2007?

**Problem 1.** Prove that for all real numbers, $x_1, x_2, y_1, y_2, z_1, z_2$, with $x_1 > 0$, $x_2 > 0$, $x_1y_1 - z_1^2 > 0$, $x_2y_2 - z_2^2 > 0$, the inequality

$$\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2} \le \frac{1}{x_1y_1-z_1^2} + \frac{1}{x_2y_2-z_2^2}$$

is satisfied. Give necessary and sufficient conditions for equality.

**Problem 2.** Define a sequence $\{u_n\}_{n=0}^{\infty}$ by

$$u_0 = 2, \quad u_1 = \frac{5}{2},$$
$$u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1, \quad \text{for } n = 1, 2, \ldots$$

Prove that for all positive $n$,

$$\log_2 \lfloor u_n \rfloor = \frac{2^n - (-1)^n}{3},$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding $x$.

**Problem 3.** Show that, for any positive integer $n$,

$$\sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \left( \frac{n-2r}{n} \binom{n}{r} \right)^2 = \frac{1}{n} \binom{2n-2}{n-1},$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding $x$.

**Problem 4.** Justify the statement that

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \cdots}}}}}.$$

**Problem 5.** Show that there are no four consecutive binomial coefficients, $\binom{n}{r}, \binom{n}{r+1}, \binom{n}{r+2}, \binom{n}{r+3}$ ($n, r$ positive integers, $r + 3 \le n$), which are in arithmetic progression.

**Problem 6.** Show that the power series representation for the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^2 (x-1)^{2n}$$

cannot have three consecutive zero coefficients.

**Problem 7.** Let $\frac{r}{s} = 0.k_1k_2k_3\ldots$ be the decimal expansion of a rational number. (If the rational number is terminating, $k_i = 0$ for $i \geq n$.) Prove that at least two of the numbers

$$\begin{aligned}
\sigma_1 &= 10\frac{r}{s} - k_1, \\
\sigma_2 &= 10^2\frac{r}{s} - (10k_1 + k_2), \\
\sigma_3 &= 10^3\frac{r}{s} - (10^2k_1 + 10k_2 + k_3), \\
&\vdots
\end{aligned}$$

are equal.

**Problem 8.** If $s_n = 1 + q + q^2 + \cdots + q^n$ and

$$S_n = 1 + \frac{1+q}{2} + \left(\frac{1+q}{2}\right)^2 + \cdots + \left(\frac{1+q}{2}\right)^n,$$

prove that

$$\binom{n+1}{1} + \binom{n+1}{2}s_1 + \binom{n+1}{3}s_2 + \cdots + \binom{n+1}{n+1}s_n = 2^n S_n.$$

**Problem 9.** Is it possible for the expression

$$\begin{aligned}
&(a_1 + a_2 + \cdots + a_{999} + a_{1000})^2 \\
&\qquad = a_1^2 + a_2^2 + \cdots + a_{999}^2 + a_{1000}^2 \\
&\qquad\qquad + 2a_1a_2 + 2a_1a_3 + \cdots + 2a_{999}a_{1000}
\end{aligned}$$

(where some of the numbers $a_1, a_2, \ldots, a_{1000}$ are positive and the rest are negative) to contain the same number of positive and negative terms in $a_ia_j$? Investigate the analogous problem for the expression

$$(a_1 + a_2 + \cdots + a_{9\,999} + a_{10\,000})^2$$

**Problem 10.** Let $N$ be any natural number and let $r$ be the number of integers in the sequence $1, 2, 3, \ldots, N-1$ that are relatively prime to $N$. Prove that if $a$ is any integer which is relatively prime to $N$, then $a^r - 1$ is divisible by $N$.

**Problem 11.** Let $n$ be a natural number. Use Problem 10 to show that $2^k - 1$, where $k = 5^n - 5^{n-1}$, is divisible by $5^n$.

Then prove that there exists no $k$ less than $5^n - 5^{n-1}$ such that $2^k - 1$ is divisible by $5^n$.

**Problem 12.** Let $r$, $s$, and $t$ be non-negative integers such that $r + s \leq t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

**Problem 13.** Let $N$ be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = 1111\ldots11.$$

Find the thousandth digit after the decimal point in $\sqrt{N}$.