

# Goldbach Conjecture and the least prime number in an arithmetic progression

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## Abstract

In this Note, we try to study the relations between the Goldbach Conjecture and the least prime number in an arithmetic progression. We give a new weakened form of the Goldbach Conjecture. We prove that this weakened form and a weakened form of the Chowla Hypothesis imply that every sufficiently large even integer may be written as the sum of two distinct primes.

## Résumé

**La conjecture de Goldbach et le plus petit nombre premier dans une progression arithmétique**

Dans ce document, nous essayons d'étudier les relations entre la conjecture de Goldbach et le plus petit nombre premier dans une progression arithmétique. Nous donnons une nouvelle forme faible de la conjecture de Goldbach. Nous prouvons que cette forme affaiblie et une forme affaiblie de l'hypothèse de Chowla impliquent que tout entier pair suffisamment grand peut être écrit comme une somme de deux nombres premiers distincts.

**Keywords:** Goldbach Conjecture, least prime number, arithmetic progression, Chowla Hypothesis, Generalized Riemann Hypothesis

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# 1 Introduction

Goldbach's famous conjecture states that every even integer  $2n \geq 4$  is the sum of two primes. Since it is trivial that for infinitely many even integers:  $2p = p + p$  (for every prime  $p$ ), we give a slightly different form of this conjecture: every even integer  $2n \geq 8$  is the sum of two distinct primes. Thus, one can state Conjecture 1 below, which is also called a weakened form of Goldbach's Conjecture or the necessary condition of Goldbach's Conjecture.

**Conjecture 1.** For integer  $n > 5$ , there exists a natural number  $r$  such that  $2n - p_r$  is coprime to each of  $2n - p_1, \dots, 2n - p_{r-1}, 2n - p_{r+1}, \dots, 2n - p_k$ , where  $p_1, \dots, p_{r-1}, p_r, p_{r+1}, \dots, p_k$  are all old primes smaller than  $n$ ,  $p_r$  satisfies  $(p_r, n) = 1$  and  $1 \leq r \leq k = \pi(n - 1) - 1$ .

Let  $k, l$  denote positive integers with  $(k, l) = 1$  and  $1 \leq l \leq k - 1$ . Denote by  $p(k, l)$  the least prime  $p \equiv l \pmod{k}$ . Let  $p(k)$  be the maximum value of  $p(k, l)$  for all  $l$  with  $(k, l) = 1$  and  $1 \leq l \leq k - 1$ . In 1992, Heath-Brown [2] proved  $p(k) \ll k^{5.5}$ . This is the best known result on  $p(k)$ . Recently, Heath-Brown told the author that Xylouris (<http://arxiv.org/abs/0906.2749>) has improved his result to  $p(k) \ll k^{5.2}$ . Chowla [1] has observed that  $p(k) \ll k^{2+\epsilon}$  for every  $\epsilon > 0$  assuming the Generalized Riemann Hypothesis. He further conjectured  $p(k) \ll k^{1+\epsilon}$  for every  $\epsilon > 0$ . Based on the conjecture of Chowla, one might state the following, Conjecture 2:

**Conjecture 2.** For every sufficiently large positive integer  $k$ , namely when  $k > c_1$ ,  $p(k) < k^{1.5}$ , where  $c_1$  is a positive constant.

The object of this Note is to study the relations between the Goldbach Conjecture and the least prime number in an arithmetic progression. We obtained the following Theorem 1 which gives a sufficient condition for the Goldbach Conjecture. As we know, even under Riemann hypothesis or if the generalized Riemann hypothesis holds, nobody has proved up until now that the Goldbach Conjecture is true. Therefore, needless-to-say, that refining the results of Heath-Brown and Xylouris, and proving Conjecture 1, should be given much attention.

**Theorem 1.** *If Conjecture 1 and Conjecture 2 hold, then every sufficiently large even integer may be written as the sum of two distinct primes.*

## 2 The proof of Theorem 1

**Proof.** One can prove that for every prime  $p \geq 48673$ , and any integer  $a$  with  $1 \leq a < p^{1.5}$ , there is a prime  $q$  coprime to  $a$  and such that  $4q^3 < p$ .

By the prime number theorem in an arithmetic progression, it is easy to prove that for any prime  $p$  with  $p \leq \max\{c_1, 48673\}$ , ( $c_1$  is the positive constant in Conjecture 2), there exists a positive constant  $c_2 > 6$  such that for every positive integer  $n > c_2$ , when  $(p, n) = 1$ , there exist two distinct odd primes  $p_1$  and  $p_2$  satisfying  $2n \equiv p_1 \equiv p_2 \pmod{p}$  and  $p_1, p_2 \in Z_n^* = \{x | 1 \leq x \leq n, (x, n) = 1\}$ .

Let  $n$  be an integer  $> c_2$ . Since we assume Conjecture 1, there exists  $r > 1$  such that  $(p_r, n) = 1$  and  $2n - p_r$  is coprime to every  $2n - p$  when  $p$  ranges through the odd primes  $\leq n$  and different from  $p_r$ . We will show that  $2n - p_r$  is prime. If this is the case, then Theorem 1 is proved, so let us suppose we can write  $2n - p_r = pm$ , where  $p$  is the least prime factor of  $2n - p_r$ . Thus,  $2n > p^2$ .

We have  $p > \max\{c_1, 48673\}$ . Indeed, if  $p$  is smaller, we can find two odd primes say  $q_1$  and  $q_2$ , not more than  $n$  and prime to  $2n$ , such that  $2n \equiv q_1 \equiv q_2 \pmod{p}$ . At most one of them, say  $q_1$ , can be equal to  $p_r$ . This means that  $2n - p_r$  is not coprime to  $2n - q_2$ , contrarily to our hypothesis on  $p_r$ .

Note that  $p_r \neq p$  since  $(p_r, n) = 1$ . If  $p_r < p$ , then  $p + p_r < p^{1.5}$  and there is a prime  $q$  coprime to  $p + p_r$  and such that  $4q^3 < p$ . Since we suppose that Conjecture 2 holds, hence there is a prime  $x$  such that  $x \equiv p + p_r \pmod{pq}$  and  $x < (pq)^{1.5} < \frac{p^2}{2} < n$ . Clearly,  $p_r \neq x$ . But  $p | (2n - p_r, 2n - x)$ . It is a contradiction by our assumption on  $p_r$ .

Hence  $p_r > p$ . We write  $p_r = pl + v$  with  $1 \leq v < p$ . If  $l \geq \sqrt{p}$ , there is a prime  $y$  such that  $y \equiv v \pmod{p}$  and  $y < p^{1.5} < p_r$  (since we suppose Conjecture 2). But we have also  $p | (2n - p_r, 2n - y)$ , it is contrary to our assumption on  $p_r$  again. So we have  $l < \sqrt{p}$ ,  $lv < p^{1.5}$  and there is a prime  $q$  coprime to  $lv$  and such that  $4q^3 < p$ . Note that there is a prime  $z$  such that  $z \equiv v \pmod{pq}$  and  $z < (pq)^{1.5} < \frac{p^2}{2} < n$  (since we suppose that Conjecture 2 holds). Obviously, we have  $z \neq p_r$  since  $(q, l) = 1$ . But  $p | (2n - p_r, 2n - z)$ . The contradiction implies that  $2n - p_r$  is a prime number. This completes the proof of Theorem 1.

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### References

- [1] S. Chowla, On the least prime in an arithmetic progression, J. Indian Math. Soc. 1 (2) (1934) 1-3.
- [2] D.R. Heath-Brown, Zero-free regions for Dirichlet L-functions, and the least prime in an arithmetic progression, Proc. London Math. Soc. (3) 64 (2) (1992) 265-338.