Not So Easy

Set I

π day

Problem 1. Let ABCDEF be a convex hexagon with AB = BC = CD, DE = EF = FA, and $\angle$BCD = $\angle$EFA = $\frac{\pi}{3}$. Let G and H be two points interior to the hexagon, such that angles AGB and DHE are both $\frac{2\pi}{3}$. Prove that AG + GB + GH + DH + HE $\geq$ CF

Problem 2. There is given a convex quadrilateral ABCD. Prove that there exists a point P inside the quadrilateral such that $\angle$PAB + $\angle$PDC = $\angle$PBC + $\angle$PAD = $\angle$PCD + $\angle$PBA = $\angle$PDA + $\angle$PCB = $\frac{\pi}{2}$ if and only if the diagonals AC and BD are perpendicular.

Problem 3. Let ABCD be a fixed convex quadrilateral with BC = DA and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy BE = DF. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

Problem 4. ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that $S \geq 2T$.

Problem 5. Each pair of opposite sides of a convex hexagon has the following property: The distance between their midpoints is equal to $\frac{\sqrt{3}}{2}$ times the sum of their lengths.

Prove that all the angles of the hexagon are equal.

Problem 6. Let $\triangle$ABC be a triangle, and let P a varying point on the arc BC of the circumcircle of $\triangle$ABC. Prove that the circle through P and the incenters of $\triangle$PAB and $\triangle$PAC passes through a fixed point independent of P.

Problem 7. $\triangle$ABC is a triangle such that AC $\neq$ BC. Let $\triangle A'B'C$ is a triangle obtained by rotating $\triangle$ABC around C. Let M, E, F be the midpoints of segments BA$'$, AC, CB$'$, respectively. Find $\angle$EMF.

Problem 8. A convex quadrilateral ABCD has perpendicular diagonals. The perpendicular bisectors of AB and CD meet at a unique point P inside ABCD. Prove that ABCD is cyclic if and only if triangles ABP and CDP have equal areas.

Problem 9. Let the sides of two rectangles be $\{a, b\}$ and $\{c, d\}$ with $a < c \leq d < b$ and $ab < cd$.

Prove that the first rectangle can be placed within the second one if and only if $(b^2 - a^2)^2 \leq (bd - ac)^2 + (bc - ad)^2$.

Problem 10. Let ABCDEF be a convex hexagon such that AB||DE, BC||EF, and CD||AF. Let R_A, R_C, R_E be the circumradii of $\triangle$FAB, $\triangle$BCD, $\triangle$DEF respectively, and let P denote the perimeter of the hexagon.

Prove that, $R_A + R_C + R_E \geq \frac{p}{2}$.

Problem 11. Let ABCD be a convex quadrilateral, and let R_A, R_B, R_C, and R_D denote the circumradii of the triangles $\triangle$DAB, $\triangle$ABC, $\triangle$BCD, and $\triangle$CDA respectively. Prove that $R_A + R_C > R_B + R_D$ if and only if $\angle A + \angle C > \angle B + \angle D$.