Winter Camp 2002

Inversion:

<u>Def</u>: Let Γ be a circle centered at O and with radius r. The image of P ($\neq O$) under <u>inversion in Γ</u> is the unique point P' on $\overrightarrow{OP}$ that satisfies

$$|OP'| = \frac{r^2}{|OP|}$$

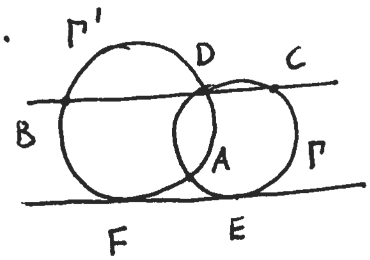
Theory: Prove the following results. Some relevant results come up in the proofs.

<u>Thm 1</u>: The image of a line (not through O) is a circle through O.

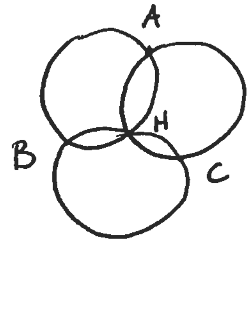
<u>Thm 2</u>: The image of a circle (not through O) is a circle

<u>Thm 3</u>: Angles are preserved under inversion. (Clarify what is meant by this.)

Exercises:

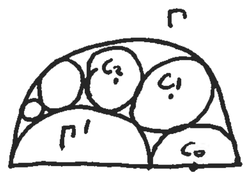
1. 

Two circles Γ and Γ' intersect at A and D, as shown. A line is tangent to Γ and Γ' at E and F, respectively. A parallel line through D intersects Γ and Γ' at C and B, respectively. Show that the circumcircles of $\triangle BDE$ and $\triangle CDF$ intersect again on the line AD.

2. 

Three congruent circles intersect at a point H, and again in pairs at A, B, and C. Prove that H is the orthocenter of $\triangle ABC$.

2a. Prove that the circumcircle of $\triangle ABC$ is congruent to the original circles. This is the 4 coin problem that was used as the logo for the '99 IMO hosted by Romania.

3. 

Two circles Γ and Γ' are internally tangent with Γ' in Γ. A mutually tangent circle has center C_0 on the common diameter of Γ and Γ' ... as shown. In the resulting curvilinear triangle, a sequence of mutually tangent circles are inscribed, with centers $c_1, c_2, \dots$. Let h_n be the distance from c_n to the common diameter of Γ and Γ'. Prove that

$$h_n = n\, d_n$$

where d_n is the diameter of the circle centered at c_n.
(This result is due to Pappus.)

More theory: These results say something about how angles and lengths are distorted under inversions.

<u>Thm</u> 4: Let A' and B' be the images of A and B under inversion.
Then

$$\measuredangle OAB = \measuredangle OB'A'. \qquad \text{(Compare w/ Thm 3.)}$$

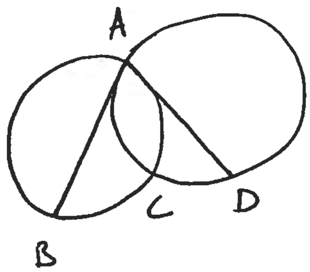
<u>Thm</u> 5. Let A' and B' be the images of A and B. Then

$$|A'B'| = \frac{|AB|\, r^2}{|OA|\,|OB|}.$$

4. (Ptolemy's inequality) Let $ABCD$ be an arbitrary quadrilateral.
Then

$$|AC|\,|BD| \le |AB|\,|CD| + |BC|\,|DA|$$

with equality if and only if $ABCD$ is cyclic

5. 

Two circles intersect at A and C. The tangents at A intersect the circles again at B and D. Prove

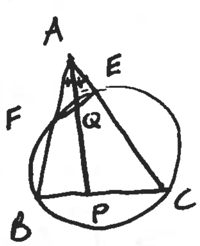
$$|AB||CD| = |AC||AD|.$$

Feuerbach's Theorem.

<u>Thm</u> (9 point circle): Let H be the orthocenter of $\triangle ABC$. Let A', B', and C' be the midpoints of the sides. Let $D, E,$ and F be the bases of the altitudes. Let A'', B'', and C'' be the midpoints of AH, BH, and CH. Then $A', B', C', D, E, F, A'', B'', C''$ all lie on a circle.

<u>Thm</u> (Feuerbach's Thm): The nine point circle is tangent to the incircle and the three excircles of $\triangle ABC$.

The following exercises lead to a proof of Feuerbach's Thm.

6. 

Let a circle through B and C intersect $\triangle ABC$ again at E and F as shown. Let the angle bisector of A intersect BC at P and EF at Q.

Prove $\measuredangle AQE = \measuredangle APB$

7. ~~Let the P, E and F be as~~ Let E and F be the bases of the altitudes at B and C, respectively. Let the angle bisector at A intersect BC at P. Prove that the line through P that is parallel to EF is tangent to both the incircle and the excircle opposite A.

8. Let the incircle to $\triangle ABC$ be tangent to BC at R, and let the excircle opposite A be tangent to BC at S. Let A' be the midpoint of BC. Prove

$$|A'R| = |A'S| = \frac{|b-c|}{2}$$

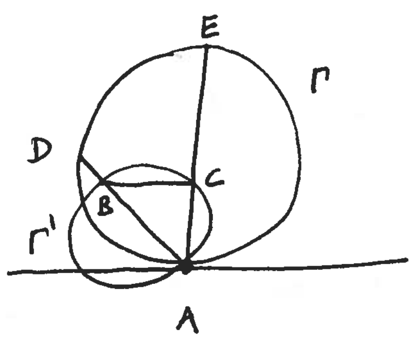
9. Let D be the base of the altitude at A. Let the angle bisector at A intersect BC at P. Prove that P is the image of D under inversion in the circle centered at A' and with radius $|A'R|$.

10. Prove that the tangent to the 9-point circle at C' is parallel to DE.

11. Prove Feuerbach's Thm.

More inversion problems:

12. Two circles Γ and Γ' intersect at A. The diameter of Γ at A intersects Γ' at C. The line through C parallel to the tangent of Γ at A intersects Γ' at B. The line AB intersects Γ again at D. Prove $BCED$ is a cyclic quadrilateral.



13. Two circles are tangent to each other and to a line. A circle with center O and radius r is inscribed in the resulting curvilinear triangle. Let A be the point of tangency of the original circles. Prove

$$|AO| \le 3r.$$