IMO Winter Training – January 2002

Functional Equations

Let $\mathbb{N}$ denote the set of positive integers.

1. Find $f : \mathbb{R} \to \mathbb{R}$ if $f(x)f(y) - f(x+y) = x + y$ for all $x, y \in \mathbb{R}$.

2. Find all functions f from $\mathbb{Q}$ to $\mathbb{Q}$ which satisfy the following two conditions: (i) $f(1) = 2$, and (ii) $f(xy) = f(x)f(y) - f(x+y) + 1$ for all x, y in $\mathbb{Q}$.

3. Find $f : \mathbb{N} \to \mathbb{N}$ if $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in \mathbb{N}$.

4. The function f satisfies
$$f(x) + f\left(\frac{1}{1-x}\right) = x$$
for all $x \in \mathbb{R}$, $x \neq 0, 1$. Find $f(x)$.

5. Find all functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which satisfy:
   (i) $f(x, x) = x$,
   (ii) $f(x, y) = f(y, x)$, and
   (iii) $(x+y)f(x, y) = yf(x, x+y)$

   for all $x, y \in \mathbb{N}$.

6. Let $f : [0, \infty) \to \mathbb{R}$, such that $f(1) = 1$ and $f(x^2 + y^2) = f(x+y)$ for all $x, y \geq 0$. Prove that $f(x) = 1$ for all x.

7. The function $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(f(m) + f(n)) = m + n$ for all $m, n \in \mathbb{N}$. Find all possible values of $f(2002)$.

8. A non-negative integer $f(n)$ is assigned to each positive integer n in such a way that the following conditions are satisfied:
   (i) $f(mn) = f(m) + f(n)$, for all positive integers m and n,
   (ii) $f(n) = 0$ whenever the units digit of n (in base 10) is a '3', and
   (iii) $f(10) = 0$.

   Prove that $f(n) = 0$ for all positive integers n.

9. Let n be a fixed positive integer, $n \geq 3$, and let f be a function assigning to each point in the plane a real number. If $A_1, A_2, \ldots, A_n$ are the vertices of a regular n-gon, then
$$f(A_1) + f(A_2) + \cdots + f(A_n) = 0.$$
Prove that $f(P) = 0$ for all points P.

10. Let $f : \mathbb{N} \to \mathbb{N}$, such that $f(n) + f(f(n)) = 6n$ for all $n \in \mathbb{N}$. Find $f(n)$.

11. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$,
    (i) $f(x+1) = f(x) + 1$, and
    (ii) $f(x^2) = f^2(x)$.

12. A sequence (u_n) is defined by $u_0 = 2$, $u_1 = 5/2$, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$ for $n = 1, 2, \ldots$. Prove that for positive integers n,

$$\lfloor u_n \rfloor = 2^{[2^n - (-1)^n]/3}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

13. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $xf(x) + f(1 - x) = x^3 - x$ for all $x \in \mathbb{R}$.

14. The set of all positive integers is the union of disjoint subsets $\{f(1), f(2), \ldots\}$, $\{g(1), g(2), \ldots\}$, where

$$f(1) < f(2) < \cdots,$$
$$g(2) < g(2) < \cdots, \text{ and}$$
$$g(n) = f(f(n)) + 1 \text{ for all } n \geq 1.$$

Determine $f(240)$.

15. The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n, $f(m + n) - f(m) - f(n) = 0$ or 1, $f(2) = 0$, $f(3) > 0$, and $f(9999) = 3333$. Determine $f(1982)$.

16. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:

    (i) $f(xf(y)) = yf(x)$ for all positive x, y,
    (ii) $f(x) \to 0$ as $x \to \infty$.

17. Find all functions f, defined on the non-negative real numbers and taking non-negative real numbers, such that:

    (i) $f(xf(y))f(y) = f(x + y)$ for all x, $y \geq 0$,
    (ii) $f(2) = 0$,
    (iii) $f(x) \neq 0$ for $0 \leq x < 2$.

18. Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for every n.

19. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all x, y in $\mathbb{Q}^+$.

20. A function f is defined for all positive integers, such that $f(1) = 1$ and

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n)$$

for all $n > 1$. Find $f(2002)$.