TJHSST BLACK BOOK OF PROBLEM SOLVING: GEMS

1. Pearls

Problem 1.1. A 6×6 chess board is tiled with 2×1 dominos. Prove that you can cut the board into two parts by a straight line that does not cut dominos.

Problem 1.2. There are 100 points in the plane. Prove that you can cover all the points with a family of circles such that the sum of their diameters is less than 100 and the distance between any two of the circles is more than one.

Problem 1.3. Prove that from every set of 200 integers you can choose a subset of 100 with total sum divisible by 100.

Problem 1.4. If an $a \times b \times c$ brick is tiled by $1 \times 2 \times 4$ bricks show that it can also be filled with all of these bricks standing in the same way.

Problem 1.5 (Djibouti 1992)**.** There are 1992 trinagles in the plane such that any three triangles share at least one point in common. Prove that there are at least 991 triangles that share a common point.

Problem 1.6. Let K be a convex set in the plane which is centrally symmetric with respect to the origin O. Suppose K has an area greater than 4. Prove that K contains a lattic points other than O.

Problem 1.7. Let $f(x) = a_n x^n + \ldots + a_0$ be a polynomial of degree $n \geq 2$ with integer coefficients. Assume that it has only real roots that are all distinct and lie in $(0, 1)$. Prove that $|a_0| \geq 2^n + 1$.

Problem 1.8 (2001 RMO)**.** In a collection of 2n+1 persons, for any group of n persons there exists a person, not in the group, who knows each person of the group. Prove that there exists a person, who knows everybody.

Problem 1.9. (1998 IMO) There is a contest with a contestants and b judges where b is odd. Each judge gives a 0 or 1 to each contestant and every pair of judges agree about at most k contestants. Prove

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

Problem 1.10. On a blackjack machine one can win or lose one dollar at a time. Suppose two players playing once per minute during a period find that eventually both of them win exactly $2N$ dollars. Show that there was a time interval during which both of them won exactly N dollars.

Problem 1.11 (Iran 1998)**.** The edges of a regular 2^n-gon are colored red and blue in some fashion. A step consists of recoloring each edge which is the same color as both of its neighbors in red, and recoloring each other edge in blue. Prove that after 2^{n-1} steps all of the edges will be red, and show that this need not hold after fewer steps.

Problem 1.12. There are n points $P_1, P_2 \dots, P_n$ in the interval $[-1, 1]$ where the points are consecutively arranged from left to right. Let τ_i be the product of all distances from P_i to the other $n-1$ points. Prove that

$$\sum_{i=1}^{n} \frac{1}{\tau_i} \geq 2^{n-2}.$$

Problem 1.13. Can the square be cut into a finite number of non-convex quadrilaterals?

Problem 1.14. The edges of K_n are colored by $p+1$ colors in such a way that each triangle has all sides of the same color or all sides are of a different color. Prove $p^2 \geq n$ and that if p is prime then the bound is sharp.

Problem 1.15. Ten gangsters are standing on a flat surface, and the distances among them are all distinct. At twelve o'clock, when the church bells start chiming, each of them shoots at the one among the other nine gangsters who is closest to him. What is the smallest number of gangsters to be killed?

Problem 1.16. (1997 Tournament of Towns) A large equilateral triangle is cut into n^2 congruent little equilateral triangles by forming n parallel strips to each side of the triangle. Two little triangles are called independent if they are not contained in the same strip. Determine the maximum number of pairwise independent triangles for $n = 9, 10$.

Problem 1.17. Given the square table $n \times n$ with $(n-1)$ marked entries. Prove that it is possible to move all the marked entries below the diagonal by moving rows and columns.

Problem 1.18. (IMO 1995 Canada) p - an odd prime. $A = \{1, 2 \dots 2p\}$. Find the number of p element subsets of A whose sum is divisible by p.

Problem 1.19. Let S be a finite set of points in the plane with diameter 1, that is for every $p, q \in S$, we have $|p - q| \leq 1$. Prove that there is a circle of radius $\sqrt{3}/3$ covering all of S.

Problem 1.20. There are n football teams in a league. The conference organizers wish to split the season into $k > 1$ round-robin tournaments such that every team plays every other team exactly once. What is the minimal such k for which this is possible?

2. Rubies

Problem 2.1. Define a Czech cube to be a center cube with one cube attached to each face. Prove that all of R^3 can be tiled by Czech cubes.

Problem 2.2 (1997 Sweden)**.** Let M be a set of real numbers. Assume that M is the union of a finite number of disjoint intervals and that the total length of the intervals is greater than 1. Prove that M contains at least one pair of distinct numbers whose difference is an integer.

Problem 2.3. Asterisks are placed in some cells of an m by n rectangular table, where $m < n$, so that there is at least one asterisk in each column. Prove that there exists an asterisk such that there are more asterisks in its row than in its column.

Problem 2.4. What is the minimal positive integer n such that among n integers one can choose exactly 17 whose sum is divisible by 17?

Problem 2.5 (1996-7 St. Petersburg)**.** Prove that the set of all 12-digit numbers cannot be divided into groups of 4 numbers so that the numbers in each group have the same digits in 11 places and four consecutive digits in the remaining place.

Problem 2.6. There is a parabola $y = x^2$ drawn on the coordinate plane. The axes are deleted. Can you restore them with the help of compass and ruler?

Problem 2.7 (Helly's Theorem in the Plane)**.** There are n convex figures in the plane such that any three figures share at least one point in common. Prove that there is a point that is common to all of the figures

Problem 2.8. Let $p \geq 3$ be a prime number. Define a sequence (a_n) by

$$a_n = n$$

for $n = 0, 1, 2, \ldots, p-1$

$$a_n = a_{n-1} + a_{n-p}$$

for $n \geq p$. Determine the remainder left by a_{p^3} on division by p.

Problem 2.9. Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

Problem 2.10. Given several squares with the total area 1. Prove that you can place them in the square of the area 2 without any intersections.

Problem 2.11. $f(x)$ is an integer polynomial of finite degree. $a_1, a_2 \ldots a_k$ are integers such that for all $x \in \mathbb{Z}$ there exists $i \in [1, k]$ such that $a_i | f(x)$. Prove that there exists an $l \in [1, k]$ such that for all integers x, $a_l | f(x)$.

Problem 2.12. α, β, γ are the angles of a triangle. Prove $\cos\alpha + \cos\beta + \cot\gamma \leq \frac{3}{2}$.

Problem 2.13 (Jan Mycielski)**.** A solid cube $20 \times 20 \times 20$ is built out of bricks each $2 \times 2 \times 1$. All bricks are laid with their faces parallel to the cubes faces, though bricks need not be laid flat. Prove that at least one straight line perpendicular to a face of the cube pierces its interior but no bricks interior.

Problem 2.14 (1995 Short List)**.** Let k be a positive integer. Prove that there are infinitely many perfect squares of the form $n2^k - 7$, where n is a positive integer.

Problem 2.15. Let $A_1, A_2, \ldots A_n$ be any n points of the plane. Show that on any segment of length l there is a point M such that

$$\overline{MA_1} \cdot \overline{MA_2} \cdots \overline{MA_n} \geq 2\left(\frac{l}{4}\right)^n.$$

How should the n points $A_1, A_2, \ldots A_n$ be chosen so that on a given segment PQ of length l there is no point M for which

$$\overline{MA_1} \cdot \overline{MA_2} \cdots \overline{MA_n} > 2\left(\frac{l}{4}\right)^n$$

Problem 2.16. Can we split the nonnegative integers into two sets A and B so that every integer n is expressible in the same number of ways as the sum of two distinct members of A, as it is the sum of two distinct members of B?

Problem 2.17 (Three Jug Problem)**.** Three jugs are given with water in them, each containing an integer number of pints. It is alloed to pour into any jug as much water as it already contains, from any other jug. Prove that after several such pourings it is possible to empty one of the jugs. (Assume that the jugs are sufficiently large; each of them can contain all the water available.)

Problem 2.18. Congress has appointed n people to serve on its mathematics advisory commission. The mathematics advisory commission has formed n subcommittees (from members of the advisory commission). No two subcommittees have identical membership. Prove that it is possible to find a member of the advisory committee whose departure leaves all of the subcommittees distinct.

Problem 2.19 (The Politician)**.** Suppose that there is a group of n people such that every two people have exactly one mutual friend. Prove that there is one person who is everyone's friend.

Problem 2.20. Show that there exist arbitrarily long sequences consisting of the digits 0,1,2,3, such that no digit or sequence of digits orrurs twice in succesion.

3. Emeralds

Problem 3.1 (Erdos)**.** Prove that if you have $n+1$ positive integers less than or equal to $2n$, some pair of them are relatively prime.

Problem 3.2. There are n cars waiting at distinct points of a circular race track. At the starting signal each car starts. Each car may choose arbitrarily which of the two possible directions to go. Each car has the same constant speed. Whenever two cars meet they both change direction (but not speed). Show that at some time each car is back at its starting point.

Problem 3.3. We call an ordered pair of sets (A, B) good if

$$A \subseteq B \text{ and } B \subseteq \{1, 2 \ldots, n\}.$$

How many good sets are there for each positive integer n?

Problem 3.4 (Balkan 2001)**.** A cube of dimensions $3 \times 3 \times 3$ is divided into 27 unit cells, each of dimensions $1 \times 1 \times 1$. One of the cells is empty, and all others are filled with unit cubes which are, at random, labelled $1, 2, ..., 26$. A legal move consists of a move of any of the unit cubes to its neighbouring empty cell. Does there exist a finite sequence of legal moves after which the unit cubes labelled k and $27 - k$ exchange their positions for all $k = 1, 2, ..., 13$? (Two cells are said to be neighbours if they share a common face.)

Problem 3.5. (a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart? (b) What if three is replaced by nine?

Problem 3.6. A 7 by 7 square is made up of 16 (1×3) tiles and 1 (1×1) tile. Determine all possible positions of the (1×1) tile.

Problem 3.7 (Gomory)**.** Prove that if any two squares of opposite color are removed from a chessboard the the remaining squares can be tiled by 1×2 dominoes.

Problem 3.8. Points X, Y, Z are on the sides of triangle ABC such that $AX = 2XB, BY = 2YC$, and $CZ = 2ZA$. Compute the area of triangle XYZ given that $AB = 13, BC = 14, CA = 15$.

Problem 3.9. Six pine trees grow on the shore of a circular lake. It is known that a treasure is submersed at the midpoint T between the intersection points of the altitudes of two triangles, the vertices of one being at three of the 6 pines, and the vertices of the second one at the other three pines. At how many points T must one dive to find the treasure?

Problem 3.10. Can the positive integers be split into n arithmetic progressions with distinct common differences?

Problem 3.11. There is an infinite arithmetic progression with the positive integer elements with one element a perfect square. Prove that the progression contains an infinite number of perfect squares.

Problem 3.12. Prove that there exists a number divisible by 5^{1000} not containing a single zero in its decimal notation.

Problem 3.13 (Orchard Problem)**.** A tree is planted at each lattice point in a circular orchard which has center at the origina and radius < 50. If the radius of the trees exceed $1/50$ of a unit, show that from the origin one is unable to see out of the orchard no mattter in what direction he looks; show that if the radius of the trees is reduced to $< \sqrt{\frac{1}{2501}}$ then one can see beyond the edge of the orchard.

Problem 3.14 (Boring Polynomials)**.** A polynomial $p(x)$ is called boring if for all rational $x, p(x)$ is rational and for all irrational $x, p(x)$ is irrational. Determine all boring polynomials.

Problem 3.15. For a given positive integer n, what is the minimal positive integer $f(n)$ such that among $f(n)$ integers one can choose exactly n whose sum is divisible by n?

Problem 3.16. Let $x_1, x_2, \ldots x_n$ be real numbers of absolute value at least 1. Prove that no more than $\binom{n}{\lfloor n/2 \rfloor}$ of the sums $\pm x_1 \pm x_2 \pm \cdots \pm x_n$ fall in an open interval of length 2.

Problem 3.17 (1988 Short List)**.** An even number of people have a discussion around a circular table. After break they sit again around the circular table in a different order. Prove that there is at least one pair of people such that the number of participants sitting between them before and after the break is the same.

Problem 3.18. Prove that the number of tilings of a $2n \times 2n$ rectangle by 1×2 dominoes is either a perfect square or twice a perfect square.

Problem 3.19 (Lighthouses)**.** There are n lighthouses situated in the oceans of Flatland. Each lighthouse has a lamp whose beam sweeps out an angle $2\pi/n$. Prove that the lamps can be rotated so that all of Flatland is illuminated.

Problem 3.20. Prove that for any integer $n > 2$ there exists a set of 2^{n-1} points in the plane such that no three lie on a line and no $2n$ are the vertices of a $2n-$gon.

4. Amethyst

Problem 4.1 (Erdos and Suranyi)**.** Prove that every integer n can be represented in infinitely many ways as

$$n = \pm 1^2 \pm 2^2 \pm \cdots \pm k^2$$

for a convenient k and a suitable choice of the signs $+$ and $-$.

Problem 4.2 (2000 TT)**.** For what largest integer n can one find n points on the surface of the cube, not all lying on one face and forming the vertices of a regular plane n-gon.

Problem 4.3. An equilateral triangle ABC has a point P in its interior such that $AP = 3, BP = 4$, and $CP = 5$. What is the area of ABC?

Problem 4.4 (Crazy Dice)**.** A pair of crazy dice are two cubes labled with intgers such that they are not labled with the same nubmers as an ordinary pair of dice, but the probability of rolling any number with the pair of crazy dice is the same as rolling it with ordinary dice. Find a pair of crazy dice.

Problem 4.5. Show that there exist arbitrarily long sequences of 1's and 2's in which no digit or sequence of digits occurs three times in succession.

Problem 4.6 (TT)**.** We are given 101 rectangles with sides of integer lengths not exceeding 100. Prove that among these 101 rectangles there are 3 rectangles, say A, B and C, such that A will fit inside B and B inside C.

Problem 4.7. A number is called harmonic if it is of the form $2^n 3^m$. Are there any pairs consecutive harmonic numbers greater than $(8, 9)$?

Problem 4.8. The entry in the ith row and jth column of an $n \times n$ matrix equals $a_i + b_j$, where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots b_n$ are distinct real numbers. The products of the numbers in each row of the matrix are equal. Prove that the products of the numbers in each column are also equal.

Problem 4.9 (ARML 2000)**.** There is a $m \times n$ array of dots. An ARML (m, n) polygon is a simple closed polygon with the following properties:

(a) It can be constructed by joining all the points in the m by n array with unit line segments that are horizontal or vertical. This can be done without lifting a pencil that trace out the path of the polygon.
(b) Each point is joined by a unit line segment to exactly to other points.

Let $f(n)$ be the number of ARML (4,n) polygons. Determine a linear recurrence relation for $f(n)$.

Problem 4.10 (2000 Bulgaria)**.** Find all polynomials $P(x)$ with real coefficients such that $P(x)P(x+1) = P(x^2)$ for all real x.

Problem 4.11 (Egyptian Fractions)**.** Determine the largest number less than 1 that can be expressed as the sum of n distinct fractions of the form $\frac{1}{m}$ where m is a positive integer.

Problem 4.12 (Perfect Rulers)**.** The ruler with 4 marks at $0, 1, 4, 6$ measure all possible $\binom{4}{2} = 6$ distances. Since it has n marks and can measure all integer distances from 1 to $\binom{n}{2}$, it is called a perfect ruler. Determine all other perfect rulers.

Problem 4.13. An integer is written in each cell of an infinite grid, and any two cells having a common side contain numbers differing by at most 1. Prove that there are n distinct cells containing the same number.

Problem 4.14. Inscribe in a given acute triangle ABC a triangle UVW whose perimeter is as small as possible.

Problem 4.15. (a) Consider n arithmetic progrssions, each consisting entirely of integers, and extended indefinitely in both dirrection. Show that if any two of the progressions have a term in common, then they all have a term in common. Show that if the progressions are allowed to assume non-integer values, then the conclusion is false. (b) Consider n arithmetic progressions, each extending indefinitely in both directions. If each three of them have a term in common, prove that they all have a term in common.

Problem 4.16. Can a countably infinite set have an uncountable collection of nonempty subsets such that the intersection of any two of them is infinite?

Problem 4.17 (Conway/Reznick)**.** Find all finite sets of points in the plane such that if two lines determined by 4 points in the set intersect in a single point, then that point is in the set.

Problem 4.18 (A problem on neighbors)**.** A square of side 1 is divided into polygons. Suppose that each of these polygons has a diameter (greatest distance between any two points) less that $1/30$. Show that there is a polygon P with at least six neighbors, that is, polygons touching P in at least one point.

Problem 4.19. Prove that the product of n consecutive positive integers is never a perfect square.

Problem 4.20 (1972 Moscow)**.** Given n straight lines in general position in the plane (no three lines pass through the same point and no two are parallel) they divide the plane into sevveral parts. Prove that among these parts there are at least

(a) $n/3$ triangles
(b) $(n-2)/2$ triangles
(c) $n-2$ triangles.

5. Topaz

Problem 5.1. Every member of a certain parliament has not more than 3 enemies. Prove that it is possible to divide it onto two subparliaments so, that everyone will have not more than one enemy in his subparliament. (A is the enemy of B if and only if B is the enemy of A.)

Problem 5.2. Prove that a graph with $2n$ vertices and n^2+1 edges must contain a triangle.

Problem 5.3 (1996 TT)**.** Consider the product of the following 100 numbers: $1!, 2!, \ldots, 100!$. Can one delete one of the factors so that the product of all the other numbers will be a perfect square?

Problem 5.4 (Conway)**.** Produce the edges of a triangle ABC to distances a beyond A, b beyond B, c beyond C where a, b, c are the edgelenths of the triangle. Prove that the 6 points so constructed lie on a circle.

Problem 5.5. There are n circles in the plane with equal radii. A valid move is one such that all pairwise distances decrease. Prove that the area of the union of the circles can not increase under a valid move.

Problem 5.6. An $n \times n$ chessboard is numbered by the numbers $1, 2, \ldots, n^2$ (and every number occurs once). Prove that there exist two neighbouring (with common edge) squares such that their numbers differ by at least n.

Problem 5.7. Draw n straight lines in the plane. Show that the regions in which these line divide the plane can be colored with two colors in such a way that no two adjacent regions have the same color.

Problem 5.8. $2n$ points are marked on the circumference of a circle. In how many different ways can these points be joined in pairs by n chords which do not intersect within the circle?

Problem 5.9 (1993 Short List)**.** A positive integer n has property P if, for all a, $n^2 \mid a^n - 1$ whenever $n \mid a^n - 1$. Show that every prime number has property P. Show that infinitely many other numbers have property P.

Problem 5.10. Let $m > 1$ be a positive odd integer. Find the smallest positive integer n such that $2^{1989} \mid m^n - 1$.

Problem 5.11. Prove that there exist 2001 convex polyhedra such that any three of them do not have any common points but any two of them touch each other (i.e., have at least one common boundary point but no common inner points).

Problem 5.12. In the plane we are given n points, not all collinear. Show that at least n straight lines are required to join all possible pairs of points.

Problem 5.13 (Kontsevich 1981 TT)**.** Initially there is a single pebble at $(0, 0)$. On each move you are allowed to replace a pebble at position (i, j) by two pebbles at positions $(i+1, j)$ and $(i, j+1)$ provided that there is not a pebble in either of those squares. Prove that there is always a pebble in one of the 10 squares (i, j) with $i + j \leq 3$.

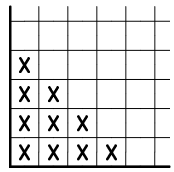

FIGURE 1. The unavoidable squares

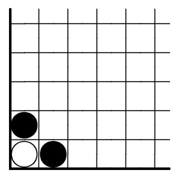

FIGURE 2. The initial pebble (white) is replaced by two new pebbles (black)

Problem 5.14 (Quantum M247)**.** Does there exist a nonlinear differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that f takes rational numbers to rational numbers and irrational numbers to irrational numbers?

Problem 5.15. Let $b(n)$ be the number of ways of expressing n as the sum of powers of 2 using each exponent at most twice and define $b(0) = 1$. For example, $b(13) = 3$, because $13 = 8+4+1 = 8+2+2+1 = 4+4+2+2+1$. Prove that all the ratios $b(n)/b(n+1)$ are in lowest terms, and every positive rational number occurs exactly once in this list.

Problem 5.16 (1998 St. Petersburg)**.** In the plane are given several squares with parallel sides, such that among any n of them, there exist four having a common point. Prove that the squares can be divided into at most $n-3$ groups, such that all of the squares in a group have a common point.

Problem 5.17 (Three Circles)**.** Let there be three circles of different radii lying completely outside each other. To exclude a trivial case, assume also that their centers are not collinear, i.e. the three centers do not lie on the same straight line. Under these conditions, six external tangents to two of the three circles, taken pairwise, intersect at three points. Prove that the three points are collinear.

Problem 5.18 (Erdős)**.** A finite number of pennies are placed flat on the plane so that no two overlap and no three touch each other. Prove that these pennies can be painted with at most three colors so that touching pennies bear different colors.

Problem 5.19. Deterime the sum of the series

$$\sum_{m\geq 0} \frac{1}{F_{2^m}},$$

where F_m is the m_{th} Fibonacci number.

Problem 5.20 (Mazurkiewicz)**.** Does there exist a subset of the plane such that every straight line intersects the set at exactly two points?

6. Diamonds

Problem 6.1. Given $2n$ points in a plane with no three of them collinear. Show that they can be divided into n pairs such that the n segments joining each pair do not intersect.

Problem 6.2 (A. Liu)**.** Put 77 chocolate bars into as few bags as possible such that you can give either 11 children 7 candy bars each or 7 children with 11 candy bars each.

Problem 6.3. A convex quadrilateral has equal diagonals. Equilateral triangles are constructed on the outside of each side. The centers of the triangles on opposite sides are joined. Show that the two joining lines are perpendicular.

Problem 6.4 (Reclusive Primes)**.** Prove that or any given number N, there exists a prime number that is at least N greater than the previous prime number and at least N smaller than the following one

Problem 6.5. Let $p > 3$ a prime and consider a partition of the set $1, 2, \ldots p-1$ into three disjoint subsets A, B, C. Prove that there exist x, y, and z all in different subsets such that $y + z - x$ is divisible by p.

Problem 6.6 (Iran 1998)**.** The edges of a regular 2^n-gon are colored red and blue in some fashion. A step consists of recoloring each edge which is the same color as both of its neighbors in red, and recoloring each other edge in blue. Prove that after 2^{n-1} steps all of the edges will be red, and show that this need not hold after fewer steps.

Problem 6.7. Randomly partion $[2n]$ into two sets of n integers. What is the probablitity that a brick with side lengths in the first side can be fit into the other brick with edges of both bricks parallel to each other?

Problem 6.8 (Swiss Cheese)**.** A block of swiss cheese is a rectangular prism with a finite number of non-interesting spheres removed. Prove that a block of swiss cheese can be cut into convex polyhedron such that each of the polyhedra contains exactly 1 sphere.

Problem 6.9 (Combinatorially Isosceles Coloring)**.** Given a planar graph, all of whose faces are triangles, prove that the edges can be colored such that every triangle has two red edges and one blue edge.

Problem 6.10. Let $\mathcal{P}$ be a convex lattice polygon with vertices $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ at lattice points taken in counter-clockwise order. Then its dual, $\mathcal{P}^\vee$ is the lattice polygon with vertices $\mathbf{q_i}$, where $\mathbf{q_i} = \mathbf{p}_{i+1} - \mathbf{q_i}$ and indices are considered modulo n, i.e. $\mathbf{p}_{n+1} = \mathbf{p}_1$. Let $\mathcal{P}$ be a convex lattice polygon with $(0,0)$ as its only interior lattice point, and let $\mathcal{P}^\vee$ be its dual. Prove that $l(\mathcal{P}) + l(\mathcal{P}^\vee) = 12$, where $l(\mathcal{Q})$ denotes the total number of lattice points on the boundary of $\mathcal{Q}$.

Problem 6.11. Suppose the letters O and T are drawn with infinitesimally thin lines. Show that uncountably infinitely many copies (in diverse sizes) of the letter O can be placed in the plane without overlapping, but only countably infinitely many copies of T can be so placed.

Problem 6.12. For what n can a regular n-gon be embedded in $\mathbb{Z}^d$?

Problem 6.13. It is known a polynomial over $\mathbb{Z}$ that $p(n) > n$ for every positive integer n. Consider $x_1 = 1, x_2 = p(x_1), \ldots$ We know that, for any positive integer N, there exists a term of the sequence divisible by N. Prove that $p(x) = x + 1$.

Problem 6.14 (Balkan 2000)**.** Find the maximal number of rectangles of the dimensions $1 \times 10\sqrt{2}$, which is possible to cut off from a rectangle of the dimensions 50×90 , by using cuts parallel to the edges of the initial rectangle.

Problem 6.15 (1998 Russia, A. Belov)**.** A cellular figure Φ consists of cells 1×1 has the following property: if the cells of an $m \times n$ rectangle are filled with numbers of positive sum the figure Φ can be positioned in the rectangle so that the sum under the cells of Φ is positive. (we are allowed to rotate Φ) Prove that a number of such figures can be put on the rectangle so that each cell will be covered by the same number of figures

Problem 6.16 (Goldbach)**.** Prove that every nonlinear polynomial in $\mathbb{Z}[X]$ can be written as the sum of two irreducible polynomials.

Problem 6.17. There are n convex red, blue, and green figures in the plane. If the intersection of any three sets of all different colors has a non-empty intersection, prove that there is a color such that the intersection of all figures of that color is non-empty.

Problem 6.18. Prove that every integer can be written as the sum of two square-free integers.

Problem 6.19 (Scissor Congruences)**.** Prove that it is impossible to decompose a regular tetrahedron of volume 1 into a finite number of tetrahedrons and then reassemble the tetrahedrons into a unit cube.

Problem 6.20 (China 2000 Hongbin Yu)**.** Given n collinear points, consider the distances between the points. Suppose each distance appears at most twice. Prove that there are at least $\lfloor n/2 \rfloor$ distances that appear once.