IMO Winter Camp 2010 Buffet Contest

Algebra

A1 From an infinite arithmetic sequence $a_1, a_2, a_3 \cdots$ of positive real numbers, some (possibly infinitely many) terms are deleted, obtaining an infinite geometric sequence $1, r, r^2, r^3, \cdots$ for some real number $r > 0$. Prove that r is an integer.

A2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

A3 Find all finite sets A of distinct non-negative real numbers for which:

(a) the set A contains at least four numbers.

(b) for any 4 distinct numbers $a, b, c, d \in A$, the number $ab + cd \in A$.

Number Theory

N1 Let a, b be rational numbers such that $a + b$ and $a^2 + b^2$ are integers. Prove that a, b are both integers.

N2 Find all pairs of positive integers (a, b) such that the sequence of positive integers $a_1, a_2, a_3, \cdots$, formed by $a_1 = a, a_2 = b$ and

$$a_n = \frac{a_{n-1} + a_{n-2}}{\gcd(a_{n-1}, a_{n-2})},$$

for $n \geq 3$, is bounded. (A bounded sequence is a sequence for which there exists a positive real number M such that $a_n \leq M$ for all $n \in \mathbb{N}$.)

N3 For a positive integer n, let $f(n)$ be the largest prime divisor of n. Prove that there are infinitely many positive integers n such that

$$f(n) < f(n+1) < f(n+2).$$

Combinatorics

C1 Let $n \geq 2$ be a positive integer. An $n \times n$ grid is filled with the integers $1, 2, \cdots, n^2$. Let t be the maximum of the (positive) difference of the entries of two neighbouring squares, where two squares are said to be neighbours if they share at least one vertex. Determine the minimum possible value of t in terms of n.

C2 A chessboard is tiled with 32 dominoes. Each domino covers two adjacent squares, a white and a black square. Show that the number of horizontal dominoes with the white square on the left of the black square equals the number of horizontal dominoes with the white square on the right of the black square.

C3 Let $n > 1$ be a positive integer. On each of $2n$ points around a circle we place a disk with one white side and one black side. We may perform the following move: select a black disk, and reverse its two neighbors. Find all initial configurations from which some sequence of such moves leads to a position where all disks but one are white.

Geometry

G1 The altitude from A of triangle ABC intersects the side BC at D. A circle is tangent to BC at D, intersects AB at M and N, and intersects AC at P and Q. Prove that

$$\frac{AM + AN}{AC} = \frac{AP + AQ}{AB}.$$

G2 Let $\mathcal{P}$ be a convex 2010-gon. The 1005 diagonals connecting opposite vertices and the 1005 lines connecting the midpoints of opposite sides are concurrent. (i.e. all 2010 lines are concurrent.) Prove that the opposite sides of $\mathcal{P}$ are parallel and have the same length.

G3 Two circles meet at A and B. Line ℓ passes through A and meets the circles again at C and D respectively. Let M and N be the midpoints of arcs BC and BD which do not contain A, and let K be the midpoint of CD. Prove that $\angle MKN = 90°$.

IMO Winter Camp 2010 Buffet Contest Solutions

Algebra

A1 From an infinite arithmetic sequence $a_1, a_2, a_3 \cdots$ of positive real numbers, some (possibly infinitely many) terms are deleted, obtaining an infinite geometric sequence $1, r, r^2, r^3, \cdots$ for some real number $r > 0$. Prove that r is an integer.

Solution: The arithmetic sequence contains 1. Let $1, d+1, 2d+1, \cdots$ be the given arithmetic sequence starting at 1. This sequence contains $1, r, r^2$. Then $r = 1 + md$ and $r^2 = 1 + nd$ for some integers m, n. If $d = 0$, then $r = 1$, which is clearly an integer. Otherwise, $m = \frac{r-1}{d}$ and $n = \frac{r^2-1}{d}$ are integers. But $\frac{r^2-1}{d} = \frac{r-1}{d} \cdot (r+1)$. This implies $r+1$ is a rational number. Hence, r is rational. But r is a term in the arithmetic sequence $1, d+1, 2d+1, \cdots$. This implies this arithmetic sequence has a rational common difference and consequently, contains only rational numbers. Let f be the denominator of the rational common difference. Then all terms in the arithmetic sequence have denominator at most f.

If r is not an integer, then if we write $r = \frac{a}{b}$ with $gcd(a,b) = 1$ and $b > 1$, then the denominators of $1, r, r^2, r^3, \cdots$ are strictly increasing, and will eventually exceed f. But then these terms whose denominator is larger than f cannot be in the given arithmetic sequence. This is a contradiction. Therefore, r must be an integer. $\square$

Source: IberoAmerican Mathematical Olympiad 2000

A2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

Solution: The answers are $f(x) \equiv 0, x^2$.

Subsituting $y = -f(x)$ yields

$$f(0) = f(x^2 + f(x)) - 4f(x)^2, \forall x \in \mathbb{R}. \tag{1}$$

Substituting $y = x^2$ yields

$$f(f(x) + x^2) = f(0) + 4f(x)x^2, \forall x \in \mathbb{R}. \tag{2}$$

Comparing equations (1) and (2) yields

$$4f(x)^2 = 4f(x)x^2$$

for all $x \in \mathbb{R}$. Equivalently, $f(x)(f(x) - x^2) = 0$. Therefore, for each $x \in \mathbb{R}$, $f(x) = 0$ or $f(x) = x^2$. I claim that exactly one of these conditions hold for all $x \in \mathbb{R}$. Suppose $f(a) = 0$ for some $a \neq 0$. I claim that $f \equiv 0$. Then substituting $x = a$ yields

$$f(y) = f(a^2 - y),$$

for all $y \in \mathbb{R}$. Suppose $f(y) = y^2$ for some $y \neq 0, \frac{a^2}{2}$. Then $f(a^2 - y) = y^2 \neq 0$. Hence, $(f(a^2 - y) = (a^2 - y)^2$, implying $(a^2 - y)^2 = y^2$. This simplifies to $a^2(a^2 - 2ay) = 0$. This contradicts $a \neq 0$ and $y \neq \frac{a^2}{2}$. Therefore, $f(y) = 0$ for all $y \neq \frac{a^2}{2}$. Choose z such that $z \neq \pm a$. By the same argument, $f(y) = 0$ for all $y \neq \frac{z^2}{2}$.

Since $\frac{a^2}{2} \neq \frac{z^2}{2}$, $f(\frac{a^2}{2}) = 0$. Hence, $f \equiv 0$.

Therefore, $f(x) \equiv 0, x^2$ are the only candidate solutions. It remains to verify that both solutions work. If $f(x) \equiv 0$, then $f(f(x) + y) = 0$ and $f(x^2 - y) + 4f(x)y = 0$. If $f(x) \equiv x^2$, then $f(f(x) + y) = (x^2 + y)^2$ and $f(x^2 - y) + 4f(x)y = (x^2 - y)^2 + 4x^2y = (x^2 + y)^2$, as desired. $\square$

Source: Iranian Mathematical Olympiad 1999

Comments: In the step $f(x)(f(x) - x^2) = 0$, a costly mistake is to conclude here that $f(x) \equiv 0$ and $f(x) \equiv x^2$ are the only solutions. It is still possible that f is a function that takes on 0 for some non-trivial values of x and x^2 for the other values of x. You must handle this with care in similar problems. Try the following problem from the 2008 International Mathematical Olympiad where you have to handle a similar situation.

Exercise: Find all functions $f : (0, \infty) \to (0, \infty)$ (so f is a function on the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2},$$

for all positive real numbers w, x, y, z such that $wx = yz$.

A3 Find all finite sets A of distinct non-negative real numbers for which:

(a) the set A contains at least four numbers.

(b) for any 4 distinct numbers $a, b, c, d \in A$, the number $ab + cd \in A$.

Solution: The answer is $\{0, \frac{1}{x}, 1, x\}$ for some $x > 0$.

We first prove a lemma.

Lemma: Let a, b, c, d be non-negative real numbers such that $a < b < c < d$. Then $ad + bc < ac + bd < ab + cd$.

Proof of Lemma: This follows from the fact that $(ab + cd) - (ac + bd) = (d - a)(c - b) > 0$ and $(ac + bd) - (ad + bc) = (b - a)(d - c) > 0$. *End Proof of Lemma*[1]

Suppose $n = |A| \geq 5$. Let $a_1, \cdots, a_n \in A$ with $a_1 < a_2 < \cdots < a_n$. Then by the Lemma,

$$a_1a_4 + a_2a_3 < a_1a_2 + a_3a_4 < a_1a_2 + a_3a_5 < \cdots < a_1a_2 + a_3a_n,$$

and

$$a_1a_2 + a_3a_n < a_1a_2 + a_4a_n < \cdots < a_1a_2 + a_{n-1}a_n,$$

and

$$a_1a_2 + a_{n-1}a_n < a_1a_3 + a_{n-1}a_n < \cdots < a_1a_{n-2} + a_{n-1}a_n$$

and

$$a_1a_{n-2} + a_{n-1}a_n < a_2a_{n-2} + a_{n-1}a_n < \cdots < a_{n-3}a_{n-2} + a_{n-1}a_n.$$

There are $(n - 2) + (n - 4) + (n - 4) + (n - 4) = 4n - 14$ elements in these inequalities. Since $n \geq 5$, $4n - 14 > n$. Hence, the $4n - 14$ terms are pairwise distinct and all in A. This is impossible since $|A| = n$

[1] This result is simply the rearrangement inequality on four variables.

and $4n - 14 > n$.

Therefore, $|A| = 4$. Let $A = \{a, b, c, d\}$ with $a < b < c < d$. Let $x = ad + bc, y = ac + bd, z = ab + cd$. Then by the Lemma, $x < y < z$. By property (b), $x, y, z \in A$. Since $A = \{a, b, c, d\}$, $\{x, y, z\} \subseteq \{a, b, c, d\}$. Therefore, $x = a$ or $x = b$.

If $x = a$, then $ad + bc = a$. Hence, $bc = a(1 - d)$. Since $bc > 0$, $d < 1$. Note that $ac + bd = b$ or c, implying $ac + bd \geq b$. Then $b - a \leq (ac + bd) - (ad + bc) = (b - a)(d - c)$. Hence, $d - c \geq 1$, contradicting $d < 1$. Hence, $x \neq a$. Since $x < y < z$ and $x, y, z \in A$ and $A = \{a, b, c, d\}$ and $x \neq a$, we conclude that $x = b, y = c, z = d$.

Hence, $ad + bc = b, ac + bd = c, ab + cd = d$. Rewriting these equations give us $ad = b(1 - c), bd = c(1 - a), ab = d(1 - c)$. Subtracting the first equation from the third equation yields $(1 - c + a)(d - b) = 0$. Since $d > b$, $c - a = 1$. Substituting this into the first equation yields $a(d + b) = 0$. Since $d > b$, $a = 0$ and $c = 1$. Substituting this into the second equation yields $bd = 1$. Therefore, $b = \frac{1}{d}$. Since $b < d$, $\frac{1}{d} < 1 < d$. It is easy to verify that that the set $\{0, \frac{1}{d}, 1, d\}$ satisfy the given condition for $d > 1$. This finishes the problem. $\square$

Source: Bulgarian Mathematical Olympiad 1998

Number Theory

N1 Let a, b be rational numbers such that $a + b$ and $a^2 + b^2$ are integers. Prove that a, b are both integers.

Solution 1: Let $a = \frac{x}{z}$ for some integers x, z with $z, > 0$ and $gcd(x, z) = 1$. Then $b = k - \frac{x}{z} = \frac{kz - x}{z}$ for some integer k. Since $gcd(kz - x, z) = gcd(x, z) = 1$, the denominator of b in lowest terms is also z. Hence, we can let $b = \frac{y}{z}$ with $gcd(y, z) = 1$.

Since $a + b$ and $a^2 + b^2$, $(a + b)^2 - (a^2 + b^2) = 2ab$ is also an integer. Hence, $\frac{2xy}{z^2}$ is an integer. Since $gcd(x, z) = gcd(y, z) = 1, gcd(xy, z^2) = 1$. Hence, z^2 divides 2. This implies $z = 1$. Therefore, a, b are integers. $\square$

Solution 2: Using the same notation as Solution 1, we note that $2(a^2 + b^2) - (a + b)^2 = (a - b)^2$ is an integer. Since a, b is rational, $a - b$ is rational. Hence, $a - b$ is an integer. Since $a + b$ is an integer, $2a, 2b$ are integers. If at least one of a, b is an integer, then since $a + b$ is an integer, the other of a, b is also an integer and we are done. Otherwise, a, b are both not integers, implying a, b have denominator 2 when expressed in lowest terms. Let $a = \frac{x}{2}, b = \frac{y}{2}$, where x, y are odd integers. Since $a^2 + b^2$ is an integer, 4 divides $x^2 + y^2$. But $x^2, y^2 \equiv 1 \bmod 4$ since x, y are odd. Hence, $x^2 + y^2 \equiv 2 \bmod 4$, implying $4 \nmid x^2 + y^2$. This is a contradiction. Therefore, a, b are integers. $\square$

Source: Russian Math Olympiad 2006 Variant

N2 Find all pairs of positive integers (a, b) such that the sequence of positive integers $a_1, a_2, a_3, \cdots$, formed by $a_1 = a, a_2 = b$ and

$$a_n = \frac{a_{n-1} + a_{n-2}}{\gcd(a_{n-1}, a_{n-2})},$$

for $n \geq 3$, is bounded. (A bounded sequence is a sequence for which there exists a positive real number M such that $a_n \leq M$ for all $n \in \mathbb{N}$.)

Solution: The answer is $(a, b) = (2, 2)$.

First note if for two consecutive terms a_k, a_{k+1} satisfy $\gcd(a_k, a_{k+1}) = 1$, then $\gcd(a_{k+2}, a_{k+1}) = \gcd(a_k + a_{k+1}, a_{k+1}) = gcd(a_k, a_{k+1}) = 1$. Iteratively, we have that $\gcd(a_n, a_{n+1}) = 1$ for all $n \geq k$. Hence, for all $n \geq k+2$, $a_n = a_{n-1} + a_{n-2}$ and so the sequence is strictly increasing since each term is a positive integer. Hence, the sequence is not bounded. Henceforth, we will assume that no two consecutive terms in the sequence are coprime.

Then for all positive integers n, we have $a_{n+2} \leq \frac{a_n + a_{n+1}}{2} \leq \max(a_n, a_{n+1})$ with equality iff $a_n = a_{n+1}$ and $\gcd(a_n, a_{n+1}) = 2$. Hence $\max(a_n, a_{n+1})$ is non-increasing. Since this number is always an integer, eventually this max becomes constant, so for some m, $\max(a_m, a_{m+1}) = \max(a_{m+1}, a_{m+2}) = \max(a_{m+2}, a_{m+3}) = \cdots = t$. If $a_m \neq a_{m+1}$, then $a_{m+2} < t, a_{m+3} < t$ so $\max(a_{m+2}, a_{m+3}) < t$, which is impossible.

Hence $a_m = a_{m+1} = a_{m+2} = \cdots = t$ and $\gcd(a_m, a_{m+1}) = 2$. Therefore, $t = 2$. Finally, we now find the values of $a_{m-1}, a_{m-2}, \cdots, a_1$. Since $2 = a_{m+1} = \frac{a_{m-1} + a_m}{\gcd(a_{m-1}, a_m)} = \frac{a_{m-1} + 2}{\gcd(a_{m-1}, 2)} \geq \frac{a_{m-1} + 2}{2}$. Hence, $a_{m-1} \leq 2$. If $a_{m-1} = 1$, then $\gcd(a_{m-1}, a_m) = 1$, which is not allowed. Therefore, $a_{m-1} = 2$. Hence, iteratively, we have that $a_{m-1} = a_{m-2} = \cdots = a_1 = 2$.

So the only such sequence is the sequence where all terms are equal to 2. Hence, $(a, b) = (2, 2)$ is the only solution. $\square$

Source: Russian Math Olympiad 1999

N3 For a positive integer n, let $f(n)$ be the largest prime divisor of n. Prove that there are infinitely many positive integers n such that

$$f(n) < f(n+1) < f(n+2).$$

Solution 1: Let $p \geq 5$ be an odd prime number and let $m_k = p^{2^k} - 1$ and $n_k = p^{2^k} + 1$ $(= m_k + 2)$. Clearly, $f(m_k + 1) = p$. Note also that

$$n_k - 2 = m_k = (p-1)(p+1)(p^2+1)\cdots(p^{2^{k-1}} - 1) = (p-1)n_0 n_1 \cdots n_{k-1}.$$

Note that $n_k = p^{2^k} + 1 \equiv 2 \bmod 4$. Hence, $4 \nmid n_k$ and $n_k > 4$. Therefore, n_k contains an odd prime factor. Suppose q is an odd prime such that $q \mid n_k, n_i$ for some $0 \leq i < k-1$. Then $q|2$, which is impossible. Therefore, n_k contains an odd prime factor which is not a prime factor of $n_0, n_1, \cdots, n_{k-1}$. Hence, let k be the smallest non-negative integer such that $f(n_k) > p$. Since $n_0 = p+1$ and $p+1$ is composite, implying $f(n_0) < p$. Therefore, $k \geq 1$. Hence, $f(m_k + 2) > p = f(m_k + 1)$.

Finally, note that $m_k = (p-1)n_0 n_1 n_2 \cdots n_{k-1}$. By the choice of k and the fact that $p \nmid n_i$, we have $f(n_i) < p$ for each $i \in \{0, 1, \cdots, k-1\}$. $p+1$ is composite, which implies $f(p+1) < p$. Hence, $f(m_k) < p$. Therefore, $f(m_k) < f(m_k+1) < f(m_k+2)$. We now choose another prime p such that $p > m_k$ to generate another integer n larger than m_k such that $f(n) < f(n+1) < f(n+2)$. We repeat this process similarly to generate infinitely many such positive integers. This completes the problem. $\square$

Solution 2: Suppose the statement is false, i.e. there are only finitely many integers n such that $f(n) < f(n+1) < f(n+2)$. Consider the number $n = 2^k$ for some positive integer k. Note that $f(n) = 2$ and $f(n+1) > 2 = f(n)$. Since there are only finitely many k such that $f(2^k) < f(2^k+1) < f(2^k+2)$, there exists a positive integer m such that $k \geq m$ implies $f(2^k) < f(2^k+1)$ and $f(2^k+2) \leq f(2^k+1)$. Since $gcd(2^k+1, 2^k+2) = 1$, $f(2^k+2) < f(2^k+1)$.

For $k \geq m$, note that $f(2^k+2) = f(2^{k-1}+1)$, since every odd factor of 2^k+2 (which is ≥ 2) is also an odd factor of $2^{k-1}+1$. Therefore, for $k \geq m$, $f(2^{k-1}+1) < f(2^k+1)$. Hence, $f(2^m+1), f(2^{m+1}+1), f(2^{m+2}+1), \cdots$ is a strictly increasing sequence of integers. (*)

Let $k \geq m$. Consider $n = 2^{2(2k-1)} + 1 = (2^{2k-1}+1)^2 - 2 \cdot 2^{2k-1} = (2^{2k-1} - 2^k + 1)(2^{2k-1} + 2^k + 1)$. Therefore, $f(2^{2(2k-1)}+1) = f(2^{2k-1}+2^k+1)$ or $f(2^{2(2k-1)}+1) = f(2^{2k-1}-2^k+1)$. If the former holds, then

$$f(2^{2(2k-1)}+1) = f(2^{2k-1}+2^k+1) = f(2(2^{2k-1}+2^k+1)) = f((2^k+1)^2) = f(2^k+1).$$

Since $2(2k-1) > k \geq m$, this contradicts (*). Therefore, $f(2^{2(2k-1)}+1) = f(2^{2k-1}-2^k+1)$ for all $k \geq m$. Since $f(2^{2(2k-1)}+1) > f(2^k+1)$ (since $k \geq m$), we have that $f(2^{2k-1}-2^k+1) > f(2^k+1)$. Let $n = 2^{2k} - 2^{k+1} = 2^{k+1}(2^{k-1}-1)$. Then $f(n) = f(2^{k-1}-1)$, $f(n+1) = f(2^{2k}-2^{k+1}+1) = f((2^k-1)^2) = f(2^k-1)$ and $f(n+2) = f(2^{2k}-2^{k+1}+2) = f(2^{2k-1}-2^k+1) > f(2^k-1) = f(n+1)$. If $f(n) < f(n+1)$ for infinitely many choices of k, then we are done. Otherwise, there exists a positive integer M such that $M > m$ and $k \geq M$ implies $f(n) \geq f(n+1)$, i.e. $f(2^{k-1}-1) \geq f(2^k-1)$. Since $(2^k-1) - 2(2^{k-1}-1) = 1$, $gcd(2^{k-1}-1, 2^k-1) = 1$. Hence, $f(2^{k-1}-1) > f(2^k-1)$ for all $k \geq M$. Hence, for $f(2^M-1), f(2^{M+1}-1), f(2^{M+2}-1), \cdots$ is an infinite strictly decreasing sequence of positive integers, which is absurd. Hence, the statement of the problem is indeed true. $\square$

Source: Brazilian Mathematical Olympiad 1995

Combinatorics

C1 Let $n \geq 2$ be a positive integer. An $n \times n$ grid is filled with the integers $1, 2, \cdots, n^2$ (with each number used exactly once). Let t be the maximum of the (positive) difference of the entries of two neighbouring squares, where two squares are said to be neighbours if they share at least one vertex. Determine the minimum possible value of t in terms of n.

Solution: The answer is $t = n + 1$.

The number of squares in the shortest sequence of successive neighbouring squares between any two squares is at most $n-1$. Since both $1, n^2$ are in the grid and differ by $n^2 - 1$, there are two neighbouring squares that differ by at least $\frac{n^2-1}{n-1} = n+1$. Hence, $t \geq n+1$. Consider the grid filled in order row by row $1, 2, \cdots, n^2$, i.e. place $(i-1)n + j$ in the i^{th} row and the j^{th} column. Two entries in such a configuration differ by at most $n+1$. Hence, $t = n+1$. $\square$

Source: Unknown

C2 A chessboard is tiled with 32 dominoes. Each domino covers two adjacent squares, a white and a black square. Show that the number of horizontal dominoes with the white square on the left of the black square equals the number of horizontal dominoes with the white square on the right of the black square.

Solution: Let S_i be the set of horizontal dominoes contained in columns i and $i+1$ (obviously, all S_i are disjoint). Then $S_i = L_i \cup R_i$, where L_i is the set of such dominoes with a white square on the left, and R_i is the set of such dominoes with a white square on the right. Since each column has an even number of rows, and an even number of those rows is taken up by vertical dominoes (each of them takes 2 rows), the horizontal dominoes must also occupy an even number of rows, so $|S_{i-1} \cup S_i|$ is even for each i. Since $|S_1|$ is even, we must have that $|S_i|$ is even for each i.

Consider the horizontal dominoes that intersect the first column of the board. The space between any consecutive pair of horizontal dominoes is taken up by vertical dominoes. Each of those occupies 2 rows, so there is an even number of rows between the horizontal dominoes. Thus, if one of those dominoes is in L_1, the other is in R_1, and vice-versa. Since $|S_1|$ is even, we must have $|L_1| = |R_1|$.

We prove by induction that $|L_i| = |R_i|$ for each i. The base case is $i = 1$, shown above. Suppose $|L_{i-1}| = |R_{i-1}|$. The set of horizontal dominoes that intersect column i is $S_{i-1} \cup S_i$. As in the $i = 1$ case, any consecutive pair of these dominoes has an even number of rows between them. Thus, the number of dominoes that use a black square in column i ($|L_{i-1} \cup R_i|$) is the same as the number of dominoes that use a white square in column i ($|R_{i-1} \cup L_i|$). However, since $|L_{i-1}| = |R_{i-1}|$, we must have that $|L_i| = |R_i|$.

Therefore, the total number of horizontal dominoes with a white square on the left ($| \cup_i L_i|$) equals the total number of horizontal dominoes with a white square on the right ($| \cup_i R_i|$). $\square$

Source: IMO Training 2006, General Problems

C3 Let $n > 1$ be a positive integer. On each of $2n$ points around a circle we place a disk with one white side and one black side. We may perform the following move: select a black disk, and reverse its two neighbors. Find all initial configurations from which some sequence of such moves leads to a position where all disks but one are white.

Solution: The answer is when the number of black disks initially is odd.

Note that each move preserves the parity of the number of black disks on the circle. Hence, if the initial number of black disks is even, then we can never have one black disk remaining.

Suppose there is an odd number of black disk. We define a *group* to be a maximal set of black disks that appear on consecutive points on the circle. Let k be the number of groups initially. I claim that if $k > 1$, after a finite number of moves, we can decrease the number of groups. Since there are an odd number of black disks, one group contains an odd number of disks. Suppose there are $2m - 1$ disks in this group for some positive integer m. We label the disks $1, 2, \cdots, 2m - 1$ in clockwise order. By assumption, the disk next to 1 (which is not 2), which we now call 0, is white and the disk next to $2m-1$ (which is not $2m-2$), which we will call $2m$, is also white. Note that since $k > 1$, disks 0 and $2m$ are distinct. We perform a move on disks 1, 3, $5, \cdots, 2m - 1$, in this order. In doing so, the disks $2, 4, \cdots, 2m - 2$ are each flipped twice, and therefore, remain black. The only disks flipped are 0 and $2m$, which are now flipped to black. Hence, this group increased by size at least 2. If it increased by more than 2, then this group merged with another group and the number of groups decreased. Otherwise, this group still has odd cardinality and we can repeat this process. Since there is another group, eventually, this group will merge with another group. Hence, we have decreased the number of groups. We repeat this process (by choosing another odd group). Eventually, the number of groups becomes one.

By performing the steps in the previous paragraph, we have one group of black disks. There are an odd number of disks in this group, say $2m - 1$ for some positive integer m. We again label the disks $1, 2, \cdots, 2m - 1$. If $m = 1$, then only one black disk remain and we are done. Otherwise, we choose the disks $2, 4, \cdots, 2m - 2$. Hence, disks $3, 5, \cdots, 2m - 3$ are flipped twice, and thus remain the same colour. But disks 1 and $2m - 1$ are changed to white. Hence, the group size decreased by 2. We can repeat this procedure until one black disk remains. We are done. $\square$

Source: Japanese Mathematical Olympiad 1998

Geometry

G1 The altitude from A of triangle ABC intersects the side BC at D. A circle is tangent to BC at D, intersects AB at M and N, and intersects AC at P and Q. Prove that

$$\frac{AM + AN}{AC} = \frac{AP + AQ}{AB}.$$

Solution 1: By Power of a Point, we have that $BM \cdot BN = BD^2$ and $CP \cdot CQ = CD^2$. Hence, $(AB - AM)(AB - AN) = BD^2 \Rightarrow AB^2 - AB(AM + AN) + AM \cdot AN = BD^2 \Rightarrow AB^2 - BD^2 = AB(AM + AN) - AM \cdot AN$. Similarly, $AC^2 - CD^2 = AC(AP + AQ) - AP \cdot AQ$. But $AB^2 - BD^2 = AD^2 = AC^2 - CD^2$ by Pythagorean Theorem and $AM \cdot AN = AP \cdot AQ$ by Power of a Point. Hence, $AB(AM + AN) = AC(AP + AQ)$, as desired. $\square$

Solution 2: Let O be the centre of the circle. Since the circle is tangent to BC at D and $AD \perp BC$, O lies on AD. Let U, V be the feet of the perpendicular on AB, AC from O, respectively. Therefore, U, V are midpoints of MN, PQ, respectively. This implies $AM + AN = 2 \cdot AU$ and $AP + AQ = 2 \cdot AV$. Hence, it suffices to show that

$$\frac{AU}{AC} = \frac{AV}{AB}.$$

Since $AU = AO \cos \angle BAO, AC = \frac{AD}{\cos \angle CAO}, AV = AO \cos \angle CAO, AB = \frac{AD}{\cos \angle BAO}$, this equation is true, as desired $\square$. [2]

Source: IMO Correspondence program, 1995-96

G2 Let $\mathcal{P}$ be a convex 2010-gon. The 1005 diagonals connecting opposite vertices and the 1005 lines connecting the midpoints of opposite sides are concurrent. (i.e. all 2010 lines are concurrent.) Prove that the opposite sides of $\mathcal{P}$ are parallel and have the same length.

Solution: We first prove a lemma.

Lemma: Let $ABCD$ be a convex quadrilateral, M the midpoint of AB, N the midpoint of CD. Suppose AC, BD, MN are concurrent at a point P. Then $AB || CD$ and $\Delta PAB \sim \Delta PCD$.

Proof of Lemma: Let l be a line passing through B parallel to CD and intersecting PA at a point A'. Let M' be the midpoint of $A'B$. Since $\Delta PA'B \sim \Delta PCD$, $\Delta PAM' \sim \Delta PCN$. Hence, $\angle A'PM' = \angle CPN$, i.e. M' lies on PN. Suppose $A \neq A'$. Then $M \neq M'$. Since M also lies on PN, MM' is parallel to AP. But they intersect at P, contradicting the fact that they are parallel. Therefore, $A = A'$. Therefore, $AB || CD$. Subsequently, $\angle PBA = \angle PDC$ and $\angle PAB = \angle PCD$, we have that $\Delta PAB \sim \Delta PCD$. *End Proof of Lemma*

Let $A_1, A_2, \cdots, A_{2010}$ be the vertices appearing clockwise of the 2010-gon and M_i the midpoint of A_i and A_{i+1}. Then A_i and A_{i+1005} are opposite vertices. Let P be the concurrent point of the 2010 lines. Since $A_iA_{i+1005}, A_{i+1}A_{i+1006}, M_iM_{1005}$ are concurrent at P, consider the quadrilateral $A_iA_{i+1}A_{i+1005}A_{i+1006}$. Since the polygon is convex, this quadrilateral is convex. By the Lemma, we have that A_iA_{i+1} is parallel to $A_{i+1005}A_{i+1006}$. Hence, opposite sides of the polygon are parallel.

By the similarities, we have that

$$\frac{PA_1}{PA_{1006}} = \frac{PA_2}{PA_{1007}} = \frac{PA_3}{PA_{1008}} = \cdots = \frac{PA_{2010}}{PA_{1005}}.$$

More importantly, we have

$$\frac{PA_i}{A_{i+1005}} = \frac{PA_{i+1005}}{PA_{i+2010}} = \frac{PA_{i+1005}}{PA_i},$$

since the indices are taken modulo 2010. Hence, $|PA_i| = |PA_{i+1005}|$. Therefore, since by the Lemma that $\Delta PA_iA_{i+1} \sim \Delta PA_{i+1005}A_{i+1006}$, we have $|A_iA_{i+1}| = |A_{i+1005}A_{i+1006}|$. Hence, the opposite sides of the

[2]We can also prove $\frac{AU}{AC} = \frac{AV}{AB}$ by noting $\Delta AOU \sim \Delta ABD$ and $\Delta AOV \sim \Delta ACD$.

polygon are equal in length. $\square$

Source: Brazilian Mathematical Olympiad 2006, modified in the obvious way

Alternate Proof of Lemma: It suffices to prove $\frac{AP}{BP} = \frac{CP}{DP}$. By Sine Law, we have that

$$\frac{AP}{BP} = \frac{AP}{AM} \cdot \frac{BM}{BP} = \frac{\sin\angle AMP}{\sin\angle APM} \cdot \frac{\sin\angle BPM}{\sin\angle BMP} = \frac{\sin\angle BPM}{\sin\angle APM},$$

since $\angle AMP + \angle BMP = 180°$. Similarly, we have

$$\frac{CP}{DP} = \frac{\sin\angle DPN}{\sin\angle CPN} = \frac{\sin\angle BPM}{\sin\angle APM}.$$

Therefore, $\frac{AP}{BP} = \frac{CP}{DP}$. *End Proof of Lemma.*

G3 Two circles meet at A and B. Line ℓ passes through A and meets the circles again at C and D respectively. Let M and N be the midpoints of arcs BC and BD which do not contain A, and let K be the midpoint of CD. Prove that $\angle MKN = 90°$.

Solution 1: Let X, Y be the midpoints of BC, BD, respectively. Note that $MX \perp BC$ and $NY \perp BD$. Also note that $KY||BC$ and $KX||BD$. Hence, $KXBY$ is a parallelogram.

Since $|NB| = |ND|$ and $|MB| = |MC|$, $\angle BNY = \frac{1}{2}\angle BND = \frac{1}{2}\angle BAC = \angle MAC = \angle MBC$. Since $\angle BXM = \angle NYB = 90°$, we conclude that $\Delta BXM \sim \Delta NYB$. Observe that

$$\frac{|MX|}{|KY|} = \frac{|MX|}{|XB|} = \frac{|BY|}{|YN|} = \frac{|XK|}{|YN|}.$$

Also, $\angle MXK = 90° + \angle BXK = \angle 90° + \angle KYB = \angle KYN$. Hence, $\Delta MXK \sim \Delta KYN$. Finally,

$$\angle MKN = \angle XKY - \angle XKM - \angle NKY = \angle XKY - \angle XKM - \angle XMK$$

$$= \angle XKY - (180° - \angle MXK) = \angle XKY - 180 + 90 + \angle BXK = 180 - 180 + 90 = 90°,$$

(with the second last assertion following from the fact that $KY||XB$), as desired. $\square$

Solution 2: (Inversion solution:) Since M, N are midpoints of the arcs BC and BD, respectively, AM bisects $\angle BAC$ and AN bisects $\angle BAD$. Therefore, $\angle MAN = 90°$. To prove $\angle MKN = 90°$, it suffices to prove that $AMNK$ is cyclic.

We will invert the diagram about the point A with radius 1. For each object X, let X' be its image under the inversion. Since $ABMC$ is cyclic, B', M', C' are collinear. Since AM bisects $\angle BAC$, AM' bisects $\angle B'AC'$. Similarly, B', N', D' are collinear and AN' bisects $\angle B'AD'$. The point K' lies on $C'D'$ external to segment $C'D'$. To prove that M', N', K' are collinear, by Menelaus' Theorem, it suffices to prove that

$$\frac{|C'M'|}{|M'B'|} \cdot \frac{|B'N'|}{|N'D'|} \cdot \frac{|D'K'|}{|K'C'|} = 1.$$

By angle bisector theorem, we have that $\frac{|C'M'|}{|M'B'|} = \frac{|AC'|}{|AB'|}$ and $\frac{|B'N'|}{|N'D'|} = \frac{|AB'|}{|AD'|}$. Hence, it suffices to show that

$$\frac{|C'A|}{|AD'|} = \frac{|C'K'|}{|K'D'|}.$$

By properties of inversion, we know that

$$|K'C'| = \frac{|KC| \cdot |AC'|}{|AK|},\ |K'D'| = \frac{|KD| \cdot |AK'|}{|AD|}.$$

Since K is the midpoint of CD, $|KC| = |KD|$. Hence,

$$\frac{|K'C'|}{|K'D'|} = \frac{|AC'| \cdot |AD|}{|AK| \cdot |AK'|} = \frac{|AC'| \cdot |AD|}{|AD'| \cdot |AD|} = \frac{|AC'|}{|AD'|},$$

as desired. $\square$

Solution 3: (Harmonic Division solution:) Using the notation of Solution 2, it suffices to prove that

$$\frac{|C'A|}{|AD'|} = \frac{|C'K'|}{|K'D'|},$$

i.e. (K', A) divides (C', D') harmonically. But since K is the midpoint of CD, (K, ∞) divides (C, D) harmonically. Since the property that a harmonic quadruple is preserved under inversion of a point on the same line as the quadruple[3], this implies (K', A) divides (C', D') harmonically, as desired. $\square$

Source: Romanian Team Selection Test 1999

[3]The proof of this fact is similar to the last step of Solution 2.