Functional Equation

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Chapter 0

Preliminaries

0.1 Notations

- $\mathbb{N}$: the set of all positive integers
- $\mathbb{Z}$: the set of all integers
- $\mathbb{Q}$: the set of all rational numbers
- $\mathbb{R}$: the set of all real numbers
- $\mathbb{N}_0$: defined by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
- $\mathbb{R}^+$: the set of all positive reals

0.2 Character

Chapter 1

Basic Concepts

Solving functional equations is a main part in Olympiad Algebra. We will first introduce the basic ideas. For example, apply injectivity to reduce the equation. Or establish some inequalities to find the value of given point. Then, we're going to solve problems in past contests. Finally, we'll learn supplement materials, which are used to bash.

1.1 What are functional equations ?

We begin with a system of equations

$$\begin{aligned} 2x - 5y &= 8 \\ 3x + 9y &= -12 \end{aligned}$$

This is a special case of functional equations. To be precise, we translate the question to

Example 1. *Find all functions* $f : \{1, 2\} \to \mathbb{R}$ *such that*

$$\begin{aligned} 2f(1) - 5f(2) &= 8 \\ 3f(1) + 9f(2) &= -12 \end{aligned}$$

In fact, when the domain of f is a finite subset A of $\mathbb{N}$, it's just a system of equations with variables $\{f(a)\}_{a \in A}$. So, it's nature to consider finding all functions with particular properties as a generalization. Here is an example,

Example 2. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$2f(x) - 5f(y) = 8, \forall x, y \in \mathbb{R}$$

As dealing with systems of equations, we want to choose several suitable equations and solve. For instance, for $f(0)$, the simplest identity is

$$2f(0) - 5f(0) = 8$$

which gives $f(0) = -\frac{8}{3}$.

Example 3. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) + xf(1-x) = x$$

for all $x \in \mathbb{R}$

Solution. *For any a, the identities in which $f(a)$ appears are*

$$f(a) + af(1-a) = a$$
$$f(1-a) + (1-a)f(a) = 1-a$$

Direct to get $f(a) = \frac{a^2}{1-a+a^2}$. It's over? No, we should verify our answer. As following:

$$\begin{aligned} f(x) + xf(1-x) &= \frac{x^2}{1-x+x^2} + \frac{x(1-x)^2}{1-(1-x)+(1-x)^2} \\ &= \frac{x^2}{1-x+x^2} + \frac{x-2x^2+x^3}{1-x+x^2} \\ &= x \end{aligned}$$

Remark 1. *The question, now, is why we need to check our answer. That's because, in most problems, we won't find all identities in which a given variable appears. The value may contradicts to other equations.*

Remark 2. *In fact, our central idea is:*

- *If $f(a)$ is unknown for some a, choose an equation in which $f(a)$ appears*
- *If the variables (f(something)) in the chosen equation are known, go to the next step. Otherwise, back to Step 1.*
- *Solve the system of equations*

1.2 Assertion

Example 4. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x-y) = f(x) + f(y) - 2xy$$

holds for all $x, y \in \mathbb{R}$

Solution. *There are infinite equations being the concern of a given variable. We must choose some of them and solve it. Well, it's clear that for all $a \in \mathbb{R}$, we have*

$$f(0) = 2f(a) - 2a^2$$

Once we know $f(0)$, we get $f(a)$. In fact, the above equation also tells us $f(0) = 0$ (why?) So the only possibility of the function is $f(a) = a^2$ for all a. Indeed,

$$f(x-y) = (x-y)^2 = x^2 + y^2 - 2xy = f(x) + f(y) - 2xy$$

Note that what we have done is to choose some equations and solve systems of equations. Equivalently, we assert values to variables. In the previous example, taking

$$f(0) = 2f(a) - 2a^2$$

and asserting $x = a, y = a$ are the same. As the result, I'll say "set $x = ?, y = ?$" instead of "choose the following equation". Assertion is the easiest and most important step in solving functional equations. Consequently, let's do more examples. We'll see that for most problems, $f(0)$ is important.

Example 5. *(Komal) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x+y) + f(x)f(y) = x^2y^2 + 2xy \tag{1.1}$$

for all $x, y \in \mathbb{R}$

Solution. *Set $y = 0$ into (1.1):*

$$(1 + f(0)) f(x) = 0$$

If $f(0) \neq -1$, then $f(x) = 0$ for all $x \in \mathbb{R}$, which is absurd. So $f(0)$ must be -1.

Now, we set $(x, y) = (a, a), (a, -a), (2a, -a)$ to get

$$\begin{aligned} f(2a) + f(a)^2 &= a^4 + 2a^2 \\ f(a) f(-a) - 1 &= a^4 - 2a^2 \\ f(a) + f(2a) f(-a) &= 4a^4 - 4a^2 \end{aligned}$$

Those implies $f(a) = a^2 - 1$, which is indeed a answer to (1.1)

In this problem, we have solved the system of equations. In many "easy" functional equations, solving the system of equations is the key-point.

Example 6. *Find all functions $f:\mathbb{N}\to\mathbb{N}$ so that*

$$f(x+y)=f(x)+f(y)+3(x+y)\sqrt[3]{f(x)f(y)}$$

holds for all $x,y\in\mathbb{N}$

Solution. *Notice that*

$$f(2x)=2f(x)+6x\sqrt[3]{f(x)^2}$$

Therefore, we can find a function $g:\mathbb{N}\to\mathbb{N}$ defined by $g(x)=\sqrt[3]{f(x)}$ (why) Then g satisfies the following equation

$$g(x+y)^3=g(x)^3+g(y)^3+3(x+y)g(x)g(y)$$

for all $x,y\in\mathbb{N}$. If we know the values of $g(1)$ and $g(2)$, it's easy to show g is the identity map by mathematical induction. As the previous example suggested, when one encounters the relation between $h(x),h(y),h(xy),h(x+y)$ for some function h, it's quite useful to get $h(1),h(2)$ by taking the following assertions:

$$(x,y)=(1,1),(1,2),(1,3),(2,2)$$

The remains are left as an exercise.

Example 7. *(2015 Baltic Way) Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying for all reals x,y,*

$$|x|f(y)+yf(x)=f(xy)+f\left(x^2\right)+f(f(y))$$

Solution. *First, consider the assertion $(x,y)=(1,1)$, we get*

$$2f(1)=2f(1)+f(f(1))\to f(f(1))=0$$

Then, take $x=0,y=1$:

$$f(0)=2f(0)+f(f(1))$$

So $f(0)=0$. It follows that for $a\geq 0$, $f(a)=0$ by setting $(x,y)=(\sqrt{a},0)$
Finally, since $f(f(b))=0$ for all $b\in\mathbb{R}$ (why) , $f(c)=-f(-1)c$ for all negatives c (why)

1.3 Injectivity and Surjectivity

Sometimes, it's not clear to show, after assertion, what f should be. Perhaps, there are too many f, we aren't sure that whether $f(a)$ appears in equations. Therefore, we hope to reduce the numbers of f. There are two ways: injectivity and surjectivity.

Definition 1. *A function $f : \mathbb{A} \to \mathbb{B}$ is called injective if $f(a_1) = f(a_2)$ implies $a_1 = a_2$*

Obviously, whenever we prove that a function is injective, the equation

$$f(\text{something } 1) = f(\text{something } 2)$$

becomes useful.

Example 8. *Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy $f(0) = 1$ and*

$$f(f(n)) = f(f(n+2)+2) = n$$

for all integers n

Solution. *Notice that $f(f(n)) = n$ implies f is injective (why) Therefore,*

$$f(f(n)) = f(f(n+2)+2) \to f(n) = f(n+2)+2$$

Then, by mathematical induction, $f(2m) = 1-2m$ for all $m \in \mathbb{Z}$ Now, given an odd integer k, write $k = 2l+1$. Our goal is to find the value of $f(2l+1)$. However, note that

$$f(f(n)) = n \quad \text{and} \quad f(2m) = 1-2m$$

It follows that

$$f(2l+1) = f(1-2(-l)) = f(f(2(-l))) = -2l = 1-(2l+1)$$

So we conclude that $f(n) = 1-n$ for all $n \in \mathbb{Z}$

Remark 3. *For convenience, let $P(x_1, x_2, \cdots, x_n)$ be the assertion of a functional equation with n variables. Assume there are two distinct elements a_1, a_2 in the domain of the function f with $f(a_1) = f(a_2)$. We should derive a contradiction by assertions. For many cases, just compare two equations. For instance, $P(a_1, x_2, \cdots, x_n)$ and $P(a_2, x_2, \cdots, x_n)$.*

In the previous remark, we see a standard way for proving injectivity: contradiction method. Although the following example can be solved by exchanging x, y, we still give a proof with injectivity.

Example 9. *Find all functions* $f : \mathbb{Z} \to \mathbb{Z}$ *such that*

$$f(x + f(y)) = f(x) + y$$

for all $x, y \in \mathbb{Z}$

Solution. *If* $f(a) = f(b)$ *for some* $a, b \in \mathbb{Z}$. *Compare* $y = a, b$ *in the equation:*

$$f(x) + a = f(x + f(a)) = f(x + f(b)) = f(x) + b \to a = b$$

In other words, f *is injective. As mentioned before, to use the injectivity, it suffices to choose an equation of the form*

$$f(\textit{something 1}) = f(\textit{something 2})$$

Indeed, one may assert $y = 0$ *and get*

$$f(x + f(0)) = f(x)$$

It follows that $f(0) = 0$. *We then see* $f(f(y)) = y$ *by taking* $x = 0$ *in the equation. Plugging back to the equation,*

$$f(x + y) = f(x) + f(y)$$

In particular, $f(x+1) = f(x) + f(1)$, *which implies*

$$f(x) = xf(1)$$

Most of them do not satisfy the original equation, the only possibilities are

$$f(x) = x, \forall x \in \mathbb{Z} \quad \textit{or} \quad f(x) = -x, \forall x \in \mathbb{Z}$$

Remark 4. *Here is a useful result: If* $g \circ f$ *is injective, then so is* f

Proof. Use the contradiction method, again. Suppose f is not injective, then there are two elements a, b such that $f(a) = f(b)$. So we have

$$g(f(a)) = g(f(b)) \to a = b$$

since $g \circ f$ is assumed to be injective. Thus, we conclude that f must be injective. □

For more information, https://en.wikipedia.org/wiki/Injective_function

Definition 2. *A function* $f : \mathbb{A} \to \mathbb{B}$ *is called surjective if for any* $b \in \mathbb{B}$*, there exists* $a \in \mathbb{A}$ *satisfying* $f(a) = b$*. Equivalently, the pre-image of any element of* $\mathbb{B}$ *is not empty.*

Example 10. *Find all functions* $f, g : \mathbb{R} \to \mathbb{R}$ *such that* g *is injective and*

$$f(g(x) + y) = g(x + f(y))$$

for all x, y *in* $\mathbb{R}$

Solution. *Since* g *is injective, we tend to fix* LHS*. Thus, set* $y = -g(x)$ *to get*

$$f(0) = g(x + f(-g(x)))$$

There must be a constant c *such that*

$$x + f(-g(x)) = c, \forall x \in \mathbb{R}$$

This equation, in particular, shows that f *is surjective. Now, the original condition tells us* g *is also surjective. Let* a *be a real number with* $g(a) = 0$*. Then*

$$f(y) = g(a + f(y)) \to y - a = g(y), \forall y \in \mathbb{R}$$

by using the surjectivity of f*. Substitute into the original condition,*

$$f(x + y - a) = x + f(y) - a$$

which is easy to get f *is a linear function (left as an exercise)*

Indeed, for every surjective function, we can replace f (something) to an arbitrary number whenever something can be any number of the codomain.

Definition 3. *A function* $f : \mathbb{A} \to \mathbb{B}$ *is called bijective if* f *is both injective and surjective.*

Sometimes, it's not obvious that a function is injective because the condition is complicate. However, if we know that the function is surjective, then we can reduce the condition to a simpler form. Then, maybe it's possible to show the injectivity.

Example 11. *(2002 A1) Find all functions* f *from reals to reals such that*

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all x, y *in* $\mathbb{R}$

Solution. *Since there is already a term "$2x$", which is a surjective function. So if we want to show f is surjective, we may hope x is "free". Therefore, one should choose a suitable y such that one of the terms $f(f(x)+y)$ and $f(f(y)-x)$ is a constant.*

$$P(x,-f(x)) \to f(f(-f(x))-x) = f(0)-2x$$

So f is surjective. Now, suppose that a,b are reals such that $f(a)=f(b)$. Then,

$$P(a,y), P(b,y) \to 2a+f(f(y)-a) = 2b+f(f(y)-b)$$

We hope $a=b$, which is equivalent to $f(f(y)-a)=f(f(y)-b)$ for some y. Note that, so far, we have only one known pair (x,y) such that $f(x)=f(y)$, namely (a,b)

Choose y to be the real with $f(y)=a+b$, the conclusion follows. Finally, one would like to set $x=0$ (why)

$$f(f(0)+y) = f(f(y)) \to f(y) = y+f(0)$$

by using injectivity. Those functions are solutions as

$$\begin{aligned} LHS = f(x+f(0)+y) &= x+y+2f(0) \\ &= 2x+f(y+f(0)-x) = 2x+f(f(y)-x) = RHS \end{aligned}$$

Example 12. *(2015 Swiss) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for arbitrary $x,y\in\mathbb{R}$,*

$$(y+1)f(x)+f(xf(y)+f(x+y)) = y$$

Solution. *First of all, rewrite the condition,*

$$f(xf(y)+f(x+y)) = (1-f(x))y-f(x)$$

So f is surjective (why) Now, assume $a,b\in\mathbb{R}$ s.t. $f(a)=f(b)$. Consider

$$P(0,a) \to (a+1)f(0)+f(f(a)) = a$$

$$P(0,b) \to (b+1)f(0)+f(f(b)) = b$$

If $f(0)\neq 1$, then f is injective. However, $f(0)\neq 1$ since

$$P(0,c) \to (c+2)f(0) = c$$

where c is a real with $f(c)=0$. We conclude that f is bijective. Next,

$$P(c,0) \to f(cf(0)) = 0 = f(c)$$

It follows that $f(0)=0$. From here, one can easily find the desired function f is

$$f(x) = -x, \quad \forall x\in\mathbb{R}$$

1.4 Cauchy's Functional Equation

So far, one may observe that the domains of functional equations are usually "$\mathbb{R}$". Thus, I would like to introduce a result of real functions.

Definition 4. *A function $f : \mathbb{A} \to \mathbb{B}$ is called additive if*

$$f\left(a_1 + a_2\right) = f\left(a_1\right) + f\left(a_2\right)$$

holds for all $a_1, a_2, a_1 + a_2 \in \mathbb{A}$

We consider the situation that $\mathbb{A} = \mathbb{B} = \mathbb{R}$. By using induction, it's easy to check

$$f\left(n\right) = nf\left(1\right)$$

for all $n \in \mathbb{Z}$. (ref. example 9) For rationals, we have

$$f\left(\frac{p}{q}\right) = \frac{1}{q}f\left(p\right) = \frac{p}{q}f\left(1\right)$$

The first equality follows from $f\left(nx\right) = nf\left(x\right)$, which can be proved by induction, also. Now, we may expect that $f\left(x\right) = xf\left(1\right)$ for all $x \in \mathbb{R}$. However, it's false! (why?) Fortunately, we have a criterion.

Theorem 1. *Anyone of following conditions implies an additive $f : \mathbb{R} \to \mathbb{R}$ must be linear.*

- *f is continuous on some interval*
- *f is bounded on some interval*
- *f is monotonic on some interval*

Proof. For the first one, notice that if f is continuous at one point, then f is continuous everywhere. Next, for any real r, choose a sequence of rationals $\{a_n\}_{n\geq 1}$ converging to that number. Since f is continuous,

$$f\left(rx\right) = \lim_{n\to\infty} f(a_n x) = \lim_{n\to\infty} a_n f\left(x\right) = rf\left(x\right)$$

For the second one, suppose there is a real r s.t. $f\left(r\right) \neq rf\left(1\right)$. WLOG f is increasing and $f\left(r\right) > rf\left(1\right)$. Choose a rationals q so that $\frac{f(r)}{f(1)} > q > r$ then

$$f\left(r\right) < f\left(q\right) = qf\left(1\right) < f\left(r\right)$$

a contradiction. For the third, let a_n to be any rationals such that $nx - a_n \in [a, b]$. Then

$$|f\left(nx - a_n\right)| = |n\left(f\left(x\right) - xf\left(1\right)\right) - \left(a_n - nx\right)f\left(1\right)| \geq n|f\left(x\right) - xf\left(1\right)| - |\left(nx - a_n\right)f\left(1\right)|$$

which means $f\left(x\right)$ must be $xf\left(1\right)$, done! $\square$

Note that the method that we used to prove the second (or third) part of theorem 1 is useful. That's because it's usually easier to determine the function on rationals and one can use rationals to approach any real number.

Example 13. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all reals x, y, we have*

$$f(x)^2 + f(y)^2 + f\left((x+y)^2\right) = 2x^2 + 2y^2 + f(xf(y) + yf(x))$$

Solution. *The original equation is complicate, especially the term $f(xf(y) + yf(x))$. To kick it out, we may hope to have $f(0) = 0$ (why) This is indeed the case,*

$$P(0,0) \to 2f(0)^2 + f(0) = f(0)$$

implies $f(0) = 0$. Next, subtract $P(x,0)$ and $P(y,0)$ from $P(x,y)$, we have the following

$$Q(x,y) : f\left((x+y)^2\right) = f\left(x^2\right) + f\left(y^2\right) + f(xf(y) + yf(x))$$

Then, compare

$$\begin{aligned} P(x,0) &: f(x)^2 + f\left(x^2\right) = 2x^2 \\ P(-x,0) &: f(-x)^2 + f\left(x^2\right) = 2x^2 \end{aligned}$$

we see that $f(-x) = \pm f(x)$ holds for all x. Assume $f(a) = f(-a) = 0$ for some a

$$P(a,-a) \to 0 = 4a^2 \Leftrightarrow a = 0$$

Now, if $f(b) = f(-b)$ for some b

$$Q(b,-b) \to 0 = 2f\left(b^2\right)$$

so $b^2 = 0$. In other words, $b = 0$. From here, one can conclude that f must be odd. Finally, add two equations $Q(x,y)$ and $Q(x,-y)$,

$$f\left((x+y)^2\right) + f\left((x-y)^2\right) = 2f\left(x^2\right) + 2f\left(y^2\right)$$

If we set $g(x) = f\left(x^2\right)$, it becomes

$$g(x+y) + g(x-y) = 2g(x) + 2g(y)$$

The remains are left to readers:

Exercise. 1. *Show that $g(nx) = n^2 g(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$*

2. *Notice that $g(x) \leq 2x^2$. Try to prove that $g(x) = g(1)x^2$ holds for all $x \in \mathbb{R}$*

We return to an application of the theorem.

Example 14. *(2015 Korea) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that*

$$f\left(x^{2015}+f(y)^{2015}\right)=f(x)^{2015}+y^{2015}$$

holds for all reals x,y

Solution. *Start with*

$$f\left(f\left(f\left(x^{2015}+f(y)^{2015}\right)\right)\right)=f\left(f(x)^{2015}+y^{2015}\right)\right)=x^{2015}+f(y)^{2015}$$

By using the surjectivity of x^{2015}, we get $f(f(x))=x$ and thus, f is bijective. Now,

$$P(x,f(0))\to f\left(x^{2015}\right)=f(x)^{2015}+f(0)^{2015}$$

and then

$$P(x,f(y))\to f\left(x^{2015}+y^{2015}\right)=f(x)^{2015}+f(y)^{2015}=f\left(x^{2015}\right)+f\left(y^{2015}\right)-2f(0)^{2015}$$

Again, use the surjectivity of x^{2015}, the above becomes

$$f(x+y)=f(x)+f(y)-2f(0)^{2015}$$

One should be able to determine the value of a (hope so) Thus, we have

$$f(x+y)=f(x)+f(y)\quad \text{and}\quad f\left(x^{2015}\right)=f(x)^{2015}$$

We claim that f is linear: Given any real number x. We have, for any rational number q,

$$f(x+q)^{2015}=(f(x)+f(q))^{2015}=f(x)^{2015}\,qf(1)\,f(x)^{2014}+\cdots+q^{2015}f(1)^{2015}$$

On the other hand,

$$f(x+q)^{2015}=f\left((x+q)^{2015}\right)=f(x)^{2015}+2015qf\left(x^{2014}\right)+\cdots+q^{2015}f(1)^{2015}$$

Since q can be arbitrary, $f\left(x^{2014}\right)=f(1)\,f(x)^{2014}$ (why) In particular, the second condition in the theorem is fulfilled. The conclusion follows.

Example 15. *(2017 CSP) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that*

$$f(x) - f(x+y) = f\left(\frac{x}{y}\right) f(x+y)$$

holds for all $x, y \in \mathbb{R}^+$

Solution. *Because we expect the only non-trivial solution is*

$$f(x) = \frac{1}{x}, \ \forall x \in \mathbb{R}^+$$

we want to show that $f(x) \neq 0$ for all $x \in \mathbb{R}^+$
Assume $a \in \mathbb{R}^+$ so that $f(a) = 0$. Given $b < a$,

$$P(b, a-b) \to f(b) = 0$$

Also, notice that

$$P(a, large) \to f(large) = 0$$

Combine those results, we conclude that $f \equiv 0$
Now, suppose $f(x) \neq 0$ for all $x \in \mathbb{R}^+$. It's easy to see $f(1) = 1$ (why) So

$$P(x, x) \to f(x) = 2f(2x)$$

We have

$$P(x,y), P(y,x) \to \frac{f(x)}{f(y)} = \frac{f\left(\frac{x}{y}\right) + 1}{f\left(\frac{y}{x}\right) + 1}$$

and thus,

$$\frac{f\left(\frac{x}{y}\right) + 1}{2f\left(\frac{y}{x}\right) + 2} = \frac{f(x)}{2f(y)} = \frac{f(2x)}{f(y)} = \frac{\frac{1}{2}f\left(\frac{x}{y}\right) + 1}{2f\left(\frac{y}{x}\right) + 1} \Leftrightarrow f\left(\frac{x}{y}\right) f\left(\frac{y}{x}\right) = 1$$

It follows that f is multiplicative and

$$f(x) f(y) = (f(x) + f(y)) f(x+y)$$

By using theorem 1, we know that (why)

$$f(x) = \frac{1}{x}, \ \forall x \in \mathbb{R}^+$$

1.5 Establishing Inequalities

Sometimes, it's not straightforward to find a proof of $f(a) = b$ for some given a, b. Instead, we try to prove the following inequalities:

$$f(a) \geq b \text{ and } f(a) \leq b$$

which maybe much easier to handle.

Example 16. *Find all functions $f : \mathbb{N} \to \mathbb{N}$ with $f(1) = 1$ and*

$$f(x+5) \geq f(x) + 5 \quad \textit{and} \quad f(x+1) \leq f(x) + 1$$

holds for all $x \in \mathbb{N}$

Solution. *By the second inequality,*

$$f(x+5) \leq f(x) + 5$$

Combine it with the first inequality, we get

$$f(x) + 5 \leq f(x+5) \leq f(x) + 5$$

So the equality of $f(x+5) \leq f(x) + 5$ must hold, which implies $f(x+1) = f(x) + 1$. A simple induction will give $f(n) = n$ for all $n \in \mathbb{N}$

Because the condition is already an inequality, it's very straightforward to come up with such a solution.

Exercise. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) + f(y) \geq 2f(x+y)$$

holds for all $x, y \in \mathbb{R}$

Let's , now, do a "true" functional equation.

Example 17. *(2017 OAO Shortlist) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that*

$$\sum_{i=1}^{n} \frac{1}{f^i(n)} = 1$$

holds for all natural number n.

Solution. *We want to find two relative equations. Here is a simple way, compare* $P(n), P(f(n))$

$$\sum_{i=1}^{n} \frac{1}{f^i(n)} = 1 = \sum_{i=1}^{f(n)} \frac{1}{f^i(f(n))}$$

From above, we get that $f(n) > n - 1$ *(why). In other words,* $f(n) \geq n$ *for all* $n \in \mathbb{N}$. *Therefore,*

$$1 = \sum_{i=1}^{n} \frac{1}{f^i(n)} \leq \sum_{i=1}^{n} \frac{1}{n} = 1$$

It follows that $f(n) = n$ *must hold for all natural* n, *which is clearly a solution of the functional equation.*

Exercise. *(Socrates) Find all functions* $f : \mathbb{N} \to \mathbb{N}$ *such that for each positive integer* n, *there exists a positive integer* k *with*

$$\sum_{i=1}^{k} f^i(n) = kn$$

To establish an inequality, we should take large numbers into consideration. For instance,

Example 18. *(2014 Taiwan) Find all increasing functions* $f : \mathbb{N} \to \mathbb{N}$ *satisfying* $f(2) = 7$,

$$f(mn) = f(m) + f(n) + f(m)f(n)$$

for all positive integers m *and* n

Solution. *Recall the well-known identity:*

$$(x+1)(y+1) = xy + x + y + 1$$

One tends to rewrite the condition:

$$f(mn) + 1 = (f(m) + 1)(f(n) + 1)$$

Let $g(n) := f(n) + 1$, *we then have* $g(mn) = g(m)g(n)$ *holds for all* $m, n \in \mathbb{N}$. *Now,*

$$n^x > 2^y \Leftrightarrow \log_2 n > \frac{y}{x} \Leftrightarrow g(n) = g(n^x)^{\frac{1}{x}} > g(2^y)^{\frac{1}{x}} = 2^{\frac{3y}{x}}$$

This gives the lower-bound of $g(n)$, *namely* n^3. *A similar argument gives* $g(n) \leq n^3$

Note that whenever the codomain of the function is $\mathbb{R}^+$ or $\mathbb{N}$, we have a trivial inequality.

Example 19. *(2016 Poland) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$2f(f(f(x))) + 5f(f(x)) = f(x) + 6x$$

holds for all $x \in \mathbb{R}+$

Solution. *If one does not familiar to recurrence, take a look: https://en.wikipedia.org/wiki/Recurrence_relation. Given any $x \in \mathbb{R}$. Let $a_i = f^i(x)$. Then conditions $P(x), P(f(x)), \cdots$ becomes*

$$2a_{i+3} + 5a_{i+2} = a_{i+1} + 6a_i$$

for all $i \in \mathbb{N}_0$. To get the general formula of a_i, we should solve

$$2x^3 + 5x^2 - x - 6 = 0$$

The roots are $1, -2, -\frac{3}{2}$, $a_i = b + c(-2)^i + d\left(-\frac{3}{2}\right)^i$ If b or c is not zero, then $a_i < 0$ for some integer i (why) This means the sequence $\{a_i\}_{i \geq 0}$ is constant. In other words, $f(x) = x$. Note that the choice of x can be arbitrary, we conclude the answer to this problem is

$$f(x) = x, \ \forall x \in \mathbb{R}^+$$

In most problems, one needs to establish the recurrence.

Example 20. *(Socrates) Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$f(f(x) + y) + f(x + y) = 2x + 2f(y)$$

holds for all $x, y \in \mathbb{R}^+$

Solution. *We have the following equations:*

$$\begin{aligned}
P(x, f(1)) &\to f(f(x) + f(1)) + f(x + f(1)) = 2x + 2f(f(1)) \\
P(x, 1) &\to f(f(x) + 1) + f(x + 1) = 2x + 2f(1) \\
P(1, x) &\to f(x + f(1)) + f(x + 1) = 2 + 2f(x) \\
P(1, f(x)) &\to f(f(x) + f(1)) + f(1 + f(x)) = 2 + 2f((x))
\end{aligned}$$

So,

$$f(f(x)) = 2x - f(x) + (f(f(1)) + f(1) - 2)$$

Denote $f(f(1)) + f(1) - 2$ by c, then

$$f(f(x)) = 2x - f(x) + c$$

A simple induction gives

$$f^n(x) = \frac{1}{3}\left(f(x) + 2x + nc - \frac{c}{3}\right) + \frac{1}{3}(-2)^n\left(x - f(x) + \frac{c}{3}\right)$$

The remains are easy.

1.6 Exercise

To get a better feeling, I strongly suggest that the readers do lots of functional equations because we usually come up with a great idea due to our experience.

1.6.1 2014

Problem 1. *(Baltic Way) Find all functions f defined on all reals and taking real values such that*

$$f\left(f\left(y\right)\right)+f\left(x-y\right)=f\left(xf\left(y\right)-x\right)$$

for all real numbers x, y

Problem 2. *(ELMO) Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying*

$$\begin{aligned} f\left(x+f\left(y\right)\right)&=g\left(x\right)+h\left(y\right)\\ g\left(x+g\left(y\right)\right)&=h\left(x\right)+f\left(y\right)\\ h\left(x+h\left(y\right)\right)&=f\left(x\right)+g\left(y\right)\end{aligned}$$

for all real numbers x and y

Problem 3. *(IZHO) Does there exist a surjective function $f:\mathbb{R}\to\mathbb{R}$ satisfying*

$$f\left(f\left(x\right)\right)=\left(x-1\right)f\left(x\right)+2$$

for all real x

Problem 4. *(Kazakhstan) Find all functions $f:\mathbb{Q}\times\mathbb{Q}\to\mathbb{Q}$ such that for all $x, y, z\in\mathbb{Q}$,*

$$f\left(x,y\right)+f\left(y,z\right)+f\left(z,x\right)=f\left(0,x+y+z\right)$$

Problem 5. *(USA) Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that*

$$xf\left(2f\left(y\right)-x\right)+y^2f\left(2x-f\left(y\right)\right)=\frac{f\left(x\right)^2}{x}+f\left(yf\left(y\right)\right)$$

holds for all $x, y\in\mathbb{Z}$ with $x\neq 0$

1.6.2 2015

Problem 6. *(Belarus) Find all functions $f(x)$ determined on interval $[0,1]$ satisfying*

$$f(f(x)) = f(x)$$

$$\{f(x)\}\sin^2 x + \{x\}\cos f(x)\cos x = f(x)$$

Problem 7. *(Canada) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that*

$$(n-1)^2 < f(n) f(f(n)) < n^2 + n$$

for every positive integer n

Problem 8. *(Greece) Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$yf(x) + f(y) \geq f(xy)$$

for all $x, y \in \mathbb{R}$

Problem 9. *(Turkey) Find all the functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f\left(x^2\right) + 4xy^2 f(y) = \left(f(x-y) + y^2\right)(f(x+y) + f(y))$$

holds for every $x, y \in \mathbb{R}$

1.6.3 2016

Problem 10. *(Balkan) Find all injective functions $f : \mathbb{R} \to \mathbb{R}$ such that for every real number x and every positive integer n,*

$$\left|\sum_{i=1}^{n} i\left(f(x+i+1) - f(f(x+i))\right)\right| < 2016$$

Problem 11. *(EMMO) Find all functions $f : \mathbb{N} \to \mathbb{R}$ such that*

$$\sum_{k=0}^{n-1} \left\lfloor x + \frac{f(k)}{n} \right\rfloor = \lfloor xf(n) \rfloor$$

holds for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$

Problem 12. *(HMMT) Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that for $w, x, y, z \in \mathbb{N}$*

$$f(f(f(z))) f(wxf(yf(z))) = z^2 f(xf(y)) f(w)$$

Show that $f(n!) \geq n!$ for every positive integer n

Problem 13. *(Netherland) Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(xy-1) + f(x)f(y) = 2xy - 1$$

holds for all $x, y \in \mathbb{R}$

Problem 14. *(Norway) Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x)f(y) = |x-y| \cdot f\left(\frac{xy+1}{x-y}\right)$$

holds for all $x \neq y \in \mathbb{R}$

Problem 15. *(Romania) Determine all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f\left(a^2\right) - f\left(b^2\right) \leq (f(a)+b)(a-f(b))$$

holds for all $x, y \in \mathbb{R}$

1.6.4 2017

Problem 16. *(Belarus) Find all functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying the following equation*

$$f(x+f(xy)) = xf(1+f(y))$$

Problem 17. *(Greece) Find all functions* $f, g : \mathbb{R} \to \mathbb{R}$ *such that* $g(1) = -8$ *and*

$$f(x-3f(y)) = xf(y) - yf(x) + g(x)$$

holds for all $x, y \in \mathbb{R}$

Problem 18. *(IZHO)Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$\left(x+y^2\right) f(yf(x)) = xyf\left(y^2+f(x)\right)$$

holds for all $x, y \in \mathbb{R}$

Problem 19. *(Pakistan) Find all functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for all distinct* x, y, z,

$$f\left(x^2\right) - f(y)f(z) = f\left(x^y\right) f(y) f(z) \left(f\left(y^z\right) - f\left(z^x\right)\right)$$

Problem 20. *(Turkey) Given a real number* a*, try to find all functions* $f : \mathbb{R} \to \mathbb{R}$ *so that*

$$f(xy+f(y)) = yf(x) + a$$

for every $x, y \in \mathbb{R}$

Chapter 2

Cauchy FE

In chapter one, we have introduced some terminologies. Recall the statement of theorem 1: For an additive $f : \mathbb{R} \to \mathbb{R}$, any of the following conditions guarantees that f is linear.

- f is continuous on some interval
- f is bounded on some interval
- f is monotonic on some interval

We're not contented because additivity and others are strong conditions.

2.1 Taiwanese Transformation

If one makes a suitable transformation, then he/she will get another useful result.

Example 21. *All monotonic functions $f : \mathbb{R} \to \mathbb{R}^+$ such that*

$$f(x+y) = f(x) f(y)$$

are $f(x) = a^x, \forall x \in \mathbb{R}$ for some constant a.

Solution. *Define $g : \mathbb{R} \to \mathbb{R}_{>a}$ by $g(x) = \log f(x)$. It's also monotonic since it's the composition of two monotonic functions. Now, the original equation becomes*

$$g(x+y) = g(x) + g(y)$$

We deduce $g(x) = cx, \forall x \in \mathbb{R}$ for some constant c, which means

$$f(x) = a^x, \forall x \in \mathbb{R}$$

for some constant a (why)

Exercise. *Find all continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *so that*

$$f(xy) = f(x) f(y)$$

for all $x, y \in \mathbb{R}$. *PS: Such a function is called multiplicative.*

Remark 5. *A multiplicative function satisfying Cauchy condition must be zero or identity.*

Proof. Notice that if a function is multiplicative, then

$$f\left(x^2\right) = f(x)^2 \geq 0$$

So f is bounded on some interval. It follows that f is linear. □

Exercise. *Find all continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *so that*

$$f(xy) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$

Example 22. *Given a constant* a. *Find all functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *so that*

$$f(x+y) = f(x) + f(y) + a$$

holds for all $x, y \in \mathbb{R}^+$

Solution. *Why we want to transform the function? Because we hope to show that* f *is linear by using **Theorem** 1! Thus, after guessing the answer to the FE, we should apply the corresponded transformation. In this example, one may guess*

$$f(x) = cx - a, \ \forall x \in \mathbb{R}$$

for some constant $c \in \mathbb{R}$ *are the only functions satisfying the conditions. So, we tend to define* $g : \mathbb{R}^+ \to \mathbb{R}$ *such that*

$$g(x) = f(x) + a$$

Then the identity becomes

$$g(x+y) = g(x) + g(y)$$

which is trivail?

Another example,

Example 23. *Given a constant d. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ so that*

$$f(x+y+d) = f(x) + f(y)$$

holds for $x, y \in \mathbb{R}$

Solution. *This is similar to the previous one. First of all, one need to guess the answer. Assume that f is linear:*

$$f(x) = ax + b$$

for some constant $a, b \in \mathbb{R}$. Then we must have

$$ax + ay + ad + b = LHS = RHS = ax + ay + 2b \Leftrightarrow b = ad$$

In other words,

$$f(x) = cx + cd = c(x+d), \forall x \in \mathbb{R}$$

for some constant $c \in \mathbb{R}$. Define $g : \mathbb{R} \to \mathbb{R}$ by the following rule

$$g(x) = f(x+d) \to g(x+y) = g(x) + g(y)$$

Exercise. *Given two real numbers a, b. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x+y+a) = f(x) + f(y) + b$$

holds for all pairs $(x, y) \in \mathbb{R} \times \mathbb{R}$

Before the next subsection, I would like to summary our previous work. (under a suitable situation i.e., bounded)

Condition	Transformation	Solution
$f(x+y) = f(x) + f(y)$		cx
$f(x+y) = f(x) f(y)$	$g(x) = \log f(x)$	a^x
$f(xy) = f(x) f(y)$	$g(x) = \log f(a^x)$	$x^{\log_a b}$
$f(xy) = f(x) + f(y)$	$g(x) = f(a^x)$	$log_a x$
$f(x+y) = f(x) + f(y) + a$	$g(x) = f(x) + a$	$cx - a$
$f(x+y+a) = f(x) + f(y)$	$g(x) = f(x+a)$	$c(x+a)$
$f(x+y+a) = f(x) + f(y) + b$	$g(x) = f(x+a) + b$	$c(x+a) - b$

2.2 Variation

Return to additive functions, we have proved that under suitable conditions, they must be linear. In this section, we'll investigate more "suitable conditions".

Example 24. *Find all additive functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f\left(\frac{1}{x}\right) f(x) = 1, \forall x \in \mathbb{R}^*$$

Solution. *Notice that if f is a solution, then $-f$ is also satisfies the conditions. So, let's assume that $f(1) = 1$. We'll calculate $f\left(\frac{1}{x^2+x}\right)$ in two different ways.*

$$f\left(\frac{1}{x^2+x}\right) = f\left(\frac{1}{x}\right) - f\left(\frac{1}{x+1}\right) = \frac{1}{f(x)} - \frac{1}{f(x)+1} = \frac{1}{f(x)^2 + f(x)}$$

On the other hand,

$$f\left(\frac{1}{x^2+x}\right) = \frac{1}{f(x^2+x)} = \frac{1}{f(x^2) + f(x)}$$

It follows that

$$f\left(x^2\right) = f(x)^2, \ \forall x \in \mathbb{R}$$

And therefore, by the corollary, f must be the identity.

This calculation is tricky, but it helps in many variations. More examples,

Example 25. *(2008 Korea) Find all additive functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}, \forall x \in \mathbb{R}^*$$

Solution. *This one looks very similar to the previous. We still assume $f(1) = 1$ and calculate the same expression $f\left(\frac{1}{x^2+x}\right)$.*

$$f\left(\frac{1}{x^2+x}\right) = f\left(\frac{1}{x}\right) - f\left(\frac{1}{x+1}\right) = \frac{f(x)}{x^2} - \frac{f(x)+1}{(x+1)^2} = \frac{(2x+1)f(x) - x^2}{(x^2+x)^2}$$

It equals to

$$\frac{f(x^2+x)}{(x^2+x)^2} = \frac{f(x^2) + f(x)}{(x^2+x)^2}$$

which is equivalent to

$$f\left(x^2\right) + x^2 = 2xf(x)$$

Now, it's not obvious to show f is the identity. So we tend to calculate another expression

$$f\left(\left(x+\frac{1}{x}\right)^2\right)=f\left(x^2+2+\frac{1}{x^2}\right)=2xf\left(x\right)-x^2+2+\frac{2f\left(\frac{1}{x}\right)}{x}-\frac{1}{x^2}$$

From early result, it must equal to

$$2\left(x+\frac{1}{x}\right)f\left(x+\frac{1}{x}\right)-x^2-2-\frac{1}{x^2}$$

After reduction, we find that $f\left(x\right)=x,\ \forall x\in\mathbb{R}$

Example 26. *Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that*

$$f\left(x+1\right)=f\left(x\right)+1,f\left(\frac{1}{x}\right)f\left(x\right)=1,\forall x\in\mathbb{R}^*$$

2.3 Extension

Sometimes, we have a **partial** Cauchy's condition.

Example 27. *Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x+y) = f(x) + f(y), \forall x \in \mathbb{R} \quad and \quad y \in \mathbb{R}^+$$

Solution. *For any $a, b \in \mathbb{R}$, choose a large $c > 0$ with $a + b + 2c, b + 2c > 0$, we have*

$$f(a+b+2c) = f(a+b) + f(2c)$$

On the other hand,

$$f(a+b+2c) = f(a) + f(b+2c) = f(a) + f(b) + f(2c)$$

Thus, f is additive and must be linear.

To derive the Cauchy's condition from a partial Cauchy's condition, we usually take another sufficiently nice number and count a specify expression in two different ways.

Example 28. *Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ so that*

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$ satisfying $|x - y| \leq 1$.

Solution. *I'll show that $f(x+y) = f(x) + f(y)$ for all $|x - y| \leq 2$. Indeed,*

$$2f\left(\frac{x+z}{2}\right) = f(x) + f(z), 2f\left(\frac{y+z}{2}\right) = f(y) + f(z)$$

where $z = \frac{x+y}{2}$. Now, notice that

$$2f\left(\frac{x+z}{2}\right) + 2f\left(\frac{y+z}{2}\right) = 2f\left(\frac{x+y+2z}{2}\right) = 2f(2z) = 4f(z)$$

Therefore,

$$f(x) + f(y) = 4f(z) - 2f(z) = 2f(z) = f(x+y)$$

One can use induction to establish the Cauchy's condition.

As the previous example suggests, extending several times may be much easier

Example 29. *Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the equation*

$$f(z+x) + f(z+y) = f(2z+x+y), \forall x, y, z \in \mathbb{R}^+ \quad with \quad x, y < z^2$$

Solution. *I'll use induction to prove that f is "almost" additive. Given a small ϵ, we claim*

$$f(z+\epsilon)+f(w)=f(z+\epsilon+w) \quad \text{for all} \quad w>z\geq 133$$

By the condition, it's true for $w\in(z,z+z^2)$. Suppose it's true for $w\in\left(z,z+\frac{n}{2}z^2\right)$ Consider $w\in\left(z+\frac{n}{2}z^2-\epsilon,z+\frac{n+1}{2}z^2\right)$. Note that

$$\sqrt{z+\frac{n+1}{2}z^2}+\epsilon+w<\sqrt{z+\frac{n+1}{2}z^2}+\epsilon+z+\frac{n+1}{2}z^2<2z+nz^2$$

Thus,

$$f\left(\sqrt{z+\frac{n+1}{2}z^2}+\epsilon\right)+f(w)=2f(a)$$

where $a\in\left(z,z+\frac{n}{2}z^2\right)$. According to the inductive hypothesis,

$$f(z+\epsilon)+2f(a)=f\left(\sqrt{z+\frac{n+1}{2}z^2}+\epsilon\right)+2f\left(\frac{z+\epsilon+w}{2}\right)$$

As the result, $f(z+\epsilon)+f(w)=2f\left(\frac{z+\epsilon+w}{2}\right)=f(z+\epsilon+w)$, as desired.
We have already proved that $f(x)+f(y)=f(x+y)$ for large $x,y\in\mathbb{R}^+$. Thus, $f(x)=cx$ for all large $x\in\mathbb{R}^+$. Finally, because $f(2x)=2f(x),\forall x\in\mathbb{R}^+$, we get $f(x)=cx,\forall x\in\mathbb{R}^+$

In fact, in these examples, we just extend the "size" of a particular set. Formally, let

$$S_0=\{(x,y)\,|f(x+y)=f(x)+f(y) \text{ is proved}\}$$

Our goal is to find sets $S_1,S_2,\dots,S_k$ such that

$$S_0\subset S_1\subset S_2\subset\cdots\subset S_k=\mathbb{R}\times\mathbb{R}$$

and it's easy to extend S_{i-1} to S_i. To illustrate this idea, we'll solve two problems.

Example 30. *(APMO) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that*

$$(z+1)f(x+y)=f(xf(z)+y)+f(yf(z)+x)$$

holds for all $x,y\in\mathbb{R}$

Example 31. *(2014 Iran) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ st for all positive reals x,y,*

$$f\left(\frac{y}{f(x+1)}\right)+f\left(\frac{x+1}{xf(y)}\right)=f(y)$$

2.4 Applications

2.5 Problems

Problem 21. *(2014 Iran) Find all continuous functions* $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that*

$$f\left(xf\left(y\right)\right) + f\left(f\left(y\right)\right) = f\left(x\right)f\left(y\right) + 2,\ \forall x, y \in \mathbb{R}_{\geq 0}$$

Problem 22. *(2002 A4) Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$\left(f\left(x\right) + f\left(z\right)\right)\left(f\left(y\right) + f\left(t\right)\right) = f\left(xy - zt\right) + f\left(xt + yz\right)$$

for all real x, y, z, t

Problem 23. *(2003 A5) Find all functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$f\left(xyz\right) + f\left(x\right) + f\left(y\right) + f\left(z\right) = f\left(\sqrt{xy}\right)f\left(\sqrt{yz}\right)f\left(\sqrt{zx}\right)$$

and $f\left(x\right) < f\left(y\right)$ *when* $1 \leq x < y$

Problem 24. *(2009 A7) Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f\left(xf\left(x + y\right)\right) = f\left(yf\left(x\right)\right) + x^2, \forall x, y \in \mathbb{R}$$

Problem 25. *(2012 A5) Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$f\left(1 + xy\right) - f\left(x + y\right) = f\left(x\right)f\left(y\right), \forall x, y, \in \mathbb{R}$$

and $f\left(-1\right) \neq 0$.

Problem 26. *(2017 Iran) Find all functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\frac{x + f\left(y\right)}{xf\left(y\right)} = f\left(\frac{1}{y} + f\left(\frac{1}{x}\right)\right)$$

Chapter 3

Double Counting

3.1 Switching

Basically, LHS and RHS, in the equation, must hold the same properties. For instance, if LHS is an increasing function of x, then RHS is also an increasing function of x. In particular, if LHS is symmetric w.r.t the variables, then so is RHS. If this is the case, we simply exchange the variables and observe what would happened.

Example 32. *(Socrates) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f\left(f\left(x\right)+f\left(y\right)\right)=f\left(x^{2}\right)+2x^{2}f\left(y\right)+f\left(y\right)^{2}$$

holds for all $x, y \in \mathbb{R}$

Solution. *Notice that the left hand side is already symmetric. Switch x, y, we see that*

$$f\left(x^{2}\right)+2x^{2}f\left(y\right)+f\left(y\right)^{2}=f\left(y^{2}\right)+2y^{2}f\left(x\right)+f\left(x\right)^{2}$$

We now want to choose y that minimizes the number of uncertain values. After solving

$$y^{2}-y=0\Leftrightarrow y=0,1$$

we tend to "take" two equations:

$$P\left(x,1\right)\to f\left(x^{2}\right)+2x^{2}f\left(1\right)+f\left(1\right)^{2}=f\left(1\right)+2f\left(x\right)+f\left(x\right)^{2}$$
$$P\left(x,0\right)\to f\left(x^{2}\right)+2x^{2}f\left(0\right)+f\left(0\right)^{2}=f\left(0\right)+f\left(x\right)^{2}$$

From here, one can easily get the answers

$$f\left(x\right)=0,\forall x\in\mathbb{R}\quad or\quad f\left(x\right)=x^{2},\forall x\in\mathbb{R}$$

Indeed, switching the variables is equivalent to comparing $P(x,y), P(y,x)$. We usually get a simple identity, which is much easier to solve the function. The only bad news is: In most of the problems, LHS and RHS aren't symmetric.

To deal with this situation, we can assert a special form to one of the variables.

Example 33. *Find all functions $f:\mathbb{N}\to\mathbb{N}$ so that*

$$f(f(x)+y)=x+f(y), \forall x,y\in\mathbb{N}$$

Solution. *As the previous comment, to make LHS become symmetric, substitute $f(y)$ for y, then we get*

$$f(f(x)+f(y))=x+f(f(y))$$

Now, the LHS is symmetric and we can switch x and y (Hooray!)

$$x+f(f(y))=f(f(x)+f(y))=f(f(y)+f(x))=y+f(f(x))$$

This means

$$f(f(z))=z+c,\ \forall z\in\mathbb{N}$$

for some constant c (why) Recall, in chapter one, I have mentioned: Always use the little result to get another identity. In this example,

$$P(x,f(y))\to f(x+y+c)=f(x)+f(y)$$

So f is eventually linear. Like what we do in chapter three: for any given x, take a sufficient large y, we can see that $x, f(x)$ satisfy the same linear equation and we conclude that f is linear. The rest is easy.

Remark 6. *In chapter one, we solve this FE by using the injectivity.*

Example 34. *Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the following equation*

$$f(xf(y))-x=f(xy), \forall x,y\in\mathbb{R}$$

Solution. *It seems that we could do our magic trick*

$$f(xf(y))-x=f(xy)=f(yx)=f(yf(x))-y$$

This identity isn't very useful. Thus, we need to find another one. Rewrite the condition

$$f(xf(y))=f(xy)+x$$

Replace x by $f(x)$,

$$f(f(x)f(y))=f(f(x)y)+x=f(xy)+y+f(x)$$

Since now LHS is symmetric,

$$f(xy) + y + f(x) = f(yx) + x + f(y)$$

must hold. And thus, f must be linear (why) Assume $f(x) = ax + b$, we have

$$\begin{aligned} LHS &= axf(y) + b = ax(ay + b) + b = a^2xy + abx + b \\ RHS &= axy + b + x \end{aligned}$$

Compare the coefficient, $a = 1, b = 1$. In other words,

$$f(x) = x + 1,\ \forall x \in \mathbb{R}$$

Example 35. *(2016 Iran) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $a, b \in \mathbb{N}$, we have*

$$(f(a) + b) f(a + f(b)) = (a + f(b))^2$$

Solution. *Similar to example 33, we have*

$$(f(f(a)) + b) f(f(a) + f(b)) = (f(a) + f(b))^2$$

As the result,

$$f(f(n)) = n + c$$

for some constant c. Note that f has fixed point:

$$P(a, a) \to f(a + f(a)) = a + f(a)$$

It follows that $f(f(n)) = n$ for all $n \in \mathbb{N}$ Now,

$$(f(a) + f(b)) f(a + b) = (a + b)^2$$

The rest is easy. *Example* 6

Although this trick is useful, there are still many problems can't be solved in this way. For example,

Example 36. *(Based on: 2014 Romania) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$f(x + 3f(y)) = f(x) + f(y) + 2y,\ \forall x, y \in \mathbb{R}^+$$

Fortunately, if we modify our trick a little bit, the previous FE becomes trivial.

3.2 Three variables method

As the previous discussion, we hope to find a more useful trick. Our idea is simple! Usually, we got stuck after switching variables because there are many restrictions. Consequently, we introduce a "free" variable in order to ignore the restrictions.

Example 37. *(Socrates) Find all solutions $f:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the equation*

$$f(x+y)^2=f(x)^2+2f(xy)+f(y)^2, \forall x,y\in\mathbb{R}^+$$

Solution. *Despite the consequences, replace y by $y+z$:*

$$f(x+y+z)^2=f(x)^2+2f(x(y+z))+f(y+z)^2$$

Notice that the term $f(y+z)^2$ can be further simplified:

$$f(y+z)^2=f(y)^2+2f(yz)+f(z)^2$$

As the result,

$$f(x+y+z)^2=f(x)^2+f(y)^2+f(z)^2+2f(x(y+z))+2f(yz)$$

Cyclic the variables, we get

$$f(yz)+f(xy+zx)=f(zx)+f(yz+xy)=f(xy)+f(zx+yz)$$

Since there is always a solution to the system of equations: $yz=a, zx=b, xy=c$, the identity can be rewrote as

$$f(a)+f(b+c)=f(b)+f(c+a)=f(c)+f(a+b)$$

Fix b,c, we see that

$$f(a+c)-f(a)=f(b+c)-f(b)$$

In other words,

$$f(x+c)=f(x)+(f(b+c)-f(b))$$

Define $g:\mathbb{R}^+\to\mathbb{R}$ by $g(c)=f(b+c)-f(b)$, then

$$f(x+y)=x+g(y),\ \forall x,y\in\mathbb{R}^+$$

which is an easy exercise.

Remark 7. *Note that the function g is well-defined*

Example 38. *(Based on: 2014 Romania) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$f(x + 3f(y)) = f(x) + f(y) + 2y, \forall x, y \in \mathbb{R}^+$$

Solution. *Observe the equation*

$$f(x + 3f(y + z)) = f(x) + f(y + z) + 2(y + z)$$

We hope that there is another way to calculate LHS. According to the condition, we tend to replace z by $3f(z)$:

$$\begin{aligned} LHS &= f(x + 3f(y) + 3f(z) + 6z) \\ &= f(x + 3f(y) + 6z) + f(z) + 2z \\ &= f(x + 6z) + f(y) + 2y + f(z) + 2z \end{aligned}$$

On the other hand,

$$\begin{aligned} RHS &= f(x) + f(y + 3f(z)) + 2y + 6f(z) \\ &= f(x) + f(y) + f(z) + 2z + 2y + 6f(z) \end{aligned}$$

We conclude that

$$f(x + 6z) = f(x) + 6f(z)$$

holds for all $x, y \in \mathbb{R}^+$. Now,

$$f(6x + 6z) = f(6x) + 6f(z) \to 6f(z) = f(6z) + c, \ \forall z \in \mathbb{R}^+$$

for some constant c. Therefore,

$$f(x + z) = f(x) + f(z) + c$$

which implies f is linear (why) and thus,

$$f(x) = x, \ \forall x \in \mathbb{R}^+$$

Remark 8. *Because we want to calculate an expression in two different ways, we usually "iterate" our condition. Besides, since we exchange two variables, we usually need to deal with something like Cauchy FE and get the additivity.*

Example 39. *(Socrates) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that*

$$f\left(2x+2f\left(y\right)\right)=x+f\left(x\right)+2y$$

holds for all $x,y\in\mathbb{R}^+$

Solution. *Substitute $2y+2f\left(z\right)$ for y,*

$$\begin{aligned} LHS &= f\left(2x+2f\left(2y+2f\left(z\right)\right)\right)\\ &= f\left(2x+2y+2f\left(y\right)+4z\right)\\ &= x+y+2z+f\left(x+y+2z\right)+2y \end{aligned}$$

On the other hand,

$$\begin{aligned} RHS &= f\left(2x+2f\left(2y+2f\left(z\right)\right)\right)\\ &= x+f\left(x\right)+2\left(2y+2f\left(z\right)\right) \end{aligned}$$

After simplification,

$$f\left(x+y+2z\right)+2z=f\left(x\right)+y+2f\left(z\right)$$

By switching the variables, we see that f is linear and thus,

$$f\left(x\right)=x,\ \forall x\in\mathbb{R}^+$$

Remark 9. *In fact, we can solve this problem by switching x,y directly. We have*

$$f\left(2f\left(x\right)+2f\left(y\right)\right)=f\left(x\right)+f\left(f\left(x\right)\right)+2y$$

Thus,

$$f\left(x\right)+f\left(f\left(x\right)\right)=2x+c$$

for some constant c. According to Example 20, we conclude that

$$f\left(x\right)=x,\ \forall x\in\mathbb{R}^+$$

However, switching doesn't help for the following problem:
Determine all functions $f:\mathbb{R}^+\to\mathbb{R}$ so that

$$f\left(2x+2f\left(y\right)\right)=x+f\left(x\right)+2y$$

holds for all positive reals x,y

In fact, I'm inspired by

Example 40. *(2007 A4) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ so that*

$$f(x + f(y)) = f(x + y) + f(y), \forall x, y \in \mathbb{R}^+$$

Solution. *The solution is due to Andreas Dwi Maryanto Gunawan, from Indonesia. For any positive reals z, we have that*

$$f(x + f(y)) + z = f(x + y) + f(y) + z$$

Then, take f on both side

$$f(f(x + f(y)) + z) = f(f(x + y) + f(y) + z)$$

Use the condition to reduce expressions:

$$f(x + f(y) + z) + f(x + f(y)) = f(x + y + f(y) + z) + f(x + y)$$

Again, we have

$$f(x + y + z) + f(y) + f(x + y) + f(y) = f(x + 2y + z) + f(y) + f(x + y)$$

$$\Rightarrow f(x + y + z) + f(y) = f(x + 2y + z)$$

The rest is a routine work.

Remark 10. *In general, if our condition is*

$$f(x + g(y)) = g(y) + h(x, y)$$

for some functions g, h. Then we can try to compare $P(x, y + g(z))$ and $P(x + h(y, z), y)$

Exercise. *(Socrates) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ so that for all $x, y \in \mathbb{R}^+$*

$$f(x + f(y)) = 3f(x) - 2x + f(y)$$

Sometimes, we need to use this method twice.

Exercise. *(2016 Taiwan) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$f(x + y + f(y)) = 4030x - f(x) + f(2016y), \forall x, y \in \mathbb{R}^+$$

3.3 Double Counting

Indeed, switching and three variables method are kinds of double counting. The central concept is: for a particular expression, we calculate the value in two different ways.

Example 41. *(2015 APMO) Let $S = \{2, 3, 4, \cdots\}$ denote the set of integers that are greater than or equal to 2. Does there exists a function $f : S \to S$ such that*

$$f(a)\,f(b) = f\left(a^2 b^2\right) \quad \text{for all } a, b \in S \text{ with } a \neq b$$

Solution. *Notice that we have*

$$\begin{aligned} f(2^1)f(2^2)f(2^3)f(2^4)f(2^5) &= f(2^3)f(2^4)f(2^5)f(2^6) \\ &= f(2^5)f(2^6)f(2^{14}) \\ &= f(2^{14})f(2^{22}) \\ &= f(2^{72}) \end{aligned}$$

On the other hand,

$$\begin{aligned} f(2^1)f(2^2)f(2^3)f(2^4)f(2^5) &= f(2^1)f(2^2)f(2^4)f(2^{16}) \\ &= f(2^2)f(2^4)f(2^{34}) \\ &= f(2^4)f(2^{72}) \end{aligned}$$

So $f(2^4) = 1$, which is absurd! Thus, we conclude that there is not such a function.

Remark 11. *If we apply the following transformation: Let $g : \log S \to \log S$ defined by*

$$g(x) = \log f(e^x)$$

Then we find that the condition becomes

$$g(a) + g(b) = g(2a + 2b)$$

Consequently,

$$g(a) + g(b) + g(2c) = g(2a + 2b) + g(2c) = g(4a + 4b + 4c)$$

Switching the variables, we get

$$g(2d) = g(d) + c, \ \forall d \in \log S$$

And so,

$$2g(d) = g(4d) = g(2d) + c = g(d) + 2c \to g(d) = 2c$$

It follows that $g \equiv 0$, which is impossible!

Of course, we need to have some identities to "open" the terms. Notice that we, initially, have only the condition. Therefore, we usually use the given condition to do the trick.

Example 42. *(2014 Poland) Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying*

$$\underbrace{f(f(f(\dots f(f}_{n}(q))\dots))) = f(nq)$$

for all integers $n \geq 1$ and rational numbers $q > 0$

Solution. *We want to calculate the following expression*

$$\underbrace{f(f(f(\dots f(f}_{n}(q))\dots)))$$

in another way. By the definition of composition,

$$LHS = \underbrace{f(f(f(\dots f(f}_{a}(\underbrace{f(f(f(\dots f(f}_{n-a}(q)\dots))))\dots)))$$

It follows that

$$f\left(a\left(n+1-a\right)q\right) = f\left(nq\right)$$

holds for all $a \in \{1, 2, \cdots, n\}$. In particular,

$$f\left(a\left(a^2 - a + 1 - a\right)q\right) = f\left(\left(a^2 - a\right)q\right) \to f\left(q\right) = f\left(mq\right),\ \forall m \in \mathbb{N}$$

by simple substitution. Now, for any $x, y \in \mathbb{Q}^+$, write x, y as $\frac{q}{r}, \frac{q'}{r'}$, respectively. Then

$$f\left(x\right) = f\left(q'r \cdot x\right) = f\left(qq'\right) = f\left(qr' \cdot y\right) = f\left(y\right)$$

We conclude that f is constant, as desired.

Remark 12. *Since we expect the only function is constant, we may want to show that for any $x, y \in \mathbb{Q}+$, there are some $a < n \in \mathbb{N}$ and $q \in \mathbb{Q}^+$ such that*

$$\begin{cases} x = a\left(n-a\right)q \\ y = nq \end{cases}$$

This is not so obvious. The readers can try to give it a proof or a counter example.

In most situations, it suffices to find out "short" (because we hope to get a simple result) identities in the beginning.

Example 43. *(Socrates) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ so that*

$$f\left(x^3\right) - f\left(y^3\right) = (x-y)\left(f\left(x^2\right) + f(xy) + f\left(y^2\right)\right), \forall x, y \in \mathbb{R}^+$$

Solution. *WLOG, assume $f(1) = 1$. Notice that*

$$f\left(x^3\right) - f\left(z^3\right) = \left(f\left(x^3\right) - f\left(y^3\right)\right) + \left(f\left(y^3\right) - f\left(z^3\right)\right)$$

Therefore, we have

$$\begin{aligned}\left(x^2-1\right)\left(f\left(x^4\right) + f\left(x^2\right) + 1\right) = &\left(x^2 - x\right)\left(f\left(x^4\right) + f\left(x^3\right) + f\left(x^2\right)\right) \\ &+ (x-1)\left(f\left(x^2\right) + f(x) + 1\right)\end{aligned}$$

After simplification,

$$f\left(x^4\right) = x(x-1)f\left(x^2\right) + \left(x^2 - x + 1\right)f(x) + x(x-1)$$

For convenience, let g, h be two polynomials such that

$$f\left(x^3\right) = g\left(x, f(x), f\left(x^2\right)\right) \quad \textit{and} \quad f\left(x^4\right) = h\left(x, f(x), f\left(x^2\right)\right)$$

Our idea is to calculate $f\left(x^{12}\right)$ in two ways. Since $\left(x^4\right)^3 = x^{12} = \left(x^3\right)^4$, it follows

$$\begin{aligned}f\left(x^{12}\right) &= h\left(x^3, f\left(x^3\right), f\left(x^6\right)\right) \\ &= h\left(x^3, g\left(x, f(x), f\left(x^2\right)\right), g\left(x^2, f\left(x^2\right), f\left(x^4\right)\right)\right) \\ &= h\left(x^3, g\left(x, f(x), f\left(x^2\right)\right), g\left(x^2, f\left(x^2\right), h\left(x, f(x), f\left(x^2\right)\right)\right)\right)\end{aligned}$$

and

$$\begin{aligned}f\left(x^{12}\right) &= g\left(x^4, f\left(x^4\right), f\left(x^8\right)\right) \\ &= g\left(x^4, h\left(x, f(x), f\left(x^2\right)\right), h\left(x^2, f\left(x^2\right), f\left(x^4\right)\right)\right) \\ &= g\left(x^3, h\left(x, f(x), f\left(x^2\right)\right), h\left(x^2, f\left(x^2\right), h\left(x, f(x), f\left(x^2\right)\right)\right)\right)\end{aligned}$$

From here, one can show that

$$f\left(x^2\right) + f(x) = x^2 + x, \ \forall x \in \mathbb{R}^+$$

Finally,

$$f\left(x^3\right) = (x-1)\left(f\left(x^2\right) + f(x) + 1\right) + 1 = (x-1)\left(x^2 + x + 1\right) + 1 = x^3$$

Usually, we use two operations which are commute to realize our idea.

3.4 Exercise

We begin with standard exercises.

Problem 27. *(2006 MOP) Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying*

$$f\left(x+f\left(y\right)\right)=x+f\left(f\left(y\right)\right)$$

for all real numbers x and y

Problem 28. *Determine all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that*

$$f(x+y+f(y))+f(x+z+f(z))=f(2f(z))+f(2y)+f(2x)$$

holds for all $x,y,z\in\mathbb{R}^+$

Problem 29. *(Socrates) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that*

$$f\left(x+f\left(x\right)+y\right)=f\left(y\right)+2x,\ \forall x,y\in\mathbb{R}^+$$

Problem 30. *(Socrates) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that*

$$f\left(x+2f\left(y\right)\right)=f\left(x+y\right)+y,\ \forall x,y\in\mathbb{R}^+$$

Problem 31. *(Mohammed Jafari) Determine all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that*

$$f\left(x+f\left(x\right)+2y\right)=f\left(2x\right)+y+f\left(y\right)$$

holds for all $x,y\in\mathbb{R}^+$

Problem 32. *(2007 Romania) Let $f:\mathbb{Q}\to\mathbb{R}$ be a function such that*

$$\left|f\left(x\right)-f\left(y\right)\right|\leq\left(x-y\right)^2$$

for all $x,y\in\mathbb{Q}$. Prove that f is constant.

Problem 33. *(Own) Find all functions $f:\mathbb{N}\to\mathbb{N}$ satisfying that for all $a,b\in\mathbb{N}$,*

$$f\left(a+f\left(b\right)\right)=f\left(a\right)^{f(b)}$$

Problem 34. *(MDS) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that*

$$f\left(x+f\left(y\right)\right)+x=f\left(x+f\left(x\right)\right)+y$$

holds for all positive reals x,y

In fact, for most problems, double counting is used to find some useful identities. Try to bash the following problems! :)

Problem 35. *(Crux) Find all real functions f such that*

$$f(xy) = f(f(x) + f(y))$$

holds for all real x, y

Problem 36. *(2017 Iran) Find all functions $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions: for all positive reals x, y, z,*

$$f(f(x,y), z) = x^2y^2 f(x,z)$$

and

$$f(x, 1 + f(x,y)) \geq x^2 + xyf(x,x)$$

Problem 37. *(2010 A5) Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:*

$$f\left(f(x)^2 y\right) = x^3 f(xy)$$

Problem 38. *(Socrates) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$f\left(x^2 + yf(z) + f(x)\right) = xf(x) + zf(y) + x,\ \forall x, y, z \in \mathbb{R}^+$$

Problem 39. *(2005 A2) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which have the property:*

$$f(x)f(y) = 2f(x + yf(x))$$

for all positive real numbers x and y

Problem 40. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x)f(yf(x) - 1) = x^2 f(y) - f(x),\ \forall x, y \in \mathbb{R}$$

Problem 41. *(2013 USA) Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the equation*

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c$$

for all $a, b, c \geq 2$

Chapter 4

Calculus (Supplement)

In this chapter, we'll fulfill some weird imaginations. I believe that for many people, at least for me, when they first know what a functional equation is, they will claim that only continuous functions or polynomials can be the solution. Yet, this imagination will be broken as soon as they learned some weird solutions, e.g. solutions of Cauchy functional equation. However, most of the functional equations have "regular" solutions, and the word "regular" may refer to continuous or differentiable. Therefore, I'll state some definitions, theorems and properties in calculus and demonstrate how to obtain these properties in a functional equation. Also, the proves of those theorems and properties can be easily found in any calculus textbook, which means I'll omit the proves here. In addition, Since this technique is about calculus, we always need to derive some inequalities from the equation, and hence it only works on some $\mathbb{R}^+$ functional equations and few $\mathbb{R}$ functional equations.

4.1 Do You Know How to Count?

Question. *Given two sets, when would we say they have the same size? Does* $\{$*dog, cat, rabbit, duck*$\}$ *have the same size as* $\{1, 2, 4, 5\}$ *? How about the set of all positive integers and integers? the set of all real numbers?*

Answer. *First we notice that if each set is finite, they have the same size iff there is a bijection between two sets; if one is larger then the other, there is a injection from the smaller one to the larger one. So we want to generalize this to the infinite case.*

Definition 5. *Given two sets* A, B*, we say* $|A| = |B|$ *if there is a bijection* $f : A \to B$

Property. *For any sets A, B, C, we have*

1. $|A| = |A|$
2. *If* $|A| = |B|$*, then* $|B| = |A|$
3. *If* $|A| = |B|$ *and* $|B| = |C|$*, then* $|A| = |C|$

Definition 6. *Given two sets A, B, we say $|B| \geq |A|$ if there is an injection $f : A \to B$*

Remark 13. *If there is an injection $f : A \to B$, then there is a surjection $g : B \to A$. The converse is true if we assume axiom of choice.*

Property. *For any sets A, B, C, we have*

1. *If* $|A| \geq |B|, |B| \geq |C|$*, then* $|A| \geq |C|$
2. *If* $|A| \geq |B|, |B| \geq |A|$*, then* $|A| = |B|$

Definition 7. *If $|A| \leq |\mathbb{N}|$, we say A is countable.*

Property. *If A is an infinite set , and A is countable, then $|A| = |\mathbb{N}|$*

Exercise. *Prove*

- $|\mathbb{N}| = |\mathbb{N} \cup \{dog\}|$
- $|\mathbb{N}| = |\mathbb{Z}|$
- $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$
- $|\mathbb{N}| = |\mathbb{Q}|$,

Theorem 2. $|\mathbb{R}| \gneq |\mathbb{N}|$

Proof. Although we don't actually know what real numbers are, and the construction will be derived in the next section, we'll first assume real numbers to be a integer plus a decimal part, but we say that 999... after some digit is same plus 1 in the previous digit. For example,

$$2.999\ldots = 3$$

In fact, we can prove that the size of real numbers in $(0, 1]$ is (strictly) larger than $\mathbb{N}$. Let's assume the statement is false, which means that we can label the real numbers in $(0, 1]$ by integers. Assume the i-th number is a_i, and write it in decimal system with infinite decimal (not end in 000...). Next, we can choose a real number $x \in (0, 1]$ such that the i-th digit is different from the i-th digit of a_i, and x is not a finite decimal. Therefore, x was not counted, and we get a contradiction! $\square$

4.2 Construction of Numbers

Before learning calculus, we should first know what the numbers we're familiar with are, so in this section, I'll first introduce the construction of integers, rational numbers, and real numbers, and deal with some basic properties of them.

Definition 8. *(Peano axioms) The set $\mathbb{N}$ are constructed as the following:*

1. *1 is a positive integer.*
2. *For every positive integer a, there is a unique successor a', which is also a positive integer.*
3. *Two positive integers are equivalent iff their successors are equivalent.*
4. *1 is not a successor of any positive integer.*
5. *For any statement about positive integers, if one can prove that it's true for 1, and for n' under the assumption that it's true for n, then the statement is true for $\mathbb{N}$*

Definition 9. *A relation $\sim$ on S is a collection of $(a, b) \in S \times S$ (denote as $a \sim b$), if a relation $\sim$ satisfies the following:*

1. $x \sim x$
2. $x \sim y \Rightarrow y \sim x$
3. $x \sim y,\ y \sim z \Rightarrow x \sim z$

then we say $\sim$ is an equivalent relation.

Definition 10. *Let $S := \{(a, b) | a, b \in \mathbb{Z}, b \neq 0\}$, and we define an equivalent relation $\sim$ by*

$$(a, b) \sim (c, d) \Leftrightarrow ad - bc = 0$$

then the set of rational numbers is defined to be $\mathbb{Q} = S/\sim$

Definition 11. *Define a sequence $a_1, a_2, \ldots$ converges to a as the following:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N},\ s.t.\ n > N \Rightarrow |a_n - a| < \varepsilon$$

denoted by $\lim_{n\to\infty} a_n = a$, and a is called the limit of the sequence.

Definition 12. *Let $\{a_n\}$ stands for the sequence $a_1, a_2, ...,$ and let*

$$S := \{(\{a_n\}, \{b_n\}) | \ \forall n \in \mathbb{N}, \ a_n, b_n \in \mathbb{Q}, \ b_n > a_n,$$

$$\{a_n\} \text{ non-decreasing}, \ \{b_n\} \text{ non-increasing}, \lim_{n\to\infty}(b_n - a_n) = 0\}$$

and define an equivalent relation $\sim$ by

$$(\{a_n\}, \{b_n\}) \sim (\{c_n\}, \{d_n\}) \Leftrightarrow \lim_{n\to\infty}(c_n - a_n) = 0$$

then the set of all real numbers is defined to be $\mathbb{R} = S/\sim$

Definition 13. *If $\{a_n\}$ satisfies that*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \ s.t. \ m, n > N \Rightarrow |a_n - a_m| < \varepsilon$$

then we call it a Cauchy sequence.

Property. *Every convergent sequence of real numbers is a Cauchy sequence.*

Theorem 3. *(Completeness of real numbers) All real-valued Cauchy sequence $\{a_n\}$ converges in $\mathbb{R}$, i.e. there is a real number a such that $\lim_{n\to\infty} a_n = a$*

4.3 Limit and Continuity

Definition 14. *We say that a real-valued function $f(x)$ converges to a when x tends to infinity if*

$$\forall \varepsilon > 0, \exists N \in \mathbb{R}, \ s.t. \ x > N \Rightarrow |f(x) - a| < \varepsilon$$

denoted by $\lim_{x\to\infty} f(x) = a$, and a is called the limit of $f(x)$ as x tends to infinity.

Definition 15. *We say that a real-valued function $f(x)$ converges to a when x tends to b if*

$$\forall \varepsilon > 0, \exists \delta > 0, \ s.t. \ 0 < |x - b| < \delta \Rightarrow |f(x) - a| < \varepsilon$$

denoted by $\lim_{x\to b} f(x) = a$, and a is called the limit of $f(x)$ as x tends to b.

Remark 14. *From $0 < |x - b| < \delta$ in the previous definition, we should notice that $f(b)$ does not necessarily equal a.*

Property. *The limit does not necessarily exist, but once it exists, it must be unique.*

Property. *If both* $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist, then*

1. $\lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n \pm b_n$
2. $\lim_{n\to\infty} a_n \times \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n \times b_n$
3. *Moreover, if* $\lim_{n\to\infty} b_n \neq 0$*, we have* $\lim_{n\to\infty} a_n \div \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n \div b_n$

Theorem 4. *(Squeeze theorem) If* $a_n \leq b_n \leq c_n$*, and*

$$\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n, \lim_{n\to\infty} c_n$$

all exist, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n$.
Moreover, if $a_n \leq b_n \leq c_n$*, and both*

$$\lim_{n\to\infty} a_n, \lim_{n\to\infty} c_n$$

exist and are equal, then $\lim_{n\to\infty} b_n$ *exists and equals the others.*

Similarly, we also have the functional version of the previous properties, and their proofs are almost the same.

Property. *Let* $b \in [-\infty, \infty]$*, if both* $\lim_{x\to b} f(x), \lim_{x\to b} g(x)$ *exist, then*

1. $\lim_{x\to b} f(x) \pm \lim_{x\to b} g(x) = \lim_{x\to b} f(x) \pm g(x)$
2. $\lim_{x\to b} f(x) \times \lim_{x\to b} g(x) = \lim_{x\to b} f(x) \times g(x)$
3. *Moreover, if* $\lim_{x\to b} g(x) \neq 0$*, then* $\lim_{x\to b} f(x) \div \lim_{x\to b} g(x) = \lim_{x\to b} f(x) \div g(x)$

Theorem 5. *(Squeeze Theorem) Let* $b \in [-\infty, \infty]$*, if* $f(x) \leq g(x) \leq h(x)$*, and*

$$\lim_{x\to b} f(x), \lim_{x\to b} g(x), \lim_{x\to b} h(x)$$

all exist, then $\lim_{x\to b} f(x) \leq \lim_{x\to b} g(x) \leq \lim_{x\to b} h(x)$
Moreover, if $f(x) \leq g(x) \leq h(x)$*, and both*

$$\lim_{x\to b} f(x), \lim_{x\to b} h(x)$$

exist and are equal, then $\lim_{x\to b} g(x)$ *exists and equals the others.*

After understanding the definition of limit, we can define least upper bound and greatest lower bound, and take limits of them in order to understand more about a function or sequence.

Definition 16. *Given a set $S \subseteq \mathbb{R}$, we say that S has an upper bound if there exists $M \in \mathbb{R}$, such that for all $x \in S$, $x \le M$, and M is called an upper bound of S; similarly we can define a lower bound of S, and if S has both upper and lower bound, we say S is bounded.*

Property. *A non-decreasing sequence with an upper bound must converge; similarly a non-increasing sequence with lower bound must converge.*

Definition 17. *Given a set $S \subseteq \mathbb{R}$, if S has an upper bound, then among all upper bounds of S, there is a smallest one, and this value is called the least upper bound (supremum) of S, denoted by $\sup S$; similarly we can define the greatest lower bound (infimum) of S, denoted by $\inf S$.*

Definition 18. *Define the limit superior ($\limsup$) as the following:*

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} (\sup_{m>n} a_m)$$
$$\limsup_{x\to\infty} f(x) = \lim_{x\to\infty} (\sup_{y>x} f(y))$$
$$\limsup_{x\to b} f(x) = \lim_{\varepsilon\to 0^+} (\sup_{x\in[b-\varepsilon,b+\varepsilon],x\neq b} f(x))$$

Similarly we can define the limit inferior ($\liminf$).

Property. *If $\limsup = \liminf$, then the limit exists and equals the others.*

Definition 19. *Define a function $f(x)$ to be continuous at x_0 as the following:*

$$\lim_{x\to x_0} f(x) = f(x_0)$$

or in other words, the limit of $f(x)$ when x tends to x_0 is $f(x_0)$. We say $f(x)$ is continuous on a set if $f(x)$ is continuous at all points in the set.

Property. *Polynomials, rational functions, exponential function, logarithm, and trigonometric functions are continuous at all points where they can be defined.*

Property. *If both $f(x), g(x)$ are continuous functions, then $f(g(x))$ is continuous at all points wherever it's well-defined.*

Theorem 6. *(Intermediate Value Theorem) If f is continuous on $[a, b]$, then for any given y between $f(a)$ and $f(b)$, there must exist a $c \in [a, b]$ such that $f(c) = y$.*

Property. *A continuous function f on $[a, b]$ must reach its supremum.*

Theorem 7. *(Bolzano Weierstrass Theorem)Given an infinite sequence $a_1, a_2, \ldots$, where $a_i \in [a, b]$ holds for all i, then we can find a subsequence $a_{i_1}, a_{i_2}, \ldots$ which converges.*

Theorem 8. *A non-decreasing (or non-increasing) function on $(a,b) \subseteq \mathbb{R}$ is continuous at all but countable points.*

Definition 20. *A real-valued function f is uniform continuous if*

$$\forall \varepsilon > 0, \exists \delta > 0,\ s.t.\ 0 < |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Property. *A function which is continuous on a closed interval must be uniform continuous.*

Example 44. *Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying that for all $x, y \in \mathbb{R}^+$,*

$$xf(x+y)(yf(x)+1) = 1$$

Proof. First, for all $a < b$, we can always find $0 < c < a$. By choosing $(x,y) = (c, a-c), (c, b-c)$, we can obtain $f(a) > f(b)$. Therefore, f is a decreasing function, so we can apply theorem 8. In other words, f is continuous on all but countable points. In particular, there is at least one point t such that f is continuous at t. Given any $r > t$, by taking $y = r - t$, we can obtain

$$\lim_{x\to t} f(x+r-t) = \lim_{x\to t} \frac{1}{x(yf(x)+1)} = \frac{1}{t(yf(t)+1)} = f(r)$$

Hence, f is continuous at r. Similarly, we can prove that $f(r)$ is continuous for all $r < t$. Therefore, we have

$$f(x) = \lim_{y\to 0^+} f(x+y) = \lim_{y\to 0^+} \frac{1}{x(yf(x)+1)} = \frac{1}{x}$$

Also, we can check it is a solution $\square$

Example 45. *Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying that for all $a, b, c \in \mathbb{R}^+$,*

$$a, b, c \textit{ forms a triangle } \Leftrightarrow f(a), f(b), f(c) \textit{ forms a triangle}$$

Proof. If $\limsup_{x\to 0^+} f(x)$ is not zero or doesn't exist, then we can find $t, k > 0$ and a sequence $a_1, a_2, ...$ such that

1. $\lim_{n\to\infty} a_n = 0$
2. For all $n \in \mathbb{N} f(a_n) > k$
3. $f(t) < 2k$

By choosing $(a, b, c) = (t, a_n, a_n)$, we can get a contradiction as long as n large enough. On the other hand, $\liminf_{x \to 0^+} f(x) \geq 0$, hence by combining this with the previous result, we proved that $\lim_{x \to 0^+} f(x) = 0$ Given $f(a) > \varepsilon > 0$, we can set $(a, b, c) = (a, \delta, x)$, where δ is small enough such that $f(\delta) < \varepsilon$. As a result, for all $x \in (a - \delta, a + \delta)$, we have $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$, which means f is continuous.

If there exists (a, b, x_0) with $x_0 \geq a + b$ and $f(x_0) \leq |f(a) - f(b)|$, by intermediate value theorem we can obtain that for all $x \geq a + b$, we have $f(x) \leq |f(a) - f(b)|$. By taking $y \geq 2a + 2b$, and let $w \in \{a, b\}$ be the one which its f-value is larger, we can plug in (w, w, y) and get a contradiction from the fact that w, w, y can't form a triangle but $f(w), f(w), f(y)$ can. Hence $f(a + b) \geq f(a) + f(b)$

By plugging in $((a, b, a + b - \varepsilon))$, we can get that $f(a + b - \varepsilon) < f(a) + f(b)$. Let $\varepsilon \to 0$, since f is a continuous function, we obtain that $f(a + b) \leq f(a) + f(b)$. Therefore, f is additive, and by combining this with $f \geq 0$ we can get that $f(x) = cx$ for some constant c, which is indeed a solution. □

Example 46. *(2016 APMO) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y, z \in \mathbb{R}^+$,*

$$(z + 1)f(x + y) = f(xf(z) + y) + f(yf(z) + x)$$

Proof. If there is a positive sequence $a_1, a_2, ...$ such that

1. $\lim_{n\to\infty} a_n = 0$

2. $\lim_{n\to\infty} f(a_n) = \infty$

We can WLOG assume that all $a_n < 1$ by forgetting first few terms. By plugging $y = 1 - x, z = a_n$ in the original equation we can get

$$f(xf(a_n) + (1 - x)) < (a_n + 1)f(1) < 2f(1)$$

It tells us that $f((1, f(a_n)))$ bounded by $2f(1)$. By taking n goes to infinite, we can get that $f((1, \infty))$ is bounded by $2f(1)$. On the other hand, if we take $x > 1, y > 1$ and z very large in the original equation, RHS is smaller than $4f(1)$, but LHS can be arbitrary large, hence we get a contradiction!

As a result, such sequence a_n don't exist. In other words, $\limsup_{x\to 0^+} f(x)$ exists. Assume $\limsup_{x\to 0^+} f(x) = k$, and we can find a sequence $x_n > 0$ such that

1. $\lim_{n\to\infty} x_n = 0$

2. $\lim_{n\to\infty} f(x_n) = k$

By plugging in $x = y = \frac{x_n}{2}$ and by taking $n \to \infty$, we can obtain that

$$(z+1)k = \lim_{n\to\infty}(z+1)f(x_n) = \lim_{n\to\infty} 2f(x_n f(z) + x_n) \leq 2k$$

Therefore, $k = 0$, and hence $\lim_{z\to 0^+} f(z) = 0$.
By forgetting one term in the RHS, we have that

$$f(x+y) \geq \frac{f(xf(z)+y)}{1+z}$$

If there exists $a > b$ with $f(a) < f(b)$, we can first solve $x + y = a, xf(z) + y = b$,

$$x = \frac{a-b}{1-f(z)}, y = \frac{b-af(z)}{1-f(z)}$$

Also, because of $\lim_{z\to 0} f(z) = 0$ we can choose z very small and $f(z)$ very small, and obtain a contradiction! Therefore, f is an increasing function.

We can take $z \to 0$ and let $g(x) = \lim_{h\to 0^+} f(x+h)$, hence we have that $f(x+y) = g(x)+g(y)$.

Moreover, take y as $y + \varepsilon$ and let $\varepsilon \to 0^+$, we have that $g(x + y) = g(x) + g(y)$. Join with the fact that $g \geq 0$, we obtain that $g(x) = cx$ and hence $f(x) = cx$. Finally, we can get that $c = 1$ and check it's indeed a solution. $\square$

4.4 Differentiation

Differentiation is a method to understand what happens to a function near some point, and its definition is just the limit of the slope of its secant lines.

Definition 21. *A real-valued function $f(x)$ is said to be differentiable at x_0 if*

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exists and we denote the value by $\frac{df}{dx}(x_0)$ or $f'(x_0)$, which is the derivative of $f(x)$ at x_0

Property. *If f is differentiable at x_0, then f is continuous at x_0*

Property. *If both f, g are differentiable, then*

1. $(f \pm g)' = f' \pm g'$
2. $(fg)' = f'g + g'f$
3. *Moreover, if $g(x_0) \neq 0$, then*$(\frac{f}{g})' = \frac{f'g-g'f}{g^2}$

Property. *If f is differentiable at x_0, and g is differentiable at $f(x_0)$, then*

$$(g \cdot f)'(x_0) = g'(f(x_0))f'(x_0)$$

Property. *If f and g are the inverse function of each other, f is differentiable at x_0 and $f'(x_0) \neq 0$, then g is differentiable at $f(x_0)$ and*

$$g'(f(x_0)) = \frac{1}{f'(x_0)}$$

Property. *The following differentiations are all taken w.r.t. x:*

1. $(x^a)' = ax^{a-1}$
2. $(e^x)' = e^x$ *(where $e := \lim_{n\to\infty}(1+\frac{1}{n})^n$)*
3. $(\log x)' = \frac{1}{x}$ *(where $\log = \log_e$)*
4. $(\sin x)' = \cos x$
5. $(\cos x)' = -\sin x$

Theorem 9. *(Mean Value Theorem of Differentiation) If f is differentiable on $[a, b]$, then there exists $c \in [a, b]$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

4.5 Integration

Here, integral means Riemann integral , but actually we follow the definition of Darboux.

Definition 22. *Define S to be a partition of $[a, b]$ if $S = \{x_0, ..., x_n | a = x_0 < x_1 < ... < x_n = b\}$, and $|S|$ means $\max_{i=1}^n(x_i - x_{i-1})$, which is the length of the greatest interval in S*

Definition 23. *Define $U(f, S) := \Sigma_{i=1}^n f_i(x_i - x_{i-1})$, where f_i is the least upper bound of f in $[x_{i-1}, x_i]$, and similarly we can define $L(f, S)$ by changing the least upper bound into greatest lower bound. Furthermore, we can define $U(f) := \inf_S U(f, S)$, and $L(f) := \sup_S L(f, S)$*

Definition 24. *A real-valued function f on $[a, b]$ is said to be (Riemann) integrable if*

$$U(f) = L(f)$$

and we can define the integral of f on $[a, b]$ by

$$\int_a^b f(x)dx := U(f)$$

Property. *If f is continuous on $[a, b]$, then f is integrable on $[a, b]$*

Property. *If f is integrable on $[a, b]$, then f is integrable on $[c, d] \subseteq [a, b]$*

Definition 25. *Define the indefinite integral of f, F by*

$$F(x) = \int_a^x f(u)du$$

Property. *If f is integrable on both $[a, b], [b, c]$, then f is integrable on $[a, c]$ and*

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Definition 26. *If $b > a$, then $\int_b^a f(x)dx := -\int_a^b f(x)dx$*

Theorem 10. *(Mean Value Theorem of Integration) If f is a continuous function on $[a, b]$, then there exists $c \in [a, b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx$$

Theorem 11. *(Fundamental Theorem of Calculus) If f is continuous, then the derivative of the integral of f is still f*

Theorem 12. *(Fundamental Theorem of Calculus) If f is differentiable and its derivative is continuous, there the difference between f and the indefinite integral of f' can only be a constant.*

Property. *(Change of Variables) If f is continuous on $[a, b]$; g is a monotone and differentiable $[a, b]$-valued function, then*

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)du$$

where g^{-1} is the inverse function of g

Property. *(Integration by Parts) If f, g are both integrable on $[a, b]$, then*

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

4.6 Ordinary Differential Equation

ODE refers to a functional equation consisting of functions and their derivatives with only one variable. However in this section, I'll only go through some common form with only one function in the equation to be solved, and the function is required to be differentiable in the region; furthermore, how many times we could differentiate are usually consistent with the equation.

Property. *If an O.D.E. is of the form $a_n u^{(n)} + ... + a_1 u' + a_0 u = 0$, where $u^{(k)}$ means the k-th derivative of u, and let $x_1, ..., x_r$ be all solutions of $a_n x^n + ... + a_1 x + a_0 = 0$, where x_i has multiplicity m_i, then all solutions of the previous O.D.E. are of the form*

$$u(x) = \Sigma_{i=1}^{r} P_i(x) e^{x_i x}$$

where P_i is a polynomial with its degree not greater then m_i

Property. *If an O.D.E. is of the form $a_n u^{(n)} + ... + a_1 u' + a_0 u = f$, and g is a solution of it , then all solutions are of the form*

$$u(x) = g(x) + \Sigma_{i=1}^{r} P_i(x) e^{x_i x}$$

where P_i is a polynomial with its degree not greater then m_i

Property. *If a differential equation is of the form $\frac{du}{dx} = P(u)Q(x)$, then we can first rewrite the equation as*

$$\frac{du}{P(u)} = Q(x)dx$$

and then take integral on both side,

$$\int \frac{du}{P(u)} = \int Q(x)dx$$

then we can find out u, while we should notice that there is a constant difference after the indefinite integral.

4.7 Lebesgue Monotone Differentiable Theorem

In this section, I'm going to talk about Lebesgue monotone differentiable theorem. However, it's a strong theorem and its proof is too complex. Therefore, I won't prove it here and will instead demonstrate the usage of it.

Definition 27. *We say that a set $S \subseteq \mathbb{R}$ has measure zero if*

$$\forall \varepsilon > 0, \exists [a_i, b_i], i = 1, 2, ..., \ s.t. \ S \subseteq \bigcup_{i=1}^{\infty} [a_i, b_i], \Sigma_{i=1}^{\infty} b_i - a_i < \varepsilon$$

Property. *The union of countably many arbitrary measure zero sets is still measure zero, which tells us an arbitrary countable set is measure zero.*

Definition 28. *We say that a property holds almost everywhere (a.e.) if it fails to hold only on a measure zero set.*

Theorem 13. *(Lebesgue Monotone Differentiable Theorem) If f is non-decreasing (or non-increasing), then $f$is differentiable a.e.*

Example 47. *Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying that for all $x, y \in \mathbb{R}^+$,*

$$f(x)f(y) = f(x + yf(x))$$

Proof. If there exists $f(a) < 1$, then we can choose $(x, y) = (a, \frac{a}{1-f(a)})$ and get that $f(a) = 1$, which gives a contradiction!

Hence, for all x, we have that $f(x) \geq 1$. Therefore, f is an increasing function. Moreover, we can rewrite the equation as (1):

$$\frac{f(x + yf(x)) - f(x)}{yf(x)} = \frac{f(y) - 1}{y}$$

By applying Lebesgue monotone differentiable theorem, we can find a point x_0 where f is differentiable at. By setting $x = x_0$, $y \to 0^+$, we can obtain that

$$\lim_{y \to 0^+} \frac{f(y) - 1}{y}$$

exists. Let's assume the value is c, hence we have that

$$\lim_{y \to 0^+} \frac{f(x + yf(x)) - f(x)}{yf(x)} = c$$

holds for all x. Also, if we plug $x = a, y = \frac{b-a}{f(a)}$ in (1), where $b > a$, we can get that

$$\frac{f(b) - f(a)}{b - a} = \frac{f(\frac{b-a}{f(a)}) - 1}{\frac{b-a}{f(a)}}$$

By taking $a \to b^-$, we have that $\frac{b-a}{f(a)} \to 0^+$. Therefore,

$$\lim_{a \to b^-} \frac{f(b) - f(a)}{b - a} = c$$

Hence, f is differentiable on $\mathbb{R}^+$, and

$$f'(x) = c$$

Therefore, $f(x) = cx + d$, and we can check which pairs of c, d satisfy the original equation. □

Example 48. *Given a positive integer $n \geq 2$, find all $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying that for all $x, y \in \mathbb{R}^+$,*

$$f(x^n) - f(y^n) = (x - y) \sum_{k=0}^{n-1} f(x^k y^{n-1-k})$$

Proof. First, we should notice that f is an increasing function. Next, we are going to prove f is continuous everywhere. Let $y \to x$ in the equation, and since $\sum_{k=0}^{n-1} f(x^k y^{n-1-k})$ is bounded when y is close to x, RHS goes to 0. Therefore, LHS suggests us that f is continuous at x^n, and since the image of x^n is $\mathbb{R}^+$, f is continuous everywhere.

Then we divide by $x^n - y^n$ on both side and get that

$$\frac{f(x^n) - f(y^n)}{x^n - y^n} = \frac{\sum_{k=0}^{n-1} f(x^k y^{n-1-k})}{\sum_{k=0}^{n-1} x^k y^{n-1-k}}$$

By taking y goes to x, we can obtain that f is differentiable everywhere and

$$f'(x^n) = \frac{f(x^{n-1})}{x^{n-1}}$$

Yet, we don't know how to solve such kind of ODE which is involved in different points. It is natural to think about changing the RHS, and to achieve this we can take $y \to 0^+$ and assume $\lim_{x \to 0^+} f(x) = k$, which exists since it is decreasing and have a lower bound, we can obtain that

$$f(x^n) - k = x(f(x^{n-1}) + (n-1)k)$$

and the fact $k = 0$ can be derived from the previous equation by plugging in $x = 1$. Therefore,

$$\frac{f(x^{n-1})}{x^{n-1}} = \frac{f(x^n)}{x^n}$$

The rest thing is just solving the ODE on $\mathbb{R}^+$

$$f'(x) = \frac{f(x)}{x}$$

which tells that $f(x) = cx$ for some constant x. Also, we can check they are solutions $\square$

4.8 Problems

Problem 42. *Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying that for all $x, y \in \mathbb{R}^+$,*

$$f(y)f(xf(y)) = f(x + y)$$

Problem 43. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying that for all $x, y \in \mathbb{R}$,*

$$f(x^2 + f(x)y) = f(x)^2 + xf(y)$$

Chapter 5

Functional Equation in $\mathbb{N}/\mathbb{Z}$

Recently, there are more and more functional equations in integers. For those problems, our previous ideas may not be enough to derive a complete solution. For instance, a linear function $f(x) = ax + b$ is surjective if $f : \mathbb{R} \to \mathbb{R}$, but it's not necessarily surjective when $f : \mathbb{Z} \to \mathbb{Z}$. Therefore, we are going to study such problems in details.

5.1 Mathematical Induction

There are many great properties for integers. In my opinion, the well-ordering principle is one of the most important things. One simple way to solve FEs in positive integers is:

1. Bash out $f(1), f(2), \ldots, f(k)$ where k is a "small" positive integer
2. Use induction to find solutions

Example 49. *(2015 Canada) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,*

$$(n-1)^2 < f(n) f(f(n)) < n^2 + n$$

Solution. *It's easy to see that $f(1) = 1$. Suppose we have proved $f(k) = k$ for all $k < n$, we must have $f(m) \geq n$ for all $m \geq n$. That's because if $f(m) = k$ for some $k < n$,*

$$(m-1)^2 < f(m) f(f(m)) = k^2 = (n-1)^2$$

We get a contradiction. Now,

$$nf(n) \leq f(n) f(f(n)) < n^2 + n$$

which means $f(n) < n + 1$. So $f(n) = n$, as desired.

Example 50. *(Base on: 2015 Mexico) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that*

$$f(a+b+ab) = f(ab)$$

holds for all positive integers a, b

Solution. *It's natural to guess the answer is the constant function. Note that we have*

$$f(2) = f(5) = f(11) = f(6) = f(13) = f(27) = f(9) = f(3) = f(1)$$

Assume that $f(1) = f(2) = \cdots = f(k-1)$ for some $k \geq 3$, then

$$\begin{cases} k \equiv 0 \mod 3, & f(k) = f\left(\frac{k}{3}\right) = f(1) \\ k \equiv 1 \mod 3, & f(k) = f(2k+1) = f\left(\frac{2k+1}{3}\right) = f(1) \\ k \equiv 2 \mod 3, & f(k) = f\left(\frac{n-2}{3}\right) = f(1) \end{cases}$$

Notice our inductive step doesn't work for $k = 2$, that's why we need to bash $f(2)$. To find the number k that we need to bash $f(1), \ldots, f(k)$ for induction, one can do the inductive step first and use the constraints to find out k.

We demonstrate this idea in the next example.

Example 51. *(Own) Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that*

$$f(f(x) + f(y)) + 2f(xy) = xf(y+1) + yf(x+1), \ \forall x, y \in \mathbb{N}_0$$

Solution. *Stick to one of our principles, substitute zeros for variables whenever it's possible. Set $x = y = 0$, we have*

$$f(2f(0)) + 2f(0) = 0$$

Since f is non-negative, $f(0)$ must be 0. Now, in the original condition, take $y = 0$:

$$f(f(x)) = xf(1)$$

It's easy to check that $f \equiv 0$ when $f(1) = 0$. For the other case, f is injective and thus, $f(1)$ should be 1 (why). So far, the only useful identity (for induction) is

$$f(f(x) + 1) + 2f(x) = 2x + f(x+1)$$

IDEA: *If $f(x) = x$, the above equation gives nothing. However, if $f(x+1) = x+1$,*

$$f(f(x)+1) + 2f(x) = 3x+1$$

Assume $f(1) = 1, \ldots, f(x-1) = x-1$, then the only possibility for $f(x)$ is x (why) This suggests us to proceed the mathematical induction

$$P(1), \ldots, P(2k) \to P(2k+2) \to P(2k+1)$$

where $P(i)$ means that $f(i) = i$ holds. So it seems that we need $P(2)$ and the corresponding equation ($y = 2$ in the original condition)

$$f(f(x)+2) + 2f(2x) = xf(3) + 2f(x+1)$$

From above, we require $P(3)$ and $2k \geq k+3$. That's because we hope to get $P(2k+2)$ by setting $x = k+1$. In summary, it suffices to bash $P(3), \ldots, P(6)$

To get the value of $f(3)$, one should notice that $4f(3) = 3f(4)$, combines this fact with

$$f(f(3)+1) + 2f(3) = 6 + f(4)$$

we must have $f(3) = 3$. Since if $f(3) > 3$,

$$f(f(3)+1) + \frac{2}{3}f(3) \geq 3+4 = 7 > 6$$

which is impossible. Finally, choose $(x, y) = (3, 2)$,

$$f(5) + 2f(6) = 17$$

According to the injectivity of f, $f(5), f(6) \geq 5$ and therefore, $f(5) = 5, f(6) = 6$

Let's left this exercise as the end of the section. Try induction!

Exercise. *(2017 Taiwan) Determine all surjective functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying*

$$f(xyz + xf(y) + yf(z) + zf(x)) = f(x)f(y)f(z)$$

holds for all $x, y, z \in \mathbb{Z}$

5.2 Number Theory Facts

5.2.1 About Divisibility

We briefly review two basic knowledges of divisibility:

- If $a \mid b$ for some $a, b \in \mathbb{Z}$ then $\mid a \mid \leq \mid b \mid$ or $b = 0$
- Every prime p^k has only $k+1$ factors, namely $1, p, \cdots, p^k$

The first one extremely important because we can usually use

$$\boxed{a \mid b \longrightarrow a \mid b - ka}$$

to rewrite the divisibility condition to the form

$$g\left(m, n, f\left(m\right), f\left(n\right)\right) \mid h\left(m, f\left(m\right)\right)$$

for some functions g, h. Then, we may have $h\left(m, f\left(m\right)\right) = 0$ whenever g is unbounded for a fixed m. See the following problem.

Example 52. *(2017 Balkan) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that*

$$n + f\left(m\right) \mid f\left(n\right) + nf\left(m\right)$$

holds for all $m, n \in \mathbb{N}$

Solution. *We have the equivalence*

$$\begin{aligned} & n + f\left(m\right) \mid f\left(n\right) + nf\left(m\right) \\ \iff & n + f\left(m\right) \mid f\left(n\right) - n^2 \end{aligned}$$

If there exists a s.t $f\left(a\right) \neq a^2$, then f is bdd. Obviously, $f\left(1\right)$ is 1 (why). So, in particular,

$$\begin{aligned} & n + f\left(1\right) \mid f\left(n\right) - n^2 \\ \iff & n + f\left(1\right) \mid f\left(n\right) - 1 \end{aligned}$$

It follows $f \equiv 1$. In conclusion,

$$\begin{cases} f\left(x\right) = x^2, \ \forall x \in \mathbb{N} \\ f\left(x\right) = 1, \ \forall x \in \mathbb{N} \end{cases}$$

Example 53. *(2013 N1) Find all functions $f:\mathbb{Z}_{>0}\to\mathbb{Z}_{>0}$ such that*

$$m^2+f(n)\mid mf(m)+n$$

holds for all positive integers m and n

Solution. *In the relation, take $m=n$:*

$$\begin{aligned} & m^2+f(m)\mid mf(m)+m \\ \longrightarrow\quad & m^2+f(m)\le mf(m)+m \end{aligned}$$

So when $m\ge 2$, $f(m)\ge m$. Then, take $m=f(n)$,

$$f(n)^2+f(n)\mid f(n)f(f(n))+n$$

which implies $f(n)\mid n$ and thus, $f(n)\le n$
Clearly, we must have $f(n)=n$ holds for all $n\in\mathbb{N}$

The second small fact is also useful. The reason is, once we have

$$f(p)\mid p^k$$

We have only few possibilities for $f(p)$. Then we plug p into one variable of the condition to determine the other values.

Solution. *This time, set $m=f(p),n=p$ where p is a prime,*

$$f(p)^2+f(p)\mid f(p)f((p))+p$$

By our assumption, $f(p)=1$ or p. Notice that for sufficient large p, $f(p)=1$. Otherwise,

$$p^2+f(n)\mid p+n$$

which is absurd. Plug $n=p$ in the original relation (p large):

$$\begin{aligned} & m^2+p\mid mf(m)+p \\ \Longrightarrow\quad & m^2+p\mid mf(m)-m^2 \end{aligned}$$

Consequently, $f(m)=m,\ \forall m\in\mathbb{N}$

Exercise. *(2016 MEMO) Find all functions $f:\mathbb{N}\to\mathbb{N}$ so that*

$$f(a)+f(b)\mid 2(a+b-1),\ \forall a,b\in\mathbb{N}$$

In fact, a common strategy for divisibility problems is

- Bash $f(1)$
- Try to find $f(p)$ for large primes p
- Use those values to find f (usually by establishing inequalities)

Example 54. *(forget) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that*

$$a + f(b) \mid a^2 f(a) + b^3$$

holds for all $a, b \in \mathbb{N}$

Solution. *As the second solution to 2013 N1, we tends to assert $a = f(p), b = p$,*

$$2f(p) \mid f(p)^2 f(f(p)) + p^3$$

So $f(p) = 1, p, p^2$ or p^3. Because $f(1) = 1$ (figure out!),

$$p + 1 \mid p^2 f(p) + 1$$

It remains two cases $f(p) = p$ or p^3 However, if $f(q) = q^3$ for some prime q, then

$$a + q^3 \mid a^2 f(a) + q^3$$

This means q is "small" (think a minute)
In other words, we know that $f(p) = p$ for "large" p. The remain part is left as an exercise.

Exercise. *(2016 N6) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that*

$$f(m) + f(n) - mn \mid mf(m) + nf(n)$$

holds for all positive integers m and n

The following problem looks somehow crazy. But it's actually a standard one for training induction and divisibility.

Exercise. *(2008 N5) For every $n \in \mathbb{N}$, let $d(n)$ denote the number of (positive) divisors of n. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the following properties:*

$$d(f(x)) = x, \ \forall x \in \mathbb{N}$$

and

$$f(xy) \mid (x-1)\, y^{xy-1} f(x), \ \forall x, y \in \mathbb{N}$$

5.2.2 Euler Theorem

Solve FEs in integers as FEs in reals, we sometimes get

$$\begin{cases} f(cn) = cf(n) \\ f(n+d) = f(n) + e \end{cases}$$

for some constants c, d, e. Then,

$$c^k f(n) = f\left(c^k n\right) = f\left(n + \frac{\left(c^k - 1\right) n}{d} \cdot d\right) = f(n) + \frac{\left(c^k - 1\right) n}{d} \cdot e$$

if $d \mid c^k - 1$. Notice that if c, d are co-prime, we can apply Euler theorem to find such a k and thus, it's easy to conclude

$$f(n) = \frac{en}{d}, \ \forall n \in \mathbb{N}$$

Theorem 14. *(Euler) If n and a are co-prime positive integers, then*

$$a^{\phi(n)} \equiv 1 \mod n$$

Example 55. *(2014 A4) Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying*

$$f(f(m) + n) + f(m) = f(n) + f(3m) + 2014$$

for all integers m and n

Solution. *For convenience, let $f(0) = c$. Then*

$$f(n + c) = f(n) + 2014$$

IDEA: It's natural to proceed induction and get

$$f(kc) = f((k-1)c) + 2014 = \cdots = c + 2014k$$

We want to compare $P(n, m)$ and $P(n, m + kc)$ for some k. Notice that m is "wrapped" in two f, we need that k is a multiple of c.

The simplest choice is $k = c$ and

$$f\left(n + f\left(c^2\right)\right) + f\left(c^2\right) = f(n) + f\left(3c^2\right) + 2014 \longrightarrow c = 1007$$

IDEA: If we hope to use the method mentioned in the beginning of this section, then we should perform taiwanese transformation $f \to g$ so that $g(0) = 0$ and show

$$\begin{cases} g(cn) = cf(n) \\ g(n+d) = g(n) + e \end{cases}$$

Define $g = f - 2014$, then the condition can be rewritten as

$$g(g(m) + n) + g(m) = g(n) + g(3m)$$

By induction, it's easy to show

$$\begin{aligned} g(n) + 2014g(m) &= g(n + 1007g(m)) \\ &= g(n) + 1007g(3m) - 1007g(m) \end{aligned}$$

which implies $g(3m) = 3g(m)$, as desired.

5.2.3 Perhaps Useful

IMO, there are many number theory facts that seems to be helpful for solving FEs. However, I can't find any suitable examples. So I decide to list a part of them in this short section.

Theorem 15. *(Chinese Remainder Theorem) Given $m_1, \ldots, m_r \in \mathbb{Z}$ such that $(m_i, m_j) = 1$ for all $i \neq j$. Then for any $c_1, \ldots, c_r \in \mathbb{Z}$ There exists C satisfying*

$$C \equiv c_i \pmod{m_i}, \ \forall i$$

Theorem 16. *(Wilson) For a integer $p > 1$,*

$$p \mid (p-1)! + 1 \Leftrightarrow p \text{ is a prime}$$

Theorem 17. *(Burton) An integer n has a primitive root if it is of the form*

$$2, 4, p^a \text{ or } 2p^a$$

where p is an odd prime and $a \geq 1$

5.3 Recursive Chain

For problems that concern the relation between $f, f^2, \ldots, f^k$ for some k. It's natural to consider the recursive chains:

$$\{x, f(x), \ldots\}, \quad x = 1, 2, \ldots$$

Definition 29. *We say that s is an **ancestor** of t if there is a positive integer of u such that*

$$\underbrace{f(f(\ldots f}_{u}(s)\ldots)) = t$$

*In this case, we call that t is a **posterity** of s*

Remark 15. *If a is an ancestor of $b \neq a$ and is a posterity of b, then f contains a cycle.*

Example 56. *(2017 Taiwan) Find all injective functions $f : \mathbb{N} \to \mathbb{N}$ such that*

$$f^{f(a)}(b)\, f^{f(b)}(a) = f(a+b)^2$$

holds for all $a, b \in \mathbb{N}$

Solution. *Obviously, the pre-image of 1 is an empty set (why) and one shall notice that*

$$f^{f(n)-1}(n) = 2n$$

for all $n \in \mathbb{N}$. Indeed, just set $a = b = n$ into the condition and use the injectivity of f. In particular, there is an integer c such that $f(c) = 2$. Then

$$2c = f^{f(c)-1}(c) = f(c) = 2$$

So $c = 1$. From here, by induction, it's easy to see that 1 is an ancestor of any power of 2. Now, for any positive integer n, take a natural integer m so that $f(n+m)$ is a power of 2 (why we can make such a choice?) Then, by our condition,

$$f^{f(m)}(n) \mid f(n+m)^2$$

Thus, $f^{f(m)}(n)$ is a power of 2. It follows that 1 is an ancestor of n

Exercise. *Finish the solution of the above example*

Example 57. *(2017 IMOC) Given a positive integer k. Find all functions $f : \mathbb{N} \to \mathbb{N}$ s.t*

$$f^n(n) = n + k, \ \forall n \in \mathbb{N}$$

IDEA: As we solve combinatorics, we shall work for small cases: For $k = 1$, It's quite clear that m is an ancestor of n whenever $m < n$. So if $f(n) \neq n$ for some $n \in \mathbb{N}$, then we'll have that the recursive chain of n is a non-trivial cycle i.e

$$|\{n, f(n), \dots\}| < \infty$$

However, this is impossible because every recursive chain is unbounded. Now, for $k = 2$: If $n > m$ such that $n \equiv m \pmod 2$, then n is a posterity of m. Similar to the previous case, we should have

$$f(n) \not\equiv n \pmod 2, \ \forall n \in \mathbb{N}$$

Consider a partition of $\mathbb{N}$:

$$A = \{1, 3, 5, \dots\} = \{2m - 1 \mid m \in \mathbb{N}\}$$
$$B = \{2, 4, 6, \dots\} = \{2m \mid m \in \mathbb{N}\}$$

Then f maps A to B and maps B to A. It seems fine at the first sight. But it implies

$$n \in A \longrightarrow f^2(n) \in A$$

Combine with the fact that $f^n(n) = n + 2$, we conclude that f contains a cycle (why), which is again absurd.

Solution. *Consider a partition of* $\mathbb{N}$*:*

$$A_1 = \{km + 1 \mid m \in \mathbb{N}_0\}$$
$$A_2 = \{km + 2 \mid m \in \mathbb{N}_0\}$$
$$\vdots$$
$$A_{k-1} = \{km + (k - 1) \mid m \in \mathbb{N}_0\}$$

We hope to generalize our idea, that is, find a positive integer $n \in A_1$ and

$$m < n \text{ satisfying } f^m(n) \in A_1$$

Indeed, just choose a large n such that

$$\min\{n, f(n), \dots\} > k$$

then n has the desired property (why)

Example 58. *(2017 Korea) Find all functions $f:\mathbb{N}\to\mathbb{N}$ satisfying the following conditions:*

- *For every $n\in\mathbb{N}$, $f^n(n)=n$*
- *For every $m,n\in\mathbb{N}$, $\mid f(mn)-f(m)f(n)\mid< 2017$*

Exercise. *Prove that if unbounded function $f:\mathbb{N}\to\mathbb{N}$ satisfies*

$$\mid f(mn)-f(m)f(n)\mid< M,\ \forall m,n\in\mathbb{N}$$

for some M, then f is multiplicative.

Solution. *Since f is multiplicative, it suffices to determine $f(p)$ for all primes p. As usual, consider the recursive chain of p,*

$$\{p,f(p),\ldots,f^{p-1}(p)\}$$

If $f(p)=p$, then we have nothing to do. If $f(p)\neq p$, then all the elements in the chain must be distinct (why). Moreover, $f^{p-1}(p)$ should be a multiple of p. That's because the recursive chains of p and $f^{p-1}(p)$ are the same i.e.

$$\{f^{p-1}(p),f^{p}(p),\ldots,f^{f^{p-1}(p)+p-2}(p)\}=\{p,f(p),\ldots,f^{p-1}(p)\}$$

Assume that $f^{p-1}(p)=cp$ for some $c>1$,

$$p=f^{p}(p)=f(cp)=f(c)f(p)>f(p)$$

However, we have $f(p)$ is a multiple of p, again. Thus, $f(p)=p$ holds for all p.

Remark 16. *For any finite recursive chain,*

$$\{x,f(x),\ldots\}$$

One could consider the minimal $s\in\mathbb{N}$ s.t $f^{s}(x)=x$ to get some useful results.

Exercise. *(2018 Japan) Let $S=\{1,2,\ldots,999\}$. Consider a function $f:S\to S$ such that for any $n\in S$,*

$$f^{n+f(n)+1}(n)=f^{nf(n)}(n)=n$$

Prove that there exists $a\in S$, such that $f(a)=a$

5.4 v2 method

The main idea is

1. Establish a v_2 relation
2. Use the inequality to determine some special function values

Example 59. *(2017 Taiwan) Find all surjective functions $f:\mathbb{Z}\to\mathbb{Z}$ satisfying*

$$f\left(y+xf\left(y\right)\right)=f\left(x\right)f\left(y\right)$$

holds for all $x,y,z\in\mathbb{Z}$

Solution. *First of all, it's easy to see that $f(0)=1$. Suppose that $f(a)=2$, then*

$$f\left(2x+a\right)=2f\left(x\right)$$

We now claim that

$$v_2\left(f\left(x\right)\right)\geq v_2\left(x+a\right),\ \forall x\in\mathbb{Z}$$

Indeed, this inequality holds when $x+a$ is odd. Assume that it holds for all $v_2(x+a)\leq k$. For $v_2(x+a)=k+1$,

$$LHS=v_2\left(f\left(2\cdot\frac{x-a}{2}+a\right)\right)=v_2\left(f\left(\frac{x-a}{2}\right)\right)\geq v_2\left(\frac{x+a}{2}\right)=k+1=RHS$$

as desired. Next, we try to show that $f(2ka)=2k+1$. Suppose not, let l to be an integer such that $f(l)=2l+1$, then take $(x,y)=((2^n-1)a,l)$ in the condition:

$$f\left(2^n\left(2k+1\right)a-l-2ka-a\right)=\left(2k+1\right)2^n$$

By using the v_2 inequality,

$$v_2\left(LHS\right)\geq v_2\left(2^n\left(2k+1\right)a-l-2ka\right)$$

We hope that $v_2\left(2^n\left(2k+1\right)a-l-2ka\right)>n$ so we choose $n=v_2(l-2ka)$. So far, we get

$$f\left(2ka\right)=2k+1,\ \forall k\in\mathbb{Z}$$

Hence, $f(ka)=k+1,\ \forall k\in\mathbb{Z}$ and $a=\pm1$. In conclusion,

$$f\left(x\right)=\begin{cases}x+1,\ \forall x\in\mathbb{Z}\\ -x+1,\ \forall x\in\mathbb{Z}\end{cases}$$

Example 60. *(2018 Taiwan) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that*

$$f(x + f(y)) f(y + f(x)) = (2x + f(y - x))(2y + f(x - y))$$

Exercise. *Suppose that F is a solution of this functional equation.*

- *Figure out $f(0)$*
- *Show that $f(x + f(x)) = 2x$ for all $x \in \mathbb{Z}$*
- *Prove that f is odd and injective.*

Solution. *Assume the results. We have*

$$2f(x) = f(x) + f(f(x)) = f(2x)$$

We have two cases:

If $f(\mathbb{Z})$ contains an odd integer, then $f(x) = x,\ \forall x \in \mathbb{Z}$:

Suppose d is an odd (why) integer so that $f(d)$ is odd. Substitute $x + d$ for y in the original FE, since

$$RHS = (2x + f(d))(2x + 2d - f(d)) \equiv 1 \pmod 2$$

by our assumption. So $f(x + d + f(x))$ must also be odd and it follows (why)

$$v_2(f(x)) = v_2(x)$$

Now, substitute $\frac{x-f(x)}{2}$ for y. After simplification, one shall have

$$f\left(x + f\left(\frac{x - f(x)}{2}\right)\right) = 2x - f(x)$$

Notice that $2x - f(x) = f(f(x))$. By injectivity, $x + f\left(\frac{x-f(x)}{2}\right) = f(x)$. Take v_2 on both side,

$$v_2(f(x) - x) = v_2\left(f\left(\frac{x - f(x)}{2}\right)\right) = v_2\left(\frac{x - f(x)}{2}\right)$$

From here, it's easy to get $f(x) = x$ for all $x \in \mathbb{Z}$

If $f(\mathbb{Z})$ contains only even integers, then $f(x) = -2x,\ \forall x \in \mathbb{Z}$

Apply Taiwanese transformation, define $g : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = -2g(x)$. Then we can rewrite our previous results

1. *$2g(g(x)) = g(x) + x$ holds for all integer x*
2. *$g(2g(x) - x)$ holds for all $x \in \mathbb{Z}$*
3. *g is odd, injective and satisfies $g(2x) = 2g(x),\ \forall x \in \mathbb{Z}$*

Furthermore, the original FE becomes

$$g(2g(x) - y)\, g(2g(y) - x) = (x + g(x - y))\,(y - g(y - x))$$

In this equation, replace y by $g(x)$:

$$LHS = g(g(x))\, g(2g(g(x)) - x) = g(g(x))^2 = \left(\frac{x + g(x)}{2}\right)^2$$

and

$$RHS = (x + g(x - g(x)))\,(g(x) + g(g(x) - x)) \le \left(\frac{x + g(x)}{2}\right)^2$$

So we must have

$$x + g(x - g(x)) = g(x) + g(g(x) - x)$$

which means

$$g(x - g(x)) = \frac{g(x) - x}{2},\ \forall x \in \mathbb{Z}$$

Again, by $g(2x) = 2g(x)$,

$$v_2(g(x - g(x))) \ge v_2(g(x - g(x)))$$

by induction. So $g(x) = x$ is the only possibility, which is indeed a solution.

Remark 17. *This problem can be solved without using v_2, sad. Anyway, if one has*

$$f(px + q) = pf(x)$$

for some prime p and $q \in \mathbb{Z}$, then one could consider to establish a v_p relation.

5.5 Problems

Problem 44. *Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $a, b, c \in \mathbb{Z}$,*

$$f\left(a^3 + b^3 + c^3\right) = f(a)^3 + f(b)^3 + f(c)^3$$

Problem 45. *(2003 USA) Find all functions $f : \mathbb{N} \to \mathbb{N}$ so that*

$$f(m+n) f(m-n) = f\left(m^2\right)$$

holds for all positive integers m, n

Problem 46. *(2012 IMO) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $a, b, c \in \mathbb{Z}$ with $a + b + c = 0$, we have*

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a) f(b) + 2f(b) f(c) + 2f(c) f(a)$$

Problem 47. *(2014 A6) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that or all $n \in \mathbb{Z}$,*

$$n^2 + 4f(n) = f(f(n))^2$$

Problem 48. *(2004 N3) Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying*

$$f(m)^2 + f(n) \mid \left(m^2 + n\right)^2$$

for $m, n \in \mathbb{N}$

Problem 49. *(2011 N3) Let $n \geq 1$ be an odd integer. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ so that*

$$f(x) - f(y) \mid x^n - y^n$$

holds for all $x, y \in \mathbb{Z}$

Problem 50. *(2016 Turkey) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$,*

$$f(mn) = f(m) f(n)$$

and

$$m + n \mid f(m) + f(n)$$

Problem 51. *(2012 A5) Let $f : \mathbb{N} \to \mathbb{N}$ be a function. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.*

Problem 52. *(2007 N5) Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for $m, n \in \mathbb{N}$ and every prime p,*

$$p \mid f(m+n) \Leftrightarrow p \mid f(m) + f(n)$$

Problem 53. *(2015 A5) Find all functions $f : \mathbb{Z} \to 2\mathbb{Z}+1$ satisfying*

$$f(x + f(x) + y) + f(x - f(x) - y) = f(x+y) + f(x-y)$$

holds for every $x, y \in \mathbb{Z}$

Problem 54. *(2018 China) Functions $f, g : \mathbb{Z} \to \mathbb{Z}$ satisfy*

$$f(g(x) + y) = g(f(y) + x)$$

for any integers x, y. Show that if f is bounded, prove that g is periodic.

Problem 55. *(2015 N6) Consider a function $f : \mathbb{N} \to \mathbb{N}$ with the following two properties*

1. *If $m, n \in \mathbb{N}$, then $\frac{f^n(m)-m}{n} \in \mathbb{N}$*
2. *The set $\mathbb{N} \setminus \{f(n) \mid n \in \mathbb{N}\}$ is finite*

Prove that the sequence $f(1)-1, f(2)-2, f(3)-3, \ldots$ is periodic.

Problem 56. *(2018 China) Let M, a, b, r be non-negative integers with $a, r \geq 2$, and suppose there exists a function $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the following conditions:*

- *For all $n \in \mathbb{Z}$, $f^r(n) = an + b$*
- *For all $n \geq M$, $f(n) \geq 0$*
- *For all $n > m > M$, $n - m \mid f(n) - f(m)$*

Show that a is a perfect r-th power

Problem 57. *(2013 A5) Find all the functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfying the relation*

$$f(f(f(n))) = f(n+1) + 1$$

for all $n \in \mathbb{Z}_{\geq 0}$

Problem 58. *(2018 Kazakhstan) Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that*

$$f(mf(n)) = f(m) f(m+n) + n$$

holds for all $m, n \in \mathbb{N}$

Problem 59. *(2018 Taiwan) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for $x, y \in \mathbb{N}$,*

$$f\left(x + yf\left(x\right)\right) = x + f\left(x\right) f\left(y\right)$$

Problem 60. *(2011 Iran) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a function for which the expression*

$$af(a) + bf(b) + 2ab$$

is always a perfect square for all $a, b \in \mathbb{N}$. Prove that $f(a) = a$ for all $a \in \mathbb{N}$

Problem 61. *(2013 N6) Determine all functions $f : \mathbb{Q} \to \mathbb{Z}$ satisfying*

$$f\left(\frac{f(x) + a}{b}\right) = f\left(\frac{x + a}{b}\right)$$

for all $x \in \mathbb{Q}, a \in \mathbb{Z}$ and $b \in \mathbb{N}$

Problem 62. *(2016 Taiwan) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that*

$$f\left(f\left(x\right) + f\left(y\right)\right) + f\left(x\right) f\left(y\right) = f\left(x + y\right) f\left(x - y\right)$$

for all integers $x, y \in \mathbb{Z}$