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DIVISION ALGORITHM

Let $f(x)$ be a polynomial of degree n.
$g(x)$ be a polynomial of degree m.
Then, there exist uniquely determined polynomials $q(x)$ and $r(x)$ with $\deg r(x) < m$ such that

$$f(x) = g(x)q(x) + r(x).$$

In particular, if $g(x) = x - a$

$$f(x) = (x-a)q(x) + r \implies f(a) = r.$$

__Corollary:__ a is a root of $f(x) \iff f(a) = 0$

We have $f(x) = (x-a)q(x) + f(a)$.

What is the remainder when we divide $f(x)$ by $(x-a)(x-b)$? $(a \neq b)$

$$f(x) = (x-a)(x-b)q(x) + ux + v$$

so
$$f(a) = ua + v$$
$$f(b) = ub + v$$

$$\iff \quad u = \frac{f(b) - f(a)}{b-a} \qquad v = \frac{bf(a) - af(b)}{b-a}$$

> __At end__
> in n of degree k
> any poly can be written as the linear comb. of ~~the~~ the binomial coeffs $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{k}$.
> This might be useful to you sometime.
> coeffs are kth differences

What happens if $a = b$?

Let $f(x) = c_n x^n + c_{n-1}x^{n-1} + \dots + c_2 x^2 + c_1 x + c_0$

Let $y = x - a$ so $x = a + y$.

$$f(x) = c_n(a+y)^n + c_{n-1}(a+y)^{n-1} + \dots + c_2(a+y)^2 + c_1(a+y) + c_0$$
$$= (c_n a^n + c_{n-1}a^{n-1} + \dots + c_2 a^2 + c_1 a + c_0) + (nc_n a^{n-1} + (n-1)c_{n-1}a^{n-2} + \dots + c_1)y + \dots$$

If $f(x) = c_n x^n + c_{n-1}x^{n-1} + \dots + c_2 x^2 + c_1 x + c_0$

we define the derivative $f'(x) = nc_n x^{n-1} + \dots + 2c_2 x + c_1$

Properties of the derivative

$$(f+g)'(x) = f'(x) + g'(x)$$
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$
$$(kf)'(x) = kf'(x)$$
$$(f\circ g)'(x) = f'(g(x)).g'(x)$$

$$f(x) = f(a) + f'(a)(x-a) + q(x)(x-a)^2.$$

We can define derivatives of higher order.

$$f^{(n)}(x) = f^{(n-1)'}(x) \quad \text{for } n \geq 1 \quad \text{where } f^{(0)}(x) = f(x).$$

Exercises: ① Establish

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

② What happens with the remainder if we divide $f(x)$ by $(x-a_1)(x-a_2)\cdots(x-a_k)$?

③ a is a root of multiplicity $k \iff f(x) = (x-a)^k q(x)$ with $q(a) \neq 0$.

Prove that a is a root of multiplicity $k \iff$

$$0 = f(a) = f'(a) = \cdots = f^{(k-1)}(a)$$

④ Establish that a polynomial of degree n has at most n roots. (and exactly n roots counting multiplicity)

Vector space

A set V of elements is a vector space iff there is an operation $+$ of addition and multiplication by scalars. such that

(1) There is a 0 with $x+0=0+x=0$ $(\forall x)$

(2) For each x, there is $(-x)$ with $x+(-x)=(-x)+x=0$.

(3) $x+y=y+x$ $(\forall x \in V)$

(4) $kx \in V$ for scalars k, $x \in V$

(5) $k(x+y)=kx+ky$ $\quad (k+l)x = kx+lx$.

Examples: n-tples in space $(x_1, x_2, \ldots, x_n)$ $\quad \mathbb{R}^n$

complex numbers.

A basis of a vector space is a set $\{x_1, x_2, \ldots, x_n\}$ for which every vector can be written uniquely in the form

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \quad \text{for scalars } c_i$$

A set $\{u_1, \ldots, u_m\}$ of vectors is linearly independent

$$\Longleftrightarrow \quad c_1x_1+c_2x_2+\cdots+c_nx_n = 0 \text{ implies } c_1=c_2=\cdots=c_n=0$$

RESULTS: ① Any two bases of a vector space has the same number of elements; this number is called the dimension of a vector space.

② If $\{u_1, \ldots, u_m\}$ is a linearly independent set and m is the dimension of the vector space, then $\{u_1, \ldots, u_m\}$ is a basis of the vector space.

Example: $\{(1,0,0,0,\ldots,0), (0,1,0,\ldots,0), (0,0,1,0,\ldots,0), \ldots (0,0,0,\ldots,0,1)\}$ is a basis of $\mathbb{R}^n$.

PARTIAL FRACTIONS

$$\frac{6x^2-25x+23}{x^3-6x^2+11x-6}$$

Consider set of functions of form $\left\{ \dfrac{ax^2+bx+c}{x^3-6x^2+11x-6} \right\}$.

It is a vector space of dimension 3 and basis

$$\left\{ \frac{1}{x^3-6x^2+11x-6}, \frac{x}{x^3-6x^2+11x-6}, \frac{x^2}{x^3-6x^2+11x-6} \right\}$$

This vector space also contains $\dfrac{1}{x-1}, \dfrac{1}{x-2}, \dfrac{1}{x-3}$ and these constitute a linearly independent set

Proof: $\dfrac{u}{x-1} + \dfrac{v}{x-2} + \dfrac{w}{x-3} \equiv 0$

$$\iff u(x-2)(x-3) + v(x-1)(x-3) + w(x-1)(x-2) \equiv 0$$

$$\iff 2u = 0, \ -v = 0, \ 2w = 0 \iff u = v = w = 0$$

So with $f(x) = 6x^2 - 25x + 23$

$g(x) = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$

we have $\dfrac{f(x)}{g(x)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$ for some A, B, C

$$g(x) = (x-1)(x-2)(x-3)$$

$$g'(x) = (x-1)(x-2) + (x-1)(x-3) + (x-2)(x-3) = 3x^2 - 12x + 11$$

$$f(x) = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$$f(1) = Ag'(1) \qquad f(2) = Bg'(2) \qquad f(3) = Cg'(3)$$

$$\therefore A = \frac{f(1)}{g'(1)} = \frac{4}{2} = 2 \qquad B = \frac{f(2)}{g'(2)} = \frac{-3}{-1} = 3 \qquad C = \frac{f(3)}{g'(3)} = \frac{2}{2} = 1$$

$$f(x) = \frac{2}{x-1} + \frac{3}{x-2} + \frac{1}{x-3}$$

$$u_{n+2} = 3u_{n+1} - 2u_n \quad (n \geq 0)$$

u_0, u_1 are given

GENERATING FUNCTION

$$\text{Let } f(x) = \sum_{n=0}^{\infty} u_n x^n = u_0 + u_1 x + \sum_{n=0}^{\infty} u_{n+2} x^{n+2}$$

$$= u_0 + u_1 x + \sum_{n=0}^{\infty} (3u_{n+1} - 2u_n) x^{n+2}$$

$$= u_0 + u_1 x + 3x \sum_{n=0}^{\infty} u_{n+1} x^{n+1} - 2x^2 \sum_{n=0}^{\infty} u_n x^n$$

$$= u_0 + u_1 x + 3x (f(x) - u_0) - 2x^2 f(x)$$

$$\Rightarrow (2x^2 - 3x + 1) f(x) = u_0 + (u_1 - 3u_0) x$$

$$\Rightarrow \quad f(x) = \frac{u_0 + (u_1 - 3u_0)x}{(1-2x)(1-x)}$$

$$= \frac{u_1 - u_0}{1-2x} + \frac{2u_0 - u_1}{1-x}$$

$$= (u_1 - u_0)(1 + 2x + 2^2x^2 + \cdots)$$
$$+ (2u_0 - u_1)(1 + x + x^2 + x^3 + \cdots)$$

$$= u_0 + u_1 x + \left[(u_1 - u_0)2^2 + (2u_0 - u_1)1^2\right] x^2 + \cdots$$

$$+ \left[(u_1 - u_0)2^n + (2u_0 - u_1)1^n\right] x^n + \cdots$$

The set of sequences of the form

$$(u_0, u_1, u_2, \ldots \quad) \quad \text{where } u_{n+2} = 3u_{n+1} - 2u_n \quad (n \geq 0)$$

is a vector space with basis

$$(1, 0, -2, -6, -14, -30, \ldots)$$
$$(0, 1, 3, 7, 15, 31, \ldots)$$

We look for a new basis consisting of sequences of the form

$$(1, r, r^2, r^3, r^4, \ldots)$$

Such a sequence belongs to the vector space $\Leftrightarrow r^{n+2} = 3r^{n+1} - 2r^n$

$$\Leftrightarrow \quad r^2 - 3r + 2 = 0$$

$$\Leftrightarrow \quad r = 1 \text{ n } r = 2.$$

So every sequence can be expressed in the form

$$u_0(1, 0, -2, -6, -14, -30, \ldots) + u_1(0, 1, 3, 7, 15, 31, \ldots)$$
$$= (u_0, u_1, 3u_1 - u_0, 7u_1 - u_0, 15u_1 - 14u_0, \ldots) = (u_0, u_1, \ldots, (2^n-1)u_1 - (2^n-2)u_0,$$

and in the form

$$v_0(1, 1, 1, 1, \ldots, 1^n, \ldots) + v_1(1, 2, 4, 8, \ldots, 2^n, \ldots)$$

$$u_0 = v_0 + v_1 \qquad\qquad v_0 = 2u_0 - u_1$$
$$u_1 = v_0 + 2v_1 \quad \text{so} \quad v_1 = u_1 - u_0$$
$$\vdots$$

$$u_n = (2^n - 1)u_1 - (2^n - 2)u_0 = v_0 1^n + v_1 2^n = (2u_0 - u_1)1^n + (u_1 - u_0)2^n$$

Generally: $u_{n+k} = c_{k-1}u_{n+k-1} + \cdots + c_0 u_n$ is a recursion of order k.

$x^k - c_{k-1}x^{k-1} - \cdots - c_0 = (x - r_1)^{s_1} \cdots (x - r_k)^{s_k}$ is its auxiliary polynomial

$$u_n = \sum_{i=1}^{k} v_i (\cdot n^{s_i - 1} + \cdot n^{s_i - 2} + \cdots + \cdot n + \cdot) r_i^n$$

$$= \sum_{i=}^{n} v_i \, p_i(n) r_i^n \qquad \text{where } \deg p_i = s_i - 1$$