New Zealand Mathematical Olympiad Committee

Geometric Inequalities

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1 Introduction

These notes present some standard tools for proving geometric inequalities, together with several typical examples of their use.

We assume throughout that our triangles, quadrilaterals and other polygons are *non-degenerate*; that is, having no three vertices collinear. Unless otherwise specified, a point *inside* a polygon will be *strictly inside*; that is, not on the polygon's boundary.

2 Elementary facts

These first four facts are basic.

Tool 1 (Triangle Inequality)**.** The sidelengths of a triangle ABC satisfy $AB + BC > AC$.

Tool 2. The area of a triangle ABC is not greater than $\frac{1}{2}AB \cdot AC$.

Tool 3. In a triangle ABC the inequality $AC < BC$ holds if and only if $\angle ABC < \angle BAC$.

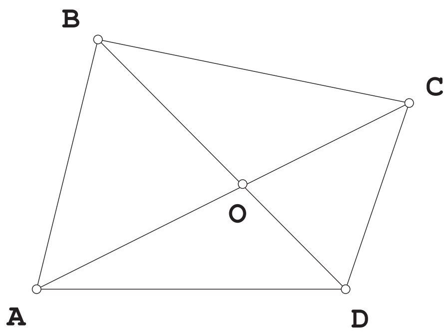
Tool 4. The length of the orthogonal projection of a segment s onto a line ℓ is not greater than the length of s.

Our next results are equally useful but slightly less obvious. For these we give proofs.

Tool 5. Let $ABCD$ be a convex quadrilateral. Then $AB + CD < AC + BD$.

That is, the combined length of the diagonals of a convex quadrilateral is greater than the combined length of any pair of its opposite sides.

Proof. Since the quadrilateral is convex, the two diagonals intersect. Let O be their point of intersection.



Then, twice applying the Triangle Inequality, we get

$$AB < AO + BO, \qquad CD < OC + OD.$$

Adding these two inequalities together gives

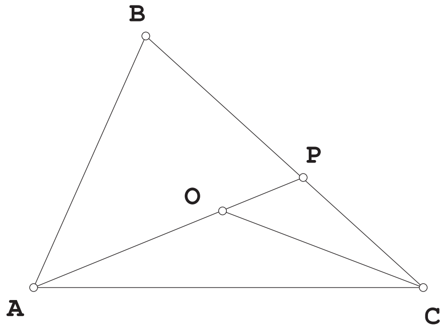
$$AB + CD < AO + BO + OC + OD = AC + BD.$$

□

Tool 6. Let O be a point inside a triangle ABC. (For this inequality, the point O can lie on a side of ABC, so long as it is not actually one of the vertices.) Then

$$AO + OC < AB + BC.$$

Proof: Let P be the intersection of AO with BC.



Twice applying the Triangle Inequality gives

$$AO + OC < AO + OP + PC = AP + PC < AB + BP + PC = AB + AC.$$

Example 1. Let O be a point inside a triangle ABC. Show that

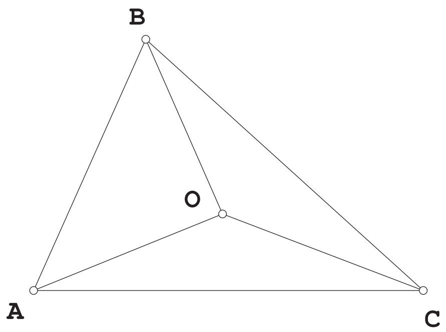
$$p < AO + BO + CO < 2p,$$

where p is the semiperimeter (that is, half the perimeter) of ABC.

Solution. The left inequality follows from three applications of the Triangle Inequality. Indeed, we have

$$\begin{array}{rcl} AB & < & OA + OB, \\ BC & < & OB + OC, \\ CA & < & OC + OA. \end{array}$$

Adding all these together, we get $2p < 2(AO + BO + CO)$ as required.



For the right inequality we need Tool 6. We get:

$$\begin{array}{rcl} OA + OB & < & CA + CB, \\ OB + OC & < & AB + AC, \\ OC + OA & < & BC + BA. \end{array}$$

Adding them all together gives $2(AO + BO + CO) < 4p$. $\square$

Comment: The idea here is to "symmetrize" some known inequalities. Most symmetric inequalities are proved in this way.

The next tool is used when a certain inequality must be proved for all values a, b, and c, which are known to be the lengths of the sides of a triangle. This tool allows us to replace a, b, and c, which are not arbitrary, by arbitrary positive x, y, and z.

Tool 7. Numbers a, b and c are the lengths of the sides of a triangle if and only if there exist positive numbers x, y, and z such that $a = x + y$, $b = y + z$ and $c = z + x$.

Proof. If a, b and c are the sides of a triangle, then let

$$\begin{aligned} x &= \frac{a+c-b}{2} > 0, \\ y &= \frac{a+b-c}{2} > 0, \\ z &= \frac{b+c-a}{2} > 0. \end{aligned}$$

This numbers clearly satisfy $a = x + y$, $b = y + z$ and $c = z + x$. On the other hand, if such x, y, z exist, then all triangle inequalities for the numbers $a = x + y$, $b = y + z$ and $c = z + x$ are satisfied. For example, $a + b = (x + y) + (y + z) = (x + z) + 2y = c + 2y > c$ as $y > 0$. □

Example 2 (IMO 1983)**.** Let a, b and c be the lengths of the sides of some triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0,$$

and determine when equality occurs.

This was the most difficult problem of its IMO. However, Tool 7 makes it trivial, in the sense that it reduces it to some very well-known ordinary inequalities.

Solution. We substitute $a = x + y$, $b = y + z$ and $c = z + x$. Since a, b and c are the sides of a triangle, the numbers x, y and z are positive. Simplifying, we get

$$x^3y + y^3z + z^3x \geq x^2yz + xy^2z + xyz^2.$$

This is just a case of Muirhead's inequality. For those unfamiliar with Muirhead we suggest the following alternative argument. Since the inequality is homogeneous, it suffices to prove it only for those x, y, z for which $xyz = 1$. In this case, our inequality takes the form

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} \geq x + y + z,$$

which follows directly from the Rearrangement Inequality: indeed, without loss of generality, we may assume $x \geq y \geq z$ and hence $x^2 \geq y^2 \geq z^2$, while we have $\frac{1}{x} \leq \frac{1}{y} \leq \frac{1}{z}$.

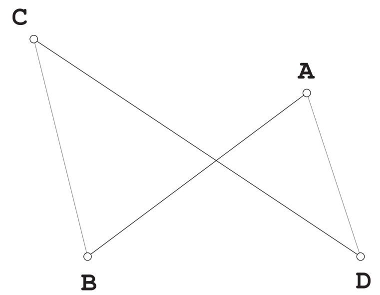
Equality occurs when $x = y = z$; that is, when $a = b = c$. □

3 Three examples

Example 3. A set of $2n$ distinct points lie in the plane, n coloured red and n blue. Show that it is possible to connect each red point to a blue point, with no two red points paired with the same blue point, so that no two of the n segments thus created intersect.

Solution. This problem might not seem to be about geometric inequalities – but that impression is premature!

There are $n!$ ways of pairing the red and blue points. How can we hit on the right one? Our solution gives an algorithm for doing so. Start with an arbitrary pairing configuration. Suppose that two of the segments in this configuration intersect, say AB and CD where points A and C are red and points B and D are blue.



Then we can obtain a new pairing configuration by instead connecting A with D and C with B. By Tool 5, the the sum of the n connecting segments in the new pairing is less than that in the pairing we started with.

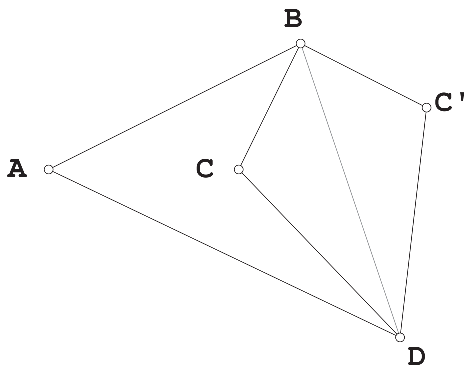
We repeat this process as long as we can. Since the sum of the configuration's n connecting segments decreases at each step, we never return to a configuration we have encountered previously in the process. So, since the number of possible configuration is $n!$, which is finite, this process must terminate. This happens only when we reach a pairing in which no two segments intersect.

Here's a shorter and less constructive version of the same argument. Consider a configuration with minimal total combined length of segments. Such a minimal configuration exists, because we have only a finite number of possibilities – the minimal configuration might not be unique, but that does not matter. If two segments of this configuration were to intersect, then we could swap endpoints as before to obtain a new configuration with smaller total combined segment length, a contradiction. So no two segments in a minimal configuration can intersect. □

Example 4 (IMO Shortlist)**.** The area of a quadrilateral with sidelengths a, b, c, and d is equal to S. Prove that

$$S \leq \frac{a+c}{2} \cdot \frac{b+d}{2}.$$

Solution. First, suppose the quadrilateral $ABCD$ is not convex. Then one of its diagonals, say BD (see Figure) has no common points with the interior of the quadrilateral. Reflecting C about BD gives a convex quadrilateral $ABC'D$ with the same sides but of greater area.



Hence we can assume without loss of generality that the quadrilateral is convex.

Now let us divide our quadrilateral by one of the diagonals into two triangles and use Tool 2. These triangles will have areas not greater than, respectively, $ab/2$ and $cd/2$. Thus $2S \leq ab + cd$.

Doing the same with the other diagonal yields $2S \leq bc+da$. Adding these two inequalities (we are symmetrizing again!) gives

$$4S \leq ab + cd + bc + da = (a+c)(b+d),$$

which is equivalent to what we have to prove. □

Example 5 (IMO 1964)**.** Let a, b, c be the sides of a triangle. Prove that

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc.$$

Solution. For this sort of problem Tool 7 is just magic. Let us use it again. Indeed, substituting $a = x + y$, $b = y + z$ and $c = z + x$ we will have a new inequality

$$2(x+y)^2 z + 2(y+z)^2 x + 2(z+x)^2 y \leq 3(x+y)(y+z)(z+x),$$

which has to be proved for $x \geq 0$, $y \geq 0$, $z \geq 0$. This is a great improvement, since these constraints on indeterminates are much easier to handle. Actually at this stage we can forget the original geometric context, and approach this as usual for inequalities. Simplifying, we get

$$x^2 y + y^2 z + z^2 x + xy^2 + yz^2 + zx^2 \geq 6xyz;$$

this splits into the three inequalities $(x^2 + y^2)z \geq 2xyz$, $(y^2 + z^2)x \geq 2xyz$, and $(z^2 + x^2)y \geq 2xyz$, which all follow from AM-GM. □

4 Ptolemy's inequality

Our final and strongest fact is

Tool 8 (Ptolemy's inequality)**.** Let A, B, C, D be arbitrary points in the plane, no three of which are collinear. Then

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$

Equality holds if and only if the points A, B, C, D are concyclic, with exactly one of B, D lying on each of the two arcs determined by the points A, C.

Example 6. Let ABC be an equilateral triangle and P be an arbitrary point in the plane. Prove that the lengths of PA, PB, PC are always the lengths of the sides of a triangle, and determine when this triangle is degenerate.

Solution. By Ptolemy's inequality

$$PA \cdot BC + PC \cdot AB \geq PB \cdot AC,$$

and since $AB = AC = BC$, this becomes

$$PA + PC \geq PB$$

with equality when P is on the circumcircle of ABC. Similarly, we can get

$$PA + PB \geq PC.$$

The triangle becomes degenerate only when P is on the circumcircle of ABC. □

Example 7. Let $M, A_1, A_2, \ldots, A_n$ $(n \geq 3)$ be distinct points in the plane such that

$$A_1A_2 = A_2A_3 = \cdots = A_{n-1}A_n = A_nA_1.$$

Prove the inequality

$$\frac{1}{MA_1 \cdot MA_2} + \frac{1}{MA_2 \cdot MA_3} + \cdots + \frac{1}{MA_{n-1} \cdot MA_n} \geq \frac{1}{MA_1 \cdot MA_n},$$

and determine all cases when equality occurs.

Solution 1. For each integer k between 1 and $n-1$ inclusive, applying Ptolemy's inequality to the quadrilateral $MA_1A_kA_{k+1}$ gives

$$MA_1 \cdot A_kA_{k+1} + A_1A_k \cdot MA_{k+1} \geq A_1A_{k+1} \cdot MA_k,$$

(which for $k = 1$ degenerates into an equality). We divide this expression by $MA_1 \cdot MA_k \cdot MA_{k+1}$ to get

$$\frac{A_kA_{k+1}}{MA_k \cdot MA_{k+1}} \geq \frac{A_1A_{k+1}}{MA_1 \cdot MA_{k+1}} - \frac{A_1A_k}{MA_1 \cdot MA_k}.$$

Summing these $n-1$ inequalities, we obtain

$$\frac{A_1A_2}{MA_1 \cdot MA_2} + \frac{A_2A_3}{MA_2 \cdot MA_3} + \cdots + \frac{A_{n-1}A_n}{MA_{n-1} \cdot MA_n} \geq$$

$$\frac{A_1A_n}{MA_1 \cdot MA_n} - \frac{A_1A_1}{MA_1 \cdot MA_1} = \frac{A_1A_n}{MA_1 \cdot MA_n}.$$

As $A_1A_2 = A_2A_3 = \cdots = A_nA_1$ we get the desired inequality.

Equality holds overall if and only if it holds for each inequality that was added; that is, for all $k = 1, 2, \ldots, n-1$. This happens when the points A_1, A_k, A_{k+1}, M lie on a circle in that particular order. Equivalently, this happens when the points $M, A_1, \ldots, A_n$ lie on a circle in that order. In this case $A_1, A_2, \ldots, A_n$ is a regular n-gon and M belongs to the (shorter) arc $\widehat{A_1A_n}$. □

Solution 2. Consider an inversion i with pole M and any coefficient r. Let $A_1', A_2', \ldots, A_n'$ be the images of $A_1, A_2, \ldots, A_n$, respectively, under this inversion. Applying the Triangle Inequality (in fact, several times) for the points $A_1', A_2', \ldots, A_n'$, we get

$$A_1'A_2' + A_2'A_3' + \cdots + A_{n-1}'A_n' \geq A_1'A_n'. \tag{1}$$

It is well-known (or easy to prove) how distances between images of points under inversion can be expressed. Here, if X, Y are any two points different from M, with images X', Y' under i, then

$$X'Y' = \frac{r^2 \cdot XY}{MX \cdot MY}.$$

This formula can be applied to any pair of points in $A_1, A_2, \ldots, A_n$ because all these points are different from M. So we rewrite (1) in the form

$$\frac{r^2 \cdot A_1A_2}{MA_1 \cdot MA_2} + \frac{r^2 \cdot A_2A_3}{MA_2 \cdot MA_3} + \cdots + \frac{r^2 \cdot A_{n-1}A_n}{MA_{n-1} \cdot MA_n} \geq \frac{r^2 \cdot A_1A_n}{MA_1 \cdot MA_n}.$$

Since $A_1A_2 = A_2A_3 = \cdots = A_{n-1}A_n = A_nA_1$, the latter yields

$$\frac{1}{MA_1 \cdot MA_2} + \frac{1}{MA_2 \cdot MA_3} + \cdots + \frac{1}{MA_{n-1} \cdot MA_n} \geq \frac{1}{MA_1 \cdot MA_n}, \tag{2}$$

as desired.

It is clear from the above argument that (1) is strict precisely when (2) is. So to finish off let us check when (1) is an equality. This happens if and only if ${A'}_1, {A'}_2, \ldots, {A'}_n$ are points on a line ℓ arranged in this order. Now, the preimage of ℓ under i can be either

1. (if ℓ passes through the pole M) the same line; or,
2. (if ℓ does not contain M) a circle C through M.

For the first case, ${A'}_1, {A'}_2, \ldots, {A'}_n$ lie on ℓ in that order precisely if $M, A_1, A_2, \ldots, A_n$ lie in the order

$$A_k, A_{k-1}, \ldots, A_1, M, A_n, \ldots, A_{k+1}$$

on ℓ, for some k. However, given the condition $A_1A_2 = A_2A_3 = \cdots = A_{n-1}A_n = A_nA_1$, this is impossible.

In the other case, the condition is satisfied if and only if $M, A_1, A_2, \ldots, A_n$ are concyclic, with the points $A_1, A_2, \ldots, A_n$ arranged in that order in one of the two possible directions (clockwise or anticlockwise), and with M lying on the arc $\widehat{A_1A_n}$ of C that does not contain any of the points $A_2, A_3, \ldots, A_{n-1}$.

We conclude that equality occurs if and only if the points $M, A_1, A_2, \ldots, A_n$ lie on a circle, arranged in that order with respect to one of the two possible directions. □

Comment 1: The argument does not demand that the given points lie in a plane. Nothing changes if they are in three-dimensional space.

Comment 2: It follows from the above solution that arbitrary distinct points $M, A_1, A_2, \ldots, A_n$ satisfy the inequality

$$\frac{A_1A_2}{MA_1 \cdot MA_2} + \frac{A_2A_3}{MA_2 \cdot MA_3} + \cdots + \frac{A_{n-1}A_n}{MA_{n-1} \cdot MA_n} \geq \frac{A_1A_n}{MA_1 \cdot MA_n},$$

and that the only equality cases are those listed above.

5 Problems

1. (USSR 1984) A straight line contains the four points A, B, C, D in that order. Prove that for every point E not on the line,
$$AE + ED + |AB - CD| > BE + EC.$$
2. (Czechoslovakian MO 1976) One convex polygon is situated inside another convex polygon. Prove that the perimeter of the inner polygon is smaller than the perimeter of the outer one.
3. How many sides can have a convex polygon if all its diagonals have equal length?
4. A point M is given inside a triangle ABC whose area is S. Prove that
$$4S \leq AM \cdot BC + BM \cdot AC + CM \cdot AB.$$
5. (Yugoslavian MO 1975) The midpoints of adjacent sides of a convex polygon are joined by segments which become the sides of a new polygon. Show that its area is not less than half the area of the original polygon.
6. A parallelogram is inside a triangle. Prove that the area of the parallelogram is not greater than half the area of the triangle.
7. (IMO 1966) Points M, K, L lie on the sides AB, BC, CA respectively of a triangle ABC. Prove that at least one of the triangles MAL, KBM, LCK has area not greater than a quarter of the area of ABC.
8. (a) A convex polygon has area S and perimeter P. Prove that it contains a circle of radius greater than S/P.
   (b) A convex polygon of area S_1 and perimeter P_1 contains a convex polygon of area S_2 and perimeter P_2. Prove that $2S_1/P_1 > S_2/P_2$.
9. Suppose that a, b, and c are the lengths of the sides of a triangle. Show that
$$(a + b - c)(a - b + c)(-a + b + c) \leq abc.$$
10. (IMO Shortlist 1997) Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$ and $EF = FA$. Prove that
$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$
11. Let $A_1A_2A_3A_4$ be a square and P be an arbitrary point in the plane. Of the distances PA_1, PA_2, PA_3, and PA_4, let M be the maximal and m the minimal. Prove that
$$PA_1 + PA_2 + PA_3 + PA_4 \geq (1 + \sqrt{2})M + m,$$
with equality when P is on the circumcircle of the square and strict inequality otherwise.
12. (IMO 1995) Let $ABCDEF$ be a hexagon such that $AB = BC = CD$, $DE = EF = FA$, and $\angle BCD = \angle EFA = 60°$. Let G and H be two arbitrary points. Show that
$$AG + BG + GH + DH + EH \geq CF.$$

13. A broken line of length L lies inside a square of sidelength 1. The positive number ϵ satisfies: for each point X inside the square, there is some point on the broken line whose distance from X is less than ϵ. Prove that

$$L \geq \frac{1}{2\epsilon} - \frac{\pi}{2}\epsilon.$$

14. A heptagon $A_1A_2\ldots A_7$ is inscribed in a circle, such that the centre of the circle is contained in the heptagon. Prove that

$$\angle A_1 + \angle A_3 + \angle A_5 < 450^\circ.$$

15. (IMO Shortlist 1996) Let $ABCD$ be a convex quadrilateral, and let R_A, R_B, R_C and R_D denote the circumradii of the triangles DAB, ABC, BCD and CDA respectively. Prove that $R_A + R_C > R_B + R_D$ if and only if $\angle A + \angle C > \angle B + \angle D$.

April 21, 2009

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