WOOT 2010-11
Double-Good Catalan

Richard Rusczyk

Back when I first started looking at Olympiad problems as a student, I remember that most of the solutions fell into one of two categories. The first category consisted of completely incomprehensible solutions. These used notation or invoked theorems that I had never seen before. While they at least gave me something to try to research, they didn't help much. (This was back before the internet—researching notation or theorems wasn't simply a matter of going to the AoPS site and asking on the forum.) The second category was even more maddening—each solution was clear and easy to understand, but there was absolutely no indication as to how I might have come up with the solution on my own unless I had seen essentially the same problem before.

Through WOOT, we hope to help you on both fronts. Many of the classes will be designed to expand your mathematical toolboxes. However, these tools are less important than our second purpose, which is to teach how to attack problems that are not just like problems you already know how to do. This latter skill is the heart of problem solving. It's much harder to teach and much harder to learn, but it's far more important than the individual tools we use to solve specific problems.

Hopefully, you will start to see more than just the final solution when you look at the solution to an Olympiad problem. This final solution is often much like the tip of an iceberg. It's the visible part that sits on top of a mountain of invisible effort that led to it. The problem solving is not in the tip—it's in all the mass below.

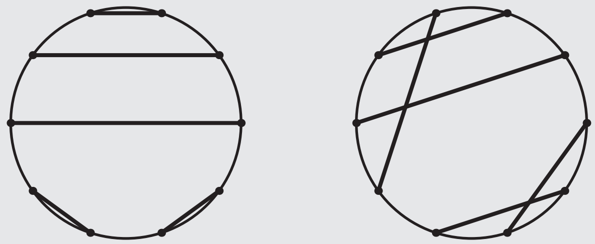
Most classes and texts focus primarily on the tip of the iceberg, with an occasional glimpse below the waterline. However, in this article, I'll instead dissect a single problem I solved for inclusion in our *Intermediate Counting & Probability* text. The problem is very difficult (I think), and it took me a long time to solve it. I'll start this article with the background material I knew prior to starting on the problem. Then, I'll explain how I attacked the problem, including all the blind alleys, false leads, harebrained ideas, and, finally, my successful approach.

1 Background: The Catalan Numbers

This section is excerpted from Art of Problem Solving's textbook Intermediate Counting & Probability by David Patrick. This book is an excellent resource for mastering advanced counting techniques, including those that are very valuable for the AIME, for national Olympiads, and for various areas of higher mathematics.

These first two sections are structured just like the Intermediate Counting & Probability text. They start with a series of problems. You should try to solve these problems before continuing. Then, we present the solutions to the problems, along with important observations about the results.

Problem 1.1: In how many ways can 10 people sitting around a circular table simultaneously shake hands (so that there are 5 handshakes going on), such that no two people cross arms? For example, the handshake arrangement on the left side below is valid, but the arrangement on the right side is invalid.



Valid Invalid

(a) Compute by hand the number of handshake arrangements for 2, 4, or 6 people sitting around a table.

(b) It's a bit hard to do it by hand for 8 people (you can try if you like), so we'll look for a more clever approach. Pick one person (out of the 8); how many people can he shake hands with?

(c) For each possible handshake for the first person in (b), in how many ways can the rest of the table shake hands?

(d) Use your answers from (b) and (c) to count the number of 8-person handshake arrangements.

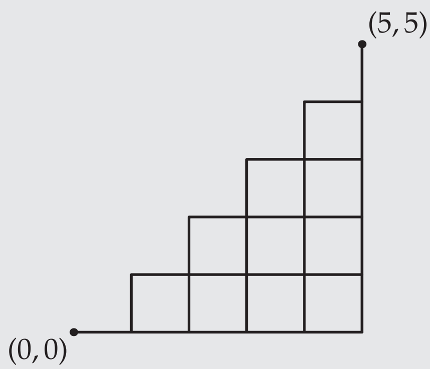
(e) Can you extend your reasoning from (b)-(d) above to solve the 10-person problem?

Problem 1.2: How many ways are there to arrange 5 open parentheses "(" and 5 closed parentheses ")" such that the parentheses "balance," meaning that, as we read left-to-right, there are never more ")"s than "("s? For example, the arrangement ((()())()) is valid, but the arrangement (()()))(() is invalid.

(a) Compute by hand the number of arrangements for 1, 2, and 3 pairs of parentheses. Do your answers look familiar?

(b) Try to find a 1-1 correspondence between arrangements of n pairs of parentheses and handshake arrangements of $2n$ people (from Problem 1.1).

Problem 1.3: How many 10-step paths are there from $(0,0)$ to $(5,5)$ on the grid below?



(a) It may be tempting to answer $\frac{1}{2}\binom{10}{5} = 126$. Explain why this is incorrect.

(b) Compute by hand the number of paths on the half-grid to $(1,1)$, $(2,2)$, and $(3,3)$. Notice anything familiar?

(c) Try to find a 1-1 correspondence between solutions to this problem and solutions to one of the two previous problems.

We'll explore several problems that look very different on the surface, but that actually all have the same underlying structure. As we work through these problems, try to keep them all in the back of your mind, with an eye towards the features in the various problems that are similar.

Sidenote: **Eugène Catalan 1814–1894**

The Catalan numbers (which we will be exploring in this section) are named after the 19^{th}-century mathematician Eugène Catalan. He is also known for his conjecture (made in 1844) that 8 and 9 are the only consecutive positive integers that are perfect powers ($8 = 2^3$ and $9 = 3^2$). This conjecture remained unproven until 2002, when it was proved by Preda Mihăilescu.

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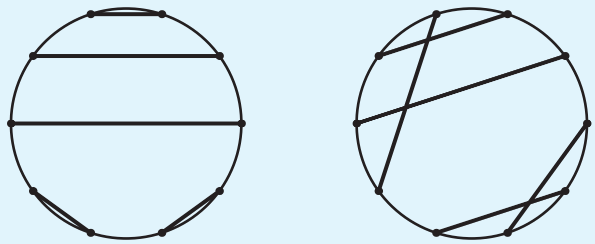
Problem 1.1: In how many ways can 10 people sitting around a circular table simultaneously shake hands (so that there are 5 handshakes going on), such that no two people cross arms? For example, the handshake arrangement on the left side below is valid, but the arrangement on the right side is invalid.



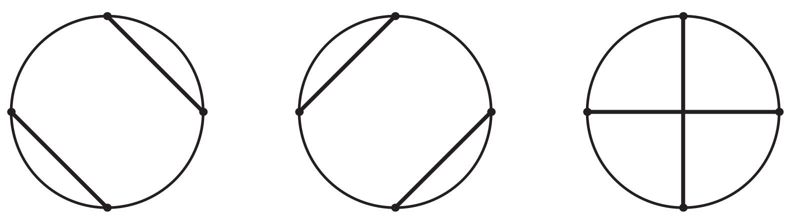
Valid Invalid

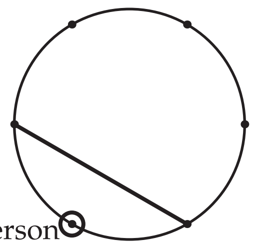
Solution for Problem 1.1: As we often do, we can experiment on smaller versions of the same problem, in order to get some idea for what's going on in general.

If there are 2 people, then there is obviously only one way for them to shake hands.

If there are 3 people, then there's no way that they can all shake hands, because there will always be an odd person left out. In general, we must always have an even number of people.

If there are 4 people, then there are 3 ways for them to shake hands (pick one of the people, and choose one of the other 3 people to shake hands with him; the other two people are then forced to shake with each other). But one of these ways is illegal: the pairs of people sitting across from each other cannot shake hands, since their arms would cross. So there are only 2 legal handshake configurations. In the figure below, we see the two legal handshake configurations on the left, and the 3$^{\text{rd}}$ (illegal) configuration on the right.



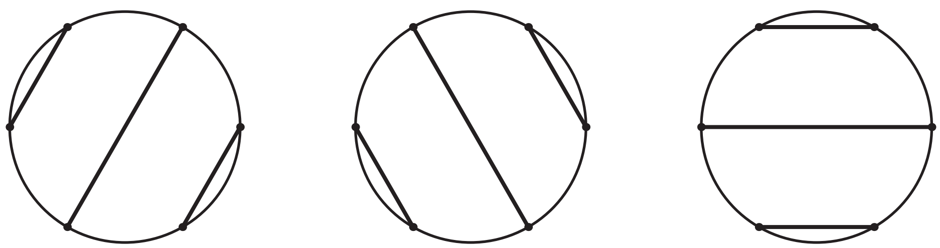


This person can't shake with anybody

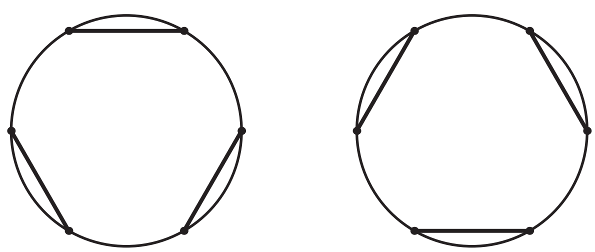
If there are 6 people, then things get a bit more complicated. The first thing to note is that no one can shake hands with the person sitting 2 positions away from them on the left or on the right, because if they did, they'd "cut off" a person who would not be able to shake hands with anyone, as in the figure on the left.

This leaves us with two cases.

Case 1: Some pair of people who are directly across from each other shake hands. There can only be one such pair, since two or more such pairs would cross each other at the center of the table. There are 3 choices for a pair of opposite people, and once we have chosen such a pair, the rest of the handshakes are fixed (the two people on each side of the central handshake must shake with each other). These three cases are shown in the figure below:

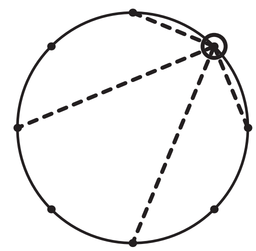


Case 2: Everybody shakes hands with one of his/her neighbors. There are two possibilities, depending on whether a specific person shakes hands to the left or to the right, as shown in the figure below:

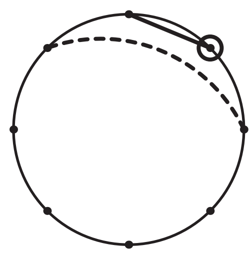


So there are a total of $3 + 2 = 5$ ways for 6 people to legally shake hands.

When we get up to 8 people, it's starting to get too complicated to list all the configurations. So let's look at it from a particular person's point-of-view. As in the 6-person case above, we cannot leave an odd number of people on either side of this person's handshake. So our initial person cannot shake hands with anybody that is an even number of people away. In the figure to the right, we show a circled initial person, and his allowed handshakes are shown by dashed lines. Note that each of these handshakes ends at a person who is an odd number of people away from our initial (circled) person.

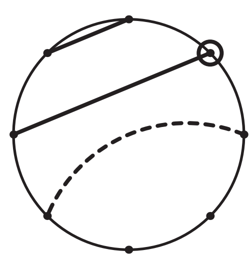


If the initial person shakes hands with a neighbor, we can think of the remaining 6 people as being on a smaller circle, as in the figure below:



These 6 remaining people have 5 ways to shake, just as in the 6-person problem. Since the initial person has 2 neighbors with whom to shake, this means that there are $2(5) = 10$ handshake arrangements that start with our initial person shaking hands with a neighbor.

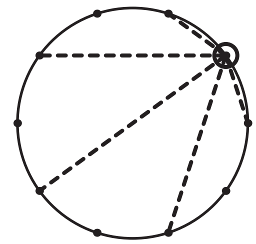
Otherwise, our initial person has to shake with a person who is 3 positions to his left or right. Once this is done, the two people who are "cut off" from the rest must shake with each other, and the other 4 people form a 4-person mini-table that can shake in 2 ways:



This gives another $2(2) = 4$ handshake arrangements, since there are 2 choices of the person that is 3 away from the original person, and then 2 choices to finish the handshaking at the 4-person mini-table.

Therefore, there are $10 + 4 = 14$ ways for 8 people to shake hands.

Finally, we can use this same strategy for 10 people. Choose an initial person. This person has 5 choices for whom to shake hands with, as shown in the picture to the right. If he shakes with one of his neighbors (2 choices), then the remaining 8 people form a mini-table that can shake in 14 ways. If he shakes with a person 3 positions away (2 choices), then 2 people are cut off (and must shake), and the other 6 people form a mini-table that can shake in 5 ways. If he shakes with the person directly opposite (1 choice), then each side of the table has a group of 4 people, each of which can shake in 2 ways.



Therefore, the number of handshake arrangements for 10 people is

$$2(14) + 2(5) + 1(2)(2) = 28 + 10 + 4 = 42.$$

□

Before we go on, let's list the numbers that we found while working through the previous problem:

Number of people	2	4	6	8	10
Number of handshake configurations	1	2	5	14	42

Keep these numbers in mind as we continue through this section.

Problem 1.2: How many ways are there to arrange 5 open parentheses "(" and 5 closed parentheses ")" such that the parentheses "balance," meaning that, as we read left-to-right, there are never more ")"s than "("s? For example, the arrangement ((()())()) is valid, but the arrangement (()()))(() is invalid.

Solution for Problem 1.2: As we often do, let's experiment with small values.

If we have 1 set of parentheses, then we only have one possibility: ().

If we have 2 sets of parentheses, we can either nest them as (()), or we can list both pairs one after the other as ()(). So there are 2 possibilities.

If we have 3 sets of parentheses, then a little experimentation will show that there are 5 possibilities:

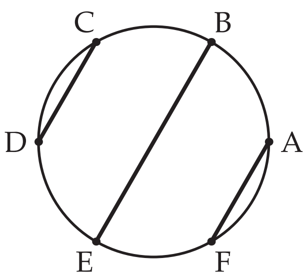
()()(), (())(), ()(()), (()()), ((())).

Hmmm. . . , 1, 2, 5, ... Do you recognize these numbers? They are the same numbers that we got for the number of non-crossing handshakes of people sitting at a round table in Problem 1.1. Perhaps there is a connection between the two problems.

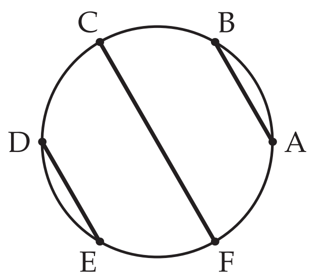
Concept: When you see the same answer for two different problems, look for a connection, or better yet, for a 1-1 correspondence between them.

Since the parentheses come in pairs, it's natural to think that in any 1-1 correspondence between parenthesis-arrangements and valid handshakes around a table, each set of parentheses will represent two people shaking hands. The fact that the parentheses must be properly nested will somehow correspond to the condition that handshakes cannot cross.

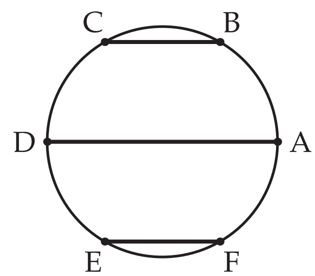
For instance, we can list all of the arrangements of 3 pairs of parentheses, and their corresponding handshake arrangements. We'll label both the parentheses and the people with the letters A through F, and note how each pair of parentheses corresponds to a pair of people that are shaking hands.



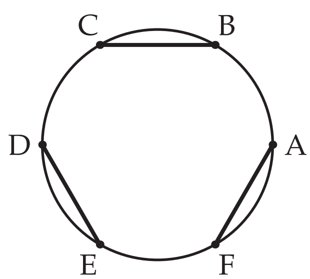
((( )))
A B C D E F



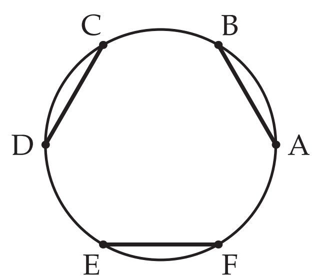
( ) (( ))
A B C D E F



(( )) ( )
A B C D E F



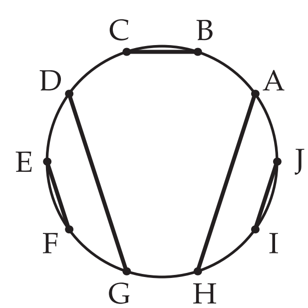
(( ) ( ))
A B C D E F



( ) ( ) ( )
A B C D E F

Let's see this further in an example with 5 sets of parentheses and 10 people around a table. We'll label the people around the table A through J, and the parentheses will also be labeled with A through J as we read from left to right. Each matching pair of parentheses corresponds to a handshake. A sample correspondence is shown below.

(( ) (( ))) ( )
A B C D E F G H I J $\leftrightarrow$



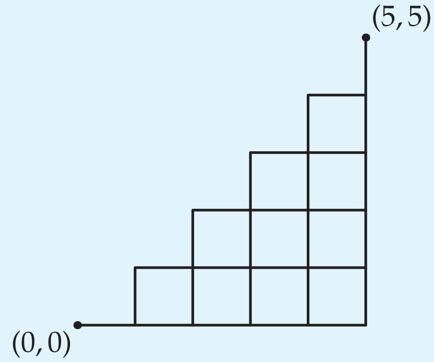
This leads to a 1-1 correspondence

$$\left\{\begin{matrix}\text{parenthesis arrangements of } n\\ \text{pairs of parentheses}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{handshake arrangements of } 2n\\ \text{people around a table}\end{matrix}\right\}.$$

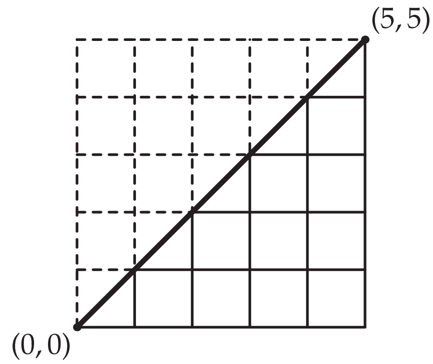
Thus the answer to our problem is the same as the number of handshake arrangements of 10 people, which is 42. □

That's two problems so far involving the sequence 1,2,5,12,42,. . . . You should therefore not be surprised by what you will find in the next problem.

Problem 1.3: How many 10-step paths are there from $(0,0)$ to $(5,5)$ on the grid below?



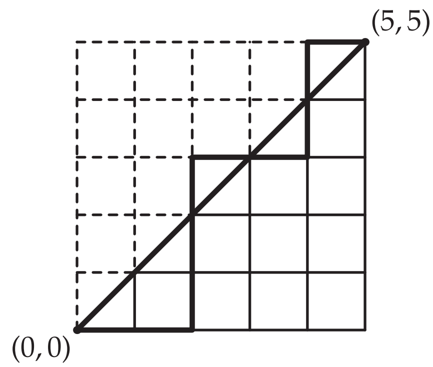
Solution for Problem 1.3: The first thing that we notice is that the grid shown is exactly the part of the full 5×5 grid that is below the main diagonal, as shown below:



This might suggest the following quick "solution":

Bogus Solution: We know that there are $\binom{10}{5} = 252$ paths on the full grid. Since we only have the lower-half of the grid to work with, that means that we have $\frac{252}{2} = 126$ paths on the lower-half of the grid.

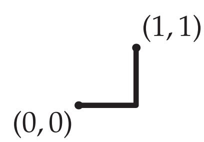
This is of course absurd, as there are many paths that pass through both halves of the grid, like the one shown below:



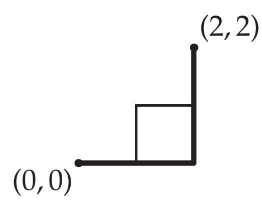
So how can we count the paths that only go below the main diagonal?

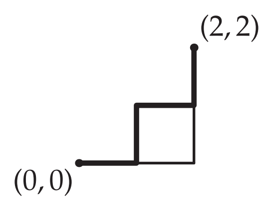
Once again, let's count the paths in some smaller cases.

If the half-grid is 1×1, then there's only one path:

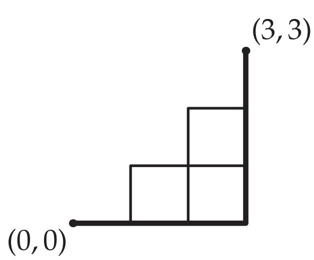


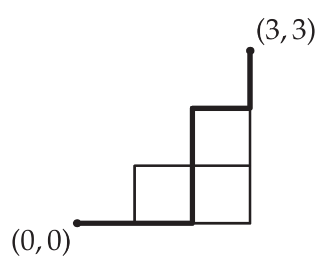
If the half-grid is 2×2, then there are 2 paths:

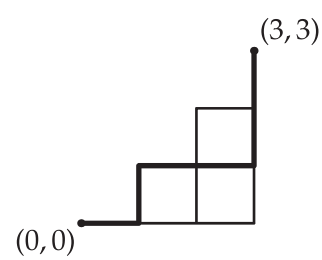


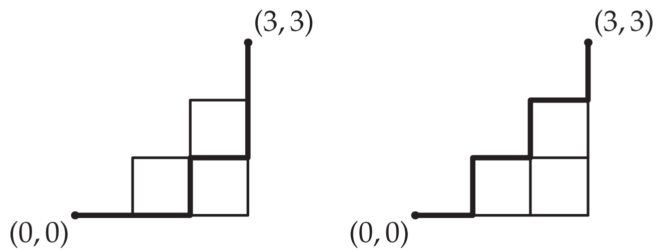


If the half-grid is 3×3, then there are 5 paths:









There are those numbers again: 1,2,5,. . . . So we'll once again look for a 1-1 correspondence between this problem and one of the previous problems. Since each path from $(0, 0)$ to (n, n) consists of n moves up and n moves to the right, we think to try to find a correspondence between these paths and lists of n "("s and n ")"s.

Indeed, we can make a 1-1 correspondence

$$\{\text{balanced expressions with } n \text{ pairs of parentheses}\} \quad \leftrightarrow \quad \{\text{paths on an } n \times n \text{ grid below the diagonal}\},$$

by letting each "(" represent a move to the right and each ")" represent a move up. As long as there are more "("s than ")"s, there will be more rights than ups, and the path will never cross above the main diagonal of the $n \times n$ grid.

Therefore, there are 42 paths on the 5×5 half-grid, since there are 42 possible nested expressions with 5 pairs of parentheses. □

2 Formulas for the Catalan Numbers

This section is excerpted from Art of Problem Solving's new textbook Intermediate Counting & Probability by David Patrick.

Problem 2.1: Can you write a recurrence relation for the Catalan numbers?

Problem 2.2: Compare the n^{th} Catalan number with the binomial coefficient $\binom{2n}{n}$. Do you notice any pattern?

Problem 2.3: Find a 1-1 correspondence between:

$$\left\{\begin{matrix}\text{paths from } (0,0) \text{ to } (n,n) \text{ that go above the main} \\ \text{diagonal}\end{matrix}\right\} \quad \leftrightarrow \quad \{\text{paths from } (0,0) \text{ to } (n-1, n+1)\}.$$

Problem 2.4: Find a formula for the n^{th} Catalan number.

As we've seen in the problems in the previous section, the n^{th} Catalan number can be defined as:

- the number of ways that $2n$ people sitting around a table can shake hands, so that no two handshakes cross arms;
- the number of ways to write n "("s and n ")"s such that the parentheses are balanced;
- the number of $2n$-step paths on a rectangular grid from $(0,0)$ to (n,n) that do not cross above the main diagonal.

It would be nice if we could easily compute the Catalan numbers. For now, let's focus on the recursive definition.

Problem 2.1: What is the recurrence relation for the Catalan numbers?

Solution for Problem 2.1: We've actually already seen it in the problems in the previous section. For each of the problems in the previous section, we can break down the problem of size n into cases, where each case is composed of two smaller problems whose sizes add to $n-1$.

For instance, in Problem 1.1, we start with a table with $2n$ people. Once we place the initial handshake, we are left with two smaller tables with $2(n-1)$ people combined.

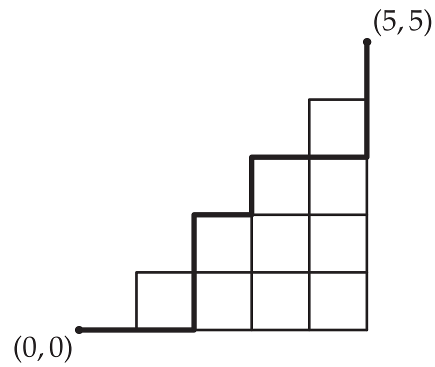
In Problem 1.2, we can look at the first parenthesis on the left and its corresponding closing parenthesis. This splits the rest of the parentheses into two groups: those that are inside this first pair, and those that are to the right of this pair. For example, in the following 10-pair nesting, the first set of parentheses (in bold) splits the rest of the parentheses into a 6-pair group (inside the bold parentheses) and a 3-pair group (to the right of the bold parentheses):

$$\boldsymbol{\Big(}(()())(())()\boldsymbol{\Big)}(())()$$

The first set of parentheses will always split the remaining $n-1$ pairs into two groups of balanced parentheses, although one of the groups may be empty.

We can use the 1-1 correspondence between Problem 1.2 and Problem 1.3 to see how to set up the recursion for the paths on the half-grid from $(0,0)$ to (n,n). The idea is that the end of the *first* complete set of parentheses corresponds to the place where the path *first* touches the diagonal after leaving $(0,0)$. For example, we show a 5-parentheses nesting and its corresponding path in the figure below. The first set of parentheses is shown in bold, and it corresponds to the path's first touching of the main diagonal at $(2,2)$.

$\big(()\big)()(()) \quad \leftrightarrow$



The path is now broken into 2 paths on 2 smaller half-grids.

In all of these problems, the solution is the n^{th} Catalan number C_n, and we arrive at the solution by breaking up the problem into a sum of two smaller problems. Specifically, we see that C_n is the sum of all possible products of the form $C_k C_l$ where $k + l = n - 1$. That is,

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0 = \sum_{k=0}^{n-1} C_k C_{n-1-k}.$$

The sequence starts at $C_0 = 1$. $\square$

We can once again verify this recursion for the numbers that we've already computed:

$$\begin{aligned}
C_0 &= 1,\\
C_1 &= C_0C_0 = 1,\\
C_2 &= C_0C_1 + C_1C_0 = 1 + 1 = 2,\\
C_3 &= C_0C_2 + C_1C_1 + C_2C_0 = 2 + 1 + 1 = 5,\\
C_4 &= C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 = 5 + 2 + 2 + 5 = 14,\\
C_5 &= C_0C_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_0 = 14 + 5 + 4 + 5 + 14 = 42.
\end{aligned}$$

Let's continue and compute the next couple of Catalan numbers:

$$\begin{aligned}
C_6 &= C_0C_5 + C_1C_4 + C_2C_3 + C_3C_2 + C_4C_1 + C_5C_0 = 42 + 14 + 10 + 10 + 14 + 42 = 132,\\
C_7 &= C_0C_6 + C_1C_5 + C_2C_4 + C_3C_3 + C_4C_2 + C_5C_1 + C_6C_0 = 132 + 42 + 28 + 25 + 28 + 42 + 132 = 429.
\end{aligned}$$

So now we have a recursive formula for the Catalan numbers. However, it is somewhat unsatisfying. Not only it is recursive, but each Catalan number depends on *all* of the preceding Catalan numbers, not just the one or two immediately prior. It would be much nicer to have a closed-form formula into which we could plug some value of n and have C_n just pop out. But where can we begin to find such a formula?

Problem 1.3 looks most promising, as it's most related to a problem that we feel like we understand well and know how to find a formula for, namely paths on a grid from $(0, 0)$ to (n, n). We know that, without any restrictions, there are $\binom{2n}{n}$ such paths. So that's a good place to start.

Problem 2.2: Compare the n^{th} Catalan number with the binomial coefficient $\binom{2n}{n}$. Do you notice any pattern?

Solution for Problem 2.2: Let's list the first 7 Catalan numbers and the first 7 values of $\binom{2n}{n}$ and see if we notice anything.

n	1	2	3	4	5	6	7
C_n	1	2	5	14	42	132	429
$\binom{2n}{n}$	2	6	20	70	252	924	3432

It's not too clear how to find a pattern between these two rows of numbers, but the one column that might jump out at you is the $n = 4$ column with the numbers 14 and 70, since $70 = 5(14)$. This might cause you to notice that $\binom{2n}{n}$ always appears to be a multiple of C_n. Let's expand our chart:

n	1	2	3	4	5	6	7
C_n	1	2	5	14	42	132	429
$\binom{2n}{n}$	2	6	20	70	252	924	3432
$\frac{\binom{2n}{n}}{C_n}$	2	3	4	5	6	7	8

Now we have strong experimental evidence that $C_n = \frac{1}{n+1}\binom{2n}{n}$. □

Of course, observing a pattern is not a proof. Let's further examine the "paths on a grid" problem and see what else we might determine. We know that $\binom{2n}{n}$ counts the number of paths from $(0, 0)$ to (n, n) on a rectangular grid. We also know that C_n counts the number of these paths that don't go above the diagonal. So $\binom{2n}{n} - C_n$ counts the number of paths that *do* go above the diagonal. Since we suspect that $C_n = \frac{1}{n+1}\binom{2n}{n}$, we suspect that the number of paths that go above the diagonal should be:

$$\binom{2n}{n} - \frac{1}{n+1}\binom{2n}{n} = \frac{n}{n+1}\binom{2n}{n}.$$

We do a bit of algebraic manipulation with this quantity:

$$\frac{n}{n+1}\binom{2n}{n} = \frac{n(2n)!}{(n+1)n!n!} = \frac{(2n)!}{(n+1)!(n-1)!} = \binom{2n}{n-1}.$$

This last quantity gives us an idea for a 1-1 correspondence:

Problem 2.3: Find a 1-1 correspondence between:

$$\left\{\begin{matrix}\text{paths from } (0,0) \text{ to } (n,n) \text{ that go above the main} \\ \text{diagonal}\end{matrix}\right\} \leftrightarrow \{\text{paths from } (0,0) \text{ to } (n-1,n+1)\}.$$

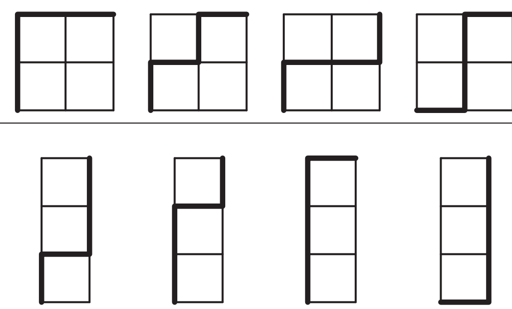
Solution for Problem 2.3: This can be a bit tricky to see, so let's play with the $n = 2$ case.

There are $\binom{4}{2} = 6$ paths from $(0,0)$ to $(2,2)$, and we know that $C_2 = 2$ of them stay on or below the main diagonal, so the other 4 go above the diagonal. We also know that there are $\binom{4}{1} = 4$ paths from $(0,0)$ to $(1,3)$. (Good—there are the same number of paths in each category, which is a necessity for there to be a 1-1 correspondence.)

Let's draw the 4 paths in each category, and see if we can match them up. (I'm going to help you out and list them in the order that we will match them—see if you can find the correspondence.)

Paths from $(0,0)$ to $(2,2)$ that go above the main diagonal

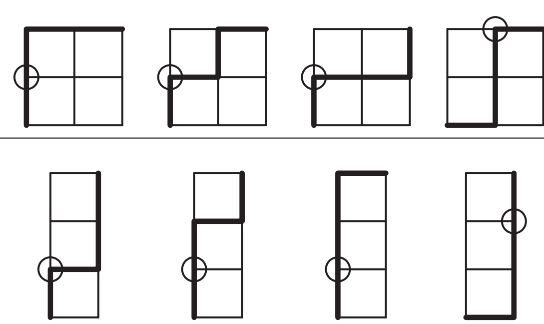
Paths from $(0,0)$ to $(1,3)$



In each column, let's start at $(0,0)$, and let's mark (with a circle) the point on each path where the two paths differ. In other words, the path from $(0,0)$ to the circled point is the same in both paths, but after the circled point, one path goes up whereas the other goes right.

Paths from $(0,0)$ to $(2,2)$ that go above the main diagonal

Paths from $(0,0)$ to $(1,3)$



We see that in each column, the path from $(0,0)$ to the circled point in both pictures is the same. However, what's more interesting is what happens after the circled point. Compare the paths after the circled point in both pictures of a column. They're mirror images of each other!

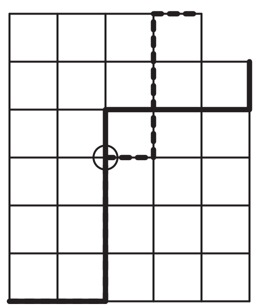
To be more precise, let's list the paths using "r" for a step to the right and "u" for a step up. We'll place in bold all of the steps after the circled point.

Paths from $(0,0)$ to $(2,2)$ that go above the main diagonal	u**urr**	ur**ur**	urr**u**	r**uur**
Paths from $(0,0)$ to $(1,3)$	u**ruu**	uu**ru**	urr**u**→ uu**ur**	r**uuu**

Note that the unbolded parts of the paths—the parts between $(0,0)$ and the circled point—are identical, and the bolded parts of the paths—the parts between the circled point and the end—are exactly reversed.

This suggests a general strategy for finding a 1-1 correspondence. Given a path from $(0,0)$ to (n,n) that goes above the diagonal, circle the *first* point at which the path crosses above the diagonal. Then, reverse all steps past the circled point: change ups to rights and rights to ups.

Here's an example where $n = 5$. The original path is shown as solid, and the new path (after the transformation described above) is shown as dashed.



Note that the solid path, before the circled point, has one more up step than right step. After the circled point, the solid path has one more right step than up step (since the circled point lies one "up" step above the diagonal). After the reversal transformation, the dashed path has, after the circled point, one more up step than right step. Hence, starting at $(0,0)$, the combined new path has 2 more up steps than right steps. Since it still has $2n$ steps in total, it must have $n+1$ up steps and $n-1$ right steps, and thus the path ends at $(n-1, n+1)$.

This process is clearly reversible, and hence we have a 1-1 correspondence:

$$\left\{\begin{matrix}\text{paths from } (0,0) \text{ to } (n,n) \text{ that go above the main} \\ \text{diagonal}\end{matrix}\right\} \leftrightarrow \{\text{paths from } (0,0) \text{ to } (n-1, n+1)\}.$$

□

Problem 2.4: Find a formula for the n^{th} Catalan number.

Solution for Problem 2.4: We know that the n^{th} Catalan number is the number of paths from $(0,0)$ to (n,n) that don't go above the diagonal. However, we know from Problem 2.3 that the paths that do go above the diagonal are in 1-1 correspondence with paths to $(n-1,n+1)$. Since there are $\binom{2n}{n}$ paths from $(0,0)$ to (n,n) and $\binom{2n}{n-1}$ paths from $(0,0)$ to $(n-1,n+1)$, we have that

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}.$$

This does not exactly look like what we conjectured in Problem 2.2, so let's try to simplify it a bit. We start by writing out the expressions for the binomial coefficients:

$$C_n = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!}.$$

We can factor out like terms and simplify:

$$C_n = \frac{(2n)!}{(n-1)!n!}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{(2n)!}{(n-1)!n!} \cdot \frac{1}{n(n+1)} = \frac{(2n)!}{(n+1)!n!}.$$

Removing an $(n+1)$ term from the denominator gives us our result:

$$C_n = \frac{1}{n+1}\left(\frac{(2n)!}{n!n!}\right) = \frac{1}{n+1}\binom{2n}{n}.$$

□

Important: The formula for the n^{th} Catalan number is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

3 The Problem

A *double-good* nesting of order n is an arrangement of $2n$ ")"s and n "("s such that as we read left-to-right, the number of ")"s that have appeared at any point is no more than 2 times the number of "("s that have appeared to that point. For example, the complete list of the double-good nestings of order 2 is

())())
()()))
(())))

Prove that the number of double-good nestings of order n is $\dfrac{1}{2n+1}\dbinom{3n}{n}$.

This is one of the hardest problems in the *Intermediate Counting & Probability* text, so don't feel bad if you don't make much progress. It took me several hours to find the solution. In the second half of this WOOT article, which we'll release after you've had a week to work on the problem, I'll describe my exploration of the problem, together with the solution I found.

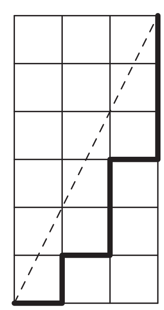
4 Finding the Solution

The problem looks a lot like Problem 1.2, the Catalan problem about parentheses. Moreover, the count, $\frac{1}{2n+1}\binom{3n}{n}$, looks a lot like the closed form for the Catalan numbers, $\frac{1}{n+1}\binom{2n}{n}$. So, immediately I thought, "Hey, this looks just like a Catalan problem, and the result is just like the Catalan numbers, so I'll use what I used to prove that the Catalan numbers are $\frac{1}{n+1}\binom{2n}{n}$. This'll be easy!"

Concept: If it looks just like another problem you know how to do, try using your solution to the known problem to tackle the unknown problem.

I knew how to find the closed form for the Catalan numbers using the block-walking approach in Section 2. So, my first step was to turn the double-good problem into a block-walking problem. We saw that the Catalan parentheses problem and block-walking problems are the same by replacing each "(" with a rightward step and each ")" with an upward step. If we do the same here, we are counting paths that have n rightward steps and $2n$ upward steps. But what condition do we want to satisfy?

In the original double-good problem, the number of ")"s that have appeared at any point must be no more than 2 times the number of "("s that have appeared to that point. So, at any point in walking to the point $(n, 2n)$, we must have taken no more upward steps than twice the number of rightward steps. In other words, just as we stay on or below the line $y = x$ in the Catalan block-walking problem, we must stay on or below the line $y = 2x$ in our double-good block-walking problem.

()())())) $\leftrightarrow$ 

So far, so good: we now want to show that the number of paths from $(0, 0)$ to $(n, 2n)$ that never go above the line $y = 2x$ is $\frac{1}{2n+1}\binom{3n}{n}$.

I then stepped through the Catalan solution to plan my next step to the double-good problem. To find a closed form for the Catalan numbers, we counted the number of paths to (n, n) that failed to stay on or below $y = x$. We found that there were $\binom{2n}{n-1}$ such "failed" paths. Then, we subtracted this count from $\binom{2n}{n}$ to find that there were $\frac{1}{n+1}\binom{2n}{n}$ acceptable paths. So, my next step in the double-good block-walking problem was to determine how many of the $\binom{3n}{n}$ paths to $(n, 2n)$ are "failed" paths that go above $y = 2x$. Since we want to show that there are $\frac{1}{2n+1}\binom{3n}{n}$ paths that do not go above $y = 2x$, we expect to find that the number of

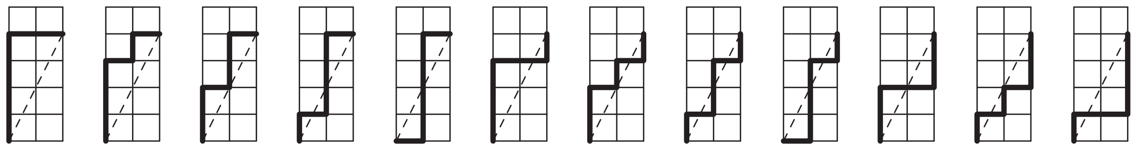


Figure 1: Failed paths to $(n, 2n)$ for $n = 2$.

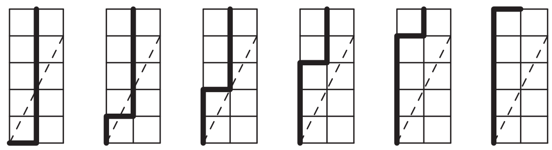


Figure 2: Failed paths to $(n-1, 2n+1)$ for $n = 2$.

paths that go above $y = 2x$ is

$$\binom{3n}{n} - \frac{1}{2n+1}\binom{3n}{n} = \frac{2n}{2n+1}\binom{3n}{n}.$$

In the Catalan problem, we were able to manipulate the expression for the number of "failed" paths into a simple combination. We try the same here:

$$\frac{2n}{2n+1}\binom{3n}{n} = \frac{2n}{2n+1} \cdot \frac{(3n)!}{n!(2n)!} = 2 \cdot \frac{n}{2n+1} \cdot \frac{(3n)!}{n!(2n)!} = 2 \cdot \frac{(3n)!}{(n-1)!(2n+1)!} = 2\binom{3n}{n-1}.$$

Aha! That's a simple expression to aim for! Now, I was sure I was on my way. I just had to show that there are $2\binom{3n}{n-1}$ paths to $(n, 2n)$ that go above $y = 2x$, and I would be finished. Again, I turned to my Catalan counting for inspiration. There, we found a 1-1 correspondence between paths that go above $y = x$ and paths to $(n-1, 2n+1)$. Here, we want to show that the number of paths that go above $y = 2x$ on the way to $(n, 2n)$ equals $2\binom{3n}{n-1}$. Since $2\binom{3n}{n-1}$ is twice the number of paths from $(0, 0)$ to $(n-1, 2n+1)$, we now wish to show that twice the number of paths that go above $y = 2x$ on the way to $(n, 2n)$ equals the number of paths from $(0, 0)$ to $(n-1, 2n+1)$. In other words, we want to find a 2-to-1 correspondence between the "failed" paths to $(n, 2n)$ and the paths to $(n-1, 2n+1)$.

To try to find this correspondence, I drew all the possible "failed paths" to $(n, 2n)$ and to $(n-1, 2n+1)$ for $n = 2$. Here they are:

Concept: Don't be afraid to get your hands dirty. Very dirty. Try small examples, look for patterns.

Unfortunately, I didn't find a general way to produce a 2-1 correspondence. I came up with all sorts of crazy ideas to manipulate each of the paths to $(1, 5)$ and produce two failed paths to $(2, 4)$. No luck. I tried all sorts of different rules. Every once in a while I thought I had something—I thought I found a rule that I could apply to each of the paths to $(1, 5)$ to make a pair of paths to $(2, 4)$, and thereby produce all of the paths to $(2, 4)$. But each time, I'd either try the rule on $n = 3$ and fail, or I'd see that the rule couldn't be reversed. In other words, I couldn't use the paths to $(2, 4)$, reverse my rule, and produce each path to $(1, 5)$ twice.

I was slow in accepting that I was at a dead end. My biggest clue that I was barking at the wrong tree is that the analogy between the earlier Catalan problem and the double-good problem was no longer exact. Drawing a 1-1 correspondence between two sets is a much more straightforward proposition than drawing a 2-1 correspondence. At this point, I should have accepted that I was facing a fundamentally new problem that would call for fundamentally new tools. But I didn't.

Instead, I plowed ahead into wishful thinking.

Concept: Don't be afraid of wishful thinking—a lot of beautiful solutions are found by entertaining seemingly crazy ideas.

I went 3-D. I started thinking of paths from $(0, 0, 0)$ to (n, n, n). In retrospect, this was really nuts. But the idea had two advantages. First, if it worked, it would be way, way cool. Second, if it worked cleanly, I might be able to generalize it to triple-good, quadruple-good, etc., with solutions in higher and higher dimensions.

Concept: Seek beauty in your solutions, if for no better reason than beauty itself.

Concept: Try approaches that might be generalized easily before those that cannot. They lead to more powerful results, and can give a deeper understanding to the problem.

Maybe you already see the fatal flaw in my nutty idea of going 3-D. The double-good problem is inherently two dimensional: we have "(" and we have ")". We don't have two different kinds of ")" that we can equate to y and z, respectively, in the coordinate plane. But I tried anyway, coming up with all sorts of crazy rules. Odd-numbered ")" signs would be steps in the y-direction and even-numbered ones would be in the z-direction. I won't get into the difficulties that this gives in trying to define the paths, let alone the difficulties of even thinking about the problem in 3-D. (In retrospect, the idea that I could "generalize" this in higher dimensions is really crazy, given how hard it was for me to even describe the problem in 3-D.)

Concept: The author William Faulkner once explained of writing that it was essential that you be willing to "kill your darlings." Problem solving is no different. This happens in every creative endeavor: You come up with a crazy idea that you think will be genius, just brilliant, amazing. But it isn't. Here are some clues that your brilliant idea isn't so hot after all:

- You go to great lengths to rationalize how it will work (like mapping the first ")" to y and the second to z).
- You can't explain why it should work (try it with the 3-D).
- The main reason you want it to work is that it would be *just so cool* if it did.

Don't get too enamored of your ideas. If you have a nutty idea that would be *just so cool*, take a little time to explore it. After a little time, reassess. If the idea has become more and more complicated, like the 3-D approach does, or if there's still no good idea to think the idea will work, then its time to kill that darling and move on to more pedestrian ideas.

That said, if you can make this 3-D idea work, let me know. It would be *just so cool*.

After a good sleep, I let the 3-D idea go. I then moved on to decidedly more pedestrian ideas: I tried to cheat.

Concept: Life is short. Sometimes you're far better off giving up, looking at the solutions to learn, and moving on. Generally, you'll learn much more in the first 90-120 minutes working on a problem than you will the next 5 hours. So, giving up and moving on helps you maximize your learning by moving you on to better learning experiences.

This last tip is obvious for some people, but others are tenacious (a good thing) to the point of being obstinate (a bad thing). Personally, I can sometimes get so obsessed by problems (as I did with this double-good problem) that the obsession interferes with other work. At times in my life, I've even put a moratorium on looking at problems for which I didn't have a solution available somewhere. I would then give myself a time limit, and look at the solution if I failed to find one before time was up. I'd be disappointed briefly with not finding the solution myself, but then I'd finally be broken of the obsession, and free to go find another one.

So, I tried to cheat. I went to the Encyclopedia of Sequences (http://oeis.org/) and looked up the sequence $\left\{\frac{1}{2n+1}\binom{3n}{n}\right\}$, hoping to find some insight I could use to build a proof. The most promising possibility was ternary trees. (Ternary trees aren't important, but you can look them up online if you're curious.) I spent 30-60 minutes thinking about ternary trees, trying to find a way to count them. I didn't get too far. Moreover, I hadn't even begun thinking about why counting ternary trees is equivalent to the double-good problem.

Perhaps that alone is a not-too-easy problem. So, realizing that this might create more problems rather than give me any solutions, I set the ternary trees aside. In fact, I again set the whole problem aside for a few hours.

But, of course, the problem called me back. When I started this time, I decided to pitch everything I had already tried and attempt a whole new approach.

> **Concept:** Frustrated? Take a little time off, then come back and try something completely different.

Instead of trying to turn my double-good problem into the block-walking Catalan problem that I knew how to do, I decided to try to turn the block-walking problem I understood into the double-good problem.

> **Concept:** Not only can you tackle problems by turning them into problems you know how to do, sometimes you can do the reverse—turn a problem you know how to do into the problem you don't know how to do.

So, my initial goal was simply to turn the block-walking proof of the general form of the Catalan numbers into a proof using sequences of parentheses. See if you can do it on your own:

> **Problem 4.1:** Call an arrangement of n "("s and n ")"s *good* if, as we read left-to-right, there are never more ")"s than "("s. For example, the arrangement (((()())()) is good, but the arrangement (()()))(() is not. Find a counting argument to show that there are $\frac{1}{n+1}\binom{2n}{n}$ good arrangements without making a 1-1 correspondence between good arrangements and the Catalan block-walking problem.

Solution for Problem 4.1: I used the block-walking proof as a guide. Instead of counting the good arrangements, I counted the bad ones. We counted the block-walking paths that went above $y = x$ by reflecting the path after the first step that goes above $y = x$. We do essentially the same to count the bad arrangements. We find the first ")" that makes an arrangement bad. For example, in the string ())(, it is the second ")" that makes the arrangement bad, and in (())())(, it is the fourth ")" that makes the arrangement bad. Then, our "reflection" consists in simply reversing each of the subsequent parentheses. Here are a few examples:

$$)(\leftrightarrow))$$
$$()())(\leftrightarrow ()()))$$
$$()))()((\leftrightarrow ()))()())$$

This process is easily reversible—just find the first place in each arrangement on the right where the arrangement "goes bad" and reverse all the subsequent parentheses.

If we have originally used k "("s before an arrangement first "goes bad," then we have used $k+1$ ")"s at the point where the arrangement first "goes bad." So, we have $n-k$ "("s and $n-k-1$ ")"s still to place after the arrangement goes bad. After we reverse all of these parentheses, we have $n-k$ ")"s and $n-k-1$ "("s after the point where the arrangement goes bad. So, the whole new arrangement has $k+1+n-k=n+1$ ")"s and $k+n-k-1=n-1$ "("s. We therefore have a 1-1 correspondence:

$$\{\text{bad arrangements with } n \text{ "("s and } n \text{ ")"s}\} \quad \leftrightarrow \quad \{\text{bad arrangements with } n-1 \text{ "("s and } n+1 \text{ ")"s}\}.$$

Can we count the second set? Yes! Every arrangement of $n-1$ "("s and $n+1$ ")"s is bad. There are $\binom{2n}{n-1}$ arrangements of $n-1$ "("s and $n+1$ ")"s, so there are $\binom{2n}{n-1}$ bad arrangements of n "("s and n ")"s.

There are $\binom{2n}{n}$ total arrangements of n "("s and n ")"s, so the number of good arrangements is

$$\begin{aligned}\binom{2n}{n}-\binom{2n}{n-1} &= \frac{(2n)!}{n!n!}-\frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!}{(n-1)!n!}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \frac{(2n)!}{(n-1)!n!}\cdot\frac{1}{n(n+1)} = \frac{(2n)!}{(n+1)!n!}\\ &= \frac{1}{n+1}\binom{2n}{n}.\end{aligned}$$

□

And now, back to our double-good story. I now had a way to get the closed form for the Catalan numbers; could I use it as inspiration for the double-good arrangements? Using the Catalan proof above as a guide, I started by looking at arrangements of n "("s and $2n$ ")"s that are *not* double-good. Naturally, I'll call them double-bad.

Above, we found a 1-1 correspondence between bad arrangements with n "("s and n ")"s and bad arrangements with $n-1$ "("s and $n+1$ ")"s. However, I knew from my earlier work that I didn't want a 1-1 correspondence for the double-good problem; I was pretty sure I needed a 2-1 correspondence. I needed to show that there are two double-bad arrangements with n "("s and $2n$ ")"s for every double-bad arrangement of $n-1$ "("s and $2n+1$ ")"s. All arrangements of $n-1$ "("s and $2n+1$ ")"s are double-bad, so we can count these easily: there are $\binom{3n}{n-1}$ of them. So, if we find our 2-1 correspondence, we will know that there are $2\binom{3n}{n-1}$ double-bad arrangements with n "("s and $2n$ ")"s leaving

$$\begin{aligned}\binom{3n}{n}-2\binom{3n}{n-1} &= \binom{3n}{n}-2\frac{(3n)!}{(n-1)!(2n+1)!} = \binom{3n}{n}-2\cdot\frac{n\cdot(3n)!}{n\cdot(n-1)!\cdot(2n+1)\cdot(2n)!}\\ &= \binom{3n}{n}-\frac{2n}{2n+1}\frac{(3n)!}{n!(2n)!} = \binom{3n}{n}-\frac{2n}{2n+1}\binom{3n}{n}\\ &= \frac{1}{2n+1}\binom{3n}{n}.\end{aligned}$$

(Yes, I really did work out these details again.)

Concept: Suppose you want to prove a statement $\mathcal{S}$ and you find a statement $\mathcal{T}$ that you think will allow you to prove $\mathcal{S}$ quickly. Take time to confirm that $\mathcal{T}$ will allow you to prove $\mathcal{S}$ *before spending a ton of time trying to prove* $\mathcal{T}$.

Double-Good Catalan

Richard Rusczyk

I now faced the problem of showing that there are twice as many double-bad arrangements with n "("s and $2n$ ")"s as there are double-bad arrangements of $n-1$ "("s and $2n+1$ ")"s. You can guess where I started: writing out examples, and looking for the 2-1 correspondence. I started with $n = 2$:

Double-bad with 2 "("s and 4 ")"s	Double bad with 1 "(" and 5 ")"s
()))(	
()))()	()))))
))))((	
)))()(	)))))(
)))(()	
))())(	))))()
))()()	
))(())	)))())
)()))(	
)())()	))()))
)()())	
)(()))	)())))

It seems natural to pair off the two on the left that start good with the one on the right that starts good. These are the arrangements above the bottom horizontal line in the table. But I couldn't find any way to correspond other pairs on the left with arrangements on the right. I didn't try too hard, though. As soon as I started looking at it, I realized that this was essentially the same thing I tried over and over with the block-walking. Having failed there, I decided there wasn't much point to repeating that failure here.

> **Concept:** Being smart doesn't just mean doing lots of smart things—it also means not doing too many unnecessary stupid things.

Feeling like I was barking up the same wrong tree again, I searched my counting toolbox for tools I understood well that I could try on this problem. I reached for recursion.

> **Concept:** When you're completely stuck, try tools that you know well, even if it's not immediately obvious how to use them on the problem. If you know the tools well enough, you might find a way to make them work.

I let a_n be the number of double-bad arrangements of n "("s and $2n$")"s. My experimentation above at least gave me confidence that I knew how to take care of arrangements that start off double-bad. If an arrangement starts off double-bad, that means it starts with ")". Then, the n "("s can be in any of the remaining $3n-1$ positions. We can choose these positions in $\binom{3n-1}{n}$ ways, so there are $\binom{3n-1}{n}$ double-bad arrangements that start with ")". But what about the arrangements that start off good? We let b_n be number of double-bad arrangements that start with "(" (i.e., those that start good), so we have

$$a_n = \binom{3n-1}{n} + b_n.$$

Now, my target is b_n. But what do I want to show b_n is equal to?

Concept: Working backwards can help focus your efforts.

I wanted to show that $a_n = 2\binom{3n}{n-1}$, so I used my equation relating a_n and b_n to figure out what I wanted b_n to be. I needed to show that b_n is

$$a_n - \binom{3n-1}{n} = 2\binom{3n}{n-1} - \binom{3n-1}{n} = \frac{2\cdot(3n)!}{(n-1)!(2n+1)!} - \frac{(3n-1)!}{n!(2n-1)!}.$$

Well, that's a mess. I then tried to manipulate this to produce a simple combination, since simple combinations often inspire a simple counting argument.

Concept: Seek simplicity

Here's what I found:

$$\begin{aligned}
a_n - \binom{3n-1}{n} &= 2\binom{3n}{n-1} - \binom{3n-1}{n} = \frac{2\cdot(3n)!}{(n-1)!(2n+1)!} - \frac{(3n-1)!}{n!(2n-1)!}\\
&= \frac{2n\cdot(3n)!}{n!(2n+1)!} - \frac{(2n)(2n+1)(3n-1)!}{n!(2n+1)!} = \frac{2n\cdot(3n)\cdot(3n-1)!}{n!(2n+1)!} - \frac{(4n^2+2n)(3n-1)!}{n!(2n+1)!}\\
&= (6n^2-(4n^2+2n))\frac{(3n-1)!}{n!(2n+1)!} = (2n^2-2n)\frac{(3n-1)!}{n!(2n+1)!}\\
&= \frac{(2)(n)(n-1)(3n-1)!}{n!(2n+1)!} = \frac{2(3n-1)!}{(n-2)!(2n+1)!}\\
&= 2\binom{3n-1}{n-2}.
\end{aligned}$$

This was a bit of a surprise: I wanted to show that the number of double-bad sequences that start off good, i.e., those start with a "(", is $2\binom{3n-1}{n-2}$. This seemed like a simple enough expression; I figured there must be a simple explanation.

Concept: If you stumble on a surprise, investigate it closely. There are few coincidences in mathematics.

Unfortunately, I couldn't find the simple explanation. I had made some progress by separating arrangements that start bad and those that start good, so I decided to try separating further. I split the double-bad arrangements based on how long they stayed good before going bad. For example, if an arrangement has only one "(" before going bad, then the arrangement starts ())). Then, we have $n-1$ "("s to place among the remaining $3n-4$ slots, which we can do in $\binom{3n-4}{n-1}$ ways. So, there are $\binom{3n-4}{n-1}$ double-bad arrangements that start ())).

But what if there are two "("s before the arrangement goes bad? Then, we have a problem. We know that such an arrangement starts with 2 "("s and 4 ")"s in a double-good arrangement, then has another ")". But, we don't have an easy way to count the number of double-good arrangements of 2 "("s and 4 ")"s without listing–doing so is the whole point of this problem!

Suppose, however, I could count them. I let g_n be the number of double-good arrangements with n "("s and $2n$ ")"s, so that there are g_2 double-good arrangements of 2 "("s and 4 ")"s. Then, to form a double-bad arrangement that starts with one of these double-good arrangements followed by a ")", I must choose $n-2$ of the remaining $3n-7$ slots for "("s. This I can do in $\binom{3n-7}{n-2}$ ways, so there are $g_2\binom{3n-7}{n-2}$ double-bad arrangements that start with a double-good arrangements of 2 "("s and 4 ")"s followed by ")". Similarly, I split all the double-bad arrangements into groups based on how long they stay double-good, and I found

$$b_n = g_1\binom{3n-4}{n-1} + g_2\binom{3n-7}{n-2} + g_3\binom{3n-10}{n-3} + \cdots .$$

This looked pretty scary. I could put this into my formula for a_n above, and note that $a_n + g_n = \binom{3n}{n}$, since a_n is the number of double-bad arrangements and g_n the number of double-good ones. Then, I'd have a recursion for g_n, but certainly not a pretty one. Maybe with some magical combinatorial identities and induction, I might get to a solution.

But I couldn't shake the feeling that somehow, I was very close now. I also thought that I probably wouldn't need a bunch of identities and recursion to finish. I was intrigued by all those nice combinations that popped out when I started splitting into cases based on how long a double-bad arrangement stayed double-good before going bad. My instincts were screaming "You're close! You're close! It has to be here somewhere!" So, I went back to examining special cases once again, and I went back to looking also at double-bad arrangements with $n-1$ "("s and $2n+1$ ")"s, hoping to use my new grouping to find my correspondence.

I first looked at $n = 2$ again:

Double-bad with 2 "("s and 4 ")"s	Double bad with 1 "(" and 5 ")"s
()))(	
()))()	()))))
))))((	
)))()(	)))))(
)))(()	
))())(	))))()
))()()	
))(())	)))())
)()))(	
)()())	))()))
)()())	
)(()))	)())))

Staring at the arrangements that start off bad (the arrangements below the line), I noticed that there are $\binom{5}{2}$ on the left and $\binom{5}{1}$ on the right. On the left, we choose 2 of 5 slots for "("s after the initial ")", and on the right, we choose 1 of 5 slots for "(" after the initial ")". And we have $\binom{5}{2} = 2\binom{5}{1}$.

OK, what about $n = 3$? I started listing arrangements, but that looked terrifying. So, I thought about them. If a double-bad arrangement of 3 "("s and 6 ")"s starts ())), then there are 5 slots left for the remaining 2 "("s, for a total of $\binom{5}{2}$ such arrangements. Turning to double-bad arrangements of $3 - 1 = 2$ "("s and 7 ")"s that start ())), we have 5 slots left for the remaining 1 "(", for a total of $\binom{5}{1}$ such arrangements. And we have $\binom{5}{2} = 2\binom{5}{1}$.

What if we have $n = 3$ and start with ")"? If a double-bad arrangement of 3 "("s and 6 ")"s starts with ")", then there are 8 slots left for the 3 "("s, for a total of $\binom{8}{3}$ such arrangements. If a double-bad arrangement of 2 "("s and 7 ")"s starts with ")", then there are 8 slots left for the 2 "("s, for a total of $\binom{8}{2}$ such arrangements. And lo and behold, we have $\binom{8}{3} = 2\binom{8}{2}$. This can't be a coincidence.

I tried a few more specific cases with higher n, and sure enough, each time, I found twice as many double-bad arrangements of n "("s and $2n$ ")"s as double-bad arrangements of $n - 1$ "("s and $2n + 1$ ")"s.

I still didn't have my proof, but I sure had a lot of evidence that I could group double-bad arrangements of n "("s and $2n$ ")"s based on how long they stayed double-good, and each group would have twice as many arrangements as the corresponding group of double-bad arrangements of $n - 1$ "("s and $2n + 1$ ")"s.

Concept: Try several examples to test a conjecture before trying to prove it. There's little more frustrating than spending an hour or two trying to prove something that isn't true.

Not only did I now feel very close to a solution, I also thought that this was an approach I could generalize—I could probably use it to count triple-good arrangements, quadruple-good arrangements, and so on.

Concept: When faced with a problem that can be easily generalized, try focusing on solution methods that can be generalized, too.

My initial examples either started off bad or started off ())). What if an arrangement has a longer double-good string at the start? I couldn't count the number of longer double-good strings, so could I still use this method? I tried a specific case once again for guidance. With $n = 4$, I considered double-bad arrangements that started (()))))—i.e., there are 2 "("s and 4 ")"s in a double-good arrangement followed by a ")". If a double-bad arrangement of 4 "("s and 8 ")"s starts with (()))))), then there are 5 slots left for the remaining 2 "("s, for a total of $\binom{5}{2}$ such arrangements. If a double-bad arrangement of $4 - 1 = 3$ "("s and $2 \cdot 4 + 1 = 9$ ")"s starts with (()))))), then there are 5 slots left for the remaining "(", for a total of $\binom{5}{1}$ such arrangements. And I was back in familiar territory: we have $\binom{5}{2} = 2\binom{5}{1}$.

Finally, the lightbulb went off: it doesn't matter how many possible double-good arrangements there are to start with before an arrangement goes bad! For any given double-good start, I conjectured that there are twice as many double-bad arrangements of n "("s and $2n$ ")"s with that start followed by ")" as there are double-bad arrangements of $n - 1$ "("s and $2n + 1$ ")"s with that start followed by ")". This I knew how to tackle.

I had already done several examples, so all that was left was to generalize those examples. I'd start with a double-good arrangement of k "("s and $2k$ ")"s followed by ")", then count the number of double-bad arrangements of n "("s and $2n$ ")"s with this start. Then, I'd count the number of double-bad arrangements of $n - 1$ "("s and $2n + 1$ ")"s with that start. Finally, I'd show that twice the latter number equals the former. From this, I could deduce that there are twice as many total double-bad arrangements of n "("s and $2n$ ")"s as there are total double-bad arrangements of $n - 1$ "("s and $2n + 1$ ")"s. From there, the rest of the proof would follow as planned.

I quickly scratched out the notes, and turned them over to Dave and Naoki, who were writing the solutions manual for the *Intermediate Counting & Probability* text. Then finally, I could let the problem go and move on.

See if you can construct the proof now on your own before reading the solution in the next section. If you can construct the proof on your own, try using it as a guide to generalize the problem—find a way to count triple-good arrangements, quadruple-good arrangements, and so on.

5 Solution

This solution appears in the *Intermediate Counting & Probability Solutions Manual* by David Patrick and Naoki Sato.

Consider an arrangement of n left parentheses and $2n$ right parentheses. Then the arrangement is double-good if for all positive integers k, as we read left-to-right, at most $2(k - 1)$ right parentheses have appeared before the k^{th} left parenthesis. If the arrangement is not double-good, then call it *k-bad* if the first time this condition fails is after the k^{th} left parenthesis but before the $(k + 1)^{\text{st}}$ left parenthesis. For example,

()))() is 1-bad, ((())))() is 2-bad, and)()()) is 0-bad. We also extend the definition of k-bad to an arrangement of any number of left parentheses and right parentheses, not just those where the number of right parentheses is double the number of left parentheses.

First, we prove the following lemma.

> **Lemma**. For all $0 \le k < n$, the number of k-bad arrangements with n left parentheses and $2n$ right parentheses is twice the number of k-bad arrangements with $n-1$ left parentheses and $2n+1$ right parentheses.

Proof. If an arrangement is k-bad, then the arrangement begins with a double-good arrangement of k "("s and $2k$ ")"s, followed by a right parenthesis. So, we have used k left parentheses and $2k+1$ right parentheses, which leaves the last $3n-3k-1$ parentheses to be determined.

For such an arrangement with n left parentheses and $2n$ right parentheses, the remaining $n-k$ left parentheses can be arranged among the last $3n-3k-1$ parentheses arbitrarily, so there are $\binom{3n-3k-1}{n-k}$ possible ways to arrange the last $3n-3k-1$ parentheses. Hence there are $\binom{3n-3k-1}{n-k}$ k-bad arrangements with n "("s and $2n$ ")"s that begin with a given double-good arrangement of k "("s and $2k$ ")"s.

Similarly, each k-bad arrangement with $n-1$ left parentheses and $2n+1$ right parentheses begins with a double-good arrangement of k "("s and $2k$ ")"s. There are $\binom{3n-3k-1}{n-k-1}$ possible ways to arrange the last $3n-3k-1$ parentheses, and thus there are $\binom{3n-3k-1}{n-k-1}$ k-bad arrangements with $n-1$ "("s and $2n+1$ ")"s that begin with a given double-good arrangement of k "("s and $2k$ ")"s.

We have:

$$\begin{aligned}
\binom{3n-3k-1}{n-k} &= \frac{(3n-3k-1)!}{(n-k)!(2n-2k-1)!} \\
&= \frac{(3n-3k-1)!(2n-2k)}{(n-k)(n-k-1)!(2n-2k)(2n-2k-1)!} \\
&= \frac{2n-2k}{n-k} \cdot \frac{(3n-3k-1)!}{(n-k-1)!(2n-2k)!} \\
&= 2 \cdot \frac{(3n-3k-1)!}{(n-k-1)!(2n-2k)!} \\
&= 2\binom{3n-3k-1}{n-k-1}.
\end{aligned}$$

Let $\mathcal{D}$ be an arbitrary double-good arrangement of k "("s and $2k$ ")"s. Our argument above shows that there are twice as many k-bad arrangements of n "("s and $2n$ ")"s that start with $\mathcal{D}$ as there are k-bad arrangements of $n-1$ "("s and $2n+1$ ")"s that start with $\mathcal{D}$. Since all k-bad arrangements begin with some double-good arrangement of k "("s and $2k$ ")"s, we conclude (summing over all possible $\mathcal{D}$) that there are twice as many k-bad arrangements of n "("s and $2n$ ")"s as there are k-bad arrangements of $n-1$ "("s and $2n+1$ ")"s.

Thus the Lemma is proved.

Since the Lemma holds for all $0 \le k < n$, it follows that the number of arrangements with n left parentheses and $2n$ right parentheses that are not double-good is twice the number of arrangements with $n-1$ left parentheses and $2n+1$ right parentheses that are not double-good. But *all* of the latter arrangements are not double-good, and there are $\binom{3n}{n-1}$ such arrangements.

Thus, we have shown that the number of arrangements with n left parentheses and $2n$ right parentheses that are not double-good is $2\binom{3n}{n-1}$. Therefore, the number of arrangements that are double-good is

$$\begin{aligned}\binom{3n}{n} - 2\binom{3n}{n-1} = \frac{(3n)!}{n!(2n)!} - 2\cdot\frac{(3n)!}{(n-1)!(2n+1)!} &= \frac{(3n)!}{(n-1)!(2n)!}\left(\frac{1}{n} - \frac{2}{2n+1}\right) \\ &= \frac{(3n)!}{(n-1)!(2n)!}\cdot\frac{1}{n(2n+1)} = \frac{1}{2n+1}\cdot\frac{(3n)!}{n!(2n)!} = \boxed{\frac{1}{2n+1}\binom{3n}{n}}.\end{aligned}$$

6 Moral of the Story

In my very long and varied exploration of the double-good problem, you'll find no genius. No moments of brilliant insight that led me to discoveries mere mortals couldn't hope to find. You won't even find any fancy mathematics. Nothing more powerful than combinations is needed to solve this problem.

What you'll see is a lot of dead-ends, and a lot of time spent chasing fruitless, and in some cases insane, ideas. You'll also see a great many of very powerful general problem solving strategies. Look for patterns. Work backwards. Try ideas out of left field. Compare to known problems. Move on when it's clear an idea isn't working out. Try tools that you've mastered. Take breaks and start afresh.

But above all, what you'll see are the most valuable problem solving tools: tenacity and experience. Desire and effort will always beat genius in the long run. I'm not a big believer in genius. All the "geniuses" I've worked with had one trait in common: the more I knew them, the more I saw that their "genius" was the product of a great deal of time spent thinking.

As you continue with WOOT, try not to view everything you learn as a giant collection of gimmicks. Instead, focus on why the "gimmicks" work, and the clues that exist in problems to tell you when to use them. Notice that in the solution I finally found to the double-good problem, it wasn't knowing what the Catalan numbers are that gave me a guide to the solution. It was understanding the proof of the closed form for the Catalan numbers, and thinking about the motivations behind the proof, that was my guide for the double-good problem.

In the real world, you can always look the "how" and "what" up. Understanding the "why" and mastering the "when"—these are the most important skills you can develop.