Some Own Problems In Number Theory

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Here are some problems proposed by me, and the problems or their solutions have not been approved by someone else. So if any fault occurs, I shall take the whole responsibility. In this case, please inform me. Among the problems,many were posted by me on **AoPS** fora, so I thank the users who posted replies and solutions there. A notable fact is that I put the problems not in order to difficulty, just randomly - which is, in my opinion, more interesting.

1 Notations

- $\mathbb{N} = \{1, 2, 3, \ldots, n, \ldots\}$: the set of positive integers.
- $\mathbb{Z} = \{1, 2, 3, \ldots, n, \ldots\}$: the set of integers.
- $\mathbb{N}_0 = \{0, 1, 2, ...\}$: the set of nonnegative integers.
- $a \in A$: a is an element of the set A.
- $a \mid b$: b is divisible by a.
- $a \nmid b$: b is not divisible by a.
- $a \mid b \wedge c$: b and c are both divisible by a.
- $\gcd(a, b)$: the Greatest Common Divisor of a and b.
- a and b are coprime : $\gcd(a, b) = 1$.
- $\varphi(m)$: the number of positive integers x, not exceeding m with the property $\gcd(x, m) = 1$.
- $\lfloor x \rfloor$: the largest integer not exceeding x.
- a is squarefree : there does not exist $x \in \mathbb{N}$ such that $x^2 \mid a$.
- $a \in \mathbb{N}$ is a perfect number : sum of positive divisors of a is $2a$.
- $ord_m(a) = x$: x is the order of a modulo m.

2 Problems

1. Prove that there does not exist a pair$(n, m) \in \mathbb{N}$ so that $n+3m$ and n^2+3m^2 are both perfect cubes. Find all such pairs if $(m, n) \in \mathbb{Z}$.

2. Find all primes p such that the number $11^p + 10^p$ is a perfect power of a positive integer.

3. In a single person game,Alex plays maintaining the following rules:
She is asked to consider the set of all natural numbers less than n on a board.Then she starts from 1 and whenever she gets an integer co-prime to n,she writes 1 on the board,otherwise she writes 0.That is she will write a binary sequence with either 1 or 0.
She denotes the number of $1'$s in this binary sequence of n by $\Phi_1(n)$ and the number of $0'$s by $\Phi_0(n)$.
Now,she wins if she can choose an n having at least 2 prime factors in the first choice such that $\Phi_1(n)|n$.Prove the following:
$\star$ 1 :There exist infinitely many n such that she can win in the first move.
$\star$ 2 :If she chooses an n having more than 3 prime factors,she can't never win.
$\star$ 3 :If $n = \prod_{i=1}^{n} p_i^{a^i}$,then $\prod_{i=1}^{n} p_i^{a_i-1} | \Phi_0(n) + 1$.
$\star$ 4 :Find all such n such that she can win.

4. Prove that for all odd $p \not| c, ord_{p^k}(c) = ord_p(c).p^{k-1}$.

5. Let $F_n = 2^{2^n} + 1$ be the n^{th} Fermat number. Prove that

$$2^{2^m+2^n} \mid F_n^{F_m-1} - 1, \quad \forall m, n \in \mathbb{N}.$$

6. Let $a > 2$ be an integer. Show that $a^{a-1} - 1$ is never squarefree.

7. Show that there are infinitely many pairs of positive integers (a, b) such that if the number $\frac{a^5+b^5}{a^3b^3+1}$ is an integer, then it is a perfect cube.

8. Let $p \equiv 2 \pmod 3$ be a prime number. Show that there exists a complete set of residue class of p such that the sum of its elements is divisible by p^2.

9. Prove that $81|10^{n+1} - 10 - 9n$ for all $n \in \mathbb{N}_0$.

10. Find all positive integers n such that $n|2^{n!} - 1$.

11. Find all positive integers n such that

1. $n \mid 2^n + 1$,
2. $n \mid 3^n + 1$.

12. Let p be a prime. Determine all perfect numbers having p factors.

13. Prove that a number which has only one prime factor, can't be a perfect number.

14. Find all pairs (a, b) of positive integers such that $ab \mid a^3 + b^3$.

15. Solve in positive integers the equation

$$a^7 + b^7 = 823543 \cdot (ac)^{1995}.$$

16. Find all $n \in \mathbb{N}$ such that

1. the number $n^2 - 27n + 182$ is a perfect square.
2. the number $n^2 - 27n + 183$ is a perfect square

17. Find all $(a, b) \in \mathbb{N}_0$ such that the number $7^a + 11^b$ is a perfect square.

18. Consider a complete set of residues modulo p. Show that we can partition this set into two subsets with equal number of elements such that the sum of elements in each set is divisible by p.

19. Let a_i, m, and n be positive integers such that $a_i + m$ is a prime for all $1 \le i \le n$. Let $N = \prod_{i=1}^{n} p_i^{a_i}$ and let S be the number of ways of expressing N as a product of m positive integers. Prove that $m^n \mid S$.

20. Let n be a positive integer. Prove that there exists $k \in \mathbb{N}$ such that

$$\frac{n}{\lfloor \sqrt[m]{n} \rfloor} > \frac{n+k}{\lfloor \sqrt[m]{n+k} \rfloor}.$$

21. Find all integers (a, b, c, d) such that $abc - d = 1$ and $bcd - a = 2$.

22. Find all positive integer values that $\frac{x^2+y^2+1}{xy+1}$ can take where x and y positive integers.

References

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