I. Induction

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June 16, 2003

1 News Flash From Zuming!

- Remind Po to take all the markers from CBA 337
- Tonight's study session for Red/Blue is in **Bessey 104**
- Future Red lectures are in **NM B-7**, the "Naval Military Base #7"
- Future Red tests and study sessions are in **Bessey 104**
- All test reviews will be in **Bessey 104**

2 Warm-Up

1. A coil has an inductance of 2mH, and a current through it changes from 0.2A to 1.5A in a time of 0.2s. Find the magnitude of the average induced emf in the coil during this time.

   Solution: Just kidding.

2. Prove that the number of binary sequences with an even number of 1's is equal to the number of binary sequences with an odd number of 1's.

   Solution: Easy induction.

3 Equality

1. Let F_n be the Fibonacci sequence. Prove that $F_n^2 = F_{n-1} + F_{n+1} \pm 1$. Determine when it's $+1$ and when it's -1.

   Solution: It's actually $F_n^2 = F_{n-1} + F_{n+1} + (-1)^n$. Inductive step is just evaluating $F_n F_{n+2}$ and expanding out $F_{n+2} = F_n + F_{n+1}$.

2. (Titu98) Let a be a real number such that $\sin a + \cos a$ is a rational number. Prove that for all $n \in \mathbb{N}$, $\sin^n a + \cos^n a$ is rational.

   Solution: Clearly true for $n = 2$; then to go from $n \mapsto n + 1$, just multiply the 1-case by the n case; use the fact that $n = 1, 2$ show that $\sin \cos \in \mathbb{Q}$.

3. You've seen maps in geography books. Did you know that they could all be colored with just 4 colors? (That is, "colored" in the sense that no two adjacent countries are the same color. Note that if two countries share a corner, they do not count as being "adjacent".) Prove it!

   Solution: Well, this is actually the 4-color theorem, which was proved via computer. So don't try too hard.

4. (Ricky03 from Internet) I'm playing the color-country game against Bob. We take turns; on my turn, I draw in a country. On Bob's turn, he chooses any color for the country, but he must make sure that no adjacent countries share the same color. Is it possible for me to force Bob to use more than 4 colors?

   Solution: Yes it is. Prove the following statement by induction: for any N, I can force Bob to use N colors, and furthermore, when Bob uses the N-th color, it is on a country that has an unshared border. This is clearly true for $N = 2$. For the inductive step, suppose we have it for N, and we are trying to get $N + 1$. Well, then first make Bob use N colors. Now, start a new blob of countries in a remote area, and force Bob to use 1 color in that blob. (He must use 1 of the N.) Next, start another new blob, and force Bob to use another of the N colors in that blob (possible by induction hypothesis). Proceed until we have N new blobs with each of the N colors in the exteriors; then engulf everything in a huge (not simply-connected) region; that region must have a color different from the N colors. So we are done.

5. (MOP98) Let S be the set of nonnegative integers. Let $h : S \to S$ by a bijective function. Prove that there do not exist functions f, g from S to itself, f injective and g surjective, such that $f(n)g(n) = h(n)$ for all $n \in S$.

   Solution: Use contradiction; assume that f and g exist, and define $F = f \circ h^{-1}$, $G = g \circ h^{-1}$; now F is injective and G is surjective, and $f(n)g(n) = h(n) \Leftrightarrow F(n)G(n) = n$. Prove by strong induction that $F(n) = n$: at $n = 1$, we have $F(1)G(1) = 1 \Rightarrow F(1) = 1$ since we are in nonnegative integers. But then if true up to N, then $F(N + 1)G(N + 1) = N + 1 \Rightarrow F(N + 1) \in [1, N + 1]$ and by injectivity, $F(N + 1) = N + 1$. Hence G can only take on two values, 1 and something else, so not surjective. Contradiction.

6. Show that every $2^n \times 2^n$ board with one square removed can be covered by Triominoes.

   Solution: First we inductively prove that we can tile any $2^n \times 2^n$ board such that we only miss one of the corners. Then look at a general $2^n \times 2^n$ board and split it into 4 equal squares. One of the squares contains the missing block, and we can use the previous result to tile the other 3 major squares. Recursively descend.

4 Inequality

1. Prove the AM-GM inequality by induction.

   Solution: Easy for $n = 1, 2$; use n to show $2n$, and then use n to show $n - 1$. For $n \mapsto 2n$, plug in $(a_k + a_{k+1})/2$; for the other one, use $a_n = (a_1 + \cdots + a_{n-1})/(n - 1)$.

2. (Zuming97) Let $a_1 = 2$ and $a_{n+1} = a_n/2 + 1/a_n$ for $n = 1, 2, \ldots$. Prove that $\sqrt{2} < a_n < \sqrt{2} + 1/n$.

   Solution: Draw a picture to see why it is always greater than $\sqrt{2}$. Also use AM-GM to prove that we must be beyond $\sqrt{2}$. For the other side, induct and bound

$$a_{n+1} < (\sqrt{2} + 1/n)/2 + 1/\sqrt{2} = \sqrt{2} + 1/(2n)$$

3. (Zuming97) For the positive sequence $\{a_n\}$ with $a_n^2 \le a_n - a_{n+1}$, prove that $a_n < 1/(n + 2)$.

   Solution: Positive sequence, so $a_1 \le 1$, and $a_2 \le 1/4$ by AM-GM, or function theory. Now

$$a_{n+1} \le a_n(1 - a_n) \le \frac{1}{n+2}\left(1 - \frac{1}{n+2}\right) = \frac{1}{n+2}\frac{n+1}{n+2} \le \frac{1}{n+2}\frac{n+2}{n+3} = \frac{1}{n+3}$$

4. (Zuming97) For $a > 0$, prove that:

$$\sqrt{a + \sqrt{2a + \sqrt{3a + \sqrt{4a + \sqrt{5a}}}}} < \sqrt{a} + 1$$

Solution: True for $n = 1$. Let $f_k(a)$ be the k-th iteration evaluated at a. Then if it's true for some k:

$$f_{k+1}(a)^2 < f_k(2a) + a < \sqrt{2a} + 1 + a < a + 2\sqrt{a} + 1 = (\sqrt{a} + 1)^2$$

5. Prove that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$$

Solution: Bunch the terms in packs of 1, 2, 4, 8, 16, etc. Each pack will exceed 1, and there are infinitely many of them.

6. Prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$$

Solution: Similar bunching, but now the bunches are bounded by $1/2^k$, geometric series converges to 2.

5 Additional Problems

1. (MOP97) Let F_k be the Fibonacci sequence, where $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$. Prove that for every $n, k \in \mathbb{N}$:

$$F_n \leq F_k F_{n-k} + F_{k+1} F_{n-k-1} \leq F_{n+1}$$

Solution: Write-up for 1997 MOP test 11, problem 3.

2. (MOP97) Prove that for $n, k \in \mathbb{Z}, n > 0, k \geq 0$:

$$F_{n+2} - F_k F_{n-k} - F_{k+1} F_{n-k-1} = F_{k+1} F_{n-k}$$

Solution: Write-up for 1997 MOP test 11, problem 3.

3. (MOP97) Given a sequence of numbers $\{a_1, \ldots, a_n\}$, define the derived sequence $\{a'_1, \ldots, a'_{n+1}\}$ by $a'_k = S - a_{k-1} - a_k$, where

$$S = \min_{1 \leq k \leq n+1} (a_{k-1} + a_k) + \max_{1 \leq k \leq n+1} (a_{k-1} + a_k)$$

and $a_0 = a_{n+1} = 0$. Thus, if we start with the sequence $\{1\}$ of length 1 and apply the derived sequence operation again and again, we get the family of sequences:

$$\{1\}, \{1, 1\}, \{2, 1, 2\}, \{3, 2, 2, 3\}, \{5, 3, 4, 3, 5\}, \ldots$$

Show that when we apply the operation $2n$ times in succession to the initial sequence $\{1\}$ (with $n \geq 1$), we get a sequence whose middle (i.e. $(n+1)$-st) term is a perfect square.

Solution: Write-up for 1997 MOP test 11, problem 3.

4. (Titu98) Prove that for every $n \in \mathbb{N}$, there exists a finite set of points in the plane such that for every point of the set there exist exactly n other points of the set at distance equal to 1 from that point.

Solution: Ask Titu

5. (Titu98) Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(n+1) > f(f(n))$ for all $n \in \mathbb{N}$. Prove that $f(n) = n$ for all n.

6. (MOP97) Suppose that each positive integer not greater than $n(n^2 - 2n + 3)/2$ is colored one of two colors (red or blue). Show that there must be an n-term sequence $a_1 < a_2 < \cdots a_n$ satisfying

$$a_2 - a_1 \leq a_3 - a_2 \leq \cdots \leq a_n - a_{n-1}.$$

Solution: Write-up for 1997 MOP IMO test 3, problem 2.