2003 Winter Camp

Common Side Theorem

Denote by $[P]$ the area of the polygon P. If AH is the altitude of triangle ABC from A to BC, it is well-known that $[ABC] = \frac{1}{2}BC \cdot AH$. From this humble beginning, we extract a very powerful tool called the Common Side Theorem.

It should be immediately clear that triangles PAB and QAB, have a common side, namely AB. Let us first consider the special case where the two triangles have another pair of sides which happen to be collinear.

Common Side Lemma.
Let B be a point on the line PQ different from P and Q, and let A be a point not on this line. Then $\frac{[PAB]}{[QAB]} = \frac{PB}{QB}$.

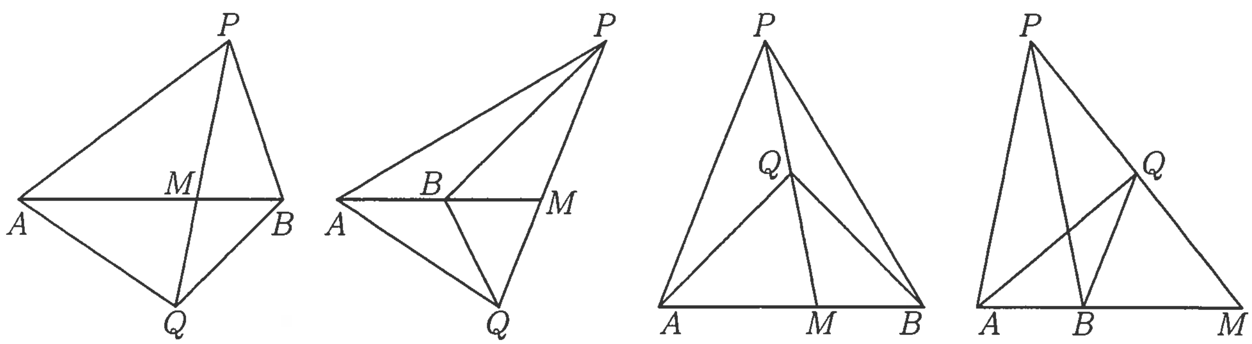
Proof:

Drop the perpendicular AH from A to the line PQ. Then

$$\frac{[PAB]}{[QAB]} = \frac{\frac{1}{2}PB \cdot AH}{\frac{1}{2}QB \cdot AH} = \frac{PB}{QB}.$$

The special case is treated first because we will use it to prove the general result.

Common Side Theorem.
Let PAB and QAB be triangles such that the lines AB and PQ meet at M. Then $\frac{[PAB]}{[QAB]} = \frac{PM}{QM}$.



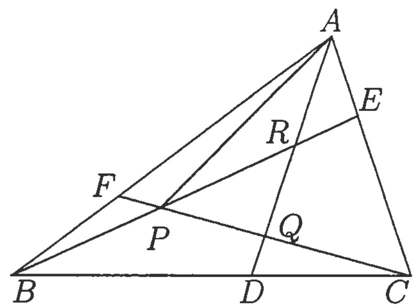
Proof:

We may assume that M is distinct from A, B, P and Q as otherwise the result reduces to the special case. By the Common Side Lemma, we have

$$\frac{[PAB]}{[QAB]} = \frac{[PAB]}{[PMB]} \cdot \frac{[PMB]}{[QMB]} \cdot \frac{[QMB]}{[QAB]} = \frac{AB}{MB} \cdot \frac{PM}{QM} \cdot \frac{MB}{AB} = \frac{PM}{QM}.$$

We now give some applications of this new tool.

Example 1.
D, E and F are points on the sides BC, CA and AB of triangle ABC, respectively, such that $BD = 2DC$, $CE = 2EA$ and $AF = 2FB$. P is the point of intersection of BE and CF, Q is the point of intersection of CF and AD, and R is the point of intersection of AD and BE. Determine $\frac{[PQR]}{[ABC]}$.



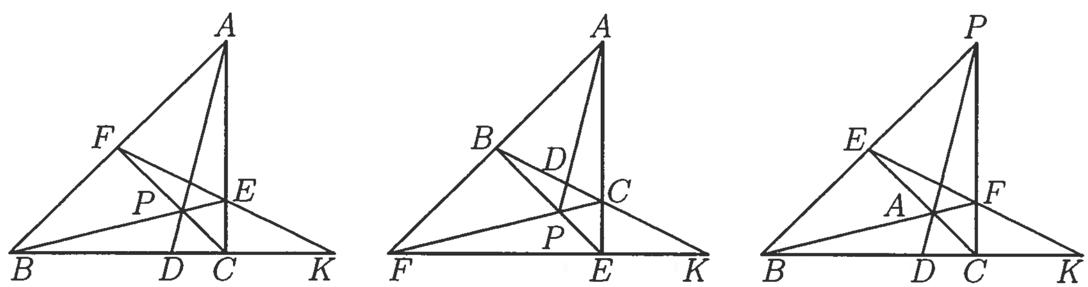
Solution:

By the Common Side Theorem, we have $\frac{[PBC]}{[PAC]} = \frac{BF}{AF} = \frac{1}{2}$ and $\frac{[PBC]}{[PBA]} = \frac{CE}{AE} = 2$. Hence

$$[ABC] = [PBC] + [PCA] + [PAB] = \left(1 + 2 + \frac{1}{2}\right)[PBC] = \frac{7}{2}[PBC].$$

Similarly, $[QCA] = [RAB] = \frac{2}{7}[ABC]$ so that $\frac{[PQR]}{[ABC]} = \frac{1}{7}$.

Example 2.
Let P be any point not collinear with any two vertices of triangle ABC. Let the lines AP, BP and CP intersect the lines BC, CA and AB at D, E and F, respectively. Let the line EF intersect the line BC at K. Prove that $\frac{BD}{DC} = \frac{BK}{KC}$.



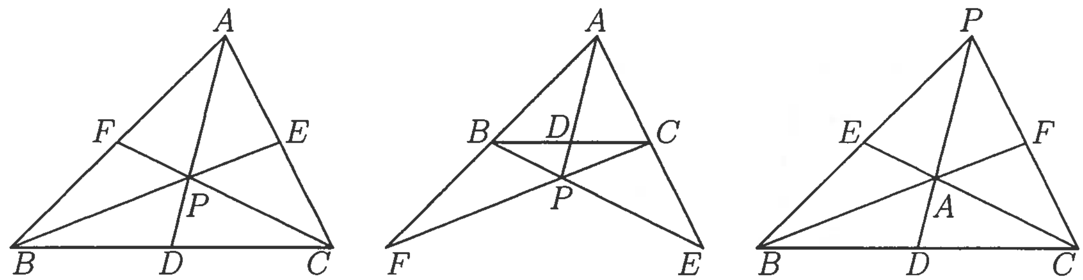
Solution:

By the Common Side Theorem, we have

$$\frac{BD}{DC} = \frac{[BPA]}{[CPA]} = \frac{[BPA]}{[BPC]} \cdot \frac{[CPB]}{[CPA]} = \frac{EA}{CE} \cdot \frac{FB}{AF} = \frac{[EFA]}{[EFC]} \cdot \frac{[FEB]}{[FEA]} = \frac{[FEB]}{[FEC]} = \frac{BK}{KC}.$$

We now give new proofs of some well-known results.

Ceva's Theorem.
Let P be any point not collinear with any two vertices of triangle ABC. Let the lines AP, BP and CP intersect the lines BC, CA and AB at D, E and F, respectively. Then $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$.

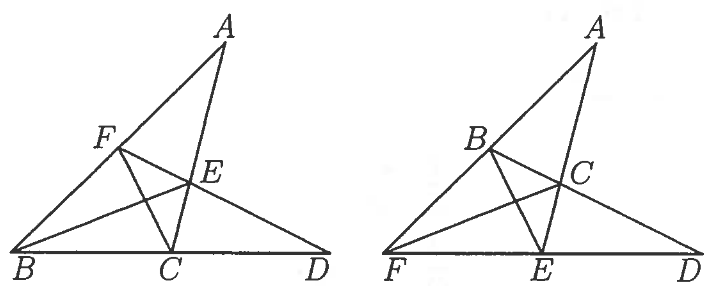


Proof:

By the Common Side Theorem, we have

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{[PAB]}{[PAC]} \cdot \frac{[PBC]}{[PBA]} \cdot \frac{[PCA]}{[PCB]} = 1.$$

Menelaus' Theorem.
A line not concurrent with any two sides of triangle ABC intersects the lines BC, CA and AB at D, E and F, respectively. Then $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$.

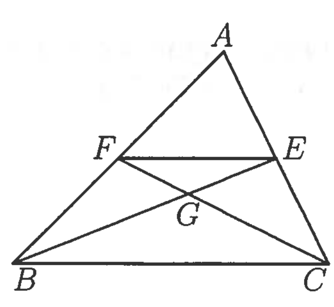


Proof:

By the Common Side Theorem, we have

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{[BEF]}{[CEF]} \cdot \frac{[CEF]}{[AEF]} \cdot \frac{[AEF]}{[BEF]} = 1.$$

Median Trisection Theorem.
The medians BE and CF of triangle ABC intersect at the centroid G. Prove that $\frac{BG}{GE} = \frac{CG}{GF} = 2$.



Proof:

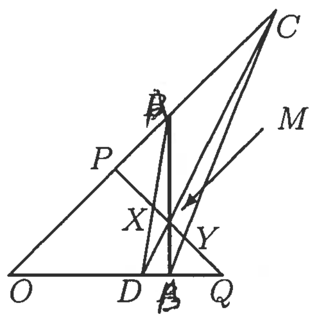
By the Common Side Theorem, we have

$$\frac{BG}{GE}=\frac{[BFC]}{[EFC]}=\frac{[BFC]}{[AFC]}\cdot\frac{[AFC]}{[EFC]}=\frac{BF}{AF}\cdot\frac{AC}{EC}=2.$$

That $\frac{CG}{GF}=2$ can be proved in a similar way.

A Degenerate Butterfly Theorem.
A and C are points on side OP while B and D are points on side OQ of triangle OPQ, such that AB and CD both pass through the midpoint M of PQ. PQ cuts AD at X and BC at Y. Prove that M is also the midpoint of XY.

Proof:



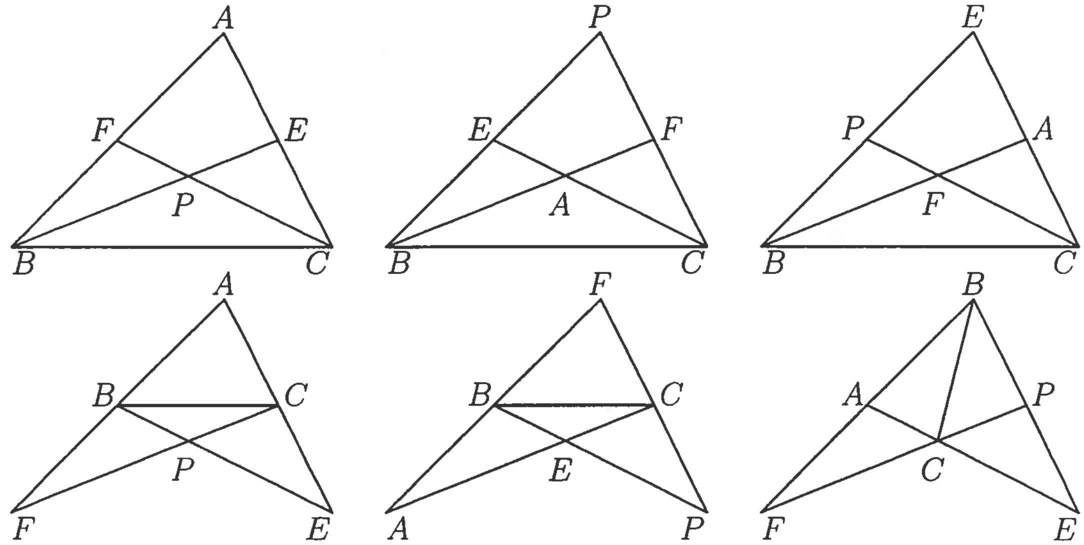
By the Common Side Theorem,

$$\begin{aligned}
\frac{PX\cdot PY}{QX\cdot QY} &= \frac{[PAD]}{[QAD]}\cdot\frac{[PBC]}{[QBC]}\\
&= \frac{[PAD]}{[OAD]}\cdot\frac{[OAD]}{[QAD]}\cdot\frac{[PBC]}{[OBC]}\cdot\frac{[OBC]}{[QBC]}\\
&= \frac{AP}{OA}\cdot\frac{OD}{DQ}\cdot\frac{CP}{OC}\cdot\frac{OB}{BQ}\\
&= \frac{[PAB]}{[OAB]}\cdot\frac{[OCD]}{[QCD]}\cdot\frac{[PCD]}{[OCD]}\cdot\frac{[OAB]}{[QAB]}\\
&= \frac{[PAB]}{[QAB]}\cdot\frac{[PCD]}{[QCD]}\\
&= \frac{PM\cdot PM}{QM\cdot QM}\\
&= 1.
\end{aligned}$$

Hence $PX(PQ-QY)=QY(PQ-PX)$. It follows that $PX=QY$, so that $MX=MY$.

Exercises

1. In each of the diagrams below, compute $\frac{PE}{BP}$ in terms of $u = \frac{CE}{EA}$ and $v = \frac{AF}{FB}$.



2. (a) Prove that in the first diagram in Ceva's Theorem, $\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1$.
   (b) Prove that in the second diagram in Ceva's Theorem, $-\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1$.
   (c) Prove that in the third diagram in Ceva's Theorem, $\frac{PD}{AD} - \frac{PE}{BE} - \frac{PF}{CF} = 1$.

3. D, E and F are points on the sides BC, CA and AB of triangle ABC, respectively, such that $BD = uDC$, $CE = vEA$ and $AF = wFB$. P is the point of intersection of BE and CF, Q is the point of intersection of CF and AD, and R is the point of intersection of AD and BE. Prove that $\frac{[PQR]}{[ABC]} = \frac{(1-uvw)^2}{(1+u+uv)(1+v+vw)(1+w+wu)}$.

Solutions to Exercises

1. We have $\frac{PE}{PB} = \frac{[CEF]}{[CBF]} = \frac{[CEF]}{[CAF]} \cdot \frac{[CAF]}{[CBF]} = \frac{CE}{AC} \cdot \frac{AF}{FB}$ by the Common Side Theorem. In the first and the fifth diagrams, E lies on CA so that $\frac{CE}{CA} = \frac{u}{u+1}$. In the second and the third diagrams, E lies on the extension of CA so that $\frac{CE}{CA} = \frac{u}{u-1}$. In the fourth and the six diagrams, E lies on the extension of AC so that $\frac{CE}{CA} = \frac{u}{1-u}$. The answers are $\frac{uv}{u+1}$, $\frac{uv}{u-1}$ and $\frac{uv}{1-u}$ respectively.

2. By the Common Side Theorem, $\frac{PD}{AD} = \frac{[PBC]}{[ABC]}$, $\frac{PE}{BE} = \frac{[PCA]}{[BCA]}$ and $\frac{PF}{CF} = \frac{[PAB]}{[CAB]}$.

   (a) The result follows from $[PBC] + [PCA] + [PAB] = [ABC]$.

   (b) The result follows from $[PCA] + [PAB] - [PBC] = [ABC]$.

   (c) The result follows from $[PBC] - [PCA] - [PAB] = [ABC]$.

3. As in Example 1, $[ABC] = (1+w+\frac{1}{v})[PBC] = (1+u+\frac{1}{w})[PCA] = (1+v+\frac{1}{u})[PAB]$. Hence

$$\begin{aligned}
\frac{[PQR]}{[ABC]} &= \frac{[ABC]}{[ABC]} - \frac{[PBC]}{[ABC]} - \frac{[PCA]}{[ABC]} - \frac{[PAB]}{[ABC]} \\
&= 1 - \frac{v}{1+v+vw} - \frac{w}{1+w+wu} - \frac{u}{1+u+uv} \\
&= (1-uvw)^2(1+u+uv)(1+v+vw)(1+w+wu).
\end{aligned}$$