The Self-Enhancing Transformations

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Abstract

This paper introduces a new technique, *Self-Enhancing Transformation (SET)*, by reformatting the well-known method of Infinite Descent and complementing it with powerful extensions. This new technique can be used not only to climb down numerical stairs, but also to climb them up, to recursively strengthen inequalities, and to various other contexts where traditional infinite descent would fail. The potential of SET is demonstrated by applying it to solve several difficult functional equations, functional inequalities, challenging divisibility problems and combinatorial problems.
Keywords: Infinite Descent, Olympiad Mathematics.

1 Introduction

1.1 Notations and abbreviations

$\mathbb{N}$	The set of natural numbers	$\forall$	For all
$\mathbb{N}_0$	The set of non-negative integers	$\|A\|$	The size of set A
$\mathbb{Z}$	The set of integers	IMO	International Mathematical Olympiad
$\mathbb{Q}$	The set of rational numbers	TST	(IMO) Team Selection Test
$\mathbb{Q}^+$	The set of positive rational numbers	NMO	National Mathematical Olympiad
$\mathbb{R}$	The set of real numbers	WLOG	Without Loss Of Generality
$\|$	Such that		

1.2 Infinite Descent

Infinite Descent is a very old and well-known problem solving technique. To prove a statement through this, first, one assumes the opposite of the statement is true (for example, one can assume an equation has a solution). Then he would build a "smaller" solution (or property) from a given solution (or property). By repeating this procedure over and over, one can indefinitely "climb down" through the numbers. But this raises an contradiction. (For example, in the case of natural numbers, as 1 is smallest positive integer, one cannot go below that) Hence the first assumption becomes false and the statement is proved to be true.

1.3 The Self-Enhancing Transformation (SET)

This is an augmented form of Infinite Descent that is not bound to descend. Instead, it keeps a bounded score function to assess a property and recursively applies a fixed function on the property. The function changes score function's value monotonically. Since the score function is assumed to be bounded, the recursion must come to a halt. Such halting nature of the function pair is utilized to arrive at a contradiction.

2 Formal Definition

For any given statement, assume the opposite of the statement is true and determine a tuple of *dependent*[1] variables $(x_1, x_2, \ldots, x_m)$ from this assumption. Let $\mathcal{T}$ be the set of all the possible values of this tuple (notice that $|\mathcal{T}|$ can be infinite). Assume $\mathcal{R}$ is the set of relationships that links the variables $(x_1, x_2 \ldots x_m)$. Each relationship $R(x_1, \ldots, x_m) \in \mathcal{R}$ is usually[2] an equation or inequality. (Note that, there can be more than one relationship.)

To define an order among the $T \in \mathcal{T}$, a bounded score function $s : \mathcal{T} \to \mathbb{R}$ is also defined. A ***transformation*** $\chi : \mathcal{T} \to \mathcal{T}$ is a function[3] that manipulates relationships in $\mathcal{R}$ to transform $T \in \mathcal{T}$ into $T' \in \mathcal{T}$. A transformation will be called ***Self-Enhancing Transformation (SET)*** if exactly one of the following conditions holds for every $T \in \mathcal{T}$:

- $s(T) < s(\chi(T))$
- $s(T) > s(\chi(T))$

From the first condition, it is easy to notice that $s(T) < s(\chi(T)) < \cdots < s(\chi^k(T))$, implying s is unbounded above. Conversely, the second condition implies that s is unbounded below. However, s was assumed to be bounded, so if either of them is true, we reach to a contradiction!

Now we provide an example to explain all these notations. We shall explain the mechanism of Infinite Descent in terms of SET.

Example: 1 (Classic Infinite Descent) *Given a specific*[4] *Diophantine equation* $E(x_1, \ldots, x_m) = 0$. *Prove* E *has no solution over* $\mathbb{N}$.

Solution:(Fermat) Let us apply SET in step by step.

Step 1: We are to prove that E has no solution. So assume E has at least one solution. Let $\mathcal{T}$ be the set of all the solution tuple $T = (x_1, \ldots, x_m)$s. The variables $x_1, \ldots, x_m$ are only related by E. So, surely $\mathcal{R} = \{E\}$.

Step 2: Now we start playing with relationships in set $\mathcal{R}$. (In this case, it contains only E.) Then we notice that given a solution, we can build another "smaller" solution tuple T' using E. This is the skeleton for transformation χ.

Step 3: But how do we measure smallness of a tuple? For a solid proof, we need a solid definition of "smaller". This is why we define the score function $s(T)$ as $\min(x_1, \ldots, x_m)$ or $\sum x_i$ (Whichever fits χ)

Step 4: $x_i \in \mathbb{N}$ implies $s(T)$ has a lower bound. Since $s(\chi(T)) < s(T)$, we can recursively apply χ on T and decrease s indefinitely. But it upsets the bound of s.

□

Implementation

Here we show the general method of applying SET:

1. Find a suitable property. Assume the opposite of it is true and determine the tuple of variables and the relationships from this assumption.
2. Find a Self-Enhancing Transformation from the relationships.
3. Set a score function to "fit" the transformation and relationships.
4. Iterate the SET until arriving at a contradiction.

[1] The variables are bound together by one or more mathematical relationships.
[2] In some cases there can be other types relationships defined by the problem statement
[3] derived from or defined in the problem statement
[4] Diophantine equations with at least one solution can not be tackled by Infinite Descent or SET

Note: The most important step is defining the right transformation. This can be only learned through practice. Furthermore, the score function only needs to be bounded in the required direction and not necessarily in both directions.

3 Application in Olympiad Problem Solving

There are numerous proofs that might appear radically different on the surface, but in fact belong to SET proof family. We provide a few examples in the following sections.

Note: The proofs were written in slightly informal language to reveal the intuition behind each step. Also, in some of the proofs, insignificant details were omitted for brevity.

3.1 Functional Equations & Inequalities[5]

Example: 2 (IMO Shortlist 2010/A5) *Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:*

$$f\left(f(x)^2 y\right) = x^3 f(xy)$$

Solution:(Adib) Following the general rule for solving nearly all functional equations and inequalities, we first guess the solution, which is $f(x) = \dfrac{1}{x} \implies xf(x) = 1$. So our goal will be to prove $xf(x) = 1$ from the given equation.

At first, prove that

$$[f(x)f(f(x))]^2 = (xf(x))^3 \tag{2.1}$$

Define a rational r to be an i^{th} *power* if $r = \left(\dfrac{p}{q}\right)^i$ for some $p, q, i \in \mathbb{N}$. By playing with (2.1), we observe that several $xf(x)$s, for different values of xs are in fact various powers. For example, $\left[\frac{1}{x} f(f(x))\right]^2 = xf(x) \implies xf(x)$ is a square (or 2^{nd} power) for each x. We can have 4^{th}, 8^{th} powers etc for other specific values. This is what motivates the following solution:

Step 1: Assume $xf(x) \neq 1$ for an x. Consider the set

$$\mathcal{T} = \{(i, xf(x)) \mid xf(x) \text{ is an } i^{\text{th}} \text{ power } \forall x \in \mathbb{Q}^+\}$$

Since we are analyzing i^{th} powers, it is a good idea keep (2.1) in $\mathcal{R}$ since its LHS and RHS are different powers. Hence, we set

$$\mathcal{R} = \left\{ [f(x)f(f(x))]^2 = (xf(x))^3 \right\}$$

Step 2: $xf(x)$ is an i^{th} power. It implies $f(x)f(f(x))$ is also an i^{th} power. Therefore, LHS of (2.1) is $2i^{\text{th}}$ power while RHS of it is $3i^{\text{th}}$ power. Hence, to be equality, both sides have to be $6i^{\text{th}}$ power. But it implies $xf(x)$ is a $\frac{6i}{3} = 2i^{\text{th}}$ power. So our transformation χ does the following:

$$\chi : (i, xf(x)) \to (2i, xf(x))$$

Step 3: Our transformation χ increments only the first variable, i, in the tuple. Hence, set $s(T) = i$. s is also bounded above since no rational number except 1 can be i^{th} powers for infinitely many i-s.

Step 4: By recursively applying χ on a fixed $T \in \mathcal{T}$, we can infinitely increment s which is a contradiction to the conclusion of the previous step. Therefore, we must have $xf(x) = 1$ since only 1 can be i^{th} power for any natural integer i. Hence, $f(x) = \dfrac{1}{x} \forall x \in \mathbb{Q}^+$.

[5] In a formal setting, one also has to perform a solution checking for functional equations and inequalities

$\square$

Example: 3 (IMO 2013/5) *Let $f : \mathbb{Q}^+ \to \mathbb{R}$ be a function satisfying the following three conditions:*

(i) *For all $x, y \in \mathbb{Q}^+$, we have $f(x)f(y) \geq f(xy)$;*

(ii) *For all $x, y \in \mathbb{Q}^+$, we have $f(x+y) \geq f(x) + f(y)$;*

(iii) *There exists a rational number $a > 1$ such that $f(a) = a$.*

Prove that $f(x) = x$ for all $x \in \mathbb{Q}^+$.

Solution:(Adib) First prove that,

1. f is strictly increasing and $f(n) \geq n \forall n \in \mathbb{N}$.
2. $a^k \geq f(a^k)\ \forall k \in \mathbb{N}$.
3. For $p, q \in \mathbb{N}$, $\frac{f(p)}{q} \geq f(\frac{p}{q}) \geq \frac{f(p)}{f(q)}$. Therefore, if $f(\mathbb{N}) = \mathbb{N}$, then $f(\mathbb{Q}^+) = \mathbb{Q}^+$.

Step 1: Suppose $n \in \mathbb{N}$. From the first property, $f(n) - n \geq 0$ while from the second property, $f(a^k) - a^k \leq 0$. This makes us wonder how $f(i) - i$ changes over $\mathbb{Q}^+$. Assume $f(n) = n + \varepsilon$, where $\varepsilon \geq 0$ and consider this set

$$\mathcal{T} = \{(n, \varepsilon) \mid n \in \mathbb{N}, f(n) = n + \varepsilon\}$$

And $\mathcal{R}$ will contain everything we have found so far.

$$\mathcal{R} = \{f(n) \geq n,\ a^k \geq f(a^k),\ \varepsilon \geq 0,\ f(x+y) \geq f(x) + f(y)\}$$

We want to prove $\varepsilon = 0\ \forall n \in \mathbb{N}$. Hence assume for an $n = m$, $\varepsilon_m > 0$.

Step 2: Since our goal is to examine the change in $f(i) - i$, we try look for a way to manipulate ε. Set $x = y = n$ in (ii) to get

$$f(2n) \geq 2f(n) = 2n + 2\varepsilon \implies f(2n) - 2n \geq 2\varepsilon$$

This operation increments $f(i) - i$. Assuming $f(2n) - 2n = 2\varepsilon + \epsilon$, with $\epsilon \geq 0$, the χ function will be

$$\chi : (n, \varepsilon) \to (2n, 2\varepsilon + \epsilon)$$

Step 3: The best choice for $s(T)$ can be,

$$s(T) = f(n) - n = \varepsilon$$

Suppose k is a very large integer so that $\lfloor a^k \rfloor > n$. (Such a k exists because $a > 1$.) We know that $a^k \geq f(a^k) \geq f\left(\lfloor a^k \rfloor\right)$ since f is strictly increasing. So,

$$\begin{aligned} a^k \geq f\left(\lfloor a^k \rfloor\right) &= f\left(\lfloor a^k \rfloor - n + n\right) \\ &\geq f\left(\lfloor a^k \rfloor - n\right) + f(n) \\ &\geq \lfloor a^k \rfloor - n + n + \varepsilon \\ &= \lfloor a^k \rfloor + \varepsilon \end{aligned}$$

Hence $s(T) = \varepsilon$ must be bounded above. In particular, for each $n \in \mathbb{N}$, we must have

$$s(T) = f(n) - n = \varepsilon \leq a^k - \lfloor a^k \rfloor < 1$$

Step 4: We recursively apply χ on $T = (m, \varepsilon_m)$. Since $\varepsilon_m > 0$, every time χ will increase s. Thus there can be no upper bound for s; contradicting our earlier findings. Therefore, $\varepsilon = 0$ and $f(n) = n \forall n \in \mathbb{N}$.

Since $f(\mathbb{N}) = \mathbb{N}$, we also have $f(\mathbb{Q}^+) = \mathbb{Q}^+$ from the third property. □

Sometimes it is not necessary to assume opposite of a statement in the beginning to apply SET. The following two examples will demonstrate this.

Example: 4 (IMO Shortlist 2013/N6) *Determine all functions $f : \mathbb{Q} \to \mathbb{Z}$ satisfying*

$$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right)$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{N}$.

Solution:(Adib) First prove that either f is a constant function, or, it has the following properties:

- $f(n) = n \ \ \forall n \in \mathbb{Z}$
- $f(x+a) = f(x) + a$, $\forall a \in \mathbb{Z}$ and $\forall x \in \mathbb{Q}$
- $f\left(\frac{1}{2}\right) \in \{0, 1\}$
- Assume $f\left(\frac{1}{2}\right) = 0$. (The other case can be proven analogously.) Now inductively prove for $k \in \mathbb{N}, 0 \leq i < 2^k$, $f\left(\frac{i}{2^k}\right) = 0$

We shall prove $f\left(\dfrac{p}{q}\right) = 0$ for $p, q \in \mathbb{N}$ with $p < q$.

Let $f\left(\dfrac{p}{q}\right) = \delta$. Set $x = \dfrac{p}{q}, a = p, b = q+1$ in the main equation to get

$$f\left(\frac{p}{q}\right) = f\left(\frac{p+\delta}{q+1}\right) = \delta \tag{4.1}$$

Now we are ready to apply SET.

Step 1: We want to prove $\delta = 0$. (4.1) asserts there are other x-s so that $f(x) = \delta$. This is an interesting property. We shall dig deeper into it. Therefore, set

$$\mathcal{T} = \left\{\left(r, s, f\left(\frac{r}{s}\right)\right) \mid r = p + n\delta, s = q + n \ \forall n \in \mathbb{N}_0\right\}$$

The relationship set $\mathcal{R}$ should be:

$$\mathcal{R} = \left\{r = p + n\delta, s = q + n, f\left(\frac{p}{q}\right) = f\left(\frac{p+\delta}{q+1}\right) = \delta\right\}$$

Step 2: We already have (4.1) to link the tuples in $\mathcal{T}$ sequentially and ensure $f\left(\dfrac{r}{s}\right) = \delta$ for all r, s. So, we can simply define (4.1) as our transformation. Hence we set

$$\chi : \left(r, s, f\left(\frac{r}{s}\right)\right) \to \left(r+\delta, s+1, f\left(\frac{r+\delta}{s+1}\right)\right)$$

Step 3: Define the score function as

$$s(T) = f\left(\frac{p - q\delta}{q+n}\right)$$

because $\delta \in \mathbb{Z}$ and

$$s(T) = f\left(\frac{p - q\delta}{q+n}\right) = f\left(\frac{p+n\delta}{q+n} - \delta\right) = f\left(\frac{p+n\delta}{q+n}\right) - \delta = f\left(\frac{r}{s}\right) - \delta = 0$$

Therefore $s(T) = 0$ for each $T \in \mathcal{T}$.

Step 4: $p - q\delta \neq 0$ (otherwise $\delta = \frac{p}{q}$ =non-integer). Suppose $T = \left(p, q, f\left(\frac{p}{q}\right)\right)$. If $\delta > 1$, then for $n = q\delta - p - q > 0$,

$$s\left(\chi^n(T)\right) = f\left(\frac{p - q\delta}{q\delta - p}\right) = f(-1) = -1$$

A contradiction. Similarly, if $\delta < 0$, then for $n = p - q\delta - q > 0$,

$$s\left(\chi^n(T)\right) = f\left(\frac{p - q\delta}{p - q\delta}\right) = f(1) = 1$$

This is also a contradiction. So $\delta \in \{0, 1\}$. However, if $\delta = 1$, then (4.1) asserts that there exists a $T \in \mathcal{T}$ so that $s = 2^k$, $r < s$ and

$$0 = f\left(\frac{r}{2^k}\right) = f\left(\frac{r}{s}\right) = \delta = 1$$

This is another contradiction. Consequently, $f\left(\frac{p}{q}\right) = \delta = 0$.

So, $f(x) = \lfloor x \rfloor$ for $0 \leq x \leq 1$. Since $f(a + \frac{p}{q}) = a + f(\frac{p}{q})\ \forall a \in \mathbb{Z}$, we get $f(x) = \lfloor x \rfloor \forall x \in \mathbb{Q}$.

Similarly, by assuming $f\left(\frac{1}{2}\right) = 1$, one can prove $f(x) = \lceil x \rceil \forall x \in \mathbb{Q}$. □

3.2 Divisibility Problems

Example: 5 (Bulgaria[4]) *Let $a, b, c \in \mathbb{N}$ such that ab divides $c(c^2 - c + 1)$ and $c^2 + 1$ divides $a + b$. Prove that the sets $\{a, b\}$ and $\{c, c^2 - c + 1\}$ must coincide.*

Solution:(Adib) WLOG assume $a \geq b$. Suppose $c(c^2 - c + 1) = rab$. Then proceed like this:

1. Start by proving,
$$rb^2 + 1 \equiv 0 \pmod{c^2 + 1} \tag{5.1}$$
2. Consider the $r \geq 2$ case.
3. Prove $c \geq 2b$ and $a > c^2 - c + 1$ from this assumption.
4. Set $d = c - b$. Thus $d \geq b$.

Now we attack the problem using SET.

Step 1: From some initial investigation, it becomes apparent that $\dfrac{d}{b}$ can be incredibly large. This is counter intuitive. Therefore, assume

$$\mathcal{T} = \left\{(a, b, c, d, n) \mid \frac{d}{b} \geq n, n \in \mathbb{N}\right\}$$

$\mathcal{R}$ will contain every relationship we have discovered till now.

$$\mathcal{R} = \left\{a \geq b,\ a > c^2 - c + 1,\ c(c^2 - c + 1) = rab,\ \frac{a + b}{c^2 + 1} \in \mathbb{N},\ n \leq \frac{d}{b}, n \in \mathbb{N}\right\}$$

Step 2: For any $(a, b, c, d, n) \in \mathcal{T}$, $c = b + d \geq (n + 1)b \implies c^2 + 1 \geq (n + 1)^2 b^2 + 1$. From (5.1),

$$rb^2 + 1 \geq c^2 + 1 \geq (n + 1)^2 b^2 + 1 \implies r \geq (n + 1)^2$$

Hence

$$\frac{c(c^2 - c + 1)}{ab} = \frac{c^2 - c + 1}{a} + \frac{d(c^2 - c + 1)}{ab} = r \geq (n + 1)^2 \tag{5.2}$$

From $\mathcal{R}$, $\frac{c^2 - c + 1}{a} < 1$. Therefore,

$$\frac{d}{b} > \frac{d(c^2 - c + 1)}{ab} = r - \frac{c^2 - c + 1}{a} > (n + 1)^2 - 1 = n^2 + 2n$$

Therefore,

$$\chi : (a, b, c, d, n) \to (a, b, c, d, n^2 + 2n)$$

Step 3: So, χ keeps a, b, c, d unchanged, but increases n to $n^2 + 2n$. So the good choice for s will be, $s(T) = n$.

Step 4: For a fixed triple (a, b, c), s has an upper bound. But applying χ recursively on any $T \in \mathcal{T}$ shows s can't have any upper bound – thus leading to a contradiction. Hence the $r \geq 2$ case is impossible; implying $r = 1$.

Now suppose $a + b > c^2 + 1$. Then

$$2a \geq a + b \geq 2(c^2 + 1) \implies \frac{c^2 - c + 1}{a} < 1$$

(5.1) implies $c \leq b$. If $c = b$, we have $a = c^2 - c + 1$, proving the problem statement. Else if $c < b$, then

$$1 = \frac{c(c^2 - c + 1)}{ab} < \frac{b(c^2 - c + 1)}{ab} < 1$$

This is again a contradiction. So $a + b = c^2 + 1$. Now we have $a + b = c + (c^2 - c + 1)$ and $ab = c(c^2 - c + 1)$. So we must have $\{a, b\} = \{c, c^2 - c + 1\}$ □

3.3 Combinatorial Problems

Example: 6 (IMO Shortlist 1994/C3) *Peter has 3 accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled.*

(a) *Prove that Peter can always transfer all his money into two accounts.*

(b) *Can he always transfer all his money into one account?*

Solution:(Pranav A. Sriram) [5] The second part of the question is trivial - if the total number of dollars is odd, it is clearly not always possible to get all the money into one account.
Now we solve the first part.

Step 1: Let $T = (A, B, C)$ be a tuple so that $A \leq B \leq C$ are the number of dollars in the accounts at a particular point of time. Assume $\mathcal{T}$ to be the set of all such tuples. Suppose Peter can never transfer all the money into two accounts. In other words, for every $T \in \mathcal{T}$, we have $C \geq B \geq A > 0$. We have only one relationship among A, B, C which was described by the problem statement as follows:

$$\mathcal{R} = \{ \text{ Move money from account } X \text{ to } Y \text{ so that money in } Y \text{ doubles } \}$$

Step 2: Suppose $B = qA + r$ with $r < A$. For brevity, we shall enumerate the account with money A, B, C as 1, 2, 3, respectively. If $A = 0$, we are done. So assume $A > 0$. $(\overline{m_1 m_2 \ldots m_k})_2$ is the binary representation of q. Transfer money k times to account 1 from accounts 2 or 3. The i^{th} transfer will be from account 2 if $m_i = 1$ and from account 3 otherwise. The number of dollars in the first account starts with A and keeps doubling in each step. Thus we end up transferring $A(m_0 + 2m_1 + \ldots + 2^k m_k) = Aq$ dollars from to account 2, and $(2^k - 1 - q)A$ dollars from account 3. So we are left with $B - Aq = r$ dollars in account 2, which now becomes the account with smallest money. Hence,

$$\chi : (A, B, C) \to \begin{cases} \left(r,\ 2^k A - A,\ C + (1 + q)A - 2^k A\right), \text{ or,} \\ \\ \left(r,\ C + (1 + q)A - 2^k A,\ 2^k A - A\right) \end{cases}$$

Step 3: But in both cases $\min(\chi(T)) \leq r$. So we define $s(T) = \min(A, B, C)$ for each $T \in \mathcal{T}$. s is indeed bounded below.

Step 4: χ reduces s from A to $r < A$ or even less. Therefore, applying χ on a fixed $T \in \mathcal{T}$ repeatedly, we can reduce s as much as we want. But this is a contradiction to the bound of s.

$\square$

Remark. In combinatorics, such score functions are called monovariants since they only change in one direction.

4 Selected Problems

1. (IMOMath[6]) Let $\mathbb{R}^* = [1, \infty)$. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ that satisfy:

   (i) $f(x) \leq 2(1+x)\ \forall x \in \mathbb{R}^*$,

   (ii) $xf(x+1) = f(x)^2 - 1\ \forall x \in \mathbb{R}^*$.

2. [7]Determine all polynomials $P(x) \in \mathbb{Z}[x]$ such that $P(x)$ is bijective over $\mathbb{R}$ and

$$P\left(P(x)\right) = P\left(x^2\right) - 2P(x) + a$$

   for a constant $a \in \mathbb{R}$ and for all $x \in \mathbb{R}$.

3. (Vietnam NMO 2012/7[8]) Find all surjective, strictly increasing functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)) = f(x) + 12x$$

4. (IMO 2011/3[3]) Let $f : \mathbb{R} \to \mathbb{R}$ follows

$$f(x+y) \leq yf(x) + f(f(x))$$

   for all $x, y \in \mathbb{R}$. Prove $f(x) = 0$ for all $x \leq 0$.

5. (China NMO 2013/2) Find all nonempty sets S of integers such that $3m - 2n \in S$ for all (not necessarily distinct) $m, n \in S$.

6. (USAMO 2015/6[9]) Consider $0 < \lambda < 1$, and let A be a multiset[6] of positive integers. Let $A_n = \{a \in A : a \leq n\}$. Assume that for every $n \in \mathbb{N}$, the set A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$.

7. (USA TST 2002/6[9]) Find all ordered pairs of positive integers (m, n) such that $mn - 1$ divides $m^2 + n^2$.

8. (IMO 2007/5[3]) Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

9. (IMO Shortlist 2013/C3) A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be *entangled*, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

   (i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

[6]A multiset is a set-like collection of elements in which order is ignored, but repetition of elements is allowed and multiplicity of elements is significant. For example, multisets $\{1, 2, 3\}$ and $\{2, 1, 3\}$ are equivalent, but $\{1, 1, 2, 3\}$ and $\{1, 2, 3\}$ differ.

(ii) At any moment, he may double the whole family of imons in his lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

10. (IMO 2010/5[3]) Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed:

    (i) Choose a non-empty box B_j, $1 \le j \le 5$, remove one coin from B_j and add two coins to B_{j+1};

    (ii) Choose a non-empty box B_k, $1 \le k \le 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

    Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

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