Winter Camp 2009
Buffet contest

A1. Show that for any positive integer n, there exists a positive integer m such that

$$(1+\sqrt{2})^n = \sqrt{m} + \sqrt{m+1}.$$

A2. For every ordered pair of positive integers (x, y), define $f(x, y)$ recursively as follows:

$$f(x,y) = \begin{cases} f(x-y, y) + 1 & \text{for } x > y, \\ f(x, y-x) + 1 & \text{for } y > x, \\ x & \text{for } x = y. \end{cases}$$

For example, $f(5,3) = f(2,3) + 1 = f(2,1) + 2 = f(1,1) + 3 = 4$. If $f(x, y) \leq 15$, show that $x + y < 2009$.

A3. Let $x, y, z \geq 0$ be such that $x + y + z = 3$. Prove that

$$\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} \geq \frac{1}{9} + \frac{2}{27} \cdot (xy + yz + zx).$$

When does equality hold?

A4. We assign a real number $t_{x,y}$ between 0 and 1 to every point on the plane (x, y) with integer coordinates. This is done in such a way that $t_{x,y} = \frac{t_{x-1,y}+t_{x,y-1}+t_{x+1,y}+t_{x,y+1}}{4}$ for all x, y. Show that all the numbers are equal.

C1. A deck contains 52 cards of 4 different suits. Vanya is told the number of cards in each suit. He picks a card from the deck, guesses its suit, and sets it aside; he repeats until the deck is exhausted. Show that if Vanya always guesses a suit having no fewer remaining cards than any other suit, he will guess correctly at least 13 times.

C2. A mathematics competition has n contestants and 5 problems. For each problem, each contestant is assigned a positive integer score which is at most seven. It turns out every pair of contestants have at most one problem whose scores are common. Find the maximum possible value of n.

C3. Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, \ldots, n\}$ with the property that the average of the elements in S is an integer. Prove that $T_n - n$ is always even.

C4. For n an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. A *tromino* is an L-shape formed by three connected unit squares. For which values of n is it possible to cover all the black squares with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?

G1. In $\triangle ABC$, points D and F are selected on sides BC and AB respectively so that $AD \cdot BC = AB \cdot CF$. Let AD and CF intersect at P. Prove that either quadrilateral $BFPD$ is cyclic or quadrilateral $FACD$ is cyclic.

G2. Let ABC be a scalene triangle and let A', B', and C' (respectively) be the points of intersection of the interior angle bisectors A, B, and C (respectively) with the opposite sides of the triangle. Now let:

- A'' be the intersection of BC with the perpendicular bisector of AA';
- B'' be the intersection of CA with the perpendicular bisector of BB';
- C'' be the intersection of AB with the perpendicular bisector of CC'.

Show that A'', B'', and C'' are collinear.

G3. Let ω_1 and ω_2 be concentric circles with ω_2 inside ω_1. Let $ABCD$ be a parallelogram with B, C, D on ω_1 and A on ω_2. If BA intersects ω_2 again at E and CE intersects ω_2 again at P, prove that $CD = PD$.

G4. Convex hexagon $ABCDEF$ has area 1. Prove that at least one triangle out of ABC, BCD, CDE, DEF, EFA, and FAB has area at most $\frac{1}{6}$.

N1. Find all positive integers n less than 1000 such that n^2 is equal to the cube of the sum of its digits.

N2. Find all integers a, b, c greater than 1 for which $ab - 1$ is divisible by c, $bc - 1$ is divisible by a, and $ca - 1$ is divisible by b.

N3. The sequence of natural numbers $a_1, a_2, a_3, \ldots,$ satisfies the condition $a_{n+2} = a_{n+1}a_n + 1$ for all n. Prove that $a_n - 22$ is composite for all $n > 10$, no matter what a_1 and a_2 are.

N4. Find all positive integers that can be written in the form

$$\frac{a^2 + b^2 + 1}{ab}$$

where a, b are positive integers.

Solutions

A1. Applying the binomial theorem to $(1+\sqrt{2})^n$, we see there exist integers a and b such that $(1+\sqrt{2})^n = a + b\sqrt{2}$ and $(1-\sqrt{2})^n = a - b\sqrt{2}$. Multiplying these, we get $a^2 - 2b^2 = \big((1+\sqrt{2})(1-\sqrt{2})\big)^n = \pm 1$.

Setting $m = \min(a^2, 2b^2)$, we have $\sqrt{m} + \sqrt{m+1} = \sqrt{a^2} + \sqrt{2b^2} = a + b\sqrt{2} = (1+\sqrt{2})^n$.

Source: Romanian Math Stars Competition 2007, #1. Also see CMO 1994, #2.

A2. Let $F_n = \{1, 1, 2, 3, 5, \ldots\}$ be the Fibonacci sequence. We prove by induction on n that if $f(x,y) = n > 1$, then $\min(x,y) \le 2F_{n-1}$ and $\max(x,y) \le 2F_n$. When $n = 2$, it is straightforward to check (x,y) must be one of $(2,2), (1,2)$, or $(2,1)$, and the result holds.

Now assume the result holds for $n = k$, and consider x, y with $f(x,y) = k+1$. If $x = y$, the result is trivial. Otherwise, assume without loss of generality that $x > y$. Then $f(x-y, y) = k$, so by our inductive hypothesis, $x = \max(x-y, y) + \min(x-y, y) \le 2F_{k-1} + 2F_k = 2F_{k+1}$, and $y \le \max(x-y, y) \le 2F_k$, completing the proof of the inductive step.

It follows that if $f(x,y) \le 15$, then $x+y = \max(x,y) + \min(x,y) \le 2F_{14} + 2F_{15} = 2F_{16} < 2009$.

A3. Since $y \ge 0$, the AM-GM inequality implies $\frac{x^3}{y^3+8} + \frac{y+2}{27} + \frac{y^2-2y+4}{27} \ge 3 \cdot \sqrt[3]{\frac{x^3}{27^2}} = \frac{x}{3}$. Similarly, $\frac{y^3}{z^3+8} + \frac{z+2}{27} + \frac{z^2-2z+4}{27} \ge \frac{y}{3}$ and $\frac{z^3}{x^3+8} + \frac{x+2}{27} + \frac{x^2-2x+4}{27} \ge \frac{z}{3}$. Adding all three inequalities, we have:

$$\begin{aligned}
\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} &\ge \frac{x+y+z}{3} + \frac{y+z+x}{27} - \frac{6}{9} - \frac{y^2+z^2+x^2}{27} \\
&= \frac{4}{9} - \frac{(x+y+z)^2 - 2xy - 2yz - 2zx}{27} \\
&= \frac{1}{9} + \frac{2}{27} \cdot (xy + yz + zx).
\end{aligned}$$

For equality to hold, we must have $\frac{y+2}{27} = \frac{y^2-2y+4}{27} \implies y^2 - 3y + 2 = 0$, so y equals 1 or 2. The same holds for z and x. Since $x + y + z = 3$, the only possibility is $x = y = z = 1$, and it is easy to check that equality does indeed hold in this case.

A4. Define $d_{x,y,n} = t_{x+n,y+n} - t_{x,y}$. Let C, ϵ be constants so that $d_{x,y,1} \ge C$ for some x, y but $d_{x,y,1} \le C + \epsilon$ for all x, y. For any x, y, note that $t_{x,y,n} = t_{x,y,1} + t_{x+1,y+1,1} + \ldots + t_{x+n-1,y+n-1,1} \le n(C+\epsilon)$.

We prove by induction that for all n, there exist x, y so that $d_{x,y,n} \ge nC - 3^n\epsilon$. For $n = 1$, the claim is trivial. Now suppose the result holds for n, and choose x, y so that $d_{x,y,n} \ge nC - 3^n\epsilon$. Using the given relation on t, we have:

$$\begin{aligned}
&\frac{d_{x-1,y,n+1} + d_{x,y-1,n+1} + d_{x+1,y,n-1} + d_{x,y+1,n-1}}{4} = d_{x,y,n} \ge nC - 3^n\epsilon \\
\implies \quad &\frac{d_{x-1,y,n+1} + d_{x,y-1,n+1}}{2} \ge 2nC - 2 \cdot 3^n\epsilon - (n-1)(C+\epsilon) \ge (n+1)C - 3^{n+1}\epsilon.
\end{aligned}$$

Therefore, one of $d_{x-1,y,n+1}$ or $d_{x,y-1,n+1}$ is at least $(n+1)C + 3^{n+1}\epsilon$, and the claim is proven.

Now suppose $d_{x,y,1} = C > 0$ for some x, y. Fix $\epsilon > 0$ and let m be the largest integer so that there exists x, y for which $d_{x,y,1} \ge C + m\epsilon$. Then, as shown above, for each n, there exist

x, y so that $d_{x,y,n} \geq nC + m\epsilon - 3^n\epsilon \geq nC - 3^n\epsilon$. Choosing n large and then ϵ small, we have $d_{x,y,n} > 1$, which is impossible.

Therefore, $d_{x,y,1} \leq 0$ for all x, y. Similarly, $d_{x,y,1} \geq 0$ for all x, y, and hence $t_{x,y} = t_{x+1,y+1}$ for all x, y. Similarly, $t_{x,y} = t_{x+1,y-1}$ for all x, y. The original relation now implies that $t_{x,y} = t_{x-1,y} = t_{x,y-1} = t_{x+1,y} = t_{x,y+1}$, and the result follows.

Source: Iberoamerican Olympiad, miscellaneous problem

Remark: You cannot assume that there exist x, y for which $d_{x,y,n}$ is maximal. However, there is a theorem in analysis saying there exist real numbers M_n such that $d_{x,y,n}$ gets arbitrarily close to M_n without exceeding M_n. If you know this theorem, the proof becomes a lot cleaner.

C1. Let M denote the maximum number of cards remaining in any single suit. As Vanya proceeds, M will only decrease if the current card is in a suit with M cards remaining, and no other suit has M cards remaining. In this case, however, Vanya will correctly guess that suit. Therefore, Vanya will guess correctly every time M decreases.

Since $M \geq \frac{52}{4} = 13$ initially and it is 0 by the end, Vanya will be correct at least 13 times.

Source: Russia, 1998

C2. There are 7 possible scores on each question. If $n \geq 50$, then at least $\left\lceil \frac{50}{7} \right\rceil = 8$ contestants got the same score on problem 1. But then two of those contestants must have gotten the same score on problem 2, which is impossible.

Now, for $1 \leq i, j \leq 7$, let $x_{i,j,k}$ denote the value in $\{1, 2, \ldots, 7\}$ that is congruent to $i + jk \pmod 7$. Consider 49 contestants $C_{i,j}$ where contestant $C_{i,j}$ receives score $x_{i,j,k}$ on problem k. Suppose that two contestants C_{i_1,j_1} and C_{i_2,j_2} got the same scores on questions k_1 and k_2. Then $i_1 - i_2 + (j_1 - j_2)k_1 \equiv i_1 - i_2 + (j_1 - j_2)k_2 \equiv 0 \pmod 7$. Subtracting, we have $(j_1 - j_2)(k_1 - k_2) \equiv 0 \pmod 7 \implies j_1 \equiv j_2 \pmod 7 \implies j_1 = j_2$. But then we must also have $i_1 = i_2$, which is a contradiction.

Therefore, it is possible to satisfy the required condition with 49 contestants, and hence 49 is the maximum possible value for n.

C3. Let S' denote the subsets of $\{1, 2, \ldots, n\}$ with at least two elements and with integer average. For each set $X \in S'$ that contains its average x, we pair it with the set $X \backslash \{x\}$, and conversely for each set $Y \in S'$ that does not contain its average y, we pair it with the set $Y \cup \{y\}$. This is a proper pairing, so S' must contain an even number of sets.

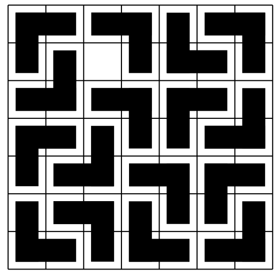
Therefore, T_n has the same parity as the number of singleton sets with integer average, of which there are exactly n.

Source: Putnam 2002, A3

C4. Let X_n denote the minimum number of trominos required to cover an $n \times n$ board in this way. We claim any $n \geq 7$ is possible, and $X_n = \frac{(n+1)^2}{4}$.

Indeed, let B denote the set of all squares that are an even number of rows and even number of columns away from the bottom-left square. There are exactly $\frac{(n+1)^2}{4}$ such squares, and they are all black. Furthermore, each tromino can cover at most one square in B, so $X_n \geq \frac{(n+1)^2}{4}$. However, if $n \in \{1, 3, 5\}$, then $3 \cdot \frac{(n+1)^2}{4} > n^2$, so it is impossible to place this many trominos on the board.

It remains to show that if $n \geq 7$, there exists a valid tiling with $\frac{(n+1)^2}{4}$ trominos. For $n = 7$, a valid tiling is shown below. Now, suppose there is a valid tiling of an $n \times n$ square using exactly $\frac{(n+1)^2}{4}$ trominos. For $k > 1$ odd, we can also tile a $2 \times k$ rectangle with $\frac{k+1}{2}$ trominos by using two tronominos for the first 2×3 rectangle, and 1 tromino for each following 2×2 rectangle. Since an $(n+2) \times (n+2)$ rectangle can be partitioned into an $n \times n$ rectangle, a $2 \times n$ rectangle, and a $2 \times (n+2)$ rectangle, it can therefore by tiled with $\frac{(n+1)^2}{4} + \frac{n+1}{2} + \frac{n+3}{2} = \frac{(n+3)^2}{4}$ trominos. The result now follows by induction.



Source: IMO Shortlist 2002, C2

G1. Let R denote the circumradius of $\triangle ABC$. By the extended sine law, we have $AD \cdot BC = AB \cdot \frac{\sin B}{\sin ADB} \cdot BC = \frac{4R^2 \cdot (\sin A) \cdot (\sin B) \cdot (\sin C)}{\sin \angle ADB}$. Similarly, $AB \cdot CF = \frac{4R^2 \cdot (\sin A) \cdot (\sin B) \cdot (\sin C)}{\sin \angle CFB}$. Equating these, we get $\sin \angle ADB = \sin \angle CFB$, which implies $\angle ADB = \angle CFB$ or $\angle ADB = 180^\circ - \angle CFB$. In the former case, $FACD$ is cyclic; in the latter case, $BFPD$ is cyclic.

G2. Assume without loss of generality that A'' lies on the same side of A' as C does. Then, $\angle A''AA' = \angle A''A'A = \angle CA'A = 180^\circ - \angle A'AC - \angle A'CA = \frac{\angle A}{2} + \angle B$. It follows that $\angle CAA'' = \angle B$, and hence AA'' is tangent to the circumcircle ω of $\triangle ABC$ at A. Therefore, A'' is the intersection of BC and the tangent to ω at A. Similar statements hold for B'' and C''.

The problem is now equivalent to Pascal's theorem on the degenerate hexagon $AABBCC$.

Source: Iberoamerican Olympiad 2004

G3. Let O denote the center of ω_1 and ω_2. The perpendicular bisectors of AE and CD are parallel and both pass through O, so they are in fact identical. Therefore, the quadrilateral $AEDC$ is symmetric about this line, and $BC = AD = EC$.

Now, let B' and C' denote the second intersections of BE and CE with ω_1. $\angle EC'B' = \angle EBC$ and $\angle B'EC' = \angle CEB$ so $\triangle C'B'E \sim \triangle BCE$, and hence, $C'B' = EB'$.

Applying the symmetry argument to $B'EAB$ and $C'EPC$, we also have $B'E = AB = DC$ and $C'E = PC$. Also, $\angle C'EB' = \angle PCD$ since $B'E$ and DC are parallel. Therefore, $\triangle C'B'E \cong \triangle PDC$, and the result follows.

Remark: there are two configurations, depending on which of A or E is closer to B, but this argument works without change in either case.

G4. Let P be the intersection of AD and BE, Q be the intersection of BE and CF, and R be the intersection of CF and AD. Also assume without loss of generality that P is on the same side of CF as A and B; i.e., P is between A and R. Then, it is easy to check that R must be between C and Q, and Q must be between E and P.

In this case, triangles ABR, BCR, CDQ, DEQ, EFP, and FAP are all disjoint, so their total area is at most 1. It follows that one of them has area at most $\frac{1}{6}$. Regardless of which triangle it is, we have found four adjacent vertices P, Q, R, S on the hexagon and a point X on segment PS for which $\triangle QRX$ has area at most $\frac{1}{6}$. Note that the area of $\triangle QRX$ is bounded between the area of $\triangle QRP$ and the area of $\triangle QRS$. Therefore, one of these triangles also has area at most $\frac{1}{6}$, and the result follows.

N1. Let s denote the sum of the digits of n. Then $s^3 = n^2 \equiv s^2 \pmod 9 \implies s^2(s-1) \equiv 0 \pmod 9$, which implies $s \equiv 0 \pmod 3$ or $s \equiv 1 \pmod 9$. Also, $s \le 9+9+9 = 27$, and s is a perfect square since $n^2 = s^3$.

This leaves only the possibilities $s = 1$ or $s = 9$, which lead to $n = 1$ and $n = 27$, both of which are valid solutions.

Source: Iberoamerican Olympiad 1999, #1

N2. Note that a, b and c are all relatively prime, since if $p|a, b$, then a cannot divide $bc - 1$. Now the given condition implies:

$$\begin{aligned} & (ab-1)(bc-1)(ca-1) \equiv 0 \pmod{abc} \\ \implies \quad & a^2b^2c^2 - a^2bc - ab^2c - abc^2 + ab + bc + ca - 1 \equiv 0 \pmod{abc} \\ \implies \quad & ab + bc + ca \equiv 1 \pmod{abc} \end{aligned}$$

Since $ab + bc + ca > 1$, it follows that $ab + bc + ca > abc$, or equivalently, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$. Now assume without loss of generality that $a \le b \le c$. If $a > 2$, then since a, b, c are relatively prime, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le \frac{1}{3} + \frac{1}{4} + \frac{1}{5} < 1$. Therefore, $a = 2$. If $b > 3$, then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le \frac{1}{2} + \frac{1}{5} + \frac{1}{7} < 1$. Therefore, $b = 3$, which leaves only the option of $c = 5$.

Conversely, it is easy to check that $(a, b, c) = (2, 3, 5)$ is a valid solution, as are all permutations of $(2, 3, 5)$.

Source: American Math Olympiad Program 1998

N3. For any $n \ge 3$, we have $a_n \equiv 0 \pmod{a_n}$ and $a_{n+1} = a_n a_{n-1} + 1 \equiv 1 \pmod{a_n}$. We now apply the recurrence to calculate the sequence $\{a_n, a_{n+1}, \ldots a_{n+6}\} \equiv \{0, 1, 1, 2, 3, 7, 22\} \pmod{a_n}$. Therefore, $a_{n+6} - 22$ must be a multiple of a_n.

For $n \ge 3$, we have $a_n = a_{n-1}a_{n-2} + 1 > 1$. It is also easy to check that $a_{n+6} - a_n > 22$. Therefore, a_n and $\frac{a_{n+6}-22}{a_n}$ are both integers greater than 1, and hence $a_{n+6} - 22$ is not prime.

Source: American Math Olympiad Program 1998

N4. Suppose k can be expressed in this form, and let (a, b) be such that $\frac{a^2+b^2+1}{ab} = k$ and $a + b$ is as small as possible.

Suppose $a < b$. Then $\frac{a^2+1}{b} = ka - b$ is an integer. Denoting this quantity by b', we have $b' \le \frac{(b-1)^2+1}{b} < b$, and

$$\frac{a^2 + (b')^2 + 1}{ab'} = \frac{a^2 + \left(\frac{a^2+1}{b}\right)^2 + 1}{a \cdot \frac{a^2+1}{b}} = \frac{a^2 + b^2 + 1}{ab} = k,$$

which contradicts the minimality of (a, b). Similarly, $b < a$ is impossible so we must have $a = b$. In this case, $\frac{a^2+b^2+1}{ab}$ is only an integer if $a = b = 1$ and $k = 3$.

Therefore, 3 is the only integer that can be expressed in this form.

Remark: The equation for b' is found by root-flipping. We interpret $\frac{a^2+b^2+1}{ab} = k$ as a quadratic equation in b, and note that if b is one root, then b' is the other one.