1998 IMO Camp

Suitable for I.M.O. team members who hate cooked mushrooms; guaranteed to scare the socks off any 7-year old.

4! Perverse Polynomial Problems

1) Determine the roots of the quartic $x^4 - 4x = 1$

2) If $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n > 0$, prove that any root r of the polynomial

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad \text{satisfies } |r| \leq 1$$

3) If all the roots of the equation

$$a_0 x^n - \binom{n}{1} a_1 x^{n-1} + \binom{n}{2} a_2 x^{n-2} - \cdots + (-1)^n a_n = 0$$

are positive, show that $a_r a_{n-r} \geq a_0 a_n$ for $1 \leq r \leq n-1$ with equality iff all the roots are equal

4) Let F_n be the n^{th} fibonacci number, with $F_1 = F_2 = 1$, and $P(x)$ a polynomial of degree 990 satisfying

$P(k) = F_k$ for $k = 992, 993, \ldots, 1982$. Show that $P(1983) = F_{1983} - 1$

5) Suppose that $a_1, a_2, \ldots, a_{2n}$ are distinct integers such that

$(x-a_1)(x-a_2)\cdots(x-a_{2n}) + (-1)^{n-1}(n!)^2 = 0$ has an integer

solution r. Show that $r = (a_1 + a_2 + \cdots + a_{2n})/2n$

6) Show that (In the original problem, $k=7$)

$$\cos\left(\frac{\pi}{2k+1}\right) \cdot \cos\left(\frac{2\pi}{2k+1}\right) \cdot \cos\left(\frac{3\pi}{2k+1}\right) \cdots \cos\left(\frac{k\pi}{2k+1}\right) = 2^{-k}$$

7) If $(1+x+x^2+x^3+x^4)^{496} = a_0 + a_1 x + \cdots + a_{1984} x^{1984}$

determine the g.c.d. of $a_3, a_8, a_{13}, \ldots, a_{1983}$

8) For any polynomial $P(x) = a_0 + a_1 x + \cdots + a_k x^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0, 1, \ \ \ \ $, Let $Q_i(x) = (1+x)^i$. Prove that if $i_1, i_2, \dots i_n$ are integers such that $0 \le i_1 < i_2 < \ \ < i_n$, then $w(Q_{i_1} + Q_{i_2} + \cdots + Q_{i_n}) \geq w(Q_{i_1})$

9) If $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2 R(x^5) = (x^4 + x^3 + x^2 + x + 1) S(x)$$

prove that $x - 1$ is a Factor of $P(x)$

10) If a and b are two of the roots of $x^4 + x^3 - 1 = 0$, show that ab is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$

11) Prove that the roots of

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

cannot all be real if $2a^2 < 5b$.

12) The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32 Determine the value of k.

13) $P(x)$ is a polynomial of degree $3n$ such that

$$P(0) = P(3) = \cdots = P(3n) = 2$$

$$P(1) = P(4) = \cdots = P(3n-2) = 1$$

$$P(2) = P(5) = \cdots = P(3n-1) = 0, \text{ and}$$

$P(3n+1) = 730$. Determine n.

14) A polynomial product of the form

$$(1-z)^{b_1}(1-z^2)^{b_2}(1-z^3)^{b_3}(1-z^4)^{b_4}\cdots(1-z^{32})^{b_{32}}$$

where the b_k are positive integers, has the surprising property that if we multiply it out (who's "we" !!?) and discard all the terms involving z to a power larger than 32, all that is left is $1-2z$. Determine b_{32}

15) Let $P(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$ be a polynomial in the complex variable z, with real coefficients c_k. Suppose that $|P(i)| < 1$. Prove that there exist real numbers a and b such $P(a+bi) = 0$ and $(a^2+b^2+1)^2 < 4b^2+1$.

16) $P(x)$ and $Q(x)$ are two polynomials that satisfy the identity $P(Q(x)) \equiv Q(P(x))$ for all real numbers x. If the equation $P(x) = Q(x)$ has no real solution, show that the equation $P(P(x)) = Q(Q(x))$ also has no real solution.

17) Show that not all roots of the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + x^2 + x + 1$ are real, where the a_i are real.

18) Let $p(z) = z^2 + az + b$, a quadratic polynomial of the complex variable z with real co-efficients a and b. Suppose $|p(z)| = 1$ whenever $|z| = 1$. Show that $a = b = 0$

19) Let $F(n)$ be the set of polynomials $a_0 + a_1 x + \cdots + a_n x^n$

For which $0 \le a_0 = a_n \le a_1 = a_{n-1} \le \cdots \le a_{\lfloor \frac{n}{2} \rfloor} = a_{\lfloor \frac{n+1}{2} \rfloor}$.

Show that, if $f(x) \in F(n)$ and $g(x) \in F(m)$, then

$$f(x) \cdot g(x) \in F(n+m)$$

20) Let $f(x)$ be a polynomial in x.
Define a series $g_i(x)$ such that $g_1(x) = f(x)$, $g_2(x) = f(f(x))$
and, in general, $g_k(x) = f(g_{k-1}(x))$.
Now define a series S_i such that S_k is the mean of
the roots of $g_k(x)$. If $S_{19} = 90$, determine S_{90}.

21) Let $f(x)$ be a quadratic with no real roots $f(x) = ax^2 + bx + c$.

The equation $f(f(x)) - x = 0$ has exactly two distinct roots

q_1 and q_2. If $q_1 - q_2 = t$, find, in terms of t, all possible

of c.

22) $F(x)$ is a polynomial in x with integer coefficients such that
$F(19) = 89$ and $F(198) = 9$. Show that $F(x)$ has no integer roots.

23) Let $p(x,y)$ be a polynomial in x and y such that

i) $p(x,y)$ is symmetric
ii) $x-y$ is a factor of $p(x,y)$

Show that $(x-y)^2$ is a factor of $p(x,y)$

24) Suppose that $f(t)$ is a polynomial of degree n such that

$f(k) = \frac{1}{k}$ for $k = 1, 2, \ldots, n+1$. Determine $f(n+2)$.