A new inequality involving primes

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Abstract

In this note, we find a new inequality involving primes and deduce several Bonse-type inequalities.

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1 Introduction

Denote the nth prime by p_n. In 1907, Bonse [1, 2] found and proved two interesting inequalities which states that for $n \geq 4$, $\prod_{i=1}^{i=n} p_i > p_{n+1}^2$ and for $n \geq 5$, $\prod_{i=1}^{i=n} p_i > p_{n+1}^3$. Based on the first inequality, he showed that a well known result which states that 30 is the largest integer N with the property that every integer a with $1 < a < N$ and $(a, N) = 1$ is prime. (This result has been further generalized. See [3, 4].) In 2007, Betts [5] obtained the inequality $p_{k+1} - p_k < p_k(p_1 p_2 \cdots p_{k-1} - p_k)/(p_{k+1} - p_k)$ by using Bonse's first inequality. Thus, naturally, people are interesting in Bonse's inequalities. More precisely, people would like to consider the inequalities about the product of the first n primes.

In 1960, Pósa [6] refined firstly Bonse's inequalities. He proved that for every integer $k > 1$ there is an n_k such that $p_{n+1}^k < p_1 p_2 \cdots p_n$ for all $n > n_k$. Moreover, the analogues of 30 are computed for the first few values of k. In 1962, Mamangakis [7] proved that for $n \geq 11$, $\prod_{i=1}^{n} p_i > p_{4n}$ and for $n \geq 46$, $\prod_{i=1}^{4n-9} p_i > p_{4n}^4$. In 1971, Reich [8] showed that for every natural number k there exists a natural number $N(k)$ such that $\prod_{i=1}^{n} p_i > p_{n+k}^2$

for all $n \geq N(k)$. In 1988, Sándor [9] showed that for $n \geq 3$, $p_1p_2\cdots p_n \geq p_1p_2\cdots p_{n-1} + p_n + p_{p_n-2}$, and for $n \geq 24$, $p_1p_2\cdots p_n \geq p_{n+5}^2 + p_{[n/2]}^2$, and for $n \geq 63$, $p_1p_2\cdots p_n \geq p_{n+3}^3 + p_{[n/3]}^6$, and so on. This refined the Bonse's inequalities again. However his approach is quite different from Bonse's. In 2000, using the Rosser-Schoenfeld and Robin estimates, Panaitopol [10] proved that $p_1p_2\cdots p_n > p_{n+1}^{n-\pi(n)}$, for all $n \geq 2$, where $\pi(n)$ is the prime-counting function. In this note, we proved the following new inequality involving primes:

Theorem 1: For integer $r \geq 20$, $p_{r+1}^{r-\pi(r)} > 2^{p_{r+1}}$ and for $1 \leq r < 20$, $p_{r+1}^{r-\pi(r)} < 2^{p_{r+1}}$.

By Theorem 1 and Panaitopol's inequality, we can deduce the following result:

Corollary 1: For integer $r \geq 10$, $p_1p_2\cdots p_r > 2^{p_{r+1}}$. For integer $0 < r < 10$ with $r \neq 8$, $p_1p_2\cdots p_r < 2^{p_{r+1}}$.

Corollary 1 improves Pósa's inequality in the following form: for given integer $k \geq 5$, $p_1p_2\cdots p_n > p_{n+1}^k$ for $n \geq 2k$. Bluntly speaking, the author likes Pósa's inequality. In [11], using Pósa's inequality, the author proved that there exists a prime q such that for all prime $p > q$, if $1 \leq a < p$, and r is the smallest prime satisfying $(r, a) = 1$, then $4r^3 < p$.

Based on Corollary 1, one could also get easily several Bonse-type inequalities for the first few values of n. For example, $\prod_{i=1}^{i=n} p_i > p_{n+1}^6$ provided $n \geq 10$, and $\prod_{i=1}^{i=n} p_i > p_{n+1}^5$ provided $n \geq 8$.

2 The Proof of Main Results

Lemma 1 [12]: For $x > 1$, $\pi(x) < \frac{1.25506x}{\log x}$.

Corollary 2: For integer $r \geq 55$, $r - \pi(r) > (r+1)\log 2$.

Proof: Firstly, we can check directly that for $63 \leq r \leq 149$, $0.3r \geq \pi(r) + 0.7$. If $r \geq 149$, then $\log r > 5$ and $7/r < 0.05$. Therefore, $\frac{12.5506}{\log r} + \frac{7}{r} < 3$. But, by Lemma 1, $\pi(r) < \frac{1.25506r}{\log r}$. So, $0.3r \geq \pi(r) + 0.7$, and for $r \geq 63$, $r - \pi(r) > (r+1) \times 0.7 > (r+1)\log 2$. When $62 \geq r \geq 55$, one can check directly $r - \pi(r) > (r+1)\log 2$. This completes the proof of Corollary 2.

Lemma 2 [12]: For $x \geq 17$, $\pi(x) > \frac{x}{\log x}$.

Proof of Theorem 1: By Lemma 2, for $p_{r+1} \geq 55$, $\pi(p_{r+1}) > \frac{p_{r+1}}{\log p_{r+1}}$. Hence, $r+1 > \frac{p_{r+1}}{\log p_{r+1}}$. By Corollary 2, for integer $r \geq 55$, $r - \pi(r) > (r+1)\log 2$. Note also that for $r \geq 55$, $p_{r+1} \geq 55$. So, $r - \pi(r) > \frac{p_{r+1}}{\log p_{r+1}} \log 2$, and for $r \geq 55$, $p_{r+1}^{r-\pi(r)} > 2^{p_{r+1}}$. When $20 \leq r \leq 54$, one can check directly that Theorem 1 is true as follows:

$$(20 - \pi(20)) \log p_{21} = 12 \times \log 73 > 51.4 > 73 \times 0.7 > 73 \log 2$$

$$(21 - \pi(21)) \log p_{22} = 13 \times \log 79 > 56.8 > 79 \times 0.7 > 79 \log 2$$

$$(22 - \pi(22)) \log p_{23} = 14 \times \log 83 > 61.8 > 83 \times 0.7 > 83 \log 2$$

$$(23 - \pi(23)) \log p_{24} = 14 \times \log 89 > 62.8 > 89 \times 0.7 > 89 \log 2$$

$$(24 - \pi(24)) \log p_{25} = 15 \times \log 97 > 68.6 > 97 \times 0.7 > 97 \log 2$$

$$(25 - \pi(25)) \log p_{26} = 16 \times \log 101 > 73.8 > 101 \times 0.7 > 101 \log 2$$

$$(26 - \pi(26)) \log p_{27} = 17 \times \log 103 > 78.7 > 103 \times 0.7 > 103 \log 2$$

$$(27 - \pi(27)) \log p_{28} = 18 \times \log 107 > 84.1 > 107 \times 0.7 > 107 \log 2$$

$$(28 - \pi(28)) \log p_{29} = 19 \times \log 109 > 89.1 > 109 \times 0.7 > 109 \log 2$$

$$(29 - \pi(29)) \log p_{30} = 19 \times \log 113 > 89.8 > 113 \times 0.7 > 113 \log 2$$

$$(30 - \pi(30)) \log p_{31} = 20 \times \log 127 > 96.8 > 127 \times 0.7 > 127 \log 2$$

$$(31 - \pi(31)) \log p_{32} = 20 \times \log 131 > 97.5 > 131 \times 0.7 > 131 \log 2$$

$$(32 - \pi(32)) \log p_{33} = 21 \times \log 137 > 103.3 > 137 \times 0.7 > 137 \log 2$$

$$(33 - \pi(33)) \log p_{34} = 22 \times \log 139 > 108.5 > 139 \times 0.7 > 139 \log 2$$

$$(34 - \pi(34)) \log p_{35} = 23 \times \log 149 > 115.0 > 149 \times 0.7 > 149 \log 2$$

$$(35 - \pi(35)) \log p_{36} = 24 \times \log 151 > 120.4 > 151 \times 0.7 > 151 \log 2$$

$$(36 - \pi(36)) \log p_{37} = 25 \times \log 157 > 126.4 > 157 \times 0.7 > 157 \log 2$$

$$(37 - \pi(37)) \log p_{38} = 25 \times \log 163 > 127.3 > 163 \times 0.7 > 163 \log 2$$

$$(38 - \pi(38)) \log p_{39} = 26 \times \log 167 > 133.0 > 167 \times 0.7 > 167 \log 2$$

$$(39 - \pi(39)) \log p_{40} = 27 \times \log 173 > 139.1 > 173 \times 0.7 > 173 \log 2$$

$$(40 - \pi(40)) \log p_{41} = 28 \times \log 179 > 145.2 > 179 \times 0.7 > 179 \log 2$$

$$(41 - \pi(41)) \log p_{42} = 28 \times \log 181 > 145.5 > 181 \times 0.7 > 181 \log 2$$

$$(42 - \pi(42)) \log p_{43} = 29 \times \log 191 > 152.3 > 191 \times 0.7 > 191 \log 2$$

$$(43-\pi(43))\log p_{44} = 30\times\log 193 > 152.6 > 193\times 0.7 > 193\log 2$$
$$(44-\pi(44))\log p_{45} = 30\times\log 197 > 158.4 > 197\times 0.7 > 197\log 2$$
$$(45-\pi(45))\log p_{46} = 31\times\log 199 > 164.0 > 199\times 0.7 > 199\log 2$$
$$(46-\pi(46))\log p_{47} = 32\times\log 211 > 171.2 > 211\times 0.7 > 211\log 2$$
$$(47-\pi(47))\log p_{48} = 32\times\log 223 > 173.0 > 223\times 0.7 > 223\log 2$$
$$(48-\pi(48))\log p_{49} = 33\times\log 227 > 179.0 > 227\times 0.7 > 227\log 2$$
$$(49-\pi(49))\log p_{50} = 34\times\log 229 > 184.7 > 229\times 0.7 > 229\log 2$$
$$(50-\pi(50))\log p_{51} = 35\times\log 233 > 190.7 > 233\times 0.7 > 233\log 2$$
$$(51-\pi(51))\log p_{52} = 36\times\log 239 > 197.1 > 239\times 0.7 > 239\log 2$$
$$(52-\pi(52))\log p_{53} = 37\times\log 241 > 202.9 > 241\times 0.7 > 241\log 2$$
$$(53-\pi(53))\log p_{54} = 37\times\log 251 > 204.4 > 251\times 0.7 > 251\log 2$$
$$(54-\pi(54))\log p_{55} = 38\times\log 257 > 210.8 > 257\times 0.7 > 257\log 2$$

When $1 \le r < 20$, one can check similarly $p_{r+1}^{r-\pi(r)} < 2^{p_{r+1}}$. This completes the proof of Theorem 1.

Proof of Corollary 1: By Panaitopol's inequality, for $r \ge 20$, we have $p_1p_2\cdots p_r > 2^{p_{r+1}}$. When $10 \le r \le 19$, one can check directly that Corollary 1 is true. The remaining case can be checked similarly. This completes the proof of Corollary 1.

Finally, we prove that Corollary 1 improves Pósa's inequality in the following form: for given integer $k \ge 5$, $p_1p_2\cdots p_n > p_{n+1}^k$ for $n \ge 2k$. Note that for $k \ge 5$, we have $n \ge 2k \ge 10$. So, by Corollary 1, we have $p_1p_2\cdots p_n > 2^{p_{n+1}}$. But by Lemma 1, we have $n+1 < \frac{1.25506p_{n+1}}{\log p_{n+1}}$. On the other hand, $\frac{n+1}{1.25506} > \frac{n}{2\log 2}$ since $\frac{1}{1.25506} > \frac{1}{2\log 2}$. Thus, $\frac{p_{n+1}}{\log p_{n+1}} > \frac{n}{2\log 2}$. So, $2^{p_{n+1}} > p_{n+1}^{n/2}$ and $p_1p_2\cdots p_n > 2^{p_{n+1}} > p_{n+1}^{n/2} \ge p_{n+1}^k$. This completes the proof. As an application, one can deduce easily that $\prod_{i=1}^{i=n} p_i > p_{n+1}^6$ provided $n \ge 10$, and $\prod_{i=1}^{i=n} p_i > p_{n+1}^5$ provided $n \ge 8$.

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