A closer look at Induction

Jacob Tsimerman

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1 Basic Induction: the basics

Fact 1. ***The Induction Principle***

Let $P(n)$ be a collection of statements, one for each natural number. If $P(1)$ is true, and for each $n \geq 1, P(n) \implies P(n+1)$, then $P(n)$ is true for every natural number n.

Induction is one of the earliest methods of proof that you learn, but it can still be a very powerful and flexible tecnhique in dealing with advanced problems.

Here's an example of 2 problems which can be solved using a fairly straightforward induction:

Example 1. *The sequence (a_i) is defined by $a_1 = \sqrt{2}$, and $a_{i+1} = \sqrt{2 + a_i}$. Prove that*

$$a_n = 2\cos\frac{\pi}{2^{n+1}}.$$

Proof. We use induction. The case for $n = 1$ (base case) follows because $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Now assuming $a_n = 2\cos\frac{\pi}{2^{n+1}}$, we have by definition

$$a_{n+1} = \sqrt{2 + a_n} = \sqrt{2 + 2\cos\frac{\pi}{2^{n+1}}}$$

so we need to show that $4\cos^2\frac{\pi}{2^{n+2}} = 2 + 2\cos\frac{\pi}{2^{n+1}}$ which follows from the double angle formula for cosine $\cos 2x = 2\cos^2 x - 1$.

□

Example 2. *Prove that for $n \geq 1$ we have the inequality $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$.*

Proof. We use induction. The case of $n = 1$ reduces to $1 \leq 2$ which is true. Now, assume the statement is true for n. Then by assumption,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$$

Now the statement for $n+1$ will follow if we can show that $2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}$. Multiplying both sides by $\sqrt{n+1}$, this becomes equivalent to:

$$2\sqrt{n^2+n} \leq 2n+1$$

which is clearly true by squaring both sides.

□

Note the difference in the use of induction in the previous 2 problems. The first problem HAD to work out since we only used induction to reduce the statement to an equivalent statement that was easier to prove (the identity of cosines). In the second problem, we used our induction hypothesis to simplify the inequality, but we had no guarantee that the resulting inequality is still true! For this reason you should always be careful when using induction on inequalities to make sure you don't try proving a false statement! As an example of how this can obviously fail, try this problem:

Example 3. *Prove that* $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2$ *for each natural number* n.

2 Strong Induction

Fact 2. ***Strong Induction Principle***
Let $P(n)$ *be a collection of statements such that* $P(1)$ *is true, and for each natural number* n, *we have the implication* $P(1), P(2), \ldots, P(n-1) \Longrightarrow P(n)$. *Then* $P(n)$ *is true for all natural numbers* n.

Whenever you are doing induction, you should always be thinking of strong induction. There are simply no drawbacks, and often assuming the strong induction hypothesis is very convenient for a problem. Here is an example:

Example 4. *The Fiobonacci sequence is defined as* $F_1 = 1, F_2 = 2$ *and for* $n \geq 1$, $F_{m+2} = F_{m+1} + F_m$ *Show that every natural number* n *can be written as a sum of distinct fiobonacci numbers, with no 2 Fiobonacci numbers being consecutive (*F_n *and* F_{n+1} *are consecutive).*

Proof. We proceed by strong induction. Let $P(n)$ be the statement that n can be written as a sum of distinct fiobonacci numbers with no 2 consecutive. It is easy to verify $P(1)$ and $P(2)$. We now proceed by strong induction, so assume $P(i)$ for all $1 \leq i \leq n$. Now consider the number $n+1$ and the highest F_m such that $F_m \leq n+1$. Then if $n+1 = F_m$ we are done.

Otherwise, $n+1-F_m$ is a positive integer less than $n+1$, and so $P(n+1-F_m)$ is true by assumption. Therefore, we can write $n+1-F_m = F_{i_1} + F_{i_2} + \cdots + F_{i_k}$ for distinct Fiobonacci numbers, no 2 of which are consecutive. Now we have written $n+1$ as a sum of distinct fiobonacci numbers

$$n+1 = F_{i_1} + F_{i_2} + \cdots + F_{i_k} + F_m$$

such that no 2 are consecutive execpt possibly the largest 2, if $F_{i_k} = F_{m-1}$. But the latter case would imply that $n \geq F_m + F_{m-1} = F_{m+1}$, contradicting the choice of m. This completes the induction.

□

3 Real Induction!

Fact 3. ***Real Induction Principle***

Induction is awesome and should be used to its full potential!

In the previous example, even after using strong induction there was still more work to be done. This is almost always going to be the case. The philosophy is to try and use your induction hypothesis to its maximum potential, getting as much out of it as you can and then hopefully the problem will be more approachable. This is called (somewhat redundantly, by a small, elite club of people which now includes you as a member!) Real induction. Here is a good example of this principle at work:

Example 5. *There are n lamps in a room, with certain lamps connected by wires. Initially all lamps are off. You can press the on/off button on any lamp A, but this also switches the state of all the lamps connected to lamp A from on to off and vice versa. Prove that by pressing enough buttons you can make all the lamps on. (Connections are such that if lamp A is connected to lamp B, then lamp B is also connected to lamp A.)*

Proof. We use induction on n. The base case of $n = 1$ is trivial (just turn the lamp on). Now assume the case of $n - 1$ lamps. Now look at the set of n lamps and ignore some lamp A. Then by induction, we can turn the remaining lamps on by pressing the buttons on some subset of them. Now if at the end of doing this A is also on, we are done. So we can assume that at the end of doing this A is off. Since A was an arbitrary lamp, we can assume that by pressing a sequence of buttons we can flip the states of all lamps except one of our choosing. Now, taking A and B to be 2 different lamps and flipping the states first of all lamps different from A and then all lamps different from B, we see that we can flip the states of only A and B. So this means we can flip the states of any number of even lamps. Now we have 2 cases:

- **n is even**: Then we are already done, since we can flip the states of any number of even lamps.
- **n is odd**: In this case, there must be some lamp A connected to an even number of lamps (prove this!) so first press the button on lamp A. Now, including A, an odd number of lamps are on, so an even number of lamps that are off remain. Flip their states to finish the proof!

□

Note that instead of using the induction hypothesis once to finish off the solution, we used it once per vertex in order to get a huge amount of infromation (flipping the state of any even set of lamps, that's clearly huge) so that finishing off the proof was much more straightforward.

Here are a couple other tricks to keep in mind which can often help you use induction to turn the tables!

- Often you can set up your induction in more than one way, and finding the right way makes the problem much simpler. See Problem 4.
- Sometimes trying to prove more by adding a stronger induction hypothesis makes it easier to carry out the induction. See Problem 16.

4 Problems

The following is a complilation of problems, all of which can be solved by using induction together with a few other insights. Some are fairly straightforward while the hardest ones are as hard as the hardest IMO problems (literally)! They are supposed to be arranged in increasing order of difficulty, but try them all, and you may find some of the later ones easier. ENJOY!

1. Show that if $\sin x \neq 0$ and n is a natural number then
$$\cos x \cdot \cos 2x \cdots \cos 2^{n-1}x = \frac{\sin 2^n x}{2^n \sin x}.$$
2. There are n cars placed around a circular track with enough fuel between them to make a complete loop around the track. Show that there is a car which can make it around the track by collecting the fuel from each car that it passes as it moves.
3. There are 2010 ninjas in the village of Konoha (what? Ninjas are cool.) Certain ninjas are friends, but it is known that there do not exist 3 ninjas such that they are all pairwise friends. Find the maximum possible number of pairs of friends.(If ninja A is friends with ninja B, then ninja B is also friends with ninja A.)
4. Given a set S of n distinct positive integers, and a (possibly empty) subset $T \subset S$, let $\sigma(T) = \sum_{t \in T}$. Let $f(S)$ be the number of distinct integers that you get by computing $\sigma(T)$ for each subset $T \subset S$. For each value of n, what is the maximum value of $f(S)$?
5. A positive integer n is called **good** if it can be written as $n = a_1 + a_2 + \cdots + a_k$ for not necessarily distinct positive integers a_i such that
$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}.$$

Given that all the numbers $37 \leq n \leq 73$ are good, prove that n is good for all positive integers $n \geq 73$.

6. Me and David play a game on an infinite sheet of paper. On my turn, I draw in a new country to be some connected shape which does not intersect any other countries except for possibly at the border. On David's turn, he chooses a color for that country, but he must make sure that no neighbouring countries share the same color. David, knowing the four color theorem, thinks that I can never make him use more than 4 colors. Silly David. Show that I can make David use at least 40 different colors.

7. Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(n+1) > f(f(n))$ for all natural numbers n. Prove $f(n) = n$ for all n.

8. Let x be a real number such that $\cos x + \sin x$ is rational. Prove that for all integers $n, \cos^n x + \sin^n x$ is rational.

9. Let $f_1(x) = x^2 - 1$, and for each positive integer $n \geq 2$ define $f_n(x) = f_{n-1}(f_1(x))$. How many distinct real roots does the polynomial f_{2009} have?

10. You are given n companies, $1, 2, \ldots, n$, intially in different piles. A *merger* is a collection of $n-1$ moves where each move takes 2 piles and makes them 1, so that at the end of the $n-1$ moves, only 1 piles remains. Two mergers are considered equivalent if they produce the same piles of companies along the way. For instance, the 2 mergers $1, 2, 3, 4 \to 12, 3, 4 \to 12, 34 \to 1234$ and $1, 2, 3, 4 \to 1, 2, 34 \to 12, 34 \to 1234$ are considered equivalent. Determine the number of inequivalent mergers.

11. Prove that for any natural number $n \in \mathbb{N}$, the sequence

    $a_1 = 1, a_{i+1} = 2^{a_i} (\bmod n)$

    $$2, 2^2, 2^{2^2}, \ldots (\bmod n)$$

    is eventually constant.

12. *JBMO 2004* Consider a convex polygon with $n \geq 4$ sides, and a triangulation of that polygon. Paint a triangle black if it has 2 sides which are also sides of the original polygon, and white if it contains no sides of the original polygon. Prove that there are 2 more black triangles than white ones.

13. Every road in the country Graphsville is one way, and every pair of cities is connected by exactly one road. Show that there is a city which can be reached by every other city either directly or by going through at most one other city.

14. If one square of a $2^n \times 2^n$ chessboard is removed, the remaining squares can be covered by L-shaped triminoes.

    Challenge: A $3n + 1 \times 3n + 1$ board has one square removed. Prove it can be tiled by L-shaped triminoes.

15. Consider all possible subsets of $\{1, 2, \ldots, N\}$ which contain no neighbouring elements. Prove that the sum of the squares of the products of all numbers in these subsets is $(N + 1)! - 1$.

16. The Fiobonacci sequence is defined as $F_1 = F_2 = 1$ and for all whole numbers $n, F_{n+2} = F_{n+1} + F_n$. Prove that $U_n = \frac{F_{2n}}{F_n}$ is always an integer, and moreover $U_{n+2} = U_{n+1} + U_n$.

17. In an $m \times n$ matrix of distinct positive integers, we color the $p \leq m$ largest numbers in each column white, and the $q \leq n$ largest numbers in each row black. Prove that at least pq numbers have been colored twice.

18. Each square of a $2^n - 1 \times 2^n - 1$ square board contains either $+1$ or -1. Such an arrangement is deemed successful if each number is the product of its neighbours. Find the number of successful arrangements.

19. You are given $\binom{n}{2}$ stones, divided into piles of various sizes. Each minute, you take one stone from each existing pile, and group them together into a new pile. Prove that eventually, you will have one pile of size i for each $1 \leq i \leq n$.

20. The polynomials $P_n(x, y)$ are defined by $P_1(x, y) = 1$ and

    $$P_{n+1}(x, y) = (x + y - 1)(y + 1)P_n(x, y + 2) + (y - y^2)P_n(x, y)$$

    Prove that $P_n(x, y) = P_n(y, x)$ for all $n \in \mathbb{N}$.

21. *IMO 2009/6* Let $a_1, a_2, \cdots, a_n$ be distinct positive integers, and let M be any set of $n-1$ positive integers which does not contain $S = a_1 + a_2 + \cdots + a_n$. A Grasshopper is to jump along the real axis from 0 to S making n jumps to the right of lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on a point in M.

    Hint: Don't panic! With some Real Induction, this problem is quite doable. To get you started, consider the largest a_i.