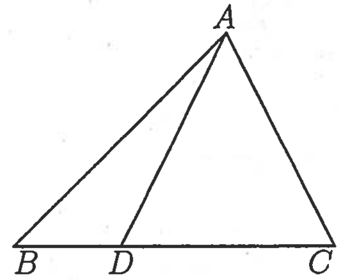
2000

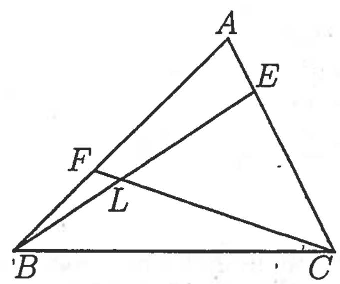
REVIEW OF ELEMENTARY GFOMETRY

One line drawn from a vertex.



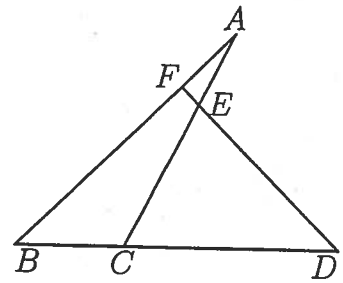
1. If $BD^2 - DC^2 = BA^2 - AC^2$, prove that AD is perpendicular to BC. What can be said if D is on the extension of BC instead?
2. If $\frac{BD}{DC} = \frac{BA}{AC}$, prove that $\angle BAD = \angle CAD$. What can be said if D is on the extension of BC instead?
3. If $BA^2 + AC^2 = 2AD^2 + BD^2 + CD^2$, prove that $BD = CD$.
4. If $BA^2 + AC^2 = 2AD^2 + 2BD^2$, is it necessarily true that $BD = DC$?
5. If $\frac{BD}{DC} = \frac{m}{n}$, prove that $mAB^2 + nAC^2 = (m+n)AD^2 + nBD^2 + mCD^2$.
6. If AD bisects the area of ABC, where is D? Given a point P on AB, closer to A than to B. Show how to bisect the area of ABC by a line through P.

Lines drawn from two vertices.



1. If $\frac{CL}{LF} = \frac{BL}{LE}$, prove that $\frac{AF}{FB} = \frac{AE}{EC}$.
2. If $CL = 2FL$ and $BL = 2LE$, prove that BE and CF are medians.
3. If FLB and ELC have equal area, prove that FLE nd CLB are similar.
4. If $AF = \frac{2}{3}AB$ and $AE = \frac{2}{3}AC$, prove that $BL = 2LE$ and $CL = 2LF$.
5. Determine $\frac{BL}{LE}$ and $\frac{CL}{LF}$ given that $\frac{AF}{FB} = \frac{m}{n}$ and $\frac{AE}{EC} = \frac{p}{q}$.
6. If $BFEC$ is cyclic, prove that AEF and ABC are similar.
7. If $AFLE$ and $BFEC$ are both cyclic, prove that BE is perpendicular to AC and CF is perpendicular to AB. If $AFLE$ is cyclic, must $BFEC$ be cyclic?

One transversal drawn.



1. If $AE = \frac{2}{3}AC$ and AEF has one quarter the area of ABC, prove that $AF = \frac{2}{3}AB$. What fraction of the area of ABC does $ACED$ have?

2. If $AF = FB$ and $AE = 2EC$, determine $\frac{BD}{CD}$.

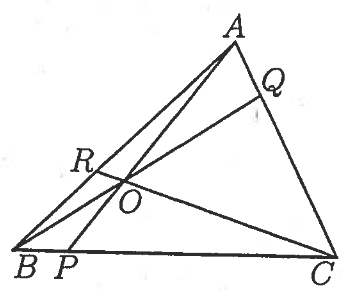
3. If $AE = EC$, prove that $\frac{AF}{FB} = \frac{CD}{BD}$.

4. Prove that $\frac{BD}{CD} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$. State and prove the converse of this result.

5. If $BCEF$ is cyclic, prove that $\angle FDB = \angle ACD - \angle ABD$.

6. If the circumcircles of AEF and DEC meet at O, prove that $ABCO$ and $FBDO$ are both cyclic.

Concurrent lines through the vertices.



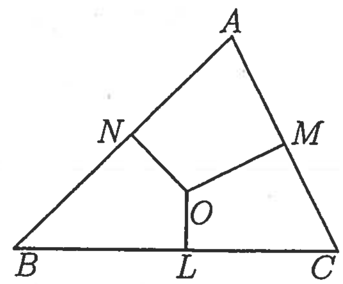
1. Name two pairs of triangles such that the ratio of their areas is the same as that of their bases BP and PC. Deduce that the ratio of the areas of AOB and AOC is equal to $\frac{BP}{PC}$.

2. Prove that if $\frac{BP}{PC} = \frac{1}{2}$ and $\frac{CQ}{QA} = \frac{3}{2}$, then AOB has the same area as POC but half the area of AOC. Deduce that $\frac{AR}{RB} = \frac{4}{3}$ and $\frac{AO}{OP} = \frac{2}{1}$. Taking the area of ABC to be 1, determine the area of each of AOR, ROB, BOP, POC, COQ and QOA. Determine also $\frac{RO}{OC}$ and $\frac{QO}{OD}$.

3. Prove that $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = 1$. State and prove the converse of this result.

4. If $AROQ$ and $BPOR$ are cyclic, prove that $CQOP$ is also cyclic, and that O is the orthocentre of ABC.

Concurrent perpendiculars to the sides.

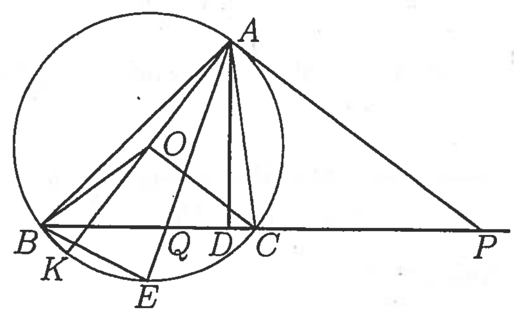


1. Prove that $BL^2 - LC^2 + CM^2 - AM^2 + AN^2 - NB^2 = 0$. State and prove the converse of this result.

2. Prove that $\angle BOC - \angle MLN = \angle CAB$.

3. Suppose that $\angle LOL' = \angle MOM' = \angle NON'$, where L', M' and N' be points on LC, MA and NB respectively. Prove that $OL'M'$ and OLM are similar, as are $L'M'N'$ and LMN.

Circumcircle. (Circumcentre O)



1. Prove that $\angle OAB = 90° - \angle BCA$.

2. What is the condition that the circle BOC should go through the orthocentre of ABC? Prove that in that case, the circle also goes through the incentre of ABC.

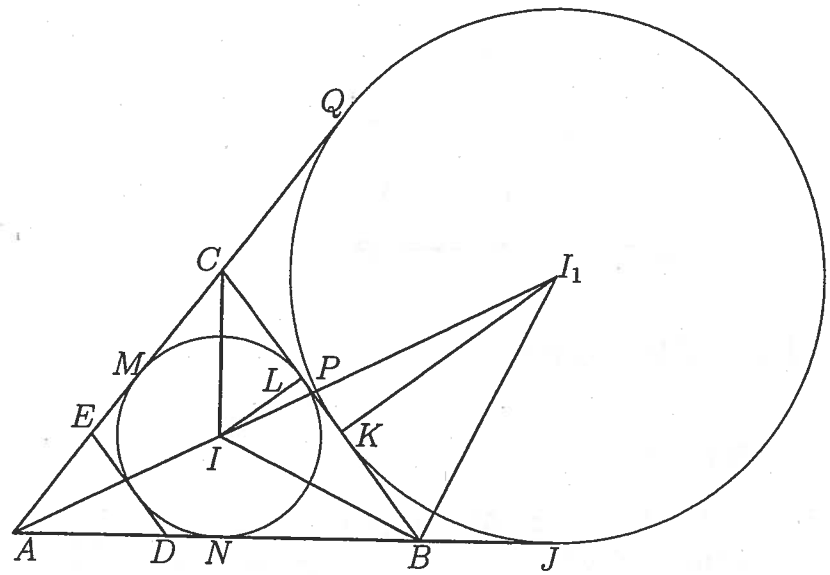
3. If BOC has half the area of BAC, prove that BAC is a right triangle.

4. If AD is an altitude of ABC and AK is a diameter of the circumcircle, prove that ADC and ABK are similar and deduce that $AK \cdot AD = AB \cdot AC$.

5. The bisector of $\angle CAB$ meets BC at Q and the circumcircle at E. Prove that ACQ and AEB are similar and that $AQ \cdot AE = AB \cdot AC$. Deduce that $AQ^2 + BQ + QC = AB \cdot AC$.

6. If the tangent to the circumcircle at A meets BC at P, prove that $\frac{BP}{PC} = \frac{AB^2}{AC^2}$.

Incircle and excircle. (Incentre I and Excentre I_1)



Notation:
Let $BC = a$, $CA = b$, $AB = c$ and $s = \frac{1}{2}(a+b+c)$. Let r be the inradius of ABC and r_1 be the exradius of ABC opposite A.

1. Let the incircle touch BC, CA and AB at L, M and N respectively. Prove that $AN = s-a$, $BL = s-b$ and $CM = s-c$.

2. Let the excircle opposite A touches BC and the extensions of AC and AB at K, Q and J respectively. Prove that $AJ = s$ and deduce that $BK = s-c = CL$.

3. Prove that the area of ABC is equal to rs.

4. Prove that the area of ABC is equal to $r_1(s-a)$.

5. Prove that the area of ABC is equal to $\sqrt{s(s-a)(s-b)(s-c)}$.

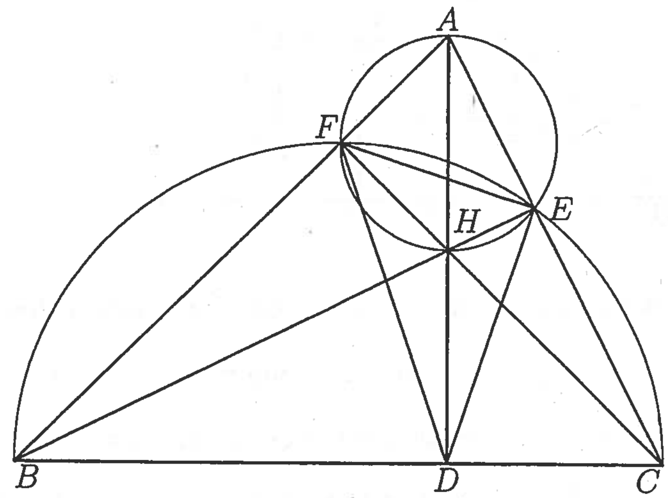
6. Prove that $\frac{AI}{AI_1} = \frac{IP}{I_1P}$, where P is the point of intersection of II_1 and BC.

7. If the tangent to the incircle parallel to BC cuts AB at D and AC at E, prove that the ratio of the areas of ADE and ABC is equal to $\frac{r}{r_1}$.

The orthocentric quadrangle.

Terminology:

A set of four points joined pairwise by six lines is called a quadrangle. BE and CF are altitudes of ABC, cutting each other at H. AH cuts BC at D. The quadrangle $ABCH$ is said to be orthocentric since BH is perpendicular to CA and CH is perpendicular to AB. H is called the orthocentre and DEF the pedal triangle of ABC.



1. Prove that $BFEC$ and $AFHE$ are cyclic and deduce that AH is also perpendicular to BC.

2. Prove that the angles of the pedal triangle DEF are bisected by HA, HB and HC, and that they are equal to $180° - 2\angle A$, $180° - 2\angle B$ and $180° - 2\angle C$.

3. Verify that A, B and C are the respective orthocentres of HBC, HCA and HAB.

4. Verify that DEF is also the pedal triangle of HBC, of HCA and of HAB.

5. Prove that H, A, B and C are the incentre and excentres of DEF.

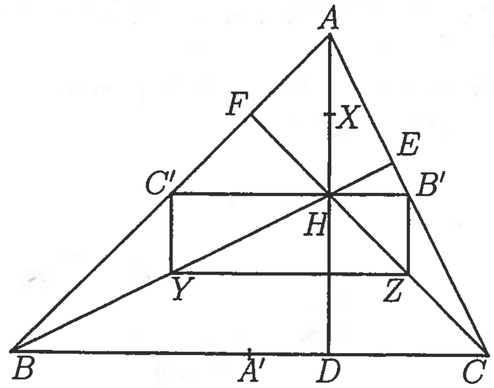
6. If DEF is isosceles, prove that either ABC is isosceles or two of its angles differ by 90°.

7. Investigate the orthocentric quadrangle and the pedal trinalge if ABC is a right triangle or an obtuse triangle.

The nine-point circle. (Orthocentre H)

Notation:

A', B', C', X, Y and Z are the respective midpoints of BC, CA, AB, AH, BH and CH.



1. Prove that $B'C'YZ$ is a rectangle and that its circumcircle passes through E and F.

2. Prove that $C'A'ZX$ is a rectangle and that its circumcircle passes through F and D.

3. Prove that $A'B'XY$ is a rectangle and that its circumcircle passes through D and E.

4. Explain why $B'C'YZ$, $C'A'ZX$ and $A'B'XY$ have the same circumcircle, which also passes through D, E and F. It is called the nine-point circle of ABC.

5. Prove that if O is the circumeentre of ABC, the centre of the nine-point circle N is the midpoint of OH.

6. Prove that the radius of tha nine-point circle is half the circumradius of ABC.

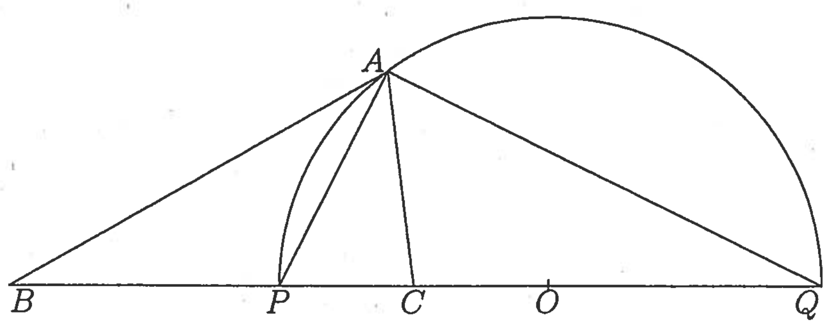
7. If BO meets the circumcircle of ABC again at K, prove that $AHCK$ is a parallelogram. Deduce that $OA' = \frac{1}{2}AH$. If AA' meets OH at G, prove that $AG = 2GA'$, so that G is the centroid and $OG = \frac{1}{3}OH$. The line passing through O, G, N and H is called the Euler line of ABC.

8. Prove that the excentres I_1, I_2 and I_3 of ABC lie on a circle of radius twice the circumradius of ABC. Moreover, prove that its centre J is such that O is the midpoint of IJ, where I is the incentre of ABC.

Circle of Apollonius.

Notation:

P and Q are points on BC such that $\angle CAB$ is bisected by AP internally and AQ externally.

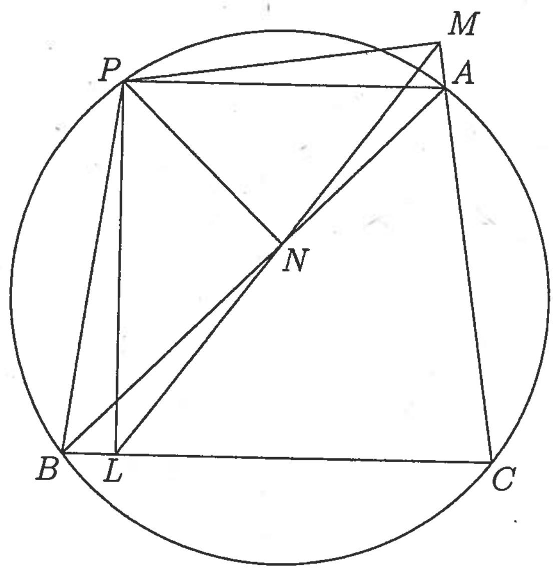


1. Prove that $\frac{BP}{PC} = \frac{BQ}{CQ}$ and that the circle on PQ as diameter goes through A.
2. Conversely, if $\frac{BP}{PC} = \frac{BQ}{CQ}$ and $\angle PAQ = 90°$, prove that AP bisects $\angle CAB$.
3. If O is the midpoint of PQ, prove that $OC \cdot OB = OP^2$.
4. If B and C are fixed points and A is a variable point such that $\frac{AB}{AC}$ has constant value, prove that the points P and Q are fixed and the locus of A the circle on PQ as diameter, called an Apollonius' Circle.
5. What happens to the Apollonius' circle as $\frac{AB}{AC}$ approaches 1?
6. A circle centre O is drawn with PQ a diameter on which B and C are taken so that $OC \cdot OB = OP^2$. If A is any point of the circle, prove that $\angle CAB$ is bisected internally by AP and externally by AQ.
7. An arbitrary circle through B and C cuts the circle centre O at the point R. Prove that OR is a tangent to that circle.
8. $ABCD$ is a square. Determine the point P on the extension of DC such that $\frac{AP}{BP}$ is maximum.

The Simson line.

Notation:

P is a point on the circumcircle of ABC. PL, PM and PN are perpendiculars dropped from P onto BC, CA and AB respectively.



1. Prove that $PNLB$ and $PNAM$ are cyxclic.
2. Prove that $\angle PAM = \angle PBL = \angle PNM$.
3. Prove that $\angle PNL + \angle PNM = 180°$, so that L, M and N are collinear. The line joining them is called the Simson line of ABC.
4. Prove that if the perpendiculars at collinear points L, M and N on the sides of ABC are concurrent at P, then P lies on the circumcircle of ABC. 5. Prove that $\angle PLM$ is that subtended by the arc PA at the circumcircle. Investigate the change in $\angle PLM$ as P starting from A moves round the circumcircle to B. In particular, prove that the Simson lines of opposite ends of a diameter are perpendicular to each other.