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# TRANSFORMATIONS ARM 2

## ISOMETRIES (preserve distance, angle, area)

TRANSLATIONS: determined by distance, direction
ROTATIONS: determined by centre of rotation and angle
REFLECTIONS: determined by axis

} and any combination (composition) of these.

① Any isometry that fixes three noncollinear points is the identity.

② An isometry is uniquely determined by what it does to three noncollinear points.

③ If $\Delta ABC$ is congruent to $\Delta A'B'C'$, then there is an isometry that takes $A \to A'$, $B \to B'$, $C \to C'$

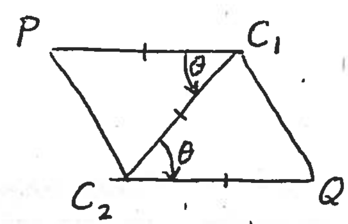
④ Every isometry can be written as a product of reflections.

(i) The composite of two reflections in distinct parallel axes is a translation perpendicular to the axes through twice the distance between them

(ii) The composite of two reflections in intersecting axes is a rotation whose centre is the point of intersection of the axes and whose angle is double the angle between the axes.

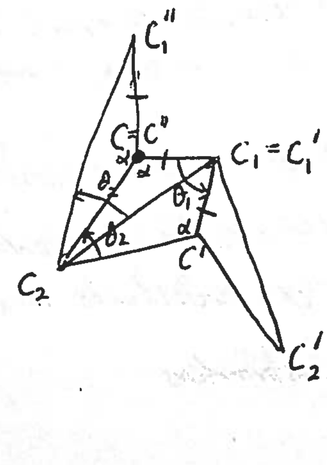
⑤ A straight line is carried by a rotation to a second straight line intersecting the first at the angle of rotation.

⑥ The {composite / product} of two rotations with different centres whose angles sum to 360° is a translation.




$P \to C_2$
$C_1 \to Q$
$C_1P \to QC_2$

⑦ The composite of two rotations with different centres $C_1$ and $C_2$ and respective angles $\theta_1$ and $\theta_2$ with $\theta_1 + \theta_2 \neq 360°$ is a rotation through angle $\theta_1 + \theta_2$




$2\alpha + \theta_1 + \theta_2 = 360°$

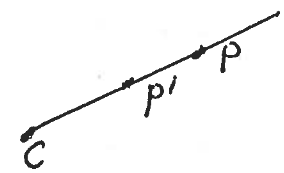
# TRANSFORMATIONS

## SIMILARITIES (scale: preserves angles, collinearity)

A CENTRAL SIMILARITY (DILATION, DILATATION, HOMOTHETY) is determined by its centre and its factor. A central similarity with factor 1 is the identity, and with factor $-1$ is a rotation about the centre through $180°$.

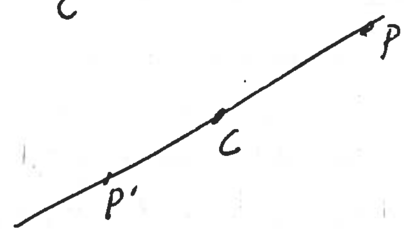
A dilatation with centre $C$ and positive factor $\lambda$ carries a point $P$ to a point $P'$ on the ray $CP$ produced such that

$$|CP'| = \lambda |CP|.$$




A dilatation with centre $C$ and negative factor $\lambda$ carries a point $P$ to a point $P'$ on the line $PC$ produced such that $C$ is between $P$ and $P'$ and

$$|CP'| = |\lambda| |CP|$$

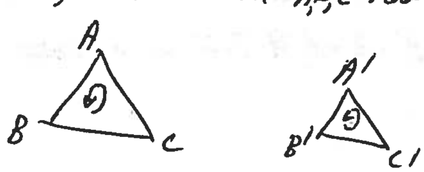



A dilatation takes any figure to a similar figure and any line $l$ to a line $l'$ that is parallel to $l$.

A dilatation with factor $\lambda$ alters linear dimensions in the proportion $|\lambda|$, areal dimensions in the proportion $\lambda^2$ and volume in the proportion $|\lambda|^3$.

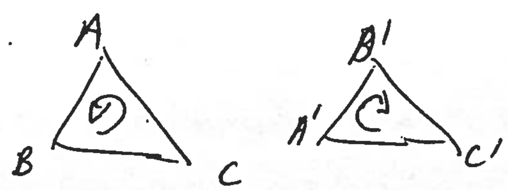
## DIRECT & OPPOSITE TRANSFORMATIONS

A direct transformation preserves sense in that, if for a triangle $ABC$ with $A, B, C$ read in a counterclockwise direction, the triangle $A'B'C'$ obtained from $ABC$ also has its vertices recorded counterclockwise.




Translations, rotations and dilatations with positive or negative factor and direct transformations.

An opposite transformation changes the orientation of a triangle from counterclockwise to clockwise.




Reflections are opposite transformations

# COMPOSITIONS INVOLVING DILATATIONS

THE COMPOSITE OF TWO DILATATIONS.

Centre $(0,0)$ and factor $\lambda$ followed by centre $(1,0)$ and factor $\mu$.

$$(x,y) \to (\lambda x, \lambda y) \to (1,0) + \mu(\lambda x - 1, \lambda y) = (1-\mu+\lambda\mu x, \lambda\mu y).$$

If $\lambda\mu = 1$, we get a translation parallel to segment joining centres of dilatation through direct distance $1-\mu$.

If $\lambda\mu \neq 1$, the composite has fixed point $\left(\frac{1-\mu}{1-\lambda\mu}, 0\right)$

Pick new coordinates making this the origin: $X = x - \frac{1-\mu}{1-\lambda\mu}$, $Y = y$

$$X' = 1-\mu+\lambda\mu x - \frac{1-\mu}{1-\lambda\mu} = \lambda\mu x - \frac{\lambda\mu(1-\mu)}{1-\lambda\mu} \qquad Y' = \lambda\mu y$$

$$(X,Y) \to (\lambda\mu X, \lambda\mu Y)$$

The composite is a dilatation with centre $\left(\frac{1-\mu}{1-\lambda\mu}, 0\right)$ and factor $\lambda\mu$.

[Note special cases: $\mu = 1$; $\lambda = 1$]

THE COMPOSITE OF A DILATATION AND A REFLECTION

Centre of dilatation $(0,0)$; axis of reflection $x=1$

$$(x,y) \to (\lambda x, \lambda y) \to (2-\lambda x, \lambda y)$$

If $\lambda = -1$, the mapping is $(x,y) \to (-x,-y) \to (2+x, -y)$
which is a glide reflection (translation followed by a reflection)

If $\lambda \neq -1$, there is a fixed point $\left(\frac{2}{1+\lambda}, 0\right)$.

If $x = \frac{2}{1+\lambda} + X$, $y = Y$, then $(x,y) \to \left(2 - \frac{2\lambda}{1+\lambda} - \lambda X, \lambda y\right) \to \left(\frac{2}{1+\lambda} - \lambda X, \lambda Y\right)$

so the combined transformation is essentially $(X,Y) \to (-\lambda X, \lambda Y)$,
a dilatation followed by a reflection in an axis through the centre of the dilatation.

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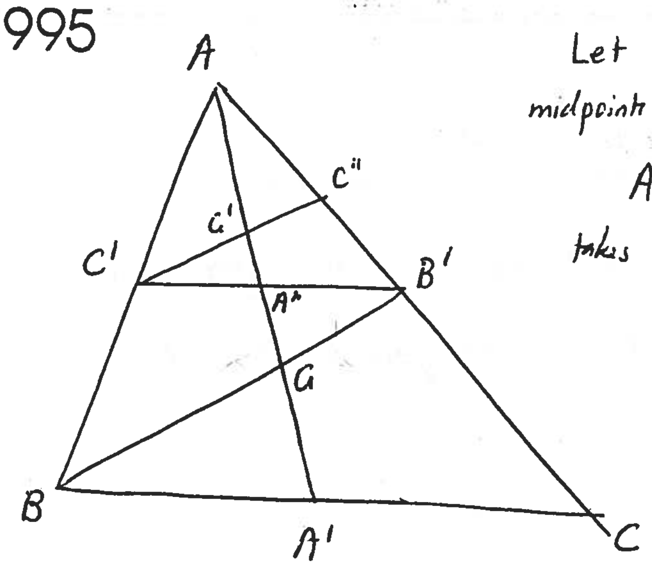
EXAMPLE 1




Let $ABC$ be a given triangle, whose side midpoints are $A', B', C'$ and whose centroid is $G$ as illustrated.

A dilatation with centre $A$ and factor $1/2$ takes

$B \to C'$
$C \to B'$
$B' \to C''$
$G \to G'$
$A' \to A''$

We have $C'G' = \frac{1}{2} BG$

$C'A'' = A''B' = \frac{1}{2} BA' = \frac{1}{2} A'C$

$C'C'' \parallel BB'$

Since a rotation of $180°$ about $A''$ takes $A'' \to A''$, $C' \to B'$, $CC'$ to a line through $B'$ and $A''G'$ to a line through $A''G$, $G'$ must go to $G$ and so $C'G' = B'G$.

Hence $BG = 2C'G' = 2B'G$

$AG' = \frac{1}{2} AG \Rightarrow AG' = G'G = A''G + A''G = 2A''G'$

$\Rightarrow AG = 2AG' = 4A''G' = 2A'G$

and so $G$ (and $G'$) trisect $AA'$.

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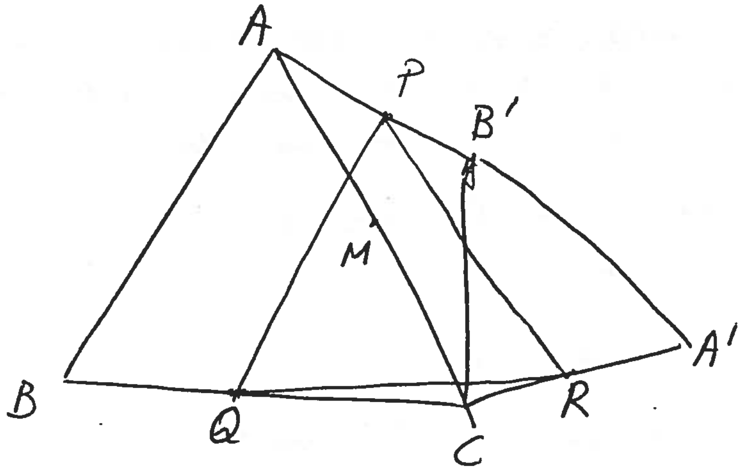
EXAMPLE 2



ABC and A'B'C are similarly oriented equilateral triangles and P, Q, R are the respective midpoints of AB', BC and A'C respectively.
Prove that $\Delta PQR$ is equilateral




First solution. Consider a dilatation with centre A and factor $\frac{1}{2}$. It take $B' \rightarrow P$ and $C \rightarrow M$, the midpoint of AC. Also $PM = \frac{1}{2}B'C = \frac{1}{2}CA' = RC$ and $PM \parallel CB'$ so PM produced makes an angle of $60°$ with A'C produced.

Consider a $60°$ rotation with centre Q. Then $Q \rightarrow Q$, $C \rightarrow M$. Also CR goes to a line through M making an angle of $60°$ with CR, so CR must go to MP. Thus $\Delta QMP$ is the image of $\Delta QCR$ $(R \rightarrow P)$, and so $\Delta QMP \equiv \Delta QCR$.
Hence $PQ = RQ$ and $\angle PQR = 60° \implies \Delta PQR$ is equilateral.

Second solution. $\vec{OP} = \frac{1}{2}\vec{OA} + \frac{1}{2}\vec{OB'}$ and $\vec{OQ} = \frac{1}{2}\vec{OB} + \frac{1}{2}\vec{OC} \implies \vec{PQ} = \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{B'C}$

Similarly $\vec{PR} = \frac{1}{2}\vec{AC} + \frac{1}{2}\vec{B'A'}$

Now $\vec{AC}$ is the image of $\vec{AB}$ under a $60°$ rotation and $\vec{B'A'}$ is the image of $\vec{B'C}$ under a $60°$ rotation.

Hence $\vec{PR}$ is the image of $\vec{PQ}$ under a $60°$ rotation, so $\angle QPR = 60°$ and $|PQ| = |PR|$.

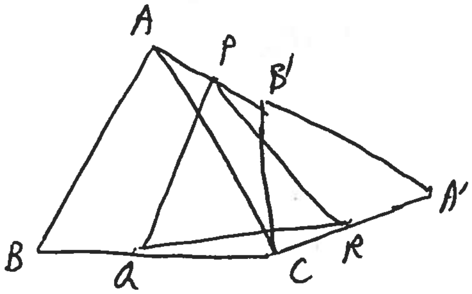
Therefore $\Delta PQR$ is equilateral.

Third solution. Consider the following composite of three transformations.

(1) dilatation with centre $C$ and factor $2$

(2) $60^\circ$ counterclockwise rotation about $B$.

(3) dilatation with centre $C$ and factor $\frac{1}{2}$.




Note that (1) and (3) preserve the direction of any line, and (2) rotates a line through $60^\circ$. The effect of (1), (2), (3) together is to preserve distance and rotate lines through $60^\circ$

The composite takes $Q \to B \to B \to Q$.

Now (1) sends $R$ to $A'$.

Suppose (2) sends $A'$ to $A''$.

Since $AA''$ is the image of $CA'$ under (2),

$AA'' = CA'$ and

$AA''$ make an angle of $60^\circ$ with $CA'$

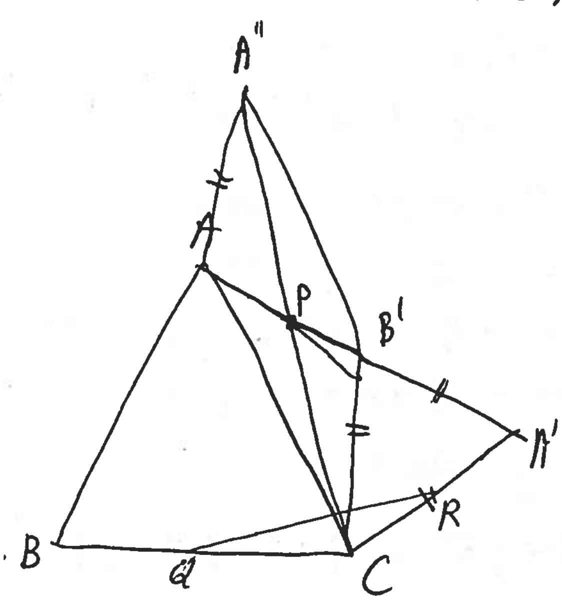
But $B'C$ makes an angle of $60^\circ$ with $CA'$

Hence $AA'' \parallel CB'$ and $AA'' = CB'$

so $ACB'A''$ is a parallelogram, so

$A''C$ bisects $AB'$ in $P$.

Thus (3) takes $A''$ to $P$.




Thus, the composite takes

$R \to A' \to A'' \to P$

and so takes $QR$ to $QP$.

Hence $\angle PQR = 60^\circ$ and $QR = QP$, from which the result follows.

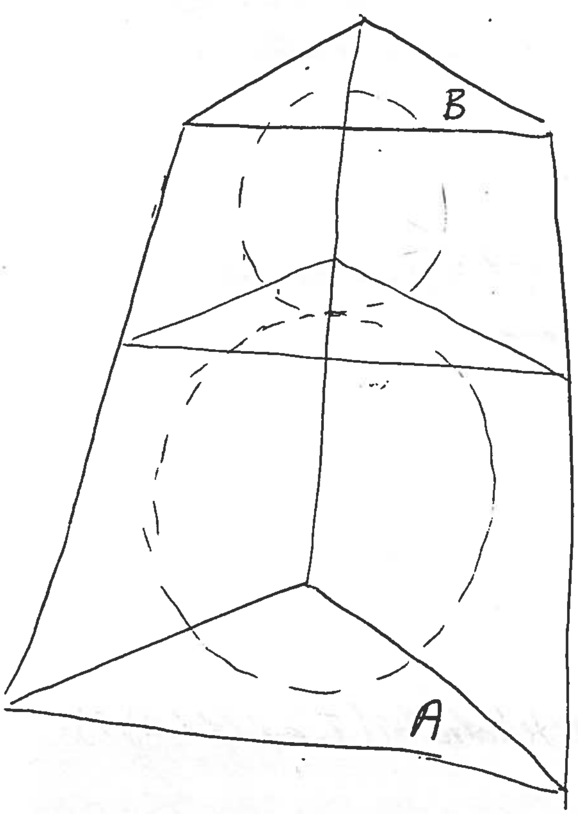
EXAMPLE 3



A frustum of a certain triangular pyramid has a lower base of area $A$, upper base of area $B < A$, and the sum of the areas of its lateral faces is $P$. The frustum is such that it can be divided by a plane parallel to the bases into two smaller frusta in each of which a sphere can be inscribed.

Prove that

$$P = \left(\sqrt{A} + \sqrt{B}\right)\left(\sqrt[4]{A} + \sqrt[4]{B}\right)^2$$



EXAMPLE 3

**CUB 3**

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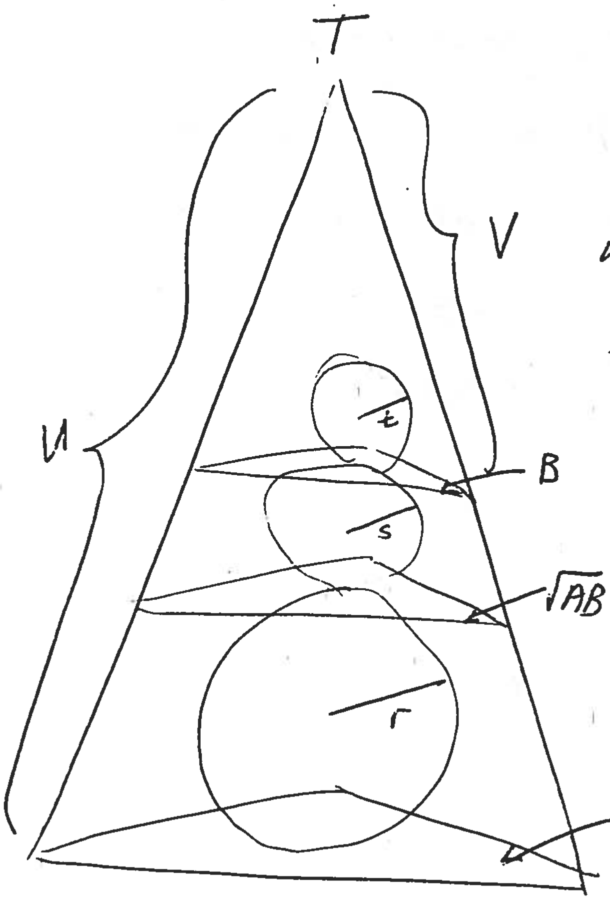
Solution:




Extend the slant edges of the frustum to T to complete the tetrahedron.

Note that a dilation with centre T with factor $\sqrt{\frac{B}{A}}$ takes the base of area A to the base of area B. The sphere touching the base of area A and the three slant sides of the frustum is the inscribed sphere of the tetrahedron (T, base of area A) and this gets carried to the inscribed sphere of the tetrahedron (T, base of area B). Let the respective radii of these spheres be r and t.

This dilatation is the square of the dilatation taking the base of area A to the intermediate base. This takes the sphere of radius r to a sphere of radius s.

It is these two spheres described in the problem.

This dilation has factor $\sqrt[4]{\frac{B}{A}} = \sqrt{\frac{t}{r}} = \frac{s}{r} = \frac{t}{s}$

The volume of the tetrahedron (T, base of area B) is $\frac{1}{3}t \times$ (surface area)

$= \frac{1}{3}t\,(B+V)$ where V is the lateral surface area.

This volume is also $\frac{1}{3}s\,(V-B)$

so $\dfrac{V-B}{B+V} = \dfrac{t}{s} = \sqrt{\dfrac{t}{r}} = \dfrac{\sqrt[4]{B}}{\sqrt[4]{A}}$

$\Longrightarrow \quad (V-B)\sqrt[4]{A} = (B+V)\sqrt[4]{B}$

$\Longrightarrow \quad B(\sqrt[4]{A} + \sqrt[4]{B}) = V(\sqrt[4]{A} - \sqrt[4]{B})$.

$P = U - V$ (where U is the lateral surface area of tetrahedron (T, area of base A))

$= \frac{V}{B}(A-B)$ (since $\frac{U}{A} = \frac{V}{B}$ by the dilation)

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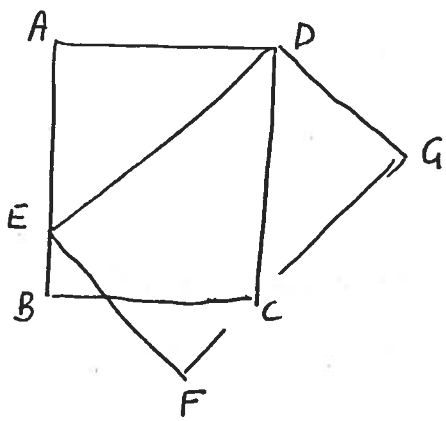
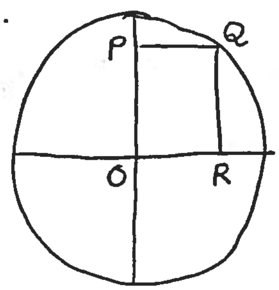
# PROBLEMS ON TRANSFORMATIONS.

**ARM 2**

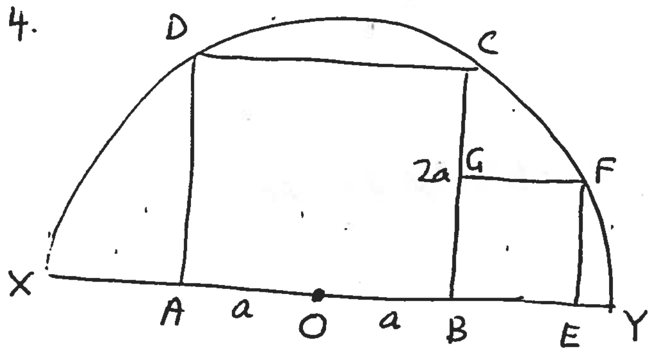
1. By means of two straight cuts, subdivide a $9 \times 16$ rectangle into three pieces that can be reassembled into a square

2. 

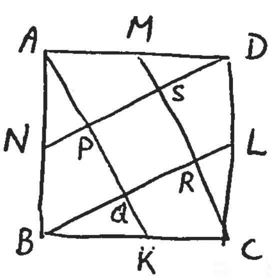

A rectangular sheet is laid atop another rectangular sheet of the same size as indicated. Does the top sheet cover more or less than half the area of the bottom sheet? (You need to ascertain where C lies relative to FG.)

3. 


OPQR is a rectangle whose sides OP and OR lie along diameters of a circle of radius r. Determine the length of PR.

4. 


ABCD and BEFG are squares inside a semicircle XDCFY whose centre O is the midpoint of side AB. If $|AB| = |BC| = |CD| = |DA| = 2a$, determine the side length of the square BEFG. (Note that the smaller square is uniquely determined by the larger one.)

5. 


ABCD is a square whose side midpoints are K, L, M, N as indicated. AK, BL, CM, DN intersect pairwise in the points P, Q, R, S.

(a) Explain why PQRS is a square.

(b) Determine the ratio $\dfrac{\text{Area}(PQRS)}{\text{Area}(ABCD)}$.

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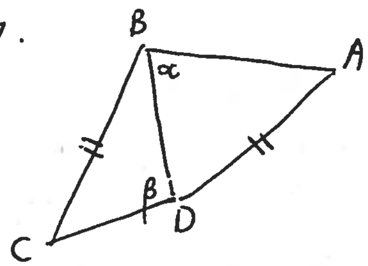
# PROBLEMS ON TRANSFORMATIONS

**ARM 2**

6. Let C be a circle and P be a given point in the plane. Each line through P which intersects C determines a chord of C. Show that the midpoints of these chords lie on a circle. (1991 CMO)

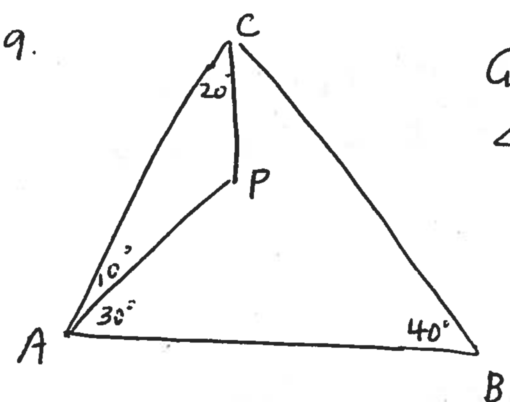
7.




Given that $\angle ABD + \angle BDC = 180^\circ$ and $AD = BC$, prove that $\angle BAD = \angle BCD$.

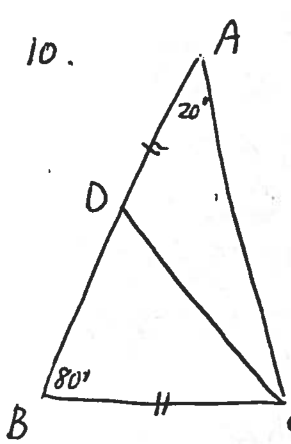
8. Let H, G, O be the respective orthocentre, centroid and circumcentre of a triangle ABC. Prove that HGO is collinear with G between H and O and that $HG = 2GO$.

9.



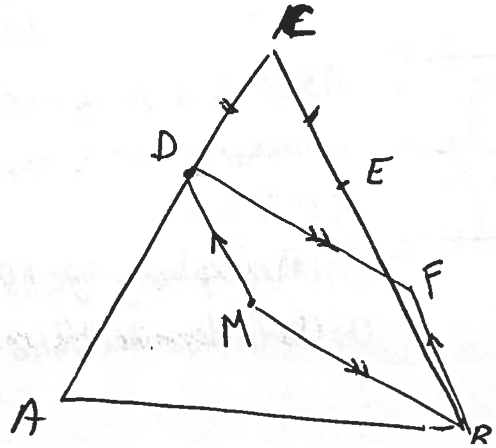

Given that $\angle PAC = 10^\circ$, $\angle PCA = 20^\circ$, $\angle PAB = 30^\circ$ and $\angle ABC = 40^\circ$, determine $\angle BPC$.

10.

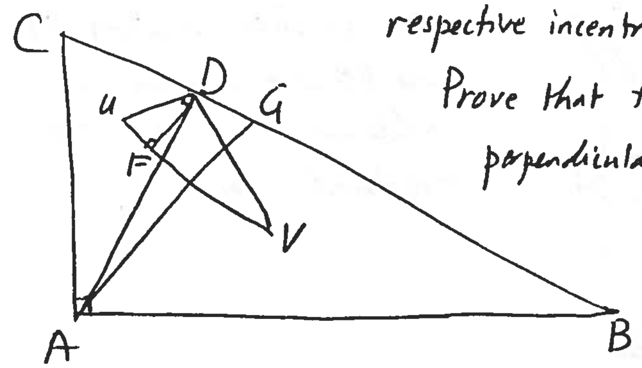



Given that $\angle BAC = 20^\circ$, $\angle ABC = 80^\circ$ and $AD = BC$, determine $\angle ADC$.

11. Given that $\Delta ABC$ is equilateral with centroid M, $CD = CE$ and DMBF is a parallelogram, prove that $\Delta MEF$ is equilateral.



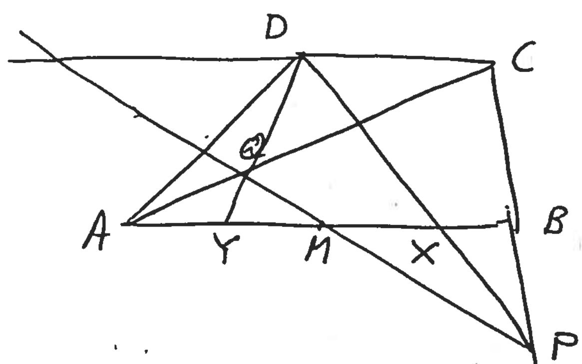
12. In triangle $ABC$, $\angle A = 90^\circ$ and $AD \perp BC$. Let $U$ and $V$ be the respective incentres of triangles $ACD$ and $ABD$. Prove that the angle bisector $AG$ of $\angle CAB$ is perpendicular to $UV$.

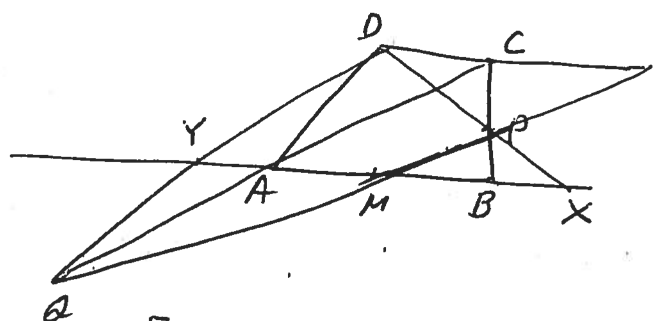



13. $ABCD$ is a trapezoid with $AB \parallel CD$, and $M$ is the midpoint of $AB$. $P$ is a point on $BC$ distinct from $B$ and $C$. Let

$X$ be the intersection of $PD$ and $AB$
$Q$ be the intersection of $PM$ and $AC$
$Y$ be the intersection of $DQ$ and $AB$

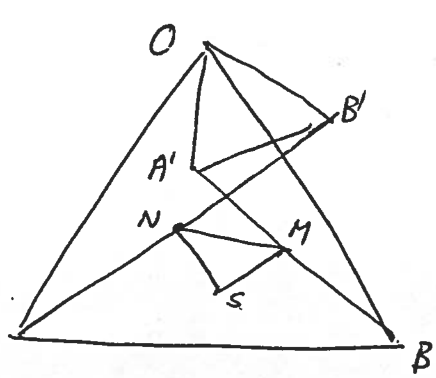
Prove that $M$ is the midpoint of $XY$



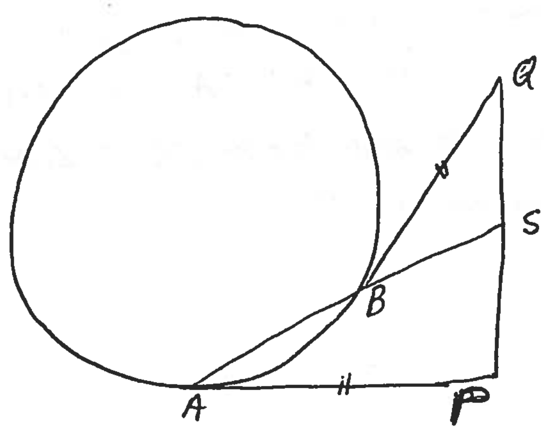




14. Construct a parallelogram $ABCD$ given $A$, $C$, $E$ and the distances $r$ and $s$ of $B$ and $D$ respectively from $E$. (When is the construction feasible?)

15. Let $OAB$ and $OA'B'$ be equilateral triangles with the same orientation, $S$ be the centroid of $\Delta OAB$ and $M$, $N$ be the respective midpoints of $A'B$ and $AB'$.
Prove that $\Delta SMB'$ is similar to $\Delta SNA'$.



16.




At points $A$ and $B$ on a circle, equal tangents $AP$ and $BQ$ are drawn as indicated. Prove that $AB$ produced bisects $PQ$

# HINTS

# ARM 2

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1. What is the side length of the square? Make a cut to produce this length. Make a translation of one of the pieces before doing the second cut.

2. Consider reflection in the angle bisector of $\angle EDC$, and look at the position of C.

4. Consider a reflection in the axis passing through O and making an angle of $45^\circ$ with XOY.

5. (a) Consider a $90^\circ$ rotation about the centre of the square.
   (b) Consider a rotation of $\Delta MSD$ about M.

6. Consider a circle concentric with C passing through P, and consider a dilatation with centre P.

7. Turn $\Delta BDC$ over so BD lies on DB.
   OR Move $\Delta BCD$ to $\Delta ADE$ and note $\angle ABD + \angle AED = 180^\circ$

8. Consider a dilatation with centre G and factor $-\frac{1}{2}$.

9. Note that $CA = CB$. Consider a reflection in the right bisector of BC which interchanges P and a point Q. What is $\angle PCQ$?

10. Consider an isometry that takes A to B, D to C and C to a point E. What can be said about $\Delta ABE$?

11. Consider the translation that takes $M \to B$, $D \to F$, $A \to A'$, $C \to C'$. Observe that $\Delta MAA'$ and $\Delta MCC'$ are equilateral. Consider also a $60^\circ$ clockwise rotation about M; where does it take E?

12. Note that $\Delta ADC$ is similar to $\Delta ABD$ being related by a rotation followed by a dilatation both with centre D. Prove that $\Delta DUV$ is similar to $\Delta ACB$, and consider the similarity that relates them.

13. Let N be the intersection of CD and PQ.
    Let $H_Q$ be the dilatation with centre Q taking $A \to C$, $M \to N$ and $H_P$ be the dilatation with centre P taking $C \to B$, $N \to M$
    Examine the effect of $H_P \circ H_Q$.

14. Consider the central reflection in ($180^\circ$ rotation about) the intersection M of the diagonals of the parallelogram

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HINTS

# SUI 3

15. Consider the dilatation with centre B and factor 2 followed by a 60° rotation with centre O, applied to S and M. Consider similar transformation with B replaced by A and M by N. Get similarity by SAS.

16 Consider the rotation about the centre of the circle that relates AP and BQ, and relates BS and AT (for a point T).
OR Consider a dilatation with centre T that carries $B \to Q$, $A \to R$ (say).
What is the factor of the dilatation with centre P that takes R to A?
What does this dilatation do to Q?