Number Theory Problems From China Olympiad

Masum Billal

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Chinese Olympiads are divided into some sectors-China Mathematical Competition, China Mathematical Olympiad, China Girls Mathematical Olympiad, China Western Mathematical Olympiad. The first part consists of the problems, and the latter contains the solutions.

Contents

Chapter 1

Problems

1.1. *China Mathematical Competition*

1. Let p be an odd prime. If k is a positive integer s.t. [1] $\sqrt{k^2 - pk}$ is a positive integer, find the value of k.

2. A natural number is called *lucky number* if the sum of its digits is 7. Arranging all lucky numbers we get the sequence

$$a_1 < a_2 < ...$$

If $a_n = 2005$, then what is the value of a_{5n}?

3. Given a sequence of numbers $a_0, a_1, ...$ satisfying

$$a_0 = 1, a_{n+1} = \frac{7a_n + \sqrt{45a_n^2 - 36}}{2}$$

Prove the following:

- $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$.
- $a_n a_{n+1} - 1$ is a perfect square.

4. Before The World Cup tournament, the football coach of F country will let seven players, $A_l, A_2, ..., A_7$ join three training matches (90 minutes each) in order to assess them. Suppose, at any moment during a match, one and only one of them enters the field, and the total time (which is measured in minutes) on the field for each one of A_1, A_2, A_3 and A_4 is divisible by 7 and the total time for each of A_5, A_6 and A_7 is divisible by 13. If there is no restriction about the number of times of substitution of players during each match, then how many possible cases are there within the total time for every player on the field?

5. Let the three sides of a triangle be integers l, m, n respectively, satisfying $l > m > n$ and

$$\left\{\frac{3^l}{10000}\right\} = \left\{\frac{3^m}{10000}\right\} = \left\{\frac{3^n}{10000}\right\}$$

where $\{x\} = x - [x]$ and $[x]$ is the integral part of x. Find the minimum perimeter of the triangle.

[1] s.t. is the short form of *such that*.

6. For $n \geq 4$, let $f(n)$ be the minimum integer s.t. for any positive integer m, in any subset with $f(n)$ elements of the set

$$\{m, m+1, ..., m+n-1\}$$

there are at least 3 mutual co-prime elements.

7. For each positive integer, define a function

$$f(n) = \begin{cases} 0 \text{ if n is a perfect-square} \\ \left[\frac{1}{\sqrt{n}}\right] \text{ otherwise} \end{cases}$$

Find the value of

$$\sum_{k=1}^{200} f(k)$$

1.2. $\boxed{\textit{China Mathematical Olympiad}}$

8 (2002)**.** Suppose a set S satisfies the following conditions:

1. Every element in S is a positive integer and not greater than 100
2. For any two different elements a and b in S, there is an element c in S such that the greatest common divisor of a and c is equal to 1, and the greatest common divisor of b and c is also 1
3. For any two different elements a and b in S, there is an element d, which is different from a and b, such that the greatest common divisor of a and d, and that of b and d are greater than 1

Find the maximum number of elements.

Proposed by-Yao Jiangang

9. Find all triples of positive integers (a, m, n) s.t. $a, m \geq 2$ and

$$a^m + 1 | a^n + 203$$

Proposed by-Chen Yonggao

10. Let c be a positive integer and $x_1, x_2, ...$ satisfying

$$x_n = x_{n-1} + \left[\frac{2x_{n-1} - (n+2)}{n}\right] + 1$$

for $n = 2, 3, ...$ where $[x]$ denotes the largest integer less than or equal to x. Determine x_n in terms of n and c.

Proposed by-Huang Yumin

11. Prove that, for every integer n, except for some finite, can be represented as a sum of 2004 positive integers so that,

$$n = \sum_{i=1}^{2004} a_i; 1 \leq a_i < a_{i+1}; a_i | a_{i+1}$$

Proposed by-Chen Yonggao

12. Find all non-negative integer solutions to the following equation:

$$2^x 3^y - 5^z 7^w = 1$$

Proposed by-Chen Yonggao

13. Suppose, positive integers $a_1, a_2, ..., a_{2006}$ satisfy: any two of

$$\frac{a_1}{a_2}, \frac{a_2}{a_3}, ..., \frac{a_{2005}}{a_{2006}}$$

are unequal. Find the least number of distinct positive integers in $\{a_1, a_2, \ldots, a_{2006}\}$.

Proposed by-Chen Yonggao

14. k, m, n are positive integers s.t.

$$k^2 + k + 3 = mn$$

Prove that, at least one of the following Diophantine equations has a solution with x, y odd:

$$x^2 + 11y^2 = 4m$$

$$x^2 + 11y^2 = 4n$$

Proposed by-Li Weigu

1.3. China Girls Mathematical Olympiad

15. Find all positive integers n such that[2]

$$20n + 2 | 2003n + 2002$$

Proposed by-Wu Weichao

16. Find all positive integers k such that , for any positive numbers a, b, c satisfying,

$$k(ab + bc + ca) > 5(a^2 + b^2 + c^2)$$

there must exist a triangle with a, b, c as the lengths of three sides.

Proposed by-Qian Zhangwang

[2] $a|b$ means a divides b

17. Find all pairs of positive integers x, y with

$$x^y = y^{x-y}$$

Proposed by-Pan Chengbiao

18. Let n be any positive integer and S_n be the set of its divisors of n. Prove that, at most half of the elements in S_n have 3 as their last digit.

Proposed by-Zuming Feng

19. We say a positive integer n is good if there exists a permutation $\{a_1, a_2, ..., a_n\}$ of $\{1, 2, ..., n\}$ s.t. $a_k + k$ is a perfect square for all $1 \le k \le n$. Determine all good numbers in the set $\{11, 13, 15, 17, 19\}$.

Proposed by-Su Chun

20. A deck of 32 cards has 2 different jokers each of which is numbered 0. There are 10 red cards numbered 1 through 10 and similarly for blue and green cards. One chooses a number of cards from the deck. If a card in hand is numbered k, then the value of the card is 2^k , and the value of the hand is the sum of the values of cards in hand. Determine the number of hands having the value 2004.

Proposed by-Tao Pingsheng

21. Let p and q be two co-prime positive integers, and let n be a non-negative integer. Determine the number of integers that can be written in the form $ip + jq$, where i and j are non-negative integer with $i + j \le n$.

Proposed by-Li Weigu

22. An integer n is called *good* if $n \ge 3$ there are n lattice points[3] in a $2d$ plane s.t.

- If line segment P_iP_j has a rational length, then there is a P_k s.t. both P_iP_k and P_jP_k have an irrational length.
- If line segment P_iP_j has an irrational length, then there is a P_k s.t. both P_iP_k and P_jP_k have a rational length.

Determine:

1. the minimum good number.
2. if 2005 a good number.

Proposed by-Zuming Feng

23. Let $m, n \in \mathbb{N}$ s.t. $m > n \ge 2$. $S = \{1, 2, ..., m\}$ and $T = \{a_1, a_2, ..., a_n\}$ is a subset of S s.t. every number in S is not divisible by any two distinct numbers in T. Prove that,

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} < \frac{m+n}{n}$$

Proposed by-Zhang Tongjun

[3] A point $P(x, y)$ is called a *lattice point* if x, y are both integers.

1.4. China Western Olympiad

24. Find all positive integers n such that

$$n^4 - 4n^3 + 22n^2 - 36n + 18$$

is a perfect square.

Proposed by-Pan Chengbiao

25. Consider a square on the complex plane. The complex numbers corresponding to its four vertices are the four roots of some equation of the fourth degree with one unknown and integer coefficients

$$x^4 + px^3 + qx^2 + rx + s = 0$$

Find the minimum value of such a square.

Proposed by-Xiong Bin

26. Assume that α, β are the roots of the equation

$$x^2 - x - 1 = 0$$

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

1. Prove that $a_{n+2} = a_{n+1} + a_n$.
2. Find all positive integers $a < b$ s.t.

$$b | a_n - 2na^n$$

for all n.

Proposed by-Li Shenghong

27. Let n be a given positive integer. Find the least positive integer u_n , such that for any positive integer d, the number of integers divisible by d in every u_n consecutive positive odd numbers is not less than the number of integers divisible by d in

$$\{1, 3, ..., 2n - 1\}$$

Proposed by-Chen Yonggao

28 (2003, 5)**.** Let k be a given positive integer and $\{a_n\}$ a number sequence with $a_0 = 0$ and

$$a_{n+1} = ka_n + \sqrt{(k^2 - 1)a_n^2 + 1}$$

Prove that, every term of the sequence is an integer and $2k | a_n$ for all n.

Proposed by-Zhang Zhenjie

29 (2004, 1)**.** Find all integers n s.t.

$$n^4 + 6n^3 + 11n^2 + 2n + 31$$

is a perfect square.

Proposed by-Xu Wanyi

30 (2004, 4)**.** Let $d(n)$ and $\varphi(n)$ denote the number of divisors of n and the number of positive integers less than or equal to n which are co-prime to n. Find all c s.t. there exists a n with

$$d(n) + \varphi(n) = n + c$$

Proposed by-Feng Zhigang

31 (2004, 5)**.** The sequence $\{a_n\}$ satisfies

$$a_1 = a_2 = 1$$

and

$$a_{n+2} = \frac{1}{a_{n+1}} + a_n$$

Find a_{2004}.

Proposed by-Wu Weichao

32 (2005, 1)**.** Assume that

$$\alpha^{2005} + \beta^{2005}$$

can be expressed as a polynomial in α, β. Find the sum of the coefficients of the polynomial.

Proposed by-Zhu Huawei

33 (2005, 3)**.** Let $S = \{1, 2, ..., 2005\}$. If there is at least one prime number in any subset of S consisting of n co-prime numbers, find the minimum value of n.

Proposed by-Tang Lihua

Chapter 2

Solutions

2.1. $\boxed{\textit{China Mathematical Competition}}$

Solution. Assume that,

$$k^2 - kp = k(k-p) = n^2$$

From Euclidean algorithm, we have $\gcd(k, k-p) = \gcd(k,p) = 1$ or $\gcd(k, k-p) = p$.

Case 1($\gcd(k, k-p) = 1$)*:* Let $k = pl$. Then

$$p^2 l(l-1) = n^2$$

So, $p|n$,[1] say $n = pm$. We get,

$$l(l-1) = m^2$$

Since $\gcd(l, l-1) = \gcd(l, 1) = 1$, both of them must be squares. If $l = x^2, l-1 = y^2$ then $x^2 - y^2 = 1 \Rightarrow (x+y)(x-y) = 1$. Therefore, $x+y = 1, x-y = 1$ forcing $x = 1, y = 0, l = 1$. We have the solution $k = p$, which doesn't give us a positive integer.

Case 2($\gcd(k, k-p) = 1$)*:* We need $k = x^2, k-p = y^2$ which yields $x^2 - y^2 = p \Rightarrow (x+y)(x-y) = p$. Since $x+y > x-y$, $x+y = p, x-y = 1$. Hence, $x = \dfrac{p+1}{2}, k = \left(\dfrac{p+1}{2}\right)^2$.

Solution. We know that, the sum of k non-negative integers can be n in $\binom{n+k-1}{k-1}$ ways. If n has k digits, and

$$n = \overline{d_k d_{k-1} \dots d_1},\ \text{then } d_1 + \dots + d_k = 7$$

where $d_k \geq 1$, so the number of solutions to this equation is $F_k = \binom{k+5}{6}$. Now, it's obvious that, 2005 is the minimum lucky number starting with 2. It is easy to check that, 2005 is the 65-th lucky number. The problem reduces to finding a_{325}. $F_4 = \binom{9}{3} = 84, F_5 = \binom{10}{6} = 210$. Also, since

$$F_1 + F_2 + F_3 + F_4 + F_5 = 330$$

[1] $a|b$ means a divides b.

we just need to find the last 5 or 6 numbers with 5 digits. Checking by hand, they are

$$70000, 61000, 60100, 60010, 60001, 52000$$

The answer is $\boxed{52000}$.

2.2. $\boxed{\textit{China Mathematical Olympiad}}$

2.3. $\boxed{\textit{China Girls Mathematical Olympiad}}$

2.4. $\boxed{\textit{China Western Mathematical Olympiad}}$

References

1. *Mathematical Olympiad In China-Problems And Solutions,* by Xiong Bin, Lee Peng Yee