WINTER CAMP 2004
INEQUALITIES

A BRIEF SUMMARY OF BASIC INEQUALITIES.

1. **The triangle inequality**
   If a, b, c are real numbers, then $\left| |a-c|-|b-c| \right| \le |a-b| \le \left| |a-c|+|b-c| \right|$.

2. **The arithmetic-geometric-harmonic means inequality**
   If $x_1, x_2, x_3, \ldots, x_n$ are positive numbers, then
$$\frac{x_1+x_2+x_3+\ldots+x_n}{n} \ge \sqrt[n]{x_1 x_2 x_3 \ldots x_n} \ge \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_n}}$$
   with equality if and only if $x_1=x_2=x_3=\ldots=x_n$.

3. **The general means inequality**

   Let $x_1, x_2, x_3, \ldots, x_n$ be positive numbers. We define $M_r=\left(\frac{x_1^r+x_2^r+x_3^r+\ldots+x_n^r}{n}\right)^{1/r}$ for $r \ne 0$ and

   $M_0=\sqrt[n]{x_1 x_2 x_3 \ldots x_n}$. If $r>s$ then $M_r \ge M_s$, with equality if and only if $x_1=x_2=x_3=\ldots=x_n$.

4. **The general weighted means inequality**
   Let $x_1, x_2, x_3, \ldots, x_n, w_1, w_2, w_3, \ldots, w_n$ be positive numbers with $w_1+w_2+w_3+\ldots+w_n=1$.
   We define $WM_r=\left(w_1x_1^r+w_2x_2^r+w_3x_3^r+\ldots+w_nx_n^r\right)^{1/r}$ for $r \ne 0$ and $WM_0=x_1^{w_1} x_2^{w_2} x_3^{w_3} \ldots x_n^{w_n}$.
   If $r>s$ then $WM_r \ge WM_s$, with equality if and only if $x_1=x_2=x_3=\ldots=x_n$.

5. **The Minkowski inequality**
   If $x_1, x_2, x_3, \ldots, x_n, y_1, y_2, y_3, \ldots, y_n$ are all ≥ 0 and $p \ge 1$, then
$$\left(\sum_{k=1}^{n}(x_k+y_k)^p\right)^{1/p} \le \left(\sum_{k=1}^{n}x_k^p\right)^{1/p}+\left(\sum_{k=1}^{n}y_k^p\right)^{1/p}$$
   with equality if and only if there exists λ such that $y_k=\lambda x_k$ for $k=1,2,3, \ldots, n$.
   The inequality is reversed if $0<p<1$.

6. **The Cauchy-Schwarz inequality**
   If $v_1, v_2, v_3, \ldots, v_n$ and $w_1, w_2, w_3, \ldots, w_n$ are real numbers, then
   $\left|v_1w_1+v_2w_2+v_3w_3+\ldots+v_nw_n\right| \le \sqrt{v_1^2+v_2^2+v_3^2+\ldots+v_n^2}\sqrt{w_1^2+w_2^2+w_3^2+\ldots+w_n^2}$, with equality if and only if there exists λ such that $w_k=\lambda v_k$ for $k=1,2,3, \ldots, n$.

7. **The Hölder inequality**
   If $x_1, x_2, x_3, \ldots, x_n, y_1, y_2, y_3, \ldots, y_n, p, q$ are all ≥ 0 and $p+q=1$, then
$$\sum_{i=1}^{n}x_i^p y_i^q \le \left(\sum_{i=1}^{n}x_i\right)^p\left(\sum_{i=1}^{n}y_i\right)^q$$
   with equality if and only if there exists λ such that $y_k=\lambda x_k$ for $k=1,2,3, \ldots, n$.

8. **The rearrangement inequality**
Suppose that $x_1 \le x_2 \le x_3 \le \ldots \le x_n$ and $y_1 \le y_2 \le y_3 \le \ldots \le y_n$, and let $z_1, z_2, z_3, \ldots, z_n$ be any permutation of the numbers $y_1, y_2, y_3, \ldots, y_n$, then

$$\sum_{i=1}^{n} x_i y_{n+1-i} \le \sum_{i=1}^{n} x_i z_i \le \sum_{i=1}^{n} x_i y_i .$$

9. **The Chebyshev inequality**
Suppose that $0 \le x_1 \le x_2 \le x_3 \le \ldots \le x_n$ and $0 \le y_1 \le y_2 \le y_3 \le \ldots \le y_n$, then

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \le n \sum_{i=1}^{n} x_i y_i .$$

<u>EXERCISES.</u>

1. Prove each of the following inequalities.

   a) If $0 \le x \le \pi/2$ then $2x \le \pi \sin x \le \pi x$. (Jordan)

   b) If $x > -1$ and $0 < r < 1$, then $(1+x)^r \le 1 + rx$. (Bernoulli)

   c) If a, b, p, q are all positive and $p+q=1$, then $ab \le p\,a^{1/p} + q\,b^{1/q}$. (Young)

   d) If a, b, c are all positive, then $\dfrac{a}{b+c} + \dfrac{b}{a+c} + \dfrac{c}{a+b} \ge \dfrac{3}{2}$. (Nesbitt)

2. Prove the rearrangement inequality.
3. Prove the Chebyshev inequality.
4. Find the volume of the largest rectangular box that fits inside the ellipsoid $x^2 + 3y^2 + 9z^2 = 9$, with faces parallel to the coordinate planes.
5. What is the maximum possible value of the expression $\dfrac{1+a+2b+3c}{\sqrt{1+2(a^2+b^2+c^2)}}$?

   What are the values of a, b and c for which the maximum value is reached ?
6. If a, b and c are positive numbers, what is the minimum possible value of the expression

$$\frac{1+a+2b+3c}{\left(1+\sqrt[3]{a}+2\sqrt[3]{b}+3\sqrt[3]{c}\right)^3} \ ?$$

   What are the values of a, b and c for which the minimum value is reached ?
7. Prove that for any positive a, b and c, $(a+b)(b+c)(a+c) \ge 8abc$.
8. Let $n > 3$ be an integer and let $x_1, x_2, x_3, \ldots, x_n$ be positive numbers such that $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$.

   Prove that $\dfrac{x_1}{1+x_2^2} + \dfrac{x_2}{1+x_3^2} + \ldots + \dfrac{x_n}{1+x_1^2} \ge \dfrac{4}{5}\left(x_1\sqrt{x_1} + x_2\sqrt{x_2} + \ldots + x_n\sqrt{x_n}\right)^2$.
9. Let $x_1, x_2, x_3, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \ldots + \frac{x_n}{1+x_1^2+\ldots+x_n^2} < \sqrt{n} .$$

10. **IMO 1975. A1.**

Let $x_1, x_2, x_3, \dots, x_n$ and $y_1, y_2, y_3, \dots, y_n$ be real numbers such that $x_1 \le x_2 \le \dots \le x_n$ and $y_1 \le y_2 \le \dots \le y_n$. Prove that, if $z_1, z_2, z_3, \dots, z_n$ is any permutation of $y_1, y_2, y_3, \dots, y_n$, then

$$\sum_{i=1}^{n} (x_i - y_i)^2 \le \sum_{i=1}^{n} (x_i - z_i)^2 .$$

11. **IMO 1978. B2**

Let $a_1, a_2, a_3, \dots, a_n$ be a sequence of distinct positive integers. Prove that, for all natural numbers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \ge \sum_{k=1}^{n} \frac{1}{k} .$$

12. **IMO 1984. A1**

Prove that $0 \le xy + yz + zx - 2xyz \le 7/27$, where x, y, z are non-negative real numbers such that $x + y + z = 1$.

SOME RECENT IMO PROBLEMS.

13. **IMO 2003. B2.**

Let $n > 2$ be a positive integer and let $x_1, x_2, \dots, x_n$ be real numbers with $x_1 \le x_2 \le \dots \le x_n$.

a) Show that $\left(\sum_{i=1}^{n} \sum_{j=1}^{n} \left| x_i - x_j \right| \right)^2 \le \frac{2}{3}(n^2 - 1) \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2$.

b) Show that equality holds if and only if $x_1, x_2, \dots, x_n$ is an arithmetic progression.

14. **IMO 2001. A2.**

Let a, b and c be positive real numbers. Prove that $\dfrac{a}{\sqrt{a^2 + 8bc}} + \dfrac{b}{\sqrt{b^2 + 8ca}} + \dfrac{c}{\sqrt{c^2 + 8ab}} \ge 1$.

15. **IMO 2000. A2.**

Let a, b and c be positive real numbers such that $abc = 1$.
Prove that $(a - 1 + 1/b)(b - 1 + 1/c)(c - 1 + 1/a) \le 1$.

16. **IMO 1999. A2.**

Let $n \ge 2$ be a fixed integer.

a) Determine the least constant C such that the inequality $\sum_{1 \le i < j \le n} x_i x_j (x_i^2 + x_j^2) \le C \left(\sum_{1 \le i \le n} x_i \right)^4$ holds for all real numbers $x_1, x_2, \dots, x_n \ge 0$.

b) For this constant C, determine when the equality holds.

17. **IMO 1997. A3.**

Let $x_1, x_2, \dots, x_n$ be real numbers satisfying the conditions $|x_1 + x_2 + \dots + x_n| = 1$ and $|x_i| \le \dfrac{n+1}{2}$ for $i = 1, 2, \dots, n$. Show that there exists a permutation $y_1, y_2, \dots, y_n$ of $x_1, x_2, \dots, x_n$ such that

$$|y_1 + 2y_2 + \dots + ny_n| \le \frac{n+1}{2} .$$