IMO Winter Training – January 2002

Sequences and Series

$a_n = \frac{\pi}{2}a_{n-1} - a_{n-2}$

1. The sequence $a_0, a_1, \ldots$ satisfies $a_n = ka_{n-1} - a_{n-2}$ for all $n \geq 2$. Show that (a_n) is periodic if k is:

   (a) 1. (b) -1. (c) $\sqrt{2}$. (d) $\sqrt{3}$. (e) $\frac{1+\sqrt{5}}{2}$.

   $1, 1, \frac{\pi}{2}-1, \frac{\pi^2}{4}-\frac{\pi}{2}-1, \ldots$

2. Let F_n denote the n^{th} Fibonacci number. Prove that for all $n \geq 1$,

   (a) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

   (b) $F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}$. $1+1 = 1 \cdot 2$ $1+1+4 = 2 \cdot 3$

   (c) The Lucas sequence (L_n) is defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for all $n \geq 2$. Find formulas for $L_1 + L_2 + \cdots + L_n$ and $L_1^2 + L_2^2 + \cdots + L_n^2$.

3. Find the following:

   (a) $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n-1}{n!}$.

   (b) $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$.

$x_{n+1} = \frac{x_n^2 - 1}{x_{n-1}}$

4. Let (x_n) be a sequence of non-zero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$, for $n = 1, 2, 3, \ldots$. Prove that there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

$(ax_{k-1} - x_{k-2})^2 - 1$

5. Let $n \geq 0$. Let $\tau = (1+\sqrt{5})/2$ and $A = \tau^{2n+1}$. Finally, let F be the fractional part of A, i.e. $F = A - \lfloor A \rfloor$. Prove that $AF = 1$.

$(ax_{k-2} - x_{k-3})$

6. Let n be a non-negative integer. Prove that the least integer greater than $(1+\sqrt{3})^{2n}$ is divisible by 2^{n+1}.

$= x_{n-1}(ax_{n-1} - x_{n-2})$

$ax_{n-1}x_{n-2} + 1 - x_{n-2}^2$

7. The faces of a regular tetrahedron are labelled 1, 2, 3, and 4. It is then placed on the plane, with the 1 initially on the bottom, and continuously rolled randomly to a different side. Find the probability p_n that the 1 is on the bottom after n rolls.

$x_n x_{n-2} + 1$

8. Let n be a positive integer. Show that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

x_{n-1}^2

9. Let n be a positive integer, and define $f(n) = 1! + 2! + \cdots + n!$. Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

   for all $n \geq 1$.

10. The sequence (a_n) satisfies a linear recurrence whose characteristic polynomial is $p(x)$. Let $s_n = a_1 + a_2 + \cdots + a_n$. Show that (s_n) satisfies the linear recurrence whose characteristic polynomial is $(x-1)p(x)$.

11. (a) Let $a_n = \binom{n}{m}$, where m is a fixed non-negative integer. Prove that

$$\Delta^k a_0 = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Let a_n be a polynomial in n. Prove that

$$a_n = a_0\binom{n}{0} + \Delta a_0\binom{n}{1} + \Delta^2 a_0\binom{n}{2} + \cdots$$

   for all integers n.

12. Let $a_0 = 4$, $a_1 = 16$, and $a_n = 14a_{n-1} - a_{n-2} - 24$ for $n \geq 2$. Prove that a_n is a perfect square for all n.

13. The sequence (a_n) satisfies $a_0 = 4$ and

$$a_n = \frac{15 - a_{n-1}}{7 - a_{n-1}}$$

for $n \geq 1$. Find a closed formula for a_n.

14. Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$, $T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}$. The first few terms are 2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

15. Let $a_1 = 1$, and $16a_{n+1} = 1 + 4a_n + \sqrt{1 + 24a_n}$ for $n \geq 1$. Prove that a_n is rational for all n.

16. Evaluate:

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2 - n + 1}\right).$$

17. Find

(a)

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}.$$

(b)

$$\sum_{k=1}^{n} \frac{k}{k^4 + k^2 + 1}.$$

18. Let a be the greatest root of the equation $x^3 - 3x^2 + 1 = 0$. Show that $\lfloor a^{1788} \rfloor$ and $\lfloor a^{1988} \rfloor$ are both divisible by 17.

19. Let (u_n) be the sequence defined by $u_0 = 0$, $u_1 = 1$, and $u_n = ku_{n-1} - u_{n-2}$ for $n \geq 2$, where k is an integer.

   (a) Prove that $u_1 + u_3 + \cdots + u_{2m-1} = u_m^2$ for all $m \geq 1$.

   (b) Prove that $u_1 + u_2 + \cdots + u_m$ divides $u_1^3 + u_2^3 + \cdots + u_m^3$ for all $m \geq 1$.

20. Let n be a positive integer, $n \geq 2$. Let F_n denote the n^{th} Fibonacci number. Prove that

$$F_n^2 = \prod_{k=1}^{n-1} \left[3 + 2\cos\left(\frac{2\pi k}{n}\right)\right].$$