Art of Problem Solving

WOOT 2010–11

Practice AIME 1 Solutions

Answer Key

1. 320	6. 533	11. 385
2. 700	7. 135	12. 180
3. 593	8. 323	13. 117
4. 762	9. 336	14. 344
5. 090	10. 057	15. 022

1. Compute $2010 \cdot 2016 \cdot 2028 - 2008 \cdot 2020 \cdot 2026$.

   Solution. Let $x = 2010$. Then

$$\begin{aligned}&2010 \cdot 2016 \cdot 2028 - 2008 \cdot 2020 \cdot 2026\\ &= x(x+6)(x+18) - (x-2)(x+10)(x+16)\\ &= x^3 + 24x^2 + 108x - (x^3 + 24x^2 + 108x - 320)\\ &= 320.\end{aligned}$$

2. Find the number of ordered quadruples (a, b, c, d) of positive integers such that $abcd = 5000$.

   Solution. Since $5000 = 2^3 \cdot 5^4$, $a = 2^{e_1}5^{f_1}$, $b = 2^{e_2}5^{f_2}$, $c = 2^{e_3}5^{f_3}$, and $d = 2^{e_4}5^{f_4}$ for some nonnegative integers e_1, e_2, e_3, e_4, f_1, f_2, f_3, and f_4. Then

$$abcd = 2^{e_1+e_2+e_3+e_4} \cdot 5^{f_1+f_2+f_3+f_4} = 2^3 \cdot 5^4,$$

   so $e_1 + e_2 + e_3 + e_4 = 3$ and $f_1 + f_2 + f_3 + f_4 = 4$.

   The number of ordered quadruples of nonnegative integers (e_1, e_2, e_3, e_4) that satisfy $e_1+e_2+e_3+e_4 = 3$ is $\binom{6}{3} = 20$, and the number of ordered quadruples of nonnegative integers (f_1, f_2, f_3, f_4) that satisfy $f_1+f_2+f_3+f_4 = 4$ is $\binom{7}{3} = 35$. Therefore, the number of ordered quadruples (a, b, c, d) is $20 \cdot 35 = 700$.

3. Let a, b, c, and d be positive integers such that $a^5 = b^6$, $c^3 = d^4$, and $d - a = 61$. Find $c - b$.

   Solution. Since $a^5 = b^6$, and 5 and 6 are relatively prime, a is a perfect 6^{th} power, which means a is also a perfect cube. Let $a = x^3$, where x is a positive integer. Since 3 and 4 are relatively prime, d is a perfect cube. Let $d = y^3$, where y is a positive integer.

   Then

$$d - a = y^3 - x^3 = (y - x)(y^2 + xy + y^2) = 61.$$

   Since 61 is prime and $x^2 + xy + y^2 \geq 3$, $y - x = 1$ and $x^2 + xy + y^2 = 61$. Then $y = x + 1$, so

$$x^2 + x(x + 1) + (x + 1)^2 = 3x^2 + 3x + 1 = 61,$$

   which simplifies as $3x^2 + 3x - 60 = 3(x^2 + x - 20) = 3(x - 4)(x + 5) = 0$. Since x is positive, $x = 4$, so $y = 5$.

   Then $b^6 = a^5 = x^{15} = 4^{15} = 2^{30}$, so $b = 2^5 = 32$, and $c^3 = d^4 = y^{12}$, so $c = y^4 = 5^4 = 625$. Therefore, $c - b = 625 - 32 = 593$.

4. For a positive integer n, let $f(n)$ denote the largest power of 17 dividing n. For example, $f(5) = 17^0 = 1$ and $f(2 \cdot 3^4 \cdot 17^2) = 17^2$. Determine the positive integer n such that

$$f(1) + f(2) + f(3) + \cdots + f(n) = 2010.$$

Solution. Note that $17^3 = 4913$, so

$$f(1) + f(2) + \cdots + f(4913) > 4913 > 2010.$$

Therefore, the n we seek is less than 17^3.

Among the integers 1, 2, ..., n, $\lfloor n/17 \rfloor$ are divisible by 17, which means that

$$n - \left\lfloor \frac{n}{17} \right\rfloor$$

are not divisible by 17. Similarly, $\lfloor n/17^2 \rfloor$ are divisible by 17^2, which means that

$$\left\lfloor \frac{n}{17} \right\rfloor - \left\lfloor \frac{n}{17^2} \right\rfloor$$

are divisible by 17, but not by 17^2. And since $n < 17^3$, none are divisible by 17^3, so

$$\begin{aligned}
&f(1) + f(2) + \cdots + f(n) \\
&= n - \left\lfloor \frac{n}{17} \right\rfloor + 17 \left(\left\lfloor \frac{n}{17} \right\rfloor - \left\lfloor \frac{n}{17^2} \right\rfloor \right) + 17^2 \left\lfloor \frac{n}{17^2} \right\rfloor \\
&= n + 16 \left\lfloor \frac{n}{17} \right\rfloor + 272 \left\lfloor \frac{n}{17^2} \right\rfloor .
\end{aligned}$$

Let $n = 17^2 a + 17b + c$, where $0 \le a,\ b,\ c \le 16$. (In other words, let $n = \overline{abc}_{17}$.) Then

$$\begin{aligned}
f(1) + f(2) + \cdots + f(n) &= n + 16 \left\lfloor \frac{n}{17} \right\rfloor + 272 \left\lfloor \frac{n}{17^2} \right\rfloor \\
&= 289a + 17b + c + 16(17a + b) + 272a \\
&= 833a + 33b + c \\
&= 2010.
\end{aligned}$$

Solving for a, we find

$$a = \frac{2010 - 33b - c}{833}.$$

Then

$$a \le \frac{2010}{833} < 3,$$

so $a \le 2$, and

$$a \ge \frac{2010 - 33 \cdot 16 - 16}{833} = \frac{1466}{833} > 1,$$

so $a \geq 2$. Therefore, $a = 2$.

Then $33b + c = 2010 - 833 \cdot 2 = 344$. Solving for b, we find

$$b = \frac{344 - c}{33}.$$

Then

$$b \leq \frac{344}{33} < 11,$$

so $b \leq 10$, and

$$b \geq \frac{344 - 16}{33} = \frac{328}{33} > 9,$$

so $b \geq 10$. Therefore, $b = 10$, and $c = 344 - 33 \cdot 10 = 14$. Hence, $n = 289 \cdot 2 + 17 \cdot 10 + 14 = 762$.

5. Find the smallest positive integer n for which

$$|x - 1| + |x - 2| + |x - 3| + \cdots + |x - n| \geq 2010$$

for all real numbers x.

Solution. Let

$$f(x) = |x - 1| + |x - 2| + \cdots + |x - n|.$$

To find the minimum value of $f(x)$, we divide into the cases where n is even and n is odd.

If n is even, then let $n = 2k$. By the Triangle inequality,

$$|x - i| + |x - (2k + 1 - i)| = |x - i| + |(2k + 1 - i) - x| \geq 2k - 2i + 1$$

for all i, $1 \leq i \leq 2k$, so

$$\begin{aligned} |x - 1| + |x - 2k| &\geq 2k - 1, \\ |x - 2| + |x - (2k - 1)| &\geq 2k - 3, \\ &\dots, \\ |x - k| + |x - (k + 1)| &\geq 1. \end{aligned}$$

Summing these inequalities, we get

$$\begin{aligned} f(x) &= |x - 1| + |x - 2| + \cdots + |x - 2k| \\ &\geq 1 + 3 + \cdots + (2k - 1) \\ &= k^2. \end{aligned}$$

Furthermore, equality occurs for $k \leq x \leq k + 1$. Hence, if $f(x) \geq 2010$ for all real numbers x, then we must have $k^2 \geq 2010$. Since $44^2 = 1936$ and $45^2 = 2025$, $k \geq 45$, and $n \geq 90$.

If n is odd, then let $n = 2k+1$. By the same reasoning as above,

$$\begin{aligned} |x-1| + |x-(2k+1)| &\ge 2k, \\ |x-2| + |x-2k| &\ge 2k-2, \\ &\dots, \\ |x-k| + |x-(k+2)| &\ge 2, \\ |x-(k+1)| &\ge 0. \end{aligned}$$

Summing these inequalities, we get

$$\begin{aligned} f(x) &= |x-1| + |x-2| + \dots + |x-(2k+1)| \\ &\ge 0 + 2 + \dots + 2k \\ &= k(k+1). \end{aligned}$$

Furthermore, equality occurs when $x = k+1$. Hence, if $f(x) \ge 2010$ for all real numbers x, then we must have $k(k+1) \ge 2010$. Since $44 \cdot 45 = 1980$ and $45 \cdot 46 = 2070$, $k \ge 45$, and $n \ge 91$.

Therefore, the smallest possible value of n is 90.

6. A sequence of positive integers is constructed as follows: First, we write down a 1. Then given the first 2^n terms of the sequence, where n is a nonnegative integer, the next 2^n terms of the sequence are generated by writing the first 2^n terms in reverse order, then adding 2^n to each of these new terms. Thus, the first few steps of the construction produce the numbers

$$\begin{aligned} &1, \\ &1, 2, \\ &1, 2, 4, 3, \\ &1, 2, 4, 3, 7, 8, 6, 5, \end{aligned}$$

and so on. Find the 1000^{th} term of the sequence.

Solution. Let a_n denote the n^{th} term of the sequence. By definition, if $2^n + 1 \le k \le 2^{n+1}$, then

$$a_k = a_{2^{n+1}+1-k} + 2^n.$$

Since $2^9 + 1 \le 1000 \le 2^{10}$,

$$a_{1000} = a_{2^{10}+1-1000} + 2^9 = a_{25} + 512.$$

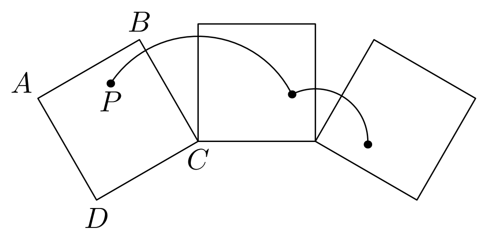
Since $2^4 + 1 \le 25 \le 2^5$,

$$a_{25} = a_{2^5+1-25} + 2^4 = a_8 + 16.$$

And since $a_8 = 5$, $a_{1000} = 5 + 16 + 512 = 533$.

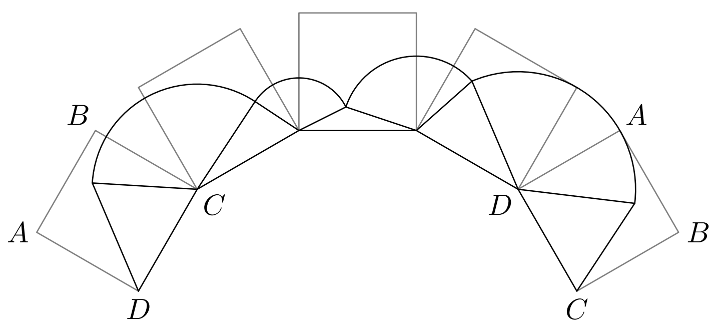
7. Let P be a point inside square $ABCD$, which has side length 5, such that P has a distance of 1 from side AB and a distance of 2 from side BC. The square is initially placed so that side CD coincides with the side of a regular dodecagon (12-sided polygon), also of side length 5.

   The square is then rolled around the dodecagon, until it returns to its original position, such that the point P stays fixed relative to the square, tracing a path γ. A portion of γ is shown below.



   The area of the region between γ and the dodecagon can be expressed in the form $a + b\pi$, where a and b are integers. Find $a + b$.

   Solution. After square $ABCD$ turns four times, side CD again coincides with a side of the regular dodecagon, so we may triple the path that P describes in four turns to obtain γ.



   We see that the region between γ and the dodecagon consists of 12 triangles and 12 circular sectors.

   Among these 12 triangles, three are congruent to triangle PAB, three are congruent to triangle PBC, three are congruent to triangle PCD, and three are congruent to triangle PDA, so the sum of the areas of all 12 triangles is equal to 3 times the area of square $ABCD$, or $3 \cdot 25 = 75$.

   The external angle at each vertex of the dodecagon is 30°, so the angle of each circular sector is $90^\circ + 30^\circ = 120^\circ$. Among these 12 circular sectors, three have radius PA, three have radius PB, three have radius PC, and three have radius PD. We can calculate that $PA^2 = 10$, $PB^2 = 5$, $PC^2 = 20$, and $PD^2 = 25$, so the sum of the areas of all 12 circular sectors is $10\pi + 5\pi + 20\pi + 25\pi = 60\pi$.

   Therefore, the area of the region between γ and the dodecagon is $75 + 60\pi$, and so the final answer is $75 + 60 = 135$.

8. Find the number of positive integers n that satisfy

$$\left\lfloor \frac{n}{35} \right\rfloor = \left\lfloor \frac{n}{37} \right\rfloor.$$

Note: For a real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Solution. For a given nonnegative integer k,

$$\left\lfloor \frac{n}{35} \right\rfloor = k$$

if and only if

$$k \le \frac{n}{35} < k+1,$$

which is equivalent to $35k \le n < 35k + 35$, or $35k \le n \le 35k + 34$. Similarly,

$$\left\lfloor \frac{n}{37} \right\rfloor = k$$

if and only if

$$k \le \frac{n}{37} < k+1,$$

which is equivalent to $37k \le n < 37k + 37$, or $37k \le n \le 37k + 36$. Since $35k \le 37k$ and $35k + 34 \le 37k + 36$, the positive integer n satisfies both conditions if and only if $37k \le n \le 35k + 34$. Hence, as long as $37k \le 35k + 34$, the number of positive integers n satisfying

$$\left\lfloor \frac{n}{35} \right\rfloor = \left\lfloor \frac{n}{37} \right\rfloor = k$$

is equal to $35k + 34 - 37k + 1 = 35 - 2k$, except when $k = 0$, because we must exclude the case $n = 0$. The inequality $37k \le 35k + 34$ is satisfied for $0 \le k \le 17$, so the total number of such positive integers n is

$$\begin{aligned}
\sum_{k=0}^{17} (35 - 2k) - 1 &= \sum_{k=0}^{17} 35 - 2\sum_{k=0}^{17} k - 1 \\
&= 18 \cdot 35 - 17 \cdot 18 - 1 \\
&= 18^2 - 1 \\
&= 323.
\end{aligned}$$

9. Let $[T]$ denote the area of triangle T. In triangle ABC, let P be on side AB and let Q be on side AC, and let CP and BQ intersect at R. Find $[ABC]$ if $[BRP] = 5$, $[BRC] = 7$, and $[CRQ] = 9$.

Solution. Let $x = [APR]$ and $y = [AQR]$. If two triangles have the same height, then their areas are proportional to their bases, so

$$\frac{x+5}{y} = \frac{BR}{RQ} = \frac{7}{9}$$

and

$$\frac{y+9}{x} = \frac{CR}{RP} = \frac{7}{5}.$$

Solving this system of equations, we find $x = 135$ and $y = 180$, so $[ABC] = 135+180+5+7+9 = 336$.

10. If the acute angles α and β satisfy

$$\begin{aligned} 2\sin 2\beta &= 3\sin 2\alpha, \\ \tan\beta &= 3\tan\alpha, \end{aligned}$$

then $\cos^2(\alpha-\beta)$ can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m+n$.

Solution. We can rewrite the given equations as

$$\begin{aligned} 2\sin\beta\cos\beta &= 3\sin\alpha\cos\alpha, \\ \frac{\sin\beta}{\cos\beta} &= 3\cdot\frac{\sin\alpha}{\cos\alpha}. \end{aligned}$$

Multiplying and dividing these equations gives $2\sin^2\beta = 9\sin^2\alpha$ and $2\cos^2\beta = \cos^2\alpha$, respectively. Adding these equations, we get

$$\begin{aligned} 2\sin^2\beta + 2\cos^2\beta &= 9\sin^2\alpha + \cos^2\alpha \\ \Rightarrow \quad 2 &= 8\sin^2\alpha + 1, \end{aligned}$$

so $\sin^2\alpha = 1/8$. Then $\sin^2\beta = (9\sin^2\alpha)/2 = 9/16$.

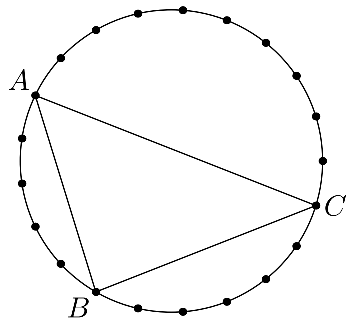
Since α and β are acute, $\sin\alpha$ and $\sin\beta$ are positive, so $\sin\alpha = 1/(2\sqrt{2})$ and $\sin\beta = 3/4$. Also, $\cos^2\alpha = 7/8$, so $\cos\alpha = \sqrt{7}/(2\sqrt{2})$, and $\cos^2\beta = 7/16$, so $\cos\beta = \sqrt{7}/4$. Therefore,

$$\begin{aligned} \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \frac{\sqrt{7}}{2\sqrt{2}}\cdot\frac{\sqrt{7}}{4} + \frac{1}{2\sqrt{2}}\cdot\frac{3}{4} \\ &= \frac{5}{4\sqrt{2}}, \end{aligned}$$

which means $\cos^2(\alpha-\beta) = 25/32$, so the final answer is $25+32 = 57$.

11. Let P be a regular 21-gon. How many acute triangles have all three of their vertices among the vertices of P?

Solution 1. Let A, B, and C be three distinct vertices of the 21-gon. Then each arc $\widehat{BC}$, $\widehat{AC}$, and $\widehat{AB}$ spans a certain number of sides of the 21-gon. For example, in the figure below, arcs $\widehat{BC}$, $\widehat{AC}$, and $\widehat{AB}$ span 6, 10, and 5 sides, respectively.



In general, let arcs $\widehat{BC}$, $\widehat{AC}$, and $\widehat{AB}$ span x, y, and z sides, respectively. Then x, y, and z are positive integers such that $x + y + z = 21$. Furthermore,

$$\angle BAC = \frac{x\pi}{21},$$

so $\angle BAC$ is acute if and only if $\frac{x\pi}{21} < \frac{\pi}{2}$, or $x < 21/2$. Since x is an integer, $x \le 10$. Similarly, $\angle ABC = \frac{y\pi}{21}$ and $\angle ACB = \frac{z\pi}{21}$. Hence, triangle ABC is acute if and only if $x \le 10$, $y \le 10$, and $z \le 10$.

Thus, each acute triangle ABC corresponds to a triple of positive integers $\{x, y, z\}$ such that $x+y+z = 21$, and $x \le 10$, $y \le 10$, and $z \le 10$. We list all such triples below:

$$\begin{array}{cc} \{1,\ 10,\ 10\}, & \{5,\ 6,\ 10\}, \\ \{2,\ 9,\ 10\}, & \{5,\ 7,\ 9\}, \\ \{3,\ 8,\ 10\}, & \{5,\ 8,\ 8\}, \\ \{3,\ 9,\ 9\}, & \{6,\ 6,\ 9\}, \\ \{4,\ 7,\ 10\}, & \{6,\ 7,\ 8\}, \\ \{4,\ 8,\ 9\}, & \{7,\ 7,\ 7\}. \end{array}$$

We then determine how many triangles correspond to each triple.

The triple $\{7, 7, 7\}$ corresponds to an equilateral triangle. There are 7 such triangles.

The triples $\{1, 10, 10\}$, $\{3, 9, 9\}$, $\{5, 8, 8\}$, and $\{6, 6, 9\}$ all correspond to non-equilateral isosceles triangles. There are 21 triangles ABC for each triple, giving us $4 \cdot 21 = 84$ triangles.

The remaining triples 7 triples all correspond to scalene triangles. There are 2 ways to orient each such triangle, so there are $2 \cdot 21$ triangles ABC for each triple, giving us $7 \cdot 2 \cdot 21 = 294$ triangles.

Therefore, there are a total of $7 + 84 + 294 = 385$ acute triangles.

Solution 2. Since 21 is an odd number, no triangle ABC is right, so we may count the number of acute triangles by counting all triangles and subtracting the number of obtuse triangles.

Let triangle ABC be obtuse, and without loss of generality, let BC be the longest side, so the number of sides x that arc $\widehat{BC}$ spans is between 11 and 19, inclusive. Then there are 21 ways to place side

BC, and there are $20-x$ ways to choose vertex A for each placement of side BC, so the number of obtuse triangles is

$$21\sum_{x=11}^{19}(20-x) = 21(9+8+\cdots+1) = 21\cdot\frac{9\cdot 10}{2} = 945.$$

Therefore, the number of acute triangles is

$$\binom{21}{3} - 945 = 1330 - 945 = 385.$$

12. A sequence of partitions is constructed as follows: We begin with the set $\{1,2,3,4,5\}$. Each subsequent partition is then obtained by splitting any set containing more than one element into two non-empty subsets. For example, the sequence of partitions may proceed as

$$\begin{aligned}&\{1,2,3,4,5\}\\ &\to \{1,2,5\}\cup\{3,4\}\\ &\to \{1,5\}\cup\{2\}\cup\{3,4\}\\ &\to \{1,5\}\cup\{2\}\cup\{3\}\cup\{4\}\\ &\to \{1\}\cup\{2\}\cup\{3\}\cup\{4\}\cup\{5\}.\end{aligned}$$

How many different sequences start with $\{1,2,3,4,5\}$ and end with $\{1\}\cup\{2\}\cup\{3\}\cup\{4\}\cup\{5\}$?

Solution 1. For positive integers a_1, a_2, $\dots$, a_k that sum to 5, let $f(a_1,a_2,\dots,a_k)$ denote the number of ways of starting from a partition in which the sizes of the subsets are a_1, a_2, $\dots$, a_k, and ending with the partition $\{1\}\cup\{2\}\cup\{3\}\cup\{4\}\cup\{5\}$. We seek $f(5)$.

We can split the set $\{1,2,3,4,5\}$ into either one subset with one element and one subset with four elements, of which there are $\binom{5}{1}=5$ ways, or one subset with two elements and one subset with three elements, of which there are $\binom{5}{2}=10$ ways, so

$$f(5) = 5f(1,4) + 10f(2,3).$$

To compute $f(1,4)$, suppose we start with a partition in which there is one subset with one element and one subset with four elements. We can only split the subset with four elements, into either one subset with one element and one subset with three elements, of which there are $\binom{4}{1}=4$ ways, or two subsets with two elements, of which there are $\frac{1}{2}\binom{4}{2}=3$ ways, so

$$f(1,4) = 4f(1,1,3) + 3f(1,2,2).$$

To compute $f(2,3)$, suppose we start with a partition in which there is one subset with two elements and one subset with three elements. Then we can either split the subset with two elements, of which there is only one way, or the subset with three elements, of which there are $\binom{3}{1}=3$ ways, so

$$f(2,3) = f(1,1,3) + 3f(1,2,2).$$

Similarly, we can compute that

$$\begin{aligned} f(1,1,3) &= 3f(1,1,1,2) \\ f(1,2,2) &= 2f(1,1,1,2) \\ f(1,1,1,2) &= f(1,1,1,1,1) \\ f(1,1,1,1,1) &= 1, \end{aligned}$$

so

$$\begin{aligned} f(1,1,1,2) &= 1, \\ f(1,1,3) &= 3f(1,1,1,2) = 3, \\ f(1,2,2) &= 2f(1,1,1,2) = 2, \\ f(1,4) &= 4f(1,1,3) + 3f(1,2,2) = 18, \\ f(2,3) &= f(1,1,3) + 3f(1,2,2) = 9, \\ f(5) &= 5f(1,4) + 10f(2,3) = 180. \end{aligned}$$

Solution 2. Consider the reverse process of starting with the partition $\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\}$ and ending with the partition $\{1,2,3,4,5\}$. Then each step consists of choosing two subsets and forming their union.

At the first step, the partition $\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\}$ consists of five subsets, so there are $\binom{5}{2}$ ways to obtain the second partition. This second partition will always consist of four subsets, so there are $\binom{4}{2}$ ways to obtain the third partition, and then $\binom{3}{2}$ ways to obtain the fourth partition, and then $\binom{2}{2}$ ways to obtain the fifth and final partition $\{1,2,3,4,5\}$. Therefore, the number of different sequences is

$$\binom{5}{2}\binom{4}{2}\binom{3}{2}\binom{2}{2} = 10 \cdot 6 \cdot 3 \cdot 1 = 180.$$

13. Let a and b be positive integers. Mr. X has a die, whose faces are labeled 1, 2, $\dots$, a. Mrs. Y also has a die, whose faces are labeled 1, 2, $\dots$, b. For each die, each face has an equal probability of appearing when rolled. Mr. X rolls his die once, but Mrs. Y rolls her die twice. The probability that Mrs. Y's higher roll is greater than Mr. X's roll is $49/100$. Determine $a + b$.

Solution. If $b \ge a$, then Mrs. Y clearly has the advantage of rolling a higher number, and the probability of her obtaining the higher roll would be at least $1/2$. Therefore, $b < a$.

Let $1 \le k \le b$. Then there are $2k - 1$ possible rolls in which Mrs. Y's higher roll is equal to k: Either the first roll is equal to k, which means the second roll is one of the numbers 1, 2, $\dots$, k, or the second roll is equal to k, which means the first roll is one of the numbers 1, 2, $\dots$, k. But we have double-counted the possibility that Mrs. Y rolls k twice, leading to the total of $2k - 1$. Therefore, the probability that Mrs. Y's higher roll is k is $(2k - 1)/b^2$.

And if Mrs. Y's higher roll is k, then Mr. X's roll must be one of 1, 2, $\dots$, $k - 1$, which occurs with probability $(k - 1)/a$. Therefore, the probability that Mrs. Y's higher roll is greater than Mr. X's roll is

$$\begin{aligned}\sum_{k=1}^{b}\frac{2k-1}{b^2}\cdot\frac{k-1}{a} &= \frac{1}{ab^2}\sum_{k=1}^{b}(2k-1)(k-1)\\ &= \frac{1}{ab^2}\sum_{k=1}^{b}(2k^2-3k+1)\\ &= \frac{1}{ab^2}\left(2\cdot\frac{b(b+2)(2b-1)}{6}-3\cdot\frac{b(b+1)}{2}+b\right)\\ &= \frac{1}{ab^2}\cdot\frac{b(b-1)(4b+1)}{6}\\ &= \frac{(b-1)(4b+1)}{6ab}.\end{aligned}$$

Then

$$\frac{(b-1)(4b+1)}{6ab}=\frac{49}{100} \quad\Rightarrow\quad 50(b-1)(4b+1)=147ab.$$

Since b is relatively prime to both $b-1$ and $4b+1$, b must divide 50. Furthermore, since 7 divides 147 and 7 is relatively prime to 50, 7 divides $(b-1)(4b+1)$. Among the factors of 50, $(b-1)(4b+1)$ is divisible by 7 only for $b=1$, 5, and 50. We have that

$$a=\frac{50(b-1)(4b+1)}{147b},$$

and substituting these values, we find that the only solution in positive integers is $b=50$ and $a=67$, so the final answer is $a+b=117$.

14. The distinct complex numbers a, b, and c satisfy

$$\begin{aligned}a^3 &= 7b^2+7c^2+1,\\ b^3 &= 7a^2+7c^2+1,\\ c^3 &= 7a^2+7b^2+1.\end{aligned}$$

Find the product abc.

Solution 1. Subtracting the second equation from the first equation, we get

$$\begin{aligned} & a^3-b^3=-7(a^2-b^2)\\ \Rightarrow\quad & (a-b)(a^2+ab+b^2)=-7(a-b)(a+b).\end{aligned}$$

Since $a\neq b$, we can divide both sides by $a-b$ to get $a^2+ab+b^2=-7a-7b$. By subtracting the other pairs of equations, we obtain the equations

$$\begin{aligned}a^2+ab+b^2 &= -7a-7b,\\ a^2+ac+c^2 &= -7a-7c,\\ b^2+bc+c^2 &= -7b-7c.\end{aligned}$$

In this new system, subtracting the second equation from the first equation, we get

$$\begin{aligned} ab - ac + b^2 - c^2 &= 7c - 7b \\ \Rightarrow \quad a(b-c) + (b-c)(b+c) &= -7(b-c) \\ \Rightarrow \quad (b+c)(a+b+c) &= -7(b-c). \end{aligned}$$

Since $b \neq c$, we can divide both sides by $b - c$ to get $a + b + c = -7$. Adding the three equations in the new system, we get

$$2a^2 + 2b^2 + 2c^2 + ab + ac + bc = -14(a + b + c) = 98.$$

But

$$a^2 + b^2 + c^2 + 2(ab + ac + bc) = (a + b + c)^2 = 49,$$

so

$$2a^2 + 2b^2 + 2c^2 + 4(ab + ac + bc) = 98.$$

Therefore, $ab + ac + bc = 0$ and $a^2 + b^2 + c^2 = 49$.

Finally,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) = (-7) \cdot 49 = -343.$$

Adding the three original equations, we get

$$a^3 + b^3 + c^3 = 14(a^2 + b^2 + c^2) + 3 = 14 \cdot 49 + 3 = 689.$$

Therefore, $abc = (689 + 343)/3 = 344$.

Solution 2. From the given equations,

$$a^3 + 7a^2 = b^3 + 7b^2 = c^3 + 7c^2 = 7(a^2 + b^2 + c^2) + 1.$$

Let $p = a^2 + b^2 + c^2$. Then a, b, and c are all roots of the cubic equation

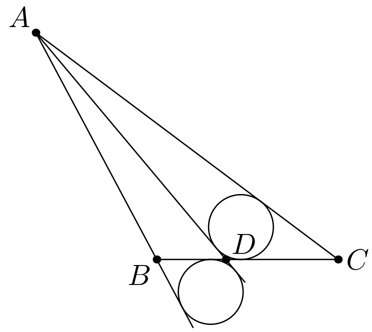
$$x^3 + 7x^2 - 7p - 1 = 0,$$

so by Vieta's formulas, $a + b + c = -7$, $ab + ac + bc = 0$, and $abc = 7p + 1$. Then

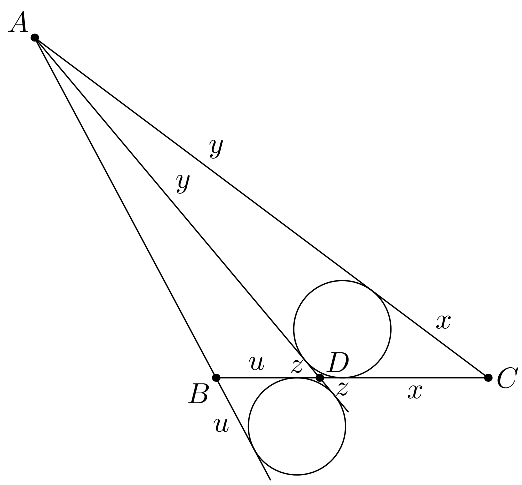
$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc) = 49,$$

so $abc = 7p + 1 = 7(a^2 + b^2 + c^2) + 1 = 7 \cdot 49 + 1 = 344$.

15. In triangle ABC, $AB = 17$, $AC = 25$, and $BC = 12$. Point D is chosen on side BC such that the incircle of triangle ACD and the excircle of triangle ABD, opposite vertex A, have the same radius. This common radius may be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.



Solution. Let x, y, and z denote the lengths of the tangents from C, A, and D to the incircle of triangle ACD, respectively. Let u denote the length of the tangent from B to the A-excircle of triangle ABD. Since the two circles have the same radius, z is also the length of the tangent from D to the A-excircle of triangle ABD. The length of the tangent from A to the A-excircle of triangle ABD is equal to both $AB + u$ and $y + 2z$, so $AB = y + 2z - u$.



Let r denote the common radius. Then from triangle ACD,

$$[ACD] = r \cdot \frac{AC + AD + CD}{2} = r \cdot \frac{x + y + y + z + x + z}{2} = r(x + y + z),$$

and from triangle ABD,

$$[ABD] = r \cdot \frac{AB + AD - BD}{2} = r \cdot \frac{y + 2z - u + y + z - (u + z)}{2} = r(y + z - u),$$

so

$$\begin{aligned}[ABC] &= [ACD] + [ABD] \\ &= r(x + 2y + 2z - u) \\ &= r[(x + y) + (y + 2z - u)] \\ &= r(AC + AB).\end{aligned}$$

Therefore,

$$r = \frac{[ABC]}{AB + AC}.$$

From Heron's formula, $[ABC] = 90$, so

$$r = \frac{90}{17 + 25} = \frac{90}{42} = \frac{15}{7},$$

and the final answer is $15 + 7 = 22$.