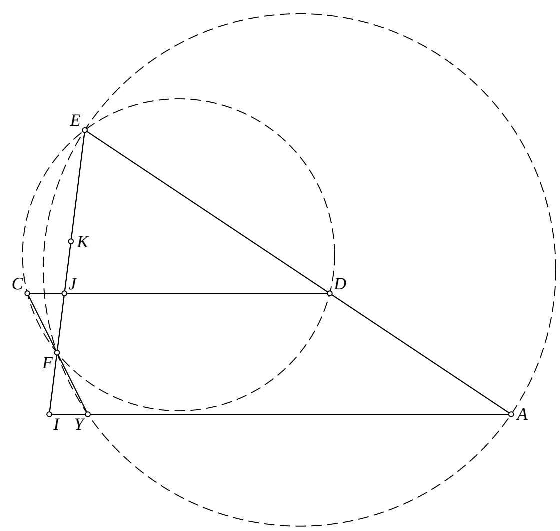
Canada Day Mock Olympiad 2009
Solutions

1. *(IMO Short lost 2008, G2)*
We will assume D is between A and E, as shown in the diagram below. It could also be that A is between D and E, which leads to a different configuration, but the same argument applies in that configuration with very little change.[1]



Since $\angle YCE = \angle ADF$, we have $\angle FCE = \angle YCE = \angle ADF$, and hence $FCED$ is a cyclic quadrilateral. Therefore, $\angle FEA = \angle FED = \angle FCD$. Since CD and YA are parallel, this implies in turn that $\angle FEA = 180° - \angle FYA$. Hence, $FEAY$ is also a cyclic quadrilateral. In particular, power of a point now implies the following two identities: $CJ \cdot DJ = FJ \cdot EJ(1)$ and $FI \cdot EI = YI \cdot AI(2)$.

Furthermore, I belongs to the circumcircle of CDK if and only if $IJ \cdot KJ = CJ \cdot DJ$, which is equivalent to $IJ \cdot KJ = FJ \cdot EJ$ by (1). Writing $IJ = IF + FJ$, $EJ = FE - FJ$, and $KJ = \frac{1}{2}FE - FJ$, this occurs if and only if $FJ = \frac{FI \cdot FE}{2FI + FE}$ (3).

Similarly, K belongs to the circumcircle of triangle AYJ if and only if $JI \cdot KI = YI \cdot AI$,

[1] You can usually avoid the issue of different configurations by using *signed* angles and distances, but in this case, you have to make sure you get the sign right when converting the given condition $\angle DAE = \angle CBF$ into a condition on signed angles.

which is equivalent to $JI \cdot KI = FI \cdot EI$. Writing $JI = FI + FJ$, $KI = FI + \frac{1}{2}FE$, and $EI = FI + FE$, this occurs if and only if $FJ = \frac{FI \cdot FE}{2FI+FE}$ (4).

The result follows from the fact that conditions (3) and (4) are identical.

2. *(IMO Short list 2008, N3)*

We first prove the result for small values of $n = 0, 1, 2, 3$. Since $a_i \geq \gcd(a_i, a_{i+1}) > a_{i-1}$, the sequence is increasing, so $a_1 \geq 2$. For each $i \geq 1$, we also have $a_{i+1} \geq \gcd(a_i, a_{i+1}) + a_i \geq a_i + a_{i-1} + 1$, hence $a_2 \geq 4, a_3 \geq 7$. $a_3 = 7$ is impossible, because then $2 \leq a_1 < \gcd(a_2, 7)$, which contradicts the fact that $a_2 < 7$. So $a_3 \geq 8$ and the claim is proven for $n = 0, 1, 2, 3$. We now proceed be induction.

Let $n \geq 3$ and assume that $a_i \geq 2^i$ for $i = 0, 1, \ldots, n$. We must show that $a_{n+1} \geq 2^{n+1}$. We will in fact show that $a_{n+1} \geq \min(2a_n, 4a_{n-1}, 8a_{n-2}, 16a_{n-3})$, which is clearly sufficient. Suppose this is not the case. Then we must have $a_{n+1} < 2a_n$, so $\gcd(a_{n+1}, a_n) < \frac{a_{n+1}}{2}$, and

$$\frac{a_{n+1}}{4} < a_{n-1} < \gcd(a_n, a_{n+1}) < \frac{a_{n+1}}{2}.$$

So we must have that $\gcd(a_{n+1}, a_n) = \frac{a_{n+1}}{3}$ and $a_n = \frac{2a_{n+1}}{3}$.

Now look at a_{n-2}. By assumption,

$$a_{n-2} > \frac{a_{n+1}}{8} > \frac{3a_{n-1}}{8} > \frac{a_{n-1}}{3}.$$

But we also know that $a_{n-2} < \gcd(a_{n-1}, a_n)$, and so $\gcd(a_{n-1}, a_n) = \frac{a_{n-1}}{2}$. Hence

$$\frac{a_n}{6} < \frac{a_{n+1}}{8} < a_{n-2} < \gcd(a_{n-1}, a_n) = \frac{a_{n-1}}{2} < \frac{a_n}{4},$$

so we conclude that $\gcd(a_{n-1}, a_n) = \frac{a_n}{5}$, and $a_{n-1} = \frac{2a_n}{5} = \frac{4a_{n+1}}{15}$.

Finally, we see that

$$\frac{a_{n+1}}{16} < a_{n-3} < \gcd(a_{n-2}, a_{n-1}) = \gcd\left(a_{n-2}, \frac{4a_{n+1}}{15}\right).$$

From before, we know that $\frac{4a_{n+1}}{45} < \frac{a_{n+1}}{8} < a_{n-2} < \gcd(a_{n-1}, a_n) = \frac{2a_{n+1}}{15}$, hence a_{n-2} doesn't divide $\frac{4a_{n+1}}{15}$, and so

$$\gcd\left(a_{n-2}, \frac{4a_{n+1}}{15}\right) \leq \frac{a_{n-2}}{2} < \frac{a_{n+1}}{15},$$

but then

$$a_{n-3} < \gcd\left(a_{n-2}, \frac{4a_{n+1}}{15}\right) \leq \frac{4a_{n+1}}{75} < \frac{a_{n+1}}{16}$$

which is a contradiction. This completes the induction, so the claim is proven.

3. *(IMO Short list 2008, A7)*

Note that $\frac{2(a-b)(a-c)}{a+b+c} = \frac{(a-c)^2}{a+b+c} + \frac{(a-c)(a-2b+c)}{a+b+c}$ and $\frac{2(c-d)(c-a)}{c+d+a} = \frac{(c-a)^2}{c+d+a} + \frac{(c-a)(c-2d+a)}{c+d+a}$. Therefore, $\frac{2(a-b)(a-c)}{a+b+c} + \frac{2(c-d)(c-a)}{c+d+a}$ can be written as

$$\begin{aligned} & (a-c)^2 \cdot \left(\frac{1}{a+b+c} + \frac{1}{c+d+a}\right) + (a-c) \cdot \left(\frac{a-2b+c}{a+b+c} - \frac{a-2d+c}{a+d+c}\right) \\ = \; & (a-c)^2 \cdot \left(\frac{2a+b+2c+d}{(a+b+c)(c+d+a)}\right) + (a-c) \cdot (d-b) \cdot \left(\frac{3a+3c}{(a+b+c)(c+d+a)}\right). \end{aligned}$$

Similarly, $\frac{2(b-c)(b-d)}{b+c+d} + \frac{2(d-a)(d-b)}{d+a+b}$ can be written as

$$(b-d)^2 \cdot \left(\frac{a+2b+c+2d}{(b+c+d)(d+a+b)}\right) - (a-c)\cdot(d-b)\cdot\left(\frac{3b+3d}{(b+c+d)(d+a+b)}\right).$$

Letting $S = a+b+c+d$, it therefore suffices to prove the following:

$$\begin{aligned} &(a-c)^2 \cdot \left(\frac{S+a+c}{(S-b)(S-d)}\right) + (b-d)^2 \cdot \left(\frac{S+b+d}{(S-a)(S-c)}\right) \\ &\quad \geq 3(a-c)(b-d)\cdot\left(\frac{(a+c)(S-a)(S-c)-(b+d)(S-b)(S-d)}{(S-a)(S-b)(S-c)(S-d)}\right). \end{aligned}$$

Assume without loss of generality that $ac(a+c) \geq bd(b+d)$. The right-hand side simplifies to $3(a-c)(b-d)\cdot\left(\frac{ac(a+c)-bd(b+d)}{(S-a)(S-b)(S-c)(S-d)}\right) \leq \frac{3\cdot|(a-c)(b-d)\cdot ac(a+c)|}{(S-a)(S-b)(S-c)(S-d)}$. By the AM-GM inequality, the left-hand side is at least

$$\begin{aligned} & 2\cdot|(a-c)(b-d)|\cdot\frac{\sqrt{(S+a+c)(S+b+d)(S-a)(S-b)(S-c)(S-d)}}{(S-a)(S-b)(S-c)(S-d)} \\ \geq\quad & 2\cdot|(a-c)(b-d)|\cdot\frac{\sqrt{2(a+c)\cdot(a+c)\cdot c\cdot(a+c)\cdot a\cdot(a+c)}}{(S-a)(S-b)(S-c)(S-d)}. \end{aligned}$$

where we used the fact here that $b, d \geq 0$ in each factor. Furthermore $(a+c)^2 \geq 4ac$, so this is at least

$$2\cdot|(a-c)\cdot(b-d)|\cdot\frac{\sqrt{8(a+c)^2\cdot a^2c^2}}{(S-a)(S-b)(S-c)(S-d)} \quad = \quad \frac{4\sqrt{2}\cdot|(a-c)(b-d)\cdot ac(a+c)|}{(S-a)(S-b)(S-c)(S-d)}.$$

This is clearly at least $\frac{3\cdot|(a-c)(b-d)\cdot ac(a+c)|}{(S-a)(S-b)(S-c)(S-d)}$, which completes the proof. For equality to hold at the end, we must have $a=c$ or $b=d$. For equality to hold during our initial application of AM-GM, we must then have both $a=c$ and $b=d$. Conversely, it is clear that if $a=c$ and $b=d$, then equality does indeed hold.