Mock Olympiad #5

July 10, 2009

1. An eccentric mathematician has a ladder with n rungs that he always ascends and descends in the following way: When he ascends, each step he takes covers a rungs of the ladder, and when he descends, each step he takes covers b rungs of the ladder, where a and b are fixed positive integers. By a sequence of ascending and descending steps he can climb from ground level to the top rung of the ladder and come back to ground level again. Find, with proof, the minimum value of n, expressed in terms of a and b.

2. Let A', B', C', D', E', F' be midpoints of the sides AB, BC, CD, DE, EF, FA of convex hexagon $ABCDEF$. Also let p denote the perimeter of $ABCDEF$ and p' denote the perimeter of $A'B'C'D'E'F'$. If all inner angles of hexagon $A'B'C'D'E'F'$ are equal, prove that

$$p \geq \frac{2 \cdot \sqrt{3}}{3} \cdot p_1.$$

When does equality hold?

3. Let X be a set of $2k$ elements and $\mathcal{F}$ a family of subsets of X each of cardinality k such that each subset of X of cardinality $k-1$ is contained in precisely one member of $\mathcal{F}$. Show that $k+1$ is prime.

1 Solutions

1. *(IMO 1990 Short list, #13)*

   We claim the minimum value of n is $a+b-\gcd(a,b)$. We will prove this first in the case where a and b are relatively prime.

   Claim: If the mathematician makes m moves beginning at some arbitrary rung and ends up where he started, then $m \geq a+b$ (assuming a, b are relatively prime).

   Proof of claim: If he ascends A times and descends B times, we must have $Aa = Bb$. Since a and b are relatively prime, it follows that $a|B$ and $b|A$, so $B \geq a$, $A \geq b$, and $m = A + B \geq a + b$. $\square$

   Consider a ladder with $n < a+b-1$ rungs, and suppose the mathematician can make a sequence of moves and end up where he started. By the above claim, this number of moves must be at least $a+b > n+1$. However, there are only $n+1$ different positions for the mathematician, so this is impossible.

   Next we show a ladder with $n = a+b-1$ rungs is always possible. From any position, the mathematician always has a legal move: if he is on rung $0, 1, \ldots, b-1$, he can ascend, and if he is on rung $b, b+1, \ldots, a+b-1$, he can descend. Therefore, he can start on the ground floor and continue to make moves until he returns to a position he has already been to. By the original claim, he can only repeat a position after visiting all other $a+b-1$ positions first. Therefore, he must return to the ground floor before repeating another position, and he must have visited the top rung during that time. Thus, $n = a+b-1$ rungs is indeed always possible, which completes our solution in the case where a, b are relatively prime.

   Finally, suppose a and b have greatest common divisor $g > 1$. Then the mathematician can only visit rungs with position a multiple of g. Ascending is equivalent to climbing up $\frac{a}{g}$ such rungs, and descending is equivalent to going down $\frac{b}{g}$ such rungs. Since $\frac{a}{g}$ and $\frac{b}{g}$ are relatively prime, we know the minimum possible n for which is possible to visit the top rung and return to the start in this scenario satisfies $\frac{n}{g} = \frac{a}{g} + \frac{b}{g} - 1$. Equivalently, $n = a+b-\gcd(a,b)$.

2. *(Vietnam TST 2004, #5)*

   Let $A_0, B_0, \ldots, F_0$ be the projections of $A, B, \ldots, F$ onto the segments $F'A', A'B', \ldots, E'F'$ respectively. Then,

$$\begin{aligned} A_0A' + A'B_0 &= AA' \cos \angle AA'A_0 + A'B \cos \angle BA'B_0 \\ &= \frac{AB}{2} \cdot \left(\cos \angle AA'A_0 + \cos(60^\circ - \angle AA'A_0) \right). \end{aligned}$$

   Since $\cos\theta$ is concave on the range $[0^\circ, 60^\circ]$, it follows from Jensen's inequality that

$$A_0A' + A'B_0 \leq AB \cdot \cos 30^\circ = \frac{AB \cdot \sqrt{3}}{2}.$$

   Applying similar arguments for $B', C', \ldots, F'$, and adding the resulting inequalities, we have $p' \leq \frac{\sqrt{3}}{2} \cdot p$, as required.

For equality to hold, $\angle AA'A_0 = \angle BA'B_0 = 30^\circ$, and so on. Therefore, $ABCDEF$ is equiangular and triangles $A'BB', B'CC', \ldots, FAA'$ are all isosceles, so $AA' = A'B = BB' = \ldots = FF' = F'A$, and $ABCDEF$ is equilateral. Thus, equality holds if and only if $ABCDEF$ is regular.

3. *(China TST 2009, #4.3 and India TST 1998)*

   Claim: Let r be a positive integer such that $2 \leq r \leq k-1$ and let S be a subset of size $k-r$. Then there are
   $$\frac{(k+r)(k+r-1)\cdots(k+2)}{r!}$$
   sets from $\mathcal{F}$ that contains S.

   Proof of claim: There are $k+r$ elements not in S. Therefore, there are $(k+r)(k+r-1)\cdots(k+2)$ ways to add elements to S (in order) to make S size $k-1$. Let the resulting set be S'. By the property of $\mathcal{F}$, there exists a unique element not in S' that we can add to S' so that the resulting set is in $\mathcal{F}$. For each set F in $\mathcal{F}$ that contains S, there are $r!$ ways to add r elements to S to make the resulting set in F. Therefore, there are
   $$\frac{(k+r)(k+r-1)\cdots(k+2)}{r!}$$
   sets from $\mathcal{F}$ that contains S. This proves the claim. $\square$

   We now have that
   $$\frac{(k+r)(k+r-1)\cdots(k+2)}{r!} \tag{1}$$
   is an integer for all $2 \leq r \leq k-1$. Let p be a prime factor of $k+1$. Then $p \mid k+1$ and p does not divide any of the numbers $k+2, k+3, \cdots, k+p$, which means
   $$\frac{(k+p)(k+p-1)\cdots(k+2)}{p!} \tag{2}$$
   is not an integer. If $k+1$ is not prime however, then $2 \leq p \leq k-1$, and (2) contradicts (1) when $r = p$. We conclude that $k+1$ must be prime.