Complex Numbers in Geometry

Solutions to Exercises

1. Given $|z| = 1$, find $|z-1|^2 + |z+1|^2$.

   Solution. We have that

$$\begin{aligned} |z-1|^2 + |z+1|^2 &= (z-1)(\overline{z-1}) + (z+1)(\overline{z+1}) \\ &= (z-1)(\overline{z}-1) + (z+1)(\overline{z}+1) \\ &= z\overline{z} - z - \overline{z} + 1 + z\overline{z} + z + \overline{z} + 1 \\ &= 2z\overline{z} + 2 \\ &= 2|z|^2 + 2 \\ &= 4. \end{aligned}$$

   We can interpret this result geometrically as follows: Let A, B, and Z be the points corresponding to the complex numbers 1, -1, and z, respectively. Since Z lies on the unit circle, $\angle AZB = 90^\circ$, so

$$|z-1|^2 + |z+1|^2 = AZ^2 + BZ^2 = AB^2 = 4.$$

2. Let a and b be distinct complex numbers. Show that z lies on the perpendicular bisector of a and b if and only if

$$(\overline{a} - \overline{b})z + (a-b)\overline{z} = |a|^2 - |b|^2.$$

   Solution. We know that z lies on the perpendicular bisector of a and b if and only if z is equidistant from a and b, so we get

$$\begin{aligned} & |z-a|^2 = |z-b|^2 \\ \Leftrightarrow\quad & (z-a)(\overline{z-a}) = (z-b)(\overline{z-b}) \\ \Leftrightarrow\quad & (z-a)(\overline{z}-\overline{a}) = (z-b)(\overline{z}-\overline{b}) \\ \Leftrightarrow\quad & z\overline{z} - \overline{a}z - a\overline{z} + a\overline{a} = z\overline{z} - \overline{b}z - b\overline{z} + b\overline{b} \\ \Leftrightarrow\quad & (\overline{a}-\overline{b})z + (a-b)\overline{z} = |a|^2 - |b|^2. \end{aligned}$$

3. Describe all triangles whose vertices a, b, and c satisfy

$$\frac{1}{b-c} + \frac{1}{c-a} + \frac{1}{a-b} = 0.$$

   Solution. Multiplying the given equation by $(b-c)(c-a)(a-b)$, we get

$$\begin{aligned} & (c-a)(a-b) + (a-b)(b-c) + (b-c)(c-a) = 0 \\ \Leftrightarrow\quad & a^2 + b^2 + c^2 - ab - ac - bc = 0. \end{aligned}$$

   Hence, the only triangles that satisfy the given condition are equilateral triangles.

4. The points $(0,0)$, $(a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of ab. (AIME, 1994)

Solution. Let $z_1 = 0$, $z_2 = a + 11i$, and $z_3 = b + 37i$. We are given that z_1, z_2, and z_3 form the vertices of an equilateral triangles, so

$$z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_1 z_3 - z_2 z_3 = 0,$$

which simplifies as

$$(a^2 - ab + b^2 - 1083) - (15a - 63b)i = 0.$$

Hence,

$$\begin{aligned} a^2 - ab + b^2 &= 1083, \\ 15a &= 63b. \end{aligned}$$

From the second equation,

$$b = \frac{15}{63}a = \frac{5}{21}a,$$

or

$$a = \frac{21}{5}b,$$

so the first equation becomes

$$a^2 - ab + b^2 = \frac{21}{5}ab - ab + \frac{5}{21}ab = 1083.$$

Therefore,

$$ab = \frac{1083}{21/5 - 1 + 5/21} = \frac{1083}{361/105} = 315.$$

5. Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let m/n be the probability that $\sqrt{2+\sqrt{3}} \le |v+w|$, where m and n are relatively prime positive integers. Find $m+n$. (AIME, 1997)

Solution. Given such complex numbers v and w, multiplying them by any 1997^{th} root of unity does not change the value of $|v+w|$ (and v and w are still 1997^{th} roots of unity). Hence, without loss of generality, we can assume that $v = 1$. Let $w = e^{i\theta}$, where $\theta = 2k\pi/1997$ for some integer k, $1 \le k \le 1996$. (Note that $k \neq 0$, since v and w are distinct.)

Then

$$\begin{aligned} |v+w| &= |1 + e^{i\theta}| \\ &= |(1+\cos\theta) + i\sin\theta| \\ &= \sqrt{(1+\cos\theta)^2 + (\sin\theta)^2} \\ &= \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta} \\ &= \sqrt{2 + 2\cos\theta}. \end{aligned}$$

Hence,

$$\begin{aligned} |v+w| &\geq \sqrt{2+\sqrt{3}} \\ \Leftrightarrow \quad \sqrt{2+2\cos\theta} &\geq \sqrt{2+\sqrt{3}} \\ \Leftrightarrow \quad 2+2\cos\theta &\geq 2+\sqrt{3} \\ \Leftrightarrow \quad \cos\theta &\geq \frac{\sqrt{3}}{2}. \end{aligned}$$

Since $0 < \theta < 2\pi$, $\cos\theta \geq \sqrt{3}/2$ if and only if $0 < \theta \leq \frac{\pi}{6}$ or $\frac{11\pi}{6} \leq \theta < 2\pi$.

The inequality $0 < \theta \leq \frac{\pi}{6}$ is equivalent to

$$0 < \frac{2k\pi}{1997} \leq \frac{\pi}{6},$$

or

$$0 < k \leq \frac{1997}{12}.$$

This is satisfied by $k = 1, 2, 3, \dots, 166$, for 166 values.

The inequality $\frac{11\pi}{6} \leq \theta < 2\pi$ is equivalent to

$$\frac{11\pi}{6} \leq \frac{2k\pi}{1997} < 2\pi,$$

or

$$\frac{21967}{12} \leq k < 1997.$$

This is satisfied by $k = 1831, 1832, 1833, \dots, 1996$, for another 166 values.

Therefore, the probability that $\sqrt{2+\sqrt{3}} \leq |v+w|$ is

$$\frac{166+166}{1996} = \frac{332}{1996} = \frac{83}{499},$$

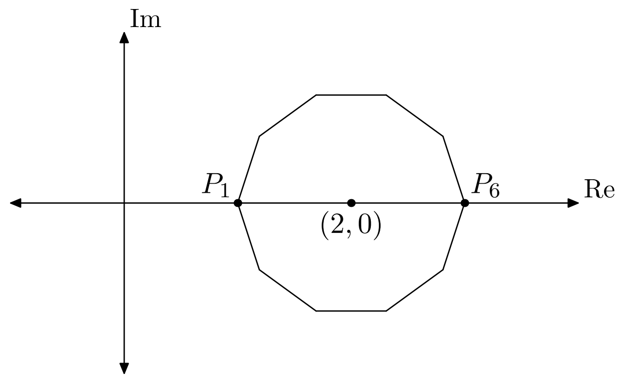
and the answer is $83 + 499 = 582$.

6. Regular decagon $P_1P_2\cdots P_{10}$ is drawn in the coordinate plane, with $P_1 = (1,0)$ and P_6 at $(3,0)$. If P_n is the point (x_n, y_n), compute the numerical value of the product

$$(x_1 + y_1 i)(x_2 + y_2 i)(x_3 + y_3 i)\cdots(x_{10} + y_{10} i).$$

(ARML, 1994)

Solution. Let $p_k = x_k + y_k i$ for $1 \leq k \leq 10$.



Then p_1, p_2, $\ldots$, p_{10} are the roots of the polynomial $(z-2)^{10}=1$, which expands as

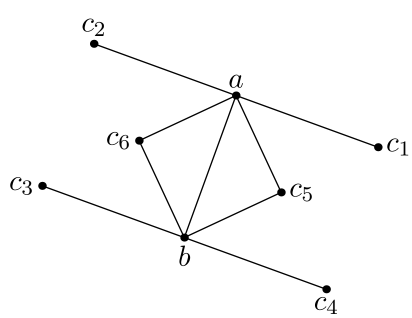
$$z^{10}-2\binom{10}{1}z^9+4\binom{10}{2}z^8-\cdots+2^{10}-1=0.$$

By Vieta's Formulas, the product of the roots is

$$p_1p_2\cdots p_{10}=2^{10}-1=1023.$$

7. In the complex plane, z, z^2, z^3 form, in some order, three of the vertices of a non-degenerate square. Let a and b represent the smallest and largest possible areas of the squares, respectively. Compute the ordered pair (a,b). (ARML, 2008)

   Solution. Given complex numbers a and b, there are six possible complex numbers c such that a, b, and c form three of the vertices of a square, namely c_1, c_2, c_3, c_4, c_5, and c_6, as shown below.



The complex number c_1 can be obtained by rotating b 90° counter-clockwise around a, so $c_1-a=i(b-a)$, or

$$c_1=(1-i)a+ib.$$

Similarly,

$$\begin{aligned} c_2 &= (1+i)a - ib, \\ c_3 &= ia + (1-i)b, \\ c_4 &= -ia + (1+i)b. \end{aligned}$$

The complex number c_5 is the midpoint of a and c_4, so

$$c_5 = \frac{a + c_4}{2} = \frac{1-i}{2}a + \frac{1+i}{2}b.$$

Similarly,

$$c_6 = \frac{a + c_3}{2} = \frac{1+i}{2}a + \frac{1-i}{2}b.$$

We have that

$$c_1 + c_2 = (1-i)a + ib + (1+i)a - ib = 2a,$$

and

$$c_1 c_2 = [(1-i)a + ib][(1+i)a - ib] = 2a^2 - 2ab + b^2,$$

so by Vieta's Formulas, c_1 and c_2 are the roots of the quadratic

$$c^2 - 2ac + 2a^2 - 2ab + b^2 = 0.$$

Taking $a = z$, $b = z^2$, and $c = z^3$, this equation simplifies to

$$z^6 - z^4 - 2z^3 + 2z^2 = z^2(z-1)^2(z^2 + 2z + 2) = 0.$$

Since the square is non-degenerate, z must be a root of $z^2 + 2z + 2 = 0$. The roots of this quadratic are $z = -1 \pm i$. The area of the square is $|a - b|^2 = |z - z^2|^2$. For both roots, the area of the square is $|z - z^2|^2 = 10$.

Similarly,

$$c_3 + c_4 = ia + (1-i)b - ia + (1+i)b = 2b,$$

and

$$c_3 c_4 = [ia + (1-i)b][-ia + (1+i)b] = a^2 - 2ab + 2b^2,$$

so by Vieta's Formulas, c_3 and c_4 are the roots of the quadratic

$$c^2 - 2bc + a^2 - 2ab + 2b^2 = 0.$$

Taking $a = z$, $b = z^2$, and $c = z^3$, this equation simplifies to

$$z^6 - 2z^5 + 2z^4 - 2z^3 + z^2 = z^2(z-1)^2(z^2+1) = 0.$$

Since the square is non-degenerate, z must be a root of $z^2 + 1 = 0$. The roots of this quadratic are $z = \pm i$. The area of the square is $|a - b|^2 = |z - z^2|^2$. For both roots, the area of the square is $|z - z^2|^2 = 2$.

Finally,

$$c_5 + c_6 = \frac{1-i}{2}a + \frac{1+i}{2}b + \frac{1+i}{2}a + \frac{1-i}{2}b = a + b,$$

and

$$c_5 c_6 = \left(\frac{1-i}{2}a + \frac{1+i}{2}b\right)\left(\frac{1+i}{2}a + \frac{1-i}{2}b\right) = \frac{a^2+b^2}{2},$$

so by Vieta's Formulas, c_5 and c_6 are the roots of the quadratic

$$c^2 - (a+b)c + \frac{a^2+b^2}{2} = 0.$$

Taking $a = z$, $b = z^2$, and $c = z^3$, this equation simplifies to

$$\frac{1}{2}(2z^6 - 2z^5 - z^4 + z^2) = \frac{1}{2}z^2(z-1)^2(2z^2+2z+1) = 0.$$

Since the square is non-degenerate, z must be a root of $2z^2 + 2z + 1 = 0$. The roots of this quadratic are $z = -\frac{1}{2} \pm \frac{1}{2}i$. The area of the square is $\frac{1}{2}|a-b|^2 = \frac{1}{2}|z - z^2|^2$. For both roots, the area of the square is $\frac{1}{2}|z-z^2|^2 = \frac{5}{8}$.

Therefore, the smallest possible area of the square is $a = 5/8$, and the largest possible area of the square is $b = 10$.

8. Let A, B, and C be three points, with respective affixes a, b, and c. Show that the signed area of triangle ABC is given by

$$\frac{i}{4}(a\overline{b} + b\overline{c} + c\overline{a} - \overline{a}b - \overline{b}c - \overline{c}a).$$

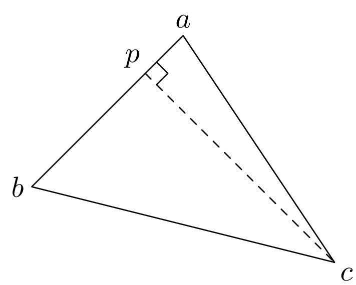
(The area is signed because this formula returns a positive real number when triangle ABC is oriented counter-clockwise, and a negative real number when triangle ABC is oriented clockwise.)

Solution. Let P be the projection of C onto AB, so

$$p = \frac{(\overline{a}-\overline{b})c + (a-b)\overline{c} + \overline{a}b - a\overline{b}}{2(\overline{a}-\overline{b})} = \frac{-a\overline{b} + \overline{a}b + a\overline{c} + \overline{a}c - b\overline{c} - \overline{b}c}{2(\overline{a}-\overline{b})}$$

Then

$$p - c = \frac{-a\overline{b} + \overline{a}b + a\overline{c} - \overline{a}c - b\overline{c} + \overline{b}c}{2(\overline{a}-\overline{b})}.$$



Assume that triangle ABC is oriented counter-clockwise, so $p - c = t(a - b)i$ for some positive real number t. Let $a - b = b_T e^{i\theta}$ and $p - c = h_T i e^{i\theta}$, where b_T and h_T represent the base and height of the triangle (as positive real numbers), respectively. Then

$$h_T = \frac{p - c}{ie^{i\theta}},$$

and $\overline{a - b} = \overline{b_T e^{i\theta}}$, so $\overline{a} - \overline{b} = b_T e^{-i\theta}$, which means

$$b_T = (\overline{a} - \overline{b})e^{i\theta}.$$

Therefore, the area of triangle ABC is

$$\begin{aligned}
\frac{1}{2} b_T h_T &= \frac{1}{2} \cdot (\overline{a} - \overline{b})e^{i\theta} \cdot \frac{p - c}{ie^{i\theta}} \\
&= \frac{1}{2i}(\overline{a} - \overline{b})(p - c) \\
&= \frac{1}{2i} \cdot \frac{-a\overline{b} + \overline{a}b + a\overline{c} - \overline{a}c - b\overline{c} + \overline{b}c}{2} \\
&= \frac{1}{4i}(-a\overline{b} + \overline{a}b + a\overline{c} - \overline{a}c - b\overline{c} + \overline{b}c) \\
&= \frac{i}{4}(a\overline{b} + b\overline{c} + c\overline{a} - \overline{a}b - \overline{b}c - \overline{c}a).
\end{aligned}$$

As a corollary, it follows that the complex numbers a, b, and c are collinear if and only if

$$a\overline{b} + b\overline{c} + c\overline{a} - \overline{a}b - \overline{b}c - \overline{c}a = 0.$$