2001 Winter Camp

Recurrence Relations

1. The Fibonacci sequence is defined by $F_0 = 1, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Find a formula for F_n.

2. Suppose that $a_1 = 2$, and $a_{k+1} = 3a_k + 1$ for all $k \geq 1$. Determine a formula for $a_1 + a_2 + \ldots + a_n$. (See if you can do it *without* finding an explicit formula for a_n!)

3. Let $\{a_n\}$ be a sequence with $a_0 = 8, a_1 = 4, a_2 = 3$, and $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$ for $n \geq 3$. Find a formula for a_n.

4. Let $x_{n+1} = 4x_n - x_{n-1}$, $x_0 = 0$, and $x_1 = 1$.

   Let $y_{n+1} = 4y_n - y_{n-1}$, $y_0 = 1$, and $y_1 = 2$.

   Prove that for each $n \geq 0$, $y_n^2 = 3x_n^2 + 1$.

   (1988 CMO, Question 4)

5. If $a+b+c = 3$, $a^2+b^2+c^2 = 5$, and $a^3+b^3+c^3 = 12$, determine the value of $a^4+b^4+c^4$.

6. Suppose that a, b, x, y are real numbers such that $ax+by = 3, ax^2+by^2 = 7$, $ax^3+by^3 = 16$, and $ax^4 + by^4 = 42$. Determine the value of $ax^5 + by^5$.

   (1990 AIME, Question 15)

7. Let a, b, c be real numbers such that $a + b + c = 0$. Let $S_n = \frac{a^n+b^n+c^n}{n}$. Prove that

   (i) $S_5 = S_2 \cdot S_3$.

   (ii) $S_7 = S_2 \cdot S_5$.

8. Let $T_0 = 2, T_1 = 3, T_2 = 6$, and for $n \geq 3$, $T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}$. Find a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

   (1990 Putnam A1)

9. Let $x_1, x_2, x_3, \ldots$ be a sequence of non-zero real numbers satisfying $x_n = \frac{x_{n-2}-x_{n-1}}{2x_{n-2}-x_{n-1}}$ for $n \geq 3$. Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

   (1979 Putnam, A3)

10. Let $f(n) = n + \lfloor \sqrt{n} \rfloor$. Prove that for each positive integer m, the sequence m, $f(m)$, $f(f(m))$, $f(f(f(m))), \ldots$ contains at least one perfect square.

    (1983 Putnam, B4)

11. Prove that there exists a unique function $f : \mathbb{R}^+ \to \mathbb{R}$ such that $f(f(x)) = 6x - f(x)$ and $f(x) > 0$ for all $x > 0$.

12. Determine the value of $\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_8} + \frac{1}{F_{16}} + \ldots$

    (Hint: play around with $F_{k-1}F_{2k} - FkF_{2k-1}$.)

13. Let $a_n = n + \lfloor (\sqrt{2}+1)^n \rfloor$. Find all positive integers n for which a_n is even.

14. Prove that for every natural number n, $\left\lfloor \left(\frac{7+\sqrt{37}}{2}\right)^n \right\rfloor$ is a multiple of 3.

15. Let t be the greatest positive root of $x^3 - 3x^2 + 1 = 0$. Prove that 17 divides $\lfloor t^{1988} \rfloor$.
(Proposed for the 1988 IMO)

16. (Josephus Problem). Arrange the numbers $1, 2, \ldots, n$ consecutively about the circumference of a circle, in clockwise order. Now remove number 2, and proceed clockwise by removing every other number among those that remain, until only one number is left. For example, for $n = 5$, the numbers are removed in the order $2, 4, 1, 5$, and 3 remains alone. Let $f(n)$ denote the final number which remains. Prove that for all $n \geq 1$, we have $f(2n) = 2f(n) - 1$, and $f(2n+1) = 2f(n) + 1$.

17. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \ldots, x_{1995}$ satisfying the two conditions:

    (i) $x_0 = x_{1995}$.

    (ii) $x_{i-1} + 2/x_{i-1} = 2x_i + 1/x_i$ for each $i = 1, 2, \ldots, 1995$.

    (1995 IMO, Question 4)

18. Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, x_2, \ldots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions:

    (i) $x_0 = x_n = 0$.

    (ii) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

    Show that there exists a pair (i, j) of distinct indices with $(i, j) \neq (0, n)$ such that $x_i = x_j$.
    (1996 IMO, Question 6)

19. A sequence $\{a_n\}$ is defined by $a_0 = 2, a_1 = \frac{5}{2}$, and $a_{n+1} = a_n(a_{n-1}^2 - 2) - a_1$ for all $n \geq 1$. Prove that for each positive integer n,

$$\lfloor a_n \rfloor = 2^{\frac{2^n - (-1)^n}{3}}.$$

    (1976 IMO)

20. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that for all $n \geq 1$, $a_{2n-1} = 0$, and $a_{2n} = \frac{x^{n-1} - y^{n-1}}{\sqrt{2}}$, where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$.
(1979 IMO)