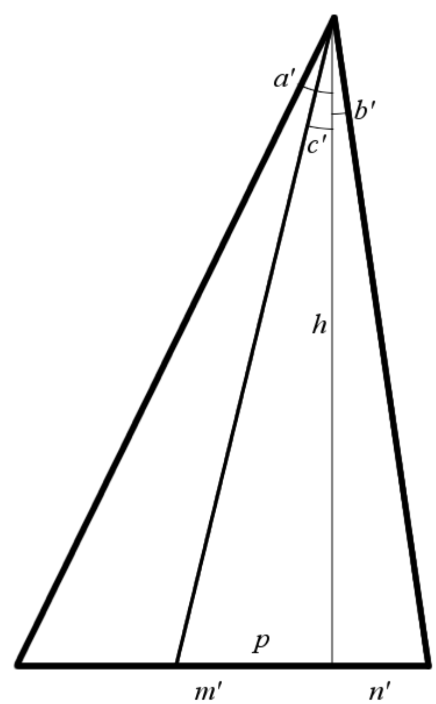
$$\frac{\cancel{\cos(\theta)}\sin^{\cancel{2}}(\theta)}{\cancel{\sin(\theta)}\cos^{\cancel{2}}(\theta)}=\tan(\theta)$$

Surely you can do the rest.

The Cotangent Rule:



The diagram shows a triangle with a line segment from its apex to its base splitting it into two. The contangent rule states that $(m+n)\cot(c) = m\cot(a) - n\cot(b)$ The trick, of course, is to prove this. At first it's hard to see the connection. But by transforming the problem it becomes solvable. We have formulas that apply trig functions to right triangles, so we transform this into a right triangle problem. We do this by constructing a line perpendicular to the base of the triangle that passes through its apex. This gives us new lengths and angles, which we designate with "primed" symbols. Each of the primed symbols is related by a simple equation to its unprimed counterpart. We solve the problem in the primed context, then use those simple equation to related it back to the original unprimed problem.



The new diagram shows the transformed problem. The new symbol, p, is the distance along the base from where the original line segment intersects the base to where the perpendicular, h, intersects it. m' and n' are the lengths along the base to the left and right of the perpendicular. c' is the angle between the splitting line segment in the first diagram and the perpendicular. Between the original problem and the transformed problem we have the following relationships along the base: $m = m' - p$, and $n = n' + p$. The angles constituting the apex have these relationships:

$$a = a' - c' \quad \text{and} \quad b = b' + c'$$

Look carefully at the diagram and observe the various right triangles that the new perpendicular line segment forms. Recall the rule of right triangles that the tangent of either of its non-right angles is equal to the length of its opposite side divided by that of the adjacent side. Hence

$$\tan(a') = \frac{m'}{h}$$

$$\tan(b') = \frac{n'}{h}$$

$$\tan(c') = \frac{p}{h}$$

Recall also from page 2 of this booklet the formula for the tangent of a sum. Applying the relationships we've already established,

$$\tan(a) = \frac{\tan(a') - \tan(c')}{1 + \tan(a')\tan(c')} = \frac{\frac{m'-p}{h}}{1 + \frac{m'p}{h^2}}$$

$$\tan(b) = \frac{\tan(b') + \tan(c')}{1 - \tan(b')\tan(c')} = \frac{\frac{n'+p}{h}}{1 - \frac{m'p}{h^2}}$$

On the right side of each of the above equations, multiply top and bottom by h^2 and replace $m' - p$ and $n' + p$ with m and n respectively:

$$\tan(a) = \frac{hm}{h^2 + m'p}$$

$$\tan(b) = \frac{hn}{h^2 - n'p}$$

Now recall that the cotangent of an angle is the reciprocal of its tangent. So flipping both equations and multiplying through by m and n respectively:

$$m\cot(a) = \frac{h^2 + m'p}{h}$$

$$n\cot(b) = \frac{h^2 - n'p}{h}$$

Taking the difference between these two equations:

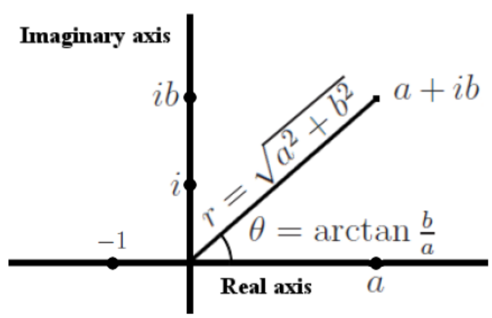
$$m \cot(a) - n \cot(b) = \frac{p(m' + n')}{h}$$

But as we've already seen, $\frac{p}{h} = \tan(c')$, and because c and c' are complementary angles (that is they add up to a right angle) we know that, $\tan(c') = \cot(c)$. Observe also that since $m + n$ and $m' + n'$ both add up to the base of the original triangle, it follows that $m' + n' = m + n$. Making all of those substitutions completes the proof.

Trig functions and complex numbers

If you have never studied complex numbers, you probably believe that -1 has no square root. And among the real numbers it indeed has no square root. But in the 16^{th} century Italian mathematicians, Niccolo Fontana Tartaglia and Gerolamo Cardano, asked the question, what happens if we *imagine* the existence of a square root of -1. They called this imaginary value, i, such that $i^2 = (-i)^2 = -1$. Of course you have to be able to add and multiply this imaginary value with any of the real numbers. This gives rise to a whole self-consistent number system known as *the complex numbers*. In general a complex number is in the form, $a + ib$, where a and b are both real numbers. Clearly the real numbers is a subset of the complex numbers where b is held to zero. In the example, a is called the real part of $a + ib$, and ib is called its imaginary part. To add two complex numbers you simply add the real parts to get the real part of the sum and add the imaginary parts to get the imaginary part of the sum. To multiply two complex numbers we use the distributive law.

$$(a + ib)(c + id) = ac+iad+ibc+i^2bd = ac+iad+ibc-bd = ac-bd+i(ad + bc)$$



You can diagram the complex numbers in the same way you diagram the $x-y$ plane, where the horizontal axis is the real axis and the vertical axis is the imaginary axis. The figure shows $a+ib$ (where, in this case, both a and b are positive) plotted on the *complex plane*. The distance from 0 (the origin) to $a+ib$ is shown as r. The angle that $a + ib$ makes with the positive real axis is shown as θ. Note that if $a + ib$ is not in quadrant I or quadrant IV, you have to add the quadrant adjustment of π or $-\pi$ for quadrants II or III respectively to the arctangent value.

From the parameters shown in the diagram, we see that we can represent $a + ib$ as $r\cos(\theta) + ir\sin(\theta)$. Likewise if another complex number, $c + id$, is s distance from the origin and makes angle ϕ with the real axis (that is $s = \sqrt{c^2 + d^2}$ and $\phi = \arctan \frac{d}{c}$ with the proper quadrant adjustment), then we can represent this complex number as $s\cos(\phi) + is\sin(\phi)$. Observe what happens when we multiply the two complex numbers together.

$$\begin{aligned}(a+ib)(c+id) &= ac-bd+i(ad+bc)\\ &= rs\cos(\theta)\cos(\phi)-rs\sin(\theta)\sin(\phi)\\ &\qquad +i(rs\cos(\theta)\sin(\phi)+rs\sin(\theta)\cos(\phi))\\ &= rs\cos(\theta+\phi)+irs\sin(\theta+\phi)\end{aligned}$$

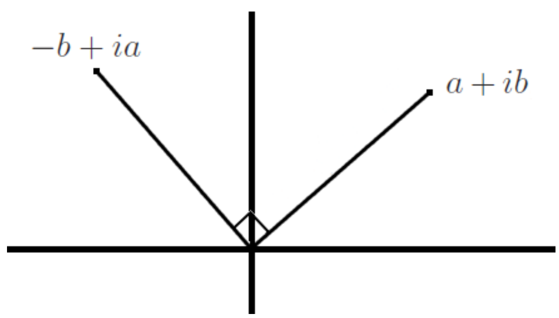
We arrive at the last line of the above by applying the sum formulas for sine and cosine. What this means is that when you multiply two complex numbers, the distance the product is from the origin will be the product of the distances from the origin of the two factors, and the angle the product makes with the real axis will be the sum of the angles to the real axis of the two factors. From these relationships it is easy to see that

$$(r\cos(\theta)+ir\sin(\theta))^n = r^n\cos(n\theta)+ir^n\sin(n\theta)$$

which is known as *De Moivre's formula*.

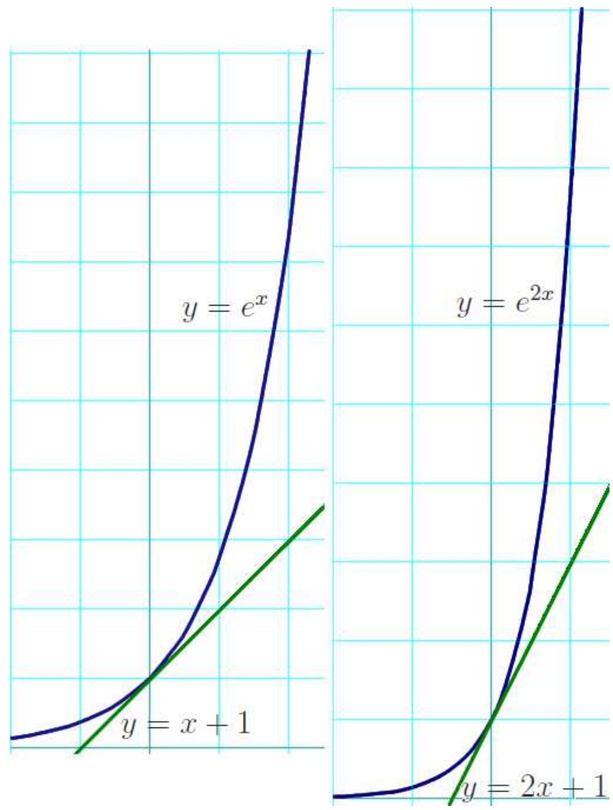
Notice that if you define $f(x)=\cos(x)+i\sin(x)$, then $f(x+y)=f(x)f(y)$. Recall that this is a property of exponential functions.

Observe that i is distance of 1 from the origin and makes angle, $\frac{\pi}{2}$ radians, with the real axis. If you multiply $a+ib$ by i you get $-b+ia$. You can see in the figure that this has the effect rotating $a+ib$ $\frac{\pi}{2}$ radians (that is $90°$) counterclockwise as would be expected from the multiplication formula for complex numbers we have just seen.



Did you know that sine and cosine are actually exponential functions? What are the critical characteristics of the function, $y=e^{kx}$, for some arbitrary constant, k? They are these: a) This function passes through the point, $(0,1)$ regardless of the value of k. b) The slope of the tangent line at $(0,1)$ is equal to k.

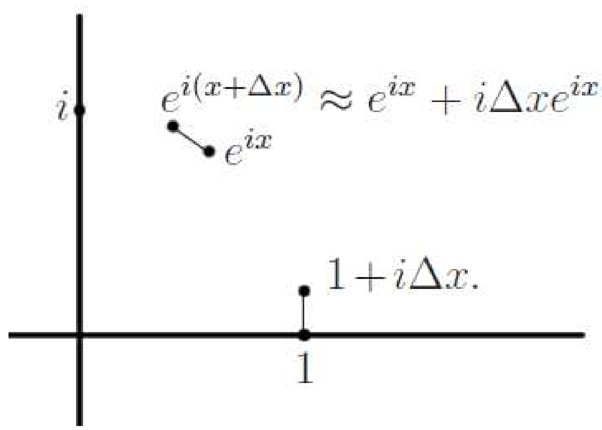
The figure shows this to be so for the examples of $k = 1$ and $k = 2$. c) At any x, the slope of the tangent line is equal to ke^{kx}. If you have already studied some calculus you will know why the latter fact is always true. So what do these three facts imply about $k = i$? In the figure when $k = 1$, if you want to approximate $e^{\Delta x}$, where Δx is very small, you can do so by adding Δx to e^0. Hence $e^{\Delta x} \approx 1 + \Delta x$. Likewise if you want to approximate $e^{2\Delta x}$, you could do so with $e^{2\Delta x} \approx 1+2\Delta x$. And in general, if you want to approximate $e^{k\Delta x}$, you could do so with $e^{k\Delta x} \approx 1 + k\Delta x$. Note that these approximations get better and better for smaller and smaller Δx.



By this pattern we see that if $k = i$, then we approximate $e^{i\Delta x}$ with $1 + i\Delta x$. That is from $(0, 1)$ if you increase x by just a little, e^{ix} advances by that same little bit but in the i direction. Note that the distance to the origin of $1 + i\Delta x$ is within Δx^2 of 1.

By the third fact, if we know the value of e^{2x} for some value of x, then we approximate $e^{2(x+\Delta x)} \approx e^{2x} + 2\Delta x e^{2x}$, and in general, $e^{k(x+\Delta x)} \approx e^{kx} + k\Delta x e^{kx}$.

Now suppose at some value for x we know where e^{ix} is. We would approximate $e^{i(x+\Delta x)} \approx e^{ix} + i\Delta x e^{ix}$. Both approximations – for $e^{i\Delta x}$ and for $e^{i(x+\Delta x)}$ – are shown in the diagram. For the latter, recall the observation several paragraphs back



that multiplying by i rotates a point on the complex plane $\frac{\pi}{2}$ counterclockwise. You can see that the short line segment of length, Δx, that starts at e^{ix} is at right angles to the line segment connecting e^{ix} to the origin.

The point of all this is that as x increases, the value of e^{ix} always moves at right angles to the line segment connecting e^{ix} to the origin. That means that e^{ix} follows a circular path, centered at the origin, around the complex plane. We know that the distance to the origin of e^{ix} at $x = 0$ is 1, so we know that the radius of the circle is also 1. In addition, at x the function, e^{ix}, has advanced an arc length around the circle from $(0, 1)$ of exactly x. This leads to the following identity:

$$e^{ix} = \cos(x) + i\sin(x)$$

This equation is known as *Euler's formula*. In electrical and mechanical engineering, as well as in physics, it arises all the time. By applying symmetry identities to Euler's formula you also have

$$e^{-ix} = \cos(x) - i\sin(x)$$

By taking the sum and difference these two and dividing each by 2

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad\qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

This is how sine and cosine are, in reality, exponential functions. So what is an immediate way we can put these formulas to work? Suppose you wanted to know what $\cos^n(\theta)$ was in terms of multiples of θ.

$$\cos^n(\theta) = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^n$$

We can apply the binomial formula to the right-hand side:

$$\begin{aligned}\cos^n(\theta) &= 2^{-n}\sum_{k=0}^{n}\binom{n}{k}e^{ik\theta}e^{-i(n-k)\theta}\\ &= 2^{-n}\sum_{k=0}^{n}\binom{n}{k}e^{i(2k-n)\theta}\end{aligned}$$

Recall that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

So applying this to $n = 6$ we have

$$\cos^6(\theta) = \frac{1}{64}\left(e^{-6i\theta} + 6e^{-4i\theta} + 15e^{-2i\theta} + 20e^0 + 15e^{2i\theta} + 6e^{4i\theta} + e^{6i\theta}\right)$$

Rearranging terms

$$\begin{aligned}
\cos^6(\theta) &= \frac{1}{32}\left(\frac{e^{6i\theta} + e^{-6i\theta}}{2} + 6\,\frac{e^{4i\theta} + e^{-4i\theta}}{2} + 15\,\frac{e^{2i\theta} + e^{-2i\theta}}{2} + 10\right) \\
&= \frac{1}{32}\left(\cos(6\theta) + 6\cos(4\theta) + 15\cos(2\theta) + 10\right)
\end{aligned}$$

See if you can use the same method to arrive at

$$\cos^4(\theta) = \frac{1}{8}\left(\cos(4\theta) + 4\cos(2\theta) + 3\right)$$

Standing the expansion for $\cos^6$ on its head along with formulas similarly derived for $\cos^4(\theta)$ and $\cos^2(\theta)$, we can readily arrive at a formula for the cosine of the hex-angle. We can get the $\cos(6\theta)$ term by multiplying our expansion of $\cos^6$ by 32. We can eliminate the $\cos(4\theta)$ term from that by subtracting 48 times the expansion for $\cos^4$.

$$\begin{array}{rcrcrcrcr}
32\cos^6(\theta) & = & \cos(6\theta) & + & 6\cos(4\theta) & + & 15\cos(2\theta) & + & 10 \\
-48\cos^4(\theta) & = & & & -6\cos(4\theta) & - & 24\cos(2\theta) & - & 18 \\
\hline
& = & \cos(6\theta) & & & - & 9\cos(2\theta) & - & 8
\end{array}$$

Now we can eliminate the $\cos(2\theta)$ term from the above result by adding 18 times the expansion for $\cos^2$.

$$\begin{array}{lcrcrcr}
32\cos^6(\theta) - 48\cos^4(\theta) & = & \cos(6\theta) & - & 9\cos(2\theta) & - & 8 \\
+18\cos^2(\theta) & = & & & 9\cos(2\theta) & + & 9 \\
\hline
& & \cos(6\theta) & & & + & 1
\end{array}$$

Now subtract 1 from this result to get

$$32\cos^6(\theta) - 48\cos^4(\theta) + 18\cos^2(\theta) - 1 = \cos(6\theta)$$

Deriving a formula for $\sin^n(\theta)$ is a higher degree of difficulty than what we just did.

$$\sin^n(\theta) = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^n = (2i)^{-n}\sum_{k=0}^{n}\binom{n}{k}e^{ik\theta}\left(-1\right)^{n-k}e^{-i(n-k)\theta}$$

or, combining terms on the right

$$\sin^n(\theta) = (2i)^{-n} \sum_{k=0}^{n} \binom{n}{k} e^{i(2k-n)} (-1)^{n-k}$$

You can see that we have the nasty business of raising i to the $-n$ power. But this, it turns out, is not hard at all. Powers of i follow a repeating pattern of length, 4. $i^0 = 1$; $i^1 = i$; $i^2 = -1$; $i^3 = -i$; then, at $n = 4$ it repeats from the beginning. When n is negative, its the same pattern going backward. The real nastiness here is that when you resolve this back to trig functions, as we did with the cosine power, you get a sum of cosines when n is even and a sum of sines when n is odd. As an exercise, work it out for $n = 3$ and $n = 4$ to see why this happens.

Challenge problem: Use the exponential forms of sine and cosine to prove that

$$y = \arctan(x) = \frac{1}{2i} \ln\left(\frac{i-x}{i+x}\right) \qquad \text{note that ln is natural log}$$

Step 1: Observe from the above that $x = \tan(y)$. Derive a formula for $x = \tan(y)$ in terms of exponentials

$$x = \tan(y) = \frac{\sin(y)}{\cos(y)} = \frac{\dfrac{e^{iy} - e^{-iy}}{2i}}{\dfrac{e^{iy} + e^{-iy}}{2}}$$

$$= \frac{1}{i} \frac{e^{iy} - e^{-iy}}{e^{iy} + e^{-iy}} \qquad \text{now multiply top and bottom by } \; e^{iy}$$

$$= \frac{1}{i} \frac{e^{2iy} - 1}{e^{2iy} + 1} \qquad \text{observing that } \; \frac{1}{i} = -i$$

$$x = \tan(y) = -i \, \frac{e^{2iy} - 1}{e^{2iy} + 1}$$

Step 2: We now need to solve for y in terms of x. To avoid algebra errors, let $u = e^{iy}$. We have

$$x = -i\,\frac{u^2 - 1}{u^2 + 1}$$

Step 3: Multiply through by $u^2 + 1$ to get rid of the fraction.

$$x(u^2 + 1) = u^2x + x = -iu^2 + i$$

Step 4: Get all the terms with a factor of u^2 to one side of the equation and all the terms without to the other.

$$iu^2 + u^2x = i - x$$

Step 5: Factor $i + x$ out of the left side.

$$u^2\,(i + x) = i - x$$

Step 6: Divide through by $i + x$.

$$u^2 = \frac{i - x}{i + x}$$

Step 6: Back-substitute $u = e^{iy}$.

$$e^{2iy} = \frac{i - x}{i + x}$$

Step 7: Take the natural log of both sides.

$$2iy = \ln\left(\frac{i - x}{i + x}\right)$$

Step 8: Divide by $2i$. Since $y = \arctan(x)$, this gets you to the final result of

$$y = \arctan(x) = \frac{1}{2i}\ln\left(\frac{i - x}{i + x}\right) \qquad \text{which is what we set out to prove.}$$

The tractor-pull problem: Suppose you have n tractors arranged at equal increments around a circle (to form the vertices of a regular n-gon). Each tractor is chained to a post at the center of the circle. The tractors all pull directly outward from the circle, each with equal force. Intuitively we expect that such equal forces arranged in this way will all cancel resulting in a net force on the post of zero. Indeed when n is even, the symmetry of the arrangement makes it clear that they do all cancel. But it's not so clear when n is odd. Using complex numbers, though, we can easily prove that no matter what integer, $n > 1$, you choose, the forces always cancel exactly. Here's the proof. Place the first tractor at $1 + i0$. Place the next at $z = \cos(\theta) + i \sin(\theta)$, where $\theta = \frac{2\pi}{n}$. Place the remaining $n-2$ tractors at $\cos(k\theta) + i \sin(k\theta)$, where k ranges from 2 to $n-1$. By De Moivre's formula, the n tractors are at points, z^k, where $z = \cos(\theta) + i \sin(\theta)$ and k ranges from zero to $n-1$. We also know that if you go a full n steps of $\frac{2\pi}{n}$ around the circle, you end up where you started. If you start at $1 + i0$, you will end up there again after encountering n tractors. Hence we know that $z^n = 1 + i0$. All we need to do now is prove that the sum of z^k, for k going from 0 to $n-1$ is always zero.

$$z^0 + z^1 + z^2 + \cdots + z^{n-1} = \sum_{k=0}^{n-1} z^k = 0 \quad \text{to be proved}$$

What happens if we multiply the polynomial above by $z-1$?

$$\begin{array}{rrrr} & z^1 + z^2 + \cdots + & z^{n-1} + & z^n \\ z^0 + & z^1 + z^2 + \cdots + & z^{n-1} & \\ \hline -z^0 + & & & z^n \end{array}$$

That is, the sum of all those powers of z times $z-1$ is equal to $z^n - 1$. But we already established that $z^n = 1$. Hence $z^n - 1 = 0$. It follows that one of the two factors, either $z-1$ or the sum of all the powers of z, must be zero. But $z = \cos(\theta) + i \sin(\theta)$, where $\theta = \frac{2\pi}{n}$. If $0 < \theta < 2\pi$, it is impossible for z to be equal to 1. Therefore $z-1$ cannot be zero, and it follows that the sum of the powers of z from zero to $n-1$ must be zero.