WOOT 2010–11
Practice Olympiad 5 Solutions

1. For a positive integer n, let

$$S_n = 5.5 + 5.55 + 5.555 + \cdots + 5.\underbrace{555\ldots5}_{n\ 5\text{'s}}.$$

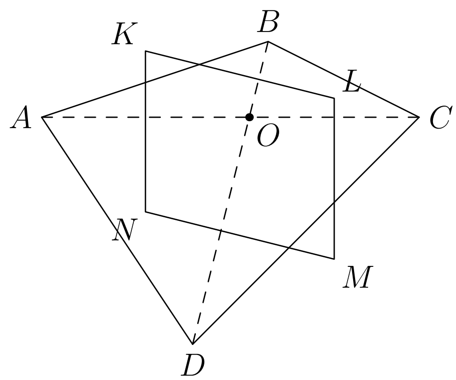
Show that $555 < S_{100} < 556$.

Solution. Each term in the sum is less than 5.56, so $S_{100} < 100 \cdot 5.56 = 556$. Also,

$$\begin{aligned} S_{100} &> 5.5 + 5.55 + \underbrace{5.555 + 5.555 + \cdots + 5.555}_{98\text{ times}} \\ &= 5.5 + 5.55 + 98 \cdot 5.555 \\ &= 555.44 \\ &> 555. \end{aligned}$$

2. Let $ABCD$ be a convex quadrilateral. Prove that there exists exactly one point O such that the circumcenters of the triangles AOB, BOC, COD, and DOA form the vertices of a (nondegenerate) parallelogram.

Solution. Let K, L, M, and N be the circumcenters of triangles AOB, BOC, COD, and DOA, respectively.



Since K is the circumcenter of triangle AOB, K lies on the perpendicular bisector of BO. Also, L is the circumcenter of triangle BOC, so L lies on the perpendicular bisector of BO. Hence, KL is perpendicular to BO. Similarly, both M and N lie on the perpendicular bisector of DO, so MN is perpendicular to DO. It follows that KL and MN are parallel if and only if BO and DO are parallel, i.e. O lies on BD.

Likewise, LM and KN are parallel if and only if O lies on AC. Therefore, quadrilateral $KLMN$ is a parallelogram if and only if O is the intersection of diagonals AC and BD.

3. Renee writes down a list of 50 positive integers, all between 1 and 50 inclusive, such that the sum of the integers is 100. Show that the list can be partitioned into two lists, such that the the sum of the numbers in each list is 50.

Solution. Let the 50 positive integers be x_1, x_2, ..., x_{50}, and for $1 \le k \le 49$, let $s_k = x_1+x_2+\cdots+x_k$. Each sum s_k is an integer from 1 to 99. If $s_k = 50$ for some k, then we are done. Otherwise, each sum s_k is an integer from 1 to 99, excluding 50.

If we can find two sums s_i and s_j that differ by 50, where $i < j$, then

$$s_j - s_i = x_{i+1} + x_{i+2} + \cdots + x_j = 50,$$

and again we are done.

So, consider the 49 pairs $\{1, 51\}$, $\{2, 52\}$, ..., $\{49, 99\}$. We have 49 sums s_k, and each s_k is an element in one of these pairs. If both elements in a pair appear among the s_k, then they differ by 50, and we are done. Otherwise, every pair contains exactly one element that appears among the s_k.

Suppose we switch two consecutive terms x_k and x_{k+1} that are distinct. Then each sum remains the same, except for the sum s_k, which changes from

$$s_k = x_1 + x_2 + \cdots + x_{k-1} + x_k$$

to

$$s_k = x_1 + x_2 + \cdots + x_{k-1} + x_{k+1}.$$

Since all the x_k are at most 50, the absolute difference $|x_{k+1} - x_k|$ is less than 50. Hence, the sum s_k moves from one pair to a different pair; this latter pair gives us two sums that differ by 50.

The only detail remaining is to see if we can find two consecutive terms x_k and x_{k+1} that are distinct. If every two consecutive terms are equal, then all terms are equal, which implies that every term is equal to 2. This case is trivial.

4. Determine all triples of nonnegative integers (x, y, z) satisfying $8^x + 15^y = 17^z$.

Solution. Taking the given equation modulo 2, we get

$$8^x + 1 \equiv 1 \pmod{2},$$

so $8^x \equiv 0 \pmod{2}$, which means x is positive.

Taking the given equation modulo 4, we get

$$(-1)^y \equiv 1 \pmod{4},$$

so y is even. Let $y = 2b$, so $8^x + 15^{2b} = 17^z$.

Taking this equation modulo 7, we get

$$2 \equiv 3^z \pmod{7}.$$

Cubing both sides, we get $27^z \equiv 8 \pmod{7}$, or $(-1)^z \equiv 1 \pmod{7}$, so z is even. Let $z = 2c$, so $8^x + 15^{2b} = 17^{2c}$. Then

$$8^x = 17^{2c} - 15^{2b} = (17^c + 15^b)(17^c - 15^b).$$

This equation tells us that both $17^c + 15^b$ and $17^c - 15^b$ must be powers of 2, so let $17^c + 15^b = 2^m$ and $17^c - 15^b = 2^n$, where $m > n$. Then

$$2^m - 2^n = 2 \cdot 15^b.$$

The left-hand side $2^m - 2^n = 2^n(2^{m-n} - 1)$ has exactly n factors of 2, and the right-hand side has exactly one factor of 2, so $n = 1$. Hence,

$$17^c - 15^b = 2.$$

If $b \geq 2$, then taking this equation modulo 9, we get

$$(-1)^c \equiv 2 \pmod{9}.$$

But $(-1)^c$ can only be congruent to 1 or -1 modulo 9, so b can only be equal to 0 or 1. If $b = 0$, then we get $17^c = 3$, which has no solutions, so $b = 1$. Then $17^c = 17$, so $c = 1$. Then $y = 2$, $z = 2$, and $8^x = 17^2 - 15^2 = 64$, so $x = 2$. Therefore, the only solution is $(x, y, z) = (2, 2, 2)$.

5. Let $ABCD$ be a convex quadrilateral such that $AB + CD = \sqrt{2}AC$ and $BC + DA = \sqrt{2}BD$. Prove that $ABCD$ is a parallelogram.

Solution. We claim that for any convex quadrilateral $ABCD$,

$$(AB + CD)^2 + (BC + DA)^2 \geq 2AC^2 + 2BD^2.$$

We prove this using vectors.

Expanding the inequality, we get

$$AB^2 + 2AB \cdot CD + CD^2 + BC^2 + 2BC \cdot DA + DA^2 \geq 2AC^2 + 2BD^2.$$

Expressing this in terms of vectors, we get

$$\begin{aligned} &|\overrightarrow{A}|^2 - 2\overrightarrow{A} \cdot \overrightarrow{B} + |\overrightarrow{B}|^2 + 2|\overrightarrow{A} - \overrightarrow{B}||\overrightarrow{C} - \overrightarrow{D}| + |\overrightarrow{C}|^2 - 2\overrightarrow{C} \cdot \overrightarrow{D} + |\overrightarrow{D}|^2 \\ &+ |\overrightarrow{B}|^2 - 2\overrightarrow{B} \cdot \overrightarrow{C} + |\overrightarrow{C}|^2 + 2|\overrightarrow{B} - \overrightarrow{C}||\overrightarrow{D} - \overrightarrow{A}| + |\overrightarrow{D}|^2 - 2\overrightarrow{D} \cdot \overrightarrow{A} + |\overrightarrow{A}|^2 \\ &\geq 2|\overrightarrow{A}|^2 - 4\overrightarrow{A} \cdot \overrightarrow{C} + 2|\overrightarrow{C}|^2 + 2|\overrightarrow{B}|^2 - 4\overrightarrow{B} \cdot \overrightarrow{D} + 2|\overrightarrow{D}|^2, \end{aligned}$$

which simplifies to

$$\begin{aligned} &|\overrightarrow{A} - \overrightarrow{B}||\overrightarrow{C} - \overrightarrow{D}| + |\overrightarrow{B} - \overrightarrow{C}||\overrightarrow{D} - \overrightarrow{A}| \\ &\quad \geq \overrightarrow{A} \cdot \overrightarrow{B} + \overrightarrow{C} \cdot \overrightarrow{D} + \overrightarrow{B} \cdot \overrightarrow{C} + \overrightarrow{D} \cdot \overrightarrow{A} - 2\overrightarrow{A} \cdot \overrightarrow{C} - 2\overrightarrow{B} \cdot \overrightarrow{D}. \end{aligned}$$

We have that

$$(\overrightarrow{A} - \overrightarrow{B}) \cdot (\overrightarrow{D} - \overrightarrow{C}) = |\overrightarrow{A} - \overrightarrow{B}||\overrightarrow{D} - \overrightarrow{C}| \cos\theta,$$

where θ is the angle between $\overrightarrow{A} - \overrightarrow{B} = \overrightarrow{BA}$ and $\overrightarrow{D} - \overrightarrow{C} = \overrightarrow{CD}$. Hence,

$$\overrightarrow{A} \cdot \overrightarrow{D} - \overrightarrow{A} \cdot \overrightarrow{C} - \overrightarrow{B} \cdot \overrightarrow{D} + \overrightarrow{B} \cdot \overrightarrow{C} \leq |\overrightarrow{A} - \overrightarrow{B}||\overrightarrow{D} - \overrightarrow{C}|.$$

Equality occurs if and only if AB and CD are parallel.

Similarly,

$$\overrightarrow{A} \cdot \overrightarrow{B} - \overrightarrow{B} \cdot \overrightarrow{D} - \overrightarrow{A} \cdot \overrightarrow{C} + \overrightarrow{C} \cdot \overrightarrow{D} \leq |\overrightarrow{B} - \overrightarrow{C}||\overrightarrow{A} - \overrightarrow{D}|,$$

with equality if and only if BC and AD are parallel. Adding these inequalities, we get

$$\begin{aligned} &\overrightarrow{A} \cdot \overrightarrow{B} + \overrightarrow{C} \cdot \overrightarrow{D} + \overrightarrow{B} \cdot \overrightarrow{C} + \overrightarrow{D} \cdot \overrightarrow{A} - 2\overrightarrow{A} \cdot \overrightarrow{C} - 2\overrightarrow{B} \cdot \overrightarrow{D} \\ &\quad \leq |\overrightarrow{A} - \overrightarrow{B}||\overrightarrow{C} - \overrightarrow{D}| + |\overrightarrow{B} - \overrightarrow{C}||\overrightarrow{D} - \overrightarrow{A}|, \end{aligned}$$

and the original inequality follows. Furthermore, equality occurs if and only if $ABCD$ is a parallelogram.

Since $AB + CD = \sqrt{2}AC$ and $BC + DA = \sqrt{2}BD$ imply

$$(AB + CD)^2 + (BC + DA)^2 = 2AC^2 + 2BD^2,$$

we conclude that $ABCD$ is a parallelogram.

Art of Problem Solving

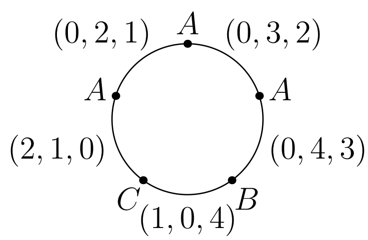
WOOT 2010–11

Practice Olympiad 5 Solutions

6. On a circle are marked 999 points. How many ways are there to assign to each point one of the letters A, B, or C, so that on the arc between any two points marked with the same letter, there are an even number of letters differing from these two?

Solution. There are three arrangements in which only one letter appears, one for each letter. Otherwise, since 999 is an odd number, each letter that does appear must appear an odd number of times. This means that each letter A, B, and C must appear at least once.

To each arc between a pair of letters, we assign the triple (a, b, c), where a denotes the number of letters one meets before the next A, starting at that arc and going counter-clockwise, and similarly for b and c. An example is shown below.

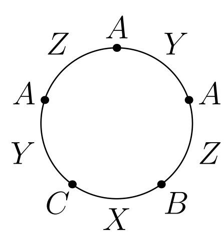


We claim that each triple (a, b, c) has exactly two even numbers. First, in every triple, there is always exactly one 0, corresponding to the letter next to the triple going counter-clockwise, so not all three numbers can be odd.

Suppose a triple (a, b, c) has exactly one even number, say $a = 0$, so the letter next to the triple going counter-clockwise is A, and b and c are odd. We consider the letter next to the triple going clockwise. If this letter is B, then there would be b letters other than B appearing between two points marked with Bs, and b is odd, so this letter cannot be B. Similarly, this letter cannot be C, so this letter must be A. Then the next triple, going clockwise, is $(0, b + 1, c + 1)$, which are all even numbers.

Conversely, if all the numbers in a triple are even, then the next triple going clockwise consists of one even number (a 0 that corresponds to the next letter) and two odd numbers. Hence, going around the circle, the triples alternate between having one even number and two odd numbers, and three even numbers. But there are 999 triples, which is an odd number, so this cannot occur. Therefore, every triple must consist of two even numbers and one odd number.

Now, replace every triple (a, b, c) where a is odd with the letter X. Similarly, replace every triple where b is odd with the letter Y, and where c is odd with the letter Z. The following diagram shows what happens with the example given above.



We claim that the resulting arrangement has the following two properties: (1) No two adjacent arcs have the same label, and (2) for each pair of adjacent arcs labeled Y and Z, X and Z, and X and Y, the point in between is marked with the letter A, B, and C, respectively.

To prove (1), suppose that two adjacent arcs have the same label, say X and X. Let the corresponding triples be (a, b, c) and (a', b', c'), going clockwise, so a and a' are odd. Since a' is odd, the point in between the two arcs cannot be marked with the letter A, which means that $a' = a + 1$, contradiction. So no two adjacent arcs can have the same label.

To prove (2), consider two arcs labeled Y and Z, going clockwise. (All other cases follow by symmetry.) Let the corresponding triples be (a, b, c) and (a', b', c'), respectively, so b and c' are odd. Since c' is odd, the point in between the two arcs cannot be marked with the letter C, so the letter must be either A or B. If the letter is B, then $a' = a + 1$, contradiction, since both a and a' are even, so the letter must be A.

Conversely, given an arrangement of labels X, Y, and Z that satisfy properties (1) and (2), we may uniquely recover the letters A, B, and C. Thus, we have a bijection, and the problem now is to count the number of arrangements of labels X, Y, and Z that satisfy properties (1) and (2). This number is equal to the number of sequences of the letters X, Y, and Z, of length 999, such that no two adjacent letters are the same, and the first letter and last letter are different.

More generally, let S_n be the set of sequences of the letters X, Y, and Z, of length n, such that no two adjacent letters are the same, and the first letter and last letter are different, and let T_n be the number of such sequences where the first letter and last letter are the same. Let $s_n = |S_n|$ and $t_n = |T_n|$.

Given a sequence in T_n, we can only append two different letters at the right, to obtain two sequences in S_{n+1}. Similarly, given a sequence in S_n, again we can only append two different letters at the right, one of which is a sequence in S_{n+1}, the other in T_{n+1}. Hence,

$$\begin{aligned} s_{n+1} &= s_n + 2t_n, \\ t_{n+1} &= s_n. \end{aligned}$$

for all $n \geq 1$. Therefore,

$$s_{n+1} = s_n + 2s_{n-1}$$

for all $n \geq 2$. We also have $s_1 = 0$ and $s_2 = 6$, so solving the recursion gives

$$s_n = 2^n + 2(-1)^n.$$

Then $s_{999} = 2^{999} - 2$, so the total number of ways to mark the 999 points with letters is $2^{999} - 2 + 3 = 2^{999} + 1$.

7. Jerry and Donna are playing the following game: start with the polynomial

$$x^4 + \_x^3 + \_x^2 + \_x + 1,$$

where the coefficients of x^3, x^2, and x are missing. Jerry goes first, and in turn they choose an empty coefficient and fill in any real number. Donna wins if the resulting polynomial has a real root; otherwise Jerry wins. Prove that Donna has a winning strategy.

Solution. After Jerry's first move, at least one of the two coefficients of odd powers must still be unfilled. Hence, we can write the polynomial in the form

$$p(x) = q(x) + \_x^s + \_x^t,$$

where s and t are distinct integers, with t odd. Donna will win by filling in the coefficient of x^s with a real number a (to be determined); we then show that for any real number b, the polynomial

$$p(x) = q(x) + ax^s + bx^t$$

always has a real root.

We have that

$$\begin{aligned} 2^{-t}p(2) + p(-1) &= 2^{-t}[q(2) + a \cdot 2^s + b \cdot 2^t] + [q(-1) + a \cdot (-1)^s - b] \\ &= 2^{-t}q(2) + q(-1) + a[2^{s-t} + (-1)^s]. \end{aligned}$$

Note that the bs cancel, and that $2^{s-t} + (-1)^s$ is nonzero, since $s \neq t$. Setting this expression to 0 and solving for a, we find

$$a = -\frac{2^{-t}q(2) + q(-1)}{2^{s-t} + (-1)^s}.$$

If Donna fills in the coefficient of x^s with this value of a, then $2^{-t}p(2) + p(-1) = 0$, which means that either $p(2) = p(-1) = 0$, in which case the polynomial $p(x)$ obviously has a real root, or $p(2)$ and $p(-1)$ are of opposite sign, in which case the polynomial $p(x)$ has a real root between -2 and 1. In either case, $p(x)$ always has a real root, regardless of Jerry's choice of b.