Sequences & Series

This handout covers basic techniques for solving problems involving sequences and series. We start with a discussion of arithmetic and geometric sequences, and then move to telescoping sums.

1 Arithmetic, Geometric, and Arithmetico-Geometric Sequences

In an *arithmetic sequence*, each subsequent term is obtained by adding a constant to the previous term. Let

$$S = a + (a+d) + \cdots + [a + (n-1)d].$$

We then list the terms in S in reverse order:

$$\begin{array}{ccccccccc} S & = & a & + & (a+d) & + & \cdots & + & [a+(n-1)d], \\ S & = & [a+(n-1)d] & + & [a+(n-2)d] & + & \cdots & + & a. \end{array}$$

If we add these equations, we get

$$\begin{aligned} 2S &= [2a+(n-1)d] + [2a+(n-1)d] + \cdots + [2a+(n-1)d] \\ &= n[2a+(n-1)d], \end{aligned}$$

so

$$S = \frac{n[2a+(n-1)d]}{2}.$$

This sum can also be written as

$$\frac{n(a+l)}{2},$$

where a is the first term of the series and $l = a + (n-1)d$ is the last term.

Problem 1.1. In an arithmetic progression, the ratio of the sum of the first r terms to the sum of the first s terms is r^2/s^2 ($r \neq s$). Find the ratio of the 8^{th} term to the 23^{rd} term. (ARML, 1983)

Solution: Let the progression be $a, a+d, a+2d, \ldots$. Then for all $r \neq s$,

$$\frac{\frac{r[2a+(r-1)d]}{2}}{\frac{s[2a+(s-1)d]}{2}} = \frac{r[2a+(r-1)d]}{s[2a+(s-1)d]} = \frac{r^2}{s^2},$$

so

$$\frac{2a+(r-1)d}{2a+(s-1)d} = \frac{r}{s}.$$

Cross-multiplying, this becomes

$$2as + (r-1)sd = 2ar + (s-1)rd.$$

Moving all the terms to one side, we get

$$2ar - 2as - rd + sd = 0,$$

which factors as $(2a - d)(r - s) = 0$. Since $r \neq s$, $d = 2a$. Then the 8^{th} term is $a + (8 - 1)d = a + 7 \cdot 2a = 15a$, and the 23^{rd} term is $a + (23 - 1)d = a + 22 \cdot 2a = 45a$, so the ratio is $(15a)/(45a) = 1/3$. □

In a *geometric sequence*, each subsequent term is obtained by multiplying the previous term by a constant. Let

$$S = a + ar + \cdots + ar^{n-1},$$

where $r \neq 1$. Then

$$rS = ar + ar^2 + \cdots + ar^n.$$

Subtracting the first equation, we get

$$rS - S = ar^n - a,$$

which we can re-write as

$$(r - 1)S = a(r^n - 1),$$

so

$$S = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}.$$

If $|r| < 1$, then the sum of the infinite geometric series is

$$S = a + ar + ar^2 + \cdots = \frac{a}{1 - r}.$$

Problem 1.2. An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has sum 10 times the sum of the original series. Find the common ratio of the original series. (AIME, 2005)

Solution: Let the original geometric series be $a + ar + ar^2 + \cdots$, so

$$a + ar + ar^2 + \cdots = \frac{a}{1 - r} = 2005.$$

Also,

$$a^2 + a^2r^2 + a^2r^4 + \cdots = \frac{a^2}{1 - r^2} = 10 \cdot 2005.$$

Thus, we have the system of equations

$$\begin{aligned} a &= 2005(1 - r), \\ a^2 &= 10 \cdot 2005(1 - r^2). \end{aligned}$$

If we divide the second equation by the first we get

$$a = 10(1 + r).$$

When we substitute this into the sum of the series we get

$$\frac{10(1 + r)}{1 - r} = 2005.$$

Solving for r, we find $r = 399/403$. □

In an *arithmetico-geometric* sequence, each term is the product of a term from an arithmetic sequence and a term from a geometric sequence.

Problem 1.3. Find

$$\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \cdots .$$

Solution: More generally, let

$$S = r + 2r^2 + 3r^3 + \cdots .$$

Then

$$rS = r^2 + 2r^3 + 3r^4 + \cdots .$$

Subtracting this equation from the first equation, we get

$$S - rS = r + r^2 + r^3 + \cdots ,$$

which becomes

$$(1 - r)S = \frac{r}{1 - r},$$

so

$$S = \frac{r}{(1 - r)^2}.$$

In particular, for $r = 1/5$, $S = 5/16$. □

If you have a series that relates to a geometric series, then multiplying the series by the common ratio is often a useful strategy.

2 Telescoping Sums

We introduce a technique for dealing with general sums, starting with a well-known example.

Sequences & Series

Problem 2.1. Find
$$\sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

Solution: We can write
$$\frac{1}{k(k+1)} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$
Then
$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).\end{aligned}$$

We find that many of the terms cancel, and we are left with
$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

□

This kind of sum, with terms that cancel, is called a *telescoping sum*. As demonstrated in this example, if you see fractions in a sum, then you should strongly consider using telescoping sums.

Problem 2.2. Prove
$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$$
(USAMO, 1992)

Solution: We have a sum that involves fractions, so we think about how we can make it telescope. Furthermore, the general term, namely
$$\frac{1}{\cos k^\circ \cos(k+1)^\circ}$$
looks a lot like the general term in the previous example, namely
$$\frac{1}{k(k+1)}.$$
Analogously, we could consider the expression
$$\frac{1}{\cos k^\circ} - \frac{1}{\cos(k+1)^\circ},$$

but this simplifies as

$$\frac{\cos(k+1)^\circ - \cos k^\circ}{\cos k^\circ \cos(k+1)^\circ},$$

which is not we are looking for, because the numerators do not match.

To give us something to work with in the numerator, we can re-write the given equation as

$$\frac{\sin 1^\circ}{\cos 0^\circ \cos 1^\circ} + \frac{\sin 1^\circ}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{\sin 1^\circ}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin 1^\circ}.$$

The general term is now

$$\frac{\sin 1^\circ}{\cos k^\circ \cos(k+1)^\circ}.$$

Somehow, we must relate the numerator $\sin 1^\circ$ to the factors $\cos k^\circ$ and $\cos(k+1)^\circ$ in the denominator.

We can do so by writing 1° as the difference of $(k+1)^\circ$ and k°:

$$\sin 1^\circ = \sin[(k+1)^\circ - k^\circ].$$

Then by the difference of angles formula for sine,

$$\begin{aligned} \sin 1^\circ &= \sin[(k+1)^\circ - k^\circ], \\ &= \sin(k+1)^\circ \cos k^\circ - \cos(k+1)^\circ \sin k^\circ, \end{aligned}$$

so

$$\begin{aligned} \frac{\sin 1^\circ}{\cos k^\circ \cos(k+1)^\circ} &= \frac{\sin(k+1)^\circ \cos k^\circ - \cos(k+1)^\circ \sin k^\circ}{\cos k^\circ \cos(k+1)^\circ} \\ &= \frac{\sin(k+1)^\circ}{\cos(k+1)^\circ} - \frac{\sin k^\circ}{\cos k^\circ}. \end{aligned}$$

Summing from $k = 0$ to 88, we see that the sum telescopes to become

$$\begin{aligned} \frac{\sin 1^\circ}{\cos 0^\circ \cos 1^\circ} + \frac{\sin 1^\circ}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{\sin 1^\circ}{\cos 88^\circ \cos 89^\circ} &= \frac{\sin 89^\circ}{\cos 89^\circ} - \frac{\sin 0^\circ}{\cos 0^\circ} \\ &= \frac{\cos 1^\circ}{\sin 1^\circ}, \end{aligned}$$

so

$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ},$$

as desired. □

Let's clean that solution up a little.

Solution: We can multiply the desired equation by $\sin 1^\circ$ to get

$$\frac{\sin 1^\circ}{\cos 0^\circ \cos 1^\circ} + \frac{\sin 1^\circ}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{\sin 1^\circ}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin 1^\circ}.$$

Since $\sin 1^\circ$ is nonzero, this is equivalent to the original equation and this is what we will prove.

In order to get a telescoping sum, we use the difference of angles formula for sine to create a sum in the numerators,

$$\begin{aligned} \sin 1^\circ &= \sin[(k+1)^\circ - k^\circ], \\ &= \sin(k+1)^\circ \cos k^\circ - \cos(k+1)^\circ \sin k^\circ, \end{aligned}$$

so

$$\begin{aligned} \frac{\sin 1^\circ}{\cos k^\circ \cos(k+1)^\circ} &= \frac{\sin(k+1)^\circ \cos k^\circ - \cos(k+1)^\circ \sin k^\circ}{\cos k^\circ \cos(k+1)^\circ} \\ &= \frac{\sin(k+1)^\circ}{\cos(k+1)^\circ} - \frac{\sin k^\circ}{\cos k^\circ}. \end{aligned}$$

Summing from $k = 0$ to 88, we see that the sum telescopes to become

$$\begin{aligned} \frac{\sin 1^\circ}{\cos 0^\circ \cos 1^\circ} + \frac{\sin 1^\circ}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{\sin 1^\circ}{\cos 88^\circ \cos 89^\circ} &= \frac{\sin 89^\circ}{\cos 89^\circ} - \frac{\sin 0^\circ}{\cos 0^\circ} \\ &= \frac{\cos 1^\circ}{\sin 1^\circ}, \end{aligned}$$

so

$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ},$$

as desired.

A close cousin of the telescoping sum is the telescoping product.

Problem 2.3. Simplify the product

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{9^2}\right)\left(1 - \frac{1}{10^2}\right).$$

(AHSME, 1986)

Sponsored by the D.E. Shaw group, Two Sigma Investments LLC, Jane Street Capital, Citadel LLC, and Dropbox

Solution: The product is equal to

$$\begin{aligned}
&\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\dots\left(1-\frac{1}{9^2}\right)\left(1-\frac{1}{10^2}\right)\\
&=\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\cdots\left(1-\frac{1}{9}\right)\left(1+\frac{1}{9}\right)\left(1-\frac{1}{10}\right)\left(1+\frac{1}{10}\right)\\
&=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\dots\cdot\frac{9}{8}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{9}{10}\cdot\frac{11}{10}.
\end{aligned}$$

The factors 3/2 and 3/2 cancel, as do the factors 4/3 and 3/4, and so on, up until the factors of 10/9 and 9/10, and we are left with $\frac{1}{2}\cdot\frac{11}{10}=\frac{11}{20}$. □

More generally, how do we know if we can turn a given series into a telescoping sum? If we are already given the closed form, then it is actually quite easy.

Problem 2.4. Show that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

Solution: Let

$$S_n = \sum_{k=1}^{n} k^2,$$

and let

$$T_n = \frac{n(n+1)(2n+1)}{6}.$$

Then the goal is to show that $S_n = T_n$ for all n.

By definition, S_n is the sum of the terms $1^2, 2^2, \dots, n^2$, so $S_{n-1} + n^2 = S_n$, or

$$S_n - S_{n-1} = n^2.$$

If $S_n = T_n$, then $T_n - T_{n-1}$ should also be equal to n^2. Indeed,

$$\begin{aligned}
T_n - T_{n-1} &= \frac{n(n+1)(2n+1)}{6} - \frac{(n-1)n(2n-1)}{6}\\
&= \frac{2n^3+3n^2+n}{6} - \frac{2n^3-3n^2+n}{6}\\
&= n^2.
\end{aligned}$$

But now we have exactly what we need to make the sum telescope: The k^{th} term in the sum, namely k^2, is equal to $T_k - T_{k-1}$. Hence,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} k^2 \\ &= 1^2 + 2^2 + \cdots + n^2 \\ &= (T_1 - T_0) + (T_2 - T_1) + \cdots + (T_n - T_{n-1}) \\ &= T_n - T_0 \\ &= T_n, \end{aligned}$$

as desired. □

More generally, suppose we are given a sequence $a_1, a_2, a_3, \ldots$ of real numbers, and asked to prove that

$$a_1 + a_2 + \cdots + a_n = T_n,$$

where T_n represents an explicit formula in terms of n with $T_1 = S_1$ (or $T_0 = 0$). Then, as shown in the problem above, it suffices to show that $a_n = T_n - T_{n-1}$.

But what if we are not given the explicit formula T_n beforehand? In such a case, we can employ the usual problem solving heuristics of looking at small cases and finding a pattern.

Problem 2.5. Find

$$\sum_{k=1}^{\infty} \arctan \frac{1}{2k^2}.$$

Solution: To analyze an infinite series, we can look at its partial sums, which in this case are given by

$$S_n = \sum_{k=1}^{n} \arctan \frac{1}{2k^2}.$$

To compute these partial sums, we must find a way to add arctangents. Let's start with adding two arctangents, say

$$\arctan x + \arctan y,$$

where x and y are positive. Let $\alpha = \arctan x$ and $\beta = \arctan y$, so $0 < \alpha, \beta < \frac{\pi}{2}$. Then $\tan \alpha = x$ and $\tan \beta = y$. Furthermore, by the addition formula for tangent,

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta},$$

so

$$\tan(\arctan x + \arctan y) = \frac{x + y}{1 - xy}. \qquad (*)$$

At this point, it is tempting to simply take the arctan of both sides, to get

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right). \qquad (**)$$

But this is not always true, because in general, $\tan\theta = z$ does not imply $\theta = \arctan z$. For example, if we take $x = 2$ and $y = 3$ in $(*)$, then we get

$$\tan(\arctan 2 + \arctan 3) = \frac{2+3}{1-2\cdot 3} = -1.$$

If we were to take the arctan of both sides, then we would get

$$\arctan 2 + \arctan 3 = \arctan(-1) = -\frac{\pi}{4}.$$

But this cannot be true, because $0 < \arctan 2 < \frac{\pi}{2}$ and $0 < \arctan 3 < \frac{\pi}{2}$, which means

$$0 < \arctan 2 + \arctan 3 < \pi.$$

The only angle θ in the interval $0 < \theta < \pi$ that satisfies $\tan\theta = -1$ is $\theta = \frac{3\pi}{4}$, so the correct conclusion is

$$\arctan 2 + \arctan 3 = \frac{3\pi}{4}.$$

Thus, we must be somewhat careful in working with the arctangent function.

In $(*)$, suppose we also know that $xy < 1$, which means $\frac{x+y}{1-xy}$ is positive. Then $\tan(\arctan x + \arctan y)$ is positive, so $0 < \arctan x + \arctan y < \frac{\pi}{2}$. In this case, we can safely take the arctan of both sides in $(*)$, and so the formula in $(**)$ does hold. Thus, we have found a way to add arctans.

We can use this formula to compute the first few partial sums S_n as follows:

$$\begin{aligned}
S_1 &= \arctan\frac{1}{2}, \\
S_2 &= \arctan\frac{1}{2} + \arctan\frac{1}{8} = \arctan\left(\frac{\frac{1}{2}+\frac{1}{8}}{1-\frac{1}{2}\cdot\frac{1}{8}}\right) = \arctan\frac{2}{3}, \\
S_3 &= \arctan\frac{2}{3} + \arctan\frac{1}{18} = \arctan\left(\frac{\frac{2}{3}+\frac{1}{18}}{1-\frac{2}{3}\cdot\frac{1}{18}}\right) = \arctan\frac{3}{4}, \\
S_4 &= \arctan\frac{3}{4} + \arctan\frac{1}{32} = \arctan\left(\frac{\frac{3}{4}+\frac{1}{32}}{1-\frac{3}{4}\cdot\frac{1}{32}}\right) = \arctan\frac{4}{5},
\end{aligned}$$

and so on. It appears that

$$\sum_{k=1}^{n} \arctan\frac{1}{2k^2} = \arctan\frac{n}{n+1}.$$

Sine we've already checked the first couple of terms, to confirm that this is the correct formula it suffices to show that

$$\arctan \frac{1}{2n^2} = \arctan \frac{n}{n+1} - \arctan \frac{n-1}{n}$$

for all positive integers n. Equivalently,

$$\arctan \frac{n-1}{n} + \arctan \frac{1}{2n^2} = \arctan \frac{n}{n+1}$$

for all positive integers n.

From our arctangent addition formula,

$$\begin{aligned}
\arctan \frac{n-1}{n} + \arctan \frac{1}{2n^2} &= \arctan \left(\frac{\frac{n-1}{n} + \frac{1}{2n^2}}{1 - \frac{n-1}{n} \cdot \frac{1}{2n^2}} \right) \\
&= \arctan \frac{n(2n^2 - 2n + 1)}{2n^3 - n + 1} \\
&= \arctan \frac{n(2n^2 - 2n + 1)}{(n+1)(2n^2 - 2n + 1)} \\
&= \arctan \frac{n}{n+1}.
\end{aligned}$$

Hence,

$$\arctan \frac{n-1}{n} + \arctan \frac{1}{2n^2} = \arctan \frac{n}{n+1}$$

for all positive integers n.

Therefore,

$$\begin{aligned}
\sum_{k=1}^{n} \arctan \frac{1}{2k^2} &= \sum_{k=1}^{n} \left(\arctan \frac{k}{k+1} - \arctan \frac{k-1}{k} \right) \\
&= \arctan \frac{1}{2} + \left(\arctan \frac{2}{3} - \arctan \frac{1}{2} \right) + \left(\arctan \frac{3}{4} - \arctan \frac{2}{3} \right) + \cdots \\
&\quad + \left(\arctan \frac{n-1}{n} - \arctan \frac{n-2}{n-1} \right) + \left(\arctan \frac{n}{n+1} - \arctan \frac{n-1}{n} \right) \\
&= \arctan \frac{n}{n+1}.
\end{aligned}$$

Finally, as $n \to \infty$, $n/(n+1) \to 1$, so

$$\sum_{k=1}^{\infty} \arctan \frac{1}{2k^2} = \arctan 1 = \frac{\pi}{4}.$$

$\square$

More formally we could put this together in the following way.

Solution: We first prove by induction that

$$\sum_{k=1}^{n} \arctan \frac{1}{2k^2} = \arctan \frac{n}{n+1}.$$

The base case, $n = 1$, is

$$\arctan \frac{1}{2 \cdot 1^2} = \frac{1}{2},$$

which holds trivially.

For the induction step, we assume that

$$\sum_{k=1}^{n-1} \arctan \frac{1}{2k^2} = \arctan \frac{n-1}{n}.$$

Now we would like to show that

$$\sum_{k=1}^{n} \arctan \frac{1}{2k^2} = \arctan \frac{n-1}{n} + \arctan \frac{1}{2n^2}.$$

The tangent of a sum identity gives

$$\tan(\arctan x + \arctan y) = \frac{x+y}{1-xy}$$

and if we apply that here we get

$$\begin{aligned} \tan\left(\arctan \frac{n-1}{n} + \arctan \frac{1}{2n^2}\right) &= \frac{\frac{n-1}{n} + \frac{1}{2n^2}}{1 - \frac{n-1}{n} \cdot \frac{1}{2n^2}} \\ &= \frac{n(2n^2 - 2n + 1)}{2n^3 - n + 1} \\ &= \frac{n(2n^2 - 2n + 1)}{(n+1)(2n^2 - 2n + 1)} \\ &= \frac{n}{n+1}. \end{aligned}$$

Since $\arctan \frac{n-1}{n}$ and $\arctan \frac{1}{2n^2}$ both take values in the range $(0, \frac{\pi}{4})$, their sum lives in the domain of arctangent, $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, applying arctangent to the left side of this equation returns the original argument. That means we can apply arctangent to the each side of the equation above to get

$$\arctan \frac{n-1}{n} + \arctan \frac{1}{2n^2} = \arctan \frac{n}{n+1}$$

for all positive integers n.

As n grows to infinity, the value of $\frac{n}{n+1}$ approaches 1 so the value of $\arctan \frac{n}{n+1}$ approaches $\frac{\pi}{4}$. That forces

$$\sum_{k=1}^{\infty} \arctan \frac{1}{2k^2} = \frac{\pi}{4}.$$

3 Linearly Recurrent Sequences

There is an important class of sequences known as *linearly recurrent* sequences. These are sequences in which the terms $x_0, x_1, x_2, \dots$ satisfy a recurrence of the form

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k}, \tag{$*$}$$

where $a_1, a_2, \dots, a_k$ are constants. A surprisingly large proportion of sequences can be described by a linear recurrence. We start with the simplest sequence that is defined by a linear recurrence.

Problem 3.1. The sequence (x_n) (which stands for the sequence $x_0, x_1, x_2, \dots$) is defined by $x_0 = c$ and $x_n = rx_{n-1}$ for all $n \geq 1$. Find x_n.

Solution: Clearly, (x_n) is a geometric sequence, with

$$x_n = cr^n.$$

□

This example may seem trivial, but it is important because it turns out that every sequence that satisfies a linear recurrence can be expressed as a combination of geometric sequences (or sequences that relate to geometric sequences).

We can then ask when the geometric sequence $x_n = cr^n$ satisfies the linear recurrence given in ($*$). Substituting, we see that this occurs if and only if

$$cr^n = ca_1 r^{n-1} + ca_2 r^{n-2} + \cdots + ca_k r^{n-k},$$

or

$$cr^n - ca_1 r^{n-1} - ca_2 r^{n-2} + \cdots - ca_k r^{n-k} = 0$$

for all $n \geq k$. This equation is trivially satisfied if $c = 0$ or $r = 0$, so assume that $c \neq 0$ and $r \neq 0$. We can then divide both sides by cr^{n-k}, to get

$$r^k - a_1 r^{k-1} - a_2 r^{k-2} - \cdots - a_k = 0.$$

Hence, the geometric sequence $x_n = cr^n$ satisfies $(*)$ if r is a root of the polynomial

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k.$$

We call this polynomial the *characteristic polynomial* of the linear recurrence defined in $(*)$. The next step is to figure out how to use the roots of this polynomial to solve the sequence, which we illustrate with an example.

Problem 3.2. The sequence (x_n) is defined by $x_0 = 2$, $x_1 = 3$, and $x_n = 3x_{n-1} - 2x_{n-2}$ for all $n \geq 2$. Find x_n.

Solution: The characteristic polynomial of the linear recurrence $x_n = 3x_{n-1} - 2x_{n-2}$ is $x^2 - 3x + 2$, which factors as $(x-1)(x-2)$. Therefore, the roots of the characteristic polynomial are 1 and 2. This gives us two classes of sequences, $c \cdot 1^n$ and $c2^n$, that solve the recurrence. Specifically, we know that the sequences

$$(1, 1, 1, 1, \ldots, 1, \ldots)$$

and

$$(1, 2, 4, 8, \ldots, 2^n, \ldots)$$

and all their multiples satisfy the linear recurrence $x_n = 3x_{n-1} - 2x_{n-2}$. However, no multiple of either sequence has $x_0 = 2$ and $x_1 = 3$. Hopefully we can combine these two in some way to get a solution with the correct initial conditions.

Suppose we have two sequences (u_n) and (v_n) that satisfy the same linear recurrence, so

$$\begin{aligned} u_n &= 3u_{n-1} - 2u_{n-2}, \\ v_n &= 3v_{n-1} - 2v_{n-2}. \end{aligned}$$

If we add the corresponding terms in the sequences (u_n) and (v_n), then this new sequence also satisfies the linear recurrence, because

$$u_n + v_n = 3(u_{n-1} + v_{n-1}) - 2(u_{n-2} + v_{n-2}).$$

These computations show that if we have sequences that satisfy a linear recurrence, then any linear combination of them also satisfy the linear recurrence.

Since we know that the sequences

$$(1, 1, 1, 1, \ldots, 1, \ldots)$$

and

$$(1, 2, 4, 8, \ldots, 2^n, \ldots)$$

satisfy the linear recurrence $x_n = 3x_{n-1} - 2x_{n-2}$. Therefore, any linear combination of these sequences, like

$$(c_1 + c_2, c_1 + 2c_2, c_1 + 4c_2, c_1 + 8c_2, \dots, c_1 + 2^n c_2, \dots),$$

also satisfies the linear recurrence $x_n = 3x_{n-1} - 2x_{n-2}$. That means all we have to do is match the first two terms $x_0 = 2$ and $x_1 = 3$, because the initial terms and the linear recurrence uniquely define the sequence.

To fit the condition $x_0 = 2$, we set $n = 0$ to get $c_1 + c_2 = 2$. To fit the condition $x_1 = 3$, we set $n = 1$ to get $c_1 + 2c_2 = 3$. Thus, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 2, \\ c_1 + 2c_2 &= 3. \end{aligned}$$

Solving this system of equations, we find $c_1 = c_2 = 1$, so we obtain the formula $x_n = 2^n + 1$. On a final note, we know that this formula works for all n, because it satisfies the linear recurrence and the initial conditions, and any sequence that is specified by a linear recurrence and initial conditions is uniquely defined. □

Of course the formal solution for such a problem is much simpler.

Solution: Note that the sequence is uniquely determined by the recurrence once the first two terms are given. Therefore we must find a sequence that satisfies this recurrence as well as the initial conditions. We claim that

$$x_n = 2^n + 1$$

is such a sequence.

Note that $x_0 = 2$ and $x_1 = 3$. Now we compute

$$\begin{aligned} 3x_{n-1} - 2x_{n-2} &= 3(2^{n-1} + 1) - 2(2^{n-2} + 1) \\ &= (6 - 2)2^{n-2} + 1 \\ &= 2^n + 1 \\ &= x_n, \end{aligned}$$

showing that $x_n = 3x_{n-1} - 2x_{n-2}$.

The relevant points for this solution are that we first argue that the solution is unique given the recurrence and initial conditions and then we present a sequence that satisfies these properties. It may look like magic to the reader and it is fairly worthless if we've made an error anywhere, but it is satisfactory as far as solving the problem is concerned.

We can solve a general linear recurrence as follows: Let the sequence (x_n) be defined by the initial terms x_0, $x_1, \dots, x_{k-1}$, and the linear recurrence

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k}$$

for all $n \geq k$. Let $r_1, r_2, \ldots, r_k$ be the roots of the characteristic polynomial

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k.$$

If these roots are distinct, then set

$$x_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n. \tag{**}$$

We can solve for the constants $c_1, c_2, \ldots, c_k$ by setting $n = 0, 1, \ldots, k-1$, to obtain the system of equations

$$\begin{aligned} c_1 + c_2 + \cdots + c_k &= x_0, \\ c_1 r_1 + c_2 r_1 + \cdots + c_k r_k &= x_1, \\ &\ldots, \\ c_1 r_1^{k-1} + c_2 r_2^{k-1} + \cdots + c_k r_k^{k-1} &= x_{k-1}. \end{aligned}$$

It is not obvious, but if $r_1, r_2, \ldots, r_k$ are distinct, then this system of equations always has a unique solution in $c_1, c_2, \ldots, c_k$. (We will not prove this claim here.)

For example, if the characteristic polynomial is $(x+2)(x+1)(x-5)$, then

$$x_n = c_1(-2)^n + c_2(-1)^n + c_3 5^n$$

for some constants c_1, c_2, and c_3. If the roots are not distinct (that is, we have repeated roots), then we must adjust this formula. If a root r has multiplicity $m > 1$, then the term cr^n is replaced by

$$c_1 r^n + c_2 n r^n + \cdots + c_m n^{m-1} r^n.$$

As an example of this, if the characteristic polynomial is $(x+2)^2(x+1)(x-5)^3$, then

$$x_n = c_1(-2)^n + c_2 n(-2)^n + c_3(-1)^n + c_4 5^n + c_5 n 5^n + c_6 n^2 5^n$$

for some constants $c_1, c_2, \ldots, c_6$. Again, we will not prove this here, but we will give an indication of where this form comes from in the Generating Functions handout, later in the course.

Problem 3.3. The sequence (x_n) is defined by $x_0 = 5$, $x_1 = 9$, $x_2 = 43$, and $x_n = 3x_{n-1} - 4x_{n-3}$ for all $n \geq 3$. Find x_n.

Solution: The characteristic polynomial of the linear recurrence $x_n = 3x_{n-1} - 4x_{n-3}$ is $x^3 - 3x^2 + 4$, which factors as $(x+1)(x-2)^2$. Therefore, the roots of the characteristic polynomial are -1, 2, and 2. Hence,

$$x_n = c_1(-1)^n + c_2 2^n + c_3 n 2^n$$

for some constants c_1, c_2, and c_3.

Setting $n = 0$, 1, and 2, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 5, \\ -c_1 + 2c_2 + 2c_3 &= 9, \\ c_1 + 4c_2 + 8c_3 &= 43. \end{aligned}$$

Solving this system of equations, we find $c_1 = 3$, $c_2 = 2$, and $c_2 = 4$. Therefore,

$$x_n = 3(-1)^n + 2 \cdot 2^n + 4n \cdot 2^n = 3(-1)^n + 2^{n+1} + n2^{n+2}.$$

□

Problem 3.4. Derive a formula for the n^{th} Fibonacci number F_n.

Solution: The characteristic polynomial of the recurrence $F_n = F_{n-1} + F_{n-2}$ is $x^2 - x - 1$. By the quadratic formula, the roots of $x^2 - x - 1$ are

$$x = \frac{1 \pm \sqrt{5}}{2},$$

so let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then $F_n = c_1\alpha^n + c_2\beta^n$ for some constants c_1 and c_2. Setting $n = 0$ and $n = 1$, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 0, \\ \alpha c_1 + \beta c_2 &= 1. \end{aligned}$$

Solving for c_1 and c_2, we find $c_1 = \frac{1}{\alpha-\beta} = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\alpha-\beta} = -\frac{1}{\sqrt{5}}$, so

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

□

We can also use the same technique to solve certain sequences that are not quite linearly recurrent, but close to it.

Problem 3.5. The sequence (x_n) is defined by $x_0 = 1$ and $x_n = 3x_{n-1} + 2n$ for all $n \geq 1$. Find x_n.

Solution: We can rewrite the recurrence as

$$x_n - 3x_{n-1} = 2n.$$

Strictly speaking, this is not a linear recurrence because of the term $2n$, so we focus on trying to eliminate it. If we substitute $n - 1$ for n, then we obtain another equation that contains the term $2n$:

$$\begin{aligned} x_n - 3x_{n-1} &= 2n, \\ x_{n-1} - 3x_{n-2} &= 2n - 2. \end{aligned}$$

Subtracting these equations, we get

$$x_n - 4x_{n-1} + 3x_{n-2} = 2.$$

We have successfully eliminated the $2n$, but we still do not have a linear recurrence because of the constant term 2.

To eliminate the constant term 2, we can substitute $n-1$ for n again:

$$x_n - 4x_{n-1} + 3x_{n-2} = 2,$$
$$x_{n-1} - 4x_{n-2} + 3x_{n-3} = 2.$$

Subtracting these equations, we get

$$x_n - 5x_{n-1} + 7x_{n-2} - 3x_{n-3} = 0.$$

This equation does represent a linear recurrence, which we know how to solve.

The characteristic polynomial of this linear recurrence is

$$x^3 - 5x^2 + 7x - 3 = (x-1)^2(x-3),$$

so

$$x_n = c_1 1^n + c_2 n 1^n + c_3 3^n = c_1 + c_2 n + c_3 3^n$$

for some constants c_1, c_2, and c_3. Setting $n = 0, 1$, and 2, we obtain the system of equations

$$c_1 + c_3 = x_0 = 1,$$
$$c_1 + c_2 + 3c_3 = x_1 = 5,$$
$$c_1 + 2c_2 + 9c_3 = x_2 = 19.$$

(Note that a_1 and a_2 are not given in the problem, but we can compute them with the recurrence that is given in the problem.) Solving this system of equations, we find $c_1 = -\frac{3}{2}$, $c_2 = -1$, and $c_3 = \frac{5}{2}$. Therefore,

$$x_n = -\frac{3}{2} - n + \frac{5}{2} \cdot 3^n = \frac{5 \cdot 3^n - 2n - 3}{2}.$$

□

More generally, we may have recurrence of the form

$$x_n - a_1 x_{n-1} - a_2 x_{n-2} - \cdots - a_k x_{n-k} = f(n),$$

where f is an arbitrary function. A recurrence of this form is sometimes called an *inhomogeneous recurrence*. If $f(n)$ itself satisfies a linear recurrence (such as $f(n) = 2n$, $f(n) = 2^n + 1$, or $f(n) = F_n$), then using the same technique of repeatedly shifting the index n, we can convert the inhomgeneous recurrence to a linear recurrence.

We have seen that we can solve a linear recurrence to obtain a solution of the form

$$x_n = c_1 n^{m_1} r_1^n + c_2 n^{m_2} r_2^n + \cdots + c_k n^{m_k} r_k^n.$$

But we can also run this process in reverse: Given a sequence of this form, we can say that it satisfies some linear recurrence. This idea of deriving the linear recurrence, rather than starting with it, can turn out to be very powerful.

Problem 3.6. Show that $F_n^2 + F_{n+1}^2 = F_{2n+1}$ for all $n \geq 0$.

Solution: Let $a_n = F_n^2 + F_{n+1}^2 - F_{2n+1}$. We know that

$$F_n = c_1\alpha^n + c_2\beta^n,$$

where $c_1 = \frac{1}{\sqrt{5}}$, $c_2 = -\frac{1}{\sqrt{5}}$, $\alpha = \frac{1+\sqrt{5}}{2}$, and $\beta = \frac{1-\sqrt{5}}{2}$. Then

$$\begin{aligned}
a_n &= F_n^2 + F_{n+1}^2 - F_{2n+1} \\
&= (c_1\alpha^n + c_1\beta^n)^2 + (c_1\alpha^{n+1} + c_2\beta^{n+1})^2 - (c_1\alpha^{2n+1} + c_2\beta^{2n+1}) \\
&= c_1^2\alpha^{2n} + 2c_1c_2\alpha^n\beta^n + c_2^2\beta^{2n} + c_1^2\alpha^{2n+2} + 2c_1c_2\alpha^{n+1}\beta^{n+1} + c_2^2\beta^{2n+2} - c_1\alpha^{2n+1} - c_2\beta^{2n+1} \\
&= c_1^2\alpha^{2n} + 2c_1c_2\alpha^n\beta^n + c_2^2\beta^{2n} + \alpha^2c_1^2\alpha^{2n} + 2\alpha\beta c_1c_2\alpha^n\beta^n + \beta^2c_2^2\beta^{2n} - \alpha c_1\alpha^{2n} - \beta c_2\beta^{2n} \\
&= (c_1^2 + \alpha^2c_1^2 - \alpha c_1)\alpha^{2n} + (c_2^2 + \beta^2c_2^2 - \beta c_2)\beta^{2n} + (2c_1c_2 + 2\alpha\beta c_1c_2)\alpha^n\beta^n.
\end{aligned}$$

At this point, we can plug in the values c_1, c_2, α, and β. However, if we look at this expression, we see that we can write it in the form

$$a_n = C_1(\alpha^2)^n + C_2(\beta^2)^n + C_3(\alpha\beta)^n,$$

where C_1, C_2, and C_3 are constants. Hence, the sequence (a_n) satisfies a linear recurrence, whose characteristic polynomial is

$$(x - \alpha^2)(x - \beta^2)(x - \alpha\beta).$$

We know that α and β are the roots of $x^2 - x - 1$, so by Vieta's formulas, $\alpha + \beta = 1$ and $\alpha\beta = -1$. Squaring $\alpha + \beta = 1$, we get $\alpha^2 + 2\alpha\beta + \beta^2 = 1$, so $\alpha^2 + \beta^2 = 1 - 2\alpha\beta = 3$. Squaring $\alpha\beta = -1$, we get $\alpha^2\beta^2 = 1$. Therefore, the characteristic polynomial is

$$\begin{aligned}
(x - \alpha^2)(x - \beta^2)(x - \alpha\beta) &= [x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2](x + 1) \\
&= (x^2 - 3x + 1)(x + 1) \\
&= x^3 - 2x^2 - 2x + 1.
\end{aligned}$$

Hence,

$$a_n = 2a_{n-1} + 2a_{n-2} - a_{n-3}$$

for all $n \geq 3$. Computing the initial terms of the sequence, we find $a_0 = 0$, $a_1 = 0$, and $a_2 = 0$. Therefore, $a_n = 0$ for all $n \geq 0$. But $a_n = F_n^2 + F_{n+1}^2 - F_{2n+1}$, so $F_n^2 + F_{n+1}^2 = F_{2n+1}$ for all $n \geq 0$. □

Problem 3.7. Let x, y, and z be real numbers such that $x + y + z = 0$. Prove that

$$\frac{x^2 + y^2 + z^2}{2} \cdot \frac{x^3 + y^3 + z^3}{3} = \frac{x^5 + y^5 + z^5}{5}.$$

Solution: Let $S_n = x^n + y^n + z^n$. If we multiply S_n by $x + y + z$ we get

$$\begin{aligned}(x+y+z)S_n &= (x^{n+1} + y^{n+1} + z^{n+1}) + (yx^n + zx^n + zy^n + xy^n + xz^n + yz^n) \\ &= S_{n+1} + (xy + yz + zx)S_{n-1} - (xyz)S_{n-2}.\end{aligned}$$

Therefore the sequence (S_n) satisfies a linear recurrence, whose characteristic polynomial is

$$t^3 - (x+y+z)t^2 + (xy + xz + yz)t - xyz = (t-x)(t-y)(t-z).$$

Let $A = xy + xz + yz$ and $B = xyz$. We are given that $x + y + z = 0$, so the characteristic polynomial can also be written as

$$t^3 + At - B.$$

Therefore,

$$S_n = -AS_{n-2} + BS_{n-3}$$

for all $n \geq 3$. To compute the terms of the sequence (S_n), we also require the initial terms of the sequence. We see that $S_0 = x^0 + y^0 + z^0 = 3$, $S_1 = x + y + z = 0$, and

$$S_2 = x^2 + y^2 + z^2 = (x+y+z)^2 - 2(xy + xz + yz) = -2A.$$

(If $x = 0$, then $y + z = 0$, and both sides of the given equation become 0. Otherwise, we may assume that x, y, and z are all nonzero, so S_0 is well-defined.)

Now we can use the linear recurrence to compute the terms of the sequence (S_n) up to S_5:

$$\begin{aligned}S_3 &= -AS_1 + BS_0 = 3B, \\ S_4 &= -AS_2 + BS_1 = 2A^2, \\ S_5 &= -AS_3 + BS_2 = -5AB.\end{aligned}$$

Therefore,

$$\frac{S_2}{2} \cdot \frac{S_3}{3} = -AB = \frac{S_5}{5}.$$

□

Problem 3.8. Let $P(x, y) = x^2y + xy^2$ and $Q(x, y) = x^2 + xy + y^2$. For $n = 1, 2, 3, \ldots$, let

$$\begin{aligned}F_n(x, y) &= (x+y)^n - x^n - y^n \\ G_n(x, y) &= (x+y)^n + x^n + y^n.\end{aligned}$$

One observes that $G_2 = 2Q$, $F_3 = 3P$, $G_4 = 2Q^2$, $F_5 = 5PQ$, $G_6 = 2Q^3 + 3P^2$. Prove that, in fact, for each n either F_n or G_n is expressible as a polynomial in P and Q with integer coefficients. (Putnam, 1976)

Solution: As suggested by the examples given in the problem, we claim that if n is even, then G_n can be expressed as a polynomial in P and Q with integer coefficients. Since

$$\begin{aligned}G_{2n} &= (x+y)^{2n} + x^{2n} + y^{2n} \\ &= [(x+y)^2]^n + (x^2)^n + (y^2)^n,\end{aligned}$$

the sequence (G_{2n}) satisfies a linear recurrence, whose characteristic polynomial is the product of the characteristic polynomials for the summands of G:

$$[t-(x+y)^2](t-x^2)(t-y^2).$$

This expands as

$$\begin{aligned}&[t^2-(x+y)^2][t-(x^2+y^2)t+x^2y^2]\\&\quad = t^3-(2x^2+2xy+2y^2)t^2+(x^4+2x^3y+3x^2y^2+2xy^3+y^4)t-x^2y^2(x+y)^2.\end{aligned}$$

We see that $2x^2+2xy+2y^2=2Q$, $x^2y^2(x+y)^2=(x^2y+xy^2)^2=P^2$, and

$$x^4+2x^3y+3x^2y^2+2xy^3+y^4=(x^2+xy+y^2)^2=Q^2,$$

so the characteristic polynomial can be written as

$$t^3-2Qt^2+Q^2t-P^2.$$

Therefore,

$$G_{2n}=2QG_{2n-2}-Q^2G_{2n-4}+P^2G_{2n-6}$$

for all $n\ge 4$. Furthermore, the initial terms of the sequence (G_{2n}) are $G_0=3$, $G_2=2Q$, and $G_4=2Q^2$. Hence, by a straightforward induction argument, G_{2n} can be expressed as a polynomial in P and Q with integer coefficients for all $n\ge 0$.

Similarly, we claim that if n is odd, then F_n can be expressed as a polynomial in P and Q. Since

$$\begin{aligned}F_{2n+1}&=(x+y)^{2n+1}-x^{2n+1}-y^{2n+1}\\&=(x+y)[(x+y)^2]^n-x(x^2)^n-y(y^2)^n,\end{aligned}$$

the sequence (F_{2n+1}) satisfies a linear recurrence, whose characteristic polynomial is the same as the characteristic polynomial for the sequence (G_{2n}), namely

$$[t-(x+y)^2](t-x^2)(t-y^2)=t^3-2Qt^2+Q^2t-P^2.$$

Therefore,

$$F_{2n+1}=2QF_{2n-1}-Q^2F_{2n-3}+P^2F_{2n-5}$$

for all $n\ge 3$. Furthermore, the initial terms of the sequence (F_{2n+1}) are $F_1=0$, $F_3=3P$, and $F_5=5PQ$. Hence, by a straightforward induction argument, F_{2n+1} can be expressed as a polynomial in P and Q with integer coefficients for all $n\ge 0$.

In summary, G_n can be expressed as a polynomial in P and Q with integer coefficients if n is even, and F_n can be expressed as a polynomial in P and Q with integer coefficients if n is odd. $\square$

Review Problems

1. If the integer k is added to each of the numbers 36, 300 and 596, one obtains the squares of three consecutive terms of an arithmetic sequence. Find k. (AIME, 1989)

2. Let $T_n = 1 + 2 + 3 + \cdots + n$ and

$$P_n = \frac{T_2}{T_2 - 1} \cdot \frac{T_3}{T_3 - 1} \cdot \frac{T_4}{T_4 - 1} \cdots \frac{T_n}{T_n - 1}$$

for $n = 2, 3, 4, \ldots$. Find P_{1991}. (AHSME, 1991)

3. Let F_n denote the n^{th} Fibonacci number. Prove that for all $n \geq 1$,

(a) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

(b) $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.

Hint: Both sums can be made to telescope.

4. Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

(Putnam, 1977)

5. Calculate the sum

$$\sum_{k=1}^{n} \frac{k}{k^4 + k^2 + 1}.$$

6. Evaluate the sum

$$\sum_{n=1}^{1994} (-1)^n \frac{n^2 + n + 1}{n!}.$$

(Canada, 1994)

7. Prove that for every positive integer n, and for every real number x *not* of the form $\frac{k\pi}{2^t}$, where $0 \leq t \leq n$ and k is an integer,

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x.$$

Sponsored by the D.E. Shaw group, Two Sigma Investments LLC, Jane Street Capital, Citadel LLC, and Dropbox

(IMO, 1966)

8. Let x, y, and z be real numbers such that $x + y + z = 0$. Prove that

$$\frac{x^2+y^2+z^2}{2} \cdot \frac{x^5+y^5+z^5}{5} = \frac{x^7+y^7+z^7}{7}.$$

9. Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

(AIME, 1990)

10. Let (x_n) be a sequence such that $x_0 = x_1 = 5$ and

$$x_n = \frac{x_{n-1} + x_{n+1}}{98}$$

for all positive integers n. Prove that $(x_n + 1)/6$ is a perfect square for all n.

11. Let a, b, and c be the roots of the equation $x^3 - x^2 - x - 1 = 0$. Show that a, b, and c are distinct, and that

$$\frac{a^{1982} - b^{1982}}{a - b} + \frac{b^{1982} - c^{1982}}{b - c} + \frac{c^{1982} - a^{1982}}{c - a}$$

is an integer. (Canada, 1982)

Challenge Problems

12. For $0 < x < 1$, express

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$$

as a rational function of x. (Putnam, 1977)

13. For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Putnam, 1980)

14. An integer sequence is defined by $a_0 = 0$, $a_1 = 1$, and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$. Prove that 2^k divides

a_n if and only if 2^k divides n. (IMO Short List, 1988)

15. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$ and

$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1})$$

for all $n = 1, 2, 3, \ldots,$ where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$. (IMO, 1979)

16. A sequence (a_n) is defined by $a_0 = a_1 = 0$, $a_2 = 1$, and $a_{n+3} = a_{n+1} + 1998a_n$ for all $n \geq 0$. Prove that $a_{2n-1} = 2a_na_{n+1} + 1998a_{n-1}^2$ for every positive integer n. (Komal)