Algebra

$\boxed{\text{A5}}$ Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C so that $\dfrac{BC}{AB} = \lambda$.

Determine all values of λ such that, for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

$\boxed{\text{A1}}$ Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Combinatorics

$\boxed{\text{C5}}$ A number of n rectangles are drawn in the plane. Each rectangle has parallel sides and the sides of distinct rectangles lie on distinct lines. The rectangles divide the plane into a number of regions. For each region R let $v(R)$ be the number of vertices. Take the sum $\sum v(R)$ over the regions which have one or more vertices of the rectangles in their boundary. Show that this sum is less than $40n$.

$\boxed{\text{C1}}$ A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.

How many ways are there to put the cards in the three boxes so that the trick works?

Geometry

G1 In the plane we are given two circles intersecting at X and Y. Prove that there exist four points with the following property:

(P) For every circle touching the two given circles at A and B, and meeting the line XY at C and D, each of the lines AC, AD, BC, BD passes through one of these points.

G2 Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

G5 Let ABC be an acute-angled triangle, and let w be the circumcircle of triangle ABC.

The tangent to the circle w at the point A meets the tangent to the circle w at C at the point B'. The line BB' intersects the line AC at E, and N is the midpoint of the segment BE.

Similarly, the tangent to the circle w at the point B meets the tangent to the circle w at the point C at the point A'. The line AA' intersects the line BC at D, and M is the midpoint of the segment AD.

a) Show that $\measuredangle ABM = \measuredangle BAN$. **b)** If $AB = 1$, determine the values of BC and AC for the triangles ABC which maximise $\measuredangle ABM$.

G6 Let $ABCD$ be a convex quadrilateral. The perpendicular bisectors of its sides AB and CD meet at Y. Denote by X a point inside the quadrilateral $ABCD$ such that $\measuredangle ADX = \measuredangle BCX < 90^\circ$ and $\measuredangle DAX = \measuredangle CBX < 90^\circ$. Show that $\measuredangle AYB = 2 \cdot \measuredangle ADX$.

G8 Let AH_1, BH_2, CH_3 be the altitudes of an acute angled triangle ABC. Its incircle touches the sides BC, AC and AB at T_1, T_2 and T_3 respectively. Consider the symmetric images of the lines H_1H_2, H_2H_3 and H_3H_1 with respect to the lines T_1T_2, T_2T_3 and T_3T_1. Prove that these images form a triangle whose vertices lie on the incircle of ABC.

Number Theory

N3 Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides $2^n + 1$?

N4 Find all triplets of positive integers (a, m, n) such that $a^m + 1 \mid (a + 1)^n$.

N2 For every positive integers n let $d(n)$ the number of all positive integers of n. Determine all positive integers n with the property: $d^3(n) = 4n$.