Winter Camp 2004

Geometry

Thursday 08 January 2004

Triangles

1. Triangle ABC is given and squares $ABHG$ and $ACFE$ are placed externally on its sides. The altitude AD of triangle ABC is extended to meet GE at M. Prove that M is the midpoint of GE.

2. Let C_1 be any point on the side AB of triangle ABC and draw C_1C. Let A_1 be the intersection of BC extended and the line through A parallel to C_1C; similarly, let B_1 be the intersection of AC extended and the line through B parallel to C_1C. Prove that

$$\frac{1}{AA_1} + \frac{1}{BB_1} = \frac{1}{CC_1}$$

3. Triangle ABC is equilateral and BC is extended to E. Then equilateral triangle CED is constructed on the same side of CE as ABC. M and N are the midpoints of BD and AE, respectively. Prove that triangle CMN is also equilateral.

4. Let ABC be a triangle with $\angle B = 2\angle A$. Prove that $b^2 = a^2 + ac$.

5. Let ABC be a triangle with $\angle B = 3\angle A$. Prove that $ac^2 = (a+b)(b-a)^2$.

6. Let ABC be a triangle with $\angle B = 4\angle A$. Prove that $ab^2c^3 = (b^2 - a^2 + ac)(a^2 - b^2 + ac)^2$.

7. Show that in ΔABC, with $a \geq b \geq c$, the sides are in arithmetic progression if and only if

$$2\cot\frac{B}{2} = 3\left(\tan\frac{C}{2} + \tan\frac{A}{2}\right)$$

8. ABC is an isosceles right triangle with right angle at A. Determine the minimum value of $BP + CP - \sqrt{3}AP$ where P is any point in the plane of the triangle.

Locus Problems

1. Consider the cube $ABCDA'B'C'D'$ (with face $ABCD$ directly above face $A'B'C'D'$).
   (a) Find the locus of the midpoints of segments XY, where X is any point of AC and Y is any point of $B'D'$.
   (b) Find the locus of points Z which lie on the segments XY of part (a) with $ZY = 2XZ$.

2. Consider ΔOAB with acute angle AOB. Through a point $M \neq O$ perpendiculars are drawn to OA and OB, the feet of which are P and Q, respectively. The point of intersection of the altitudes of ΔOPQ is H. What is the locus of H if M is permitted to range over
   (a) the side AB
   (b) the interior of ΔOAB

3. A square of side $2a$, lying always in the first quadrant of the xy-plane, moves so that two consecutive vertices are always on the x- and y-axes, respectively. Find the locus of the centre of the square.

sipos@yorku.ca
won't come

mbraverm@cs.yorku.ca
cs.utoronto.ca

Quadrilaterals

1. $PQRS$ is a quadrilateral of area A. O is a point inside it. Prove that if $2A = OP^2 + OQ^2 + OR^2 + OS^2$, then $PQRS$ is a square and O is its centre.

2. Prove that for any quadrilateral with sides a, b, c, d, we have $a^2 + b^2 + c^2 > \frac{1}{3}d^2$.

3. Point M lies inside square $ABCD$. If $MA = c$, $MB = b$ and $MC = a + b$, with $a, b, c > 0$ and $a^2 + b^2 = c^2$, determine $\angle BMC$.

4. Let M be the midpoint of side AB of cyclic quadrilateral $ABCD$ and let P be the intersection of the line segments MC and BD. The line through C parallel to AP intersects BD at Q. Prove that if $\angle CAD = \angle BAP = \frac{1}{2}\angle BMC$, then $BP = QD$.

Circles and Triangles

1. Six equal circles are stacked in the following way. Three are placed on a straight line, with possibly unequal gaps between them. A row of two more is placed on top, then a final one at the summit. Prove that the centre of the top circle lies directly above the midpoint of the segment joining the centres of the outer two circles on the bottom row.

2. In triangle ABC, if eight times the square of the circumradius is equal to the sum of the squares of the sides, show that the triangle is right angled and $\sin^2 A + \sin^2 B + \sin^2 C = 2$.

3. A circle is inscribed in triangle ABC with sides a, b, c. Tangents to the circle parallel to the sides of the triangle are constructed. Each of these tangents cuts off a triangle from ΔABC. In each of these triangles, a circle is inscribed. Find the sum of the areas of all four inscribed circles (in terms of a, b, c).

4. The circle k intercepts the sides BC, CA, AB of triangle ABC in points A_1, A_2, B_1, B_2, C_1, C_2. The perpendiculars to BC, CA, AB through A_1, B_1, C_1, respectively, meet at a point M. Prove that the three perpendiculars to BC, CA, AB through A_2, B_2, C_2, respectively, also meet in one point.

5. Consider an isosceles triangle. Let R be the radius of its circumscribed circle and r the radius of its inscribed circle. Prove that the distance d between the centres of these two circles is $d = \sqrt{R(R-2r)}$.

6. Let a, b, c be the lengths of the sides of a triangle, let $s = \frac{1}{2}(a+b+c)$, and let r be the radius of the inscribed circle. Show that

$$\frac{1}{(s-a)^2} + \frac{1}{(s-b)^2} + \frac{1}{(s-c)^2} \geq \frac{1}{r^2}$$

7. The incircle of triangle ABC touches BC at F. E is the midpoint of BC and D is the midpoint of AF. Prove that the incentre lies on DE.