Additional Inequalities

1 Schur's Inequality

Let x, y, and z be nonnegative real numbers, and let n be a positive integer. Then

$$x^n(x-y)(x-z) + y^n(y-x)(y-z) + z^n(z-x)(z-y) \geq 0.$$

Equality occurs if and only if $x = y = z$, or two of them are equal and the other is 0.

Proof. Without loss of generality, assume that $x \geq y \geq z$. Then the inequality can be re-written as

$$(x-y)[x^n(x-z) - y^n(y-z)] + z^n(x-z)(y-z) \geq 0.$$

Since $x \geq y$, $x^n(x-z) \geq y^n(y-z)$, and $z^n(x-z)(y-z) \geq 0$, the inequality holds.

In particular, for $n = 1$, Schur's inequality states that

$$x(x-y)(x-z) + y(y-x)(y-z) + z(z-x)(z-y) \geq 0,$$

or

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2.$$

Schur's inequality pops up occasionally, in different forms. All of the following problems are equivalent to Schur's inequality (for $n = 1$):

- Suppose a, b, c are the sides of a triangle. Prove that

  $$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc.$$

  (IMO, 1964)

- Let a, b, and c be any positive numbers. Prove that

  $$abc \geq (-a+b+c)(a-b+c)(a+b-c).$$

  (British Mathematical Olympiad, 1981)

- For x, y, $z \geq 0$, establish the inequality

  $$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z)$$

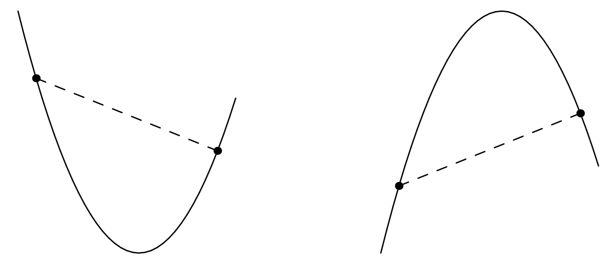
  and determine when equality holds. (Canada, 1992)

2 Jensen's Inequality

Let $f : I \to \mathbb{R}$, where I is some interval. Then f is *convex* if for every sub-interval $[a, b] \subseteq I$,

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b)$$

for all $t \in [0, 1]$. This means that the graph of f on $[a, b]$ lies below the chord joining the points $(a, f(a))$ and $(b, f(b))$.



A convex function A concave function

The function f is *concave* if the inequality holds in the reverse direction.

Jensen's inequality states that if $f : I \to \mathbb{R}$ is a convex function, then for all $x_1, x_2, \dots, x_n \in I$,

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \le \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

If f is concave, then the inequality holds in the reverse direction.

There is also a weighted version of Jensen's inequality: Let $w_1, w_2, \dots, w_n > 0$ be real numbers such that $w_1 + w_2 + \cdots + w_n = 1$. If $f : I \to \mathbb{R}$ is a convex function, then for all $x_1, x_2, \dots, x_n \in I$,

$$f(w_1x_1 + w_2x_2 + \cdots + w_nx_n) \le w_1f(x_1) + w_2f(x_2) + \cdots + w_nf(x_n).$$

(Again, if f is concave, then the inequality holds in the reverse direction.)

3 Rearrangement Inequality

Let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers. Then over all permutations $(y_1, y_2, \dots, y_n)$ of $(x_1, x_2, \dots, x_n)$, the expression

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

is maximized when the y_i are sorted similarly as the x_i (i.e. $y_1 \le y_2 \le \cdots \le y_n$), and minimized when the y_i are sorted oppositely as the x_i (i.e. $y_n \le y_{n-1} \le \cdots \le y_1$).

4 Chebyshev's Inequality

For any real numbers $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$,

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \le \frac{1}{n}\sum_{i=1}^{n} a_ib_i.$$

5 Hölder's Inequality

Let p and q be positive real numbers such that $1/p + 1/q = 1$. Then for any real numbers $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$,

$$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}.$$

Hölder's inequality is a generalization of the Cauchy-Schwarz inequality.

6 Minkowski's Inequality

For any real numbers $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$, and $p > 1$,

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/p} \le \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{1/p}.$$

Minkowski's inequality is a generalization of the triangle inequality.