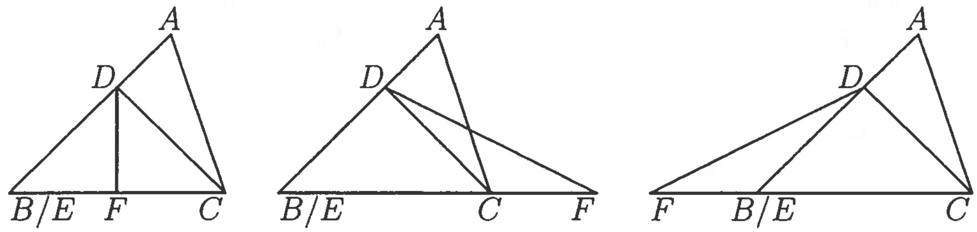
2003 Winter Camp

Common Angle Theorem

Two triangles are said to have a common angle if an angle of one triangle is equal or supplementary to an angle of the other. Such pairs of triangles occur frequently in geometric diagrams. A basic result relates the ratio of their areas to the ratio of the products of the pair of their sides enclosing the common angle.

Common Angle Theorem.
If $\angle ABC = \angle DEF$ or $\angle ABC + \angle DEF = 180°$, then $\frac{[ABC]}{[DEF]} = \frac{AB \cdot BC}{DE \cdot EF}$.



Proof:

Superimpose the two triangles so that B coincides with E and is collinear with A and D. Then it is also collinear with C and F. By the Common Side Lemma, $\frac{[ABC]}{[DEF]} = \frac{[ABC]}{[DBC]} \cdot \frac{[DEC]}{[DEF]} = \frac{AB}{DE} \cdot \frac{BC}{EF}$.

We first use this tool to derive some simple basic results.

Converse of the Isosceles Triangle Theorem.
If $\angle ABC = \angle ACB$, then $AB = AC$.

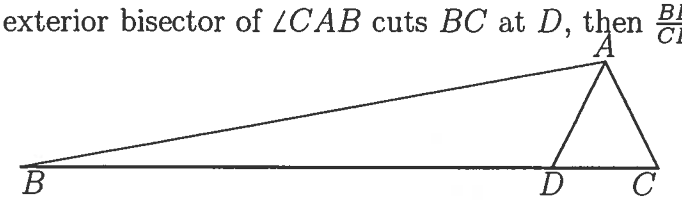
Proof:

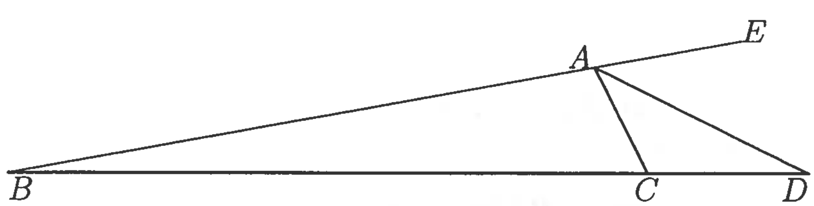
In triangles ABC and ACB, $\angle ABC = \angle ACB$. By the Common Angle Theorem

$$1 = \frac{[ABC]}{[ABC]} = \frac{AB \cdot BC}{AC \cdot CB} = \frac{AB}{AC}.$$

Hence $AB = AC$.

Angle Bisectors Theorem.
If either the interior or exterior bisector of $\angle CAB$ cuts BC at D, then $\frac{BD}{CD} = \frac{AB}{AC}$.





Proof:

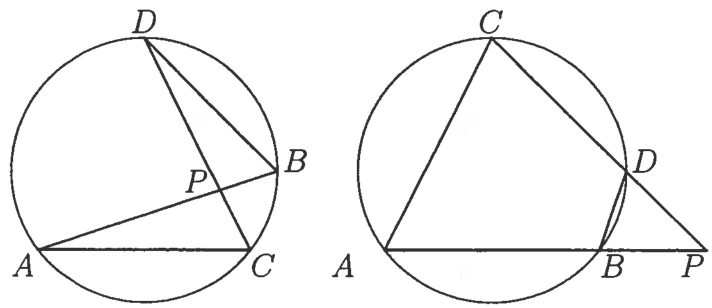
We have $\angle BAD = \angle CAD$ in the case of the interior bisector and, in the case of the exterior bisector, $\angle BAD + \angle CAD = \angle BAD + \angle DAE = 180°$. It follows from the Common Angle Theorem that $\frac{BD}{CD} = \frac{[BAD]}{[CAD]} = \frac{AB \cdot AD}{AC \cdot AD} = \frac{AB}{AC}$.

AA Similarity Theorem.
If $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$, then $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

Proof:

By the Common Angle Theorem, $\frac{AB \cdot BC}{DE \cdot EF} = \frac{[ABC]}{[DEF]} = \frac{BC \cdot CA}{EF \cdot FD}$. Hence $\frac{AB}{DE} = \frac{CA}{FD}$. Since we also have $\angle CAB = 180° - \angle ABC - \angle BCA = 180° - \angle DEF - \angle EFD = \angle FDE$, we can prove in the same way that $\frac{AB}{DE} = \frac{BC}{EF}$.

Two Chords Theorem.
If P is the points of intersection of two chords AB and CD of a circle, then $PA \cdot PB = PC \cdot PD$.

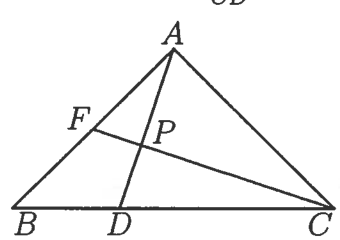


Proof:

We have $\angle PAC = \angle PDB$ and $\angle PCA = \angle PBD$. It follows from the Common Angle Theorem that $\frac{AC \cdot AP}{DB \cdot DP} = \frac{[PAC]}{[PDB]} = \frac{AC \cdot CP}{DB \cdot BP}$. Hence $PA \cdot PB = PC \cdot PD$.

Next, we solve some problems using the Common Angle Theorem.

Example 1.
In triangle ABC, $AB = AC$ and $\angle CAB = 90°$. F is the midpoint of AB. The perpendicular from A to CF cuts CF at P and BC at D. Determine $\frac{BD}{CD}$.

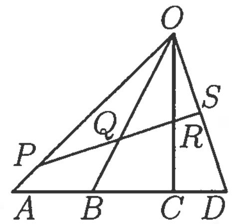


Solution:

We have $\angle ACF = 90° - \angle CAD = \angle BAD$. Similarly, $\angle FCA = \angle CAD$. It follows from the Common Angle Theorem that

$$\frac{BD}{CD} = \frac{[BAD]}{[CAD]} = \frac{[BAD]}{[FAP]} \cdot \frac{[FAP]}{[CAD]} = \frac{AB \cdot AD}{AC \cdot CF} \cdot \frac{AF \cdot CF}{AD \cdot AC} = \frac{1}{2}.$$

Example 2.
Four rays from a point O intersect a line at A, B, C and D, and another line at P, Q, R and S, respectively. Prove that $\frac{AB\cdot CD}{AD\cdot BC}=\frac{PQ\cdot RS}{PS\cdot QR}$.



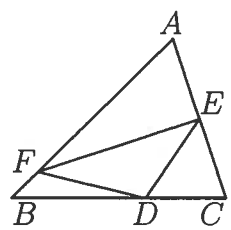
Solution:

By the Common Side Lemma and the Common Angle Theorem,

$$\begin{aligned}\frac{PQ\cdot RS}{PS\cdot QR}\cdot\frac{AD\cdot BC}{AB\cdot CD} &= \frac{[OPQ]}{[OPS]}\cdot\frac{[ORS]}{[OQR]}\cdot\frac{[OAD]}{[OAB]}\cdot\frac{[OBC]}{[OCD]}\\ &= \frac{[OPQ]}{[OAB]}\cdot\frac{[ORS]}{[OCD]}\cdot\frac{[OAD]}{[OPS]}\cdot\frac{[OBC]}{[OQR]}\\ &= \frac{OP\cdot OQ}{OA\cdot OB}\cdot\frac{OR\cdot OS}{OC\cdot OD}\cdot\frac{OA\cdot OD}{OP\cdot OS}\cdot\frac{OB\cdot OC}{OQ\cdot OR}\\ &= 1.\end{aligned}$$

The desired result follows immediately.

Example 3.
D, E and F are arbitrary points on the sides BC, CA and AB of triangle ABC. Prove that at least one of $[AEF]$, $[BFD]$ and $[CDE]$ is at most $\frac{1}{4}$ of $[ABC]$.



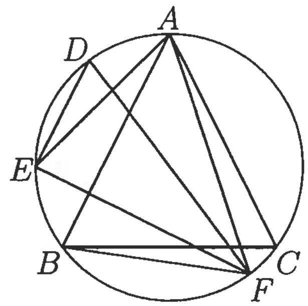
Solution:

Note that $x(1-x)=\frac{1}{4}-(x-\frac{1}{2})^2\leq\frac{1}{4}$ for all real numbers x between 0 and 1. By the Common Angle Theorem,

$$\begin{aligned}\frac{[AEF]}{[ABC]}\cdot\frac{[BFD]}{[ABC]}\cdot\frac{[CDE]}{[ABC]} &= \frac{AE\cdot AF}{AB\cdot AC}\cdot\frac{BF\cdot BD}{BC\cdot BA}\cdot\frac{CD\cdot CE}{CA\cdot CB}\\ &= \frac{AF}{AB}\left(1-\frac{CE}{CA}\right)\cdot\frac{BD}{BC}\left(1-\frac{AF}{AB}\right)\cdot\frac{CE}{CA}\left(1-\frac{BD}{BC}\right)\\ &\leq \left(\frac{1}{4}\right)^3.\end{aligned}$$

The desired result follows immediately.

Example 4.
ABC and DEF are two triangles inscribed in the same circle. Prove that $\frac{[ABC]}{[DEF]} = \frac{AB\cdot BC\cdot CA}{DE\cdot EF\cdot FD}$.



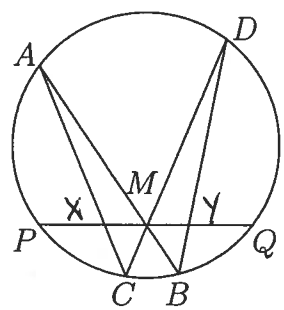
Solution:

By the Common Angle Theorem,

$$\begin{aligned}\frac{[ABC]}{[DEF]} &= \frac{[ABC]}{[ABF]}\cdot\frac{[ABF]}{[AEF]}\cdot\frac{[AEF]}{[DEF]}\\ &= \frac{AC\cdot BC}{AF\cdot BF}\cdot\frac{AB\cdot BF}{AE\cdot EF}\cdot\frac{AE\cdot AF}{DE\cdot DF}\\ &= \frac{AB\cdot BC\cdot CA}{DE\cdot EF\cdot FD}.\end{aligned}$$

We conclude with the derivation of two well-known results.

The Butterfly Theorem.
Two chords AB and CD of a circle passes through the midpoint M of a third chord PQ. If PQ cuts AC at X and BD at Y, then M is also the midpoint of XY.



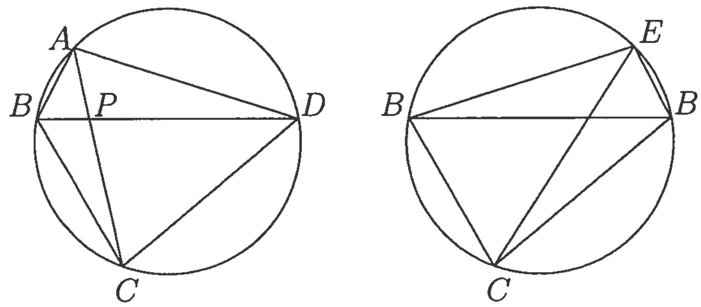
Proof:

We have $AX\cdot CX = PX\cdot QX$, $BY\cdot DY = PY\cdot QY$ and $PM = QM$. We have

$$\begin{aligned}1 &= \frac{[MAX]}{[MDY]}\cdot\frac{[MDY]}{[MCX]}\cdot\frac{[MCX]}{[MBY]}\cdot\frac{[MBY]}{[MAX]}\\ &= \frac{MA\cdot AX}{MD\cdot DY}\cdot\frac{MD\cdot MY}{MC\cdot MX}\cdot\frac{MC\cdot CX}{MB\cdot BY}\cdot\frac{MB\cdot MY}{MA\cdot MX}\\ &= \frac{MY^2\cdot PX\cdot QX}{MX^2\cdot PY\cdot QY}\\ &= \frac{MY^2(PM^2-MX^2)}{MX^2(PM^2-MY^2)}\end{aligned}$$

by the Common Angle Theorem. It follows that $MX = MY$.

Ptolemy's Theorem.
In any cyclic quadrilateral $ABCD$, $AB \cdot CD + AD \cdot BC = AC \cdot BD$.



Proof:

Let P be the point of intersection of AC and BD. Reflect A across the perpendicular bisector of BD to E. Then $\angle BPA = \angle PAD + \angle PDA = \angle CBD + \angle EBD = \angle EBC$. By the Common Angle Theorem,

$$\begin{aligned}
\frac{AB \cdot CD + AD \cdot BC}{BE \cdot BC} &= \frac{ED \cdot CD + BE \cdot BE}{BE \cdot BC} \\
&= \frac{[CDE] + [BCE]}{[BCE]} \\
&= \frac{[PAB] + [PBC] + [PCD] + [PDA]}{[BCE]} \\
&= \frac{PA \cdot PB + PB \cdot PC + PC \cdot PD + PD \cdot PA}{BE \cdot BC} \\
&= \frac{AC \cdot BD}{BE \cdot BC}.
\end{aligned}$$

The desired result follows immediately.

Exercises

1. Prove that the opposite sides of a parallelogram are equal.

2. $ABCD$ and $DEFG$ are squares constructed outside triangle ADE. Prove that

   (a) $[ECD] = [GAD]$;

   (b) $[EAD] = [GCD]$.

3. CD is the altitude from the right angle C to the hypotenuse AB.

   (a) Prove that $CD^2 = AD \cdot BD$.

   (b) Prove that $CA^2 = AB \cdot AD$.

   (c) Prove that $CB^2 = BA \cdot BD$.

   (d) Deduce Pythagoras' Theorem from (b) and (c).

4. Prove that if the tangent to a circle at a point T on the circle meets the extension of a chord AB at P, then $PT^2 = PA \cdot PB$.

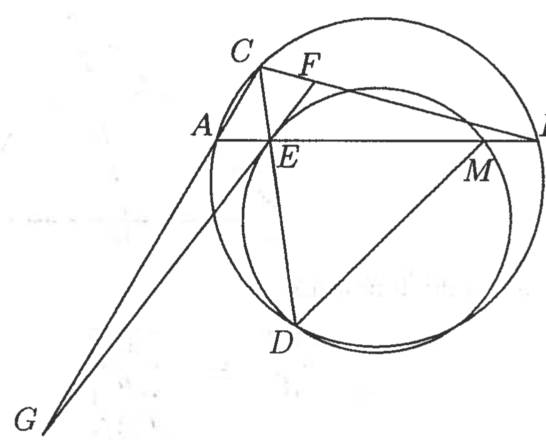
5. Let E and F be points on the sides CA and AB of triangle ABC, respectively, such that EF meets the median AM at N.

   (a) Prove that $\frac{FN}{NE} = \frac{AB}{CA}$ if $AE = AF$.

   (b) Prove that $\frac{AM}{AN} = \frac{1}{2}(\frac{AC}{AE} + \frac{AB}{AF})$.

6. Each of the diagonals AD, BE and CF of a convex hexagon $ABCDEF$ bisects its area. Prove that these three diagonals are concurrent.

7. AB and CD are two chords of a circle intersecting at E. M is an arbitrary point on BE. The tangent at E to the circumcircle of triangle MED intersects BC at F and CA at G. Prove that $\frac{GE}{EF} = \frac{AM}{MB}$.



Solutions to Exercises

1. Let $ABCD$ be a parallelogram. Since AB is parallel to DC, $[BAD] = [BAC]$. Similarly, $[BAC] = [BDC]$. By the Common Angle Theorem, $\frac{AB}{DC} = \frac{AB \cdot BD}{DC \cdot BD} = \frac{[BAD]}{[BDC]} = \frac{[BAD]}{[BAC]} \cdot \frac{[BAC]}{[BDC]} = 1$.

2. (a) Note that $\angle CDE = 90° + \angle ADE = \angle ADG$. Hence $\frac{[ECD]}{[GAD]} = \frac{CD \cdot DE}{AD \cdot DG} = 1$ by the Common Angle Theorem.

   (b) Note that $\angle CDG = 180° - \angle ADE$. Hence $\frac{[EAD]}{[GCD]} = \frac{AD \cdot DE}{CD \cdot DG} = 1$ by the Common Angle Theorem.

3. Note that $\angle CAD = 90° - \angle ACD = \angle BCD$. Similarly, $\angle CBD = \angle ACD$.

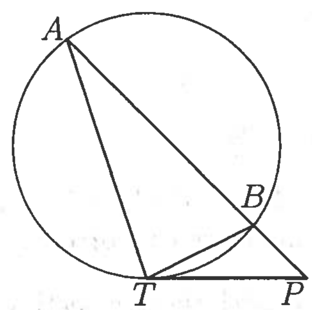
   (a) By the Common Angle Theorem, $\frac{AC \cdot CD}{CB \cdot BD} = \frac{[ACD]}{[CBD]} = \frac{CA \cdot AD}{BC \cdot CD}$. Hence $CD^2 = AD \cdot BD$.

   (b) By the Common Angle Theorem, $\frac{AC \cdot CD}{CB \cdot BA} = \frac{[ACD]}{[ABC]} = \frac{AD \cdot DC}{AC \cdot CB}$. Hence $CA^2 = AB \cdot AD$.

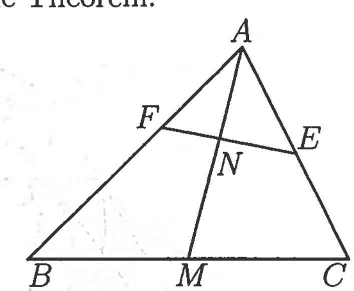
   (c) By the Common Angle Theorem, $\frac{BC \cdot CD}{BA \cdot AC} = \frac{[BCD]}{[ABC]} = \frac{CD \cdot DB}{AC \cdot CB}$. Hence $CB^2 = BA \cdot BD$.

   (d) From (b) and (c), $CA^2 + CB^2 = AB(AD + BD) = AB^2$.

4. We have $\angle BTP = \angle BAT$ and $\angle PTA = \angle BTP + \angle ATB = \angle BAT + \angle ATB = \angle PBT$. By the Common Angle Theorem, $\frac{PA \cdot AT}{PT \cdot TB} = \frac{[PAT]}{[PTB]} = \frac{AT \cdot TP}{PB \cdot BT}$. Hence $PA \cdot PB = PT^2$.



5. (a) We have $\frac{FN}{NE} = \frac{[FAN]}{[NAE]} = \frac{[FAN]}{[ABM]} \cdot \frac{[ACM]}{[NAE]} = \frac{AF \cdot AN}{AB \cdot AM} \cdot \frac{AC \cdot AM}{AE \cdot AN} = \frac{AB}{AC}$ by the Common Side Lemma and the Common Angle Theorem.



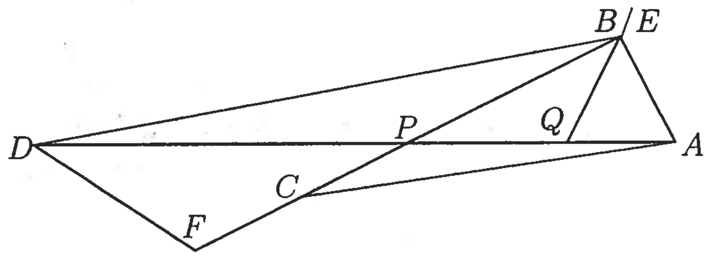
   (b) By the Common Angle Theorem,

$$\begin{aligned}\frac{AF \cdot AE}{AB \cdot AC} &= \frac{[FAE]}{[ABC]} \\ &= \frac{[FAN]}{2[ABM]} + \frac{[NAE]}{2[ACM]} \\ &= \frac{1}{2}\left(\frac{AF}{AB} + \frac{AE}{AC}\right)\frac{AN}{AM}.\end{aligned}$$

   It follows that $\frac{AM}{AN} = \frac{1}{2}(\frac{AF}{AB} + \frac{AE}{AC})\frac{AB \cdot AC}{AF \cdot AE} = \frac{1}{2}(\frac{AC}{AE} + \frac{AB}{AF})$.

Converse of the Common Angle Theorem

Theorem 1.
Suppose $\angle ABC > \angle DEF$ and $\angle ABC + \angle DEF < 180°$. Then $\frac{[ABC]}{[DEF]} > \frac{AB \cdot BC}{DE \cdot EF}$.



Proof:

Superimpose the two triangles so that B coincides with E and the ray BC coincides with the ray EF. Since $\angle ABC + \angle DEF < 180°$, this ray intersects AD at some point P. Let Q be the point on AD such that BQ bisects $\angle DEA$. Since $\angle ABC > \angle DEF$, Q lies on the segment AP. By the Common Side Lemma and the Angle Bisectors Theorem,

$$\frac{[ABC]}{[DEF]} = \frac{[ABC]}{[ABP]} \cdot \frac{[ABP]}{[DEP]} \cdot \frac{[DEP]}{[DEF]} = \frac{BC}{BP} \cdot \frac{AP}{DP} \cdot \frac{EP}{EF} > \frac{BC}{EF} \cdot \frac{AQ}{DQ} = \frac{AB \cdot BC}{DE \cdot EF}.$$

Theorem 2.
Suppose $\angle ABC > \angle DEF$ and $\angle ABC + \angle DEF > 180°$. Then $\frac{[ABC]}{[DEF]} < \frac{AB \cdot BC}{DE \cdot EF}$.

Proof:

Extend AB to X and DE to Y such that $AB = BX$ and $DE = EY$. Then $[ABC] = [XBC]$ and $[DEF] = [YEF]$. Note that we have $\angle YEF = 180° - \angle DEF > 180° - \angle ABC = \angle XBC$. Moreover, $\angle YEF + \angle XBC = 360° - (\angle ABC + \angle DEF) < 180°$. Then $\frac{[DEF]}{[ABC]} = \frac{[YEF]}{[XBC]} > \frac{YE \cdot EF}{XB \cdot BC} = \frac{DE \cdot EF}{AB \cdot BC}$ by Theorem 1. It follows that $\frac{[ABC]}{[DEF]} < \frac{AB \cdot BC}{DE \cdot EF}$.

Combining Theorems 1 and 2, we have the following result.

Converse of the Common Angle Theorem.
We have $\frac{[ABC]}{[DEF]} = \frac{AB \cdot BC}{DE \cdot EF}$ if and only if either $\angle ABC = \angle DEF$ or $\angle ABC + \angle DEF = 180°$.

We give two simple applications.

Isosceles Triangle Theorem.
In triangle ABC, if $AB = AC$, then $\angle ACB = \angle ABC$.

Proof:

Note that $\angle ABC + \angle ACB < 180°$. We have $\frac{[ABC]}{[ACB]} = 1 = \frac{AB \cdot BC}{AC \cdot CB}$. By the Converse of the Common Angle Theorem, $\angle ABC = \angle ACB$.

Angle-Side Inequality.
In triangle ABC, $\angle ACD < \angle ABC$ if and only if $AB < AC$.

Proof:

Note that $\angle ABC + \angle ACB < 180°$. By the Converse of the Common Angle Theorem, we have $\angle ACD < \angle ABC$ if and only if $1 = \frac{[ABC]}{[ACB]} > \frac{AB \cdot BC}{AC \cdot CB}$. The latter condition is equivalent to $AB < AC$.