The NIMO Compendium

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Contents

Part A.

Problems

I. Monthly Contest

1. September 17, 2012

8:00 PM – 8:30 PM ET

1. (*Eugene Chen*) Dan the dog spots Cate the cat 50m away. At that instant, Cate begins running away from Dan at 6 $\frac{\text{m}}{\text{s}}$, and Dan begins running toward Cate at $8\frac{\text{m}}{\text{s}}$. Both of them accelerate instantaneously and run in straight lines. Compute the number of seconds it takes for Dan to reach Cate.

2. (*Aaron Lin*) A permutation $(a_1, a_2, a_3, \ldots, a_{100})$ of $(1, 2, 3, \ldots, 100)$ is chosen at random. Denote by p the probability that $a_{2i} > a_{2i-1}$ for all $i \in \{1, 2, 3, \ldots, 50\}$. Compute the number of ordered pairs of positive integers (a, b) satisfying $\frac{1}{a^b} = p$.

3. (*Aaron Lin*) For positive integers $1 \le n \le 100$, let
$$f(n) = \sum_{i=1}^{100} i|i-n|.$$
Compute $f(54) - f(55)$.

4. (*Aaron Lin*) In $\triangle ABC$, $AB = AC$. Its circumcircle, Γ, has a radius of 2. Circle Ω has a radius of 1 and is tangent to Γ, $\overline{AB}$, and $\overline{AC}$. The area of $\triangle ABC$ can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c, where b is squarefree and $\gcd(a, c) = 1$. Compute $a + b + c$.

5. (*Lewis Chen*) If $w = a + bi$, where a and b are real numbers, then $\Re(w) = a$ and $\Im(w) = b$. Let $z = c + di$, where $c, d \ge 0$. If
$$\Re(z) + \Im(z) = 7,$$
$$\Re(z^2) + \Im(z^2) = 17,$$
then compute $\left|\Re\left(z^3\right) + \Im\left(z^3\right)\right|$.

6. (*Lewis Chen*) A square is called *proper* if its sides are parallel to the coordinate axes. Point P is randomly selected inside a proper square S with side length 2012. Denote by T the largest proper square that lies within S and has P on its perimeter, and denote by a the expected value of the side length of T. Compute $\lfloor a \rfloor$, the greatest integer less than or equal to a.

7. (*Aaron Lin*) Point P lies in the interior of rectangle $ABCD$ such that $AP + CP = 27$, $BP - DP = 17$, and $\angle DAP \cong \angle DCP$. Compute the area of rectangle $ABCD$.

8. (*Lewis Chen*) The positive integer-valued function $f(n)$ satisfies $f(f(n)) = 4n$ and $f(n+1) > f(n) > 0$ for all positive integers n. Compute the number of possible 16-tuples $(f(1), f(2), f(3), \ldots, f(16))$.

2. October 17, 2012

8:00 PM – 8:30 PM ET

1. (*Evan Chen*) Compute the largest integer $N \leq 2012$ with four distinct digits.

2. (*Unknown*) A *normal magic square* of order n is an arrangement of the integers from 1 to n^2 in a square such that the n numbers in each row, each column, and each of the two diagonals sum to a constant, called the *magic sum* of the magic square. Compute the magic sum of a normal magic square of order 8.

3. (*Aaron Lin*) A polygon $A_1A_2A_3 \ldots A_n$ is called *beautiful* if there exist indices i, j, and k such that $\measuredangle A_iA_jA_k = 144^\circ$. Compute the number of integers $3 \leq n \leq 2012$ for which a regular n-gon is beautiful.

4. (*Eugene Chen*) When flipped, coin A shows heads $\frac{1}{3}$ of the time, coin B shows heads $\frac{1}{2}$ of the time, and coin C shows heads $\frac{2}{3}$ of the time. Anna selects one of the coins at random and flips it four times, yielding three heads and one tail. The probability that Anna flipped coin A can be expressed as $\frac{p}{q}$ for relatively prime positive integers p and q. Compute $p+q$.

5. (*Lewis Chen*) In $\triangle ABC$, $AB = 30$, $BC = 40$, and $CA = 50$. Squares A_1A_2BC, B_1B_2AC, and C_1C_2AB are erected outside $\triangle ABC$, and the pairwise intersections of lines A_1A_2, B_1B_2, and C_1C_2 are P, Q, and R. Compute the length of the shortest altitude of $\triangle PQR$.

6. (*Aaron Lin*) In $\triangle ABC$ with circumcenter O, $\measuredangle A = 45^\circ$. Denote by X the second intersection of $\overrightarrow{AO}$ with the circumcircle of $\triangle BOC$. Compute the area of quadrilateral $ABXC$ if $BX = 8$ and $CX = 15$.

7. (*Lewis Chen*) The sequence $\{a_i\}_{i\geq 1}$ is defined by $a_1 = 1$ and
$$a_n = \lfloor a_{n-1} + \sqrt{a_{n-1}} \rfloor$$
for all $n \geq 2$. Compute the eighth perfect square in the sequence.

8. (*Evan Chen*) Compute the number of sequences of real numbers $a_1, a_2, a_3, \ldots, a_{16}$ satisfying the condition that for every positive integer n,
$$a_1^n + a_2^{2n} + \cdots + a_{16}^{16n} = \begin{cases} 10^{n+1} + 10^n + 1 & \text{for even } n \\ 10^n - 1 & \text{for odd } n \end{cases}.$$

3. November 24, 2012

8:00 PM – 8:45 PM ET

1. (*Isabella Grabski*) Hexagon $ABCDEF$ is inscribed in a circle. If $\measuredangle ACE = 35^\circ$ and $\measuredangle CEA = 55^\circ$, then compute the sum of the degree measures of $\angle ABC$ and $\angle EFA$.

2. (*Aaron Lin*) Compute the number of positive integers $n < 2012$ that share exactly two positive factors with 2012.

3. (*Lewis Chen*) Compute the sum of the distinct prime factors of 10101.

4. (*Lewis Chen*) The *subnumbers* of an integer n are the numbers that can be formed by using a contiguous subsequence of the digits. For example, the subnumbers of 135 are 1, 3, 5, 13, 35, and 135. Compute the number of primes less than 1,000,000,000 that have no non-prime subnumbers. One such number is 37, because 3, 7, and 37 are prime, but 135 is not one, because the subnumbers 1, 35, and 135 are not prime.

5. (*Lewis Chen*) The hour and minute hands on a certain 12-hour analog clock are indistinguishable. If the hands of the clock move continuously, compute the number of times strictly between noon and midnight for which the information on the clock is not sufficient to determine the time.

6. (*Evan Chen*) In rhombus $NIMO$, $MN = 150\sqrt{3}$ and $\measuredangle MON = 60^\circ$. Denote by S the locus of points P in the interior of $NIMO$ such that $\angle MPO \cong \angle NPO$. Find the greatest integer not exceeding the perimeter of S.

7. (*Lewis Chen*) For every pair of reals $0 < a < b < 1$, we define sequences $\{x_n\}_{n\ge 0}$ and $\{y_n\}_{n\ge 0}$ by $x_0 = 0$, $y_0 = 1$, and for each integer $n \ge 1$:

$$\begin{aligned} x_n &= (1-a)x_{n-1} + ay_{n-1}, \\ y_n &= (1-b)x_{n-1} + by_{n-1}. \end{aligned}$$

   The *supermean* of a and b is the limit of $\{x_n\}$ as n approaches infinity. Over all pairs of real numbers (p, q) satisfying $\left(p - \frac{1}{2}\right)^2 + \left(q - \frac{1}{2}\right)^2 \le \left(\frac{1}{10}\right)^2$, the minimum possible value of the supermean of p and q can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

8. (*Lewis Chen*) Concentric circles Ω_1 and Ω_2 with radii 1 and 100, respectively, are drawn with center O. Points A and B are chosen independently at random on the circumferences of Ω_1 and Ω_2, respectively. Denote by ℓ the tangent line to Ω_1 passing through A, and denote by P the reflection of B across ℓ. Compute the expected value of OP^2.

9. (*Lewis Chen*) Let $f(x) = x^2 - 2x$. A set of real numbers S is *valid* if it satisfies the following:
   (a) If $x \in S$, then $f(x) \in S$.
   (b) If $x \in S$ and $\underbrace{f(f(\dots f}_{k\ f\text{'s}}(x)\dots)) = x$ for some integer k, then $f(x) = x$.

   Compute the number of 7-element valid sets.

10. (*Evan Chen*) For reals $x_1, x_2, x_3, \dots, x_{333} \in [-1, \infty)$, let $S_k = x_1^k + x_2^k + \dots + x_{333}^k$ for each k. If $S_2 = 777$, compute the least possible value of S_3.

4. December 17, 2012

8:00 PM – 8:45 PM ET

1. (*Eugene Chen*) Compute the average of the integers $2, 3, 4, \ldots, 2012$.

2. (*Eugene Chen*) For which positive integer n is the quantity $\frac{n}{3} + \frac{40}{n}$ minimized?

3. (*Lewis Chen*) In chess, there are two types of minor pieces, the bishop and the knight. A bishop may move along a diagonal, as long as there are no pieces obstructing its path. A knight may jump to any lattice square $\sqrt{5}$ away as long as it isn't occupied.

   One day, a bishop and a knight were on squares in the same row of an infinite chessboard, when a huge meteor storm occurred, placing a meteor in each square on the chessboard independently and randomly with probability p. Neither the bishop nor the knight were hit, but their movement may have been obstructed by the meteors.

   The value of p that would make the expected number of valid squares that the bishop can move to and the number of squares that the knight can move to equal can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

4. (*Lewis Chen*) Let $S = \{(x, y) : x, y \in \{1, 2, 3, \ldots, 2012\}\}$. For all points (a, b), let $N(a, b) = \{(a-1, b), (a+1, b), (a, b-1), (a, b+1)\}$. Kathy constructs a set T by adding n distinct points from S to T at random. If the expected value of $\sum_{(a,b)\in T} |N(a, b) \cap T|$ is 4, then compute n.

5. (*Eugene Chen*) A number is called *purple* if it can be expressed in the form $\frac{1}{2^a 5^b}$ for positive integers $a > b$. The sum of all purple numbers can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

6. (*Lewis Chen*) The polynomial $P(x) = x^3 + \sqrt{6}x^2 - \sqrt{2}x - \sqrt{3}$ has three distinct real roots. Compute the sum of all $0 \le \theta < 360$ such that $P(\tan \theta^\circ) = 0$.

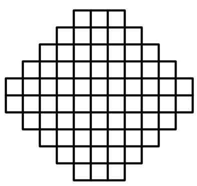
7. (*Aaron Lin*) In quadrilateral $ABCD$, $AC = BD$ and $\measuredangle B = 60^\circ$. Denote by M and N the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. If $MN = 12$ and the area of quadrilateral $ABCD$ is 420, then compute AC.

8. (*Lewis Chen*) Bob has invented the Very Normal Coin (VNC). When the VNC is flipped, it shows heads $\frac{1}{2}$ of the time and tails $\frac{1}{2}$ of the time - unless it has yielded the same result five times in a row, in which case it is guaranteed to yield the opposite result. For example, if Bob flips five heads in a row, then the next flip is guaranteed to be tails.

   Bob flips the VNC an infinite number of times. On the nth flip, Bob bets 2^{-n} dollars that the VNC will show heads (so if the second flip shows heads, Bob wins \$0.25, and if the third flip shows tails, Bob loses \$0.125).

   Assume that dollars are infinitely divisible. Given that the first flip is heads, the expected number of dollars Bob is expected to win can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

9. (*Lewis Chen*) In how many ways can the following figure be tiled with 2×1 dominos?



10. (*Aaron Lin*) In cyclic quadrilateral $ABXC$, $\measuredangle XAB = \measuredangle XAC$. Denote by I the incenter of $\triangle ABC$ and by D the projection of I on $\overline{BC}$. If $AI = 25$, $ID = 7$, and $BC = 14$, then XI can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

5. January 24, 2013

8:00 PM – 8:30 PM ET

1. (*Evan Chen*) Tim is participating in the following three math contests. On each contest his score is the number of correct answers.
   - The Local Area Inspirational Math Exam consists of 15 problems.
   - The Further Away Regional Math League has 10 problems.
   - The Distance-Optimized Math Open has 50 problems.

   For every positive integer n, Tim knows the answer to the nth problems on each contest (which are pairwise distinct), if they exist; however, these answers have been randomly permuted so that he does not know which answer corresponds to which contest. Unaware of the shuffling, he competes with his modified answers. Compute the expected value of the sum of his scores on all three contests.

2. (*Eugene Chen*) The cost of five water bottles is \$13, rounded to the nearest dollar, and the cost of six water bottles is \$16, also rounded to the nearest dollar. If all water bottles cost the same integer number of cents, compute the number of possible values for the cost of a water bottle.

3. (*Evan Chen*) In triangle ABC, $AB = 13$, $BC = 14$ and $CA = 15$. Segment BC is split into $n + 1$ congruent segments by n points. Among these points are the feet of the altitude, median, and angle bisector from A. Find the smallest possible value of n.

4. (*Aaron Lin*) The infinite geometric series of positive reals $a_1, a_2, \ldots$ satisfies

$$1 = \sum_{n=1}^{\infty} a_n = -\frac{1}{2013} + \sum_{n=1}^{\infty} \mathrm{GM}(a_1, a_2, \ldots, a_n) = \frac{1}{N} + a_1$$

   where $\mathrm{GM}(x_1, x_2, \ldots, x_k) = \sqrt[k]{x_1 x_2 \cdots x_k}$ denotes the geometric mean. Compute N.

5. (*Evan Chen*) Compute the number of five-digit positive integers $\overline{vwxyz}$ for which

$$(10v + w) + (10w + x) + (10x + y) + (10y + z) = 100.$$

6. (*Lewis Chen*) Tom has a scientific calculator. Unfortunately, all keys are broken except for one row: `1`, `2`, `3`, `+` and `-`. Tom presses a sequence of 5 random keystrokes; at each stroke, each key is equally likely to be pressed. The calculator then evaluates the entire expression, yielding a result of E. Find the expected value of E. (Note: Negative numbers are permitted, so `13-22` gives $E = -9$. Any excess operators are parsed as signs, so `-2-+3` gives $E = -5$ and `-+-31` gives $E = 31$. Trailing operators are discarded, so `2++-+` gives $E = 2$. A string consisting only of operators, such as `-++-+`, gives $E = 0$.)

7. (*Evan Chen*) For each integer $k \geq 2$, the decimal expansions of the numbers 1024, 1024^2, …, 1024^k are concatenated, in that order, to obtain a number X_k. (For example, $X_2 = 10241048576$.) If

$$\frac{X_n}{1024^n}$$

   is an odd integer, find the smallest possible value of n, where $n \geq 2$ is an integer.

8. (*Evan Chen*) Let $AXYZB$ be a convex pentagon inscribed in a semicircle with diameter AB. Suppose that $AZ - AX = 6$, $BX - BZ = 9$, $AY = 12$, and $BY = 5$. Find the greatest integer not exceeding the perimeter of quadrilateral $OXYZ$, where O is the midpoint of $\overline{AB}$.

6. February 24, 2013

8:00 PM – 8:45 PM ET

1. (*Anonymous*) Find the sum of all primes that can be written both as a sum of two primes and as a difference of two primes.

2. (*Ivan Koswara*) Let f be a function from positive integers to positive integers where $f(n) = \frac{n}{2}$ if n is even and $f(n) = 3n + 1$ if n is odd. If a is the smallest positive integer satisfying
$$\underbrace{f(f(\cdots f}_{2013\ f\text{'s}}(a)\cdots)) = 2013,$$
find the remainder when a is divided by 1000.

3. (*Kevin Sun*) Find the integer $n \geq 48$ for which the number of trailing zeros in the decimal representation of $n!$ is exactly $n - 48$.

4. (*Alexander Dai*) While taking the SAT, you become distracted by your own answer sheet. Because you are not bound to the College Board's limiting rules, you realize that there are actually 32 ways to mark your answer for each question, because you could fight the system and bubble in multiple letters at once: for example, you could mark AB, or AC, or ABD, or even $ABCDE$, or nothing at all!

   You begin to wonder how many ways you could mark off the 10 questions you haven't yet answered. To increase the challenge, you wonder how many ways you could mark off the rest of your answer sheet without ever marking the same letter twice in a row. (For example, if ABD is marked for one question, AC cannot be marked for the next one because A would be marked twice in a row.) If the number of ways to do this can be expressed in the form $2^m p^n$, where $m, n > 1$ are integers and p is a prime, compute $100m + n + p$.

5. (*Ahaan Rungta*) Zang is at the point $(3, 3)$ in the coordinate plane. Every second, he can move one unit up or one unit right, but he may never visit points where the x and y coordinates are both composite. In how many ways can he reach the point $(20, 13)$?

6. (*ssilwa*) For each positive integer n, let $H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$. If
$$\sum_{n=4}^{\infty} \frac{1}{nH_nH_{n-1}} = \frac{M}{N}$$
for relatively prime positive integers M and N, compute $100M + N$.

7. (*Matthew Babbitt*) In $\triangle ABC$ with $AB = 10$, $AC = 13$, and $\measuredangle ABC = 30^\circ$, M is the midpoint of $\overline{BC}$ and the circle with diameter $\overline{AM}$ meets $\overline{CB}$ and $\overline{CA}$ again at D and E, respectively. The area of $\triangle DEM$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $100m + n$.

8. (*Ivan Koswara*) Find the number of positive integers n for which there exists a sequence $x_1, x_2, \cdots, x_n$ of integers with the following property: if indices $1 \leq i \leq j \leq n$ satisfy $i + j \leq n$ and $x_i - x_j$ is divisible by 3, then $x_{i+j} + x_i + x_j + 1$ is divisible by 3.

9. (*Calvin Lee*) Let $ABCD$ be a square of side length 6. Points E and F are selected on rays AB and AD such that segments EF and BC intersect at a point L, D lies between A and F, and the area of $\triangle AEF$ is 36. Clio constructs triangle PQR with $PQ = BL$, $QR = CL$ and $RP = DF$, and notices that the area of $\triangle PQR$ is $\sqrt{6}$. If the sum of all possible values of DF is $\sqrt{m} + \sqrt{n}$ for positive integers $m \geq n$, compute $100m + n$.

10. (*Varun Mohan*) Let $x \neq y$ be positive reals satisfying $x^3 + 2013y = y^3 + 2013x$, and let $M = \left(\sqrt{3} + 1\right)x + 2y$. Determine the maximum possible value of M^2.

7. May 27, 2013

8:00 PM – 8:40 PM ET

1. (*Lewis Chen*) At ARML, Santa is asked to give rubber duckies to 2013 students, one for each student. The students are conveniently numbered $1, 2, \cdots, 2013$, and for any integers $1 \le m < n \le 2013$, students m and n are friends if and only if $0 \le n - 2m \le 1$.

   Santa has only four different colors of duckies, but because he wants each student to feel special, he decides to give duckies of different colors to any two students who are either friends or who share a common friend. Let N denote the number of ways in which he can select a color for each student. Find the remainder when N is divided by 1000.

2. (*Eugene Chen*) In $\triangle ABC$, points E and F lie on $\overline{AC}, \overline{AB}$, respectively. Denote by P the intersection of $\overline{BE}$ and $\overline{CF}$. Compute the maximum possible area of $\triangle ABC$ if $PB = 14$, $PC = 4$, $PE = 7$, $PF = 2$.

3. (*Lewis Chen*) Richard has a four infinitely large piles of coins: a pile of pennies (worth 1 cent each), a pile of nickels (5 cents), a pile of dimes (10 cents), and a pile of quarters (25 cents). He chooses one pile at random and takes one coin from that pile. Richard then repeats this process until the sum of the values of the coins he has taken is an integer number of dollars. (One dollar is 100 cents.) What is the expected value of this final sum of money, in cents?

4. (*Evan Chen*) Find the positive integer N for which there exist reals $\alpha, \beta, \gamma, \theta$ which obey

$$\begin{aligned} 0.1 &= \sin\gamma\cos\theta\sin\alpha, \\ 0.2 &= \sin\gamma\sin\theta\cos\alpha, \\ 0.3 &= \cos\gamma\cos\theta\sin\beta, \\ 0.4 &= \cos\gamma\sin\theta\cos\beta, \\ 0.5 &\ge \left|N - 100\cos 2\theta\right|. \end{aligned}$$

5. (*Lewis Chen*) For every integer $n \ge 1$, the function $f_n : \{0, 1, \cdots, n\} \to \mathbb{R}$ is defined recursively by $f_n(0) = 0$, $f_n(1) = 1$ and

$$(n-k)f_n(k-1) + kf_n(k+1) = nf_n(k)$$

   for each $1 \le k < n$. Let $S_N = f_{N+1}(1) + f_{N+2}(2) + \cdots + f_{2N}(N)$. Find the remainder when $\lfloor S_{2013} \rfloor$ is divided by 2011. (Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

8. September 24, 2013

8:00 PM – 8:40 PM ET

1. (*Evan Chen*) Let a, b, c, d, e be positive reals satisfying

$$\begin{aligned} a+b &= c \\ a+b+c &= d \\ a+b+c+d &= e. \end{aligned}$$

   If $c = 5$, compute $a+b+c+d+e$.

2. (*Aaron Lin*) A positive integer N has 20 digits when written in base 9 and 13 digits when written in base 27. How many digits does N have when written in base 3?

3. (*Evan Chen*) Integers a, b, c are selected independently and at random from the set $\{1, 2, \dots, 10\}$, with replacement. If p is the probability that $a^{b-1}b^{c-1}c^{a-1}$ is a power of two, compute $1000p$.

4. (*Aaron Lin*) On side $\overline{AB}$ of square $ABCD$, point E is selected. Points F and G are located on sides $\overline{AB}$ and $\overline{AD}$, respectively, such that $\overline{FG} \perp \overline{CE}$. Let P be the intersection point of segments $\overline{FG}$ and $\overline{CE}$. Given that $[EPF] = 1$, $[EPGA] = 8$, and $[CPFB] = 15$, compute $[PGDC]$. (Here $[\mathcal{P}]$ denotes the area of the polygon $\mathcal{P}$.)

5. (*Aaron Lin*) Let x, y, z be complex numbers satisfying

$$\begin{aligned} z^2 + 5x &= 10z \\ y^2 + 5z &= 10y \\ x^2 + 5y &= 10x \end{aligned}$$

   Find the sum of all possible values of z.

6. (*Lewis Chen*) Let $f(n) = \varphi(n^3)^{-1}$, where $\varphi(n)$ denotes the number of positive integers not greater than n that are relatively prime to n. Suppose

$$\frac{f(1)+f(3)+f(5)+\dots}{f(2)+f(4)+f(6)+\dots} = \frac{m}{n}$$

   where m and n are relatively prime positive integers. Compute $100m+n$.

7. (*Aaron Lin*) Dragon selects three positive real numbers with sum 100, uniformly at random. He asks Cat to copy them down, but Cat gets lazy and rounds them all to the nearest tenth during transcription. If the probability the three new numbers still sum to 100 is $\frac{m}{n}$, where m and n are relatively prime positive integers, compute $100m+n$.

8. (*Evan Chen*) The diagonals of convex quadrilateral $BSCT$ meet at the midpoint M of $\overline{ST}$. Lines BT and SC meet at A, and $AB = 91$, $BC = 98$, $CA = 105$. Given that $\overline{AM} \perp \overline{BC}$, find the positive difference between the areas of $\triangle SMC$ and $\triangle BMT$.

9. November 13, 2013

8:00 PM – 8:40 PM ET

1. (*Jeremy Lu*) A sequence $a_0, a_1, a_2, \ldots$ of real numbers satisfies $a_0 = 999$, $a_1 = -999$, and $a_n = a_{n-1}a_{n+1}$ for each positive integer n. Compute $|a_1 + a_2 + \cdots + a_{1000}|$.

2. (*Ahaan S. Rungta*) Let f be a non-constant polynomial such that
$$f(x-1) + f(x) + f(x+1) = \frac{f(x)^2}{2013x}$$
for all nonzero real numbers x. Find the sum of all possible values of $f(1)$.

3. (*Michael Ren*) Let $a_1, a_2, \ldots, a_{1000}$ be positive integers whose sum is S. If $a_n!$ divides n for each $n = 1, 2, \ldots, 1000$, compute the maximum possible value of S.

4. (*Ahaan S. Rungta / Amir Hossein*) Consider a set of 1001 points in the plane, no three collinear. Compute the minimum number of segments that must be drawn so that among any four points, we can find a triangle.

5. (*Matthew Lerner-Brecher*) Let d and n be positive integers such that d divides n, $n > 1000$, and n is not a perfect square. The minimum possible value of $|d - \sqrt{n}|$ can be written in the form $a\sqrt{b} + c$, where b is a positive integer not divisible by the square of any prime, and a and c are nonzero integers (not necessarily positive). Compute $a + b + c$.

6. (*Yang Liu*) Let ABC be a triangle with $AB = 42$, $AC = 39$, $BC = 45$. Let E, F be on the sides $\overline{AC}$ and $\overline{AB}$ such that $AF = 21, AE = 13$. Let $\overline{CF}$ and $\overline{BE}$ intersect at P, and let ray AP meet $\overline{BC}$ at D. Let O denote the circumcenter of $\triangle DEF$, and R its circumradius. Compute $CO^2 - R^2$.

7. (*Joshua Xiong*) Tyler has two calculators, both of which initially display zero. The first calculators has only two buttons, [+1] and [×2]. The second has only the buttons [+1] and [×4]. Both calculators update their displays immediately after each keystroke.

   A positive integer n is called *ambivalent* if the minimum number of keystrokes needed to display n on the first calculator equals the minimum number of keystrokes needed to display n on the second calculator. Find the sum of all ambivalent integers between 256 and 1024 inclusive.

8. (*Michael Ren*) Let $ABCD$ be a convex quadrilateral with $\angle ABC = 120^\circ$ and $\angle BCD = 90^\circ$, and let M and N denote the midpoints of $\overline{BC}$ and $\overline{CD}$. Suppose there exists a point P on the circumcircle of $\triangle CMN$ such that ray MP bisects $\overline{AD}$ and ray NP bisects $\overline{AB}$. If $AB + BC = 444$, $CD = 256$ and $BC = \frac{m}{n}$ for some relatively prime positive integers m and n, compute $100m + n$.

10. December 3, 2013

8:00 PM – 8:40 PM ET

1. (*Evan Chen*) Richard likes to solve problems from the IMO Shortlist. In 2013, Richard solves 5 problems each Saturday and 7 problems each Sunday. He has school on weekdays, so he "only" solves 2, 1, 2, 1, 2 problems on each Monday, Tuesday, Wednesday, Thursday, and Friday, respectively – with the exception of December 3, 2013, where he solved 60 problems out of boredom. Altogether, how many problems does Richard solve in 2013?

2. (*Lewis Chen*) How many integers n are there such that $(n+1!)(n+2!)(n+3!)\cdots(n+2013!)$ is divisible by 210 and $1 \le n \le 210$?

3. (*Evan Chen*) At Stanford in 1988, human calculator Shakuntala Devi was asked to compute $m = \sqrt[3]{61{,}629{,}875}$ and $n = \sqrt[7]{170{,}859{,}375}$. Given that m and n are both integers, compute $100m + n$.

4. (*Lewis Chen*) Let $S = \{1, 2, \cdots, 2013\}$. Let N denote the number 9-tuples of sets $(S_1, S_2, \dots, S_9)$ such that $S_{2n-1}, S_{2n+1} \subseteq S_{2n} \subseteq S$ for $n = 1, 2, 3, 4$. Find the remainder when N is divided by 1000.

5. (*Evan Chen*) In a certain game, Auntie Hall has four boxes B_1, B_2, B_3, B_4, exactly one of which contains a valuable gemstone; the other three contain cups of yogurt. You are told the probability the gemstone lies in box B_n is $\frac{n}{10}$ for $n = 1, 2, 3, 4$.

   Initially you may select any of the four boxes; Auntie Hall then opens one of the other three boxes at random (which may contain the gemstone) and reveals its contents. Afterwards, you may change your selection to any of the four boxes, and you win if and only if your final selection contains the gemstone. Let the probability of winning assuming optimal play be $\frac{m}{n}$, where m and n are relatively prime integers. Compute $100m + n$.

6. (*Lewis Chen*) Given a regular dodecagon (a convex polygon with 12 congruent sides and angles) with area 1, there are two possible ways to dissect this polygon into 12 equilateral triangles and 6 squares. Let T_1 denote the union of all triangles in the first dissection, and S_1 the union of all squares. Define T_2 and S_2 similarly for the second dissection. Let S and T denote the areas of $S_1 \cap S_2$ and $T_1 \cap T_2$, respectively. If $\frac{S}{T} = \frac{a+b\sqrt{3}}{c}$ where a and b are integers, c is a positive integer, and $\gcd(a, c) = 1$, compute $10000a + 100b + c$.

7. (*Evan Chen*) Let $ABCD$ be a convex quadrilateral for which $DA = AB$ and $CA = CB$. Set $I_0 = C$ and $J_0 = D$, and for each nonnegative integer n, let I_{n+1} and J_{n+1} denote the incenters of $\triangle I_nAB$ and $\triangle J_nAB$, respectively. Suppose that

$$\angle DAC = 15^\circ, \quad \angle BAC = 65^\circ \quad \text{and} \quad \angle J_{2013}J_{2014}I_{2014} = \left(90 + \frac{2k+1}{2^n}\right)^\circ$$

   for some nonnegative integers n and k. Compute $n + k$.

8. (*Lewis Chen*) The number $\frac{1}{2}$ is written on a blackboard. For a real number c with $0 < c < 1$, a *c-splay* is an operation in which every number x on the board is erased and replaced by the two numbers cx and $1 - c(1 - x)$. A *splay-sequence* $C = (c_1, c_2, c_3, c_4)$ is an application of a c_i-splay for $i = 1, 2, 3, 4$ in that order, and its *power* is defined by $P(C) = c_1c_2c_3c_4$.

   Let S be the set of splay-sequences which yield the numbers $\frac{1}{17}, \frac{2}{17}, \dots, \frac{16}{17}$ on the blackboard in some order. If $\sum_{C \in S} P(C) = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

11. January 31, 2014

8:00 PM – 8:40 PM ET

1. (*Lewis Chen*) Define $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Let the sum of all H_n that are terminating in base 10 be S. If $S = m/n$ where m and n are relatively prime positive integers, find $100m + n$.

2. (*Evan Chen*) In the game of Guess the Card, two players each have a $\frac{1}{2}$ chance of winning and there is exactly one winner. Sixteen competitors stand in a circle, numbered $1, 2, \ldots, 16$ clockwise. They participate in an 4-round single-elimination tournament of Guess the Card. Each round, the referee randomly chooses one of the remaining players, and the players pair off going clockwise, starting from the chosen one; each pair then plays Guess the Card and the losers leave the circle. If the probability that players 1 and 9 face each other in the last round is $\frac{m}{n}$ where m, n are positive integers, find $100m + n$.

3. (*Lewis Chen*) Call an integer k *debatable* if the number of odd factors of k is a power of two. What is the largest positive integer n such that there exists n consecutive debatable numbers? (Here, a power of two is defined to be any number of the form 2^m, where m is a nonnegative integer.)

4. (*Evan Chen*) Let a, b, c be positive reals for which

$$\begin{aligned}(a+b)(a+c) &= bc + 2\\(b+c)(b+a) &= ca + 5\\(c+a)(c+b) &= ab + 9\end{aligned}$$

   If $abc = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

5. (*Evan Chen*) In triangle ABC, $\sin A \sin B \sin C = \frac{1}{1000}$ and $AB \cdot BC \cdot CA = 1000$. What is the area of triangle ABC?

6. (*Lewis Chen*) Suppose we wish to pick a random integer between 1 and N inclusive by flipping a fair coin. One way we can do this is through generating a random binary decimal between 0 and 1, then multiplying the result by N and taking the ceiling. However, this would take an infinite amount of time. We therefore stopping the flipping process after we have enough flips to determine the ceiling of the number. For instance, if $N = 3$, we could conclude that the number is 2 after flipping $.011_2$, but $.010_2$ is inconclusive.

   Suppose $N = 2014$. The expected number of flips for such a process is $\frac{m}{n}$ where m, n are relatively prime positive integers, find $100m + n$.

7. (*Evan Chen*) Let $P(n)$ be a polynomial of degree m with integer coefficients, where $m \le 10$. Suppose that $P(0) = 0$, $P(n)$ has m distinct integer roots, and $P(n) + 1$ can be factored as the product of two nonconstant polynomials with integer coefficients. Find the sum of all possible values of $P(2)$.

8. (*Eugene Chen*) The side lengths of $\triangle ABC$ are integers with no common factor greater than 1. Given that $\angle B = 2\angle C$ and $AB < 600$, compute the sum of all possible values of AB.

12. February 24, 2014

8:00 PM – 8:40 PM ET

1. (*Aaron*) You drop a 7cm long piece of mechanical pencil lead on the floor. A bully takes the lead and breaks it at a random point into two pieces. A piece of lead is unusable if it is 2cm or shorter. If the expected value of the number of usable pieces afterwards is $\frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

2. (*Eugene Chen*) Let ABC be an equilateral triangle. Denote by D the midpoint of $\overline{BC}$, and denote the circle with diameter $\overline{AD}$ by Ω. If the region inside Ω and outside $\triangle ABC$ has area $800\pi - 600\sqrt{3}$, find the length of AB.

3. (*Aaron Lin*) In land of Nyemo, the unit of currency is called a *quack*. The citizens use coins that are worth 1, 5, 25, and 125 quacks. How many ways can someone pay off 125 quacks using these coins?

4. (*Eugene Chen*) Let S be the set of integers which are both a multiple of 70 and a factor of 630,000. A random element c of S is selected. If the probability that there exists an integer d with $\gcd(c, d) = 70$ and $\operatorname{lcm}(c, d) = 630{,}000$ is $\frac{m}{n}$ for some relatively prime integers m and n, compute $100m + n$.

5. (*Lewis Chen*) Triangle ABC has sidelengths $AB = 14$, $BC = 15$, and $CA = 13$. We draw a circle with diameter $\overline{AB}$ such that it passes $\overline{BC}$ again at D and passes $\overline{CA}$ again at E. If the circumradius of $\triangle CDE$ can be expressed as $\frac{m}{n}$ where m, n are coprime positive integers, determine $100m + n$.

6. (*Lewis Chen*) Let $N = 10^6$. For which integer a with $0 \le a \le N - 1$ is the value of
$$\binom{N}{a+1} - \binom{N}{a}$$
maximized?

7. (*Ivan Koswara*) Find the sum of all integers n with $2 \le n \le 999$ and the following property: if x and y are randomly selected without replacement from the set $\{1, 2, \dots, n\}$, then $x + y$ is even with probability p, where p is the square of a rational number.

8. (*Evan Chen*) Let a, b, c, d be complex numbers satisfying
$$\begin{aligned} 5 &= a + b + c + d \\ 125 &= (5-a)^4 + (5-b)^4 + (5-c)^4 + (5-d)^4 \\ 1205 &= (a+b)^4 + (b+c)^4 + (c+d)^4 + (d+a)^4 + (a+c)^4 + (b+d)^4 \\ 25 &= a^4 + b^4 + c^4 + d^4 \end{aligned}$$
Compute $abcd$.

13. March 24, 2014

8:00 PM – 8:40 PM ET

1. (*Kevin Sun*) Let $\eta(m)$ be the product of all positive integers that divide m, including 1 and m. If $\eta(\eta(\eta(10))) = 10^n$, compute n.

2. (*Rajiv Movva*) Two points A and B are selected independently and uniformly at random along the perimeter of a unit square with vertices at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. The probability that the y-coordinate of A is strictly greater than the y-coordinate of B can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $100m+n$.

3. (*Yonah Borns-Weil*) Find the number of positive integers n with exactly 1974 factors such that no prime greater than 40 divides n, and n ends in one of the digits 1, 3, 7, 9. (Note that $1974 = 2 \cdot 3 \cdot 7 \cdot 47$.)

4. (*Ahaan Rungta*) A black bishop and a white king are placed randomly on a 2000×2000 chessboard (in distinct squares). Let p be the probability that the bishop attacks the king (that is, the bishop and king lie on some common diagonal of the board). Then p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute m.

5. (*Akshaj*) Let a positive integer n be *nice* if there exists a positive integer m such that

$$n^3 < 5mn < n^3 + 100.$$

   Find the number of *nice* positive integers.

6. (*Alex Gu*) Let $P(x)$ be a polynomial with real coefficients such that $P(12) = 20$ and

$$(x-1) \cdot P(16x) = (8x-1) \cdot P(8x)$$

   holds for all real numbers x. Compute the remainder when $P(2014)$ is divided by 1000.

7. (*Michael Tang*) Let N denote the number of ordered pairs of sets (A, B) such that $A \cup B$ is a size-999 subset of $\{1, 2, \dots, 1997\}$ and $(A \cap B) \cap \{1, 2\} = \{1\}$. If m and k are integers such that $3^m 5^k$ divides N, compute the the largest possible value of $m+k$.

8. (*Akshaj*) Triangle ABC lies entirely in the first quadrant of the Cartesian plane, and its sides have slopes 63, 73, 97. Suppose the curve $\mathcal{V}$ with equation $y = (x+3)(x^2+3)$ passes through the vertices of ABC. Find the sum of the slopes of the three tangents to $\mathcal{V}$ at each of A, B, C.

14. May 15, 2014

8:00 PM – 8:40 PM ET

1. (*Evan Chen*) Let A, B, C, D be four points on a line in this order. Suppose that $AC = 25$, $BD = 40$, and $AD = 57$. Compute $AB \cdot CD + AD \cdot BC$.

2. (*Lewis Chen*) In the Generic Math Tournament, 99 people participate. One of the participants, Alfred, scores 16th in Algebra, 30th in Combinatorics, and 23rd in Geometry (and does not tie with anyone). The overall ranking is computed by adding the scores from all three tests. Given this information, let B be the best ranking that Alfred could have achieved, and let W be the worst ranking that he could have achieved. Compute $100B + W$.

3. (*Lewis Chen*) In triangle ABC, we have $AB = AC = 20$ and $BC = 14$. Consider points M on $\overline{AB}$ and N on $\overline{AC}$. If the minimum value of the sum $BN + MN + MC$ is x, compute $100x$.

4. (*Lewis Chen*) Define the infinite products

$$A = \prod_{i=2}^{\infty} \left(1 - \frac{1}{n^3}\right) \text{ and } B = \prod_{i=1}^{\infty} \left(1 + \frac{1}{n(n+1)}\right).$$

   If $\frac{A}{B} = \frac{m}{n}$ where m, n are relatively prime positive integers, determine $100m + n$.

5. (*Evan Chen*) Find the largest integer n for which 2^n divides

$$\binom{2}{1}\binom{4}{2}\binom{6}{3} \cdots \binom{128}{64}.$$

6. (*Lewis Chen*) 10 students are arranged in a row. Every minute, a new student is inserted in the row (which can occur in the front and in the back as well, hence 11 possible places) with a uniform $\frac{1}{11}$ probability of each location. Then, either the frontmost or the backmost student is removed from the row (each with a $\frac{1}{2}$ probability).

   Suppose you are the eighth in the line from the front. The probability that you exit the row from the front rather than the back is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

7. (*Lewis Chen*) Ana and Banana play a game. First, Ana picks a real number p with $0 \le p \le 1$. Then, Banana picks an integer h greater than 1 and creates a spaceship with h hit points. Now every minute, Ana decreases the spaceship's hit points by 2 with probability $1 - p$, and by 3 with probability p. Ana wins if and only if the number of hit points is reduced to exactly 0 at some point (in particular, if the spaceship has a negative number of hit points at any time then Ana loses). Given that Ana and Banana select p and h optimally, compute the integer closest to $1000p$.

8. (*Evan Chen*) Let x be a positive real number. Define

$$A = \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!}, \quad B = \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)!}, \quad \text{and} \quad C = \sum_{k=0}^{\infty} \frac{x^{3k+2}}{(3k+2)!}.$$

   Given that $A^3 + B^3 + C^3 + 8ABC = 2014$, compute ABC.

II. Summer Contest

1. Summer 2011

6:00 PM – 6:15 PM ET

1. (*Unknown*) A jar contains 4 blue marbles, 3 green marbles, and 5 red marbles. If Helen reaches in the jar and selects a marble at random, then the probability that she selects a red marble can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2. (*Unknown*) The sum of three consecutive integers is 15. Determine their product.

3. (*Isabella Grabski*) Define $\lfloor x \rfloor$ as the largest integer less than or equal to x. Define $\{x\} = x - \lfloor x \rfloor$. For example, $\{3\} = 3 - 3 = 0$, $\{\pi\} = \pi - 3$, and $\{-\pi\} = 4 - \pi$. If $\{n\} + \{3n\} = 1.4$, then find the sum of all possible values of $100\{n\}$.

4. (*Lewis Chen*) Find the number of ordered pairs of integers (a, b) that satisfy the inequality

$$1 < a < b + 2 < 10.$$

5. (*Isabella Grabski*) In equilateral triangle ABC, the midpoint of $\overline{BC}$ is M. If the circumcircle of triangle MAB has area 36π, then find the perimeter of the triangle.

6. (*Lewis Chen*) If the answer to this problem is x, then compute the value of $\frac{x^2}{8} + 2$.

7. (*Aaron Lin*) Let $P(x) = x^2 - 20x - 11$. If a and b are natural numbers such that a is composite, $\gcd(a, b) = 1$, and $P(a) = P(b)$, compute ab.

   Note: $\gcd(m, n)$ denotes the greatest common divisor of m and n.

8. (*Lewis Chen*) Triangle ABC with $\measuredangle A = 90^\circ$ has incenter I. A circle passing through A with center I is drawn, intersecting $\overline{BC}$ at E and F such that $BE < BF$. If $\frac{BE}{EF} = \frac{2}{3}$, then $\frac{CF}{FE} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

9. (*Eugene Chen*) The roots of the polynomial $P(x) = x^3 + 5x + 4$ are r, s, and t. Evaluate $(r+s)^4(s+t)^4(t+r)^4$.

10. (*Eugene Chen*) The edges and diagonals of convex pentagon $ABCDE$ are all colored either red or blue. How many ways are there to color the segments such that there is exactly one monochromatic triangle with vertices among A, B, C, D, E; that is, triangles whose edges are all the same color?

11. (*Unknown*) How many ordered pairs of positive integers (m, n) satisfy the system

$$\gcd(m^3, n^2) = 2^2 \cdot 3^2,$$
$$\text{LCM}[m^2, n^3] = 2^4 \cdot 3^4 \cdot 5^6,$$

   where $\gcd(a, b)$ and $\text{LCM}[a, b]$ denote the greatest common divisor and least common multiple of a and b, respectively?

12. (*Lewis Chen*) In triangle ABC, $AB = 100$, $BC = 120$, and $CA = 140$. Points D and F lie on $\overline{BC}$ and $\overline{AB}$, respectively, such that $BD = 90$ and $AF = 60$. Point E is an arbitrary point on $\overline{AC}$. Denote the intersection of $\overline{BE}$ and $\overline{CF}$ as K, the intersection of $\overline{AD}$ and $\overline{CF}$ as L, and the intersection of $\overline{AD}$ and $\overline{BE}$ as M. If $[KLM] = [AME] + [BKF] + [CLD]$, where $[X]$ denotes the area of region X, compute CE.

13. (*Lewis Chen*) For real θ_i, $i = 1, 2, \ldots, 2011$, find the maximum value of the expression

$$\sin^{2012}\theta_1\cos^{2012}\theta_2+\sin^{2012}\theta_2\cos^{2012}\theta_3+\cdots+\sin^{2012}\theta_{2010}\cos^{2012}\theta_{2011}+\sin^{2012}\theta_{2011}\cos^{2012}\theta_1.$$

14. (*Eugene Chen*) In circle ω_1 with radius 1, circles $\phi_1, \phi_2, \ldots, \phi_8$, with equal radii, are drawn such that for $1 \le i \le 8$, ϕ_i is tangent to ω_1, ϕ_{i-1}, and ϕ_{i+1}, where $\phi_0 = \phi_8$ and $\phi_1 = \phi_9$. There exists a circle ω_2 such that $\omega_1 \neq \omega_2$ and ω_2 is tangent to ϕ_i for $1 \le i \le 8$. The radius of ω_2 can be expressed in the form $a - b\sqrt{c} - d\sqrt{e - \sqrt{f}} + g\sqrt{h - j\sqrt{k}}$ such that $a, b, \ldots, k$ are positive integers and the numbers $c, f, k, \gcd(h, j)$ are squarefree. What is $a+b+c+d+e+f+g+h+j+k$?

15. (*Lewis Chen*) Let
$$N = \sum_{a_1=0}^{2}\sum_{a_2=0}^{a_1}\sum_{a_3=0}^{a_2}\cdots\sum_{a_{2011}=0}^{a_{2010}}\left[\prod_{n=1}^{2011} a_n\right].$$
Find the remainder when N is divided by 1000.

2. Summer 2012

7:00 PM – 7:15 PM ET

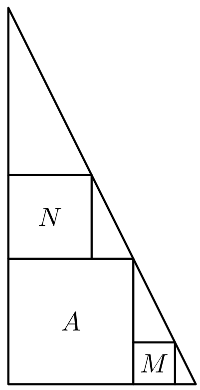
1. (*Eugene Chen*) Let $f(x) = (x^4 + 2x^3 + 4x^2 + 2x + 1)^5$. Compute the prime p satisfying $f(p) = 418{,}195{,}493$.

2. (*Isabella Grabski*) Compute the number of positive integers n satisfying the inequalities
$$2^{n-1} < 5^{n-3} < 3^n.$$

3. (*Lewis Chen*) Let
$$S = \sum_{i=1}^{2012} i!.$$
The tens and units digits of S (in decimal notation) are a and b, respectively. Compute $10a + b$.

4. (*Lewis Chen*) The degree measures of the angles of nondegenerate hexagon $ABCDEF$ are integers that form a non-constant arithmetic sequence in some order, and $\angle A$ is the smallest angle of the (not necessarily convex) hexagon. Compute the sum of all possible degree measures of $\angle A$.

5. (*Aaron Lin*) In the diagram below, three squares are inscribed in right triangles. Their areas are A, M, and N, as indicated in the diagram. If $M = 5$ and $N = 12$, then A can be expressed as $a + b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime. Compute $a + b + c$.



6. (*Eugene Chen*) When Eva counts, she skips all numbers containing a digit divisible by 3. For example, the first ten numbers she counts are 1, 2, 4, 5, 7, 8, 11, 12, 14, 15. What is the 100^{th} number she counts?

7. (*Aaron Lin*) A permutation $(a_1, a_2, a_3, \dots, a_{2012})$ of $(1, 2, 3, \dots, 2012)$ is selected at random. If S is the expected value of
$$\sum_{i=1}^{2012} |a_i - i|,$$
then compute the sum of the prime factors of S.

8. (*Aaron Lin*) Points A, B, and O lie in the plane such that $\measuredangle AOB = 120^\circ$. Circle ω_0 with radius 6 is constructed tangent to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$. For all $i \geq 1$, circle ω_i with radius r_i is constructed such that $r_i < r_{i-1}$ and ω_i is tangent to $\overrightarrow{OA}$, $\overrightarrow{OB}$, and ω_{i-1}. If
$$S = \sum_{i=1}^{\infty} r_i,$$

then S can be expressed as $a\sqrt{b}+c$, where a, b, c are integers and b is not divisible by the square of any prime. Compute $100a+10b+c$.

9. (*Aaron Lin*) A quadratic polynomial $p(x)$ with integer coefficients satisfies $p(41) = 42$. For some integers $a, b > 41$, $p(a) = 13$ and $p(b) = 73$. Compute the value of $p(1)$.

10. (*Lewis Chen*) A *triangulation* of a polygon is a subdivision of the polygon into triangles meeting edge to edge, with the property that the set of triangle vertices coincides with the set of vertices of the polygon. Adam randomly selects a triangulation of a regular 180-gon. Then, Bob selects one of the 178 triangles in this triangulation. The expected number of 1° angles in this triangle can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a+b$.

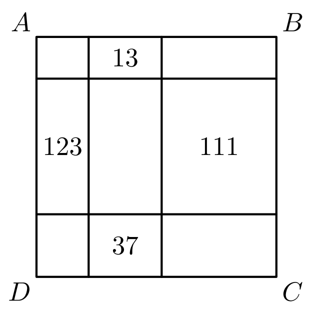
11. (*Lewis Chen*) Let a and b be two positive integers satisfying the equation
$$20\sqrt{12} = a\sqrt{b}.$$
Compute the sum of all possible distinct products ab.

12. (*Lewis Chen*) The NEMO (National Electronic Math Olympiad) is similar to the NIMO Summer Contest, in that there are fifteen problems, each worth a set number of points. However, the NEMO is weighted using Fibonacci numbers; that is, the n^{th} problem is worth F_n points, where $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. The two problem writers are fair people, so they make sure that each of them is responsible for problems worth an equal number of total points. Compute the number of ways problem writing assignments can be distributed between the two writers.

13. (*Evan Chen*) For the NEMO, Kevin needs to compute the product
$$9 \times 99 \times 999 \times \cdots \times 999999999.$$
Kevin takes exactly ab seconds to multiply an a-digit integer by a b-digit integer. Compute the minimum number of seconds necessary for Kevin to evaluate the expression together by performing eight such multiplications.

14. (*Lewis Chen*) A set of lattice points is called *good* if it does not contain two points that form a line with slope -1 or slope 1. Let $S = \{(x, y) \mid x, y \in \mathbb{Z}, 1 \leq x, y \leq 4\}$. Compute the number of non-empty good subsets of S.

15. (*Lewis Chen*) In the diagram below, square $ABCD$ with side length 23 is cut into nine rectangles by two lines parallel to $\overline{AB}$ and two lines parallel to $\overline{BC}$. The areas of four of these rectangles are indicated in the diagram. Compute the largest possible value for the area of the central rectangle.



3. Summer 2013

5:00 PM – 5:15 PM ET

1. (*Evan Chen*) What is the maximum possible score on this contest? Recall that on the NIMO 2013 Summer Contest, problems 1, 2, ..., 15 are worth 1, 2, ..., 15 points, respectively.

2. (*Evan Chen*) If $\frac{2+4+6}{1+3+5} - \frac{1+3+5}{2+4+6} = \frac{m}{n}$ for relatively prime integers m and n, compute $100m+n$.

3. (*Aaron Lin*) Jacob and Aaron are playing a game in which Aaron is trying to guess the outcome of an unfair coin which shows heads $\frac{2}{3}$ of the time. Aaron randomly guesses "heads" $\frac{2}{3}$ of the time, and guesses "tails" the other $\frac{1}{3}$ of the time. If the probability that Aaron guesses correctly is p, compute $9000p$.

4. (*Evan Chen*) Find the sum of the real roots of the polynomial

$$\prod_{k=1}^{100} \left(x^2 - 11x + k\right) = \left(x^2 - 11x + 1\right)\left(x^2 - 11x + 2\right) \dots \left(x^2 - 11x + 100\right).$$

5. (*Evan Chen*) A point (a, b) in the plane is called *sparkling* if it also lies on the line $ax+by = 1$. Find the maximum possible distance between two sparkling points.

6. (*Lewis Chen*) Let ABC and DEF be two triangles, such that $AB = DE = 20$, $BC = EF = 13$, and $\angle A = \angle D$. If $AC - DF = 10$, determine the area of $\triangle ABC$.

7. (*Evan Chen*) Circle ω_1 and ω_2 have centers $(0, 6)$ and $(20, 0)$, respectively. Both circles have radius 30, and intersect at two points X and Y. The line through X and Y can be written in the form $y = mx + b$. Compute $100m + b$.

8. (*Evan Chen*) A pair of positive integers (m, n) is called *compatible* if $m \ge \frac{1}{2}n + 7$ and $n \ge \frac{1}{2}m + 7$. A positive integer $k \ge 1$ is called *lonely* if (k, ℓ) is not compatible for any integer $\ell \ge 1$. Find the sum of all lonely integers.

9. (*Evan Chen*) Compute $99(99^2 + 3) + 3 \cdot 99^2$.

10. (*Evan Chen*) Let $P(x)$ be the unique polynomial of degree four for which $P(165) = 20$, and

$$P(42) = P(69) = P(96) = P(123) = 13.$$

Compute $P(1) - P(2) + P(3) - P(4) + \cdots + P(165)$.

11. (*Aaron Lin*) Find $100m + n$ if m and n are relatively prime positive integers such that

$$\sum_{\substack{i,j \ge 0 \\ i+j \text{ odd}}} \frac{1}{2^i 3^j} = \frac{m}{n}.$$

12. (*Eugene Chen*) In $\triangle ABC$, $AB = 40$, $BC = 60$, and $CA = 50$. The angle bisector of $\angle A$ intersects the circumcircle of $\triangle ABC$ at A and P. Find BP.

13. (*Lewis Chen*) In trapezoid $ABCD$, $AD \parallel BC$ and $\angle ABC + \angle CDA = 270^\circ$. Compute AB^2 given that $AB \cdot \tan(\angle BCD) = 20$ and $CD = 13$.

14. (*Evan Chen*) Let p, q, and r be primes satisfying

$$pqr = 18999999999999999999999999999999999999999999999999999962.$$

Compute $S(p) + S(q) + S(r) - S(pqr)$, where $S(n)$ denote the sum of the decimals digits of n.

15. (*Lewis Chen*)

> Ted quite likes haikus,
> poems with five-seven-five,
> but Ted knows few words.
>
> He knows $2n$ words
> that contain n syllables
> for every int n.
>
> Ted can only write
> N distinct haikus. Find N.
> Take mod one hundred.

Ted loves creating haikus (Japanese three-line poems with 5, 7, 5 syllables each), but his vocabulary is rather limited. In particular, for integers $1 \le n \le 7$, he knows $2n$ words with n syllables. Furthermore, words cannot cross between lines, but may be repeated. If Ted can make N distinct haikus, compute the remainder when N is divided by 100.

4. Summer 2014

5:00 PM – 5:15 PM ET

1. (*Evan Chen*) Compute $1 + 2 \cdot 3^4$.

2. (*Evan Chen*) How many $2\times2\times2$ cubes must be added to a $8\times8\times8$ cube to form a $12\times12\times12$ cube?

3. (*Evan Chen*) A square and equilateral triangle have the same perimeter. If the triangle has area $16\sqrt{3}$, what is the area of the square?

4. (*Evan Chen*) Let n be a positive integer. Determine the smallest possible value of $1 - n + n^2 - n^3 + \cdots + n^{1000}$.

5. (*Lewis Chen*) We have a five-digit positive integer N. We select every pair of digits of N (and keep them in order) to obtain the $\binom{5}{2} = 10$ numbers 33, 37, 37, 37, 38, 73, 77, 78, 83, 87. Find N.

6. (*Lewis Chen*) Suppose x is a random real number between 1 and 4, and y is a random real number between 1 and 9. If the expected value of
$$\lceil \log_2 x \rceil - \lfloor \log_3 y \rfloor$$
can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers, compute $100m + n$.

7. (*Evan Chen*) Evaluate
$$\frac{1}{729} \sum_{a=1}^{9} \sum_{b=1}^{9} \sum_{c=1}^{9} \left(abc + ab + bc + ca + a + b + c\right).$$

8. (*Aaron Lin*) Aaron takes a square sheet of paper, with one corner labeled A. Point P is chosen at random inside of the square and Aaron folds the paper so that points A and P coincide. He cuts the sheet along the crease and discards the piece containing A. Let p be the probability that the remaining piece is a pentagon. Find the integer nearest to $100p$.

9. (*Lewis Chen*) Two players play a game involving an $n \times n$ grid of chocolate. Each turn, a player may either eat a piece of chocolate (of any size), or split an existing piece of chocolate into two rectangles along a grid-line. The player who moves last loses. For how many positive integers n less than 1000 does the second player win?

   (Splitting a piece of chocolate refers to taking an $a \times b$ piece, and breaking it into an $(a-c) \times b$ and a $c \times b$ piece, or an $a \times (b - d)$ and an $a \times d$ piece.)

10. (*Evan Chen*) Among 100 points in the plane, no three collinear, exactly 4026 pairs are connected by line segments. Each point is then randomly assigned an integer from 1 to 100 inclusive, each equally likely, such that no integer appears more than once. Find the expected value of the number of segments which join two points whose labels differ by at least 50.

11. (*Evan Chen*) Consider real numbers $A, B, \ldots, Z$ such that
$$EVIL = \frac{5}{31},\ LOVE = \frac{6}{29},\ \text{and } IMO = \frac{7}{3}.$$
If $OMO = \frac{m}{n}$ for relatively prime positive integers m and n, find the value of $m + n$.

12. (*Evan Chen*) Find the sum of all positive integers n such that

$$\frac{2n+1}{n(n-1)}$$

has a terminating decimal representation.

13. (*Lewis Chen*) Let α and β be nonnegative integers. Suppose the number of strictly increasing sequences of integers $a_0, a_1, \dots, a_{2014}$ satisfying $0 \le a_m \le 3m$ is $2^\alpha(2\beta+1)$. Find α.

14. (*Evan Chen*) Let ABC be a triangle with circumcenter O and let X, Y, Z be the midpoints of arcs BAC, ABC, ACB on its circumcircle. Let G and I denote the centroid of $\triangle XYZ$ and the incenter of $\triangle ABC$.

    Given that $AB = 13$, $BC = 14$, $CA = 15$, and $\frac{GO}{GI} = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

15. (*Lewis Chen*) Let $A = (0,0)$, $B = (-1,-1)$, $C = (x,y)$, and $D = (x+1,y)$, where $x > y$ are positive integers. Suppose points A, B, C, D lie on a circle with radius r. Denote by r_1 and r_2 the smallest and second smallest possible values of r. Compute $r_1^2 + r_2^2$.

III. April Fun Round

1. April 2013

April 1, 2013

1. (*George Xing, et al.*) Find the value of 645.

2. (*Evan Chen*) At a certain school, the ratio of boys to girls is $1 : 3$. Suppose that:
   - Every boy has most 2013 distinct girlfriends.
   - Every girl has at least n boyfriends.
   - Friendship is mutual.

   Compute the largest possible value of n.

3. (*Evan Chen*) Bored in an infinitely long class, Evan jots down a fraction whose numerator and denominator are both 70-character strings, as follows:
$$r = \frac{loooloolloolloololllloloollollolllloollloloolooololooolololooooollllol}{lolooloolollollolloooooloooloololloolllooollololoooollllooololllolooool}.$$
   If $o = 2013$ and $l = \frac{1}{50}$, find $\lceil roll \rceil$.

4. (*Evan Chen*) Let a, b, c be the answers to problems 4, 5, and 6, respectively. In $\triangle ABC$, the measures of $\angle A$, $\angle B$, and $\angle C$ are a, b, c in degrees, respectively. Let D and E be points on segments AB and AC with $\frac{AD}{BD} = \frac{AE}{CE} = 2013$. A point P is selected in the interior of $\triangle ADE$, with barycentric coordinates (x, y, z) with respect to $\triangle ABC$ (here $x + y + z = 1$). Lines BP and CP meet line DE at B_1 and C_1, respectively. Suppose that the radical axis of the circumcircles of $\triangle PDC_1$ and $\triangle PEB_1$ pass through point A. Find $100x$.

5. (*Evan Chen*) Consider $\triangle\natural\flat\sharp$. Let $\flat\sharp$, $\sharp\natural$ and $\natural\flat$ be the answers to problems 4, 5, and 6, respectively. If the incircle of $\triangle\natural\flat\sharp$ touches $\natural\flat$ at $\odot$, find $\flat\odot$.

6. (*Evan Chen*) Let n and k be integers satisfying $\binom{2k}{2} + n = 60$. It is known that n days before Evan's 16th birthday, something happened. Compute $60 - n$.

7. (*Evan Chen and Lewis Chen*) Let p be the largest prime less than 2013 for which
$$N = 20 + p^{p^{p+1}-13}$$
   is also prime. Find the remainder when N is divided by 10^4.

8. (*Lewis Chen*) A person flips 2010 coins at a time. He gains one penny every time he flips a prime number of heads, but must stop once he flips a non-prime number. If his expected amount of money gained in dollars is $\frac{a}{b}$, where a and b are relatively prime, compute $\lceil \log_2(100a + b) \rceil$.

9. (*Evan Chen*) Haddaway once asked, "what is love?". The answer can be written in the form $\frac{m}{n}$, where m and n are positive integers such that $m^2 + n^2 < 2013$. Find $100m + n$.

10. (*Evan Chen*) There exist primes p and q such that

$$pq = 1208925819614629174706176 \times 2^{4404} - 4503599560261633 \times 134217730 \times 2^{2202} + 1.$$

Find the remainder when $p + q$ is divided by 1000.

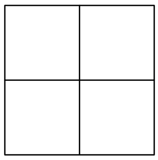
11. (*Evan Chen, Eugene Chen, Lewis Chen*) `http://goo.gl/wVR25`

12. (*Evan Chen, Lewis Chen*) If X_i is the answer to problem i for $1 \le i \le 12$, find the minimum possible value of $\sum_{n=1}^{12} (-1)^n X_n$.

2. April 2014

April 1, 2014

1. (*Evan Chen*) **Binary Sudoku (2 points)**
   How many ways are there to fill the 2×2 grid below with 0's and 1's such that no row or column has duplicate entries?

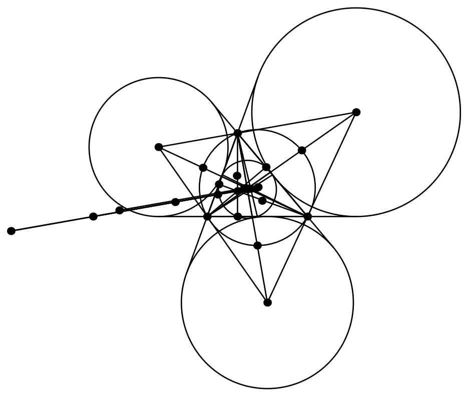


2. (*Evan Chen*) **Angry and Hungry (3 points)**
   I'm thinking of a five-letter word that rhymes with "angry" and "hungry". What is it?

3. (*Evan Chen*) **Engineer's Induction (5 points)**

4. (*Evan Chen*) **Do You Even Lift the Exponent? (7 points)**
   Let n be largest number such that
   $$\frac{2014^{100!} - 2011^{100!}}{3^n}$$
   is still an integer. Compute the remainder when 3^n is divided by 1000.

5. (*Evan Chen*) **Triangle Centers (11 points)**
   Let ABC be a triangle with $AB = 130$, $BC = 140$, $CA = 150$. Let G, H, I, O, N, K, L be the centroid, orthocenter, incenter, circumenter, nine-point center, the symmedian point, and the de Longchamps point. Let D, E, F be the feet of the altitudes of A, B, C on the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Let X, Y, Z be the A, B, C excenters and let U, V, W denote the midpoints of $\overline{IX}$, $\overline{IY}$, $\overline{IZ}$ (i.e. the midpoints of the arcs of (ABC).) Let R, S, T denote the isogonal conjugates of the midpoints of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Let P and Q denote the images of G and H under an inversion around the circumcircle of ABC followed by a dilation at O with factor $\frac{1}{2}$, and denote by M the midpoint of $\overline{PQ}$. Then let J be a point such that $JKLM$ is a parallelogram. Find the perimeter of the convex hull of the self-intersecting 17-gon $LETSTRADEBITCOINS$ to the nearest integer. A diagram has been included but may not be to scale.



6. (*Evan Chen*) **Chinese Remainder Theorem (13 points)**

We know $\mathbb{Z}_{210} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$. Moreover,

$$\begin{aligned} 53 &\equiv 1 \pmod 2 \\ 53 &\equiv 2 \pmod 3 \\ 53 &\equiv 3 \pmod 5 \\ 53 &\equiv 4 \pmod 7. \end{aligned}$$

Let

$$M = \begin{pmatrix} 53 & 158 & 53 \\ 23 & 93 & 53 \\ 50 & 170 & 53 \end{pmatrix}.$$

Based on the above, find $\overline{(M \mod 2)(M \mod 3)(M \mod 5)(M \mod 7)}$.

7. (*Evan Chen*) **Foreign Language (17 points)**
Evaluate the following:
`http://internetolympiad.org/archive/2014/AprilFools/foreign_lang.txt`.

8. (*Evan Chen*) **Silver Cyanide (19 points)**
Three of the below entries, with labels a, b, c, are blatantly incorrect (in the United States).
What is $a^2 + b^2 + c^2$?

041. The Gentleman's Alliance Cross
042. Glutamine (an amino acid)
051. Grant Nelson and Norris Windross
052. A compact region at the center of a galaxy
061. The value of `'wat'-1`.[1]
062. Threonine (an amino acid)
071. Nintendo Gamecube
072. Methane and other gases are compressed
081. A prank or trick
082. Three carbons
091. Australia's second largest local government area
092. Angoon Seaplane Base
101. A compressed archive file format
102. Momordica cochinchinensis
111. Gentaro Takahashi
112. Nat Geo
121. Ante Christum Natum
122. The supreme Siberian god of death
131. Gnu C Compiler
132. My TeX Shortcut for $\angle$.

9. (*Evan Chen*) **Yaler Repus (23 points)**
This is an ARML Super Relay! I'm sure you know how this works! You start from #1 and #15 and meet in the middle. We are going to require you to solve all 15 problems, though – so for the entire task, submit the sum of all the answers, rather than just the answer to #8.

Also, uhh, we can't actually find the slip for #1. Sorry about that. Have fun anyways!

[1] See `https://www.destroyallsoftware.com/talks/wat`.

2. Let $T =$ TNYWR. Find the number of way to distribute 6 indistinguishable pieces of candy to T hungry (and distinguishable) schoolchildren, such that each child gets at most one piece of candy.
3. Let $T =$ TNYWR. If d is the largest proper divisor of T, compute $\frac{1}{2}d$.
4. Let $T =$ TNYWR and flip 4 fair coins. Suppose the probability that at most T heads appear is $\frac{m}{n}$, where m and n are coprime positive integers. Compute $m + n$.
5. Let $T =$ TNYWR. Compute the last digit of T^T in base 10.
6. Let $T =$ TNYWR and flip 6 fair coins. Suppose the probability that at most T heads appear is $\frac{m}{n}$, where m and n are coprime positive integers. Compute $m + n$.
7. Let $T =$ TNYWR. Compute the smallest prime p for which $n^T \not\equiv n \pmod{p}$ for some integer n.
8. Let M and N be the two answers received, with $M \le N$. Compute the number of integer quadruples (w, x, y, z) with $w + x + y + z = M\sqrt{wxyz}$ and $1 \le w, x, y, z \le N$.
9. Let $T =$ TNYWR. Compute the smallest integer n with $n \ge 2$ such that n is coprime to $T + 1$, and there exists positive integers a, b, c with $a^2 + b^2 + c^2 = n(ab + bc + ca)$.
10. Let $T =$ TNYWR and flip 10 fair coins. Suppose the probability that at most T heads appear is $\frac{m}{n}$, where m and n are coprime positive integers. Compute $m + n$.
11. Let $T =$ TNYWR. Compute the last digit of T^T in base 10.
12. Let $T =$ TNYWR and flip 12 fair coins. Suppose the probability that at most T heads appear is $\frac{m}{n}$, where m and n are coprime positive integers. Compute $m + n$.
13. Let $T =$ TNYWR. If d is the largest proper divisor of T, compute $\frac{1}{2}d$.
14. Let $T =$ TNYWR. Compute the number of way to distribute 6 indistinguishable pieces of candy to T hungry (and distinguishable) schoolchildren, such that each child gets at most one piece of candy.

Also, we can't find the slip for #15, either. We think the SFBA coaches stole it to prevent us from winning the Super Relay, but that's not going to stop us, is it? We have another #15 slip that produces an equivalent answer. Here you go!

15. Let A, B, C be the answers to #8, #9, #10. Compute $\gcd(A, C) \cdot B$.

IV. Winter Olympiad

1. Winter Olympiad 2011

January 2011

1. A point (x, y) in the first quadrant lies on a line with intercepts $(a, 0)$ and $(0, b)$, with $a, b > 0$. Rectangle M has vertices $(0, 0)$, $(x, 0)$, (x, y), and $(0, y)$, while rectangle N has vertices (x, y), (x, b), (a, b), and (a, y). What is the ratio of the area of M to that of N?

   (Eugene Chen)

2. Two sequences $\{a_i\}$ and $\{b_i\}$ are defined as follows: $\{a_i\} = 0, 3, 8, \ldots, n^2 - 1, \ldots$ and $\{b_i\} = 2, 5, 10, \ldots, n^2 + 1, \ldots$. If both sequences are defined with i ranging across the natural numbers, how many numbers belong to both sequences?

   (Isabella Grabski)

3. Billy and Bobby are located at points A and B, respectively. They each walk directly toward the other point at a constant rate; once the opposite point is reached, they immediately turn around and walk back at the same rate. The first time they meet, they are located 3 units from point A; the second time they meet, they are located 10 units from point B. Find all possible values for the distance between A and B.

   (Isabella Grabski)

4. In the following alpha-numeric puzzle, each letter represents a different non-zero digit. What are all possible values for $b + e + h$?

$$\begin{array}{cccc} & \text{a} & \text{b} & \text{c} \\ & \text{d} & \text{e} & \text{f} \\ + & \text{g} & \text{h} & \text{i} \\ \hline 1 & 6 & 6 & 5 \end{array}$$

   (Eugene Chen)

5. We have eight light bulbs, placed on the eight lattice points in space that are $\sqrt{3}$ units away from the origin. Each light bulb can either be turned on or off. These lightbulbs are unstable, however. If two light bulbs that are at most 2 units apart are both on simultaneously, they both explode. Given that no explosions take place, how many possible configurations of on/off light bulbs exist?

   (Lewis Chen)

6. Circle $\odot O$ with diameter $\overline{AB}$ has chord $\overline{CD}$ drawn such that $\overline{AB}$ is perpendicular to $\overline{CD}$ at P. Another circle $\odot A$ is drawn, sharing chord $\overline{CD}$. A point Q on minor arc $\overline{CD}$ of $\odot A$ is chosen so that $\measuredangle AQP + \measuredangle QPB = 60°$. Line l is tangent to $\odot A$ through Q and a point X on l is chosen such that $PX = BX$. If $PQ = 13$ and $BQ = 35$, find QX.

   (Aaron Lin)

7. The number $\left(2 + 2^{96}\right)!$ has 2^{93} trailing zeroes when expressed in base B.

   a) Find the minimum possible B.

b) Find the maximum possible B.

c) Find the total number of possible B.

(Lewis Chen)

8. Define $f(x)$ to be the nearest integer to x, with the greater integer chosen if two integers are tied for being the nearest. For example, $f(2.3) = 2$, $f(2.5) = 3$, and $f(2.7) = 3$. Define $[A]$ to be the area of region A. Define region R_n, for each positive integer n, to be the region on the Cartesian plane which satisfies the inequality $f(|x|) + f(|y|) < n$. We pick an arbitrary point O on the perimeter of R_n, and mark every two units around the perimeter with another point. Region S_{nO} is defined by connecting these points in order.

a) Prove that the perimeter of R_n is always congruent to $4 \pmod 8$.

b) Prove that $[S_{nO}]$ is constant for any O.

c) Prove that $[R_n] + [S_{nO}] = (2n - 1)^2$.

(Lewis Chen)

2. Winter Olympiad 2012

January 2012

1. In a 10 by 10 grid of dots, what is the maximum number of lines that can be drawn connecting two dots on the grid so that no two lines are parallel?

   (Aaron Lin)

2. If r_1, r_2, and r_3 are the solutions to the equation $x^3 - 5x^2 + 6x - 1 = 0$, then what is the value of $r_1^2 + r_2^2 + r_2^3$?

   (Eugene Chen)

3. The expression $\circ 1 \circ 2 \circ 3 \circ \cdots \circ 2012$ is written on a blackboard. Catherine places a $+$ sign or a $-$ sign into each blank. She then evaluates the expression, and finds the remainder when it is divided by 2012. How many possible values are there for this remainder?

   (Aaron Lin)

4. Parallel lines ℓ_1 and ℓ_2 are drawn in a plane. Points $A_1, A_2, \dots, A_n$ are chosen on ℓ_1, and points $B_1, B_2, \dots, B_{n+1}$ are chosen on ℓ_2. All segments A_iB_j are drawn, such that $1 \le i \le n$ and $1 \le j \le n+1$. Let the number of total intersections between these segments (not including endpoints) be denoted by Q. Given that no three segments are concurrent, besides at endpoints, prove that Q is divisible by 3.

   (Lewis Chen)

5. In convex hexagon $ABCDEF$, $\angle A \cong \angle B$, $\angle C \cong \angle D$, and $\angle E \cong \angle F$. Prove that the perpendicular bisectors of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ pass through a common point.

   (Lewis Chen)

6. The positive numbers a, b, c satisfy $4abc(a+b+c) = (a+b)^2(a+c)^2$. Prove that $a(a+b+c) = bc$.

   (Aaron Lin)

7. For how many positive integers $n \le 500$ is $n!$ divisible by 2^{n-2}?

   (Eugene Chen)

8. A convex 2012-gon $A_1A_2A_3 \dots A_{2012}$ has the property that for every integer $1 \le i \le 1006$, $\overline{A_iA_{i+1006}}$ partitions the polygon into two congruent regions. Show that for every pair of integers $1 \le j < k \le 1006$, quadrilateral $A_jA_kA_{j+1006}A_{k+1006}$ is a parallelogram.

   (Lewis Chen)

3. Winter Olympiad 2013

March 2013

1. Find the remainder when $2+4+\cdots+2014$ is divided by $1+3+\cdots+2013$. Justify your answer.

   (Evan Chen)

2. Square $\mathcal{S}$ has vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$. Points P and Q are independently selected, uniformly at random, from the perimeter of $\mathcal{S}$. Determine, with proof, the probability that the slope of line PQ is positive.

   (Isabella Grabski)

3. Let ABC be a triangle. Prove that there exists a unique point P for which one can find points D, E and F such that the quadrilaterals $APBF$, $BPCD$, $CPAE$, $EPFA$, $FPDB$, and $DPEC$ are all parallelograms.

   (Lewis Chen)

4. Let $\mathcal{F}$ be the set of all 2013×2013 arrays whose entries are 0 and 1. A transformation $K : \mathcal{F} \to \mathcal{F}$ is defined as follows: for each entry a_{ij} in an array $A \in \mathcal{F}$, let S_{ij} denote the sum of all the entries of A sharing either a row or column (or both) with a_{ij}. Then a_{ij} is replaced by the remainder when S_{ij} is divided by two.

   Prove that for any $A \in \mathcal{F}$, $K(A) = K(K(A))$.

   (Aaron Lin)

5. In convex hexagon $AXBYCZ$, sides AX, BY and CZ are parallel to diagonals BC, XC and XY, respectively. Prove that $\triangle ABC$ and $\triangle XYZ$ have the same area.

   (Evan Chen)

6. A strictly increasing sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers is said to be [i]large[/i] if, for every real number L, there exists an integer n such that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} > L$. Do there exist large sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ such that the sequence $\{a_i + b_i\}_{i=1}^{\infty}$ is not large?

   (Lewis Chen)

7. Let a, b, c be positive reals satisfying $a^3 + b^3 + c^3 + abc = 4$. Prove that

$$\frac{(5a^2+bc)^2}{(a+b)(a+c)} + \frac{(5b^2+ca)^2}{(b+c)(b+a)} + \frac{(5c^2+ab)^2}{(c+a)(c+b)} \geq \frac{(a^3+b^3+c^3+6)^2}{a+b+c}$$

   and determine the cases of equality.

   (Evan Chen)

8. For a finite set X define

$$S(X) = \sum_{x \in X} x \text{ and } P(x) = \prod_{x \in X} x.$$

   Let A and B be two finite sets of positive integers such that $|A| = |B|$, $P(A) = P(B)$ and $S(A) \neq S(B)$. Suppose for any $n \in A \cup B$ and prime p dividing n, we have $p^{36} \mid n$ and $p^{37} \nmid n$. Prove that

$$|S(A) - S(B)| > 1.9 \cdot 10^6.$$

   (Evan Chen)

4. Winter Olympiad 2014

January 2014

1. Find, with proof, all real numbers x satisfying $x = 2\left(2\left(2\left(2\left(2x-1\right)-1\right)-1\right)-1\right)-1$.

   (Evan Chen)

2. Determine, with proof, the smallest positive integer c such that for any positive integer n, the decimal representation of the number $c^n + 2014$ has digits all less than 5.

   (Evan Chen)

3. The numbers $1, 2, \ldots, 10$ are written on a board. Every minute, one can select three numbers a, b, c on the board, erase them, and write $\sqrt{a^2+b^2+c^2}$ in their place. This process continues until no more numbers can be erased. What is the largest possible number that can remain on the board at this point?

   (Evan Chen)

4. Prove that there exist integers a, b, c with $1 \le a < b < c \le 25$ and

   $$S(a^6 + 2014) = S(b^6 + 2014) = S(c^6 + 2014)$$

   where $S(n)$ denotes the sum of the decimal digits of n.

   (Evan Chen)

5. Let ABC be an acute triangle with orthocenter H and let M be the midpoint of $\overline{BC}$. (The *orthocenter* is the point at the intersection of the three altitudes.) Denote by ω_B the circle passing through B, H, and M, and denote by ω_C the circle passing through C, H, and M. Lines AB and AC meet ω_B and ω_C again at P and Q, respectively. Rays PH and QH meet ω_C and ω_B again at R and S, respectively. Show that $\triangle BRS$ and $\triangle CRS$ have the same area.

   (Aaron Lin)

6. Let $\varphi(k)$ denote the numbers of positive integers less than or equal to k and relatively prime to k. Prove that for some positive integer n,

   $$\varphi(2n-1) + \varphi(2n+1) < \frac{1}{1000}\varphi(2n).$$

   (Evan Chen)

7. Let ABC be a triangle and let Q be a point such that $\overline{AB} \perp \overline{QB}$ and $\overline{AC} \perp \overline{QC}$. A circle with center I is inscribed in $\triangle ABC$, and is tangent to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ at points D, E, and F, respectively. If ray QI intersects $\overline{EF}$ at P, prove that $\overline{DP} \perp \overline{EF}$.

   (Aaron Lin)

8. Define the function $\xi : \mathbb{Z}^2 \to \mathbb{Z}$ by $\xi(n,k) = 1$ when $n \le k$ and $\xi(n,k) = -1$ when $n > k$, and construct the polynomial

   $$P(x_1, \ldots, x_{1000}) = \prod_{n=1}^{1000} \left(\sum_{k=1}^{1000} \xi(n,k) x_k \right).$$

   (a) Determine the coefficient of $x_1 x_2 \ldots x_{1000}$ in P.

   (b) Show that if $x_1, x_2, \ldots, x_{1000} \in \{-1, 1\}$ then $P(x_1, x_2, \ldots, x_{1000}) = 0$.

   (Evan Chen)

Part B.

Answers and Solutions

V. Monthly Contest

1. September 17, 2012

8:00 PM – 8:30 PM ET
(1) 25 (2) 6 (3) 2080 (4) 339 (5) 73 (6) 1676 (7) 220 (8) 243

2. October 17, 2012

8:00 PM – 8:30 PM ET
(1) 1987 (2) 260 (3) 401 (4) 273 (5) 124 (6) 230 (7) 16384 (8) 1091328

3. November 24, 2012

8:00 PM – 8:45 PM ET
(1) 270 (2) 504 (3) 60 (4) 9 (5) 132 (6) 764 (7) 307 (8) 10004 (9) 258 (10) 999

4. December 17, 2012

8:00 PM – 8:45 PM ET
(1) 1007 (2) 11 (3) 102 (4) 2013 (5) 109 (6) 1140 (7) 37 (8) 34783 (9) 1683 (10) 17524

5. January 24, 2013

8:00 PM – 8:30 PM ET
(1) 50 (2) 11 (3) 27 (4) 4052169 (5) 164 (6) 1866 (7) 5 (8) 23

6. February 24, 2013

8:00 PM – 8:45 PM ET
(1) 5 (2) 496 (3) 62 (4) 2013 (5) 210 (6) 611 (7) 103838 (8) 8 (9) 1806 (10) 16104

7. May 27, 2013

8:00 PM – 8:40 PM ET
(1) 768 (2) 84 (3) 1025 (4) 54 (5) 26

8. September 24, 2013

8:00 PM – 8:40 PM ET
(1) 40 (2) 39 (3) 136 (4) 36 (5) 40 (6) 702 (7) 304 (8) 336

9. November 13, 2013

8:00 PM – 8:40 PM ET
(1) 1332 (2) 6039 (3) 1716 (4) 499500 (5) 38 (6) 300 (7) 34776 (8) 3290111

10. December 3, 2013

8:00 PM – 8:40 PM ET
(1) 1100 (2) 120 (3) 39515 (4) 369 (5) 203 (6) 40003 (7) 2021 (8) 4817

11. January 31, 2014

8:00 PM – 8:40 PM ET
(1) 9920 (2) 164 (3) 17 (4) 4532 (5) 5 (6) 332156 (7) 152 (8) 4899

12. February 24, 2014

8:00 PM – 8:40 PM ET
(1) 1007 (2) 80 (3) 82 (4) 106 (5) 3308 (6) 499499 (7) 598 (8) 70

13. March 24, 2014

8:00 PM – 8:40 PM ET
(1) 450 (2) 716 (3) 10000 (4) 1333 (5) 53 (6) 545 (7) 1006 (8) 237

14. May 15, 2014

8:00 PM – 8:40 PM ET
(1) 1000 (2) 167 (3) 3514 (4) 103 (5) 193 (6) 828 (7) 382 (8) 183

VI. Summer Contest

1. Summer 2011

6:00 PM – 6:15 PM ET
(1) 17 (2) 120 (3) 145 (4) 28 (5) 36 (6) 4 (7) 99 (8) 7 (9) 256 (10) 260 (11) 2 (12) 91 (13) 1005 (14) 31 (15) 95

2. Summer 2012

7:00 PM – 7:15 PM ET
(1) 2 (2) 5 (3) 13 (4) 24 (5) 36 (6) 120 (7) 2086 (8) 227 (9) 2842 (10) 185 (11) 840 (12) 256 (13) 870 (14) 1224 (15) 180

3. Summer 2013

5:00 PM – 5:15 PM ET
(1) 120 (2) 712 (3) 5000 (4) 330 (5) 2 (6) 126 (7) 303 (8) 91 (9) 999999 (10) 20 (11) 504 (12) 40 (13) 260 (14) 8 (15) 28

4. Summer 2014

5:00 PM – 5:15 PM ET
(1) 163 (2) 152 (3) 36 (4) 1 (5) 37837 (6) 1112 (7) 215 (8) 57 (9) 999 (10) 1037 (11) 579 (12) 52 (13) 10 (14) 104 (15) 2523

VII. April Fun Round

1. April 2013

April 1, 2013

(1) 645 (2) 671 (3) 1 (4) 69 (5) 45 (6) 66 (7) 1781 (8) 2017 (9) 629 (10) 358 (11) 80 (12) 1

2. April 2014

April 1, 2014

(1) 2 (2) 3 (3) 5 (4) 375 (5) 420 (6) 4097 (7) 54 (8) 21586 (9) 5656

VIII. Winter Olympiad

1. Winter Olympiad 2011

January 2011

1. A point (x, y) in the first quadrant lies on a line with intercepts $(a, 0)$ and $(0, b)$, with $a, b > 0$. Rectangle M has vertices $(0, 0)$, $(x, 0)$, (x, y), and $(0, y)$, while rectangle N has vertices (x, y), (x, b), (a, b), and (a, y). What is the ratio of the area of M to that of N?

   (Eugene Chen)

   Solution. Throughout this solution the area of region $\Re$ will be denoted by $[\Re]$.

   Let $O = (0, 0)$, $A = (a, 0)$, $B = (0, b)$, $C = (a, b)$, $X = (x, 0)$, $Y = (0, y)$, $X' = (x, b)$, $Y' = (a, y)$, and $Z = (x, y)$. It follows that $[M] = [OXZY] = xy$. Because $\triangle AXZ \sim \triangle AOB$, we know
   $$\frac{AX}{XZ} = \frac{AO}{OB} \implies \frac{AX}{y} = \frac{a}{b} \implies AX = \frac{ay}{b}$$
   and similarly $BY = \dfrac{bx}{a}$.

   But $AX = ZY'$ and $BY = ZX'$ so
   $$[N] = [ZY'CX'] = ZX' \cdot ZY' = \frac{ay}{b} \cdot \frac{bx}{a} = xy = [M].$$
   Hence, $\dfrac{[M]}{[N]} = \boxed{1}$.

2. Two sequences $\{a_i\}$ and $\{b_i\}$ are defined as follows: $\{a_i\} = 0, 3, 8, \ldots, n^2 - 1, \ldots$ and $\{b_i\} = 2, 5, 10, \ldots, n^2 + 1, \ldots$. If both sequences are defined with i ranging across the natural numbers, how many numbers belong to both sequences?

   (Isabella Grabski)

   Solution. If there are common members of the sequences, there exist natural numbers k and l such that
   $$k^2 - 1 = l^2 + 1.$$
   This may be rewritten as
   $$k^2 - l^2 = 2$$
   $$(k + l)(k - l) = 2.$$
   Because k and l are natural numbers, it follows that $k + l = 2$. But then $k - l = 1$, yielding $k = \dfrac{3}{2}$, a contradiction. Hence, no numbers are members of both sequences.

3. Billy and Bobby are located at points A and B, respectively. They each walk directly toward the other point at a constant rate; once the opposite point is reached, they immediately turn around and walk back at the same rate. The first time they meet, they are located 3 units from point A; the second time they meet, they are located 10 units from point B. Find all possible values for the distance between A and B.

   (Isabella Grabski)

Solution. Let X be the first meeting point and Y the second. Denote the distance AB by x. Billy and Bobby can meet at at most one point before they reach the point they are traveling to. However, the distance between A and B depends on when they reach the opposite end. We proceed with casework.

Case 1: Billy and Bobby reach B and A, respectively, before meeting for a second time.
In this case, Billy travels AX before meeting for the first time while Bobby walks XB. Before they meet for a second time, Billy walks $XB + BY$ and Bobby walks $XA + AY$ in the same time. Because they walk at constant rates, we have:

$$\frac{AX}{XB} = \frac{XB + BY}{XA + AY}$$
$$\frac{3}{x-3} = \frac{x-3+10}{3+x-10}$$

Solving this, we find $x = 0$ or $x = -1$, neither of which can be walking distances in this problem. So there are no solutions in this case.

Case 2: Billy reaches point B first and meets Bobby at point Y before Bobby reaches A.
In a similar fashion, we find $x = -12$ or $x = 5$. However, -12 is negative, and 5 is less than 10, so neither can be possible total lengths of the path.

Case 3: Bobby reaches point A first and meets Billy at point Y before Billy reaches B.
This yields $x = 4$ or $x = 15$. Because $4 < 10$, the path cannot be 4 units long.
Thus, the only possible length of the path is $\boxed{15}$ units.

4. In the following alpha-numeric puzzle, each letter represents a different non-zero digit. What are all possible values for $b + e + h$?

$$\begin{array}{cccc} & \text{a} & \text{b} & \text{c} \\ & \text{d} & \text{e} & \text{f} \\ + & \text{g} & \text{h} & \text{i} \\ \hline 1 & 6 & 6 & 5 \end{array}$$

(Eugene Chen)

Solution. We can rewrite the equation as

$$100(a + d + g) + 10(b + e + h) + (c + f + i) = 1665.$$

Because $100(a+d+g)$ and $10(b+e+h)$ have units digit 0, it follows that $c+f+i$ has units digit 5. But

$$c + f + i \geq 1 + 2 + 3 = 6 > 5,$$
$$c + f + i \leq 7 + 8 + 9 = 24 < 25,$$

so $c + f + i = 15$. We can again rewrite the equation as

$$100(a + d + g) + 10(b + e + h) = 1650$$
$$10(a + d + g) + (b + e + h) = 165.$$

Because $10(a + d + g)$ has units digit 0, it follows that $b + e + h$ has units digit 5. By the same logic as above, we obtain $b + e + h = 15$.

It remains to show that $b + e + h = 15$ is achievable. But this may be achieved by setting $a = 1, b = 2, c = 4, d = 5, e = 6, f = 3, g = 9, h = 7, i = 8$. Hence, the only possible value for $b + e + h$ is $\boxed{15}$.

5. We have eight light bulbs, placed on the eight lattice points in space that are $\sqrt{3}$ units away from the origin. Each light bulb can either be turned on or off. These lightbulbs are unstable, however. If two light bulbs that are at most 2 units apart are both on simultaneously, they both explode. Given that no explosions take place, how many possible configurations of on/off light bulbs exist?

   (Lewis Chen)

   Solution. The lightbulbs are located at the points $(\pm 1, \pm 1, \pm 1)$ which determine a cube. Note that a configuration will explode only if both lightbulbs on an edge are simultaneously on.

   We proceed by casework on the number of lightbulbs on.

   If 0 lightbulbs are on then there is only 1 configuration of the lightbulbs, which indeed satisfies the conditions of the problem.

   If 1 lightbulb is on, then there are 8 configurations of the lightbulbs, all of which satisfy the conditions of the problem.

   If 2 lightbulbs are on, then there are $\binom{8}{2} = 28$ configurations of the lightbulbs. However, if both endpoints of an edge are lit then the configuration is not satisfactory so there are $28 - 12 = 16$ solutions for this case.

   To count the number of configurations where 3 lightbulbs are on, first choose a lightbulb to be lit. Notice that if the opposite lightbulb is lit, then no more can be lit; hence, the opposite lightbulb is not lit. Additionally, no lightbulbs that share an edge with the original lightbulb can be lit. So there are 3 lightbulbs left to choose from, and each choice of 2 from these 3 satisfies the conditions, for $\binom{3}{2} = 3$ solutions. But the original lightbulb can be chosen in 8 ways, and each case is counted three times (once for each lightbulb as the starting point), so the number of configurations from this case is $3 \cdot \frac{8}{3} = 8$.

   We count the ways for four lightbulbs to be on in the same way as the previous case. After choosing a starting lightbulb, there is $\binom{3}{3} = 1$ way to light the rest, for a total of $1 \cdot \frac{8}{4} = 2$ configurations.

   In total, there are $1 + 8 + 16 + 8 + 2 = \boxed{35}$ configurations.

6. Circle $\odot O$ with diameter $\overline{AB}$ has chord $\overline{CD}$ drawn such that $\overline{AB}$ is perpendicular to $\overline{CD}$ at P. Another circle $\odot A$ is drawn, sharing chord $\overline{CD}$. A point Q on minor arc $\overline{CD}$ of $\odot A$ is chosen so that $\measuredangle AQP + \measuredangle QPB = 60^\circ$. Line l is tangent to $\odot A$ through Q and a point X on l is chosen such that $PX = BX$. If $PQ = 13$ and $BQ = 35$, find QX.

   (Aaron Lin)

   Solution. Note that $\triangle ACB$ is a right triangle, so CP is the geometric mean of BP and AP. Thus, by the Pythagorean Theorem,

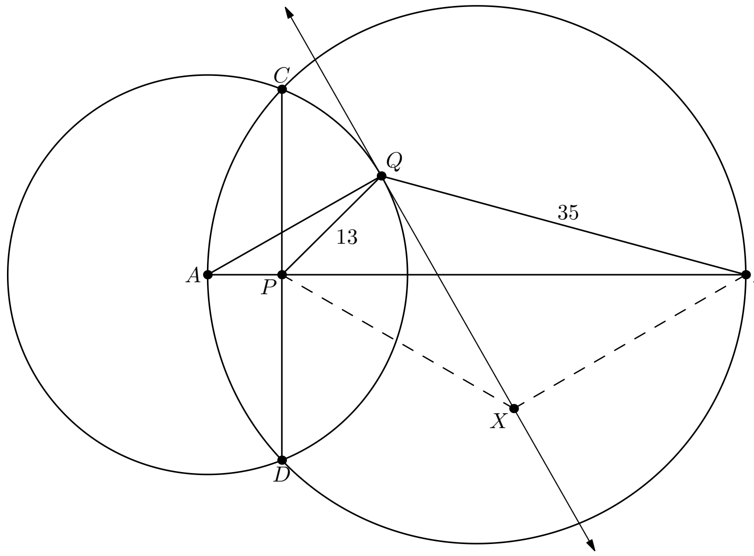
$$AC^2 = CP^2 + AP^2 = (AP \cdot BP) + AP^2 = AP(BP + AP) = AP \cdot AB.$$

   Because $AC = AQ$ it follows that $AQ^2 = AP \cdot AB$. Thus, $\overline{AQ}$ is tangent to the circumcircle of $\triangle BQP$. Since line l is perpendicular to $\overline{AQ}$ through Q, l must pass through the circumcenter of $\triangle BQP$. Hence, X, which is equidistant from points B and P, must be the circumcenter of $\triangle BQP$.

Furthermore, the equation $AP^2 = AP \cdot AB$ implies that $\triangle APQ \sim \triangle AQB$, so $\angle AQP \cong \angle ABQ \cong \angle PBQ$. Thus, $\mathrm{m}\angle AQP + \mathrm{m}\angle BPQ = \mathrm{m}\angle PBQ + \mathrm{m}\angle BPQ = 60°$, so $\mathrm{m}\angle BQP = 120°$. From the Law of Cosines it follows that $BP = 43$. By the Law of Sines,

$$QX = \frac{BP}{2\sin\angle BQP} = \frac{43}{\sqrt{3}}$$

Hence $QX = \boxed{\dfrac{43\sqrt{3}}{3}}$.



7. The number $\left(2+2^{96}\right)!$ has 2^{93} trailing zeroes when expressed in base B.
   a) Find the minimum possible B.
   b) Find the maximum possible B.
   c) Find the total number of possible B.

   (Lewis Chen)

Solution. Definition: The method of finding the number of trailing zeroes of $N!$ in prime base p is as follows:

$$\left\lfloor \frac{N}{p} \right\rfloor + \left\lfloor \frac{\left\lfloor \frac{N}{p} \right\rfloor}{p} \right\rfloor + \left\lfloor \frac{\left\lfloor \frac{\left\lfloor \frac{N}{p} \right\rfloor}{p} \right\rfloor}{p} \right\rfloor + \ldots.$$

Lemma. 3^a divides $(2+2a)!$ if and only if $2+2a = 3^m + 3^n$ for some integer m, n.

Proof: Note that

$$a \le \sum_{i=1}^{\infty} \left\lfloor \frac{2a+2}{3^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{2a+2}{3^i} = a+1.$$

Thus,
$$\sum_{i=1}^{\infty}\left\{\frac{2a+2}{3^i}\right\}=1.$$
Write
$$2a+2=\sum_{k=0}^{\infty}d_k3^k, d_k\in\{0,1,2\}$$
so
$$\sum_{i=1}^{\infty}\left\{\frac{\sum_{k=0}^{\infty}d_k3^k}{3^i}\right\}=\sum_{i=1}^{\infty}\frac{\sum_{k=0}^{i-1}d_k3^k}{3^i}=1$$
or
$$\sum_{k=0}^{\infty}\sum_{i=k+1}^{\infty}\frac{d_k3^k}{3^i}=\sum_{k=0}^{\infty}\frac{d_k}{2}=1.$$
Thus, $2a+2=3^m+3^n$, as desired.

In base 2, $(2+2^{96})!$, when written in base 2, has exactly
$$\left(1+2^{95}+2^{94}+2^{93}+\cdots+2^2+2^1+2^0\right)=2^{96}$$
trailing zeroes. Hence, when written in base $2^8=256$, there are exactly $\frac{2^{96}}{8}=2^{93}$ zeroes.
In base $p\geq 11$, there are:
$$\left\lfloor\frac{2+2^{96}}{p}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{p}\right\rfloor}{p}\right\rfloor+\cdots\leq\left(2+2^{96}\right)\left(\frac{1}{11}+\frac{1}{11^2}+\frac{1}{11^3}+\dots\right)=\left(2+2^{96}\right)\left(\frac{1}{10}\right)<2^{93}$$
trailing zeroes. Hence, the maximum possible prime divisor of the base is 7.
In base 7, there are
$$\left\lfloor\frac{2+2^{96}}{7}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{7}\right\rfloor}{7}\right\rfloor+\cdots\leq\left(2+2^{96}\right)\left(\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+\dots\right)\leq\left(2+2^{96}\right)\left(\frac{1}{6}\right)<2^{94},$$
so the maximum prime power of 7 is 1. To prove that this is indeed the maximum, notice that
$$\left\lfloor\frac{2+2^{96}}{7}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{7}\right\rfloor}{7}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{7}\right\rfloor}{7}\right\rfloor}{7}\right\rfloor+\cdots>\left\lfloor\frac{2+2^{96}}{7}\right\rfloor>\frac{2+2^{96}}{7}-1>\frac{2^{96}}{8}=2^{93}.$$
Similarly, the maximum prime power of 5 is 1. The prime power of 3 can be 3. To see that it cannot be 4, note that:
$$2+2^{96}\equiv 3\pmod{9}.$$
From the lemma, $2+2^{96}=3^a+3^b\equiv 3\pmod 9$, so exactly one of a,b equals 3. But then $2^{96}-1=3^a$, but $2^3-1=7$ divides $2^{96}-1$, a contradiction. Hence, the maximum prime power of 3 is 3.

In base $B=p_1^{e_1}p_2^{e_2}\dots p_k^{e_k}$, note that $\left(2+2^{96}\right)!$ has exactly
$$\min\left(\left\{\left\lfloor\frac{1}{e_i}\left\lfloor\frac{2+2^{96}}{p_i}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{p_i}\right\rfloor}{p_i}\right\rfloor+\left\lfloor\frac{\left\lfloor\frac{\left\lfloor\frac{2+2^{96}}{p_i}\right\rfloor}{p_i}\right\rfloor}{p_i}\right\rfloor+\dots\right\rfloor\right\}\right)$$

trailing zeroes. Since this must equal exactly 2^{93}, and the value for $p_i = 3, 5, 7$ cannot equal exactly 2^{93}, B must divide 2^8 exactly. We can then choose a nonnegative integer at most the maximum for each of the other prime bases.

It follows that the answers are:

a) $2^8 = \boxed{256}$

b) $2^8 \cdot 3^3 \cdot 5^1 \cdot 7^1 = \boxed{241920}$

c) $(3+1)(1+1)(1+1) = \boxed{16}$

8. Define $f(x)$ to be the nearest integer to x, with the greater integer chosen if two integers are tied for being the nearest. For example, $f(2.3) = 2$, $f(2.5) = 3$, and $f(2.7) = 3$. Define $[A]$ to be the area of region A. Define region R_n, for each positive integer n, to be the region on the Cartesian plane which satisfies the inequality $f(|x|) + f(|y|) < n$. We pick an arbitrary point O on the perimeter of R_n, and mark every two units around the perimeter with another point. Region S_{nO} is defined by connecting these points in order.

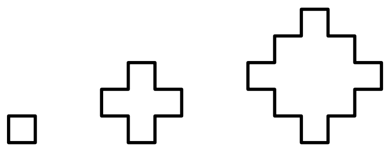
   a) Prove that the perimeter of R_n is always congruent to $4 \pmod 8$.

   b) Prove that $[S_{nO}]$ is constant for any O.

   c) Prove that $[R_n] + [S_{nO}] = (2n-1)^2$.

   (Lewis Chen)

Solution. By graphing out the given function in the problem, the function traces out a polygon composed of the union of all unit squares with centers of at most $n-1$ rectilinear distance from the origin. In particular, it creates a polygon with all sides of length 1, at right angles with each other and to the coordinate axes, as shown:



and so on.

a) Taking the projection to either the x-axis or y-axis, we get a length of $2n-1$. Since all sides are orthogonal to the coordinate axes, every length contributes either 0 or 1 to this amount. Since we can take the projection to the left, right, top, or bottom, the perimeter is $(1)(4)(2n-1) + 8n - 4 \equiv 4 \pmod 8$.

b) Since all sides are at either right angles or reflex right angles to each other, the sides alternate from parallel to perpendicular to the x-axis. Therefore, by marking every two units around, we select every other side, so we pick either all the parallel or all the perpendicular sides. Note that if we rotate the diagram by 90 degrees around the origin, we map the parallel sides to the perpendicular sides, so we may assume WLOG that point O lies on a segment parallel to the x-axis. Above the line $y = 0$, it forms an isosceles right triangle with height of $n-1$ to the hypotenuse; similarly, below the line $y = 0$ it forms a right triangle with height of $n-1$. Joining the two forms a parallelogram with a constant base and constant height, so the area $[S_{nO}]$ is invariant.

c) We compute the area of R_n first: a straightforward computation yields $[R_n] = 2(n-1)^2 + 2(n-1) = 2n^2 - 2n$. Additionally, $[S_{nO}] = n^2 + (n-1)^2 = 2n^2 - 2n + 1$, so $[R_n] + [S_{nO}] = 4n^2 - 4n + 1 = (2n-1)^2$, as claimed.

2. Winter Olympiad 2012

January 2012

1. In a 10 by 10 grid of dots, what is the maximum number of lines that can be drawn connecting two dots on the grid so that no two lines are parallel?

   (Aaron Lin)

   Solution. We are looking for the total number of distinct slopes of the lines connecting two points on the grid. This is equal to twice the number of positive slopes, plus two (to account for vertical and horizontal lines).

   Because slope is equal to $\frac{\Delta y}{\Delta x}$, all positive slopes can be written in the form $\frac{a}{b}$ for $1 \leq a, b \leq 9$. Each ordered pair (a, b) produces a distinct slope unless $\gcd(a, b) > 1$. By listing, we find that there are 55 distinct positive slopes. Thus, there are a total of $2(55)+2 = \boxed{112}$ different slopes, and thus a maximum of 112 non-parallel lines can be drawn.

2. If r_1, r_2, and r_3 are the solutions to the equation $x^3 - 5x^2 + 6x - 1 = 0$, then what is the value of $r_1^2 + r_2^2 + r_2^3$?

   (Eugene Chen)

   Solution. By Vieta's Formulas, $r_1+r_2+r_3 = 5$ and $r_1r_2+r_2r_3+r_3r_1 = 6$. Thus, $r_1^2+r_2^2+r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_2r_3 + r_3r_1) = 5^2 - 2(6) = \boxed{13}$.

3. The expression $\circ 1 \circ 2 \circ 3 \circ \cdots \circ 2012$ is written on a blackboard. Catherine places a $+$ sign or a $-$ sign into each blank. She then evaluates the expression, and finds the remainder when it is divided by 2012. How many possible values are there for this remainder?

   (Aaron Lin)

   Solution. Regardless of sign, each odd number contributes an odd amount to the sum and each even number contributes an even amount to the sum. Because there are 1006 odd numbers, the expression must evaluate to an even number, so all odd remainders cannot be achieved.

   Now, we show that each even number can be achieved. For all $1 \leq i \leq 1006$, take the sum $1+2+3+\cdots+(i-1)-i+(i+1)+\cdots+2012$. This sum is equal to $\frac{2012\cdot 2013}{2} - 2i \equiv 1006-2i \pmod{2012}$. But $503 - i$ takes all values from 1 to 1006 modulo 1006 for $1 \leq i \leq 1006$, so $1006 - 2i$ takes on all even values modulo 2012. Hence, all even remainders can be achieved, so the answer is $\boxed{1006}$.

4. Parallel lines ℓ_1 and ℓ_2 are drawn in a plane. Points $A_1, A_2, \ldots, A_n$ are chosen on ℓ_1, and points $B_1, B_2, \ldots, B_{n+1}$ are chosen on ℓ_2. All segments A_iB_j are drawn, such that $1 \leq i \leq n$ and $1 \leq j \leq n + 1$. Let the number of total intersections between these segments (not including endpoints) be denoted by Q. Given that no three segments are concurrent, besides at endpoints, prove that Q is divisible by 3.

   (Lewis Chen)

   Solution. Each set of four points $A_aA_bB_cB_d$ defines exactly one intersection point, and each intersection point is defined by a set of four points $A_aA_bB_cB_d$. Thus, there are $\binom{n}{2}\binom{n+1}{2}$ total intersections between these segments. This is equal to $\frac{(n-1)n^2(n+1)}{4}$. Because $n - 1, n, n + 1$ are three consecutive integers, one of them must be divisible by 3. It follows that $Q = \frac{(n-1)n^2(n+1)}{4}$ is divisible by 3.

5. In convex hexagon $ABCDEF$, $\angle A \cong \angle B$, $\angle C \cong \angle D$, and $\angle E \cong \angle F$. Prove that the perpendicular bisectors of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ pass through a common point.

   (Lewis Chen)

   Solution. Assume $\overline{AF}$, $\overline{BC}$, and $\overline{DE}$ are pairwise nonparallel. Denote the intersection of $\overleftrightarrow{AF}$ and $\overleftrightarrow{BC}$ by X, the intersection of $\overleftrightarrow{BC}$ and $\overleftrightarrow{DE}$ by Y, and the intersection of $\overleftrightarrow{DE}$ and $\overleftrightarrow{AF}$ by Z. Because $\angle A \cong \angle B$, it follows that $\angle XAB \cong \angle XBA$ and thus $\triangle XAB$ is isosceles. Then, the perpendicular bisector of $\overline{AB}$ is the angle bisector of $\angle AXB = \angle ZXY$. Similarly, the perpendicular bisectors of $\overline{CD}$ and $\overline{EF}$ are the angle bisectors of $\angle XYZ$ and $\angle YZX$, respectively. The angle bisectors of $\angle ZXY$, $\angle XYZ$, and $\angle YZX$ are concurrent at the incenter of $\triangle XYZ$, so the perpendicular bisectors of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ are concurrent in this case.

   If two of the sides are parallel, then assume, without loss of generality, that $\overline{AF} \parallel \overline{BC}$. Define Y and Z as they were defined in the previous part. Because $\overline{AZ} \parallel \overline{BY}$, quadrilateral $ABYZ$ is a trapezoid. Again, the perpendicular bisectors of $\overline{CD}$ and $\overline{EF}$ are the angle bisectors of $\angle BYZ$ and $\angle YZA$, respectively. Denote by P the midpoint of $\overline{YZ}$ and by Q the intersection of the angle bisectors of $\angle BYZ$ and $\angle YZA$. Now, because $\text{m}\angle AZY + \text{m}\angle BYZ = 180^\circ$, we have $\text{m}\angle QZY + \text{m}\angle QYZ = 90^\circ$, from which $\angle YQZ$ is right. It follows that Q lies on the circle with diameter $\overline{YZ}$, so $PY = PZ = PQ$. Thus, $\angle PQY \cong \angle QYP$, and because $\overline{YQ}$ bisects $\angle BYZ$, we have $\angle PQY \cong \angle BYQ$. Hence, $\overline{PQ} \parallel \overline{BY}$. Because P is the midpoint of $\overline{YZ}$, $\overline{PQ}$ must be the midline of trapezoid $ABYZ$. But the perpendicular bisector of $\overline{AB}$ is also the midline of trapezoid $ABYZ$, so Q lies on the perpendicular bisector of $\overline{AB}$ as desired.

6. The positive numbers a, b, c satisfy $4abc(a+b+c) = (a+b)^2(a+c)^2$. Prove that $a(a+b+c) = bc$.

   (Aaron Lin)

   Solution. Let $x = a(a+b+c)$ and $y = bc$. Then, $x+y = (a+b)(a+c)$. The given equation implies:

$$\begin{aligned} 4xy &= (x+y)^2 \\ x^2 - 2xy + y^2 &= 0 \\ (x-y)^2 &= 0 \\ x &= y. \end{aligned}$$

   Thus, $a(a+b+c) = bc$ as desired.

7. For how many positive integers $n \le 500$ is $n!$ divisible by 2^{n-2}?

   (Eugene Chen)

   Solution. By Legendre's Theorem, $v_p(n!) = \frac{n - s_p(n)}{p-1}$, where p is a prime, $v_p(n)$ is the exponent of the prime p that divides n, and $s_p(n)$ is the sum of the digits of n when written in base p.

   Choosing $p = 2$ yields $v_2(n!) = n - s_2(n)$. The given condition holds iff $v_2(n!) = n - s_2(n) \ge n - 2$, or $s_2(n) \le 2$. Thus, the sum of the digits of n in base 2 is 1 or 2. There are 9 positive integers n not greater than 500 such that the sum of the digits in the binary representation of n is 1, and 36 positive integers n not greater than 500 such that the sum of the digits in the binary representation of n is 2. The answer is $9 + 36 = \boxed{45}$.

The answers 44 and 45 were both accepted for this problem, as the case $n = 1$ can be argued to be either valid or invalid.

8. A convex 2012-gon $A_1A_2A_3 \dots A_{2012}$ has the property that for every integer $1 \le i \le 1006$, $\overline{A_iA_{i+1006}}$ partitions the polygon into two congruent regions. Show that for every pair of integers $1 \le j < k \le 1006$, quadrilateral $A_jA_kA_{j+1006}A_{k+1006}$ is a parallelogram.

   (Lewis Chen)

   Solution.

   Lemma. $\angle A_{i-1}A_iA_{i+1} \cong \angle A_{i+1005}A_{i+1006}A_{i+1007}$.

   Proof. Consider the partition through $A_{i+503}A_{i+1509}$. Regardless of whether the two resultant polygons are rotations or reflections of each other, $\angle A_{i-1}A_iA_{i+1}$ and $\angle A_{i+1005}A_{i+1006}A_{i+1007}$ are opposite $\overline{A_{i+503}A_{i+1509}}$, because if it is a reflection, then $\angle A_{i-1}A_iA_{i+1} \cong \angle A_{i+1007}A_{i+1006}A_{i+1005}$, and if it is a rotation, then $\angle A_{i-1}A_iA_{i+1} \cong m < A_{i+1005}A_{i+1006}A_{i+1007}$.

   Lemma. $A_iA_{i+1} = A_{i+1006}A_{i+1007}$.

   Proof. For the sake of contradiction, suppose the contrary; that is, suppose that $A_iA_{i+1} \neq A_{i+1006}A_{i+1007}$. Then, consider the partitioning line $\overline{A_{i+1}A_{i+1007}}$. It follows that the two resultant polygons must be reflections of each other, because our assumption is contradicted if they are rotations. Thus, $A_iA_{i+1} = A_{i+1}A_{i+2}$, and $A_{i+1006}A_{i+1007} = A_{i+1007}A_{i+1008}$. In particular, $A_{i+1}A_{i+2} \neq A_{i+1007}A_{i+1008}$.

   In general, from $A_kA_{k+1} \neq A_{k+1006}A_{k+1007}$, consider the partitioning line $\overline{A_{k+1}A_{k+1007}}$ to conclude that $A_kA_{k+1} = A_{k+1}A_{k+2}$, $A_{k+1006}A_{k+1007} = A_{k+1007}A_{k+1008}$, and thus $A_{k+1}A_{k+2} \neq A_{k+1007}A_{k+1008}$.

   After repeating this argument 1006 times, we may conclude that $A_iA_{i+1} = A_{i+1}A_{i+2} = A_{i+2}A_{i+3} = \dots = A_{i+1006}A_{1007}$, contradicting our assumption. Hence, the initial assumption must be false, and the lemma must be true.

   Lemma. $A_iA_j = A_{i+1006}A_{j+1006}$.

   Proof. Consider the polygons $A_iA_{i+1}A_{i+2} \dots A_j$ and $A_{i+1006}A_{i+1007}A_{i+1008} \dots A_{j+1006}$. By Lemma 2, they share at least all but one congruent corresponding side length. By Lemma 1, they share all congruent corresponding angles. Hence, they are congruent polygons, so the lemma is true.

   By Lemma 3, $A_iA_j = A_{i+1006}A_{j+1006}$ and $A_jA_{i+1006} = A_{j+1006}A_i$, so $A_iA_jA_{i+1006}A_{j+1006}$ is a parallelogram, as desired.

3. Winter Olympiad 2013

March 2013

1. Find the remainder when $2 + 4 + \cdots + 2014$ is divided by $1 + 3 + \cdots + 2013$. Justify your answer.

   (Evan Chen)

   Solution. Let $A = 2 + 4 + \cdots + 2014$ and $B = 1 + 3 + \cdots + 2013$. Then

   $$A - B = \underbrace{1 + 1 + \cdots + 1}_{1007\ 1's} = 1007.$$

   Other solutions may explicitly compute $A = 1007 \cdot 1008$ and $B = 1007^2$ to arrive at the same conclusion.

   5 points for any valid method or approach. Decent attempts may earn 2 points.

   Since $B > 1007$, the remainder is 1007.

   2 points for the numerical answer.

   Remark. A good solution should make the remark that $B > 1007$; however, a solution which does not mention this will not be penalized. The possible marks for this approach are 0, 2, 5, 7.

2. Square $\mathcal{S}$ has vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. Points P and Q are independently selected, uniformly at random, from the perimeter of $\mathcal{S}$. Determine, with proof, the probability that the slope of line PQ is positive.

   (Isabella Grabski)

   Solution 1. The answer is $\frac{1}{2}$.

   1 point for a correct numerical answer.

   Let $A = (1, 0)$, $B = (0, 1)$, $C = (-1, 0)$ and $D = (0, -1)$. Consider a point $P \in \overline{CD}$ (the other cases are analogous). We claim that even for a fixed P, the probability is $\frac{1}{2}$.

   1 point for this claim.

   Let P_x be the point on $\overline{AD}$ with PP_x parallel to the x-axis, and define P_y analogously. Then it is easy to see that the region where Q can lie is precisely the polygonal line P_yBAP_x.

   4 points for defining P_x and P_y and finding the locus of Q. Serious but unsuccessful attempts to use points P_x and P_y will be awarded 1 point.

   Since $P_yC = PC = P_xA$, it easy to see that the desired region is precisely one-half the perimeter of $\mathcal{S}$. Hence the claim holds, and the answer is therefore $\frac{1}{2}$.

   1 point for this finishing touch.

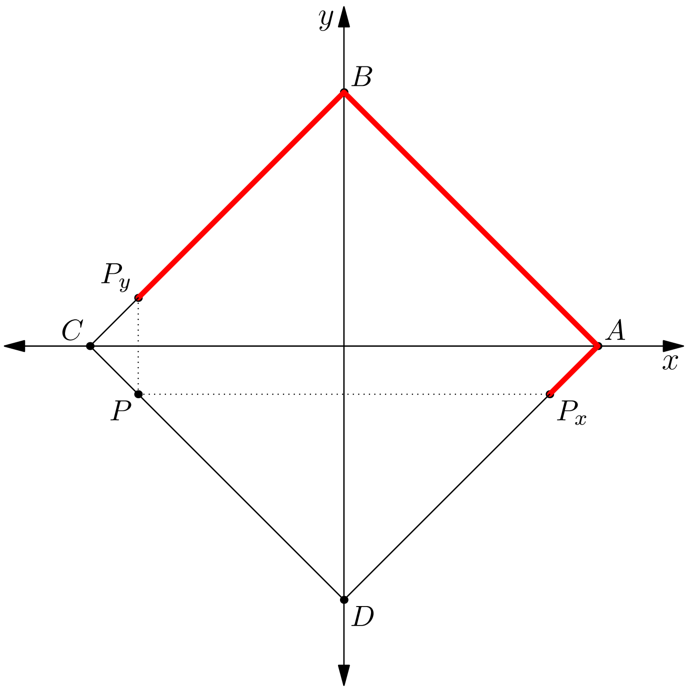


Figure VIII.1.: Desired region for problem 2, highlighted in red.

> **Remark.** In grading this approach, we have the rules $1+1+1 = 1+1+1+1 = 2$. Basically, if the locus Q is not successfully found in this approach, at most 2 points will be given. A score of 4 is strictly possible but seems extremely rare. The possible marks for this approach are $0, 1, 2, 5, 6, 7$.

Solution 2. We will show the answer is $\frac{1}{2}$.

> **1 point for the correct numerical answer.**

Remark that the probability that a line has either undefined or zero slope is 0. Therefore it suffices to consider lines with either positive or negative slopes.

> **1 point for this remark.**

Notice that by reflecting a line PQ over the x-axis we obtain a line $P'Q'$ whose slope is the negative of line PQ. We may ignore the cases where the slope is 0 or not defined. Hence, by symmetry, the answer is $\frac{1}{2}$.

> **5 points for finishing.**

> **Remark.** The possible marks for this approach are $0, 1, 2, 5, 6, 7$.

Note: It's important to note (although no points will be deducted for failing to do so) that symmetry is very different from having a $1-1$ correspondence. The real importance is to note that reflection preserves the length of segments; it is not sufficient to find a bijection from lines with positive slope to lines with negative slope, because the sets involved are infinite.

For the sake of example, consider the problem:

x is chosen randomly in the interval $[0, 1)$. What is the probability that $x \le \frac{1}{3}$?

It is easy to construct a bijection between $\left[0, \frac{1}{3}\right)$ and $\left[\frac{1}{3}, 1\right)$; namely, $x \mapsto 2x + \frac{1}{3}$. But this certainly does not imply the answer is $\frac{1}{2}$!

When we say we pick a point "uniformly and at random from the perimeter of $\mathcal{S}$", this is equivalent to saying that the probability a point is selected from some interval is proportional to the length of that interval. That's why the length-preserving property (which may be expressed more perversely as just "symmetry") plays a crucial role in this problem.

3. Let ABC be a triangle. Prove that there exists a unique point P for which one can find points D, E and F such that the quadrilaterals $APBF$, $BPCD$, $CPAE$, $EPFA$, $FPDB$, and $DPEC$ are all parallelograms.

   (Lewis Chen)

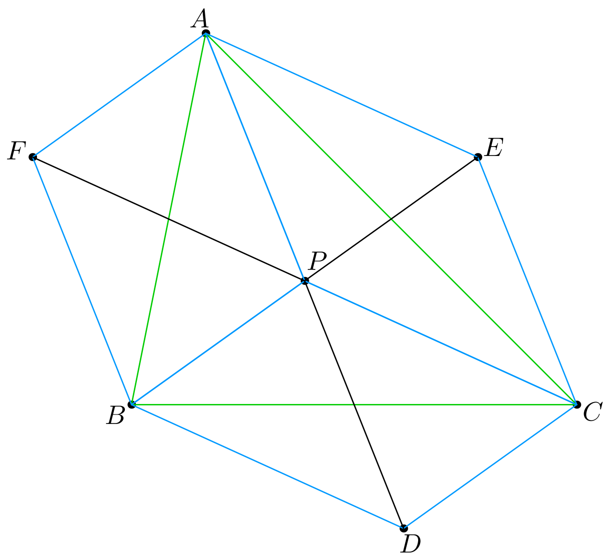


Figure VIII.2.: Parallelograms

Solution. We claim that the centroid of $\triangle ABC$ is the unique point with the property.

A generous 2 points for this claim. (It may be given implicitly later.)

First, suppose P, D, E, F have the desired property. Clearly, $\overline{CP} \parallel \overline{BD}$ and $\overline{FP} \parallel \overline{PD}$. This implies that C, P, F are collinear. Similarly, A, P, D and B, P, E are collinear.

2 points for this observation.

Now, $\overline{FP}$ bisects $\overline{AB}$ because $APBF$ is a parallelogram. Hence $\overline{CP}$ is a median of $\triangle ABC$. Similarly, $\overline{AP}$ and $\overline{BP}$ are medians, so P must be the centroid of $\triangle ABC$.

2 points for this observation.

Conversely, if P is the centroid, then select D, E, F to be the reflections of P over the midpoints of sides BC, CA, AB; this forces $APBF$, $BPCD$ and $CPAE$ to be parallelograms. Then $\overline{BP} \parallel \overline{FA}$ and $\overline{CP} \parallel \overline{EA}$, so $AFPE$ is a parallelogram. Similarly, $BDPF$ and $CEPD$ are parallelograms, as desired.

1 point for this direction.

Remark. If a solution begins by assuming P is the centroid, but fails to verify P is unique, it may still earn points for the correct observations. Basically, because both directions are essentially the same in the synthetic solution, a complete proof of one direction earns $2+2+2=6$ points. Completing the other direction earns the last point. The possible marks for this approach are $0, 2, 4, 6, 7$.

Solution 2. The solution is by vectors. Note the $WXYZ$ is a parallelogram if and only if

$$\overrightarrow{W}+\overrightarrow{Y}=\overrightarrow{X}+\overrightarrow{Z}.$$

1 point for any serious attempt to use vectors/analytic methods.

The condition is equivalent to finding $\overrightarrow{P}$ for which there exists $\overrightarrow{D}, \overrightarrow{E}, \overrightarrow{F}$ such that

$$\begin{aligned}
\overrightarrow{A}+\overrightarrow{B}&=\overrightarrow{P}+\overrightarrow{F}\\
\overrightarrow{B}+\overrightarrow{C}&=\overrightarrow{P}+\overrightarrow{D}\\
\overrightarrow{C}+\overrightarrow{A}&=\overrightarrow{P}+\overrightarrow{E}\\
\overrightarrow{E}+\overrightarrow{F}&=\overrightarrow{P}+\overrightarrow{A}\\
\overrightarrow{F}+\overrightarrow{E}&=\overrightarrow{P}+\overrightarrow{B}\\
\overrightarrow{D}+\overrightarrow{D}&=\overrightarrow{P}+\overrightarrow{C}
\end{aligned}$$

1 point for obtaining this system of equations.

Adding twice the sum of the first three equations with the following three, we obtain that

$$\overrightarrow{P}=\frac{\overrightarrow{A}+\overrightarrow{B}+\overrightarrow{C}}{3}$$

i.e. P is the centroid of $\triangle ABC$.

4 points for solving for deriving that P is the centroid (or just showing the explicit form above).

It is easy to verify this works.

1 point for this last remark.

Remark. It seems difficult to reach later stages of this solution without previous results. The possible marks for this approach are $0, 1, 2, 6, 7$.

4. Let $\mathcal{F}$ be the set of all 2013×2013 arrays whose entries are 0 and 1. A transformation $K : \mathcal{F} \to \mathcal{F}$ is defined as follows: for each entry a_{ij} in an array $A \in \mathcal{F}$, let S_{ij} denote the sum of all the entries of A sharing either a row or column (or both) with a_{ij}. Then a_{ij} is replaced by the remainder when S_{ij} is divided by two.

   Prove that for any $A \in \mathcal{F}$, $K(A) = K(K(A))$.

   (Aaron Lin)

Solution. We will show the result holds for any $n \times n$ grid, where n is odd.

> **1 point if this claim is made. A correct solution which does not explicitly note this, of course, will not be penalized.**

Let the entries of A be a_{ij}, $1 \le i, j \le n$. Let the entries of $K(A)$ be b_{ij}, and let the entries of $K(K(A))$ be c_{ij}.

By symmetry, it suffices to check that $c_{11} = b_{11}$.

> **1 point if this claim is made. Again, a correct solution which does not utilize this will still receive the 1 point.**

Now, let $R = a_{11} + a_{12} + \cdots + a_{1n}$ and $C = a_{11} + a_{21} + \cdots + a_{n1}$. Then,

$$\begin{aligned}
c_{11} &\equiv -b_{11} + \sum_{k=1}^{n} b_{k1} + \sum_{k=1}^{n} b_{1k} \pmod 2 \\
&= b_{11} + \sum_{k=2}^{n} b_{k1} + \sum_{k=2}^{n} b_{1k} \\
&= b_{11} + \sum_{k=2}^{n} \left(C + \sum_{j=2}^{n} a_{kj} \right) + \sum_{k=2}^{n} \left(R + \sum_{j=2}^{n} a_{jk} \right) \\
&= b_{11} + \sum_{k=2}^{n} \sum_{j=2}^{n} a_{kj} + \sum_{k=2}^{n} \sum_{j=2}^{n} a_{jk} + (n-1)C + (n-1)R \\
&= b_{11} + 2 \sum_{2 \le x, y \le n} a_{xy} + (n-1)C + (n-1)R \\
&\equiv b_{11} \pmod 2
\end{aligned}$$

which implies $c_{11} = b_{11}$ as desired.

> **5 points for this computation. 2 points may be awarded for a decent attempt with double summation.**

> **Remark.** In grading this approach, we have the following rules in addition: $1 + 1 = 1$, $1 + 2 = 2$, and $1 + 1 + 2 = 2$. Essentially any solution which does not successfully carry out the double summation earns at most 2 points, and any solution which fails to attempt double summation earns at most 1 point. On the other hand, any solution which correctly sums the expression implicitly earns the remaining points. The possible marks for this approach are $0, 1, 2, 7$.

5. In convex hexagon $AXBYCZ$, sides AX, BY and CZ are parallel to diagonals BC, XC and XY, respectively. Prove that $\triangle ABC$ and $\triangle XYZ$ have the same area.

   (Evan Chen)

Solution. Let $[\mathcal{P}]$ denote the area of a polygon $\mathcal{P}$.

The important claim is that if $\overline{KL} \parallel \overline{MN}$, then $[KLM] = [KLN]$. This is a simple consequence of the formula $A = \frac{1}{2}bh$.

> **1 point for this remark. No points will be deducted if this claim is cited as well-known.**

Then, we find that

$$\begin{aligned} [ABC] &= [XBC] \quad (\text{since } \overline{AX} \parallel \overline{BC}) \\ &= [XYC] \quad (\text{since } \overline{BY} \parallel \overline{XC}) \\ &= [XYZ] \quad (\text{since } \overline{CZ} \parallel \overline{XY}) \end{aligned}$$

as desired.

> **6 points for completing the solution.**

> **Remark.** The possible marks for this approach are 0, 1, 7.

6. A strictly increasing sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers is said to be [i]large[/i] if, for every real number L, there exists an integer n such that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} > L$. Do there exist large sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ such that the sequence $\{a_i + b_i\}_{i=1}^{\infty}$ is not large?

   (Lewis Chen)

Solution. The answer is yes.

> **1 point for correctly answering yes and some attempt at a construction. Solutions which claim no sequences exist earn 0 points.**

We present one of many possible constructions.

The idea is to select a_i and b_i such that $a_i + b_i = 2^i$, which will make $\{a_i + b_i\}_{i=1}^{\infty}$ not large.

Let the a_i and b_i alternate in runs, where one sequence increases by one at each step, as shown in the table below.

i	1	2	3	4	5	6	$\cdots$	16	17	18	$2^{16}-8$	
a_i	1	2	3	4	19	50	$\cdots$	$2^{16}-24$	$2^{16}-23$	$\cdots$	$2^{17}-48$	$\cdots$
b_i	1	2	5	12	13	14	$\cdots$	24	$2^{16}+23$	$\cdots$	$2^{2^{16}-8}-2^{17}+48$	$\cdots$
						length 12				length $2^{16}-24$		

Table VIII.1.: a_i and b_i

Formally, we define the sequences by $t_1 = 2$, $a_1 = b_1 = 1$, $a_2 = b_2 = 2$, and

$$t_{n+1} = t_n + \max\{a_{t_n}, b_{t_n}\}$$

$$a_{n+1} = \begin{cases} a_n + 1 & \text{if } \exists k : t_{2k-1} + 1 \le n \le t_{2k} \\ 2^{n+1} - b_n & \text{otherwise} \end{cases}$$

$$b_{n+1} = \begin{cases} b_n + 1 & \text{if } \exists k : t_{2k} + 1 \le n \le t_{2k+1} \\ 2^{n+1} - a_n & \text{otherwise.} \end{cases}$$

Note that

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} > \underbrace{\frac{1}{2n}+\cdots+\frac{1}{2n}}_{n \text{ terms}} = \frac{1}{2}.$$

This construction guarantees that

$$\sum_{i=t_{2k-1}+1}^{t_{2k}} \frac{1}{a_i} > \frac{1}{2}.$$

Similar equations hold for b_i. Therefore $\{a_i\}$ and $\{b_i\}$ are large. So we're done.

6 points for a valid construction. Serious but unsuccessful attempts with good ideas can earn 1 or 2 points. Essentially correct constructions with minor calculation errors earn 5 points.

Remark. In this approach, $1+1=1+2=2$. The possible marks for this approach are 0, 1, 2, 6, 7.

7. Let a, b, c be positive reals satisfying $a^3+b^3+c^3+abc=4$. Prove that

$$\frac{(5a^2+bc)^2}{(a+b)(a+c)}+\frac{(5b^2+ca)^2}{(b+c)(b+a)}+\frac{(5c^2+ab)^2}{(c+a)(c+b)} \ge \frac{(a^3+b^3+c^3+6)^2}{a+b+c}$$

and determine the cases of equality.

(Evan Chen)

Solution. The equality cases are $a=b=c=1$ and the cyclic permutations of

$$(a,b,c)=\left(\frac{2}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}}\right).$$

1 point for stating the correct equality cases (even without proof).

By the Cauchy-Schwarz inequality,

$$\sum_{\text{cyc}} \frac{(5a^3+abc)^2}{a^2(a+b)(a+c)} \ge \frac{\left(\sum_{\text{cyc}} 5a^3+abc\right)^2}{\sum_{\text{cyc}} a^2(a+b)(a+c)} = \frac{\left(5a^3+5b^3+5c^3+3abc\right)^2}{(a+b+c)(a^3+b^3+c^3+abc)}$$

which simplifies to the desired right-hand side.

3 points for solving the inequality. No points are awarded for just mentioning Cauchy.

Equality occurs if and only if

$$\frac{5a^3+abc}{a^2(a+b)(a+c)} = \frac{5b^3+abc}{b^2(b+c)(b+a)} = \frac{5c^3+abc}{c^2(c+a)(c+b)}$$

Multiplying by $abc(a+b)(b+c)(c+a)$ we observe this is equivalent to

$$bc(5a^2+bc)(b+c) = ca(5b^2+ca)(c+a) = ab(5c^2+ab)(a+b)$$

Let us assume without loss of generality that $c \ge \max\{a, b\}$. We now find that

$$\begin{aligned} 0 &= b(5a^2 + bc)(b + c) - a(5b^2 + ac)(a + c) \\ &= 5abc(a - b) + c(b^3 - a^3) + c^2(b^2 - a^2) \\ &= c(b - a)(-5ab + a^2 + ab + b^2 + c(a + b)) \\ &= c(b - a)(a^2 + b^2 + c(a + b) - 4ab) \end{aligned}$$

But,

$$a^2 + b^2 + ca + cb - 4ab \ge 2(a^2 + b^2) - 4ab \ge 0$$

with equality only when $a = b = c$. This forces $a = b$; otherwise the two factors are both nonzero. Now, if we set $t = \frac{c}{a} = \frac{c}{b}$ we find that

$$0 = t(5 + t)(t + 1) - (5t^2 + 1)(2) = t^3 - 4t^2 + 5t - 2 = (t - 2)(t - 1)^2$$

which gives the equality cases claimed above.

3 points for correctly establishing the equality cases, as above. 2 points can be given if there are minor calculation errors; 1 point for significant progress with good ideas (e.g. breaking symmetry).

Remark. The real difficulty in this problem is not proving the inequality itself but finding and proving the additional equality cases. A solution which does not note the unusual equality case is therefore worth 0 or 3 points. The possible marks for a solution with the correct equality cases are 1, 4, 5, 6, 7.

8. For a finite set X define

$$S(X) = \sum_{x \in X} x \text{ and } P(x) = \prod_{x \in X} x.$$

Let A and B be two finite sets of positive integers such that $|A| = |B|$, $P(A) = P(B)$ and $S(A) \neq S(B)$. Suppose for any $n \in A \cup B$ and prime p dividing n, we have $p^{36} \mid n$ and $p^{37} \nmid n$. Prove that

$$|S(A) - S(B)| > 1.9 \cdot 10^6.$$

(Evan Chen)

Solution. Let $A = \left\{a_1^{36}, a_2^{36}, \cdots, a_n^{36}\right\}$ and $B = \left\{b_1^{36}, b_2^{36}, \cdots, b_n^{36}\right\}$. Notice that $a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_n$ and the a_i, b_i are squarefree.

The crucial component of the solution is the claim

Claim. For any prime p such that $p - 1 \mid 36$, we have $S(A) \equiv S(B) \pmod{p}$.

2 points if this claim is made.

Proof. Let $A_p = \{a \in A \mid p \text{ divides } a\}$ and define B_p analogously. The condition that the a_i and b_i are squarefree, together with $P(A) = P(B)$, imply that $|A_p| = |B_p|$.

1 point if it is noted that $|A_p| = |B_p|$, even outside the context of this claim.

Now by Fermat's Little Theorem, we see that

$$a^{p-1} \equiv \begin{cases} 1 & a \not\equiv 0 \pmod p \\ 0 & a \equiv 0 \pmod p \end{cases}.$$

So $S(A) \equiv n - |A_p| \pmod p, S(B) \equiv n - |B_p| \pmod p \implies S(A) \equiv S(B) \pmod p$. ■

3 points for completing this proof.

Now $S(A) - S(B) \equiv 0 \pmod p$ for $p \in \{2, 3, 5, 7, 13, 19, 37\}$. Hence, $S(A) - S(B)$ is divisible by

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 = 1919190 > 1.9 \cdot 10^6$$

which implies the conclusion upon remarking that $S(A) - S(B) \neq 0$.

1 point for this finishing touch.

Remark. In this problem, the rule $2 + 1 = 2$ applies. The possible marks for this approach are $0, 1, 2, 6, 7$.

4. Winter Olympiad 2014

January 2014

1. Find, with proof, all real numbers x satisfying $x = 2(2(2(2(2x-1)-1)-1)-1)-1$.

   (Evan Chen)

   Solution 1. Define $f(x) = 2x-1$. Because the equation is linear, it has at most one solution.

   5 points for this claim. 1 point for trying to consider this function in a nontrivial way.

   Because 1 is a fixed point (i.e. $f(1) = 1$), $f(f(f(f(f(1))))) = 1$ implies $x = 1$ is a solution, and hence the only one.

   2 points for this correct answer.

   Remark. Here $1 + 2 = 2$. The possible marks for this approach are 0, 1, 2, 5, 7.

   Solution 2. Expanding, the right-hand side is seen to equal $32x - 31$.

   5 points for expanding correctly. 1 point can be given for a failed attempt at expanding.

   Setting this equal to x gives $32x - 31$, or $x = 1$.

   2 points for this correct answer.

   Remark. The combination $1 + 2$ seems unrealizable. The possible marks for this approach are 0, 1, 2, 7.

2. Determine, with proof, the smallest positive integer c such that for any positive integer n, the decimal representation of the number $c^n + 2014$ has digits all less than 5.

   (Evan Chen)

   Solution. The answer is $c = 10$.

   1 point for this answer.

   First, we will prove $c \geq 10$ is necessary to work. For $1 \leq c \leq 5$, $n = 1$ gives the numbers 2015, 2016, $\ldots$, 2019, none of which work. On the other hand, for $6 \leq c \leq 9$, $n = 2$ gives the numbers 2050, 2063, 2078, 2095, none of which work.

   3 points for proving $c \geq 10$ is necessary.

Next, we will prove $c = 10$ is a working example for all n. When $1 \le n \le 3$, we have 2024, 2114 and 3014. When $n \ge 4$, we find that

$$10^n + 2014 = 1\underbrace{000\ldots000}_{n-4\text{ zeros}}2014$$

which also works. This shows that $c = 10$ is the answer.

3 points for proving $c = 10$ works. Deduct 2 points for failing to acknowledge the case $1 \le n \le 3$. Do not award any points for merely claiming $c = 10$ "clearly works".

Remark. In grading this approach, $1 + 3 = 3$, but $1 + 3 + 1 = 5$. The combination $3 + 3 = 6$ seems very hard to achieve. The possible marks for this approach are 0, 1, 3, 5, 7.

Note. In any optimization problem, there are always two parts. The first is to establish a bound – in this case, we proved $c \ge 10$ by exhausting the smaller cases. The second step is to prove that the bound is tight – in this case, we had to check $c = 10$ actually worked. We hope this problem was an illustrative/instructive example for what it means to rigorously show a value is minimal or maximal.

3. The numbers $1, 2, \ldots, 10$ are written on a board. Every minute, one can select three numbers a, b, c on the board, erase them, and write $\sqrt{a^2 + b^2 + c^2}$ in their place. This process continues until no more numbers can be erased. What is the largest possible number that can remain on the board at this point?

   (Evan Chen)

Solution. The key observation is that the sum of the squares of all numbers is preserved in each step.

3 points for this step. Deduct 2 points if assumptions are made about the order in which the numbers are taken.

Because we erase three numbers and replace it with one number in each operation, there will be two positive numbers in the end. The sum of the squares of the 10 original numbers is $1^2 + 2^2 + \cdots + 10^2 = 385$. Because the minimum possible value for one of the final two numbers is 1, the maximum possible value for the other is $\sqrt{385 - 1^2} = \sqrt{384} = 8\sqrt{6}$.

1 point for noting two numbers remain, and 1 point for noting is at least one. 2 points for the correct final answer.

Remark. In grading this problem, we have three rules in addition.

- Many teams do not note the SOS invariant. Apply $1+2 = 1+1+2 = 2$ and $1+1 = 1$. In other words, such solutions earn at most 2.
- Some solutions make observations about the sum of squares but do so by making assumptions about what numbers are taking, leading to $1/3$ in the first part. These solutions earn at most 4 points; we apply $1+1+1+2 = 4$ to them. Moreover, $1+1 = 1+1+1 = 1$.
- In rare cases, a solution may note the SOS invariant but fail to yield any final answer. I do not recall seeing any such solution, but apply $3+1 = 3+1+1 = 3$ here. Graders may be more generous if an incorrect final answer was clearly due to a simple arithmetic error.

The possible marks for this problem are 0, 1, 2, 3, 4, 5, 6, 7.

4. Prove that there exist integers a, b, c with $1 \le a < b < c \le 25$ and

$$S(a^6 + 2014) = S(b^6 + 2014) = S(c^6 + 2014)$$

where $S(n)$ denotes the sum of the decimal digits of n.

(Evan Chen)

Solution. It is not hard to show that $25^6 + 2014 < 500{,}000{,}000$ (say by $25^6 = 625^3 < 700^3 = 343 \cdot 10^6$). We claim one can select a, b, c not divisible by three.

2 point for claiming this is possible with a, b, c not divisible by three.

There are 17 possible choices of a, b, c. Now $x^6 + 2014 \equiv 8 \pmod 9$ for each $x = a, b, c$ by Euler's Theorem. Hence the possible values of $S(x^6 + 2014) \equiv 8 \pmod 9$ are $\{8, 15, \dots, 71\}$ (using the bound above).

2 points for considering modulo 9 and using $S(x) \equiv x \pmod 9$.

There are eight such values, hence three of them coincide by Pigeonhole.

3 points for this finishing remark, along with the bound. 1 point can be awarded for mentioning Pigeonhole.

Remark. The 1 point is not additive, so $2+1 = 2$ and $2+2+1 = 4$. All other marks are additive. The 3 seems hard to obtain without the previous steps. The possible marks for this problem are 0, 1, 2, 3, 4, 7.

Solution 2. There are several other triples which can be explicitly shown to work, for example $(1, 4, 10)$ and $(5, 7, 8)$.

2 points for stating a triple and 5 points for showing it works.

Remark. The possible marks for this approach are 0, 2, 7.

5. Let ABC be an acute triangle with orthocenter H and let M be the midpoint of $\overline{BC}$. (The *orthocenter* is the point at the intersection of the three altitudes.) Denote by ω_B the circle passing through B, H, and M, and denote by ω_C the circle passing through C, H, and M. Lines AB and AC meet ω_B and ω_C again at P and Q, respectively. Rays PH and QH meet ω_C and ω_B again at R and S, respectively. Show that $\triangle BRS$ and $\triangle CRS$ have the same area.

   (Aaron Lin)

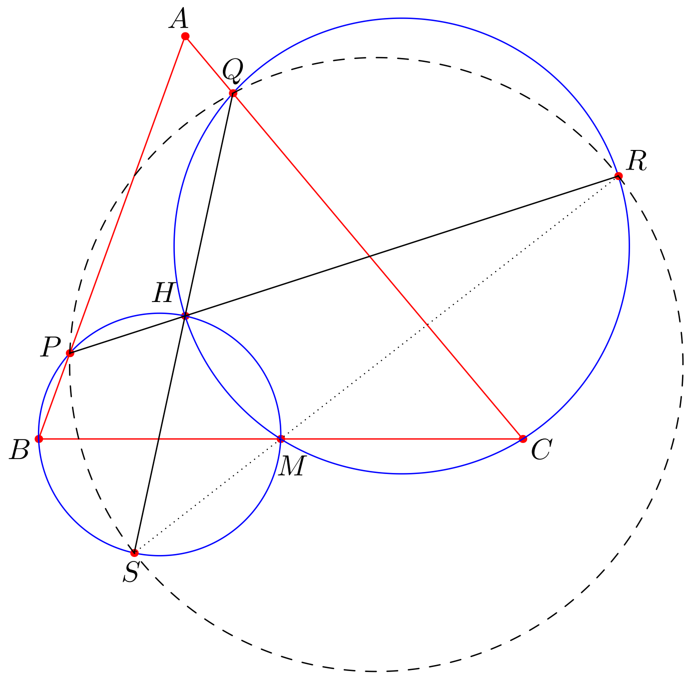


Figure VIII.3.: The points R, M, S are collinear.

Solution. In what follows all angles are directed modulo π.

It is easy to see that $\measuredangle ABH = \measuredangle HCA$ (they are $90 - \measuredangle A$). Hence

$$\measuredangle PSQ = \measuredangle PSH = \measuredangle PBH = \measuredangle ABH = \measuredangle HCA = \measuredangle HCQ = \measuredangle HRQ = \measuredangle PRQ.$$

Therefore $PQRS$ is cyclic.

2 points for proving this. A claim without proof gets 1 point.

Next, we claim M, R, S are collinear. This follows from

$$\measuredangle HMS = \measuredangle HPS = \measuredangle RPS = \measuredangle RQS = \measuredangle RQH = \measuredangle HMR.$$

3 points for proving this. A claim without proof gets 1 point.

Now note that triangles BMR and CMR have the same area, as do BMS and CMS (since M is the midpoint of $\overline{BC}$). Hence triangles BRS and CRS have the same area as desired.

2 points for this with all other steps.

Remark. In this problem, $1+1=1$, $2+1=2$, but $1+3=3$; i.e. reducing the problem to $PQRS$ cyclic earns three points. Failing to account for configuration issues leads to a 1-point deduction on full solutions. The possible marks for this approach are 0, 1, 2, 5, 6 (configuration dependencies), 7.

6. Let $\varphi(k)$ denote the numbers of positive integers less than or equal to k and relatively prime to k. Prove that for some positive integer n,

$$\varphi(2n-1) + \varphi(2n+1) < \frac{1}{1000}\varphi(2n).$$

(Evan Chen) **Note.** This is not the version of the problem that appeared on the contest. The contest version erroneously asks a contestant to prove that $\varphi(2n-1) + \varphi(2n+1) > 2\varphi(2n)$.

Solution. Let $p_1, p_2, \ldots$ be the sequence of odd prime numbers. It is well known that $\prod_i \left(1 - p_i^{-1}\right)$ tends to zero. That means we can find indices k and ℓ for which

$$\prod_{i=k+1}^{\ell} \left(1 - p_i^{-1}\right) < \prod_{i=1}^{k} \left(1 - p_i^{-1}\right) < \varepsilon$$

for any $\varepsilon > 0$. By the Chinese Remainder Theorem, we can find several n such that $2n-1 \equiv 0 \pmod{p_i}$ for each $i = 1, 2, \ldots, k$ and $2n+1 \equiv 0 \pmod{p_i}$ for each $i = k+1, k+2, \ldots, \ell$, and by Dirichlet's Theorem, we may select such an n prime (note that $n \not\equiv 0 \pmod{p}_i$ for any of these i).

Putting such an n, the right-hand side is merely $\frac{1}{1000}n$ while the left is less than

$$(2n-1)\prod_{i=1}^{k} \left(1 - p_i^{-1}\right) + (2n+1)\prod_{i=k+1}^{\ell} \left(1 - p_i^{-1}\right) < 4n\varepsilon$$

which is less than $\frac{1}{1000}n$ when ε is sufficiently small.

7. Let ABC be a triangle and let Q be a point such that $\overline{AB} \perp \overline{QB}$ and $\overline{AC} \perp \overline{QC}$. A circle with center I is inscribed in $\triangle ABC$, and is tangent to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ at points D, E, and F, respectively. If ray QI intersects $\overline{EF}$ at P, prove that $\overline{DP} \perp \overline{EF}$.

(Aaron Lin)

Solution. Let ray QP meet the cirumcircle of ABC at X. First, we remark that X is concyclic with quadrilateral $AFIE$. Indeed,

$$\angle AFI = 90^\circ = \angle AXQ = \angle AXI.$$

1 point for this claim.

Consider an inversion through the incircle. (It is expected many students will use similar triangles to phrase the following steps instead of inversion. We choose to present the inversive solution because it is more natural.) It sends points A, B, C to A^*, B^*, C^*, the midpoints of $\overline{EF}$, $\overline{FE}$, and $\overline{FD}$. The circumcircle of $AFIE$ is sent to line EF, and consequently because I, P, X are collinear we see that X maps to P.

1 points for attempting to use similarity in this fashion, or inverting through the incircle. No points for just mentioning inversion.

Because A, B, C, P are concyclic, so are A^*, B^*, C^* and P.

2 points for this observation, through inversion or otherwise.

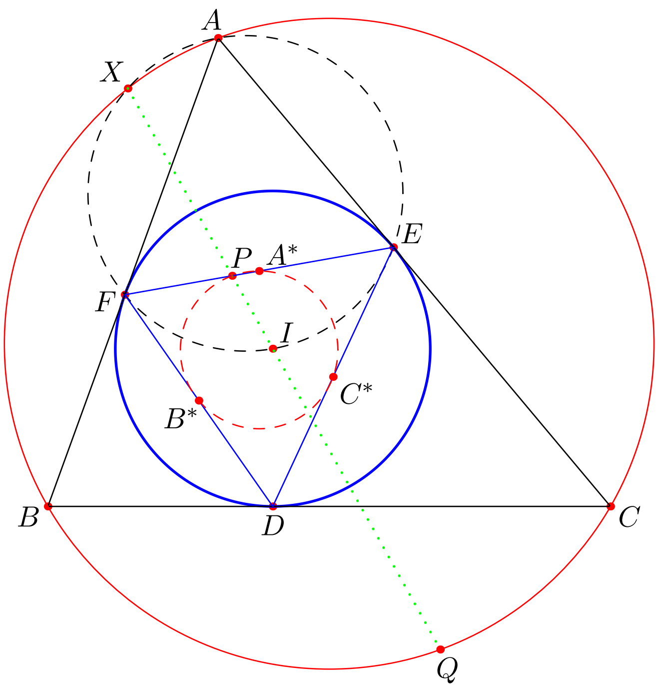


Figure VIII.4.: P lies on the nine-point circle of triangle DEF.

Now the circumcircle of $A^*B^*C^*$ is the nine-point circle of $\triangle DEF$, so $\overline{DP} \perp \overline{EF}$ as desired.

3 points for finishing.

Remark. In grading this approach, $1+1=1$ and $1+2=1+1+2=2$. Basically, incomplete solutions earn at most 2 points. The possible marks for this approach are 0, 1, 2, 7.

8. Define the function $\xi : \mathbb{Z}^2 \to \mathbb{Z}$ by $\xi(n,k) = 1$ when $n \le k$ and $\xi(n,k) = -1$ when $n > k$, and construct the polynomial

$$P(x_1, \dots, x_{1000}) = \prod_{n=1}^{1000} \left(\sum_{k=1}^{1000} \xi(n,k) x_k \right).$$

(a) Determine the coefficient of $x_1 x_2 \dots x_{1000}$ in P.

(b) Show that if $x_1, x_2, \dots, x_{1000} \in \{-1, 1\}$ then $P(x_1, x_2, \dots, x_{1000}) = 0$.

(Evan Chen)

Solution. For part (a), the answer is zero.

Let S_{1000} denote the set of permutations on 1000 elements. For a permutation π of $\eta(\pi)$ denote the set

$$\eta(\pi) = \left\{ 1 \le y \le 1000 \mid y \ge \pi^{-1}(y) \right\}.$$

In particular, $1000 \in \eta(\pi)$. It is then easy to see that the desired coefficient is just

$$\sum_{\pi \in S_{1000}} \left((-1)^{|\eta(\pi)|} \right).$$

Define $\Xi : S_{1000} \to S_{1000}$ as follows. If $\pi \in S_{1000}$ and $\Xi(\pi) = \sigma$, then

$$\sigma(n) = \begin{cases} 1000 & \text{if } \pi(1001-n) = 1000 \\ 1000 - \pi(1001-n) & \text{otherwise} \end{cases}.$$

This is clearly a bijection. Note also that $\sigma(1001 - \pi^{-1}(y)) = 1000 - y$.

1 point for this setup and attempting to construct a bijection.

Now observe that for any $1 \le y \le 999$, we have

$$\begin{aligned} y \in \eta(\pi) &\iff y \ge \pi^{-1}(y) \\ &\iff y > \pi^{-1}(y) - 1 \\ &\iff 1000 - y > 1001 - \pi^{-1}(y) \\ &= \sigma^{-1}(1000-y) \\ \iff 1000 - y \notin\in \eta(\sigma) \end{aligned}$$

Hence if $\Xi(\pi) = \sigma$, then $\eta(\pi) \cap \eta(\sigma) = \{1000\}$ and $\eta(\pi) \cup \eta(\sigma) = \{1, 2, \dots, 1000\}$. This now implies the sum is zero, because then $|\eta(\pi)|$ and $|\eta(\sigma)|$ have opposite parities.

2 points for completing the bijection and finishing the problem. No partial credit in this part.

Remark. No points are awarded for guessing the correct answer or for only bijection terms to permutations. The possible marks for this part are 0, 1, 3.

The second part is a consequence of the so-called Discrete Intermediate Value Theorem. Define

$$S_n = \frac{1}{2} \sum_{k=1}^{1000} \xi(n,k) x_k$$

for $n = 0, 1, 2, \dots, 1000$, and each is an integer. Observe that

$$|S_{n+1} - S_n| = 1.$$

for any $n = 0, 1, 2, \dots, 999$. On the other hand $S_{1000} = -S_0$.

1 point for both of these observations.

If $S_0 = S_{1000} = 0$ then the conclusion is clear. Otherwise, there must be some intermediate index k with $S_k = 0$, finishing the problem.

3 points for applying discrete IVT successfully to yield the conclusion. 1 point can be awarded for mentioning discrete IVT.

Remark. Here $1 + 1 = 1$. The possible marks for this part are 0, 1, 4.

Remark. In grading this problem, a partial mark in one part and a full mark in the other are not additive. The possible marks for this approach are 0, 1, 3, 4, 7.

Note. Part (a) actually follows from part (b) using a contradiction of the **combinatorial nullstellensatz**. Here is an adopted proof. Assume for contradiction that the coefficient in part (a) is actually $c \neq 0$, and let $\overline{P}$ be the polynomial formed by replacing every instance of x_i^2 with 1 for every value of i. Now, $\overline{P}$ must have the term $cx_1x_2 \dots x_{1000}$, as any reduced terms have degree strictly less than 1000 and so cannot cancel it. The resulting polynomial unfortunately, has degree at most 1 in any x_i. A simple induction shows that P must be the zero polynomial, contradicting the fact that $c \neq 0$. The conclusion thus follows.

Making only minor modifications to the above proof, one can derive the theorem of Alon.

> Let f be a polynomial of degree $t_1 + \cdots + t_n$ with coefficients in some field F. Suppose $S_1, S_2, \dots, S_n$ are nonempty subsets of F such that $|S_i| \geq t_i + 1$ for all i. Then there exists $s_1 \in S_1$, $s_2 \in S_2$, $\dots$, $s_n \in S_n$ for which
>
> $$f(s_1, s_2, \dots, s_n) \neq 0$$
>
> as long as the coefficient of $x_1^{t_1} x_2^{t_2} \dots x_n^{t_n}$ is nonzero.

The above deduction is the special case where $F = \mathbb{R}$ and $S_1 = S_2 = \cdots = S_{1000} = \{-1, 1\}$.