Mock Olympiad #3 Solutions

July 6, 2009

1. *(IMO Short list 2008, C2)*
   Let X_n be the number of permutations of $\{1, 2, \ldots, n\}$ that have the given property, and let Y_n be the number of permutations of $\{2, 3, \ldots, n+1\}$ that have the given property. Note that $2(a_1 + a_2 + \ldots + a_k) \equiv 2\Big((a_1 - 1) + (a_2 - 1) + \ldots + (a_k - 1)\Big)(\text{mod } k)$, so we can always subtract one from each $a_1, a_2, \ldots, a_n$ without changing the problem. In particular, this means that $X_n = Y_n$.

   It is easy to check that $X_1 = 1, X_2 = 2$, and $X_3 = 6$. From now on, we will assume $n \geq 4$. We have $2(a_1 + a_2 + \ldots + a_n) = 2(1 + 2 + l \cdots + n) = n(n+1) \equiv 0(\text{mod } n)$, so a permutation need only satisfy the given condition for $k = 1, 2, \ldots, n-1$. Examining the condition for $n-1$, we see that
   $$(n-1)|2(a_1 + a_2 + \cdots + a_n - a_n),$$
   so
   $$(n-1)|n(n+1) - 2a_n.$$
   This says that $2a_n = 2(\text{mod } n-1)$, so either $a_n = 1, n$ or n is odd and $a_n = \frac{n+1}{2}$. Suppose we're in the latter case. Then examining the condition for $n-2$, we get
   $$(n-2)|2(a_1 + a_2 + \cdots + a_n - a_n - a_{n-1}),$$
   so
   $$(n-2)|n^2 - 1 - 2a_{n-1}.$$
   But this implies $2a_{n-1} = 3(\text{mod } n-2)$, and since $n-2$ is odd, this has the unique solution $a_{n-1} = \frac{n+1}{2}$, which contradicts $a_{n-1} \neq a_n$. So we must have $a_n = 1$ or $a_n = n$. This means that $X_n = X_{n-1} + Y_{n-1} = 2X_{n-1}$. For $n \geq 3$, we therefore have $X_n = 3 \cdot 2^{n-2}$.

2. *(Mongolia 2008 TST, #2.1)*
   Claim: For any positive real numbers a, b, c, d satisfying $d^2 = a^2 + b^2 + c^2$, we have $f(a) + f(b) + f(c) = f(d)$.

   Proof: Set $z = \frac{d-a}{2}, x = \frac{b^2}{4z}, y = \frac{c^2}{4z}$. These values are all positive since $d > a$. Furthermore,

$$x+y-z=\frac{b^2+c^2}{4z}-z=\frac{(d^2-a^2)-(d^2+a^2-2ad)}{2(d-a)}=a,$$

$$2\sqrt{xz}=2\sqrt{z\cdot\frac{b^2}{4z}}=b,\qquad 2\sqrt{yz}=2\sqrt{z\cdot\frac{c^2}{4z}}=c,\qquad\text{and}$$

$$x+y+z=\frac{b^2+c^2}{4z}+z=\frac{(d^2-a^2)+(d^2+a^2-2ad)}{2(d-a)}=d.$$

Since $x+y-z=a>0$, we can substitute x,y,z into the given equation to get $f(a)+f(b)+f(c)=f(d)$, which completes the proof of the claim. $\square$

Now let $g(x)=f(\sqrt{x})$. We have proven that $g(a)+g(b)+g(c)=g(a+b+c)$ for all $a,b,c>0$. Taking $a=b=c=x$, we have $g(3x)=3g(x)$. Taking $a=b=3x$ and $c=x$, we then have $g(7x)=7g(x)$. Taking $a=b=2x$ and $c=3x$, we then have $g(2x)=2g(x)$. Now, taking $a=b=x$ and $c=(n-2)x$, we have $g(nx)=2g(x)+g((n-2)x)$, so it follows that $g(nx)=ng(x)$ for all positive integers n. For any positive integers p,q, we therefore have $g(\frac{p}{q})=\frac{1}{q}\cdot g(p)=\frac{p}{q}\cdot g(1)$ (*).

Also, if $a<d$, then $g(d)-g(a)=g(b)+g(c)>0$, so g is increasing. Since the rationals are dense in the reals, it now follows from $(*)$ that $g(x)=Cx$ for some constant $C>0$, and hence $f(x)=Cx^2$. Conversely, if $f(x)=Cx^2$, then $f(x+y-z)+f(2\sqrt{xz})+f(2\sqrt{yz})=(x+y-z)^2+(2\sqrt{xz})^2+(2\sqrt{yz})^2=(x+y+z)^2=f(x+y+z)$.

Remark: It isn't necessary to calculate x,y,z explicitly to prove the claim. One may also proceed as follows: First fix the products xz and yz, and let z vary. Then if z is very small, $x+y-z$ will approach infinity, and if z is very big, $x+y-z$ will approach minus infinity, so by continuity of $x+y-z$ as a function of z, $x+y-z$ will take every possible value. If you find this sort of argument confusing and want to understand it better, feel free to speak to one of the trainers!

3. *(IMO Short list 2002, N4*

If $p=5$, then $a=2$ satisfies the given condition. From this point forward, we will assume $p\geq 7$.

Claim 1: If $x\in[1,p-1]$, then $x^{p-1}\not\equiv(p-x)^{p-1}(\text{mod }p^2)$.

Proof: By the binomial theorem,

$$(p-x)^{p-1}\equiv x^{p-1}-p(p-1)\cdot x^{p-2}\not\equiv x^{p-1}(\text{mod }p^2)$$

since $p\not| x,p-1$. $\square$

Claim 2: If $x\in[1,p-1]$, then $x^{p-1}\not\equiv(x+2p)^{p-1}(\text{mod }p^2)$.

Proof: Again, the binomial theorem gives

$$(x+2p)^{p-1}\equiv x^{p-1}+2p(p-1)\cdot x^{p-2}\not\equiv x^{p-1}(\text{mod }p^2)$$

since $p\not| x,p-1$. $\square$

Let us call a number x "good" if $x^{p-1}\equiv 1(\text{mod }p^2)$. By pairing up x and $p-x$ for all $x\in\left[1,\frac{p-1}{2}\right]$, Condition 1 implies that at most $\frac{p-1}{2}$ numbers in $[1,p-1]$ are good. Furthermore, we already know that 1 is good. Now, if two consecutive numbers in $[1, p]$ are not good, then the problem is done. Otherwise, the good numbers in $[1, p]$ must be exactly $1, 3, 5, \ldots, p-2$. Now, $(3p-6)^{p-1} \equiv 3^{p-1} \cdot (p-2)^{p-1} \equiv 1 (\text{mod } p^2)$, so $3p-6$ is good. However, we already know $p-6$ is good, so this contradicts Claim 2. The proof is now complete.