Olympiad Combinatorics

Pranav A. Sriram

August 2014

Copyright notices

All USAMO and USA Team Selection Test problems in this chapter are copyrighted by the Mathematical Association of America's American Mathematics Competitions.

© Pranav A. Sriram. This document is copyrighted by Pranav A. Sriram, and may not be reproduced in whole or part without express written consent from the author.

About the Author

Pranav Sriram graduated from high school at The International School Bangalore, India, and is currently pursuing undergraduate studies at Stanford University.

7. EXTREMAL COMBINATORICS

Introduction

Extremal combinatorics is, in essence, the combinatorics of superlatives. Problems in this field typically revolve around finding or characterizing the maximum, minimum, best, worst, biggest or smallest object, number or set satisfying a certain set of constraints. This chapter and the next two will take us to the heart of combinatorics, and will represent a deep dive into the intersection of Olympiad mathematics, classical combinatorics, and modern research in the field. Extremal combinatorics is an actively researched area, with deep connections to fundamental problems in theoretical computer science, operations research and statistical learning theory. Entire books have been devoted to the subject (and rightfully so), so we will not be able to do complete justice to this field in a single chapter. However, the powerful arsenal of tools we have built up in the first six chapters has already done much of our work for us: indeed, pretty much every technique we have seen so far has a role to play in extremal combinatorics. This chapter, then, will develop specialized, niche methods for extremal combinatorics, as well as demonstrate how to effectively exploit combinatorial structure to apply classical techniques like induction effectively in the context of extremal problems.

Injections and Bijections

One simple way to compare the cardinalities of sets is to find mappings between them. For instance, suppose I have two sets S and T, and a function f mapping elements from S to elements of T. If f is *injective* – that is, $f(x) = f(y)$ if and only if $x = y$ – then it follows that T must have *at least* as many elements as S. Moreover, if f is also *surjective* - that is, all elements in T are mapped to by some element in S - then it follows that $|S| = |T|$. A function that is injective and surjective is called a *bijection*.

The basic idea in this section will be to construct functions between carefully chosen sets. The choice of these sets will enable us to exploit information given in the problem in order to conclude that our function is injective or bijective. This conclusion will give us quantitative results relating the sizes of the sets, which will hopefully reduce to the result we are trying to prove.

Example 1 [APMO 2008]
Students in a class form groups. Each group contains exactly three members and any two distinct groups have at most one member in common. Prove that if there are 46 students in the class, then there exists a set of at least 10 students in which no group is properly contained.

Answer:
Let T be the set of 46 students. Take the largest set S of students such that no group is properly contained in S. Now take any student X not in S. By the maximality of S, there exists a group containing X and two students of S (otherwise we could add X to S, contradicting maximality). This suggests the following mapping: if (A, B, X) is this group, define a mapping from $T \setminus S$ to pairs in S such that $f(X) = (A, B)$. This mapping is injective because if $f(Y) = f(Z)$ for some $Y \neq Z$, then both (Y, C, D) and (Z, C, D) are groups for

some (C, D), contradicting the fact that any two groups have at most one common student. The injectivity implies $| T \setminus S | \le \binom{|S|}{2}$, or $(46-|S|) \le \binom{|S|}{2}$. Simplifying gives $|S| \ge 10$. ■

Example 2 [IMO Shortlist 1988]
Let $N = \{1, 2, \dots, n\}$, with $n \ge 2$. A collection $F = \{A_1, A_2, \dots, A_t\}$ of subsets of N is said to be *separating*, if for every pair $\{x, y\}$ there is a set $A_i \in F$ so that $A_i \cap \{x, y\}$ contains just one element. F is said to be *covering*, if every element of N is contained in at least one set $A_i \in F$. What is the smallest value of t in terms of n so that there is a family $F = \{A_1, A_2, \dots, A_t\}$ which is simultaneously separating and covering?

Answer:
Associate each element m of N with a binary string $x_1x_2\dots x_t$, where $x_i = 1$ if m is in set A_i and 0 if m is not in A_i. The condition that F is separating simply means that distinct elements of N will be mapped to distinct binary strings. The condition that F is covering means that no element of N will be mapped to $(0, 0, \dots, 0)$.

Thus we have n distinct binary strings of length t, none of which is the all 0 string. This implies $n \le 2^t-1$. Conversely, if we indeed have $n \le 2^t-1$, then a construction is easy by reversing the above process: first label each element with a different binary string and then place it into the appropriate sets. Thus $n \le 2^t-1$ is necessary and sufficient, so $t = \lfloor \log_2 n \rfloor+1$ is the answer. ■

Remark: This idea of associating elements with binary strings is more than just a useful trick on Olympiads – in fact, it plays an important role in a whole branch of combinatorics known as algebraic combinatorics, where these binary "strings" are actually treated as vectors. Algebraic manipulations of these vectors (which often take place mod 2 or in some other field) can produce surprising combinatorial results.

Example 3 [IMO 2006-2]
A diagonal of a regular 2006-gon is called *odd* if its endpoints divide the boundary into two parts, each composed of an odd number of sides. Sides of the 2006-gon are also regarded as odd diagonals. Suppose the 2006-gon has been dissected into triangles by 2003 nonintersecting diagonals. Define a *good triangle* as an isosceles triangles with two odd sides. Find the maximum number of good triangles.

Answer:
Note a good triangle has two odd sides and an even side; hence the pair of equal sides must be the odd sides.

Experimentation with 2006 replaced by small even numbers hints that the general answer for a regular $2n$-gon is n. This is attainable by drawing all diagonals of the form $A_{2k}A_{2k+2}$, where A_1, A_2, …, A_{2n} are the vertices of the $2n$-gon. Now we show this is indeed the maximum.

Consider a $2n$-gon ***P***. To simplify notation, draw the circumcircle of ***P***. For a side AB in a triangle ABC, "arc AB" will denote the arc of the circumcircle not containing C. Arc AB is a "good arc" if AB is odd in a good triangle ABC.

Our basic idea is to construct a mapping f from sides of ***P*** to good triangles such that each good triangle is mapped to by at least 2 sides, and no side is mapped to more than one good triangle. This will immediately imply the result.

Consider a side XY of ***P***. Let AB denote the smallest good arc containing vertices X and Y, if it exists. (Note that {A, B} may be equal to {X, Y}.) Let C be the third vertex of the good triangle ABC. Then we will map XY to ABC: f(XY) = ABC.

All we need to show is that each good triangle is mapped to by at least two sides. In fact, for a good triangle DEF, with DE and EF odd, we will show that at least one side of ***P*** with vertices in arc DE is mapped to triangle DEF; the same argument will hold for EF and we will hence have two sides mapped to DEF.

Suppose to the contrary that no side with vertices in arc DE is mapped to DEF. Consider some side RS of ***P*** with vertices R and S in arc DE. Let *f*(RS) = D'E'F' for some D', E', F' lying on arc DE with D'E' and E'F' the odd sides of good triangle D'E'F'. Then by symmetry, the side of ***P*** that is the reflection of RS across the perpendicular bisector of D'F' will also be mapped to D'E'F'.

In this manner, sides in arc DE can be paired up, with each pair of sides being mapped to the same triangle. But there are an odd number of sides in arc DE, so they cannot all be paired up. Contradiction. ■

The Alternating Chains Technique

(Yes, I made that name up)

The basic idea that we will use in some form or the other for the next few problems is a simple consequence of the pigeonhole principle. Suppose you have n points on a line, and you are allowed to mark some of them such that no consecutive points are marked. Then the maximum number of points you can mark is $\lceil n/2 \rceil$, and this can be achieved by marking alternate points. If the n points were on a circle and not a line segment, then the maximum would be $\lfloor n/2 \rfloor$. These obvious statements can be cleverly applied in several combinatorial settings.

Example 4 [IMO Shortlist 1990]

Let $n \geq 3$ and consider a set E of $2n$-1 distinct points on a circle. Suppose that exactly k of these points are to be colored red. Such a coloring is *good* if there is at least one pair of red points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

Answer:
Let j be maximum number of colored points a *bad* coloring can have. Then $k = j+1$, so it suffices to find j.

Let the points be $A_1, A_2, \ldots, A_{2n-1}$. Join vertices A_i and A_{i+n+1} by an edge for each i (indices modulo $2n$-1). This decomposes E into disjoint cycles. The coloring is bad if and only if no two red vertices are joined by an edge. In other words, no two consecutive vertices in the cycle are both red. How many cycles are there? Using elementary number theory, it is easy to show that the number of cycles is equal to gcd (n+1, 2n-1).

Since $2n$-1 = 2(n+1)–3, *gcd* (n+1, 2n-1) = 3 if n+1 is divisible by 3, and *gcd* (n+1, 2n-1) = 1 otherwise. If *gcd* (n+1, 2n-1) = 1, we get only one cycle containing all 2n-1 points. Then j = n-1 by our earlier discussion. Hence $k = j+1 = n$.

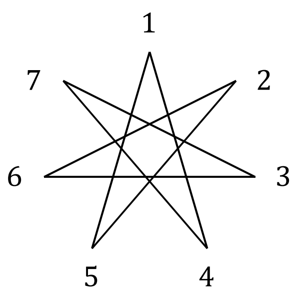

Figure 7.1.

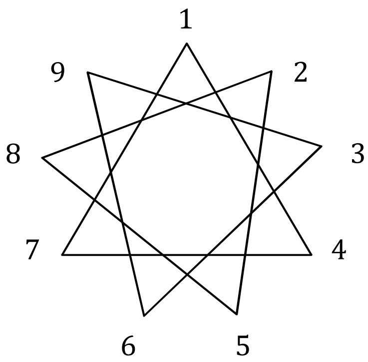

Figure 7.2.

If $gcd(n+1, 2n-1) = 3$, then we get 3 cycles, each containing $\frac{2n-1}{3}$ vertices. Each cycle then can have at most $\lfloor (2n-1)/6 \rfloor$ red points in a bad coloring. Thus at most $3\lfloor (2n-1)/6 \rfloor$ points in total can be colored in a bad coloring. Hence $k = 3\lfloor (2n-1)/6 \rfloor+1$ if 3 divides $n+1$, and $k = n$ otherwise. ■

Example 5 [USAMO 2008, Problem 3]
Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that $|x| + |y+½| \leq n$. A path is a sequence of distinct points (x_1, y_1), (x_2, y_2), ..., (x_k, y_k) in S_n such that, for i = 1, 2, ..., k-1, the distance between (x_i, y_i) and (x_{i+1}, y_{i+1}) is 1 (in other words, the points (x_i, y_i) and (x_{i+1}, y_{i+1}) are neighbors in the lattice of points with integer coordinates). Prove that the points in S_n cannot be partitioned into fewer than n paths (a partition of S_n into m paths is a set P of m nonempty paths such that each point in S_n appears in exactly one of the m paths in P).

Answer:
Color the points of each row of S_n alternately red and black, starting and ending with red. Any two neighboring points are of opposite color, unless they are from the middle two rows.

Consider a partition of S_n into m paths, for some m. For each of the m paths, split the path into two paths wherever there are consecutive red points. Now no path has consecutive red points. Further, since there are n pairs of consecutive red points in S_n (from the middle two rows), we have split paths at most n times. Thus we now have at most $m+n$ paths (each split increases the number of paths by one).

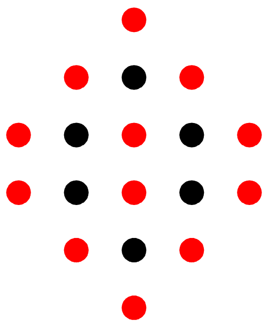

Figure 7.3. Example for n = 3

Now, there are $2n$ more red points than black points in S_n, but each of the $m+n$ paths contains at most one more red point than black point (since no path contains consecutive red points). Thus we obtain $m+n \geq 2n$, or $m \geq n$, proving the result. ■

In the next example, in order to use the chain decomposition idea we need some starting point, that is, some information or assumption that lends enough structure to the problem for us to exploit. To this end, we use the extremal principal by considering the smallest element in a certain set.

Example 6 [USAMO 2009-2]

Let n be a positive integer. Determine the size of the largest subset S of $N = \{-n, -n+1, \ldots, n-1, n\}$ which does not contain three elements

(a, b, c) (not necessarily distinct) satisfying $a + b + c = 0$.

Answer:

Obviously 0 is not in S since $0+0+0 = 0$. We claim the answer is n if n is even and $(n+1)$ if n is odd. These bounds can clearly be achieved by taking all the odd numbers in N, since the sum of three odd numbers can never be 0. To show this is the maximum, let j be the element of smallest absolute value in S (if both j and $-j$ are present, consider the positive one). Assume WLOG that $j > 0$, and let T denote the set of elements with absolute value at least j. Note that all elements of S are in T.

Case 1: $(-j)$ is not in S. Consider the pairs $(j, -2j)$, $(j+1, -(2j+1))$, $(j+2, -(2j+2))$, …., $(n-j, -n)$. In each of these pairs the sum of the numbers is $(-j)$, so at most one of the two elements is in S (otherwise the sum of the two elements plus j would be 0). There are exactly $(n-2j+1)$ pairs, so at most $(n-2j+1)$ of the paired numbers are in S.

Furthermore, there are exactly $2j-1$ unpaired numbers in $T \setminus \{-j\}$: j positive unpaired numbers (namely n, $n-1$, …, $n-j+1$), and $(j-1)$ negative unpaired numbers (namely $-(j+1)$, $-(j+2)$, …, $-(2j-1)$). Thus the maximum number of elements in S is $(2j-1) + (n-2j+1) = n$.

Case 2: $(-j)$ is also in S. Now we use the chain decomposition idea. If a and b are elements in T, then a is joined to b by an edge if and only if $a+b = j$ or $a+b = -j$. This ensures that no two elements joined by an edge can both be in S. Each element x in T is joined to at least one and at most two other elements of T ($(j-x)$ and $(-j-x)$). Hence the elements of T have been partitioned into disjoint chains.

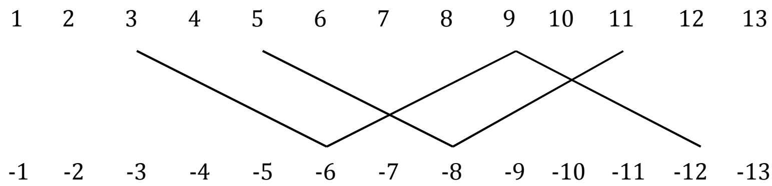

Figure 7.4: n = 13, j = 3. There are 6 chains, of which 2 are shown.

Now the rest is just counting. There are exactly $2j$ chains, namely the chains starting with $\pm j$, $\pm(j+1)$, ..., $\pm(2j-1)$. Let the lengths of these chains be l_1, l_2, ..., l_{2j}. From a chain of length l_i, at most $\left\lceil \frac{l_i}{2} \right\rceil$ elements can be in S. Thus the total number of elements we can take is at most

$$\sum_{i=1}^{2j} \left\lceil \frac{l_i}{2} \right\rceil \leq \sum_{i=1}^{2j} \frac{l_i+1}{2} = j + \frac{\sum_{i=1}^{2j} l_i}{2} = j + \frac{2n-2(j-1)}{2} = n+1.$$

Here we used $\sum_{i=1}^{2j} l_i = 2n - 2(j-1)$ since both sides are equal to the number of elements of absolute value at least j.

Now we are done if n is odd. If n is even, we need to tighten our bound to n. For this we note that the inequality $\sum_{i=1}^{2j} \left\lceil \frac{l_i}{2} \right\rceil \leq \sum_{i=1}^{2j} \frac{l_i+1}{2}$ is strict if there is a chain of even length, since $\left\lceil \frac{x}{2} \right\rceil$ is strictly less than $\frac{x+1}{2}$ for even x. It now suffices to prove the existence of a chain of even length if n is even; this is pretty simple and is left to the reader. ■

Two problems on boards

We now look at two problems based on $n \times n$ boards that initially look quite similar but are actually very different. Through these two problems we will demonstrate two important ways of thinking about and exploiting the structure of boards.

The following example uses the idea of examining individual objects' contributions toward some total. We saw a similar idea in examples 5 and 6 from chapter 3.

Example 7 [USAMO 1999-1]
Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

a) Every square that does not contain a checker shares a side with one that does;
b) Given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $\frac{n^2-2}{3}$ checkers have been placed on the board.

Answer:
Suppose we have an empty board, and we want to create an arrangement of k checkers satisfying (a) and (b). Call a square *good* if it contains a checker or shares a side with a square containing a checker. By (a), every square must eventually be *good*. Let us place the checkers on the board as follows: place one checker on the board to start, and then in each step place one checker adjacent to one that has already been placed. Since any arrangement of checkers that satisfies the problem must be

connected by (b), we can form any arrangement of checkers in this manner.

In the first step at most 5 squares become *good* (the square we placed the checker on and its neighbors). In each subsequent step, at most 3 squares that are **not** already *good* become *good*: the square we just put a checker on and the square next to it are already good, leaving 3 neighbors that could become *good*. Thus, at the end of placing k checkers, at most $5+3(k-2) = 3k+2$ squares are good. But we know all n^2 squares are *good* at the end, so $n^2 \leq 3k+2$, proving the result. ■

Remark 1: Initially the problem appears difficult due to the fact that a given square may be *good* due to more than one checker. This makes it hard to calculate "individual contributions", that is, the number of squares that are good because of a certain checker. We get around this problem by imagining the k checkers being *added sequentially*, rather than simply "being there." This allowed us to measure the "true contribution" of a checker by not counting its neighbors that are already *good*. This was just a simple example, but the ideas of introducing an element of time and adopting a dynamic view of a static problem have powerful applications in combinatorics and algorithms.

Remark 2: The only property of the board we are using is that any square has at most 4 neighboring squares. Thus we can actually think of the board as a graph with vertices representing squares and two vertices being connected if and only if they correspond to adjacent squares on the board. This interpretation of problems involving $n \times n$ boards is often very useful, and we have already used this idea in example 3 of chapter 3. With the present problem, we can easily generalize to graphs with maximum degree Δ.

At first glance, the next example looks very similar to the previous one. However, it is significantly more difficult, and the solution uses a clever coloring. Coloring is another extremely important way of exploiting the structure of boards.

Example 8 [IMO 1999, Problem 3]

Let n be an even positive integer. We say that two different cells of an $n \times n$ board are **neighboring** if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

Answer:

Let $n = 2k$. Color the board black and white in layers as shown in figure. Note that any square (black or white) neighbors exactly two black squares. Hence, since the number of black squares is $2k(k+1)$, we must mark at least $k(k+1)$ squares. On the other hand, this bound can be achieved by marking alternate black squares in each layer, in such a way that each white cell neighbors exactly one marked black square. ■

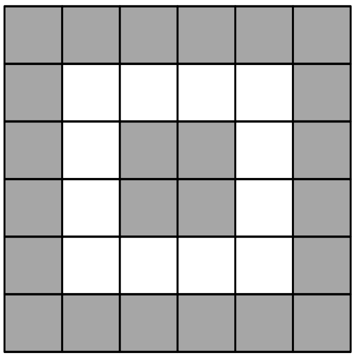

Figure 7.5: Example for n = 6.

The Classification Method

Suppose we have a set S of objects, and we want to show that there exists a large subset S' of these objects such that S' satisfies a particular condition. The idea behind the classification method is to partition (split) S into sets S_1, S_2, …, S_k such that $S_1 \cup S_2 \cup \ldots \cup S_k = S$, and furthermore, each of the sets S_1, S_2, …, S_k satisfies the given condition. Then, by the pigeonhole principle, at least one of these sets will have size at least $|S|/k$, thereby proving the existence of a subset of S of size at least $|S|/k$ satisfying the given condition.

Example 9 [IMO Shortlist 2001, C6]
For a positive integer n define a sequence of zeros and ones to be *balanced* if it contains n zeros and n ones. Two balanced sequences a and b are *neighbors* if you can move one of the $2n$ symbols of a to another position to form b. For instance, when $n = 4$, the balanced sequences 01101001 and 00110101 are neighbors because the third (or fourth) zero in the first sequence can be moved to the first or second position to form the second sequence. Prove that there is a set S of at most $\frac{1}{n+1}\binom{2n}{n}$ balanced sequences such that every balanced sequence is equal to or is a neighbor of at least one sequence in S.

Answer:
Call such a set S a *dominating set*. Our idea is to partition the set of $\binom{2n}{n}$ balanced sequences into (n+1) classes, so that the set of sequences in any class form a dominating set. Then we will be done by the pigeonhole principle, since some class will have at most $\frac{1}{n+1}\binom{2n}{n}$ balanced sequences.

To construct such a partition, for any balanced sequence A let $f(A)$ denote the sum of the positions of the ones in A (mod (n+1)). For example, $f(100101) \equiv 1+4+6 \pmod 4 \equiv 3 \bmod 4$. A sequence is in class i if and only if $f(A) \equiv i$ mod $(n+1)$. It just remains to show that every class is indeed a dominating set, that is, for any class C_i and any balanced sequence A not in C_i, A has a neighbor in C_i.

This isn't difficult: if A begins with a one, observe that moving this one immediately to the right of the kth zero gives a sequence B satisfying $f(B) \equiv f(A) + k$ mod $(n+1)$. Hence simply choose $k \equiv I - f(A)$ mod $(n+1)$, and then by shifting the first one to the right of the kth zero we end up with a sequence B satisfying $f(B) \equiv i$ mod $(n+1)$. Hence B is a sequence in C_i. The case when A begins with a zero is similar. Thus each class is indeed a dominating set and we are done by the first paragraph. ■

Remark: It is worth mentioning that the reason for naming S a "dominating set" is that this problem has a very nice graph theoretic interpretation. Dominating sets are graph structures that we will encounter in chapters 8 and 9.

Two Graph Theory Problems

Graphs are a rich source of extremal problems. We present two here, and you will see several more in the next chapter. The main thing to keep in mind when dealing with such problems is dependencies – how are all the quantities in question related? How does the size of one impact the size of another? Indeed, the entire purpose of a graph is to model connections and dependencies, so the structure of a graph invariably proves useful for answering these types of extremal questions.

Example 10 [Swell coloring]
Let K_n denote the complete graph on n vertices, that is, the graph with n vertices such that every pair of vertices is connected by an edge. A *swell coloring* of K_n is an assignment of a color to each of

the edges such that the edges of any triangle are either all of distinct colors or all the same color. Further, more than one color must be used in total (otherwise trivially if all edges are the same color we would have a swell coloring). Show that if K_n can be swell colored with k colors, then $k \geq \sqrt{n} + 1$.

Answer:
Let $N(x, c)$ denote the number of edges of color c incident to a vertex x. Fix x_0, c_0 such that $N(x_0, c_0)$ is maximum and denote this maximum value by N. There are $n-1$ edges incident to x_0, colored in at most k colors, with no color appearing more than N times. Hence $Nk \geq n-1$.

Now consider vertex y such that edge yx_0 is **not** of color c_0. Also let $x_1, x_2, \ldots, x_N$ be the N vertices joined to x_0 by color c_0. Note that for any i and j in $\{1, 2, \ldots, N\}$, x_ix_j is of color c_0 since x_0x_i and x_0x_j are of color c_0.

Suppose yx_i is of color c_0 for some i in $\{1, 2, \ldots, N\}$. Then triangle yx_ix_0 contradicts the swell coloring condition, since two sides (yx_i and xx_i) are the same color c_0 but the third side isn't. Hence the color of yx_i is not c_0 for $i = 1, 2, \ldots, N$.

Now suppose yx_i and yx_j are the same color for some distinct i and j in $\{0, 1, 2, \ldots, N\}$. Then x_ix_j also must be this color. But x_ix_j is of color c_0, which implies yx_i and yx_j are also of color c_0, contradicting our earlier observation.

It follows that $yx_0, yx_1, yx_2, \ldots, yx_N$ are all different colors, and none of them is c_0. This implies that there are at least $N+2$ distinct colors, so $k \geq N+2$. Since we already showed $Nk \geq n-1$, it follows that $k(k-2) \geq n-1$, from which the desired bound follows. ■

Remark: Basically, one can think about the above proof as follows: either there is a big clique of one color, or there isn't. If there isn't, then we need many colors to avoid big monochromatic cliques. If there is, then anything outside this clique needs many different colors to be connected to the clique.

Example 11 [Belarus 2001]
Given n people, any two are either friends or enemies, and friendship and enmity are mutual. I want to distribute hats to them, in such a way that any two friends possess a hat of the same color but no two enemies possess a hat of the same color. Each person can receive multiple hats. What is the minimum number of colors required to always guarantee that I can do this?

Answer:
Set up a graph in the usual way, with vertices standing for people and edges between two people if and only if they are friends. Note that if we have a complete bipartite graph with $\lfloor n/2 \rfloor$ vertices on one side and $\lceil n/2 \rceil$ vertices on the other, then we need at least $\lfloor n/2 \rfloor \lceil n/2 \rceil$ colors. This is because we need one for each pair of friends and no color could belong to more than two people (otherwise some two people on the same side of the bipartition would have the same color, which is not possible since they are enemies). We claim this is the worst case, that is, given $\lfloor n/2 \rfloor \lceil n/2 \rceil$ colors we can always satisfy the given conditions. We will use strong induction, the base cases n = 1, 2, 3 being easy to check.

Obviously if the graph has fewer than $\lfloor n/2 \rfloor \lceil n/2 \rceil$ edges we are done, since we can assign a separate color for each pair of friends. Now if the graph has more than $\lfloor n/2 \rfloor \lceil n/2 \rceil$ edges, then by the contrapositive of corollary 1 in chapter 6 (example 8), the graph contains a triangle. Use one color for the triangle (that is, give each member of the triangle a hat of that color). Using at most n−3 colors, we can ensure that each person not in the triangle who is friends with some member(s) of the triangle has a common color with them. Now among the remaining n−3 people, we need at most $\lfloor (n-3)/2 \rfloor \lceil (n-3)/2 \rceil$ more colors by the induction hypothesis. Hence in total we use at most

$(1+(n-3)+\lfloor (n-3)/2 \rfloor \lceil (n-3)/2 \rceil) \leq \lfloor n/2 \rfloor \lceil n/2 \rceil$ colors. ■

Remark: The important part of the previous example is guessing the worst case scenario. Intuitively, when there are too many or too few edges, we don't need many colors, because we would then have either very few enemies or very few friends. This leads us to guess that the worst case is "somewhere in the middle". In these cases, bipartite graphs should be your first suspects (followed by multipartite graphs). This gives us the intuition needed to complete the solution- the simplest structure we can exploit in a graph with "too many edges" is that it will have triangles.

Induction and Combinatorics of Sets

In this section we will use induction to solve extremal problems on sets. We first establish two simple lemmas.

Lemma 7.1: Let S be a set with n elements, and let F be a family of subsets of S such that for any pair A, B in F, $A \cap B \neq \emptyset$. Then $|F| \leq 2^{n-1}$.

Proof: For any set A in F, the complement of A, that is, $S \backslash A$, cannot be in F. So at most ½ of the total number of subsets of S can be in F. Equality is achieved by taking all subsets of S containing a fixed element x in S.

Lemma 7.2: Let S be a set with n elements, and let F be a family of subsets of S such that for any pair A, B in F, S is not contained by $A \cup B$. Then $|F| \leq 2^{n-1}$.

Proof: The proof is identical to that of lemma 7.1. Equality holds by taking all subsets of S excluding a fixed element x of S.

Example 12 [Iran TST 2008]

Let $S = \{1, 2, \ldots, n\}$, and let F be a family of 2^{n-1} subsets of S.

Suppose for all A, B and C in S, $A \cap B \cap C \neq \emptyset$. Show that there is an element in S belonging to all sets in F.

Answer

Suppose there are sets X and Y in F such that $|X \cap Y| = 1$. Then we are trivially done since all sets in F must contain this element by the intersection condition of the problem. Now assume that F is 2-intersecting; that is, $|X \cap Y| \geq 2$ for each X and Y in F. We prove by induction on n that in this case, $|F| < 2^{n-1}$, showing that this case cannot occur.

Base cases $n = 1$ and 2 are trivial, so assume the result holds for $n-1$ and that we are trying to prove it for n. We can write F as $F = F_n \cup F_{n-1}$, where F_n consists of all sets in F containing n and F_{n-1} consists of all sets in F **not** containing n.

By the induction hypothesis, $|F_{n-1}| < 2^{n-2}$. Now define $F_n' = \{S\backslash n \mid S \in F_n\}$. In other words, define F_n' is obtained from F_n by deleting n from all sets in F_n. Since F_n is 2-intersecting, F_n' is still intersecting, so by lemma 7.1, $|F_n'| \leq 2^{n-2}$. Note that $|F_n'| = |F_n|$, so we get $|F| = |F_n| + |F_{n-1}| = |F_n'| + |F_{n-1}| < 2^{n-2} + 2^{n-2} < 2^{n-1}$, as desired. ■

Example 13 [Kleitman's lemma] (U*)

A set family F is said to be *downwards closed* if the following holds: if X is a set in F, then all subsets of X are also sets in F. Similarly, F is said to be *upwards closed* if whenever X is a set in F, all sets containing X are also sets in F. Let F_1 and F_2 be downwards closed families of subsets of $S = \{1, 2, ..., n\}$, and let F_3 be an upwards closed family of subsets of S.

(a) Show that $|F_1 \cap F_2| \geq \frac{|F_1||F_2|}{2^n}$.

(b) Show that $|F_1 \cap F_3| \leq \frac{|F_1||F_3|}{2^n}$. .

Answer

a) Induct on n. The base case $n = 1$ is trivial, so assume the result holds for $(n-1)$ and that we are trying to prove it for n. Let X_1 be the family of sets in F_1 containing n and Y_1 be the family of sets not containing n. Delete n from each set in X_1 to obtain a new family X_1'. Note that $|X_1'| = |X_1|$ and $|X_1'|+|Y_1| = |F_1|$. Also note that X_1 and Y_1 are still downwards closed. Analogously define X_2, Y_2 and X_2' for the family F_2.

Observe that $|F_1 \cap F_2| = |Y_1 \cap Y_2| + |X_1' \cap X_2'| \geq \frac{|Y_1||Y_2|}{2^{n-1}} + \frac{|X_1'||X_2'|}{2^{n-1}}$ (applying the induction hypothesis). Since F_1 and F_2 are downwards closed, X_1' is a subset of Y_1, and similarly X_2' is a subset of Y_2. Hence $|Y_1| \geq |X_1'|$, $|Y_2| \geq |X_2'|$, so Chebyshev's inequality (or just basic algebra) yields $\frac{|Y_1||Y_2|}{2^{n-1}} + \frac{|X_1'||X_2'|}{2^{n-1}} \geq (|Y_1| + |X_1'|)(|Y_2| + |X_2'|)/2^n = \frac{|F_1||F_2|}{2^n}$.

b) The proof is similar to that in part (a), but with inequality signs reversed. ■

Lemmas (i) and (ii) in this section are fairly straightforward. Note that in both cases, the imposition of a certain constraint decreases the number of sets we can have by a factor of ½ (we can include 2^{n-1} sets out of 2^n total possibilities). A natural question is what happens if we impose both conditions simultaneously – will the number of possible sets in the family be decreased by a factor of 4? Interestingly, Kleitman's lemma answers this question in the affirmative.

Example 14 (U*)

Let $F = \{A_1, A_2, \ldots, A_k\}$ be a family of subsets of $S = \{1, 2, \ldots, n\}$ ($n > 2$), such that for any distinct subsets A_i and A_j, $A_i \cap A_j \neq \emptyset$ and $A_i \cup A_j \neq S$. Show that $k \leq 2^{n-2}$.

Answer

F can be extended to a downward closed system D by adding all subsets of the sets in F. Similarly, F can be extended to an upward closed system U by adding all subsets of S that contain some set in

F. Note that $F = U \cap D$. Since F is intersecting, so is U (since in creating U we only added "big" sets). Hence by lemma 7.1, $|U| \le 2^{n-1}$. Similarly, since the union of no two sets in F covers S, the same holds for D. Hence by lemma 7.2, $|D| \le 2^{n-1}$. Then by part (b) of the previous problem,
$k = |F| = |U \cap D| \le |U||D|/2^n \le 2^{n-1} \times 2^{n-1}/2^n = 2^{n-2}$. ■

Example 15 [stronger version of USA TST 2011]
Let $n \ge 1$ be an integer, and let S be a set of integer pairs (a, b) with $1 \le a < b \le 2^n$. Assume $|S| > n2^n$. Prove that there exist four integers $a < b < c < d$ such that S contains all three pairs (a, c), (b, d) and (a, d).

Answer:
We induct on n. The base cases being trivial, suppose the result holds for $(n-1)$. Let S' be the set of pairs (a, b) in S with $a < b \le 2^{n-1}$. If $|S'| \ge (n-1)2^{n-1}$, we would be done by applying the induction hypothesis to S'.

Similarly, let S'' be the set of pairs (a, b) with $2^{n-1} < a < b$. If $|S''| \ge (n-1)2^{n-1}$ we would again be done by applying the induction hypothesis to S'', treating the pair (a, b) as if it were the pair $(a-2^{n-1}, b-2^{n-1})$. (Take a moment to fully understand this.)

Now suppose neither of these cases arises. Then more than $(n2^n-2(n-1)2^{n-1}) = 2^n$ pairs (a, b) would have to satisfy $a \le 2^{n-1} < b$.

We call a pair (a, b) in S a *B-champion* if $a \le 2^{n-1} < b$, and b is the smallest number greater than 2^{n-1} with which a occurs in a pair. Note that there is at most 1 *B-champion* for fixed a, and at most 2^{n-1} choices for a. Thus there are at most 2^{n-1} *B-champions*. Similarly, define an *A-champion* to be a pair (a, b) in S if $a \le 2^{n-1} < b$

such that a is the largest number less than or equal to 2^{n-1} with which B is paired. The same argument shows that there are at most 2^{n-1} *A-champions*.

Since there are more than 2^n pairs (a, b) with $a \leq 2^{n-1} < b$, at least one of these pairs, say (x, y) is neither an A-champion nor a B-champion. Then there must exist z such that $2^{n-1} < z < y$ and (x, z) is in S (since (x, y) is not a B- champion). Similarly, there exists w such that $x < w \leq 2^{n-1}$ and (y, w) is in S. Hence $x < w < z < y$, and (x, z), (w, y) and (x, y) are all in S, proving the statement of the problem. The induction step, and hence the proof, is complete. ■

Remark 1: Observe the structure of this solution- we first tried to find a suitable subset of S to which we could apply the induction hypothesis; that is, we tried to break the problem down. We then solved the problem for the cases in which this didn't work. By dealing with an easy case of a problem first, we do more than just get the easy case out of the way. We actually learn some important conditions a case must satisfy to *not* be easy- and this information is crucial for handling the hard case. This subtly illustrates the following problem solving tenet - *the first step in solving a hard problem often lies in identifying what makes the problem hard.*

Remark 2: The definition of *A-champions* and *B-champions* in the above solution initially appears to come out of nowhere. However, after carefully reading the whole solution, the purpose behind it becomes clear: we are trying to find a pair (x, y) with w and z "squished between" x and y such that (x, z) and (w, y) are pairs in S. The only way this can happen is if (x, y) is neither an *A-champion* nor a *B-champion*.

Example 16 [The Sunflower Lemma]
A sunflower with k petals and a core X is a family of sets $S_1, S_2, \ldots, S_k$ such that $S_i \cap S_j = X$ for each $i \neq j$. (The reason for the name is that the Venn diagram representation for such a family resembles

a sunflower.) The sets $S_i \setminus X$ are known as petals and must be nonempty, though X can be empty. Show that if F is a family of sets of cardinality s, and $|F| > s!(k\text{-}1)^s$, then F contains a sunflower with k petals.

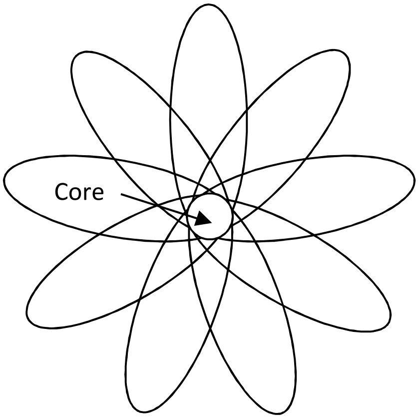

Figure 7.6: A Sunflower.

Answer:
Induct on s. The result is trivial for $s = 1$, since then k singleton sets will form the sunflower (its core will be empty but that's okay, no one said the sunflower has to be pretty). Passing to the inductive step, let $s \geq 2$ and take a maximal family $A = \{A_1, A_2, \ldots, A_t\}$ of pairwise disjoint sets in F. If $t \geq k$ we are done, since this family will form our required sunflower (with empty core). Now suppose $t \leq k - 1$ and let $B = A_1 \cup A_2 \cup \ldots \cup A_t$. Note that $|B| = st \leq s(k-1)$. Also, by the maximality of A, it follows that B intersects all sets in F (otherwise we could add more sets to A). Hence by the pigeonhole principle, some element x of B must be contained in at least

$$\frac{|F|}{|B|} > \frac{s!(k-1)^s}{s(k-1)} = (s-1)!(k-1)^{s-1}$$

sets in F. Deleting x from these sets and applying the induction hypothesis to these sets (which now contain s–1 elements each), we see that there exists a sunflower with k petals. Adding x back to all these sets doesn't destroy the sunflower since it just goes into the core, so we get the desired sunflower. ■

Example 17 [IMO Shortlist 1998, C4]

Let $\boldsymbol{U} = \{1, 2, \ldots, n\}$, where $n \geq 3$. A subset $\boldsymbol{S}$ of $\boldsymbol{U}$ is said to be *split* by an arrangement of the elements of $\boldsymbol{U}$ if an element not in $\boldsymbol{S}$ occurs in the arrangement somewhere between two elements of $\boldsymbol{S}$. For example, 13542 splits {1, 2, 3} but not {3, 4, 5}. Prove that for any family $\boldsymbol{F}$ of $(n-2)$ subsets of $\boldsymbol{U}$, each containing at least 2 and at most $n-1$ elements, there is an arrangement of the elements of $\boldsymbol{U}$ which splits all of them.

Answer:

We induct on n. As always, we unceremoniously dismiss the base case as trivial and pass to the induction step, assuming the result for $n-1$ and prove it for n. We first prove a claim.

Claim: There exists an element a in $\boldsymbol{U}$ that is contained in all subsets of $\boldsymbol{F}$ containing $n - 1$ elements, but in at most one of the 2-element subsets.

Proof: A simple counting argument suffices. Let $\boldsymbol{F}$ contain k $(n-1)$-element subsets and m 2-element subsets. Note that $k+m$ is at most the total number of subsets in $\boldsymbol{F}$, which is $n-2$. Hence $(k+m) \leq (n-2)$. The intersection of the k $(n-1)$-element subsets contains exactly $(n-k)$ elements. This is because for each of these subsets there is exactly one element it doesn't contain.

But $n-k \geq m+2$ and at most m elements can be in more than one of the two-element sets. Thus one of these elements that is in the intersection of all the $(n-1)$-element subsets is in at most one of the 2-element sets, proving the claim.

Now let $\boldsymbol{A}$ be the 2-element subset that contains a if it exists; otherwise let it be an arbitrary subset of $\boldsymbol{F}$ containing a. Now exclude a from all subsets in $\boldsymbol{F} \setminus \boldsymbol{A}$. We get at most $n-3$ subsets of $\boldsymbol{U} \setminus \{a\}$ containing at least 2 and at most $n-2$ elements. Applying the inductive hypothesis, we can arrange the elements of $\boldsymbol{U} \setminus a$ so as to split all subsets of $\boldsymbol{F} \setminus \boldsymbol{A}$. Replace a anywhere away from $\boldsymbol{A}$ and we are done. ■

Exercises

1. **[Generalization of USAMO 1999, Problem 4]**
   Find the smallest positive integer m such that if m squares of an $n \times n$ board are colored, then there will exist 3 colored squares whose centers form a right triangle with sides parallel to the edges of the board.

2. **[Erdos-Szekeres Theorem] (U*)**
   Show that any sequence of n^2 distinct real numbers contains a subsequence of length n that is either monotonically increasing or monotonically decreasing.

3. **[USA TST 2009, Problem 1]**
   Let m and n be positive integers. Mr. Fat has a set $\boldsymbol{S}$ containing every rectangular tile with integer side lengths and area of a power of 2. Mr. Fat also has a rectangle R with dimensions $2^m \times 2^n$ but with a 1×1 square removed from one of the corners. Mr. Fat wants to choose $(m+n)$ rectangles from $\boldsymbol{S}$, with respective areas 2^0, 2^1, …, 2^{m+n-1} and then tile R with the chosen rectangles. Prove that this can be done in at most $(m+n)!$ ways.

4. **[Generalization of APMO 2012, Problem 2]**
   Real numbers in [0, 1] are written in the cells of an $n \times n$ board. Each gridline splits the board into two rectangular parts. Suppose that for any such division of the board into two parts along a gridline, at least one of the parts has weight at most 1, where the weight of a part is the sum of all numbers written in

cells belonging to that part. Determine the maximum possible sum of all the numbers written on the board.
[Challenge: generalize to k dimensional boards.]

5. [Indian postal coaching 2011]
Consider 2011^2 points arranged in the form of a 2011 x 2011 grid. What is the maximum number of points that can be chosen among them so that no four of them form the vertices of either an isosceles trapezium or a rectangle whose parallel sides are parallel to the grid lines?

6. [China Girls Math Olympiad 2004]
When the unit squares at the 4 corners are removed from a 3×3 square, the resulting shape is called a cross. Determine the maximum number of non-overlapping crosses that can be placed within the boundary of a 10×11 board.

7. [IMO Shortlist 2010, C2]
Let $n > 3$ be a positive integer. A set of n distinct binary strings of length n is said to be *diverse* if there exists an $n \times n$ array whose rows are these n binary strings in some order, and all entries along the main diagonal of this array are equal. Find the smallest integer m, such that among any m binary strings of length n, there exist n strings forming a diverse set.

8. [Iran TST 2007]
Let $\boldsymbol{A}$ be the largest subset of $\{1, 2, \dots, n\}$ such that for each $x \in \boldsymbol{A}$, x divides at most one other element in $\boldsymbol{A}$. Show that $2n/3 \leq |\boldsymbol{A}| \leq \left\lceil \frac{3n}{4} \right\rceil$.

9. [IMO 2014, Problem 2]
Let n be a positive integer, and consider an $n \times n$ board. Suppose some rooks are placed on this board such that each row contains exactly one rook and each column contains exactly one rook. Find the largest integer k such that for any such configuration as described above, there necessarily

exists a $k \times k$ square which does not contain a rook on any of its k^2 squares.

10. [IMO 2013, Problem 2]

A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is *good* for a Colombian configuration if the following conditions are satisfied:

(i) No line passes through any point of the configuration

(ii) No region contains points of both colors

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

11. [IMO Shortlist 2005, C3]

Consider an $m \times n$ rectangular board consisting of mn unit squares. Two of its unit squares are called adjacent if they have a common edge, and a path is a sequence of unit squares in which any two consecutive squares are adjacent. Two paths are called non-intersecting if they don't share any common squares. Suppose each unit square of the rectangular board is colored either black or white.

Let N be the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge. Let M be the number of colorings of the board for which there exist at least two non-intersecting black paths from the left edge of the board to its right edge.

Prove that $N^2 \geq M \times 2^{mn}$.

12. [Balkan Math Olympiad 1994]

Find the smallest number $n \geq 5$ for which there can exist a set of n people, such that any two people who are friends have no common friends, and any two people who are not friends have exactly two common acquaintances.

13. Given a set ***S*** of n points in 3-D space, no three on a line, show that there exists a subset ***S'*** of ***S*** containing at least $n^{1/4}$ points, such that no subset of the points in **S'** form a regular polygon.

14. [IMO Shortlist 2000, C4]
Let n and k be positive integers such that $n/2 < k \leq 2n/3$. Find the smallest number m for which it is possible to mark m squares on an $n \times n$ board such that no row or column contains a block of k adjacent unoccupied squares.

15. [IMO Shortlist 1988]
The code for a safe is a three digit number with digits in {1, 2, …, 8}. Due to a defect in the safe, it will open even if the number we enter matches the correct code in two positions. (For example, if the correct code is 245 and we enter 285, it will open.) Determine the smallest number of combinations that must be tried in order to guarantee opening the safe.

16. [Bulgaria 1998]
Let n be a given positive integer. Determine the smallest positive integer k such that there exist k binary (0-1) sequences of length $2n+2$, such that any other binary sequence of length $2n+2$ matches one of the k binary sequences in at least $n+2$ positions.

17. [IMO Shortlist 1996, C3]
Let k, m, n be integers satisfying $1 < n \leq m-1 \leq k$. Determine the maximum size of a subset ***S*** of the set $\{1, 2, \ldots, k\}$ such that no n distinct elements of ***S*** add up to m.

18. [IMO Shortlist 1988]
49 students took part in a math contest with three problems. Each problem was worth 7 points, and scores on each problem were integers from 0 to 7. Show that there exist two

students A and B such that A scored at least as many points as B on each of the three problems.

19. [USAMO 2007, Problem 3]

Let $\boldsymbol{S}$ be a set containing (n^2+n-1) elements, for some positive integer n. Suppose that the n-element subsets of $\boldsymbol{S}$ are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

20. [IMO Shortlist 2009, C6]

On a 999 × 999 board a limp rook can move in the following way: from any square it can move to any of its adjacent squares, that is, a square having a common side with it, and every move must be a turn, that is, the directions of any two consecutive moves must be perpendicular. A non-intersecting route of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called cyclic, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.

How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

21. [USAMO 2002, Problem 6]

Some trominoes (3×1 tiles) are to be placed on an $n \times n$ board without overlaps or trominoes sticking out of the board. Let $b(n)$ denote the minimum number of trominoes that must be placed so that no more can be placed according to the above rules. Show that there exist constants c and d such that $n^2/7 - cn \le b(n) \le n^2/5 + dn$.

22. [IberoAmerican Math Olympiad 2009, Problem 6]

6000 points on the circumference of a circle are marked and colored with 10 colors such that every group of 100 consecutive points contains all ten colors. Determine the smallest positive integer k such that there necessarily exists a group of k consecutive points containing all ten colors.

23. [IMO Shortlist 2011, C6]
Let n be a positive integer, and let $\boldsymbol{W} = \ldots x_{-1}\ x_0\ x_1 \ldots$ be an infinite periodic word, consisting of just letters a and/or b. Suppose that the minimal period N of $\boldsymbol{W}$ is greater than 2^n.
A finite nonempty word $\boldsymbol{U}$ is said to appear in $\boldsymbol{W}$ if there exist indices $k \leq l$ such that $\boldsymbol{U} = x_k \ldots x_l$. A finite word U is called ubiquitous if the four words $\boldsymbol{Ua}$, $\boldsymbol{Ub}$, $\boldsymbol{aU}$ and $\boldsymbol{bU}$ all appear in $\boldsymbol{W}$. Prove that there are at least n ubiquitous finite nonempty words.

24. [IMO Shortlist 2007, C8]
Consider a convex polygon $\boldsymbol{P}$ with n vertices. A triangle whose vertices lie on vertices of $\boldsymbol{P}$ is called good if all its sides have equal length. Prove that there are at most $2n/3$ good triangles.

25. [Stronger version of IMO Shortlist 2008, C6]
For $n > 2$, let $\boldsymbol{S}_1$, $\boldsymbol{S}_2, \ldots, \boldsymbol{S}_{2^n}$ be 2^n subsets of $\boldsymbol{A} = \{1, 2, \ldots, 2^{n+1}\}$ that satisfy the following property: There do not exist indices a and b with $a < b$ and elements $x, y, z \in \boldsymbol{A}$ with $x < y < z$ and $y, z \in \boldsymbol{S}_a$ and $x, z \in \boldsymbol{S}_b$. Prove that at least one of the sets S_1 , $S_2, \ldots, S_{2^n}$ contains at most $2n+1$ elements. (Note: the original problem had a weaker bound of $4n$ instead of $2n+1$.)

26. [IMO Shortlist 2005, C8]
In a certain n-gon, some $(n-3)$ diagonals are colored black and some other $(n-3)$ diagonals are colored red, so that no two diagonals of the same color intersect strictly inside the polygon, although they can share a vertex. (Note: a side is not a diagonal.) Find the maximum number of intersection points between diagonals colored differently strictly inside the polygon, in terms of n.

27. [IMO Shortlist 2011, C7]

On a 2011×2011 square table we place a finite number of napkins that each cover a square of 52 by 52 cells. Napkins can overlap, and in each cell we write the number of napkins covering it, and record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?