# CHAPTER 1

# CEVA'S THEOREM AND MENELAUS'S THEOREM

The purpose of this chapter is to develop a few results that may be used in later chapters. We will begin with a simple but useful theorem concerning the area ratio of two triangles with a common side. With this theorem in hand, we prove the famous Ceva's theorem and Menelaus's theorem. The converses of these two theorems guarantee the existence of the centroid, incenter and orthocenter of any given triangle. As we will see in the examples, Menelaus's theorem can be used to prove the Simson's theorem. Based on this, we will then go on to discuss the Ptolemy's theorem. These theorems are of the same importance.

**Notation.** Given a triangle *ABC*, we denote the length of three sides by $a = BC$, $b = CA$, $c = AB$. The length of three medians are denoted by $m_a$, $m_b$, $m_c$, the length of three altitudes by $h_a$, $h_b$, $h_c$, and the length of three angle bisectors by $t_a$, $t_b$, $t_c$. The subscripts of these symbols indicate which median/altitude/angle bisector we are talking about. Also, the area of a triangle *ABC* will be denoted by $(ABC)$. These are all standard notations used in many books. One more notation that is less standard: the semi-perimeter of a given triangle is usually denoted by $p$. In the case there is no risk of confusion, we will use these notations throughout this book without explanation anymore.

## 1.1 A simple theorem on area ratio

Area is one of the most intuitive concepts in mathematics. On one hand it is simple, people learn it since they were in primary school. On the other hand, it leads to an important notion called measure, which is a corner-stone of measure theory and even various branches of modern mathematics. In our situation, we are concerned in the techniques of using area to solve problems in geometry (especially those in Olympic level).

**Theorem 1.1-1**（共邊定理）

If the lines *AB*, *PQ* intersect at *M*, then

$$\frac{(ABP)}{(ABQ)} = \frac{PM}{QM}.$$

- Theorem 1.1-1 does not make any assumption on the positions of the points *A*, *B*, *P*, *Q*. As we will see in the proof, there are four possible cases depending on the positions of these points.

Before proving the theorem, let's recall that the area of a triangle $ABC$ is given by $(ABC)=\frac{1}{2}ah_a$. It means that if $h_a$ is fixed then the area is directly proportional to $a$. For example, in Figure 1 we have

$$\frac{(ACD)}{(BCD)}=\frac{AD}{BD}.$$

Making use of this observation we can have a short proof of Theorem 1.1-1.

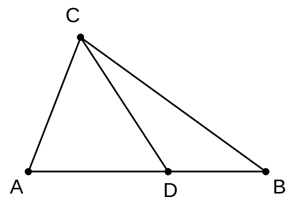


Figure 1

**Proof**

Without loss of generality we assume all triangles involved are not degenerated. Now, one has

$$\begin{aligned}\frac{(ABP)}{(ABQ)}&=\frac{(ABP)}{(AMP)}\cdot\frac{(AMP)}{(AMQ)}\cdot\frac{(AMQ)}{(ABQ)}\\&=\frac{AB}{AM}\cdot\frac{PM}{QM}\cdot\frac{AM}{AB}\\&=\frac{PM}{QM}\end{aligned}$$

Q.E.D.

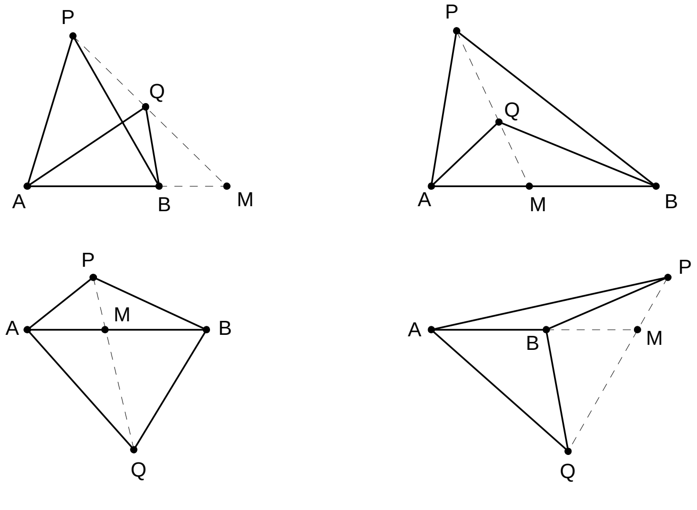


Figure 2

In order to familiarize ourselves with Theorem 1.1-1, we look at a few examples.

**Example 1.1-1**

Let $P$ be an interior point of triangle $ABC$, the rays $AP$, $BP$, $CP$ meet the sides $BC$, $CA$, $AB$ at points $D$, $E$, $F$ respectively. Prove that $\frac{PD}{AD}+\frac{PE}{BE}+\frac{PF}{CF}=1$.

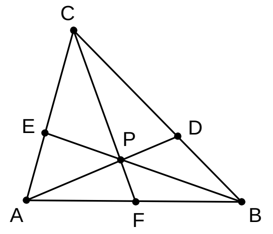


Figure 3

**Solution**

$$\begin{aligned}\frac{PD}{AD}+\frac{PE}{BE}+\frac{PF}{CF}&=\frac{(PBC)}{(ABC)}+\frac{(APC)}{(ABC)}+\frac{(ABP)}{(ABC)}\\&=\frac{(ABC)}{(ABC)}\\&=1\end{aligned}$$

Q.E.D.

**Example 1.1-2 (IMO 1998 Hong Kong Preliminary Selection Contest)**

In $\Delta ABC$, $E$, $F$, $G$ are points on $AB$, $BC$, $CA$ respectively such that $AE : EB = BF : FC = CG : GA = 1 : 3$. $K$, $L$, $M$ are the intersection points of the lines $AF$ and $CE$, $BG$ and $AF$, $CE$ and $BG$, respectively. Suppose the area of $\Delta ABC$ is 1; find the area of $\Delta KLM$.

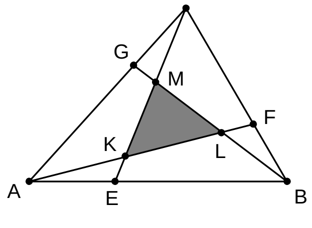


Figure 4

**Solution**

Let $s$ = ($ABL$). By Theorem 1.1-1 we have ($CAL$) = 3$s$ and ($BCL$) = $s$ / 3. Note that

$$(ABL) + (BCL) + (CAL) = (ABC) = 1,$$

so we have $s+\frac{s}{3}+3s=1$ and therefore $s=\frac{3}{13}$. We have proved that $(ABL)=\frac{3}{13}$. Similar argument shows $(BCM)=(CAK)=\frac{3}{13}$. Hence,

$$\begin{aligned}(KLM)&=(ABC)-(ABL)-(BCM)-(CAK)\\&=1-\frac{3}{13}-\frac{3}{13}-\frac{3}{13}\\&=\frac{4}{13}\end{aligned}$$

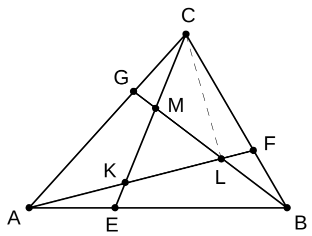


Figure 5

**Example 1.1-3**

Refer to Figure 6, there is a convex quadrilateral $ABCD$. The lines $DA$ and $CB$ intersect at $K$, the lines $AB$ and $DC$ intersect at $L$, the lines $AC$ and $KL$ intersect at $G$, the lines $DB$ and $KL$ intersect at $F$. Prove that

$$\frac{KF}{FL}=\frac{KG}{GL}.$$

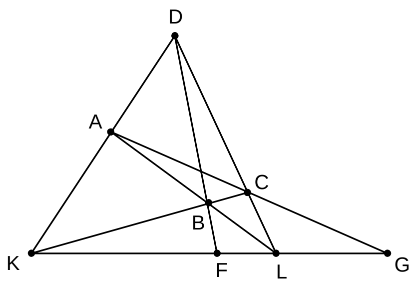


Figure 6

**Solution**

Apply Theorem 1.1-1 repeatedly,

$$\begin{aligned}\frac{KF}{LF}&=\frac{(KBD)}{(LBD)}\\&=\frac{(KBD)}{(KBL)}\cdot\frac{(KBL)}{(LBD)}\\&=\frac{CD}{CL}\cdot\frac{AK}{AD}\\&=\frac{(ACD)}{(ACL)}\cdot\frac{(ACK)}{(ACD)}\\&=\frac{(ACK)}{(ACL)}\\&=\frac{KG}{LG}\end{aligned}$$

➢ We will come back to this example later with two different proofs. One using Ceva's theorem and Menulaus' theorem (to be introduced in the next section), while another one involves the notion of cross ratio（交比） in projective geometry.

**Exercise**

1. Let $ABC$ be a triangle and $D$, $E$ are points on the segment $BC$, $CA$ respectively such that $AE = \lambda AC$ and $BD = \mu BC$. Find, in terms of $\lambda$ and $\mu$, the ratio $AF : FD$.

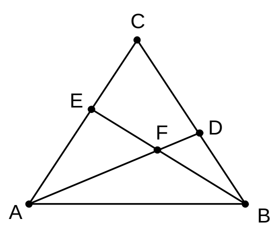


Figure 7

2. （定比分點公式）Suppose $P$, $Q$ are two points on the same side of the line $AB$. $R$ is a point on the segment $PQ$ such that $PR = \lambda PQ$. Prove that $(ABR) = (1 - \lambda)(ABP) + \lambda(ABQ)$.

3. Refer to Figure 8, $ABCD$ is a convex quadrilateral. $AC$ and $BD$ intersect at $E$. $P$, $Q$ are the

midpoints of $AC$ and $BD$ respectively. Given that $AE = \lambda AC$ and $BE = \mu BD$.

(a) Find the ratios $AR : RD$ and $BS : SC$ (in terms of $\lambda$ and $\mu$).

(b) Suppose the area of $ABCD$ is 1. What is the area of $ABSR$?

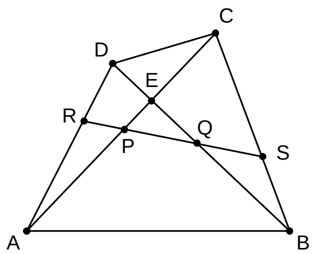


Figure 8

4. Given a convex quadrilateral $ABCD$. Let $P_1$, $P_2$ be the trisection points of the segment $AB$ and $Q_1$, $Q_2$ be the trisection points of the segment $CD$ as shown in Figure 9. Prove that

$$\frac{(P_1P_2Q_2Q_1)}{(ABCD)} = \frac{1}{3},$$

where $(XYUV)$ denotes the area of the quadrilateral $XYUV$.

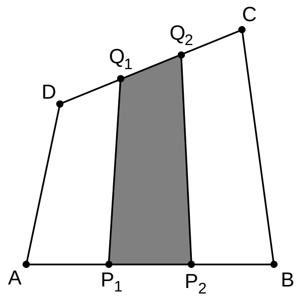


Figure 9

Refer to Figure 10, we trisect $BC$, $DA$ by the points $R_1$, $R_2$, $S_1$, $S_2$. Prove that

$$\frac{(KLMN)}{(ABCD)} = \frac{1}{9}.$$

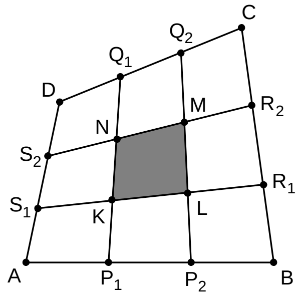


Figure 10

## 1.2 Ceva's theorem, Menelaus's theorem and their converses

We are in a position to introduce two important theorems (and their converses) in elementary geometry, which are powerful tools for proving collinear points and concurrent lines.

**Theorem 1.2-1 (Ceva's theorem)**

Let *ABC* be a triangle and *D*, *E*, *F* be points on the lines *BC*, *CA*, *AB* respectively. If *AD*, *BE*, *CF* are concurrent (i.e. meet at a point *P*), then

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=+1.$$

The + sign emphasizes directed segments were used here.

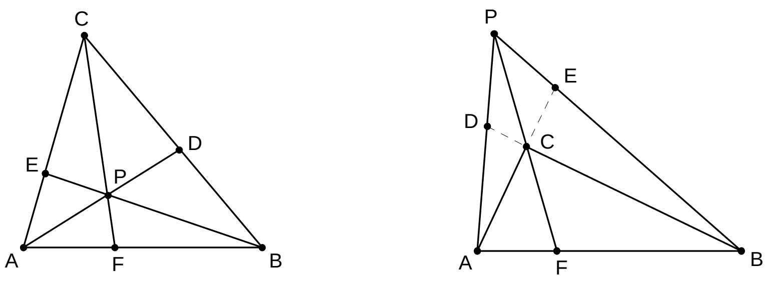


Figure 11

**Proof**

The theorem can be proved easily by Theorem 1.1-1 as follows:

$$\left|\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}\right|=\frac{(APC)}{(PBC)}\cdot\frac{(ABP)}{(APC)}\cdot\frac{(PBC)}{(ABP)}=1,$$

and the sign is obviously positive.

Q.E.D.

- Since Theorem 1.1-1 doesn't depend on the positions of the points involved, the proof above is valid even for the case where $P$ lies outside the triangle $ABC$.

**Theorem 1.2-2 (Converse of Ceva's theorem)**

Let $ABC$ be a triangle and $D$, $E$, $F$ be points on the lines $BC$, $CA$, $AB$ respectively. Suppose that

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=+1.$$

Then $AD$, $BE$, $CF$ are either concurrent or mutually parallel (sometimes we say the lines are concurrent at **the point at infinity**).

The proof of Theorem 1.2-2 is left to the reader as exercise.

**Theorem 1.2-3 (Menelaus's theorem)**

Let $ABC$ be a triangle and $D$, $E$, $F$ be points on the lines $BC$, $CA$, $AB$ respectively. If $D$, $E$, $F$ are collinear, then

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=-1.$$

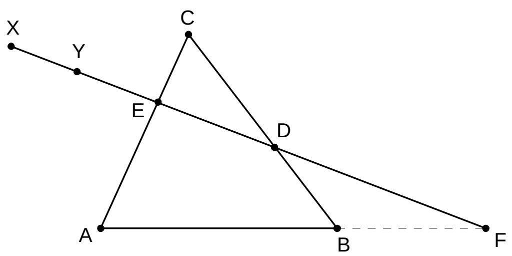


Figure 12

**Proof**

Let $X$, $Y$ be two arbitrary (distinct) points on the line $DEF$. Then

$$\left|\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}\right|=\frac{(AXY)}{(BXY)}\cdot\frac{(BXY)}{(CXY)}\cdot\frac{(CXY)}{(AXY)}=1.$$

Again, it is clear that the sign is negative in this case.

Q.E.D.

Menelaus's theorem also has a converse:

**Theorem 1.2-4. (Converse of Menelaus's theorem)**

Let *ABC* be a triangle and *D*, *E*, *F* be points on the lines *BC*, *CA*, *AB* respectively. Suppose that

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=-1.$$

Then *D*, *E*, *F* are collinear.

Here are some corollaries of the theorems above:

- Three medians of any given triangle are concurrent. The point of intersection is called the **centroid**（重心）of the triangle.

- Three altitudes of any given triangle are concurrent. The point of intersection is called the **orthocenter**（垂心）of the triangle.

- Three angle bisectors of any given triangle are concurrent. The point of intersection is called the **incenter**（內心）of the triangle. Moreover, the external bisectors of any two angles of a triangle are concurrent with the internal bisector of the third angle. The point of intersection is called an **excenter**（旁心）of the triangle. Note that a triangle has three excenters.

Usually, circumcenter, centroid, orthocenter and incenter are denoted by the letters *O*, *G*, *H*, *I* respectively. As we will see in chapter 2, for any given triangle the circumcenter *O*, the centroid *G* and the orthocenter *H* are collinear and $OG : GH = 1 : 2$. The line *OGH* is called the **Euler line**（歐拉線）of the triangle.

The following are some applications of Ceva's theorem, Menelaus's theorem and their converses. Readers should be careful when applying these theorems we don't consider directed segments since the sign of an expression is usually obvious.

**Example 1.2-1 (IMO 1982-5)**

The diagonals $AC$ and $CE$ of the regular hexagon $ABCDEF$ are divided by the inner points $M$ and $N$, respectively, so that $AM / AC = CN / CE = r$. Determine $r$ if $B$, $M$, $N$ are collinear.

**Solution**

Join $BE$ which intersects $AC$ at $P$. Apply Menelaus's theorem to the triangle $CPE$ and the line $BMN$, one has

(2.1) $$\frac{CM}{MP}\cdot\frac{PB}{BE}\cdot\frac{EN}{NC}=1.$$

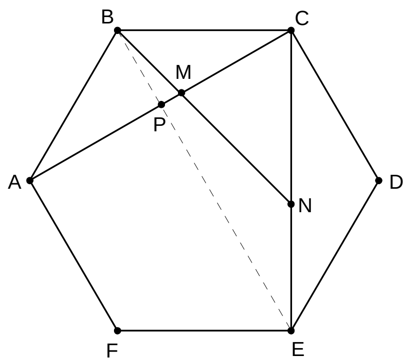


Figure 13

Note that

(i) $\frac{CM}{MP}=\frac{1-r}{r-\frac{1}{2}}=\frac{2-2r}{2r-1}$;

(ii) $PB=AB\cos\angle ABP=\frac{1}{2}AB=\frac{1}{4}BE\Rightarrow\frac{PB}{BE}=\frac{1}{4}$;

(iii) $\frac{EN}{NC}=\frac{1-r}{r}$.

Substitute (i), (ii), (iii) into (2.1),

$$\frac{2-2r}{2r-1}\cdot\frac{1}{4}\cdot\frac{1-r}{r}=1$$

which implies $r=\frac{\sqrt{3}}{3}$.

**Example 1.2-2 (Alternative solution to Example 1.1-3)**

Refer to Figure 14, there is a convex quadrilateral $ABCD$. The lines $DA$ and $CB$ intersect at $K$, the lines $AB$ and $DC$ intersect at $L$, the lines $AC$ and $KL$ intersect at $G$, the lines $DB$ and $KL$ intersect at $F$. Prove that

$$\frac{KF}{FL}=\frac{KG}{GL}.$$

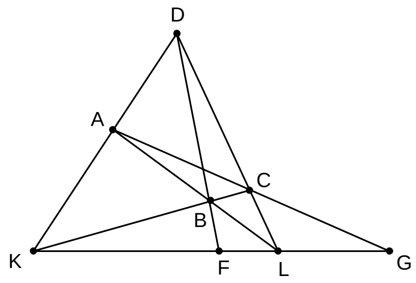


Figure 14

**Solution**

Apply Ceva's theorem to triangle $DKL$ and the point $B$, we have

(2.2) $$\frac{DA}{AK}\cdot\frac{KF}{FL}\cdot\frac{LC}{CD}=1.$$

Apply Menelaus's theorem to triangle $DKL$ and the line $ACG$, we have

(2.3) $$\frac{DA}{AK}\cdot\frac{KG}{GL}\cdot\frac{LC}{CD}=1.$$

Divide (2.2) by (2.3), the result follows.

Q.E.D.

Before looking at the third solution to this question, let's recall that the **cross ratio** of four (distinct) collinear points $A$, $B$, $C$, $D$ is defined by

$$\{AB,CD\}=\frac{AC\times BD}{CB\times DA}.$$

An ordered quadruple ($A$, $B$, $C$, $D$) of four distinct collinear points is called a **harmonic quadruple** （調和四元組）if $\{AB, CD\} = 1$. One may verify that

$$\{AB,CD\}=\{CD,AB\} \quad \text{and} \quad \{AB,CD\}=\frac{1}{\{AB,DC\}}.$$

So, an ordered quadruple ($A$, $B$, $C$, $D$) is harmonic if and only if $\{AB, CD\} = \{AB, DC\}$.

**Theorem 1.2-5 (Invariant under perspectivity)**

Let $L_1$, $L_2$ be two distinct lines on the plane. If $A$, $B$, $C$, $D$ are distinct points on $L_1$ and $A'$, $B'$, $C'$, $D'$ are distinct points on $L_2$, and if the lines $AA'$, $BB'$, $CC'$, $DD'$ are concurrent, then

$$\{AB,CD\}=\{A'B',C'D'\}.$$

Equivalently,

$$\frac{AC\times BD}{CB\times DA}=\frac{A'C'\times B'D'}{C'B'\times D'A'}.$$

Theorem 1.2-5 says that cross ratio is an invariant under perspectivity（中心投影）.

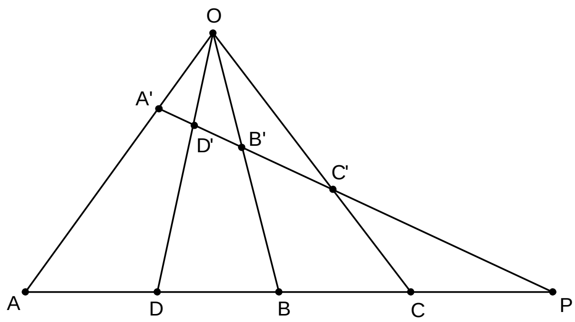


Figure 15

**Proof**

Let $AA'$, $BB'$, $CC'$, $DD'$ intersect at $O$ and $P$ be the intersection of $L_1$, $L_2$ (when $L_1 // L_2$ we regard $P$ as the point at infinity, this proof is still valid).

Apply Menelaus's theorem to the triangles $APA'$, $A'PA$, $B'PB$, $BPB'$ with the intersecting lines $CC'$, $DD'$, $CC'$, $DD'$ respectively,

$$\begin{cases}\dfrac{AC}{CP}\cdot\dfrac{PC'}{C'A'}\cdot\dfrac{A'O}{OA}=1\\[2ex]\dfrac{A'D'}{D'P}\cdot\dfrac{PD}{DA}\cdot\dfrac{AO}{OA'}=1\\[2ex]\dfrac{B'C'}{C'P}\cdot\dfrac{PC}{CB}\cdot\dfrac{BO}{OB'}=1\\[2ex]\dfrac{BD}{DP}\cdot\dfrac{PD'}{D'B'}\cdot\dfrac{B'O}{OB}=1\end{cases}$$

Multiply these four equalities together, we obtain

$$\frac{AC \times A'D' \times B'C' \times BD}{C'A' \times DA \times CB \times D'B'} = 1.$$

It follows that

$$\frac{AC \times BD}{CB \times DA} = \frac{A'C' \times B'D'}{C'B' \times D'A'}.$$

Q.E.D.

Theorem 1.2-5 has its origin in projective geometry which we will not pursue. We give an example to show how Theorem 1.2-5 gives a beautiful solution of Example 1.1-3.

**Example 1.2-3 (The third solution to Example 1.1-3)**

Refer to Figure 16, there is a convex quadrilateral *ABCD*. The lines *DA* and *CB* intersect at *K*, the lines *AB* and *DC* intersect at *L*, the lines *AC* and *KL* intersect at *G*, the lines *DB* and *KL* intersect at *F*. Prove that

$$\frac{KF}{FL} = \frac{KG}{GL}.$$

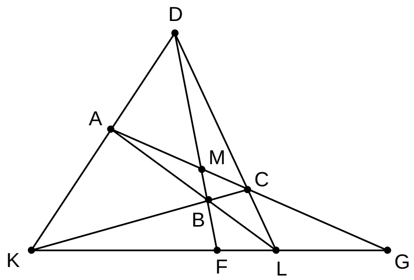


Figure 16

**Proof**

Let *AG*, *DF* meet at *M*. Consider the perspectivity of *KG* onto *DF* with center *A*, by Theorem 1.2-5 one has

(2.4) $$\{KL, FG\} = \{DB, FM\}.$$

Next, we consider the perspectivity of *DF* onto *LF* with center *C*. By the same theorem one has

(2.5) $$\{DB, FM\} = \{LK, FG\}.$$

Now, combining (2.4) and (2.5) gives

$$\{KL, FG\} = \{LK, FG\},$$

saying that ($K$, $L$, $F$, $G$) is a harmonic quadruple.

Q.E.D.

**Exercise**

1. Prove Theorem 1.2-2 and Theorem 1.2-4.

2. (The 26th and 31st IMO shortlisted problem) Let $M$ be an interior point of triangle $ABC$. $AM$ meets $BC$ at $D$, $BM$ meets $CA$ at $E$, $CM$ meets $AB$ at $F$. Prove that $(DEF) \le \frac{1}{4}(ABC)$.

3. （張角公式）Suppose $PA$, $PB$, $PC$ be three rays for which $\angle APC = \angle APB + \angle BPC < 180°$. Prove that $A$, $B$, $C$ are collinear if and only if

$$\frac{\sin \angle APC}{PB} = \frac{\sin \angle APB}{PC} + \frac{\sin \angle BPC}{PA}.$$

Using this result, find an alternative solution to Example 1.2-1.

4. (Pascal's theorem) Let $A$, $B$, $C$, $D$, $E$, $F$ be arbitrary (distinct) points on a given circle. Prove that the intersections of $AB$ with $DE$, $CD$ with $FA$, and $EF$ with $BC$ are collinear if they exist.

5. (Pappus's theorem) If $A$, $C$, $E$ are three points on one line, $B$, $D$, $F$ on another, and if the three lines $AB$, $CD$, $EF$ meet $DE$, $FA$, $BC$, respectively, then the three points of intersection $L$, $M$, $N$ are collinear.

6. (Desargues's theorem) If two triangles are perspective from a point, and if their pairs of corresponding sides meet, then three points of intersection are collinear.

## 1.3 Simson's theorem and Ptolemy's theorem

The next theorem, involving circles, possibly should not be put in this chapter. However, it is one of the famous applications of Menelaus's theorem.

**Theorem 1.3-1 (Simson's theorem)**

Let $ABC$ be a triangle. Suppose $P$ is a point on the circumcircle of triangle $ABC$. Let $D$, $E$, $F$ be the feet of perpendicular from $P$ to $BC$, $CA$, $AB$ respectively. Then $D$, $E$, $F$ are collinear.

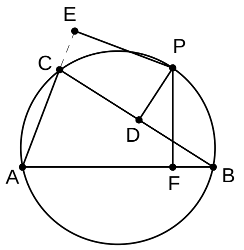


Figure 17

**Proof**

To show $D$, $E$, $F$ are collinear, we need to verify

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1.$$

Note that $AF = PA\cos\angle PAF$, $FB = PB\cos\angle PBF$, $BD = PB\cos\angle PBD$, $DC = PC\cos\angle PCD$, $CE = PC\cos\angle PCE$, $EA = PA\cos\angle PAE$. Therefore,

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{\cos\angle PAF\times\cos\angle PBD\times\cos\angle PCE}{\cos\angle PBF\times\cos\angle PCD\times\cos\angle PAE}.$$

Note also that $\angle PAF = \angle PCD$, $\angle PBD = \angle PAE$, $\angle PCE = \angle PBF$, the result follows.

Q.E.D.

- The line $DEF$ is called the **Simson line** (or simply **simson**) of point $P$ with respect to triangle $ABC$. The converse of Simson's theorem is also true. This is left to the reader as exercise.

**Example 1.3-1**

Refer to Figure 18, $D$, $E$, $F$ are respectively the feet of perpendicular from $A$ to $BC$, $B$ to $CA$, and $C$ to $AB$. Draw perpendicular lines from $D$ to $AB$, $AC$, $BE$, $CF$ and let $P$, $Q$, $M$, $N$ be the feet of perpendicular respectively. Prove that $P$, $Q$, $M$, $N$ are collinear.

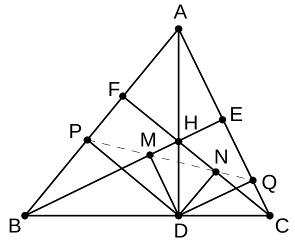


Figure 18

**Solution**

It is clear that *BDHF* is a cyclic quadrilateral, the Simson line of *D* passes *P*, *M*, *N*. In other words, *P*, *M*, *N* are collinear. Similar argument shows *Q*, *M*, *N* are also collinear.

Q.E.D.

**Example 1.3-2 (IMO 1998 shortlisted problem)**

Let *ABC* be a triangle, *H* its orthocenter, *O* its circumcenter, and *R* its circumradius. Let *D* be the reflection of *A* across *BC*, *E* be that of *B* across *CA*, and *F* that of *C* across *AB*. Prove that *D*, *E* and *F* are collinear if and only if $OH = 2R$.

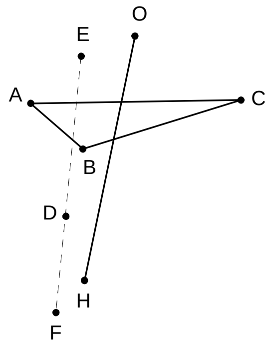


Figure 19

**Solution**

Let *PQR* be the triangle with *ABC* as its medial triangle, i.e. *A* is the midpoint of *QR*, *B* is that of *RP* and *C* that of *PQ*. From *O* draw perpendicular lines to *QR*, *RP* and *PQ* with feet of perpendicular $D'$, $E'$ and $F'$ respectively. It can be proved, by considering a suitable homothety, that *D*, *E* and *F* are collinear if and only if $D'$, $E'$ and $F'$ are collinear. We postpone the proof until chapter 5 in which we discuss geometric transformations. Readers who know homothety may try to prove it at this point.

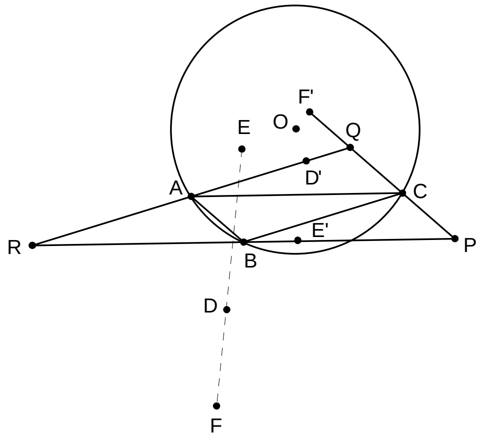


Figure 20

By Simson's theorem (and its converse), $D'$, $E'$ and $F'$ are collinear if and only if $O$ lies on the circumcircle of triangle $PQR$. Note that the circumcenter of triangle $PQR$ is the orthocenter of triangle $ABC$, namely $H$. So $O$ lies on the circumcircle of triangle $PQR$ if and only if $OH$ equals the circumradius of triangle $PQR$, which is $2R$.

Q.E.D.

**Theorem 1.3-2. (Ptolemy's theorem)**

For any four points $A$, $B$, $C$, $D$ in general position (i.e. no two of them coincide, no three of them are collinear),

$$AB \times CD + AD \times BC \geq AC \times BD.$$

Equality holds if and only if $ABCD$ is a cyclic quadrilateral.

**Proof**

Let $L$, $M$, $N$ be respectively the feet of perpendicular from $D$ to $BC$, $CA$, $AB$. Since $\angle CLD = \angle CMD = 90°$, the points $L$, $C$, $D$, $M$ are concyclic. Figure 21 shows one of the possible cases. In any case we have

$$LM = CD \sin \angle BCA = \frac{CD \times AB}{2R},$$

where $R$ denotes the circumradius of triangle $ABC$. Similarly, we have

$$MN = \frac{AD \times BC}{2R} \quad \text{and} \quad LN = \frac{BD \times AC}{2R}.$$

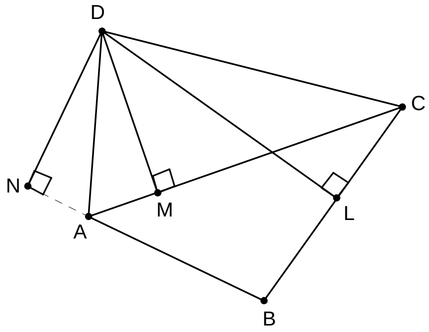


Figure 21

By triangle inequality, $LM + MN \ge LN$. Using the expressions of *LM*, *MN*, *LN* we obtained, it leads to

$$\frac{CD \times AB}{2R} + \frac{AD \times BC}{2R} \ge \frac{BD \times AC}{2R}.$$

The required inequality is proved. Equality holds if and only if *L*, *M*, *N* are collinear, by Simson's theorem (and its converse) this happens if and only if *D* lies on the circumcircle of triangle *ABC*.

Q.E.D.

**Example 1.3-3**

Let *ABC* be an equilateral triangle and *P* be a point on the circumcircle of triangle *ABC* which lies on the arc $\overset{\frown}{BC}$. Prove that *PA* = *PB* + *PC*.

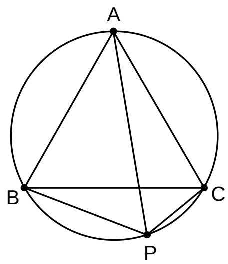


Figure 22

**Solution**

The result follows immediately by applying Ptolemy's theorem to the cyclic quadrilateral *ABPC*.

Q.E.D.

Here we are going to give an alternative solution to Example 1.3-3 which uses only congruent triangles. Readers are encouraged to look at it carefully since it illustrates a standard technique which proves useful in problem involving broken segments.

**Alternative Solution**

Extend *BP* to point *D* such that *PD* = *PC*. Since *PD* = *PC* and $\angle CPD = \angle CAB = 60°$, *CPD* is an equilateral triangle. Consider triangles *APC* and *BDC*. We have *AC* = *BC*, $\angle CAP = \angle CBD$, and $\angle APC = \angle ABC = 60° = \angle BDC$. So, $\Delta APC \cong \Delta BDC$ and hence $PA = DB = PB + PC$.

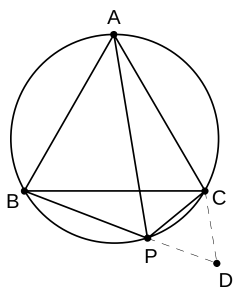


Figure 23

Q.E.D.

**Example 1.3-4**

If a circle passing through point *A* cuts two sides and a diagonal of a parallelogram *ABCD* at points *P*, *Q*, *R* as shown in Figure 24, then $AP \times AB + AR \times AD = AQ \times AC$.

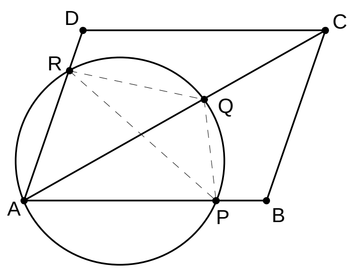


Figure 24

**Solution**

Apply Ptolemy's theorem to the cyclic quadrilateral *APQR*, we have

(3.1) $$AP \times RQ + AR \times PQ = AQ \times RP.$$

Observe that $\Delta ABC \sim \Delta RQP$. We multiply the constant $AB / RQ$ to (3.1), it gives

$$AP \times AB + AR \times CB = AQ \times AC .$$

Replace $CB$ by $AD$, we have $AP \times AB + AR \times AD = AQ \times AC$ .

Q.E.D.

**Example 1.3-5**

Let $A$, $B$, $C$, $D$ be adjacent vertices of a regular 7-sided polygon, in that order. Prove that

$$\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD} .$$

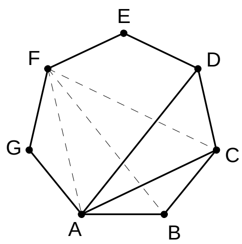


Figure 25

**Solution**

Refer to Figure 25, let $E$, $F$, $G$ be the remaining vertices of the 7-sided polygon with the indicated order. Apply Ptolemy's theorem to the cyclic quadrilateral $ABCF$:

$$AC \times BF = AB \times CF + BC \times FA .$$

Substitute $BF$ by $AD$, $CF$ by $AD$, $BC$ by $AB$ and $FA$ by $AC$ in the above equality, we obtain

$$AC \times AD = AB \times AD + AB \times AC .$$

Dividing both sides by $AB \times AC \times AD$, the result follows.

Q.E.D.

**Example 1.3-6 (1998-99 Iranian Math Olympiad, IMO 2000 Hong Kong Team Selection Test)**

$ABC$ is a triangle with $BC > CA > AB$. $D$ is a point on side $BC$, and $E$ is a point on $BA$ produced beyond $A$ so that $BD = BE = CA$. Let $P$ be a point on side AC such that $E$, $B$, $D$, $P$ are concyclic, and let $Q$ be the second intersection point of $BP$ with the circumcircle of $\Delta ABC$. Prove that $AQ + CQ = BP$ .

**Solution**

We claim that $\Delta AQC \sim \Delta EPD$ . This is because $\angle CAQ = \angle CBQ = \angle DEP$ and $\angle AQC = 180° - \angle ABD = \angle EPD$. On the other hand, by Ptolemy's theorem, we have

$$BP \times DE = BE \times DP + BD \times EP .$$

So $BP = BE \times \dfrac{DP}{DE} + BD \times \dfrac{EP}{DE} = CA \times \dfrac{CQ}{CA} + CA \times \dfrac{AQ}{CA} = AQ + CQ$ .

Q.E.D.

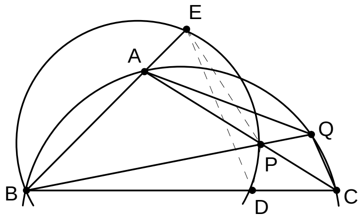


Figure 26

**Exercise**

1. State and prove the converse of Simson’s theorem.

2. Suppose four lines intersect with each other and therefore any three lines among them determine a triangle. There are four such triangles. Prove that the circumcircles of these triangles have a common point.

3. Let *ABCD* be a square. If *P* is a point on the circumcircle of *ABCD* which lies on the arc $\overset{\frown}{AD}$, prove that the value $(PA + PC) / PB$ does not depend on the position of *P*.

4. Let *ABCDE* be a regular pentagon inscribed in a circle *O*. *P* is a point on *O* which lies on the arc $\overset{\frown}{AB}$. Prove that $PA + PB + PD = PC + PE$.

5. (IMO 1995) Let *ABCDEF* be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$ and $\angle BCD = \angle EFA = 60°$. Let *G* and *H* be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120°$. Show that $AG + GB + GH + DH + HE \geq CF$ .