1998 IMO Camp

Combinatorics Problems

1. Find the number of 0-1 strings of length n containing no block consisting of an odd number of zeros between two nonempty blocks of ones.

2. How many strings of length n can be formed with the alphabet $\{0,1,2,3,4\}$ if neighboring digits differ by 1 (in absolute value)?

3. Given $k \geq 1$, let a_n be the number of 0-1 strings of length n that do not have k consecutive zeros, and let b_n be the number of 0-1 strings that have neither $k+1$ consecutive zeros nor $k+1$ consecutive ones. Prove that $b_{n+1} = 2a_n$.

4. Call a finite set of positive integers S fat if every element of S is at least $|S|$. By convention, the empty set is fat.
   (a) Prove that the total number of fat subsets of $[n] = \{1, 2, \ldots, n\}$ is
   $$F_{n+2} = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-k+1}{k},$$ the $n+2$-th Fibonacci number.

   (b) Prove that there are also F_{n+2} alternating subsets of $[n]$.
   (c) How many subsets of $[n]$ are both alternating and fat?

5. Show that the number of subsets of $[n]$ containing exactly one pair of consecutive integers is
$$\sum_{k=1}^{n-1} F_k F_{n-k} = \frac{2n F_{n+1} - (n+1) F_n}{5}$$
where F_k is the k-th Fibonacci number.

6. Find the sequence (a_n) if $a_0 = 1$ and

$$\sum_{k=0}^{n} a_k a_{n-k} = 1 \quad \text{for } n \geq 1.$$

7. Prove that the number of partitions of n into parts not divisible by d is the same as the number of partitions of n in which no part occurs d or more times.

8. Prove that the number of partitions of n in which all even parts are distinct is the same as the number of partitions of n in which each part is repeated at most three times.

9. For each $n \geq 1$, find the sum of the products of Fibonacci numbers $F_{k_1} \cdot F_{k_2} \cdot \cdots \cdot F_{k_r}$ where the sum is over all 2^{n-1} compositions $n = k_1 + k_2 + \cdots + k_r$ (for example, for $n = 3$ the desired sum is $F_3 + F_1 \cdot F_2 + F_2 \cdot F_1 + F_1 \cdot F_1 \cdot F_1 = 5$)

10. Find the number of permutations of $[n]$ that have no r-cycle.

11. Prove:

$$\sum_{k=r}^{n} (-1)^{k-r} \binom{k}{r} \binom{n-k}{k} 2^{n-2k} = \binom{n+1}{2r+1}$$

12. (a) How many ballot sequences with n A's and n B's are there where A and B are never tied until the last vote?

(b) How many such sequences are there where A and B are tied at exactly one point before the last vote?