WOOT, Week 3: Sequences & Series

Sequences & Series, Problem 1

A sequence of numbers a_1, a_2, a_3, $\ldots$ satisfies (i) $a_1 = \frac{1}{2}$, (ii) $a_1 + a_2 + \cdots + a_n = n^2 a_n$ for all $n \ge 1$. Determine the value of a_n.

Sequences & Series, Problem 2

Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \ge 3$, $T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}$. The first few terms are 2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

Sequences & Series, Problem 3

Find
$$\sum_{k=1}^{n} \frac{k}{(k+1)!}$$
and
$$\sum_{k=1}^{n} \frac{k+1}{(k-1)! + k! + (k+1)!}.$$

Sequences & Series, Problem 4

Let $a_1 = a_2 = 1$ and $a_{n+1} = (a_n^2 + 2)/a_{n-1}$ for all $n \ge 2$. Prove that a_n is an integer for all n.

Sequences & Series, Problem 5

The sequence (T_n) is defined by $T_1 = 2$ and $T_{n+1} = T_n^2 - T_n + 1$ for all $n > 0$. Prove that
$$\sum_{i=1}^{\infty} \frac{1}{T_i} = 1.$$

Sequences & Series, Problem 6

Show that
$$\sum_{k=0}^{n} \binom{n}{k}^{-1} = \frac{n+1}{2^n} \sum_{k=0}^{n} \frac{2^k}{k+1}.$$

Sequences & Series, Problem 7

Let a be a positive integer, and let (a_n) be defined by $a_0 = 0$, and
$$a_{n+1} = (a_n + 1)a + (a+1)a_n + 2\sqrt{a(a+1)a_n(a_n+1)}$$
for $n \ge 1$. Show that for each positive integer n, a_n is a positive integer.

Sequences & Series, Problem 8

Prove that the average of the numbers $n \sin n^\circ$ $(n = 2, 4, 6, \ldots, 180)$ is $\cot 1^\circ$.

Sequences & Series, Problem 9

Write the sum

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k^3 + 9k^2 + 26k + 24}$$

in the form $\frac{p(n)}{q(n)}$, where p and q are polynomials with integer coefficients.

Sequences & Series, Problem 10

For a nonnegative integer n, let t_n be 0 or 1, depending on whether the binary representation of n has an even or odd number of 1s, respectively. (This sequence is known as the *Thue-Morse sequence*). Show that

$$\prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{(-1)^{t_n}} = \frac{1}{\sqrt{2}}.$$