Functional Equation Hints

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Problems: Get one term to appear in two ways

1. It is pseudo-symmetry. What happens if you swap x and y?

2. Suppose $f(x) = 0$ for some $x \neq 0$. Prove that $f(y) = 0$ for all $y \neq \frac{x^2}{2}$. This problem isn't really about writing one term in multiple ways; it's just about finishing the worked example.

3. In both cases, we can write the equation as $f(x)+f(g(x)) = h(x)$. In (a), we have $g(g(x)) = x$, so try substituting in both x and $g(x)$. In (b), we have $g(g(g(x))) = x$ and the same idea works. *(Source: Korea 1999 for (b))*

4. A variant on fudging: Choose x and y so that xy is fixed and vary $x + y$. You should be able to prove $f(x) = f(y)$ if $x \leq 2\sqrt{y}$, and now let y be very large.

5. Fudging: Suppose there exists some y such that $\lfloor f(y) \rfloor \neq 0$. Then, vary x to conclude $f(x) = f(0)$ for $x \in [0, 1)$. Now pick x large and vary y to show $f(z) = f(0)$ for all $z > 0$. *(Source: IMO 2010)*

Problems: Induction, squeezing, and Cauchy

1. Set $n = 1$ to get $f(m + 1) = f(m) + f(1) + m$, and look for a formula. For (b), the answer is of the same form, but see what happens when you change the coefficients.

2. For each positive integer n, show that $|f(x) - f(y)| \leq nK \left(\frac{x-y}{n}\right)^2$, and then squeeze.

3. Prove that $f(x^3) = xf(x)^2$. Then if $f(nx) = nf(x)$ for all x, you can show $f(\sqrt[3]{n} \cdot x) = \sqrt[3]{n} \cdot f(x)$. There is a square root so you have to watch out for sign though. Is is possible that $f(x) < 0$ for $x > 0$? Once this is proven, try $y = \sqrt[3]{n} \cdot x$.

4. There are several steps here. First write $f(x + 1)$ in two different ways to calculate $f(1)$. Then calculate $f(n)$ for all positive integers n. To extend to the rational numbers, try writing $f\left(\frac{a}{b}\right) = f\left(\left(b + \frac{a}{b}\right) - b\right)$.

5. Let $f(0) = k$. Find a formula for $f(f(y))$ and let $y = f(z)$ to get $f(x^2+z+k^2) = f(x)^2+f(z)$. If you can show $k = 0$, then this is reduced to Example 3. *(Source: IMO 1992)*

6. Let $P(x)$ be that quadratic polynomial that equals $f(x)$ for $x = -1, 0, 1$. (Do you see why such a polynomial has to exist?) Show by induction that $f(x) = P(x)$ for all integers. Now repeat the argument for integer multiples of $\frac{1}{n}$.

Problems: Injectivity and surjectivity

1. First show f is injective. Now let $x = f(z)$.
2. Set $x = y$ and let y vary to see f is surjective. Now let $y = 0$.
3. Set $y = -f(x)$. For any z, can you choose x so that $x + f(-f(x)) = z$? *(Source: Mathlinks)*
4. First show f is injective. Now look at $f(g(f(x))$ and conclude $f(x^3) = f(x)^2$. Is this possible for an injective function? *(Source: IMO Shortlist 1997)*
5. First show f is surjective. Now choose x, y so that $f(x) = 0$ and $f(y) = z$ for some arbitrary z. You should be able to conclude that f is linear. *(Source: IMO Shortlist 2002)*

More Problems: Hints

1. Set $x = y$.
2. Use induction to show $f(nx) = n^2 f(x)$. *(Source: Nordic 2008)*
3. From the first two conditions, you can prove $f(n+2) = f(n) - 2$. You know $f(0)$. What is $f(1)$?
4. You may be tempted to use Cauchy's equation, but it's not that hard. First find $f(0)$ and then try $y = -x$.
5. It's quite similar to #4. Set $x = a, y = 0, z = 0$ to get $f(a) \le f(0)$. Can you now show $f(a) \ge f(0)$? *(Source: Russia 2000)*
6. For (a), try focusing on perfect squares. For (b), what happens when $h(n) = 1$, what happens when $h(n) = 2$, etc.?
7. First show f is injective. Now show $f(f(x+y) + f(1) + f(1)) = f(f(x) + f(y) + f(2))$ to get $f(x+y) = f(x) + f(y) + C$ for some constant C. Then show $f(x) = ax + b$ for some a, b.
8. Use pseudo-symmetry to show $f(x) = u(x) - 1$ and then $u(x+y) = u(x)u(y)$. Show $u(x) > 0$ for all x, and let $v(x) = \log u(x)$ to reduce to Cauchy's equation, and solve with monotonicity. *(Source: Romania 1998)*
9. Try self-cancelation with $m = 2n - f(n)$. We would be done if we could show that $f(t) = 0$ only when $t = 0$. But if $f(t) = 0$, then $f(m-t) = f(m)$ so f would have to be periodic and hence bounded. Is that possible? *(Source: Italy 2006)*
10. It's not as scary as it looks. Use the first two conditions to get a formula for $\frac{f(kx,ky,kz)}{f(x,y,z)}$. Now, given arbitrary x, y, z, find a k such that $x + ky = k^2 z$. *(Source: Balkan 2013)*
11. First show f is bijective. Setting $x = f^{-1}(0)$ should let you prove $f(0) = 1$ and eventually $f(n) = n + 1$ for all integers n. Now try $y = -2$. When is it possible to have $f(z) = 2z$? *(Source: Brazil 2006)*

12. It is all about fudging. First show $f(0) = 0$, then $f^{-1}(0) = 0$, then $f(a) = f(b)$ only if $a = b$. *(Source: APMO 2010)*

13. Let $k = f(1)$. First try to prove $k \cdot f(mn) = f(m) \cdot f(n)$. Use this to prove $k | f(n)$ for all n, and hence $g(n) = \frac{f(n)}{k}$ is a strictly better solution. Finally show f is a bijection on primes. The answer is $2^3 \cdot 3 \cdot 5 = 120$. *(Source: IMO 1998)*

14. Prove by induction that $f(n+1) > f(f(n))$ for all n. This is enough to prove $f(n) = n$. For this part, try showing $f(n) \geq n$. *(Source: Titu Andreescu? See also IMO 1997)*

15. The hardest part is showing $f(1) = 1$. Show $f(-x) = \pm f(x)$ and then look at $x, x+1, -x, 1-x$. If $f(1) \neq \pm 1$, then $f(x)$ can take on only finitely many values, which is impossible. From here, show f is increasing and $f(q) = q$ for $q \in \mathbb{Q}$. *(Source: Bulgaria 2005)*

16. When is $f(x) = 0$? From there, you should be able to show that $f(q) = q$ and $f(qx) = qf(x)$ for all rational numbers q. In the interest of fudging, try multiplying m by q and dividing n by q. If done right, you should get $f(x+y) = f(x) + f(y)$ and then $f(x^2) = f(x)^2$ to reduce to Cauchy's. *(Source: Mathlinks)*