Some Mind-Boggling Problems

(At least for me!)

1. Substitute different digits for different letters in the given alphametics such that so the corresponding addition is true:

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  USA
+USSR
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PEACE
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(Famous problem discovered during the Cold War)

2. A circle and a point P lie in a vertical plane. A particle moves along a straight line from P to a point Q on the circle under the influence of gravity. That is the distance travelled from P in time t is $\frac{1}{2}gt\sin\alpha$, where g is a constant and α is the angle between PQ and the horizontal. Describe, geometrically the point Q for which the time taken to move from P to Q is minimized.

(South African Mathematics Olympiad, 1997)

3. Prove AM-GM in the following way: Let AM-GM-n denote AM-GM for n variables.
   1. Prove that AM-GM-2 holds.(Geometrically)
   2. Prove that AM-GM-n implies AM-GM-2n
   3. Prove that AM-GM-n implies AM-GM-(n-1)

(Faustus)

4. A point in the plane with both integer Cartesian coordinates is called a lattice point. Consider the following one player game. A finite set of selected lattice points and finite set of selected segments is called a position in this game if the following hold:
   a) the endpoints of each selected segment are lattice points
   b) each selected segment is parallel to a coordinate axis, or to the line $y = x$, or to the line $y = -x$
   c) each selected segment contains exactly five lattice points and all of them are selected
   d) each two selected segments have at most one common point.

   A move in this game consists of selecting a lattice point and a segment such that the new set of selected lattice points and selected segment is a position. Prove or disprove that there exists an initial position such that the game has infinitely many moves.

(Austrian-Polish Mathematics Competition, 2002)

5. Find all non-negative integer solution of the equation: $2^x 3^y - 5^z 7^w = \pm 1$.

(Faustus)

6. Let $a_1, a_2, a_3, \dots, a_n$ be the odd numbers, none of which has a prime divisors greater than 5, prove that
$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \frac{15}{8}$$

(Mathscope)

7. The squares of a squared paper are enumerated as shown on the picture. Devise a polynomial $p(m,n)$ in two variables such that for any $m,n \in \mathbb{N}$ the number written in the square with coordinates $(m,n)$is equal to $p(m,n)$.

$\ddots$					
10	$\ddots$				
6	9	$\ddots$	$\ddots$		
3	5	8	12	$\ddots$	
1	2	4	7	11	$\ddots$

(Baltic Way, 1990)

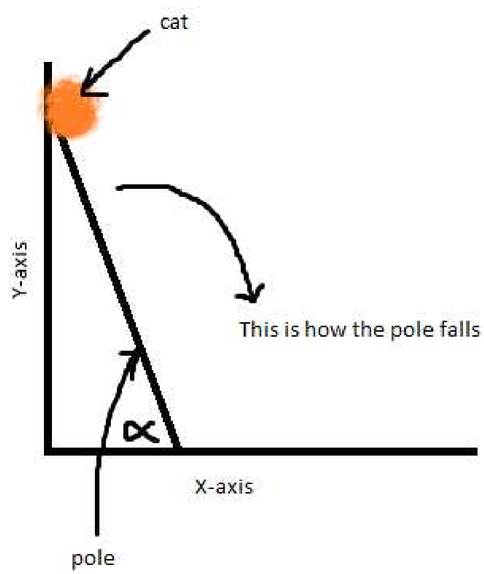
8. Show that $\varphi(a^n + b^n) \equiv 0 \ (mod\ n)$ for relatively prime positive integers a and b.

(Pisolve)

9. Show that if two numbers are chosen at random from the Fibonacci sequence, the probability P that they are relatively prime satisfies the inequality $\frac{8}{\pi^2} > P > \frac{7}{\pi^2}$

(Jack Garfunkel)

10. Suppose a ladder or a pole of any finite length is supported by a wall (y-axis) and it makes an arbitrary angle $\alpha(\alpha < \pi/2)$ with the ground (x-axis). A cat is sitting on top of it. Suddenly the pole starts falling and the frightened cat starts descending. When the pole reaches the ground, it is found the cat had just reached the end of the pole. Find the locus of the cat.



(Faustus)

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Enjoy! Peace