Art of Problem Solving

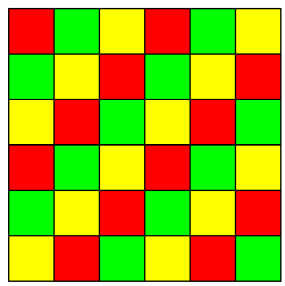
WOOT 2010–11

Practice Olympiad 3 Solutions

1. A square of side n is divided into n^2 unit squares, each colored red, yellow, or green. Find the minimum value of n such that for any such coloring, we can find a row or a column containing at least three squares of the same color.

   Solution. The minimum value of n is 7. By the Pigeonhole Principle, if a row contains 7 unit squares (or more), then there has to be at least three squares of the same color in that row.

   However, we can color a square of side 6 so that no row or column contains at least three squares of the same color, as follows:



   Also, it is clear we can use the same coloring scheme for any positive integer $n \leq 6$. Hence, the minimum value of n is 7.

2. The function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x) \le x$$

and

$$f(x+y) \le f(x) + f(y)$$

for all real numbers x and y. Prove that $f(x) = x$ for all real numbers x.

Solution. In the first inequality, taking $x = 0$, we get $f(0) \le 0$. In the second inequality, taking $x = y = 0$, we get $f(0) \le 2f(0)$, so $f(0) \ge 0$. Hence, $f(0) = 0$.

Taking $y = -x$ in the second inequality, we get

$$0 \le f(x) + f(-x).$$

But from the first inequality, $f(x) \le x$ and $f(-x) \le -x$, so

$$0 \le f(x) + f(-x) \le x + (-x) = 0.$$

Therefore, we must have equality, so $f(x) = x$ for all real numbers x.

3. Determine the largest positive integer that is a factor of

$$n^4(n-1)^3(n-2)^2(n-3)$$

for all positive integers n.

Solution 1. Let $f(n) = n^4(n-1)^3(n-2)^2(n-3)$, and let d be the greatest positive integer that divides $f(n)$ for all positive integers n. Then in particular, d divides each of the numbers

$$\begin{aligned} f(4) &= 4^4 \cdot 3^3 \cdot 2^2 \cdot 1 = 2^{10} \cdot 3^3, \\ f(5) &= 5^4 \cdot 4^3 \cdot 3^2 \cdot 2 = 2^7 \cdot 3^2 \cdot 5^4, \\ f(7) &= 7^4 \cdot 6^3 \cdot 5^2 \cdot 4 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7^4. \end{aligned}$$

Therefore, d divides $\gcd(f(4), f(5), f(7)) = 2^5 \cdot 3^2 = 288$. We claim that $f(n)$ is divisible by 288 for all positive integers n.

If n is even, then $n-2$ is also even, which means that $f(n)$ is divisible by $2^4 \cdot 2^2 = 2^6$. Otherwise, n is odd. If $n \equiv 1 \pmod 4$, then $n-1$ is divisible by 4 and $n-3$ is divisible by 2, so $f(n)$ is divisible by $4^3 \cdot 2 = 2^7$. If $n \equiv 3 \pmod 4$, then $n-1$ is divisible by 2 and $n-3$ is divisible by 4, so $f(n)$ is divisible by $2^3 \cdot 4 = 2^5$. Hence, $f(n)$ is divisible by 2^5 for all positive integers n.

Also, for all positive integers n, one of n, $n-1$, or $n-2$ must be divisible by 3, which implies that $f(n)$ is always divisible by 3^2.

Therefore, $f(n)$ is divisible by $2^5 \cdot 3^2 = 288$ for all positive integers n. We conclude that the answer is 288.

Solution 2. As in Solution 1, d must divide 288. But

$$n^4(n-1)^3(n-2)^2(n-3) = 288\binom{n}{4}\binom{n}{3}\binom{n}{2}\binom{n}{1},$$

which is always divisible by 288, so the answer is 288.

4. Let n be a positive integer. For $1 \le k \le n$, let a_k denote the number of pairs (x, y) of nonnegative integers satisfying $kx + (k+1)y = n - k + 1$. Show that $a_1 + a_2 + \cdots + a_n = n$.

Solution 1. Rearranging the equation $kx + (k+1)y = n - k + 1$, we get

$$k(x + y + 1) + y = n + 1.$$

Setting $z = x + y + 1$, this equation becomes

$$kz + y = n + 1.$$

Since $0 \le y < x + y + 1 = z$, we can describe k and y as the quotient and remainder, respectively, that results from dividing $n + 1$ by z. Thus, given the value $z = x + y + 1$, there are unique values k and y that satisfy the original equation $kx + (k+1)y = n - k + 1$.

If $z = 1$, then k is equal to $n + 1$ (and y is equal to 0), which is not allowed because $k \le n$. Also, if $z \ge n + 2$, then $k = 0$, which is also not allowed because $k \ge 1$. On the other hand, if $2 \le z \le n + 1$, then k, as the quotient that results from dividing $n + 1$ by z, lies between 1 and n. Therefore, there are a total of n possible values of z that lead to a solution where $1 \le k \le n$.

Furthermore, for each possible value of $z = x + y + 1$, k and y are uniquely determined, which means that x is also uniquely determined. Therefore, the total number of solutions (x, y) where $1 \le k \le n$ is $a_1 + a_2 + \cdots + a_n = n$.

Solution 2. We can write the equation $kx + (k+1)y = n - k + 1$ as

$$kx + (k+1)y + k = n + 1.$$

Hence, a_k, the number of solutions in nonnegative integers x and y to this equation, is the coefficient of t^{n+1} in the generating function

$$\begin{aligned}
&(t^k + t^{2k} + t^{3k} + \cdots)(t^{k+1} + t^{2(k+1)} + t^{3(k+2)} + \cdots)t^k \\
&= \frac{1}{1 - t^k} \cdot \frac{1}{1 - t^{k+1}} \cdot t^k \\
&= \frac{t^k}{(1 - t^k)(1 - t^{k+1})}.
\end{aligned}$$

We can express this as

$$\begin{aligned}
\frac{t^k}{(1 - t^k)(1 - t^{k+1})} &= \frac{1}{(1-t)^2} \left[\frac{t^k}{(1 + t + t^2 + \cdots + t^{k-1})(1 + t + t^2 + \cdots + t^k)} \right] \\
&= \frac{1}{(1-t)^2} \left[\frac{(1 + t + t^2 + \cdots + t^k) - (1 + t + t^2 + \cdots + t^{k-1})}{(1 + t + t^2 + \cdots + t^{k-1})(1 + t + t^2 + \cdots + t^k)} \right] \\
&= \frac{1}{(1-t)^2} \left(\frac{1}{1 + t + t^2 + \cdots + t^{k-1}} - \frac{1}{1 + t + t^2 + \cdots + t^k} \right) \\
&= \frac{1}{(1-t)^2} \left(\frac{1 - t}{1 - t^k} - \frac{1 - t}{1 - t^{k+1}} \right) \\
&= \frac{1}{(1-t)(1-t^k)} - \frac{1}{(1-t)(1-t^{k+1})}.
\end{aligned}$$

Hence, $a_1 + a_2 + \cdots + a_n$ is the coefficient of t^{n+1} in

$$\begin{aligned}
&\sum_{k=1}^{n} \left[\frac{1}{(1-t)(1-t^k)} - \frac{1}{(1-t)(1-t^{k+1})} \right] \\
&= \left[\frac{1}{(1-t)(1-t)} - \frac{1}{(1-t)(1-t^2)} \right] + \left[\frac{1}{(1-t)(1-t^2)} - \frac{1}{(1-t)(1-t^3)} \right] \\
&\quad + \cdots + \left[\frac{1}{(1-t)(1-t^n)} - \frac{1}{(1-t)(1-t^{n+1})} \right] \\
&= \frac{1}{(1-t)^2} - \frac{1}{(1-t)(1-t^{n+1})}.
\end{aligned}$$

The coefficient of t^{n+1} in

$$\frac{1}{(1-t)^2} = 1 + 2t + 3t^2 + \cdots$$

is $n+2$, and the coefficient of t^{n+1} in

$$\begin{aligned}
\frac{1}{(1-t)(1-t^{n+1})} &= (1 + t + t^2 + \cdots)(1 + t^{n+1} + t^{2(n+1)} + \cdots) \\
&= 1 + t + t^2 + \cdots + t^n + 2t^{n+1} + \cdots
\end{aligned}$$

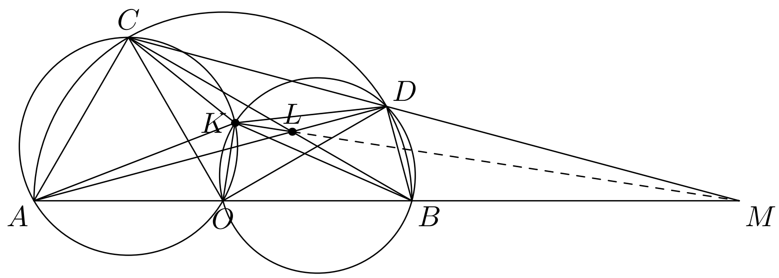
is 2. Therefore, the coefficient of t^{n+1} in

$$\frac{1}{(1-t)^2} - \frac{1}{(1-t)(1-t^{n+1})}$$

is $(n+2) - 2 = n$.

5. A semicircle has center O and diameter AB. Let M be a point on AB extended past B. A line through M intersects the semicircle at C and D, so that D is closer to M than C. The circumcircles of triangles AOC and DOB intersect at O and K. Show that $\angle MKO = 90°$.

Solution. Let L be the intersection of AD and BC.



Since quadrilaterals $AOKC$ and $BOKD$ are cyclic, $\angle OKA = \angle OCA = \angle OAC$ and $\angle OKB = \angle ODB = \angle OBD$. Then

$$\begin{aligned}\angle AKB &= \angle OKA + \angle OKB \\ &= \angle OAC + \angle OBD \\ &= \frac{1}{2}(\overset{\frown}{CDB} + \overset{\frown}{ACD}) \\ &= \frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{CD}) \\ &= 90° + \frac{\overset{\frown}{CD}}{2}.\end{aligned}$$

But

$$\angle ALB = 90° + \frac{\overset{\frown}{CD}}{2},$$

so quadrilateral $AKLB$ is cyclic.

Then

$$\begin{aligned}\angle LKO &= \angle LKA - \angle OKA \\ &= (180° - \angle LBO) - \angle OKA \\ &= 180° - \frac{\overset{\frown}{AC}}{2} - \frac{\overset{\frown}{BC}}{2} \\ &= 90°.\end{aligned}$$

Also,

$$\begin{aligned}\angle DKC &= 360^\circ - \angle DKO - \angle CKO \\ &= \angle DBO + \angle CAO \\ &= 90^\circ + \frac{\widehat{CD}}{2} \quad \text{(from our work above)} \\ &= \angle DLC,\end{aligned}$$

so quadrilateral $DLKC$ is also cyclic.

Let ω_1, ω_2, and ω_3 denote the circumcircles of quadrilaterals $ABCD$, $AKLB$, and $DLKC$, respectively. Then the radical axis of ω_1 and ω_2 is AB, the radical axis of ω_1 and ω_3 is CD, and the radical axis of ω_2 and ω_3 is KL. All three radical axes concur at the radical center of ω_1, ω_2, and ω_3, which must be M. Therefore, M lies on KL, which means $\angle MKO = 90^\circ$.

6. Let $\{A_1, A_2, \ldots, A_{2010}\}$ and $\{B_1, B_2, \ldots, B_{2010}\}$ be two partitions of the set $\{1, 2, \ldots, n\}$, such that for all $1 \le i, j \le 2010$, if $A_i \cap B_j = \emptyset$, then $|A_i| + |B_j| \ge 2010$. Find the minimum value of n for which such partitions exist.

   Note: We say that $\{S_1, S_2, \ldots, S_k\}$ is a partition of the set S if $S_i \cap S_j = \emptyset$ for all $i \ne j$, and $S_1 \cup S_2 \cup \cdots \cup S_k = S$.

   Solution. The minimum value of n is $1005 \cdot 2010$.

   First, we show that there exist such partitions of $\{1, 2, \ldots, 1005 \cdot 2010\}$. Let $\{A_1, A_2, \ldots, A_{2010}\}$ and $\{B_1, B_2, \ldots, B_{2010}\}$ be any partitions where each part contains 1005 elements. Then $|A_i| + |B_j| = 1005 + 1005 \ge 2010$ for all i and j, so the condition is clearly satisfied.

   Now, let $\{A_1, A_2, \ldots, A_{2010}\}$ and $\{B_1, B_2, \ldots, B_{2010}\}$ be two partitions of $\{1, 2, \ldots, n\}$ that satisfy the given conditions. We claim that $n \ge 1005 \cdot 2010$. Without loss of generality, assume that A_1 has the smallest cardinality among all of the sets $A_1, A_2, \ldots, A_{2010}, B_1, B_2, \ldots, B_{2010}$.

   Let $k = |A_1|$. If $k \ge 1005$, then

$$n = |A_1| + |A_2| + \cdots + |A_{2010}| \ge 2010k \ge 1005 \cdot 2010,$$

   and we are done. Otherwise, $k < 1005$.

   Suppose that exactly m of the 2010 sets $B_1, B_2, \ldots, B_{2010}$ have non-empty intersection with A_1. Each of these m sets have cardinality at least k. Also, since the sets B_i are disjoint, $m \le k$. From the given condition, each of the remaining $2010 - m$ sets does not intersect with A_1, so each such set must have cardinality at least $2010 - k$. Hence,

$$\begin{aligned} n &= |B_1| + |B_2| + \cdots + |B_{2010}| \\ &\ge mk + (2010 - m)(2010 - k) \\ &= 2010^2 - 2010k + 2m(k - 1005). \end{aligned}$$

   Since $k < 1005$ and $m \le k$,

$$\begin{aligned} n &\ge 2010^2 - 2010k + 2k(k - 1005) \\ &= 2010^2 - 2k(2010 - k). \end{aligned}$$

   By the AM-GM inequality,

$$k(2010 - k) \le \frac{2010^2}{4},$$

   so

$$n \ge 2010^2 - 2 \cdot \frac{2010^2}{4} = \frac{2010^2}{2} = 1005 \cdot 2010.$$

7. The sequence a_0, a_1, a_2, ... is defined by $a_0 = 1$ and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{3a_n}\right).$$

Let

$$A_n = \frac{3}{3a_n^2 - 1}.$$

Prove that A_n is a perfect square and that it has at least n distinct prime divisors for all $n \geq 1$.

Solution. Note that

$$\begin{aligned}
\frac{A_{n+1}}{A_n} &= \frac{3a_n^2 - 1}{3a_{n+1}^2 - 1} \\
&= \frac{3a_n^2 - 1}{\frac{3}{4}(a_n^2 + \frac{2}{3} + \frac{1}{9a_n^2}) - 1} \\
&= \frac{12a_n^2(3a_n^2 - 1)}{9a_n^4 - 6a_n^2 + 1} \\
&= \frac{12a_n^2(3a_n^2 - 1)}{(3a_n^2 - 1)^2} \\
&= \frac{12a_n^2}{3a_n^2 - 1} \\
&= 4a_n^2 \cdot \frac{3}{3a_n^2 - 1} \\
&= 4a_n^2 A_n,
\end{aligned}$$

so $A_{n+1} = 4a_n^2 A_n^2$. Also, from the given relationship,

$$a_n^2 = \frac{1}{3}\left(\frac{3}{A_n} + 1\right) = \frac{A_n + 3}{3A_n},$$

so

$$A_{n+1} = 4a_n^2 A_n^2 = 4 \cdot \frac{A_n + 3}{3A_n} \cdot A_n^2 = \frac{4}{3}A_n(A_n + 3).$$

We prove by induction that A_n and $(A_n + 3)/3$ are perfect squares for all $n \geq 1$. Since $A_1 = 9$ and $(A_1 + 3)/3 = 4$, the claim is true for $n = 1$.

Assume that A_k and $(A_k + 3)/3$ are perfect squares for some positive integer $k \geq 1$. Then

$$A_{k+1} = \frac{4}{3}A_k(A_k + 3) = 4A_k \cdot \frac{A_k + 3}{3}$$

is a perfect square. Also,

$$\begin{aligned}\frac{A_{k+1}+3}{3} &= \frac{4A_k(A_k+3)/3+3}{3} \\ &= \frac{4A_k^2+12A_k+9}{9} \\ &= \left(\frac{2A_k+3}{3}\right)^2.\end{aligned}$$

Since $(A_k+3)/3 = A_k/3+1$ is an integer, $A_k/3$ is an integer, so $2A_k/3+1 = (2A_k+3)/3$ is an integer. Hence,

$$\frac{A_{k+1}+3}{3} = \left(\frac{2A_k+3}{3}\right)^2$$

is a perfect square. Therefore, the claim is true for $n = k+1$, and by induction, for all $n \geq 1$.

We also prove by induction that A_n has at least n distinct prime divisors for all $n \geq 1$. The number $A_1 = 9$ has at least one prime divisor, so the claim is true for $n = 1$.

Assume that A_k has at least k distinct prime divisors for some positive integer $k \geq 1$. We know that

$$A_{k+1} = 4A_k \cdot \frac{A_k+3}{3}.$$

Since

$$3 \cdot \frac{A_k+3}{3} - A_k = 3,$$

the greatest common divisor of $(A_k+3)/3$ and A_k must be 1 or 3.

We know that $A_k/3$ is a positive integer, i.e. A_k is divisible by 3. But A_k is a perfect square, so A_k is divisible by 9, which means $A_k/3$ is divisible by 3. Hence,

$$\frac{A_k+3}{3} = \frac{A_k}{3}+1$$

is not divisible by 3.

Therefore, the greatest common divisor of $(A_k+3)/3$ and A_k is 1, i.e. $(A_k+3)/3$ and A_k are relatively prime, which means that $(A_k+3)/3$ has a prime divisor that does not divide A_k. Therefore,

$$A_{k+1} = 4A_k \cdot \frac{A_k+3}{3}$$

has at least $k+1$ distinct prime divisors, which completes the induction.