Practice Olympiad 1 Solutions

1. For positive integers n, the sequence a_1, a_2, $\ldots$ is defined by $a_1 = 1$,

$$a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + \cdots + a_{n-1})$$

for $n > 1$. Determine the value of a_{2010}.

Solution. We can rewrite the given equation as

$$a_1 + a_2 + \cdots + a_{n-1} = \frac{n-1}{n+1} \cdot a_n.$$

Replacing n with $n+1$, we get

$$a_1 + a_2 + \cdots + a_n = \frac{n}{n+2} \cdot a_{n+1}.$$

Subtracting these equations, we get

$$a_n = \frac{n}{n+2} \cdot a_{n+1} - \frac{n-1}{n+1} \cdot a_n,$$

so

$$\frac{n}{n+2} \cdot a_{n+1} = a_n + \frac{n-1}{n+1} \cdot a_n = \frac{2n}{n+1} \cdot a_n,$$

or

$$a_{n+1} = \frac{n+2}{n} \cdot \frac{2n}{n+1} \cdot a_n = \frac{2(n+2)}{n+1} \cdot a_n$$

for all $n \geq 1$.

We claim that $a_n = (n+1)2^{n-2}$ for all positive integers n. We prove this by induction.

For the base case $n = 1$, $(n+1)2^{n-2} = 2 \cdot 2^{-1} = 1$, and $a_1 = 1$, so the result holds. Assume that the result holds for some positive integer $n = k$, so $a_k = (k+1)2^{k-2}$. Then

$$\begin{aligned} a_{k+1} &= \frac{2(k+2)}{k+1} \cdot a_k \\ &= \frac{2(k+2)}{k+1} \cdot (k+1)2^{k-2} \\ &= (k+2)2^{k-1}. \end{aligned}$$

Hence, the result holds for $n = k+1$, and by induction, for all positive integers n. In particular, $a_{2010} = 2011 \cdot 2^{2008}$.

Note. We can also derive the answer using a telescoping product:

$$\begin{aligned}
a_{2010} &= \frac{2 \cdot 2011}{2010} \cdot a_{2009} \\
&= \frac{2 \cdot 2011}{2010} \cdot \frac{2 \cdot 2010}{2009} \cdot a_{2008} \\
&= \frac{2 \cdot 2011}{2010} \cdot \frac{2 \cdot 2010}{2009} \cdot \frac{2 \cdot 2009}{2008} \cdot a_{2007} \\
&= \cdots \\
&= \frac{2 \cdot 2011}{2010} \cdot \frac{2 \cdot 2010}{2009} \cdot \frac{2 \cdot 2009}{2008} \cdots \frac{2 \cdot 4}{3} \cdot \frac{2 \cdot 3}{2} \cdot a_1 \\
&= 2011 \cdot 2^{2008}.
\end{aligned}$$

2. Find all positive integers n such that $1 + 2^2 + 3^3 + 4^n$ is a perfect square.

Solution 1. Let $1 + 2^2 + 3^3 + 4^n = x^2$, where x is an integer, which simplifies to $4^n + 32 = x^2$. Then

$$32 = x^2 - 4^n = x^2 - 2^{2n} = (x - 2^n)(x + 2^n).$$

Since 32 is a power of 2, both $x - 2^n$ and $x + 2^n$ must also be powers of 2.

Let $x - 2^n = 2^a$ and $x + 2^n = 2^b$, where a and b are nonnegative integers, so $a \le b$ and $a + b = 5$. Taking the difference of these equations, we get $2 \cdot 2^n = 2^b - 2^a$, so $2^b - 2^{n+1} = 2^a$, or

$$2^{b-a} - 2^{n+1-a} = 1.$$

The only powers of 2 that differ by 1 are 2 and 1, so $2^{b-a} = 2$ and $2^{n+1-a} = 1$, which means $b - a = 1$ and $n + 1 - a = 0$. Combined with the equation $a + b = 5$, we find $a = 2$, $b = 3$, and $n = 1$. For $n = 1$, $1 + 2^2 + 3^3 + 4^n = 36 = 6^2$, so the only solution is $n = 1$.

Solution 2. If $n \ge 3$, then we can write

$$1 + 2^2 + 3^3 + 4^n = 4^n + 32 = 2^{2n} + 2^5 = 2^5(2^{2n-5} + 1).$$

The exponent $2n - 5$ is at least 1, so 2^{2n-5} is even, which means $2^{2n-5} + 1$ is odd. Hence, the number $1 + 2^2 + 3^3 + 4^n = 2^5(2^{2n-5} + 1)$ has an odd number of factors of 2 (namely 5), so it cannot be a perfect square. Thus, it suffices to check the cases $n = 1$ and $n = 2$.

For $n = 1$, $4^n + 32 = 36 = 6^2$, and for $n = 2$, $4^n + 32 = 48$. Therefore, the only solution is $n = 1$.

Solution 3. We see that $1 + 2^2 + 3^3 + 4^n = 2^{2n} + 32 > 2^{2n}$ for all positive integers n. Also, for $n \ge 4$, $2 \cdot 2^n + 1 \ge 2 \cdot 16 + 1 > 32$, so

$$2^{2n} + 32 < 2^{2n} + 2 \cdot 2^n + 1 = (2^n + 1)^2.$$

Hence, for $n \ge 4$, $2^{2n} + 32$ lies between two consecutive perfect squares, which means that it cannot be a perfect square itself. Thus, it suffices to check the cases $n = 1$, 2, and 3.

For $n = 1$, $2^{2n} + 32 = 36 = 6^2$, for $n = 2$, $2^{2n} + 32 = 48$, and for $n = 3$, $2^{2n} + 32 = 96$. Therefore, the only solution is $n = 1$.

3. For a partition π of $\{1, 2, 3, \ldots, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. For example, for the partition π given by $\{1, 4, 6, 7\} \cup \{2, 8\} \cup \{3\} \cup \{5, 9\}$, $\pi(5) = 2$, $\pi(6) = 4$, and $\pi(3) = 1$.

   Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, \ldots, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

   Solution 1. First, we claim that among the values $\pi(1)$, $\pi(2)$, $\ldots$, $\pi(9)$, some value appears at least four times.

   For the sake of contradiction, suppose that no value appears at least four times, so each value appears at most three times. Then in the partition π, there are at most three parts containing one element, at most one part containing two elements, at most one part containing three elements, and no parts containing n elements for any $n \geq 4$. Hence, the partition π contains at most $3 + 2 + 3 = 8$ elements, contradiction.

   Therefore, among the values $\pi(1)$, $\pi(2)$, $\ldots$, $\pi(9)$, some value appears at least four times. Let a_1, a_2, a_3, a_4 be distinct elements in $\{1, 2, 3, \ldots, 9\}$ such that $\pi(a_1) = \pi(a_2) = \pi(a_3) = \pi(a_4)$.

   Next, we claim that among the values $\pi'(1)$, $\pi'(2)$, $\pi'(3)$, $\ldots$, $\pi'(9)$, there are at most three distinct values. If there are (at least) four distinct values, then these four values are at least 1, 2, 3, and 4 (in increasing order), which means that the partition π' contains at least $1 + 2 + 3 + 4 = 10$ elements, contradiction. Hence, there are at most three distinct values among $\pi'(1)$, $\pi'(2)$, $\pi'(3)$, $\ldots$, $\pi'(9)$.

   In particular, among the values $\pi'(a_1)$, $\pi'(a_2)$, $\pi'(a_3)$, and $\pi'(a_4)$, there are at most three distinct values, which means that two of them are equal, say $\pi'(a_i) = \pi'(a_j)$. We also have $\pi(a_i) = \pi(a_j)$, which proves the result.

   Solution 2. As in Solution 1, there are at most three distinct values among the values $\pi(1)$, $\pi(2)$, $\ldots$, $\pi(9)$, and at most three distinct values among the values $\pi'(1)$, $\pi'(2)$, $\ldots$, $\pi'(9)$.

   For the sake of contradiction, suppose that there do not exist distinct elements x and y such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. Then the nine pairs $(\pi(x), \pi'(x))$, $1 \leq x \leq 9$, are distinct. But the range of π and π' each are at most three distinct values, so each possible pair occurs exactly once.

   If $z = \pi(x)$ for some x, then z occurs exactly three times in the range of π. However, from the point of view of the partition π, z occurs exactly kz times in the range of π, where k is the number of distinct parts of size z. Hence, $kz = 3$, so $z = 1$ or $z = 3$. In other words, the only possible values of $\pi(x)$ are 1 and 3. This is a contradiction, because the range of π has exactly three distinct values.

   Therefore, there exist distinct elements x and y such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

4. (a) Show that for every positive integer n, there exists a polynomial P_n such that

$$t^n + \frac{1}{t^n} = P_n\left(t + \frac{1}{t}\right)$$

for all $t \neq 0$.

(b) Find all positive integers n for which there exists a polynomial Q_n such that

$$t^n - \frac{1}{t^n} = Q_n\left(t - \frac{1}{t}\right)$$

for all $t \neq 0$.

Solution. (a) We prove the result using strong induction. For $n = 1$, we can take $P_1(x) = x$. For $n = 2$,

$$t^2 + \frac{1}{t^2} = \left(t + \frac{1}{t}\right)^2 - 2,$$

so we can take $P_2(x) = x^2 - 2$.

Now assume that the result is true for $n = 1, 2, \ldots, k$, for some positive integer $k \geq 2$. Then

$$\begin{aligned} \left(t + \frac{1}{t}\right)\left(t^k + \frac{1}{t^k}\right) &= t^{k+1} + \frac{1}{t^{k+1}} + t^{k-1} + \frac{1}{t^{k-1}} \\ \Rightarrow \quad t^{k+1} + \frac{1}{t^{k+1}} &= \left(t + \frac{1}{t}\right)\left(t^k + \frac{1}{t^k}\right) - \left(t^{k-1} + \frac{1}{t^{k-1}}\right), \end{aligned}$$

so we can take $P_{k+1}(x) = xP_k(x) - P_{k-1}(x)$, which clearly makes P_{k+1} a polynomial, and the result is true for $n = k + 1$. Hence, by strong induction, such a polynomial P_n exists for all positive integers n.

(b) Setting $t = 2$, we get

$$Q_n\left(\frac{3}{2}\right) = 2^n - \frac{1}{2^n} = \frac{4^n - 1}{2^n}.$$

Setting $t = -1/2$, we get

$$Q_n\left(\frac{3}{2}\right) = \left(-\frac{1}{2}\right)^n - (-2)^n = \frac{(-1)^n(1 - 4^n)}{2^n}.$$

Hence, such a polynomial Q_n can exist only if $(-1)^n = -1$, i.e. n is odd. We prove that such a polynomial Q_n exists for all odd n using strong induction.

For $n = 1$, we can take $Q_1(x) = x$. For $n = 3$,

$$\begin{aligned} \left(t - \frac{1}{t}\right)^3 &= t^3 - 3t + \frac{3}{t} - \frac{1}{t^3} \\ \Rightarrow \quad t^3 - \frac{1}{t^3} &= \left(t - \frac{1}{t}\right)^3 + 3\left(t - \frac{1}{t}\right), \end{aligned}$$

so we can take $Q_3(x) = x^3 + 3x$.

Now assume that the result is true for $n = 1, 3, \ldots, k$, for some odd positive integer $k \geq 3$. Then

$$\begin{aligned}\left(t - \frac{1}{t}\right)^2 \left(t^k - \frac{1}{t^k}\right) &= \left(t^2 - 2 + \frac{1}{t^2}\right)\left(t^k - \frac{1}{t^k}\right) \\ &= t^{k+2} - 2t^k + t^{k-2} - \frac{1}{t^{k-2}} + \frac{2}{t^k} - \frac{1}{t^{k+2}} \\ &= t^{k+2} - \frac{1}{t^{k+2}} - 2\left(t^k - \frac{1}{t^k}\right) + t^{k-2} - \frac{1}{t^{k-2}},\end{aligned}$$

so

$$t^{k+2} - \frac{1}{t^{k+2}} = \left[\left(t - \frac{1}{t}\right)^2 + 2\right]\left(t^k - \frac{1}{t^k}\right) - \left(t^{k-2} - \frac{1}{t^{k-2}}\right).$$

Thus, we can take $Q_{k+2}(x) = (x^2+2)Q_k(x) - Q_{k-2}(x)$, which clearly makes Q_{k+2} a polynomial, and the result is true for $n = k+2$. Hence, by strong induction, such a polynomial exists for all odd positive integers n.

5. 49 students solve a set of three problems. The score for each problem is an integer between 0 and 7. Prove that there are two students, such that for each problem, the first student scored at least as many as the second student.

Solution 1. We identify each student with a triple (a, b, c) of nonnegative integers, where a, b, and c are the scores that the student received on the first, second, and third problem, respectively, so $0 \le a$, b, $c \le 7$. We say that two such triples (a, b, c) and (a', b', c') are *comparable* if either $a \le a'$, $b \le b'$, and $c \le c'$, or $a \ge a'$, $b \ge b'$, and $c \ge c'$. Thus, two students satisfy the given condition if and only if their corresponding triples are comparable.

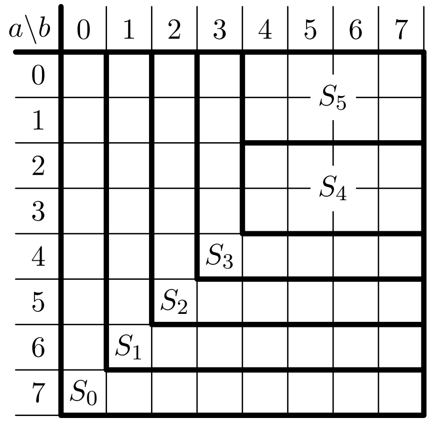
If there are two triples of the form (a, b, c) and (a, b, c'), then they are comparable, so assume that no two triples have the same a- and b-values. Then we can plot the 49 triples in a table, where a is plotted on the vertical, b is plotted on the horizontal, and c is entered in the square itself (so every square has at most one entry). For example, the triples $(1, 5, 3)$, $(2, 2, 5)$, $(5, 1, 4)$, and $(6, 4, 1)$ are plotted below.

$a \backslash b$	0	1	2	3	4	5	6	7
0								
1						3		
2			5					
3								
4								
5		4						
6					1			
7								

We partition the 64 squares of the table into six sets as follows: For $0 \le n \le 3$, let

$$S_n = \{(0, n), (1, n), \ldots, (7 - n, n), (7 - n, n + 1), \ldots, (7 - n, 7)\}.$$

Let $S_4 = \{(a, b) : 2 \le a \le 3, 4 \le b \le 7\}$ and $S_5 = \{(a, b) : 0 \le a \le 1, 4 \le b \le 7\}$.

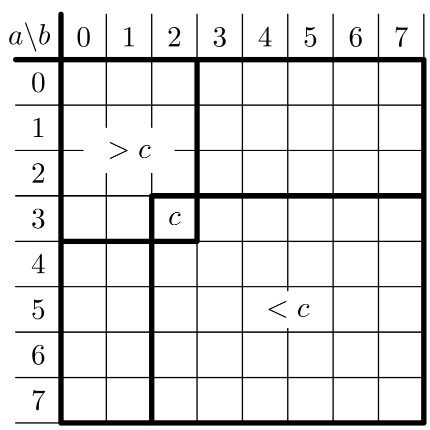


Since $49 = 8 \cdot 6 + 1$, by the Pigeonhole Principle, some set S_n contains at least 9 entries. Since both sets S_4 and S_5 contain eight squares, the set S_n must be S_0, S_1, S_2, or S_3.

There are only 8 possible entries, so two entries in S_n are equal. The triples corresponding to these entries are comparable.

Solution 2. Let S be a set of triples (a, b, c) of nonnegative integers, where $0 \le a,\ b,\ c \le 7$, such that no two triples in S are comparable. We claim that S contains at most 48 triples.

We plot the triples in S in a table, as in Solution 1. If the square (a, b) contains the entry c, then because no two triples in S are comparable, all entries appearing above and/or to the left of it must be greater than c, and all values appearing below and/or to the right of it must be less than c.



Hence, any other appearances of the entry c are restricted to the squares either strictly above and to the right, or strictly below and to the left. If $a + b \le 7$, then the entry c appears at most a times strictly above and to the right (one for each row), and at most b times strictly below and to the left (one for each column), for an upper bound of $a + b + 1$.

If $a+b \geq 7$, then the entry c appears at most $7-b$ times strictly above and to the right (one for each column), and at most $7-a$ times strictly below and to the left (one for each row), for an upper bound of $1+(7-b)+(7-a) = 15-a-b$. Therefore, the entry c can appear at most $f(a,b)$ times, where $f(a,b) = \min\{a+b+1, 15-a-b\}$.

Let $s(c)$ denote actual number of appearances of the entry c in the table. For a fixed value of c, $s(c) \leq f(a,b)$ for all a, b such that $(a,b,c) \in S$, so

$$s(c) \leq \min_{(a,b,c)\in S} f(a,b).$$

For any positive real numbers x_1, x_2, $\dots$, x_n,

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} \geq \min\{x_1, x_2, \dots, x_n\},$$

i.e. the harmonic mean is greater than or equal to the minimum. Applying this inequality to the $s(c)$ values $1/f(a,b)$ such that $(a,b,c) \in S$, we get

$$\frac{s(c)}{\sum_{(a,b,c)\in S}\frac{1}{f(a,b)}} \geq \min_{(a,b,c)\in S} f(a,b).$$

Combining this with the previous inequality, we get

$$s(c) \leq \frac{s(c)}{\sum_{(a,b,c)\in S}\frac{1}{f(a,b)}},$$

so

$$\sum_{(a,b,c)\in S}\frac{1}{f(a,b)} \leq 1.$$

Therefore,

$$\sum_{c=0}^{7}\sum_{(a,b,c)\in S}\frac{1}{f(a,b)} \leq 8.$$

Looking at the values of $1/f(a,b)$ for $0 \leq a$, $b \leq 7$, we see that the 8 smallest values of $1/f(a,b)$ are $1/8$ (for which $a+b=7$), the 14 next smallest values are $1/7$ (for which $a+b=6$ or $a+b=8$), and so on. The sum of the 48 smallest values of $1/f(a,b)$ is

$$\frac{8}{8}+\frac{14}{7}+\frac{12}{6}+\frac{10}{5}+\frac{4}{4} = 8.$$

Therefore, S contains at most 48 triples.

6. Let n be a positive integer, and let $a_1, a_2, \dots, a_n$ be distinct positive integers. Prove that

$$(a_1^7 + a_2^7 + \cdots + a_n^7) + (a_1^5 + a_2^5 + \cdots + a_n^5) \ge 2(a_1^3 + a_2^3 + \cdots + a_n^3)^2.$$

Solution 1. We prove the inequality by induction. For the base case $n = 1$, the inequality becomes $a_1^7 + a_1^5 \ge 2a_1^6$, which follows from the AM-GM inequality.

Assume that the inequality holds for $n = k$, for some positive integer k. We want to prove the inequality for $n = k + 1$. Let $a_1, a_2, \dots, a_{k+1}$ be distinct positive integers. By symmetry, without loss of generality, we may assume that $a_1 < a_2 < \cdots < a_{k+1}$.

Since $a_1, a_2, \dots, a_{k+1}$ are distinct positive integers,

$$a_1^3 + a_2^3 + \cdots + a_k^3 \le 1^3 + 2^3 + \cdots + (a_{k+1} - 1)^3 = \frac{a_{k+1}^2(a_{k+1} - 1)^2}{4},$$

so

$$4a_{k+1}^3(a_1^3 + a_2^3 + \cdots + a_k^3) \le a_{k+1}^5(a_{k+1} - 1)^2 = a_{k+1}^7 - 2a_{k+1}^6 + a_{k+1}^5.$$

Then

$$a_{k+1}^7 + a_{k+1}^5 \ge 4a_{k+1}^3(a_1^3 + a_2^3 + \cdots + a_k^3) + 2a_{k+1}^6.$$

Also, by the induction hypothesis,

$$(a_1^7 + a_2^7 + \cdots + a_k^7) + (a_1^5 + a_2^5 + \cdots + a_k^5) \ge 2(a_1^3 + a_2^3 + \cdots + a_k^3)^2.$$

Adding these inequalities, we get

$$\begin{aligned}
&(a_1^7 + a_2^7 + \cdots + a_{k+1}^7) + (a_1^5 + a_2^5 + \cdots + a_{k+1}^5) \\
&\ge 2(a_1^3 + a_2^3 + \cdots + a_k^3)^2 + 4a_{k+1}^3(a_1^3 + a_2^3 + \cdots + a_k^3) + 2a_{k+1}^6 \\
&= 2(a_1^3 + a_2^3 + \cdots + a_{k+1}^3)^2.
\end{aligned}$$

Hence, the inequality holds for $n = k + 1$, and by induction, for all positive integers n.

Solution 2. (hurdler) We want to prove that for distinct positive integers $a_1, a_2, \dots, a_n$,

$$\begin{aligned}
&\sum_{i=1}^n a_i^7 + \sum_{i=1}^n a_i^5 \ge 2\left(\sum_{i=1}^n a_i^3\right)^2 \\
\Leftrightarrow\quad &\sum_{i=1}^n a_i^7 + \sum_{i=1}^n a_i^5 \ge 2\sum_{i=1}^n a_i^6 + 4\sum_{1 \le i < j \le n} a_i^3 a_j^3 \\
\Leftrightarrow\quad &\sum_{i=1}^n (a_i^7 - 2a_i^6 + a_i^5) \ge 4\sum_{1 \le i < j \le n} a_i^3 a_j^3.
\end{aligned}$$

Dividing both sides by 4 and rearranging, we get

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{a_i^5(a_i-1)^2}{4} \geq \sum_{1\leq i<j\leq n} a_i^3 a_j^3 \\
\Leftrightarrow\quad &\sum_{i=1}^{n} a_i^3 \cdot \frac{a_i^2(a_i-1)^2}{4} \geq \sum_{1\leq i<j\leq n} a_i^3 a_j^3 \\
\Leftrightarrow\quad &\sum_{i=1}^{n} a_i^3[1^3+2^3+\cdots+(a_i-1)^3] \geq \sum_{1\leq i<j\leq n} a_i^3 a_j^3.
\end{aligned}$$

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By symmetry, without loss of generality, we may assume that $a_1 < a_2 < \cdots < a_n$. The left-hand side of the last inequality is

$$\begin{aligned}
&a_1^3[1^3+2^3+\cdots+(a_1-1)^3] \\
&+a_2^3[1^3+2^3+\cdots+a_1^3+\cdots+(a_2-1)^3] \\
&+a_3^3[1^3+2^3+\cdots+a_1^3+\cdots+a_2^3+\cdots+(a_3-1)^3] \\
&+\cdots \\
&+a_n^3[1^3+2^3+\cdots+a_i^3+\cdots+(a_n-1)^3].
\end{aligned}$$

This sum contains every term of the form $a_i^3 a_j^3$, where $1 \leq i < j \leq n$. Hence, the inequality holds.

7. Show that the set of positive integers that cannot be represented as a sum of distinct perfect squares is finite.

Solution. (mgao) We claim that all positive integers greater than

$$\sum_{i=1}^{116}(116i-1)^2$$

can be expressed as the sum of distinct perfect squares.

Let n be a positive integer such that

$$n > \sum_{i=1}^{116}(116i-1)^2,$$

and let r be the remainder when n is divided by 116, so $r \equiv n \pmod{116}$ and $0 \le r < 116$. Let $s = r$ if $1 \le r < 116$, and let $s = 116$ if $r = 0$. Then

$$n - \sum_{i=1}^{s}(116i-1)^2 \equiv r - \sum_{i=1}^{s}(-1)^2 \equiv r - s \equiv 0 \pmod{116},$$

so

$$n = 116q + \sum_{i=1}^{s}(116i-1)^2$$

for some nonnegative integer q.

Let the binary representation of q be

$$q = \sum_{i=0}^{M} 2^{2i}a_i + \sum_{i=0}^{N} 2^{2i+1}b_i,$$

where $a_i \in \{0,1\}$ and $b_i \in \{0,1\}$ for all i. Then $a_i = a_i^2$ and $b_i = b_i^2$ for all i. Hence,

$$\begin{aligned}
116q &= 116\sum_{i=0}^{M} 2^{2i}a_i + 116\sum_{i=0}^{N} 2^{2i+1}b_i \\
&= 116\sum_{i=0}^{M} 2^{2i}a_i + 232\sum_{i=0}^{N} 2^{2i}b_i \\
&= (10^2+4^2)\sum_{i=0}^{M}(2^i a_i)^2 + (14^2+6^2)\sum_{i=0}^{N}(2^i b_i)^2 \\
&= \sum_{i=0}^{M}[(10^2 2^i a_i)^2 + (4^2 2^i a_i)^2] + \sum_{i=0}^{N}[(14^2 2^i b_i)^2 + (6^2 2^i b_i)^2],
\end{aligned}$$

so

$$\begin{aligned} n &= 116q + \sum_{i=1}^{s}(116i-1)^2 \\ &= \sum_{i=0}^{M}[(10^2 2^i a_i)^2 + (4^2 2^i a_i)^2] + \sum_{i=0}^{N}[(14^2 2^i b_i)^2 + (6^2 2^i b_i)^2] + \sum_{i=1}^{s}(116i-1)^2 \\ &= A+B+C+D+E, \end{aligned}$$

where

$$A = \sum_{i=0}^{M}(10^2 2^i a_i)^2, \quad B = \sum_{i=0}^{M}(4^2 2^i a_i)^2, \quad C = \sum_{i=0}^{N}(14^2 2^i b_i)^2, \quad D = \sum_{i=0}^{N}(6^2 2^i b_i)^2, \quad E = \sum_{i=1}^{s}(116i-1)^2.$$

We claim that the squares in all these sums are distinct. (If any of the squares are 0, then they can be deleted.)

Each square in the sum E is different from all the other squares, because each square in E is odd, and all the other squares are even. And of the squares in the sums A, B, C, and D, only the squares in A are divisible by 5, only the squares in B are powers of 2, only the squares in C are divisible by 7, and only the squares in D are divisible by 3.

Hence, except for possible zeros that can be deleted, all the squares in the sum $n = A+B+C+D+E$ are distinct. Therefore, every positive integer n such that

$$n > \sum_{i=1}^{116}(116i-1)^2$$

can be expressed as the sum of distinct perfect squares. It follows that the set of positive integers that cannot be represented as a sum of distinct perfect squares is finite.