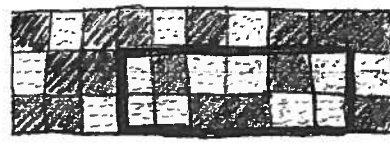
Individual Mock Olympiad

Jan 5th, 2003

1. Determine the smallest positive integer n for which the following statement is true:

   If every square of a 3 by n chessboard is coloured either red or blue, then there must exist a rectangle for which all four of its corner squares are the same colour.



2. Let ABC be a triangle and D be a point on side AB. The incircles of triangles ACD and BCD touch each other on CD. Prove that the incircle of $\triangle ABC$ touches AB at D.

3. As usual, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Let $n \geq 2$ be a positive integer. Prove that

$$\sum_{k=2}^{n} \left\lfloor \frac{n^2}{k} \right\rfloor = \sum_{k=n+1}^{n^2} \left\lfloor \frac{n^2}{k} \right\rfloor$$

4. Let a, b, c be the sides of a triangle.
   - $\boxed{i)}$ Determine the largest real number m for which $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq m$
   - $\boxed{ii)}$ Determine the smallest real number M for which $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} < M$

5. There were n people present at a New Year's Party. Determine the value of n, given the following information:

   a) n is not a multiple of 11, and $5 \leq n \leq 30$.
   b) Each pair of strangers had exactly two common acquaintances, and each pair of acquaintances had no common acquaintances.