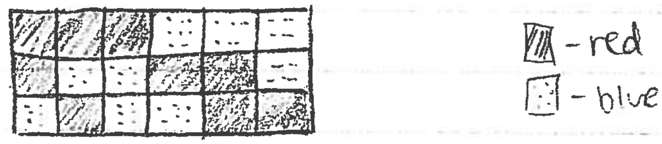
2003 Winter Camp

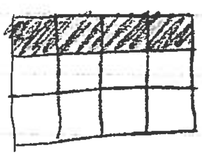
Individual Mock Olympiad Solutions

1. We prove that $n=7$. First, we note that the following colouring of a 3×6 chessboard yields no rectangle, all of whose corner squares are the same colour.

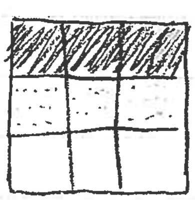


To finish the problem, we now prove that if every square of a 3×7 chessboard is coloured either red or blue, then there must exist a rectangle, all of whose corner squares are the same colour.

By the Pigeonhole Principle, some color is represented at least four times in the first row (i.e. of the seven squares in the first row, some color appears at least four times). Without loss, assume that this color is red (indicated by ▨). Now disregard three of the columns so that we are left with a 3×4 chessboard, where the first row is entirely red (see diagram).

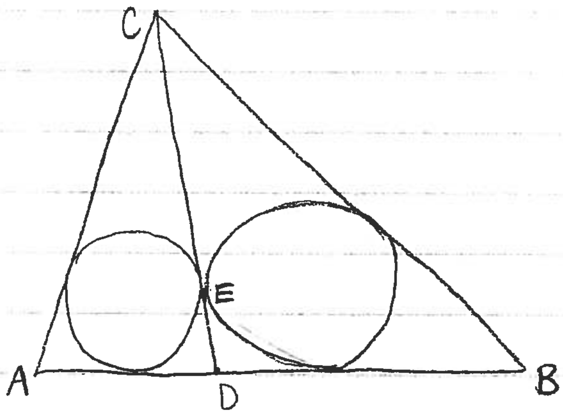


If there are 2 or more red squares in the second row, then we are done (we have a rectangle, all of whose corner squares are red). So suppose at most one square in the second row is red. Hence, there are at least three columns in our original chessboard where the top square is red and the middle square is blue. Consider the bottom squares of these three columns. By the Pigeonhole Principle, at least two of these bottom squares are the same colour, and this gives us our desired rectangle (combining them with the two red squares in the top row or the two blue squares in the middle).



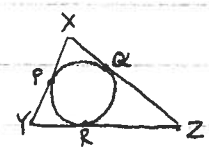
We conclude that $n=7$.

2.



We are given that the incircles of $\triangle ACD$ and $\triangle CDB$ touch each other on CD. Let E be this point.

<u>Lemma:</u>

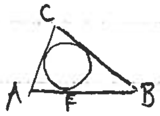


Let the incircle of $\triangle XYZ$ meet XY at P, as illustrated in the diagram. Then, $XP = \dfrac{XY + XZ - YZ}{2}$

<u>Proof:</u> Clearly $XP = XQ$ (equal tangents). Also, $YP = YR$ and $ZQ = ZR$.
So $XY + XZ - YZ = (XP + PY) + (XQ + QZ) - (YR + YZ) = 2XP$, which proves the lemma.

Thus, by our lemma, we have $DE = \dfrac{AD + CD - AC}{2}$ and $DE = \dfrac{BD + CD - BC}{2}$.
From these two equations, we have $AD - AC = BD - BC$, or $\underline{AD - BD = AC - BC}$.

Let the incircle of $\triangle ABC$ touch AB at F. Then by our lemma, we have $AF = \dfrac{AB + AC - BC}{2}$, and $BF = \dfrac{AB + BC - AC}{2}$



$\therefore \underline{AF - BF = AC - BC}$.

Hence, D and F are points on AB such that $AD - BD = AF - BF$.
This implies that D and F are the same point.
We conclude that the incircle of $\triangle ABC$ touches AB at D.

3. For each positive integer t, define $S_t = \{K : n+1 \le K \le n^2 \text{ and } \lfloor \frac{n^2}{K} \rfloor \ge t\}$.
For example, if $n=5$, then $S_2 = \{6,7,8,9,10,11,12\}$. Note $S_t = \phi$ for $t \ge n$.

Note that for each t, S_t consists of all the integers from $n+1$ to $\lfloor \frac{n^2}{t} \rfloor$. (since $\lfloor \frac{n^2}{K} \rfloor \ge t$ if $K \le \lfloor \frac{n^2}{t} \rfloor$ and $\lfloor \frac{n^2}{K} \rfloor < t$ if $K > \lfloor \frac{n^2}{t} \rfloor$). Thus, there are exactly $\lfloor \frac{n^2}{t} \rfloor - (n+1) + 1 = \lfloor \frac{n^2}{t} \rfloor - n$ integers in set S_t. So, $|S_t| = \lfloor \frac{n^2}{t} \rfloor - n$.

Now, $\sum_{K=n+1}^{n^2} \lfloor \frac{n^2}{K} \rfloor = 1 \times \{\#\text{ of integers } K \text{ with } n+1 \le K \le n^2 \text{ and } \lfloor \frac{n^2}{K} \rfloor = 1\}$
$+ 2 \times \{$ " $\lfloor \frac{n^2}{K} \rfloor = 2\}$
$+ 3 \times \{$ " $\lfloor \frac{n^2}{K} \rfloor = 3\}$
$\vdots$
$+ (n-1) \times \{$ " $\lfloor \frac{n^2}{K} \rfloor = n-1\}$

$$\begin{aligned}
\text{So, } \sum_{K=n+1}^{n^2} \left\lfloor \frac{n^2}{K} \right\rfloor &= 1(|S_1| - |S_2|) + 2(|S_2| - |S_3|) + 3(|S_3| - |S_4|) + \dots + (n-1)(|S_{n-1}| - |S_n|) \\
&= |S_1| + |S_2| + |S_3| + \dots + |S_{n-1}| - (n-1)|S_n| \\
&= |S_1| + |S_2| + |S_3| + \dots + |S_{n-1}|, \text{ since } |S_n| = 0. \\
&= \left(\left\lfloor \frac{n^2}{1} \right\rfloor - n\right) + \left(\left\lfloor \frac{n^2}{2} \right\rfloor - n\right) + \left(\left\lfloor \frac{n^2}{3} \right\rfloor - n\right) + \dots + \left(\left\lfloor \frac{n^2}{n-1} \right\rfloor - n\right) \\
&= n^2 + \sum_{K=2}^{n-1} \left\lfloor \frac{n^2}{K} \right\rfloor - n \cdot (n-1) \\
&= \left(\sum_{K=2}^{n-1} \left\lfloor \frac{n^2}{K} \right\rfloor\right) + n = \left(\sum_{K=2}^{n-1} \left\lfloor \frac{n^2}{K} \right\rfloor\right) + \left\lfloor \frac{n^2}{n} \right\rfloor = \sum_{K=2}^{n} \left\lfloor \frac{n^2}{K} \right\rfloor.
\end{aligned}$$

Therefore, $\sum_{K=2}^{n} \lfloor \frac{n^2}{K} \rfloor = \sum_{K=n+1}^{n^2} \lfloor \frac{n^2}{K} \rfloor$, as required.

4. $\boxed{a)}$ We prove that $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}$, with equality iff $a=b=c$. This establishes the fact that $m=\frac{3}{2}$.

<u>Lemma:</u> $(a+b+c)^2\geq 3(ab+bc+ca)$, with equality iff $a=b=c$.

<u>Proof:</u> $(a-b)^2+(b-c)^2+(c-a)^2\geq 0$, with equality iff $a=b=c$.

Expanding, we get $2(a^2+b^2+c^2-ab-bc-ca)\geq 0 \Leftrightarrow a^2+b^2+c^2-ab-bc-ca\geq 0$

$\Leftrightarrow a^2+b^2+c^2+2ab+2bc+2ca\geq 3(ab+bc+ca) \Leftrightarrow (a+b+c)^2\geq 3(ab+bc+ca)$.

Thus, we have proven the lemma. Equality occurs iff $a=b=c$.

By Cauchy-Schwarz, $\left(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\right)\left(a(b+c)+b(a+c)+c(a+b)\right)\geq(a+b+c)^2$

$\Rightarrow \frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}\geq\frac{3(ab+bc+ca)}{2(ab+bc+ca)}=\frac{3}{2}$, by our lemma.

Thus, $\boxed{m=3/2}$.

$\boxed{b)}$ We claim that $M=2$. First we note that $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}$ can be made arbitrarily close to 2 (which proves that $M\geq 2$). To see this, let $a=b=x$ and $c=1$, where x is sufficiently large (clearly, 1-x-x are the sides of a Δ). And the sum $\frac{x}{x+1}+\frac{x}{x+1}+\frac{1}{x+x}=\frac{2x}{x+1}+\frac{1}{2x}=2-\frac{2}{x+1}+\frac{1}{2x}=2-\frac{4-\frac{1}{x}}{2(x+1)}$ can be made arbitrarily close to 2 by setting x large enough. Now we prove that $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}<2$.

where $s=\frac{a+b+c}{2}$

Since a,b,c are the sides of a Δ, we know that $x=s-a=\frac{b+c-a}{2}>0$. Similarly, $y=s-b>0$ and $z=s-c>0$. With this substitution, our inequality becomes equivalent to $\frac{s-x}{s+x}+\frac{s-y}{s+y}+\frac{s-z}{s+z}<2$. Since this inequality is homogeneous, we can assume WLOG that $\boxed{s=1}$. Hence, we are required to prove that $\frac{1-x}{1+x}+\frac{1-y}{1+y}+\frac{1-z}{1+z}<2$, which can be rewritten as $\frac{2x}{1+x}+\frac{2y}{1+y}+\frac{2z}{1+z}>1$. (where $x,y,z>0$).

And this inequality is true because $\frac{2x}{1+x}+\frac{2y}{1+y}+\frac{2z}{1+z}>1$

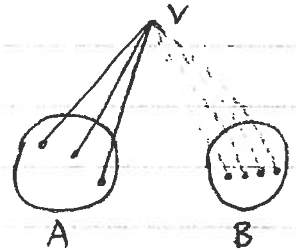
$\Leftrightarrow 2x(1+y)(1+z)+2y(1+x)(1+z)+2z(1+x)(1+y)>(1+x)(1+y)(1+z)$

$\Leftrightarrow 2(x+y+z)+4(xy+yz+zx)+6xyz>1+(x+y+z)+(xy+yz+zx)+xyz$

$\Leftrightarrow 3(xy+yz+zx)+5xyz>0$ since $x+y+z=s=1$.

$\therefore \boxed{M=2}$

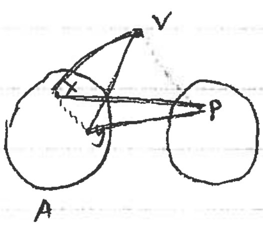
5. Represent the people by dots (i.e. vertices), and draw a solid edge between two vertices if those two people are acquaintances, and a dotted edge otherwise. Consider any vertex V. Let A be the set of vertices that are acquaintances with V (i.e. connected by a solid edge), and let B be all the other vertices.



We shall now prove that $|B| = \binom{|A|}{2}$. ← note: $|S|$ represents the size of set S.

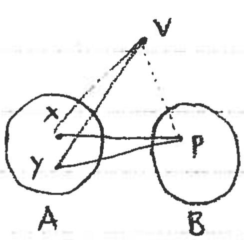
Since each pair of acquaintances has no common acquaintance, each of the $\binom{|A|}{2}$ pairs of vertices in A must be joined by a dotted line, i.e. x and y are strangers, for all $x, y \in A$.

Now, each of the $\binom{|A|}{2}$ pairs of strangers in A have exactly two common acquaintances. For any such pair (x,y), V is a common acquaintance, so there must be exactly one vertex $p \in B$ that is acquainted with both x and y. Note that p cannot be the common acquaintance of any other pair in A, or else the pair of strangers (V, p) will have at least three common acquaintances.



Hence, each of the $\binom{|A|}{2}$ pairs of strangers in A must be acquainted with a <u>unique</u> vertex in B. Thus, $|B| \geq \binom{|A|}{2}$.

Consider the $|B|$ vertices in B. For every vertex $p \in B$, the pair of strangers (V, p) must have two common acquaintances, x and y. Note that x and y must be in A, so each pair (V,p) gets matched to a pair (x,y) in A. Suppose vertices p and q in B both get matched to (x,y). Then the pair of strangers (x,y) have at least three common acquaintances (namely V, p, and q).

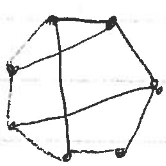


Hence, each of the $|B|$ vertices in B must be matched up to a <u>unique</u> pair of vertices in A. Thus, $|B| \leq \binom{|A|}{2}$.

Combining the above results, we conclude that $|B| = \binom{|A|}{2}$.

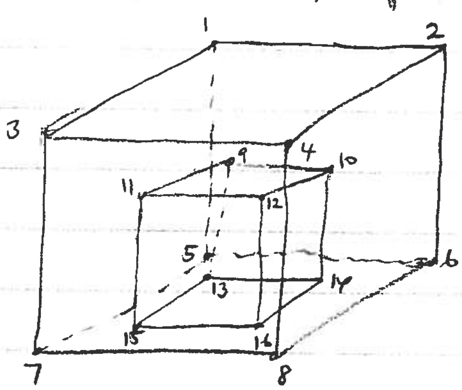
Now, $n = 1 + |A| + |B|$, so $n = 1 + |A| + \binom{|A|}{2}$.
Therefore, n must be of the form $1 + m + \binom{m}{2}$, where m is an integer.
The condition $5 \le n \le 30$ implies that $n = 7, 11, 16, 22$, or 29. Since $11 \nmid n$, we have $n = 7, 16$, or 29.

Note that if $n=7$, then $|A| = 3$, i.e. vertex v has exactly 3 acquaintances. Repeating the argument on every other vertex, we see that every other vertex also has exactly 3 acquaintances. In other words, we must have a graph on 7 vertices, where each vertex is connected to exactly 3 others. But then $7 \times 3 = 2 \times (\#\text{ of edges in the graph})$, which



gives a fractional number of edges. And that is a contradiction. Hence, $n \neq 7$. Similarly, $n \neq 29$ since $|A| = 7$ in this case, and $\frac{29 \times 7}{2}$ is not an integer. This proves that the only candidate for n is $\underline{n=16}$.

To conclude the proof, we must verify that a solution does exist for $n=16$



Consider the following "cube within a cube" diagram where each vertex in the inner cube is connected to two diagonally opposite vertices in the outer cube: namely, the pairs of edges are $(9,1), (9,8), (10,2), (10,7), (11,3), (11,6), (12,4), (12,5), (13,5), (13,4), (14,6), (14,3), (15,7), (15,2), (16,8)$, and $(16,1)$.

There are no triangles in the graph, so each pair of acquaintances have no common acquaintances. Let us show that every pair of strangers have exactly two common acquaintances. By symmetry, it suffices to verify this for every pair of strangers containing vertex 1. The following is the desired list: $(1,4)$ is acquainted with 2 and 3, $(1,6)$ with 2 and 5, $(1,7)$ with 3 and 5, $(1,8)$ with 9 and 16, $(1,10)$ with 2 and 9, $(1,11)$ with 3 and 9, $(1,12)$ with 5 and 16, $(1,13)$ with 5 and [illegible], $(1,14)$ with 3 and 16, and $(1,15)$ with 2 and 16. Thus, we have found a solution for $n=16$, and so we conclude that the unique solution is $\underline{n=16}$.