New Zealand Mathematical Olympiad Committee

Recurrence relations

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1 Introduction

Have you met the *Fibonacci sequence* before? It starts out 1, 1, 2, 3, 5, 8, 13, ..., and to find the next number or *term* you just add the last two: so the next term here is $8 + 13 = 21$.

We can write this mathematically as

$$f_{n+1} = f_n + f_{n-1}, \tag{1}$$

which says that the $(n + 1)$th term is the sum of the nth and $(n - 1)$th. An equation such as this that tells us how to find the next term of a sequence in terms of one or more of the preceding terms is known as a *recurrence relation.* Recurrence relations can come in various flavours that are more or less difficult to solve, and the one we have here is a *second order linear homogeneous recurrence relation with constant co-efficients.* That's a bit of a mouthful, but sometimes, the more words you use the easier something gets, and we'll see later that such recurrences can be solved quite easily. First, though, I'll explain what all those terms mean.

The *order* of a recurrence relation tells us how far back in the sequence we have to go in order to calculate the next term. The Fibonacci sequence depends only on the previous two terms, so it's second order; the recurrence

$$a_n = a_{n-2} + a_{n-3}$$

is third order, because I need to go back three terms in order to find a_n, even though I only use two out of three of those terms. Note that the order of a recurrence relation need not be defined: for example, the Catalan numbers (a famous sequence that has a habit of cropping up all over the place) satisfy the recurrence

$$C_n = C_1C_{n-1} + C_2C_{n-2} + \cdots + C_{n-1}C_1, \tag{2}$$

in which each term depends on *all* of the earlier terms.

A *linear* recurrence relation is one which may be written in the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + b_n,$$

where $c_1, c_2, \ldots, c_k, b_n$ may be functions of n but *not* of the a_i. Equation (2) does not have this form, because it has terms of the form C_iC_j, so it is an example of a *nonlinear* recurrence. As a general rule in mathematics, nonlinearity almost always makes things harder!

A linear recurrence is *homogeneous* if b_n is zero for all n, and *non-homogeneous* otherwise. For example, the recurrence $q_n = 2q_{n-1} + 4^{n-1}$ is non-homogeneous, with non-homogeneous term

4^{n-1}. Finally, a linear recurrence relation has *constant co-efficients* if the c_i are independent of n. The linear recurrence relations given so far all satisfy this condition (note that for this purpose we don't care about the non-homogeneous term); an example of one that has *non-constant co-efficients* is the recurrence $D_n = (n-1)(D_{n-1} + D_{n-2})$, which is satisfied by the derangement numbers. Unfortunately, the method for solving linear recurrence relations that we'll give below only applies to the constant co-efficient case.

Before we move on to finding and solving recurrence relations, a few more words are needed. A recurrence relation only tells us how to find the *next* term of the sequence; it doesn't tell us how to start things off. To pin the sequence down completely we must also specify the *initial conditions*. For example, the Fibonacci sequence is defined by the recurrence (1), *together* with the initial conditions $f_0 = f_1 = 1$. If we use the same recurrence $l_n = l_{n-1} + l_{n-2}$ with the initial conditions $l_0 = 2$, $l_1 = 1$ we get the *Lucas numbers* instead.

In general, for a kth order recurrence the initial conditions must specify the first k terms.

2 Finding a recurrence relation

We'll illustrate the kinds of ideas used in finding a recurrence relation by looking at several examples. But first, a little language. You can probably guess what a *binary sequence* of length n is: it's a sequence of n 0s and 1s. Similarly, a *ternary sequence* is a sequence of 0s, 1s and 2s, and a *quaternary sequence* is a sequence of 0s, 1s, 2s and 3s.

Example 1. *Let t_n be the number of ternary sequences of length n that do not have any non-zero terms adjacent. Find a recurrence relation and initial conditions for t_n.*

Solution: Clearly $t_1 = 3$, and $t_2 = 5$. To find t_3, we might start listing the sequences by their first digit. If this is 0, it can be followed by any of the five sequences of length two; but if it is 1 or 2, it must be followed by a 0, and then by any of the three sequences of length 1. So $t_3 = 5 + 2 \times 3 = 11$. Hey! This idea works just as well for n larger than three: if the first digit is 0, the rest can be any of the t_{n-1} allowable sequence of length $n-1$; and if it's 1 or 2, it must be followed by 0, and then by any of the t_{n-2} allowable sequences of length $n-2$. So the recurrence we're looking for must be

$$t_n = t_{n-1} + 2t_{n-2}.$$

This is second order, so we need two consecutive values as initial conditions, which we've already found above. □

Before moving on, let's see if we can make sense of t_0, as it's sometimes convenient (but usually not essential) to use this as one of our initial conditions. This should count the number of ternary sequences of length zero with no nonzero entries adjacent. Well, there's only one ternary sequence of length zero—the empty sequence, " "—and it doesn't have any nonzero terms adjacent, so t_0 must be 1. This fits with what we found above, because $t_1 + 2t_0 = 3 + 2 = 5 = t_2$, so we could instead take $t_0 = 1$, $t_1 = 3$ as our initial conditions.

Example 2. *Let q_n be the number of quaternary sequences of length n with an odd number of zeros. Find a recurrence relation and initial conditions for q_n.*

Solution: Working out the first few terms we find $q_0 = 0$ (the empty sequence has no zeros, and so an even rather than an odd number of them), $q_1 = 1$, and $q_2 = 6$. To find a recurrence, consider the first term of the sequence. If this is non-zero, the rest of the sequence must be a quaternary sequence of length $n-1$ with an odd number of zeros, of which there are q_{n-1}. Altogther we get $3q_{n-1}$ sequences that start with 1, 2 or 3. On the other hand, if a sequence starts with 0, the remaining $n-1$ terms must include an *even* number of zeros. There are $4^{n-1} - q_{n-1}$ such sequences (why?), so altogether we get

$$q_n = 3q_{n-1} + (4^{n-1} - q_{n-1}) = 2q_{n-1} + 4^{n-1}.$$

This is first order, so the initial condition $q_0 = 0$ is enough. Note also that it is a non-homogeneous recurrence (in fact, it is the example of such given above). □

Example 3. *Find a recurrence relation for D_n, the number of derangements of n objects.*

Solution: Recall that a *derangement* of 1 to n is a permutation of 1 to n in which no number is in its correct position. To find a recurrence for D_n, consider where 1 ends up. It can't end up in the first position, leaving the $n-1$ possibilities second, third, ..., nth place. Suppose for the moment it ended up in the second position, and consider where 2 went to. If it happened to go to the first position, what's left must be a derangement of 3 to n, of which there are D_{n-2}; and if it *didn't*, we can treat the permutation as a derangement of 2 to n (why?). Altogether, this gives $D_{n-1} + D_{n-2}$ derangements in which 1 is in the second position. Applying the same argument to the remaining $n-2$ positions 1 could go to we get

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

□

A word of caution. When finding a recurrence relation, some care is required to make sure that you've counted each thing once and once only.

3 Solving linear recurrence relations with constant coefficients

We'll look first at homogeneous recurrences. Not only are they simpler, but it turns out we'll need to know how to solve them in order to handle the non-homogeneous case too. Note that we cannot reasonably expect to be able to solve *all* non-homogeneous recurrence relations—the non-homogeneous term could be *very* complicated!—but we will be able to give a complete solution in the homogeneous case, and a method that will handle many non-homogeneous recurrences.

3.1 The homogeneous case

If the recurrence is first order, the solution is easily found by inspection: the solution to $a_n = ra_{n-1}$ is clearly $a_n = a_0 r^n$. We'll use this as inspiration in solving higher order recurrences. The second order case contains all the important ideas, so we'll concentrate on that and just say a few words at the end about third and higher order recurrences.

3.1.1 Second order recurrence relations

Consider the recurrence relation

$$a_n = pa_{n-1} + qa_{n-2}, \tag{3}$$

where p and q are constants. Having nothing more to go on than the solution to the first order recurrence found above, let's take a stab at the solution and guess that there's one of the form $a_n = \alpha^n$. There certainly will be if q happens to equal $0\ldots$ so perhaps it's not such a stretch.

To find out what α could be we substitute our guess at the solution into the recurrence, to get

$$\alpha^n = p\alpha^{n-1} + q\alpha^{n-2}.$$

After dividing through by α^{n-2}, this tells us that α^n is a solution to the recurrence if and only if α is a root of the quadratic

$$\alpha^2 - p\alpha - q = 0. \tag{4}$$

This equation is called the *characteristic equation* of the recurrence, and the quadratic appearing on the left-hand side is called the *characteristic polynomial*. Since quadratics typically have two roots we will typically get two solutions of equation (3), α_1^n and α_2^n.

To continue we'll make use of the following lemma, which tells us that as soon as we've found *one* solution to a linear problem, we've found many. This is what makes linear problems so much easier than nonlinear ones. I'll leave you to check the details, by substituting the alleged solution into the recurrence relation.

Lemma 1. *If a'_n and a''_n are two solutions to a homogeneous linear recurrence relation, then so is $a_n = ca'_n + da''_n$ for any constants c, d.*

By the lemma, when the characteristic equation has distinct roots α_1 and α_2, $c_1\alpha_1^n + c_2\alpha_2^n$ is a solution to (3) for any constants c_1, c_2. This is good: the initial conditions have two degrees of freedom (a_0 and a_1 can be chosen arbitrarily), so our solution will have to have two degrees of freedom as well if we're to have any hope of satisfying *any* choice of initial conditions thrown at us. Things don't look so rosy, however, when (4) has only a single root α_0. In this case the lemma only gives us the solution $a_n = c\alpha_0^n$, which isn't going to cut it: we need another solution if we want to be able to meet all possible initial conditions. Fortunately, it turns out that when α_0 is repeated, $n\alpha_0^n$ is the second solution we need. The complete solution is thus given by the following theorem.

Theorem 2. *Suppose that the characteristic equation* (4) *has distinct roots α_1 and α_2. Then the solution to the recurrence relation* (3) *is*

$$a_n = c_1\alpha_1^n + c_2\alpha_2^n,$$

where c_1 and c_2 are constants that may be found using the initial conditions.

If the characteristic equation has a repeated root α_0, then the solution to (3) *is*

$$a_n = (c_1 + c_2 n)\alpha_0^n,$$

where c_1 and c_2 are again constants found using the initial conditions.

Exercise 1. When the characteristic equation has a repeated root $\alpha = \alpha_0$, verify that $n\alpha_0^n$ is a solution to the recurrence.

Hint: You'll need to use the fact that $\alpha^2 - p\alpha - q = (\alpha - \alpha_0)^2 = \alpha^2 - 2\alpha_0 - \alpha_0^2$.

Example 4 (Example 1, continued)**.** Recall that t_n satisfies $t_n = t_{n-1} + 2t_{n-2}$, $t_0 = 1$, $t_1 = 3$. The characteristic equation is $\alpha^2 - \alpha - 2 = (\alpha - 2)(\alpha + 1) = 0$, so the general solution is

$$t_n = c_1 2^n + c_2(-1)^n.$$

To find c_1 and c_2 we substitute $n = 0$ and 1, to get

$$\begin{aligned} c_1 + c_2 &= 1, \\ 2c_2 - c_2 &= 3, \end{aligned}$$

and we solve these equations simultaneously to get $c_1 = 4/3$, $c_2 = -1/3$. The solution is therefore $t_n = (4(2^n) - (-1)^n)/3$, which you can check gives the correct values for the next few terms of the sequence.

3.1.2 Higher order recurrence relations

The method used above may be applied to higher order recurrence relations. If the recurrence relation has order k, then the characteristic polynomial will be of degree k. This will typically have k roots $\alpha_1, \ldots, \alpha_k$, so we will typically get k "independent" solutions to the recurrence relation, and a general solution of the form

$$a_n = c_1\alpha_1^n + \cdots + c_k\alpha_k^n.$$

This expression contains k constants $c_1, \ldots, c_k$, which will allow us to satisfy any possible choice of initial conditions $a_0, \ldots, a_{k-1}$. Of course, finding the roots α_i in the first place will be much harder now!

If one of the roots α_i is repeated, then we will again need to find some additional solutions to the recurrence relation if we want to be able to meet any choice of initial conditions. It turns out that if α_i is a root of multiplicity m (this means that $(\alpha - \alpha_i)^m$ divides the characteristic polynomial), then $\alpha_i^n, n\alpha_i^n, n^2\alpha_i^n, \ldots, n^{m-1}\alpha_i^n$ are the extra solutions we need.

3.1.3 A word on roots of polynomials

You may have been taught at school that not every polynomial of degree k has k roots. If so, you're probably wondering how we would solve a recurrence relation such as $a_n = 2a_{n-1} - 2a_{n-2}$: the characteristic equation can be written $(\alpha - 1)^2 = -1$, which has no real root.

The answer is to allow the roots to be complex numbers, rather than restrict ourselves to using real numbers only. If we use complex numbers, the *Fundamental Theorem of Algebra* guarantees that every polynomial of degree k has exactly k roots, provided you count them "with multiplicity". This means that the method given above works in all possible cases—in the example above where it appears to fail, the roots are $\alpha = 1 \pm i$, and the general solution is $a_n = c_1(1 + i)^n + c_2(1 - i)^n$. The real difficulty lies in finding the roots in the first place, which will generally get much harder as the degree of the polynomial goes up!

3.2 Non-homogeneous linear equations

Non-homogeneous linear equations can sometimes be solved using the "Method of Undetermined Co-efficients". This is really just a fancy way of saying "educated guess and check". To keep things simple, I'll assume we're trying to solve an equation of the form

$$a_n = pa_{n-1} + qa_{n-2} + b_n, \tag{5}$$

where p and q are constants, and b_n is a function of n. The method I'll explain works when b_n is an exponential (e.g. $b_n = 3^n$), a polynomial (e.g. $b_n = n^2 + 1$), or a combination of these, and a few other situations besides.

The key here is that if a'_n and a''_n are two solutions to (5), then their *difference* $a'_n - a''_n$ is a solution to the homogeneous recurrence (3) (check this!). This means that we can try to solve (5) by

1. finding the general solution $c_1\alpha_1^n + c_2\alpha_2^n$ to the "associated homogeneous equation" (3), as above;

2. finding a *single* solution β_n to the non-homogeneous equation (this is usually called a "particular solution");

3. adding the two to get
$$a_n = c_1\alpha_1^n + c_2\alpha_2^n + \beta_n$$
(or $\alpha_n = (c_1 + c_2 n)\alpha_0^n + \beta_n$, in the repeated root case).

The step we don't yet know how to do is obviously the second one. The idea behind the Method of Undetermined Co-efficients is to guess that this will have roughly the same form as the non-homogeneous term b_n. The "roughly" part will be expressed in terms of some un-known co-efficients that we will solve for (or *determine*) in order to arrive at the solution we want. This is best illustrated through examples, so I'll do three and leave it at that.

Example 5. *Find the general solution to the recurrence* $a_n = a_{n-1} + 2a_{n-2} + n$.

Solution: Here the non-homogeneous term is a polynomial of degree 1, so we will guess that there is a particular solution that is also a degree one polynomial, namely $a_n = \beta_n = An + B$. Substituting this into the recurrence $a_n = a_{n-1} + 2a_{n-2} + n$ gives

$$An + B = (A(n-1) + B) + 2(A(n-2) + B) + n,$$

or

$$-2An + (5A - 2B) = n.$$

Matching co-efficients on each side we get $-2A = 1$ and $5A - 2B = 0$, or $A = -1/2$, $B = -5/4$. Using the general solution to the associated homogeneous equation found in Example 4, the general solution is

$$a_n = c_1 2^n + c_2(-1)^n - n/2 - 5/4.$$

□

Note: Had we guessed only $a_n = An$ instead of $An + B$, we'd have wound up with the unsatisfiable equation $-2An+5A = n$. This is often a sign that something is missing from your guess; in this case, the $5A$ suggests we should include a constant term B, as we did above, in order to cancel it out. As a general rule, if the non-homogeneous term is a polynomial of degree n, then your guess for the particular solution should be too.

Example 6 (Example 2, cont'd)**.** *Solve* $q_n = 2q_{n-1} + 4^{n-1}$, $q_0 = 0$.

Solution: The non-homogeneous term is a multiple of 4^n, so we will guess that the particular solution is too, and try $q_n = \beta_n = A4^n$. Substituting, we get

$$A4^n = 2A4^{n-1} + 4^{n-1} = (2A + 1)4^{n-1},$$

or $4A = 2A + 1$. This gives $A = 1/2$. The homogeneous recurrence $q_n = 2q_{n-1}$ has solution $q_n = c2^n$, so the general solution is

$$q_n = c2^n + 4^n/2.$$

Using the initial condition $q_0 = 0$ we finally get $q_n = (4^n - 2^n)/2$, which gives $q_1 = 1$, $q_2 = 6$, as found above.

□

Example 7. *Find a particular solution to the recurrence* $a_n = a_{n-1} + 6a_{n-2} + 3^n$.

Solution: As a first guess, let's try $a_n = \beta_n = A3^n$. Substituting, we get

$$a_n = A3^n = A3^{n-1} + 6A3^{n-2} + 3^n = (A + 2A + 1)3^{n-1}.$$

This leads to $A3^n = (3A+1)3^{n-1}$, or $3^{n-1} = 0$. That's no good! The problem here is that 3^n is in fact a solution to the associated non-homogeneous equation: the characteristic polynomial is $\alpha^2 - \alpha - 6 = (\alpha - 3)(\alpha + 2)$. When this happens, the trick is to try multiplying your guess by n, so let's try $\beta_n = An3^n$.

$$a_n = An3^n = A(n - 1)3^{n-1} + 6A(n - 2)3^{n-2} + 3^n = (3An - 5A + 3)3^{n-1}.$$

Now we have $3^n = 5A3^{n-1}$, which leads to the particular solution $\beta_n = 3^{n+1}n/5$. □

By now you'll have grasped the underlying idea, and also seen some of the dodges used to avoid some of the pitfalls that can crop up. It may all seem a little *ad hoc*, and so perhaps a little unsatisfying, but it often works, so it's worth having in your bag of tricks.

4 Problems

1. Write down the first five terms of each recurrence relation, and then find the solution.

   (a) $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = 1$, $a_1 = 5$.

   (b) $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 2$.

2. Find a closed formula for the nth Fibonacci number. Use it to show that the ratio f_{n+1}/f_n of successive Fibonacci numbers approaches $\frac{1+\sqrt{5}}{2}$ (the golden ratio) as $n \to \infty$.

3. How many ways are there to tile a $2 \times n$ rectangle using 1×2 and 2×2 tiles, if the 1×2 tiles come in three colours but the 2×2 tiles only come in one? The 1×2 tiles may be placed either horizontally or vertically.

4. Let s_n be the number of ternary sequences of length n that do not contain two consecutive 0s or two consecutive 1s. Find a formula for s_n by finding and solving a second order linear recurrence relation.

   Hint: If the recurrence you initially write down isn't second order, you may have to manipulate it until it is. Once you've done this, try re-deriving your second order recurrence directly.

5. Let r_n be the number of ways a $1 \times n$ rectangle may be tiled using 1×1, 1×2, and 1×3 rectangles, if consecutive 1×1 rectangles are not permitted. Find but do not solve a recurrence relation and initial conditions for the sequence $r_0, r_1, r_2, r_3, \ldots$.

6. Solve the given non-homogeneous recurrence relations using the method of undetermined co-efficients.

   (a) $a_n = 4a_{n-1} + 3 \cdot 2^n$, $a_0 = 1$.

   (b) $a_n = 6a_{n-1} - 9a_{n-2} + 2n$, $a_0 = 1$, $a_1 = 0$.

7. Determine the number of ternary sequences of length n that contain the subsequence 00.

8. How many ternary sequences have no 1 *anywhere* to the right of a 0?

9. Find a recurrence relation and initial conditions for the number of ways of dividing $2n$ people into n 2-person committees.

   The number of committees can also be found directly, using standard counting techniques. Try doing it both ways, and check that your answers agree.

10. (a) Use the recurrence relation $D_n = (n-1)(D_{n-1} + D_{n-2})$ to show that the derangement numbers also satisfy the first order non-homogeneous recurrence relation

$$D_n = nD_{n-1} + (-1)^n.$$

   (b) Show by induction that $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

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