1.

Some Combinatorial Problems

Winter Camp 2006
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1) Consider the task of choosing a K-member team, with a team captain, from a pool of n players to prove: $K\binom{n}{K} = n\binom{n-1}{K-1}$

2) Evaluate: $\sum_{K=0}^{n} (2K+1)\binom{n}{K}$

3) Show that $\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3} = n^3$ combinatorially

and hence evaluate: $1^3 + 2^3 + \cdots + n^3$.

4) Show that: $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$

5) Show that: $\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$

6) Show that: $(n-r)\binom{n+r-1}{r}\binom{n}{r} = n\binom{n+r-1}{2r}\binom{2r}{r}$

7) Given $n > 4$ points in a plane, prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals which can be formed with these points as their vertices.

8) How many integer solutions are there to $x_1 + x_2 + x_3 = 0$ with $x_i \geq -5$?

9) In how many ways can 9 identical balls be placed in 5 distinguishable cans?

10) In how many ways can 26 identical marbles be distributed among 4 different size cups (maximum capacities of 14, 16, 18, and 20 marbles) if no cup is to be empty?

11) How many proper irreducible fractions have a denominator of 2006?

12) How many vertices of a 2006-gon can lie on a single straight line?

13) A 10×10 square table is filled in with the numbers: zero, one, or negative one in each cell. The cells in each row, column, and diagonal are added. Prove that, among all these sums, at least two are identical.

14) David had 100 cards marked with the numbers from 1 to 100, but he lost 79 of them. Prove that he can still find two pairs from those remaining which have equal sums (□+□=□+□).

15) Alex had a similar set of cards, but he only lost 31 out of his set. Prove that, among his remaing cards, there are three whose sum equals another of his remaining cards. (□+□+□=□).

16) A rectangle consists of 5 rows of 41 squares, each painted either pink or purple. Prove that it is possible to find 3 rows and 3 columns so that the 9 square at their intersections are all the same colour.

17) Is it possible to label the vertices of a 28-gon with the letters from A to H (repeated as necessary) so that each side is assigned a different pair of letters (from its endpoints). Note: pairs XY and YX are considered identical, and "pairs" XX and YY are not allowed.

18) Given 33 different positive integers under 100, prove that it is always possible to choose two with a difference of either 8 or 9 or 17.

19) Prove that the digits of any six digit number can be rearranged so that the sum of the first three digits differs by at most 9 from the sum of the remaining digits.

20) The unit cube: $C = \{(x, y, z) \mid 0 \le x, y, z \le 1\}$ is cut along the planes $x = y$, $y = z$, and $z = x$. How many pieces are there?

21) A set of $2n+1$ real numbers has the property that the sum of any n of them is less than the sum of the remaining $n+1$. Show that they are all positive.

22) In a 10×10 table all the integers from 1 to 100 are written. From each row, the third largest number is selected. Show that the sum of the selected numbers exceeds the sum of the numbers in some row.

23) Given $2n+3$ points in the plane, no three collinear and no four on a circle, prove that there exists a circle through three of the points with exactly n of the remaining points in its interior.

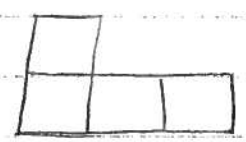
24) Is it possible to move a Knight on a 5×5 chessboard so that it returns to it's original position after having visited each square of the board exactly once?

25) There is one stone at each vertex of a square. You may change the number of stones according to this rule:

You may take away any number of stones from one vertex while adding double that number to one of the adjacent vertices.

Is it possible to get exactly 2000, 2003, 2005 and 2006 stones at the four corners?

26) Prove that a 4×11 rectangle cannot be covered by L-shaped 3×2 pieces:



27) What is the smallest number of squares on an 8×8 chess board which would have to be painted so that no 3×1 rectangle could be placed on the board without covering a painted square?