IMO Winter Training Camp 1999
Combinatorics – Generating Functions

Some Basic Notation

Combinatorics deals with the problem of counting the number of elements in a finite set. For example: how many functions $f\colon \{1,2,3\} \to \{2,4,5,6\}$ are strictly increasing? We will use some notation from set-theory. If S is a finite set, we will write $|S|$ for the number of elements in that set. For two sets A and B, we write $A \cap B$ for their intersection and $A \cup B$ for their union. We write $[n] = \{1,2,3,\cdots,n\}$. If we have a whole family of sets $A_1, A_2, A_3, A_4, \cdots, A_n$, we will also write $A_i, i \in [n]$. Then we use

$$\bigcup_{i\in[n]} A_i$$

for the union and

$$\bigcap_{i\in[n]} A_i$$

for the intersection. We also write $A \times B = \{(a,b); a \in A, b \in B\}$ for the Cartesian product. Here are some basic rules for counting the number of elements in such sets:

Rule A If $A \cap B = \emptyset$ then $|A \cup B| = |A| + |B|$.

Could you extend this rule for three or four sets?

Rule B 1. $|A \times B| = |A| \cdot |B|$
2. $|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$

Rule C The number of subsets of $[n]$ with k distinct elements is $\binom{n}{k}$.

Exercise 1 Show that the number of subsets of $[2n]$ having the same number of even elements as odd elements is $\binom{2n}{n}$

Exercise 2 Find the number of k-tuples $(S_1, S_2, \cdots, S_k)$ satisfying

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq [n]$$

Exercise 3 How many k-element subsets of $[n]$ contain no two consecutive integers?

0-1 Strings

Definition A **0-1 string** of length n is an n-tuple $(a_1, a_2, \cdots, a_n)$ in which each element is either 0 or 1.

The number of 0-1 strings of length n is obviously 2^n. In these notes you will find some techniques to count the number of 0-1 strings of a particular kind. 0-1 Strings are often used to count the number of elements in a different kind of set by constructing a bijection (a coding) between that set and a particular set of 0-1 strings.

Exercise 4 [Ballot Problem] Show that there are $\frac{1}{n+1}\binom{2n}{n}$ 0-1 strings consisting of n ones and n zeros with the property that, when read from left to right the number of ones never lags behind the number of zeros. (Such a sequence is called a **ballot sequence.**)

The number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is called the n-th **Catalan number**. The first six Catalan numbers are $C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42$. They show up in many important combinatorial problems. Two examples follow in the next exercises. The main idea behind solving these exercises is to construct a bijection with a set of ballot sequences.

Exercise 5 Find the number of non decreasing functions $f: [n] \to [n]$ that satisfy $f(k) \leq k$ for $k = 1, 2, \cdots, n$.

Exercise 6 Show that the number of permutations π of $[n]$ with no triple $i < j < k$ such that $\pi(i) > \pi(k) > \pi(j)$ is C_n, the nth Catalan number.

Recurrence Relations

If you have a problem where you want to count the number of elements of each set A_i in a family of finite sets $A_1, A_2, A_3, \cdots$, it is often useful to try to find an expression of the form $|A_n| = f(|A_{n-1}|, \cdots, |A_{n-p}|)$. Look for example at the following problem: A mail carrier has to deliver the mail in a street with n houses. He has a letter for each house and decided to make a practical joke: he wants to deliver each letter to a wrong mail box. In how many ways can he do that? Let D_n be the number of ways in which he can do this. So $D_1 = 0$ (if there is only one house, there is no way in which he can put the letter in the wrong mailbox) and $D_2 = 1$ (he can interchange the letters in precisely one way). In order to find the general solution, assume that he puts the last letter, letter n, into mailbox i. Now there are two cases: (i) he can put letter i into mailbox n or (ii) he puts letter i into some other mailbox. In the first case we have reduced the problem to the problem of delivering $n-2$ letters into $n-2$ wrong mailboxes. He can do that in D_{n-2} ways. There were $n-1$ possible choices for mailbox i, so this gives us in total $(n-1)D_{n-2}$ ways of delivering the letters. In case (ii) there is a letter j with $j \neq i$ which he delivers to mailbox n. In this case we can reduce the problem to the problem of delivering the letters in a street with $n-1$ houses by assuming that the mail carrier would have delivered letter j to house i in that case. We still had $n-1$ choices for mailbox i (to deliver letter n), so in this case there are $(n-1)D_{n-1}$ ways to deliver the letters. In conclusion we find that:

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

This is an example of a **recurrence relation**. In general, a recurrence relation is a sequence (a_n) of the form

$$a_n = f(a_{n-1}, a_{n-2}, \cdots, a_{n-p})$$

for some constant p. For such a recurrence relation we need the **initial values** $a_1, \cdots, a_p$ to determine all the values of the a_n. The recurrence relation is called linear when the function f is linear:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_p a_{n-p}$$

where the $c_1, \cdots, c_p$ are constants. If the recurrence relation is linear, there is often a quick way to get the complete solution for all the a_n in a closed form (i.e. as a function of n). To get the idea, let us first consider the special case where $a_n = c_1 a_{n-1}$. If $a_1 = 1$, we find that $a_n = c^{n-1}$. So let us try the solution $a_n = \lambda^n$ for a general linear recurrence relation. Then λ would have to satisfy

$$\lambda^n = c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_p\lambda^{n-p}$$

so (since $\lambda \neq 0$):

$$\lambda^p - c_1\lambda^{p-1} - c_2\lambda^{p-2} - \cdots - c_p = 0$$

This is called the **characteristic equation** of the recurrence relation. If the roots $\lambda_1, \cdots, \lambda_p$ of this equation are all real and distinct, there exist numbers $r_1, r_2, \cdots, r_p$ such that

$$a_n = \sum_{k=1}^{p} r_k \lambda_k^n$$

satisfies the recurrence relation and the initial values.

Example The Fibonacci numbers are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with initial values $F_1 = 1$ and $F_2 = 1$. Show that:

$$F_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right]$$

Exercise 7 [Terquem's Problem] Find the number of subsets of $[n]$ having the property that, when the elements are put in decreasing order, the smallest one is odd, the next smallest one is even, and so on. (Sets with this property are called **alternating**. By convention, the empty set is alternating.)

If the characteristic equation doesn't have n distinct real roots or the recurrence is not linear, use the method of the next section.

Generating Functions

Let $a_0, a_1, a_2, \cdots$ be the sequence of numbers you want to calculate. We call the function

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

the **generating function** of the sequence. This kind of function is called a formal power series. We use the notation $[x^n]G(x)$ to indicate the coefficient of x^n in the power series $G(x)$. So for the power series above we have:

$$a_n = [x^n]G(x)$$

The most important power series for our purposes is the **geometric series:**

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

From this power series we can derive other power series which can also be written as rational function in x. For example:

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

$$\sum_{n=0}^{\infty} a^n x^n = \frac{1}{1-ax}$$

for any constant a. The fact that this function can also be written as rational functions will be essential to find the coefficients of the generating functions we introduced above. We will illustrate this by calculating the closed form for the Fibonacci series: If

$$G(x) = \sum_{n=1}^{\infty} F_n x^n$$

then

$$G(x) = x + x^2 + \sum_{n=3}^{\infty} F_n x^n$$

$$\begin{aligned}
&= x + x^2 + \sum_{n=3}^{\infty}(F_{n-1} + F_{n-2})x^n \\
&= x + x^2 + \sum_{n=3}^{\infty} F_{n-1}x^n + \sum_{n=3}^{\infty} F_{n-2}x^n \\
&= x + x^2 + x\sum_{n=3}^{\infty} F_{n-1}x^{n-1} + x^2\sum_{n=3}^{\infty} F_{n-2}x^{n-2} \\
&= x + x^2 + x(G(x) - x) + x^2G(x)
\end{aligned}$$

So

$$G(x) = \frac{x}{1 - x - x^2}$$

Note that $1 - x - x^2 = (1 - \frac{1+\sqrt{5}}{2}x)(1 - \frac{1-\sqrt{5}}{2}x)$. Use partial fractions to write

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left[\frac{1}{1-ax} - \frac{1}{1-bx}\right]$$

where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. Use a variation of the geometric series to write

$$G(x) = \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}(a^n - b^n)x^n = \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}(a^n - b^n)x^n$$

So

$$F_n = [x^n]G(x) = \frac{1}{\sqrt{5}}(a^n - b^n)$$

Exercise 8 Let a_n be the number of words of length n consisting of symbols 0 and 1 such that neither 101 not 111 occur as 3-digit blocks. Express a_n in terms of Fibonacci numbers.

Exercise 9 For each $n \geq 1$, find the sum of the products $F_{k_1}F_{k_2}F_{k_3}\cdots F_{k_r}$, where the sum is over all 2^{n-1} compositions $n = k_1+k_2+\cdots+k_r$. (For example, for $n = 3$ the desired sum is $F_3 + F_1F_2 + F_2F_1 + F_1F_1F_1 = 5$.)

Partitions and Permutations

Two concepts which occur frequently in combinatorial problems are partitions and permutations.

Definition A **partition** of the set $[n]$ is a division of $[n]$ into disjoint subsets: $[n] = S_1 \cup S_2 \cup \cdots \cup S_k$ and $S_i \cap S_j = \emptyset$ when $i \neq j$. This is also called a partition into k blocks.

The number of partitions of $[n]$ into k nonempty blocks is the **Stirling number of the second kind** $\left\{ {n \atop k} \right\}$. They satisfy the following recurrence relation:

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

You can see this as follows: if you have a partition of $[n]$ into k blocks the element n can be in a block by itself share a block with at least one other number. If it is by itself the remaining blocks form a partition of $[n-1]$ into $k-1$ blocks. These partitions are being counted by the first term on the right hand side. If n shares a block the partition without the element n forms a partition of $[n-1]$ into k blocks. For any given such partition there are k ways of making it into a partition of $[n]$ by adding n to one of the blocks. These partitions are represented by the second term of the sum in the right hand side of the equation. We also define that:

$$\left\{ {0 \atop 0} \right\} = 1, \quad \left\{ {0 \atop k} \right\} = 0 \text{ for } k \geq 1.$$

Note that this recurrence relation has two variables which are changing, both n and k. We solve this by creating a family of generating functions, indexed by k:

$$G_k(x) = \sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} x^n$$

I.e.

$$\left\{ {n \atop k} \right\} = [x^n] G_k(x)$$

These G_k satisfy the following recurrence relation:

$$G_0(x) = 1 \text{ and } G_k(x) = G_{k-1}(x) + kxG_k(x)$$

So

$$G_k(x) = \frac{x G_{k-1}(x)}{1-kx}$$

So you can prove by induction that

$$G_k(x) = \frac{\cancel{1}\, x^k}{(1-x)(1-2x)\cdots(1-kx)}$$

From this expression you can find the Stirling numbers of the second kind by applying partial fractions and induction. (This is **Exercise 10.**)

Definition A **permutation** of $[n]$ is a bijective function $\pi \colon [n] \to [n]$.

Let π be a permutation and consider the sequence of iterates

$$\pi(1), \pi(\pi(1)), \pi(\pi(\pi(1))), \cdots$$

Since π is bijective and the set $[n]$ is finite, this sequence has to return to 1 after a number of steps. A sequence like that is called a **cycle**. If you take an element of n which is not in this cycle, you get a cycle of iterates which is disjoint from the first one. So every permutation of $[n]$ gives rise to a partition of $[n]$ into cycles.

The number of permutations of n with k cycles is the (unsigned) **Stirling number of the first kind** $\left[\begin{matrix} n \\ k \end{matrix} \right]$. The recurrence relation for these numbers is:

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right] + (n-1) \left[\begin{matrix} n-1 \\ k \end{matrix} \right]$$

You can see this by considering what happens with the element n under the permutation. The first term in the sum counts the permutations where n is in a cycle by itself. If n is in a cycle with other elements, removing n would give you a permutation of $[n-1]$ with k cycles. Given any such permutation there are $n-1$ possible places to insert n into the cycles which will give you different permutations. So these permutations are counted by the second term of the sum. In this case the generating functions are polynomials indexed by n:

$$H_n(x) = \sum_{k=1}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] x^k.$$

So

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = [x^k] H_n(x).$$

Then $H_1(x) = x$ and $H_n(x) = (x + n - 1)H_{n-1}(x)$. By induction:

$$\sum_{k=1}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] x^k = H_n(x) = x(x+1)\cdots(x+n-1).$$

The expression on the right is called **arising factorial** and is denoted by $x^{\overline{n}}$.

Exercise 11 Find the value of the sum

$$\sum_{S \in [n]^k} \frac{1}{\pi(S)}$$

where $\pi(S)$ denotes the product of the elements of S and the sum is over all k-element subsets of $[n]$. By convention $\pi(\emptyset) = 1$.

Exercise 12 Prove that the number of permutations of $[n]$ with k left-to-right maxima is $\left[\begin{matrix} n \\ k \end{matrix} \right]$.

More Generating Functions

You don't always need to have a recurrence relation in order to use generating functions. You can make a direct connection between the generating function and the set of combinatorial objects you try to count if you have a function w from the set S to the nonnegative integers. For $\sigma \in S$,the number $w(\sigma)$ is called the **weight** of σ. The generating function is defined to be

$$G_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$$

The coefficient of x^n in this power series is equal to the number of elements in S with weight n.

Example Let S be the set of all permutations of [3] and let $w(\sigma)$ be the number of cycles in σ. The table of elements of S with their weights is

Element	Weight
$(1,2,3)$	3
$(1,3,2)$	2
$(2,1,3)$	2
$(2,3,1)$	1
$(3,1,2)$	1
$(3,2,1)$	2

So the generating function is

$$G_S(x) = 2x + 3x^2 + x^3.$$

The Sum and Product Rules from Set-theory have their counterparts for generating functions:

Rule A [Sum] If $S = A \cup B$ where $A \cap B = \emptyset$ then

$$G_S(x) = G_A(x) + G_B(x).$$

Rule B If $S = A \times B$ and if for all $\sigma \in S$ the weight of $\sigma = (a, b)$ is $w(a)+w(b)$ then

$$G_S(x) = G_A(x)G_B(x).$$

Exercise 13 Generalize these rules for unions and products of k sets.

The general binomial series expansion is important for some of the following examples.

$$(1-x)^{-a} = \sum_{n=0}^{\infty} \binom{n+a-1}{n} x^n, \quad |x| < 1.$$

Exercise 14 Let S be the set of all k-tuples $(a_1, a_2, \cdots, a_k)$ where the a_i's are positive integers and the weight of the element $\sigma = (a_1, a_2, \cdots, a_k)$ in S is $\sum_{i=1}^{n} a_i$. Find the generating function.

More 0-1 Strings

The new method of using generating functions is very useful for counting sets of 0-1 strings. To see how that works, we first need to introduce some notation for strings in general. The symbol ϵ will be used to indicate the empty string. If a and b are strings, then we write ab for the string obtained by concatenating a and b. For example if $a = 1110$ and $b = 01011$, then $ab = 111001011$. If A and B are sets of strings, then $AB = \{ab; a \in A, b \in B\}$. And A^* will denote the set of all strings obtained by concatenating any number of strings from A:

$$A^* = \{\epsilon\} \cup A \cup AA \cup AAA \cup \cdots$$

When we want to apply generating functions we will in general use the length of a string as its weight. So the coefficient of x^n in the generating function will usually represent the number of strings of length n in the set. If $G_A(x)$ and $G_B(x)$ are generating functions, then it is not true in general that $G_{AB}(x) = G_A(x)G_B(x)$. For example, let $A = \{01, 011\}$ and $B = \{\epsilon, 1\}$. Then $AB = \{01, 011, 0111\}$ and $G_A(x) = x^2 + x^3$ and $G_B(x) = 1 + x$. But $G_{AB}(x) = x^2 + x^3 + x^4 \neq x^2 + 2x^3 + x^4 = G_A(x)G_B(x)$. The reason is that the element $011 \in AB$ can be obtained in two different ways as a concatenation of elements from A and B,but we only count it once. If we avoid this problem, concatenation of sets of strings corresponds to multiplication of generating functions:

Rule C [Concatenation] If A and B are sets of strings such that every string in AB has a *unique* representation of the form ab for $a \in A$ and $b \in B$, then $G_{AB}(x) = G_A(x)G_B(x)$.

Example If $A = \{0, 00, 000, 0000, \cdots\}$ and $B = \{11, 1111, 111111, \cdots\}$, then $G_A(x) = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$ and $G_B(x) = \frac{1}{1-x^2}$. So

$$G_{AB}(x) = G_A(x)G_B(x) = \frac{x}{(1-x)(1-x^2)}.$$

The concatenation rule and the sum rule can be combined to find a generating function for A^* if each element has a unique representation as a concatenation of elements of A:

$$\begin{aligned} G_{A^*}(x) &= 1 + G_A(x) + G_{AA}(x) + G_{AAA}(x) + \cdots \\ &= 1 + G_A(x) + (G_A(x))^2 + (G_A(x))^3 + \cdots \\ &= \frac{1}{1 - G_A(x)} \end{aligned}$$

So the only thing we have reduced the problem of counting 0-1 strings to the problem of describing them in terms of concatenation in such a way that every string has a unique representation.

Example If $A = 00^*111^*$, then $G_A(x) = \frac{x}{1-x}\frac{x^2}{1-x} = \frac{x^3}{(1-x)^2}$ and $G_{A^*}(x) = \frac{1}{1-G_A(x)} = \frac{(1-x)^2}{(1-x)^2-x^3}$.

Exercise 15 Find the generating function for the set of all 0-1 strings with no two consecutive zeros.

Exercise 16 Find the number of 0-1 strings of length n with exactly m zeros followed immediately by ones.

Exercise 17 Find the generating function for the set of 0-1 strings where neither the patterns 101 nor 111 occur.

Exercise 18 (a) How many ballot sequences with n A's and n B's are there where A and B are never tied until the last vote?
(b) How many such sequences are there where A and B are tied at exactly one point before the last vote?

Exercise 19 Find the number of 0-1 strings of length n containing no block consisting of an odd number of zeros between two nonempty blocks of ones.

Exercise 20 Let Q_n denote the number of ways of placing nonattacking rooks on the n-by-n chessboard so that the arrangement is symmetric about the diagonal from the lower left corner to the upper right corner. Find a recurrence relation for Q_n, ~~and use this to write Q_n as a function of n.~~