Primes and Sequences

CMS Winter Camp 2015

January 7, 2015

Warmups

1. Do there exist 2013 consecutive composite numbers?

2. Do there exist 2013 consecutive numbers that cannot be written as the sum of two squares?

3. Do there exist 2013 consecutive numbers that cannot be written as the sum of two cubes?

4. (IMO 1989/5) Do there exist 2013 consecutive numbers, none of which is a prime power?

5. Let $p(x)$ be an arbitrary non-constant integer polynomial. Let P be the set of primes which divide $p(n)$ for some n. Is P necessarily infinite?

6. Let $p(x)$ be an arbitrary non-constant integer polynomial. Do there exist 2013 consecutive values of p (i.e. $p(n), p(n+1), \ldots, p(n+2012)$) that are all composite?

7. Show that there exist infinitely many positive integers n such that the largest prime divisor of n^2+1 is greater than $2n$.

8. (IMO 2008/3) Show that there exist infinitely many positive integers n such that the largest prime divisor of n^2+1 is greater than $2n+\sqrt{2n}$.

Problems

1. Show that there exist infinitely many positive integers n such that the largest prime divisor of n^3+1 is greater than or equal to $2n-1$.

2. (ToT 2001) Do there exist 2013 consecutive numbers such that exactly 10 of them are prime?

3. Let $f(n) = a^n + b^n$, for $a > b \geq 1$. Let P be the set of primes which divide $f(n)$ for some n. Is P necessarily infinite?

4. (IberoAmerican 1987) Define the sequence p_n by $p_1 = 2$ and for all $n \geq 2$, p_n is the largest prime divisor of the expression $p_1 p_2 \ldots p_{n-1} + 1$. Prove that $p_n \neq 5$ for all n.

5. Let $p(n)$ be an integer polynomial such that $p(n) > 1$ for all $n > 0$, and for each natural number n, let $f(n)$ be the largest prime divisor of $p(n)$. Do there exist infinitely many n such that $f(n+1) > f(n)$?

6. (ISL 2011 N2) Consider a polynomial $P(x) = \prod_{j=1}^{9}(x+d_j)$ where $d_1, d_2, \ldots, d_9$ are nine distinct integers. Prove that there exists an integer N such that for all integers $x \geq N$, the number $P(x)$ is divisible by a prime number greater than 20.

7. Let $f(n)$ be the largest prime divisor of n, and let a_n be a strictly increasing sequence of positive integers. Prove that the set $\{f(a_i+a_j) | i \neq j\}$ is unbounded.

8. (ARO 2011 10.7) For positive integers $a > b > 1$, let

$$x_n = \frac{a^n - 1}{b^n - 1}$$

Find the least d such that for any a, b, the sequence x_n does not contain d consecutive numbers that are prime numbers.

9. (ARO 2009 10.8) Let x, y be two integers with $2 \leq x, y \leq 100$. Prove that $x^{2^n} + y^{2^n}$ is not a prime for some positive integer n.

10. (ARO 2012 11.4) For a positive integer n, define $S_n = 1! + 2! + \cdots + n!$. Prove that there exists an integer n such that S_n has a prime divisor greater than 10^{2012}.

11. (ARO 2011 11.7) Let $P(a)$ be the largest prime divisor of a^2+1. Prove that there exist infinitely many positive integers a, b, and c such that $P(a) = P(b) = P(c)$.

12. (Brazil 1995) For a positive integer $n > 1$, let $P(n)$ denote the largest prime divisor of n. Prove that there exist infinitely may positive integers n for which

$$P(n) < P(n+1) < P(n+2)$$

13. (Iran 2004) Let $a_1, a_2, \ldots, a_n$ be positive integers, not all equal. Prove that there are infinitely many primes p such that for some k, $p | a_1^k + a_2^k + \cdots + a_n^k$.

14. (USAMO 2006/3) For an integer m, let $p(m)$ be the largest prime divisor of m (by convention, set $p(\pm 1) = 1$ and $p(0) = \infty$). Find all polynomials f with integer coefficients such that the sequence

$$\{p(f(n^2)) - 2n\}_{n \geq 0}$$

is bounded above. (In particular, this requires $f(n^2) \neq 0$ for $n \geq 0$).

15. (ISL 2005 N7) Let $P(x)$ be an integer polynomial of degree at least 2. Prove that there exists a positive integer m such that $P(m!)$ is composite.

16. (Stormer's Theorem[1]). Let S_k be the set of integers whose prime factors are all less than k. Then S_k contains finitely many pairs of consecutive integers.

17. (Zgimondy's Theorem[2]). Choose relatively prime integers $a > b > 0$ such that $a + b$ is not a power of two and $(a, b) \neq (2, 1)$. Then for all $n > 1$, there is a prime p which divides $a^n - b^n$ but not $a^k - b^k$ for any $k < n$.

[1]Surprisingly, this has applications in music theory.
[2]This is quite hard to prove, but can sometimes be a useful sledgehammer for solving problems like these.