1998 IMO Camp

Functional Equations Problems

① Determine as many solutions as possible to each of the following:

(a) $f(x)f(x+1) = f(2x+1)$

(b) $f(x)f(x+1) = \frac{1}{2}[f(1)+f(2x+1)]$

(c) $f(x)f(x+1) = f(x)-f(x+1)$

(d) $f(x)f(x+1) = f[f(x)+x]$.

② Suppose that the functions $f: \mathbb{R}\to\mathbb{R}$ and $g: \mathbb{R}\to\mathbb{R}$ satisfy $f[x+g(y)] = 2x+y+5$. Determine an explicit expression for $g[x+f(y)]$

③ Find all nonconstant polynomials $p(x)$, with real coefficients, satisfying $p(x^2) = p(x)p(x+1)$

④ Suppose $f, g: \mathbb{R}\to\mathbb{R}$ satisfy

$$f(x+y) = f(x)f(y) - g(x)g(y)$$
$$g(x+y) = g(x)f(y) + g(y)f(x)$$

and $f(0)\neq 0$. Also suppose that

$$[f(x)]^2 + [g(x)]^2 \leq K,$$

for some K. Prove that

$$[f(x)]^2 + [g(x)]^2 = 1$$

⑤ Find all $f: \mathbb{R}\to\mathbb{R}$ satisfying $|f(x)-f(y)| \leq K(x-y)^2$

6. Consider a polynomial $f(x)$ with real coefficients, having the property $f[g(x)] = g[f(x)]$ for every polynomial $g(x)$ with real coefficients. Determine $f(x)$.

7. Prove or disprove: there exists a function $f(n)$ defined for all positive integers n, taking values in the positive integers, such that $f^{f(n)}(n) = n+1$, for all n. Here: $f^1(n) = f(n)$, $f^2(n) = f(f(n))$, $f^3(n) = f(f^2(n))$, etc.

8. Find all continuous $f: \mathbb{R} \to \mathbb{R}$ such that $f(x_1) + f(x_2) + f(x_3) = f(y_1) + f(y_2) + f(y_3)$ whenever $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0$.

9. Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ satisfying $f(x+y) = f[x + f(y)]$.

10. Find all $g: \mathbb{R} \to \mathbb{R}$ such that

$$g(x+y) + g(x-y) = 2g(x)g(y)$$

and

$$\lim_{x \to \infty} g(x) = 0$$