Advanced Topics in Inequalities

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Introduction

Algebraic inequalities are an important part of algebra, and even in other fields, where classical algebraic inequalties like Cauchy-Schwarz are used in number theory, such as in Gallagher's large sieve. We direct the interested reader to the splendid collection of problems, *Straight From The Book*.

Here, however, we will be covering different techniques in inequalities than normal, namely, the separation technique, the tangent figure trick, Cauchy flipping, integration reduction in inequalities, and denominator nullification. Note that none of these are standard names, so do not cite them in an olympiad!

2

The Separation Technique

2.1 Introduction

Often times in olympiad inequalities, if the problem is homogenous, or has a condition that can easily homogenize the inequality (ex. $abc = 1$), we can homogenize the problem and then bash with Muirhead and Schur. However, some inequalities resist all attempts at homogenization, and some are even unhomogenizable. If problems have intractable conditions like $a^2 + b^2 + c^2 + abc = 4$ or in fact none at all, the resulting problems can be extremely difficult, and dealing with them can be nearly impossible.

2.2 A motivating example

Consider the following heterogenous inequality, proposed by Ahann Rugata:

Example 2.1.

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca),$$

where $a, b, c > 0$.

Solution. WLOG b and c are on the same side of 1 so $(b-1)(c-1) \geq 0$

$$\begin{aligned}(a^2 + b^2 + c^2 + 2abc + 1) - (2ab + 2ac + 2bc) &= 2abc - 2ab - 2ac + 2a + b^2 - 2bc + c^2 + a^2 - 2a + 1 \\ &= 2a(b-1)(c-1) + (b-c)^2 + (a-1)^2 \\ &\geq 0.\end{aligned}$$

□

The key step in the previous problem is that we randomly assume that two of a, b, and c are on the same side of a number. This concept allows us to deal with abc, which otherwise would cause many problems in sharp inequalities. The problem is that the strongest symmetric inequality with abc is Schur's inequality. Schur's is a problem sometimes, because it is very weak compared to other inequalities. Next, we will present a very weird problem, as in that many appproaches fail. The condition seems completely intractable, yet the solution is still beautiful with the separation method.

2.3 A strange inequality

Example 2.2. (Mathematical Reflections) Let $a_1, a_2, \cdots, a_n \in [0, 1]$ and c be a constant such that

$a_1 + a_2 + \cdots + a_n = n + 1 - c$.

For any permutation $b_1, b_2, \cdots, b_n$ of $a_1, a_2, \cdots, a_n$, prove that we have

$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq n + 1 - c^2$.

This inequality is not particularly difficult, but its condition is extremely strange. However, with the separation trick, this problem quickly becomes trivial.

Solution. Once again, the difficulty is in lower bounding a_kb_k. However, this is doable with the what else, the separation trick!

For any a_kb_k, we have that $(a_k - 1)(b_k - 1) \geq 0$ because they are both less than one. Then, we have that $a_kb_k \geq a_k + b_k - 1$. Putting this back in, we have

$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq (a_1 + a_2 \cdots + a_n) + (b_1 + b_2 \cdots + b_n) - n$

$= n + 2 - 2c$

Now it remains to prove that

$n + 2 - 2c \geq n + 1 - c^2$

Fortunately, this is trivial as it reduces to

$(c - 1)^2 \geq 0$, so we are done. □

2.4 Exercises

Problem 2.1. (USAMO 2001/3) If a, b, and c are positive reals such that $a^2 + b^2 + c^2 + 2abc = 4$, prove that $0 \leq ab + ac + bc - abc \leq 2$.

Problem 2.2. (Franklyn Wang, Akshaj Kadaveru) For all real numbers a, b, and c, prove that

$4(1 + a^2)(1 + b^2)(1 + c^2) \geq 3(a + b + c)^2$.

Problem 2.3. (Vasc) Prove that if $a, b, c \geq 0$ and $x = a + \frac{1}{b}$, $y = b + \frac{1}{c}$, $z = c + \frac{1}{a}$, then

$xy + yz + zx \geq 2 + x + y + z$.

3

Tangent Figure Trick

3.1 Introduction

The tangent figure trick is most often used when one side involves the sum of one-variable expressions. The technique involves bounding each expression with it's tangent figure at the equality case. Usually this figure is a line, but other times polynomials or even strange, rational functions are needed! This technique will be made more clear with examples.

3.2 A motivating example

Example 3.1. (USAMO 2003/5)$\sum \frac{(2a+b+c)^2}{2a^2+(b+c)^2} \le 8$ for positive real a, b, c.

Solution. WLOG, assume $a+b+c=3$.

Then the LHS becomes $\sum \frac{(a+3)^2}{2a^2+(3-a)^2} = \sum \frac{a^2+6a+9}{3a^2-6a+9} = \sum \left(\frac{1}{3} + \frac{8a+6}{3a^2-6a+9}\right)$.

Notice $3a^2-6a+9 = 3(a-1)^2+6 \ge 6$, so $\frac{8a+6}{3a^2-6a+9} \le \frac{8a+6}{6}$.

So $\sum \frac{(a+3)^2}{2a^2+(3-a)^2} \le \sum \left(\frac{1}{3} + \frac{8a+6}{6}\right) = 1 + \frac{8(a+b+c)+18}{6} = 8$ as desired. $\square$

Motivation. The question is of course, what is the motivation for the constants used? The answer is that they are all derived from the local linear approximation, and that the derivative of the function at the equality case is used (in this case $\frac{(a+3)^2}{2a^2+(3-a)^2}$).

Let's try writing this in a more instructive form:

$\frac{8a+6}{6} = \frac{4(a-1)}{3} + \frac{8}{3}$

Aha! Here, that $(a-1)$ serves as an extra number, and $\frac{4}{3}$ is the slope of the line. In other words, the graph of the function lies entirely below this line, and this linearization quickly finishes. Nevertheless, remember that perhaps its not always a line that the function lies under. Maybe it is a complicated algebraic function. But in any case, this concept of locally bounding each term is very powerful, and is the idea behind the tangent figure trick and the related concept of isolated fudging. $\square$

3.3 Exercises

Problem 3.1. (Titu Andresscu) Given that a, b, and c are positive reals such that $a+b+c \geq 3$, prove that

$$\frac{1}{a^2+b+c} + \frac{1}{a+b^2+c} + \frac{1}{a+b+c^2} \leq 1.$$

Problem 3.2. (Japan 1997) Show that

$$\frac{(a+b-c)^2}{(a+b)^2+c^2} + \frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} \leq 3/5$$

where a, b, and $c \in \mathbb{R}^+$.

Problem 3.3. (PuMaC 2014 Individual Finals A2) Given $a, b, c \in \mathbb{R}^+$, and that $a^2+b^2+c^2 = 3$. Prove that

$$\frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} \geq 1.$$

Problem 3.4. (Akshaj Kadaveru) Given that a, b, and c are positive reals such that $a+b+c = 3$, show that

$$(2a^2+3)(2b^2+3)(2c^2+3) \geq 125$$

4

Cauchy Flipping

4.1 Introduction

We return to an exercise from the previous chapter, using a radically different strategy.

4.2 The Power of Cauchy Flipping

Example 4.1. (PuMaC 2014 Individual Finals A2) Given $a, b, c \in \mathbb{R}^+$, and that $a^2+b^2+c^2 = 3$. Prove that

$$\frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} \geq 1$$

Solution. (By Michael Kural) If we use $AM - GM$ on the denominator (keeping in mind that the equality case is $a = b = c = 1$), we get $a^3 + 2 \geq 3a$, so $\frac{1}{a^3+2} \leq \frac{1}{3a}$. However, this is going the wrong direction, so instead we try to move the fractions to the other side. The easiest way to do so is to subtract each fraction from 1:

$$\begin{aligned}
&\sum \frac{1}{a^3+2} \geq 1 \\
&\qquad \Leftrightarrow 2 \geq \sum 1 - \frac{1}{a^3+2} \\
&\qquad \Leftrightarrow 2 \geq \sum \frac{2a^3}{a^3+2} \\
&\qquad \Leftrightarrow 1 \geq \sum \frac{a^3}{a^3+2}
\end{aligned}$$

and now we can use AM-GM as we would like. Since $a^3 + 2 \geq 3a$,

$$\sum \frac{a^3}{a^3+2} \leq \sum \frac{a^3}{3a} = \frac{1}{3} \sum a^2 = 1$$

so we're done. $\square$

4.3 Exercises

Problem 4.1. (Bulgaria) Let a, b, c be positive reals such that $a + b + c = 3$. Prove that

$$\frac{a}{1+b^2} + \frac{b}{1+c^2} + \frac{c}{1+a^2} \geq \frac{3}{2}$$

Problem 4.2. (T.Q. Anh) If a, b, c are positive reals such that $ab + ac + bc = 3$, prove that

$$\frac{a}{2a+b^2} + \frac{b}{2b+c^2} + \frac{c}{2c+a^2} \leq 1$$

5

Denominator Nullification

5.1 Introduction

One of the most difficult parts of inequalities involving large amounts of fractions is when the denominators are not the same. Dealing with these inequalities usually requires clever Titu's lemma or smoothing. Yet sometimes, wishfully thinking is good. If we can modify all the denominators to be the same thing, we can multiply it out. This idea seems trivial, but the problems it can solve are far from trivial!

5.2 An IMO/3

Example 5.1. (IMO 2005/3) Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

Taking the negative and adding 3, we have

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{x^2 + y^5 + z^2} + \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^5} \leq 3.$$

Normally, we would never induce an $\leq$ sign, but in denominator nullification, this is required. Our sacrifice is richly rewarded. Making the denominators equal, we have
$(x^5 + y^2 + z^2)(\frac{1}{x} + y^2 + z^2) \geq (x^2 + y^2 + z^2)^2$
So

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} \leq \frac{(x^2 + y^2 + z^2)(\frac{1}{x} + y^2 + z^2)}{(x^2 + y^2 + z^2)^2} = \frac{\frac{1}{x} + y^2 + z^2}{(x^2 + y^2 + z^2)}$$

The finish is now clear. Summing over, it suffices to prove that

$$\frac{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}{x^2 + y^2 + z^2} \leq 1$$

Fortunately, this is trivial as it reduces to

$$\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{x^2+y^2+z^2} \le \frac{(xyz)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})}{x^2+y^2+z^2} = \frac{xy+yz+xz}{x^2+y^2+z^2} \le 1$$

where the last step is straightforward.

5.3 Exercises

Problem 5.1. (Tuan Le) Prove that for $xyz \ge 1$, and $x, y, z \in \mathbb{R}^+$, the following inequality holds:

$$\frac{x}{x^3+y^2+z} + \frac{y}{y^3+z^2+x} + \frac{z}{z^3+x^2+y} \le 1$$

6

Inequality Separation

6.1 Introduction

Frequently, inequalities on one side are just sums or products of functions, while the other side is more symmetrical. Many strategies have been concocted to deal with these difficult inequalities, such as isolated fudging and the tangent figure trick, but usually the best approach is quite a bit different: Working backwards from the symmetric part. Heuristically speaking, if our inequality is proving

$$f(x)f(y)f(z) \geq g,$$

we want to reduce the problem to proving

$$f(x) \geq k(x),$$

where it is easy to somehow show that

$$k(x)k(y)k(z) \geq g.$$

This is best demonstrated through an example.

6.2 A Difficult Problem

Example 6.1. (USAMO 2004/5) Let $a, b, c > 0$. Prove that $(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \geq (a+b+c)^3$.

Solution. It is easy to see that by Holder's inequality,

$$(a^3+1+1)(1+b^3+1)(1+1+c^3) \geq (a+b+c)^3$$

so it suffices to show that

$$a^5 - a^2 + 3 \geq 2 + a^3$$

Fortunately, this is trivial as a^2 and a^3 are always on the same side of one and this inequtality reduces to

$$(a^3-1)(a^2-1) \geq 0$$

which is immediate. □

Motivation. There are two key steps in this problem. One of them is the use of Holder's on $(a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3$. Keep this in mind! It will come back on the exercises, so be wary. Another one is the whole idea of separation into factors. □

6.3 Exercises

Problem 6.1. (APMO 2004/5) $\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right) \geq 9\left(ab+bc+ca\right)$ holds for all positive reals a, b, c.

Problem 6.2. (Vasc) Prove the inequality $\frac{a^3+1}{a^2+1} \cdot \frac{b^3+1}{b^2+1} \cdot \frac{c^3+1}{c^2+1} \cdot \frac{d^3+1}{d^2+1} \geq \frac{abcd+1}{2}$ for $a, b, c, d \in \mathbb{R}^+$.