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Use One-to-One Correspondence to solve problems

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In many problems, a key step to find a solution is to use the fact that for some quantity g in the problem, a corresponding quantity g^* exits. The actual application of this simple principle can take many different forms some of which are:

1. Since for each g, there exists a g^*, we have $\cdots$
2. Algebraically manipulating g and g^* could simplify some complicated looking expressions in the problem.
3. (Principle of 1-1 correspondence)
   If A and B are two sets and there is a bijection $f : A \to B$, then $|A| = |B|$.

Ex1. Every real polynomial of odd degree must have at least one real root.
(Reason: $z \longleftrightarrow \overline{z}$).

Ex2. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$

Ex3. Evaluate $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx \qquad \left(\text{Answer} : \frac{\pi^2}{4}\right)$

Ex4. (Gauss' ingeneous idea)
$1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.
(Here, $g = k, g^* = n - k + 1, k = 1, 2, \cdots, n$)

Ex5. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

Ex6. In how many ways can n identical candies be consumed if you have to eat at least one a day?

Ex7. In how many ways can p 1's and q 0's be lined up in a row so that no two 0's are adjacent ($q \leq p + 1$)?

Ex8. Suppose n tennis players compete in a tournament ($n \geq 2$). Each player plays one game against each of the other $n - 1$ players. Let w_i and l_i denote the numbers of games won and lost by the ith player, $i = 1, 2, \cdots, n$. (assuming no tie). Show that (i) $\sum w_i = \sum l_i$ and (ii) $\sum w_i^2 = \sum l_i^2$.

Ex9. For $n \in \mathbb{N}$, let $\tau(n)$ denote the number of (positive) divisors of n. Show without using the known formula for $\tau(n)$, that $\tau(n)$ is odd if and only if n is a square.

Ex10. For $n \in \mathbb{N}$, let $\sigma(n)$ denote the sum of all the divisors of n. Show that $\sigma(n) \geq \tau(n)\sqrt{n}$.

Ex11. Let $n \in \mathbb{N}$ be odd. Prove that the number of divisors d of n with $d \geq \sqrt{n}$ equals the number of representations of n as $n = a^2 - b^2$ where $a, b \in \mathbb{Z}$ with $a > b \geq 0$.
(i.e. $15 = 4^2 - 1^2 = 8^2 - 7^2$ and the two divisors of 15 which are at least $\sqrt{15}$ are 5 and 15.)

Ex12. Choose n points on a circle ($n \geq 3$). Connect every pair of points with a chord. Suppose no three of the chords are concurrent. Determine the number of

(i) intersection points inside the circle.

(ii) triangles all of whose three vertices are inside the circle.

Ex13. In the figure, the side length of the large equilateral triangle is 3 and $f(3)$, the number of parallelograms contained in the grid, is 15. For the general analogous situation, find a formula for $f(n)$ for a triangle of side length n. (1991 CMO)

Ex14. For the annual Sino-Japanese "Go" tournament, each country sends a team of seven players, C_i's and J_i's, respectively. All the players of each country are of different ranks (strength) so $C_1 < C_2 < \cdots < C_7$ and $J_1 < J_2 < \cdots < J_7$. The tournament starts with a match between C_1 and J_1. Each match is determined by one game only, with no tie. The winner then takes on teh next higher ranked player of the opponent country. The tournament continues until all the seven players of the country are eliminated and then the other country is declared the winner. (For people who are familiar with the ancient Chinese Chess game of "Go", a better and perhaps more descriptive translation would be "the surrounding chess".) What is the total number of possible sequences of outcomes if each country sends in n players?

Ex15. Serge was solving the equation $f(19x - \frac{96}{x}) = 0$ and found 11 different solutions. Prove that if he tried hard, he would be able to find at least one more solution.

Ex16. For $n \in \mathbb{N}$, let $f(n)$ denote the number of divisors d of n^2 such that $d < n$ and $d \nmid n$. Express $f(n)$ using the τ−function. In particular, find $f(n)$ for $n = 2^{13} \cdot 3^{11} \cdot 5^7$.
$\left(\text{Answer : } \frac{1}{2}(\tau(n^2) - 2\tau(n) + 1);\ 3314\right)$
(XL Math Olympiad of the Republic of Moldova.)

Ex17*. Show that $\sum_{i=1}^{n} \tau(i) = \left(2 \sum_{i=1}^{\lfloor\sqrt{n}\rfloor} \lfloor \frac{n}{i} \rfloor\right) - \lfloor\sqrt{n}\rfloor^2$.

Ex18. Show that among any 52 integers chosen, there must be two such that 100 divides their sum or difference. (1995 Icelandic Math Olympiad.)

Ex19*. Determine the number of ways of expressing $n \in \mathbb{N}$ as the sum of three positive integers if order matters. (e.g. 6=1+2+3=3+1+2 are considered different.)
(First Moscow Math Olympiad.)

Ex21*. Determine the number, $a(n)$, of ordered partitions of $n \in \mathbb{N}$ as a sum of 1's and 2's.
(e.g. $a(3) = 3$ since 3=1+1+1=1+2=2+1).
(17 th Putnam Competition.)

Ex22. Without using the binomial theorem, show that $\sum_{k=1}^{n} \binom{n}{k} n^k = (1+n)^n - 1$.

Ex23*. Let n and r be positive integers such that $n \geq 2$. Let $g = gcd(n,r)$.
Prove that $\sum_{i=1}^{n-1} \langle \frac{ri}{n} \rangle = \frac{1}{2}(n-g)$ where $=< x >= x - \lfloor x \rfloor$ is the fractional part of x.
(1995 Japan Math Olympiad).