2003 Winter Camp

IMO Inequality Problems

Note: solutions to these problems, as well as solutions to every problem in the history of the IMO, are available at http://www.kalva.demon.co.uk/imo.html

1. Let a, b, and c be the sides of a triangle. Let T be its area.
   Show that $a^2 + b^2 + c^2 \geq 4\sqrt{3}T$. When does equality hold?
   (1961 IMO, Question 2).

2. Let a, b, and c be the sides of a triangle. Prove that
   $$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc.$$
   Determine when equality occurs.
   (1964 IMO, Question 2).

3. Let $x_1 \geq x_2 \geq \ldots \geq x_n$, and $y_1 \geq y_2 \geq \ldots \geq y_n$ be real numbers. Prove that if $\{z_i\}$ is a permutation of $\{y_i\}$, then $\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2$.
   (1975 IMO, Question 1).

4. Let $\{a_k\}$ be a sequence of distinct positive integers. Prove that for all integers $n \geq 1$,
   $$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$
   (1978 IMO, Question 5)

5. Let a, b, and c be the lengths of the sides of a triangle. Prove that
   $$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$
   Determine when equality occurs.
   (1983 IMO, Question 6)

6. Prove that if x, y, and z are non-negative real numbers satisfying $x + y + z = 1$, then
   $$0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}.$$
   (1984 IMO, Question 1)

7. Show that the set of real numbers x which satisfy the inequality

$$\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}+\ldots+\frac{70}{x-70}\geq\frac{5}{4}.$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

(1988 IMO, Question 4)

8. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, ..., n\}$ such that whenever $a_i + a_j \leq n$ for some i, j (possibly the same), we have $a_i + a_j = a_k$ for some k. Prove that

$$\frac{a_1+a_2+\ldots+a_m}{m}\geq\frac{n+1}{2}.$$

(1994 IMO, Question 1)

9. Suppose that a, b, and c are positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

(1995 IMO, Question 2)

10. Let $n \geq 2$ be a fixed integer. Find the smallest constant C such that for all non-negative reals $x_1, x_2, \ldots, x_n$,

$$\sum_{1\leq i<j\leq n} x_ix_j\left(x_i^2+x_j^2\right)\leq C\left(\sum_{1\leq i\leq n} x_i\right)^4.$$

For this constant C, determine when equality occurs.

(1999 IMO, Question 2)

11. Let a, b, and c be positive real numbers with $abc = 1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1.$$

(2000 IMO, Question 2)