Article/CJB/2011/176

Orthologic Triangles

Christopher Bradley

Abstract: Triangles ABC and DEF are orthologic if the perpendiculars from the vertex of one of them to the corresponding sides of the other are concurrent. The property is symmetric but not transitive.

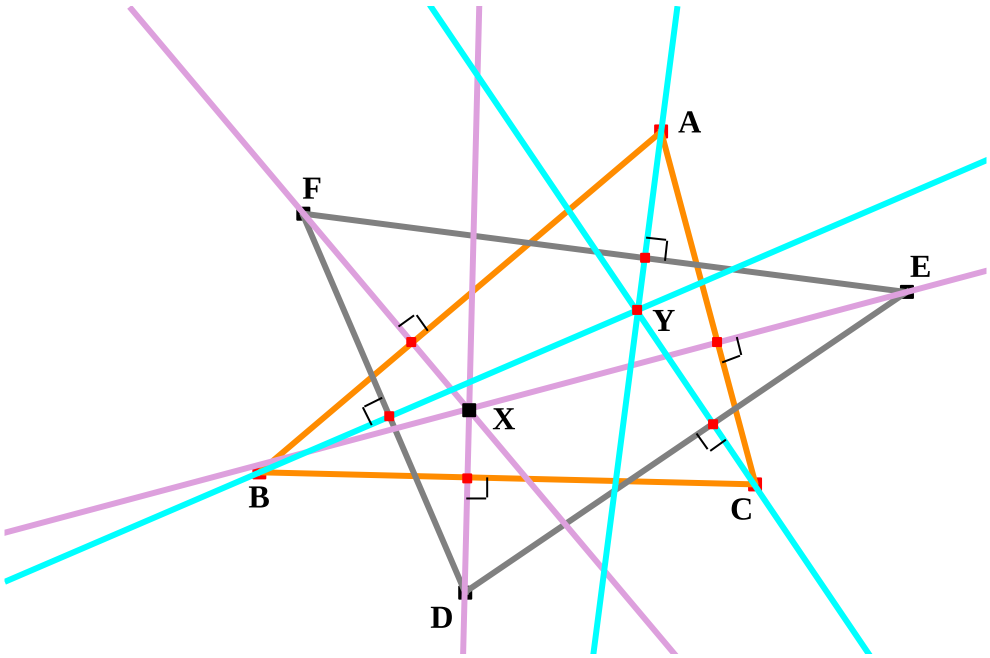

Fig. 1

A pair of orthologic triangles

1. The vertices and sides of the two triangles

The Cartesian co-ordinates of the vertices are taken to be:

$$A(a, p), B(b, q), C(c, r), D(d, u), E(e, v), F(f, w). \quad (1.1)$$

The equations of the sides of triangle DEF are

DE: $(u - v)x + (e - d)y + dv - eu = 0,$ (1.2)

EF: $(v - w)x + (f - e)y + ew - fv = 0,$ (1.3)

FD: $(w - u)x + (d - f)y + fu - dw = 0.$ (1.4)

2. Perpendiculars from A, B, C onto EF, FD, DE respectively

The perpendicular from A onto EF has equation

$$(w - v)x - (e - f)y = a(w - v) - p(e - f). \tag{2.1}$$

The perpendicular from B onto FD has equation

$$(u - w)x - (f - d)y = b(u - w) - q(f - d). \tag{2.2}$$

The perpendicular from C onto DE has equation

$$(v - u)x - (d - e)y = c(v - u) - r(d - e). \tag{2.3}$$

3. The condition for orthologic triangles

It is easy now to find the condition that the three lines in Section 2 are concurrent is

$$a(v - w) + b(w - u) + c(u - v) + p(e - f) + q(f - d) + r(d - e) = 0. \tag{3.1}$$

Since this is the same equation when a, b, c are replaced by d, e, f and p, q, r are replaced by u, v, w it follows that if ABC is orthologic to DEF then DEF is orthologic to ABC. (There are too many variables for transitivity to be deduced from only one pair of such equations. Orthologic triangles date back to Steiner (1827).

Flat 4,
Terrill Court,
12-14, Apsley Road,
BRISTOL BS8 2SP