① Naoki

IMO Winter Camp 2006 - Geometry

Jan. 4, 2006

1. Given triangle ABC, prove that the following are equivalent:
   (i) The Euler line of $\triangle ABC$ is parallel to BC.
   (ii) $\tan B \tan C = 3$
   (iii) $2\tan A = \tan B + \tan C$.

2. Let I be the incentre of triangle ABC, and let AI intersect the circumcircle at P. Show that
$$\frac{AI}{IP} = \frac{b+c-a}{a}.$$

3. Let AB, AC be the two tangents to a given circle from an external point A. Extend AO to the circle at D, where O is the centre of the circle. Let AEF be a secant which is parallel to CD. Let $M = BD \cap EF$. Prove that M is the midpoint of EF.

4. An acute triangle ABC is inscribed in a circle centred at D. The circle, drawn through the points A, B, D intersects the sides AC and BC at the points M and N, respectively. Prove that the radii of the circumcircles of the triangles ABD and MNC are equal.

5. Let H be the orthocentre of an obtuse triangle ABC, and A_1, B_1, C_1 arbitrary points on the sides BC, AC, AB, respectively. Prove that the tangents from the point H to the circles with diameters AA_1, BB_1, and CC_1 are equal.

6. Given triangle ABC, let C_1 and C_2 be two circles, with radii r_1 and r_2, respectively, which are externally tangent at the incentre I and internally tangent to the circumcircle. Prove that
$$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2}.$$
(r is the inradius of $\triangle ABC$.)

7. Let ABC be an acute-angled triangle. Let A_2, B_2, C_2 be the midpoints of altitudes AA_1, BB_1, CC_1, respectively. Find $\angle B_2 A_1 C_2 + \angle C_2 B_1 A_2 + \angle A_2 C_1 B_2$.

8. Let AD, BE, and CF be three cevians of an acute triangle which meet at the point X, such that $AX \cdot XD = BX \cdot XE = CX \cdot XF$. Prove that X is the orthocentre of triangle ABC.

9. Let D, E, F be the feet of the altitudes of triangle ABC, and let P be a point. Prove that the circumcircles of triangles PAD, PBE, and PCF concur at a point on the line PH.

10. Let I be the incentre of triangle ABC, and let I_A, I_B, I_C be the excentres. Prove that the midpoints of II_A, II_B, and II_C lie on the circumcircle of triangle ABC.

11. Chord AB is drawn in a circle, and a point X is chosen on AB. A circle is then inscribed in each "half" of the circle split by AB, tangent to AB at X. Prove that the ratio of the radii of the circles is independent of the location of X.

12. Two squares, not necessarily of equal size, intersect to form an octagon $A_1A_2A_3A_4A_5A_6A_7A_8$. Prove that $A_1A_5 \perp A_3A_7$.