Functional Equations

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1 A Reminder

Remember, before trying to prove something it is best to try to disprove it. Additionally, before you try to show that $a \implies b$, it might be good to find a situation in which this is not true.

2 Definitions

1. A function is *one-to-one* if $f(x) = f(y) \implies x = y$.
2. A function $f : S \to T$ is *onto* if $\forall t \in T \exists s \in S$ s.t. $f(s) = t$.
3. A function $f : S \to T$ is a *one-to-one correspondence* between S and T if it is both one-to-one and onto.
4. Alternately, we can substitute the words injective, surjective, and bijective for the terms one-to-one, onto, and one-to-one correspondence, which often makes what you mean more clear.

3 Problems

1. Prove that there is a rational between any two real numbers.
2. Find all monotonically increasing functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x+y) = f(x)+f(y)$.
3. Find all functions satisfying $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$.
4. Find all continuous functions satisfying $f(x + y) = f(x) + f(y)$.
5. Find all polynomials satisfying $(x - 16)p(2x) = 16(x - 1)p(x)$.
6. Find all functions $f : \mathbb{R} \to [0, \infty)$ such that for all $x, y \in \mathbb{R}$,
$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$$
7. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying

   (a) For every $n \in \mathbb{N}$, $f(n + f(n)) = f(n)$
   (b) For some $n_0 \in \mathbb{N}, f(n_0) = 1$
   Show that $f(n) = 1$ for all $n \in \mathbb{N}$.

[1] with large amounts of influence from Reid Barton's files http://web.mit.edu/rwbarton/Public/mop/ and corresponding MOP lecture.

8. (USAMO 2000/1) Prove that there does not exist any function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{f(x) + f(y)}{2} \geq f(\frac{x+y}{2}) + |x - y|$$

9. (USAMO 2002/4) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

10. Let $n > 2$ be an integer and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that

$$f(A_1) + f(A_2) + \ldots + f(A_n) = 0$$

for all regular n-gons $A_1A_2\ldots A_n$. Prove that f is the zero function.

11. Let $f : [0, 1] \to \mathbb{R}$ be a function such that
(a) $f(1) = 1$,
(b) $f(x) \geq 0$ for all $x \in [0, 1]$,
(c) if x,y and $x + y$ all lie in $[0, 1]$, then $f(x + y) \geq f(x) + f(y)$.
Prove that $f(x) \leq 2x$ for all $x \in [0, 1]$.

12. Find all functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

(a) if $x < y$, then $f(x) < f(y)$
(b) for all $x, y \in \mathbb{R}$, $f(xy) = g(y)f(x) + f(y)$

13. Find all functions $f(f(x) + y) = f(x^2 - y) + 4f(x)y$ holds for all $x, y \in \mathbb{R}$.

14. Prove that there does not exist a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x)^2 \geq f(x + y)(f(x) + y)$$

15. Let $\mathbb{Q}^+$ denote the set of positive rationals. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$ $f(x + 1) = f(x) + 1$ and $f(x^2) = f(x)^2$.