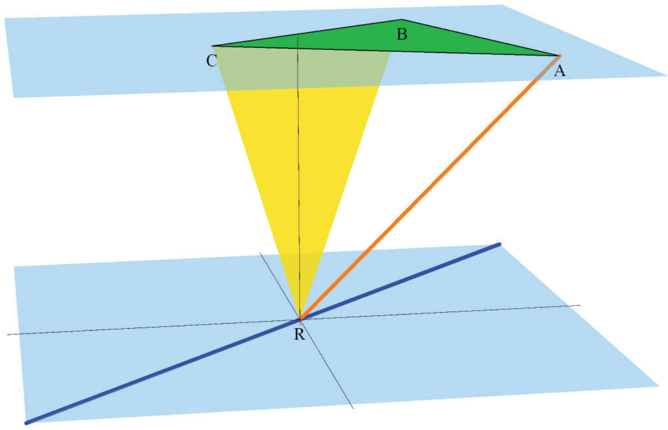
Projective geometry

Projective geometry is an extension of Euclidean geometry, endowed with many nice properties incurred by affixing an extra 'line at infinity'. Certain theorems (such as Desargues' and Pascal's theorems) have projective geometry as their more natural setting, and the wealth of projective transformations can simplify problems in ordinary Euclidean geometry.

The real projective plane

In Euclidean geometry, we assign a coordinate pair $(x,\ y)$ to each point in the plane. In projective geometry, we augment this with an extra coordinate, so three values are used to represent a point: $(x,\ y,\ z)$. Moreover, scalar multiples are considered equivalent; $(x,\ y,\ z)$ and $(\lambda\, x,\ \lambda\, y,\ \lambda\, z)$ represent the same point. $R = (0,\ 0,\ 0)$ is not part of the projective plane, but can be regarded as a 'projector', from which all points, lines, circles *et cetera* emanate.

Since scalar multiples of points are considered equivalent, we can identify points in $\mathbb{RP}^2$ (the real projective plane) with lines through the origin. Projective lines are identified with planes through the origin, and are of the form $a\,x + b\,y + c\,z = 0$. Note that this equation is *homogeneous*: all terms are of first degree. In general, all algebraic curves are represented by homogeneous polynomials in x, y and z.



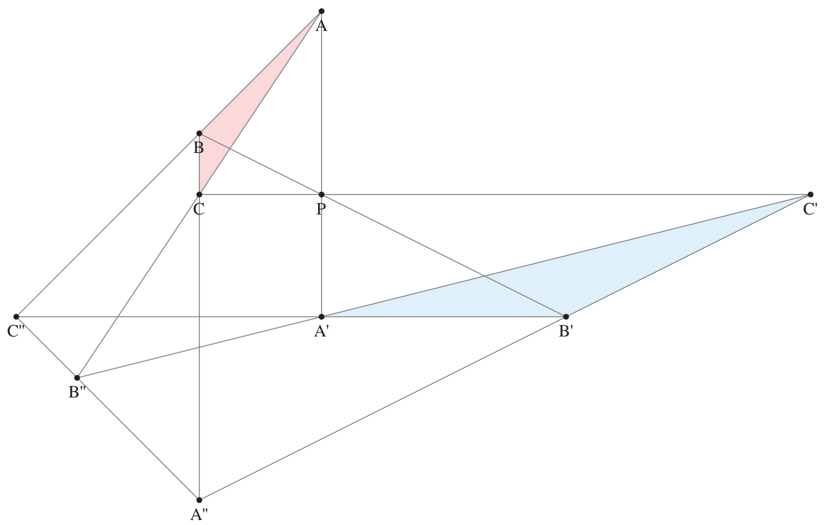
In the diagram above, a triangle $A\,B\,C$ is shown in the *reference plane* ($z = 1$). The point A is 'projected' from R along the orange line, intersecting the reference plane at A. Similarly, the 'plane' containing the yellow triangle represents the line $B\,C$. Horizontal 'lines' such as the blue one do not intersect the reference plane, so correspond to points 'at infinity'. The horizontal 'plane' (parallel to the reference plane) through R represents the line at infinity ($z = 0$). Parallel lines on the projective plane can be considered to meet at a point on the line at infinity.

1. Prove that any two distinct lines intersect in precisely one point.

2. Show that the equation of a line through points $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$ is given by $\det\begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} = 0$.

You have shown that any two points share a common line, and any two lines share a common point. This suggests a fundamental interchangeability between lines and points, known as *projective duality*. We shall explore this later.

3. Let $A\,B\,C$ and $A'\,B'\,C'$ be two triangles. Let $A\,B$ meet $A'\,B'$ at C'', and define A'' and B'' similarly. Show that $A\,A'$, $B\,B'$ and $C\,C'$ are concurrent if and only if A'', B'' and C'' are collinear. **[Desargues' theorem]**



If two triangles exhibit this relationship, they are said to be *in perspective*. The point P is the *perspector*, and the line $A'' B'' C''$ is the *perspectrix*. As Desargues' theorem is the projective dual of its converse, you only need to prove the statement in one direction. Surprisingly, it is actually easier to prove Desargues in three dimensions (where the triangles are in different planes); the two-dimensional result then follows by projecting it onto the plane.

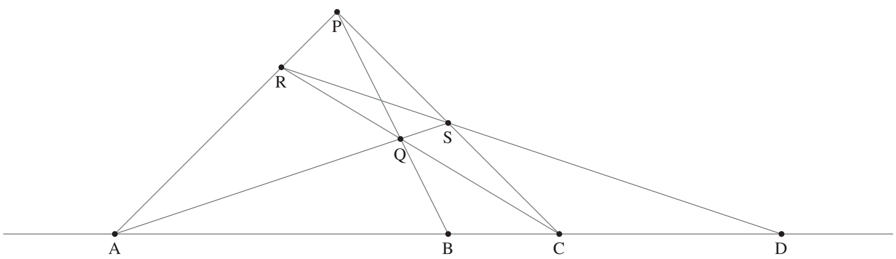
4. Let $A B C$ be a triangle and X be a point inside the triangle. The lines $A X$, $B X$ and $C X$ meet the circle $A B C$ again at P, Q and R, respectively. Choose a point U on $X P$ which is between X and P. Suppose that the lines through U which are parallel to $A B$ and $C A$ meet $X Q$ and $X R$ at points V and W respectively. Prove that the points R, W, V and Q lie on a circle. **[BMO2 2011, Question 1]**

Cross-ratio and harmonic ranges

You may have encountered homogeneous projective coordinates before in the form of areal (or barycentric) coordinates. Even though qualitative properties such as collinearity and concurrency can be defined in terms of any projective coordinates, to compare distances they must first be *normalised*, *i.e.* projected onto the reference plane.

Firstly, we define a vector, $\underline{n}$, perpendicular to the reference plane. (For Cartesian coordinates, we generally allow $z = 1$ to be the reference plane, so $\underline{n} = (0, 0, 1)$. For areal coordinates, $x + y + z = 1$ is the reference plane, so $\underline{n} = (1, 1, 1)$. In general, if the reference plane is given by $a x + b y + c z = 1$, the vector $\underline{n} = (a, b, c)$. For a given vector $\underline{x}$ in the projective plane, it is normalised by the operation $\underline{x} \to \frac{\underline{x}}{\underline{n} \cdot \underline{x}}$

5. If A, B, C and D are collinear points represented by normalised vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$, respectively, show that $\frac{\overrightarrow{A B}}{\overrightarrow{C D}} = \frac{\underline{a} - \underline{b}}{\underline{c} - \underline{d}}$.



Suppose we have three points, A, B and C, which are collinear. We can select an arbitrary point P, and a further arbitrary point Q lying on the line $P B$. Let $Q C$ meet $P A$ at R, and $Q A$ meet $P C$ at S. Finally, let $R S$ meet line

ABC at D.

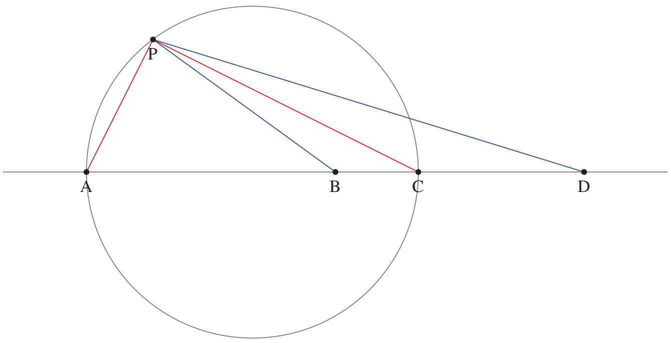
6. Prove that $\frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{\overrightarrow{BC} \cdot \overrightarrow{DA}} = -1$, irrespectively of the locations of P and Q. *(Hint: this can be done by applying two similar theorems in quick succession.)*

It is a remarkable fact that the location of D does not depend on that of P and Q. B and D are described as *projective harmonic conjugates* with respect to the line segment AC. For any four collinear points, the quantity $(A, C; B, D) = \frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{\overrightarrow{BC} \cdot \overrightarrow{DA}}$ is known as the *cross-ratio*. If $(A, C; B, D) = -1$, we say that they form a *harmonic range*.

■ $(A, C; B, D) = -1$ is equivalent to A and C being inverse points with respect to the circle on diameter BD, which is in turn equivalent to the circles on diameters AC and BD being orthogonal. **[Equivalent definitions of harmonic range]**

There is another remarkable and useful fact concerning harmonic ranges. Let P be a point not on the line $ABCD$. Then any two of the following four properties implies the other two:

- $\angle APC = \frac{\pi}{2}$;
- PA is an angle bisector of $\angle BPD$;
- PC is the other angle bisector of $\angle BPD$;
- $(A, C; B, D) = -1$.



By Thales' theorem, the first of these conditions is equivalent to P lying on the circle with diameter AC. Hence, if P lies on either of the intersection points of the (orthogonal) circles on diameters AC and BD, the four lines through P divide the plane into eight equal octants of angle $\frac{\pi}{4}$.

Projective transformations

Returning to the idea of representing points in the projective plane as three-element vectors, we can consider the group of operations represented by linear maps $\underline{x} \to M\,\underline{x}$, where M is a non-singular matrix. These are known as *projective transformations*, or *collineations*.

7. Show that applying a projective transformation to a line results in another line, and thus that collinear points remain collinear.

A generalisation is that degree-d algebraic curves are mapped to degree-d algebraic curves by projective transformations. Hence, conics are preserved. Moreover, it is possible to choose a projective transformation carefully to map a given conic to any other conic, with three real degrees of freedom remaining.

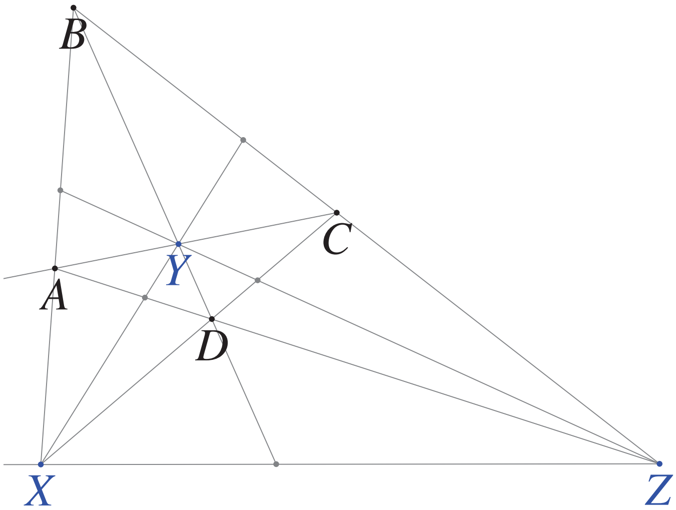
8. Show that applying a projective transformation with matrix M to a point $x = p + \lambda\, q$ results in a point with normalised coordinates $\frac{M\,\underline{p} + \lambda\, M\,\underline{q}}{n \cdot M\,\underline{p} + n \cdot \lambda\, M\,\underline{q}}$.

9. Hence show that the cross-ratio $(A, C; B, D)$ of four collinear points is preserved under projective transformations.

10. For any four points A, B, C and D, no three of which are collinear, show that there exists a unique projective transformation mapping them to $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$, respectively.

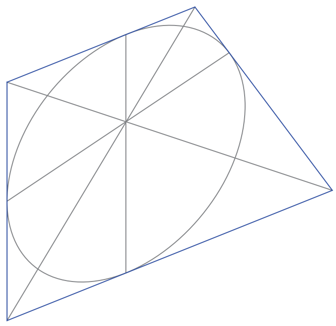
11. Hence show that there exists a unique projective transformation mapping any four points (no three of which are collinear) to any other four points (no three of which are collinear).

The last of these theorems enables one to simplify a projective problem by converting any quadrilateral into a parallelogram. This enables one to find all of the harmonic ranges in the *complete quadrangle* displayed below. Try to spot as many as you can!

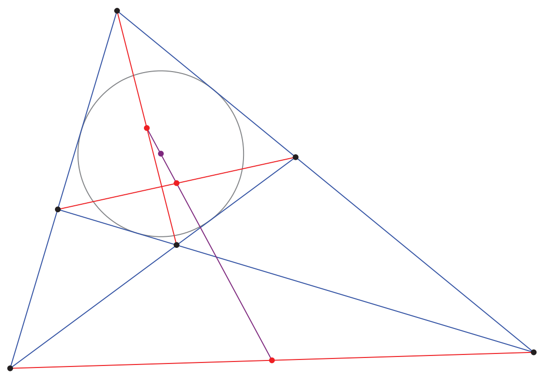


12. The diagonals of the quadrilateral $ABCD$ meet at X. The circumcircles of ABX and CDX intersect again at Y; the circumcircles of BCX and DAX intersect again at Z. The midpoints of the diagonals AC and BD are denoted M and N, respectively. Prove that M, N, X, Y and Z are concyclic. **[Sherry Gong, Trinity 2012]**

Another configuration occurring in many instances is a quadrilateral with an inscribed conic. With a projective transformation, we can convert the quadrilateral into a parallelogram. To simplify matters even further, a (possibly imaginary) affine transformation is capable of turning the conic into a circle. The symmetry of the configuration implies that the diagonals of the quadrilateral are concurrent with the lines joining opposite points of tangency.

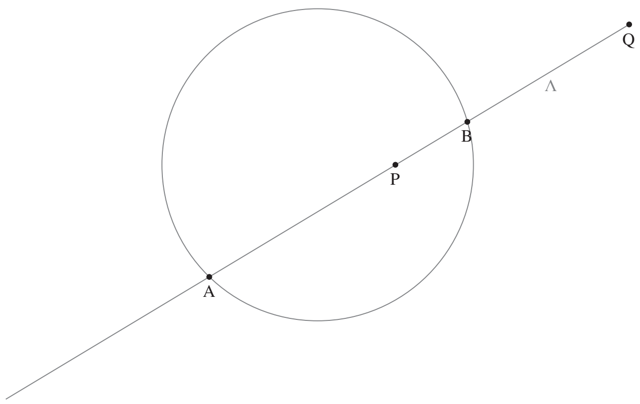


When viewed as a *complete quadrilateral* (four lines in general position intersecting in six points), we have three of these concurrency points and plenty of harmonic ranges! The centre of the inscribed conic also happens to lie on a line passing through the three midpoints of pairs opposite vertices, by *Newton's theorem*. This line is thus the locus of the centres of all possible inscribed conics.



Polar reciprocation and conics

One convenient way to demonstrate projective duality is by creating a bijection between lines and points, such that collinear points map to concurrent lines. There are several ways in which this can be defined, but one of the most elegant is polar reciprocation:



13. Consider a point P in the unit circle. Draw a line Λ through P, intersecting the unit circle at A and B. Let Q be the projective harmonic conjugate of P with respect to AB. Show that the locus of Q as Λ varies is a straight line.

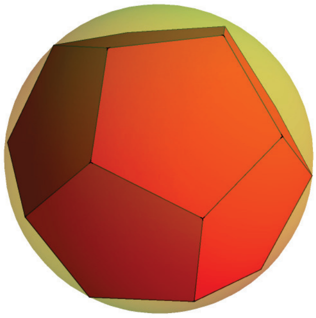
This locus is known as the *polar* of P, and P is its *pole*. Moreover, there is nothing special about the unit circle in projective geometry, so this construction generalises to any conic.

14. Let ABC be a scalene triangle, and let Γ be its nine-point circle. Γ intersects BC at points P and Q; the tangents from Γ at P and Q intersect at A'. Points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent. **[NST4 2011, Question 3]**

The easiest route to solving the above problem is to prove the much more general theorem that a triangle and its polar reciprocal triangle (with respect to any conic) are in perspective. This is (like many results in projective geometry!) known as *Chasles' theorem*.

Polar reciprocation generalises to three dimensions, where we can reciprocate about a sphere (or, more generally, quadric surface).

15. What is the polar reciprocal of a dodecahedron with respect to its circumsphere S?



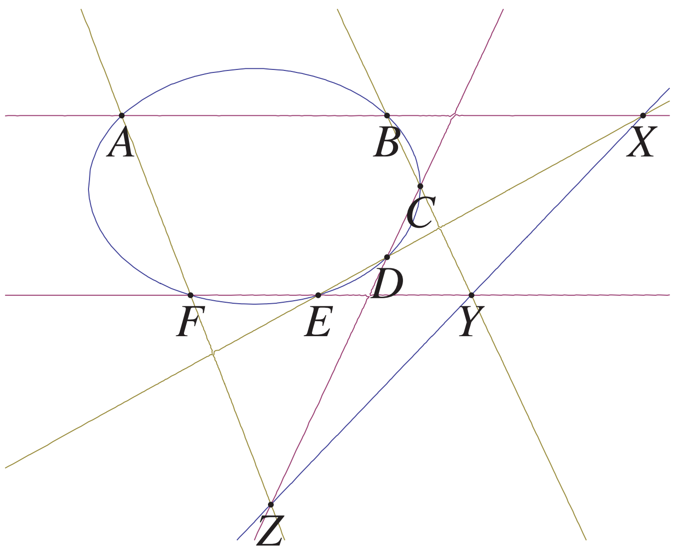
More generally, the dual of the regular polyhedron with Schläfli symbol $\{a, b\}$ has the Schläfli symbol $\{b, a\}$. For higher dimensions, we simply reflect the symbol. The tetrahedron, with the palindromic Schläfli symbol $\{3, 3\}$, is thus self-dual, as is the square tiling with Schläfli symbol $\{4, 4\}$. More generally, a simplex has Schläfli symbol $\{3, 3, 3, \ldots, 3, 3\}$ and a hypercubic tessellation has Schläfli symbol $\{4, 3, 3, \ldots, 3, 4\}$, both of which are palindromic. For four-dimensional solids, the 4-simplex $\{3, 3, 3\}$ is not the only self-dual regular polychoron; we also have the '24-cell' with Schläfli symbol $\{3, 4, 3\}$ (meaning that three octahedral cells are clustered around each edge).

Circular points at infinity

In projective Cartesian coordinates, the equation of a circle is of the form $x^2 + y^2 + b\,x\,z + c\,y\,z + d\,z^2 = 0$. Note that the points $(1, i, 0)$ and $(i, 1, 0)$ satisfy this equation, where i is the imaginary unit. Hence, all circles can be considered to pass through two imaginary 'circular points' on the line at infinity. Indeed, a circle can be *defined* as any conic passing through both circular points. In the n-dimensional complex projective plane ($\mathbb{CP}^n$), we have a $(n-2)$-sphere on the line at infinity contained by all $(n-1)$-spheres.

Apart from the circular points, other imaginary points on the complex projective plane $\left(\mathbb{CP}^2\right)$ are scarcely useful, so there is no need to worry about them.

16. Let $A\,B\,C\,D\,E\,F$ be a hexagon, the vertices of which lie on a conic. Let $A\,B$ and $D\,E$ meet at X; $B\,C$ and $E\,F$ meet at Y; and $C\,D$ and $F\,A$ meet at Z. Prove that X, Y and Z are collinear. **[Pascal's theorem]**



17. Write down the projective dual of Pascal's Theorem. **[Brianchon's theorem]**

18. Let A, C and E lie on a straight line, and B, D and F lie on another straight line. Let AB and DE meet at X; BC and EF meet at Y; and CD and FA meet at Z. Prove that X, Y and Z are collinear. **[Pappus' theorem]**

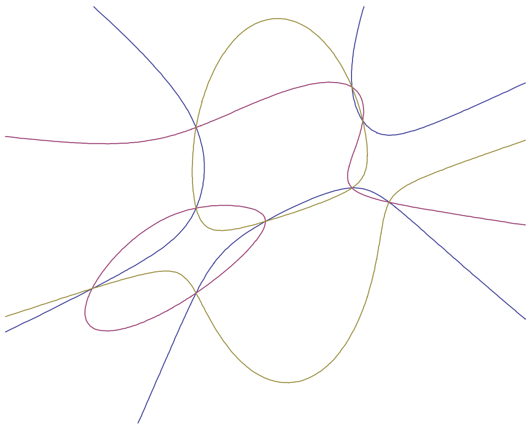
19. Let ABC be a triangle, and let the tangent to its circumcircle at A meet BC at D. Let l be a line meeting AD internally at P, the circumcircle at Q and T, the sides AB and AC internally at R and S respectively, and BC at U. Suppose that $PQRSTU$ lie in that order on l. Show that if $QR = ST$ then $PQ = UT$. **[UK IMO Squad Practice Exam 2011, Question 2]**

20. Suppose Γ_1 and Γ_2 are two disjoint ellipses, with Γ_1 inside Γ_2. If there is at least one triangle with its sides tangent to Γ_1 and vertices on Γ_2, show that there are infinitely many. **[Poncelet's porism]**

Actually, this theorem generalises to polygons with any number of sides. However, it is very difficult to prove with elementary methods.

- Suppose Γ_1 and Γ_2 are two disjoint ellipses, with Γ_1 inside Γ_2. If there is at least one n-gon with its sides tangent to Γ_1 and vertices on Γ_2, then there are infinitely many. **[Poncelet's porism]**

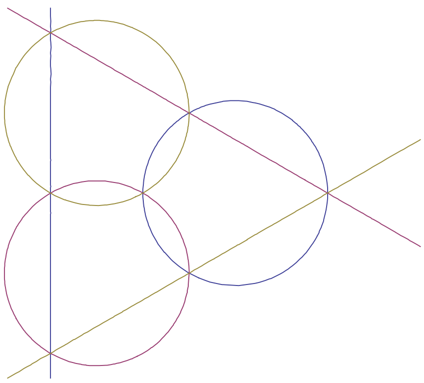
You may have noticed that Pappus' theorem is a special case of Pascal's theorem (and indeed Brianchon's theorem, as it is self-dual) when the conic degenerates into a pair of straight lines. Both of these theorems can be considered to be special cases of the Cayley-Bacharach theorem.



- Three cubic curves each pass through the same eight points, no four of which are collinear and no seven of which are conconic. The three cubics then share a ninth point. **[Degree-3 Cayley-Bacharach theorem]**

21. If two circles intersect, the *radical axis* is the line passing through both intersection points. For three mutually intersecting circles, prove that the three radical axes are concurrent. **[Radical axis theorem]**

22. Three circles MNP, NLP and LMP have a common point P. A point A is chosen on circle MNP (other than M, N or P). AN meets circle NLP at B and AM meets circle LMP at C. Prove that BC passes through L. **[UK MOG 2011, Question 1]**



Cayley-Bacharach generalises to curves of arbitrary degree, but the generalised version is difficult to prove. The quartic version is given below.

- Three quartic curves pass through the same thirteen points, no five of which are collinear, no nine of which are conconic and no twelve of which are concubic. The three quartics then share a further three points. **[Degree-4 Cayley-Bacharach theorem]**

23. Let the eight vertices of an octagon lie on a conic, and alternately colour the edges red and blue. Prove that the remaining eight heterochromatic intersections of (the extensions of) the edges lie on another conic. **[Generalised Pascal's theorem]**

24. Two cyclic quadrilaterals, $ABCD$ and $A'B'C'D'$, share the same circumcircle. The four intersections of corresponding edges (e.g. AB with $A'B'$) are labelled P, Q, R and S. Show that if P, Q and R are collinear, then S also lies on this line. **[Two butterflies theorem]**

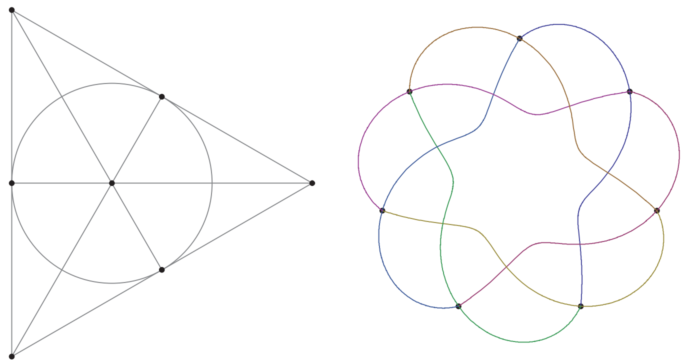
25. Let PQ be the chord of a circle Γ, and let M be the midpoint of PQ. Chords AB and CD are drawn through M. Let AC and BD intersect PQ at R and S, respectively. Prove that M is the midpoint of RS. **[Butterfly theorem]**

26. Let $ABCD$ be a quadrilateral. AC and BD intersect at E. X and Y are two points in the plane, and the line XY intersects AC at F and BD at G. R is the harmonic conjugate of F with respect to AC; S is the harmonic conjugate of G with respect to BD. The conics $ABEXY$ and $CDEXY$ intersect a fourth time at P; the conics $BCEXY$ and $DAEXY$ intersect a fourth time at Q. Prove that $PQRSEXY$ are conconic. **[Sam Cappleman-Lynes, 2012]**

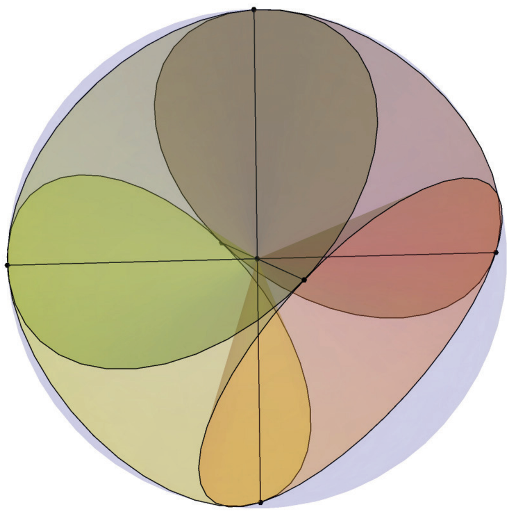
27. A number of line segments ($l_1, l_2, \cdots, l_n$) are drawn in general position on the plane, such that every pair of line segments intersects. A line Λ cuts all of the line segments. For each l_i, the endpoint on the left of Λ is called A_i, and the other endpoint is called B_i. An ant walks along a line segment l_i in the direction $A_i \to B_i$. Whenever it hits B_i, it teleports to A_i. Whenever it meets an intersection point ($l_i \cap l_j$), it moves onto the other line segment l_j and continues moving (in the direction $A_j \to B_j$, still). Prove that there exists an initial position of the ant such that it visits every line segment infinitely often.

Finite projective planes

The applications of projective geometry in olympiad problems involve infinite projective planes, namely $\mathbb{RP}^2$ and $\mathbb{CP}^2$. The construction where we take three coordinates (x, y, z) and consider scalar multiples to be equivalent generalises, enabling one to define a projective geometry over any field. For instance, the finite field of order 2 results in a projective plane with seven points: (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0) and (1, 1, 1). This finite projective plane, considered to be the simplest non-trivial geometry, is called the *Fano plane*:



The seven lines are shown (in the left diagram) as six straight lines and one circle; it is impossible to embed it in the real projective plane using only straight lines. Although not obvious from the left diagram, the right diagram demonstrates that all points are equivalent. There are, in fact, no fewer than 168 symmetries, corresponding to the rotation group PSL(2, 7) of Klein's quartic. Fixing a single vertex reduces the number to $\frac{168}{7} = 24$ symmetries, which are apparent in the following embedding in three-space. One point is at the centre of an octahedron formed by the other six points. The three orthogonal axes of the octahedron, together with the circumcircles of alternate faces, form the seven lines of the Fano plane.



Coincidentally, this resembles an embedding (the *Roman surface*) of the real projective plane into $\mathbb{R}^3$.

28. The seven vertices of the Fano plane are each coloured with one of c colours. How many different colourings are possible, taking into account the symmetries?

Solutions

1. Distinct lines correspond to planes through R. They must intersect in a line through R, which corresponds to a point on the projective plane.

2. The volume of the tetrahedron $R\,A\,B\,C$, with coordinates $(0, 0, 0)$, (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), respectively, is given by $\frac{1}{6}\det\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$. If this is zero, the points R, A, B and C must be coplanar. As we can assume without loss of generality that A, B and C lie on the reference plane, and R does not, we can deduce that A, B and C must be collinear. (The converse is obviously also true.) This condition that three points are collinear can be converted into the equation of a line, by allowing C to be a variable point.

3. Assume P exists, and consider the case where the two triangles lie in different planes. The two planes must then intersect on a line, where A'', B'' and C'' must obviously lie. The result for two dimensions is obtained by projecting it onto the plane. Note that the converse is the projective dual, so proving it in one direction is sufficient.

4. Applying Desargues' theorem results in the revelation that $W\,V$ is parallel to $B\,C$. Hence, $W\,X\,V$ is similar (indeed, homothetic) to $C\,X\,B$, so the result follows by applying the converse of the intersecting chords theorem to point X.

5. Consider the reference plane and use the Pythagorean distance formula.

6. Applying Ceva's theorem and Menelaus' theorem result in the two equations: $\frac{\overrightarrow{AB}}{\overrightarrow{BC}} \cdot \frac{\overrightarrow{CS}}{\overrightarrow{SP}} \cdot \frac{\overrightarrow{PR}}{\overrightarrow{RA}} = 1$ and $\frac{\overrightarrow{DA}}{\overrightarrow{CD}} \cdot \frac{\overrightarrow{CS}}{\overrightarrow{SP}} \cdot \frac{\overrightarrow{PR}}{\overrightarrow{RA}} = -1$. Dividing the two equations yields the desired result.

7. A projective transformation of the projective plane $\mathbb{P}^2$ is a linear transformation of the Euclidean space $\mathbb{R}^3$ in which it can be considered to reside. As planes are preserved by linear transformations, lines are preserved by projective transformations.

8. Firstly, we have $M\,\underline{x} = M\left(\underline{p} + \lambda\,\underline{q}\right) = M\,\underline{p} + \lambda\,M\,\underline{q}$. However, this must be normalised in the obvious way, yielding $\frac{M\,\underline{p}+\lambda\,M\,\underline{q}}{n\cdot\left(M\,\underline{p}+\lambda\,M\,\underline{q}\right)}$. The dot product is distributive over addition.

9. Consider the four points $\underline{a} = \underline{p} + \alpha\,\underline{q}$, $\underline{b} = \underline{p} + \beta\,\underline{q}$, $\underline{c} = \underline{p} + \gamma\,\underline{q}$ and $\underline{d} = \underline{p} + \delta\,\underline{q}$. Calculating the cross-ratio before and after the projective transformation will result in the same value of $\frac{(\alpha-\beta)\,(\gamma-\delta)}{(\beta-\gamma)\,(\delta-\alpha)}$.

10. Instead, we will find the inverse matrix. As we want the three unit vectors to be mapped to a, b and c, we write the (unnormalised) coordinates of each of them in each column of the matrix. To map $(1, 1, 1)$ to $\underline{d}$, we need to multiply each of the three columns by nonzero real constants, which are the solutions to three simultaneous linear equations. For obvious reasons, there is a single such solution.

11. Firstly, transform the four original points to $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$. Now, we can transform these to the four final points. Multiplying the two matrices together results in a single transformation.

12. Inverting about X results in a projective linear configuration, where we have to show that the intersections of opposite sides of quadrilateral $ABCD$ are collinear with the projective harmonic conjugates of P with respect to each of the diagonals. We can projectively transform $ABCD$ to the vertices of a square, rendering the problem trivial.

13. Form two points Q_1 and Q_2 in this manner. Apply a projective transformation taking them to the line at infinity, so P is the midpoint of two chords cutting an ellipse. Applying an affine transformation makes this ellipse into a circle, and P lies on two diameters so must be the centre. Obviously, the polar of P is the line at infinity. Applying the inverse projective transformations to this configuration will result in the original configuration, and the polar will remain a straight line.

14. Instead, we will prove Chasles' theorem (that a triangle and its polar reciprocal are in perspective) here, as the original problem is a special case. Due to Desargues' theorem, we can prove that the corresponding sides of the two triangles intersect at collinear points. Without loss of generality, we can assume two of these points are at infinity, and our objective is to show that the third also lies at infinity. We can then apply a projective transformation to take the two intersections of the conic with the line at infinity (which exist, due to Bezout's theorem) to the circular points at infinity (thus preserving the line at infinity, so this is an affine transformation), so we can assume that the conic is a circle. In this case, we have that the polar of A is parallel to BC, thus OA is perpendicular to BC. The polar of B is parallel to AC, thus OB is perpendicular to AC. Hence, O is the orthocentre of ABC, so OC is perpendicular to AB. This means that the polar of C is parallel to AB, so the triangles are in perspective.

15. The polar reciprocal (or 'dual') is a regular icosahedron with insphere S.

16. By projective transformations, we can assume the conic is a circle and the lines AB and DE are parallel, as are the lines AF and DC. From this, we draw the lines l_1 and l_2 (through O, the centre of the circle) such that B and E are reflections of A and D, respectively, in l_1, and that F and C are the reflections of A and D, respectively, in l_2. Hence, BE is a rotated copy of FC, so they are congruent. This means that BC and FE must indeed be parallel.

17. If a hexagon is circumscribed around a conic, its three main diagonals are concurrent.

18. Consider the degenerate case of Pascal's theorem where the conic is two lines.

19. We can reflect about the perpendicular bisector of RS (also the perpendicular bisector of QT) to transform this into a more projective problem. We will indicated a reflected version of point A with A', et cetera. We want to prove that the tangent at A meets the line $B'C'$ somewhere on the line l. To do this, apply Pascal's theorem to the degenerate hexagon $AAA'C'B'B$. As AA' and BB' intersect at the point at infinity on l, and AB and $A'C'$ intersect at R (also on l), we have that the tangent at A indeed meets $B'C'$ on l. By definition, this must be at point P and point U', so P is the reflection of U in the perpendicular bisector of RS. The problem becomes trivial.

20. In general, the ellipses must intersect four times on the complex projective plane; we can transform two of those points to the circular points at infinity, resulting in two circles. The problem then becomes equivalent to showing that infinitely many triangles share the same incircle (or excircle) and circumcircle. This is a consequence of Euler's formula, $OI^2 = R^2 - 2Rr$, and dimension counting.

21. We can consider the union of one circle and the radical axis of the other two to be a cubic. Repeating for the other two circles yields three cubics which intersect in at least eight points (the two circular points, plus the six pairwise circle intersections) so must intersect in the ninth. By applying Bezout's theorem, this additional intersection point cannot lie on any of the circles, so must lie on all three radical axes.

22. Same argument as in the previous question, but applied to the diagram displayed near the question.

23. Let Q_1 be the union of red lines, Q_2 be the union of blue lines, and Q_3 be the union of the main conic with the conic passing through five of the other eight heterochromatic intersections. By the quartic version of Cayley-Bacharach, Q_3 must pass through the other three intersections.

24. This is essentially the same argument as before, but with the realisation that a conic passing through three collinear points implies that it is a degenerate conic (union of two lines). Four of the eight 'other heterochromatic intersections' must lie on one line, and the other four lie on another line. The result then follows.

25. Reflect the 'butterfly' $ACBD$ in the perpendicular bisector of PQ to create another butterfly, $A'C'B'D'$. Colour AB, $B'C'$, CD and $D'A'$ red, and the remaining four edges blue. Since four of the heterochromatic intersections must be collinear (lying on the mirror line), so must the other four. Hence, $D'B'$ and AC intersect at R, and the mirror image must be S. (This is a degenerate case of the 'two butterflies theorem'.)

26. Apply a projective transformation to send X and Y to the circular points at infinity. The problem is then reduced to question 12.

27. This is the projective dual of Geoff Smith's 'windmill problem' from IMO 2011. An official solution can be easily obtained from the Internet.

28. There is one identity permutation. If we choose to fix one vertex and consider the three-dimensional embedding, we have eight rotations which permute the remaining six vertices in two 3-cycles. As there were seven vertices to initially choose from, this results in 56 pairs of 3-cycles. We can also rotate by π about any of the six 'diagonal' axes, resulting in another 42 permutations, each comprising a 2-cycle and 4-cycle. A rotation by π about an orthogonal axes gives a pair of 2-cycles and fixes three collinear points; this gives 21 further permutations. There must be a 7-cycle due to the floral embedding of the Fano plane. When translated into the three-dimensional embedding, this becomes totally asymmetric, so we have 24 7-cycles in this conjugacy class. Moreover, reversing their direction gives 24 more 7-cycles, completing the list of 168 symmetries. Applying Burnside's lemma, we get $\frac{1}{168}\left(c^7 + 56\,c^3 + 42\,c^3 + 21\,c^5 + 48\,c\right)$ distinct colourings.