The Riemann sphere

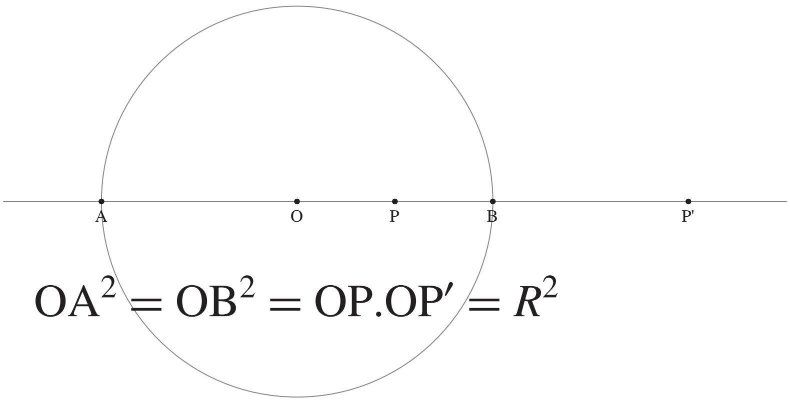
We can augment the complex plane by adding an extra point, ∞, to the plane. This produces the *complex projective line*, $\mathbb{CP}^1$. It can be considered to be a projective space represented by a pair of complex coordinates, (x, y), where scalar (complex) multiples are considered equivalent. Hence, for all points except ∞, we can normalise the coordinates as $\left(\frac{x}{y}, 1\right)$. We identify these points with points on the ordinary complex plane by letting $z = \frac{x}{y}$. If $y = 0$, this results in the point at infinity, ∞. We consider a line to be a *generalised circle* passing through ∞.

Two-dimensional inversion

Inversion is essentially a reflection in a circle. Points outside the circle are interchanged with points inside the circle in an involution of the plane.

- For a circle Γ with centre O, we define the *inverse point* of P to be the point P' on the line OP (at the same side as P from O) such that $OP \cdot OP' = R^2$, where R is the radius of Γ. **[Definition of inversion]**

If O is the origin, the inverse point can be found by the simple transformation $z \to \frac{R^2}{z^*}$.

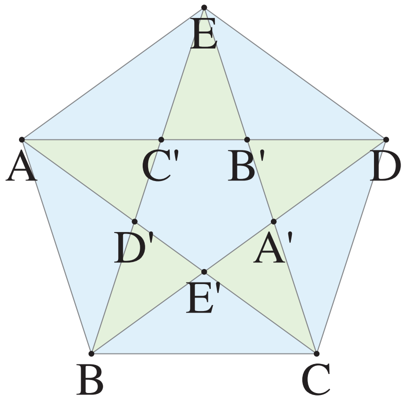


1. Show that O and ∞ are inverse points with respect to any circle centred on O.

2. Demonstrate that $(A, B; P, P')$ is a harmonic range.

3. If P and Q invert to P' and Q', show that P, P', Q and Q' are concyclic.

4. Show that $P'Q' = \frac{PQ \cdot R^2}{PO \cdot QO}$. **[Inversion distance formula]**



5. Let $A\,B\,C\,D\,E$ be a regular pentagon with side length 1. Let the diagonals $B\,D$ and $C\,E$ intersect at A', and define B', C', D' and E' similarly. Show that:
   - $B\,E = \phi$;
   - $B\,D' = \frac{1}{\phi}$;
   - $D'\,C' = \frac{1}{\phi^2}$;
   - $B\,C' = 1$;

   where ϕ is the positive root of the equation $x^2 - x - 1 = 0$. ($\phi = \frac{1+\sqrt{5}}{2} \simeq 1.618034$.)

Möbius transformations

We define a *Möbius transformation* to be a transformation of the form $z \to \frac{a\,z+b}{c\,z+d}$, where a, b, c, $d \in \mathbb{C}$. It is also necessary to include the condition that $a\,d - b\,c \neq 0$, to remove degenerate singular non-invertible cases. We can assume without loss of generality, therefore, that $a\,b - c\,d = 1$.

6. Show that the composition of any two Möbius transformations is another Möbius transformation.

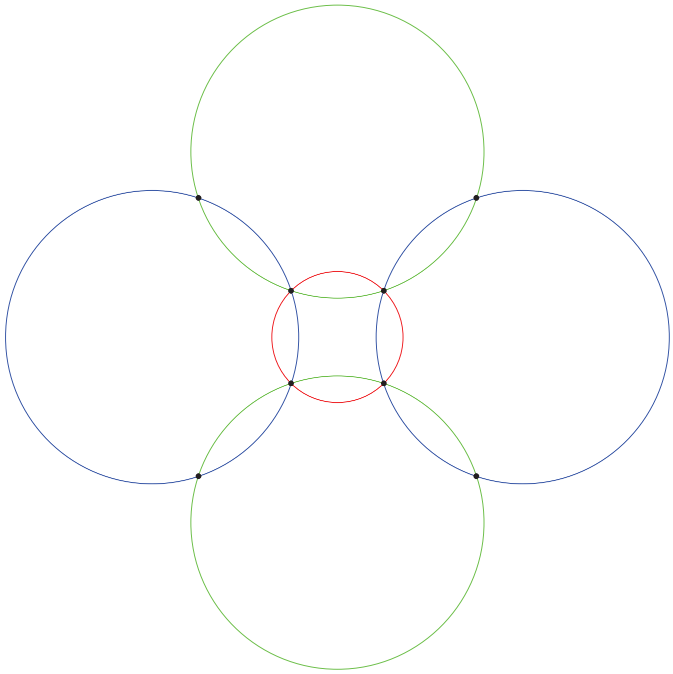
7. Prove that the composition of any inversion followed by any reflection is a Möbius transformation.

8. Show that, for four points w, x, y, $z \in \mathbb{C}$, the value of $\frac{(w-x)\,(y-z)}{(x-y)\,(z-w)}$ remains invariant when a Möbius transformation is applied.

Indeed, this follows naturally from the fact that Möbius transformations are projective transformations of the complex projective line. Hence, it is possible to find a unique Möbius transformation mapping any three points to any other three points.

9. Demonstrate that generalised circles remain as generalised circles under any Möbius transformation, and thus under inversion.

Like all non-trivial rational functions of z, Möbius transformations are *conformal maps*, which means angles between curves are (in general) preserved. Inversion reverses the direction of directed angles, but preserves the magnitude. This property can be derived from the fact that generalised circles remain as generalised circles.



10. Suppose $P_1 P_2 P_3 P_4$ is a cyclic quadrilateral. The circle Γ_n passes through P_n and P_{n+1}, with subscripts considered modulo 4. Circles Γ_n and Γ_{n+1} intersect again at Q_{n+1}. Prove that $Q_1 Q_2 Q_3 Q_4$ are either concyclic or collinear. **[Miquel's theorem]**

11. Let ABC be a triangle with a right-angle at C. Let CN be an altitude. A circle Γ is tangent to line segments BN, CN and the circumcircle of ABC. If D is where Γ touches BN, prove that CD bisects angle $\angle BCN$. **[NST2 2011, Question 3]**

Ivan's 25 circles

Consider a cyclic quadrilateral $ABCD$ with circumcentre O. Let AB intersect CD at P. Similarly, we define $Q = BC \cap DA$ and $R = AC \cap BD$.

12. Prove that P is the pole of QR, and hence that O is the orthocentre of PQR. **[Brocard's theorem]**

Denote the feet of the altitudes from P, Q and R with X, Y and Z, respectively. Then, we obtain six cyclic quadrilaterals from the fact that O is the orthocentre of PQR.

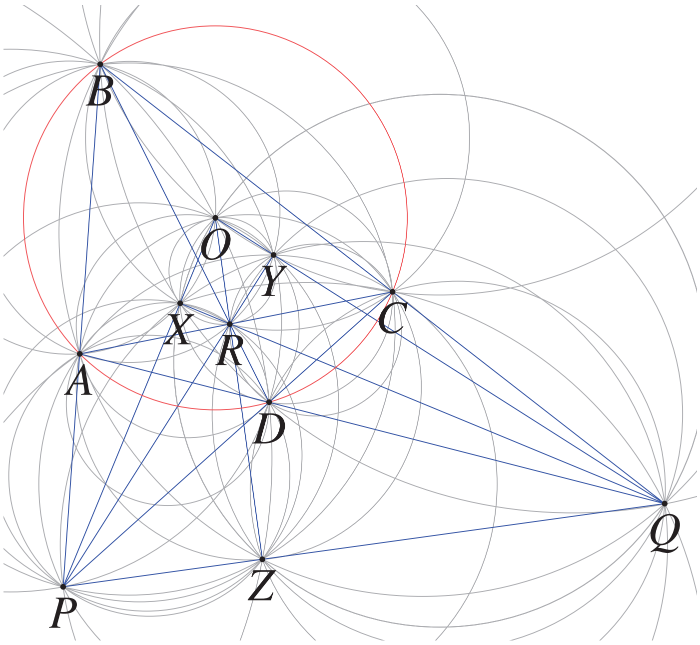
13. Prove that inversion about the circumcircle of $ABCD$ interchanges O with ∞, P with X, Q with Y and R with Z.

14. Hence prove that $OABX$ are concyclic.

By symmetry, this gives us six cyclic quadrilaterals, increasing the total to twelve (excluding $ABCD$). There are still another twelve circles on the diagram to be found.

15. Show also that $BCXR$ are concyclic.

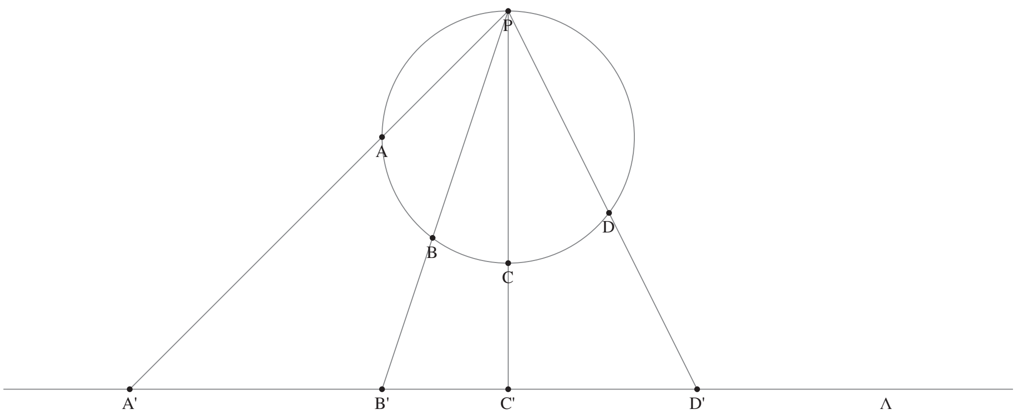
By symmetry, this increases the total of cyclic quadrilaterals to 24 (excluding $ABCD$). These circles were discovered by Ivan Guo.



As well as inverting about O, one can also invert about P, Q or R to permute the vertices.

Harmonic quadrilaterals

16. Let $ABCD$ be a cyclic quadrilateral. Choose a point P on the circumcircle of $ABCD$ and a line Λ outside the circle. The line PA meets Λ at A'; points B', C' and D' are defined similarly. Show that the cross-ratio $(A', C'; B', D')$ does not depend on the locations of P and Λ.



If $(A', C'; B', D') = -1$, then the cyclic quadrilateral $ABCD$ is known as a *harmonic quadrilateral*. Harmonic quadrilaterals have many nice properties:

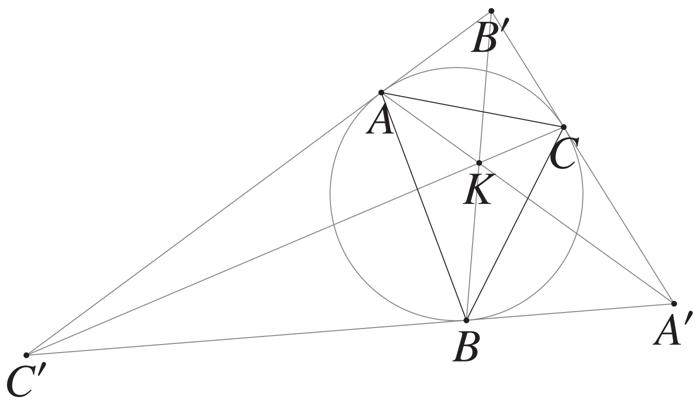
17. Let $ABCD$ be a convex cyclic quadrilateral. Let the tangents to the circumcircle at A and C meet at E; let the tangents to the circumcircle at B and D meet at F. Let the diagonals AC and BD intersect at P. Then show that the following properties are equivalent:

- $ABCD$ is a harmonic quadrilateral;
- $AB \cdot CD = BC \cdot DA = \frac{1}{2} AC \cdot BD$;
- E, B and D are collinear;
- B, D and K are collinear, where K is the symmedian point of ABC;
- $(P, E; B, D) = -1$.

18. If a quadrilateral $ABCD$ is represented by complex numbers a, b, c and d in the Argand plane, show that it is harmonic if and only if $(a-b)(c-d)+(b-c)(d-a)=0$.

19. Deduce that harmonic quadrilaterals/ranges remain harmonic after inversion.

The collinearity of E, B, K and D gives us an elegant construction of the symmedian point: let the tangents to the circumcircle at B and C intersect at A', and define B' and C' similarly. The symmedian point is then the intersection of AA', BB' and CC'.



Generalised spheres

We started by defining Möbius transformations and inversion in the environment of complex numbers. However, complex numbers are restricted to two (real) dimensions, so cannot be used to generalise the ideas to n-dimensional space. Instead, we will need to consider this from a more Euclidean perspective.

When discussing objects in n-dimensional space, we use the following conventions:

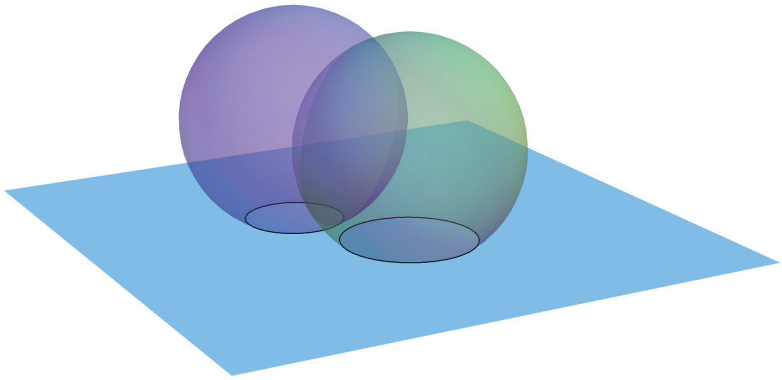
- A *n-ball* comprises all points in $\mathbb{R}^n$ within a distance of R from a point O.
- The surface of a n-ball is a $(n-1)$-*sphere*.
- The set of points in $\mathbb{R}^n$ obeying a single linear equation is a $(n-1)$-*plane*.
- A *generalised* $(n-1)$-*sphere* can be a $(n-1)$-sphere or a $(n-1)$-plane.

20. Show that the intersection of two generalised $(n-1)$-spheres in $\mathbb{R}^n$ is either empty, a single point, or a generalised $(n-2)$-sphere lying in a $(n-1)$-plane of $\mathbb{R}^n$. **[Intersection of generalised spheres]**

21. Let Γ be a $(n-1)$-sphere and P a point in $\mathbb{R}^n$. Lines l_1 and l_2 pass through P. l_1 intersects Γ at A and B; l_2 intersects Γ at C and D. Prove that $PA \cdot PB = PC \cdot PD$. **[$n$-dimensional intersecting chords theorem]**

22. If a point $P \in \mathbb{R}^n$ has equal power with respect to non-concentric $(n-1)$-spheres Γ_1 and Γ_2, then it lies on their *radical axis*. Show that the radical axis of two identical $(n-1)$-spheres is a $(n-1)$-plane.

This forms the basis of an ingenious proof by Géza Kós that the radical axis of any two (non-concentric) circles on the plane Λ is a line. Firstly, erect two spheres of equal radius, Γ_1 and Γ_2, on the circles.



The radical axis of the two circles is the intersection of the radical plane of Γ_1 and Γ_2 with the plane Λ. This construction clearly generalises to two non-concentric $(n-1)$-spheres in $\mathbb{R}^n$, by embedding the situation into $\mathbb{R}^{n+1}$ with equiradial n-spheres.

n-dimensional inversion

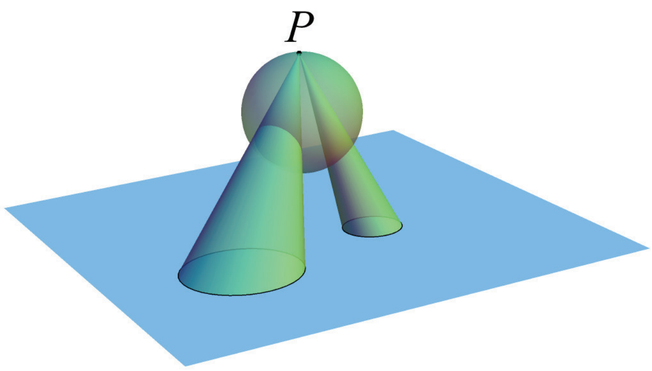
We are now in a suitable position to define n-dimensional inversion and investigate its properties.

- Let Γ be a $(n-1)$-sphere in $\mathbb{R}^n$ with centre O and radius R. For any point $P \in \mathbb{R}^n$, the *inverse point* P' is defined to lie on the line OP at the same side of O such that $OP \cdot OP' = R^2$. **[Definition of n-dimensional inversion]**

23. Prove that generalised $(n-1)$-spheres map to generalised $(n-1)$-spheres under inversion.

24. Draw a generalised $(n-2)$-sphere on the surface of some generalised $(n-1)$-sphere in $\mathbb{R}^n$. Prove that, after inversion, this will remain a generalised $(n-2)$-sphere. **[Backward compatibility of inversion]**

The oblique cones in the diagram below intersect both the plane and the sphere in circular cross-sections, as the sphere and plane are inverses with respect to P. This idea of projecting the sphere onto a plane from a point on the sphere is known as *stereographic projection*.



The diagram above enables us to easily define Möbius transformations. Let P be an arbitrary point outside the plane Λ. An arbitrary Möbius transformation of the plane Λ is the composition of:

- A two-dimensional translation and/or homothety of the plane Λ;
- An inversion about the unit sphere centred on P, transforming Λ into a sphere Ω (the Riemann sphere);
- A rotation of the sphere Ω about its centre (P does not move with the sphere);

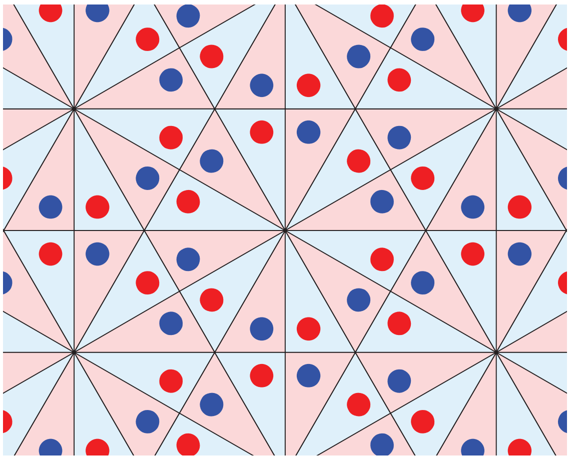
- An inversion about the unit sphere centred on P, transforming Ω back into Λ.

There are six degrees of freedom in this transformation, so they must represent all Möbius transformations, and nothing else. This definition of a Möbius transformation clearly generalises to $\mathbb{R}^n$ (with $\frac{1}{2}(n+1)(n+2)$ degrees of freedom), whereas the complex number definition does not.

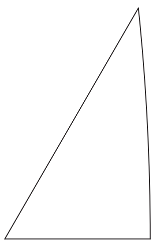
25. Suppose two smooth curves are drawn in $\mathbb{R}^n$, which intersect at a point. Prove that the angle of intersection is preserved (or, more accurately, reversed) after inversion. **[Anti-conformal map]**

Kaleidoscopes

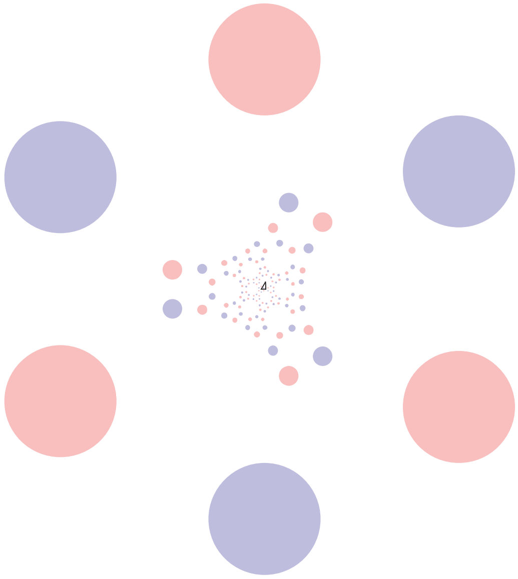
You may have encountered kaleidoscopes, where mirrors are used to form lots of repeated copies of a 'fundamental region'. For example, we can generate patterns with the same symmetries as the hexagonal tiling by using a triangle of mirrors with internal angles of $\frac{1}{2}\pi$, $\frac{1}{3}\pi$ and $\frac{1}{6}\pi$. Mathematicians regard this as a group of symmetries, *generated* by the three reflections. If we call the reflections α, β, γ, then we have the relations $\alpha^2 = \beta^2 = \gamma^2 = I$, where I is the identity element; a reflection is its own inverse. Our three rotations, $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$, together generate half of the symmetry group, namely the group of direct congruences. We also have $(\alpha\beta)^2 = (\beta\gamma)^3 = (\gamma\alpha)^6 = I$, as applying a rotation of 2π is equivalent to the identity.



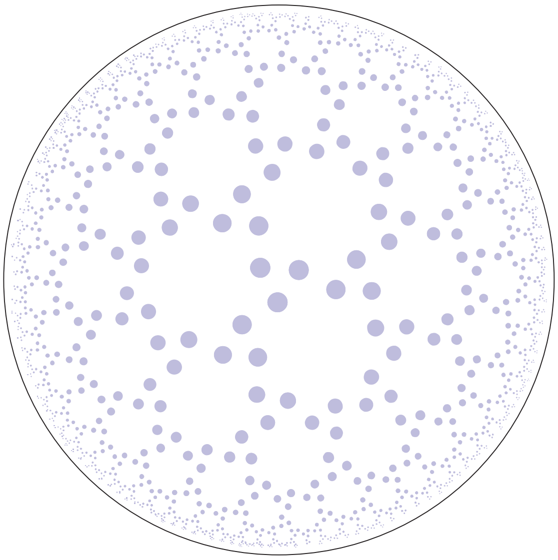
In Euclidean geometry, the interior angles of a triangle necessarily sum to π. However, if we allow circular arcs instead of straight lines, this condition can be relaxed. For example, we can have a curvilinear triangle with interior angles of $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{5}\pi$:



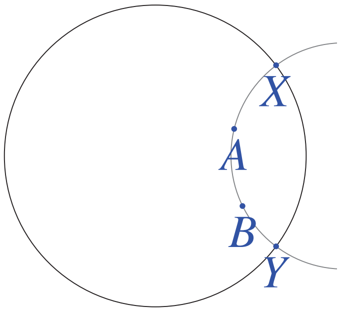
Reflection in the circular arc is simply inversion, and reflections in the straight lines are ordinary reflections. In this manner, we generate a spherical reflection group, namely the symmetry group of a dodecahedron. The compositions $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$ are no longer necessarily rotations, but are instead Möbius transformations. For this group, $(\alpha\beta)^2 = (\beta\gamma)^3 = (\gamma\alpha)^5$.



The above picture of icosahedral symmetry may look distorted, not least because we have flattened the Riemann sphere into a plane by stereographic projection. Finally, we can produce hyperbolic tilings by ensuring the interior angles of the fundamental triangle sum to less than π (by using concave arcs); this visualisation of the hyperbolic plane is known as the *Poincaré disc model*.



We can define 'distances' and 'angles' in hyperbolic geometry. As hyperbolic space must be invariant under any Möbius transformation mapping the unit circle to itself, distances and angles must also be preserved. We already know that complex (or cyclic) cross-ratio is invariant under Möbius transformations. The (directed) hyperbolic distance between two points, A and B, is given by the logarithm of the cyclic cross-ratio $(A, B; X, Y)$, where X and Y are the two intersections of the circle $A\,B\,B'\,A'$ (where $'$ indicates inversion in the unit circle) with the unit circle. Note that $A\,B\,B'\,A'$ is orthogonal to the unit circle.



- The hyperbolic distance between two points is given by $AB = |\ln(A, B; X, Y)|$. **[Definition of hyperbolic distance]**

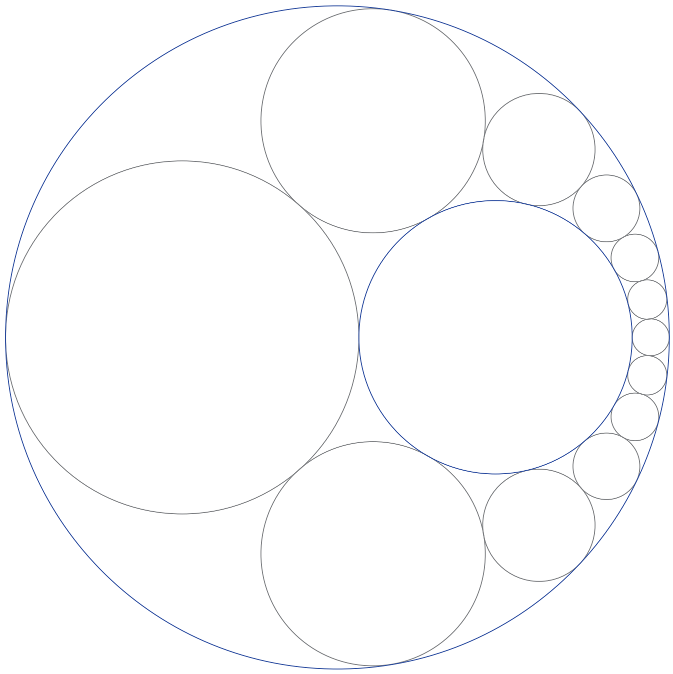
A hyperbolic line, *i.e.* the shortest path between two points A and B, is the arc AB of the circle $ABB'A'$. Hence, the triangle inequality applies: for three points A, B, C, $AB + BC \geq AC$, with equality if and only if they are collinear and in the correct order. As angles are preserved under Möbius transformations, the angles in the Poincaré disc model are the same as those in the hyperbolic plane.

- The hyperbolic angle between two hyperbolic lines is identical to the ordinary angle between the corresponding circular arcs on the Poincaré disc model. **[Definition of hyperbolic angle]**

With these principles, it is possible to explore the rich world of Bolyai-Lobachevskian geometry. Four of Euclid's postulates (basic assumptions from which all of geometry can be derived) hold in hyperbolic geometry, whereas the fifth postulate does not. The fifth postulate is equivalent to the interior angles of a triangle summing to π.

Steiner's porism and Soddy's hexlet

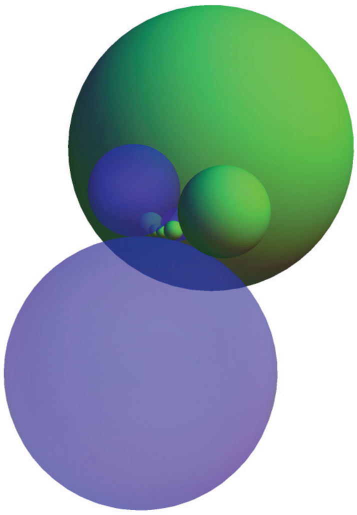
We have already encountered Poncelet's porism, which states that if there is one n-gon inscribed in one ellipse and circumscribed about another, then there are infinitely many. Another example of a porism (a term which people struggle to define) is *Steiner's porism*, again named after Jakob Steiner.



26. Let Γ_1 and Γ_2 be two circles, such that Γ_1 is contained within Γ_2. A set of $n \geq 3$ circles $\{C_1, C_2, \ldots, C_n\}$ is known as a *Steiner chain* of length n if each C_i is tangent externally to C_{i+1}, C_{i-1} (where subscripts are considered modulo n) and Γ_1, and is tangent internally to Γ_2. Show that if there exists one Steiner chain of length n for two given circles, then there exist infinitely many. **[Steiner's porism]**

There is an analogous three-dimensional porism, which is less general but more interesting, called *Soddy's hexlet*. This also appeared as a Japanese Sangaku problem.

27. Let Γ_1, Γ_2 and Γ_3 be three mutually tangent spheres. A set of $n \geq 3$ spheres $\{S_1, S_2, \ldots, S_n\}$ is known as a *Soddy chain* of length n if each S_i is tangent externally to S_{i+1}, S_{i-1} (where subscripts are considered modulo n), Γ_1, Γ_2 and Γ_3. Show that infinitely many Soddy chains exist of length 6, and no Soddy chains exist for $n \neq 6$. **[Soddy's hexlet]**



The six green spheres $\{S_1, \ldots, S_6\}$ are tangent to a quartic doughnut-shaped surface known as a *Dupin cyclide*, which is an inverted torus.

Solutions

1. Assume, without loss of generality, that O is the origin of the complex plane. Then, we have $0 \to \frac{1}{0^*} = \frac{1}{0} = \infty$.

2. We can compute the linear cross-ratio $\frac{\overrightarrow{AP}}{\overrightarrow{PB}} \cdot \frac{\overrightarrow{BP'}}{\overrightarrow{P'A}} = \frac{(R+p)\left(\frac{R^2}{p} - R\right)}{(R-p)\left(-\frac{R^2}{p} - R\right)} = \frac{\frac{R^3}{p} - R\,p}{-\frac{R^3}{p} + R\,p} = -1$.

3. This is a trivial corollary of the intersecting chords theorem.

4. $P'\,Q'\,O$ is similar to $Q\,P\,O$, so we have $\frac{P'Q'}{P'O} = \frac{P\,Q}{Q\,O}$. By using the identity $P\,O \cdot P'\,O = R^2$, we obtain the desired formula.

5. Let the length of the diagonals be denoted by x. By Ptolemy's theorem on $B\,C\,D\,E$, we have $1 + x = x^2$, so $x = \phi$. Note that D' and C are inverse points with respect to the unit circle centred on A. Hence, we obtain $B\,D' = \frac{B\,C}{A\,B \cdot A\,C} = \frac{1}{\phi}$; $B\,C' = \frac{B\,D}{A\,B \cdot A\,D} = \frac{\phi}{\phi} = 1$; $D'\,C' = \frac{C\,D}{A\,C \cdot A\,D} = \frac{1}{\phi^2}$.

6. Möbius transformations are projective transformations of the complex projective line, thus the composition is simply the product of their matrices.

7. Translations composed with rotations and dilations are Möbius transformations, as they have the form $z \to a\,z + b$. Hence, we need only consider the case where the inversion is in the unit circle, and thus has the form $z \to \frac{1}{z^*}$. Composing this with the general form of an indirect similarity, $z \to a\,z^* + b$, results in the transformation $z \to \frac{a}{z} + b$, which is clearly a Möbius transformation.

8. Möbius transformations can be regarded as a projective transformations of the complex projective line, so must necessarily preserve complex cross-ratio.

9. The condition for four points to be concyclic or collinear is that the complex cross-ratio, $\frac{(a-b)\,(c-d)}{(b-c)\,(d-a)}$, is real. As it is preserved under Möbius transformations, so must the property that four points are concyclic or collinear. Obviously, a reflection also preserves this property, so inversions (compositions of reflections and Möbius transformations) must also do so.

10. Inverting about P_1 reduces the problem to the pivot theorem.

11. By Thales' theorem, $A\,B$ is a diameter of the circumcircle, so is orthogonal to it. Invert about C. $A'\,B'$ and $N'\,\infty'$ are diameters of the same circle (the inverse of line $A\,B$), so $A'\,N'\,B'\,\infty'$ forms a rectangle. The inverse of Γ is tangent to the diagonals and circumcircle of the rectangle $A'\,N'\,B'\,\infty'$, so lies on one of the lines of symmetry of the rectangle. The point D' must thus be the midpoint of the minor arc $B'\,N'$, as D lies between B and N in the original diagram. As arcs $B'\,C'$ and $C'\,N'$ are congruent, they must subtend equal angles at ∞'. As angles are preserved under inversion, we are finished.

12. Let $Q\,R$ intersect $A\,B$ and $C\,D$ at E and F, respectively. From applying a projective transformation to convert $A\,B\,C\,D$ into a square, we can deduce that $(P, E; A, B)$ and $(P, F; C, D)$ are harmonic ranges. This is sufficient for $Q\,R$ to be the polar of P. Hence, $O\,P$ is perpendicular to $Q\,R$. By symmetry, O must be the orthocentre of $P\,Q\,R$.

13. O is the centre of the circle of inversion, so is exchanged with the point at infinity. X is the intersection of OP with the polar of P, so they must be inverse points. By symmetry, we obtain the other two pairs of inverse points.

14. Inverting about the circle $ABCD$ maps O and X to ∞ and P, respectively. ABP is a straight line and thus passes through ∞, so the original four points were concyclic.

15. Note that lines BC, OY and XR concur at Q. Applying the converse of the radical axis theorem to circles $OYBC$ and $ROXY$, we obtain the concyclicity of $XRBC$.

16. It obviously does not depend on Λ, as we can view P as a projector, and moving Λ is simply applying one-dimensional projective transformations to the line $A'B'C'D'$. Hence, the cross-ratio is dependent only upon the angles APB, BPC and CPD. Due to basic circle theorems, these are invariant as P moves on the circumcircle of $ABCD$.

17. Inversion about P and considering similar triangles derives the first of these results. To show that the collinearity of E, B and D implies that the quadrilateral is harmonic, use the intersecting chords theorem to show that AED is similar to BEA, and that DEC is similar to CEB. We then have $DA/AB = DE/AE = DE/CE = CD/BC$, thus the products of opposite sides are equal. The converse follows trivially. Showing that P is in harmonic range with E, B and D stems from the fact that P lies on AC, which is the polar of E. To demonstrate that K lies on BD, it is sufficient to show that P is the foot of the symmedian from B to AC. This is equivalent to the statement that $AP/AB^2 = CP/CB^2$. To prove this, we can exploit similar triangles to show that $AP/AB = DP/CD = (PA \cdot PC/AB)/(BC \cdot DA/AB)$, whence it follows that $AP\big/\big(AB^2\big) = (PA \cdot PC)/(PB \cdot BC \cdot DA) = (PA \cdot PC)/(PB \cdot BA \cdot DC) = CP\big/\big(CB^2\big)$. This shows that P has the required areal coordinates to be the foot of the symmedian.

18. Consider when the complex cross-ratio $(a-b)(c-d)/(b-c)(d-a) = -1$. From here, we use polar coordinates, resulting in $\arg(a-b)/\arg(c-b) = \arg(a-d)/\arg(c-d)$ and $|a-b|\ |c-d| = |b-c|\ |d-a|$. These are equivalent to $\angle ABC = \angle ADC$ and $AB \cdot CD = BC \cdot DA$, respectively. Clearly, the first of these is the 'angles in the same segment' criterion for concyclicity, and the latter is condition for a cyclic quadrilateral to be harmonic.

19. A composition of an inversion and a reflection is a Möbius transformation, *i.e.* a map of the form $z \to (az+b)/(cz+d)$. This can be regarded as a projective transformation of the complex projective line, so must necessarily preserve complex cross-ratio.

20. For two $(n-1)$-planes, the intersection is either empty (if they are parallel) or the $(n-2)$-plane formed by solving their algebraic equations simultaneously. For two $(n-1)$-spheres, consider all 2-planes passing through both centres. If the circles are disjoint, so are the original spheres. If the circles are tangent, the original spheres share a single point. If the circles intersect in two points, then let the radical axis meet the line of centres at P. Clearly, all intersection points of the original two spheres must lie on the $(n-1)$-plane through P perpendicular to the line of centres, and must all be equidistant from P. For the case of one $(n-1)$-sphere and one $(n-1)$-plane, we can reflect the sphere in the plane and reduce it to the previous case.

21. Consider the plane containing l_1 and l_2, and apply the two-dimensional intersecting chords theorem to this configuration.

22. The power of a point is equal to $OP^2 - R^2$. When the spheres are of equal radii, the equation for the radical plane becomes $OP_1{}^2 = OP_2{}^2$, which is the locus of all points equidistant from the centres of the two spheres. This must be the plane of reflective symmetry of the configuration.

23. Draw the line l through O orthogonal to the generalised $(n-1)$-sphere, intersecting it at A and B (one of which may be at infinity if the generalised sphere is a plane). Let the inverse points be A' and B', which also lie on l. Every plane containing l must intersect the generalised $(n-1)$-sphere in a generalised circle of diameter AB; this inverts to a generalised circle of diameter $A'B'$ using ordinary two-dimensional inversion. The union of all such generalised circles is the generalised $(n-1)$-sphere of diameter $A'B'$.

24. Consider the generalised $(n-2)$-sphere to be the intersection of two $(n-1)$-spheres. The result follows from the fact that the intersection is either empty (impossible), a single point (impossible) or a generalised $(n-2)$-sphere.

25. We can reduce this to the 3-dimensional problem, as we need only consider the 3-plane containing the tangents to the two curves, l_1 and l_2, and the centre of inversion, O. Assume they intersect at P with an angle α. After inversion, the plane $l_1 l_2$ becomes a generalised 2-sphere, and generalised circles l_1 and l_2 intersect at P' with an angle β. If the generalised 2-sphere is a sphere, we invert about the point diametrically opposite to P', preserving P' whilst mapping l_1 and l_2 back to lines. Clearly, the lines now intersect at an angle $-\beta$. This orientation-preserving generalised-circle-preserving map from the Riemann sphere to itself must necessarily be a Möbius transformation, and thus preserve angles. Hence, $\alpha = -\beta$, and we are done.

26. Let the line of centres l intersect Γ_1 at B and C, and Γ_2 at A and D, such that $A\,B\,C\,D$ are in that order along l. Note that the two circles intersect l orthogonally. Let $A'B'$ and $C'D'$ be two unit lengths on the same line l', and separate them such that $(A', D'; B', C') = (A, D; B, C)$. Apply a Möbius transformation to map A to A', B to B' and C to C'. As the cross-ratios are equal, D must necessarily map to D'. The two circles are now both orthogonal to l', so the centres both lie on this line. As $A'D'$ and $B'C'$ share a midpoint, the centres must coincide and the circles are concentric. We can now ‘rotate’ the Steiner chain within the annulus by an arbitrary angle, similar to a ball bearing. Applying the inverse Möbius transformation will restore Γ_1 and Γ_2.

27. Invert about the tangency point of Γ_1 and Γ_2, resulting in two parallel planes sandwiching the sphere Γ_3. All spheres tangent to Γ_1 and Γ_2 must have diameter equal to the separation of the planes. Hence, their centres must be coplanar with Γ_3 and have equal radius. Hence, we can consider this to be a two-dimensional problem of packing a closed loop of n circles of unit radius around a circle of unit radius. As the centres of Γ_3, S_i and S_{i+1} must form an equilateral triangle, this forces the angles to be $\frac{\pi}{3}$ and thus there must be a regular hexagon of six spheres. There are infinitely many orientations in which this can be done.