Sequences & Series

Solutions to Review Problems

1. If the integer k is added to each of the numbers 36, 300 and 596, one obtains the squares of three consecutive terms of an arithmetic sequence. Find k. (AIME, 1989)

Solution: Let the terms of the arithmetic sequence be $a-d$, a, and $a+d$. Then

$$\begin{aligned}(a-d)^2 &= k+36,\\ a^2 &= k+300,\\ (a+d)^2 &= k+596.\end{aligned}$$

Adding the first and third equations and dividing by 2, we get $a^2+d^2 = k+316$. Subtracting the second equation from this equation, we get $d^2 = 16$. Subtracting the first equation from the third equation, we get $4ad = 560$, so $ad = 560/4 = 140$. Hence,

$$a^2 = \frac{140^2}{d^2} = \frac{140^2}{16} = 1225.$$

Finally, from the second equation, $k = a^2 - 300 = 925$.

2. Let $T_n = 1+2+3+\cdots+n$ and

$$P_n = \frac{T_2}{T_2-1}\cdot\frac{T_3}{T_3-1}\cdot\frac{T_4}{T_4-1}\cdots\frac{T_n}{T_n-1}$$

for $n = 2, 3, 4, \ldots$. Find P_{1991}. (AHSME, 1991)

Solution: We have that $T_n = n(n+1)/2$, so

$$\frac{T_n}{T_n-1} = \frac{n(n+1)/2}{n(n+1)/2-1} = \frac{n(n+1)}{n^2+n-2} = \frac{n(n+1)}{(n-1)(n+2)}.$$

Hence,

$$\begin{aligned}P_{1991} &= \frac{T_2}{T_2-1}\cdot\frac{T_3}{T_3-1}\cdot\frac{T_4}{T_4-1}\cdots\frac{T_{1991}}{T_{1991}-1}\\ &= \frac{2\cdot 3}{1\cdot 4}\cdot\frac{3\cdot 4}{2\cdot 5}\cdot\frac{4\cdot 5}{3\cdot 6}\cdots\cdot\frac{1990\cdot 1991}{1989\cdot 1992}\cdot\frac{1991\cdot 1992}{1990\cdot 1993}\\ &= \frac{3\cdot 1991}{1993}\\ &= \frac{5973}{1993}.\end{aligned}$$

3. Let F_n denote the n^{th} Fibonacci number. Prove that for all $n \geq 1$,

(a) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

(b) $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.

Hint: Both sums can be made to telescope.

Solution: (a) The sum telescopes as

$$\begin{aligned} F_1 + F_2 + \cdots + F_n &= (F_2 - F_0) + (F_3 - F_1) + \cdots + (F_{n+1} - F_{n-1}) \\ &= F_n + F_{n+1} - F_0 - F_1 \\ &= F_{n+2} - 1. \end{aligned}$$

(b) The sum telescopes as

$$\begin{aligned} F_1^2 + F_2^2 + \cdots + F_n^2 &= F_1(F_2 - F_0) + F_2(F_3 - F_1) + \cdots + F_n(F_{n+1} - F_{n-1}) \\ &= F_1F_2 - F_0F_1 + F_2F_3 - F_1F_2 + \cdots + F_nF_{n+1} - F_{n-1}F_n \\ &= F_nF_{n+1} - F_0F_1 \\ &= F_nF_{n+1}. \end{aligned}$$

4. Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

(Putnam, 1977)

Solution: Let

$$a_n = \frac{n^3 - 1}{n^3 + 1} = \frac{(n-1)(n^2 + n + 1)}{(n+1)(n^2 - n + 1)},$$

so the first few factors are

$$\begin{aligned} a_2 &= \frac{(2-1)(2^2 + 2 + 1)}{(2+1)(2^2 - 2 + 1)} = \frac{1 \cdot 7}{3 \cdot 3}, \\ a_3 &= \frac{(3-1)(3^2 + 3 + 1)}{(3+1)(3^2 - 3 + 1)} = \frac{2 \cdot 13}{4 \cdot 7}, \\ a_4 &= \frac{(4-1)(4^2 + 4 + 1)}{(4+1)(4^2 - 4 + 1)} = \frac{3 \cdot 21}{5 \cdot 13}, \end{aligned}$$

and so on. Note that

$$a_{n+1} = \frac{n[(n+1)^2 + (n+1) + 1]}{(n+2)[(n+1)^2 - (n+1) + 1]} = \frac{n(n^2 + 3n + 3)}{(n+2)(n^2 + n + 1)},$$

so the factor of n^2+n+1 in a_n cancels the factor of n^2+n+1 in a_{n+1}.

To compute the infinite product, we first compute the finite product

$$\prod_{n=2}^{m} \frac{n^3-1}{n^3+1},$$

which telescopes as

$$\begin{aligned}\prod_{n=2}^{m} \frac{n^3-1}{n^3+1} &= a_2a_3a_4\cdots a_m \\ &= \frac{1\cdot 7}{3\cdot 3}\cdot\frac{2\cdot 13}{4\cdot 7}\cdot\frac{3\cdot 21}{5\cdot 13}\cdots\frac{(m-2)(m^2-m+1)}{m(m^2-3m+3)}\cdot\frac{(m-1)(m^2+m+1)}{(m+1)(m^2-m+1)} \\ &= \frac{2(m^2+m+1)}{3m(m+1)} \\ &= \frac{2m^2+2m+2}{3m^2+3m}.\end{aligned}$$

Dividing the numerator and denominator by m^2, we get

$$\prod_{n=2}^{m} \frac{n^3-1}{n^3+1} = \frac{2+\frac{2}{m}+\frac{2}{m^2}}{3+\frac{3}{m}}.$$

Hence, letting m approach infinity, we find that the infinite product is

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \frac{2}{3}.$$

5. Calculate the sum

$$\sum_{k=1}^{n} \frac{k}{k^4+k^2+1}.$$

Solution: First, k^4+k^2+1 factors as $(k^2+k+1)(k^2-k+1)$. Using partial fractions, we find

$$\frac{k}{k^4+k^2+1} = \frac{1}{2}\left(\frac{1}{k^2-k+1}-\frac{1}{k^2+k+1}\right).$$

Then by the observation in the previous solution, the given sum telescopes as

$$\begin{aligned}\sum_{k=1}^{n} \frac{k}{k^4+k^2+1} &= \frac{1}{2}\left(1-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{7}\right)+\cdots+\frac{1}{2}\left(\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right)\\ &= \frac{1}{2}\left(1-\frac{1}{n^2+n+1}\right)\\ &= \frac{n^2+n}{2(n^2+n+1)}.\end{aligned}$$

6. Evaluate the sum

$$\sum_{n=1}^{1994}(-1)^n\frac{n^2+n+1}{n!}.$$

(Canada, 1994)

Solution: The sum telescopes as

$$\begin{aligned}\sum_{n=1}^{1994}(-1)^n\frac{n^2+n+1}{n!} &= \sum_{n=1}^{1994}\left[(-1)^n\frac{n^2}{n!}+(-1)^n\frac{n+1}{n!}\right]\\ &= \sum_{n=1}^{1994}\left[(-1)^n\frac{n}{(n-1)!}+(-1)^n\frac{n+1}{n!}\right]\\ &= \left(-\frac{1}{0!}-\frac{2}{1!}\right)+\left(\frac{2}{1!}+\frac{3}{2!}\right)+\left(-\frac{3}{2!}-\frac{4}{3!}\right)+\cdots+\left(\frac{1994}{1993!}+\frac{1995}{1994!}\right)\\ &= \frac{1995}{1994!}-\frac{1}{0!}\\ &= \frac{1995}{1994!}-1.\end{aligned}$$

7. Prove that for every positive integer n, and for every real number x *not* of the form $\frac{k\pi}{2^t}$, where $0 \le t \le n$ and k is an integer,

$$\frac{1}{\sin 2x}+\frac{1}{\sin 4x}+\cdots+\frac{1}{\sin 2^n x}=\cot x-\cot 2^n x.$$

(IMO, 1966)

Solution: Let $T_k = \cot x - \cot 2^k x$, and let $\theta_k = 2^{k-1}x$. Then

$$\begin{aligned} T_k - T_{k-1} &= (\cot x - \cot 2^k x) - (\cot x - \cot 2^{k-1} x) \\ &= \cot 2^{k-1} x - \cot 2^k x \\ &= \frac{\cos 2^{k-1} x}{\sin 2^{k-1} x} - \frac{\cos 2^k x}{\sin 2^k x} \\ &= \frac{\cos \theta_k}{\sin \theta_k} - \frac{\cos 2\theta_k}{\sin 2\theta_k}. \end{aligned}$$

By the double angle formulas, $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$, so

$$\begin{aligned} T_k - T_{k-1} &= \frac{\cos \theta_k}{\sin \theta_k} - \frac{\cos 2\theta_k}{\sin 2\theta_k} \\ &= \frac{\cos \theta_k}{\sin \theta_k} - \frac{2\cos^2\theta_k - 1}{2\cos\theta_k \sin\theta_k} \\ &= \frac{\cos \theta_k}{\sin \theta_k} - \frac{2\cos^2\theta_k}{2\cos\theta_k \sin\theta_k} + \frac{1}{2\cos\theta_k \sin\theta_k} \\ &= \frac{\cos \theta_k}{\sin \theta_k} - \frac{\cos \theta_k}{\sin \theta_k} + \frac{1}{2\cos\theta_k \sin\theta_k} \\ &= \frac{1}{2\cos\theta_k \sin\theta_k} \\ &= \frac{1}{\sin 2\theta_k} \\ &= \frac{1}{\sin 2^k x}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \dots + \frac{1}{\sin 2^n x} &= (T_1 - T_0) + (T_2 - T_1) + \dots + (T_n - T_{n-1}) \\ &= T_n - T_0 \\ &= (\cot x - \cot 2^n x) - (\cot x - \cot x) \\ &= \cot x - \cot 2^n x. \end{aligned}$$

8. Let x, y, and z be real numbers such that $x + y + z = 0$. Prove that

$$\frac{x^2 + y^2 + z^2}{2} \cdot \frac{x^5 + y^5 + z^5}{5} = \frac{x^7 + y^7 + z^7}{7}.$$

Solution: Using the same notation as in the handout,

$$S_6 = -AS_4 + BS_3 = -2A^3 + 3B^2,$$
$$S_7 = -AS_5 + BS_4 = 7A^2B,$$

so

$$\frac{S_2}{2} \cdot \frac{S_5}{5} = A^2B = \frac{S_7}{7}.$$

Note: Problem 2 on the 1982 USAMO generalizes this result as follows: Let $S_r = x^r + y^r + z^r$, with x, y, z real. It is known that if $S_1 = 0$,

$$\frac{S_{m+n}}{m+n} = \frac{S_m}{m} \cdot \frac{S_n}{n} \qquad (*)$$

for $(m, n) = (2, 3)$, $(3, 2)$, $(2, 5)$, or $(5, 2)$. Determine *all* other pairs of integers (m, n) if any, so that $(*)$ holds for all real numbers x, y, z such that $x + y + z = 0$.

9. Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$ax + by = 3,$$
$$ax^2 + by^2 = 7,$$
$$ax^3 + by^3 = 16,$$
$$ax^4 + by^4 = 42.$$

(AIME, 1990)

Solution: Let $S_n = ax^n + by^n$. Then the sequence (S_n) satisfies a linear recurrence, whose characteristic polynomial is

$$(t - x)(t - y) = t^2 - (x + y)t + xy.$$

Let $A = x + y$ and $B = xy$, so the characteristic polynomial can also be written as $t^2 - At + B$. Then

$$S_n = AS_{n-1} - BS_{n-2}$$

for all $n \geq 3$. Setting $n = 3$ and $n = 4$, we obtain the system of equations

$$7A - 3B = 16,$$
$$16A - 7B = 42.$$

Solving for A and B, we find $A = -14$ and $B = -38$. Therefore,

$$ax^5 + by^5 = S_5 = AS_4 - BS_3 = (-14)(42) - (-38)(16) = 20.$$

10. Let (x_n) be a sequence such that $x_0 = x_1 = 5$ and

$$x_n = \frac{x_{n-1} + x_{n+1}}{98}$$

for all positive integers n. Prove that $(x_n + 1)/6$ is a perfect square for all n.

Solution: From the given equation,

$$x_{n+1} - 98x_n + x_{n-1} = 0$$

for all $n \geq 1$. At this point, we could solve for x_n, but we take another approach. We compute the first few terms of the sequence:

n	x_n	$(x_n + 1)/6$
0	5	1
1	5	1
2	485	81
3	47525	7921
4	4656965	776161

We see that the few terms of the sequence $(x_n + 1)/6$ are perfect squares, and that their square roots are 1, 1, 9, 89, 881. Since the sequence (x_n) satisfies a linear recurrence, we suspect that these square roots may satisfy a linear recurrence as well.

First, we try a linear recurrence where each term depends on the previous two terms. If the coefficients of these two terms in the linear recurrence are A and B, then

$$\begin{aligned} A + B &= 9, \\ 9A + B &= 89. \end{aligned}$$

Solving this system of equations, we find $A = 10$ and $B = -1$. These coefficients produce a linear recurrence that is consistent with the other square roots that we have computed.

Hence, we define the sequence (y_n) by $y_0 = y_1 = 1$ and $y_n = 10y_{n-1} - y_{n-2}$ for all $n \geq 2$. Clearly, y_n is an integer for all $n \geq 0$. The characteristic polynomial for this linear recurrence is $t^2 - 10t + 1$. Let the roots of this quadratic be α and β, so by Vieta's Formulas, $\alpha + \beta = 10$ and $\alpha\beta = 1$. Also,

$$y_n = c_1\alpha^n + c_2\beta^n$$

for some constants c_1 and c_2. Now, let

$$z_n = 6y_n^2 - 1.$$

We want to show that $z_n = x_n$ for all n.

We have that

$$\begin{aligned} z_n &= 6y_n^2 - 1 \\ &= 6(c_1\alpha^n + c_2\beta^n)^2 - 1 \\ &= 6c_1^2\alpha^{2n} + 12c_1c_2\alpha^n\beta^n + 6c_2^2\beta^{2n} - 1 \\ &= 12c_1c_2 - 1 + 6c_1^2(\alpha^2)^n + 6c_2^2(\beta^2)^n. \end{aligned}$$

We see that the sequence (z_n) satisfies a linear recurrence, whose characteristic polynomial is

$$(t-1)(t-\alpha^2)(t-\beta^2) = (t-1)[t^2 - (\alpha^2+\beta^2) + \alpha^2\beta^2].$$

Squaring $\alpha + \beta = 10$, we get $\alpha^2 + 2\alpha\beta + \beta^2 = 100$, so $\alpha^2 + \beta^2 = 100 - 2\alpha\beta = 98$. Squaring $\alpha\beta = 1$, we get $\alpha^2\beta^2 = 1$. Thus, the characteristic polynomial is

$$(t-1)(t^2 - 98t + 1) = t^3 - 99t^2 + 99t - 1,$$

which means

$$z_n - 99z_{n-1} + 99z_{n-2} + z_{n-3} = 0$$

for all $n \geq 3$.

The given sequence (x_n) satisfies $x_n - 98x_{n-1} + x_{n-2} = 0$ for all $n \geq 2$. The characteristic polynomial for this linear recurrence is $t^2 - 98t + 1$. We found that the roots of this quadratic are α^2 and β^2, so

$$x_n = d_1\alpha^{2n} + d_2\beta^{2n}$$

for some constants d_1 and d_2. We can also write

$$x_n = 0 \cdot 1^n + d_1\alpha^{2n} + d_2\beta^{2n}.$$

Hence, the sequence (x_n) also satisfies the linear recurrence whose characteristic polynomial is

$$(t-1)(t^2 - 98t + 1) = t^3 - 99t^2 + 99t - 1.$$

(More generally, if a sequence (x_n) satisfies a linear recurrence whose characteristic polynomial is $p(x)$, then the sequence (x_n) satisfies the linear recurrence whose characteristic polynomial is any multiple of $p(x)$.) Therefore,

$$x_n - 99x_{n-1} + 99x_{n-2} + x_{n-3} = 0$$

for all $n \geq 3$. Furthermore, $x_0 = z_0 = 5$, $x_1 = z_1 = 5$, and $x_2 = z_2 = 485$. We conclude that $x_n = z_n$ for all n, which means $(x_n + 1)/6 = (z_n + 1)/6 = y_n^2$ is a perfect square for all n.

11. Let a, b, and c be the roots of the equation $x^3 - x^2 - x - 1 = 0$. Show that a, b, and c are distinct, and that

$$\frac{a^{1982} - b^{1982}}{a-b} + \frac{b^{1982} - c^{1982}}{b-c} + \frac{c^{1982} - a^{1982}}{c-a}$$

is an integer. (Canada, 1982)

Solution: By Vieta's Formulas, $a + b + c = 1$, $ab + ac + bc = -1$, and $abc = 1$.

Suppose that two of the roots are equal. Without loss of generality, assume that $b = c$. Then from the equations above, $a + 2b = 1$, $2ab + b^2 = -1$, and $ab^2 = 1$. From the first equation, $a = 1 - 2b$. Substituting this expression for a into the equation $2ab + b^2 = -1$, we get $2(1-2b)b + b^2 = -1$, which simplifies as

$$3b^2 - 2b - 1 = (b-1)(3b+1) = 0,$$

so $b = 1$ or $b = -1/3$. But neither of these values satisfy the cubic $x^3 - x^2 - x - 1 = 0$, contradiction. Therefore, the roots a, b, and c are distinct.

Now, let

$$\begin{aligned} S_n &= \frac{a^n - b^n}{a-b} + \frac{b^n - c^n}{b-c} + \frac{c^n - a^n}{c-a} \\ &= \left(\frac{1}{a-b} - \frac{1}{c-a}\right)a^n + \left(\frac{1}{b-c} - \frac{1}{a-b}\right)b^n + \left(\frac{1}{c-a} - \frac{1}{b-c}\right)c^n. \end{aligned}$$

Then the sequence (S_n) satisfies a linear recurrence, whose characteristic polynomial is

$$(x-a)(x-b)(x-c) = x^3 - x^2 - x - 1.$$

Hence,

$$S_n - S_{n-1} - S_{n-2} - S_{n-3} = 0$$

for all $n \geq 3$.

Furthermore, the first few terms of the sequence are

$$\begin{aligned} S_0 &= \frac{a^0 - b^0}{a-b} + \frac{b^0 - c^0}{b-c} + \frac{c^0 - a^0}{c-a} = 0, \\ S_1 &= \frac{a-b}{a-b} + \frac{b-c}{b-c} + \frac{c-a}{c-a} = 3, \\ S_2 &= \frac{a^2 - b^2}{a-b} + \frac{b^2 - c^2}{b-c} + \frac{c^2 - a^2}{c-a} = (a+b) + (b+c) + (c+a) = 2a + 2b + 2c = 2. \end{aligned}$$

Since the coefficients of the recurrence are integers, it follows that S_n is an integer for all $n \geq 0$. In particular,

$$S_{1982} = \frac{a^{1982} - b^{1982}}{a-b} + \frac{b^{1982} - c^{1982}}{b-c} + \frac{c^{1982} - a^{1982}}{c-a}$$

is an integer.

Challenge Problems

12. For $0 < x < 1$, express

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$$

as a rational function of x. (Putnam, 1977)

Solution: We give two solutions to this problem.

Solution 1: For a nonnegative integer k, let

$$S_k = \sum_{n=0}^{k} \frac{x^{2^n}}{1 - x^{2^{n+1}}}.$$

Computing the first few partial sums S_k, we find

$$S_0 = \frac{x}{1 - x^2},$$

and

$$\begin{aligned}
S_1 &= \frac{x}{1 - x^2} + \frac{x^2}{1 - x^4} \\
&= \frac{x(1 + x^2)}{(1 + x^2)(1 - x^2)} + \frac{x^2}{1 - x^4} \\
&= \frac{x + x^3}{1 - x^4} + \frac{x^2}{1 - x^4} \\
&= \frac{x + x^2 + x^3}{1 - x^4} \\
&= \frac{x(1 + x + x^2)}{1 - x^4} \\
&= \frac{x(1 - x)(1 + x + x^2)}{(1 - x)(1 - x^4)} \\
&= \frac{x(1 - x^3)}{(1 - x)(1 - x^4)} \\
&= \frac{x - x^4}{(1 - x)(1 - x^4)},
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= \frac{x - x^4}{(1 - x)(1 - x^4)} + \frac{x^4}{1 - x^8} \\
&= \frac{(x - x^4)(1 + x^4)}{(1 - x)(1 + x^4)(1 - x^4)} + \frac{x^4(1 - x)}{(1 - x)(1 - x^8)} \\
&= \frac{x + x^5 - x^4 - x^8}{(1 - x)(1 - x^8)} + \frac{x^4 - x^5}{(1 - x)(1 - x^8)} \\
&= \frac{x - x^8}{(1 - x)(1 - x^8)}.
\end{aligned}$$

It appears that

$$S_k = \frac{x - x^{2^{k+1}}}{(1 - x)(1 - x^{2^{k+1}})}$$

for all $k \ge 0$, so let

$$T_k = \frac{x - x^{2^{k+1}}}{(1-x)(1-x^{2^{k+1}})}.$$

Then

$$\begin{aligned}
T_k - T_{k-1} &= \frac{x - x^{2^{k+1}}}{(1-x)(1-x^{2^{k+1}})} - \frac{x - x^{2^k}}{(1-x)(1-x^{2^k})} \\
&= \frac{x - x^{2^{k+1}}}{(1-x)(1-x^{2^{k+1}})} - \frac{(1+x^{2^k})(x - x^{2^k})}{(1-x)(1+x^{2^k})(1-x^{2^k})} \\
&= \frac{x - x^{2^{k+1}}}{(1-x)(1-x^{2^{k+1}})} - \frac{x - x^{2^k} + x^{2^k+1} - x^{2^{k+1}}}{(1-x)(1-x^{2^{k+1}})} \\
&= \frac{x^{2^k} - x^{2^k+1}}{(1-x)(1-x^{2^{k+1}})} \\
&= \frac{x^{2^k}(1-x)}{(1-x)(1-x^{2^{k+1}})} \\
&= \frac{x^{2^k}}{1-x^{2^{k+1}}}
\end{aligned}$$

for all $k \ge 1$.

Hence,

$$\begin{aligned}
S_k &= \sum_{n=0}^{k} \frac{x^{2^n}}{1 - x^{2^{n+1}}} \\
&= \frac{x}{1-x^2} + \sum_{n=1}^{k} \frac{x^{2^n}}{1 - x^{2^{n+1}}} \\
&= \frac{x}{1-x^2} + \sum_{n=1}^{k} (T_n - T_{n-1}) \\
&= \frac{x}{1-x^2} + (T_1 - T_0) + (T_2 - T_1) + \cdots + (T_k - T_{k-1}) \\
&= \frac{x}{1-x^2} + T_k - T_0 \\
&= \frac{x}{1-x^2} + \frac{x - x^{2^{k+1}}}{(1-x)(1-x^{2^{k+1}})} - \frac{x - x^2}{(1-x)(1-x^2)} \\
&= \frac{x}{1-x^2} + \frac{x - x^{2^{k+1}}}{(1-x)(1-x^{2^{k+1}})} - \frac{x(1-x)}{(1-x)(1-x^2)} \\
&= \frac{x}{1-x^2} + \frac{x - x^{2^{k+1}}}{(1-x)(1-x^{2^{k+1}})} - \frac{x}{1-x^2} \\
&= \frac{x - x^{2^{k+1}}}{(1-x)(1-x^{2^{k+1}})}.
\end{aligned}$$

As $k \to \infty$,

$$\frac{x - x^{2^{k+1}}}{(1-x)(1-x^{2^{k+1}})} \to \frac{x}{1-x},$$

so

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}} = \frac{x}{1-x}.$$

Solution 2: The given sum is

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}} = \frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \frac{x^8}{1-x^{16}} + \cdots.$$

Using the formula for an infinite geometric series, we can write

$$\frac{x}{1-x^2} = x(1 + x^2 + x^4 + x^6 + \cdots) = x + x^3 + x^5 + x^7 + \cdots,$$

which gives us every power of x of the form x^{2k+1}, where k is a nonnegative integer. Also,

$$\frac{x^2}{1-x^4} = x^2(1 + x^4 + x^8 + x^{12} + \cdots) = x^2 + x^6 + x^{10} + x^{14} + \cdots,$$

which gives us every power of x of the form $x^{2(2k+1)}$. In general,

$$\begin{aligned}\frac{x^{2^n}}{1-x^{2^{n+1}}} &= x^{2^n}(1 + x^{2^{n+1}} + x^{2\cdot 2^{n+1}} + x^{3\cdot 2^{n+1}} + \cdots) \\ &= x^{2^n} + x^{2^n+2^{n+1}} + x^{2^n+\cdot 2^{n+1}} + x^{2^n+3\cdot 2^{n+1}} + \cdots \\ &= x^{2^n} + x^{3\cdot 2^n} + x^{5\cdot 2^n} + x^{7\cdot 2^n} + \cdots,\end{aligned}$$

which gives us every power of x of the form $x^{(2k+1)\cdot 2^n}$.

Every positive integer can be written uniquely in the form $(2k+1)\cdot 2^n$, where k and n are nonnegative integers, so

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}} = x + x^2 + x^3 + \cdots = \frac{x}{1-x}.$$

13. For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Putnam, 1980)

Solution: The first few terms of the sequence are

$$\begin{aligned}u_1 &= 2u_0 - 0 = 2a, \\ u_2 &= 2u_1 - 1 = 4a - 1, \\ u_3 &= 2u_2 - 4 = 8a - 6.\end{aligned}$$

From the given recurrence,

$$u_{n+1} - 2u_n = n^2$$

for all $n \geq 0$. Shifting the index n by 1, we get

$$\begin{aligned}u_{n+1} - 2u_n &= n^2, \\ u_n - 2u_{n-1} &= (n-1)^2.\end{aligned}$$

Subtracting these equations, we get

$$u_{n+1} - 3u_n + 2u_{n-1} = n^2 - (n-1)^2 = 2n - 1$$

for all $n \geq 1$.

Shifting the index n by 1 again, we get

$$u_{n+1} - 3u_n + 2u_{n-1} = 2n - 1,$$
$$u_n - 3u_{n-1} + 2u_{n-2} = 2(n-1) - 1.$$

Subtracting these equations, we get

$$u_{n+1} - 4u_n + 5u_{n-1} - 2u_{n-2} = 2n - 1 - [2(n-1) - 1] = 2$$

for all $n \geq 2$.

Shifting the index n by 1 again, we get

$$u_{n+1} - 4u_n + 5u_{n-1} - 2u_{n-2} = 2,$$
$$u_n - 4u_{n-1} + 5u_{n-2} - 2u_{n-3} = 2.$$

Subtracting these equations, we get

$$u_{n+1} - 5u_n + 9u_{n-1} - 7u_{n-2} + 2u_{n-3} = 0$$

for all $n \geq 3$.

Hence, the sequence (u_n) satisfies a linear recurrence, whose characteristic polynomial is

$$x^4 - 5x^3 + 9x^2 - 7x + 2 = (x-1)^3(x-2).$$

Therefore,

$$u_n = c_1 + c_2 n + c_3 n^2 + c_4 2^n$$

for some constants c_1, c_2, c_3, and c_4. Setting $n = 0, 1, 2,$ and 3, we obtain the system of equations

$$\begin{aligned} c_1 + c_4 &= a, \\ c_1 + c_2 + c_3 + 2c_4 &= 2a, \\ c_1 + 2c_2 + 4c_3 + 4c_4 &= 4a - 1, \\ c_1 + 3c_2 + 9c_3 + 8c_4 &= 8a - 6. \end{aligned}$$

Solving this system of equations, we find $c_1 = 3$, $c_2 = 2$, $c_3 = 1$, and $c_4 = a - 3$, so

$$u_n = 3 + 2n + n^2 + (a-3)2^n$$

for all $n \geq 0$. In this expression, for large n, the term 2^n grows the fastest. (In other words, 2^n dominates 1, n, and n^2 for large n.) The coefficient of 2^n is $a - 3$, so $u_n > 0$ for all $n \geq 0$ if and only if $a \geq 3$.

14. An integer sequence is defined by $a_0 = 0$, $a_1 = 1$, and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$. Prove that 2^k divides a_n if and only if 2^k divides n. (IMO Short List, 1988)

Solution: The characteristic polynomial of the sequence (a_n) is $x^2 - 2x - 1$. By the quadratic formula, the roots of this quadratic are $1 \pm \sqrt{2}$, so let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Then

$$a_n = c_1\alpha^n + c_2\beta_n$$

for some constants c_1 and c_2. Setting $n = 0$ and $n = 1$, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 0, \\ \alpha c_1 + \beta c_2 &= 1. \end{aligned}$$

Solving this system of equations, we find $c_1 = \frac{1}{\alpha-\beta} = \frac{1}{2\sqrt{2}}$ and $c_2 = -\frac{1}{\alpha-\beta} = -\frac{1}{2\sqrt{2}}$, so

$$a_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$$

for all $n \geq 0$.

Since $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$,

$$a_n \equiv a_{n-2} \pmod 2$$

for all $n \geq 2$. Since $a_1 = 1$, a_n is odd for all odd n. In particular, this shows the result for odd n.

Now it would help if we could relate a_{2n} to a_n as this would allow us to bootstrap from the case for a_{odd}. To this end, let

$$b_n = \frac{a_{2n}}{2a_n} = \frac{(\alpha^{2n} - \beta^{2n})/(2\sqrt{2})}{2(\alpha^n - \beta^n)/(2\sqrt{2})} = \frac{\alpha^{2n} - \beta^{2n}}{2(\alpha^n - \beta^n)} = \frac{\alpha^n + \beta^n}{2}$$

for $n \geq 1$. Then the sequence (b_n) satisfies a linear recurrence, whose characteristic polynomial is $(x - \alpha)(x - \beta) = x^2 - 2x - 1$, so

$$b_n - 2b_{n-1} - b_{n-2} = 0$$

for all $n \geq 3$. Also, $b_1 = a_2/(2a_1) = 1$ and $b_2 = a_4/(2a_2) = 3$. It follows that b_n is an integer for all $n \geq 1$. Furthermore,

$$b_n = 2b_{n-1} + b_{n-2} \equiv b_{n-2} \pmod 2$$

for all $n \geq 2$. Since $b_1 = 1$ and $b_2 = 3$ are odd, b_n is odd for all $n \geq 1$. (Specifically this tells us that a_{2n} has one more factor of 2 than a_n, from the definition of b_n.)

Given a positive integer n, we can write n uniquely in the form $n = 2^e \cdot t$, where e is a nonnegative integer and t is an odd positive integer. Then

$$\begin{aligned} a_n &= a_{2^e \cdot t} \\ &= 2^e \cdot \frac{a_{2^e \cdot t}}{2a_{2^{e-1} \cdot t}} \cdot \frac{a_{2^{e-1} \cdot t}}{2a_{2^{e-2} \cdot t}} \cdots \frac{a_{2t}}{2a_t} \cdot a_t \\ &= 2^e b_{2^{e-1} \cdot t} b_{2^{e-2} \cdot t} \cdots b_t a_t. \end{aligned}$$

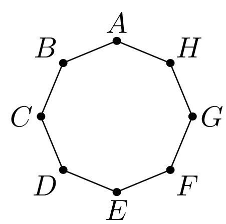
All the factors b_i are odd, and a_t is odd since t is odd, so a_n has exactly e factors of 2. In other words, n and a_n always have the same number of factors of 2. It follows that 2^k divides a_n if and only if 2^k divides n.

15. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$ and

$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1})$$

for all $n = 1, 2, 3, \ldots$, where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$. (IMO, 1979)

Solution: Label the vertices of the octagon as shown.



Clearly, the frog can be only at one of the vertices A, C, E, or G after an even number of jumps, so $a_{2n-1} = 0$ for all $n \ge 1$.

For all $n \ge 1$, let r_n be the number of distinct paths of exactly n jumps starting at A and ending at C, and let s_n be the number of distinct paths of exactly n jumps starting at A and ending at A. By symmetry, r_n is also the number of distinct paths of exactly n jumps starting at A and ending at G.

If the frog is at vertex E after n jumps, then it must have been either at vertex C or G two jumps before (but not at vertex E, because once the frog reaches E, it stays there), so

$$a_n = 2r_{n-2}$$

for all $n \ge 2$.

If the frog is at vertex C after n jumps, then it must have been either at vertex A or C two jumps before (but not at vertex E), and there are two ways to go from C back to C after two jumps, so

$$r_n = 2r_{n-2} + s_{n-2}$$

for all $n \ge 2$.

Finally, if the frog is at vertex A after n jumps, then it must have been either at vertex A, C, or G two jumps before, and there are two ways to go from A back to A after two jumps, so

$$s_n = 2r_{n-2} + 2s_{n-2}$$

for all $n \geq 2$. Thus, we have the linear recurrences

$$a_n = 2r_{n-2}, \tag{1}$$
$$r_n = 2r_{n-2} + s_{n-2}, \tag{2}$$
$$s_n = 2r_{n-2} + 2s_{n-2} \tag{3}$$

for all $n \geq 2$. The first few values of a_n, r_n, and s_n are as follows:

n	a_n	r_n	s_n
0	0	0	1
2	0	1	2
4	2	4	6
6	8	14	20
8	28	48	68

From equation (2),

$$s_{n-2} = r_n - 2r_{n-2}$$

for all $n \geq 2$. Then $s_n = r_{n+2} - 2r_n$ for all $n \geq 0$. Substituting these expressions into equation (3), we get

$$r_{n+2} - 2r_n = 2r_{n-2} + 2(r_n - 2r_{n-2}),$$

or

$$r_{n+2} - 4r_n + 2r_{n-2} = 0$$

for all $n \geq 2$. Then

$$2r_{n+2} - 8r_n + 4r_{n-2} = 0$$

for all $n \geq 2$, so from equation (1),

$$a_{n+4} - 4a_{n+2} + 2a_n = 0$$

for all $n \geq 2$. Now we've eliminated the sequences r_n and s_n and can focus only on a_n.

Let $d_n = a_{n/2}$ for all even integers $n \geq 1$, so

$$d_{n+2} - 4d_{n+1} + 2d_n = 0$$

for all $n \geq 1$. Hence, the sequence (d_n) satisfies a linear recurrence, whose characteristic polynomial is $t^2 - 4t + 2$. By the quadratic formula, the roots of this quadratic are $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$. Then

$$d_n = Ax^n + By^n$$

for some constants A and B. Setting $n = 1$ and $n = 2$, we obtain the system of equations

$$xA + yB = 0,$$
$$x^2A + y^2B = 2.$$

Solving this system of equations, we find

$$A = \frac{2}{x(x-y)}$$
$$B = \frac{2}{y(y-x)}$$

which we shouldn't simplify yet since some things ought to cancel nicely. Specifically,

$$\begin{aligned} a_{2n} &= d_n \\ &= Ax^n + Bx^n \\ &= \frac{2}{x(x-y)} \cdot x^n + \frac{2}{y(y-x)} \cdot y^n \\ &= 2 \cdot \frac{x^{n-1} - y^{n-1}}{x-y}. \end{aligned}$$

Since $x - y = (2 + \sqrt{2}) - (2 - \sqrt{2}) = 2\sqrt{2}$, we get

$$a_{2n} = \frac{x^{n-1} - y^{n-1}}{\sqrt{2}}$$

for all $n \geq 1$ as we wanted.

16. A sequence (a_n) is defined by $a_0 = a_1 = 0$, $a_2 = 1$, and $a_{n+3} = a_{n+1} + 1998a_n$ for all $n \geq 0$. Prove that $a_{2n-1} = 2a_na_{n+1} + 1998a_{n-1}^2$ for every positive integer n. (Komal)

Solution: We provide two solutions to this problem.

Solution 1 We suspect that there's nothing special about 1998, so instead we will try to prove the more general statement. Let

$$a_{n+3} = a_{n+1} + ka_n$$

for all $n \geq 0$. We claim that

$$a_{2n-1} = 2a_na_{n+1} + ka_{n-1}^2$$

for all $n \geq 1$. The sequence (a_n) satisfies a linear recurrence, whose characteristic polynomial is $x^3 - x - k$. Let α, β, and γ be the roots of this cubic, so

$$a_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n$$

for some constants c_1, c_2, and c_3. Let

$$\begin{aligned} r_n &= a_{2n-1} - 2a_n a_{n+1} - ka_{n-1}^2 \\ &= c_1\alpha^{2n-1} + c_2\beta^{2n-1} + c_3\gamma^{2n-1} - 2(c_1\alpha^n + c_2\beta^n + c_3\gamma^n)(c_1\alpha^{n+1} + c_2\beta^{n+1} + c_3\gamma^{n+1}) \\ &\quad - k(c_1\alpha^{n-1} + c_2\beta^{n-1} + c_3\gamma^{n-1})^2. \end{aligned}$$

Expanding this expression, we find that

$$r_n = d_1\alpha^{2n} + d_2\beta^{2n} + d_3\gamma^{2n} + d_4\alpha^n\beta^n + d_5\alpha^n\gamma^n + d_6\beta^n\gamma^n$$

for some constants d_1, d_2, d_3, d_4, d_5, and d_6. Hence, the sequence (r_n) satisfies a linear recurrence, whose characteristic polynomial is

$$(x - \alpha^2)(x - \beta^2)(x - \gamma^2)(x - \alpha\beta)(x - \alpha\gamma)(x - \beta\gamma).$$

This is a sixth degree polynomial, so the sequence (r_n) satisfies a linear recurrence, where each term depends on the previous six terms.

We compute the first few terms of the sequence (a_n):

n	a_n
0	0
1	0
2	1
3	0
4	1
5	k
6	1
7	$2k$
8	$k^2 + 1$
9	$3k$
10	$3k^2 + 1$
11	$k^3 + 4k$

We can then compute the first few terms of the sequence (r_n):

$$\begin{aligned} r_1 &= a_1 - 2a_1a_2 - ka_0^2 = 0, \\ r_2 &= a_3 - 2a_2a_3 - ka_1^2 = 0, \\ r_3 &= a_5 - 2a_3a_4 - ka_2^2 = k - k = 0, \\ r_4 &= a_7 - 2a_4a_5 - ka_3^2 = 2k - 2k = 0, \\ r_5 &= a_9 - 2a_5a_6 - ka_4^2 = 3k - 2k - k = 0, \\ r_6 &= a_{11} - 2a_6a_7 - ka_5^2 = k^3 + 4k - 4k - k^3 = 0. \end{aligned}$$

The first six terms of the sequence (r_n) are all 0, so $r_n = 0$ for all $n \geq 1$. Hence, $a_{2n-1} = 2a_n a_{n+1} + ka_{n-1}^2$ for all $n \geq 1$.

Solution 2 Consider an infinite sequence of cities $C_0, C_1, C_2, \ldots$, where there is one road from C_i to C_{i+2}, and 1998 roads from C_i to C_{i+3}, for all $i \geq 0$. Let r_n be the number of possible paths from C_0 to C_n. Then $r_0 = 1$, $r_1 = 0$, and $r_2 = 1$.

Let $n \geq 0$. The only way to reach city C_{n+3} is to either go through city C_{n+1} or city C_n. There is one way from C_{n+1} to C_{n+3}, and 1998 ways from C_n to C_{n+3}, so

$$r_{n+3} = r_{n+2} + 1998r_n.$$

Furthermore, $r_0 = a_2 = 1$, $r_1 = a_3 = 0$, and $r_2 = a_4 = 1$. It follows that $r_n = a_{n+2}$ for all $n \geq 0$. Hence, the problem has become showing that

$$r_{2n-3} = 2r_{n-2}r_{n-1} + 1998r_{n-3}^2$$

for all $n \geq 3$. (We also must verify that $a_{2n-1} = 2a_n a_{n+1} + 1998a_{n-1}^2$ for $n = 1$ and 2, but this is easy.)

Every path from C_0 to C_{2n-3} satisfies exactly one of the following conditions:

(a) The path passes through C_{n-1}.

(b) The path passes through C_{n-2}.

(c) The path passes through neither C_{n-1} nor C_{n-2}.

In case (a), there are r_{n-1} paths from C_0 to C_{n-1}, and r_{n-2} paths from C_{n-1} to C_{2n-3}, so there are $r_{n-2}r_{n-1}$ such paths.

In case (b), there are r_{n-2} paths from C_0 to C_{n-2}, and r_{n-1} paths from C_{n-2} to C_{2n-3}, so there are again $r_{n-2}r_{n-1}$ such paths.

In case (c), the path must pass through C_{n-3}, then go to C_n. There are r_{n-3} paths from C_0 to C_{n-3}, 1998 paths from C_{n-3} to C_n, and r_{n-3} paths from C_n to C_{2n-3}, so there are $1998r_{n-3}^2$ such paths.

Hence, the total number of paths from C_0 to C_{2n-3} is equal to

$$r_{2n-3} = 2r_{n-1}r_{n-2} + 1998r_{n-3}^2.$$