Solutions

A single spherical silver nanoparticle

2.1	Volume of the nanoparticle: $V = \frac{4}{3}\pi R^3 = 4.19 \times 10^{-24}\ \mathrm{m}^3$. Mass of nanoparticles: $M = V\,\rho_{\mathrm{Ag}} = 4.39 \times 10^{-20}$ kg Number of ions: $N = N_A \frac{M}{M_{\mathrm{Ag}}} = 2.45 \times 10^5$. Charge density $\rho = \frac{eN}{V} = 9.38 \times 10^9\ \mathrm{C\,m^{-3}}$ Electron concentration $n = \frac{N}{V} = 5.85 \times 10^{28}\ \mathrm{m}^{-3}$, so charge density $\rho = en$ Total charge of free electrons $Q = eN = 3.93 \times 10^{-14}$ C, Total mass of free electrons $m_0 = m_e N = 2.23 \times 10^{-25}$ kg.	0.7

The electric field in a charge-neutral region inside a charged sphere

2.2	For a sphere with radius R and constant charge density ρ, for any point inside the sphere designated by radius-vector $\mathbf{r} = r\mathbf{e}_r$ $(r < R)$ Gauss's law yields directly $4\pi r^2 \varepsilon_0 \boldsymbol{E}_+ = \frac{4}{3}\pi r^3 \rho\, \boldsymbol{e}_r$, where $\boldsymbol{e}_r$ is the unit radial vector pointing away from the center of the sphere. Thus, $\boldsymbol{E}_+ = \frac{\rho}{3\varepsilon_0}\boldsymbol{r}$. Likewise, inside another sphere of radius R_1 and charge density $-\rho$ the field is $\boldsymbol{E}_- = \frac{-\rho}{3\varepsilon_0}\boldsymbol{r}'$, where $\boldsymbol{r}'$ is the radius-vector of the point in the coordinate system with the origin in the center of this sphere. Merging the two charge configurations gives the setup we want with $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{r}_d$. So inside the charge-free region $\lvert \boldsymbol{r} - \boldsymbol{x}_{\mathrm{p}} \rvert < R_1$ the field is $\boldsymbol{E} = \boldsymbol{E}_+ + \boldsymbol{E}_- = \frac{\rho}{3\varepsilon_0}\mathbf{r} + \frac{-\rho}{3\varepsilon_0}(\boldsymbol{r} - \boldsymbol{x}_d)$ or $\boldsymbol{E} = \frac{\rho}{3\varepsilon_0}\boldsymbol{x}_d$ with the pre-factor $A = \frac{1}{3}$	1.2

The restoring force on the displaced electron cloud

2.3	With $\boldsymbol{x}_{\mathrm{p}} = x_{\mathrm{p}}\,\boldsymbol{e}_x$ and $x_{\mathrm{p}} \ll R$ we have from above that approximately the filed induced inside the particle is $\boldsymbol{E}_{\mathrm{ind}} = \frac{\rho}{3\varepsilon_0}\boldsymbol{x}_{\mathrm{p}}$. The number of electrons that produced $\boldsymbol{E}_{\mathrm{ind}}$ is negligibly smaller than the number of electrons inside the particle, so $\boldsymbol{F} \cong Q\boldsymbol{E}_{\mathrm{ind}} = (-eN)\frac{\rho}{3\varepsilon_0}\boldsymbol{x}_{\mathrm{p}} = -\frac{4\pi}{9\varepsilon_0}R^3 e^2 n^2\, x_p \boldsymbol{e}_x$ (like for a harmonic oscillator). The work done on the electrons to shift the electron cloud is $W_{\mathrm{el}} = -\int_0^{x_{\mathrm{p}}} F(x')\,\mathrm{d}x' = \frac{1}{2}\left(\frac{4\pi}{9\varepsilon_0}R^3 e^2 n^2\right) x_{\mathrm{p}}^2$	1.0

The spherical silver nanoparticle in an external constant electric field

2.4	Inside the metallic particle in the steady state the electric field must be equal to 0. The induced field (from 2.2 or 2.3) compensates the external field: $\boldsymbol{E}_0 + \boldsymbol{E}_{\mathrm{ind}} = 0$, so $x_{\mathrm{p}} = \frac{3\varepsilon_0}{\rho}E_0 = \frac{3\varepsilon_0}{en}E_0$. Charge displaced through the yz-plane is the total charge of electrons in the cylinder of	0.6

	radius R and height x_p: $-\Delta Q = -\rho\, \pi R^2 x_\mathrm{p} = -\pi R^2\, ne\, x_\mathrm{p}$.	

The equivalent capacitance and inductance of the silver nanoparticle

2.5a	The electric energy W_el of a capacitor with capacitance C holding charges $\pm\Delta Q$ is $W_\mathrm{el} = \frac{\Delta Q^2}{2C}$. The energy of such capacitor is equal to the work (see 2.3) done to separate the charges (see 2.4), thus $C = \frac{\Delta Q^2}{2W_{el}} = \frac{9}{4}\varepsilon_0 \pi R = 6.26 \times 10^{-19}$ F.	0.7
2.5b	Equivalent scheme for a capacitor reads: $\Delta Q = CV_0$. Combining charge from (2.4) and capacitance from (2.5a) gives $V_0 = \frac{\Delta Q}{C} = \frac{4}{3} R\, E_0$.	0.4

2.6a	The kinetic energy of the electron cloud is defined as the kinetic energy of one electron multiplied by the number of electrons in the cloud $W_\mathrm{kin} = \frac{1}{2} m_e v^2 N = \frac{1}{2} m_e v^2 \left(\frac{4}{3}\pi R^3\, n\right)$. The current I is the charge of electrons in the cylinder of area πR^2 and height $v\Delta t$ divided by time Δt, thus $I = -e\, nv\, \pi R^2$.	0.7
2.6b	The energy carried by current I in the equivalent circuit with inductance L is $W = \frac{1}{2} L\, I^2$ is, in fact, the kinetic energy of electrons W_kin. Taking the energy and current from (2.6a) results $L = \frac{4\, m_e}{3\pi R n e^2} = 2.57 \times 10^{-14}$ H.	0.5

The plasmon resonance of the silver nanoparticle

2.7a	From the LC-circuit analogy we can directly derive $\omega_p = (LC)^{-1/2} = \sqrt{ne^2/3\varepsilon_0 m_e}$. Alternatively it is possible to use the harmonic law of motion in (2.3) and get the same result for the frequency	0.5
2.7b	$\omega_\mathrm{p} = 7.88 \times 10^{15}$ rad/s, for light with angular frequency $\omega = \omega_\mathrm{p}$ wavelength is $\lambda_\mathrm{p} = 2\pi c/\omega_\mathrm{p} = 239$ nm.	0.4

The silver nanoparticle illuminated with light at the plasmon frequency

2.8a	The velocity of an electron $v = \frac{dx}{dt} = -\omega x_0 \sin\omega t = v_0 \sin\omega t$. For harmonic motion it is enough to average over period of oscillations. The time-averaged kinetic energy on the electron $\langle W_k\rangle = \langle\frac{m_e v^2}{2}\rangle = \frac{m_e}{2}\langle v^2\rangle$. During time t_0 each electron hits the ions t_0/τ times. So The energy lost in the whole nanoparticle during one period of oscillations is $W_{heat} = N\langle\frac{m_e v^2}{2}\rangle = \frac{4}{3}\pi R^3 n \langle\frac{m_e v^2}{2}\rangle$. Time-averaged Joule heating power $P_\mathrm{heat} = \frac{1}{\tau} W_\mathrm{kin} = \frac{1}{2\tau} m_e \langle v^2\rangle \left(\frac{4}{3}\pi R^3\, n\right)$. The expression for current is taken from (2.6a), squared and averaged $\langle I^2\rangle = (en\, \pi R^2)^2\, \langle v^2\rangle = \left(\frac{3Q}{4R}\right)^2 \langle v^2\rangle$.	1.0

2.8b	The oscillating current $I = I_0 \sin \omega t = \pi R^2 nev_0 \sin \omega t$ produces the heat in the resistance R_{heat} equal to $P_{\text{heat}} = R_{heat}\langle I^2\rangle$, what together with results from (2.8a) leads to $R_{\text{heat}} = \frac{W_{\text{kin}}}{\tau I^2} = \frac{2m_e}{3\pi ne^2 R\tau} = 2.46\ \Omega$.	1.0

2.9	$R_{\text{scat}} = \frac{P_{\text{scat}}}{\langle I^2\rangle}$ and $\langle v^2\rangle = \frac{1}{2}\omega_{\text{p}}^2 x_0^2$ yields $R_{\text{scat}} = \frac{Q^2 x_0^2 \omega_{\text{p}}^4}{12\pi\varepsilon_0 c^3}\frac{16R^2}{9Q^2\langle v^2\rangle} = \frac{8\omega_0^2 R^2}{27\pi\varepsilon_0 c^3} = 2.45\ \Omega$.	1.0

2.10a	Ohm's law for a *LCR* serious circuit is $I_0 = \frac{V_0}{\sqrt{(R_{heat}+R_{scat})^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}$. At the resonance frequency time-averaged voltage squared is $\langle V^2\rangle = Z_R^2\langle I^2\rangle = (R_{\text{heat}} + R_{\text{scat}})^2\langle I^2\rangle$. And from (2.5b) $\langle V^2\rangle = \frac{1}{2}V_0^2 = \frac{8}{9}R^2E_0^2$, so Ohm's law results in $\langle I^2\rangle = \frac{8R^2E_0^2}{9(R_{\text{heat}}+R_{\text{scat}})^2}$. Now time-averaged power loses are $P_{\text{heat}} = R_{\text{heat}}\langle I^2\rangle = \frac{8R_{\text{heat}}R^2}{9(R_{\text{heat}}+R_{\text{scat}})^2}E_0^2$ and $P_{\text{scat}} = \frac{8R_{\text{scat}}R^2}{9(R_{\text{heat}}+R_{\text{scat}})^2}E_0^2 = \frac{R_{\text{scat}}}{R_{\text{heat}}}\langle P_{\text{heat}}\rangle$.	1.2
2.10b	Starting with the electric field amplitude $E_0 = \sqrt{2S/(\varepsilon_0 c)} = 27.4\ \text{kV/m}$, we calculate $P_{\text{heat}} = 6.82\ \text{nW}$ and $P_{\text{scat}} = 6.81\ \text{nW}$.	0.3

Steam generation by light

2.11a	Total number of nanoparticles in the vessel: $N_{\text{np}} = h^2 a\, n_{\text{np}} = 7.3 \times 10^{11}$. Then the total time-averaged Joule heating power: $P_{\text{st}} = N_{\text{np}}P_{\text{heat}} = 4.98\ \text{kW}$. This power goes into the steam generation: $P_{\text{st}} = \mu_{\text{st}}L_{\text{tot}}$, with $L_{\text{tot}} = c_{\text{wa}}(T_{100} - T_{\text{wa}}) + L_{\text{wa}} + c_{\text{st}}(T_{\text{st}} - T_{100}) = 2.62 \times 10^6\ \text{J kg}^{-1}$. Thus the mass of steam produced in second μ_{st}: $\mu_{\text{st}} = \frac{P_{\text{st}}}{L_{\text{tot}}} = 1.90 \times 10^{-3}\ \text{kg s}^{-1}$.	0.6
2.11b	The power of light incident on the vessel $P_{\text{tot}} = h^2 S = 0.01\text{m}^2 \times 1\ \text{MW m}^{-2} = 10.0\ \text{kW}$, and the power directed for steam production by nanoparticles is given in 2.11a. Thus $\eta = \frac{P_{\text{st}}}{P_{\text{tot}}} = \frac{4.98\ \text{kW}}{10.0\ \text{kW}} = 0.498$.	0.2

	Total	**12.0**