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AN EXPONENTIAL DIOPHANTINE EQUATION

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Let p be an odd prime with $p > 3$. In this paper we give all positive integer solutions (x, y, m, n) of the equation $x^2 + p^{2m} = y^n$, $\gcd(x, y) = 1$, $n > 2$ satisfying $2 \mid n$ or $2 \nmid n$ and $p \not\equiv (-1)^{(p-1)/2} \pmod{4n}$.

1. INTRODUCTION

Let $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ be the sets of all integers, positive integers and rational numbers respectively. Let p be a prime. There have been many papers concerned with solutions (x, y, m, n) of the equation

$$x^2 + p^m = y^n,\ x, y, m, n \in \mathbb{N},\ \gcd(x, y) = 1,\ n > 2. \tag{1}$$

All solutions of (1) for $p \in \{2, 3\}$ have been determined. The known results include the following:

1. (Nagell [12].) If $p = 2$, then the only solution of (1) with $m = 2$ is $(x, y, m, n) = (11, 5, 2, 3)$.
2. (Cohn [3].) If $p = 2$, then the only solution of (1) with $2 \nmid m$ are $(x, y, m, n) = (5, 3, 1, 3)$ and $(7, 3, 5, 4)$.
3. (Le [5, 6].) If $p = 2$, then (1) has no solutions (x, y, m, n) satisfying $2 \mid m$ and $m > 2$.
4. (Arif and Muriefah [1].) If $p = 3$, then the only solution of (1) with $2 \nmid m$ is $(x, y, m, n) = (10, 7, 5, 3)$.
5. (Luca [9].) If $p = 3$, then the only solution of (1) with $2 \mid m$ is $(x, y, m, n) = (46, 13, 4, 3)$.

In this paper we investigate the solutions (x, y, m, n) of (1) for m even. Then (1) may be written as

$$x^2 + p^{2m} = y^n,\ x, y, m, n \in \mathbb{N},\ \gcd(x, y) = 1,\ n > 2. \tag{2}$$

We prove the following two results.

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Theorem 1. *If $p > 3$, then all the solutions (x, y, m, n) of (2) with $2 \mid m$ are given as follows:*

(i) $p = 239$, $(x, y, m, n) = (28560, 13, 1, 8)$.

(ii) $p = E(q)$, $(x, y, m, n) = \left(\left((E(q))^2 - 1\right)/2,\ F(q), 1, 4\right)$, *where q is an odd prime, and*

$$E(q) = \frac{1}{2}\left(\left(1+\sqrt{2}\right)^q + \left(1-\sqrt{2}\right)^q\right), \quad F(q) = \frac{1}{2\sqrt{2}}\left(\left(1+\sqrt{2}\right)^q - \left(1-\sqrt{2}\right)^q\right). \tag{3}$$

Theorem 2. *If $p > 3$ and $p \not\equiv (-1)^{(p-1)/2} \pmod{4n}$, then (2) has no solutions (x, y, m, n) with $2 \nmid n$.*

By the above theorems, we can completely determine all solutions of (2) for the case that p is either a Fermat prime or a Mersenne prime.

Corollary 1. *If p is a Fermat prime with $p > 3$, then (2) has no solutions (x, y, m, n).*

Corollary 2. *If $p = 7$, then the only solution of (2) is $(x, y, m, m) = (24, 5, 1, 4)$. If p is a Mersenne prime with $p > 7$, then (2) has no solutions (x, y, m, n).*

2. Preliminaries

Lemma 1. [11, pp.12–13] *Every solution (X, Y, Z) of the equation*

$$X^2 + Y^2 = Z^2,\ X, Y, Z \in \mathbb{N},\ \gcd(X, Y) = 1,\ 2 \mid X \tag{4}$$

can be expressed as

$$X = 2AB,\ Y = A^2 - B^2,\ Z = A^2 + B^2, \tag{5}$$

where A, B are positive integers satisfying

$$A > B,\ \gcd(A, B) = 1,\ 2 \mid AB. \tag{6}$$

Lemma 2. [11, pp.122–123] *Let n be an odd integer with $n > 1$. Then every solution (X, Y, Z) of the equation*

$$X^2 + Y^2 = Z^n,\ X, Y, Z \in \mathbb{N},\ \gcd(X, Y) = 1 \tag{7}$$

can be expressed as

$$Z = A^2 + B^2,\ X + Y\sqrt{-1} = \lambda_1\left(A + \lambda_2 B\sqrt{-1}\right)^n,\ \lambda_1, \lambda_2 \in \{-1, 1\}, \tag{8}$$

where A, B are coprime positive integers.

LEMMA 3. [7] *The only solutions of the operation*

$$X^2 + 1 = 2Y^4, \ X, \ Y \in \mathbb{N} \tag{9}$$

are $(X, Y) = (1, 1)$ and (239,13).

LEMMA 4. [8] *Let D be a positive integer which is not a square. Then the equation*

$$X^4 - DY^2 = -1, \ X, Y \in \mathbb{N} \tag{10}$$

has at most one solution (X, Y). Moreover, if (X, Y) is a solution of (10), then the fundamental solution $U_1 + V_1\sqrt{D}$ of the Pell equation

$$U^2 - DV^2 = -1, \ U, V \in \mathbb{N} \tag{11}$$

satisfies

$$U_1 = dt^2, \ X^2 + Y\sqrt{D} = \left(U_1 + V_1\sqrt{D}\right)^d, \ d, t \in \mathbb{N}, \ 2 \nmid d, \ d \text{ is square free.} \tag{12}$$

LEMMA 5. [13] *The equation*

$$X^2 + 1 = 2Y^r, \ X, Y, r \in \mathbb{N}, \ X > Y > 1, \ r > 1, \ 2 \nmid r \tag{13}$$

has no solutions (X, Y, r).

LEMMA 6. [4, Lemma 15] *The equation*

$$X^{2r} + 1 = 2Y^2, \ X, Y, r \in \mathbb{N}, \ X > 1, \ Y > 1, \ r > 1, \ 2 \nmid r \tag{14}$$

has no solutions (X, Y, r).

Let α, β be algebraic integers. If $\alpha + \beta$ and $\alpha\beta$ are nonzero coprime integers and α/β is not a root of unity, then (α, β) is called a Lucas pair. Further, let $a = \alpha + \beta$ and $c = \alpha\beta$. Then we have

$$\alpha = \frac{1}{2}\left(a + \lambda\sqrt{b}\right), \ \beta = \frac{1}{2}\left(a - \lambda\sqrt{b}\right), \ \lambda \in \{-1, 1\}, \tag{15}$$

where $b = a^2 - 4c$. We call (a, b) the parameters of the Lucas pair (α, β). Two Lucas pairs (α_1, β_1) and (α_2, β_2) are equivalent if $\alpha_1/\alpha_2 = \beta_1/\beta_2 = \pm 1$. Given a Lucas pair (α, β), one defines the corresponding sequence of Lucas numbers by $u_t = u_t(\alpha, \beta) = (\alpha^t - \beta^t)/(\alpha - \beta)$ for $t = 0, 1, 2, \ldots$. For equivalent Lucas pairs (α_1, β_1) and (α_2, β_2), we have $u_t(\alpha_1, \beta_1) = \pm u_t(\alpha_2, \beta_2)$ for any $t \geqslant 0$. A prime p is a primitive divisor of $u_t(\alpha, \beta)$ if $p \mid u_t$ and $p \nmid bu_1 \cdots u_{t-1}$.

Lemma 7. [10] *Let (α,β) be a Lucas pair with parameters (a,b). If p is a primitive divisor of $u_t(\alpha,\beta)$ $(t>2)$, then $p-\left(\frac{b}{p}\right)\equiv 0 \pmod t$ where $\left(\frac{b}{p}\right)$ is the Legendre symbol.*

A Lucas pair (α,β) such that $u_t(\alpha,\beta)$ has no primitive divisors will be called a t-defective Lucas pair.

Lemma 8. [14] *Let t satisfy $4<t<30$ and $t\neq 6$. Then, up to equivalence, all parameters of t-defective Lucas pairs are given as follows:*

(i) $t=5$, $(a,b)=(1,5),(1,-7),(2,-40),(1,-11),(1,-15),(12,-76),(12,-1364)$;
(ii) $t=7$, $(a,b)=(1,-7),(1,-19)$;
(iii) $t=8$, $(a,b)=(2,-24),(1,-7)$;
(iv) $t=10$, $(a,b)=(2,-8),(5,-3),(5,-47)$;
(v) $t=12$, $(a,b)=(1,5),(1,-7),(1,-11),(2,-56),(1,-15),(1,-19)$;
(vi) $t\in\{13,18,30\}$, $(a,b)=(1,-7)$.

A positive integer t is called totally non-defective if no Lucas pair is t-defective.

Lemma 9. [2] *If $t>30$, then t is totally non-defective.*

3. Proofs

Proof of Theorem 1: Let (x,y,m,n) be a solution of (2). Since $p>3$ and $n>2$, we have $2\mid x$ and $2\nmid y$. If $2\mid n$, since $\gcd\left(y^{n/2}+x,\ y^{n/2}-x\right)=1$, then from (2) we get $y^{n/2}+x=p^{2m}$ and $y^{n/2}-x=1$. This implies that

$$p^{2m}+1=2y^{n/2}, \tag{16}$$

$$p^{2m}-1=2x. \tag{17}$$

Since $n/2>1$, by Lemma 5, we see from (16) that $n/2$ has no odd prime divisors. So we have $n=2^{s+1}$, where s is a positive integer.

When $s=1$, (16) can be written as

$$p^{2m}+1=2y^2. \tag{18}$$

Then $(u,v)=(p^m,y)$ is a solution of the Pell equation

$$u^2-2v^2=-1,\ u,v\in\mathbb{N}. \tag{19}$$

Since $1+\sqrt{2}$ is the fundamental solution of (19), we get

$$\begin{aligned} p^m&=\frac{1}{2}\left(\left(1+\sqrt{2}\right)^l+\left(1-\sqrt{2}\right)^l\right),\\ y&=\frac{1}{2\sqrt{2}}\left(\left(1+\sqrt{2}\right)^l-\left(1-\sqrt{2}\right)^l\right),\ l\in\mathbb{N},\ 2\nmid l. \end{aligned} \tag{20}$$

On the other hand, if m has an odd prime divisor r, then $(X,Y) = (p^{m/r}, y)$ is a solution of (14). However, by Lemma 6, this is impossible. Therefore, if $m > 1$, then m is a power of 2 and $(X,Y) = (p^{m/2}, y)$ is a solution of (10) for $D = 2$. But, by Lemma 4, this is impossible too. So we have $m = 1$. Then the positive integer l in (20) must be an odd prime. Thus, by (17) and (20), we obtain the solution (ii).

When $s > 1$, we see from (16) that $(X,Y) = (p^m, y^{n/8})$ is a solution of (9). Therefore, by Lemma 3, we get the solution (i). Thus, the theorem is proved. □

PROOF OF THEOREM 2: Let (x,y,m,n) be a solution of (2) with $2 \nmid n$. Then $(X,Y,Z) = (x, p^m, y)$ is a solution of (7). By Lemma 2, we get

$$x + p^m\sqrt{-1} = \lambda_1\left(A + \lambda_2 B\sqrt{-1}\right)^n, \ \lambda_1, \lambda_2 \in \{-1, 1\}, \tag{21}$$

where A, B are positive integers satisfying

$$A^2 + B^2 = y, \ \gcd(A,B) = 1. \tag{22}$$

From (21), we get

$$p^m = \lambda_1\lambda_2 B \sum_{i=0}^{(n-1)/2} \binom{n}{2i+1} A^{n-2i-1}\left(-B^2\right)^i. \tag{23}$$

Let

$$\alpha = A + B\sqrt{-1}, \quad \beta = A - B\sqrt{-1}. \tag{24}$$

We see from (22) and (24) that (α, β) is a Lucas pair with parameters $(2A, -4B^2)$. Further, let $u_t(\alpha,\beta)$ $(t = 0, 1, 2, \dots)$ denote the corresponding Lucas numbers. By (23), we get

$$p^m = \pm B u_n(\alpha, \beta). \tag{25}$$

Notice that $\left(\frac{-4B^2}{p}\right) = \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$, where $\left(\frac{*}{p}\right)$ is the Legendre symbol. By Lemma 7, if p is a primitive divisor of $u_n(\alpha,\beta)$, then $p - (-1)^{(p-1)/2} \equiv 0 \pmod{n}$. Since $2 \nmid n$ and $p - (-1)^{(p-1)/2} \equiv 0 \pmod 4$, we get $p \equiv (-1)^{(p-1)/2} \pmod{4n}$. Therefore, by (25), if the solution (x,y,m,n) satisfies $p \not\equiv (-1)^{(p-1)/2} \pmod{4n}$, then $u_n(\alpha,\beta)$ has no primitive divisors. By Lemmas 8 and 9, we deduce that $n = 3$ and $p \mid B$. Then, by (23), we get

$$B = p^s, \ 3A^2 - B^2 = \pm p^{m-s}, \ s \in \mathbb{N}, \ s \leqslant m. \tag{26}$$

Since $\gcd(A,B) = 1$, we see from (26) that $p = 3$. thus, if $p > 3$, then (2) has no solutions (x,y,m,n) satisfying $2 \nmid n$ and $p - (-1)^{(p-1)/2} \not\equiv 0 \pmod{4n}$. The theorem is proved. $\square$

PROOF OF COROLLARY 1: Let p be a Fermat prime. Then we have

$$p = 2^{2^s} + 1, \quad s \in \mathbb{N}. \tag{27}$$

Since $p - (-1)^{(p-1)/2} = 2^{2^s}$, by Theorem 2, then (2) has no solutions (x,y,m,n) with $2 \nmid n$.

On the other hand, since $p \neq 239$, by the proof of Theorem 1, if (x,y,m,n) is a solution of (2) with $2 \mid n$, then we have $m = 1$, $n = 4$ and

$$p^2 + 1 = 2y^2. \tag{28}$$

Substitute (27) into (28), and we get

$$2^{2^{s+1}-2} + \left(2^{2^s-1} + 1\right)^2 = y^2. \tag{29}$$

Therefore, by Lemma 1, we obtain from (29) that

$$2^{2^s-1} = 2AB, \; 2^{2^s-1} + 1 = A^2 - B^2, \; y = A^2 + B^2, \tag{30}$$

where A, B are positive integers satisfying (6). From (30), since $\gcd(A,B) = 1$, we get from the first equation $s > 1$, $A = 2^{2^s-2}$ and $B = 1$. However, by the second equation in (30), we get

$$1 \equiv 2^{2^s-1} + 1 = 2^{2^{s+1}-4} - 1 \equiv 3 \pmod 4, \tag{31}$$

which is a contradiction. Thus, the corollary is proved. $\square$

PROOF OF COROLLARY 2: Let p be a Mersenne prime. Then we have

$$p = 2^r - 1, \; r \text{ is an odd prime}, \tag{32}$$

if $p \geqslant 7$. Since $p-(-1)^{(p-1)/2} = 2^r$, by Theorem 2, then (2) has no solutions (x,y,m,n) with $2 \nmid n$.

By Theorem 1, if $r = 3$, then $p = 7$ and the only solution of (2) with $2 \mid n$ is $(x,y,m,n) = (24,5,1,4)$. Since $p \neq 239$, by the proof of Theorem 1, if $r > 3$ and (x,y,m,n) is a solution of (2) with $2 \mid n$, then $m = 1$, $n = 4$ and (28) holds. Substitute (32) into (28), and we get

$$2^{2r-2} + \left(2^{r-1} - 1\right)^2 = y^2. \tag{33}$$

By Lemma 1, we obtain from (33) that

$$2^{r-1} = 2AB, \; 2^{r-1} - 1 = A^2 - B^2, \; y = A^2 + B^2, \tag{34}$$

whence we obtain $A = 2^{r-2}$ and $B = 1$, since $\gcd(A,B) = 1$, but these do not satisfy the second equation in (34), when $r > 3$. Thus, if $p > 7$, then (2) has no solutions (x,y,m,n). The corollary is proved. $\square$

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