Applications of Pell's Equation

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Pell's Equation

The Equation: $x^2 - dy^2 = 1, \quad d \in \mathbf{Z}^+, \quad d \neq \square.$

x	3	17	99	577	3363	19601
y	2	12	70	408	2378	13860

Solutions to $x^2 - 2y^2 = 1$

x	2	7	26	97	362	1351
y	1	4	15	56	209	780

Solutions to $x^2 - 3y^2 = 1$

New Solutions from Old

$$(x_1^2 - dy_1^2)(x_2^2 - dy_2^2) = (x_1x_2 + dy_1y_2)^2 - d(x_1y_2 + x_2y_1)^2.$$

From one solution with $x > 0$ and $y > 0$, there are infinitely many.

Theorem (Lagrange, 1768)

For nonsquare $d \in \mathbf{Z}^+$, Pell's equation $x^2 - dy^2 = 1$ has infinitely many integral solutions.

Reducing to the Pell Equation

Theorem

For each nonzero $k \in \mathbf{Z}$, there are a finite number of solutions to $x^2 - dy^2 = k$, say $(x_1, y_1), \ldots, (x_n, y_n)$, such that every solution to $x^2 - dy^2 = k$ arises from

$$x = ax_i + dby_i, \quad y = bx_i + ay_i,$$

where $a^2 - db^2 = 1$.

Example

$x^2 - 3y^2 = -2 \iff x = a \pm 3b, y = b \pm a$, where $a^2 - 3b^2 = 1$.

Example

$x^2 - dy^2 = 1 \iff x = a, y = b$, where $a^2 - db^2 = 1$. (Here $x_1 = 1, y_1 = 0$.)

Irregular Behavior

Least positive solution to $x^2 - dy^2 = 1$ behaves erratically as d grows.

d	60	61	62	108	109	110
x	31	1766319049	63	1351	158070671986249	21
y	4	226153980	8	130	15140424455100	2

Fermat (1657) challenged mathematicians to solve $x^2 - dy^2 = 1$ for general $d \in \mathbf{Z}^+$, and if failing that to at least try $x^2 - 61y^2 = 1$ and $x^2 - 109y^2 = 1$, where he chose small coefficients "pour ne vous donner pas trop de peine" (so you don't have too much work).

Rational or Integral Solutions?

The British mathematicians first thought Fermat was asking for all *rational* solutions of $x^2 - dy^2 = 1$. That is easy:

$$x = \frac{1 + dm^2}{1 - dm^2}, \quad y = \frac{2m}{1 - dm^2}$$

for $m \in \mathbf{Q}$, along with $(-1, 0)$. Fermat was unimpressed:

> Ma proposition n'est que pour trouver des nombres entiers, qui satisfassent à la question, car, en cas de fractions, le moindre arithméticien en viendroit à bout. (My proposition is to find integers which satisfy the question, for in the case of fractions the lowest type of arithmetician could find the solution.)

It is not easy to see when the formulas for rational x and y have values in $\mathbf{Z}$, except for $m = 0$.

Application 1: Double Equations (from Diophantus)

Example

Find $t \in \mathbf{Z}$ such that $10t + 9 = x^2$ and $5t + 4 = y^2$.

$$t = \frac{x^2 - 9}{10} \Rightarrow 5\frac{x^2 - 9}{10} + 4 = y^2 \Rightarrow x^2 - 2y^2 = 1.$$

x	3	17	99	577	3363	19601
y	2	12	70	408	2378	13860
t	0	28		33292	1130976	

Fermat carefully read Diophantus. Maybe such examples got him interested in Pell's equation.

Application 2: Rational Approximations to Square Roots

We can't write $\sqrt{d} = \frac{x}{y}$ with $x, y \in \mathbf{Z}$: $\sqrt{d}$ is irrational. But

$$x^2 - dy^2 = 1 \Longrightarrow \left(\frac{x}{y}\right)^2 = d + \frac{1}{y^2} \approx d,$$

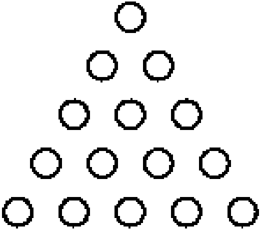
so Pell solutions lead to good rational approximations to $\sqrt{d}$.

Example

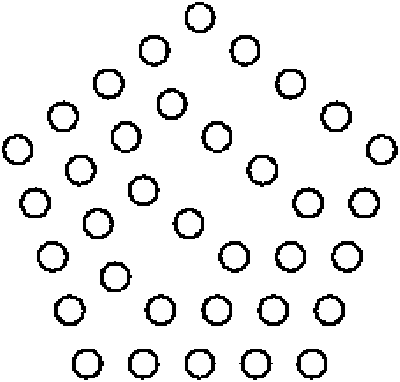
The fourth solution to $x^2 - 2y^2 = 1$ is $(x, y) = (577, 408)$ and $577/408 \approx 1.4142156$, while $\sqrt{2} \approx 1.4142135$.

Archimedes estimated $\sqrt{3} \approx 1.7320508$ by $265/153 \approx 1.7320261$ and $1351/780 \approx 1.7320512$; these x/y satisfy $x^2 - 3y^2 = -2$ and $x^2 - 3y^2 = 1$.

Polygonal Numbers



Triangular: 1, 3, 6, 10, 15



Pentagonal: 1, 5, 12, 22, 35

Formulas for Polygonal Numbers

$$T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

$$S_n = 1 + 3 + \cdots + (2n-1) = n^2.$$

$$P_n = 1 + 4 + \cdots + (3n-2) = \frac{3n^2 - n}{2}.$$

n	1	2	3	4	5	6	7	8	9	10
T_n	1	3	6	10	15	21	28	36	44	54
S_n	1	4	9	16	25	36	49	64	81	100
P_n	1	5	12	22	35	51	70	92	117	145

Application 3: Simultaneous Polygonal Numbers

Triangular-square number: $T_m = S_n$. So far, $T_1 = S_1$, $T_8 = S_6$. In general,

$$\begin{aligned} T_m &= S_n \\ \frac{m^2+m}{2} &= n^2 \\ m^2+m &= 2n^2 \\ \left(m+\frac{1}{2}\right)^2 - \frac{1}{4} &= 2n^2 \\ (2m+1)^2 - 1 &= 2(2n)^2 \\ (2m+1)^2 - 2(2n)^2 &= 1. \end{aligned}$$

We need to solve $x^2 - 2y^2 = 1$ with $x, y > 0$ and x odd, y even. Data suggest the parity (even/odd) constraints are automatic.

Simultaneous Polygonal Numbers

If $x^2 - 2y^2 = 1$, let's show x is odd and y is even:

$$x^2 = 2y^2 + 1 \Rightarrow x^2 \text{ is odd } \Rightarrow x \text{ is odd.}$$

Writing $x = 2m + 1$,

$$4m^2 + 4m + 1 = 2y^2 + 1 \Rightarrow y^2 = 2m^2 + 2m \text{ is even } \Rightarrow y \text{ is even.}$$

x	3	17	99	577	3363	19601
y	2	12	70	408	2378	13860
$m = (x-1)/2$	1	8	49	288	1681	9800
$n = y/2$	1	6	35	204	1189	6930
$T_m = S_n$	1	36	1225	41616	1413721	48024900

$$T_{49} = S_{35} = 1225 \text{ means } 1 + 2 + \cdots + 49 = 35 \cdot 35.$$

Simultaneous Polygonal Numbers

Square-pentagonal number: $S_m = P_n$. So far, $S_1 = P_1$.

$$\begin{aligned} S_m &= P_n \\ m^2 &= \frac{3n^2 - n}{2} \\ 2m^2 &= 3n^2 - n \\ &= 3\left(\left(n - \frac{1}{6}\right)^2 - \frac{1}{36}\right) \\ 6(2m)^2 &= (6n-1)^2 - 1 \\ (6n-1)^2 - 6(2m)^2 &= 1. \end{aligned}$$

Must solve $x^2 - 6y^2 = 1$ with $x, y > 0$ and $x = 6n - 1$, y even.

Simultaneous Polygonal Numbers

Solve $x^2 - 6y^2 = 1$ where $x = 6n - 1$, $y = 2m$. Will always have $x = 6n \pm 1$ and y even, but not always $x = 6n - 1$.

x	5	49	485	4801	47525	470449
y	2	20	198	1960	19402	192060
m	1		99		9701	
n	1		81		7921	
$S_m = P_n$	1		9801		94109401	

Can we solve $T_m = S_n = P_k$ other than 1? Will return to this later. Using hexagonal instead of square,

$$40755 = T_{285} = P_{165} = H_{143}.$$

Archimedes' cattle problem: count bulls and cows of the Sun god

$$w = \left(\frac{1}{2}+\frac{1}{3}\right) b + y, \; b = \left(\frac{1}{4}+\frac{1}{5}\right) s + y, \; s = \left(\frac{1}{6}+\frac{1}{7}\right) w + y$$

$$w' = \left(\frac{1}{3}+\frac{1}{4}\right)(b+b'), \; b' = \left(\frac{1}{4}+\frac{1}{5}\right)(s+s'),$$

$$s' = \left(\frac{1}{5}+\frac{1}{6}\right)(y+y'), \; y' = \left(\frac{1}{6}+\frac{1}{7}\right)(w+w'),$$

$w+b$ is square and $s+y$ is triangular.

Excluding last part, for any integer $k>0$ solution is

$$\begin{aligned}(w,b,s,y) &= (10366482k, 7460514k, 7358060k, 4149387k),\\ (w',b',s',y') &= (7206360k, 4893246k, 3515820k, 5439213k).\end{aligned}$$

For last part, $k = 4456749y^2$ and $410286423278424y^2+1 = x^2$.

Archimedes' cattle problem: count bulls and cows of the Sun god

$$
\begin{aligned}
x^2 - 410286423278424y^2 &= 1 \\
x^2 - 4729494(9314y)^2 &= 1
\end{aligned}
$$

Solution first "found" by Amthor in 1880, who determined the number of bulls and cows is an integer with 206,545 digits and initial digits 776. More fully, the number begins and ends as

$$776027140648\ldots719455081800.$$

Since it has been calculated that it would take the work of a thousand men for a thousand years to determine the complete number, it is obvious that the world will never have a complete solution.

Letter to the New York Times, January 18, 1931

Application 4: Sums of Consecutive Integers

Example

$1+2=3, \quad 1+2+\cdots+14=15+\cdots+20.$

$$\begin{aligned}
1+2+\cdots+k &= (k+1)+\cdots+\ell \\
\frac{(1+k)k}{2} &= \frac{(k+1+\ell)(\ell-k)}{2} \\
2k^2+2k &= \ell^2+\ell \\
2\left(\left(k+\frac{1}{2}\right)^2-\frac{1}{4}\right) &= \left(\ell+\frac{1}{2}\right)^2-\frac{1}{4} \\
2((2k+1)^2-1) &= (2\ell+1)^2-1 \\
(2\ell+1)^2-2(2k+1)^2 &= -1.
\end{aligned}$$

Sums of Consecutive Integers

Solving

$$1 + 2 + \cdots + k = (k+1) + \cdots + \ell$$

is the same as solving the negative Pell equation

$$x^2 - 2y^2 = -1, \quad x, y > 0, \quad x = 2\ell + 1, \quad y = 2k + 1.$$

Exercise: Show $x^2 - 2y^2 = -1 \Rightarrow x$ and y are odd.

x	1	7	41	239	1393	8119
y	1	5	29	169	985	5741
$k = (y-1)/2$	0	2	14	84	492	2870
$\ell = (x-1)/2$	0	3	20	119	696	4059

This recovers $1 + 2 = 3$ and $1 + 2 + \cdots + 14 = 15 + \cdots + 20$.
Now we have more solutions, such as

$$1 + 2 + \cdots + 84 = 85 + \cdots + 119.$$

Sums of Consecutive Integers

Let's skip an integer.

Example

$1+2+3+4+5=7+8.$

$$\begin{aligned}
1+2+\cdots+(k-1) &= (k+1)+\cdots+\ell \\
\frac{k(k-1)}{2} &= \frac{(k+1+\ell)(\ell-k)}{2} \\
2k^2 &= \ell^2+\ell \\
2k^2 &= \left(\ell+\frac{1}{2}\right)^2-\frac{1}{4} \\
2(2k)^2 &= (2\ell+1)^2-1 \\
(2\ell+1)^2-2(2k)^2 &= 1.
\end{aligned}$$

Sums of Consecutive Integers

Solving

$$1 + 2 + \cdots + (k-1) = (k+1) + \cdots + \ell$$

is the same as solving $x^2 - 2y^2 = 1$ with $x = 2\ell + 1$ and $y = 2k$. Automatically x is odd and y is even, so no constraints.

x	3	17	99	577	3363	19601
y	2	12	70	408	2378	13860
$k = y/2$	1	6	35	204	1189	6930
$\ell = (x-1)/2$	1	8	49	288	1681	9800

This recovers $1 + 2 + 3 + 4 + 5 = 7 + 8$ and produces all the others, such as $1 + 2 + \cdots + 34 = 36 + \cdots + 49$.

Sums of Consecutive Integers

Let's skip m integers. Solving

$$1+2+\cdots+(k-m)=(k+1)+\cdots+\ell$$

for k and ℓ is the same as solving

$$(2\ell+1)^2-2(2k+1-m)^2=2m^2-1.$$

Need to solve

$$x^2-2y^2=2m^2-1,\ x \text{ odd and } y\equiv 1+m \bmod 2 \text{ with } x>y+m.$$

Nonsolution: $x=1, y=m$. Instead look at $(1+m\sqrt{2})(1+\sqrt{2})$.
First solution: $x=2m+1, y=m+1$. NO: $x=y+m$.
Next: $x=10m+7, y=7m+5 \Rightarrow k=4m+2, \ell=5m+3$.

$$1+2+\cdots+(3m+2)=(4m+3)+\cdots+(5m+3).$$

Sums of Consecutive Integers

Make sum of consecutive integers equal to product of outer terms:

$$k + (k+1) + \cdots + \ell \stackrel{?}{=} k\ell.$$

Example

$3 + 4 + 5 + 6 = 3 \cdot 6.$

$$\begin{aligned}
k + (k+1) + \cdots + \ell &= k\ell \\
\frac{(k+\ell)(\ell - k + 1)}{2} &= k\ell \\
\ell^2 + \ell - k^2 + k &= 2k\ell \\
\ell^2 - 2k\ell - k^2 + \ell + k &= 0 \\
(\ell - k)^2 - 2k^2 + \ell + k &= 0.
\end{aligned}$$

Sums of Consecutive Integers

$$k + (k+1) + \cdots + \ell \stackrel{?}{=} k\ell.$$

Set $u = \ell - k$, so

$$\begin{aligned}
(\ell - k)^2 - 2k^2 + \ell + k &= 0 \\
u^2 - 2k^2 + (u+k) + k &= 0 \\
u^2 + u - 2(k^2 - k) &= 0 \\
\left(u + \frac{1}{2}\right)^2 - \frac{1}{4} - 2\left(\left(k - \frac{1}{2}\right)^2 - \frac{1}{4}\right) &= 0 \\
(2u+1)^2 - 2(2k-1)^2 &= -1.
\end{aligned}$$

Need to solve $x^2 - 2y^2 = -1$ with $x = 2u + 1 = 2(\ell - k) + 1$ and $y = 2k - 1$. Must have x and y odd, so no constraints.

Sums of Consecutive Integers

$$k + (k+1) + \cdots + \ell = k\ell \iff (2(\ell - k) + 1)^2 - 2(2k-1)^2 = -1.$$

x	1	7	41	239	1393	8119
y	1	5	29	169	985	5741
$k = (y+1)/2$	1	3	15	85	493	2871
$\ell = (x+y)/2$	1	6	35	204	1189	6930

$$\begin{aligned} 3 + 4 + 5 + 6 &= 3 \cdot 6, \\ 15 + 16 + \cdots + 35 &= 15 \cdot 35, \\ 85 + 86 + \cdots + 204 &= 85 \cdot 204. \end{aligned}$$

Application 5: Pythagorean Triangle with Consecutive Legs

Example

3,4,5

Want to solve

$$m^2 + (m+1)^2 = n^2.$$

Then n^2 is odd, so n is odd.

$$\begin{aligned}
2m^2 + 2m + 1 &= n^2 \\
2(m^2 + m) + 1 &= n^2 \\
2\left(\left(m + \frac{1}{2}\right)^2 - \frac{1}{4}\right) + 1 &= n^2 \\
(2m+1)^2 + 1 &= 2n^2 \\
(2m+1)^2 - 2n^2 &= -1.
\end{aligned}$$

Pythagorean Triangle with Consecutive Legs

$$m^2 + (m+1)^2 = n^2 \iff (2m+1)^2 - 2n^2 = -1.$$

Need to solve $x^2 - 2y^2 = -1$. Automatically x and y are odd, and want n odd, so no constraints.

x	1	7	41	239	1393	8119
y	1	5	29	169	985	5741
$m = (x-1)/2$	0	3	20	119	696	4059
$n = y$	1	5	29	169	985	5741

$$3^2 + 4^2 = 5^2$$

$$20^2 + 21^2 = 29^2$$

$$119^2 + 120^2 = 169^2$$

Pythagorean Triangle with Consecutive Legs

What about leg and hypotenuse consecutive?

$$m^2 + n^2 = (n+1)^2 \iff m^2 = 2n + 1.$$

Then m is odd. For $m = 2k + 1$, $n = (m^2 - 1)/2 = 2k^2 + 2k$.

k	1	2	3	4
$2k+1$	3	5	7	9
$2k^2+2k$	4	12	24	40
$2k^2+2k+1$	5	13	25	41

This has nothing to do with Pell's equation!

Application 6: Consecutive Heronian Triangles

Heronian triangle: integral sides and integral area.

Example

The 3, 4, 5 right triangle has area 6.

Hero's area formula (any triangle with sides a, b, c) says

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \quad s = \frac{a+b+c}{2}.$$

Find Heronian triangle with consecutive sides: $a-1, a, a+1$. Then $s = 3a/2$.

$$\begin{aligned} A^2 &= s(s-a)(s-a+1)(s-a-1) \\ A^2 &= \frac{3a}{2} \cdot \frac{a}{2} \cdot \frac{a+2}{2} \cdot \frac{a-2}{2}. \end{aligned}$$

Consecutive Heronian Triangles

Sides $a-1, a, a+1$ and area A: $(4A)^2 = 3a^2(a^2-4)$. Then a is even, say $a = 2x$, so

$$A^2 = 3x^2(x^2-1).$$

Unique factorization implies $x^2 - 1 = 3y^2$, so

$$x^2 - 3y^2 = 1, \quad a = 2x, \ A = 3xy.$$

x	2	7	26	97	362	1351
y	1	4	15	56	209	780
a	4	14	52	194	724	2702
A	6	84	1170	16926	226974	3161340

So we have triangles with sides 3, 4, 5 (area 6); 13, 14, 15 (area 84); 51, 52, 53 (area 1170); and so on.

Application 7: Sums of n and $n+1$ Consecutive Squares

For $n \in \mathbf{Z}^+$, find positive integers x and y such that

$$\underbrace{x^2 + (x+1)^2 + \cdots + (x+n-1)^2}_{n \text{ squares}} = \underbrace{y^2 + \cdots + (y+n)^2}_{n+1 \text{ squares}}.$$

At $n = 1$ this is Pythag. triangle with consecutive legs (Appn. 5). Writing $z = x - y$, some algebra turns it into

$$(y + n(1-z))^2 = n(n+1)z(z-1).$$

Let $n(n+1) = a^2 b$, b squarefree. Then $a^2 b | (y + (n-1)z)^2$, so (!) $ab | (y + (n-1)z)$. Write $y + n(1-z) = abw$, $w \in \mathbf{Z}$. Then the displayed equation above is the same as

$$a^2 b^2 w^2 = a^2 bz(z-1) \iff (2z-1)^2 - 4bw^2 = 1.$$

We are reduced to solving $v^2 - 4bw^2 = 1$ (v *must* be odd), and $4b \neq \square$. This is a fairly general Pell equation!

Sums of n and $n+1$ Consecutive Squares

Theorem (H. L. Alder, W. H. Simons (1967))

Writing $n(n+1) = a^2 b$ with squarefree b, solving

$$\underbrace{x^2 + (x+1)^2 + \cdots + (x+n-1)^2}_{n \text{ squares}} = \underbrace{y^2 + \cdots + (y+n)^2}_{n+1 \text{ squares}}$$

is the same as solving

$$v^2 - 4bw^2 = 1,$$

with

$$x = \frac{(n+1)v - (n-1)}{2} + abw, \quad y = n\frac{v-1}{2} + abw.$$

One soln: $v = 2n+1$ and $w = a$, so $x = 2n^2 + 2n + 1$ and $y = 2n^2 + n$: $5^2 = 3^2 + 4^2$, $13^2 + 14^2 = 10^2 + 11^2 + 12^2$, etc.

Sums of n and $n+1$ Consecutive Squares

For $n = 8$, $n(n+1) = 6^2 \cdot 2$, so $b = 2$ and $v^2 - 4bw^2 = v^2 - 8w^2$.

v	3	17	99	577	3363	19601
w	1	6	35	204	1189	6930
x	22	145	862	5041	29398	171361
y	20	136	812	4752	27716	161560

$$x^2 + \cdots + (x+7)^2 = y^2 + \cdots + (y+8)^2.$$

Does every squarefree b arise in such a problem? Seek n and a with

$$\begin{aligned} n(n+1) &\stackrel{?}{=} a^2 b \\ 4n(n+1) &\stackrel{?}{=} 4a^2 b \\ (2n+1)^2 - 4ba^2 &\stackrel{?}{=} 1. \end{aligned}$$

We can find n and a by solving $Y^2 - 4bX^2 = 1$.

Higher-degree Pell

For $n \geq 3$ and $d \in \mathbf{Z}^+$, seek integral solutions to $x^n - dy^n = 1$. There is a problem: no simple higher degree analogue of

$$(x_1^2 - dy_1^2)(x_2^2 - dy_2^2) = (x_1x_2 + dy_1y_2)^2 - d(x_1y_2 + x_2y_1)^2.$$

For instance,

$$\begin{aligned}(x_1^3 - dy_1^3)(x_2^3 - dy_2^3) &= (x_1x_2)^3 - d(x_1y_2 + x_2y_1)^3 + d^2(y_1y_2)^3 \\ &\quad +3dx_1x_2y_1y_2(x_1y_2 + x_2y_1).\end{aligned}$$

Same problem with rationalizing higher degree denominators:

$$\frac{1}{7+\sqrt{2}} = \frac{1}{7+\sqrt{2}} \cdot \frac{7-\sqrt{2}}{7-\sqrt{2}} = \frac{7-\sqrt{2}}{47},$$

$$\text{but} \quad \frac{1}{7+\sqrt[3]{2}} = \frac{1}{7+\sqrt[3]{2}} \cdot \frac{7-\sqrt[3]{2}}{7-\sqrt[3]{2}} = \frac{7-\sqrt[3]{2}}{49-\sqrt[3]{4}} =??$$

Degree 3

Theorem (Delone (1930), Nagell (1928))

For $d \in \mathbf{Z}$, the equation $x^3 - dy^3 = 1$ has at most one integral solution (x, y) besides $(1, 0)$.

Although there are finitely many integral solutions, there may or may not be finitely many rational solutions.

Example (Rational Solutions)

$x^3 - 2y^3 = 1$ for $(1, 0)$ and $(-1, -1)$.
$x^3 - 7y^3 = 1$ for $(1, 0), (2, 1), (1/2, -1/2), (17/73, -38/73) \ldots$.
$x^3 - 11y^3 = 1$ for $(1, 0)$.

Finiteness Theorems

Pell equations are essentially the only "interesting" equations in *two* variables with infinitely many integral solutions:

Theorem (Thue, 1909)

An integral polynomial equation of the form

$$c_n x^n + c_{n-1} x^{n-1} y + \cdots + c_0 y^n = k$$

with $k \neq 0$ *has finitely many integral solutions* (x, y) *except perhaps when the left side is a constant multiple of* $(ax + by)^n$ *or* $(ax^2 + bxy + cy^2)^{n/2}$.

In particular, for $d \in \mathbf{Z}$ and $n \geq 3$, the equation $x^n - dy^n = 1$ has only finitely many integral solutions (x, y).

Finiteness Theorems

Theorem (Siegel, 1929)

If $f(x,y)$ with integral coefficients is irreducible of degree at least 3, the equation $f(x,y) = 0$ has finitely many integral solutions except if it has a nonconstant integral polynomial solution: $f(a(t), b(t)) = 0$ where $a(t)$ and $b(t)$ are not both constant.

Example

For any $d \geq 1$, $f(x,y) = y^d - x^d(x+1)$ is irreducible but the equation $f(x,y) = 0$ has infinitely many integral solutions since $f(t^d - 1, t(t^d - 1)) = 0$.

This example doesn't contradict Thue's theorem.

Triangular-square-pentagonal number

$$T_m = S_n = P_k \iff (2m+1)^2 - 2(2n)^2 = 1, \quad (6k-1)^2 - 6(2n)^2 = 1.$$

Seek integers $x, y, z > 0$ such that $x^2 - 2y^2 = 1$, $z^2 - 6y^2 = 1$.

$$x^2 - 2y^2 = z^2 - 6y^2 \Rightarrow x^2 + (2y)^2 = z^2.$$

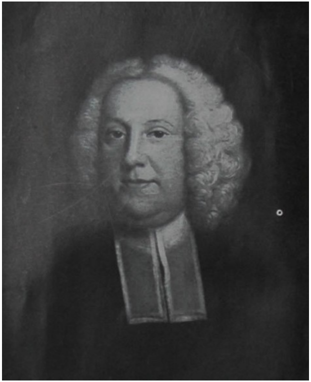
Since x odd and $(x, y) = 1$, $(x, 2y, z)$ is primitive Pythag. triple:

$$x = p^2 - q^2, \quad 2y = 2pq, \quad z = p^2 + q^2.$$

$x^2 - 2y^2 = 1 \Rightarrow (p^2 - q^2)^2 - 2(pq)^2 = 1 \Rightarrow (p^2 - 2q^2)^2 - 3q^4 = 1.$

$u^2 - 3v^4 = 1$ has only *two* solutions in $\mathbf{Z}^+$: $(2, 1)$ and $(7, 2)$. First leads back to $m = n = k = 1$, second doesn't lead back.

Why is it called Pell's equation?



John Pell (1611 – 1685)

Why is it called Pell's equation?

Mike Pellegrino (1986 –)

Why is it called Pell's equation?

Theorem (Fermat)

No triangular number larger than 1 is a fourth power.

Proof.

$m(m+1)/2 = n^4 \Rightarrow m(m+1) = 2n^4$. Since m and $m+1$ are relatively prime, $\{m, m+1\} = \{x^4, 2y^4\}$, so

$$x^4 - 2y^4 = \pm 1 \Longrightarrow y^8 \pm x^4 = \left(\frac{x^4 \pm 1}{2}\right)^2.$$

Fermat proved a sum and difference of 4th powers is not a square except $1 + 0 = 1$ and $1 - 1 = 0$. So $y = 1$ and $m = 1$. □

Why is it called Pell's equation?

Goldbach (1730) wrote about Fermat's theorem to Euler, saying that he had proved a triangular number greater than 1 is not even a square! Euler noted the error and relation to $x^2 - 2y^2 = 1$, adding

> *Pell devised for [such equations] a peculiar method described in Wallis' works.*

Wallis' *Algebra* often mentions Pell, but never in connection with $x^2 - dy^2 = 1$. Wallis cites Brouncker for work on this equation. Attempted renamings:

1. Fermat
2. Brahmagupta-Bhaskara

> *It is interesting to think that if Euler had not made this error then Brouncker, instead of being relatively unknown as a mathematician, would be universally known through 'Brouncker's equation'.* J. J. O'Connor, E. F. Robertson