Computations with Probability

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§1 Readings

My handout *Expected Uses of Probability*, §1, §2. Suggested exercises to check your understanding (from §2 of above).

Problem 1.1 (AHSME 1989)**.** Suppose that 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row $GBBGGGBGBGGGBGBGGBGG$ we have $S = 12$. Find the expected value of S.

Problem 1.2 (AIME 2006 #6)**.** Let $\mathcal{S}$ be the set of real numbers that can be represented as repeating decimals of the form $0.\overline{abc}$ where a, b, c are distinct digits. Find the sum of the elements of $\mathcal{S}$.

§2 Lecture notes

A Markov chain consists of the following:

- Possible states
- Transitions between the states, with given probability

Start on a given state, and then with certain probabilities go from one state to another.

Problem 2.1 (Drunkard's Walk)**.** Suppose you have a number line, and a person is standing at $x = 3$. And every minute, either go left / right with even probability.

(a) What's the probability they reach $x = 0$ before the point $x = 10$?

(b) Expected time until termination?

Problem 2.2 (AIME 1995)**.** If we flip a coin repeatedly, what's the probability of getting five consecutive heads before two consecutive tails?

Problem 2.3 (NIMO, by Lewis Chen)**.** Ten students are arranged in a row. Every minute, a new student is inserted in the row (which can occur in the front and in the back as well, hence 11 possible places) with a uniform $\frac{1}{11}$ probability of each location. Then, either the frontmost or the backmost student is removed from the row (each with a $\frac{1}{2}$ probability). Suppose you are the eighth in the line from the front. What is probability that you exit the row from the front rather than the back?

§3 Practice Problems

Problem 3.1 (HMMT 2013 C6)**.** Values $a_1, \dots, a_{2013}$ are chosen independently and at random from the set $\{1, \dots, 2013\}$. What is the expected number of distinct values in the set $\{a_1, \dots, a_{2013}\}$?

Problem 3.2 (HMMT 2005 C8)**.** We repeatedly roll a tetrahedral die with sides 1, 2, 3, 4 until the sum of two previous rolls is prime. What is the probability the last number is a 1?

Problem 3.3 (AMC 10B 2014 #25)**.** In a small pond there are eleven lily pads in a row labeled 0 through 10. A frog is sitting on pad 1. When the frog is on pad N, $0 < N < 10$, it will jump to pad $N-1$ with probability $\frac{N}{10}$ and to pad $N+1$ with probability $1 - \frac{N}{10}$. Each jump is independent of the previous jumps. If the frog reaches pad 0 it will be eaten by a patiently waiting snake. If the frog reaches pad 10 it will exit the pond, never to return. What is the probability that the frog will escape being eaten by the snake?

Problem 3.4 (HMMT 2012 C7)**.** You are repeatedly flipping a fair coin. What is the expected number of flips until the first time that your previous 2012 flips are 'HTHT...HT'?

Problem 3.5 (NIMO 4.3)**.** One day, a bishop and a knight were on squares in the same row of an infinite chessboard, when a huge meteor storm occurred, placing a meteor in each square on the chessboard independently and randomly with probability p. Neither the bishop nor the knight were hit, but their movement may have been obstructed by the meteors. For what value of p is the expected number of valid squares that the bishop can move to (in one move) equal to the expected number of squares that the knight can move to (in one move)?

Problem 3.6 (NIMO 7.3)**.** Richard has a four infinitely large piles of coins: a pile of pennies, a pile of nickels, a pile of dimes, and a pile of quarters. He chooses one pile at random and takes one coin from that pile. Richard then repeats this process until the sum of the values of the coins he has taken is an integer number of dollars. What is the expected value of this final sum of money, in cents?

Problem 3.7 (NIMO 5.6)**.** Tom has a scientific calculator. Unfortunately, all keys are broken except for one row: `1`, `2`, `3`, `+` and `-`. Tom presses a sequence of 5 random keystrokes; at each stroke, each key is equally likely to be pressed. The calculator then evaluates the entire expression, yielding a result of E. Find the expected value of E.

(Note: Negative numbers are permitted, so `13-22` gives $E = -9$. Any excess operators are parsed as signs, so `-2-+3` gives $E = -5$ and `-+-31` gives $E = 31$. Trailing operators are discarded, so `2++-+` gives $E = 2$. A string consisting only of operators, such as `-++-+`, gives $E = 0$.)