1 Problems

1. Sequences $(a_n), (b_n), (c_n)$, and (d_n) satisfy the following conditions:

$$\begin{aligned} a_{n+1} &= a_n + b_n, \qquad & b_{n+1} &= b_n + c_n, \\ c_{n+1} &= c_n + d_n, \qquad & d_{n+1} &= d_n + a_n, \end{aligned}$$

   for $n = 1, 2, \ldots$. If the sequences are periodic[1], prove that $a_2 = b_2 = c_2 = d_2 = 0$.

2. Let m, n be positive integers. Show that the number of positive integers x such that

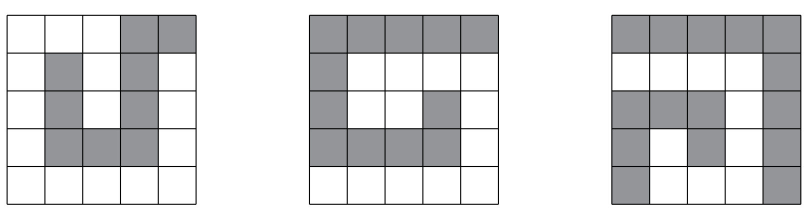
$$\sum_{i=1}^{n} \left\lfloor \frac{x}{2^i} \right\rfloor = x - m$$

   is at most 2^n.

3. In a cyclic quadrilateral $ABCD$ with $AB = AD$, points M, N lie on the sides BC and CD respectively so that $BM + ND = MN$. Lines AM and AN meet the circumcircle of $ABCD$ again at points P and Q respectively. Prove that the orthocenter of the triangle APQ lies on the segment MN.

4. Let $n \geq 3$ be an odd integer. Amy has coloured the squares in an $n \times n$ grid white and black. We will call a sequence of squares $S_1, S_2, \ldots, S_m$ a "path" if all these squares are the same colour, if S_i and S_{i+1} share an edge for all $i \in \{1, 2, \ldots, m-1\}$, and if no other squares in the sequence share an edge. Prove that if both the white squares and black squares form a single path, then one of these paths must begin or end at the center of the grid.

   For example, the grid on the left is a valid colouring, but the other two are invalid. In the middle grid, the white squares loop back and touch themselves, and in the right grid, the black squares are disconnected.



[1] A sequence (t_n) is said to be "periodic" if there exists a positive integer p for which $t_{n+p} = t_p$ for all n.

2 Solutions

1. If a sequences (t_n) is periodic, note that $|t_n|$ must be upper-bounded by some constant[2]. Specifically, if the sequence has period p, then $a_n \in \{a_1, a_2, \dots, a_p\}$ for all n, and hence $|a_n| \leq \max(a_1, a_2, \dots, a_p)$.

   In our case, this means the sequence $(a_n + b_n + c_n + d_n)$ is bounded (since $(a_n), (b_n), (c_n)$, and (d_n) are all bounded). Substituting the given recurrence in for $n \geq 1$, we have $a_{n+1} + b_{n+1} + c_{n+1} + d_{n+1} = 2(a_n + b_n + c_n + d_n)$. To prevent $|a_n + b_n + c_n + d_n|$ from getting arbitrarily large, $a_n + b_n + c_n + d_n$ must be 0 for all n.

   If $n \geq 2$, we then have:

$$\begin{aligned} a_{n+2} &= a_{n+1} + b_{n+1} \\ &= a_n + 2b_n + c_n \\ &= a_{n-1} + 3b_{n-1} + 3c_{n-1} + d_{n-1} \\ &= 2b_{n-1} + 2c_{n-1} \\ &= 2b_n. \end{aligned}$$

   Similarly $b_{n+2} = 2c_n, c_{n+2} = d_n$, and $d_{n+2} = 2a_n$. Therefore, $a_{n+8} = 2b_{n+6} = 4c_{n+4} = 8d_{n+2} = 16a_n$, and so $|a_n|$ will become arbitrarily large unless $a_2 = 0$. Thus, $a_2 = 0$. The same reasoning also implies $b_2 = c_2 = d_2 = 0$.

   Source: Yugoslavia 1992.

2. Let $f(x) = x - \sum_{i=1}^{n} \left\lfloor \frac{x}{2^i} \right\rfloor$.

   Any integer x can be written uniquely in the form $a \cdot 2^n + b$ for non-negative integers a, b with $b < 2^n$, Using this notation, we have:

$$\begin{aligned} f(x) = f\left(a \cdot 2^n + b\right) &= a \cdot 2^n + b - \sum_{i=1}^{n} \left\lfloor \frac{a \cdot 2^n + b}{2^i} \right\rfloor \\ &= a \cdot 2^n + b - \sum_{i=1}^{n} a \cdot 2^{n-i} - \sum_{i=1}^{n} \left\lfloor \frac{b}{2^i} \right\rfloor \\ &= a \cdot 2^n + f(b) - \sum_{i=1}^{n} a \cdot 2^{n-i} \\ &= a + f(b). \end{aligned}$$

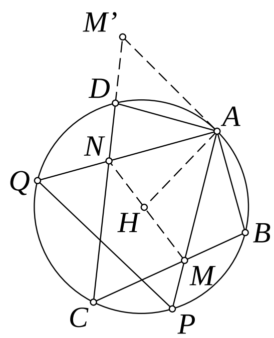
   Now fix a positive integer m. For any b, there is at most one a such that $f(x) = a + f(b) = m$. Since there are only 2^n legal values of b, there are at most 2^n values of x for which $f(x) = m$.

   Comment: You can think of $\left\lfloor \frac{x}{2^i} \right\rfloor$ as stripping off the last i binary digits from x. Given this, it is natural to consider a binary representation of x. The main insight is to group the last n digits, and handle everything else separately.

   Source: Adapted from Iran 2008.

3. Let H be the point on NM such that $NH = ND$ and $MH = MB$.

[2] This is not true for most aperiodic sequences. For example, if $t_n = n$ for all n, there is no such thing as $\max_n |t_n|$. For any constant C you choose, I can find an n such that $|t_n| > C$.



Lemma: H is the reflection of D across line AQ.

Proof of Lemma: Let M' be the point on ray $\overrightarrow{ND}$ such that $DM' = BM$. Then $\angle ADM' = \angle ABM$ since $ABCD$ is cyclic, and we are given that $AB = AD$. Therefore, $\triangle ADM' \cong \triangle ABM$ by side-angle-side.

It follows that $AM' = AM$. Furthermore, $M'N = DM' + ND = BM + ND = MN$, and so $\triangle M'AN \cong \triangle MAN$ by side-side-side. In particular, $\angle AND = \angle ANM' = \angle ANM = \angle ANH$. The lemma now follows from the fact that $NH = ND$. $\square$

By the same argument, H is the reflection of B across AP, and so $\angle PAQ = \frac{1}{2} \cdot \angle BAD$.

Finally, the Lemma implies $\angle AQH = \angle AQD = \frac{1}{2} \cdot \angle DCB = \frac{1}{2} \cdot (180° - \angle DAB) = 90° - \angle QAP$, and so PH is perpendicular to AQ. Similarly, QH is perpendicular to AP. Therefore, H is the orthocenter of $\triangle APQ$, as required.

Solution #2: Define H as in the previous solution. Let BH and DH intersect the circumcircle of $ABCD$ again at points Q' and P'. Also let $\alpha = \angle HDN = \angle DHN$ and let $\beta = \angle MBH = \angle HBM$.

Then $\angle NHQ' = \beta$ and $\angle Q'DC = \angle Q'BC = \angle HBM = \beta$. Therefore, $\angle HDQ' = \angle DHQ' = \alpha + \beta$, and so $Q'D = Q'H$. Since $ND = NH$ as well, $Q'N$ is perpendicular to DH, and also $Q'N$ bisects $\angle DQ'H = \angle DQ'B$.

This means that $Q'N$ must meet the circumcircle at the midpoint of arc BD, but this midpoint is exactly A since $AB = AD$. Therefore, A, N, Q' are collinear and hence $Q' = Q$. Similarly $P' = P$. We already showed line PHD is perpendicular to line ANQ, and similarly line QHB is perpendicular to line AMP, which means H is the orthocenter of $\triangle APQ$.

Comment: When you see one length is the sum of two others, it is often useful to divide into two lengths like we did here. A good straightedge and compass diagram should also tell you exactly where H is.

Source: International Zhautykov Olympiad 2010.

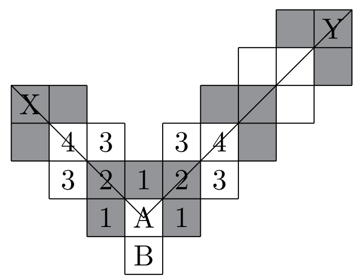
4. Let us begin by making some definitions:

   - Call the four corner vertices of the grid *corner points*.
   - Call any point on the boundary of the grid where we switch between white and black squares a *transition point*.
   - Each path must have two endpoint squares where it begins and ends. Call the centers of these four squares *origin points*.

Lemma: Let A be an origin point. Then there exist two line segments ℓ_1 and ℓ_2 connecting A to either a boundary point or a corner point, and satisfying the following conditions: (1) ℓ_1 and ℓ_2 form a 45° angle with the gridlines, (2) ℓ_1 and ℓ_2 are perpendicular to each other, and (3) ℓ_1 and ℓ_2 do not pass through any other origin points.

Proof of Lemma: First note that no 2×2 square can form a checkerboard pattern with two opposite squares being white, and the other two being black. If this happened, then the two white squares would have to be connected by a single path, but this path would disconnect the two black squares, which is not allowed.

Now consider the endpoint A of a single path. Assume without loss of generality that A is white, and let B be the one white square adjacent to A:



In the picture above, the squares labeled 1 must all be black since A is assumed to be an endpoint. The squares labeled 2 must also be black to avoid a 2×2 checkerboard. But now the squares labeled 2 already have two black neighbours, so the squares labeled 3 must be white. And then the squares labeled 4 must be white to avoid a 2×2 checkerboard. This same argument can be extended until we reach squares on the boundary, say X and Y.

Since X and Y already have two neighbours of the same colour, they must either be at a corner of the grid, or the other square they are adjacent to must be of the opposite colour. Thus, the lines through the center of A and the centers of X, Y must end at either a corner point or a transition point. Note that this is true even if A itself is on the boundary.

It is clear that these two lines are perpendicular and form a 45° angle with the gridlines. Finally, they cannot pass through another origin point because each square between A and X, Y must have two neighbours of the same colour. $\square$

Next, let us consider the boundary of the grid. Suppose it has two disjoint white components W_1 and W_2. Then there must also be two disjoint black components B_1 and B_2. W_1 and W_2 must be part of the same path, but this path would have to disconnect B_1 and B_2, which is not allowed. Therefore, the boundary contains only one white component and only one black component, and hence there are exactly two transition points.

Consider the following eight 45° diagonals: one emerging onto the grid from each corner point (call these corner diagonals), and two emerging onto the grid from each transition point (call these transition diagonals). By the Lemma, each origin point must be at the intersection of two perpendicular diagonals. Since there are only eight diagonals altogether, each diagonal can be used by only one origin point, and there are four origin points, each diagonal must be used. If two transition diagonals are used by the same origin point, then another origin point must use two corner diagonals to compensate, and hence be in the center of the grid. The problem is solved in that case.

Otherwise, every transition diagonal must intersect a corner diagonal at an origin point. Let A, B, C, D be the four corners of the grid, and let P be a transition point. Without loss of generality, we can assume that P lies on side AB. If each grid square has side length 1, then AB has odd length, so either AP or PB is even. Assume further without loss of generality that AP is even. But then the diagonal through P parallel to BD never intersects BD, and it intersects AC only on the corner of a square. Therefore, it cannot possibly intersect a corner diagonal at an origin point, and we have the desired contradiction.

Source: Adapted from Google Code Jam Finals 2010.