Art of Problem Solving

WOOT 2010–11

Linearly Recurrent Sequences

There is an important class of sequences known as *linearly recurrent* sequences. These are sequences in which the terms x_0, x_1, x_2, $\ldots$ satisfy a recurrence of the form

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k}, \tag{$*$}$$

where a_1, a_2, $\ldots$, a_k are constants. A surprisingly large proportion of sequences can be described by a linear recurrence. We start with the simplest sequence that is defined by a linear recurrence.

Problem. The sequence (x_n) (which stands for the sequence x_0, x_1, x_2, $\ldots$) is defined by $x_0 = c$ and $x_n = rx_{n-1}$ for all $n \ge 1$. Find x_n.

Solution. Clearly, (x_n) is a geometric sequence, with

$$x_n = cr^n.$$

This example may seem trivial, but it is important because it turns out that every sequence that satisfies a linear recurrence can be expressed as a combination of geometric sequences (or sequences that relate to geometric sequences).

We can then ask when the geometric sequence $x_n = cr^n$ satisfies the linear recurrence given in $(*)$. Substituting, we see that this occurs if and only if

$$cr^n = ca_1 r^{n-1} + ca_2 r^{n-2} + \cdots + ca_k r^{n-k},$$

or

$$cr^n - ca_1 r^{n-1} - ca_2 r^{n-2} + \cdots - ca_k r^{n-k} = 0$$

for all $n \ge k$. This equation is trivially satisfied if $c = 0$ or $r = 0$, so assume that $c \ne 0$ and $r \ne 0$. We can then divide both sides by cr^{n-k}, to get

$$r^k - a_1 r^{k-1} - a_2 r^{k-2} - \cdots - a_k = 0.$$

Hence, the geometric sequence $x_n = cr^n$ satisfies $(*)$ if r is a root of the polynomial

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k.$$

We call this polynomial the *characteristic polynomial* of the linear recurrence defined in $(*)$. The next step is to figure out how to use the roots of this polynomial to solve the sequence, which we illustrate with an example.

Problem. The sequence (x_n) is defined by $x_0 = 2$, $x_1 = 3$, and $x_n = 3x_{n-1} - 2x_{n-2}$ for all $n \ge 2$. Find x_n.

Solution. The characteristic polynomial of the linear recurrence $x_n = 3x_{n-1} - 2x_{n-2}$ is $x^2 - 3x + 2$, which factors as $(x-1)(x-2)$. Therefore, the roots of the characteristic polynomial are 1 and 2.

From our work above, we know that the geometric sequences of the form $x_n = c_1 1^n = c_1$ and $x_n = c_2 2^n$ satisfy the given linear recurrence. Neither of these sequences, on their own, can fit both conditions $x_0 = 2$ and $x_1 = 3$. However, we may be able to fit these conditions if we look at a sequence of the form $x_n = c_1 + c_2 2^n$.

(It is not hard to see that the sequence $x_n = c_1 + c_2 2^n$ also satisfies the given linear recurrence for any constants c_1 and c_2.)

To fit the condition $x_0 = 2$, we set $n = 0$ to get $c_1 + c_2 = 2$. To fit the condition $x_1 = 3$, we set $n = 1$ to get $c_1 + 2c_2 = 3$. Thus, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 2, \\ c_1 + 2c_2 &= 3. \end{aligned}$$

Solving this system of equations, we find $c_1 = c_2 = 1$, so we obtain the formula $x_n = 2^n + 1$. On a final note, we know that this formula works for all n, because it satisfies the linear recurrence and the initial conditions, and any sequence that is specified by a linear recurrence and initial conditions is uniquely defined. ■

We can solve a general linear recurrence as follows: Let the sequence (x_n) be defined by the initial terms x_0, x_1, $\dots$, x_{k-1}, and the linear recurrence

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \dots + a_k x_{n-k}$$

for all $n \geq k$. Let r_1, r_2, $\dots$, r_k be the roots of the characteristic polynomial

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \dots - a_k.$$

If these roots are distinct, then set

$$x_n = c_1 r_1^n + c_2 r_2^n + \dots + c_k r_k^n. \tag{$**$}$$

We can solve for the constants c_1, c_2, $\dots$, c_k by setting $n = 0$, 1, $\dots$, $k-1$, to obtain the system of equations

$$\begin{aligned} c_1 + c_2 + \dots + c_k &= x_0, \\ c_1 r_1 + c_2 r_1 + \dots + c_k r_k &= x_1, \\ &\dots, \\ c_1 r_1^{k-1} + c_2 r_2^{k-1} + \dots + c_k r_k^{k-1} &= x_{k-1}. \end{aligned}$$

It is not obvious, but if r_1, r_2, $\dots$, r_k are distinct, then this system of equations always has a unique solution in c_1, c_2, $\dots$, c_k. (We will not prove this claim here.)

For example, if the characteristic polynomial is $(x + 2)(x + 1)(x - 5)$, then

$$x_n = c_1(-2)^n + c_2(-1)^n + c_3 5^n$$

for some constants c_1, c_2, and c_3. If the roots are not distinct (that is, we have repeated roots), then we must adjust this formula. If a root r has multiplicity $m > 1$, then the term cr^n is replaced by

$$c_1 r^n + c_2 n r^n + \dots + c_m n^{m-1} r^n.$$

For example, if the characteristic polynomial is $(x + 2)^2(x + 1)(x - 5)^3$, then

$$x_n = c_1(-2)^n + c_2 n(-2)^n + c_3(-1)^n + c_4 5^n + c_5 n 5^n + c_6 n^2 5^n$$

for some constants c_1, c_2, $\dots$, c_6. Again, we will not prove this here, but we will give an indication of where this form comes from in the Generating Functions handout, later in the course.

Problem. The sequence (x_n) is defined by $x_0 = 5$, $x_1 = 9$, $x_2 = 43$, and $x_n = 3x_{n-1} - 4x_{n-3}$ for all $n \geq 3$. Find x_n.

Solution. The characteristic polynomial of the linear recurrence $x_n = 3x_{n-1} - 4x_{n-3}$ is $x^3 - 3x^2 + 4$, which factors as $(x+1)(x-2)^2$. Therefore, the roots of the characteristic polynomial are -1, 2, and 2. Hence,

$$x_n = c_1(-1)^n + c_2 2^n + c_3 n 2^n$$

for some constants c_1, c_2, and c_3.

Setting $n = 0$, 1, and 2, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 5, \\ -c_1 + 2c_2 + 2c_3 &= 9, \\ c_1 + 4c_2 + 8c_3 &= 43. \end{aligned}$$

Solving this system of equations, we find $c_1 = 3$, $c_2 = 2$, and $c_2 = 4$. Therefore,

$$x_n = 3(-1)^n + 2 \cdot 2^n + 4n \cdot 2^n = 3(-1)^n + 2^{n+1} + n2^{n+2}.$$

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Problem. Derive a formula for the n^{th} Fibonacci number F_n.

Solution. The characteristic polynomial of the recurrence $F_n = F_{n-1} + F_{n-2}$ is $x^2 - x - 1$. By the quadratic formula, the roots of $x^2 - x - 1$ are

$$x = \frac{1 \pm \sqrt{5}}{2},$$

so let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then $F_n = c_1\alpha^n + c_2\beta^n$ for some constants c_1 and c_2. Setting $n = 0$ and $n = 1$, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 0, \\ \alpha c_1 + \beta c_2 &= 1. \end{aligned}$$

Solving for c_1 and c_2, we find $c_1 = \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\alpha-\beta} = -\frac{1}{\sqrt{5}}$, so

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

■

We can also use the same technique to solve certain sequences that are not quite linearly recurrent, but close to it.

Problem. The sequence (x_n) is defined by $x_0 = 1$ and $x_n = 3x_{n-1} + 2n$ for all $n \geq 1$. Find x_n.

Solution. We can rewrite the recurrence as

$$x_n - 3x_{n-1} = 2n.$$

Strictly speaking, this is not a linear recurrence because of the term $2n$, so we focus on trying to eliminate it. If we substitute $n-1$ for n, then we obtain another equation that contains the term $2n$:

$$\begin{aligned} x_n - 3x_{n-1} &= 2n, \\ x_{n-1} - 3x_{n-2} &= 2n - 2. \end{aligned}$$

Subtracting these equations, we get

$$x_n - 4x_{n-1} + 3x_{n-2} = 2.$$

We have successfully eliminated the $2n$, but we still do not have a linear recurrence because of the constant term 2.

To eliminate the constant term 2, we can substitute $n-1$ for n again:

$$\begin{aligned} x_n - 4x_{n-1} + 3x_{n-2} &= 2, \\ x_{n-1} - 4x_{n-2} + 3x_{n-3} &= 2. \end{aligned}$$

Subtracting these equations, we get

$$x_n - 5x_{n-1} + 7x_{n-2} - 3x_{n-3} = 0.$$

This equation does represent a linear recurrence, which we know how to solve.

The characteristic polynomial of this linear recurrence is

$$x^3 - 5x^2 + 7x - 3 = (x-1)^2(x-3),$$

so

$$x_n = c_1 1^n + c_2 n 1^n + c_3 3^n = c_1 + c_2 n + c_3 3^n$$

for some constants c_1, c_2, and c_3. Setting $n = 0$, 1, and 2, we obtain the system of equations

$$\begin{aligned} c_1 + c_3 &= x_0 = 1, \\ c_1 + c_2 + 3c_3 &= x_1 = 5, \\ c_1 + 2c_2 + 9c_3 &= x_2 = 19. \end{aligned}$$

(Note that a_1 and a_2 are not given in the problem, but we can compute them with the recurrence that is given in the problem.) Solving this system of equations, we find $c_1 = -\frac{3}{2}$, $c_2 = -1$, and $c_3 = \frac{5}{2}$. Therefore,

$$x_n = -\frac{3}{2} - n + \frac{5}{2} \cdot 3^n = \frac{5 \cdot 3^n - 2n - 3}{2}.$$

■

More generally, we may have recurrence of the form

$$x_n - a_1 x_{n-1} - a_2 x_{n-2} - \cdots - a_k x_{n-k} = f(n),$$

where f is an arbitrary function. A recurrence of this form is sometimes called an *inhomogeneous recurrence.* If $f(n)$ itself satisfies a linear recurrence (such as $f(n) = 2n$, $f(n) = 2^n + 1$, or $f(n) = F_n$), then using the same technique of repeatedly shifting the index n, we can convert the inhomgeneous recurrence to a linear recurrence.

We have seen that we can solve a linear recurrence to obtain a solution of the form

$$x_n = c_1 n^{m_1} r_1^n + c_2 n^{m_2} r_2^n + \cdots + c_k n^{m_k} r_k^n.$$

But we can also run this process in reverse: Given a sequence of this form, we can say that it satisfies some linear recurrence. This idea of deriving the linear recurrence, rather than starting with it, can turn out to be very powerful.

Problem. Show that $F_n^2 + F_{n+1}^2 = F_{2n+1}$ for all $n \geq 0$.

Solution. Let $a_n = F_n^2 + F_{n+1}^2 - F_{2n+1}$. We know that

$$F_n = c_1 \alpha^n + c_2 \beta^n,$$

where $c_1 = \frac{1}{\sqrt{5}}$, $c_2 = -\frac{1}{\sqrt{5}}$, $\alpha = \frac{1+\sqrt{5}}{2}$, and $\beta = \frac{1-\sqrt{5}}{2}$. Then

$$\begin{aligned}
a_n &= F_n^2 + F_{n+1}^2 - F_{2n+1} \\
&= (c_1\alpha^n + c_1\beta^n)^2 + (c_1\alpha^{n+1} + c_2\beta^{n+1})^2 - (c_1\alpha^{2n+1} + c_2\beta^{2n+1}) \\
&= c_1^2\alpha^{2n} + 2c_1c_2\alpha^n\beta^n + c_2^2\beta^{2n} + c_1^2\alpha^{2n+2} + 2c_1c_2\alpha^{n+1}\beta^{n+1} + c_2^2\beta^{2n+2} - c_1\alpha^{2n+1} - c_2\beta^{2n+1} \\
&= c_1^2\alpha^{2n} + 2c_1c_2\alpha^n\beta^n + c_2^2\beta^{2n} + \alpha^2c_1^2\alpha^{2n} + 2\alpha\beta c_1c_2\alpha^n\beta^n + \beta^2c_2^2\beta^{2n} - \alpha c_1\alpha^{2n} - \beta c_2\beta^{2n} \\
&= (c_1^2 + \alpha^2c_1^2 - \alpha c_1)\alpha^{2n} + (c_2^2 + \beta^2c_2^2 - \beta c_2)\beta^{2n} + (2c_1c_2 + 2\alpha\beta c_1c_2)\alpha^n\beta^n.
\end{aligned}$$

At this point, we can plug in the values c_1, c_2, α, and β. However, if we look at this expression, we see that we can write it in the form

$$a_n = C_1(\alpha^2)^n + C_2(\beta^2)^n + C_3(\alpha\beta)^n,$$

where C_1, C_2, and C_3 are constants. Hence, the sequence (a_n) satisfies a linear recurrence, whose characteristic polynomial is

$$(x - \alpha^2)(x - \beta^2)(x - \alpha\beta).$$

We know that α and β are the roots of $x^2 - x - 1$, so by Vieta's formulas, $\alpha + \beta = 1$ and $\alpha\beta = -1$. Squaring $\alpha + \beta = 1$, we get $\alpha^2 + 2\alpha\beta + \beta^2 = 1$, so $\alpha^2 + \beta^2 = 1 - 2\alpha\beta = 3$. Squaring $\alpha\beta = -1$, we get $\alpha^2\beta^2 = 1$. Therefore, the characteristic polynomial is

$$\begin{aligned}
(x - \alpha^2)(x - \beta^2)(x - \alpha\beta) &= [x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2](x + 1) \\
&= (x^2 - 3x + 1)(x + 1) \\
&= x^3 - 2x^2 - 2x + 1.
\end{aligned}$$

Hence,

$$a_n = 2a_{n-1} + 2a_{n-2} - a_{n-3}$$

for all $n \geq 3$. Computing the initial terms of the sequence, we find $a_0 = 0$, $a_1 = 0$, and $a_2 = 0$. Therefore, $a_n = 0$ for all $n \geq 0$. But $a_n = F_n^2 + F_{n+1}^2 - F_{2n+1}$, so $F_n^2 + F_{n+1}^2 = F_{2n+1}$ for all $n \geq 0$. ■

Problem. Let x, y, and z be real numbers such that $x + y + z = 0$. Prove that

$$\frac{x^2 + y^2 + z^2}{2} \cdot \frac{x^3 + y^3 + z^3}{3} = \frac{x^5 + y^5 + z^5}{5}.$$

Solution. Let $S_n = x^n + y^n + z^n$. We see that the sequence (S_n) satisfies a linear recurrence, whose characteristic polynomial is

$$(t - x)(t - y)(t - z) = t^3 - (x + y + z)t^2 + (xy + xz + yz)t - xyz.$$

Let $A = xy + xz + yz$ and $B = xyz$. We are given that $x + y + z = 0$, so the characteristic polynomial can also be written as

$$t^3 + At - B.$$

Therefore,

$$S_n = -AS_{n-2} + BS_{n-3}$$

for all $n \geq 3$. To compute the terms of the sequence (S_n), we also require the initial terms of the sequence. We see that $S_0 = x^0 + y^0 + z^0 = 3$, $S_1 = x + y + z = 0$, and

$$S_2 = x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + xz + yz) = -2A.$$

(If $x = 0$, then $y + z = 0$, and both sides of the given equation become 0. Otherwise, we may assume that x, y, and z are all nonzero, so S_0 is well-defined.)

Now we can use the linear recurrence to compute the terms of the sequence (S_n) up to S_5:

$$\begin{aligned} S_3 &= -AS_1 + BS_0 = 3B, \\ S_4 &= -AS_2 + BS_1 = 2A^2, \\ S_5 &= -AS_3 + BS_2 = -5AB. \end{aligned}$$

Therefore,

$$\frac{S_2}{2} \cdot \frac{S_3}{3} = -AB = \frac{S_5}{5}.$$

■

Problem. Let $P(x, y) = x^2y + xy^2$ and $Q(x, y) = x^2 + xy + y^2$. For $n = 1, 2, 3, \ldots$, let $F_n(x, y) = (x + y)^n - x^n - y^n$ and $G_n(x, y) = (x + y)^n + x^n + y^n$. One observes that $G_2 = 2Q$, $F_3 = 3P$, $G_4 = 2Q^2$, $F_5 = 5PQ$, $G_6 = 2Q^3 + 3P^2$. Prove that, in fact, for each n either F_n or G_n is expressible as a polynomial in P and Q with integer coefficients. (Putnam, 1976)

Solution. As suggested by the examples given in the problem, we claim that if n is even, then G_n can be expressed as a polynomial in P and Q with integer coefficients. Since

$$\begin{aligned} G_{2n} &= (x+y)^{2n} + x^{2n} + y^{2n} \\ &= [(x+y)^2]^n + (x^2)^n + (y^2)^n, \end{aligned}$$

the sequence (G_{2n}) satisfies a linear recurrence, whose characteristic polynomial is

$$[t-(x+y)^2](t-x^2)(t-y^2).$$

This expands as

$$\begin{aligned} &[t^2-(x+y)^2][t-(x^2+y^2)t+x^2y^2] \\ &\quad = t^3-(2x^2+2xy+2y^2)t^2+(x^4+2x^3y+3x^2y^2+2xy^3+y^4)t-x^2y^2(x+y)^2. \end{aligned}$$

We see that $2x^2+2xy+2y^2 = 2Q$, $x^2y^2(x+y)^2 = (x^2y+xy^2)^2 = P^2$, and

$$x^4+2x^3y+3x^2y^2+2xy^3+y^4 = (x^2+xy+y^2)^2 = Q^2,$$

so the characteristic polynomial can be written as

$$t^3-2Qt^2+Q^2t-P^2.$$

Therefore,

$$G_{2n} = 2QG_{2n-2} - Q^2G_{2n-4} + P^2G_{2n-6}$$

for all $n \geq 4$. Furthermore, the initial terms of the sequence (G_{2n}) are $G_0 = 3$, $G_2 = 2Q$, and $G_4 = 2Q^2$. Hence, by a straightforward induction argument, G_{2n} can be expressed as a polynomial in P and Q with integer coefficients for all $n \geq 0$.

Similarly, we claim that if n is odd, then F_n can be expressed as a polynomial in P and Q. Since

$$\begin{aligned} F_{2n+1} &= (x+y)^{2n+1} - x^{2n+1} - y^{2n+1} \\ &= (x+y)[(x+y)^2]^n - x(x^2)^n - y(y^2)^n, \end{aligned}$$

the sequence (F_{2n+1}) satisfies a linear recurrence, whose characteristic polynomial is the same as the characteristic polynomial for the sequence (G_{2n}), namely

$$[t-(x+y)^2](t-x^2)(t-y^2) = t^3-2Qt^2+Q^2t-P^2.$$

Therefore,

$$F_{2n+1} = 2QF_{2n-1} - Q^2F_{2n-3} + P^2F_{2n-5}$$

for all $n \geq 3$. Furthermore, the initial terms of the sequence (F_{2n+1}) are $F_1 = 0$, $F_3 = 3P$, and $F_5 = 5PQ$. Hence, by a straightforward induction argument, F_{2n+1} can be expressed as a polynomial in P and Q with integer coefficients for all $n \geq 0$.

In summary, G_n can be expressed as a polynomial in P and Q with integer coefficients if n is even, and F_n can be expressed as a polynomial in P and Q with integer coefficients if n is odd. ■

Exercises

1. Let x, y, and z be real numbers such that $x + y + z = 0$. Prove that
$$\frac{x^2 + y^2 + z^2}{2} \cdot \frac{x^5 + y^5 + z^5}{5} = \frac{x^7 + y^7 + z^7}{7}.$$

2. Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations
$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$
(AIME, 1990)

3. Let (x_n) be a sequence such that $x_0 = x_1 = 5$ and
$$x_n = \frac{x_{n-1} + x_{n+1}}{98}$$
for all positive integers n. Prove that $(x_n + 1)/6$ is a perfect square for all n.

4. Let a, b, and c be the roots of the equation $x^3 - x^2 - x - 1 = 0$. Show that a, b, and c are distinct, and that
$$\frac{a^{1982} - b^{1982}}{a - b} + \frac{b^{1982} - c^{1982}}{b - c} + \frac{c^{1982} - a^{1982}}{c - a}$$
is an integer. (Canada, 1982)

5. For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Putnam, 1980)

6. An integer sequence is defined by $a_0 = 0$, $a_1 = 1$, and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$. Prove that 2^k divides a_n if and only if 2^k divides n. (IMO Short List, 1988)

7. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$ and
$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1})$$
for all $n = 1, 2, 3, \ldots$, where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$. (IMO, 1979)

8. A sequence (a_n) is defined by $a_0 = a_1 = 0$, $a_2 = 1$, and $a_{n+3} = a_{n+1} + 1998a_n$ for all $n \geq 0$. Prove that $a_{2n-1} = 2a_na_{n+1} + 1998a_{n-1}^2$ for every positive integer n. (Komal)