# Functional Equations Toolbox

## 1 General Instructions

Most of these are adapted from Dr. Small's *"Get Fit" Functional Equation Mental Workout*.

1. DON't FORGET TO PLUG IN YOUR SOLUTION TO SEE IF IT WORKS!
2. **Know the Basics** Get to know the functional relationships satisfied by the usual suspects such as linear functions, trigonometric functions, logarithms, powers, *etc.* Don't forget reciprocals and functions of the form $1/(x + a)$.
3. **Small Values** Try solving for a few values such as $f(0)$, $f(1)$, $f(-1)$ and so on. If you can find enough of these you may be able to recognize the form of the function.
4. **Special Solutions** Try plugging in a few simple functions such as linear functions or polynomials and solving for the coefficients.
5. **General Behaviour** Try to gain further insight about the general behaviour of the function:
    (a) Is it strictly positive, strictly negative, or does it have zeros for certain values of $x$?
    (b) Is the function surjective, injective, or bijective?
    (c) Is the function monotone (increasing or decreasing)?
    (d) What happens for very small or very large values of $x$? Is the function unbounded?
6. **Linearize** Watch out for the opportunity to linearize the equation.
7. **Composites** If the equation involves $f[\alpha(x)]$, investigate the splinter $\alpha^n(x)$. If $\alpha^m(x) = x$ for some $m$, it may be possible to solve for $f(x)$ by applying the equation $m$ times.
8. **Recurrence Relations** Sequences and recurrence relations may be helpful in solving functional equations involving expressions of the form $f(f(x))$.
9. **Periodicity** Watch out for periodic functions, *i.e.*, those for whch $f(x + a) = f(x)$ for some positive $a$. A statement of periodicity is itself a functional equation, and can sometimes be derived from the given functional equation.
10. **Domains** Watch out for multiplicative or additive functions defined on the natural numbers. These functions do not generally behave the same way as multiplicative or additive functions of real numbers.
11. DON'T FORGET TO PLUG IN YOUR SOLUTION TO SEE IF IT WORKS!

## 2 Functions on the integers

**Notation** We will write $\mathbb{N}$ for the positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the nonnegative integers, and $\mathbb{Z}$ for the set of all integers.

Useful properties of the positive integers:

1. each of its subsets is bounded below and has a smallest element;
2. every positive integer has a unique decomposition as a product of powers of primes;
3. induction.

These properties imply that functions defined on the positive integers satisfy certain properties that do not apply to their real counterparts.

**Result** *If $f, g: \mathbb{N} \to \mathbb{N}$ satisfy $f(n) \geq g(n)$ for all $n$ and moreover, $f$ is surjective and $g$ is bijective, then $f = g$.*

*Proof* Assume that $f \neq g$, then there is a smallest value of $n_0$ such that $f(n_0) > g(n_0)$. Let $M = g(n_0)$, then $A = \{k | g(k) \leq M\}$ has exactly $M$ elements since $g$ is bijective. On the other hand, consider $B = \{k | f(k) \leq M\}$. Since $f(n_0) > g(n_0)$, it follows that $n_0 \notin B$. Also, since $f(n) \geq g(n)$ for all $n$, it follows that $B \subset A$. So $B$ has at least one element less than $A$. Hence the values of $f$ do not exhaust all numbers less than $M + 1$, which contradicts the surjectivity of $f$. We conclude that $f = g$.

*Problem 1.* Find all functions $f: \mathbb{N}_0 \to \mathbb{N}_0$ with the property that for all $n \geq 0$,

$$f(n) \geq n + (-1)^n.$$

### 2.1 Induction

Functions defined on the positive integers can be considered as sequences and induction is often a very useful tool. Also, plugging in small values and finding patterns works very well for this type of functions.

*Problem 2* Find all functions $f: \mathbb{N} \to \mathbb{N}$ satisfying $f(1) = 1$ and $f(m + n) = f(m) + f(n) + mn$ for all $m, n \in \mathbb{N}$.

*Problem 3* Find al functions $f: \mathbb{N} \to \mathbb{N}$ with the property that

$$f(f(m) + f(n)) = m + n, \text{ for all } m \text{ and } n.$$

*Problem 4* Find all pairs of functions $f, g: \mathbb{N}_0 \to \mathbb{N}_0$ satisfying

$$f(n) + f(n + g(n)) = f(n + 1).$$

## 2.2 Domain and Range

We saw already that knowing that a function is surjective can be very helpful. Here is an example that shows that determining the range of a function can also give new insight:

*Example* Find all functions $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f(m + f(n)) = f(f(m)) + f(n) \text{ for all } m, n \geq 0.$$

*Solution* First look for easy special cases: $f = 0$ is a solution. In the rest of this solution we will assume that $f$ is not identically 0.

Plug in small values: for $m = n = 0$ we obtain $f(f(0)) = f(f(0)) + f(0)$, so $f(0) = 0$ for any solution. Setting only $m = 0$ (or only $n = 0$), we find that $f(f(n)) = f(n)$ for all $n \geq 0$.

This is a situation where it is helpful to find out what the range $T$ of the function is. By our last observation we have that $T = \{n | n = f(n)\}$. Is $T$ closed under addition or subtraction? Suppose that $m, n \in T$. Then $f(m+n) = f(m + f(n)) = f(f(m)) + f(n) = f(m) + f(n) = m + n$, so $m + n \in T$. Also, if $n > m$, then $n = f(n) = f(m+n-m) = f(f(m)+n-m) = f(f(n-m))+f(m) = f(n-m)+m$, so $f(n-m) = n-m$ and $n-m \in T$. It is now a standard result that $T$ consists of all multiples of its smallest positive element, say $a$.

We conclude from this that $f(n)$ is a multiple of $a$ for all $n \geq 0$. Write $f(n) = k_n \cdot a$, and consider $k_1, \ldots, k_{a-1}$. Any postive integer $m$ can be written as $m = qa + r$ where $0 \leq r < a$, which can be used as follows

$$f(m) = f(qa + r) = f(r + f(qa)) = f(f(r)) + f(qa) = k_r \cdot a + qa = (k_r + q)a.$$

Finally we need to check whether functions of this form satisfy the equation: Let $m = qa + r$ and $n = sa + t$, with $0 \leq r, t < a$. Then

$$\begin{aligned} f(m + f(n)) &= f(qa + r + sa + k_t a) \\ &= qa + sa + k_t a + k_r a \\ &= f(qa + k_r a) + f(sa + t) \\ &= f(f(m)) + f(n). \end{aligned}$$

Hence the desired functions are $f = 0$ and all functions of the form described above.

*Problem 5* Let $f\colon \mathbb{N} \to \mathbb{N}$ be such that

$$f(n + 1) > f(f(n)) \text{ for all } n \in \mathbb{N}.$$

Prove that $f(n) = n$ for all $n \in \mathbb{N}$.

*Problem 6* Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ with the property that for all $n \in \mathbb{N}$,

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(f(n))}{f(n+1)}.$$

# 3 Iterative Roots and Composition of Functions

Let $h(x)$ be a given function. Any solution to the equation

$$f(f(x)) = h(x)$$

is called an *iterative square root* of the function $h(x)$. Analogously, a solution to the equation

$$f^n(x) = h(x)$$

is an *iterative $n$-th root* of $h(x)$. In the particular case where $h(x) = x$, this is called the *Babbage equation.* A solution of the Babbage equation with $n = 2$ is called an *involution.* Involutions are far from unique. Some examples are $f(x) = -x$, but also $f(x) = a - x$ for any real number $a$, and in general, any function of the form

$$f(x) = g^{-1}(-g(x)),$$

will be an involution.

## 3.1 Injectivity and Surjectivity

Iterative roots $f(x)$ as above inherit many properties from the function $h(x)$. For example, if $h$ is injective, so is $f$, and if $h$ is surjective, so is $f$. Note that if the functions are assumed to be continous, injectivity implies strict monotonicity. These properties can sometimes be used to prove that a function does not have an iterative root.

**Proposition** *Let $h$ be a one-to-one function.*

1. *Suppose that $n$ is even and that there exists a pair of real numbers $x < y$ such that $h(x) > h(y)$. Then there is no continuous iterative $n$th root of $h$.*
2. *Suppose that $n$ is odd, and that $f$ is a continuous iterative $n$th root of $h$. Then $h$ is continuous and strictly increasing or strictly decreasing as $f$ is strictly increasing or strictly decreasing respectively.*

*Proof* Left to the reader.

*Problem 7* Prove that there exists no function $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = n + 1987 \text{ for all } n \in \mathbb{N}.$$

*Problem 8* Establish necessary and sufficient conditions on the constant $k$ for the existence of a continuous real-valued function $f(x)$ satisfying

$$f(f(x)) = kx^9$$

for all real $x$.

### 3.2 Construction of Iterative Roots

One way to construct iterative roots for a function $h(x)$ is as follows. If the Abel equation

$$g(h(x)) = g(x) + 1$$

has a one-to-one solution $g(x)$, define

$$f(x) = g^{-1}\left(g(x) + \frac{1}{n}\right).$$

It can be checked that $f(x)$ is an iterative $n$th root of $h(x)$.

Another way to find iterative roots is as follows. If $g(x)$ is a one-to-one solution to the equation

$$g(h(x)) = sg(x)$$

for some $s \neq 1$, then

$$f(x) = g^{-1}\left(s^{1/n} g(x)\right)$$

is an iterative $n$th root of $h(x)$.

A completely different approach to solving such equations is to apply $f$ to both sides of the equation. For example the equation $f^n(x) = h(x)$ can be transformed into

$$f(h(x)) = f(f^n(x)) = f^n(f(x) = h(f(x)).$$

However, this is a place where one has to be very careful. If the original system had a solution, so will the new system. However, the converse is certainly not true, and it is very important to plug in your solution to see whether it works!

*Problem 9* Find all integer valued functions $f(n)$ taking values in the integers satisfying the equation $f(f(n)) = n + 1$.

This method can also be applied to the composition of two or more different functions. For example, if we have the simultaneous equations

$$f(g(x) = h_1(x) \text{ and } g(f(x)) = h_2(x)$$

where $h_1$ and $h_2$ are given, and $f$ and $g$ are unknown, we can apply $g$ to the first equation to obtain

$$g(h_1(x)) = g(f(g(x))) = h_1(f(x)).$$

*Problem 10* Do there exist functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \text{ and } g(f(x)) = x^3$$

for all $x \in \mathbb{R}$?

*Problem 11* Determine whether there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = n^2 - 19n + 99,$$

for all positive integers $n$.

### 3.3 Recurrence Relations

If the function $h(x)$ is expressed in terms of $f(x)$, it can be helpful to construct a recurrence relation: $a_0 = x$ and $a_n = f(a_{n-1})$ and use the functional equation to express $a_n$ in terms of the preceding terms.

*Problem 12* Prove that there exists a unique function $f$ defined on the positive reals such that $f(f(x)) = 6x - f(x)$ and $f(x) > 0$ for all positive $x$.

# 4 Linearization

Sometimes functional equations involving $f(x^n)$ or $(f(x))^n$ can be turned into linear functional equations with the use of logarithms and exponentials in the appropriate places.

*Example* Find a solution to the equation

$$f(x^2) - f(x) = 1, \text{ for } x > 1.$$

*Solution* Let $F(x) = f(a^x)$ for some $a > 0$, or equivalently, $F(\log_a x) = f(x)$. Then for $x > 0$ the function satisfies

$$F(2x) - F(x) = 1 \text{ for all } x > 0.$$

This equation reminds us of the properties of a logarithm. The solution of this second equation is $F(x) = \log_2(x)$. So

$$f(x) = log_2 log_a(x)$$

satisfies the first equations for all $a > 0$.

*Example* Find a solution to the equation

$$f(x+1) = (f(x))^2.$$

*Solution* Assume that $f(x) \neq 0$ for all $x$. In order for the equation to be true for all $x$ we have that $f(x) > 0$ for all $x$. So we can make the equation linear by setting $F(x) = \log_a f(x)$ with $a > 0$. The equation for $F(x)$ becomes $F(x+1) = 2F(x)$, which has $F(x) = 2^x$ as a solution. So a solution to the original equation is

$$f(x) = a^{a^x}.$$

*Problem 13* Do there exist functions $f\colon \mathbb{R} \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x^2 \text{ and } g(f(x)) = x^4$$

for all $x \in \mathbb{R}$?

# 5 Nested Radicals

*Problem 14* Calculate

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\cdots}}}}.$$

*Problem 15* Calculate

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{\cdots}}}}.$$

# 6 (Dys)functional Equation Problems

1. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$f^{(19)}(n) + 97f(n) = 98n + 232.$$

2. Find all $f\colon \mathbb{Q} \to \mathbb{Q}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$ and $f(1) = 2005$.

3. Find all $f\colon \mathbb{N} \to \mathbb{N}$ such that $xf(y) + yf(x) = (x+y)f(x^2+y^2)$ for all $x, y \in \mathbb{N}$.

4. Find all functions $u\colon \mathbb{R} \to \mathbb{R}$ for which there exists a strictly monotonic function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x)u(y) + f(y) \text{ for all } x, y \in \mathbb{R}.$$

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that satisfies
   (a) $f(x+y) + f(x-y) = 2f(x)f(y)$, for all $x, y \in \mathbb{R}$.
   (b) There exists $x_0$ with $f(x_0) = -1$.

   Prove that $f$ is periodic.

6. Find a bijective function $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that for all $m, n$,

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$

7. Find all functions $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ satisfying the following conditions:
   (a) For any $m, n \in \mathbb{N}_0$,

$$2f(m^2+n^2) = (f(m))^2 + (f(n))^2.$$

   (b) For any $m, n \in \mathbb{N}_0$ with $m \geq n$,

$$f(m^2) \geq f(n^2).$$