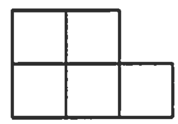
Winter Camp 2000 Comptition

1. For any positive integer n, denote by $f(n)$ the highest power of 2 which divides $n!$ and by $g(n)$ the number of 1's in the base-2 representation of n. Prove that $f(n) + g(n) = n$.

2. If a $5 \times n$ rectangle can be constructed using n copies of the figure below, prove that n is even.



3. There are some ink-blots on a white paper square with side length a. The area of each blot is not greater than 1 and every line parallel to any one of the sides of the square intersects no more than one blot. Prove that the total area of the blots is not greater than a.

4. Each of six friends has a juicy piece of gossip and is eager to spread it around. This is done through a sequence of phone calls, each between exactly two of the friends. In a conversation, each tells the other all the pieces of gossip known up to that point. What is the minimum number of phone calls in order for everyone to know all pieces of gossip?

5. A and B are arbitrary points on a circle with diameter NS. The tangents to the circle at A and B intersect at C. The tangent at S intersects the line NA at D, the line NB at E and the line NC at F. Prove that $DF = EF$.

Problems for Informal Discussion

1. A polyhedron has exactly 100 sides. At most how many of them can be cut by one plane?

2. In the function $f(x) = \dfrac{x^2 + ax + b}{x^2 + cx + d}$, the quadratic polynomials $x^2 + ax + b$ and $x^2 + cx + d$ have no common roots. Prove that the following two statements are equivalent.

   (i) There exist two real numbers $h < k$ such that for any real number x, either $f(x) < h$ or $f(x) > k$.

   (ii) The function $f(x)$ can be expressed in the form $f_1(f_2(\cdots f_{n-1}(f_n(x))\cdots))$, where each of the functions f_i is either x^2, $\frac{1}{x}$ or some linear polynomial.