2003 Winter Camp

Buffet Contest

Algebra

1. Determine all real numbers p such that the three roots of

$$5x^3 - 5(p+1)x^2 + (71p-1)x - (66p-1) = 0$$

are positive integers.

2. Let x, y and z be real numbers such that $x \geq y \geq z \geq \frac{\pi}{12}$ and $x+y+z=\frac{\pi}{2}$.

   (a) Determine the maximum value of $\cos x \sin y \cos z$.

   (b) Determine the minimum value of $\cos x \sin y \cos z$.

3. Determine the range of the real number θ such that for $0 \leq x \leq 1$,

$$x^2 \cos\theta - x(1-x) + (1-x)^2 \sin\theta > 0.$$

4. Determine the range of values of the real number a such that for any real numbers x and θ where $0 \leq \theta \leq \frac{\pi}{2}$, $(x+3+2\sin\theta\cos\theta)^2 + (x + a\sin\theta + a\cos\theta)^2 \geq \frac{1}{8}$.

5. Let $a_1,\ a_2,\ \ldots,\ a_n,\ b_1,\ b_2,\ \ldots,\ b_n$ be real numbers in [1,2] such that $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2$.

   (a) Prove that $\sum_{i=1}^{n} \frac{a_i^3}{b_i} \leq \frac{17}{10} \sum_{i=1}^{n} a_i^2$.

   (b) Determine a necessary and sufficient condition for equality to hold.

6. Let n be a positive integer and M be a positive real number. Among all arithmetic progressions $\{a_k\}$ satisfying $a_1^2 + a_{n+1}^2 \leq M$, determine the maximum value of $a_{n+1} + a_{n+2} + \cdots + a_{2n+1}$.

7. For any positive integer n, let $S_n = 1+2+\cdots+n$. Determine the maximum value of $\frac{S_n}{(n+32)S_{n+1}}$.

8. For any real number $a < 0$, there exists a largest positive real number $\ell(a)$ such that for all real numbers x satisfying $0 \leq x \leq \ell(a)$, we have $|ax^2 + 8x + 3| \leq 5$.

   (a) Determine the maximum value of $\ell(a)$.

   (b) Determine a for which $\ell(a)$ is maximum.

9. Determine the real numbers $a \leq b$ such that for $a \leq x \leq b$, the minimum value of $\frac{1}{2}(13 - x^2)$ is $2a$ and the maximum value is $2b$.

10. For positive integers a and n, let $a = qn+r$ where q and r are non-negative integers with $r < n$. Define $F_n(a) = q+r$. Determine the largest positive integer A such that for all positive integers $a \leq A$, $F_{n_6}(F_{n_5}(F_{n_4}(F_{n_3}(F_{n_2}(F_{n_1}(a)))))) = 1$ for some positive integers $n_1,\ n_2,\ n_3,\ n_4,\ n_5$ and n_6.

Combinatorics

11. If all positive integers relatively prime to 105 are arranged in ascending order, determine the 1000-th term.

12. The sum of the first n terms of the sequence $\{a_k\}$ is $2a_n - 1$ for all $n \geq 1$. The sequence $\{b_k\}$ is defined by $b_1 = 3$ and for $k \geq 1$, $b_{k+1} = a_k + b_k$. Determine the sum of the first n terms of $\{b_n\}$.

13. The sequence $\{a_n\}$ of real numbers is defined by $a_0 = a_1 = 1$ and for $n \geq 2$,

$$\sqrt{a_n a_{n-2}} - \sqrt{a_{n-1}a_{n-2}} = 2a_{n-1}.$$

Find a formula for a_n independent of $a_0,\ a_1,\ \ldots,\ a_{n-1}$.

14. The sequences $\{a_n\}$ and $\{b_n\}$ are defined by $a_0 = 1$, $b_0 = 0$, $a_{n+1} = 7a_n + 6b_n - 3$ and $b_{n+1} = 8a_n + 7b_n - 4$ for $n \geq 0$. Prove that a_n is the square of an integer for all $n \geq 0$.

15. None of the subsets $A_1,\ A_2,\ \ldots,\ A_m$ of an n-element set contains another. Prove that
(a) $\displaystyle\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \leq 1$; (b) $\displaystyle\sum_{i=1}^{m} \binom{n}{|A_i|} \geq m^2$.

16. Each entry of a 100×25 array is a non-negative real number such that the sum of the 25 numbers in each row is at most 1. The 100 numbers in each column are rearranged from top to bottom in descending order. Determine the smallest value of k such that the sum of the 25 numbers in each row from the k-th row on down will always be at most 1.

17. There are $n \geq 6$ people at a party. Each is a mutual acquaintant of at least $\lfloor \frac{n}{2} \rfloor$ others. Among any $\lfloor \frac{n}{2} \rfloor$ of them, either two of them are mutual acquaintants, or two of the remaining $n - \lfloor \frac{n}{2} \rfloor$ are mutual acquaintants. Prove that there are three people at the party such that every two of them are mutual acquaintants.

18. Any two of n friends have a phone conversation at most once. Among any $n-2$ of them, the total number of phone conversations is a constant positive power of 3. Determine all values of n for which this is possible.

19. Let n be a positive integer. We wish to construct a set of tokens, the weight of each being an integral number of grams, such that any object whose weight is an integral number of grams up to n can be balanced by a subset of the tokens. Some of the tokens may be put on the same pan on the balance as the object.

    (a) Determine in terms of n the minimum value of the number of tokens.

    (b) For what values of n is the minimal set of tokens unique?

20. On the plane are 1994 points, no three collinear. They are to be partitioned into 83 sets, each with at least three points. Three points in the same set form a triangle.

    (a) What is the minimum number of triangles?

    (b) In a partition which yields the minimum number of triangles, prove that each segment joining two points in the same set can be painted in one of four colours such that no triangles have three sides with the same colour.

Geometry

21. Only one interior angle of the convex quadrilateral $ABCD$ is obtuse. $ABCD$ is to be partitioned into n obtuse triangles whose vertices other than A, B, C and D are inside $ABCD$. Prove that this is possible if and only if $n \geq 4$.

22. Each point of the plane is painted in one of two colours. Prove that there exist two similar triangles such that the lengths of the sides of one are 1995 times the lengths of the corresponding sides of the other, and all three vertices of each triangle have the same colour.

23. The diagonal AC of the quadrilateral $ABCD$ bisects $\angle BAD$. E is a point on the side CD. BE cuts AC at F, and the line DF cuts BC at G. Prove that $\angle CAE = \angle CAG$.

24. A circle is tangent to the sides AB, BC, CD and DA of a rhombus $ABCD$ at E, F, G and H respectively. M, N, P and Q are points on AB, BC, CD and DA respectively such that MN is tangent to the arc EF and PQ is tangent to the arc GH. Prove that MQ is parallel to NP.

25. A line m passes through the centre of a circle. A, B and C are three points outside the circle and lying on a line ℓ perpendicular to m, with A farthest from m and C nearest to it. AP, BQ and CR are tangents to the circle.

    (a) Prove that $AB \cdot CR + BC \cdot AP = CA \cdot BQ$ if ℓ is tangent to the circle.

    (b) Prove that $AB \cdot CR + BC \cdot AP < CA \cdot BQ$ if ℓ intersects the circle in two points.

    (c) Prove that $AB \cdot CR + BC \cdot AP > CA \cdot BQ$ if ℓ is disjoint from the circle.

26. Two circles with unequal radii intersect at M and N. They are inside a circle with centre O and tangent to it at S and T. Prove that OM is perpendicular to MN if and only if S, N and T are collinear.

27. E and F are point on the sides BC of an acute triangle ABC, with B closer to E than to F, such that $\angle BAE = \angle CAF$. Perpendiculars FM and FN are dropped from F onto the sides AB and AC, respectively. The line AE intersects the circumcircle of ABC again at D. Prove that the area of ABC is equal to the area of the quadrilateral $AMDN$.

28. The excircle of triangle ABC opposite C is tangent to the line BC at E and the line CA at G. The excircle of ABC opposite B is tangent to the line BC at F and the line AB at H. If the lines EG and FH intersect at P, prove that AP is perpendicular to BC.

29. Let O and I be the circumcentre and the incentre of triangle ABC respectively. If $\angle B = 60°$, $\angle A < \angle C$ and the exterior bisector of $\angle A$ meets the circumcircle again at E, prove that

    (a) $IO = AE$;

    (b) $2R < IO + IA + IC < (1 + \sqrt{3})R$, where R is the circumradius of ABC.

30. The circumcentre and incentre of triangle ABC are O and I respectively. The line OI cuts the side BC at D, and AD is perpendicular to BC. Prove that the circumradius of ABC is equal to the radius of the excircle of ABC opposite A.

Buffet Contest

Algebra

1. Determine all real numbers p such that the three roots of

$$5x^3 - 5(p+1)x^2 + (71p-1)x - (66p-1) = 0$$

are positive integers.

Solution:

The given equation may be factored as $(x-1)(5x^2-5px+66p-1)=0$. Hence one of its roots is 1. Let the other two be $u \le v$. Then $u+v=p$ and $uv = \frac{66p-1}{5}$. Hence $25uv = 330(u+v)-5$ so that $(5u-66)(5v-66) = 4351 = 19 \cdot 229$. Since u and v are positive integers, $5u-66=19$ and $5v-66=229$. Hence $u=17$, $v=59$ and $p=u+v=76$.

2. Let x, y and z be real numbers such that $x \ge y \ge z \ge \frac{\pi}{12}$ and $x+y+z=\frac{\pi}{2}$.

   (a) Determine the maximum value of $\cos x \sin y \cos z$.

   (b) Determine the minimum value of $\cos x \sin y \cos z$.

Solution:

(a) Note that $\cos^2 \frac{\pi}{12} = \frac{1}{2}(1+\cos\frac{\pi}{6}) = \frac{2+\sqrt{3}}{4}$. Hence

$$\begin{aligned}
\cos x \sin y \cos z &= \frac{1}{2}(\sin(x+y)-\sin(x-y))\cos z \\
&\le \frac{1}{2}\sin(x+y)\cos z \\
&= \frac{1}{2}\cos^2 z \\
&\le \frac{1}{2}\cos^2\frac{\pi}{12} \\
&= \frac{2+\sqrt{3}}{8}.
\end{aligned}$$

This is achieved by $\cos\frac{5\pi}{24}\sin\frac{5\pi}{24}\cos\frac{\pi}{12} = \frac{1}{2}\sin\frac{5\pi}{12}\cos\frac{\pi}{12} = \frac{1}{2}\cos^2\frac{\pi}{12} = \frac{2+\sqrt{3}}{8}$.

(b) Note that $x = \frac{\pi}{2}-(y+z) \le \frac{\pi}{2} - (\frac{\pi}{12}+\frac{\pi}{12}) = \frac{\pi}{3}$. Hence

$$\begin{aligned}
\cos x \sin y \cos z &= \frac{1}{2}\cos x(\sin(y+z)+\sin(y-z)) \\
&\ge \frac{1}{2}\cos x \sin(y+z) \\
&= \frac{1}{2}\cos^2 x \\
&\ge \frac{1}{2}\cos^2\frac{\pi}{3} \\
&= \frac{1}{8}.
\end{aligned}$$

This is achieved by $\cos\frac{\pi}{3}\sin\frac{\pi}{12}\cos\frac{\pi}{12} = \frac{1}{4}\sin\frac{\pi}{6} = \frac{1}{8}$.

3. Determine the range of the real number θ such that for $0 \le x \le 1$,

$$x^2\cos\theta - x(1-x) + (1-x)^2\sin\theta > 0.$$

Solution:

Let

$$\begin{aligned} f(x) &= x^2\cos\theta - x(1-x) + (1-x)^2\sin\theta \\ &= (x\sqrt{\cos\theta} - (1-x)\sqrt{\sin\theta})^2 + x(1-x)(2\sqrt{\cos\theta\sin\theta} - 1). \end{aligned}$$

The equation $x\sqrt{\cos\theta} = (1-x)\sqrt{\sin\theta}$ has a root $c = \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}+\sqrt{\sin\theta}}$ in the open interval (0,1). Hence $2\sqrt{\cos\theta\sin\theta}-1 = \frac{f(c)}{c(1-c)} > 0$. On the other hand, if $2\sqrt{\cos\theta\sin\theta}-1 > 0$, then $f(x) > 0$ for $0 \le x \le 1$. This necessary and sufficient condition may be rewritten as $\sin 2\theta > \frac{1}{2}$. Now $\cos\theta = f(1) = 0$ and $\sin\theta = f(0) > 0$, so that $0 < \theta < \frac{\pi}{2}$ on the interval $[0, 2\pi]$. Hence $0 < 2\theta < \pi$. It follows that $\frac{\pi}{6} < 2\theta < \frac{5\pi}{6}$ so that $\frac{\pi}{12} < \theta < \frac{|5pi}{12}$. Thus the entire range is $2k\pi + \frac{\pi}{12} < \theta < 2k\pi + \frac{5\pi}{12}$ for each integrer k.

4. Determine the range of values of the real number a such that for any real numbers x and θ where $0 \le \theta \le \frac{\pi}{2}$, $(x+3+2\sin\theta\cos\theta)^2 + (x + a\sin\theta + a\cos\theta)^2 \ge \frac{1}{8}$.

Solution:

The inequality $(x+A)^2 + (x+B)^2 \ge \frac{1}{8}$ may be rewritten as $x^2 + (A+B)x + \frac{8A^2+8B^2-1}{16} \ge 0$. This holds for all real number x if and only if $(A+B)^2 - \frac{8A^2+8B^2-1}{4} \le 0$, which is equivalent to $(A-B)^2 \ge \frac{1}{4}$. Hence it is sufficient to deal with $(3+2\sin\theta\cos\theta - a\sin\theta - a\cos\theta)^2 \ge \frac{1}{4}$. Consider first the case $3 + 2\sin\theta\cos\theta - a\sin\theta - a\cos\theta \ge \frac{1}{2}$. Then

$$\begin{aligned} a &\le \frac{3+2\sin\theta\cos\theta - \frac{1}{2}}{\sin\theta+\cos\theta} \\ &= \frac{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta + \frac{3}{2}}{\sin\theta+\cos\theta} \\ &= (\sin\theta+\cos\theta) + \frac{3}{2}\cdot\frac{1}{\sin\theta+\cos\theta}. \end{aligned}$$

By the Arithmetic-Geometric-Means Inequality, the minimum value of the last expression is $2\sqrt{\frac{3}{2}} = \sqrt{6}$. Hence $a \le \sqrt{6}$. Consider now the case $3 + 2\sin\theta\cos\theta - a\sin\theta - a\cos\theta \le -\frac{1}{2}$. Then $a \ge (\sin\theta+\cos\theta) + \frac{5}{2}\cdot\frac{1}{\sin\theta+\cos\theta}$. The minimum value of the function $f(x) = x + \frac{5}{2}\cdot\frac{1}{x}$ occurs at $x = \sqrt{\frac{5}{2}}$. Now $\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta) = \sin(\theta+\frac{\pi}{4})$. For $0 \le \theta \le \frac{\pi}{2}$, $\frac{1}{\sqrt{2}} \le \sin(\theta+\frac{\pi}{4}) \le 1$. Hence $1 \le \sin\theta+\cos\theta \le \sqrt{2}$. It follows that the maximum value of $(\sin\theta+\cos\theta)+\frac{5}{2}\cdot\frac{1}{\sin\theta+\cos\theta}$ occurs at $\sin\theta+\cos\theta = 1$, so that $a \ge \frac{7}{2}$. In summary, either $a \le \sqrt{6}$ or $a \ge \frac{7}{2}$.

5. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be real numbers in [1,2] such that $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2$.

(a) Prove that $\displaystyle\sum_{i=1}^{n}\frac{a_i^3}{b_i} \le \frac{17}{10}\sum_{i=1}^{n} a_i^2$.

(b) Determine a necessary and sufficient condition for equality to hold.

Solution:

(a) Note that $\frac{1}{2} \le \sqrt{\frac{a_i^3}{b_i}} \cdot \frac{1}{\sqrt{a_i b_i}} = \frac{a_i}{b_i} \le 2$. Hence $\frac{\sqrt{a_i b_i}}{2} \le \sqrt{\frac{a_i^3}{b_i}} \le 2\sqrt{a_i b_i}$. Hence

$$\begin{aligned} 0 &\ge \left(\frac{\sqrt{a_i b_i}}{2} - \sqrt{\frac{a_i^3}{b_i}}\right)\left(2\sqrt{a_i b_i} - \sqrt{\frac{a_i^3}{b_i}}\right) \\ &= a_i b_i + \frac{a_i^3}{b_i} - \frac{5}{2}a_i^2. \end{aligned}$$

Similarly, $\frac{b_i}{2} \le a_i \le 2b_i$ leads to $0 \ge b_i^2 + a_i^2 - \frac{5}{2}a_i b_i$. It follows that

$$\begin{aligned} \sum_{i=1}^n \frac{a_i^3}{b_i} &\le \frac{5}{2}\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i b_i \\ &\le \frac{5}{2}\sum_{i=1}^n a_i^2 - \frac{2}{5}\sum_{i=1}^n (a_i^2 + b_i^2) \\ &= \frac{17}{10}\sum_{i=1}^n a_i^2. \end{aligned}$$

(b) For equality to hold, we must have $\frac{a_i}{b_i} = \frac{1}{2}$ or 2 for every i. This means that one of a_i and b_1 is 1 and the other is 2. Since $\sum_{i=1}^n a_i^2 = \sum_{i=1}^n b_i^2$, n must be even and $a_i = 1$ occurs the same number of times as $a_i = 2$.

6. Let n be a positive integer and M be a positive real number. Among all arithmetic progressions $\{a_k\}$ satisfying $a_1^2 + a_{n+1}^2 \le M$, determine the maximum value of $a_{n+1} + a_{n+2} + \cdots + a_{2n+1}$.

Solution:

Let $a_{n+1} = x$ and let the common difference of the arithmetic progression be d. Then we have $S = a_{n+1} + a_{n+2} + \cdots + a_{2n+1} = (n+1)x + \frac{n(n+1)}{2}d$ so that $\frac{S}{n+1} = x + \frac{nd}{2}$. We wish to express $a_1^2 + a_{n+1}^2$ in the form $p(\frac{s}{n+1})^2 + qT^2$. Now

$$(x - nd)^2 + x^2 = p\left(x + \frac{nd}{2}\right)^2 + (2-p)x^2 - (2-p)ndx + \left(1 - \frac{p}{4}\right)n^2d^2.$$

From $(2+p)^2 = 4(2-p)(1-\frac{p}{4})$, we have $p = \frac{2}{5}$ and $\frac{8}{5}x^2 - \frac{12}{5}ndx + \frac{9}{10}n^2d^2 = \frac{1}{10}(4x - 3nd)^2$. It follows that $M \ge \frac{2}{5}(\frac{S}{n+1})^2$ so that $S \le \sqrt{\frac{5M}{2}}(n+1)$. To show that this maximum value can be achieved, choose $x = 3k$ and $d = \frac{4k}{n}$ so that $4x - 3nd = 0$, with k to be chosen later. Then $S = 3k(n+1) + \frac{4k}{n} \cdot \frac{n(n+1)}{2} = 5k(n+1)$. Hence $a_1^2 + a_{n+1}^2 = \frac{2}{5}(\frac{5k(n+1)}{n+1})^2 = 10k^2$. Setting this equal to M, we have $k = \sqrt{\frac{M}{10}}$. It follows that if $x = 3\sqrt{\frac{M}{10}}$ and $d = \frac{4}{n}\sqrt{\frac{M}{10}}$, then $S = \sqrt{\frac{5M}{2}}(n+1)$.

7. For any positive integer n, let $S_n = 1+2+\cdots+n$. Determine the maximum value of $\frac{S_n}{(n+32)S_{n+1}}$.

Solution:

We have $\frac{(n+32)S_{n+1}}{S_n} = \frac{(n+32)(n+2)}{n} = n+34+\frac{64}{n} \ge 34+2\sqrt{64} = 50$ by the Arithmetic-Geometric-Means Inequality. Hence $\frac{S_n}{(n+32)S_{n+1}} \le \frac{1}{50}$. For $n = 8$, we have $\frac{8}{(8+32)(8+2)} = \frac{1}{50}$. Hence this is indeed the maximum value.

8. For any real number $a < 0$, there exists a largest positive real number $\ell(a)$ such that for all real numbers x satisfying $0 \le x \le \ell(a)$, we have $|ax^2 + 8x + 3| \le 5$.

   (a) Determine the maximum value of $\ell(a)$.

   (b) Determine a for which $\ell(a)$ is maximum.

   Solution:

   Let $f(x) = ax^2 + 8x + 3 = a(x + \frac{4}{a})^2 + 3 - \frac{16}{a}$. It is an upsidedown parabola with maximum $3 - \frac{16}{a}$ occurring at $x = -\frac{4}{a}$. Note that $f(0) = 3$ lies between -5 and 5. Suppose $3 - \frac{16}{a} > 5$, so that $-8 < a < 0$. Then $|f(x)| < 5$ until $f(x) = 5$ for the first time after $x = 0$. Hence $\ell(a)$ is the smaller root of the equation $ax^2 + 8x + 3 = 5$, and we have

$$\ell(a) = \frac{-8 + \sqrt{64 + 8a}}{2a} = \frac{2}{\sqrt{16 + 2a} + 4} < \frac{1}{2}.$$

   Suppose $3 - \frac{16}{a} \le 5$, so that $a \le -8$. Then $|f(x)| < 5$ until $f(x) = -5$ for the first time after $x = 0$. Hence $\ell(a)$ is the larger root of the equation $ax^2 + 8x + 3 = -5$, and we have

$$\ell(a) = \frac{-8 - \sqrt{64 - 32a}}{2a} = \frac{4}{\sqrt{4 - 2a} - 2} \le \frac{4}{\sqrt{4 - 2(-8)} - 2} = \frac{\sqrt{5} + 1}{2}.$$

   (a) Since $\frac{\sqrt{5}+1}{2} > \frac{1}{2}$, the maximum value of $\ell(a)$ is $\frac{\sqrt{5}+1}{2}$.

   (b) The maximum value of $\ell(a)$ occurs at $a = -8$.

9. Determine the real numbers $a \le b$ such that for $a \le x \le b$, the minimum value of $\frac{1}{2}(13 - x^2)$ is $2a$ and the maximum value is $2b$.

   Solution:

   Let $f(x) = \frac{1}{2}(13 - x^2)$. We consider three case. Suppose $0 \le a < b$. Then $f(x)$ is decreasing. Hence its maximum value is $f(a) = 2b$ and its minimum value is $f(b) = 2a$. Subtracting $\frac{1}{2}(13 - a^2) = 2b$ from $\frac{1}{2}(13 - b^2) = 2a$, we have $\frac{1}{2}(a - b)(a + b) = 2(a - b)$ so that $a + b = 4$. From $\frac{1}{2}(13 - a^2) = 2(4 - a)$, we have $0 = a^2 - 4a + 3 = (a - 1)(a - 3)$. Hence $a = 1$ and $b = 3$. Suppose $a < 0 < b$. Then $f(x)$ is increasing on $(a, 0)$ and decreasing on $(0, b)$. Hence its maximum value is $f(0) = \frac{13}{2} = 2b$, so that $b = \frac{13}{4}$. Its minimum value is either $f(b) = 2a$ or $f(a) = 2a$. However, $f(\frac{13}{4}) = \frac{1}{2}(13 - \frac{169}{16}) = \frac{39}{32} \ne 2a$. Thus it is $2a = f(a) = \frac{1}{2}(13 - a^2)$ so that $a^2 + 4a - 13 = 0$. Hence $a = -2 - \sqrt{7}$, the positive root $-2 + \sqrt{7}$ being rejected. Suppose $a < b \le 0$. Then $f(x)$ is increasing. Hence its minimum value is $2a = f(a) = \frac{1}{2}(13 - a^2)$, so that $a^2 + 4a + 13 = 0$. There are no real roots. In summary, the desired values are $(a, b) = (1, 3)$ or $(-2 - \sqrt{7}, \frac{13}{4})$.

10. For positive integers a and n, let $a = qn + r$ where q and r are non-negative integers with $r < n$. Define $F_n(a) = q + r$. Determine the largest positive integer A such that for all positive integers $a \le A$, $F_{n_6}(F_{n_5}(F_{n_4}(F_{n_3}(F_{n_2}(F_{n_1}(a)))))) = 1$ for some positive integers n_1, n_2, n_3, n_4, n_5 and n_6.

    Solution:

    Let x_k be the largest integer such that there exist positive integers $n_1, n_2, \ldots, n_k$ for which $F_{n_k}(F_{n_{k-1}}(\cdots(F_{n_1}(a))\cdots)) = 1$ for all positive integers $a \le x_k$. Then $A = x_6$. We shall determine x_k recursively and show that it is always even. For $k=1$, choose $n_1=2$. Then $F_2(1)=F_2(2)=1$ while $F_2(3)=2$, so that $x_1\geq 2$. On the other hand, $F_n(2)=2$ for $n\neq 2$. Hence $x_1=2$ and it is indeed even. Suppose x_k has been determined and is even. Then x_{k+1} is the largest integer such that there exists a positive integer n for which $F_n(a)\leq x_k$ for all positive integers $a\leq x_{k+1}$. Let $x_{k+1}=qn+r$ where $0\leq r\leq n-1$. Then $F_n(x_{k+1})=q+r\leq x_k$ so that $x_{k+1}\leq x_k+q(n-1)$. We also have $F_n(qn-1)=(q-1)+(n-1)\leq x_k$ so that

$$q(n-1)\leq\left\lfloor\left(\frac{q+n-1}{2}\right)^2\right\rfloor\leq\left\lfloor\left(\frac{x_k+1}{2}\right)^2\right\rfloor=\left\lfloor\frac{x_k(x_k+2)}{4}+\frac{1}{4}\right\rfloor.$$

Since x_k is even, we have $q(n-1)\leq\frac{x_k(x_k+2)}{4}$ so that $x_{k+1}\leq x_k+q(n-1)\leq\frac{x_k(x_k+6)}{4}$. Now take $n=\frac{x_k+4}{2}$. Then $\frac{x_k(x_k+6)}{4}=\frac{x_K}{2}n+\frac{x_k}{2}$. If $a\leq\frac{x_k(x_k+6)}{4}$, write $a=qn+r$ with $0\leq r\leq n-1$. If $q=\frac{x_k}{2}$, then $r\leq\frac{x_k}{2}$ so that $f_n(a)=q+r\leq x_k$. If $q\leq\frac{x_k}{2}-1$, then $r\leq n-1=\frac{x_k}{2}+1$ so that $F_n(a)=q+r\leq x_k$. It follows that $x_{k+1}=\frac{x_k(x_k+6)}{4}$. Since one of x_k and x_k+6 is a multiple of 4 and the other is even, x_{k+1} is also even. From $x_1=2$, we have $x_2=4,\ x_3=10,\ x_4=40,\ x_5=460$ and $A=x_6=53590$.

Combinatorics

11. If all positive integers relatively prime to 105 are arranged in ascending order, determine the 1000-th term.

Solution:

The number of positive integers under 105 that are relatively prime to $105 = 3 \times 5 \times 7$ is $105(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{7}) = 48$. When 1000 is divided by 48, the quotient is 20 and the remainder is 40. The positive integers relatively prime to 105 start with 1, 2, 4, 8, 11, 13, 16 and 17, with 19 being the 9-th. Thus the 40-th is $105 - 19 = 86$, and the 1000-th is $105 \times 20 + 86 = 2186$.

12. The sequence $\{a_k\}$ is such that $a_1 + a_2 + \cdots + a_n = 2a_n - 1$ for all $n \geq 1$. The sequence $\{b_k\}$ is defined by $b_1 = 3$ and for $k \geq 1$, $b_{k+1} = a_k + b_k$. Determine $b_1 + b_2 + \cdots + b_n$.

Solution:

From $a_1 = 2a_1 - 1$, we have $a_1 = 1$. From $a_n = (2a_n - 1) - (2a_{n-1} - 1)$, we have $a_n = 2a_{n-1}$ so that $a_n = 2^{n-1}$. Summing $b_{k+1} = a_k + b_k$ from $k = 1$ to n, we have $b_{n+1} = 2a_n - 1 + b_1 = 2^n + 2$. Hence $b_1 + b_2 + \cdots + b_n = 2^n - 1 + 2n$.

13. The sequence $\{a_n\}$ of real numbers is defined by $a_0 = a_1 = 1$ and for $n \geq 2$,

$$\sqrt{a_n a_{n-2}} - \sqrt{a_{n-1}a_{n-2}} = 2a_{n-1}.$$

Find a formula for a_n independent of $a_0, a_1, \ldots, a_{n-1}$.

Solution:

Rewrite the recurrence relation as $\sqrt{a_n a_{n-2}} - 2a_{n-1} = \sqrt{a_{n-1}a_{n-2}}$. Dividing by the right side, we have $b_n - 2b_{n-1} = 1$ where $b_n = \sqrt{\frac{a_n}{a_{n-1}}}$. In particular, $b_1 = \sqrt{\frac{a_1}{a_0}} = 1$ and $b_2 = 2b_0 + 1 = 3$. Now $b_{n-1} - 2b_{n-2} = 1$, and subtraction yields $b_n - 3b_{n-1} + 2b_{n-2} = 0$. The characteristic equation is $x^2 - 3x + 2 = (x-1)(x-2) = 0$ with characteristic roots 1 and 2. Hence $b_n = c_1 + c_2 2^n$. From $1 = b_1 = c_1 + 2c_2$ and $3 = b_2 = c_1 + 4c_2$, we have $c_1 = -1$ and $c_2 = 1$ so that $b_n = 2^n - 1$. Finally, for $n \geq 1$, we have

$$\begin{aligned} a_n &= a_{n-1}(2^n - 1)^2 \\ &= a_{n-2}(2^n - 1)^2(2^{n-1} - 1)^2 \\ &= \cdots \\ &= a_0(2^n - 1)^2(2^{n-1} - 1)^2 \cdots (2^2 - 1)^2(2 - 1)^2. \end{aligned}$$

Hence $a_0 = 1$ and $a_n = (2^n - 1)^2(2^{n-1} - 1)^2 \cdots (2^2 - 1)^2(2 - 1)^2$ for all $n \geq 1$.

14. The sequences $\{a_n\}$ and $\{b_n\}$ are defined by $a_0 = 1$, $b_0 = 0$, $a_{n+1} = 7a_n + 6b_n - 3$ and $b_{n+1} = 8a_n + 7b_n - 4$ for $n \geq 0$. Prove that a_n is the square of an integer for all $n \geq 0$.

Solution:

We have $a_1 = 7a_0 + 6b_0 - 3 = 4$, $b_1 = 8a_0 + 7b_0 - 4 = 4$ and $a_2 = 7a_1 + 6b_1 - 3 = 49$. Substituting $b_n = \frac{1}{6}(a_{n+1} - 7a_n + 3)$ into the other recurrence relation, we have $a_{n+2} - 14a_{n+1} + a_n = 6$. Subtracting from this $a_{n+1} - 14a_n + a_{n-1} = 6$, we have $a_{n+1} - 15a_{n+1} + 15a_n - a_{n-1} = 0$. The characteristic equation is $x^3 - 15x^2 + 15x - 1 = (x-1)(x^2 - 14x + 1) = 0$ with characteristic roots 1 and $7 \pm 4\sqrt{3}$. Hence $a_n = c_1 + c_2(7+4\sqrt{3})^n + c_3(7-4\sqrt{3})^n$. Using the initial values, we have

$$\begin{aligned}
1 &= c_1 + c_2 + c_3, && (1)\\
4 &= c_1 + c_2(7-4\sqrt{3}) + c_3(7-4\sqrt{3}), && (2)\\
49 &= c_1 + c_2(97+56\sqrt{3}) + c_3(97-56\sqrt{3}). && (3)
\end{aligned}$$

Subtracting (1) from each of (2) and (3), we obtain

$$\begin{aligned}
3 &= c_2(6+4\sqrt{3}) + c_3(6-4\sqrt{3}), && (4)\\
6 &= c_2(12+7\sqrt{3}) + c_3(12-7\sqrt{3}). && (5)
\end{aligned}$$

Subtracting $6+4\sqrt{3}$ times (5) from $12+7\sqrt{3}$ times (4), we have $-3\sqrt{3} = -12\sqrt{3}c_3$ so that $c_3 = \frac{1}{4}$. From (4), we have $c_2 = \frac{1}{4}$ and from (1), we have $c_1 = \frac{1}{2}$. It follows that

$$\begin{aligned}
a_n &= \frac{1}{2} + \frac{1}{4}(7+4\sqrt{3})^n + \frac{1}{4}(7-4\sqrt{3})^n\\
&= \frac{1}{4}(2+\sqrt{3})^{2n} + \frac{1}{2} + \frac{1}{4}(2-\sqrt{3})^{2n}\\
&= \frac{1}{4}((2+\sqrt{3})^n + (2-\sqrt{3})^n)^2.
\end{aligned}$$

Expanding the binomials inside the bracket in the last expression, all the irrational terms cancel out and the final value is an even integer. Hence a_n is the square of an integer for all $n \geq 0$.

15. None of the subsets A_1, A_2, $\ldots$, A_m of an n-element set contains another. Prove that

(a) $\sum\limits_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \leq 1$; (b) $\sum\limits_{i=1}^{m} \binom{n}{|A_i|} \geq m^2$.

Solution:

Note that $\frac{1}{\binom{n}{|A_i|}} = \frac{|A_i|!(n-|A_i|)!}{n!}$ so that (a) is equivalent to $\sum\limits_{i-1}^{m} |A_i|!(n-|A_i|)! \leq n!$. There are $n!$ permutations of the n elements. The desired result will follow if we can prove that A_i accounts for $|A_i|!(n-|A_i|)!$ of them, without overlap. 1 The $|A_i|$ elements in A_i are listed first, and they can be permuted among themselves in $|A_i|!$ ways. They are followed by the elements not in A_i, which can be permuted among themselves in $(n-|A_i|)!$ ways. Suppose the same permutation is generated by A_i and A_j. We may assume that $|A_i| \leq |A_j|$. Then A_i must be a subset of A_j, which contradicts the hypothesis. This establishes (a), and (b) may be derived from Cauchy's Inequality as follows:

$$\begin{aligned}
m^2 &= \left(\sum_{i=1}^{m} \frac{1}{\sqrt{\binom{n}{|A_i|}}}\sqrt{\binom{n}{|A_i|}}\right)^2\\
&\leq \left(\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}}\right)\left(\sum_{i=1}^{m} \binom{n}{|A_i|}\right)\\
&\leq \sum_{i=1}^{m} \binom{n}{|A_i|}.
\end{aligned}$$

16. Each entry of a 100×25 array is a non-negative real number such that the sum of the 25 numbers in each row is at most 1. The 100 numbers in each column are rearranged from top to bottom in descending order. Determine the smallest value of k such that the sum of the 25 numbers in each row from the k-th row on down will always be at most 1.

Solution:

We first construct an example to show that $k \geq 97$. Partition the array into four 25×25 subarrays. In each, set 25 entries equal to 0, with no two in the same row or the same column. Set the remaining entries equal to $\frac{1}{24}$. Then the sum of each row of the whole array is exactly 1. After rearrangement, the one hundred 0s sink to the bottom 4 rows. The sum of the entries in each of the first 96 rows is $\frac{25}{24} > 1$. It follows that $k \geq 97$. We now prove that $k = 97$. Let the entries of the 97-th row after rearrangement be x_j, $1 \leq j \leq 25$. We wish to show that their sum is at most 1. Now the bottom 3 rows after rearrangement contain among them entries from at most 75 rows before. Hence there is an original row such that each entry y_j, $1 \leq j \leq 25$, is in the top 97 rows after rearrangement. Then $x_j \leq y_j$ for $1 \leq j \leq 25$, so that $x_1 + x_2 + \cdots + x_{25} \leq y_1 + y_2 + \cdots + y_{25} \leq 1$.

17. There are $n \geq 6$ people at a party. Each is a mutual acquaintant of at least $\lfloor \frac{n}{2} \rfloor$ others. Among any $\lfloor \frac{n}{2} \rfloor$ of them, either two of them are mutual acquaintants, or two of the remaining $n - \lfloor \frac{n}{2} \rfloor$ are mutual acquaintants. Prove that there are three people at the party such that every two of them are mutual acquaintants.

Solution:

Let a and b be two mutual acquaintants. Let A be the set of mutual acquaintants of a and B be that of b. If there exists $c \in A \cap B$, then a, b and c form a desired trio. Hence we may assume that $A \cap B = \emptyset$. Since $|A| \geq \lfloor \frac{n}{2} \rfloor$ and $|B| \geq \lfloor \frac{n}{2} \rfloor$, we may assume that $|A| = \lfloor \frac{n}{2} \rfloor$ and either $|B| = n - |A|$ or $|B| = n - |A| - 1 = \lfloor \frac{n}{2} \rfloor$. In the former case, suppose there exist two mutual acquaintants in A. Then they form a desired trio with a. If no two mutual acquaintants exist in A, then we must have two in B. They form a desired trio with b. In the latter case, there exists $c \notin A \cup B$. If we still have two mutual acquaintants in A, or two in B, we can conclude as before. Suppose they do not exist. Then we must have two mutual acquaintants in $A \cup \{c\}$ as well as in $B \cup \{c\}$. This means that c is a mutual acquaintant with some $a_1 \in A$ and some $b_1 \in B$. Since $n \geq 6$, c has at least 3 mutual acquaintants. We may assume that a third one is $a_2 \in A$. Now b_1 is not a mutual acquaintant with anyone in B. Hence b_1 must be a mutual acquaintant with all but one of the others, which must include either a_1 or a_2. By symmetry, we may assume that it is a_1. Then a_1, b_1 and c form a desired trio.

18. Any two of n friends have a phone conversation at most once. Among any $n - 2$ of them, the total number of phone conversations is a constant positive power of 3. Determine all values of n for which this is possible.

Solution:

Clearly, we must have $n \geq 5$. If $n = 5$ and every two friends have a phone conversation exactly once, then the total number of phone converxations among any 3 of them is 3. Hence the scenario is possible with $n = 5$. We now prove that this is the only possible value. For $1 \leq i, j \leq n$, let $c_{i,j}$ be the number of phone conversations between the i-th and the j-th friend. Then $c_{i,i} = 0$ and $c_{i,j} = 0$ or 1. Let $t_i = c_{i,1} + c + i, 2 + \cdots + c_{i,n}$ for $1 \leq i \leq n$, and we may assume that $t_1 \geq t_2 \geq \cdots \geq t_n$. The total number of phone conversations is

$T = \frac{1}{2}(t_1 + t_2 + \cdots + t_n)$. For any i and j, $T - t_i - t_j + c_{i,j} = 3^m$ for some constant $m \geq 1$. For $2 \leq k \leq n-1$, we have

$$\begin{aligned} t_1 - t_n &= (t_1 + t_k) - (t_n - t_k) \\ &= (T - 3^m + c_{1,k}) - (T - 3^m + c_{n,k}) \\ &= c_{1,k} - c_{n,k} \\ &\leq 1. \end{aligned}$$

If $t_1 - t_n = 1$, then $c_{1,k} = 1$ and $c_{n,k} = 0$ for $2 \leq k \leq n-1$. Hence $t_1 \geq n-2$ while $t_n \leq 1$. However, $t_1 - t_n \geq n - 2 - 1 \geq 2$, which is a contradiction. It follows that t_k has a constant value t for $1 \leq k \leq n$. From $c_{i,j} = T - 2t - 3^m$, $c_{i,j}$ also has a constant value c for $1 \leq i, j \leq n$, and we must have $c = 1$ so that $t = n - 1$ and $T = \frac{n(n-1)}{2}$. Now $1 = \frac{n(n-1)}{2} - 2(n-1) - 3^m$ simplifies to $(n-2)(n-3) = 2 \cdot 3^m$. Since $n-2$ and $n-3$ are relatively prime, one of them is equal to 2 and the other 3^m. If $n - 2 = 2$, then $n - 3 = 1$ and $m = 0$. This is not permitted. Hence $n - 3 = 2$, $n - 2 = 3$ and $m = 1$. This completes the proof that $n = 5$ is the only possible value.

19. Let n be a positive integer. We wish to construct a set of tokens, the weight of each being an integral number of grams, such that any object whose weight is an integral number of grams up to n can be balanced by a subset of the tokens. Some of the tokens may be put on the same pan on the balance as the object.

   (a) Determine in terms of n the minimum value of the number of tokens.

   (b) For what values of n is the minimal set of tokens unique?

Solution:

(a) Let the weights of the tokens be the positive integers $a_1, a_2, \ldots, a_k$ in non-decreasing order. Then the weight of an object that may be balanced is $displaystylew = \sum_{i=1}^{k} x_i a_i$, where $x_i = -1$, 0 or 1 for $1 \leq i \leq k$. The possible values of w must include the $2n+1$ integers from $-n$ to n. It follows that $2n + 1 \leq 3^k$ so that $n \leq \frac{3^k - 1}{2}$. Let m be such that $\frac{3^{m-1}-1}{2} < n \leq \frac{3^m - 1}{2}$. Then $k \geq m$. We claim that we may have $k = m$, that by taking $a_i = 3^{i-1}$ for $1 \leq i \leq m$, we can balance any object of weight $w \leq n$. Note that $w + \sum_{i=1}^{m} 3^{i-1} \leq 3^m - 1$. Let its base 3 representation be $\sum_{i=1}^{m} y_i 3^{i-1}$, where $y_i = 0$, 1 or 2. Then $w = \sum_{i=1}^{m} x_i 3^{i-1}$, where $x_i = -1$, 0 or 1.

(b) Let $\frac{3^{m-1}-1}{2} < n < \frac{3^m - 1}{2}$. We know that the set of tokens of weight 1, 3, $\ldots$, 3^{m-1} work. The heaviest token is only used if $w > \frac{3^{m-1}-1}{2}$, and the balancing can be achieved up to $\frac{3^m-1}{2}$. Hence if we replace this token by one of weight $3^{m-1} - 1$, we can still balance any weight up to $\frac{3^m - 1}{2} - 1$. It follows that the minimal set of tokens is not unique. Consider now the remaining case where $n = \frac{3^m - 1}{2}$. We claim that $a_i = 3^{i-1}$ for $1 \leq i \leq m$ is the only minimal set of tokens that works. Since $\sum_{i=1}^{m} x_i a_i$, where $x_i = -1$, 0 or 1, can represent any of the $2n+1$ integers w from $-n$ to n, and $2n + 1 = 3^m$, there are no duplicate representations. Now $0 \leq w + \sum_{i=1}^{m} 3^{i-1} \leq 3^m - 1$, and it has a unique representation $\sum_{i=1}^{m} y_i a_i$ where $y_i = 0$, 1 or 2. For $i = 1$, the smallest positive integer not yet represented is 1. Hence we must have $a_1 = 1$. Suppose $a_i = 3^{i-1}$ for $1 \le i \le j$. Since $\sum_{i=1}^{j} y_1 a_i = \sum_{i=1}^{j} y_i 3^{i-1}$ are the base 3 representations of $0, 1, \ldots, 3^j - 1$, we must have $a_{j+1} = 3^j$. The desired conclusion follows from mathematical induction.

20. On the plane are 1994 points, no three collinear. They are to be partitioned into 83 sets, each with at least three points. Three points in the same set form a triangle.

 (a) What is the minimum number of triangles?

 (b) In a partition which yields the minimum number of triangles, prove that each segment joining two points in the same set can be painted in one of four colours such that no triangles have three sides with the same colour.

Solution:

(a) Suppose $m - n \ge 2$. We claim that $\binom{m}{3} + \binom{n}{3}$ is larger than $\binom{m-1}{3} + \binom{n+1}{3}$. Indeed, their difference is $\binom{m-1}{2} - \binom{n}{2} > 0$ since $m - 1 > n$. It follows that the 1994 points should be distributed among the 83 sets as evenly as possible. When 1994 is divided by 83, the quotient is 24 and the remainder is 2. Thus we should have 81 sets of size 24 and 2 sets of size 5, and the mimimum number of triangles is $81\binom{24}{3} + 2\binom{25}{3} = 168544$.

(b) All that is needed is to show that the task can be accomplished in a set with 25 points. The same method can then be applied to each of the other sets, suppressing an arbitrary point if there are only 24 points. Label the points (x, y) where $0 \le x, y \le 4$. For two points (x_1, y_1) and (x_2, y_2), if $x_1 - x_2 \equiv \pm 1 \pmod 5$, colour the segment joining them red. If $x_1 - x_2 \equiv \pm 2 \pmod 5$, colour the segment green. If $x_1 \equiv x_2 \pmod 5$, colour it yellow if $y_1 - y_2 \equiv \pm 1 \pmod 5$, and colour it blue if $y_1 - y_2 \equiv \pm 2 \pmod 5$. Consider any three points (x_1, y_1), (x_2, y_2) and (x_3, y_3). If they have different x-coordinates, then all three sides are red or green. However, $(x_1 - x_2) + (x_2 - x_3) + (x_3 - x)1 = 0$. Since no combination of three ± 1 and no combination of three ± 2 can be equal to 0, there is at least one red side and one green side. If exactly two x-coordinates are the same, the side joining these two points will be blue or yellow while the other two sides will be green or red. Finally, if they have the same x-coordinates, then all three sides rae yellow or blue, but there must be at least one of each colour since $(y_1 - y_2) + (y_2 - y_3) + (y_3 - y_1) = 0$.

Geometry

21. Only one interior angle of the convex quadrilateral $ABCD$ is obtuse. $ABCD$ is to be partitioned into n obtuse triangles whose vertices other than A, B, C and D are inside $ABCD$. Prove that this is possible if and only if $n \geq 4$.

    Solution:

    An obtuse triangle may be divided into n obtuse triangles for any $n \geq 1$. Let AB_0B_n be such that $\angle AB_0B_n > 90°$. Take points $B_1, B_2, \ldots, B_{n-1}$ on B_0B_n and join each of them to A. Then we have divided AB_0B_n into n obtuse triangles. A non-obtuse triangle cannot be divided into n obtuse triangles for $n \leq 2$. The case $n = 1$ is trivial. In the case $n = 2$, the cut must be from one vertex to the opposite side. There are no obtuse angles to begin with, and at most one can be created where the cut meets the opposite side. On the other hand, a non-obtuse triangle can be divided into 3 obtuse triangles. Let the angle at A be the largest in a non-obtuse triangle. Drop a perpendicular from A to BC until it reaches some point D inside the semicircle with diameter BC. Joining D to the vertices of ABC will result in 3 obtuse triangles. In the given quadrilateral, we can first cut off the obtuse triangle CAD. Then we cut the non-obtuse triangle CAB into 3 obtuse triangles. Thus $ABCD$ can be cut into 4 obtuse triangles. By cutting up one of the obtuse triangles in the manner described above, we can obtain n obtuse triangles for any $n \geq 4$, establishing sufficiency. We now establish necessity. If each side of $ABCD$ belongs to a different triangle, we already have $n \geq 4$. Hence some triangle must contain two adjacent sides of $ABCD$, implying that we cut along one of the diagonals. If we cut along AC, we have one obtuse triangle CAD, but it is not possible to cut the non-obtuse triangle CAB into 2 obtuse triangles. If the cut is along BD, at least one of triangles BAD and BCD is non-otbuse, and it cannot be cut into 2 obtuse triangles.

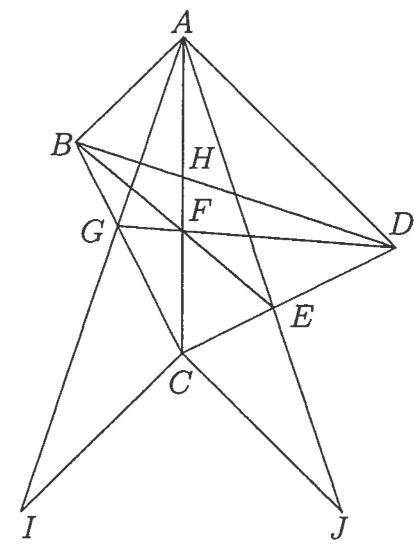
22. Each point of the plane is painted in one of two colours. Prove that there exist two similar triangles such that the lengths of the sides of one are 1995 times the lengths of the corresponding sides of the other, and all three vertices of each triangle have the same colour.

    Solution:

    We first prove that there is a monochromatic right triangle. Let P and Q be any two points of the same colour and let $PQRS$ be any rectangle. If either R or S has the same colour as P and Q, we have a monochromatic right triangle. Otherwise, any point on PS will form a monochromatic right triangle with either P and Q or R and S. So let (0,0), $(1995a, 0)$ and $(0, 1995b)$ be the vertices of a monochromatic right triangle T. Expand it into a rectangle R by adding $(1995a, 1995b)$ as the fourth vertex, and divide it into 1995^2 small rectangles of equal sizes, all similar to R. If any of them have three vertices of the same colour, then we have a monochromatic right triangle similar to T and having side lengths $\frac{1}{1995}$ the side lengths of T. Suppose this is not the case. Then $(i, 0)$, $0 \leq i \leq 1995$, cannot be alternating in colour as otherwise (0,0) and $(1995a, 0)$ will have different colours. Hence there exists a value $i, 0 \leq i \leq 1994$, such that $(ia, 0)$ and $((i+1)a, 0)$ have the same colour. Then (ia, b) and $((i+1)a, b)$ must both be of the other colour. It follows that (ia, kb) and $((i+1)a, kb)$ have the same colour for $0 \leq k \leq 1995$. Similarly, there exists a value j, $0 \leq j \leq 1994$, such that $(0, jb)$ and $(0, (j+1)b)$ have the same colour, so that (ka, jb) and $(ka, (j+1)b)$ have the same colour for $0 \leq k \leq 1995$. This is a contradiction since (ia, jb), $(ia, (j+1)b)$, $((i+1)a, jb)$ and $((i+1)a, (j+1)b)$ would all have the same colour.

23. The diagonal AC of the quadrilateral $ABCD$ bisects $\angle BAD$. E is a point on the side CD. BE cuts AC at F, and the line DF cuts BC at G. Prove that $\angle CAE = \angle CAG$.

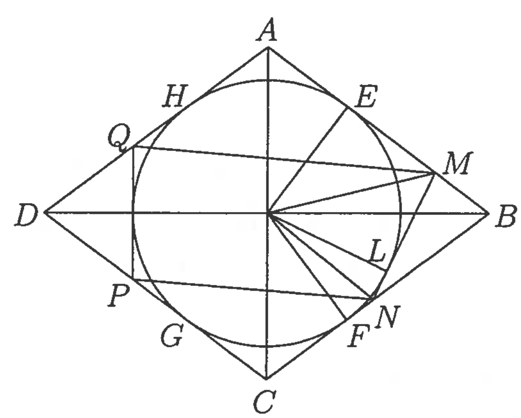
Solution:



Let BD intersect AC at H. Applyinig Ceva's Theorem to triangle BCD, $\frac{BH}{HD} \cdot \frac{DE}{EC} \cdot \frac{CG}{GB} = 1$. Since AH bisects $\angle BAD$, we have $\frac{BH}{HD} = \frac{BA}{AD}$ so that $\frac{BA \cdot CG}{GB} = \frac{AD \cdot EC}{DE}$. Draw a line through C parallel to AB, cutting the extension of AG at I. Since triangles BAG and CIG are similar, $CI = \frac{BA \cdot CG}{BG}$. Draw a line through C parallel to AD, cutting the extension of AE at J. As before, we have $CJ = \frac{AD \cdot EC}{DE}$. It follows that $CI = CJ$. Along with $AC = AC$ and $\angle ACI = 180° - \angle BAC = 180° - \angle DAC = \angle ACJ$, triangles ACI and ACJ are congruent. Hence $\angle CAG = \angle CAE$.

24. A circle is tangent to the sides AB, BC, CD and DA of a rhombus $ABCD$ at E, F, G and H respectively. M, N, P and Q are points on AB, BC, CD and DA respectively such that MN is tangent to the arc EF and PQ is tangent to the arc GH. Prove that MQ is parallel to NP.

Solution:



Let O be the centre of the circle and let L the point of tangency of the circle with MN. Since $\angle AOE = \angle COF$, $\angle EOM = \angle MOL$ and $\angle LON = \angle FON$, $2\angle AOE + 2\angle EOM + 2\angle FON = 180°$. Hence $\angle FON = 90° - \angle AOE - \angle EOM = \angle BOM$. It follows that

$$\angle AMO = \angle ABO + \angle BOM = \angle COF + \angle FON = \angle CON.$$

Along with $\angle MAO = \angle OCN$, triangles MAO and OCN are similar, so that $\frac{AM}{AO} = \frac{CO}{CN}$ or $AM \cdot CN = AO \cdot CO$. In the same way, we can prove that $AQ \cdot CP = AO \cdot CO$, so that $\frac{AM}{AQ} = \frac{CP}{CN}$. Along with $\angle MAQ = \angle PCN$, triangles MAQ and PCN are similar. It follows that MQ is parallel to NP.

25. A line m passes through the centre of a circle. A, B and C are three points outside the circle and lying on a line ℓ perpendicular to m, with A farthest from m and C nearest to it. AP, BQ and CR are tangents to the circle.

(a) Prove that $AB \cdot CR + BC \cdot AP = CA \cdot BQ$ if ℓ is tangent to the circle.

(b) Prove that $AB \cdot CR + BC \cdot AP < CA \cdot BQ$ if ℓ intersects the circle in two points.

(c) Prove that $AB \cdot CR + BC \cdot AP > CA \cdot BQ$ if ℓ is disjoint from the circle.

Solution:

Denote by M the point of intersection of ℓ and m.

(a) If ℓ is tangent to the circle, then $AP = AM$, $BQ = BM$ and $CR = CM$. We have

$$\begin{aligned} AB \cdot CR + AP \cdot BC &= AB \cdot CM + (AC + CM)BC \\ &= (AB + BC)CM + AC \cdot BC \\ &= AC(CM + BC) \\ &= AC \cdot BQ. \end{aligned}$$

This result is known as Ptolemy's Theorem for collinear points.

(b) Suppose ℓ intersects the circle at D and E. Then $AP^2 = D \cdot AE$, $BQ^2 = BD \cdot BE$ and $CR^2 = CD \cdot CE$. Using Ptolemy's Theorem for collinear points, we have

$$\begin{aligned} & AC^2BQ^2 - (AB \cdot CR + BC \cdot AP)^2 \\ = \; & AC \cdot BD(AC \cdot DE) - AB^2CD \cdot CE - BC^2AD \cdot AE \\ & -2AB \cdot BC\sqrt{CD \cdot CE \cdot AD \cdot AE} \\ = \; & AC \cdot BD(AB \cdot CE + AE \cdot BC) - AB \cdot CE(AB \cdot CD) - BC \cdot AE(BC \cdot AD) \\ & -2AB \cdot BC\sqrt{CD \cdot CE \cdot AD \cdot AE} \\ = \; & AB \cdot CE(AC \cdot BD - AB \cdot CD) + AE \cdot BC(AC \cdot BD - BC \cdot AD) \\ & -2AB \cdot BC\sqrt{CD \cdot CE \cdot AD \cdot AE} \\ = \; & AB \cdot CE(AD \cdot BC) + AE \cdot BC(AB \cdot AD) - 2AB \cdot BC\sqrt{CD \cdot CE \cdot AD \cdot AE} \\ = \; & AB \cdot BC(AD \cdot CE - 2\sqrt{AD \cdot CE}\sqrt{CD \cdot AE} + CD \cdot AE) \\ = \; & AB \cdot BC(\sqrt{AD \cdot CE} - \sqrt{CD \cdot AE})^2 \\ \geq \; & 0. \end{aligned}$$

Since $AD \cdot CE = AC \cdot DE + CD \cdot AE > CD \cdot AE$, the inequality is actually strict. Hence $AC \cdot BQ > AB \cdot CR + BC \cdot AP$.

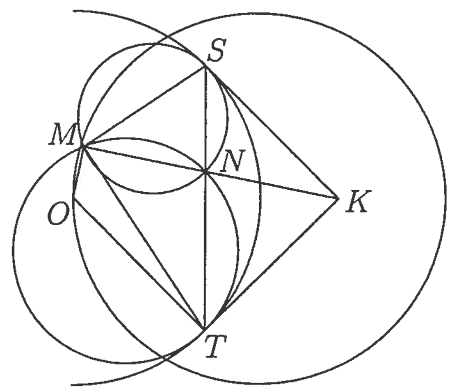
(c) Suppose ℓ is disjoint from the circle. Let MS be a tangent and let T be the point on m inside the circle such that $MT = MS$. Let O be the centre of the circle and r the radius. Then

$$AP^2 = OA^2 - r^2 = AM^2 + OM^2 - r^2 = AM^2 + MS^2 = AM^2 + MT^2 = AT^2.$$

Hence $AP = AT$. Similarly, $BQ = BT$ and $CR = CT$. Since T, A, B and C are neither collinear nor concyclic, Ptolemy's Inequality yields $AB \cdot CT + BC \cdot AT > AC \cdot BT$.

26. Two circles with unequal radii intersect at M and N. They are inside a circle with centre O and tangent to it at S and T. Prove that OM is perpendicular to MN if and only if S, N and T are collinear.

Solution:



If the common tangents at S and T are parallel, then S, O and T are collinear. It follows that N' S and T will not be collinear, and OM will not be perpendicular to MN if the two smaller circles are not of the same size. Hence we may assume that these two tangents meet at some point K. Then OK is a diameter of a circle passing through S and T. Moreover, we have $\angle SMT = \angle SMN + \angle NMT = \angle NSK + \angle NTK$. We claim that M, N and K are collinear. Otherwise, suppose the line KM intersects the first circle again at N_1 and the second circle again at N_2. Then $KM \cdot KN_1 = KS^2 = KT^2 = KM \cdot KN_2$, which implies that N_1 and N_2 coincide at N. Suppose S, N and T are collinear. Then

$$\angle SMT + \angle SKT = \angle NSK + \angle NTK + \angle SKT = 180°.$$

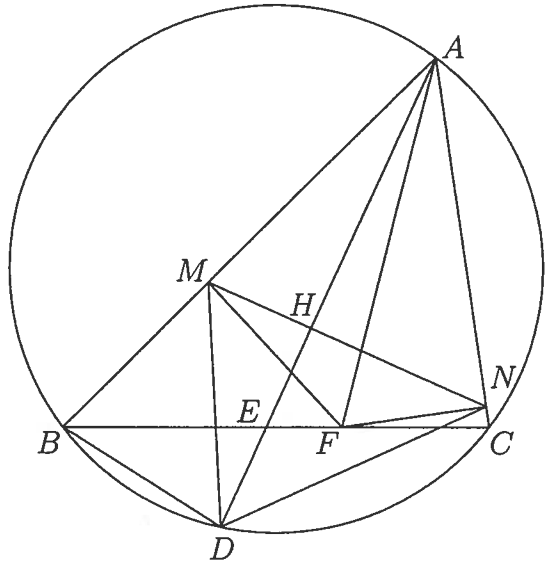
Hence M also lies on the circle with diameter OK, so that OM is perpendicular to MN. Conversely, suppose OM is perpendicular to MN. Then M also lies on the circle with diameter OK. Hence

$$\angle NSK + \angle NTK + \angle SKT = \angle SMT + \angle SKT = 180°,$$

so that S, N and T are collinear.

27. E and F are point on the sides BC of an acute triangle ABC, with B closer to E than to F, such that $\angle BAE = \angle CAF$. Perpendiculars FM and FN are dropped from F onto the sides AB and AC, respectively. The line AE intersects the circumcircle of ABC again at D. Prove that the area of ABC is equal to the area of the quadrilateral $AMDN$.

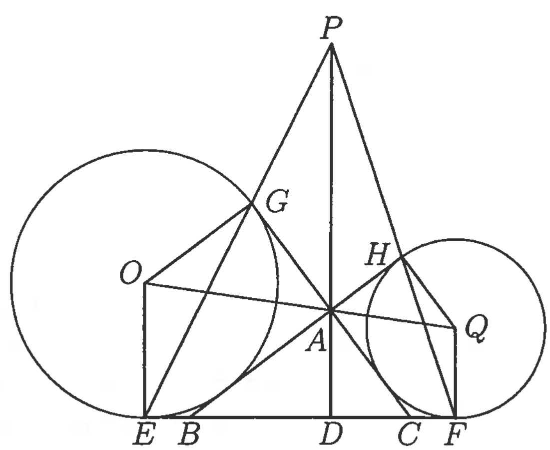
Solution:



Since $\angle AMF = 90° = \angle ANF$, $AMFN$ is cyclic so that $\angle AMN = \angle AFN$. Then we have $\angle AHM = 180° - \angle AMN - \angle MAE = 180° - \angle AFN - \angle FAN = \angle ANF = 90°$, where H is the point of intersection of AD and MN. Hence the area of $AMDN$ is equal to $\frac{1}{2}AD \cdot MN$. Now AF is a diameter of the circumcircle of triangle MAN. By the Extended Law of Sines, $MN = AF \sin CAB$. Since the area of ABC is equal to $\frac{1}{2}AB \cdot AC \sin CAB$, it remains to be proved that $AD \cdot AF = AB \cdot AC$. We have $\angle ABD = \angle ACF$ and $\angle BAD = \angle FAC$, so that triangles BAD and FAC are similar. The desired result follows from $\frac{AB}{AD} = \frac{AF}{AC}$.

28. The excircle of triangle ABC opposite C is tangent to the line BC at E and the line CA at G. The excircle of ABC opposite B is tangent to the line BC at F and the line AB at H. If the lines EG and FH intersect at P, prove that AP is perpendicular to BC.

Solution:



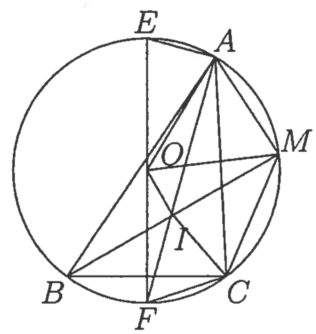
Let O be the excentre opposite C and Q be that opposite B, and let the extension of PA intersect BC at D. Note that $\angle PGA = 180° - \angle AGE = 180° - \angle GEC$. Applying the Law of the Sines to triangle PGA, we have $\frac{AG}{\sin APG} = \frac{AP}{\sin PGA} = \frac{AP}{\sin GEC}$. Applying this law to triangle PDE, we have $\frac{DE}{\sin APG} = \frac{PD}{\sin GEC}$. Hence $\frac{DE}{AG} = \frac{PD}{AP}$. In the same way, we can prove that $\frac{DF}{AH} = \frac{PD}{AP}$, so that $\frac{DE}{AG} = \frac{DF}{AH}$. Now triangles AGO and AHQ are similar, so that $\frac{AG}{AH} = \frac{AO}{AQ}$. Hence $\frac{DE}{DF} = \frac{AO}{AQ}$. Since OE and QF are parallel to each other, they are also parallel to AD. It follows that AP is perpendicular to BC.

29. Let O and I be the circumcentre and the incentre of triangle ABC respectively. If $\angle B = 60°$, $\angle A < \angle C$ and the exterior bisector of $\angle A$ meets the circumcircle again at E, prove that

   (a) $IO = AE$;

   (b) $2R < IO + IA + IC < (1+\sqrt{3})R$, where R is the circumradius of ABC.

   Solution:



Let $\alpha = \angle CAB$ and $\gamma = \angle BCA$. Then $\alpha + \gamma = 120°$. Extend AI and BI to cut the circumcircle again at F and N, respectively.

   (a) Since $\angle MOA = 2\angle MBA = 60°$ and $\angle MAO = \angle CAO + \angle CBM = \frac{\alpha}{2} + \frac{\gamma}{2} = 60°$, triangle MOA is equilateral. Hence $AM = R$. Since

$$\angle MIA = \angle MBA + \angle BAI = 30° + \frac{\alpha}{2} = \angle MAC + \angle CAI = \angle MAI,$$

   we have $MI = MA = R$. Now $\angle IMO = \angle AMB - \angle AMO = \gamma - 60°$. On the other hand, $\angle AEO = \angle ABF = 60° + \frac{\alpha}{2}$ so that $\angle AOE = 180° - 2(60° + \frac{\alpha}{2}) = \gamma - 60°$. It follows that triangles AOE and IMO are congruent, so that $AE = IO$.

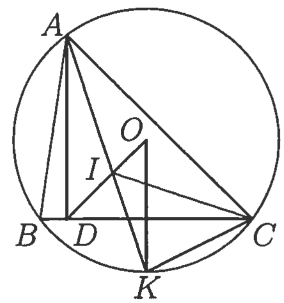
   (b) Since $\angle IFC = \angle ABC = 60°$ and $\angle ICF = \angle ICB + \angle BCF = \frac{\gamma}{2} + \frac{\alpha}{2} = 60°$, triangle ICF is equilateral. Hence $IO + IA + IC = AE + AF > EF = 2R$. On the other hand,

$$\begin{aligned} AE + AF &= 2R(\cos AFE + \sin AFE) \\ &= 2\sqrt{2}R\sin(45° + \frac{1}{2}\angle AOE) \\ &= 2\sqrt{2}R\sin(15° + \frac{\gamma}{2}) \\ &< 2\sqrt{2}R\sin 75°. \end{aligned}$$

   We have $\cos 15° = \sin(60° + 15°) = \frac{\sqrt{3}}{2}\cos 15° + \frac{1}{2}\sqrt{1-\cos^2 15°}$. This simplifies to $(2-\sqrt{3})^2\cos^2 15° = 1 - \cos^2 15°$, so that $\cos^2 15° = \frac{1}{8-4\sqrt{3}} = \frac{8+4\sqrt{3}}{16} = \frac{4+2\sqrt{3}}{8}$. Hence $\sin 75° = \cos 15° = \frac{1+\sqrt{3}}{2\sqrt{2}}$. It follows that $IO + IA + IC < (1+\sqrt{3})R$.

30. The circumcentre and incentre of triangle ABC are O and I respectively. The line OI cuts the side BC at D, and AD is perpendicular to BC. Prove that the circumradius of ABC is equal to the radius of the excircle of ABC opposite A.

Solution:



Let $BC = a$, $CA = b$, $AB = c$, R be the circumradius and r_a be the exradius opposite A. The area of triangle ABC is given by $\frac{1}{2}bc\sin A$ as well as $\frac{1}{2}r_a(b+c-a)$. It follows that $r_a = \frac{bc\sin A}{b+c-a} = \frac{2R\sin A\sin B\sin C}{\sin B+\sin C-\sin A}$. Now

$$\begin{aligned}
\sin B + \sin C - \sin A &= 2\sin\frac{B+C}{2}\cos\frac{B-C}{2} - 2\sin\frac{B+C}{2}\cos\frac{B+C}{2} \\
&= 2\sin\frac{B+C}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) \\
&= 4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.
\end{aligned}$$

Hence $r_a = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$. Let the extension of AI cut the circumcircle at K. Then K is the midpoint of the arc BC, and OK is perpendicular to BC. Hence triangles ADI and KOI are similar, so that $\frac{AI}{IK} = \frac{AD}{KO} = \frac{c\sin B}{R} = 2\sin B\sin C$. On the other hand, $\frac{AI}{IK} = \frac{[CAI]}{[CKI]}$ where $[T]$ denotes the area of triangle T. Then we have $[CAI] = \frac{1}{2}AC\cdot CI\sin\frac{C}{2}$. Note that $\angle KCI = \angle KAB + \angle ICB = \frac{A+C}{2} = \cos\frac{B}{2}$. Hence $[CKI] = \frac{1}{2}CK\cdot CI\cos\frac{B}{2}$. It follows that $2\sin B\sin C = \frac{AI}{IK} = \frac{AC}{CK}\frac{\sin\frac{C}{2}}{\cos\frac{B}{2}} = \frac{\sin B\sin\frac{C}{2}}{\sin\frac{A}{2}\cos\frac{B}{2}}$. This reduces to $4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 1$, so that $r_a = R$.