Combinatorial Number Theory

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Ricky Liu

What is combinatorial number theory? Basically any problem involving sets of numbers in some way can be described as combinatorial number theory. Alternatively, any combinatorics problem that uses the additive or multiplicative structure of natural numbers could also be described as combinatorial number theory. Since both these descriptions are inherently vague, it's perhaps best not to dwell on this discussion for now and move on to some examples.

1. Let A and B be finite sets of integers, and let $A+B$ denote the set of $a+b$, where $a \in A$, $b \in B$. Prove that $|A+B| \geq |A|+|B|-1$. When does equality occur?

2. Prove that out of any ten consecutive integers, one of them is relatively prime to all the others.

3. (Erdős) Prove that any $(n+1)$-element subset of $\{1, 2, \ldots, 2n\}$ contains two elements that are relatively prime and two elements, one of which divides the other.

4. Prove that for every positive integer n, there exists a permutation of $(1, 2, \ldots, n)$ such that the average of any two numbers in the sequence does not appear between them.

5. (IMO 1989) Prove that for every positive integer n, there exist n consecutive positive integers, none of which is a prime power.

6. Prove that the number of partitions of n into odd parts is the same as the number of partitions of n into distinct parts.

7. (BMC 1999) Suppose the natural numbers are partitioned into finitely many subsets. Show that one of these subsets contains infinitely many multiples of every positive integer.

8. Let $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_n)$ be two sequences of positive integers at most k with equal sums such that no nonempty proper subsequence of (a_i) has the same sum as a nonempty proper subsequence of (b_j). Determine the maximum possible value of $\sum_i a_i = \sum_j b_j$.

9. (USAMO 1998) Prove that for each $n \geq 2$, there exists a set S of n integers such that $(a-b)^2 \mid ab$ for distinct $a, b \in S$.

10. (Russia 1995) Does there exist a sequence of natural numbers in which each natural number occurs exactly once such that the sum of the first k terms is divisible by k for all $k \geq 1$?

11. (IMO 1991) Let $S = \{1, \ldots, 280\}$. Find the smallest positive integer n such that any n-element subset of S contains five numbers which are pairwise relatively prime.

12. (USAMO 1996) Show that there exists a subset A of the integers such that any integer can be uniquely expressed as $a + 2b$, where $a, b \in A$.

13. (IMO 2000) A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one, and a blue one, so that each box contains at least one card. A member of the audience selects two of the three boxes, chooses one card from each, and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen. How many ways are there to put all the cards into the boxes so that this trick always works?

14. (IMO 1993) Does there exist a strictly increasing function $f : \mathbf{N} \to \mathbf{N}$ such that $f(1) = 2$ and $f(f(n)) = f(n) + n$ for all $n \in \mathbf{N}$?

15. Suppose the natural numbers are partitioned into a finite number of arithmetic progressions. Prove that two of these progressions have the same common difference.

16. (USAMO 2004) Suppose $a_1, \ldots, a_n$ are integers with greatest common divisor 1. Let S be the set of integers such that $a_i \in S$ and $a_i - a_j \in S$ for all i and j, and if $x, y, x+y \in S$, then $x - y \in S$. Show that S is the set of all integers.

17. (Schur) Given a positive integer k, prove that for all n sufficiently large, for every partition of $\{1, \ldots, n\}$ into k subsets, one of these subsets contains three distinct elements x, y, and z such that $x + y = z$.

18. (TST 2001) Let A be a finite set of integers. Prove that there exists a finite set B of integers such that $A \subset B$ and $\prod_{x \in B} x = \sum_{x \in B} x^2$.

19. (IMO 1995) Let p be an odd prime. How many p-element subsets of $\{1, 2, \ldots, 2p\}$ have sum divisible by p?

20. (HMMT 2005) Let A and B be subsets of nonnegative integers such that every nonnegative integer can be uniquely expressed as $a + b$, where $a \in A$, $b \in B$. If A is finite, prove that there exists some integer n such that $a \in A$ if and only if $n - a \in A$.

21. (Putnam 1999) Let S be a finite set of positive integers greater than 1 such that for any positive integer n, there exists some $s \in S$ such that either s divides n or s is relatively prime to n. Prove that there exist two elements of S (possibly the same) whose greatest common divisor is prime.

22. (IMO 1992) For every positive integer n, let $S(n)$ be the greatest integer such that n^2 can be written as the sum of k positive squares for all $1 \leq k \leq S(n)$.
    (a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
    (b) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

23. (IMO 1997) Let $f(n)$ denote the number of ways to partition n into powers of two. Prove that for $n \geq 3$, $2^{n^2/4} < f(2^n) < 2^{n^2/2}$.

24. Let $a_1 = 1$, $a_2 = 2$, and for $n \geq 3$, let a_n be the smallest positive integer such that $a_n \neq a_i$ for $i < n$ and $\gcd(a_n, a_{n-1}) > 1$. Prove that every positive integer appears as some a_i.

25. (Sierpinski, USAMO 1982) Prove that there exists a positive integer k such that $k \cdot 2^n + 1$ is composite for all positive integers n.

26. (Erdős) Prove that there exists an infinite arithmetic progression of odd numbers, none of which can be written as the sum of a prime number and a power of two.

27. Prove that there exists an infinite set of natural numbers, none of which can be written as the sum of a prime number and two distinct powers of two.

28. (IMO Shortlist 2005) Let $n \geq 1$ be a given integer, and let $a_1, \ldots, a_n$ be a sequence of integers such that n divides the sum $a_1 + \cdots + a_n$. Show that there exist permutations σ and τ of $1, 2, \ldots, n$ such that $\sigma(i) + \tau(i) \equiv a_i \pmod{n}$ for all $i = 1, \ldots, n$.

29. (Erdős-Ginzburg-Ziv) Prove that for any sequence of $2n - 1$ integers, there exists some subsequence of length n whose sum is divisible by n.

30. (Cauchy-Davenport) Let A and B be sets of residue classes modulo a prime p, and let $A + B$ denote the set of residue classes $a + b$, where $a \in A$, $b \in B$. Prove that $|A + B| \geq \min\{p, |A| + |B| - 1\}$.