Problem Set: Continuity

Problem 1. Let $a < b < c$ be real numbers. Show that the equation

$$\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = 0$$

has two solutions x_1 and x_2 that satisfy $a < x_1 < b < x_2 < c$.

Problem 2 *(Universal Chord Theorem)*. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that for any positive integer n, there exists a real number x, $0 \le x \le 1 - \frac{1}{n}$, such that

$$f(x) = f\left(x + \frac{1}{n}\right).$$

Problem 3. Given $x_1, x_2, \ldots, x_n \in [0,1]$, show that there is some $x \in [0,1]$ such that

$$\sum_{k=1}^{n} |x - x_k| = \frac{n}{2}.$$

Problem 4. Let S be a set of strings consisting of the symbols a and b. The set S has the following properties:

(i) $\emptyset \in S$.

(ii) If $\alpha \in S$ then $a\alpha b \in S$ and $b\alpha a \in S$.

(iii) If $\alpha \in S$ and $\beta \in S$ then $\alpha\beta \in S$.

Prove that S is the set of all strings that have an equal number of as and bs.

Problem 5. Given a closed curve, show that there exist two perpendicular lines that divide the curve into four sections of equal area.

Problem 6. Let S be a set of $6n$ points in a line. Choose arbitrarily $4n$ of these points and paint them blue; the other $2n$ points are painted green. Prove that there exists a line segment that contains exactly $3n$ points from S, $2n$ of them blue and n of them green.

Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let f_n denote the n^{th} iterate of f, so $f_1 = f$ and $f_n = f \circ f_{n-1}$ for all $n \ge 2$.

Prove that if f_n has a fixed point for some n, then f has a fixed point.

Problem 8. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions

$$|x_1 + x_2 + \cdots + x_n| = 1$$

and

$$|x_i| \le \frac{n+1}{2}$$

for $i = 1, 2, \ldots, n$.

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

Problem 9. Establish necessary and sufficient conditions on the constant k for the existence of a continuous real valued function $f(x)$ satisfying $f(f(x)) = kx^9$ for all real x.