Art of Problem Solving

WOOT 2010–11

Practice AIME 1

Instructions

- The time limit is 3 hours. These 3 hours should be continuous, e.g. do not work for 2 hours on one day, and then 1 hour on another day.
- All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor a penalty for wrong answers.
- No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted**.
- Enter your answers at the following link, before **midnight ET/9 PM PT** on **Wednesday, December 15, 2010**:

  `http://www.artofproblemsolving.com/School/WOOT/aime.php`

- Do not discuss the problems before Wednesday, December 15. The answers will be posted on Thursday, December 16.

1. Compute $2010 \cdot 2016 \cdot 2028 - 2008 \cdot 2020 \cdot 2026$.

2. Find the number of ordered quadruples (a, b, c, d) of positive integers such that $abcd = 5000$.

3. Let a, b, c, and d be positive integers such that $a^5 = b^6$, $c^3 = d^4$, and $d - a = 61$. Find $c - b$.

4. For a positive integer n, let $f(n)$ denote the largest power of 17 dividing n. For example, $f(5) = 17^0 = 1$ and $f(2 \cdot 3^4 \cdot 17^2) = 17^2$. Determine the positive integer n such that

$$f(1) + f(2) + f(3) + \cdots + f(n) = 2010.$$

5. Find the smallest positive integer n for which

$$|x - 1| + |x - 2| + |x - 3| + \cdots + |x - n| \geq 2010$$

for all real numbers x.

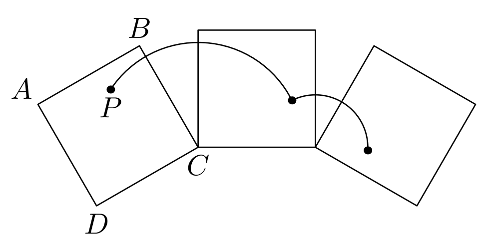
6. A sequence of positive integers is constructed as follows: First, we write down a 1. Then given the first 2^n terms of the sequence, where n is a nonnegative integer, the next 2^n terms of the sequence are generated by writing the first 2^n terms in reverse order, then adding 2^n to each of these new terms. Thus, the first few steps of the construction produce the numbers

$$\begin{array}{l} 1, \\ 1, 2, \\ 1, 2, 4, 3, \\ 1, 2, 4, 3, 7, 8, 6, 5, \end{array}$$

and so on. Find the 1000^{th} term of the sequence.

7. Let P be a point inside square $ABCD$, which has side length 5, such that P has a distance of 1 from side AB and a distance of 2 from side BC. The square is initially placed so that side CD coincides with the side of a regular dodecagon (12-sided polygon), also of side length 5.

   The square is then rolled around the dodecagon, until it returns to its original position, such that the point P stays fixed relative to the square, tracing a path γ. A portion of γ is shown below.



   The area of the region between γ and the dodecagon can be expressed in the form $a + b\pi$, where a and b are integers. Find $a + b$.

8. Find the number of positive integers n that satisfy

$$\left\lfloor \frac{n}{35} \right\rfloor = \left\lfloor \frac{n}{37} \right\rfloor.$$

Note: For a real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

9. Let $[T]$ denote the area of triangle T. In triangle ABC, let P be on side AB and let Q be on side AC, and let CP and BQ intersect at R. Find $[ABC]$ if $[BRP] = 5$, $[BRC] = 7$, and $[CRQ] = 9$.

10. If the acute angles α and β satisfy

$$\begin{aligned} 2\sin 2\beta &= 3\sin 2\alpha, \\ \tan\beta &= 3\tan\alpha, \end{aligned}$$

then $\cos^2(\alpha - \beta)$ can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

11. Let P be a regular 21-gon. How many acute triangles have all three of their vertices among the vertices of P?

12. A sequence of partitions is constructed as follows: We begin with the set $\{1, 2, 3, 4, 5\}$. Each subsequent partition is then obtained by splitting any set containing more than one element into two non-empty subsets. For example, the sequence of partitions may proceed as

$$\begin{aligned} &\{1, 2, 3, 4, 5\} \\ &\to \{1, 2, 5\} \cup \{3, 4\} \\ &\to \{1, 5\} \cup \{2\} \cup \{3, 4\} \\ &\to \{1, 5\} \cup \{2\} \cup \{3\} \cup \{4\} \\ &\to \{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\}. \end{aligned}$$

How many different sequences start with $\{1, 2, 3, 4, 5\}$ and end with $\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\}$?

13. Let a and b be positive integers. Mr. X has a die, whose faces are labeled 1, 2, $\dots$, a. Mrs. Y also has a die, whose faces are labeled 1, 2, $\dots$, b. For each die, each face has an equal probability of appearing when rolled. Mr. X rolls his die once, but Mrs. Y rolls her die twice. The probability that Mrs. Y's higher roll is greater than Mr. X's roll is $49/100$. Determine $a + b$.

14. The distinct complex numbers a, b, and c satisfy

$$\begin{aligned} a^3 &= 7b^2 + 7c^2 + 1, \\ b^3 &= 7a^2 + 7c^2 + 1, \\ c^3 &= 7a^2 + 7b^2 + 1. \end{aligned}$$

Find the product abc.

15. In triangle ABC, $AB = 17$, $AC = 25$, and $BC = 12$. Point D is chosen on side BC such that the incircle of triangle ACD and the excircle of triangle ABD, opposite vertex A, have the same radius. This common radius may be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.