THE ART AND CRAFT
OF PROBLEM SOLVING

INSTRUCTOR'S MANUAL

Second Edition

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JOHN WILEY & SONS, INC.

Contents

Chapter 1

Teaching Problem Solving

I wrote *The Art and Craft of Problem Solving* (TAACOPS) as a textbook for my problem-solving seminar at the University of San Francisco. My target audience is bright but naive college students who most likely are not attending elite universities. I believe that TAACOPS can be easily adapted to teach problem solving to a wide range of people, from gifted high school students to graduate students in education. TAACOPS was designed to be friendly enough and easy enough to use for beginning teachers of problem solving, that is, math teachers who have always wanted to teach problem solving but have not done so for lack of a proper textbook and the intimidating nature of the subject. The brief comments below are meant for these teachers. What follows are merely a few things that I have learned from my own experiences. Obviously, every teacher has a different style, so take what I say with many grains of salt. Unless otherwise stated, all page and example references are from TAACOPS, not this manual.

1.1 Preparation

If you have never taught a problem solving class, it can be rather scary. Problems by their nature are not easy to solve and teachers by their nature prefer a high level of omniscience. You will relinquish at least some of your sage status no matter how well your course goes, but doing your homework during the summer will go a long way. (Incidentally, it is best to teach your course in the fall semester if you plan to make use of the Putnam Exam. And it is nice to have a summer to prepare.)

You need to become a better problem solver before your course begins. Not a great problem solver, but preferably at least as good as your good students (perhaps not as good as your best ones). The ideas in TAACOPS work just as well with teachers and professors as they do with students, and of course you already have the advantage of knowing lots of mathematics and knowing how to concentrate. You should discover that a modest effort at mastering the strategies and tactics of Chapters 1–4 will dramatically improve your "solve rate" and your confidence. And please take the advise on p. 20 seriously. The more ideas you appropriate, the better.

As you work on problems, start collecting favorite exemplary ones, which may or may not be worked examples in the book (do use the solutions in this manual as

well). You want to have a nice store of linked problems so that you can motivate your students.

Besides immersing yourself in problems from TAACOPS, you must start assembling a small library of problems and related literature. Even though TAACOPS has over seven hundred problems, you will need a few more. Here are the essential ones (incentive: you will need some of these to find solutions to some of the problems in TAACOPS).[1]

Putnam Exams   You need the problems and solutions from at least 1980. There are three books [8, 1, 14] which have problems and solutions from 1938–64, 1964–84, and 1985–2000, respectively. The most recent book is especially good. For exams after 2000 a web search should yield exams and solutions.

USAMO and IMO Exams   There are two books [10, 15] which respectively cover the periods from 1959–77 and 1978–85. For exams after, order pamphlets from the American Mathematics Competitions (AMC) at `http://www.unl.edu/amc`.

AIME Exams   Pamphlets are available starting from 1982 from the American Mathematics Competitions.

USSR Exams   The mother of all problem books is the *USSR Olympiad Problem Book* [22], which is now available in a Dover paperback, I believe. It contains many easy-to-moderate problems, some very hard ones, and lots of detailed solutions and discussion of techniques.

Math Circles   Another Russian import is *Mathematical Circles* [4], a guidebook for teaching young kids (mostly elementary school to early high school). But the mathematics is deep and fascinating. In many ways, this is a perfect companion book to TAACOPS, for high school and junior high school teachers. Reading TAACOPS should prepare the beginning teacher to utilize *Mathematical Circles* very effectively to teach younger students.

Recreational Problems   Several books by Martin Gardner (as a great source of recreational problems).

In addition, here are some other good sources of problems which often include useful instructional material.

- The Hungarian problem books [16, 17, 18], which contain the oldest olympiad-style contests and much useful elementary material. The most recent book is especially recommended.
- For relatively easy problems, the Canadian Olympiad collection [5]. For an unusually varied set of elementary (but hard) questions, try the Leningrad Olympiad collection [6]. Another fine collection, with great variety (mostly moderate level) is *500 Mathematical Challenges* [2]. This book of mostly Canadian problems is notable for its compilation of useful tools (called "The Toolchest") in an appendix. For a small collection of imaginative college-level problems, try the *Wohascum County Problem Book* [7].

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[1]The references are cited at the end of this manual, not the bibliography of TAACOPS.

- The problem sections in the journals mentioned on p. 8 of TAACOPS.
- The many essay collections by Ross Honsberger, most having titles like *Mathematical Gems*, for example [13]. These are nicely written discussions of fascinating problems (great for beginning a class or a course).
- Ravi Vakil's enjoyable *A Mathematical Mosaic* [24] is excellent for its well-chosen "folklore" topics.

If you are worried about course enrollment (perhaps your school has never offered such a course before, or offered it only sporadically), you may want to talk up your favorite problems to students as a recruitment tool. I have had lots of success in recruiting students from Linear Algebra or Calculus II courses by coming in with a 5-minute spiel that usually includes some combination of Examples 1.1.3, 1.1.4, 1.3.12, and 1.3.16.

It is also a good idea to decide ahead of time what kind of course you want it to be. Do you want to help students do well on the Putnam Exam? Or do you want to get beginners comfortable with concentrating hard and investigating problems? I have found that low expectations are best, for then you may be pleasantly surprised. And I have never had much difficulty with a mix of abilities (see the discussion on group work below).

1.2 One Plan for a Problem-Solving Seminar

I advocate a seminar-style course, one with limited enrollment (preferably no more than 20) and longer meeting times (for example, 90 minutes twice a week instead of 60 minutes thrice weekly). Problem solving is not something that can be lectured about, at least not all the time.

The basic structure of the seminars that I have conducted is not very elaborate: usually we spend the first half of class time discussing problems (students working at the board), and I wrap up the discussion with either a canned lecture (rarely) or a variation on a theme presented by the students. For example, suppose a student solved a combinatorial problem in a nice way. I may decide to use this as an entry point to discuss generating functions (something most students have never seen before). Or I will conclude from the low level of rigor of the last proof (all too common!) that we should review some proof-logic techniques, etc. My biggest challenge in the classroom is getting shy students to come to the board and shutting up the ones who hog the board. Of course, that is the main problem with most seminar-style courses. I try to do my homework by selecting the right mix of problems that are interesting, and at just the right level of difficulty for the moment. And each year, as I learn more and read more and work on more problems, I get better at improvising.

Besides the standard seminar environment, there are a few non-traditional features, described below.

Group Work

I cannot recommend this highly enough! I spend a fair amount of time agonizing over group assignments, and organize the students into groups of 3 or 4 people. This seems

to be the ideal size; two-person groups are not fluid enough, and larger groups often cannot meet at the same time. My groups are chosen so that they will function well. I don't worry about making groups of equal "strength," although this is desirable, but instead try to make sure that the individuals interact harmoniously. The perfect group has

- People who enjoy each other's company (no romances, please).
- People with compatible schedules (often the critical property among today's overworked students).
- Fairly homogeneous math skills/ability.
- Mostly homogeneous introvert/extrovert mix, unless this offsets a mathematical imbalance. For example, a group of fairly shy people works well, certainly much better than one with several shy people plus one dominant type, *unless* the dominant person is perceived to be one of the weaker mathematicians.
- People who usually, but not exclusively, play distinct roles. For example, it is good to have one student who is great at visualization and one who leans towards computation instead. It is good to have a dreamy creative type, even if she is not that skilled/rigorous, if one of her partners is a less imaginative but better-trained logician. Many successful groups function like sports teams, with starters, who ask the questions, break the rules, and get the investigation going; power players, who come up with the crux move more often than not; and closers, who can carefully critique, improve, and ultimately complete a creative but floundering argument. The very best groups, I have found, have people who mostly play distinct roles, but who are capable of switching positions as well.

The things to avoid in groups are putting too many close friends together, and avoiding imbalanced groups where a brilliant extrovert is together with several shy weaker students. Such groups are not doomed to failure, but may require intervention. Sometimes a flawed group can be fixed by exchanging people between groups. Sometimes there is nothing that can be done. But we teachers are used to imperfection. Generally, if 4 out of 5 groups function well, I am thrilled and know that realistically this is a good success rate. See appendix B for a true story about how much a group can do, an experience which inspired me in part to write TAACOPS.

Homework

Assignments come in many flavors: reading and problems to discuss by next class, individual work, group work (one paper per group), rewrites. Since the students are beginning proof writers (usually), rewrites are critical. I usually assign a grade (A,B,C or No Credit) to each problem, with a symbol like "/R" to indicate "please rewrite it by next week and then I will change the grade." It makes the recordkeeping difficult but is worth the trouble. I assign semi-official "backburner" problems for students to work on with no deadlines attached. In general, it pays to be very relaxed about deadlines, as long as the students are working hard. I know that my seminar is succeeding when I run into a group working at the blackboard in a deserted classroom late at night, arguing and laughing. Likewise, I am in heaven when I ask about a particular problem

and several students complain that they stayed up all night and didn't work on their history paper and they still didn't solve the problem.

A typical homework load would be something like 5–10 problems due per class on average, with a good mix of easy (designed secretly to boost confidence), medium, and hard problems. Usually at least half of the homework load at the beginning of the course is group work. This is important for keeping up the confidence level. Generally groups outperform individuals, so the group is bound to solve some impressive problems. The weaker individuals will be proud of their group's accomplishments, which will help them with their own confidence. Of course it is your job to spot these individuals and nurture them.

It is important to start slowly, since it is much easier to build confidence gradually, and this is the bottleneck. It is not unlike sports: if you try to do too much, you risk injury. A few too many super-hard problems can easily crush a neophyte's ego beyond recovery. A good way to avoid this is by relying on many recreational problems at the beginning. These problems are fun and deeper than they seem. Once you have seduced your students with the handshake problem (1.1.4) or the lightbulb problem (1.3.2) or the rope problem (2.1.19) or the professors problem (2.1.21), they will be willing to go to great mathematical lengths on later problems. On the other hand, there is nothing wrong with sprinking in a few killer problems even from the very first day. These "backburner" problems should probably be identified as such. For example, I often present on the first day of the course several problems like 1.3.10, 1.3.12, and 3.3.5, and usually at least 5 to 10 minutes of each class is devoted to progress reports on these problems. Since they are designated as super-hard, some students avoid them, but others are liberated by the lack of expectations. I have experimented with not telling students that a hard problem is hard, with mixed results.

Exams

Usually 3 to 4 exams are given, all take-home except for the Putnam Exam (see below). After a few weeks of foundational work (studying most of Chapters 2 and part of Chapter 3, with smatterings of other sections), I assign the first exam.[2] It has a time limit of about a week, and is to be entirely done in groups. The group hands in one paper, and all group members get the same grade. Additionally, on the day the exam is due, we have an oral exam, in which students are randomly chosen to go to the board without notes or help from their group partners. The student may choose any problem that has not yet been discussed, and present a solution in a few minutes. The presentation is judged (I usually invite colleagues from my department for this) Olympics-style (with cards numbered from 1–5), and polite but "hostile" questions are allowed by judges or members of other groups. These questions are designed to probe for misunderstanding, etc. The process is a bit scary, but done with good humor, and is designed so that everyone is obligated to understand as much as possible. An unbalanced group with a dominant member will not do well unless this person can teach his or her comrades. In practice, the highest oral grades go to the most cohesive

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[2]See appendix A for sample exams.

groups, not the groups of the best mathematicians. And of course, the group gets only one oral grade.

Almost always, the first exam achieves its purpose. Enough of the problems were easy enough so that everyone could feel successful, and enough were sufficiently challenging so that each group could develop an almost-military élan. Virtually every student experienced or at least witnessed the epiphany of seemingly sudden inspiration that in reality was fueled by many hours of sweat.

At this point, the students have been converted: almost all of them are now active problem solvers, and need now "only" to keep practicing their skills and accumulate new tactics and tools. Exam 2 begins, usually a month-long individualized process. I meet with each student separately to negotiate a collection of problems that they will solve, a mix of Putnam problems plus problems selected from one or two topics. This list of about 15 problems varies immensely from student to student. Some people are ready to tackle hard Putnam problems (say, the two or three hardest of the 12-question exam), while others will be lucky to solve the easiest Putnam problems. That doesn't matter, as long as the student and I can come to an accurate consensus of what will be the appropriate challenge.

Regular class discussion and homework continue while the students work on this exam, and towards the end of this period I usually do the most lecturing that I do during the semester (since the students are too busy doing their own problems to work hard on group assignments or public problems).

The second exam is hard for students. They are separated from their now-beloved group members and must deal with their own weaknesses. But again, if I have done my homework, at least some of the problems that each student solved were challenging enough so that they really felt a sense of accomplishment. Needless to say, I am not always successful, but I usually err on the side of making the exam too easy rather than too hard. And for some students (especially teachers or future teachers), I include problem composition assignments along with problem solving questions.

The third exam is the Putnam Exam, which is held on the first Saturday of December. My students are required to take this exam (or else substitute something so arduous that they always choose the Putnam). I usually agree to retroactively change anyone's grade to an A+ if they solve 2 or more problems (the results are not available until March). For most students, the Putnam does not affect their grade. But it is a valuable experience, one that we try to carefully prepare for, and we spend most of the rest of the semester discussing the problems in class. I try not to foster much examination mania among my students; my stress throughout the course is that *investigation* is paramount and time limits are silly. The 6-hour-long Putnam exam consists of 12 problems, which is a huge amount of time pressure, but most of my students prudently work on just a few questions during the exam and then spend much more time afterwards investigating the harder problems. Teachers of younger students (high school, for example) may find that the students may enjoy exam competition more and group work less. The most important thing is to promote whatever causes the students to spend lots and lots of time pondering mathematics.

The final exam is similar to the first exam, a group reunion experience, only the problems have more mathematical content. And again, we end with oral presentations and often refreshments. Usually the problems are not super-hard, since the students

have barely recovered from the Putnam.

Grading

As you can imagine, this is not a fun part of the course for the instructor. The main problem, of course, is that the students' experiences and achievements are not at all comparable in a quantitative sense. I keep careful records of who got what grade on what problem when (using a spreadsheet program) and compute weighted averages and all that stuff, but in the end, I usually just ask myself, "What did this person get out of the course?" And I grade pretty leniently. Only when a student truly doesn't try will I grade harshly (C or D). Usually at least half of my students get an A, but I don't feel guilty, because I know that these people worked like crazy and experienced profound frustration mixed with sporadic triumphs. It is essentially binary, like boot camp: either you overcome adversity, or not. Most students succeed at this course, because most people like to rise to challenges.

Chapter 2

Chapter-by-Chapter Solutions

What follows are solutions, solutions sketches, or at least substantial hints to many of the problems. We omit solutions to most of the Putnam, AIME, USAMO, and IMO problems, because these exams (and solutions) are widely published and we assume (and strongly encourage!) that you have them on your bookshelf. We also do not include solutions to problems from the Bay Area Mathematical Olympiad, as these are freely available at `http://mathcircle.berkeley.edu/BMC6/pages/BAMO/bamo.html`. Also, we do not give solutions to some of the more routine questions that are typically covered in standard texts. References are mentioned, although you may have other favorites.

All page and example references are from TAACOPS (2nd edition), but the references cited use the bibliography at the end of this manual, not the bibliography on pp. 356–9 of TAACOPS.

A large number of problems in TAACOPS were drawn from 1995 Olympiad contests from many countries. I thank the officials from these countries who graciously agreed to let me (and others) disseminate their questions, and am grateful to Titu Andreescu, Elgin Johnston, and, especially, Kiran Kedlaya for helping me to solve many of them!

1.3

2   Leave one switch on, one off, and turn the third on for a minute and then off. Then check for the *three* possibilities of light is on, light is off and cold, light is off and warm.

4   Notice that the line from $(0,0)$ to $(12,16)$ passes through the center of the circle, which is $(6,8)$.

5   The maximum value is 73. Draw a picture: the first equation is that of a circle with radius 8, and center at $(7,3)$. The equation $3x + 4y = k$ defines a family of parallel lines, all with slope $-3/4$. The extreme values of k are attained when the line is tangent to the circle. Since the tangent line is perpendicular to the line drawn from the the radius to the point of tangency, it is easy to compute the coordinates of the two points of tangency and then plug into $3x + 4y$.

11   (Sketch due to Kiran Kedlaya) It suffices to show that $\cos(\cos(x)) > \sin(\sin(x))$ for all x, from which it follows that

$$\cos(\cos(\cos(\cos(x)))) > \sin(\sin(\cos(\cos(x)))) \\ > \sin(\sin(\sin(\sin(x)))),$$

and hence the given equation has no solutions.

Useful facts are

$$cos(\cos(x)) = \sin(\pi/2 - \cos(x)),$$

$\sin(x) + \cos(x) = \sqrt{2}\cos(x - \pi/4)$, and the monotonicity of sine and cosine (for angles between 0 and $\pi/2$).

An excellent reference for problems of this kind, and much more, is *Geometric Etudes in Combinatorial Geometry* by Vladimir Boltyanski and Alexander Soifer [3].

20   The idea is that triangular packing (so the centers of three circles are vertices of an equilateral triangle) is a more efficient use of space, however, to do this, you waste some space at the beginning of the rectangle. So you need a long rectangle to catch up.

2.1

17   No. Color the vertices of a square black, white, black, white.

18   The first set contains only numbers whose numerals are drawn exclusively with curves; the second contains "linear" numerals; the third contains the mixed numerals.

19   Crux idea: cut one of the ropes near the ceiling to make a small loop.

21   First prof writes a random number, shows it to the next prof who then writes the sum of her salary and the first number on her piece of paper. This process of writing the running sum is continued until the last prof shows her sum to the first prof, who then subtracts her random number. Now the sum (and hence the average) is known.

22   Both have the same fraction. This can be computed easily by algebra, but it can be *seen* by imagining that the bottles contain not fluid but differently-colored ping-pong balls. Then just remember: mass is conserved!

23   Crux idea: the two slow people must travel together at some point.

24   See problem 6.4.12.

25   Consult any standard reference for the formulas.

26   See the discussion on Eulerian paths in section 4.1.

27   (a)   30 years old.

(b)   1 (one book may have all the colored illustrations).

(c)   5 times (the outcomes are $0, 1, 2, 3, 4, 5$ heads).

(d)   The answer is 5. The original pyramid has 5 faces. Attaching the tetrahedron adds *no* new faces! The reason: when the tetrahedron is glued to the pyramid, two of the faces of the tetrahedron are flush with faces of the pyramid! Of the two other faces of the tetrahedron, one is glued to a face of the pyramid (and thus both of these faces now lie inside the new solid), and the other becomes one of the faces of the new solid. The big question is, how to show that the faces are flush? One way is to do rather difficult 3-dimensional vector calculations, with a fair amount of trig. This is a good exercise, and well worth doing. The ingenious way to see it it in a flash is to imagine two pyramids sitting next to each other so their square faces are touching side to side. It is easy to see that the "gap" between them is exactly the size and shape of a regular tetrahedron (draw a picture, or better yet, build some models!). Hence, when you place a regular tetrahedron into this gap, you get a nice prism. In other words, two of the faces of the tetrahedron will be flush with two faces of the pyramid.

(e)   All *n*. Just draw parallel strips!

2.2

9   Verify that $f^3(x) = x$.

10 $T_n = n! + 2^n$.

11 $f(n)$ is the equal to the number of 1's in the base-2 (binary) representation of *n*.

12   See the discussion of Euler's formula (3.4.3, 3.4.40).

13   If $f(n)$ is the number of areas for *n* lines, the $(n+1)$-st line will intersect each of the *n* lines, producing one new area each time, plus one additional area (the "exit" point. Thus $f(n+1) = f(n) + n + 1$. This gives an algortithm, and it is possible to use formulas for arithmetic series (see section 5.3) to get the formula $f(n) = (n^2 + n + 2)/2$.

14   By the same methods as 2.2.13 above, the answer will be essentially double: $n^2 - n + 2$.

15   For example, $1/8 + 1/56 = 1/7$. What's the pattern?

16   Motivated by the examples $1/2 + 1/3 + 1/6 = 1$ and $1/2 + 1/3 + 1/7 + 1/42 = 1$, it is easy to guess that the next example is $1/2 + 1/3 + 1/7 + 1/43 + 1/(2 \cdot 3 \cdot 7 \cdot 43) = 1$, and now it is easy to generate any solution: let $x_1 = 2, x_2 = 3$ and recursively define $x_m = x_1 \cdot x_2 \cdots x_{m-1} + 1$ until x_n is defined.

17   See 2.3.38.

18   When *n* is a perfect square.

19   As long as $n \geq 60 = (7-1)(11-1)$, we can always do it.

20   The sum stays the same! To prove this, we need to do a rather tricky induction argument (see Section 2.3).

21   (a) $n(n+1)/2$ b) $n^2(n+1)^2/4$ c) There is no "simple" formula, but see Example 5.3.2 for ideas (you'll need the binomial theorem).

22 $s(n) = 2^{n-1}$.

23 Use the fact that $(u^2)^2 + (2u)^2 + 2^2 = (u^2 + 2)^2$.

25 Let us work out the first few terms of the product. We get

$$(1 + x^3)(1 + 2x^9)(1 + 3x^{27})(1 + 4x^{81}) \cdots =$$

$$= 1 + x^3 + 2x^9 + 2x^{12} + 3x^{27} + 3x^{30} + 6x^{36} + 6x^{39} + 4x^{81} + \cdots.$$

What are the (positive) exponents k_i? All integers of the form $3^{u_1} + 3^{u_2} + \cdots + 3^{u_r}$, where the integers u_j satisfy $1 \le u_1 < u_2 < \cdots < u_r$. In other words, they will be numbers which, when written in *base-3*, only contain ones and zeros and end with a zero. In order, the first few exponents are (written in base-3) are

$$10, 100, 110, 1000, 1010, 1100, 1110, 10000, \ldots.$$

Of course, these numbers are just the *base-2* representations of the sequence $2, 4, 6, 8, \ldots$. In particular, to figure out k_{1996}, we just write $2 \cdot 1996 = 3992$ in base-2:

$$3992 = 2048 + 1024 + 512 + 256 + 128 + 16 + 8,$$

so the base-2 representation of 3994 is 111110011000, and k_{1996} is equal to 111110011000 (base-3).

In other words,

$$k_{1996} = 3^3 + 3^4 + 3^7 + 3^8 + 3^9 + 3^{10} + 3^{11}.$$

26 Solved in Example 2.4.3.

27 See [9] for a very nice discussion of this and related problems.

28 $g(n) = 2^r$, where r is the number of 1's in the base-2 representation of n.

32 Let m not be a perfect square. We can write $m = n^2 + b$, where $1 \le b \le 2n$. Since $(n + 1/2)^2 = n^2 + n + 1/4$, we see that if $b \le n$, then $\{\sqrt{m}\} = n$, and if $b > n$, then $\{\sqrt{m}\} = n + 1$. Hence, in the first case, we have $f(m) = m + n = n^2 + b + n < n^2 + 2n + 1$. In the second case, we have $f(m) = m + n + 1 = n^2 + b + n + 1 > n^2 + 2n + 1$. In neither case will $f(m)$ be a perfect square. We finish things off by noting that if $m = n^2$, then $f(m) = n^2 + n$, which is not a perfect square.

35 a) No; let $n = 41$. See also Example 7.5.1. b) The correct formula is $t(n) = 1 + \binom{n}{2} + \binom{n}{4}$. For a derivation and further discussion, see the reference cited in the problem.

2.3

21 See page 223.

22 Assume that neither U nor T are closed under multiplication. Then there exist $t_1, t_2 \in T$ and $u_1, u_2 \in U$ such that $t_1 t_2 \notin T$ and and $u_1 u_2 \notin U$. But then $t_1 t_2 \in U$ and $u_1 u_2 \in T$, since if something is not a member of T it must be a member of U and vice-versa (since all of these products are in S, which is

closed under multiplication). Now consider $t_1 t_2 u_1 u_2$. This is a product of three elements of T, since $t_1, t_2 \in T$ and $u_1 u_2 \in T$. By hypothesis, then, $t_1 t_2 u_1 u_2 \in T$. But $t_1 t_2 u_1 u_2$ is also a product of three elements of U, namely $t_1 t_2, u_1$ and u_2, and hence must be a member of U by hypothesis. But this is a contradiction: $t_1 t_2 u_1 u_2$ cannot be a member of both T and U, for these two sets are disjoint. So our original assumption that neither T nor U were closed could not be true. We conclude that at least one of these sets is closed.   ∎

23   No, it is not possible. WLOG, let there be 3 (which is odd) numbers placed on the circle; call them a, b, c. Now consider the ratios $a/b, b/c, c/a$. Each is either a prime or the reciprocal of a prime, while the product is equal to 1. The product will then be a fraction, whose numerator (before simplifying) is a product of primes, and whose denominator is a product of primes. Since there are a total of 3 primes in the numerator and denominator, we have a contradiction: the numerator and denominator will end up with different numbers of primes, and there is no way that the two quantities can be equal (so that the fraction will equal 1). As long as there are an odd number of numbers placed on the circle, it cannot be done. It is easy to do it with an even number, though.

32   Given that $x^n + 1/x^n$ is an integer, we will show that $x^{n+1} + 1/x^{n+1}$ is also an integer. By hypothesis, $x + 1/x$ is an integer, so the product

$$(x^n + x^{-n})(x + x^{-1}) = x^{n+1} + 2 + x^{-(n+1)}.$$

Hence $x^{n+1} + 1/x^{n+1}$ is an integer.

40   Crux idea: look at this picture!
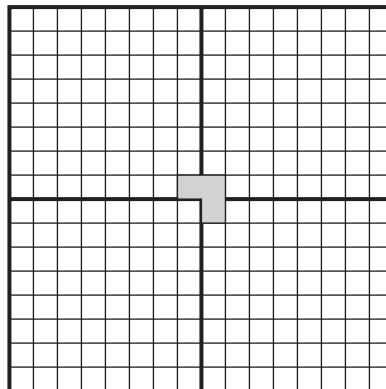
2.4

7   After drawing a distance-time graph, it is easy to see that Pat walks for 55 min.

8   Draw a picture with dots arranged in a grid pattern.

9   Sunrise was at 4:30 AM. This is a pretty hard problem to do by algebra alone; instead we will draw a time-distance graph and do some geometry. In the graph below, which is not drawn to scale, the horizontal axis measures time starting with sunrise and the vertical axis represents the distance along the road from A to B. The points $F, G,$ and H are respectively noon, 5 PM and 11:15 PM, and the line segments $\overline{AD}$ and $\overline{BH}$ represent the time-distance paths of Pat and Dana, respectively, which meet at C. We can determine the time of sunrise once we compute the length AF.
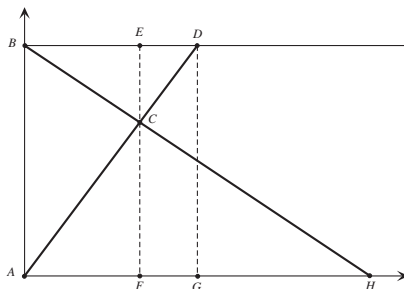
Triangles ACH and DCB are similar, so

$$\frac{BE}{ED} = \frac{FH}{AF}.$$

Since $ED = 5, FH = 11.25$ and $BE = AF$, we have $AF^2 = \dfrac{45}{4} \cdot 5$, and $AF = \dfrac{15}{2}$.

10   The bug should travel along two line segments: first from $(7, 11)$ to $O = (0,0)$, and then from O to $(-17, -3)$. This is a consequence of the following principle: the bug must avoid quandrant II completely, even though a straight line path from $(7, 11)$ to $(-17, -3)$ goes through quadrant II.

To see why this is true, let a and b be arbitrary positive numbers, and consider a path starting at $A = (0, a)$ and ending at $B = (-b, 0)$. Certainly the quickest route *within quadrant II* is the line segment $\overline{AB}$, and the length of this path is $\sqrt{a^2 + b^2}$. Now consider the alternate route $\overline{AO}$ followed by $\overline{OB}$. This path lies outside quadrant II (since quadrant II does not include the x- or y-axes) and has total length $a + b$. Compare these two lengths. By the arithmetic-geometric mean inequality, we have $a^2 + b^2 \geq 2ab$, which implies that $2a^2 + 2b^2 \geq a^2 + 2ab + b^2 = (a + b)^2$. Hence

$$a + b \leq \sqrt{2}\sqrt{a^2 + b^2}.$$

We conclude that as long as the speed in quadrant II is less than $\frac{1}{\sqrt{2}}$, then any path from A to B that passes through quadrant II will take more time than the shortest non-quadrant-II path (along the y- and x-axes). Since $\frac{1}{2} < \frac{1}{\sqrt{2}}$, our bug will save time by avoiding quadrant II.

11   During the course of the twelve-hour period from midnight to noon, the hands begin and end lined up, and line up also eleven other times: a bit after

1, then after 2, etc. The interval between successive line-up times is always the same (to see why, consider the point of view of an ant that lives on the hour hand), hence these intervals occur every $12/11$ hours. So the answer is 1 o'clock plus $1/11$ hour, or 1:05:27 and $3/11$ sec.

15 Recast the problem with counting dots. You need to characterize the "trapezoidal" numbers.

17 The answer is 8. Observe that the quantity to be minimized is just the square of the distance between the two points $(u, \sqrt{2 - u^2})$ and $(v, 9/v)$. This is the distance between an arbitrary point on a the circle $x^2 + y^2 = 2$ and a point on the upper half of the hyperbola $y = 9/x$. Thus we need to find the minimum distance between these two curves. It is easy to see (draw a picture) that this attained by picking the point $(1, 1)$ on the circle and the point $(3, 3)$ on the hyperbola.

18 Unfold the cube; shortest distance is a straight line; answer is $\sqrt{5}$.

19 Let R be the interior of the rectangle with vertices $(0,0), (b,0), (b,a), (0,a)$. The line $y = ax/b$ intersects no lattice points in R (it passes through $(0,0)$ and (b,a), but these points are not included in R, and there are no other lattice points on the line, since a and b have no common divisors). Observe that $\lfloor ai/b \rfloor$ is just the number of lattice points that lie below this line in R for $x = i$. Thus the left-hand sum is just the number of lattice points lying below the line in R . By similar reasoning, the right-hand sum is equal to the number of lattice points lying above the line. The common value must equal one-half of the total number of lattice points, which is of course $(a-1)(b-1)$. Observe that at least one of a and b must be odd, for otherwise the two numbers would share a common divisor (namely, 2). Consequently $(a-1)(b-1)$ is even and can be divided by 2.

20 See Example 8.2.1. The only difference is that now we use the graph of $y = \sqrt{1 - x}$ in place of the hyperbola.

21 The answer is $\sqrt{10^4 + 17^2}$. Each term $\sqrt{(2k - 1)^2 + a_k^2}$ can be interpreted as the length of a vector with horizontal displacement $(2k - 1)$ and vertical displacement a_k. If we place these vectors *end-to-end*, then the sum

$$\sum_{k=1}^{n} \sqrt{(2k - 1)^2 + a_k^2}$$

is the total length of a possibly jagged path which has a net horizontal displacement of

$$1 + 3 + \cdots + (2 \cdot 10 - 1) = 10^2,$$

and net vertical displacement of

$$a_1 + a_2 + \cdots + a_{10} = 17.$$

If we vary the a_i so that the path length is minimized, the path will clearly be a straight line. The length of this line is of course just $\sqrt{10^4 + 17^2}$.

23   The answer is 1235. Convert the problem into one where you are counting lattice points inside triangular and rectangular regions; Problem 2.4.19 is slightly similar.

24   Consider the general problem of n marbles $m_0, m_1, \ldots, m_{n-1}$ with arbitrary starting locations. Each marble has a "ghost path," the path it would travel if it did not bounce off its neighbors but instead passed through them. Whenever the marbles bounce, the actual path of a marble coincides with another marble's ghost path. After one minute has passed, each ghost path has returned to the original positions of each marble. Hence after one minute, the actual locations of the marbles are a permutation of the original positions. Moreover, this permutation must be a cyclic permutation, since the marbles cannot pass through one one another.

   We claim that the permutation takes m_0 to m_d, where d is the "counterclockwise excess," i.e. the difference modulo n between the number of counterclockwise marbles and the number of clockwise marbles.

   To see this, let $v_i(t)$ be the *velocity* function for marble m_i, where the velocities of $+1, -1$ denote counterclockwise and clockwise motion, respectively. Notice that for any time t,

$$\sum_{i=0}^{n-1} v_i(t) = d,$$

since the number of clockwise and the number of counterclockwise marbles never changes (even when marbles collide). There will be finitely many bounces, and in any time interval between bounces, each velocity function is a constant. Let $t_1, t_2, \ldots, t_k$ be time values inside each interval, and let each each interval have length ℓ_i. For each marble m_i, denote the net counterclockwise distance traveled from $t = 0$ to $t = 1$ by

$$s_i = v_i(t_1)\ell_1 + v_i(t_2)\ell_2 + \cdots + v_i(t_k)\ell_k.$$

Summing this over all marbles, we get

$$\sum_{i=0}^{n-1} s_i = d(\ell_1 + \ell_2 + \cdots + \ell_k) = d \cdot 1 = d.$$

The only cyclic permutation associated with this sum of net distance traveled is the one which takes m_0 to m_d.

3.1

13   The answer is $\sqrt{170}$. Let the path go from $(3,5)$ to $(0,a)$ to $(b,0)$ to $(8,2)$. This path has the same length as the path (draw a picture!) from $(-3,5)$ to $(0,a)$ to $(b,0)$ to $(8,-2)$, and this path is clearly minimized when it is a straight line from $(-3,5)$ to $(8,-2)$.

15   The product of the divisors of n is $\sqrt{n^{d(n)}}$.

16 The answer is 2^{29}. Notice that the sum of all the elements of the set $S = \{1, 2, 3, \ldots, 30\}$ is $1 + 2 + 3 + \cdots + 30 = 30 \cdot 31/2 = 465$. Let A be a subset of S and let A^c denote the *complement* of A (the elements of S which are not elements of A. The sum of the elements of A plus the sum of the elements of A^c must equal 465. Because $465 = 232 + 233$, if the sum of the elements of a A is greater than 232, the sum of the elements of A^c must be less than 232. In other words, there is a one-to-one correspondence between subsets whose element sum is greater than 232 and subsets whose element sum is not (namely, $A \leftrightarrow A^c$). Hence the number of subsets whose element sum is greater than 232 is exactly half of the total number of subsets of S, and the number of subsets of S is 2^{30}.

19 There will be an infinite amount of rotation. Consider the situation after the bugs have each rotated 1 degree. They are now, as before, lying at four vertices of a square, with exactly the same situation as at the start, in terms of angular relationships. Except for scale, it would be impossible to tell if this was not the starting configuration. Hence the bugs will rotate yet another degree, and another, and another …

20 The first player can always win by first placing the penny exactly at the center of the table and thereafter "mirroring" his opponents move with respect to the center point.

21 Label the points where the ball bounces $C, C_1, C_2, \ldots$. Draw line segment $\overline{BD_1}$ so that $BD_1 = BC$ and $\angle CBD_1 = \beta$. Now reflect the path of the billiard ball inside $\angle ABC$ about the axis $\overline{BC}$. Let E_1 be the reflection of C_1. For counting purposes, it doesn't matter whether we look at the real path $CC_1C_2\ldots$ or the reflected path beginning with CE_1, so let us pretend that the billiard ball actually begins its path by traveling from C to E_1. Once again, draw line segment $\overline{BD_2}$ with $\angle D_1BD_2 = \beta$ and $BD_2 = BD_1$. Again, reflect the new path of the ball, starting with E_1, and let the reflection of C_2 be E_2. Since $\angle CE_1D_1 = \angle C_2E_1B = \angle BE_1E_2$, the path CE_1E_2 must be a straight line.

Now we keep repeating the process, constructing line segments $\overline{BD_2}, \overline{BD_3}, \ldots$ with $\angle D_iBD_{i+1} = \beta$ and $BD_i = BD_{i+1}$ and reflecting the path, until we have a new path which is a the ray $\overrightarrow{CE_1}$. Each bounce C_i of the original path will correspond to the intersection point of the ray with $\overline{BD_i}$. To count the number of intersections, draw a circle with center B and radius AB. The path that concerns us is the line segment $\overline{CE}$ (once the ray leaves the circle, there will be no more bounces in the original path), where $\angle BCE = \alpha$. To count the number of intersections with the $\overline{BD_i}$, we need to find $\angle CBE$. Since $BE = BC$, $\angle BEC = \angle BCE$, so $\angle CBE = 180 - 2\alpha$. Hence the number of intersections is equal to $\left\lfloor \frac{(180 - 2\alpha)}{\beta} \right\rfloor + 1$.

25 The answer is $1600/3$. The temperature function is given by

$$T(x, y, z) = (x + y)^2 + (y - z)^2.$$

Now consider two similar functions: $U(x, y, z) = T(y, z, x)$ and $V(x, y, z) = T(z, x, y)$. In other words, U and V are obtained from T by cyclically permuting the vari-

ables. The average values of T, U and V taken over the surface of the sphere will be the same because of symmetry. Let us call this average value A. Then the average value of $T + U + V$ will equal $3A$. But

$$
\begin{aligned}
T(x,y,z) + U(x,y,z) + V(x,y,z) &= \\
(x+y)^2 + (y-z)^2 + (y+z)^2 &+ (z-x)^2 + (z+x)^2 + (x-y)^2 \\
&= 4x^2 + 4y^2 + 4z^2.
\end{aligned}
$$

However, on the surface of the planet, $4x^2 + 4y^2 + 4z^2 = 4 \cdot 20^2 = 1600$, a constant! Hence the average value of $T + U + V$ will be $1600 = 3A$.

26   Let $f(x) := 1/(1 + \tan x)^{\sqrt{2}}$ and let $g(x) := 1/(1 + \cot x)^{\sqrt{2}}$ Since $\cot x = \tan(\pi/2 - x)$, the graphs of the two functions from $x = 0$ to $x = \pi/2$ are just mirror images of one another (reflected about $x = \pi/4$). Hence

$$
\int_0^{\pi/2} f(x)dx = \int_0^{\pi/2} g(x)dx.
$$

On the other hand, $\cot x = 1/\tan x$, and easy algebra yields $f(x) + g(x) = 1$. Thus

$$
\int_0^{\pi/2} f(x)dx = \frac{1}{2}\int_0^{\pi/2}[f(x) + g(x)]dx = \pi/4.
$$

30a   Consult the lovely little book *Fifty Challenging Problems in Probability with Solutions* by Frederick Mosteller [19]. Only 88 pages long, yet it contains many wonderful insights about probability, a topic sadly neglected in TAA-COPS.

30b   (Jim Propp) Number the cards 1 through 52 for convenience, with k denoting the card that was originally the *kth* from the top. For $1 \le k \le 52$, let A_k denote the number of selections such that the kth card remains kth from the top. For $k \le 26$, A_k is the number of selection in which cards 1 through k all get selected, so that $A_k = \binom{52-k}{26-k}$. By symmetry, $A_k = A_{53-k}$ for $27 \le k \le 52$, so

$$
\sum_{k=1}^{52} A_k = 2\sum_{k=1}^{26}\binom{52-k}{26-k}.
$$

The second sum equals $\binom{52}{27}$ (see the hockey stick identity, Example 6.2.5 on p. 218). Multiplying by 2 and dividing by $\binom{52}{26}$, we get $52/27$.

30c   Consider a sequence with n distinct integers. The ith number ($i = 2, 3, \ldots, n$) will get swapped if and only if it it was the smallest of the first i elements. The probability of this happening is $1/i$. Hence the average number of swaps is just

$$
\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.
$$

3.2

7   Assume that the values are not all equal. Let $a > 0$ be the smallest value on the board. There must be a square containing a which is adjacent (WLOG) on the east by a square containing the value b which is strictly greater than a. But then a is equal to the average of 4 numbers, none less than a, one of which is strictly greater than a. This is a contradiction.

10   The coin with smallest diameter cannot be tangent to more than 5 others.

12   Certainly when n is even, it is not true: Just imagine a set of pairs of people standing a few inches apart, with each pair quite far from every other pair. Now, if n is odd, first eliminate all pairs as in the above case, where two people end up shooting each other. Since n is odd, some dry people remain. Now consider the person whose nearest neighbor is maximal (there may be ties). This person will stay dry, since the only way that he could get shot is if someone else is as close to him as he is to his nearest neighbor. But that contradicts the fact that for each person, the distances to the others are different.

14   We will show that no palindrome can exist by contradiction. Assume that the concatenation of the numbers from 1 to n was the palindrome

$$P := 1234567891011\cdots 4321.$$

Consider the longest run of consecutive zeros in P; note that this exists, since n is surely greater than 10. There may be several runs of consecutive zeros that are all equally long; pick the last (rightmost) one. Observe that immediately to the left of this string is a single digit, and this digit plus the zeros forms one of the numbers from 1 to n. For concreteness, suppose that the longest string of zeros was 0000. Then the rightmost such string obviously consists of the last digits of one of the numbers from 1 to n, not the middle of one, and doesn't straddle two (for example, if the number was, say, 400005, then the number 400000 would have appeared to the left of it, contradicting the fact that 0000 is longest string of zeros. Likewise, the number that ends with 0000 had to start with a single digit, for if, say, the number was 7310000 then there would have been the number 7000000 to the left of it.

So, let us suppose that the rightmost string of 0000 is the last digits of the number 70000. Then, writing the predecessor and successor numbers, these four zeros are embedded in the string 699997000070001 Assume also, that there is at least one other string of 0000 in P. Since P is a palindrome, the first 0000 must be embedded in the string 100070000799996. But that makes no sense, since the first time 0000 appears is as the last digits of the number 10000.

So the only remaining possibility is that there is only one 0000 string, which by necessity is at the exact center of P and is the last four digits of the number 10000. Writing the predecessor and successor, and letting "|" mark the exact midpoint of P, we must have the following string at the center:

$$\cdots 9999100|0010001\cdots$$

But this isn't symmetrical ($9 \neq 0$), achieving our contradiction.

15 Consider the shortest path joining 1 with n^2, where path means a walk along adjacent squares. The worst case scenario is that the path has length n (if 1 and n^2 are at opposite diagonal corners). In any event, the members of the path will be at most n distinct numbers between 1 and n^2, inclusive. If their successive differences were all less than or equal to n, that means there are $n - 1$ successive differences which bridge the gap from 1 to n^2. Since $n^2 - 1 = (n-1)(n+1)$, the largest difference must be at least $n+1$.

16 See Section 4.1.

18 This is the Eisenstein irreducibility criterion, which is discussed and proved in most modern algebra texts, for example, [12].

3.3

11 Suppose that there are n people. The possible "acquaintance numbers" range from 0 to $n - 1$ inclusive. If no one knows zero people, then we can apply pigeonhole on the $n - 1$ numbers from 1 to $n - 1$. If at least one person knows zero people, then the maximal acquaintance number is $n - 2$, so again pigeonhole will work.

16 Partition the square with 50 rectangles of size 0.02. Pigeonhole places three points in one of these rectangles. (Actually, it is not completely trivial to argue that the area of a triangle placed inside a rectangle is at most half the area of the rectangle. Try to find an airtight argument!)

18 Consecutive numbers are relatively prime.

19 Let there be n people. Each person is seated a distance d from there correct place, where $0 < d < n$ is measured counterclockwise. There are n people, but $n - 1$ different values of d. Hence at least two people share the same distance d.

20 The crux idea: perfect squares are products of numbers raised to even powers, and even powers occur when two numbers with exponents of the same parity are divided. To understand the idea, let us solve this problem for a specific case: $n = 3$. Let the n distinct numbers used be a, b, c, and let the sequence we are examining be

$$u_1, u_2, \ldots, u_N,$$

where $N \geq 2^3 = 8$. Now define the successive products

$$p_i := u_1 u_2 \cdots u_i,$$

for $i = 1, 2, \ldots, N$. Each of these N numbers can be written in the form $a^k b^\ell c^m$, where k, ℓ, m are nonnegative integers. We are interested only in the parity of the exponents: there are a total of 2^3 different possibilities for the parities of the elements of an ordered triple (k, ℓ, m). For example, one possiblility is (even, even, odd), another is (odd,odd, even). Now, if one of the p_i is of the form (even, even, even), we are done, for this would be a perfect square. Otherwise, we are using at most $2^3 - 1$ different parity possibilities for $N \geq 2^3$ different p_i. Hence

two of these, say p_i and p_j, have the same parity values for their exponents. For example, suppose both were (odd, odd, even). Then the quotient (divide the larger by the smaller!) will be a perfect square, since when we divide powers, we subtract exponents, so all three exponents will be even! This method clearly generalizes; the hard part is notation, which also obscures the ideas.

21   Define $f(x,y) = x + y\sqrt{2}$ and let

$$S = \{f(a,b) | a, b, \text{ integers with } 0 \le a, b \le m\}.$$

Because $\sqrt{2}$ is irrational, S has $(m+1)^2$ distinct elements, the largest of which is $m(1 + \sqrt{2})$. Divide the interval $[0, m(1 + \sqrt{2})]$ into $m^2 + 2m$ subintervals of length $(1 + \sqrt{2})/(m+2)$. By the pigeonhole principle, there exist two *distinct* $f(a_1, b_1) > f(a_2, b_2)$ in the same subinterval. Without loss of generality, $f(a_1, b_1) > f(a_2, b_2) > 0$. It is easy to check that $a = a_1 - a_2$ and $b = b_1 - b_2$ fit the bill.

22   Consider the $(n+1)$-member sequence $7, 77, 777, \ldots$. Upon division by n, there are at most n different remainders, so by pigeonhole, at least two of the members of the sequence have the same remainder. Thus their difference is a multiple of n, and contains only the digits 7 and 0.

24   Let t_d be the total number of games played at the end of the dth day. The members of the sequence $t_1, t_2, \ldots, t_{56}$ are distinct and lie in the range from 1 to 88, inclusive. By the same reasoning as in 3.3.22, there are 23 possible remainders upon division by 23, so by pigeonhole, there are at least $\lceil 56/23 \rceil = 3$ members of the sequence that have the same remainder, say t_a, t_b, t_c, with $a < b < c$. Then $t_b - t_a$ and $t_c - t_b$ are both multiples of 23, and they cannot both be equal to 46, since that would make $t_c \ge 92$, greater than the upper limit of 88. Without loss of generality, suppose that $t_b - t_a = 23$. Then she played exactly 23 games during days $a + 1$ through b.

3.4

18   The geeks agree on a code, used only by the last person in line: "white" means "I see an even number of black hats" and "black" means "I see an odd number of black hats." The last person in line hence has a 50% probability of surviving, but thereafter, each geek knows the parity of the number of black hats, from their own (invisible) hat onward. Since each geek can see the number of black hats ahead of them, they have full information to to deduce their own hat color. Notice that as we move forward along the line, the geeks have to keep track of which hats are which color to keep adjusting the parity. This method will even work if one of the geeks accidentally says the wrong color and is (noisily) killed.

19   The answer is "No." Place the frogs on a standard coordinate system, on three of the vertices of the square $(0,0), (0,1), (1,1), (1,0)$. It is easy to check that jumping does not change the parity of the coordinates.

20   We argue inductively, inducting on the number of terms, and, as in Example 2.3.10, we find it useful to prove a somewhat stronger statement. For two terms, the sum is

$$\frac{1}{n} + \frac{1}{n+1} = \frac{2n+1}{n(n+1)}.$$

The numerator of the right-hand side is certainly odd, and the denominator is even. The parity of numerator and denominator will not change even after reducing to lowest terms. We claim that this parity relationship holds for longer sums. Suppose that the sum of t terms is o/e, where o and e are respectively odd and even. Let the $t+1$st term be $1/k$. Then the sum of the $t+1$ terms will be

$$\frac{o}{e} + \frac{1}{k} = \frac{ok+e}{ek}.$$

Now, if k is odd, the numerator of the right-hand side will be odd, and the denominator will be even, and we are done. If k is even, then we can write $k = 2^a k', e = 2^b e'$, where k' and e' are odd. There are three cases: $a < b, a > b, a = b$. If $a < b$, then after dividing the numerator and denominator by 2^a, we get the fraction

$$\frac{ok' + 2^{b-a}e'}{ek'},$$

and the numerator is odd and the denominator is even. The other two cases are handled in the same way. Essentially, the denominator ek contains too many powers of two and the numerator cannot match it. Notice that this argument actually proves an even stronger statement: Given any finite sequence of positive integers, (not necessarily consecutive, not neccessarily distinct), as long as two of them have different parity, then the sum of the reciprocals will reduce to a fraction with odd numerator and even denominator.

22   See 7.2.2.

23   It is easy to verify that if (x,y) is a legal point, then $y-x$ will be a multiple of 11. Since $1999-3$ is not a multiple of 11, the answer is no.

24   Notice that $uv+u+v+1 = (u+1)(v+1)$, so that u,v are replaced with $(u+1)(v+1)-1$. Therefore, if the sequence contains the values $a_1, a_2, \ldots, a_n$, the quantity

$$(a_1+1)(a_2+1)(a_3+1)\cdots(a_n+1)$$

is invariant! Hence the final number will equal $100!-1$, no matter what the choices are.

25   The three quantities

$$\frac{b-c}{a-b}, \frac{c-a}{b-c}, \frac{a-b}{c-a}$$

are all integers by hypotheses, but their product is 1. Hence either two of them are equal to -1, or all three are equal to 1. In the first case, if just one of the

numbers, say the first, is equal to -1, we get $b-c = b-a$ which implies that $a = c$, contradicting the fact that a,b,c are distinct. In the second case, we get

$$2a = b+c, 2b = a+c, 2c = a+b$$

which means that each number is the average of the other two. This is impossible if the numbers are distinct. (For example, suppose that $a < c$. Then since b is the average of a and c, we get $a < b < c$. But c is the average of a and b, forcing $a < c < b$, a contradiction.)

26   It is easy to verify that if the numbers are x, y, z, the quantity $x^2 + y^2 + z^2$ is invariant. Since $3^2 + 4^2 + 12^2 \neq 4^2 + 6^2 + 12^2$, the answer is no.

27   At first, the chocolate bar is in one single piece. At each turn, the chocolate bar is broken into one more piece. When the game is over, the chocolate bar must be reduced to 48 pieces. Thus, no matter how the game is played, the first player wins (since the game will end after 47 turns)! In general, the first player wins if mn is even; the second player wins if mn is odd.

28   See the generalization below in 3.4.29, which includes this case.

29   We will prove that the minimum number of moves needed is $2n$. Without loss of generality, assume that the top-left square of the chessboard is black and that the chessboard ends up entirely black. First, consider the top boundary of the chessboard. As we move from right to left along this line, the number of color changes that we encounter at first is $2n - 1$. When we are done, this "color change count" will become zero. It is easy to see that any move will either leave the color change count unchanged, or else reduce it by 1 or 2 (since only the boundaries of the selected subregion can affect the color change count; the interior is inverted which doesn't change the count). Consequently, the minimum number of moves needed to render the top boundary monochrome will be $\lceil (2n-1)/2 \rceil = n$.

Likewise, at least n moves are required to render the left boundary line monochrome. Thus, the minimum number of moves need to make both of these lines monochrome is $n+n$ minus the number of "overlapping" moves, i.e., the number of moves which invert the colors of a rectangle whose upper-left corner is the upper-left corner of the chessboard.

We will show that, without loss of generality, the number of overlapping moves is zero. Since the upper-left corner is black, the number of overlapping moves must be even. Note that moves are commutative; the order doesn't matter. Consider a sequence of moves which renders the chessboard entirely black, which includes two overlapping moves, the inversion of rectangles R and S. The net effect of these two moves is to leave the rectangle $R \cap S$ unchanged and to invert the colors in the region $R \triangle S$. $R \cap S$ includes the upper-left corner, while $R \triangle S$ can always be partitioned into two rectangles, neither of which include the upper-left corner. Thus we conclude that any sequence of moves which includes an even number of overlapping moves can be replaced by a new sequence with an equal number of total moves, but with no overlapping moves. In other words, without loss of generality, we can assume that there are no overlapping moves. Then the minimal number of moves needed is at least $n+n = 2n$.

Finally, $2n$ moves is sufficient: select every other vertical column of squares, then every other horizontal column.

30 This is yet another application of the Handshake Lemma (see Example 3.4.7 and p. 121). Let $d_0, d_1, d_2, \ldots, d_n$ be the number of doors in room i, where we include "outside" as room 0 (having as its doors each of the doors in the house that open to the outside). Then by the same reasoning as in 3.4.7, the sum $d_0 + d_1 + d_2 + \ldots + d_n$ must be even. By hypothesis, $d_1, d_2, \ldots, d_n$ are all even, which forces d_0 to be even as well.

31 Let us call any set $\{x_1, x_2, \ldots, x_{23}\}$ of integers "balanced" if it has the property that no matter which of the x_i is chosen for the referee, then one can decompose the remaining 22 numbers into two sets of 11 which have equal sums. Clearly if a set is balanced and we multiply or divide each element by the same number, it will still be balanced. Likewise, if a set is balanced and we add or subtract the same number to each element, it will still be balanced.

Now, let us suppose we have a balanced set $\{x_1, x_2, \ldots, x_{23}\}$ of positive integers. Let S be the sum of the 23 elements. If we pick x_1 as referee, then we know that $S - x_1$ must be *even*, since the remaining 22 elements can be partitioned into two sets with equal integer sums. By the same reasoning, $S - x_2, S - x_3, \ldots, S - x_{23}$ are all even. Therefore, if the set of integers is balanced, then all the elements $x_1, x_2, \ldots, x_{23}$ are the same parity (i.e., all are even, or all are odd).

Now consider our balanced set $\{x_1, x_2, \ldots, x_{23}\}$ of positive integers. We wish to show that all elements are equal. Let a be the minimum value of the elements. If we define $b_i = x_i - a$ for $i = 1, 2, \ldots, 23$, then the new set $\{b_1, b_2, \ldots, b_{23}\}$ will also be a balanced set of *nonnegative* integers. Some of the elements will be zero, and perhaps some are not. We would like to prove that they are all zero. Since some of the elements are zero, and zero is even, then all of the elements must be even. Consequently we can form a new set $\{c_1, c_2, \ldots, c_{23}\}$, where $c_i = b_i/2$ for $i = 1, 2, \ldots, 23$. But this set also has some zero elements, hence all of its elements are even, hence we can divide them all by 2 and get yet another balanced set of nonnegative integers. We can do this forever! The only integer which one can divide by 2 endlessly and still get an even integer as a result is zero. We conclude that the elements of $\{b_1, b_2, \ldots, b_{23}\}$ are all zero, i.e., the elements of $\{x_1, x_2, \ldots, x_{23}\}$ are all equal.

32 (Kiran Kedlaya) The boy's net travel between the first and last rides is $1 + \cdots + n - 1 = n(n-1)/2$, since each distance must occur exactly once. If n is odd, this number is divisible by n, which means that the boy ends up where he started at his last ride, no matter what he does. So he misses at least one horse. On the other hand, if n is even, the sum is not divisible by n. It is easy to find an algorithm which works in this case (try a few examples).

33 This problem is rather tricky unless we start by considering the 2-dimensional case. A bit of playing around convinces us that 5 is the magic number: If 5 lattices points are chosen in the plane (all distinct, of course), then one of the line segments joining two of these points will have a lattice point in the interior. The key ideas are parity and pigeonhole. There are only four distinct parity types

for lattice points: (odd, odd), (odd, even), (even, odd), and (even, even). Hence among any 5 distinct lattice points, two must be of the same parity type, *which means that the midpoint of the line segment joining them is a lattice point!* The argument adapts easily to 3-dimensions.

34   This is a nice application of parity and the extreme principle. First, reduce everything modulo 2; i.e., replace each odd number by a 1 and each even by a 0. Since 10 is even, we need not worry about anything but parity. The first terms of the sequence then are 0,1,0,1,0,1,1,0, etc. We need to know if the subsequence 1,1,1,1,1 can ever appear. If it does, then it must appear for the *first time*. Suppose that 1,1,1,1,1 is such a subsequence. Then 0 must be the term on the left of it (otherwise it wouldn't be the first!). So now we have 0,1,1,1,1,1 to work with. Since the righmost term is 1, and it is the sum (modulo 2) of the 6 previous terms, the term to the left of the 0 must also be 1. By the same reasoning, the term to the left of this 1 must also be 1, etc. It is not hard to deduce that there must, in fact, be 5 1's to the left of the 0: we now have 1,1,1,1,1,0,1,1,1,1,1. But this contradicts the fact that the subsequence that we started with was the first occurrence of 5 consecutive 1's (we are using the fact that the sequence does not begin with 5 1's as well).

35   If the value at C were larger than 1, we would no longer have a monovariant, because moves in which a checker landed at C would increase the Conway sum.

39   (Sketch due to Kiran Kedlaya) No matter what the starting sequence, eventually all terms become 0. Use induction on the maximum absolute value of the terms, which does not increase after the transformation.

First we show that after 8 steps, all the terms will be even, using the fact that $|x_2 - x_1| \equiv x_2 + x_1 \pmod 2$. (So we can drop the absolute value brackets.)

At this point, divide each term by 2, then use the inductive hypothesis.

4.1

8   This is the same as 3.3.11.

9   Recast this as a graph in which the people are vertices and every edge between people are drawn, with an edge colored red if the people are friends, and blue if they are strangers. Then we must find a monochromatic triangle. Pick an arbitrary vertex v. Since 5 edges emanate from it, at least 3 must be the same color, say red. Now consider the three vertices a, b, c which are adjacent to v, joined by red edges. If just one of the the edges of the triangle abc is red, we are done, for if, say, ab is red, then the triangle vab is monochromatic. But if none of the edges of abc are red, then abc is monochromatic (all blue edges)!

10   This is a simple generalization of the previous problem.

11   There are two cases, v even or v odd. Suppose that v is even. Then we can write $v = 2m$, where m is an integer. If the graph is not connected, then it contains at least two connected components, one of which has at most m vertices. Consider an arbitrary vertex in this component. By hypothesis, it has

at least $(v-1)/2 = m-1/2$ neighbors. This means that the component contains at least $m+1$ vertices, which is greater than m, a contradiction. The other case is similar.

12   Certainly if the number of edges is less than or equal to $\binom{n-1}{2}$, the graph need not be connected, since $n-1$ of the vertices can form a complete graph, and the remaining vertex will be isolated. We will show that as long as the number of edges e is greater than $\binom{n-1}{2}$, the graph will be connected. While this agrees with our intuition, we need to guard against the possibility that we may be able to take $e = \binom{n-1}{2} + 1$ edges and divvy them up to form two (or more connected components). This will not happen if the inequality

$$\binom{a}{2} + \binom{b}{2} \leq \binom{a+b-1}{2} + 1$$

is true. This can be established easily by factoring, or by the following graph theoretic argument: Consider two complete graphs, with a and b vertices. If we "glue" them together on an edge, we produce a new graph with $a+b-1$ vertices. Count edges: the original configuration had $\binom{a}{2} + \binom{b}{2}$ edges, while the new configuration has at most $\binom{a+b-1}{2}$ edges. Since we lost an edge when we glued the two graphs together, we conclude that

$$\binom{a}{2} + \binom{b}{2} \leq \binom{a+b-1}{2} + 1.$$

13   As with 3.4.30, we use the handshake lemma. Let each room be a vertex, and connect two vertices if the corresponding rooms are connected by a door. Include "outside" as a single vertex (of degree 1). Then if we look at the connected component which contains the outside, the sum of the degrees of the vertices will be even, which means that at least one of the vertices in this connected component has odd degree, i.e., its corresponding room contains a TV set.

22   We argue by induction on n. Clearly, it is true for $n = 2$. We will show that if it works for an arbitrary n, it will work for $n+1$. (It is helpful to just visualize the case $n = 2$.) The $(n+1)$-cube consists of two copies of an n-cube, where we join each vertex of one copy to its corresponding sister. Now we will show that a Hamiltonian cycle is possible for the $(n+1)$-cube. Start at vertex a in the first n-cube. By the inductive hypothesis, there is a Hamiltonian cycle for this n-cube whose last vertex (before returning to a) is b. Likewise, the second cube has a corresponding Hamiltonian path from a' to b'. Now consider the path that starts at a, traverses all vertices of the first copy ending with vertex b, *then* travels to the corresponding vertex b' of the second n-cube, then travels *backwards* along the Hamiltonian path in the second cube from b' to a', and then finally travels back to the first cube's vertex at a. This is a Hamiltonian cycle of the $(n+1)$-cube.

25   By the handshake lemma, this is a graph with 9 vertices and 21 edges. We wish to show that there must be a triangle. Pick the vertex with the largest degree. This degree must be at least 5 (since the sum of the 9 degrees is 42).

Consider this vertex x plus its five guaranteed neighbors a,b,c,d,e. If there were no triangle among these 6 vertices, then none of the $\binom{5}{2} = 10$ edges can be drawn connecting any of a,b,c,d,e. Let the three remaining vertices be f,g,h. Of the 8 vertices $\{a,b,c,d,e,f,g,h\}$, there are $\binom{8}{2} - \binom{5}{2} = 18$ "legal" edges available, which consist of the $3 \cdot 5 = 15$ possible edges connecting each element of $\{f,g,h\}$ to each element of $\{a,b,c,d,e\}$, plus the three edges connecting all the elements of $\{f,g,h\}$. Since we need to use 16 edges, we are forced to either create the triangle fgh or have a triangle which includes two elements of $\{f,g,h\}$ with one from $\{a,b,c,d,e\}$.

A general theorem that handles this problem, and much more, is Turan's theorem, which gives the minimum number of edges to force an embedded K_r. *Pearls in Graph Theory* [11] is an excellent starting reference.

26 There are many possibilities; see the article [26] for many of them. Perhaps the simplest is just to let all corners be vertices and join them with edges if the corners are vertices of a rectangle and an integral distance away. Then a parity-based analysis similar to the Tibet problem (4.1.7) will force a connected graph which includes two corners of the big rectangle.

4.2

We strongly encourage you to read at least the first chapter of Tristan Needham's *Visual Complex Analysis* [20].

9a This is an immediate consequence of 4.2.8c: $z\bar{z} = |z|^2$.

9c See the solution to 4.2.20.

9d Combining 4.2.8e (conjugate of a sum/product is sum/product of conjugate) with the fact that conjugation leaves real numbers unchanged yields the fact that $\overline{P(z)} = P(\bar{z})$ for any polynomial P with real coefficients.

9f Taking magnitudes of both sides, we have $|z-1|^{10} = |z|^{10}$, so $|z-1| = |z| = |z-0|$, which means that z is equidistant from 0 and 1, i.e., $\operatorname{Re} z = 1/2$.

9g The idea is to use the fact that if z/w is real, then $\arg z = \arg w$, i.e., the vectors z and w are parallel. This can be seen by means of a picture very similar to the one used in 4.2.20 below.

12 1. and 2. are consequences of the factor theorem and fundamental theorem of algebra (see page 166), while 3. follows from the fact that the n roots of unity can be thought of as vectors emanating from the origin whose endpoints are symmetrically arranged around the circumference of a unit circle. Their sum must be zero (origin), by symmetry.

20 It is almost obvious once a picture is drawn.
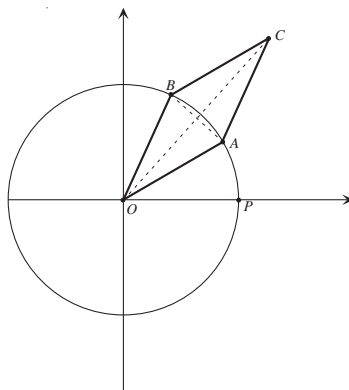
Let $A = e^{it}, B = e^{is}$. Then $C = e^{it} + e^{is}$ and $OACB$ is a rhombus. Since $BA \perp OC$ and $\angle AOB = s - t$, it is clear that $\arg C = (s+t)/2$ and $OC = 2\cos\frac{1}{2}(t-s)$.

21   This follows easily using De Moivre's theorem (4.2.6) from the fact that $x = \text{Cis}(a)$, which is obvious once you draw a picture, or by using 4.2.20.

22   The formulas are, respectively, $\sin\frac{1}{2}(n+1)a\sin\frac{na}{2}\csc\frac{a}{2}$ and $\cos\frac{1}{2}(n+1)a\sin\frac{na}{2}\csc\frac{a}{2}$. Both can be computed pretty easily (and simultaneously) by summing the geometric series $e^{ai} + e^{2ai} + \cdots + e^{nai}$, using 4.2.19, and considering real and imaginary parts. Try it!

23   Let $P(z) := z^5 + z + 1$. If we look at the exponents of P modulo 3, they are, respectively, 2, 1, 0. In other words, modulo 3, P resembles $u^2 + u + 1$. This should make you think about roots of unity, specifically, cube roots of unity. If we substitute in a cube root of unity

$$\zeta = \text{Cis}(2\pi/3),$$

we get

$$P(\zeta) = \zeta^5 + \zeta + 1 = \zeta^3\zeta^2 + \zeta + 1 = \zeta^2 + \zeta + 1 = 0,$$

by 4.2.12.3. Likewise, $P(\zeta^2) = 0$. Thus ζ and ζ^2 are zeros of $P(z)$, so by the factor theorem, $(z - \zeta)(z - \zeta^2)$ is a factor. But

$$(z - \zeta)(z - \zeta^2) = z^2 + z + 1$$

by 4.2.12.2. Thus $z^2 + z + 1$ is a factor of $P(z)$. Polynomial long division yields the factorization

$$P(z) = (z^2 + z + 1)(z^3 - z^2 + 1).$$

24   Using the same methods of 4.2.23 above, we deduce that all four complex 5th roots of unity are roots of the polynomial in question; i.e., $z^4 + z^3 + z^2 + z + 1$ is a factor of $z^6 + z^4 + z^3 + z^2 + 1$. By long division, the other factor is $z^2 - z + 1$, whose roots are easily found with the quadratic formula.

25   Let $1, \zeta, \zeta^2, \zeta^3, \ldots, \zeta^{n-1}$, be the nth roots of unity (so $\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$). If we exclude the root 1, the remaining $n - 1$ complex numbers are the roots of

$(x^n-1)/(x-1)$. In other words,

$$1+x+x^2+\cdots+x^{n-1} = (x-\zeta)(x-\zeta^2)(x-\zeta^3)\cdots(x-\zeta^{n-1}).$$

Letting $x=1$, we get

$$n = (1-\zeta)(1-\zeta^2)(1-\zeta^3)\cdots(1-\zeta^{n-1}).$$

Now take the absolute value of both sides, and use the fact (Problem 4.2.9c) that

$$|1-(\cos\theta+i\sin\theta)| = 2\sin\frac{\theta}{2}.$$

Putting this all together yields

$$n = 2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}.$$

26   The vertices of the n-gon are nth roots of unity $1,\zeta,\zeta^2,\zeta^3,\ldots,\zeta^{n-1}$, which are the n zeros of the polynomial x^n-1. Let us only consider the line segments emanating from the root 1. The product of the lengths of these line segments is

$$|1-\zeta||1-\zeta^2|\cdots|1-\zeta^{n-1}| = |(1-\zeta)(1-\zeta^2)\cdots(1-\zeta^{n-1})|.$$

But recall that

$$x^n-1 = (x-1)(x-\zeta)(x-\zeta^2)\cdots(x-\zeta^{n-1}).$$

Divide this by $x-1$,and we get

$$x^{n-1}+x^{n-2}+\cdots+x^2+x+1 = (x-\zeta)(x-\zeta^2)\cdots(x-\zeta^{n-1}).$$

Now substitute $x=1$ and we have

$$n = (1-\zeta)(1-\zeta^2)\cdots(1-\zeta^{n-1}),$$

in other words, the product of the lengths emanating from 1 equals n. By symmetry, the product of the lengths emanating from any vertex will equal n. Since there are n vertices, the product will be n^n. However, this includes overcounting by a factor of 2, i.e. we have counted the length from vertex a to vertex b twice in our prouduct. To correct for this, we just take the square root. In other words, the correct answer is $\sqrt{n^n}$.

31   Let us argue by contradiction. Suppose such an equilateral triangle existed. Without loss of generality, one vertex is $(0,0)$. Let the other vertices be the complex numbers a and b, where $a = x+iy$ and $b = u+iv$, with x,y,u,v integers. Suppose that b, as a vector emanating from the origin, lies 60 degrees counterclockwise from a. Then the condition for the triangle to be equilateral is merely that

$$b = e^{\pi i/6}a.$$

Thus

$$u+iv = (\sqrt{3}/2+i/2)(x+iy).$$

Comparing real and imaginary parts immediately leads to a contradiction, since $\sqrt{3}$ is irrational: we must have $x\sqrt{3}/2-y/2$ equal to an integer.

32 The vectors $u := a - b, v := b - c, w := c - a$ represent the sides of the triangle. We have $v = \zeta u, w = \zeta^2 u$, where $\zeta = e^{2\pi i/3}$. Notice that ζ is a cube root of unity, so $\zeta^2 + \zeta + 1 = 0$. Consequently,

$$u^2 + v^2 + w^2 = u^2(1 + \zeta + \zeta^2) = 0.$$

Substituting back for a, b, c yields $a^2 + b^2 + c^2 = ab + bc + ca$.

38 This problem can be found in [15] (supplementary algebra problem 1).

4.3

Don't forget to keep a copy of Herbert Wilf's wonderful *generatingfunctionology* [27] close at hand.

8 Write each term $\binom{n}{k}^2$ as $\binom{n}{k}\binom{n}{n-k}$. Then the sum of these terms as k ranges from 0 to n is the x^n term of $(x+1)^n(x+1)^n$.

9 This is a generalization of the above, known as the Vandermonde Convolution Formula (see 6.2.31 also). All that it is saying is, look at the x^k term in the product $(x+1)^{n+m}$. If we write $(x+1)^{n+m} = (x+1)^n(x+1)^m$, then the identity desired follows immediately from the rules for multiplying two polynomials (see also p. 180).

15a This is a straightforward application of the formula for the sum of geometric series (Section 5.3).

15c Let $\omega := e^{2\pi i/3}$ be a 3rd root of unity. Then

$$(1+1)^n + (1+\omega)^n + (1+\omega^2)^n = \sum_{r=0}^{2} \sum_{j=0}^{n} \binom{n}{j} \omega^{rj}$$

$$= \sum_{j=0}^{n} \binom{n}{j} \sum_{r=0}^{2} \omega^{rj}$$

$$= 3 \sum_{3|n} \binom{n}{j}.$$

Using pictures or 4.2.19, we can simplify $1 + \omega = \text{Cis}(\pi/3)$ and $1 + \omega^2 = \text{Cis}(-\pi/3)$, so our answer is

$$\frac{1}{3}\left((1+1)^n + (1+\omega)^n + (1+\omega^2)^n\right) = \frac{1}{3}\left(2^n + 2\cos(n\pi/3)\right).$$

18 The generating function for partitions that have no part which is a multiple of 3 is

$$A(x) = (1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + \cdots)$$
$$(1 + x^4 + x^8 + \cdots)(1 + x^5 + x^{10} + \cdots)$$
$$= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^5}\right)\cdots.$$

The generating function for partitions that have no part repeated more than twice is

$$B(x) = (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)(1+x^4+x^8)\cdots$$

$$= \left(\frac{1-x^3}{1-x}\right)\left(\frac{1-x^6}{1-x^2}\right)\left(\frac{1-x^9}{1-x^3}\right)\left(\frac{1-x^{12}}{1-x^4}\right),$$

where we use the formula for the sum of a geometric series (p. 173) or the factorization of a difference of two cubes (5.2.7). Study $A(x)$ and $B(x)$ carefully. Notice that when you cancel the numerators and denominators in $B(x)$, the numerator becomes 1, and the denominator loses all terms of the form $(1-x^r)$, where r is a multiple of 3. In otherwords, you get $A(x)$. So $A(x) = B(x)$, and consequently their coefficients (and corresponding partition functions) are equal.

19   The answer is 192.

20   Using the geometric series tool, the function is equal to

$$f(x) := 1 + (x+x^2+x^3+x^4+x^5+x^6) + (x+x^2+x^3+x^4+x^5+x^6)^2 + \cdots$$

Examine the term $(x+x^2+x^3+x^4+x^5+x^6)^r$. The coefficient of x^k will equal the number of ways that k can be represented as a sum of r values, each taken from the set $\{1,2,3,4,5,6\}$. So our function is the generating function for dice sums: the coefficient of x^k of $f(x)$ is the number of ways that k can be represented as a sum of dice.

21   By the same reasoning used in 4.3.20 above, the function

$$f(x) := (x+x^2+x^3+x^4+x^5+x^6)^2 = x^2 + 2x^3 + 3x^4 + \cdots + 2x^{11} + x^{12}$$

is the generating function for rolling two dice: the coefficient of x^k is the number of ways that you can get a sum of k by rolling two dice. The probabilities are merely the coefficients divided by 36.

So our problem amounts to asking whether there is a non-trivial factorization of $f(x)$ that is not the square of $(x+x^2+x^3+x^4+x^5+x^6)$, which also has the proper behavior: no nonnegative coefficients, coefficients sum to 6. We have

$$x+x^2+x^3+x^4+x^5+x^6 = x\left(\frac{x^6-1}{x-1}\right)$$

$$= \frac{x(x^3-1)(x^3+1)}{x-1} = x(x^2+x+1)(x^3+1)$$

$$= x(x^2+x+1)(x+1)(x^2-x+1).$$

Thus

$$f(x) = x^2(x^2+x+1)^2(x+1)^2(x^2-x+1)^2.$$

Some trial and error (using the useful fact that the sum of the coefficients of a polynomial can be computed by plugging in $x=1$) yields the factorization

$$f(x) = [x(x+1)(x^2+x+1)][x(x+1)(x^2+x+1)(x^2-x+1)^2].$$

The first factor simplifies to

$$x^4 + 2x^3 + 2x^2 + x,$$

and the second factor is

$$x^8 + x^6 + x^5 + x^4 + x^3 + x.$$

Thus the two dice are labeled respectively $\{4,3,3,2,2,1\}$ and $\{8,6,5,4,3,1\}$.

22 Here's a sketch of a solution for $N = 8$. Let the checkers be denoted by zeros (white) or ones (black) with an initial sequence $a_0, a_1, \ldots, a_7$. Define $A(x) := a_0 + a_1 x + a_2 x^2 + \cdots + a_7 x^7$ and consider the doubly infinite generating function

$$f(x) := \cdots + \frac{A(x)}{x^{16}} + \frac{A(x)}{x^8} + A(x) + A(x)x^8 + A(x)x^{16} + \cdots.$$

(If you are worried about convergence issues, you don't need to make this infinite in either direction. We need both positive and negative powers in order to simulate the circular nature of the problem, but we will only focus on the terms with exponents from 0 to 7.) Observe that if we view this as a function with coefficients in *base 2* (binary), then the coefficients with exponents 0 to 7 in $(1+x)f(x)$ are the checker values after one move! This is the crux idea, so make sure you understand it!

Hence after 8 moves, we get $(1+x)^8 f(x)$. But in base 2,

$$(1+x)^8 = 1 + x^8,$$

so we have $(1+x^8)f(x)$. But now the exponent 0 to 7 terms are just $A(x) + A(x) = 0$ in base 2!

5.1

1a Yes; $n \mapsto n+1$ works. So does $n \mapsto n + 1999$, etc.

1b Yes; one example is $n \mapsto 2n + 1$.

4a There are many other approaches (one idea: use the density concept defined in 8.2.12). Here is one method that is easily seen by an example. Let $a = 23/99, b = 37/151$. We will find an irrational number x that lies between them. First, square a and b. We have $a^2 = 0.0539\ldots$ and $b^2 = 0.06004\ldots$. Since 10^4 is a perfect square, the number $u := 550/10^4 = 0.0550$ lies between a^2 and b^2, and moreover, $x := \sqrt{u} = \sqrt{550}/10^2$. Since 550 is not a perfect square, x is irrational, and lies between a and b.

4b Let x and y be irrational numbers. Clearly, one can write them as infinite decimals and find a finite (terminating) decimal between them. For example, if $x = 5.6708903\ldots$ and $y = 5.670931\ldots$, just pick $r = 5.67091$.

5 Each function can be thought of as an m-tuple of values, where each value has n possibilities. By the basic counting principles of Section 6.1, the number of m-tuples will be n^m.

10 This problem is deceptively hard, because it requires subtle inequalities plus a number-theory idea. Let $a := \sqrt{n} + \sqrt{n+1}$. Then

$$a^2 = \left(\sqrt{n} + \sqrt{n+1} \right)^2 = 2n + 1 + 2\sqrt{n^2 + n}.$$

Now $(n+1)^2 = n^2 + 2n + 1$ implies that

$$n < \sqrt{n^2 + n} < n + 1,$$

which means that

$$2n < 2\sqrt{n^2 + n} < 2n + 2.$$

But we can refine this further. Observe that

$$\left(n + \frac{1}{2} \right)^2 = n^2 + n + \frac{1}{4},$$

which means that $n < \sqrt{n^2 + n} < n + \frac{1}{2}$ and consequently

$$a^2 = 2n + 1 + 2\sqrt{n^2 + n} < 2n + 1 + 2\left(n + \frac{1}{2} \right) = 4n + 1 + \theta,$$

where $0 < \theta < 1$. Thus

$$a = \sqrt{4n + 1 + \theta},$$

which means that

$$\lfloor a \rfloor = \left\lfloor \sqrt{4n+1} \right\rfloor.$$

But we wish to show that $a = \left\lfloor \sqrt{4n+2} \right\rfloor$. Simple number theory ideas (see Section 7.2) show that perfect squares are always of the form $4k$ or $4k + 1$, so $\sqrt{4n+2}$ is never an integer, but always shares its integer part with $\sqrt{4n+1}$. In other words,

$$\left\lfloor \sqrt{4n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor,$$

and we are done.

12 (Sketch) Let $t_k := 1 + 2 + \cdots + k$. Thus $t_1 = 1, t_2 = 3, t_3 = 6, t_4 = 10$, etc. If a_n is the nth term of the indicated sequence, then $a_n = r$, where $t_{r-1} + 1 \leq n < t_r$. Using the Gaussian pairing tool, $t_r = r(r+1)/2$. Thus

$$r(r - 1) \leq 2n \leq (r+1)r.$$

At the boundaries, we will have the two quadratic equations

$$r^2 - r + 1 = 2n \quad \text{and} \quad r^2 + r = 2n.$$

By the quadratic formula, we have

$$\frac{-1 + \sqrt{8n+1}}{2} \leq r \leq \frac{1 + \sqrt{8n-7}}{2}.$$

These two boundary points are less than one apart, consequently

$$r = \left\lfloor \frac{1 + \sqrt{8n - 7}}{2} \right\rfloor.$$

13 This problem appeared in the 1995 Hungarian Olympiad and also the 1968 IMO. There are many ways to solve it. One method is induction. Another is to look at the base-2 representation of n. A third method, due to Kiran Kedlaya, uses the easily verifiable identity

$$\left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor - \lfloor x \rfloor$$

which converts the problem into a telescoping sum (see section 5.3).

5.2

25a Factoring, we have $(x - y)(x + y) = 20$. Since x, y are integers, the only possibilities are (notice that $x - y < x + y$)

$$x - y = 1, x + y = 20,$$
$$x - y = 2, x + y = 10,$$
$$x - y = 4, x + y = 5.$$

Only the second system yields integer values; namely $x = 6, y = 4$.

25b The left-hand side contains the first 3 terms of $(x + 3)(y + 5)$. Consequently, the equation can be rewritten as

$$(x + 3)(y + 5) = 215.$$

Now we proceed as in 5.2.25a; the only solution is $x = 2, y = 38$.

26 Consider the expression

$$S := (a^2 + c^2 - 1)^2 + (b^2 + d^2 - 1)^2 + (ab + cd)^2.$$

If we can show that this equals zero, we will be done. Rather than expand blindly, let's think about what will help us: the given three equations, which can also be written as

$$(a^2 + b^2 - 1)^2 = (c^2 + d^2 - 1)^2 = (ac + bd)^2 = 0.$$

Now it is easy to check: when expanding the terms of S, we certainly get all the required 4th-degree terms, and we can rearrange the four terms of the form $-2a^2$ to suit our purposes, and likewise get all six terms of the form a^2b^2. Writing it out in detail almost obscures this analysis!

28 Let $S := xy + yz + xz$. Using 5.2.10, we have

$$1 + 2S = (x + y + z)^2 \geq 0.$$

Thus $S \geq -1/2$. This minimum value is attained for any values of x, y, z that satisfy $x + y + z = 0$ and $x^2 + y^2 + z^2 = 1$; for example $x = 1/\sqrt{6}, y = 1/\sqrt{6}, z = -2/\sqrt{6}$.

30   Messing around with 5.2.10 leads to the following factorization of the left-hand side:

$$\begin{aligned}
n^4 + 2n^3 + 2n^2 + 2n + 1 &= (n^2 + n + 1)^2 - n^2 \\
&= (n^2 + n + 1 - n)(n^2 + n + n + 1) \\
&= (n^2 + 1)(n^2 + 2n + 1) \\
&= (n^2 + 1)(n + 1)^2.
\end{aligned}$$

This quantity is equal to the perfect square m^2, which forces $n^2 + 1$ to be a perfect square. This can only happen if $n = 0$: the only solution is $n = 0, m = 1$.

32   Notice that an n-digit number consisting only of the digit d is equal to $d(10^n - 1)/9$; this is easy to see using the formula for the sum of a geometric series (Section 5.3) or just observing that the n-digit number $11 \cdots 1 = 99 \cdots 9/9$. The nth term in the sequence will then be

$$4 \cdot 10^n \left(\frac{10^n - 1}{9} \right) + 8 \left(\frac{10^n - 1}{9} \right) + 1 = \frac{4 \cdot 10^{2n} + 4 \cdot 10^n + 1}{9}$$

$$= \left(\frac{2 \cdot 10^n + 1}{3} \right)^2.$$

Notice that $(2 \cdot 10^n + 1)/3$ is an integer, since the numerator is a number whose digits sum to 3 (see 7.2.2).

33 $n^4 - 20n^2 + 4 = (n^4 - 4n^2 + 4) - 16n^2 = (n^2 - 2)^2 - (4n)^2 = (n^2 - 4n - 2)(n^2 + 4n - 2)$.

34   Use the factorization of $x^3 + y^3 + z^3 - 3xyz$ developed in Example 3.1.12. The only quantity needed that isn't explicitly given is $xy + yz + xz$, but this is just $[(x + y + z)^2 - (x^2 + y^2 + z^2)]/2$. The answer is -112.

35   Denote the first two expressions in parentheses by A and B, respectively. Then the equation becomes

$$A^3 + B^3 = (A + B)^3,$$

which quickly reduces to $AB(A + B) = 0$, i.e., $A = 0$ or $B = 0$ or $A + B = 0$ are the solutions. This leads to $x = \pm 3/4, \pm 4/3, \pm 1$.

36   Factoring, we get $(x - y)(x^2 + xy + y^2) = 7 \cdot 103$. So either $x - y = 1$ and $x^2 + xy + y^2 = 721$ or $x - y = 7$ and $x^2 + xy + y^2 = 103$. Solving each system yields the two solutions $(16, 15)$ and $(9, 2)$.

37   Cubing a trinomial is too hard! Try isolating $x + y$, and then the right-hand side becomes (using 5.2.4)

$$([x + y] + z)^3 = (x + y)^3 + z^3 + 3(x + y)z(x + y + z).$$

Let $S := x + y + z$, and our equation becomes

$$x^3 + y^3 + z^3 = (x + y)^3 + z^3 + 3(x + y)zS.$$

Subtracting z^3 factoring yields

$$x^3 + y^3 = (x + y)^3 + 3(x + y)zS.$$

The left-hand side factors (5.2.8) into $(x+y)(x^2-xy+y^2)$, hence we can subtract the left-hand side from the right-hand side and factor as follows:

$$(x+y)\{(x+y)^2+3zS-(x^2-xy+y^2)\}=0.$$

The expression in the brackets simplifies to

$$3zS+3xy=3(xz+yz+z^2+xy)=3(x+z)(y+z).$$

Hence we have

$$3(x+y)(y+z)(x+z)=0,$$

so the solutions are all (x,y,z) where at least one pair of the variables are negatives of one another; for example, $(17,-17,675)$, or $(48,201,-201)$.

Notice the nifty factorization

$$(x+y+z)^3-(x^3+y^3+z^3)=3(x+y)(x+z)(y+z).$$

We can deduce this another way using the important **define a function** tool (p. 98). Define the polynomial $P(x):=(x+y+z)^3-(x^3+y^3+z^3)$, where y and z are constants. We have $P(-y)=P(-z)=0$, which implies by the factor theorem (Section 5.4) that P(x) has the factors $(x+y)$ and $(x+z)$. By symmetry (i.e., define a polynomial in y instead) we deduce that $(y+z)$ is also a factor. Hence

$$(x+y+z)^3-(x^3+y^3+z^3)=K(x+y)(x+z)(y+z),$$

where K is a constant that can be computed by plugging in values such as $x=0$. We quickly see that $K=-3$.

5.3

7 Consider the more general arithmetic-geometric series

$$S=a+(a+d)r+(a+2d)r^2+\cdots+(a+nd)r^n.$$

Employing the telescope method again, we compute

$$rS-S=-(a+dr+dr^2+\cdots+dr^n)+(a+nd)r^{n+1}.$$

Most of the above is now just a geometric series, so we have

$$S(r-1)=-a-dr\left(\frac{r^n-1}{r-1}\right)+(a+nd)r^{n+1}$$

and S is just this quantity divided by $r-1$ (no point in simplifying it, or remembering it, for that matter).

9 Experimentation (and analogy from calculus) leads to the guess that the sum is $n(n+1)(n+2)(n+3)/4$, which is easily verified with induction. The generalization is clear: for a series whose terms are products of m consecutive numbers, starting with $1\cdot2\cdot3\cdots m$ and ending with $n(n+1)\cdots(n+m-1)$, the sum is

$$\frac{n(n+1)\cdots(n+m-1)(n+m)}{m+1}.$$

13 Each term in the sum is $k \cdot k! = (k+1)k! - k! = (k+1)! - k!$. Hence the sum telescopes to $(n+1)! - 1$.

14 Likewise,

$$\frac{k}{(k+1)!} = \frac{(k+1) - 1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}.$$

The entire sum telescopes to $1 - 1/(n+1)!$

15 Call the product P. Multiply by the catalyst $\sin\theta$, employing the formula $\sin\alpha\cos\alpha = \sin 2\alpha$:

$$\sin\theta P = \frac{1}{2}\sin 2\theta \prod_{k=1}^{n} \cos\left(2^k\theta\right)$$

$$= \frac{1}{4}\sin 4\theta \prod_{k=2}^{n} \cos\left(2^k\theta\right)$$

$$= \frac{1}{8}\sin 8\theta \prod_{k=3}^{n} \cos\left(2^k\theta\right)$$

$$\vdots$$

$$= \frac{1}{2^n}\sin\left(2^n\theta\right)\cos\left(2^n\theta\right)$$

$$= \frac{1}{2^{n+1}}\sin\left(2^{n+1}\theta\right).$$

Hence

$$P = \frac{\sin\left(2^{n+1}\theta\right)}{2^{n+1}\sin\theta}.$$

19 The problem is easier to work with, believe it or not, if we generalize, replacing $1, 2, \ldots, 2^{n-1}$ with arbitrary $a_1, a_2, \ldots, a_n$. Then (try an example or two) is is easy to conjecture that the sum is $1/(a_1 a_2 \cdots a_n)$. And this is not hard to prove with induction; the only difficulty is notation. Here is an example of how to go from $n = 2$ to $n = 3$: For $n = 2$, the assertion is the easy identity

$$\frac{1}{a}\frac{1}{a+b} + \frac{1}{b}\frac{1}{b+a} = \frac{1}{ab}.$$

For $n = 3$ we want to show that the sum of the six terms of the form

$$\frac{1}{a}\frac{1}{a+b}\frac{1}{a+b+c},$$

where we permute the order of the three variables, is equal to $1/abc$. But this sum factors into

$$\frac{1}{a+b+c}\left(\frac{1}{a}\frac{1}{a+b} + \frac{1}{b}\frac{1}{a+b} + \frac{1}{b}\frac{1}{b+c} + \frac{1}{c}\frac{1}{b+c} + \frac{1}{a}\frac{1}{a+c} + \frac{1}{c}\frac{1}{a+c}\right).$$

The inductive hypothesis simplifies this to

$$\frac{1}{a+b+c}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac}\right),$$

which easily simplifies to $1/abc$.

20   From the bug's point of view, after time 0 it walks 1/3 of the band's length. After time 1 minute, it walks 1/4 of the band's length, after time 2 minutes, it walks 1/5 of the band's length, etc. So we want to know whether the sum $1/3 + 1/4 + 1/5 + 1/6 + \dots$ will exceed 1. It will, since the harmonic series diverges. So the bug will reach the wall (in just a few minutes).

21   The sum converges. Here is a crude estimate: For each positive integer k, there are 9^k elements of S that have k digits. Each of these elements is greater than 10^{k-1} and less than 10^k. The sum of the reciprocals of these elements is bounded above by $9^k/10^{k-1}$. We can evaluate the infinite geometric series

$$\sum_{k=1}^{\infty} \frac{9^k}{10^{k-1}} = \frac{9}{1 - 9/10} = 90,$$

so the sum in question is smaller than 90 (much smaller, since our estimate was so crude). Note how the crude estimate made the sums so easy to compute!

24   See chapter 2 of [9] for a nice discussion of this and much more.

5.4

9   Let $P(x) := x^{81} + x^{49} + x^{25} + x^9 + x$. By the division algorithm, we have

$$P(x) = Q(x)(x^3 - x) + ax^2 + bx + c,$$

where $Q(x)$ is a polynomial and a, b, c are constants to be determined. Since $x^3 - x = x(x-1)(x+1)$, we can plug in $x = 0, 1, -1$ into the previous equation, yielding three equations in three unknowns (notice that $P(0) = 0, P(1) = 5, P(-1) = -5$):

$$0 = c,$$
$$5 = a + b + c,$$
$$-5 = a - b + c,$$

and we have $a = 0, b = 5, c = 0$. Hence the remainder is $5x$.

12   Let $x = \sqrt{2} + \sqrt{5}$. Squaring yields $x^2 = 7 + 2\sqrt{10}$, so $x^2 - 7 = 2\sqrt{10}$. Squaring again, we get $x^4 - 14x^2 + 49 = 40$, so $x^4 - 14x^2 + 9$ is the desired polynomial.

13   See Example 7.1.7.

14   Define $q(x) := p(x) - 1999$. The zeros of q include $0, 1$, so the factor theorem implies that

$$p(x) - 1999 = x(x-1)r(x),$$

where $r(x)$ is a polynomial with integer coefficients. Let u be an integer zero of $p(x)$. Plugging in $x = u$, we get

$$-1999 = u(u-1)r(u).$$

But 1999 is prime, so -1999 cannot be a product of several integers, two of which are consecutive! That's our contradiction; we conclude that $p(x)$ has no integer zeros.

15   We will argue by contradiction. Suppose that $p(x) = u(x)v(x)$. Then one of these factors, say $u(x)$ has degree 999 or less. If $p(x_i) = \pm 1$ for the 1999 different integers $x_1, x_2, \ldots, x_{1999}$, this forces $u(x_i) = \pm 1$ as well. But then for at least 1000 of these x_i, we must have $u(x_i)$ equaling only one value, say -1. By the identity principle (5.4.7), this forces $u(x)$ to be the constant polynomial -1, contradicting the fact that it is a non-trivial polynomial.

18   (Sketch): By looking at constant and linear polynomials, we conjecture that if the degree of $p(x)$ is n, then we need $n+1$ consecutive integer values of x. We can easily prove this by induction. Notice that if $p(k)$ is an integer for all integral k, then $p(0)$, which equals its constant term, must be an integer. If we define $q(x) := p(x+1) - p(x)$, then q has degree one less than that of $p(x)$, and we can recover all values of p from those of q, as long as we know $p(0)$.

19   Hint: think about the Eisenstein irreducibility criterion (3.2.18).

21   Let $Q(x) := P(x) - 5$. Then we have

$$Q(x) = (x-a)(x-b)(x-c)(x-d)R(x),$$

where $R(x)$ is a polynomial with integer coefficients. If $P(k) = 8$, we must have

$$(k-a)(k-b)(k-c)(k-d)R(k) = 3.$$

There is no way that 3 can be written as a product of 4 (or 5) distinct integers.

23   If a polynomial has real coefficients, its non-real roots come in complex conjugate pairs. Conversely, if a polynomial factors into terms of the form $(x-r)$, where r is real, and pairs of factors of the form $(x-z)(x-\bar{z})$, where z is complex, then the polynomial will have real coefficients. With these ideas, we will show how to solve the problem with a simpler example. Let $P(x)$ be a quadratic with real coefficients and complex roots $z, \bar{z}$. We will find a polynomial $Q(x)$ with real coefficients such that the degree of all terms of $Q(x)P(x)$ is a multiple of 3. Consider the polynomial

$$R(x) := (x^3 - z^3)(x^3 - \bar{z}^3).$$

By the remarks above, it has real coefficients, and the degree of all terms is a multiple of 3. Furthermore,

$$\frac{x^3 - z^3}{x - z} = (x - \zeta z)(x - \zeta^2 z),$$

where $\zeta = \text{Cis}(2\pi/3)$ is a complex cube root of unity. Notice that $\zeta^2 = \bar{\zeta}$. Consequently,

$$Q(x) := \frac{R(x)}{P(x)} = (x - \zeta z)(x - \zeta^2 z)(x - \zeta \bar{z})(x - \zeta^2 \bar{z})$$

will have real coefficients, since

$$\overline{\zeta z} = \zeta^2 \bar{z}, \quad \overline{\zeta^2 z} = \zeta \bar{z}.$$

This method certainly generalizes. If we want multiples of 10^9, we just define $R(x) := (x^{10^9} - z^{10^9})(x^{10^9} - \bar{z}^{10^9})$, and $Q(x) := R(x)/P(x)$ (Verify that the zeros will pair off into conjugate pairs!)

5.5

13 Since $1999!^{(2000)} = (1999!)!^{(1999)}$, and $1999!$ is much larger than 2000, then $1999!^{(2000)}$ is much much larger than $2000!^{(1999)}$.

14 The function $f(x) := (x+1)/(10x+1)$ is monotonically decreasing, with a limiting value of $1/10$ as x approaches $+\infty$. (This can be verified by looking at the derivative, or using the "?" technique of Example 5.5.9). Consequently, $f(10^{1999}) < f(10^{1998})$.

15 $1000^{2000} > 2000!$.

16 Your instinct should tell you that 1999^{1999} is larger than 2000^{1998}, since the former number involves more exponentiation yet its base is virtually the same as the latter number. Upon division, comparing the two numbers boils down to comparing 1999 with $(2000/1999)^{1998}$. This latter number is not very big, for it is bounded above by $e = 2.71828\ldots$. This is true because $\lim_{x\to\infty}(1 + 1/x)^x = e$, approaching the limit from below. While this may or may not be a familiar idea to you from calculus, here is a sketch of a low-tech way to verify the inequality: use the binomial theorem to evaluate $(1 + 1/n)^n$: We have

$$\left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{n} + \frac{1}{n^2}\binom{n}{2} + \frac{1}{n^3}\binom{n}{3} + \cdots.$$

Since

$$\binom{n}{r} < \frac{n^r}{r!},$$

we have

$$\frac{1}{n^r}\binom{n}{r} < \frac{1}{r!},$$

and consequently the binomial theorem sum above is less than

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots,$$

and this can in turn can be bounded above using a crude estimate with an infinite geometric series:

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 3.$$

24 See 5.5.16 above.

27 The crux move is to rationalize the *numerator*. We have

$$\sqrt{a+1} - \sqrt{a} = \frac{(a+1)-a}{\sqrt{a+1}+\sqrt{a}} = \frac{1}{\sqrt{a+1}+\sqrt{a}}.$$

This value is very very close, for large a, to $1/2\sqrt{a}$, so in our case the answer is $n = 2000$.

28 Use Gaussian pairing and AM-GM.

31 The nasty thing here are the square roots, so let us square the alleged inequality, leading to a potentially nicer one:

$$\frac{1}{4n} \leq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \cdot \frac{2n-1}{2n} \leq \frac{1}{2n}.$$

Call the product above P. Now we can employ the massage-telescope tool on P, by altering half of the terms: In one direction, we have

$$P < \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdots \frac{2n-1}{2n} \cdot \frac{2n}{2n} = \frac{1}{2n},$$

and in the other direction (we leave the first two terms alone) we have

$$P > \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdots \frac{2n-2}{2n-1} \cdot \frac{2n-1}{2n} = \frac{1}{4n}.$$

33 Consider the equation $x^a = a^x$, where $a > 1$ is fixed. Obviously one solution is $x = a$. However, an examination of the graphs of $y = x^a$ and $y = a^x$ indicates that there will be a second, larger solution. Taking logarithms (base to be determined later), we get $a\log x = x\log a$, or $\dfrac{x}{a} = \dfrac{\log x}{\log a}$. The symmetrical nature of this equation suggests that we reduce to one variable by setting $r = x/a$. This yields $r = \dfrac{\log r + \log a}{\log a}$. If we choose base r for our logarithms, this equation reduces to $r\log a = 1 + \log a$, so $\log a = \dfrac{1}{r-1}$. Thus $a = r^{\frac{1}{r-1}}$. In other words, if $a = r^{\frac{1}{r-1}}$, where $r > 1$, the two solutions to $x^a = a^x$ are

$$x = a \quad \text{and} \quad x = ra = r \cdot r^{\frac{1}{r-1}} = r^{\frac{r}{r-1}}.$$

34 Recast the problem geometrically. Note that $\sqrt{a^2 + b^2}$ is the length of the vector that starts at the origin ending at the point (a,b). Thus we can interpret the sum

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_n^2 + b_n^2}$$

as the total length of n vectors. If we arrange these vectors *end-to-end* then this sum is the length of a piecewise linear path. Since the shortest distance between two points is a straight line, the total length is at least as big as the straight-line path from the start $(0,0)$ to the finish $(a_1 + a_2 + \cdots + a_n, b_1 + b_2 + \cdots + b_n)$. But this is equal to

$$\sqrt{(a_1 + a_2 + \cdots + a_n)^2 + (b_1 + b_2 + \cdots + b_n)^2}.$$

38   Apply Cauchy-Schwarz to the two sequences $a_1, a_2, \ldots, a_n$ and $1, 1, \ldots, 1$. We get

$$(a_1 \cdot 1 + a_2 \cdot 1 + \cdots + a_n \cdot 1)^2 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} 1^2.$$

This reduces to

$$1 \le n \sum_{i=1}^{n} a_i^2.$$

40   Apply Cauchy-Schwarz to the two sequences $\sqrt{a}, \sqrt{b}, \sqrt{c}$ and $1, 1, 1$.

41   Apply Cauchy-Schwarz to the two sequences $\sqrt{a}, \sqrt{b}$ and $1/\sqrt{a}, 1/\sqrt{b}$, which yields

$$\left(\sqrt{a} \frac{1}{\sqrt{a}} + \sqrt{b} \frac{1}{\sqrt{b}} \right)^2 \le (a+b) \left(\frac{1}{a} + \frac{1}{b} \right),$$

in other words,

$$\frac{1}{a} + \frac{1}{b} \ge \frac{4}{a+b}.$$

Now this inequality can be applied twice more:

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} \ge \frac{4}{a+b} + \frac{4}{c} \ge \frac{16}{a+b+c},$$

and

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \ge \frac{16}{a+b+c} + \frac{16}{d} \ge \frac{64}{a+b+c+d}.$$

Incidentally, the key inequality used (worth remembering) is that for positive $a_1, a_2, \ldots, a_n$,

$$\left(\sum_{i=1}^{n} a_i \right) \left(\sum_{i=1}^{n} \frac{1}{a_i} \right) \ge n^2.$$

42   Use the relationship between zeros and coefficients (pp. 168–9) and the fact that a sum of squares of *real* numbers must be nonnegative.

43   Applying Cauchy-Schwarz to the sequences x, y, z and $1, 1, 1$ yields the inequality

$$3(x^2 + y^2 + z^2) \ge x + y + z,$$

which is true for all real x, y, z. Next, apply AM-GM with the hypothesis that $xyz = 1$ (since $x, y, z \ge 0$, AM-GM can be used) and we get

$$x^2 + y^2 + z^2 \ge \frac{x+y+z}{3} \ge \sqrt[3]{xyz} = 1.$$

47   Just apply AM-GM; the product of the *n* fractions is 1, so their arithmetic mean is at least 1, hence their sum is at least *n*.

48   Study the proof of Chebyshev's inequality and develop an algorithmic argument: Start with the sum $\sum a_i b_i$, and show that any permutation of just two members of $b_1, b_2, \ldots, b_n$ will reduce the sum. Since any permutation can be achieved by finitely many "swaps" of two elements, you are done.

6.1

20a   Consider n boys and n girls. Choosing two people from this set of $2n$ can be broken into the three mutually exclusive cases of either picking only two girls (in $\binom{n}{2}$ ways) or only two boys (in $\binom{n}{2}$ ways) or picking one boy and one girl (in n^2 ways). Notice that the case of picking one girl and one boy is redundant; the order of pick is immaterial.

22a   Plug $x = 1$ into the expansion for $(x+1)^n$.

22b   Plug $x = -1$ into the expansion for $(x+1)^n$.

24a   See Section 6.2.

24b   This is pretty hard, requiring at least an intuitive undestanding of PIE (see pp. 226–229).

25   There are 100 perfect cubes and 1000 perfect squares under a million, hence at most $100 \cdot 1000 = 100000$ distinct sums (perhaps less, if there is overlap. Since there are 900000 numbers left over, the numbers not representable as a sum of a cube and a square are in the majority.

27a   17!

27b   Either boys go first, or girls. So there are two cases, each with 8!9! arrangements, so the answer is $2 \cdot 8!9!$.

27c   The arrangement must be girl-boy-...-girl, so answer is 8!9!

28a   10!

28b   Arrange the kids in some fixed order, such as height. Pick the first person in the line, pair them up with someone. Then go down the line, pairing up the first person in the line with someone chosen from among the unpaired population. The answer is $19 \cdot 17 \cdot 15 \cdots 1$. Alternatively, take the answer in (c) below, and divide by 10! to fix the overcounting, since honeymoons are indistinguishable in (b).

28c   Assign each honeymoon location a letter a, b, c,..., j. Line up the 20 people in some order. Now assign the honeymoon letters to the people. This is equivalent to creating a word with 20 letters, consisiting of 2 a's, 2 b's, etc. So by the Mississippi formula, the answer is $20!/2^{10}$.

29 $(x_1 + x_2 + \cdots + x_n)^r$ consists of the sum of all terms of the form

$$\frac{r!}{a_1! a_2! \cdots a_n!} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

where the a_i range over all nonnegative values that sum to r.

6.2

8 Here is one nice method. There are other solutions.

1. First, Pat and Sal establish a "lexigraphic" (dictionary) order for the cards in a deck. For example, the cards can be ordered

$$A\clubsuit, 2\clubsuit, \ldots, K\clubsuit, A\diamondsuit, \ldots, K\diamondsuit, A\heartsuit, \ldots, K\heartsuit, A\spadesuit, \ldots, K\spadesuit.$$

2. Within each suit, cards are ordered from 1 to 13. Given two cards of the same suit, place them on a 13-hour clock. Notice that the no matter what the two cards are, one is at most 6 clockwise units away from the other. For example, the clockwise distance between $4\clubsuit$ and $K\clubsuit$ is 4, since you can travel 4 units clockwise from 13 to get to 4.

3. Next, Pat looks at the five cards. By the pigeonhole principle, there are at least two cards that are of the same suit. Pat choses one of these and puts it back in the deck and puts the other card on the top of the pile. Pat also computes the clockwise distance from this card to the "hidden" card. For example, suppose the five cards chosen are $3\clubsuit, J\heartsuit, 10\diamondsuit, Q\clubsuit, A\spadesuit$. Pat puts the $3\clubsuit$ back into the deck and puts the $Q\clubsuit$ on the top of the pile, and notes that the distance from $Q\clubsuit$ to $3\clubsuit$ is 4 (since $Q = 12$ and $12 + 4 \equiv 3 \pmod{13}$).

4. Now all Pat needs to do in encode a number between 1 and 6 with the remaining three cards in the pile. This can be done by agreeing in advance which of the $3! = 6$ lexigraphic permutations refers to which number between 1 and 6. For example, if we order permutations "alphabetically" like so:

$$(a,b,c), (a,c,b), (b,a,c), (b,c,a), (c,a,b), (c,b,a),$$

Then we would encode "4" by placing the bottom three cards in the lexigraphic order (b,c,a). In our example, the three remaining cards are $J\heartsuit, 10\diamondsuit$ and $A\spadesuit$ and their lexigraphic order is

$$10\diamondsuit < J\heartsuit < A\spadesuit.$$

Hence the (b,c,a) ordering would correspond to $J\heartsuit, A\spadesuit, 10\diamondsuit$.

5. So the final ordering (from top to bottom) for our example would be

$$Q\clubsuit, J\heartsuit, A\spadesuit, 10\diamondsuit.$$

9 Each person has four possible fates: no treat, only cookie, only ice cream, both treats. Thus there are 4^8 possible scenarios. But we must exclude the impossible scenarios—no one getting anything, and no one getting both treats. The first of these happens in just one way, and the second happens in 3^8 ways. But notice that the second scenario includes the first. So our answer is $4^8 - 3^8$.

10 The answer is 2^{29}. Notice that the sum of all the elements of the set $S = \{1, 2, 3, \ldots, 30\}$ is $1 + 2 + 3 + \cdots + 30 = 30 \cdot 31/2 = 465$. Let A be a subset of S and let A^c denote the complement of A The sum of the elements of A plus the

sum of the elements of A^c must equal 465. Because $465 = 232 + 233$, if the sum of the elements of a A is greater than 232, the sum of the elements of A^c must be less than 232. In other words, there is a one-to-one correspondence between subsets whose element sum is greater than 232 and subsets whose element sum is not (namely, $A \leftrightarrow A^c$). Hence the number of subsets whose element sum is greater than 232 is exactly half of the total number of subsets of S, and the number of subsets of S is 2^{30}.

11   There is a one-to-one correspondence between strictly increasing sequences starting with 1 and ending with 1000 and subsets of the set $\{2, 3, \ldots, 999\}$. So there are 2^{998} such sequences.

13   We will combine an encoding idea with an induction argument. Let n be odd, and let A be a subset of an n-element set S. Then $|A|$ and $|\overline{A}|$ have opposite parity. Hence there is a one-to-one correspondence between the subsets of S with an odd number of elements and the subsets with an even number of elements. Hence there are an equal number of both kinds.

This argument won't work is n is even, though. But now let us move inductively from n to $n+1$, where n is still odd. So consider adding a new element x to the set S. The subsets of this new set $T := \{x\} \cup S$ fall into two classes: subsets containing x, and subsets which do not. The subsets of T which have an odd number of elements are either subsets of S with an odd number of elements, or subsets of S with an even number of elements with x added in. Thus the number of subsets of T which have an odd number of elements is equal to the number of subsets of S. And of course, the argument is the same for subsets with an even number.

16   Of course we assume that $r, w \geq n$. First, place one red and one white ball in all n boxes. We are left with $r - n$ red balls, which can be freely placed in $\binom{n+r-n-1}{r-n} = \binom{r-1}{n-1}$. Likewise, the white balls can be freely placed in $\binom{w-1}{n-1}$ ways. The answer is the product $\binom{r-1}{n-1}\binom{w-1}{n-1}$.

17   First of all, there are $\binom{12}{4}$ possible arrangements, which we assume are all equally likely. It is easiest to now compute the complement: how many arrangements do not allow the SUV to park? Let e, c respectively denote an empty space and a compact car. Then we must start by surrounding each empty space with compact cars as follows: ececece. We still have 5 c's to distribute. The locations where they may be placed are: before the first e, after the final e, and between consecutive e's, a total of 5 possible slots. This is just a ball and box problem; there are $\binom{5+5-1}{5} = \binom{9}{5}$ possible ways. So the probability is $1 - \binom{9}{5}/\binom{12}{4}$.

19   It is easy come up with the conjecture of 2^{n-1} which certainly suggests some kind of simple binary coding. Imagine the number n represented by n dots in a row. Beginning with the space between the first two dots, either write a plus sign, or not. Do the same with the other spaces between dots. There are $n-1$ such spaces, so there are 2^{n-1} ways of writing the plus signs, and each way corresponds uniquely to an ordered partition of n.

21   In this case, the only distinguishable thing is the ration of biscuits; for ex-

ample, "one dog gets all 8 biscuits" is different from "one dog gets 5 biscuits and 3 other dogs get one biscuit apiece." Hence the answer is the number of unordered partitions of 8, which is 22. See Problem 4.3.17 for a generating-functionological interpretation.

22   As discussed in Problem 6.1.27, each distinct term is of the form

$$\frac{1999!}{a!b!c!}x^a y^b c^z,$$

where a, b, c are nonnegative and sum to 1999. Following Example 6.2.7, we have a ball and box problem whose answer is $\binom{1999+3-1}{1999}$.

23   This is just like 6.2.7, except now we must first place a ball in each box, leaving $n-3$ balls to freely distribute among the 3 boxes.

24   This is a modification of a 1993 AIME problem. The crux idea: each element is either in the first subset, the second, or both. Hence there are 3 choices for each element. And there is overcounting to eliminate.

28   Each intersection point is determined by four points: the endpoints of the two chords which are intersecting. Every different choice of four points yields a different intersection point. So we have a one-to-one correspondence, and the answer is $\binom{n}{4}$.

31   Let's verify it for the case $k = 6, n = 11, m = 107$. The formula then states that

$$\sum_{j=0}^{6}\binom{11}{j}\binom{107}{6-j} = \binom{11}{0}\binom{107}{6} + \binom{11}{1}\binom{107}{5}$$
$$+ \binom{11}{2}\binom{107}{4} + \cdots + \binom{11}{6}\binom{107}{0}$$
$$= \binom{118}{6}.$$

The right-hand side corresponds to the number of ways you can pick a 6-person team chosen from 118 people in a room. Assume that 11 people are men and 107 are women. Then your 6-person teams can be partitioned into the classes: 0 men and 6 women, 1 man and 5 women, 2 men and 4 women, …, 6 men and 0 women. Each term in the left-hand side has the form $\binom{11}{j}\binom{107}{6-j}$, which is exactly the number of ways of picking a 6-person team in which j are men and $6-j$ are women, chosen from a population of 11 men and 107 women!

(See also 4.3.9 for a generatingfunctionological approach.)

6.3

A nice source of mostly easy-to-moderate exercises and problems about PIE can be found in chaper 8 of [23], which also relates this material nicely to generating functions (which are implicitly used in the indicator functions argument of pp. 212–214 in TAACOPS).

8   Now it is ball and box problem, with 31 boxes and 10 balls. The number of unrestricted orders is then $\binom{31+10-1}{10}$. We must subtract from this the number of orders in which all the flavors are different. This means that we choose 10 flavors and place one ball into each of the 10 chosen flavors. The number of ways of doing this is just $\binom{31}{10}$. So our answer is $\binom{40}{10} - \binom{31}{10}$.

9   We are undercounting. Suppose the first person chosen was b_3. Then the second cannot be g_3, so we do have 6 choices. Suppose we choose b_2. The next choice cannot be g_2, but it can be g_3 now, so there are actually 6 choices, not 5. The problem is that if we now choose g_3, we have fewer choices for the next slot than if we don't choose g_3. So the numbers available are not quite independent of the choices—the multiplication method doesn't work.

10   See the discusssion of the ϕ function on pp. 259–261.

11   Use complement PIE with the properties "*not* multiple of 2", "*not* multiple of 5."

12   By Example 6.2.7, the number of non-negative solutions (with no restrictions) is $\binom{17+4-1}{17} = \binom{20}{3}$. We will subtract from this the complement: the number of solutions where $d > 12$. This is equal to the number of nonnegative solutions to $a + b + c + u = 4$, where $u := d - 13$. So our answer is $\binom{20}{3} - \binom{4+4-1}{4} = \binom{20}{3} - \binom{7}{3}$.

13   Let us do an example with $n = 8$. Suppose we have 8 letters, each addressed to a different address, and we randomly deliver them to the 8 addresses. We wish to compute the number of ways that we can do this so that no one gets the correct letter. First of all, the "things" that we are counting are permutations of the 8 letters. So using the language of complement PIE (6.3.6), $N = 8!$. Let the sets A_j be all permutations in which "letter j went to the right place," for $j = 1, 2, \ldots, 8$. First, let's compute $|A_j|$. This is just the number of permutations which "fix" the jth letter, or 7!. (Notice that this does not depend on j.) Now let us compute $|A_i \cap A_j|$ for $i \neq j$. This is the same as counting all permutations of 8 letters which fix two letters, namely the ith and jth letters. Thus $|A_i \cap A_j| = 6!$. By the same reasoning, $|A_i \cap A_j \cap A_k| = 5!$, etc.

Now it is time to count up everything. We have

$$S_1 = 8 \cdot 7! = 8!,$$

since there are 8 different A_j sets. On the other hand, there are 8*choose*2 different sets of the form $A_i \cap A_j$, corresponding to each choice of i and j. Thus

$$S_2 = \binom{8}{2} 6!.$$

Likewise,

$$S_3 = \binom{8}{3} 5!,$$

etc. When the dust settles, the number of permutations that fix no letters will be

$$8! - 8 \cdot 7! + \binom{8}{2} \cdot 6! - \binom{8}{3} \cdot 5! + \cdots + 1.$$

Remember that we can always write $\binom{n}{r}$ as $\frac{n!}{r!(n-r)!}$. When this is done, and we factor out 8!, we find that

$$D_8 = 8!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!}\right).$$

14 Partition the $n!$ permutations by the exact number of fixed points, i.e., the number of elements which are not not displaced. If there are r fixed points, we first choose the elements which will be fixed. We can do this in $\binom{n}{r}$ ways. Then the remaining $n-r$ elements must have no fixed points, and this can be done in D_{n-r} ways.

16 Among the k^n unrestricted assignments of cones to children, let A_r denote the set of assignments for which flavor r is *not* used. By the language of complement PIE, $N = k^n$ and the value we seek is

$$N - S_1 + S_2 - \cdots.$$

Now, there are k ways of choosing a flavor that won't be used, and for each flavor so negatively chosen, there will be $(k-1)^n$ possible assignments. Hence $S_1 = k(k-1)^n$. Likewise,

$$S_2 = \binom{k}{2}|A_i \cap A_j|,$$

and $|A_i \cap A_j| = (k-2)^n$, etc.

17 Hint: find a one-to-one correspondence between permutations without property P to *a proper subset* of permutations with property P.

6.4

4 Let m_n be the number of messages possible with n keystrokes. Clearly $m_1 = m_2 = 1$. In general, given n keystrokes, messages either begin with two consecutive letters or a single letter followed by a single space. There are m_{n-1} of the first type and m_{n-2} of the second. So $m_n = m_{n-1} + m_{n-2}$: a Fibonacci recurrence.

6 Let s_n be the number of such subsets. We have $s_2 = 3, s_3 = 5$ (including the empty set). Now let us compute s_{n+1}. The subsets under consideration partition into two classes: those which contain the element $n+1$, and those that do not. The second class consists of the s_n subsets of $\{1,2,\ldots,n\}$ which contain no two consecutive elements. The first class consists of subsets of $\{1,2,\ldots,n-1\}$ which contain no two consecutive elements, each with the element $n+1$ appended. There are s_{n-1} such subsets, so

$$s_{n+1} = s_n + s_{n-1},$$

yielding yet another Fibonacci recurrence.

12 Fibonacci, of course.

13   Let us compute $u(n+1)$, where the $(n+1)$st element is x. Consider the partitions of the $(n+1)$-element set including x by focusing on x. This element has to live somewhere; let us partition all partitions according to how many neighbors x has in its partition. If x is alone, there are $u(n)$ possible ways to partition the other elements. If x has one neighbor, there are n choices for this neighbor and then $u(n-1)$ ways to partition the others. If x has two neighbors, there will be $\binom{n}{2}$ choices of neigbors with $u(n-2)$ ways to partition the rest. In general, we have

$$u(n+1) = \sum_{r=0}^{n} \binom{n}{r} u(n-r),$$

where we use the convention that $u(0) = 1$ (there is only one way to partition the empty set, after all).

14   This is secretly the same as problem 6.4.9.

15   Let's verify it for $n = 5$. Imagine slots numbered $1,2,3,4,5$ into which we want to derange the letters a,b,c,d,e (so a cannot go into slot 1, b cannot go into slot 2, etc.). There are 4 choices for slot 1, namely b,c,d,e. We will show that, for each of these choices, there are $D_4 + D_3$ ways of deranging the other 4. Without loss of generality, suppose that we chose c for slot 1. Of the remaining choices, there are two cases:

1.   Slot 3 contains a. Now slots 1 and 3 have swapped their proper members, i.e., these two slots form a derangement. We need to derange b,d,e in slots $2,4,5$, and this can of course be done in D_3 ways.

2.   Slot 3 doesn't contain a. Then we need place a,b,d,e in slots $2,3,4,5$ so that b,d,e respectively do not land in $2,4,5$, and a doesn't land in slot 3. Although the numbering is unorthodox (we normally associate a with 1), this is precisely a derangement of 4 elements. So there are D_4 ways.

Hence the total number of derangements is $4(D_3 + D_4)$.

16a   There is a 1-1 correspondence between the legal partitions of $2n+1$ and the legal partitions of $2n$: just remove one of the 1's from the former to get the latter (note that any legal partition of $2n+1$ must have at least one 1).

16b   The legal partitions of any number can be broken into two mutually exclusive classes: those that contain 1's, and those that do not. The legal partitions of $2n$ which do not contain one can be placed into 1-1 correspondence with the legal partitions of n: just divide the terms of the former by 2 to get the latter. Likewise, there is a 1-1 correspondence between the legal partitions of $2n$ which do contain 1's and the legal partitions of $2n - 1$: just remove one of the 1's from the former to get the latter.

16c   Once (a) and (b) are established, this is a trivial induction (as long as you use strong induction).

7.1

An Introduction to the Theory of Numbers, by Niven, Zuckerman, and Montgomery [21] is an excellent reference to have on hand for this section and sections 7.2–7.5.

15 For example, let f_n denote the nth Fibonacci number. We have

$$(f_n, f_{n+1}) = (f_n, f_{n+1} - f_n) = (f_n, f_{n-1}) = (f_{n-1}, f_n).$$

This process can be repeated until we get $(f_n, f_{n+1}) = (f_1, f_2) = 1$.

19 Let n be an arbitrary positive integer. Then the $n-1$ numbers

$$n! + 2, n! + 3, \ldots, n! + n$$

are all composite, being divisible respectively by $2, 3, \ldots, n$.

20 See 3.4.20.

22 Each zero at the end of a number represents a 10 in the factorization of the number. Hence we need to find the greatest k such that $10^k | 1000!$. Since $10 = 2 \cdot 5$ and there will always be many more 2's than 5's in the factorization, we need only focus on the exponent k of 5 in the PPF of 1000!. Since

$$1000! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots 1000,$$

we might first think that $k = 1000/5 = 200$, corresponding to the 200 multiples of 5 between 1 and 1000. But some of these multiples are richer in 5's than others: each multiple of 25, and there are $1000/25 = 40$ of them, contribute two 5's apiece, only one of which was counted. Likewise, there are $1000/125 = 8$ multiples of 5^3 and $fl1000/625 = 1$ multiple of 5^4. This adds up to $200 + 40 + 8 + 1 = 249$, as desired.

This is easy to generalize. Let p be a prime. Then the greatest power of p in the PPF of $n!$ will be

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots.$$

It doesn't hurt to make this sum infinite, since eventually, all the terms will be zero.

23 Write

$$\binom{p^r}{k} = \frac{(p^r)!}{k!(p^r - k)!},$$

and employ the formula from 7.1.22 to compare the powers of p in the numerator and denominator.

24 Let $a := f_k$ be the kth Fibonacci number and suppose it is a multiple of n. Let $b := f_{k+1}$. Then

$$\begin{aligned}
f_{k+2} &= f_{k+1} + f_k = a + b, \\
f_{k+3} &= f_{k+2} + f_{k+1} = a + 2b, \\
f_{k+4} &= f_{k+3} + f_{k+2} = 2a + 3b, \\
f_{k+5} &= f_{k+4} + f_{k+3} = 3a + 5b,
\end{aligned}$$

and so on. It is easy to see that the coefficients of a and b are themselves Fibonacci numbers! Indeed, we have

$$f_{k+r} = f_{r-1}a + f_r b.$$

Consequently,

$$f_{2k} = f_{k-1}a + f_k b = f_{k-1}a + ab = a(f_{k-1} + b)$$

will be a multiple of n, since it in fact a multiple of f_k itself.

31a Consecutive numbers are relatively prime, so their product cannot be a square unless both numbers were squares, but consecutive numbers cannot be perfect squares (besides 0 and 1).

31b We argue by contradiction. Suppose the first number is odd. Then the third number is also odd, and relatively prime to the first two numbers. So all three numbers are relatively prime in pairs. The product cannot be a perfect square, by the same reasoning used in 7.1.28a. Now, suppose that the first number is even. The middle number must be an odd perfect square (since it is relatively prime to the other two). Thus the numbers are $a^2 - 1, a^2, a^2 + 1$. Since the product of all three is a perfect square, $(a^2 - 1)(a^2 + 1) = a^4 - 1$ must be a perfect square. But since a^4 is also a perfect square, this again implies two consecutive perfect squares, an impossibility.

32 Since $(a_n, a_{2n}) = (n, 2n) = n$ for any n, we see that $n|a_n$, so among other things, a_n is at least as large as n. Now, suppose that there is an n such that $a_n \neq n$. By the above reasoning, we have $a_n = u = rn$ for some $r > 1$. But the hypothesis of the problem yields

$$(a_n, a_u) = (n, u) = (n, rn) = n.$$

But $a_n = u$ and a_u is a multiple of u, so $(a_n, a_u) = u > n$, a contradiction.

7.2

6 All perfect squares are congruent to either 0 or $+1$ modulo 3; this follows from simply squaring the only three choices modulo 3: $0^2 \equiv 0, 1^2 \equiv 1, 2^2 \equiv 1$. Thus a^2 and b^2 cannot both be congruent to 1 mod 3, for that would make c^2 congruent to 2, which is impossible. The only possiblilites are that one or both of a is congruent to 0, forcing ab to be a multiple of 3.

7 The only cubes modulo 7 turn out to be $0, 1, -1$.

9 Again, use mod 7.

10 $N = 381654729$.

15 No. Let $a_0 = p$, a prime. Then $a_r = 2^r p + 2^r - 1$. By Fermat's little theorem, $2^{p-1} \equiv 1 \pmod{p}$, so a_{p-1} will be a multiple of p.

7.3

1   Since $1 \perp 1$, we have $f(1) = f(1 \cdot 1) = f(1)f(1)$, which means that $f(1)$ equals 0 or 1. The first possibility is excluded because the range of f is $\mathbb{N}$.

6   Let $d \mid n$, where $n = ab$ with $a \perp b$. By the FTA, we can write $d = uv$, with $u \mid a$ and $v \mid b$. The trickier part is showing that this is a one-to-one correspondence. It suffices to show that if the u, u' and v, v' are respectively divisors of a and b and $uv = u'v'$, then $u = u', v = v'$. This follows by mimicking the final steps of the proof of the FTA on p. 247.

14   Suppose $\phi(n) = 14$. We can assume without loss of generality that n is odd, for if n is even then $n = 2s$, where s is odd and $\phi(s) = 14$ as well (if $n = 2^r s$ then $\phi(n) = \phi(2^r)\phi(s)$ which is a multiple of 4, because $\phi(2^r) = 2^{r-1}$ and $\phi(s)$ is even).

   If n is odd, it cannot contain more than one prime in its PPF, since if we write $n = ab$ where $a, b > 1$ and $a \perp b$, we have $\phi(n) = \phi(a)\phi(b)$ which is a product of two even numbers.

   The only possibility is the $n = p^r$ for some odd prime p. But then $\phi(n) = p^{r-1}(p-1) = 14$, which has no solution (since $p = 3$ and $p = 7$ are the only possibilities).

22 F is the indicator function for perfect squares, i.e., $F(n) = 1$ if n is a perfect square and $F(n) = 0$ otherwise.

24   Certainly, if n is a prime, then $\phi(n) + \sigma(n) = (n-1) + (n+1) = 2n$. To prove the converse, we will show that if n is composite, then $\phi(n) + \sigma(n) > 2n$. Suppose that n is not a prime, but has only one prime divisor p. Then $\phi(n) = n - n/p$ while $\sigma(n) \geq n + n/p + 1$, so $\phi(n) + \sigma(n) > 2n$. Next, suppose that n has two distinct prime factors, p and q. Then we have

$$\phi(n) = n - \frac{n}{p} - \frac{n}{q} + \frac{n}{pq},$$

while

$$\sigma(n) \geq n + \frac{n}{p} + \frac{n}{q} + \frac{n}{pq},$$

and once again $\phi(n) + \sigma(n) > 2n$. The pattern is clear, now: if n is not prime, $\phi(n)$ will be an alternating sum starting with n, and $\sigma(n)$ will contain at least all the terms in $\phi(n)$, but all positive, so when $\phi(n)$ is added to $\sigma(n)$, some terms will cancel, but not all, leaving $2n$ plus some extra stuff.

26   (Sketch) We argue by contradiction. Assume that there exists a function $f(n)$ that is strictly increasing, multiplicative, and where $f(n) > n$ starting at some $n = n_0$. Our strategy is to get some upper bounds on $f(n)$, and then contradict them. We shall use the fact that $f(2m) = 2f(m)$ for odd m. For any k, we have

$$f(2^k) < f(2^k + 2)$$
$$= 2f(2^{k-1} + 1)$$
$$< 2f(2^{k-1} + 2)$$

$$= 2^2 f(2^{k-2} + 1)$$
$$\vdots$$
$$< 2^{k-1} f(3).$$

Thus $f(2^k)/2^k < f(3)$, which is a constant. Since k is arbitrary, this means that the function essentially grows in a linear way; it cannot get too wildly out of control, since it is strictly increasing. We will get a contradiction once we show that we can find an n such that $f(n)/n$ can be made arbitrarily large. But this is easy: suppose that the p_1 is the first prime for which $f(p_1) > p_1$. Then we have $f(p_1) \geq p_1 + 1$, and in fact, $f(k) \geq k + 1$ for all $k \geq p_1$. Since there are infinitely many primes, we can let $n = p_1 p_2 \cdots p_r$, where the p_i are successive primes and r is as big as we please. Then

$$\frac{f(n)}{n} \geq \left(\frac{p_1 + 1}{p_1} \right) \left(\frac{p_2 + 1}{p_2} \right) \cdots \left(\frac{p_r + 1}{p_r} \right),$$

and this expression diverges as $r \to \infty$ (see Example 9.4.7), achieving the desired contradiction.

27   There are many ways to do this. One method is to write $9^{99} = (10 - 1)^{99}$ and use the binomial theorem, noting that most of the terms are multiples of 100.

Another is to use Euler's extension of Fermat's little theorem (Problem 7.3.25). Since $\phi(100) = 40$, we have $9^{80} \equiv 1 \pmod{100}$, so the problem is reduced to finding 9^{19} modulo 100. Break 100 into relatively prime factors $4 \cdot 25$. It is easy to see that $9^r \equiv 1 \pmod 4$ for any r, and Euler's extension again yields $9^{20} \equiv 1 \pmod{25}$. Thus $9^{20} \equiv 1 \pmod{100}$. So, if $9^{19} \equiv x \pmod{100}$, we have $9x \equiv 1 \pmod{100}$, which is equivalent to the linear diophantine equation $9x - 100y = 1$. This can be easily solved by the methods of 7.1.13, yielding $x = 89$.

Certainly the second method is harder, but more instructive, and potentially more useful (we were lucky that $9 = 10 - 1$).

28–31   There are many excellent number theory texts, of course, but I am especially fond of the clarity of Vanden Eynden's approach to the Möbius inversion formula in [25]. This is a perfect source for the beginner. On a more sophisticated level, section 2.6 of Wilf's *generatingfunctionology* has a fascinating discussion that is also good supplementary reading for section 9.4 of TAACOPS.

7.4

6   The first equation can be solved by the factor tool. It is equivalent to $5^n = m^2 - 2500 = (m + 50)(m - 50)$, and the only solution is $n = 5$ (since there is only one pair of powers of 5 whose difference is 100. The second equation has a similar solution; $n = 8$.

7 Without loss of generality, assume $a \geq b \geq c$. Note that we must have $2 \leq (1 + 1/c)^3$. This implies that $c \leq 3$. Now we have reduced it to just a few cases that are easy to deal with.

- If $c = 1$, then $(1 + 1/a)(1 + 1/b) = 1$ which is clearly impossible.
- $c = 2$ leads to $(1 + 1/a)(1 + 1/b) = 4/3$ which implies that $4/3 \leq (1 + 1/b)^2$ which forces $b < 7$. Since $(1 + 1/a) > 1$, we must also have $b > 3$. Plugging in values yields the solutions $(7, 6, 2), (9, 5, 2), (15, 4, 2)$.
- $c = 3$ gives $(1 + 1/a)(1 + 1/b) = 3/2$. Similar analysis leads to $b < 5, b \geq c = 3$. These values yield the solutions $(8, 3, 3)$ and $(5, 4, 3)$.

In conclusion, the solutions are all cyclic permutations of

$$(7, 6, 2), (9, 5, 2), (15, 4, 2), (8, 3, 3) \text{ and } (5, 4, 3).$$

9 Multiply both sides by $x + y$ and add n^2, to get

$$xy - nx - ny + n^2 = n^2,$$

which factors into

$$(x - n)(y - n) = n^2.$$

Let $d | n^2$ and let $e := n^2/d$. Then $(x - n)(y - n) = de$ yields the solution $x = d + n, y = e + n$. There is a one-to-one correspondence between (positive) divisors of n^2 and solutions (negative divisors won't work; at least one of the divisors would have to be $\leq -n$, forcing either x or y to be zero or negative), hence the number of solutions is $d(n^2)$.

11 The equation we must solve is

$$abc = a + b + c + 2.$$

Monotonize: assume without loss of generality that $a \leq b \leq c$. Then $a + b + c \leq 3c$. Hence, we have

$$abc \leq 2 + 3c.$$

If $c > 2$, then $2 + 3c < 4c$, which means that $ab < 4$. So there are just a few cases:

1. $c \leq 2$: This leads to the solution $(2, 2, 2)$ only.
2. $c > 2$ and $ab = 1$: This implies $1 \cdot c = 1 + 1 + c + 2$, an impossibility.
3. $c > 2$ and $ab = 2$: This leads to $(1, 2, 5)$.
4. $c > 2$ and $ab = 3$: This leads to $(1, 3, 3)$.

These are the only solutions (up to permutation).

12 Hint: Show that $(x + 1)^3 < x^3 + 8x^2 - 6x + 8 < (x + 3)^3$. The only solutions are $(0, 9), (9, 11)$.

13 (Sketch) There are no solutions other than $x = y = 1$. If $y \geq 2$, a modulo 9 analysis yields forces x to be even. Then we can write $x = 2m$ and use the factor tool to analyze

$$3^y = 7^{2m} - 4 = (7^m + 2)(7^m - 2).$$

15 See problem 3.4.31.

22 Notice that 8 and 9 fit the bill. Inspired by this, we need only find solutions to $8x^2 + 1 = 9y^2$. Substituting $u = 2x, v = 3y$ transforms this into $v^2 - 2u^2 = 1$, which was solved above in 7.4.19.

7.5

13 Hint: Let $g = \text{GCD}(m,n)$ and write $m = ag, n = bg$, where $a \perp b$. Then the equation becomes

$$g(ab - a - b + 1) = g(a-1)(b-1) = 0.$$

14 (Sketch due to Kiran Kedlaya) No geometric progression of integers between 1 and 100 can have more than 7 terms. To see this, let m/n be the common ratio, where $m \perp n$, and look at prime factors and inequalities. So 12 progressions, each with at most 7 terms, only can cover 84 or fewer of the numbers from 1 to 100.

Alternate solution: Use the fact that there are 25 primes from 1 to 100. How many primes can be in a geometric progression?

17 (Sketch) Parity analysis shows that all three variables are even or else one is even and two are odd, but in this case, modulo 4 analysis produces a contradiction. Hence all three are even. We can write $x = 2a, y = 2b, z = 2c$, where $a, b, c \in \mathbb{Z}$. Substituting and simplifying yields the new equation

$$a^2 + b^2 + c^2 = 4abc,$$

and a similar mod 4 analysis forces a, b, c to all be even. Writing them as $a = 2u, b = 2v, c = 2w$ yields

$$u^2 + v^2 + w^2 = 8uvw.$$

This process can be repeated indefinitely. What this means is that x, y, z can be divided by 2 arbitrarily many times and integers will always result. The only numbers that satisfy this are $x = y = z = 0$. This method of solution is called *infinite descent* and was discovered by Fermat. See [21] for an excellent discussion.

19 There are many fun approaches to this problem. One idea is to use the formula developed in the solution to 7.1.22 for the highest power of 2 that divides $n!$ and use this to compute the highest power of 2 which divides $\binom{n}{r}$. Another idea is to use (strong) induction, starting by writing Pascal's triangle modulo 2. A third, more sophisticated approach is generatingfunctionological—think about $(1+x)^n$ modulo 2, using the fact that

$$(1+x)^2 \equiv 1 + x^2 \pmod{2}.$$

21 If $p \neq 2$, then $d(8p) = 8$. If $p = 2$, then $d(6p) = 6$. If $p \neq 3$, then $d(9p^2) = 9$. If $p = 3$, then $d(12p^2) = 12$.

24 See problem 7.3.23.

25   Yes. Just pick 1997 relatively prime squares, for example $2^2, 3^2, 5^2, 7^2, 11^2, \ldots$. The Chinese remainder theorem (7.2.16) guarantees that one can find x such that

$$x \equiv 0 \pmod{2^2}, \quad x \equiv -1 \pmod{3^2}, \quad x \equiv -2 \pmod{5^2}, \ldots.$$

Then

$$\mu(x) = \mu(x+1) = \mu(x+2) = \cdots = 0.$$

27   (Sketch) If m is odd, then $\phi(2m) = \phi(2)\phi(m) = \phi(m)$. Hence the inverse image sets consists of pairs. And note that $\mu(2m) = -\mu(m)$.

28   Yes, there are infinitely many such rows. For example,

$$\binom{203}{68} = 2\binom{203}{67} \quad \text{and} \quad \binom{203}{85} = 2\binom{203}{83}.$$

There are infinitely many rows having two *adjacent* elements in a 1:2 ratio, for

$$2\binom{n}{k} = \binom{n}{k+1}$$

reduces to $2(k+1) = n - k$, or $n = 3k + 2$. So as long as $n \equiv 2 \pmod 3$, there will be two adjacent elements in a 1:2 ratio.

Next, we search for "doubles" that are not adjacent. The next easiest case to try is

$$2\binom{n}{k} = \binom{n}{k+2},$$

which reduces to

$$2(k+2)(k+1) = (n-k)(n-k-1).$$

Substitute $u = n - k$ and $v = k + 2$; our equation becomes

$$2(v^2 - v) = u^2 - u.$$

Multiplying both sides by 4 and completing the square yields

$$2(4v^2 - 4v + 1) = 4u^2 - 4u + 1 + 1,$$

so substituting $x = 2v - 1, y = 2u - 1$ reduces the original equation to

$$2x^2 - y^2 = 1.$$

This is a Pell's equation with infinitely many solutions which can be generated in the standard way (see 7.4.17–7.4.20), or simply observe that $(1,1), (5,7)$ are solutions and that if (x,y) is a solution, then $(3x+2y, 4x+3y)$ is also a solution.

It remains to show that one of these solutions produces an n which is congruent to 2 modulo 3. Since $n = \frac{x+y}{2} - 1$, we must have $x + y \equiv 0 \pmod 6$. The first solution with this property is $(5,7)$, but this doesn't work, because it corresponds to $n = 5$, and the elements of the row are $1, 5, 10, 10, 5, 1$, so the

doubles are not distinct. Generating solutions via $(x, y) \rightarrow (3x + 2y, 4x + 3y)$ modulo 6, we have the repeating pattern

$$(1, 1), (-1, 1), (-1, -1), (1, -1), (1, 1), \ldots,$$

which shows that every other solution (x, y) has the property $x + y \equiv 0 \pmod 6$. Clearly the next solution after $(5, 7)$ with this property will correspond to an n large enough so that the doubles will be distinct. Specifically, the solution is $(169, 239)$ which corresponds to $n = 203, k = 83$.

There are no rows before $n = 203$ which have two distinct double pairs. I know of no easy way to see this (I used a computer to check).

31 Trace the "orbit" of the top card, which starts in position 1. It goes to position 2, then 4, etc. Check that after r shuffles, it goes to position $2r$ (mod 95). (Not mod 94!) The minimum number of shuffles will be the minimum r such that $2^r \equiv 1 \pmod{95}$. By Euler's theorem, r must divide $\phi(95) = 72$. It turns out that 36 is the smallest such exponent. In general, for a $2n$-card deck, the minimum number of shuffles needed is the order of 2 modulo $2n + 1$.

32 (Sketch due to Elgin Johnston) Because $m = \phi(n)$ and n is odd, we know by Euler's extension of Fermat's little theorem that n divides $2^m - 1$. There are two cases:

- $n = q^r > 3$ for some odd prime q. Then $m = q^k - q^{k-1} \geq 4$ is even and q divides $2^m - 1 = (2^{m/2} - 1)(2^{m/2} + 1)$, so it must also divide one of the factors on the right. Any prime divisor of the *other* factor meets our requirement.

- n has at least two distinct prime factors, then m is a multiple of 4 and $q - 1$ divides $m/2$ for each prime factor q of n. Hence q also divides $2^{m/2} - 1$. It follows that no prime factor of n divides $2^{m/2} + 1$, and any of them meet our requirement.

33–35 Vanden Eynden's book [25] is exceptionally clear on this material.

38 This is a simple consequence of equation (3) on p. 238.

39 This follows from the problem above by taking logs, using the Möbius inversion formula, and then exponentiating.

40 (Sketch) Call the sum in question $f(n)$. Show that it is a multiplicative function, and then show that for $n > 1$,

$$\sum_{d|n} f(d) = 1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0.$$

Since $f(1) = 1$, the mystery function $f(n)$ satisfies the same equation as $\mu(n)$ in 7.3.12. Hence $f(n) = \mu(n)$.

41 The first n for which $\Phi_n(x)$ has coefficients other than ± 1 is $105 = 3 \cdot 5 \cdot 7$.

8.2

26 The diagonal of any quadrilateral dissects it into two triangles, and you already know things about triangles and midpoints.

27   Whenever you encounter a right triangle, remember that it inscribes nicely in a circle, with the hypotenuse as diameter. Always associate hypotenuses with diameters!

28g   The given point, tangent, and circle center make a right angle. Remember: right angles can be found inscribed in circles!

28k   There are two types of tangent lines. The "outside" ones, which intersect on one side of both circles; the "inside" ones, which intersect "between" them. Here is one case: Let the radii be R, r, with $R > r$. To construct an outside tangent line to both circles, you need to look at a right triangle, with one leg of length $R - r$ whose hypotenuse is the line segment connecting the two centers.

30   Let A, B, C denote the points of tangency between ℓ_1 and ω, ω and γ, γ and ℓ_2, respectively. Let P, Q denote the centers of ω, γ, respectively. Note that P, B, Q are collinear, and that $AP \parallel QC$, and that triangles CQB and APB are isosceles. You may want to try an argument by contradiction: "Let A' denote the intersection of line CB with line AP. If $A' \neq A$, then ..."

32   Use the result of Problem 8.2.27 to help with the angle chasing.

33   A fold is a reflection across the fold line. Each point P is reflected to a point P' such that the fold line is the perpendicular bisector of PP'. These perpendicular bisector give you lots of angle chasing opportunities.

35   You can easily determine which circle the midpoints must lie on, since you are told that they lie on a circle. To prove that they lie on this particular circle, you need the right penultimate step; use 8.2.13.

8.3

17   Look at the circumcircle of the medial triangle.

19   Compare areas.

21b   Extend the sides of the trapezoid upwards to form a triangle. (This is often a good idea when looking at trapezoids.) Now you have several similar triangles to contemplate that are each similar to this new triangle, and you also have two smaller similar triangles inside the trapezoid. Surely this is enough information!

23   Use Problem 8.3.18.

24   Extend AP to intersect BC at D. We'd be done if we could show that $BD = DC$. Since $[ACP] = [ABP]$, we should look for two altitudes that are equal. This gives us a perpendicular bisector. Now draw the appropriate parallel lines.

28   Draw the angle bisector through A, which intersects BC at D'. Use the angle bisector theorem to prove that D and D' must be the same point.

30   Use a little algebra. Let k be the ratio of similitude between the two given triangles, so, for example $AB/DE = k$. Then let $r = AX/XB$.

32   It's easier to prove (d) before (c), since (d) implies (c). Just draw the appropriate diameter of the circle and look for an inscribed right triangle. Then you can read off the sine instantly. For (e), try (if C is obtuse) extending BC

to D so that BDA is a right triangle with right angle at D. Now you have right triangles to read off trig functions with ease, and you can use the pythagorean theorem to get c^2.

38 It is easier to look at the ratio $CE : EB$, which can be found with angle chasing and law of sines. You will need to use calculus, or the result of the previous problem. A very useful corrollary of this is $\lim_{\theta \to \infty}(\sin a\theta / \sin \theta) = a$, for any constant a.

40 Draw a line tangent to the escribed circle that is parallel to BC.

42 Midpoints suggest equal areas. You'll need auxiliary lines.

43 The area between the circle and the chord cutting off the arc is independent of the location of the arc, so you can ignore it, and the problem becomes one of rectangles and triangles. It is equivalent—why?—to showing that twice the area of the triangle with dark outline below is equal to the area of the enclosing rectangle, minus the area of the lower-left rectangle. This can be proven with complex numbers or trig (you will need to know the angle-addition formulas), but it can also be done much more simply.
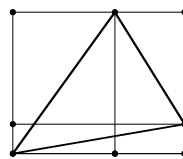
8.4

7 You'd guess—correctly—that this ratio is constant, which means you have to hunt for similar triangles. Draw appropriate diameters to get right triangles, and look for the crux angle, one that is is inscribed simultaneously in both circles. This angle will be the bridge that allows you to compare angles inscribed in one circle with angles inscribed in the other.

8 Perpendiculars dropped to sides are practically begging you to consider area. Do you know the formula for the area of an equilateral triangle in terms of its side length? If not, work it out now.

9 Use Stewart's theorem (Problem 8.3.33) and the law of cosines, which you need to prove Stewart's theorem, as well.

10 Think about the penultimate step for showing an angle to be constant. One such penultimate step would be for this angle to be inscribed in a circle, subtending a chord of constant length.

15 One penultimate step for this equality would be showing that a circle can be *inscribed* in quadrilateral $A'B'C'D'$. There are other approaches as well. And please notice that there are other cyclic quadrilaterals in this problem.

16 One way to show that these three points lie on a line is to show that $\triangle YJZ \sim ZIX$. Look for other similar triangles to prove this.

18   Make sure that you have studied Example 8.4.2. The answer, by the way, is $1/6$.

19   Parallel lines give birth to parallelograms; look for equal areas.

21   Again, study Example 8.4.2 carefully.

25   One way to show that XY is invariant is to show that it is a chord in a circle that subtends a constant angle. Look for a cyclic quadrilateral.

28   The problem asks you to prove an equality involving ratios, so you are naturally led to thinking about similar triangles. Draw a careful diagram, and you will see that it is infested with similar triangles. This is a challenging but fun problem; stick with it!

29   Note that $ABCQ$ is a cyclic quadrilateral.

31   Assume that A, E, C, G are on the same side of PQ. Then it suffices to show that CG is parallel to PQ (why?). You can prove this by finding similar triangles. Notice that $PQAE$ is a cyclic quadrilateral (why?). As usual, this will allow many similar triangles to surface, some due to the tactic of looking at chords in two circles simultaneously.

32   Consider the case where $AB < CD$. Then lines DA and CB meet at a point X. A penultimate step for the area equality that we are to prove is that P always lies on the angle bisector of $\angle AXB$. Look for similar triangles, and don't forget to involve line EF, since you have ratio information about this line segment.

33   You may want to defer working on this problem until you have read about symmetry and reflections in the next section. The $60°$ angle and the perpendiculars creates many equal angles and equal sides...

34   Angle chase to find inscribed right angles.

35   The ratios in the problem suggest looking at similar triangles. Believe it or not, this problem succumbs to simple angle chasing, but you need to be *very* careful drawing the diagram. Color pencils are helpful to mark equal angles. Dont forget the tangent version of the inscribed angle theorem, and the fact that the measure of an exterior angle of a triangle is the sum of the opposite interior angles. Finally, you may want to compare the ratios AM/BM with EG/EF first.

8.5

18   Contemplate the parallelograms in this hexagon.

19   One method is to use vectors and efficient algebra. This is worth trying. Another, more "transformational" idea is the following: Suppose you are given the midpoints A', B', C', D', E' of a pentagon $ABCDE$, such that A' is the mipoint of AB, B' is the midpoint of BC, etc. Consider the rotation $R_{A',\pi}$. This takes A to B. Likewise, $R_{B',\pi}$ takes B to C. Now consider the composition of rotations

$$R_{E',\pi} \circ R_{D',\pi} \circ R_{C',\pi} \circ R_{B',\pi} \circ R_{A',\pi}$$

with the single rotation $R_{A',\pi}$. A translation is lurking about.

20 Notice that the height is fixed. If you need more hints, look at Example 3.1.5.

21 Vectors.

23 To construct $M(T)$, try to build a triangle by translating the medians by vectors that are parallel to the sides of the original triangle. Vector notatation may be helpful with the other parts of the problem.

25 Make sure that you understand the following: suppose that point X lies on line segment AB, with $AX : AB = r$. Then

$$\vec{X} = \vec{A} + r(\vec{B} - \vec{A}).$$

For **(a)**, consider the Euler line. For **(b)**, use the fact that the incenter is the intersection of the angle bisectors of the triangle. The method of "weights" (Problem **??** may also be useful.

30 Find rotations that take one of AX, BY, CZ to another. Since rotations preserve length, you're done!

31 Look at Example 8.5.5.

32 By now you probably realize that the penultimate step is a rotation. Symmetrical "points of interest" (besides the given points) are the midpoints of the sides of the parallelogram, and, of course, the center of the parallelogram.

33 The area of a right triangle is, of course, base times height divided by two. The base and the height do not *have* to be the two legs of the triangle, though. . .

34 Suppose we successfully inscribe a square in the triangle; let its vertices be $DEFG$, with D and E on side BC. Then $FG \parallel BC$, so $\triangle AFG \sim ABC$. Now, rather than think of these triangles as merely similar, think of them as also homothetic with center at A. . .

36 The pantograph is nothing more (or less) than a homothety machine; you should have no difficulty in finding the center and scaling controls, etc.

37 There are several solutions, but one of the most straightforward uses vectors. Remember that if the dot-product is zero, the vectors are perpendicular.

39 This is really an exercise, not a problem, provided that you know how to find the center of rotation for a composition of two rotations. You can also use complex numbers; the algebra is not too hard, although the geometric details are somewhat obscured.

46 Suppose the circle ω has been found. If we invert about P, then ω' will be a *line*, which will be tangent to the images of the two given circles. Notice that we have now converted our configuration from one with three circles (hard) to one with two circles and a line (much easier).

47 This is a hard problem, originally discussed by Apollonius. One approach is to first show that there is an inversion which takes two of the circles into concentric circles. Then it is (relatively) easy to construct the desired circle; another inversion restores it to its "correct" position. A useful web site to visit that discusses this problem is `http://whistleralley.com/tangents/tangents.htm`.

48 Invert about E, showing that the image of the four reflection points lies on a rectangle.

9.2

8 p must be a rational number of the form $k/(k+1)$. To see why, suppose that p is such that it is not the case that Arnie must reach a win percentage of exactly p. Let w, g be respectively, the number of wins for Arnie and games played when Arnie's win ratio exceeds p for the first time. Then we have

$$\frac{w-1}{g-1} < p < \frac{w}{g}.$$

Clearly, irrational values of p will not lead to contradictions. On the other hand, if $p = a/b$ for relatively prime integers a, b, a little algebra yields

$$a - b < ag - wb < 0.$$

If $a - b = -1$, then we get a contradiction, since no integer lies properly between -1 and 0. However, if $a - b < -1$, it is easy to find values of g, w that work without contradiction (exercise).

9 The solve for the limit tool leads to the guess that $a_n \to \phi$, where $\phi \approx 1.602$ is the positive root of $x = 1 + 1/x$. There are many ways to prove that this limit is actually attained, and you are encouraged to try other ideas than the following: Examining the first few values of a_n, they appear to oscillate about ϕ:

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots,$$

and $1 < \phi$ while $2 > \phi$ and $3/2 < \phi$, etc. So it makes sense to look at every other term in the sequence. More precisely, if $a := a_n$, then $a_{n+2} = (2a+1)/(a+1)$. If $a < \phi$, it is straighforward to show that $a < (2a+1)/(a+1)$, since this is equivalent to $a^2 - a - 1 < 0$, and the positive root of this quadratic equation is ϕ.

Additionally, it is easy to show that if $a < \phi$, then $(2a+1)/(a+1) < \phi$. Putting these together, we conclude that if $a < \phi$, then

$$a < (2a+1)/(a+1) < \phi.$$

By similar reasoning, we can show that if $a > \phi$, then

$$\phi < (2a+1)/(a+1) < a.$$

This means that the two subsequences $a_1, a_3, a_5, \ldots$ and $a_2, a_4, a_6, \ldots$ are each bounded and monotonic, and so each converges. The solve for the limit tool yields ϕ in each case [by solving the equation $(2x+1)/(x+1) = x$,] so ϕ must be the limit of the full sequence.

We are done, but it is very instructive to try to look at error terms like $a_n - \phi$ and try to show that they go to zero, etc.

10   We know nothing about the convergence of (b_n). For example, let $a_n = 1/n!$ and $b_n = n^2$. Then $a_n/b_n \to 0$ but $b_n \to \infty$. On the other hand, if $b_n = 1/n$, now both (a_n/b_n) and (b_n) converge to zero.

11   Let $a_1 = \sqrt{2}$, and let $a_{n+1} = \sqrt{2 + a_n}$. Solving for the limit suggests a limit of 2, and it is easy to show that the sequence is bounded and monotonic (as long as $a < 2$, we have $a < \sqrt{2 + a} < 2$.)

12   A picture similar to the one on p. 321 shows that $a_n \to \sqrt{\alpha}$.

15b   This is a trick question, for this subset includes $\mathbb{Q}$!

15d   No element in S is close to 0.35.

17   Let $S := \{\langle \alpha \rangle, \langle 2\alpha \rangle, \langle 3\alpha \rangle, \ldots\}$. We will show that given any $x \in [0, 1]$, there is an element of S which is within a distance of $1/10$ from x. We will use the pigeonhole principle. Divide $[0, 1]$ into 10 equal subintervals

$$[0, 1/10], [1/10, 2/10], \ldots, [9/10, 1].$$

Now consider the 11 numbers

$$\langle \alpha \rangle, \langle 2\alpha \rangle, \langle 3\alpha \rangle, \ldots, \langle 11\alpha \rangle\}.$$

Notice that these are 11 distinct numbers, for if two of them were equal, it would force α to be rational. By the pigeonhole principle, two of these numbers must lie within the same subinterval. Without loss of generality, suppose that $\langle 5\alpha \rangle$ and $\langle 8\alpha \rangle$ both lie in the subinterval $[4/10, 5/10]$. We can write $8\alpha = N + \theta$ and $5\alpha = M + \eta$, where N, M are integers and $\theta, \eta \in (0, 1/10)$. Then $3\alpha = (N - M) + (\theta - \eta)$, which means that $\langle 3\alpha \rangle$ will lie in either $(0, 1/10)$ or $(9/10, 1)$, depending on whether $\theta > \eta$ or $\theta < \eta$. In any event, we are done. Suppose that $\langle 3\alpha \rangle \in [9/10, 1]$. Then $\langle 6\alpha \rangle$ lies to the left $\langle 3\alpha \rangle$ and less than $1/10$ away. Continuing, the sequence

$$\langle 3\alpha \rangle, \langle 6\alpha \rangle, \langle 9\alpha \rangle, \ldots$$

will fill the unit inteval with points that are less than $1/10$ from each other, which was what was desired. Clearly this method will work with any distance, not just $1/10$.

23   Here is a proof by picture, where 10 is replaced by 4. The total shaded area is equal to one-fourth of the desired sum; note that the lower-left $1/4 \times 1/4$ square will remain unshaded no matter what f is.
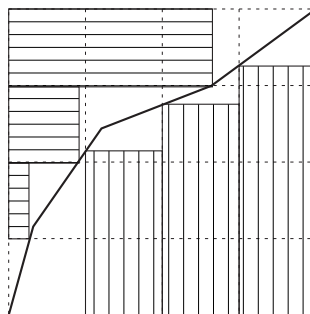
24   We are given that $f(1000) = 999$, so $f(1000)f(f(1000)) = 1$, which implies that $999f(999) = 1$, so $f(999) = 1/999$. Now, we use the continuity of f: by the intermediate value theorem, there must be a value $a \in (999, 1000)$ such that $f(a) = 500$ (because $f(999) < 500 < f(1000)$). Now it is easy: we have

$$f(a)f(f(a)) = 1,$$

which means that

$$500f(500) = 1,$$

so $f(500) = 1/500$.

25   For simplicity, we will replace 1999 by 4. The idea is the same. Now we must show that there is a value $x \in [0, 3/4]$ such that $f(x) = f(x+1/4)$. Define $g(x) := f(x) - f(x+1/4)$. We wish to show that there is a value $x \in [0, 3/4]$ such that $g(x) = 0$. We have

$$g(0) = f(0) - f(1/4),$$
$$g(1/4) = f(1/4) - f(2/4),$$
$$g(2/4) = f(2/4) - f(3/4),$$
$$g(3/4) = f(3/4) - f(1).$$

Then

$$g(0) + g(1/4) + g(2/4) + g(3/4) = f(0) - f(1) = 0.$$

Thus either all of these four values of $g(x)$ is zero, and we are done (all we need is one value), or two of these values have opposite sign, in which case we will be done after one application of the IVT.

So our solution was a combination of telescoping, the average principle (5.5.12), and the IVT.

30   To keep notation simple, we will assume that the x_n are positive and monotonically increasing. Pick $\varepsilon > 0$. Then there will be an N such that for all $n \geq N$, we have $x_n - x_{n-1} < \varepsilon$. In other words, once $n \geq N$, the sequence values increase by less than the increment of ε. Now imagine that n is quite a bit larger than N. We will have

$$x_n < x_N + (n - N)\varepsilon,$$

so

$$\frac{x_n}{n} < \frac{x_N}{n} + \frac{n - N}{n}\varepsilon < \frac{x_N}{n} + \varepsilon.$$

There is an L such that if $n \geq L$, then $x_N/n < \varepsilon$, and we have

$$\frac{x_n}{n} < 2\varepsilon$$

for $n \geq L$. Thus $x_n/n \to 0$.

34   Hint: Mimic the ideas used in Example 5.3.4, which showed that the harmonic series diverges.

9.3

12   Let $f(x) := ax + bx^2/2 + cx^3/3 + dx^4/4 + ex^5/5$. Then $f(0) = 0$ and $f(1) = a + b/2 + c/3 + d/4 + e/5 = 0$. By Rolle's theorem, $f'(x) =$ for some value of x. But $f'(x) = a + bx + cx^2 + dx^3 + ex^4$.

13b   This is the same as problem 9.2.25. Let $s(t)$ denote the position of the car north of Los Angeles at time t minutes. Then $s(0) = 0, s(300) = 400$. Define $f(t) := s(t + 1) - s(t)$. We wish to show that there exists a time t such that $f(t) = 4/3$ (miles per minute, which is equivalent to 80 miles per hour). Since $f(0) + f(1) + \cdots + f(299) = 400$, by the average principle, either all the terms equal $4/3$, or some are less than $4/3$ and some are greater. In the former case, we are done immediately, and in the latter, we are done after one application of IVT.

17   Plugging in $y = 0$ into $f(x + y) = f(x)f(y)$, we see that $f(0) = 1$ unless the function is identically zero. We have

$$\frac{f(x + h) - f(x)}{h} = \frac{f(x)f(h) - f(x)}{h} = f(x)\left(\frac{f(h) - 1}{h}\right).$$

Taking the limit as $h \to 0$, we get

$$f'(x) = f(x)f'(0) = 3f(x).$$

Thus $f(x) = e^{3x}$.

18   Using the same ideas as 8.3.15, we get $f(x) = 3\log x$.

31   One way to prove this is with mathematical induction. But an easy way to *see* why it is true is by thinking about the graph of $y = f(x)$. It is a "hat" which starts at $(0,0)$, rises with a straight line to $(\frac{1}{2}, 1)$ and then drops down with a straight line to $(1,0)$. The integral of f from 0 to 1 is just the area of this triangle, which is clearly 1/2. Now, what does the graph of $y = f_2(x) = f(f(x))$ look like? As x ranges from 0 to $\frac{1}{2}$, $f(x)$ varies linearly from 0 to 1. Thus, as x ranges from 0 to $\frac{1}{2}$, $f(f(x))$ will assume the same values that $f(x)$ does when x ranges from 0 to 1. In other words, the graph of $f(f(x))$ as x ranges from 0 to $\frac{1}{2}$ is another "hat" that goes from $(0,0)$ to $(\frac{1}{4}, 1)$ to $(\frac{1}{2}, 0)$. Then, as x ranges from $\frac{1}{2}$ to 1, $f(x)$ varies linearly from 1 down to 0, causing the graph of $f(f(x))$ to again draw a "hat" but *in reverse*. However, the original hat was symmetrical, so the final picture of the graph of $f(f(x))$ is just two hats, each with base 1/4 and height 1. Once again, the area is still 1/2.

The graph of $f_3(x)$ works in a similar way. Informally, we just apply f to the graph of f_2. What this means in practice is that any line which goes (left or right) from $y = 0$ to $y = 1$ gets transformed into a hat. So in general, the graph of $f_n(x)$ will have 2^{n-1} hats, each with base $1/2^{n-1}$ and height 1. The area is always 1/2.

32   The sum in question is equal to

$$\frac{1}{n}\left(1 + \frac{1}{1 + \frac{1}{n}} + \frac{1}{1 + \frac{2}{n}} + \cdots + \frac{1}{2 - \frac{1}{n}}\right).$$

As $n \to \infty$, this is equal to

$$\int_1^2 \frac{dx}{x} = \log 2.$$

36 f cannot have two fixed points, for then the graph of f intersects the graph of $y = x$ in two points. The line joining these two points has slope 1, and the Mean value theorem implies that there exists a value of x such that $f'(x) = 1$, a contradiction. The somewhat challenging part is finding functions that never have derivative equal to 1. One idea is to use the fact that e^x never equals zero, and thus $f(x) := x + e^x$ has derivative $1 + e^x$ which is always strictly greater than 1. Since $f(x) > x$, the function has no fixed points. However, we can translate this function vertically so that it intersects the graph of $y = x$. For example, the function $f(x) := x + e^x - 1$ will have one fixed point, at $(0, 0)$, yet the derivative will still never equal 1.

37   Let the limit be L. Take logarithms, to turn the product into a sum, then reinterpret as a definite integral:

$$\log L = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \log \left(1 + \frac{k}{n} \right)$$

$$= \int_1^2 \log x \, dx$$

$$= (x \log x - x) \Big]_1^2 = 2 \log 2 - 1.$$

Hence,

$$L = e^{2 \log 2 - 1} = 4/e.$$

39   Assuming that f is differentiable leads to $f'(x) = 0$ for all values of x, so $f(x) = 0$ for all x (since it equals zero for one value). But calculus is not needed, just the triangle inequality. We have

$$|f(x)| = |f(x) - f(0)|$$
$$= |f(x) - f(2x/3) + f(2x/3) - f(x/3) + f(x/3) - f(0)|$$
$$\leq |f(x) - f(2x/3)| + |f(2x/3) - f(x/3)| + |f(x/3) - f(0)|$$
$$\leq 3 \cdot (x/3)^2$$
$$= x^2/3.$$

Notice that there was nothing special about the role played by 3, here. Breaking it down into n sums would yield $|f(x)| \leq x^2/n$. Since n is arbitrary we can make $f(x)$ arbitrarily small; we conclude that $f(x) = 0$ for all x.

45   (Sketch) Let $a_i := x_{x+1} - x_i$. If you draw a picture, it is apparent that the sum is an approximation of $\int_0^1 x \, dx = 1/2$. The error is at its worst (and without loss of generality, positive) if $a_2, a_4, \ldots, a_{2n}$ are all tiny compared with $a_1, a_3, \ldots, a_{2n-1}$, in which case the error is (slightly less than)

$$\frac{a_1^2 + a_3^2 + \cdots + a_{2n-1}^2}{2}.$$

We will be done if we can prove the following inequality: If $b_1, b_2, \ldots, b_n$ are positive numbers such that $b_1 + b_2 + \cdots + b_n = 1$, then

$$b_1^2 + b_2^2 + \cdots + b_n^2 \le \max(b_1, b_2, \ldots, b_n).$$

This inequality can be verified by looking at a number of cases. For example, if $n = 3$, and we assume that $b_1 \le b_2 \le b_3$, then one case is $b_3 \ge 1/2$. Check that $b_1^2 + b_2^2$ is maximized when $b_1 = 0, b_2 = 1 - b_3$ which means that we must verify $(1 - b_3)^2 + b_3^2 \le b_3$. The other case is that $1/3 \le b_3 < 1/2$. Then $b_1^2 + b_2^2$ is maximized when $b_2 = b_3, b_1 = 1 - 2b_3$, and now we must verify $2b_3^2 + (1 - 2b_3)^2 < b_3$.

The general case will reduce to a number of inequalities of this kind. It isn't particularly pretty, but it works. I suspect that there is a more elegant method lurking about. Reader, can you find one?

9.4

17   It is easy to prove (by induction, for example) that the sum of the first r terms is

$$\frac{x^u + x^{u-1} + x^{u-2} + \cdots + x}{x^u + x^{u-1} + x^{u-2} + \cdots + x + 1},$$

where $u = 2^{r-1} - 1$. As $r \to \infty$, this quotient certainly approaches 1, for any x.

21a   The finite case was Problem 5.3.15. We have

$$P_M = \frac{\sin x}{2^M \sin\left(\dfrac{x}{2^M}\right)},$$

where

$$P_M := \prod_{n=1}^{M} \cos\left(\frac{x}{2^n}\right).$$

Let $h = x/2^M$. Then we have

$$P_M = \frac{h \sin x}{x \sin h} = \left(\frac{h}{\sin h}\right)\left(\frac{\sin x}{x}\right).$$

As $M \to \infty$, $h \to 0$. The result follows from the well known fact that

$$\lim_{h \to 0} \frac{h}{\sin h} = 1,$$

which can be proven by several methods (L'hôpital's Rule, power series, geometry, etc.)

23 Notice that

$$\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^2}+\cdots\right)=\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^3}+\frac{1}{8^2}+\cdots.$$

Consequently,

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^3}+\frac{1}{7^2}+\cdots=\zeta(2)-\frac{1}{4}\zeta(2)=\frac{3\pi^2}{4\cdot6}=\frac{\pi^2}{8}.$$

24 Using the methods and notation of Example 9.4.7,

$$S=S_3S_5S_7=\frac{3}{2}\frac{5}{4}\frac{7}{6}=\frac{105}{48}.$$

25a Actually, no. If the FTA were false, the product might include duplicates, so it would include extra terms. Since the harmonic series diverges, so will a sum of reciprocals of the positive integers that includes duplicates of some integers. And all we need is that the product in question diverges.

27 This is easy to evaluate by changing the order of summation (an Eulerian tactic), so that we can deal with easy geometric series instead of difficult zeta functions:

$$S=\sum_{r=2}^{\infty}\left(\sum_{k=2}^{\infty}\frac{1}{k^r}\right)=\sum_{k=2}^{\infty}\left(\sum_{r=2}^{\infty}\frac{1}{k^r}\right)=\sum_{k=2}^{\infty}\left(\frac{1/k^2}{1-1/k}\right)=\sum_{k=2}^{\infty}\frac{1}{k^2-k}.$$

The last sum is (by now) an easy telescoping exercise:

$$\sum_{k=2}^{\infty}\frac{1}{k^2-k}=\frac{1}{2\cdot1}+\frac{1}{3\cdot2}+\cdots=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots=1.$$

28 One (self-referential) way to characterize P so as to avoid redundancy (counting 16 twice, for example, once as 2^4 and again as 4^2) is to say that P is the set of all a^r where $r=2,3,4\ldots$ and a ranges through all non-perfect powers. Using the geometric series tool our sum is

$$\sum_{j\in P}\frac{1}{j-1}=\sum_{j\in P}\frac{1/j}{1-1/j}$$

$$=\sum_{j\in P}\left(\frac{1}{j}+\frac{1}{j^2}+\frac{1}{j^3}+\cdots\right)$$

$$=\sum_{\substack{a\notin P\\r\geq2}}\left(\frac{1}{a^r}+\frac{1}{(a^2)^r}+\frac{1}{(a^3)^r}+\cdots\right),$$

and this final sum is equal to the sum of all terms of the form $1/u^v$, where $u,v\geq2$, i.e., the "redundant" perfect powers. But this sum was computed in 9.4.27; it is equal to 1.

30   Let p_k denote the kth prime. Recall (see p. 237) that if $n = p_1 p_2 \cdots p_r$, then

$$\frac{\phi(n)}{n} = \prod_{k=1}^{r} \left(1 - \frac{1}{p_k} \right).$$

But

$$\prod_{k=1}^{r} \left(1 - \frac{1}{p_k} \right) = \left(\prod_{k=1}^{r} \frac{p_k}{p_k - 1} \right)^{-1}.$$

We showed earlier (Example 9.4.7) that the divergent harmonic series can be written as the infinite product

$$\prod_{k=1}^{\infty} \left(\frac{p_k}{p_k - 1} \right).$$

This means that if we pick a sufficently large number r, the product

$$\left(\frac{2}{1} \right) \left(\frac{3}{2} \right) \left(\frac{5}{4} \right) \left(\frac{7}{6} \right) \cdots \left(\frac{p_r}{p_r - 1} \right)$$

can be made arbitrarily large. Hence if we choose $n_r := p_1 p_2 \cdots p_r$, then

$$\lim_{k \to \infty} \frac{\phi(n_k)}{n_k} = 0.$$

31   A generalization of Example 9.4.7 is that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^x} = \prod_{p \, \text{prime}} \left(1 + \frac{f(p)}{p^x} + \frac{f(p^2)}{p^{2x}} + \cdots \right)$$

for any multiplicative function f (see p. 257).

Specifically,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^x} = \prod_{p \, \text{prime}} \left(1 - \frac{1}{p^x} \right).$$

Taking reciprocals, and using the geometric series tool, we have

$$\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^x} \right)^{-1} = \prod_{p \, \text{prime}} \left(1 - \frac{1}{p^x} \right)^{-1}$$

$$= \prod_{p \, \text{prime}} \left(1 + \frac{1}{p^x} + \frac{1}{p^{2x}} + \cdots \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^x} = \zeta(x).$$

Appendix A

Sample Exams

Here are two sample group exams, one given at near the beginning and the other toward the end of the semester. There is some overlap with problems from TAACOPS.

An Early Exam

1 Pat the Magician asks a member of the audience to write a 13-digit number on the blackboard, while Pat is blindfolded. Pat then asks the volunteer to reverse the digits of the number, and to subtract the smaller of the two 13-digit numbers from each other. (If the original number ended in one or more zeros, the reversed number of course will have less than 13 digits.) Then Pat asks the volunteer to circle any one of the digits in the difference, provided the digit is not zero. Then Pat asks the volunteer to say what the uncircled digits were. After hearing all of this, Pat is able to tell the amazed volunteer what the circled digit was (Pat is blindfolded during this entire trick). How does Pat do the trick? Explain *why* (not just how) the trick works. By the way, there is nothing special about 13. The trick could have been done with a number of any length.

2 A bug is crawling on the coordinate plane from $(7, 11)$ to $(-17, -3)$. The bug travels at constant speed one unit per second everywhere but quadrant II (negative x- and positive y-coordinates), where it travels at $\frac{1}{2}$ units per second. What path (with proof) should the bug take to complete her journey in minimal time?

3 Sal the Magician asks you to pick any five cards from a standard deck. You do so, and then hand them to Sal's assistant Pat. Then you pick one of the five cards, and Pat puts it back into the deck, and takes the remaining 4 cards, arranges them in some way. Sal is blindfolded, and does not witness any of this. Then Sal takes off the blindfold, takes the pile of 4 cards, reads the four cards that Pat has arranged, and is able to find the fifth card in the deck (even if you shuffle the deck after the assistant puts the card in the deck!). Assume that neither Sal nor Pat have supernatural powers, and that the deck of cards is not marked, and that the pile of 4 cards that Pat arranges does not have any funny folding, or weird angles, etc.

4 Seventeen people are at a party. It turns out that for each pair of people present, exactly one of the following statements is always true: "They haven't met," "They are good friends," or "They hate each other." Prove that there must be a trio (3) of people, all of whom are either mutual strangers, mutual good friends, or mutual enemies.

5 A group of airplanes is based on a small island. The tank of each plane holds just enough fuel to take it halfway around the world. Any desired amount of fuel can be transferred from the tank of one plane to the tank of another while the planes are in flight. The only source of fuel is on the island, and we assume that there is no time lost in refueling either in the air or on the ground. What is the smallest number of planes that will ensure the flight of one plane around the world on a great circle, assuming that the planes have the same constant ground speed and rate of fuel consumption and that all planes return safely to the island base?

6 Color the plane in 3 colors. Prove that no matter how the coloring is done, there are two points of the same color 1 unit apart.

7 Color the plane in 2 colors. Prove that no matter how the coloring is done, there exists a rectangle, all of whose vertices are the same color.

8 An "ell" is an L-shaped tile made from three 1×1 squares (see picture). For what positive integers a, b is it possible to completely tile an $a \times b$ rectangle only using ells? For example, it is clear that you can tile a 2×3 rectangle with ells, but (draw a picture) you cannot tile a 3×3 with ells.

9 Define $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} 2x & 0 \le x \le 1/2, \\ -2x+2 & 1/2 < x \le 1. \end{cases}$$

Next, define a sequence f_n of functions from $[0,1]$ to $[0,1]$ as follows: Let $f_1(x) = f(x)$ and let $f_n(x) = f(f_{n-1}(x))$ for $n > 1$. Prove that for each n,

$$\int_0^1 f_n(x)dx = 1/2.$$

10 For positive integers n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2},$$

as the $a_1, a_2, \ldots, a_n$ range through all positive values such that $a_1 + a_2 + \cdots + a_n = 17$. Find S_{1999}.

11 Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

12 Let

$$\prod_{n=1}^{1997} \left(1 + nx^{3^n}\right) = 1 + a_1 x^{k_1} + a_2 x^{k_2} + \ldots + a_m x^{k_m},$$

where $a_1, a_2, \ldots, a_m$ are nonzero and $k_1 < k_2 < \ldots < k_m$. Find a_{1997}.

A Late Exam

1 A runner runs a 6-mile course in exactly 30 minutes. Prove that there must be a 5-minute time interval during which the runner ran exactly one mile.

2 Show that among any seven distinct positive integers not greater than 126, one can find two of them, say x and y, satisfying the inequalities $1 < \frac{y}{x} \le 2$.

3 Seventeen chairs are evenly placed around a circular table on which there are name cards for seventeen guests. The guests fail to notice the cards until after they have sat down, and it turns out that no one is sitting at his or her designated seat. Prove that it is possible to rotate the table so that at least two of the guests will be (simultaneously) seated in the correct spot.

4 Let a, b, c be complex numbers satisfying

$$abc = 1 \quad \text{and} \quad a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Show that at least one of a, b, c must equal 1.

5 Let $\{x\}$ denote the closest integer to the real number x. For example, $\{3.1\} = 3$ and $\{4.7\} = 5$. Now define $f(n) := n + \{\sqrt{n}\}$. Prove that, for every positive integer m, the sequence

$$m, f(m), f(f(m)), f(f(f(m))), \ldots$$

never contains the square of an integer.

6 Evaluate

$$\lim_{n \to \infty} \frac{1}{n^4} \prod_{j=1}^{2n} (n^2 + j^2)^{1/n}.$$

7 Place the integers $1, 2, 3, \ldots, n^2$ (without duplication) in any order onto an $n \times n$ chessboard, with one integer per square. Show that there exist two adjacent entries whose difference is at least $n + 1$. (Adjacent means horizontally or vertically or diagonally adjacent.)

8 Find the maximum value of

$$f(x) = \sqrt[3]{4 - 3x + \sqrt{16 - 24x + 9x^2 - x^3}} + \sqrt[3]{4 - 3x - \sqrt{16 - 24x + 9x^2 - x^3}}$$

in the interval $0 \le x \le 1$. Hint: No calculus, please! A delicate touch will work.

9 A **great circle** is a circle drawn on a sphere that is an "equator;" i.e., its center is also the center of the sphere. There are n great circles on a sphere, no three of which meet at any point. They divide the sphere into how many regions?

10 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $f(x) \cdot f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.

11 (Russia, 1996) A **palindrome** is a number or word that is the same when read forward and backward, for example, "176671" and "civic." Can the number obtained by writing the numbers from 1 to n in order $(n > 1)$ be a palindrome?

12 (E. Johnston) Let S be the set of positive integers which do not have a zero in their base-10 representation; i.e.,

$$S = \{1, 2, \ldots, 9, 11, 12, \ldots, 19, 21, \ldots\}.$$

Does the sum of the reciprocals of the elements of S converge or diverge?

13 Let $P = \{4, 8, 9, 16, \ldots\}$ be the set of perfect powers, i.e., the set of positive integers of the form a^b, where a and b are integers greater than 1. Prove that

$$\sum_{j \in P} \frac{1}{j - 1} = 1.$$

14 Let $g(n)$ be the number of odd terms in the row of Pascal's Triangle which starts with $1, n, \ldots$. For example, $g(6) = 4$, since the row

$$1, 6, 15, 20, 15, 6, 1$$

contains 4 odd numbers.

(a) Conjecture a formula (or an easy way of computing) $g(n)$. Hint: base-2.

(b) Prove your formula. Hint: induction is possible, as always.

Appendix B

An Inspiring Story

During the fall semester of 1994, I taught a problem-solving seminar. The twelve students varied greatly in preparation and ability. I divided the class into several small groups, chosen on the basis of personal compatibility rather than mathematical well-roundedness. Thus one group contained two women who were already close friends and very serious, highly skilled math majors. And another group ended up containing my least knowledgeable students. This group consisted of two women, recent college graduates (in psychology and environmental studies) enrolled in the MAT program, and a young man who had just switched his major from history to math. None of the three had studied any math beyond a year or so of calculus and linear algebra.

One of the first strategic ideas that I shared with the class was the need to look for patterns. This is a good topic for beginners, because it is fun, easy, liberating (one is free to "mess around" and "brainstorm"), and quite productive when combined with mathematical induction. After a few of weeks of rather intensive work on this and other strategies, and after about a week of elementary work in combinatorics, I told the class the cautionary tale of The Five Circles Problem.

The problem is the following: Draw a circle, and place n points on the circumference, where n is a positive integer. Draw lines between each pair of points. What is the maximum number of regions that these lines divide the circle into?

To help the class investigate the problem, I drew five circles on the blackboard, and marked one dot on the first circumference, two on the second, etc. Then I joined the dots in each circle, having made sure that no three lines met in a point (to insure that the number of regions would be maximal). Then we counted regions. This is all standard procedure. With any new problem, the first thing to do is play around, experiment, look for a pattern.

The first circle had no lines, hence just one region (the circle itself). The second circle was divided into two regions, and the third had four regions, etc. Here is the sequence of numbers that we discovered:

$$1, 2, 4, 8, 16.$$

"At this point, I'm sure you all are ready to make a conjecture about the general formula," I said, and sure enough, everyone agreed that the formula was probably 2^{n-1}. (Of course, a few students first guessed 2^n, but then quickly recanted after plugging in values of n.) One student even ventured an argument for *why* this formula was true,

saying that the addition of a new point on the circumference would end up dividing each of the old regions into two regions, thus doubling the number each time. But as soon as she finished speaking, she said, "Wait, that can't quite make sense..."

We agreed that we saw a pattern, but had no understanding of it, and carefully looked at the next case, $n = 6$. Instead of 32 regions, there were only 31! The unexpected 31 persisted after numerous recounts. "The moral of the story," I said, "is that just finding a pattern is never enough. You have to let the pattern guide you towards an understanding of the problem. A pattern by itself may just be a mirage, as it seems to be here. In fact, those first 5 powers of two are just a coincidence. I challenge you to find the real formula, and prove that your formula is correct for all values of n. By the way, this is a pretty hard problem."

I certainly knew that it was a hard problem. I had first seen it in high school (the 5 circles problem is part of "folklore"), although I had never worked out the correct formula. Twenty years later, during the IMO training camp, a guest lecturer by the name of Bjorn Poonen discussed this problem, among others. Bjorn derived the formula for the number of regions, using a tricky argument that employed fairly esoteric ideas. Yet the formula looked very simple, containing only binomial coefficients. I asked him, "Surely something with such a simple combinatorial formula should have a simple combinatorial derivation. Is there one that you know of?" He replied that he had never found one.

When Bjorn Poonen says that he could not do something, everyone pays attention, for he is one of the most talented young problemists in the country. He is only the second person in history to have placed in the top 5 on the Putnam Exam during each of his four years in college, and he composes very original, beautiful problems for a number of math contests and journals. So if someone as gifted and accomplished as Bjorn cannot do it, it must be hard.

Nevertheless, by the end of his lecture, one of the IMO team members had come up with a very elementary derivation, a combinatorial argument which used only the most basic ideas of that subject. We were all impressed with his brilliance, for elegant, elementary arguments are often the hardest to find, especially when you have already seen another way to do a problem. Everyone agreed that it was a virtuoso accomplishment.

Naturally, I did not have great expectations that my USF students would have success with this problem. But besides saying that the problem was difficult, I did not discourage them. And sure enough, by the next class, my "weakest" group announced that they had found a solution. One of the MAT students went to the board, and rather tentatively ("I'm sure this is nonsense...") presented a formula and its derivation. Her argument wasn't smooth, but it was correct, and was essentially the same brilliant reasoning that the IMO team member discovered!

The moral of this story: It is true that the USF students worked in a group and took a few days (working for several hours intensively) while the IMO team member solved the problem alone in 40 minutes. However, the USF students are less talented, and much less experienced. They had the courage to tackle a really hard problem, and the tenacity to stick with it until they solved it. Their very impressive achievement shows that high-level problem-solving skills are "teachable;" that bright but inexperienced people can do just about anything when properly guided and encouraged.

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