2013 Mathematical Olympiad Summer Program Homework

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Chapter 1

Red and Green groups homework

1. Let n be a positive integer. Given n non-overlapping circular discs on a rectangular piece of paper, prove that one can cut the piece of paper into convex polygonal pieces each of which contains exactly one disc.

2. In the coordinate plane, color the lattice points which have both coordinates even black and all other lattice points white. Let P be a polygon with black points as vertices. Prove that any white point on or inside P lies halfway between two black points, both of which lie on or inside P.

3. For n an odd positive integer, the unit squares of an $n \times n$ chessboard are colored alternately black and white, with the four corners colored black. A *tromino* is obtained by remove a unit square from a 2×2 unit square grid. For which values of n is it possible to cover all the black squares of the given chessboard with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?

4. Determine if there exists a polynomial $Q(x)$ of degree at least 2 with nonnegative integer coefficients such that for each prime p, $Q(p)$ is also a prime.

5. Let $n \geq 2$ be an integer. Let S be a subset of $\{1, 2, \ldots, n\}$ such that S neither contains two elements one of which divides the other, nor contains two elements which are relatively prime. What is the maximal possible number of elements of such a set S?

6. Find all triples of nonnegative integers (x, y, z) for which $4^x + 4^y + 4^z$ is the square of an integer.

7. Line AB is tangent to circle ω_1 at B. Let C be a point not on ω_1 such that segment AC meets ω_1 at two distinct points. Circle ω_2 is tangent to line AC and ω_1 at C and D, respectively, such that D and B are on opposite sides of line AC. Prove that the circumcenter of triangle BCD lies on the circumcircle of triangle ABC.

8. Let ABC be an acute triangle with $AB \neq AC$, and let D be the foot of perpendicular from A to line BC. Point P is on altitude AD. Rays BP and CP meet sides AC and AB at E

and F, respectively. If $BFEC$ is cyclic, prove that P is the orthocenter of triangle ABC.

9. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers, for which there exists a real number c with $0 \le a_i \le c$ for all i, such that
$$|a_i - a_j| \ge \frac{1}{i+j}$$
for all $i \ne j$. Prove that $c \ge 1$.

10. Let $a_1, a_2, \ldots, a_{2n}$ be real numbers such that $\sum_{i=1}^{2n-1}(a_{i+1}-a_i)^2 = 1$. Determine the maximum value of
$$(a_{n+1} + a_{n+2} + \cdots + a_{2n}) - (a_1 + a_2 + \cdots + a_n).$$

Chapter 2

Black and blue groups homework

1. Let p be a prime. How many irreducible polynomials of degree 4, with coefficients (mod p), are there?

2. Let n be a positive integer. Consider a variant on the well-known Towers of Hanoi game: We have n discs on a peg stacked in order of size with the smallest disc on top. A move consists of taking the disc from the top of a peg and moving to another peg such that at no point is a larger disc on top of a smaller one. In this variant, there are four pegs, including the one the discs start on, and we want to move all the discs onto one of the empty pegs.

   Let x_n be the minimum number of moves required to achieve this. Determine whether or not there exist positive reals a, b with $b > 1$ such that $x_n \geq a \cdot b^n$ for all positive integers n.

3. The usual procedure for shuffling a deck of n cards involves first splitting the deck into two blocks and then merging the two blocks to form a single deck in such a way that the order of the cards within each block is not changed. A trivial cut in which one block is empty is allowed.

   (a) How many different permutations of a deck of n cards can be produced by a single shuffle?

   (b) How many of these single shuffle permutations can be inverted by another such single shuffle, putting the deck back in order?

4. Let n be a positive integer, and let S_n be the set of all positive integer divisors of n (including 1 and n). Prove that at most half of the elements of S_n end in the digit 3.

5. Let m and n be positive integers and p be a prime number such that $m, n > 1$ and $n | m^{p(n-1)} - 1$. Prove that $\gcd(m^{n-1} - 1, n) > 1$.

6. In triangle ABC three distinct triangles are inscribed, similar to each other, but not necessarily similar to triangle ABC, with corresponding points on corresponding sides of triangle ABC. Prove that if two of these triangles share a vertex, than the third one does as well.

7. Let ABC be an acute triangle. Let DAC, EAB, and FBC be isosceles triangles exterior to ABC, with $DA = DC, EA = EB$, and $FB = FC$, such that

$$\angle ADC = 2\angle BAC, \quad \angle BEA = 2\angle ABC, \quad \angle CFB = 2\angle ACB.$$

Let D' be the intersection of lines DB and EF, let E' be the intersection of EC and DF, and let F' be the intersection of FA and DE. Find, with proof, the value of the sum

$$\frac{DB}{DD'} + \frac{EC}{EE'} + \frac{FA}{FF'}.$$

8. Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lie on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

9. Given regular polygon $A_1A_2\cdots A_n$ inscribed in circle ω and point P inside the circle, prove that there exist vertices A_i and A_j such that $\angle A_iPA_j \geq (1 - \frac{1}{n}) \cdot 180°$.

10. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers, for which there exists a real number c with $0 \leq a_i \leq c$ for all i, such that

$$|a_i a_j| \geq \frac{1}{i+j}$$

for all $i \neq j$. Prove that $c \geq 1$.

11. Let $c > 2$ be a real number, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that

$$a(m+n) \leq 2 \cdot a(m) + 2 \cdot a(n) \text{ for all } m, n \geq 1,$$

and $a\left(2^k\right) \leq \frac{1}{(k+1)^c}$ for all $k \geq 0$. Prove that the sequence $a(n)$ is bounded.

12. Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all x, y, and z,

$$g(x,y)g(y,z) + g(y,z)g(z,x) + g(z,x)g(x,y) \geq 3.$$

where $g(x,y) = \left(f(x) + \frac{1}{f(y)} - 1\right)$.