Induction, Problem 1

Prove that

$$2(\sqrt{n+1}-1) < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

for all positive integers n.

Induction, Problem 2

Given $0 < a < b < c < d < e < 1$, prove that $abcde > a+b+c+d+e-4$

Induction, Problem 3

Let p be a prime, and let a, k be positive integers such that $p^k \mid (a-1)$. Show that $p^{n+k} \mid (a^{p^n}-1)$ for all positive integers n.

Induction, Problem 4

For any positive integer n, let $S(n)$ be the sum of digits in the decimal representation of n. Any positive integer obtained by removing one or more digits from the right-hand end of the decimal representation of n is called a stump of n. Let $T(n)$ be the sum of all stumps of n. Prove that $n = S(n) + 9T(n)$. (For example, if $n = 238$, we have $S(n) = 2+3+8 = 13$, and stumps 2 and 23, so $T(n) = 2 + 23 = 25$. We verify that $238 = 13 + 9(25)$.)

Induction, Problem 5

Find, in terms of n, the sum of the digits of

$$9 \times 99 \times 9999 \times \cdots \times \left(10^{2^n}-1\right),$$

where each factor has twice as many nines as the previous factor.

Induction, Problem 6

Let $P(z)$ be a polynomial with complex coefficients of degree 1992 with 1992 distinct zeros. Prove that there exist complex numbers $a_1, a_2, \ldots, a_{1992}$ such that $P(z)$ divides the polynomial

$$(\cdots((z-a_1)^2-a_2)^2\cdots-a_{1991})^2-a_{1992}.$$

Induction, Problem 7

For any positive integer $n \geq 2$, let S_n be the set of all fractions of the form $\frac{1}{pq}$, where p and q are relatively prime, $0n$. Show that the sum of the elements of S_n is $\frac{1}{2}$.

Induction, Problem 8

Here is a problem and a proposed solution.

Problem. Let n be a nonnegative integer. Suppose we are given a triangle and n points inside it, with no three of the given $n+3$ points collinear. We divide the triangle into smaller triangles, using the $n+3$ points as vertices. Show that we always end up with $2n+1$ triangles.

Solution. For the base case $n = 0$, there is clearly $2n+1 = 1$ triangle. For the inductive step, assume that k points inside the triangle define $2k+1$ triangles. If we add a point x, as shown, then we lose one triangle but create three more triangles, for a net addition of two triangles. Hence, there are a total of $2k+1+2 = 2k+3 = 2(k+1)+1$ triangles, which completes the induction. ■

This proposed solution has a major conceptual flaw. Identify the flaw, and fix the induction argument.

The diagram can be found at the end of this document.

Induction, Problem 9

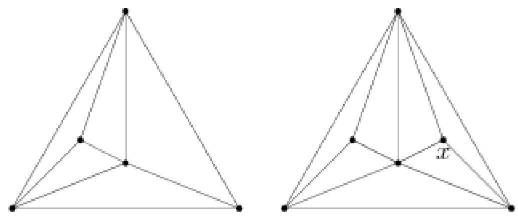
Let S be a finite nonempty set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set A.

Induction, Problem 10

Prove that every positive integer can be written as a finite sum of distinct integral powers of the golden ratio. (Recall that the golden ratio is $\tau = \frac{1}{2}(1+\sqrt{5})$.)



$k = 2$ $k = 3$