Collinearity and concurrency

Carlos Shine

Abstract

Some tips on proving that lines are concurrent or points are collinear.

There are many, many ways to prove that lines are concurrent and points are collinear. Here are some ways, but be aware that there are many others (like, for instance, projective geometry in general).

1 Collinearity

1.1 Menelao's theorem

Let ABC be a triangle and P, Q, R be points on lines BC, CA and AB, respectively. Then P, Q and R are collinear if and only if

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1.$$

1.2 Pappus' theorem

Let A_1, A_2, A_3 be points in a line r and B_1, B_2, B_3 points in a line s. Then $A_1B_2 \cap A_2B_1$, $A_1B_3 \cap A_3B_1$ and $A_2B_3 \cap A_3B_2$ are collinear.

1.3 Pascal's theorem

Let $ABCDEF$ be a hexagon inscribed in a circle. Then the intersections of opposite sides $AB \cap DE$, $BC \cap EF$ and $CD \cap FA$ are collinear. The hexagon doesn't need to be convex, and degenerate cases (for example, $A = B$) are allowed (in the mentioned case, $AB = AA$ is the tangent line through A).

2 Concurrency

2.1 Ceva's theorem

Let ABC be a triangle and P, Q, R be points on lines BC, CA and AB, respectively. Then AP, BQ and CR are concurrent if and only if

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = 1.$$

This also holds if some points are on extensions of sides.

2.2 Trig Ceva

Let ABC be a triangle and P, Q, R be points on lines BC, CA and AB, respectively. Then AP, BQ and CR are concurrent if and only if

$$\frac{\sin\angle BAP}{\sin\angle PAC} \cdot \frac{\sin\angle CBQ}{\sin\angle QBA} \cdot \frac{\sin\angle ACR}{\sin\angle RCB} = 1.$$

This also holds if some points are on extensions of sides.

2.3 Brianchon's theorem

Let $ABCDEF$ be a hexagon circumscribed to a circle. Then the lines connecting opposite vertices, that is, AD, BE and CF, are concurrent. The hexagon doesn't need to be convex, and degenerate cases are allowed.

2.4 Radical axes and center

Let Γ_1, Γ_2 and Γ_3. Then the radical axes of Γ_1, Γ_2, Γ_2, Γ_3 and Γ_3, Γ_1 are either all parallel or concurrent at the radical center of the three circles.

3 Collinearity and concurrency

3.1 Desargues' theorem

Let $A_1B_1C_1$ and $A_2B_2C_2$ be triangles in space (yes, this works in 3D!). Then the lines connecting corresponding vertices A_1A_2, B_1B_2, C_1C_2 are concurrent (or all parallel) if and only if the intersections of corresponding sides $A_1A_2 \cap B_1B_2$, $A_2A_3 \cap B_2B_3$ and $A_3A_1 \cap B_3B_1$ are collinear.

3.2 Homothety

- If two diagrams are homothetic then all lines connecting corresponding points are concurrent at the homothety center.
- In a homothety, the homothety center and two corresponding points are collinear.

3.3 Composition of homotheties

If σ_1 and σ_2 are homotheties with center O_1, O_2 respectively and ratios k_1, k_2 respectively (k_1, k_2 might be negative), $k_1k_2 \neq 1$, then the composition $\sigma_1 \circ \sigma_2$ is a homothety with center O and ratio k_1k_2, and O, O_1 and O_2 are collinear. If $k_1k_2 = 1$, then $\sigma_1 \circ \sigma_2$ is a translation.

This can be proved via Desargues' theorem. Try it!

3.4 Inversion

As much as in homothety:

- If two diagrams are inverse then all lines connecting corresponding points are concurrent at the inversion center.
- In an inversion, the inversion center and two corresponding points are collinear.

3.5 Some final hints

- Guessing the common point of concurrent lines or the line passing through collinear points can be really helpful. Another good reason to make good, big diagrams.
- Any concurrency problem is a collinearity problem: just prove that the intersection P of two of these lines lies on the other line; if this line is defined by two points A and B, then you have to prove that A, B and P are collinear.
- Good ol' angle chasing is helpful as well!

4 Problems

1. Six circles C_i, $i = 1, 2, \ldots, 6$ are all externally tangent to circle C, and C_i is externally tangent to C_{i+1} for all i (in which $C_7 = C_1$). Let P_i be the tangency point between C_i and C. Show that P_1P_4, P_2P_5 and P_3P_6 are concurrent.

2. Let $ABCDEF$ be a convex hexagon. Prove that if each diagonal AD, BE, CF bisects the hexagon area, then they are concurrent.

3. Let I be the incenter of triangle ABC. Let the line passing through I perpendicular to AI intersect side BC at A'. Points B' and C' are defined in a similar fashion. Prove that A', B' and C' lie on a line perpendicular to line IO, where O is the circumcenter of ABC.

4. Given are six distinct points on a circle. The orthocenter of a triangle with vertices on three of these points is connected to the centroid of the triangle with vertices on the three remaining points. Prove that all twenty line segments that may be defined in this fashion meet at a single point.

5. Prove one of the *Fontené's theorems*: let Q be a point inside triangle ABC. The orthogonal projections of Q onto sides BC, CA and AB are D, E and F, respectively. Let M, N and P be the midpoints of sides BC, CA and AB. Let X, Y and Z be the intersections of lines EF, NP; FD, PM; DE, MN respectively. Prove that lines DX, EY and FZ concur at one of the intersections of the circumcircles of MNP and DEF.

6. (IMO 1978, generalized) Let ABC be a triangle. A circle which is internally tangent with the circumscribed circle of the triangle is also tangent to the sides AB, AC in the points P, respectively Q. Prove that the midpoint of PQ is the center of the inscribed circle of the triangle ABC.

7. (IMO 1981) Three circles of equal radius have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point O.

8. (IMO 1982) A non-isosceles triangle $A_1A_2A_3$ has sides a_1, a_2, a_3 with the side a_i lying opposite to the vertex A_i. Let M_i be the midpoint of the side a_i, and let T_i be the point where the inscribed circle of triangle $A_1A_2A_3$ touches the side a_i. Denote by S_i the reflection of the point T_i in the interior angle bisector of the angle A_i. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

9. (IMO 1995) Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

10. (IMO 1996) Let P be a point inside a triangle ABC such that
$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$
Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE meet at a point.

11. (IMOSL 1997) Let $A_1A_2A_3$ be a non-isosceles triangle with incenter I. Let C_i, $i = 1, 2, 3$, be the smaller circle through I tangent to A_iA_{i+1} and A_iA_{i+2} (the addition of indices being mod 3). Let B_i, $i = 1, 2, 3$, be the second point of intersection of C_{i+1} and C_{i+2}. Prove that the circumcenters of the triangles $A_1B_1I, A_2B_2I, A_3B_3I$ are collinear.

12. (IMOSL 1997) Let X, Y, Z be the midpoints of the small arcs BC, CA, AB respectively (arcs of the circumcircle of ABC). M is an arbitrary point on BC, and the parallels through M to the internal bisectors of $\angle B$, $\angle C$ cut the external bisectors of $\angle C$, $\angle B$ in N, P respectively. Show that XM, YN, ZP concur.

13. (IMOSL 2000) Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E and F on sides BC, CA and AB respectively such that $OD + DH = OE + EH = OF + FH$ and the lines AD, BE and CF are concurrent.

14. (IMOSL 2001) Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1, C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.

15. (Iran TST 2002) Let I be the incenter of triangle ABC. Let A_1 and A_2 be points on side BC such that $\angle BIA_1 = \angle CIA_2 = 90°$, let B_1 and B_2 be points on side AC such that $\angle CIB_1 = \angle AIB_2 = 90°$, and let C_1 and C_2 be points on side AB such that $\angle AIC_1 = \angle BIC_2 = 90°$.

    Let A', B', and C' be the midpoints of arcs BC, AC, and AB on the circumcircle of triangle ABC. Let $A'A_1$ intersect AC at A_1', let $A'A_2$ intersect AB at A_2', let $B'B_1$ intersect AB at B_1', let $B'B_2$ intersect BC at B_2', let $C'C_1$ intersect BC at C_1', and let $C'C_2$ intersect AC at C_2'. Prove that $A_1'A_2'$, $B_1'B_2'$, and $C_1'C_2'$ are concurrent.

16. (Olympic Revenge 2002) Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Diagonals AC and BD meet at P. Let Γ_1 and Γ_2 be the circumcircles of triangles ABO and CDO, respectively. Finally, let M and N be the midpoints of arcs AB of Γ_1 and CD of Γ_2 that do not contain O. Prove that M, N and P are collinear.

17. (IMOSL 2003) Let ABC be an isosceles triangle with $AC = BC$, whose incenter is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

18. (IMOSL 2004) Let Γ be a circle and let d be a line such that Γ and d have no common points. Further, let AB be a diameter of the circle Γ; assume that this diameter AB is perpendicular to the line d, and the point B is nearer to the line d than the point A. Let C be an arbitrary point on the circle Γ, different from the points A and B. Let D be the point of intersection of the lines AC and d. One of the two tangents from the point D to the circle Γ touches this circle Γ at a point E; hereby, we assume that the points B and E lie in the same halfplane with respect to the line AC. Denote by F the point of intersection of the lines BE and d. Let the line AF intersect the circle Γ at a point G, different from A.

    Prove that the reflection of the point G in the line AB lies on the line CF.

19. (Iberoamerican 2004) Given a scalene triangle ABC. Let A', B', C' be the points where the internal bisectors of the angles $\angle CAB, \angle ABC, \angle BCA$ meet the sides BC, CA, AB, respectively. Let the line BC meet the perpendicular bisector of AA' at A''. Let the line CA meet the perpendicular bisector of BB' at B'. Let the line AB meet the perpendicular bisector of CC' at C''. Prove that A'', B'' and C'' are collinear.

20. (IMOSL 2006)[1] Circles w_1 and w_2 with centers O_1 and O_2 are externally tangent at point D and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and w_2 at D. Let AB be the diameter of w perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

21. (IMOSL 2007) Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C', respectively. Assume that the points A, B, C, A', B', C' are distinct, and lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

[1] A Brazilian problem!

22. (IMOSL 2007) Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

23. (IMO 2008) Let $ABCD$ be a convex quadrilateral with BA different from BC. Denote the incircles of triangles ABC and ADC by k_1 and k_2 respectively. Suppose that there exists a circle k tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD.

    Prove that the common external tangents to k_1 and k_2 intersects on k.

24. (Sharygin 2008) Let BM and CN be two medians of triangle ABC. Let ω_B and ω_C be circles with chords BM and CN such that smallest arcs BM from ω_B and CN from ω_C have the same angular measure. Suppose ω_B and ω_C intersect at two points P and Q. Prove that A, P and Q are collinear.

25. (Romanian Master 2009) Given four points A_1, A_2, A_3, A_4 in the plane, no three collinear, such that
$$A_1A_2 \cdot A_3A_4 = A_1A_3 \cdot A_2A_4 = A_1A_4 \cdot A_2A_3,$$
denote by O_i the circumcenter of the triangle $A_jA_kA_l$ with $\{i,j,k,l\} = \{1,2,3,4\}$. Assuming $A_i \neq O_i$ for all $i = 1,2,3,4$, prove that the four lines A_iO_i are concurrent or parallel.

26. (Olympic Revenge 2010) Let ABC be a triangle inscribed in its circumcircle Γ. Points D, F, G, E, arranged in this order, belong to the arc BC that does not contain A and are such that $\angle BAD = \angle CAE$ and $\angle BAF = \angle CAG$. Let D', F', G', E' be the intersections of AD, AF, AG, AE with BC, respectively. Lines DF' and EG' meet at X, lines $D'F$ and $E'G$ meet at Y, lines $D'G$ and $E'F$ meet at Z and lines EF' and DG' meet at W. Prove that X, Y and A are collinear, Z, W and A are collinear and that $\angle BAX = \angle CAZ$.

27. (Romanian Master 2010) Let $A_1A_2A_3A_4$ be a quadrilateral with no pair of parallel sides. For each $i = 1,2,3,4$, define ω_i to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i$, A_iA_{i+1} and $A_{i+1}A_{i+2}$ (indices are considered modulo 4 so $A_0 = A_4$, $A_5 = A_1$ and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4 and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1 and T_1T_3 are concurrent.

28. (Sharygin 2010) A point E lies on the altitude BD of triangle ABC, and $\angle AEC = 90°$. Points O_1 and O_2 are the circumcenters of triangles AEB and CEB; points F and L are the midpoints of the segments AC and O_1O_2. Prove that the points E, F and L are collinear.