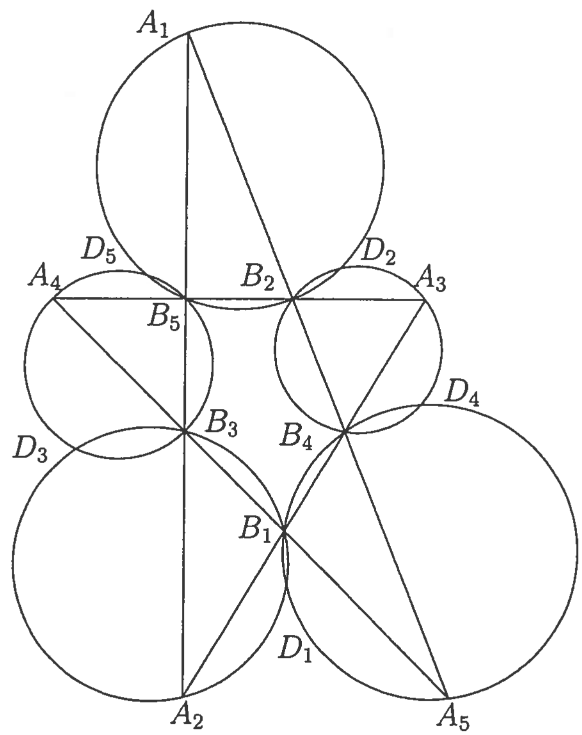
2003 Winter Camp

Team Contest

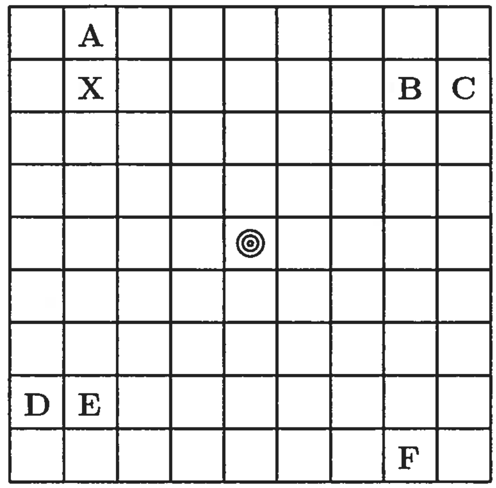
1. The sequence $\{a_n\}$ is defined by $a_0 = 0$ and $a_{n+1} = 2a_n + 1 + \sqrt{3a_n^2 + 6a_n + k}$. Determine all constant values k for which a_n is an integer for every n.

2. Let $A_1A_2A_3A_4A_5$ be a star pentagon, with A_1A_2 and A_3A_4 intersecting at B_5, A_2A_3 and A_4A_5 at B_1, A_3A_4 and A_5A_1 at B_2, A_4A_5 and A_1A_2 at B_3, and A_5A_1 and A_2A_3 at B_4. Let the circumcircles of $B_1A_2B_3$ and B_1, B_4A_5 intersect at D_1, those of $B_2A_3B_4$ and $B_2B_5A_1$ at D_2, those of $B_3A_4B_5$ and $B_3B_1A_2$ at D_3, those of $B_4A_5B_1$ and $B_4B_2A_3$ at D_4, and those of $B_5A_1B_2$ and $B_5B_3A_4$ at D_5. Prove that D_1, D_2, D_3, D_4 and D_5 are concyclic.



3. The world of the amoeba consists of the first quadrant of the plane divided into unit squares. Initially, a solitary amoeba is imprisoned in the bottom left corner square. A prison is defined as a set of squares consisting of the bottom a_i squares in the i-th cokumn for $1 \leq i \leq n$ such that $a_1 \geq a_2 \geq \cdots \geq a_n$. Such a prison is denoted by $(a_1, a_2, \ldots, a_n)$. It is unguarded, and the Great Escape is successful if the entire prison is unoccupied. In each move, an amoeba splits into two, with one going to the square directly above and one going to the square directly to the right. However, the move is not permitted if either of those two squares is already occupied. Determine all prisons with $a_2 \geq 2$ from which the Great Escape is achievable.

4. In the following board, there are seven U F O's. Each can move in any of four directions, left, right, up and down. Once it starts moving, it cannot stop unless the next square in its path is occupied by another U F O. If it reaches the border of the board before it is stopped, it simply disappears. The object of the puzzle is to get the U F O marked X into the target square at the centre of the board. Other U F O's can occupy this square temporarily, eventually moving out to make room for X. Remember that the target square by itself does not stop an U F O there.


A
X
B
C
D
E
F

Team Contest

1. The sequence $\{x_n\}$ is defined by $x_0 = 0$ and $x_{n+1} = 2x_n + 1 + \sqrt{3x_n^2 + 6x_n + k}$. Determine all constant values k for which x_n is an integer for every n.

   Solution:

   Note that x_1 is positive, so that we have $x_n < x_{n+1}$ for all n. Rewrite the given recurrence relation as $x_{n+1} - 2x_n - 1 = \sqrt{3x_n^2 + 6x_n + k}$. We obtain $x_{n+1}^2 + x_n^2 - 4x_{n+1}x_n - 2x_{n+1} - 2x_n + 1 = k$ by squaring both sides. This is an expression symmetric in x_{n+1} and x_n. It follows that

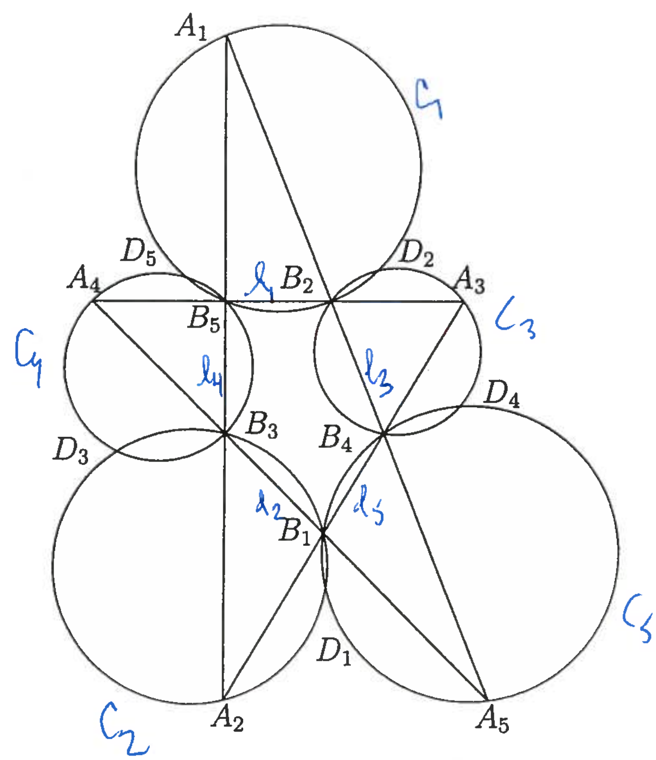
   $$x_n = 2x_{n+1} + 1 - \sqrt{3x_{n+1}^2 + 6x_{n+1} + k},$$

   the negative square root being chosen since $x_n < x_{n+1}$. Shifting the indices of the given recurrence relation by 1, we have

   $$x_{n+2} = 2x_{n+1} + 1 + \sqrt{3x_{n+1}^2 + 6x_{n+1} + k}.$$

   Adding the two displayed equations, we have $x_{n+2} = 4x_{n+1} - x_n + 2$. Hence all x_n are integers provided that $x_1 = 1 + \sqrt{k}$ is an integer, and this is the case if and only if k is the square of an integer.

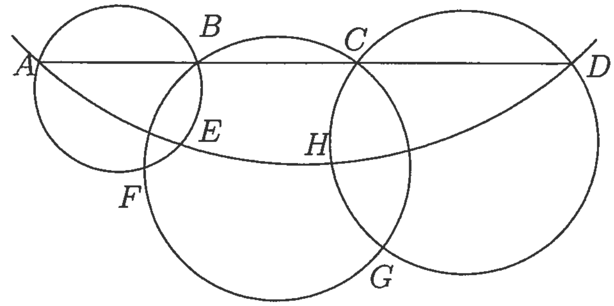
2. Let $A_1A_2A_3A_4A_5$ be a star pentagon, with A_1A_2 and A_3A_4 intersecting at B_5, A_2A_3 and A_4A_5 at B_1, A_3A_4 and A_5A_1 at B_2, A_4A_5 and A_1A_2 at B_3, and A_5A_1 and A_2A_3 at B_4. Let the circumcircles of $B_1A_2B_3$ and B_1, B_4A_5 intersect at D_1, those of $B_2A_3B_4$ and $B_2B_5A_1$ at D_2, those of $B_3A_4B_5$ and $B_3B_1A_2$ at D_3, those of $B_4A_5B_1$ and $B_4B_2A_3$ at D_4, and those of $B_5A_1B_2$ and $B_5B_3A_4$ at D_5. Prove that D_1, D_2, D_3, D_4 and D_5 are concyclic.



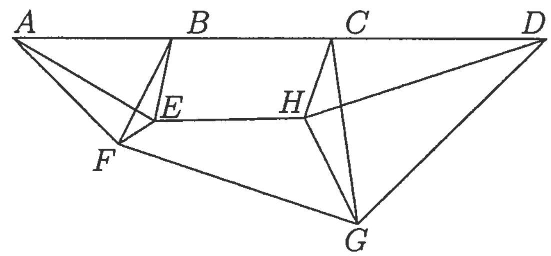
Solution:

We first prove two preliminary results.

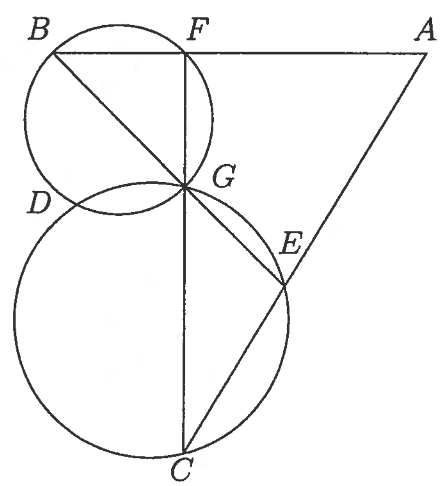
Lemma 1.
The circle C_2 intersects the C_1 at B and F and the circle C_3 at C and G. The line BC intersects C_1 again at A and C_3 again at D. The circle C_4 passing through A and D intersects C_1 again at E and C_3 again at H. Then E, F, G and H are either collinear or concyclic.



Proof:
It is not hard to prove that if three of E, F, G and H are collinear, then they are all collinear. We only deal with the case when no three of them are collinear. Let $\angle EAD = \theta$ and $\angle GHD = \phi$. Since $ABEF$ is cyclic, $\angle EFB = \theta$. Since $ADHE$ is cyclic, $\angle EHD = 180° - \theta$. Since $CDGH$ is cyclic, $\angle GCD = \phi$. Since $BCGF$ is cyclic, $\angle BFG = \phi$. Now $\angle EFG = \phi - \theta$. On the other hand, $\angle EHG = 360° - (180° - \theta) - \phi = 180° + \theta - \phi$. It follows that $EFGH$ is indeed cyclic. Analogous arguments apply if the relative positions of the points are varied.

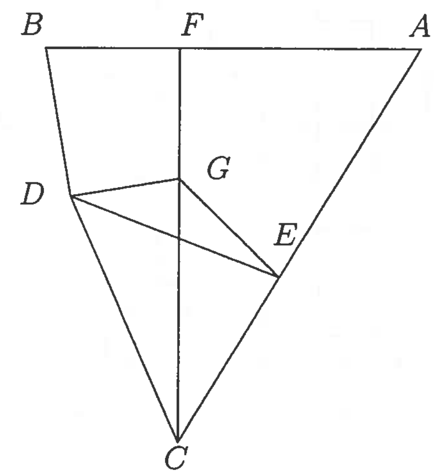


Lemma 2.
Let E be a point on the segment AC and F be a point on the segment AB. Let BE and CF intersect at G. If the circumcircles of triangles BFG and CEG intersect again at D, then the circumcircles of triangles ABE and ACF also pass through D.

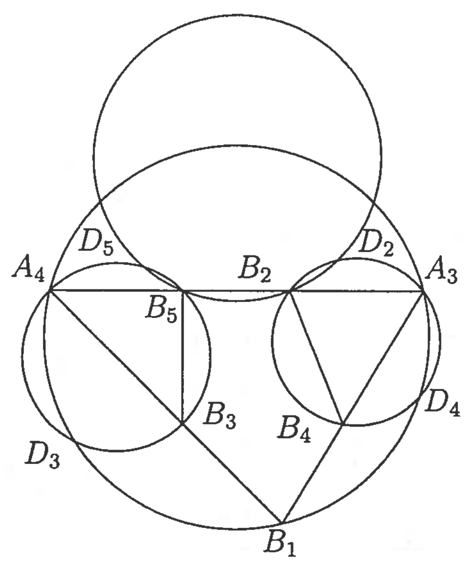


Proof:
Since $BDGF$ is cyclic, $\angle FBD = \angle DGC$. Since $CDGE$ is cyclic, $\angle CDG = \angle GEA$ and $\angle GED = \angle GCD$. Hence $\angle DBA + \angle DEA = \angle DGC + \angle GCD + \angle CDG = 180°$, so that

$ABDE$ is a cyclic quadrilateral. Similarly, so is $ACDF$.



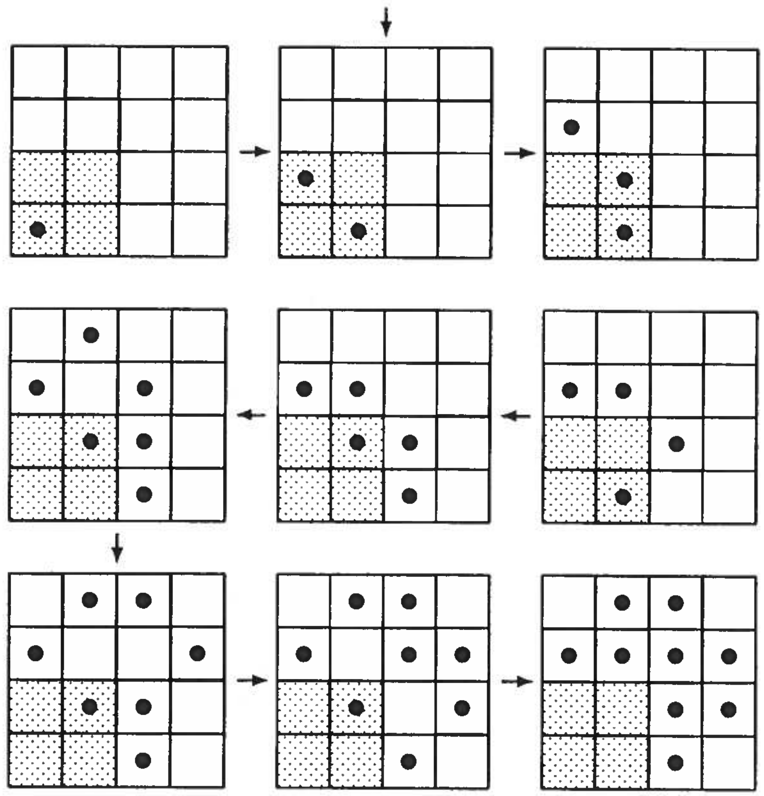
We now present the solution to the original problem. By symmetry, it is sufficient to prove that D_2, D_3, D_4 and D_5 are concyclic. By Lemma 2, D_3 is the other point of intersection of the circumcircles of triangles $A_4B_3B_5$ and $A_4B_1A_3$ while D_4 is the other point of intersection of the circumcircles of triangles $A_3B_2B_4$ and $A_4B_1A_3$. The desired conclusion now follows from Lemma 1.



3. The world of the amoeba consists of the first quadrant of the plane divided into unit squares. Initially, a solitary amoeba is imprisoned in the bottom left corner square. A prison is defined as a set of squares consisting of the bottom a_i squares in the i-th cokumn for $1 \leq i \leq n$ such that $a_1 \geq a_2 \geq \cdots \geq a_n$. Such a prison is denoted by $(a_1, a_2, \ldots, a_n)$. It is unguarded, and the Great Escape is successful if the entire prison is unoccupied. In each move, an amoeba splits into two, with one going to the square directly above and one going to the square directly to the right. However, the move is not permitted if either of those two squares is already occupied. Determine all prisons with $a_2 \geq 2$ from which the Great Escape is achievable.

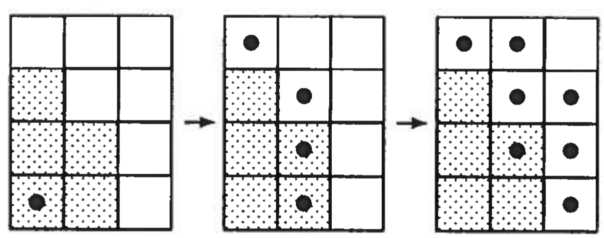
Solution:

The only prison for which $a_2 = 2$ and from which the Great Escape is achievable is (2,2), in 8 moves, as shown in the diagram below.

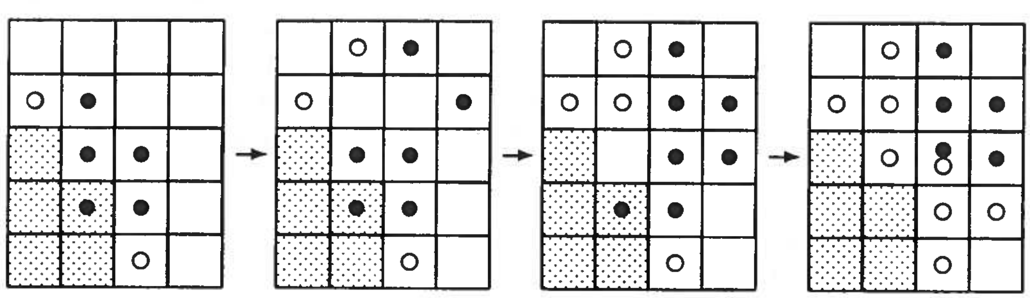


We claim that the Great Escape from the prison (3,2) is not achievable. It then follows that it is not achievable from any other prisons for which $a_2 \geq 2$.

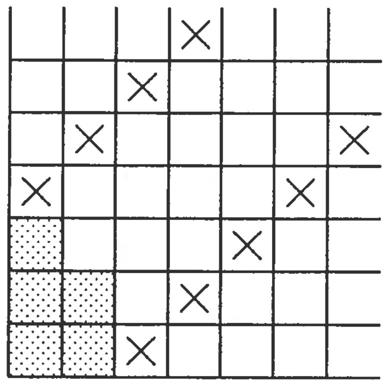
Suppose the Great Escape from (3,2) is achievable. We first point out that the order of the moves are irrelevant, as long as we allow temporary multiple occupancy of squares. Thus there is essentially one escape plan, if any exists. So we may begin an attempt by making a 3-move Northward Breakout followed by a 3-move Eastward Breakout, as shown in the diagram below.



At this point, note that the amoeba on the first column and the one on the first row should not be moved any further, since they are outside the prison and not blocking the escape paths of any other amoebas. We mark them with white circles. We now move the other five amoebas one row at a time, as shown in the diagram below.

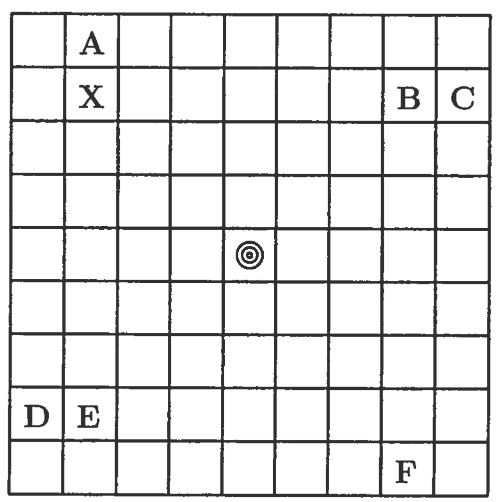


We have five more amoebas to move, and they form the same configuration as before except shifted one square diagonally in the north-east direction. It follows that in the Great Escape from (3,2), the amoebas do not venture outside the two diagonals of squares as indicated in the diagram below.



The total value of the squares between and including these two diagonals but outside the prison is $\frac{1}{4}+3(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots)=\frac{1}{4}+\frac{3}{4}=1$. Hence the Great Escape cannot be achieved in a finite number of moves.

4. In the following board, there are seven U F O's. Each can move in any of four directions, left, right, up and down. Once it starts moving, it cannot stop unless the next square in its path is occupied by another U F O. If it reaches the border of the board before it is stopped, it simply disappears. The object of the puzzle is to get the U F O marked X into the target square at the centre of the board. Other U F O's can occupy this square temporarily, eventually moving out to make room for X. Remember that the target square by itself does not stop an U F O there.



Solution:

Below is a solution which is by no means the shortest possible.

Move	UFO	Directions
1	E	u
2	X	r
3	A	d
4	B	d
5	C	l d
6	B	l
7	F	u l
8	B	u
9	D	r
10	B	d r
11	E	d r
12	B	u
13	C	l u
14	D	u
15	B	l d
16	A	r d
17	X	d l
18	E	l
19	B	u
20	F	u
21	B	r d l
22	X	d r d l