Adrian Jan 4/06

IMO Winter Camp 2006 - Number Theory Problem Set

1.) Let x, y be positive integers. Prove that $7x + 2y$ is divisible by 13 if and only if $x + 4y$ is divisible by 13.

2.) Let $a, b > 1$ be positive integers. Prove that $ab^2 + b$ cannot divide $a^2b - 1$.

3.) Let S be a set of integers such that

- If $a, b \in S$, then $a - b \in S$
- S contains two positive integers that are relatively prime.

Prove that S contains every integer.

4.) Let a, b, n be positive integers such that $n > ab - a - b$ and $gcd(a, b) = 1$. Prove that $ax + by = n$ has non-negative integer solutions.

5.) Find all integer solutions to $a^2 + b^2 + c^2 = 2007$.

6.) Let n be a positive integer not divisible by 2 or 3. Prove that $2^{-1} + 3^{-1} + 6^{-1} \equiv 1 \mod n$.

7.) Find all integer solutions to $x_1^9 + x_2^9 + \cdots + x_8^9 = 2005$.

8.) A Peng number is an integer that is the sum of two perfect squares. (Zero counts as a perfect square)

a.) Prove that if n is Peng, then so is $2n$.

b.) Prove that the product of two Peng numbers is also a Peng number.

c.) Prove that a positive integer n is Peng if and only if the number of prime factors of n congruent to 3 mod 4 is even.

9.) A calculator is broken except for the buttons sin, cos, ~~sin~~, tan, $\cos^{-1}$, $\sin^{-1}$, $\tan^{-1}$. The initial display on the calculator is zero. Assume that the calculator is infinite precision. Prove that one can obtain any non-negative rational number on the display.

10.) Find all non-negative integer solutions to $4ab - a - b = c^2$.

11.) Prove that 2005^{2005} is the sum of two squares but not the sum of two cubes.

12.) Find all positive integers n such that n divides $2^n - 1$.

13.) Prove that every integer can be written in the form $x^2 + y^2 - 5z^2$ where x, y, z are integers.

14.) Find all integer solutions to $a^4 + 4^a = p$ where a is a positive integer and p is a prime.

15.) Let a, b be positive integers that are relatively prime and of different parity. Suppose S is a set of integers that contains a and b such that if $x, y, z \in S$, then $x + y + z \in S$. Prove that S contains every integer larger than $2ab$.

16.) Prove that $x^3 + y^4 = 2^{2003}$ has no integer solutions.

17.) Prove that for any positive integer n, there exists n consecutive integers that are not prime powers.

18.) Find all positive integers n such that there exists a positive integer m such that $\tau(m^2)/\tau(m) = n$. ($\tau(m)$ is the number of positive divisors of m)

19.) Find all integers $N \geq 3$ with the following property:

If $1 \leq k \leq N$, and $gcd(k, N) = 1$, then k is prime.

20.) Find all positive integers $n, n \leq 2005$ such that n divides $2^n + 2$.