Buffet Contest Solutions

1 Algebra

1. Find all functions f from the set of non-zero real numbers to itself satisfying

$$f(x) + f(y) = f(xyf(x+y))$$

for all $x, y \neq 0$ with $x + y \neq 0$.

2. Consider the sequence $(a_n)_{n\geq 1}$ of real numbers where $a_1 \in (1,2)$ and $a_{n+1} = a_n + \frac{n}{a_n}$ for all $n \geq 1$. Prove that there is at most one pair of terms in this sequence with integer sum.

3. For a positive integer $n \geq 5$, let a_i, b_i $(i = 1, 2, \ldots, n)$ be integers satisfying the following two conditions:

   - The pairs (a_i, b_i) are distinct for $i = 1, 2, \ldots, n$;
   - $|a_1b_2 - a_2b_1| = |a_2b_3 - a_3b_2| = \ldots = |a_nb_1 - a_1b_n| = 1$.

   Prove that there exist indices i, j such that $1 < |i - j| < n - 1$ and $|a_ib_j - a_jb_1| = 1$.

2 Combinatorics

1. One or more powers of 2 are written on each of n sheets of paper. The sum of the numbers on each sheet are the same. If each number appears at most 5 times among the n sheets, what is the largest possible value for n?

2. The numbers 1 and -1 are written in the cells of a 2000×2000 grid. If the sum of the entries in the grid is positive, show that one can select 1000 rows and 1000 columns such that the sum of the numbers written in cells of their intersections is at least 1000.

3. In a country, every pair of cities is either joined by a two-way road or not joined by any road. The country has the property that for any pair of cities there is a sequence of roads one can take to get from the first city to the second city. However, if any sequence of an odd number of distinct roads beginning and ending at the same city is closed down, the country no longer has this property. Prove that the cities in the country can be partitioned into 4 districts such that there is no road between two cities in the same district.

3 Geometry

1. ABC is an isosceles triangle with $AC = BC$. Let O be its circumcenter, let I be its incenter, and let D be the point on BC such that lines OD and BI are perpendicular. Prove that lines ID and AC are parallel.

2. Consider the parallelogram $ABCD$ with obtuse angle A. Let H be the foot of the perpendicular from A to the side BC. The median from C in triangle ABC meets the circumcircle of triangle ABC at the point K. Prove that points K, H, C, D lie on the same circle.

3. Let M be the midpoint of the internal bisector AD of $\triangle ABC$. Circle ω_1 with diameter AC intersects segment BM at E and circle ω_2 with diameter AB intersects segment CM at F. Show that B, E, F, C lie on a circle.

4 Number Theory

1. A perfect number greater than 28 is divisible by 7. Prove that it is also divisible by 49.

   (Note: A number n is "perfect" if the sum of the divisors of n is equal to $2n$. For example, 28 is perfect since $1 + 2 + 4 + 7 + 14 + 28 = 2 \cdot 28$.)

2. Find (with proof) all monic polynomials $P(x)$ with integer coefficients that satisfy the following two conditions.

   (a) $P(0) = 2014$.

   (b) If x is irrational, then $P(x)$ is also irrational.

   Note: A polynomial is *monic* if its highest degree term has coefficient 1.

3. Let a_n be the leftmost digit of 2^n and let b_n be the leftmost digit of 5^n. Prove that the reverse of any k consecutive terms in the sequence a_n appears as k consecutive terms in the sequence b_n.

5 Algebra Solutions

1. Suppose $f(z) \neq \frac{1}{z}$ for some $z \neq 0$. Let $x = \frac{1}{f(z)}$ and $y = z - x$. Clearly $x \neq 0$; we also know $y \neq 0$ since we assumed $f(z) \neq \frac{1}{z}$, and we know $x + y = z \neq 0$. Therefore,

$$\begin{aligned} & f(x) + f(y) = f(xyf(x+y)) \\ \implies \quad & f\left(\frac{1}{f(z)}\right) + f(z-x) = f\left(\frac{1}{f(z)} \cdot (z-x) \cdot f(z)\right) \\ \implies \quad & f\left(\frac{1}{f(z)}\right) = 0, \end{aligned}$$

which is a contradiction. It follows that $f(x) = \frac{1}{x}$ is the only possible solution. Checking, we see it does indeed satisfy the conditions of the problem. *(Nordic 2003)*

2. Suppose $a_n > n$ for some n. Then

$$a_{n+1} = a_n + \frac{n}{a_n} = \frac{(a_n - n)(a_n - 1)}{a_n} + n + 1 > n + 1.$$

It is given that $a_n > n$ for $n = 1$, so it follows from induction that $a_n > n$ for all n. Therefore,

$$a_{n+1} = a_n + \frac{n}{a_n} < a_n + 1.$$

Define $b_n = a_n - n$. We have just shown that the sequence (b_n) is always positive and decreasing. Furthermore, $a_i + a_j$ is an integer if and only if $b_i + b_j$ is an integer, so it suffices to prove there exists at most pair (i, j) such that $b_i + b_j$ is an integer.

Now, $a_2 = \frac{a_1^2+1}{a_1} = \frac{(a_1-2)(a_1-0.5)}{a_1} + 2.5 < 2.5$, so $b_2 < 0.5$. Therefore, if $i, j > 1$, we have $b_i + b_j > 0$ and $b_i + b_j \leq 2b_2 < 1$, so $b_i + b_j$ cannot possibly be an integer.

It remains to consider $b_1 + b_i$. In this case, we know $b_1 + b_i > 0$ and $b_1 + b_i \leq 2b_1 < 2$, so this is an integer if and only if $b_i = 1 - b_1$. However, (b_n) is decreasing, so there is at most one i with this property. *(Russia 2009)*

3. Throughout this write-up, indices will always be taken mod n.

Define P_i to be the 2-dimensional vector (a_i, b_i). Let $|P_i|$ denote the length of P_i and let $K_{i,j}$ denote the area of the triangle with sides P_i and P_j. Recall that $K_{i,j} = \frac{1}{2} \cdot |a_i b_j - a_j b_i|$. Thus, we are given $K_{i,i+1} = \frac{1}{2}$ for all i and we need to show $K_{i,j} = \frac{1}{2}$ for some i, j with $|j - i| \neq 1$.

First suppose P_i and P_j lie on the same line for some i, j. Assume without loss of generality that $|P_i| \leq |P_j|$. Then, using the base times height formula for the area of a triangle, we have $K_{i,j+1} = K_{j,j+1} \cdot \frac{|P_i|}{|P_j|}$. This quantity is positive so it is at least $\frac{1}{2}$. (Recall that the formula for $K_{i,j}$ is an integer multiple of $\frac{1}{2}$). Furthermore, it is at most $K_{j,j+1} = \frac{1}{2}$, so in fact $K_{i,j+1} = \frac{1}{2}$, and similarly $K_{i,j-1} = \frac{1}{2}$. Since $n \geq 5$, we cannot have j adjacent to both $i + 1$ and $i - 1$. Therefore, one of $K_{i,j-1}$ or $K_{i,j+1}$ solves the problem.

Otherwise, let i be such that $|P_i|$ is maximal. Since P_{i-1} and P_i are not collinear, we can express P_{i+1} as a linear combination of the vectors P_{i-1} and P_i: $P_{i+1} = uP_{i-1} + vP_i$ for some real numbers u, v. Using the base times height formula for the area of a triangle again, we have $\frac{1}{2} = K_{i,i+1} = |u| \cdot K_{i-1,i} = \frac{|u|}{2}$, which implies $u = \pm 1$.

Now, $\gcd(a_i, b_i) = 1$ or else $|a_i b_{i+1} - a_{i+1} b_i|$ could not possibly be 1. Since $vP_i = -P_{i+1} \pm P_{i-1}$ has integer coordinates, it then follows that v must also be an integer. If $v = 0$, then P_{i-1} and P_{i+1} are collinear, which we already covered. If $|v| \geq 2$, then $|P_{i+1}| = |uP_{i-1} + vP_i| \geq |v| \cdot |P_i| - |u| \cdot |P_{i-1}| \geq |P_i|$. Equality holds only if P_{i-1}, P_i are collinear, which is impossible. Thus, $|P_{i+1}| > |P_i|$ in this case, and that contradicts the choice of i. The only remaining possibility is $v = \pm 1$, but then $K_{i-1,i+1} = |v| \cdot K_{i-1,i} = \frac{1}{2}$ and the problem is solved. *(Korea 2001)*

6 Combinatorics Solutions

1. Suppose that we have n sheets with the same sum S. There exists k such that $2^k \leq S < 2^{k+1}$. The sum of all numbers must then be at least $n \cdot 2^k$. On the other hand, we cannot have any power of 2 larger than 2^k or the sum on that sheet would be too large. Since each number appears at most 5 times, the sum of all numbers must be at most $5 \cdot (1 + 2 + \ldots + 2^k) < 10 \cdot 2^k$. Therefore, $n \cdot 2^k < 10 \cdot 2^k$ and hence $n < 10$.

   Conversely, if $n = 9$, we can use the following arrangement:

   (a) 2 sheets containing $1, 1, 2, 4$.
   (b) 1 sheet containing $2, 2, 4$.
   (c) 1 sheet containing $4, 4$.
   (d) 5 sheets containing 8.

   Therefore, the maximum value of n is 9. *(St. Petersburg 1998)*

2. **Lemma:** The numbers 1 and -1 are written in the cells of a grid with n rows and m columns. Let S denote the sum of the entries in the grid. If m is odd, it is possible to delete one row and end up with sum at least $\min(S + 1, n - 1)$. If m is even, it is possible to delete one row and end up with sum at least $\min(S, n - 1)$.

   Proof: If every row has positive sum, then we can drop an arbitrary row, and there will remain $n - 1$ rows, each with sum at least 1, and the claim is proven.

   Otherwise, there exists some row with non-positive sum. If m is odd, then this row actually has sum at most -1, and therefore if we drop it, the remaining sum will be at least $S + 1$. If m is even, then we still know this row has sum at most 0, and therefore if we drop it, the remaining sum will be at least S. This covers all cases, and the lemma is proven.

   Note that the same argument can be applied to removing columns as well. Now we use the lemma repeatedly to solve the problem:

   - We start with a 2000×2000 grid with $S \geq 1$.
   - By the lemma, we can delete 1 column to get a 2000×1999 subgrid with $S \geq 1$.
   - By the lemma, we can delete 1000 rows to get a 1000×1999 subgrid with $S \geq 1000$.
   - By the lemma, we can delete 999 columns to get a 1000×1000 subgrid with $S \geq 1000$, completing the proof.

(Russia 1995)

Comment: Can you solve the problem if 2000×2000 is replaced with 1500×1500? The same approach works.

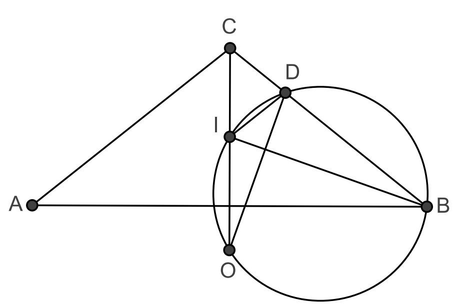
3. Let G be the graph with vertices corresponding to cities and with edges corresponding to roads between cities. In graph theoretic terms, we are given that G is connected (there is sequence of edges joining any two vertices) and that removing any odd cycle will disconnect G. We need to show it is 4-colourable (the vertices can be assigned colours 0, 1, 2, 3 such that no two vertices with the same colour have an edge between them).

   Suppose we remove edges from G one by one without disconnecting it, stopping only when it is impossible to remove any more edges. Let G_1 be the subgraph of G containing only the edges we removed, and let G_2 be the subgraph of G containing only the edges we did not remove. Since we can remove all of the edges in G_1 without disconnecting the graph, the problem conditions guarantee that G_1 has no odd cycles. Furthermore, G_2 has no cycles at all. If it did have a cycle, we could remove one of the edges in the cycle without disconnecting the graph, which contradicts the definition of G_2.

   We will now use the following very useful theorem from graph theory: if a graph has no odd cycles, then it is 2-colourable.[1]

   It follows that G_1 and G_2 are both 2-colourable. In particular, there are functions f_1, f_2 mapping cities to $\{0, 1\}$ such that $f_1(u) = f_1(v)$ only if there is no edge between u and v in G_1, and $f_2(u) = f_2(v)$ only if there is no edge between u and v in G_2. Let $f(v) = 2 \cdot f_1(v) + f_2(v)$. This assigns every vertex a colour in $\{0, 1, 2, 3\}$, and if $f(u) = f(v)$, then $f_1(u) = f_1(v)$ and $f_2(u) = f_2(v)$, which means u and v are not connected by an edge in either G_1 or G_2. It follows that f is a valid 4-colouring of G, and the problem is solved. *(Russia 2010)*

7 Geometry Solutions

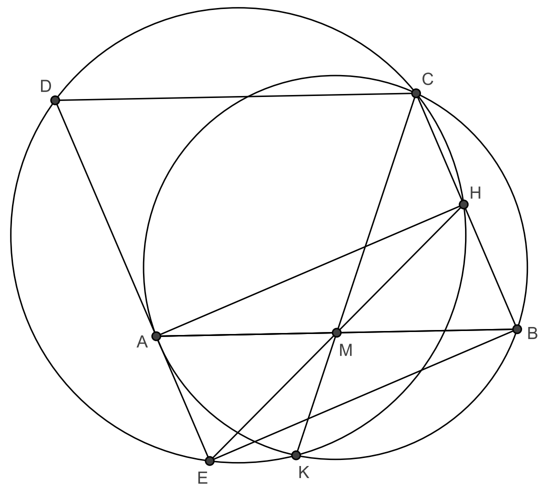


1. Since ABC is isosceles, I and O both lie on the perpendicular bisector of AB. Therefore,

[1]You can prove this as follows. First assume the graph is connected. If it is not, we can apply the proof independently to each connected component. Pick an arbitrary vertex u and colour it 0. Now, consider another vertex v. If there is both an even-length path from u to v and an odd length path from u to v, we could combine them to get an odd-length cycle. Therefore, we have either that all paths from u to v are even length, or all paths are odd length. In the former case, we colour v with 0; in the latter case, we colour it with 1. Do you see why this is a valid 2-colouring?

$\angle DOI = 90^\circ - \angle BIO = \angle ABI = \angle DBI$, from which it follows that $DBOI$ is a cyclic quadrilateral.

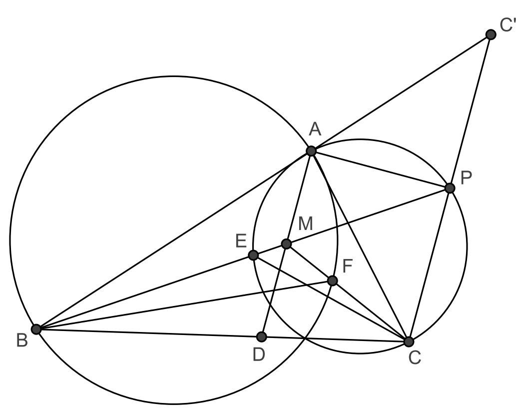
This means $\angle BDI = 180^\circ - \angle BOI = 180^\circ - \angle BOC = 2\angle BCI = \angle BCA$, and hence ID and AC are parallel. (*Russia 1996*)



2. Let M be the midpoint of AB and let E be the point such that $AHBE$ is a rectangle. Note that, since $AHBE$ is a rectangle, H, M, E are collinear.

   Now, $\angle DEH = \angle AEH = \angle ABH = \angle ABC = 180^\circ - \angle DCB = 180^\circ - \angle DCH$, so $DCHE$ is cyclic. Consider the circumcircles of $DCHE, AHBE$, and ABC. The radical axis of $DCHE$ and $AHBE$ is HE, and the radical axis of ABC and $AHBE$ is AB, so the radical center of the three circles is M. In particular, M is on the radical axis of $DCHE$ and ABC.

   Since C is also on the radical axis, we know the radical axis is equal to line CM, and so K is also on the radical axis. Since K is on circle ABC, it follows that K is also on circle $DCHE$, and the problem is solved. *(Russia 2012)*



3. Let C' be the point on line AB such that $C'C$ is parallel to AD, and let P be the midpoint of $C'C$. The homothety about B that takes AD to $C'C$ also takes M to P. This implies that P lies on line BEM.

Now, $\angle CC'A = \angle DAB = \angle DAC = \angle C'CA$, so $AC = AC'$. Since P is the midpoint of $C'C$, we conclude that $\angle APC = 90^\circ$ and hence P lies on ω_1. Since $C'C||AD$, this also implies $\angle PAD = 90^\circ$

Using the above facts, we have $\angle BEC = 180^\circ - \angle PEC = 180^\circ - \angle PAC = 90^\circ + \angle DAC = 90^\circ + \frac{\angle BAC}{2}$. The same argument applies to $\angle BFC$, so we have shown $\angle BEC = \angle BFC$, and the problem is solved. *(Ukraine 2013)*

8 Number Theory Solutions

1. Let n be a perfect number divisible by 7 but not 49. Then we can prime factorize n as $2^k \cdot 7 \cdot \prod p_i^{e_i}$ for some (possibly empty) set of primes p_i and some non-negative integer k. Recall[2] that the sum of the divisors of n is then given by $(2^{k+1} - 1) \cdot 8 \cdot \prod \left(\frac{p_i^{e_i+1}-1}{p_i-1}\right)$.

   Since n is perfect, we have

   $$(2^{k+1} - 1) \cdot 8 \cdot \prod \left(\frac{p_i^{e_i+1} - 1}{p_i - 1}\right) = 2 \cdot 2^k \cdot 7 \cdot \prod p_i^{e_i}$$

   The right-hand side must be a multiple of 8, so $k \geq 2$. Then $\frac{8}{7} \geq \frac{2^{k+1}}{2^{k+1}-1}$, which implies $(2^{k+1} - 1) \cdot 8 \geq 2 \cdot 2^k \cdot 7$. Furthermore $\frac{p_i^{e_i+1}-1}{p_i-1} > p_i^{e_i}$ for all i. This means that if $k \geq 2$,

   $$(2^{k+1} - 1) \cdot 8 \cdot \prod \left(\frac{p_i^{e_i+1} - 1}{p_i - 1}\right) \geq 2 \cdot 2^k \cdot 7 \cdot \prod p_i^{e_i}$$

   with equality only if $k = 2$ and there are no primes p_i. However, this occurs only when $n = 28$, so the problem is solved. *(Russia 2000)*

2. Clearly $P(x) = x + 2014$ satisfies the two conditions.

   We now show that is the only solution. Suppose by way of contradiction that we have a polynomial $P(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_1x + 2014$ that satisfies the two conditions and has degree $d \geq 2$. Let $M = \max(|a_1|, |a_2|, \ldots, |a_{d-1}|)$, and let p be a prime such that $p > \max(P(1), Md)$.

   Since $P(1) < p$ and $P(x) > p + 2014$ for large x, there must exist some $x > 1$ such that $P(x) = p + 2014$. We know this x must be rational, or else we would have found x such that x is irrational and $P(x)$ is rational. Thus, we have shown there is a rational number x that satisfies the following:

   $$x^d + a_{d-1}x^{d-1} + \ldots + a_1x - p = 0.$$

   By the rational root theorem, it must be that $x \in \{-p, -1, 1, p\}$. Since we know $x > 1$, it follows that $x = p$, and hence

   $$p^d + a_{d-1}p^{d-1} + \ldots + a_1p - p = 0.$$

[2] The general formula is that if $n = \prod p_i^{e_i}$, then the sum of the divisors of n is $\prod \left(\frac{p_i^{e_i+1}-1}{p_i-1}\right)$. To prove this, note that $\frac{p_i^{e_i+1}-1}{p_i-1} = 1 + p_i + p_i^2 + \ldots + p_i^{e_i}$, and see what monomials appear when you multiply everything out.

However,

$$\begin{aligned} p^d + a_{d-1}p^{d-1} + \ldots + a_1 p - p &\geq p^d - Mp^{d-1} - Mp^{d-2} - \ldots - Mp - p \\ &\geq p^d - Mdp^{d-1} > 0, \end{aligned}$$

which is a contradiction. *(Bay Area 2004)*

3. **Lemma:** For every real number $\epsilon > 0$ and every integer A, there exist integers n and m with $n > A$ and $10^m \leq 5^n < 10^m \cdot (1+\epsilon)$.

   Proof: Let B be a positive integer such that $(1+\epsilon)^B > 10$. Then $10^{\frac{1}{B}} < 1 + \epsilon$.

   Consider the numbers $\{0 \cdot \log_{10} 5\}, \{1 \cdot \log_{10} 5\}, \ldots, \{B \cdot \log_{10} 5\}$, where $\{x\}$ denotes the fractional part of x. There are $B+1$ such numbers and all lie in the range $[0,1]$. Furthermore, they are all distinct: if $\{u \cdot \log_{10} 5\} = \{v \cdot \log_{10} 5\}$, then 5^{u-v} is a power of 10, but that is impossible for $u \neq v$. Therefore, there must exist non-negative integers $u, v \leq B$ such that $0 < \{u \cdot \log_{10} 5\} - \{v \cdot \log_{10} 5\} \leq \frac{1}{B}$.

   Let $x = \{u \cdot \log_{10} 5\} - \{v \cdot \log_{10} 5\}$, and let C be a positive integer larger than $A + \frac{B}{x}$. Then there must exist some non-negative integer k such that $kx \leq \{C \cdot \log_{10} 5\} < (k+1)x$. Let $n = C + k(v-u)$. Note that $n > A$ since $|k(v-u)| \leq \frac{1}{x} \cdot B < C - A$. Furthermore,

   $$\begin{aligned} & \{n \cdot \log_{10} 5\} = \{C \cdot \log_{10} 5\} + k \cdot \{v \cdot \log_{10} 5\} - k \cdot \{u \cdot \log_{10} 5\} \\ \implies \quad & \{n \cdot \log_{10} 5\} < x \leq \frac{1}{B}. \end{aligned}$$

   Letting $m = \lfloor n \log_{10} 5 \rfloor$, we have:

   $$\begin{aligned} & m \leq n \cdot \log_{10} 5 < m + \frac{1}{B} \\ \implies \quad & 10^m \leq 5^n < 10^m \cdot 10^{\frac{1}{B}} \\ \implies \quad & 10^m \leq 5^n < 10^m \cdot (1+\epsilon), \end{aligned}$$

   which completes the proof of the lemma.

   Now consider a sequence $a_u, a_{u+1}, \ldots, a_v$. For each $i \in \{u, u+1, \ldots, v\}$, there exist integers e_i such that $(a_i + 1) \cdot 10^{e_i} - 1 \geq 2^i \geq a_i \cdot 10^{e_i}$.

   Let $\epsilon = 1 + \frac{1}{2^v}$ and $A = v$. Choose n and m according to the lemma.

   Then, for $i \in \{u, u+1, \ldots, v\}$, we have

   $$5^{n-i} \geq 10^m \cdot 5^{-i} = 10^{m-i} \cdot 2^i \geq a_i \cdot 10^{m-i+e_i},$$

   and similarly

   $$5^{n-i} < 10^{m-i} \cdot 2^i \cdot (1+\epsilon) \leq 10^{m-i} \cdot (2^i + 1) \leq (a_i + 1) \cdot 10^{m-i+e_i}.$$

   Therefore, $b_{n-i} = a_i$, and the proof is complete.