Solutions

1. Consider the dilation carrying ω to the excircle opposite to A. Point E is mapped to F, which must also be the point of tangency of the excircle to BC.

2. Let the excircle Ω be tangent to BC at F, and G a point such that FG is diameter of Ω. Let ω be the incircle of $\triangle ABC$.Then the homothety with centre A carrying ω to Ω maps E to G, so A, E, G are collinear. Hence E is the intersection of AG and DF. Therefore E lies on the line connecting the midpoints of AG, DF which is MI_a.

3. The dilation with centre P carrying ω to Γ sends K to a point M on arc AB not containing P. Line AB is sent to a line l parallel to AB and tangent to Γ at M. Angle-chasing finishes the problem.

4. (Proof from Yufei Zhao's notes on Lemmas in Eucledian Geometry). Extend KE to meet Γ at M. M is the midpoint of arc BC (see problem 3) hence A, I, M are collinear. Let EI intersect ω at F'. We will show AF' is tangent to ω.
Since $\angle EF'K, \angle MAK$ subtend arcs EK, MK in circles ω, Γ and MK is the image of KE under the homothety carrying ω onto Γ it follows that $\angle EF'K = \angle MAK$ so A, K, I, F' are concyclic.
Since $\angle BCM = \angle CBM = \angle CKM$ it follows that $\triangle MEC \sim \triangle MCK$ hence $MI^2 = MC^2 = ME \cdot MK$, so MI is tangent to the circumcircle of $\triangle KIM$. Hence $\angle AF'K = \angle AIK = \angle IEK$ so AF' is tangent to ω and $F \equiv F'$.

5. Notice that $\angle O_1DO_2 = 90°$. Let ω_1 be tangent to AD, DC at F, E and ω_2 be tangent to AD, BD at H, G. Then GH, FE intersect at I. The rest is a simple trig bash.

6. Let T be the insimilicentre of ω and Γ. By the Monge-d'Alembert Theorem A', D, T are collinear. Hence $A'D, B'E, C'F$ intersect at T.

7. $B'C'$ and BC intersect at N; they are polars of A, A' respectively. Hence AA' is the polar of N. [**This is a useful fact!**] Similarly BB' is the polar of M. Hence MN is the polar of N. The result follows.

8. Let ω be tangent to BC at D. AD intersect PQ, ω at K, S. Considering the dilation carrying the incircle of $\triangle APQ$ to ω it follows that $PK = RQ$ and $MK = MR$. Also $\angle RSK = 90°$ hence $MR = MK = MS$ and MS is tangent to ω. AD is the polar of T with respect to ω hence TS is tangent to ω. The result follows.

9. Let BI intersect EF at X', EF intersect BC at T, and D be the point of tangency of ω with BC. Then $(T, D; B, C)$ is harmonic and XB is the angle bisector of $\angle FX'D$ hence $X'C \perp BX$. Hence $X \equiv X'$ and X, Y lie on EF.
Let ID intersect EF at N'. Let P, Q be points on AB, AC so that N lies on PQ and $PQ||BC$. The projections of I onto AF, EE, FE are collinear, so by Simpson's theorem I, P, A, Q are concyclic. Since $\angle PAI = \angle QAI$ it follows that $IP = IQ$ and $N'P = N'Q$ hence A, N', M are collinear and $N' \equiv N$. So N lies on ID.
By angle chasing I is the incentre of $\triangle YXD$ and $\triangle DYX \sim \triangle ABC$. Since DN is the angle bisector of $\angle YDX$ (as it contains I it follows that $\frac{NX}{NY} = \frac{DX}{DY} = \frac{AC}{AB}$.

10. Let U, V, W be centers $\omega_a, \omega_b, \omega_c$ respectively. Let R be the intersection of EF, VW; S the intersection of ED, VW, T the intersection of FD, UV. (Some of these might be points of infinity but that's ok). Then R, S, T are the exsimilicentres between pairs of the three circles. Hence R

lies on BC, S lies on AC, T lies on AB (as they are common external tangents between the pairs of circles). By Monge's Theorem R, S, T are collinear, hence $\triangle ABC, \triangle DEF$ are perspective with respect to a line. By Desargues' theorem these triangles are perspective with respect to a point. The result follows.

11. Let $\Gamma, \omega_1(O_1), \omega_2(O_2), \omega_3(O_3), \omega_4(O_4)$ be the circumcircles of the $ABCD, \triangle APB, \triangle BPC, \triangle CPD$, $\triangle DPA$, respectively ($\omega(O_1)$ means circle ω_1 with centre O_1). Let $\omega_1 \cap \omega_3 = P, N$ and $\omega_2 \cap \omega_4 = P, M$.Then I, the point of intersection of O_1O_3 and O_2O_4 lies on the perpendicular bisectors of PM, PN hence is the centre of the circumcircle ζ of $\triangle PNM$. Let $AD \cap BC = F, AB \cap CD = G$. Then $OE \perp FG$ by Brocard's Theorem, and it suffices to show $OI \perp FG$ (as then O, I, E are collinear). By the radical axis theorem, PM, AD, BC are concurrent at F and PN, AB, CD are concurrent at G. Since F lies on the radical axes of circles ζ, ω_3 and ω_3, Γ it follows that F lies on the radical axis of ω_3, Γ. Similarly G lies on the radical axis of ω_3, Γ. So $OE \perp FG$ and the result follows.

12. By Thebault's theorem O_1, I, O_2 are collinear. After some angle chasing we get I is the midpoint of O_1O_2. Assume l passes through M. Then $\angle O_1MO_2 = 90°$. Also $\angle O_1DO_2 = 90°$ hence O_1, D, M, O_2 lie on a circle with centre I. Hence $ID = IM$. Let the sides of the triangle be a, b, c and F be the point of tangency of the incircle with BC. Then $2BF = BD + DM$ hence $a + c - b = c \cdot \frac{a^2+c^2-b^2}{2ac} + \frac{a}{2}$. Simplifying we get $c + b = 2a$. **Note**: You should not be afraid of using trig bash in your solutions. However, first try to look for a purely geometric solution; use trig bash only when you know where it is going (and not just thoughtless length calculations).

13. Let $\{K\} \equiv CI \cap FE, \{G\} \equiv BI \cap EF$. Then $BK \perp CK$ and $BG \perp CG$. Hence $\{H\} \equiv BK \cap CG$. Let J be the midpoint of EF. Let P' be the intersection by HJ and DM. It suffices to prove that P' is the midpoint of DM.
Let S be the projection of H onto EF and Y the intersection of HD and EF. Since $MD||HS$, in order to prove P is the midpoint of DM, it suffices to prove the pencil $H(M, J, Y, S)$ is harmonic, i.e. that $(M, Y'J, S)$ is harmonic. Since $MD||JI||HS$, considering the pencil $P_\infty(M, J, Y, S)$ and intersecting it with HD (where P_∞ is the intersection of MD and HS) it suffices to prove $D, Y; I, H)$ is harmonic.
Since BG, CH and ID are altitudes of $\triangle BIC$ it follows that EI is the angle bisector of $\angle KGD$. Since $\angle HEI = 90°$ it follows that $D, Y; I, H)$ is harmonic and the result follows.

14. Let $\Gamma(O)$ be the circle tangent to the lines AB, BC, AD and let $\omega_1, \omega_2, \omega_3$ be the incircles of triangles APD, BPC and CPD respectively.
Since A is the exsimilicenter of ω_1 and Γ and K is the insimilicenter of ω_1 and ω_3, by the Monge-D'Alembert theorem, the line AK intersects the line OI at the insimilicenter of Γ and ω_3. Similarly, line BK intersects OI at the same insimilicenter F of Γ and ω_3. It suffices to prove that E lies on the line OI.
By properties of tangents it follows that $AP + CD = PC + AD$ and $BP + CD = BC + PD$ so there exist circles ω_5, ω_6 inscribed in quadrilaterals $APCD, BCPD$. Let X be the exsimilicentre of ω_1, ω_3 and Y the exsimilicentre of ω_2, ω_3. By Monge-D'Alembert theorem applied to circles $\omega_1, \omega_3, \omega_5$ and to circles $\omega_2, \omega_3, \omega_5$ it follows that A, C, X and B, D, Y are collinear. Let E' be the exsimilicentre of Γ and ω_3. By the Monge's theorem applied to $\Gamma, \omega_1, \omega_3$ it follows that A, X, E' are collinear. So E' lies on AC and on OI. Similarly E' lies on BD and OI. Hence $E' \equiv F$ and E, O, I are collinear.

15. [Proof by Ivan on AOPS] Let AB, CD meet at X, AD, BC meet at Y, let k meet AB, DC, AD, BC at P, Q, R, S respectively. Using the tangency properties with respect to k we get:

$$BA + AD = BA + AR - DR = BP - DR = BS - DQ = BC + CQ - DQ = BC + CD$$

Let k_1, k_2 meet AC at J, L respectively. Then $AB + JC = BC + AJ$ and $DA + LC = DC + LA$. Adding and using $BA + AD = BC + CD$ we get $JC + LC = AL + AJ$ hence $AL = JC$.
Let the excircle of $\triangle ABC$ on the side AC be k_3, and the *excircle* of $\triangle ADC$ on the side AC be k_4. Then k_3, k_4 meet AC at L and J.
Construct the tangent of k which is parallel to AC (and on the same side of k as AC). Let that tangent meet k at Z. The dilation about B takes k_3 to k and L to Z. The negative dilation about D takes k_4 to k and J to Z. Hence BL and DJ meet at Z.
Construct the two missing tangents to k_1 and k_2 which are parallel to AC, let the points of tangency be M and N respectively. Similar dilation arguments show that B, M, L, Z are collinear and D, N, J, Z are also collinear.
Since JM and LN are parallel and are diameters of k_1 and k_2, then they meet at the centre of dilation which takes k_1 to k_2, which we know is the point Z. Hence Z is the intersection of the common external tangents of k_1, k_2.

16. Let Γ be the circumcircle of $\triangle ABC$. Let ω_1 intersect Γ at B, D and DC at D, E. Then $\angle XED = 180° - \angle XBD = \angle ACK$ so $XE||AC$. Simple angle chasing gives $\angle AXY = \angle AYX$; let $\angle AXY = \alpha$. Then $\angle YXE = \alpha$, XY is tangent to ω_1 at X so $\angle XKC = \alpha$ and $XYCD$ is cyclic. By the radical axis theorem applied to Γ and the circumcircles of $\triangle AXY$ and $XYCD$ it follows that AQ, XY, CD are concurrent at a point O. Since $XE||YC$ and XY is tangent to ω_1, ω_2 then the homothety with centre O' taking ω_1 to ω_2 takes X to Y and E to C, where O' is the exsimilicentre of ω_1, ω_2. Since $XY \cap EC = O$ it follows that O is the exsimilicentre of ω_1, ω_2.
Simple angle chasing gives the circumcircle ζ of $\triangle XYK$ is tangent to OK at K. Since $\angle XKP = \angle PXY, \angle YKP = \angle XYP$ it follows that $\angle XKY = \angle XYP + \angle PXY = \angle XYB = \angle AXY$ so AB is tangent to ζ at X. Similarly AC is tangent to ζ at Y. Hence KA is the polar of O with respect to ζ (since XY, CD are polars of K, A and intersect at O). Let KA intersect ζ at K, R and XY at S. Then $(O, S; X, Y)$ is harmonic (proved in previous problems) and if M is the midpoint of XY then $OR^2 = OK^2 = OQ \cdot OA = OX \cdot OY$ (power of a point) $= OS \times \cdot OM$ (property of harmonic division).
Consider the inversion with centre O that fixes points R, K. The line AK is carried to a circle passing through R, K, O and if this circle intersects OA, OS at Q', M' respectively then $OR^2 = OK^2 = OQ' \cdot OA = OS \times \cdot OM'$. Hence $Q \equiv Q'$ and $M \equiv M'$ and $OQRMK$ is cyclic. Also K, P, M are collinear (as M lies on the radical axis of ω_1, ω_2). Hence $\angle QKP = \angle QKM = \angle QOY$. Since $AY^2 = AR \cdot AK = AQ \cdot AO$ it follows the circumcircle of $\triangle OQY$ is tangent to AC and $\angle QKP = \angle QOY = \angle QYA = \angle QXA$ and we are done.