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On the Notion of Oriented Angles in Plane Elementary Geometry and Some of its Applications

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Summary. The usual ambiguities in ordinary treatment of angles in Euclidean plane geometry are removed by means of the notion of **oriented angles**. It is then applied to the proof of various examples of geometry theorems including the celebrated Miquel-Clifford theorem.

Key Words. Oriented Angles, Miquel-Clifford Theorems, Miquel-Clifford Point & Miquel-Clifford Circle

1. Introduction

In plane elementary geometry the usual treatment of angles causes usually troubles owing to the ambiguity of their representation. For example, in Euclid's $\ll$ *Elements* $\gg$, for four points A, B, C, D lying on the same circle, the angles $\angle(ACB)$ and $\angle(ADB)$ will be equal or complementary to each other according to whether the points C, D are on the same side or the opposite side of the chord AB or not, see Figs.1.1,1.2. This dependence of positions relying on intuition and exacteness of drawing causes much trouble in the proving of geometry theorems.

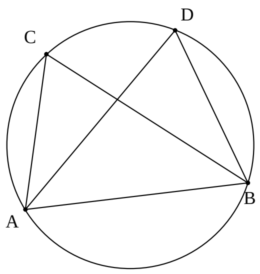

Fig. 1.1

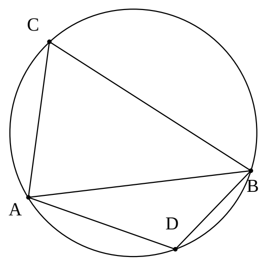

Fig. 1.2

Various kinds of remedies to this troublesome situation had been devised in the literature, for which we may cite in particular the introduction of *full angles* by Chou, Gao and Zhang, cf. their joint book [C-G-Z]. On the other hand the present author had introduced the notion of *oriented angles* to avoid the ambiguity in order to be applied to mechanical geometry theorem-proving, cf. the author's book [WU], Chap.7, §2. In the present work we shall adopt the notion of *oriented angles* in a slight different way of representation and will be applied to the proving of plane elementary geometry theorems,including in particular the celebrated Miquel-Clifford theorems involving lines and circles. Thus, in §2 we shall give the notion of *oriented angles* and the various Rules of operations about these angles. In §3 we shall show how various theorems, mainly taken from a paper of LI Hongbo (cf. [LI]) may be proved by means of the notion of oriented angles. In §4 we state the theorems of Miquel-Clifford and give an inductive proof by means of oriented angles. In the final §5 we raise some questions for further studies.

2. Notion of Oriented Angles

Consider lines and circles in a definite plane. We shall say that two lines are in *generic* position if they are neither coincident nor parallel. We say also that $n(\geq 3)$ lines are in *generic* position if any two of them are in general position and any 3 of them are not concurrent. In what follows we shall consider usually lines in *generic* position unless otherwise stated so that the modifier *generic* will be omitted.

For any two lines L_1, L_2 intersecting in a point O let α be now the angle in turning anticlockwise around O from line L_1 to line L_2. The angle $\alpha \ mod \ \pi$ determined up to integral multiples of π will then be called an *oriented angle* and will be denoted by $\angle(O, L_1, L_2)$ or simply $\angle(L_1, L_2)$ with $O = \bigwedge(L_1, L_2)$ omitted in which $\bigwedge$ means point of intersection involved.

Write for simplicity $\equiv$ instead $\equiv \ mod \ \pi$. Then the following Rules about the oriented angles are readily verified:

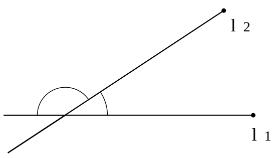

Fig. 2.1

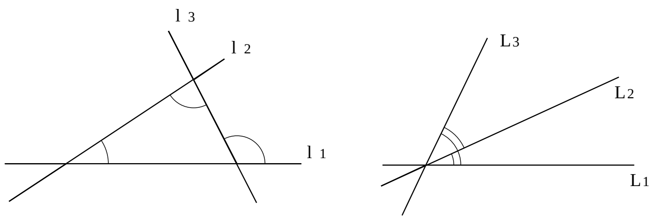

Fig. 2.2

Rule 1. (See Fig.2.1.) For any two lines L_1, L_2 we have

$$\angle(L_1, L_2) \equiv -\angle(L_2, L_1).$$

Rule 2. (See Fig.2.2.) For any 3 lines L_1, L_2, L_3 , intersecting in the same point or not,

$$\angle(L_1, L_2) + \angle(L_2, L_3) \equiv \angle(L_1, L_3).$$

Rule 3. (See Fig.2.3.) For any 3 points P_1, P_2, P_3 on a circle with center O we have

$$2\angle(P_1P_2, P_1P_3) \equiv \angle(OP_2, OP_3).$$

Rule 3' (See Fig.2.3'.) Let two circles with centers O_1, O_2 intersect at points A_1, A_2. Let B_1, B_2 be points on the two circles respectively. We have then

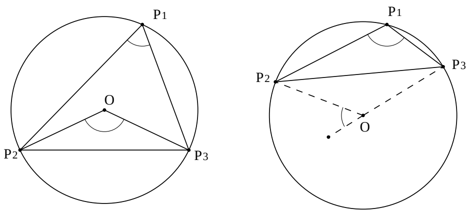

Fig. 2.3

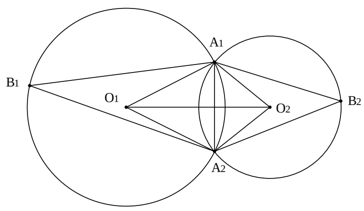

Fig. 2.3'

$$\angle(B_1A_1, B_1A_2) \equiv \angle(O_1A_1, O_1A_2).$$

Rule 4. (See Fig.2.4.) 4 points P_1, P_2, P_3, P_4 will lie on the same circle or *co-circle* if and only if

$$\angle(P_1P_3, P_1P_4) \equiv \angle(P_2P_3, P_2P_4).$$

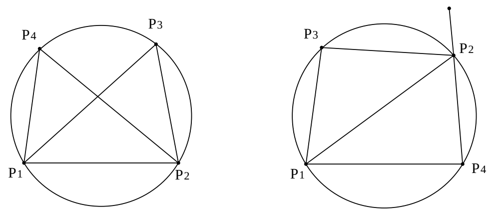

Fig. 2.4

We see that Rules 3, 3' and 4 remove all ambiguities involved in the Euclidean notion of angles for points on a circle.

We remark that a further ambiguity in the ordinary Euclidean treatment is about the bisectors of the angle formed by two intersecting lines. We may resolve this ambiguity by means of oriented angles according to the following Rule:

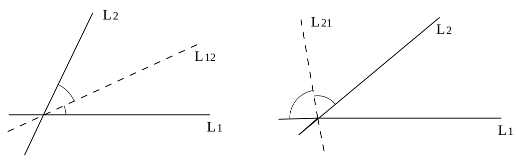

Fig. 2.5

Rule 5. (See Fig.2.5.) For two lines L_1, L_2 intersecting at a point O, there are two bisectors L_{12} of angle $\angle(L_1, L_2)$, and L_{21} of $\angle(L_2, L_1)$ characterized respectively by the conguences below:

$$\angle(L_1, L_{12}) \equiv \angle(L_{12}, L_2),\ \angle(L_2, L_{21}) \equiv \angle(L_{21}, L_1).$$

We may also add two Rules below:

Rule 6. (See Fig.2.6.) **Criterion of Parallelizability.** For 3 lines L_1, L_2, L_3 with L_3 intersecting both L_1, L_2; L_1, L_2 will be *parallel* if and only if

$$\angle(L_1, L_3) \equiv \angle(L_2, L_3).$$

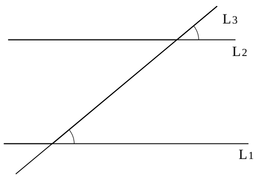

Fig. 2.6

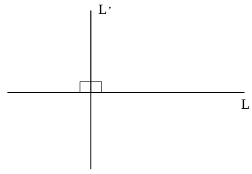

Fig. 2.7

Rule 7. (See Fig.2.7.) **Criterion of Orthogonality.** Two intersecting lines L, L' will be orthogonal to each other if and only if

$$\angle(L, L') \equiv \angle(L', L).$$

Let A, B be two points on an oriented line L. Then the directed length $AB(= -BA)$ will take the value + or - $|AB|$ according to AB is in the same or opposite direction as that of the oriented line L. However, for any 4 points A, B, C, D on the same line L, the product $AB * CD$ and the ratio $\frac{AB}{CD}$ will take the same values irrespective of the orientation way of the oriented line L. We shall take advantage of this remark to state some further rules and theorems below:

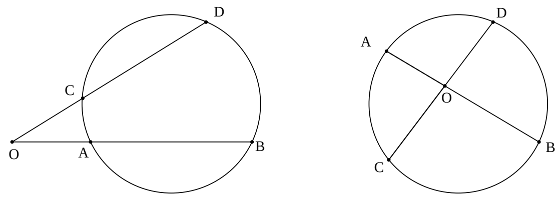

Fig. 2.8

Rule 8. (See Fig.2.8.) Through a point O two lines will meet a circle in points A, B and C, D respectively. Then irrespective of orientations of the two lines we have always

$$OA * OB = OC * OD.$$

Moreover, we may also put the above equation in either of the forms below:

$$\frac{OA}{OC} = \frac{OD}{OB}, \ \frac{OA}{OD} = \frac{OC}{OB}, \ etc.,$$

in which each fraction will take positive or negative values according to the chosen orientations of the two lines, but the equalities will always be true irrespective of the orientations chosen of the lines.

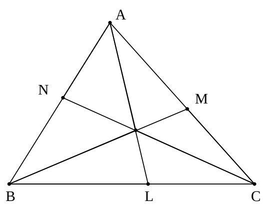

Fig. 2.9

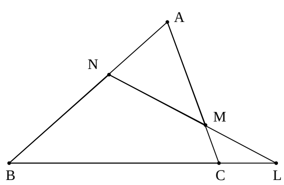

Fig. 2.10

Ceva Theorem. (See Fig.2.9.) Let L, M, N be points on the sides BC, CA, AB respectively. Then AL, BM, CN will be concurrent (or *co-point*) if and only if

$$\frac{BL}{LC} * \frac{CM}{MA} * \frac{AN}{NB} = +1.$$

Menelaus Theorem. (See Fig.2.10.) Let L, M, N be points on the sides BC, CA, AB respectively. Then L, M, N will lie on the same line (or *co-line*) if and only if

$$\frac{BL}{LC} * \frac{CM}{MA} * \frac{AN}{NB} = -1.$$

3. Some Simple Applications of Oriented Angles.

We now give some simple applications of oriented angles to the proving of plane Euclidean geometry theorems. For this purpose we shall consider the theorems exhibited in a paper of LI Hongbo (see [LI]).

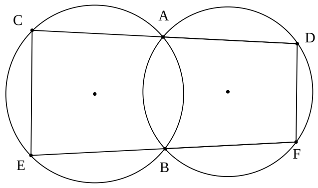

Fig. 3.1

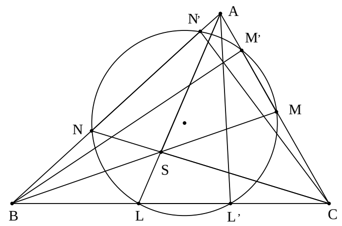

Fig. 3.2

Example 1. (See Fig.3.1.) Through the two common points A, B of two circles, two lines are drawn meeting the circles at points C, D and E, F respectively. Then $CE \parallel DF$.

Proof. By Rule 4, A, B, C, E being co-circle would imply

$$\angle(BE, BA) \equiv \angle(CE, CA), \ \ or \ \angle(EF, BA) \equiv \angle(CE, CD).$$

Similarly, A, B, D, F being co-circle implies

$$\angle(EF, BA) \equiv \angle(DF, CD).$$

Hence

$$\angle(CE, CD) \equiv \angle(DF, CD)$$

so that CE, DF are parallel by Rule 6.

Example 2. (See Fig.3.2.) If the lines joining the vertices A, B, C of a triangle to a point S meet the respectively opposite sides in L, M, N, and the circle LMN meets these sides again in L', M', N', then the lines AL', BM', CN' are concurrent.

Proof. AL, BM, CN being co-point at S we have by Ceva's Theorem

$$\frac{BL}{CL} * \frac{CM}{AM} * \frac{AN}{BN} = +1.$$

By Rule 8 we have

$$BL * BL' = BN * BN', CM * CM' = CL * CL', AM' * AM = AN' * AN.$$

From these we get readily

$$\frac{BL'}{CL'} * \frac{CM'}{AM'} * \frac{AN'}{BN'} = +1.$$

Hence by Ceva's Theorem AL', BM', CN' are co-point, as to be proved.

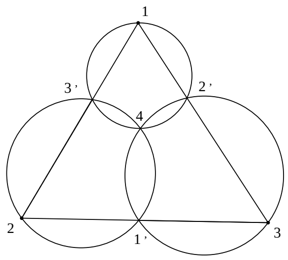

Fig. 3.3

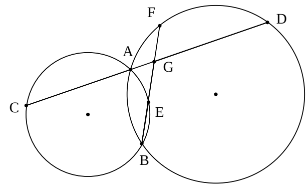

Fig. 3.4

Example 3. (See Fig.3.3.) Let there be a triangle 123 in the plane. Let 1',2',3' be points on the three sides 23,13,12 respectively. Then the three circles circumscribing triangles 12'3', 1'23', and 1'2'3 respectively meet at a common point 4.

Proof. Let the circles $\odot 21'3', \odot 31'2'$ meet at point 4 beside the point 1'. Then for points $4, 2, 1', 3'$ on the circle $\odot 21'3'$ we get by Rule 4

$$\angle(41', 43') \equiv \angle(21', 23').$$

Similarly for points $4, 3, 1', 2'$ on the same circle $\odot 31'2'$ we have

$$\angle(42', 41') \equiv \angle(32', 31').$$

It follows by Rules 1-3 that

$$\begin{aligned}\angle(42', 43') &\equiv \angle(42', 41') + \angle(41', 43') \equiv \angle(32', 31') + \angle(21', 23')\\ &\equiv \angle(31, 32) + \angle(23, 21) \equiv \angle(31, 21) \equiv \angle(12', 13').\end{aligned}$$

By Rule 4 the points $4, 1, 2', 3'$ are thus co-circle or the circle $\odot 12'3'$ passes through the point 4 too.

Example 4. (See Fig.3.4.) Let A, B be the two common points of two circles. Through A a line is drawn meeting the circles at C, D respectively. G is the midpoint of CD. Line BG intersects the two circles at E, F respectively. Then $G = mid(EF)$.

Proof. The points A, F, B, D and A, B, C, E being both co-circle we have by Rule 4

$$\angle(DC, DB) \equiv \angle(FA, FB),\ \angle(CD, CB) \equiv \angle(EA, EF),\ \angle(BD, BG) \equiv \angle(AG, AF).$$

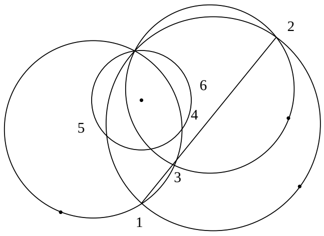

Fig. 3.5

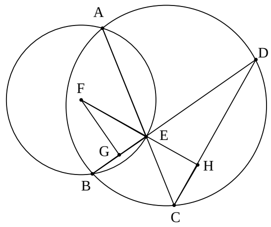

Fig. 3.6

It follows that the configuration $BCDG$ is similar to the configuration $AEFG$ with points B, C, D, G in correspondence to A, E, F, G respectively. As G is the midpoint of CD, so G is also the midpoint of EF.

Example 5. (See Fig.3.5.) If three circles having a point in common intersect pairwise at three collinear points, their common point is cocircle with the three centers.

Proof. Let the common point be O not on the line L with 3 points $1, 2, 3$ on it. Let the centers of the 3 circles $\odot O12, \odot O13, \odot O23$ be $4, 5, 6$ respectively. Then we have to prove that the 4 points $O, 4, 5, 6$ are co-circle.

In fact, from the circles $\odot O12, \odot O13$ we get by Rule 3'

$$\angle(4O, 45) \equiv \angle(2O, 23) \equiv \angle(2O, L).$$

Similarly from the circles $\odot O13, \odot O23$ we get by Rule 3'

$$\angle(6O, 65) \equiv \angle(2O, L), \ \textit{too}.$$

It follows that

$$\angle(4O, 45) \equiv \angle(6O, 65)$$

so that by Rule 4 the points $O, 4, 5, 6$ are cocircle.

Example 6. (See Fig.3.6.) Let E be the intersection of the two non-adjacent sides AC and BD of a quadrilateral $ABCD$ inscribed in a circle. Let F be the center of the circle $\odot ABE$. Then CD, EF are perpendicular to each other.

Proof. Let H be the intersection point of FE and CD. Let FG be the perpendicular from F to BE with G on BE. For circle $\odot ABE$ with center F we have by Rule 3'

$$\angle(FE, FG) \equiv \angle(AB, AE).$$

As A, B, C, D are co-circle we have by Rule 4

$$\angle(DB, DC) \equiv \angle(AB, AC).$$

It follows that

$$\angle(DH, DE) + \angle(DE, EH) \equiv \angle(FE, FG) + \angle(EG, EF) \equiv \frac{1}{2}\pi.$$

For the triangle DEH we have therefore

$$\angle(HD, HE) \equiv \frac{1}{2}\pi$$

or EH is perpendicular to CD.

Besides the above examples from the LI's paper [LI], we add now a further Example 7 for the use in the next section.

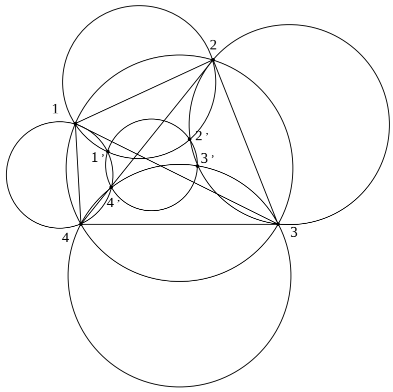

Fig. 3.7

Example 7. (See Fig.3.7.) Let the points 1,2,3,4 be co-circle. Through the pairs of points 1,2; 2,3; 3,4; 4,1 let us pass circles $\odot 12, \odot 23, \odot 34, \odot 14$ respectively. Let the pairs of circles $(\odot 12, \odot 14), (\odot 12, \odot 23), (\odot 23, \odot 34), (\odot 34, \odot 14)$ intersect besides the points 1,2,3,4 also at points 1',2',3',4' respectively. Then the points 1',2',3',4' are co-circle.

Proof. As the points 1,2,3,4 are co-circle we have by Rule 4:

$$\angle(21, 23) \equiv \angle(41, 43). \tag{3.0}$$

From the co-circleness of the quadruples of points (121'2'), (232'3'), (343'4'), (141'4') we have respectively by Rule 4:

$$\angle(21, 22') \equiv \angle(1'1, 1'2'), \tag{3.1}$$

$$\angle(22', 23) \equiv \angle(3'2', 3'3), \tag{3.2}$$

$$\angle(43, 44') \equiv \angle(3'3, 3'4'), \tag{3.3}$$

$$\angle(44', 41) \equiv \angle(1'4', 1'1). \tag{3.4}$$

Add the left-sides of (3.1),(3.2),(3.3), (3.4) together, we get by Rule 2

$$L.S. \equiv \angle(21, 23) + \angle(43, 41) \equiv 0,$$

by Rule 4, since the points 2,4,1,3 are co-circle.

It follows that the sum of the right-sides of equations (3.1), $\cdots$, (3.4) are also equal to 0, i.e.

$$\angle(1'4', 1'2') + \angle(3'2', 3'4') \equiv 0.$$

By Rule 4 again, the 4 points 1',2',3', 4' are thus co-circle.

4. Miquel-Clifford Theorems and their Proofs

Let lines be generically given in a Euclidean plane. Now two lines will intersect in a unique point and three lines will determine 3 points which determines a unique circle. Consider now 4 lines then each 3 of them will determine a circle. It was first pointed out and proved by A.Miquel that such circles, 4 in all, will be co-point which has been called *Miquel Point* of the 4 lines (See [MIQ1]). Further, if there are 5 lines then there will be 5 such Miquel points determined by the 5 sets of 4 lines chosen from the 5 ones. Miquel had proved that these 5 Miquel points will be cocircle which had been called in the literature the *Miquel Circle* of the 5 given lines.

In year 1870 W.K.Clifford published a paper (see [CLI]) showing that for each positive integer $n > 3$ there will be associated a *point* for each *even* n and a *circle* for each *odd* n which reduces to the known Miquel point and the Miquel circle in the cases $n = 4, n = 5$. Moreover, for each odd $n \geq 5$ the associated circle will pass through the $n - 1$ points associated to the $(n-1)-ple$ lines chosen from the given n lines, and for each even $n \geq 6$ the associated point will lie on the $(n - 1) - ple$ circles chosen from the given n lines. We shall accordingly call these points and circles the *Miquel-Clifford Point* and *Miquel-Clifford Circle* of the n lines according to n be even or odd.

The proof of Clifford about his theorem is however so intricate that it seems that no one had been able to understand his reasoning. Below we shall give an *elementary* proof based on our notion of *oriented angles* as exhibited in the previous paragraphs. As the case of $n = 4, 5$ are easily proved, we shall begin by proving the cases $n = 6, 7$ and then procced to an *inductive* proof from case $n - 1$ to n for n even and odd successively.

For this purpose we shall first introduce some notations. The lines in question will be denoted by $L_i, i = 1, \cdots, n$. The intersection point of two lines $L_i, L_j, i < j$ will be denoted by Q_{ij}. The Miquel-Clifford point for $2 * m$ lines $L_{i_1}, L_{i_2}, \cdots, L_{i_{2*m}}$ with $i_1 < i_2 < \cdots \imath_{2*m}$ will be denoted by $Q_{i_1 i_2 \cdots i_{2*m}}$, and the Miquel-Clifford circle for $2*m+1$ lines $L_{i_1}, L_{i_2}, \cdots, L_{i_{2*m+1}}$ with $i_1 < i_2 < \cdots < i_{2*m+1}$ will be denoted by $\odot i_1 i_2 \cdots i_{2*m+1}$.

Let us now proceed to the proof of the case $n = 6$ with 6 lines $L_1, \cdots, L_6$. For the 6 5-tuples of lines the associated Miquel-Clifford circles are $\odot 23456, \odot 13456, \odot 12456, \odot 12356$, $\odot 12346$, $\odot 12345$. We have to show that they are concurrent at a point or co-point. To see this, let the circles $\odot 23456, \odot 13456$ intersect besides the point Q_{3456} also at a point Q. We have to prove that the other 6 Miquel-Clifford circles $\odot 12456$, etc. pass through this point Q too.

As the circle $\odot 13456$ contains besides the points Q, Q_{3456} also the points Q_{1456}, Q_{1356}, we have by Rule 4:

$$\angle(QQ_{3456}, QQ_{1456}) \equiv \angle(Q_{1356}Q_{3456}, Q_{1356}Q_{1456}). \tag{4.1}$$

Similarly for 4 points $Q, Q_{3456}, Q_{2456}, Q_{2356}$ on the same circle $\odot 23456$ we have by Rule 4

$$\angle(QQ_{3456}, QQ_{2456}) \equiv \angle(Q_{2356}Q_{3456}, Q_{2356}Q_{2456}). \tag{4.2}$$

Subtracting these two congruences we get

$$\angle(QQ_{2456}, QQ_{1456}) \equiv \angle(Q_{1356}Q_{3456}, Q_{1356}Q_{1456}) - \angle(Q_{2356}Q_{3456}, Q_{2356}Q_{2456}). \tag{4.3}$$

By Rule 2 we get

$$\angle(Q_{1356}Q_{3456}, Q_{1356}Q_{1456}) \equiv \angle(Q_{1356}Q_{3456}Q_{1356}Q_{56}) - \angle(Q_{1356}Q_{1456}, Q_{1356}Q_{56}).$$

$$\angle(Q_{2356}Q_{3456}, Q_{2356}Q_{2456}) \equiv \angle(Q_{2356}Q_{3456}Q_{2356}Q_{56}) - \angle(Q_{2356}Q_{2456}, Q_{2356}Q_{56}).$$

It follows that (4.3) becomes

$$\angle(QQ_{2456}, QQ_{1456}) \equiv \angle X + \angle Y, \tag{4.4}$$

in which

$$\angle X \equiv \angle_{1356}Q_{3456}, Q_{1356}Q_{56}) - \angle(Q_{2356}Q_{3456}, Q_{2356}Q_{56}), \tag{4.5}$$

$$\angle Y \equiv \angle(Q_{2356}Q_{2456}, Q_{2356}Q_{56}) - \angle(Q_{1356}Q_{1456}, Q_{1356}Q_{56}). \tag{4.6}$$

Now the circle $\odot 356$ contains 4 points $Q_{1356}, Q_{2356}, Q_{3456}, Q_{56}$ so that by Rule 4 we have

$$\angle X \equiv 0. \tag{4.7}$$

On the other hand the circles $\odot 156, \odot 256$ contain the 4 points $Q_{1356}, Q_{1456}, Q_{1256}, Q_{56}$ and $Q_{2356}, Q_{2456}, Q_{1256}, Q_{56}$ respectively, so that we have by Rule 4

$$\angle(Q_{1356}Q_{1456}, Q_{1356}Q_{56}) \equiv \angle(Q_{1256}Q_{1456}, Q_{1256}Q_{56}), \tag{4.8}$$

$$\angle(Q_{2356}Q_{2456}, Q_{2356}Q_{56}) \equiv \angle(Q_{1256}Q_{2456}, Q_{1256}Q_{56}). \tag{4.9}$$

From (4.4)-(4.9) we get then

$$\angle(QQ_{2456}, QQ_{1456}) \equiv \angle(Q_{1256}Q_{2456}, Q_{1256}Q_{1456}).$$

By Rule 4 the 4 points $Q, Q_{1456}, Q_{1256}, Q_{2456}$ are thus co-circle, or the circle $\odot 12456$ passes through the point Q too. In the same way we prove that the circles $\odot 12356, \odot 12346, \odot 12345$ all pass through the point Q or the 6 circles in question are co-point at Q which proves the Miquel-Clifford Theorem in the case $n = 6$ with the above point Q as the Miquel-Clifford point.

Next let us consider the case of $n = 7$. We have to prove that the 7 Miquel-Clifford points $Q_{234567}, Q_{134567}, \cdots, Q_{123456}$ are co-circle.

On the circle $\odot 567$ we have the 4 points $Q_{1567}, Q_{2567}, Q_{3567}, Q_{4567}$. Now through the pairs of points $(Q_{1567}, Q_{2567}), (Q_{2567}, Q_{3567}), (Q_{3567}, Q_{4567}), (Q_{1567}, Q_{4567})$ we have respectively the circles $\odot 12567, \odot 23567, \odot 34567, \odot 14567$. The pair of circles $\odot 12567, \odot 23567$ intersect beside the point Q_{2567} also the point Q_{123567}. Similarly, the pairs of circles $(\odot 23567, \odot 34567)$, $(\odot 34567, \odot 14567)$, and $(\odot 12567, \odot 14567)$ intersect besides the points $Q_{3567}, Q_{4567}, Q_{1567}$, also at the points Q_{234567}, Q_{134567}, Q_{124567} respectively. By Example 7 in §3 the 4 points $Q_{123567}, Q_{234567}, Q_{134567}, Q_{124567}$ are thus co-circle. By considering the circles $\odot ijk, 1 \leq i < j < k \leq 7$ in the same way we see that the 7 points $Q_{234567}, Q_{134567}, Q_{124567}, Q_{123567}, Q_{123467}$, Q_{123457}, Q_{123456} are co-circle which proves the Miquel-Clifford Theorem for the case $n = 7$.

Consider now the inductive case from $n = 2*m-1$ to $n+1 = 2*m$. For this purpose let us write α for the tuple $7 \cdots 2*m$. In the above proof of the Miquel-Clifford Theorem for $n = 6$ let us replace each Q_{ijkl} or Q_{ij} by $Q_{ijkl\alpha}$ or $Qij\alpha$. Similarly for each circle $\odot hijkl$ by $\odot hijkl\alpha$, etc. Then the above proof for the case $n = 6$ will give a proof that the circles $\odot 12456\alpha, \odot 12356\alpha, \odot 12346\alpha, \odot 12345\alpha$ are co-point at some point Q. By suitable rearrangements of the indices we see that all the Miquel-Clifford Circles with indices chosen from the $2*m-1$ integers will be co-point which proves the theorem for $n = 2*m$.

The inductive proof of the Miquel-Clifford Theorem from $n = 2*m$ to $n+1 = 2*m+1$ may be done in the similar way.

This completes the proof of Miquel-Clifford Theorem.

5. Further Examples and Some Discussions

In the preceding sections we have shown how the notion of **oriented angles** permit us to prove a lot of plane geometry theorems. However, we have to point out that this is far from being a method *complete* in some sense to be precised below. For geometry theorem proving, we have given a method which is really **complete** in the following sense:

Any eukidean geometry theorem may be proved to be true or un-true. In the case of being *true* one may give the precise domain of truth under eventually some subsidiary non-degeneracy conditions which may also be precisely given. For details we refer to the author's book [WU].

Clearly the method of oriented angles is far from being *complete* in the above sense. This is also the case of the notion of *full angles* as well as the *area method*, etc. as given in the book [C-G-Z] of CHOW-GAO-ZHANG, no matter how many theorems may be proved by their methods.

On the other hand CHOW-GAO-ZHANG had shown in their book that, the theorems considered by them may be proved in an *algorithmic* and even *readable* way. Clearly, for theorems which may be proved by means of *oriented angles* as in the preceding sections, readable algorithmic proofs may also be given. As this is a laborious, tedious, and time-consuming task, we shall leave this to any one who may be interested in this.

In the present author's opinion, what is of utmost importance is the **pedagogical** effect of the notion of **oriented angles**.

In the middle school we begin learning euklidean (plane) geometry emphasized on geometry theorem-proving. In accordance to the theorem to be proved, we draw some geometric figures, try to draw further auxiliary lines or even auxuliary circles to assist the geometry reasoning in order to arrive at a proof. This marvelous combination of geometric intuition and logical reasoning is an incomparable training which seems to be, so far the author knows, impossible to be attained by any other means. It is absurd to try to deprive off such a training in middle school as some mathematicians had once tried to do so. Their failure is unavoidable which proves their absurdity in reasoning. On the other hand the present author had been much benefitted by such training of geometry reasoning aided by geometry intuition from his learning of geometry in middle school. Without such training in my youth it is hardly possible for me to get such success in mathematics researches in later years, from earlier algebraic-topology studies to mathematics-mechanization in recent years, mechanization of geometry theorem-proving in particular.

Though the study of euclidean geometry and its theorem-proving is, in the present author's opinion, indispensable in middle school teaching, the defects owing to the ambiguity caused by inadequate representation of angles should not be neglected. Thus, the present author proposes that the notion of **oriented angles** or the alike should be taken into acount for the reformation of geometry teaching now in progress in so many countries in the world.

As for the reformation of geometry-teaching in our country, it seems that there is some tendency to admit *mechanical geometry theorem-proving by means of computers* (sometimes even under the pretext that this is a contribution of the present author). However, just on the contrary the present author is firmly against such **absurd** and **dangerous** suggestions for the geometry-teaching. It is true that mechanical geometry theorem-proving by means of computers is a nice subject for mathematical researches and has to be further developed in years to come. However, such researches require somewhat deep insight of geometry as well as some matureness of mathematical research ability which is far from being possible for young students in the middle school. In view of the utmost importance for the teaching reformation of mathematics, particularly in geometry, I have to repeatably emphasize on this point. It is true that students should learn how to manipulate the computers as early as possible in the school, but surely not through the learning of mechanical geometry theorem-proving!!!

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