Inequalities Talk - Outline

Jan. 7th, 200

In today's talk, we will be discussing three different strategies we can use to solve inequality problems:

1) Symmetry
2) Homogeneity
3) Defining a Function

We will illustrate these techniques with the following problems:

① If $a_1, a_2, \ldots, a_n > 0$, prove that

$$a_1 + a_2 + \cdots + a_n \geq \frac{2a_1a_2}{a_1+a_2} + \frac{2a_2a_3}{a_2+a_3} + \cdots + \frac{2a_na_1}{a_n+a_1}.$$

② If $a, b, c, d > 0$, prove that

$$\frac{a^3+b^3+c^3}{a+b+c} + \frac{b^3+c^3+d^3}{b+c+d} + \frac{c^3+d^3+a^3}{c+d+a} + \frac{d^3+a^3+b^3}{d+a+b} \geq a^2+b^2+c^2+d^2.$$

③ If $a, b, c > 0$, prove that (1998 USAMO, #2

$$(a^3+b^3+abc)^{-1} + (b^3+c^3+abc)^{-1} + (c^3+a^3+abc)^{-1} \leq (abc)^{-1}.$$

④ If $x, y, z > 0$ and $x+y+z=1$, prove that

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

(1984 IMO, #1)

⑤ If $x, y, z > 0$ and $x+y+z=1$, prove that

$$x^2y + y^2z + z^2x \leq \frac{4}{27}.$$

(1999 CMO, #5)

⑥ Let n be a fixed integer, with $n \geq 2$.

a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_ix_j(x_i^2+x_j^2) \leq C\left(\sum_{1\leq i\leq n} x_i\right)^4$$

holds for all real numbers $x_1, x_2, \ldots, x_n \geq 0$.

b) For this constant C, determine when equality holds.

(1999 IMO, #2)