Extremal Principle Problems

1. **Extremal Principle, Problem 1**

   We are given an $m \times n$ array of real numbers. An operation consists of changing the sign (positive to negative, and vice versa) of all the entries in any row or column. Prove that we can perform a number of such operations such that in the resulting array, the sum of the entries in each row and in each column is nonnegative.

2. **Extremal Principle, Problem 2**

   In the coordinate plane, prove that the vertices of a regular pentagon cannot all be lattice points.

3. **Extremal Principle, Problem 3**

   Suppose there are 997 points given in a plane. If every two points are joined by a line segment with its midpoint colored in red, show that there are at least 1991 red points in the plane. Can you find a special case with exactly 1991 red points?

4. **Extremal Principle, Problem 4**

   A number of coins are placed on a rectangular table, not necessarily the same size. Prove that we can slide one of the coins off the table without touching any of the other coins.

5. **Extremal Principle, Problem 5**

   Two distinct squares of the 8 by 8 chessboard C are said to be adjacent if they have a vertex or side in common. Also, g is called a C-gap if for every numbering of the squares of C with all the integers 1, 2, $\ldots$, 64, there exist two adjacent squares whose numbers differ by at least g. Determine the largest C-gap g.

6. **Extremal Principle, Problem 6**

   Six points are given in the plane, such that no three points are collinear. Prove that three of the points form a triangle such that one of the angles in the triangle is at least $120°$.

7. **Extremal Principle, Problem 7**

   Find all finite sets of positive integers with at least two elements such that for any two numbers a, b $(a > b)$ belonging to the set, the number $\frac{b^2}{a-b}$ belongs to the set, too.

8. **Extremal Principle, Problem 8**

   Given 50 segments on the real line, prove that either (1) there exist 8 segments that have non-empty intersection, or (2) there exist 8 segments, no two of which intersect.

9. **Extremal Principle, Problem 9**

   For each positive integer n, let $a_n = 0$ if the number of 1s in the binary representation of n is even, and $a_n = 1$ if the number of 1s in the binary representation of n is odd. Show that there do not exist positive integers k and m such that

   $$a_{k+j} = a_{k+m+j} = a_{k+2m+j}$$

   for $0 \leq j \leq m - 1$.