New Zealand Mathematical Olympiad Committee

2011 Squad Assignment Two

Geometry

Due: Monday 28th February 2011

1. *Find all possible values of the quotient*

$$\frac{r+\rho}{a+b}$$

where r and ρ are respectively the radii of the circumcircle and incircle of the right triangle with legs a and b.

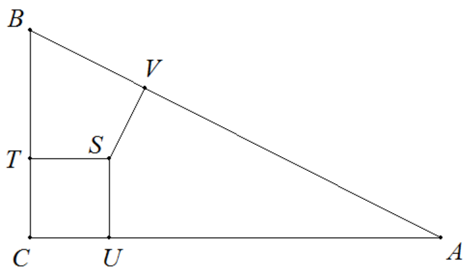
Solution: The distances between the vertices and points of tangency of the incircle (denoted according to the figure below) of a triangle ABC are

$$|AU| = |AV| = \frac{b+c-a}{2},\ |BV| = |BT| = \frac{a+c-b}{2},\ |CT| = |CU| = \frac{a+b-c}{2}.$$

The points C, T, U and S (the centre of the incircle) form a square. The side of the square is $\rho = |SU| = |CU| = \frac{a+b-c}{2}$. Moreover $r = \frac{c}{2}$ and so

$$r + \rho = \frac{c}{2} + \frac{a+b-c}{2} = \frac{a+b}{2}.$$

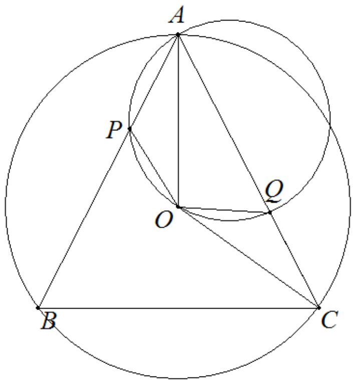
Hence there is only one possible value for $\frac{r+\rho}{a+b}$ which is $\frac{1}{2}$.



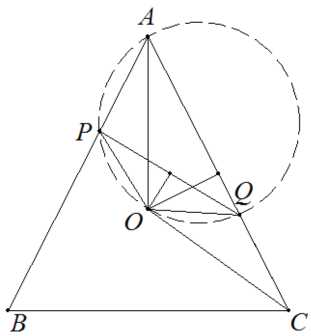
$\square$

2. *Let ABC be an isosceles triangle with $|AB| = |AC|$ and P, Q are interior points of AB and AC respectively. Prove that the circumcircle of $\triangle APQ$ passes through the circumcentre of $\triangle ABC$ if and only if $|AP| = |CQ|$*

Solution: First assume that the circumcircle of $\triangle APQ$ passes through the circumcentre of the $\triangle ABC$. Let O be the circumcentre of $\triangle ABC$ and join O with P and Q. Because $\triangle ABC$ is isosceles, $\angle PAQ = \angle OAQ$. As $|OA| = |OC|$, we have $\angle OPA = \angle OQC$. This shows that the triangles APO and CQO have the same angles. As $|OA| = |OC|$ this implies that $|AP| = |CQ|$.



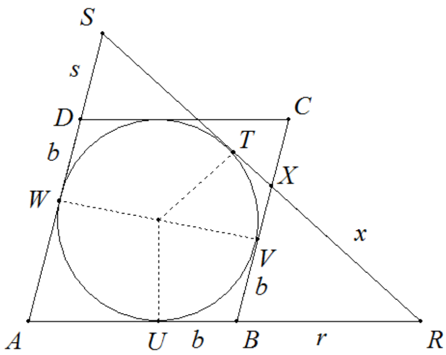
Assume now that $|AP| = |CQ|$ and let O be the intersection of the perpendicular bisectors of PQ and AC (see figure below). First note that the four points A, P, O, Q are concyclic. This is so, because $|OP| = |OQ|$, $|OA| = |OC|$ and $|AP| = |CQ|$ implies that the two triangles APO and CQO are congruent, hence $\angle APO = \angle CQO$, which implies that A, P, O, Q are on a circle.



On the other hand, we have $\angle PAO = \angle QCO$ from the congruence seen above and also $\angle QCO = \angle QAO$ from the definition of O. Therefore, AO is the angle bisector and since $|AB| = |AC|$, the line AO bisects BC at a right angle. This implies that O is the circumcentre of $\triangle ABC$. $\square$

3. *Let $ABCD$ be a rhombus and let a tangent of its incircle cut the interior of the sides BC and CD, and denote R, S the intersections of the tangent with the lines AB, AD respectively. Prove that the value of $|BR| \cdot |DS|$ is independent of the choice of the tangent.*

   Solution: Let U, V, W, T be the points of tangency of the incircle with the sides AB, BC, DA, DA, and with the tangent respectively. Further let X be the intersection point of the tangent and the side BC, and let $a = |AB| = |AD|, b = |BU| = |BV| = |DW$ be fixed quantities, while $r = |BR|$ and $s - |DS|$ are the variables dependent on the choice of the tangent. We will show that $r \cdot s = a \cdot b$.



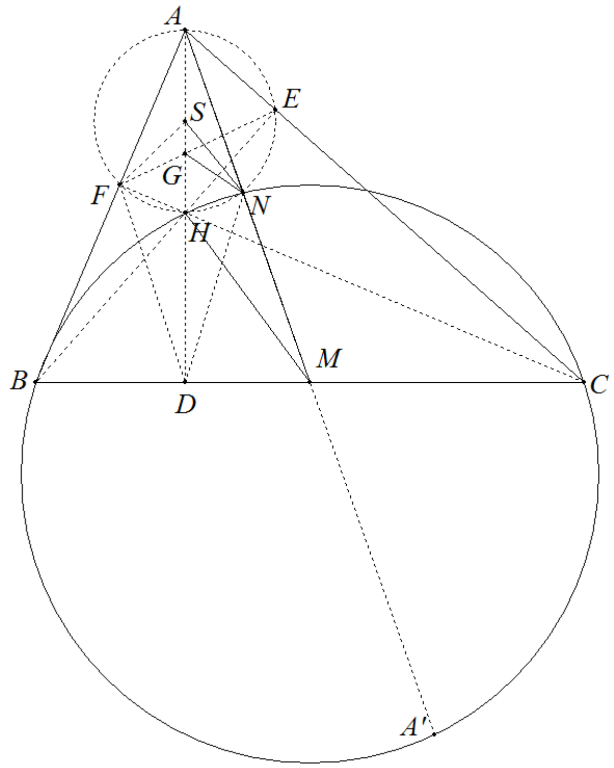
   The point R is the centre of similitude of the triangles ARS and BRX. Moreover the incircle is the incircle of $\triangle ARS$ and the excircle of $\triangle BRX$ tangent to BX. According to the well-known fact, the centre of BX is the centre of symmetry of the points of tangency of the incircle and excircle with BX. This means that the ratio $|SW| : |AR|$ in $\triangle ARS$ corresponds to the ratio $|BV| : |BR|$ in $\triangle BRX$, that is

$$\frac{b+s}{a+r} = \frac{b}{r},$$

   which is equivalent to $r \cdot s = a \cdot b$. □

4. *In an acute-angled triangle ABC, M is the midpoint of side BC, and D, E and F the feet of the altitudes from A, B and C, respectively. Let H be the orthocentre of triangle ABC, S the midpoint of AH, and G the intersection of FE and HA. If N is the intersection of the line segment AM and the circumcircle of triangle BCH, prove that $\angle HMA = \angle GNS$.*

   Solution: *First solution.* Let the line AN meet the circle BHS again at the point A'. The $ABA'C$ is a parallelogram and $\angle HCA' = \angle HCB + \angle BCA' = \angle HCB + \angle ABC = 90°$, so the points A, B, N all lie on the circle with diameter HA' and therefore $\angle ANH = 90°$.



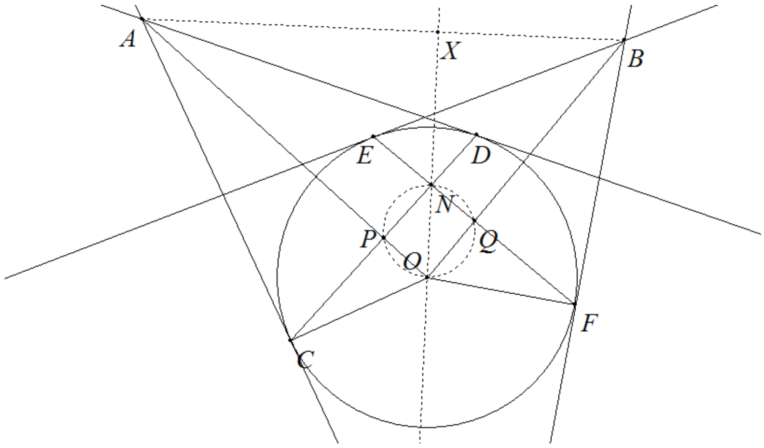
Since S is the circumcentre of $\triangle AEF$, it follows that $\angle SFG = 90^\circ - \angle EAF = \angle ACF = \angle ADF$; hence the triangles SFG and SDF are similar and $SG \cdot SD = SF^2 = SN^2$. This in turn shows that $\triangle SNG\ \triangle SDN$ and finally $\angle GNS = \angle GDN = \angle HMN$ since the quadrilateral $HDMN$ is cyclic.

Second solution. This second solution uses slightly heavier machinery but provides a somewhat more elegant proof. Quadrilaterals $BDHF$ and $DCEH$ are cyclic and $AF \cdot AB = AH \cdot AD = Æ \cdot AC$. We apply the inversion $\mathcal{I}$ with centre A and power $AF \cdot AB$. Clearly $\mathcal{I}(F) = B, \mathcal{I}(H) = D, \mathcal{I}(E) = C$, so $\mathcal{I}$ maps the line BC to the circumcircle of $\triangle FHE$, i.e. the circle with diameter AH; also, $\mathcal{I}$ maps the circumcircle of BCH to the circumcircle ω of FDE which is the nine-point circle of $\triangle ABC$. Since $M \in \omega \cap AM$ and $\mathcal{I}$ preserves line AM, it follows that $\mathcal{I}(M) = N$.

Let $\mathcal{I}(G) = G^*$ and $\mathcal{I}(S) = S^*$. Since $\mathcal{I}(EF)$ is the circumcircle of $\triangle ABC, S \in \omega$ and $\mathcal{I}(AH) = AH$, points G^* and S^* are the second intersection points of AH with the circumcircles of $\triangle ABC$ and $\triangle HBC$ respectively. Thus $\angle GNS = \angle G^*MS^* = \angle HMA$, for G^*, S^* are the reflections of H, A in BC. $\square$

5. *Points C, D, E and F lie on a circle with centre O. The two chords CD and EF intersect at a point N. The tangents at C and D intersect at A, and the tangents at E and F intersect at B. Prove that $ON \perp AB$.*

   Solution: Let P be the intersection point of CD and AO and let Q be the intersection point of EF and BO. Then the right-angled triangles CPO and ACO are similar, and hence $OA \cdot OP = OC^2$. Similarly we obtain $OB \cdot OQ = OF^2$ and since $OC = OF$, we have $OA \cdot OP = OB \cdot OQ$. It follows therefore that the triangles OAB and OQP are similar, so $\angle OQP = \angle OAB$.

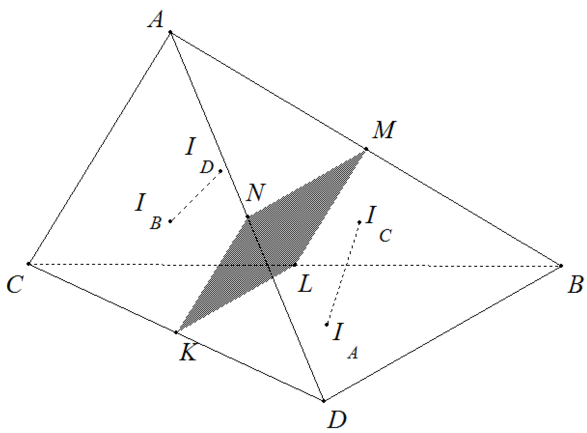


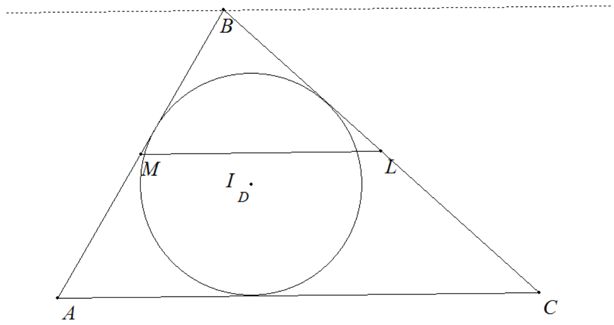
On the other hand, since $\angle OPN = \angle OQN = 90°$, we infer that the quadrilateral $OPNQ$ is cyclic. Hence we obtain $\angle ONP = \angle OQP = \angle OAB$. Let X be the point of intersection of the lines ON and AB. Then $\angle OAB = \angle ONP = 180° - \angle PNX$, which implies that the quadrilateral $APNX$ is cyclic. Therefore, since $\angle APN = 90°$, we immediately get $\angle AXN = 90°$. □

6. *Can the four incentres of the four faces of a tetrahedron be coplanar?*

Solution: They cannot. Let I_A, I_B, I_C, I_D be the incentres of the triangles BCD, ACD, ABD, ABC respectively. Assume they are coplanar. Then in their common plane they form either a convex quadrilateral, or a triangle inside of which the remaining point lies.

Case 1. Without loss of generality assume that I_A, I_B, I_C, I_D is a convex quadrilateral. Then the line segments I_AI_C and I_BI_D intersect. Let M, N, K, L be the midpoints AB, AD, CD, BC respectively. We observe that M, N, K, L are coplanar, since MN and KL are both parallel to BD, and this plane separates the tetrahedron with A, C on one side and B, D on the other.



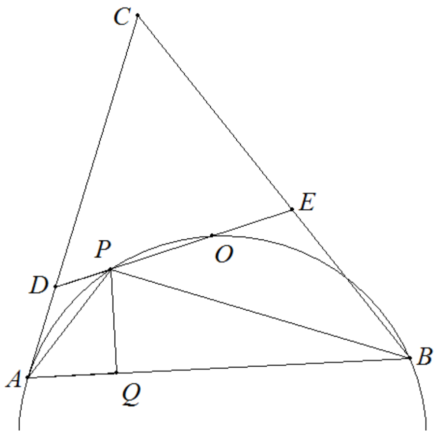


Consider triangle ABC and let ℓ be the line through B parallel to AC. Then the incircle of triangle ABC lies between ℓ and AC and is tangent to AC but not to ℓ. This implies that the centre of the incircle of triangle ABC, I_D, and B lie on different sides of the line ML, which is exactly half way between AC and ℓ. By a similar process we see that I_B and I_D lie on the same side of the plane $MNKL$ (on the opposite side of points B, D) while I_A and I_C lie on the other side. Hence the line segments I_AI_C and I_BI_D cannot intersect and we have reached a contradiction.

Case 2. It is sufficient to show that I_A cannot lie inside the triangle $I_BI_CI_D$. This follows from the fact that the points I_B, I_C, I_D lie strictly to one side of the plane BCD while I_A lies inside this plane. $\square$

7. *Let O be the circumcentre of an acute-angled triangle ABC. A line through O intersects the sides CA and CB at points D and E respectively, and meets the circumcircle of triangle ABO again at point $P \neq O$ inside the triangle. A point Q on side AB is such that*
$$\frac{AQ}{QB} = \frac{DP}{PE}.$$
Prove that $\angle APQ = 2\angle CAP$.

Solution: Denote $\angle PAD = \phi, \angle QPA = \psi, \angle BCA = \gamma$. The $\angle APB = 2\gamma$ and $\angle DAP + \angle EBP = \angle APB - \angle ACB = \gamma$, so $\angle PBE = \gamma - \phi$ and $\angle BPQ = 2\gamma - \psi$. Since $\angle APD = \angle BPE = 90° - \gamma$, we also have $\angle ADP = 90° + \gamma - \phi$ and $\angle BEP = 90° + \phi$.



The sine rule in triangles APD and PBE gives us

$$\frac{DP}{QB} = \frac{DP}{PA} \cdot \frac{PA}{PB} \cdot \frac{PB}{PE} = \frac{\sin\phi cos\phi}{\sin(\gamma-\phi)\cos(\gamma-\phi)} \cdot \frac{PA}{PB} = \frac{\sin 2\phi}{\sin(2\gamma - 2\phi)} \cdot \frac{PA}{PB}.$$

On the other hand,

$$\frac{AQ}{QB} = \frac{AQ}{AP} \cdot \frac{AP}{BP} \cdot \frac{BP}{QB} = \frac{\sin\psi}{\sin(2\gamma-\psi)} \cdot \frac{AP}{PB}.$$

Now $\frac{AQ}{QB} = \frac{DP}{PE}$ implies $f(2\phi) = f(\psi)$, where $f(x) = \frac{\sin x}{\sin(2\gamma - x)}$. However f is strictly increasing for all x because $f'(x) = \frac{\sin(2\gamma)}{(\sin(2\gamma-x))^2} > 0$, so it follows that $\psi = 2\phi$. $\square$

March 9, 2011

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