Lemmas in Olympiad Geometry Cheat Sheet

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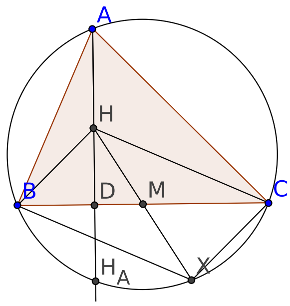
1 Introduction

Yes, the sheet "Lemmas in olympiad geometry" is hard. Proving all the lemmas from that sheet is virtually impossible (unless you're the geo god). So here's the cheat sheet.
Most of the lemmas can be found in the books 'Geometry Revisited' and 'Euclidean Geometry in Mathematical Olympiads(EGMO)'. We will reference the page number of those if it's the case.

2 Orthocenter related proprties

1. We only need to show $\angle BAH = \angle CAO$ since the other angle equalities follow by triangle symmetry. Hmm, what about angle chasing?
2. We need to show $DH = DH_A$. Can you prove $\Delta BDH \cong \Delta BDH_A$? For showing $HM = MX$, we consider the idea of phantom points[1]. Take X such that $BHCX$ is a parallelogram. Now, X lies on the $\odot ABC$ since $\angle BXC = \angle BHC = 180 - \angle BAC$. And the rest follows.

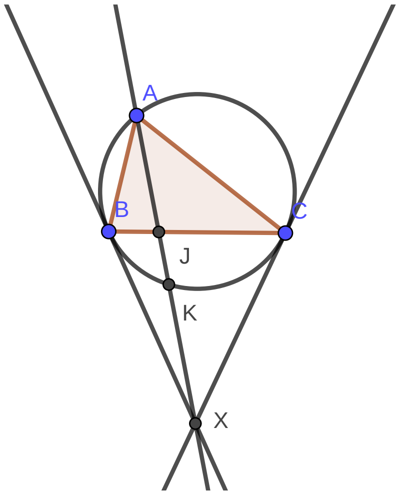


3. See page 18 of Geometry Revisited.
4. See page 8 of EGMO for the proof of H being the incenter. The excenter part follows if you consider ΔDEF as the reference triangle and construct A, B, C. The construction is the same as constructing an excenter. So, doesn't that like, make them excenters?
5. Can you show $\angle C'CA = \angle B'BA$? And how does this prove the lemma?
6. We use lemma 3 here. $AX = 2OM$ where M is the midpoint of BC. Again, $DY = 2OM$. So, $AX = DY$ and $AX \parallel DY$.
7. Page 44 and 45 of 'Geometry Revisited' has a detailed proof of this lemma.
8. The chapter 10.3 of EGMO is on this lemma.
9. Theorem 2.46 on page 37-38 in 'Geometry Revisited' shows the proof of this lemma.

[1] See page 16 of EGMO for a better understanding of phantom points

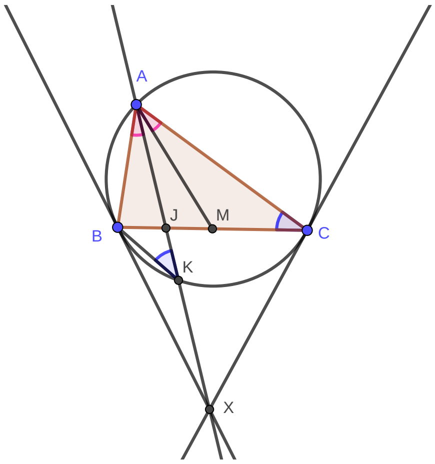
3 Symmedian Related Properties

1. Compute $\frac{sin\angle BAX}{sin\angle CAX}$ by using trig ceva on ΔABC and point X[2]. Then using sine law on ΔBAJ and ΔCAJ we get the desired result.



2. Using the alternate segment theorem(also known as tangent criterion)[3] in ΔXBK and ΔXBA we get that they're similar and thus, $BK.AX = BX.AB$. Similarly we get $CK.AX = CX.AC$. Then dividing the two equations and using the fact that $BX = CX$, we get the desired result that ABKC is a harmonic quadrilateral[4].

3. (M is the midpoint of BC) You've already computed $\frac{\sin\angle BAX}{\sin\angle CAX}$. Now just show that it's equal to $\frac{\sin\angle CAM}{\sin\angle BAM}$. Which gives us that AK is the symmedian of triangle ABC. Then proving the result is just angle chase.



4. The first one just follows from the fact that perpendicular from the centre bisects a chord.Now, since $(ABCD)$ is a harmonic quadrilateral, we can get that BJ is the symmedian of ΔABK and CJ is the symmadian of ΔACK. Now proving the second one is just angle chase.

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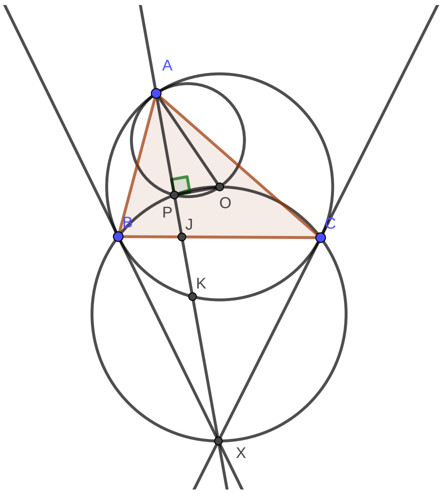
[2]See page 45, theorem 3.4 of EGMO to understand Trig Ceva

[3]see page 15 of EGMO

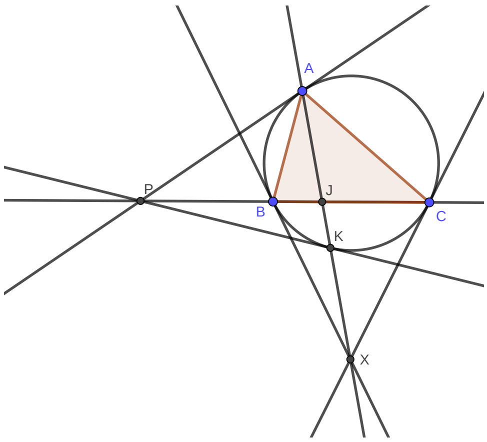
[4]A harmonic quadrilateral is a cyclic quadrilateral where the products of two pairs of opposite sides are equal. Look up page 64,173 of EGMO and this sheet on Projective Geometry by Alexander Remorov (http://alexanderrem.weebly.com/uploads/7/2/5/6/72566533/projectivegeometry.pdf) to know more about harmonic quadrilaterals and symmedians.



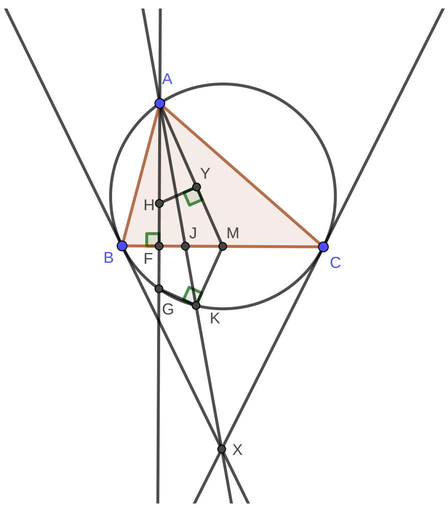
5. Proved in the previous one.

6. (Correction: It will be MO instead of MK) This is just angle chase.

7. Let the tangents at A, K meet at P. We get that CP is the symmedian of ΔACK. But we got earlier that BC is also the symmedian of ΔACK. So, BC and BP are the same i.e. P lies on BC, as desired.

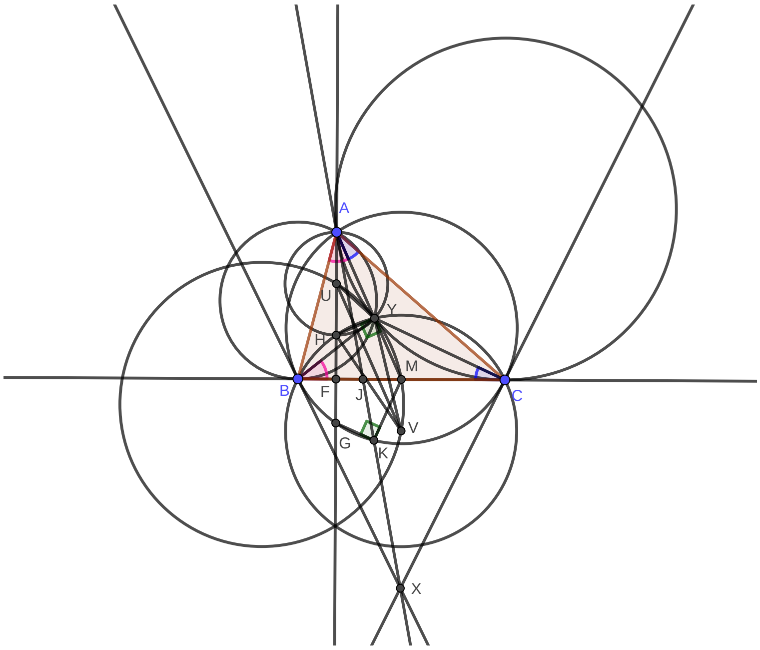


8. You can easily prove that Y lies on AM by angle chase. Now let the reflection of H under BC be G and the foot of perpendicular of A on BC be F. You can easily prove that $GK \perp KM$ by angle chase. So, by reflection $HY \perp YM$ or $HY \perp AM$. Also, $AH \perp FM$. So, $HFMY$ is cyclic so $AH.AF = AY.AM$, as desired.

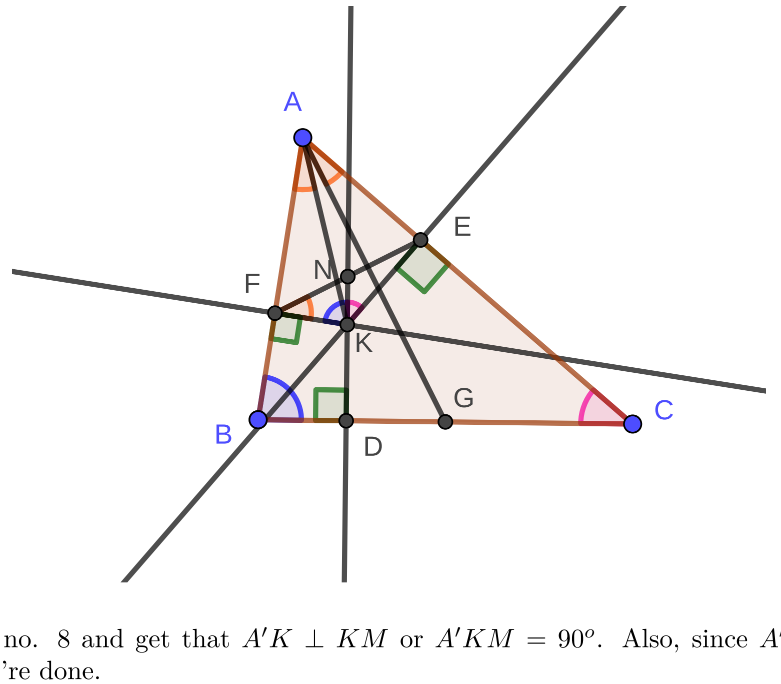


9. (BHC) is just the reflection of (ABC) under BC. So, since K belongs to (ABC), Y belongs to (BHC). Again we already got $\angle AYH = 90^o$ so Y belongs to (AH). We've shown before that Y lies on AM. Easy angle-chase gives

that $\angle MBY = \angle BAY$. So by alternate segment theorem we get that the circumcircle of ΔBAY is tangent to BC. So, the circle through AB tangent to B goes through Y. We get similar result for ΔCAY. Let the circumcentre of $(AHY), (BHYC)$ be U, V respectively. Here U is the midpoint of AH. Since quadrilateral $UHVY$ is a kite, we get that $UV \perp HY$. Again, $AY \perp HY$. So, $UV || AY$ or, $UV || YM$. Now, it is well known that $AH = 2OM$. So $UY = \frac{1}{2}AH = OM = MV$. So, $UYMV$ is a isosceles trapezoid and an isosceles trapezoid is obviously cyclic.



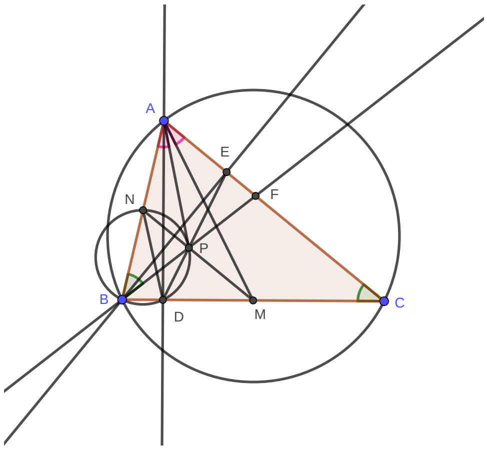
10. Use the fact that the symmedians are just reflections of the medians under the angle bisector. Then using trig ceva on ΔABC and the centroid of triangle ABC , you'll get an equation. Then using the converse of trig ceva in ΔABC and it's three symmedians and the equation found before, you'll get that the three symmedians are thus concurrent. (Note: the symmedian point is the isogonal conjugate[5] of the centroid.)Now let K be the symmedian point of ΔABC and ΔDEF be its pedal triangle with D, E, F being on BC, CA, AB respectively. Let DK meet EF at N. Now, we will first show that KN is the median of triangle KEF. Here,
$\frac{KE}{KF} = \frac{KE}{AK} \times \frac{AK}{KF} = \frac{\sin \angle EAK}{\sin \angle FAK} = \frac{\sin \angle BAM}{\sin \angle CAM} = \frac{\sin \angle ACB}{\sin \angle ABC} = \frac{\sin \angle ECD}{\sin \angle FBD} = \frac{\sin \angle EKN}{\sin \angle FKN}$.
So, KN is the median of ΔKEF. Thus, $DN = DK$ is the median of triangle DEF. Similarly EK, FK are also medians. So, K is the centroid of DEF, as desired.



11. We can use the proof in no. 8 and get that $A'K \perp KM$ or $A'KM = 90^o$. Also, since $A'DM = 90^o$, we get that $DA'KM$ is cyclic. So, we're done.

[5]Look at page 63 of EGMO to understand about isogonal conjugates.

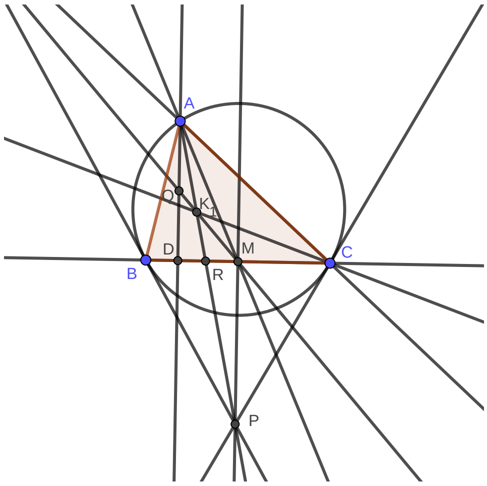
12. Let F be a point on AC such that BF is the line antiparallel to BC through B with respect to ΔABC. M,N be the midpoints of BC,BA respectively. Now, let BF meet MN at P. Then, since MN is the B-midline, P is the midpoint of BF. Now we get that ΔABF is similar to ΔABC by angle chase. So, since AP is the median of ΔABF, we get by similarity and angle chase that AP is the symmedian of ΔABC. Now, $\angle NDB = \angle ABD = \angle ABC = \angle AFB = \angle NPB$. So, $BDPN$ is cyclic. So, we get $\angle CDP = \angle BNP = \angle BAC = \angle CDE$. Thus, P lies on EF. So we're done. [6]



13. Let The tangents at B, C meet at P. M be the midpoint of BC, D be the A-foot of perpendicular, AK' meet BC at R and AR meet MQ at K_1. Then

    $-1 = (MA, MD; MQ, MP) \stackrel{M}{=} (A, R; K_1, P) \stackrel{B}{=} (BA, BR, BK_1, BP)$. [7]

    Therefore K_1 coincides with K', as desired.



4 In/excenter related properties

1. The A-excircle can be considered as a homothety[8] of the incircle where the homothetic center is A. The proof of D and D" being isotomic is given on page 11-13 of 'Geometry Revisited'.

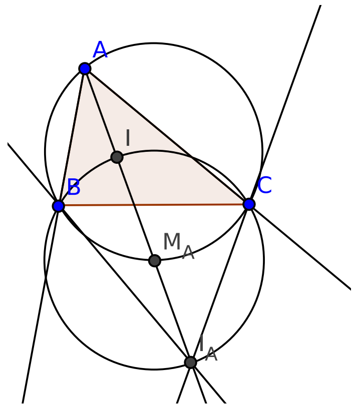
2. The hint of the previous lemma works here too. Think homothety.

3. $\angle IBI_A = \angle ICI_A = 90^o$. Which proves that I, B, I_A, C is cyclic. Now, we are done if we prove that M_A is the center of the circle. We do angle chasing for this. $\angle M_A BI = \angle M_A BC + \angle CBI = \dfrac{\angle A + \angle B}{2}$. $\angle M_A IB = 180 - \angle AIB = 180 - (180 - \dfrac{\angle A + \angle B}{2}) = \dfrac{\angle A + \angle B}{2}$. So, $BM_A = IM_A$. And similarly, $IM_A = CM_A$.

[6] If you don't know about perspectivity and projectivity, look up chapter 9 of EGMO and the Projective Geometry sheet of Alexander Remorov mentioned before.

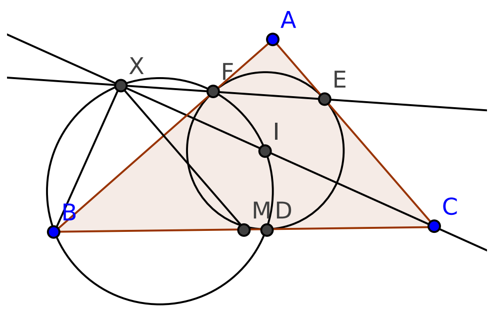
[7] $\stackrel{X}{=}$ means taking perspectivity from point X. Read chapter 9 of EGMO to learn about Projective Geometry

[8] You can learn about homothety on page 49 of EGMO



4. Hints of this lemma is given on page 63 of EGMO.

5. Let $EF \cap CI = X$. We do angle chasing to show X lies on $\odot BDIF$. $\angle IXF = 180 - \angle XEC - \angle ECX = 180 - (180 - \angle AEF) - \dfrac{\angle C}{2} = \dfrac{\angle B + \angle C}{2} - \dfrac{C}{2} = \dfrac{\angle B}{2} = \angle IBF$. So, $BDIFXiscyclic$. Now we will show that X lies on the C-midline. We have $\angle BDI = 90^o$. So, $\angle BXC = 90^o$ too. Which makes M the center of $\odot BXC$. So, $MX = MC$ implies $\angle MXC = \angle MCX = \angle XCA$. This implies, $MX \parallel AC$.



6. Hint: Consider $I_AI_BI_C$ as the reference triangle and work out what the other points are. You may look up page 49 of EGMO for further reading on the nine point circle.

7. Consider homothety. Hints are given on page 62-63 of EGMO.

8. There is a typo on this lemma. In the first sentence, change Z with K. Use La Hire for the pole part. And use ceva/menelaus for the harmonic bundle part. You may consult chapter 9 of EGMO.

9. Let the second intersection of $\odot ABC$ and $\odot AEIF$ be X. We have, $\angle AXI = 90^o$. Also, $\angle AXA' = 90^o$. So, X, I, A' is colinear. Now, show that the feet of the perpendicular from D is the image of X when inverted [9] with right to the incircle.

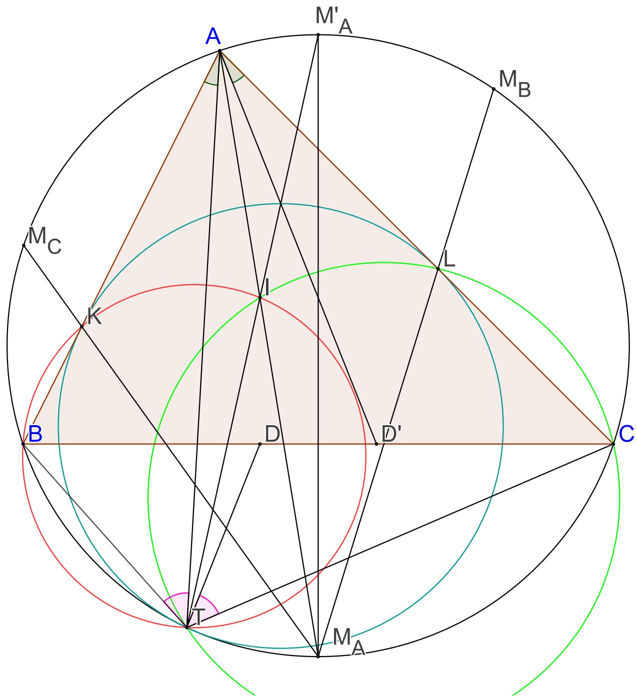
10. Can you prove that $\odot ABC$ becomes the nine point circle of the contact triangle when inverted with right to the incircle? Which means, the nine point center of the contact triangle lies on OI. And we know that the nine point center lies on the euler line.

11. By lemma 3, $M_AC = M_AI$. Since $\triangle M_ACM$ and $\triangle M_AM'_AC$ are similar, we have $M_AC^2 = M_AM.M_AM'_A = M_AI^2$. So $\triangle M_AMI$ and $\triangle M_AIM'_A$ are similar. Now do some angle chasing to prove the final result.

12. Information on curvilinear incircle is given on page 67 of EGMO.

13. Information and some proofs of mixtilinear incircle properties is given on page 68-69 of EGMO.

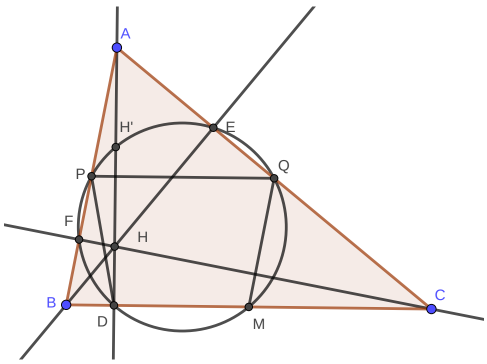
[9] See chapter 8 of EGMO



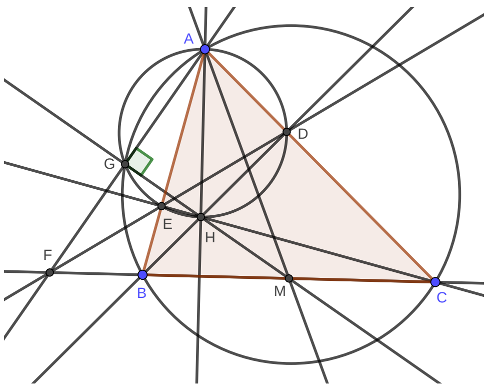
14.

5 Feet of the altitudes and the midpoints

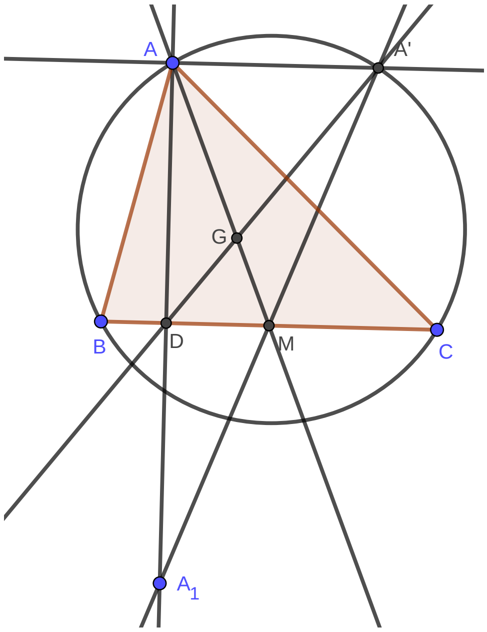
1. Let the reflection of O under M be O_1. Since $AH = 2OM$, we get $AH = OO_1$. And since $AH||OO_1$, we get that AHO_1O is a parallelogram. So O_1 is the reflection of A under N.



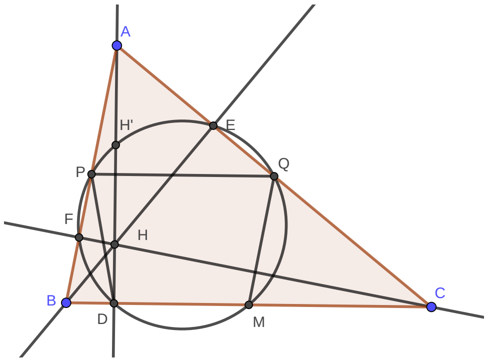
2. Let the B, C-feet of perpendiculars be D, E respectively. Let DE meet BC at P. And let AP meet (ABC) at X. Now, by miquel and angle chasing we get P lies on $(AH), (AM)$ and HM.



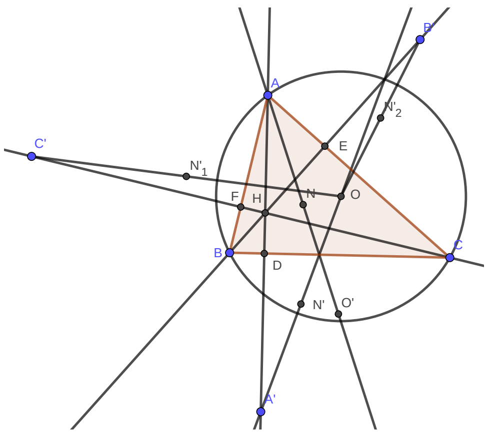
3. Let A' be a point such that AA' is parallel to BC and A_1 be the reflection of A under BC. M be the midpoint of BC. Let $A'D$ intersect AM at G'. We can easily see that $A'A_1$ goes through M. And since D is the midpoint of AA_1 and M is the midpoint of $A'A_1$, by using Menelaus theorem on $A'A_1D$ and line AM we get that $\frac{AG'}{G'M} = \frac{2}{1}$. So, G' coincides with G.



4. The first one follows from $PD = PB = \frac{AB}{2} = QM$ and $PQ||BC||DM$. And the second one follows from $H'E = H'F$ both being radius of circle $AEHF$ and $ME = MF$ both being radius of the circle $BCEF$.



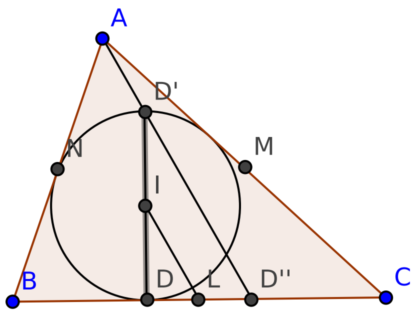
5. We've proved in no. 1 that the reflection of A under N coincides with the reflection of O under BC. Now, we just reflect A, N, O'(reflection of O under BC) under BC. Thus we get that the reflection of N under BC is the midpoint of O and the reflection of A under BC. We get the same for points B and C and thus get the desired result by homothety.



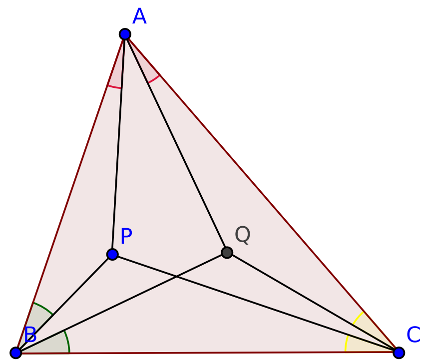
6. This is equivalent to no. 12 of Symmedian section.

6 Triangle Centers

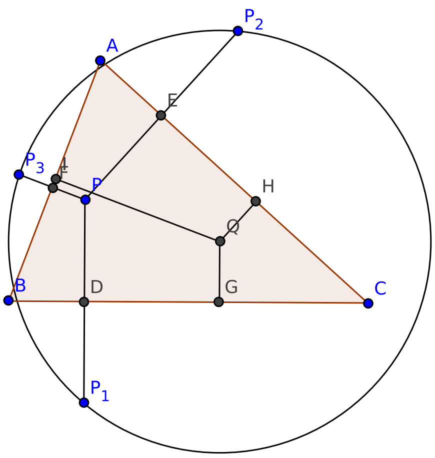
1. Since the medial triangle and $\triangle ABC$ are homothetic from G, we are done if we prove that the incenter is the nagel point of the medial triangle($\triangle LMN$). For this, we only need to prove that $IL \parallel AD'$(why?? Think homothety). From previous lemmas, we know that M is the midpoint of DD'' and I is the midpoint of DD'. And by that we are done.



2. Here, Q is the isogonal conjugate of P with right to $\triangle ABC$.



3. You can use power of a point to show that the pedal triangle of P and the pedal triangle of Q is cyclic. Use $\triangle BPF$ is similar to $\triangle BQG$ and also, $\triangle BPD$ is similar to $\triangle BQI$ to get the ratios to imply power of a point. Now, $PDGQ$ is a trapezium. So, the center of the pedal triangles' circumcircle is the midpoint of PQ. Now consider a $2x$ dilation from P to the circle. The midpoint of PQ goes to Q and the feets of P go to the reflections.



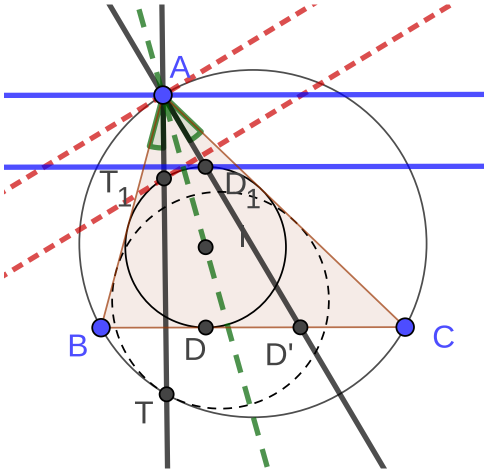
4. Use the fact that H and O are isogonal conjugates. Consider $\triangle AIH$(we use the picture above since we're too lazy to draw another picture). The circumcircle of that triangle lies on AQ. So, the orthocenter must lie on AP. Thus, $AP \perp IH$.

5. Let the isotomic conjugate of P be P'. Let the cevian triangles of the two points be respectively ΔDEF and ΔUVW respectively. Consider $BD = x, CE = y, AF = z$. Then you get $CU = x, AV = y, BW = z$. Now find the ratios $\frac{\Delta AEF}{\Delta ABC}, \frac{\Delta BDF}{\Delta ABC}, \frac{\Delta CDE}{\Delta ABC}$ by sine rule and then subtract their sum from 1. The difference will be the ratio $\frac{\Delta DEF}{\Delta ABC}$. Similarly do it for ΔXYZ. You'll get that both are the same.

6. This comes from the definition of isotomic conjugates and the first in/excenter related lemma above.

7. Let ω_a be the A-mixtilinear incircle and it touches (ABC) at T. Let D' be the A-excircle touchpoint with BC. D be the incircle touchpoint and reflection of D under the incenter be D_1 which lies on the incircle. We already know that A, D_1, D', N_a are collinear. From the no. 4 of mixtilinear incircles subsection of In/excenter related properties section, we get that AT and AD' are isogonals. Therefore since AD' goes trough N_a, AT does through it's isogonal conjugate. Now we're left to prove that AT goes through the exsimilicenter of the incircle and excircle. Let AT meets the incircle at two points and the closet to A be T_1. Now we're done if we can show that the tangent to the incircle at T_1 is parallel to the tangent at the circumcircle at A. Now, consider the A-angle bisector AI. The line tangent to T_1 is just the reflection of the line tangent to D_1 under AI. Now, the line parallel to BC through A and the tangent at A are isogonals i.e. reflections under AI also. Therefore, since we can easily see that the tangent at D_1 to the incircle is parallel to the line through A parallel to BC, by reflection we get the result we wanted. Thus proving our claim.

   Now do all these similarly replacing D_1 with D. Let the isogonal of AD intersect the incircle at two points and let the farthest one from A be T_2. And thus the second can be proved similarly.



8. Let $Q_1A = x, Q_1B = y, Q_1C = z$. Then by using sin and cosine rule we can get the following:
   $\cot\omega = \frac{a^2+y^2-z^2}{4\Delta BQ_1C} = \frac{b^2+z^2-x^2}{4\Delta CQ_1A} = \frac{c^2+x^2-y^2}{4\Delta AQ_1B} = \frac{a^2+b^2+c^2}{4\Delta ABC}$
   $\cot A = \frac{b^2+c^2-a^2}{4\Delta ABC}$, $\cot B = \frac{c^2+a^2-b^2}{4\Delta ABC}$, $\cot C = \frac{a^2+b^2-c^2}{4\Delta ABC}$
   So, $\cot\omega = \cot A + \cot B + \cot C$.

9. Go to page 117 of Geometry Revisited.

7 Miscellaneous useful properties

1. Well the statement looks creepy but its an obvious statement. It states that in any geometric transformation, for example inversion, reflection or any other type of transformation where it takes a definite point in the plane to another definite point, you will find that, if you vary a point P under a definite path and keep track of the path of it's image P' under transformation, this path of the P' of that point is exactly the image of the path of the P under transformation. (Note: This path is also called locus.) For example, if you invert a line under a definite circle, the image(inversion) of the line will be a circle. Now, if you pick a point P on that line, you will find a another point P' on the circle which is the image of P. And now if you vary P on the line P' will vary in the circle. And, the statement means that the locus of geometric transformation of P(which is the locus of P') is exactly the transformation of the whole locus(i.e. the circle on which P') lies on is the image of the line.

2. Just show that $\Delta YAB \sim \Delta YCD$ and $\Delta YAC \sim \Delta YBD$ by angle chase.

3. Consider the line joining the centre of composition of the two homotheties and the center of no. 1 homothety. Similarly consider it with the no. 2 homothety. You can see that both the lines divide their respective homotheties with same ratios. So, both the lines must be the same from the centre of composition of the two homotheties. This proves the statement.

4. Consider the line joining the centre of inversion with centre of first circle. The other circle obviously must be symmetric with respect to this line otherwise it's just illogical. So the result follows.

5. We just use cartesian coordinates. The equation of a circle is just $(x-p)^2+(y-q)^2=r^2$ where (p,q) is the coordinate of the centre of the circle and r is the radius of the circle. Now just take two equations of two circles in the cartesian plane. The power of a point O with respect to circle ω with center O_1 and radius r, $pow(O,\omega)=OO_1^2-r^2$. Now take a variable point in the cartesian plane and find out it's power with respect to ω_1 and power respect to ω_2. The ratio of the powers i.e. the ratio of the two equations is constant, let k. Now you can easily see that this can be rearranged into an equation same as that of a circle. So we're done.

6. Monge's theorem states that, if three circles lie on a plane with none lying completely inside another, if external tangents between every pair is drawn and they intersect at a point, these three points thus found are collinear. It's easy to prove just by Menelaus theorem in the triangle having the centres of the three circles as vertexes.

7. The cevian nest theorem is shown on page 57 of EGMO. The proof is given on page 248 of the same book.

8. We use the 5th lemma of this section to prove this. Consider the point circles B and C. The locus of the point P such that $\dfrac{PB}{PC}=\dfrac{AB}{AC}$ must be a circle. The intersections of A-bisectors with BC is on this circle. Moreover, this circle must be coaxial with B and C. Which means, center of the circle is on line BC. Which gets us the conclusion.

9. Brocard's theorem is a part of projective geometry. See Chapter 9 of EGMO for the proof of Brocard's theorem.

10. This is simple angle chasing.

11. See page 198 of EGMO for the proof of miquel's theorem.

12. Page 185 of EGMO cotains proof of the butterfly theorem. Page 45-46 of Geometry Revisited also has the proof of this theorem.