1998 IMO Camp

Problems

(1) Let f be a real-valued function defined for all positive x, satisfying $f(x+y) = f(xy)$ for all positive x, y. Prove that f is the constant function.

(2) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(a+x) - f(a-x) = 4ax$$

for all real a, x.

(3) Find all integer-valued functions f defined on the integers satisfying:

(a) $f(f(n)) = n$ for all integers n;

(b) $f(f(n+2)+2) = n$ for all integers n;

(c) $f(0) = 1$.

(4) Let $f(n)$ be a function defined on the set of all positive integers and having its values in the same set. Prove that, if $f(n+1) > f(f(n))$ for all positive integers n, then $f(n) = n$ for all n.

(5) Let f be a real-valued function defined for all real numbers x such that, for some positive constant a

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - [f(x)]^2}$$

(a) Prove that f is periodic

(b) For $a=1$, give an example of a nonconstant function with the required properties.

(6) Find all $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) \leq x$ and $f(x+y) \leq f(x) + f(y)$.

(7) Determine which functions f mapping the rationals to the rationals satisfy

$$f[x+f(y)] = f(x)f(y)$$

(8) For each positive integer n, let f_n be a real-valued symmetric function of n real variables, satisfying

(a) $f_n(x_1+y, \ldots, x_n+y) = f_n(x_1, \ldots, x_n) + y$

(b) $f_n(-x_1, \ldots, -x_n) = -f_n(x_1, \ldots, x_n)$

(c) $f_{n+1}[f_n(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n), x_{n+1}]$
$= f_{n+1}(x_1, \ldots, x_{n+1})$

Prove that $f_n(x_1, \ldots, x_n) = \dfrac{x_1 + \cdots + x_n}{n}$

(9) Find a non trivial solution to the equation

$$f\left(\frac{sx}{1+x}\right) = s\,f(x)$$

where $0<s<1$. Specify carefully the domain over which f is defined.

(10) Let $f(x)$ be a real-valued function defined for all real x except $x=0$ and $x=1$, satisfying $f(x) + f[(x-1)/x] = 1+x$. Find all such f.

(11) Find all complex-valued functions f of a complex variable such that

$$f(z) + z\,f(1-z) = 1+z$$

for all z.

(12) Prove that the equation $f^n(x) = x^{-1}$ defined on the nonzero real numbers has an infinite number of solutions for each positive integer $n \geq 2$. [Here $f^1(x) = f(x)$, $f^{n+1}(x) = f(f^n(x))$, etc.]

(13) Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the equation

$$f(x) \cdot f[f(x)] = 1$$

for all real x. Assuming that $f(1000) = 999$, find the value of $f(500)$.

(14) Suppose that $f: \mathbb{R} \to \mathbb{R}$ has the following properties:

(a) $f(x) \leq 1$ for all real x;

(b) $f\left(x + \frac{13}{42}\right) + f(x) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right)$

Prove that f is periodic.

(15) Let u, f and g be functions defined for all real numbers x, such that

$$\frac{u(x+1) + u(x-1)}{2} = f(x)$$

$$\frac{u(x+4) + u(x-4)}{2} = g(x)$$

Determine $u(x)$ in terms of f and g.

(16) Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ satisfying $f[x+f(x)] = f(x)$ for all real x.

(17) Suppose $g: \mathbb{R} \to \mathbb{R}$ satisfies

$$g(x+y) + g(x-y) = 2g(x)g(y)$$

$$\lim_{x \to \infty} g(x) = 0$$

Prove that $g(x) = 0$ for all x.

(18) Find all continuous $f: \mathbb{R} \to \mathbb{R}$ such that

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$$

(19) Suppose that there exists a function $h(t)$ defined for all positive t, and a positive function $f(x,y)$ defined for all positive x and y such that

$$f(tx, ty) = h(t) \cdot f(x,y)$$

where h is continuous. Prove that $h(t) = t^k$ for some choice of k.

(20) Find all continuous $f: \mathbb{R} \to \mathbb{R}$ such that

$$f\left(\sqrt{x^2+y^2}\right) = f(x) f(y)$$

for all real x and y.

(21) Find all continuous functions f such that $f(x+y) f(x-y) = [f(x) \cdot f(y)]^2$

(22) [OGRE!] Show that there exists a function $f: \mathbb{R} \to \mathbb{R}$ which is continuous and satisfies the equation $f(x) + f(2x) + f(3x) = 0$ for all x, but f is not the constant function.