2008 BLUE MOP, INEQUALITIES-I
ALİ GÜREL

(1) (Cauchy-Schartz Inequality) Using the famous inequality $x^2 \geq 0$, prove that for any real numbers a_i, b_i we have

$$\left(a_1^2 + ... + a_n^2\right)\left(b_1^2 + ... + b_n^2\right) \geq (a_1 b_1 + ... + a_n b_n)^2.$$

(2) (Victors Linis) Prove that, for any quadrilateral with sides a, b, c, d,

$$\frac{a^2 + b^2 + c^2}{d^2} > \frac{1}{3}.$$

(3) (Nesbitt Inequality) If $a, b, c > 0$, prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

(4) If a, b, c are positive real numbers satisfying $a^2 + b^2 + c^2 = 1$, find the minimal value of

$$S = \frac{a^2 b^2}{c^2} + \frac{b^2 c^2}{a^2} + \frac{c^2 a^2}{b^2}.$$

(5) (Iran-98) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

(6) (Poland-96) Let $n \geq 2$, $a_1, ..., a_n$ positive numbers whose sum is 1 and $x_1, ..., x_n$ positive numbers whose sum is also 1. Prove that

$$2\sum_{i<j} x_i x_j \leq \frac{n-2}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i}.$$

(7) (Turkey-97) Given an integer $n \geq 2$, find the minimal value of

$$\frac{x_1^5}{x_2 + x_3 + ... + x_n} + \frac{x_2^5}{x_3 + x_4 + ... + x_1} + ... + \frac{x_n^5}{x_1 + x_2 + ... + x_{n-1}},$$

where x_i are positive numbers whose sum of squares is 1.

(8) (G.C.Giri) If $a, b, c > 0$, prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a^8 + b^8 + c^8}{a^3 b^3 c^3}.$$

Problem 1, Solution by Matthew Superdock:

$$\begin{aligned}
& \sum_{1\le i<j\le n} (a_ib_j - a_jb_i)^2 \ge 0 \\
\Rightarrow\quad & \sum_{1\le i<j\le n} (a_i^2b_j^2 + a_j^2b_i^2) \ge \sum_{1\le i<j\le n} 2a_ib_i \cdot a_jb_j \\
\Rightarrow\quad & \sum_{i=1}^{n} a_i^2b_i^2 + \sum_{1\le i<j\le n} (a_i^2b_j^2 + a_j^2b_i^2) \ge \sum_{i=1}^{n} a_ib_i \cdot a_ib_i + \sum_{1\le i<j\le n} 2a_ib_i \cdot a_jb_j \\
\Rightarrow\quad & \sum_{i,j} a_i^2b_j^2 \ge \sum_{i,j} a_ib_i \cdot a_jb_j \\
\Rightarrow\quad & \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \ge \left(\sum_{i=1}^{n} a_ib_i\right)^2 \quad \square
\end{aligned}$$

Problem 2, Solution by Toan Phan: By Cauchy inequality,

$$(a^2+b^2+c^2)(1^2+1^2+1^2) \ge (a+b+c)^2 > d^2 \Rightarrow \frac{a^2+b^2+c^2}{d^2} > \frac{1}{3} \ \square$$

Problem 3, Solution by John Berman: By Cauchy-Schwartz,

$$((a+b)+(b+c)+(c+a))\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \ge 9.$$

So

$$\frac{a+b+c}{a+b}+\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a} \ge \frac{9}{2}.$$

Substituting 3 from both sides yields

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}, \quad \text{as desired } \square$$

Problem 4, Solution by Brian Hamrick: Notice that by Cauchy,

$$S^2 = \left(\frac{a^2b^2}{c^2}+\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}\right)\left(\frac{c^2a^2}{b^2}+\frac{a^2b^2}{c^2}+\frac{b^2c^2}{a^2}\right) \ge (a^2+b^2+c^2)^2 = 1.$$

This minimum is attained at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ $\square$

Problem 5, Solution by Sam Keller: We have $\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=1$. Therefore, by Cauchy

$$\sqrt{x+y+z} = \sqrt{(x+y+z)\left(\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}\right)} \ge \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1} \ \square$$

Problem 6, Solution by David B. Rush:

$$2\sum_{i<j} x_i x_j \le \frac{n-2}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i}$$

$$\Leftrightarrow \quad \left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2 \le 1 - \frac{1}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i}$$

$$\Leftrightarrow \quad \frac{1}{n-1} \le \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i} = \sum_{i=1}^{n} \frac{x_i^2}{1-a_i}.$$

By Cauchy-Schwartz,

$$1 = (\sum_{i=1}^{n} x_i)^2 \le \sum_{i=1}^{n} \frac{x_i^2}{1-a_i} \sum_{i=1}^{n} (1-a_i) = \sum_{i=1}^{n} \frac{x_i^2}{1-a_i}\,(n-1),$$

which yields the desired result $\square$

Problem 7, Solution by Damien Jiang: Let S be the given sum and $T = \sum_{cyc} x_1 x_2 + ... + x_1 x_n$. By Cauchy-Schwartz,

$$S \cdot T \ge \left(\sum_{cyc} x_1^3\right)^2.$$

By Power Mean and the given condition

$$\left(\sum_{cyc} x_1^3\right)^2 \ge \frac{1}{n}\left(\sum_{cyc} x_1^2\right)^3 = \frac{1}{n}.$$

Also, $T \le (n-1)\sum_{cyc} x_1^2$ by adding the inequalities $x_i^2 + x_j^2 \ge 2x_i x_j$ over i, j. So, as

$$ST \ge \frac{1}{n},\ S \ge \frac{1}{n(n-1)}.$$

and this value can be achieved by setting each $x_i = \frac{1}{\sqrt{n}}$ $\square$

Problem 8, Solution by Wenyu Cao:

$$\left(\sum_{cyc} \frac{a^6}{b^2c^2}\right)\left(\sum_{cyc} \frac{b^6}{c^2a^2}\right) \ge \left(\sum_{cyc} \frac{a^2b^2}{c^2}\right)^2 \Leftrightarrow \sum_{cyc} \frac{a^6}{b^2c^2} \ge \sum_{cyc} \frac{a^2b^2}{c^2}, \quad \text{and}$$

$$\left(\sum_{cyc} \frac{a^2b^2}{c^2}\right)\left(\sum_{cyc} \frac{a^2c^2}{b^2}\right) \ge (\sum_{cyc} a^2)^2 \Leftrightarrow \sum_{cyc} \frac{a^2b^2}{c^2} \ge \sum_{cyc} a^2.$$

Thus,

$$\sum_{cyc} \frac{a^6}{b^2c^2} \ge \sum_{cyc} \frac{a^2b^2}{c^2} \ge \sum_{cyc} a^2 \sum_{cyc} ab \Leftrightarrow \frac{a^8+b^8+c^8}{a^3b^3c^3} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ \square$$