INEQUALITIES : THE TOOL KIT

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Here are the basic inequalities which are very useful to solve any inequality. The inequalities are stated with some special cases.

1. **Triangle Inequality:** For all, $x_i \in \mathbb{R}$,

$$a + b \leq |a + b| \leq |a| + |b|$$

$$\sum_{i=1}^{n} x_i \leq \left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|$$

Equality: Iff all x_i have the same sign.

2. **$max \geq QM \geq AM \geq GM \geq HM \geq min$ inequality:** For all, $x_1 \in \mathbb{R}^+$,

$$\max(x_i) \geq \sqrt{\frac{\sum_{i=1}^{n} {x_i}^2}{n}} \geq \frac{\sum_{i=1}^{n} x_i}{n} \geq \sqrt[n]{\prod_{i=1}^{n} x_i} \geq \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} \geq \min(x_i)$$

$$\max(a, b, c) \geq \sqrt{\frac{a^2 + b^2 + c^2}{3}} \geq \frac{a + b + c}{3} \geq \sqrt[3]{abc} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \geq \min(a, b, c)$$

Equality: Iff all x_i are equal.

Weighted $AM \geq GM$ Inequality:

If $x_i \geq 0, \omega_i > 0$ and $\omega_1 + \omega_2 + \cdots + \omega_n = 1$, then,

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n \geq x_1^{\omega_1} . x_2^{\omega_2} \dots x_n^{\omega_n}$$

Equality: Iff all x_i are equal.

3. **Rearrangement Inequality:**

If we consider two sequence of real numbers$(a_i, b_i \in \mathbb{R})$,

$$a_1 \leq a_2 \leq \cdots \leq a_n \quad \text{and} \quad b_1 \leq b_2 \leq \cdots \leq b_n$$

For any permutation $(a'_1, a'_2, \dots, a'_n)$ of $a_1, a_2, \dots, a_n$ we have that,

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a'_1 b_1 + a'_2 b_2 + \cdots + a'_n b_n$$

Maximum and Minimum of Rearrangement inequality:

$$Max = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \quad \text{and} \quad Min = a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1$$

$$\text{So, } Max \geq a'_1 b_1 + a'_2 b_2 + \cdots + a'_n b_n \geq Min$$

Equality: Iff $a'_i = a_i$ (But the maximum minimum inequality always holds)

- **Chebyshev's Inequality:**

$$Max \geq \frac{(a_1 + \cdots + a_n)(b_1 + \cdots + b_n)}{n} \geq Min$$

Another form,

$$\frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \geq \frac{(a_1 + \cdots + a_n)}{n} \cdot \frac{(b_1 + \cdots + b_n)}{n}$$

Equality: Iff there exists some $\lambda \in \mathbb{R}$ with $a_i = \lambda b_i$

4. **Cauchy-Schwarz Inequality:**

$$\left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right) \geq \left(\sum_{i=1}^{n} x_i y_i\right)^2$$

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \geq (ax + by + cz)^2$$

Equality: Iff there exists some $\lambda \in \mathbb{R}$ with $x_i = \lambda y_i$

5. **Helpful Inequality (Angel's form):**

If $a_i \in \mathbb{R}$ and $x_i \in \mathbb{R}^+$, then,

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \cdots + \frac{a_n^2}{x_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{x_1 + x_2 + \cdots + x_n}$$

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a + b + c)^2}{x + y + z}$$

Equality: Iff $\frac{a_1}{x_1} = \frac{a_2}{x_2} = \cdots = \frac{a_n}{x_n}$

6. **Schur's Inequality:**

$$a^r(a - b)(a - c) + b^r(b - c)(b - a) + c^r(c - a)(c - b) \geq 0$$

Equality: Iff $a = b = c$ or two of a, b, c are equal and other is 0

7. **Power Mean Inequality:**

If $x_i, \omega_i \in \mathbb{R}^+$; $\omega_1 + \omega_2 + \cdots + \omega_n = 1$, and s, t non-zero reals with $s > t$, then,

$$\left(\frac{\omega_1 x_1^s + \omega_2 x_2^s + \cdots + \omega_n x_n^s}{n}\right)^{\frac{1}{s}} \geq \left(\frac{\omega_1 x_1^t + \omega_2 x_2^t + \cdots + \omega_n x_n^t}{n}\right)^{\frac{1}{t}}$$

REMARK: With, $\omega_i = \frac{1}{n}$, here $M_\infty \geq M_2 \geq M_1 \geq M_0 \geq M_{(-1)} \geq M_{(-\infty)}$ are nothing but the calssical inequalities, $\boldsymbol{max \geq QM \geq AM \geq GM \geq HM \geq min}$

8. **Weighted Power Mean Inequality:**

If x_i, ω_i are non-negative reals and $\sum \omega_i > 0$, then,

$$f(s) = \left(\frac{\omega_1 x_1^s + \omega_2 x_2^s + \cdots + \omega_n x_n^s}{\omega_1 + \omega_2 + \cdots + \omega_n}\right)^{\frac{1}{s}}$$

is in general, a non-decreasing function of $\boldsymbol{s}$.

REMARK: It can also produce the classical inequalities, $\boldsymbol{max \geq QM \geq AM \geq GM \geq HM \geq min}$

9. **Holder's Inequality:**

If $x_i, y_i \in \mathbb{R}^+$ and $a, b > 0$ such that, $\frac{1}{a} + \frac{1}{b} = 1$, then

$$\left(\sum_{i=1}^{n} x_i^a\right)^{1/a} \left(\sum_{i=1}^{n} y_i^b\right)^{1/b} \geq \sum_{i=1}^{n} x_i y_i$$

REMARK: With $a = b = 2$ we get the famous Cauchy-Schwarz Inequality.

- More generally, if a_{i_j} are positive real numbers such that $1 \leq i \leq m, 1 \leq j \leq n$. Then,

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{i_j}\right) \geq \left(\sum_{j=1}^{n} \sqrt[m]{\prod_{i=1}^{m} a_{i_j}}\right)^{m}$$

- Special case,

$$(a^3 + b^3 + c^3)(p^3 + q^3 + r^3)(x^3 + y^3 + z^3) \geq (aqx + bqy + crz)^3$$

10. **Minkowski's Inequality:**

If $x_i, y_i \in \mathbb{R}^+$ and $p >$ then,

$$\left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p} \geq \left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{1/p}$$

11. **Nesbit's Inequality:** For $a, b, c \in \mathbb{R}^+$,

$$\sum_{cyc} \frac{a}{b+c} \geq \frac{3}{2} \overset{i.e.}{\Rightarrow} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Equality: Iff $a = b = c$

12. **Bernouli's Inequality:** For all $r \geq 1$ and $x \geq -1$,

$$(1+x)^r \geq 1 + xr$$

13. **Jesen's Inequality:**

If f is convex in $[a, b]$, then for any $\omega_i \in [0,1]$ with $\sum_{i=1}^{n} \omega_i$ and $x_i \in [a, b]$, we have,

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1 x_1 + \cdots + \omega_n x_n)$$

Convexity Test: Let f be twice differentiable function on $[a, b]$. Then,

- f is convex on$[a, b]$ if $f''(x) \geq 0$ for every $x \in [a, b]$.
- f is **strictly convex** on$[a, b]$ if $f''(x) > 0$ for every x in the interior of $[a, b]$.

14. **Some Important trivial Inequalities:**

1. $x^2 + y^2 + z^2 \geq xy + yz + zx$
2. $a^2 + b^2 + c^2 + d^2 + e^2 \geq a(b + c + d + e)$
3. $(ab + bc + ca) \geq 3abc(a + b + c)$
4. $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$

5. $a^4 + b^4 + c^4 + \geq abc(a + b + c)$
6. $2(a^3 + b^3 + c^3) \geq ab(a + b) + bc(b + c) + ca(c + a)$
7. $a^3b + b^3c + c^3a \geq abc(a + b + c)$
8. $(a + b + c)^2 \geq 3(ab + bc + ca)$

Equality: Iff all variables are equal.

15. Some very useful factorization techniques:

$$xy + xk + yj + jk = (x + j)(y + k)$$
$$Equivalently,\ xy + x + y + 1 = (x + 1)(y + 1)$$

$$\sum bc(b - c) = bc(b - c) + ca(c - a) + ab(a - b) = -(b - c)(c - a)(a - b)$$

$$\sum a^2(b - c) = -(b - c)(c - a)(a - b)$$

$$\sum a(b^2 - c^2) = (b - c)(c - a)(a - b)$$

$$\sum a^3(b - c) = -(b - c)(c - a)(a - b)(a + b + c)$$

$$\sum b^2c^2(b^2 - c^2) = -(b - c)(c - a)(a - b)(b + c)(c + a)(a + b)$$

$$(ab + bc + ca)(a + b + c) - abc = (a + b)(b + c)(c + a)$$

$$(a + b)(b + c)(c + a) + abc = (ab + bc + ca)(a + b + c)$$

$$(a + b + c)^3 - a^3 - b^3 - c^3 = 3(a + b)(b + c)(c + a)$$