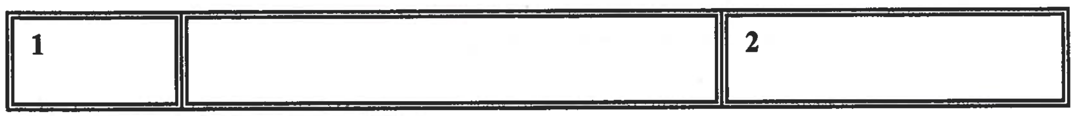
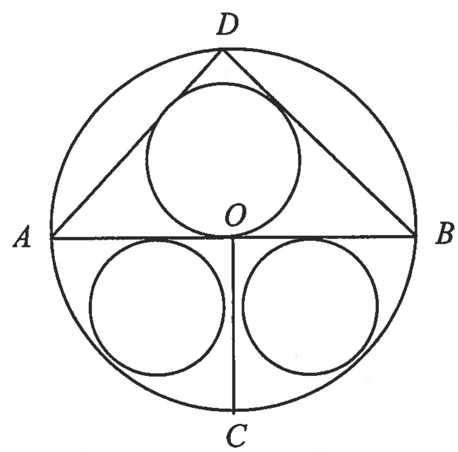
2003 Winter Camp

Warm-up Problems (Donette Jan 4)

1. If you place two marks on a six-inch stick, one just 1 inch from one end and the other two inches from the other end, then you can use this stick to measure any whole number of inches from 1 to 6. See the figure below. Place three marks on a nine-inch stick so that you can measure any whole number of inches from 1 to 9.



2. In the accompanying figure, point O is the center of the circle, and the lines AB and OC are perpendicular at O. Point D is diametrically opposite point C. The three smaller circles are inscribed in their respective regions; that is, the top circle is the incircle for the triangle and the lower two circles are tangent to the radii and to the large circle in their respective circular sections. Sow that all three smaller circles have the same radius.



3. Three lunch boxes are labeled HH, HC, and CC. One of then contains two ham sandwiches, another one ham sandwich and one cheese sandwich, and the last one two cheese sandwiches. You must straighten out the labels; that is, you must correctly identify the contents of each lunch box, but you must do it as cheaply as possible. You may open a lunch box and look at one sandwich in that box for a fee of $1. You would pay another dollar to look at the other sandwich in that box. You are able to see the labels and you know that they are all wrong. Of course, you can correctly identify the contents of the boxes for $6 if you look at every sandwich. How cheaply can you identify them all?

4. [Germany 1997] Three sides of a regular tetrahedron are coloured white, and the other one black. In the beginning, the tetrahedron lies on its black side. It is then tilted several times over its edges. After a while it takes the original position again. Can it now lie on one of its white sides?

5. [Germany 1997] An arbitrary radius divides a semicircle with diameter AB into two sectors. To each of the sectors we inscribe a smaller circle and denote the points of the circles with the line AB by S and T respectively. Prove that $\overline{ST} \geq 2r\left(\sqrt{2}-1\right)$.

6. [Germany 1997] Prove for any natural number n:
   If both $3n+1$ and $4n+1$ are squares then n is divisible by 56.

7. [Germany 1998] Prove the existence of an infinite sequence of square numbers with the following properties:
   (1) The arithmetic mean of any two neighbour elements is a square.
   (2) No two neighbour elements have a common divisor greater than 1.
   (3) The sequence is strictly monotonic increasing.

8. [Germany 1998] Prove that $n+\left\lfloor\left(\sqrt{2}+1\right)^n\right\rfloor$ is odd for any natural number n.

9. [Lithuania 1997] Find all a such that the equation $x^8+ax^4+1=0$ has four roots that form an arithmetical progression.

10. [Lithuania 1997] Is there at least one positive integer n, such that the last 1997 digits of the number 1997^n-1 are zeros?

11. [Lithuanian 1998] A function f: $N \rightarrow N$ satisfies the following conditions:
    (1) $f(xy)=f(x)+f(y)-1$ for all $x, y \in N$;
    (2) there exists only a finite number of x such that $f(x)=1$;
    (3) $f(30)=4$.
    Determine $f(14400)$.

12. What is the smallest value of the expression
    $(1-x_1)^2+(x_1-x_2)^2+(x_2-x_3)^2+\cdots+(x_{2003}-x_{2004})^2+x_{2004}^2,$
    where the $x_1, x_2, \ldots, x_{2004}$ are all real numbers?

13. [Lithuania 2000] Find all functions $f(x)$ which are defined for $x>0$ and satisfy the condition
    $$f\left(x^{2000}\right)=5f\left(x^{-2000}\right)+\sin x$$

14. In a basket ball tournament every team plays twice against each other team. A single team wins the tournament with 26 points and exactly two teams are in the last position with 20 points. How many teams participated in the tournament? (Recall that winning a game gives a team 2 points, where losing a game gives 0 points, and no game can end in a draw.)

15. A square is filled with 100 lamps, arranged in 10 rows and 10 columns. Some of them are on, and others are out. Each lamp has a push-button that, when pressed, switches all ~~maps~~ lamps of its row and its column (including the lamp itself).

    (1) Determine the states from which it is possible to light all lamps.
    (2) What is the answer if the square has 81 laps in 9 rows and 9 columns?

16. Let $P(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r)=0$. Show that the n numbers
$c_nr, c_nr^2+c_{n-1}r, c_nr^3+c_{n-1}r^2+c_{n-2}r, \ldots, c_nr^n+c_{n-1}r^{n-1}+\cdots c_1r$ are integers.