2018 Special Camp - NT pset

Thanic Nur Samin

October 2018

1. Find all n such that there exists a solution of $a^2 + b^2 = n!$ for positive integers a, b, n.
2. Show that there are infinitely many positive integer numbers n such that $n^2 + 1$ has two positive divisors whose difference is n.
3. The function $f : \mathbb{N} \to \mathbb{R}$ satisfies $f(1) = 1, f(2) = 2$ and $f(n+2) = f(n+2-f(n+1)) + f(n+1-f(n))$. Show that $0 \leq f(n+1) - f(n) \leq 1$. Find all n for which $f(n) = 1025$.
4. We call a positive integer n amazing if there exist positive integers a, b, c such that the equality
$$n = (b, c)(a, bc) + (c, a)(b, ca) + (a, b)(c, ab)$$
holds. Prove that there exist 2011 consecutive positive integers which are amazing.
Note. By (m, n) we denote the greatest common divisor of positive integers m and n.
5. Find all non-negative integers m, n, x, y such that $x^m y^n = (x + y)^2 + 1$
6. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m, n,
   - $mf(f(m)) = (f(m))^2$,
   - If $\gcd(m, n) = d$, then $f(mn) \cdot f(d) = d \cdot f(m) \cdot f(n)$,
   - $f(m) = m$ if and only if $m = 1$.
7. Call a rational number r powerful if r can be expressed in the form $\frac{p^k}{q}$ for some relatively prime positive integers p, q and some integer $k > 1$. Let a, b, c be positive rational numbers such that $abc = 1$. Suppose there exist positive integers x, y, z such that $a^x + b^y + c^z$ is an integer. Prove that a, b, c are all powerful.
8. Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be positive and distinct integers such that for every positive integer k, $x_1 x_2 x_3 \cdots x_n | (x_1 + k)(x_2 + k) \cdots (x_n + k)$.
Prove that, $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$.
9. For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.
   - Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
   - Find an integer n such that $S(n) = n^2 - 14$.
   - Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.
10. Let a, b, c some positive integers and x, y, z some integer numbers such that we have:
    - $ax^2 + by^2 + cz^2 = abc + 2xyz - 1$
    - $ab + bc + ca \geq x^2 + y^2 + z^2$

    Prove that a, b, c are all sums of three squares of integer numbers.[1]

[1] n can't be represented as sum of three squares if and only if $n = 4^t(8k + 7)$ for some t and k. Might or might not be relevant.