2000

Geometry Problems

Selected from Senior A-Level Papers in the International Mathematics Tournament of the Towns

1. We are given convex quadrilateral $ABCD$. Each of its sides is divided into n line segments of equal length. The points of division of side AB are connected with the points of division of side CD by non-intersecting line segments (which we call the first set of straight lines), and the points of division of side BC are connected with the points of division of side DA by non-intersecting line segments (which we call the second set of straight lines). This forms n^2 smaller quadrilaterals. From these we choose n quadrilaterals in such a way that any two are at least divided by one line from the first set and one line from the second set. Prove that the sum of the areas of these chosen quadrilaterals is equal to $\frac{1}{n}$ of the area of $ABCD$.

2. On sides AB, BC and CA of triangle ABC are located points P, M and K, respectively, so that AM, BK and CP intersect in one point and the sum of the vectors AM, BK and CP equals 0. Prove that K, M and P are midpoints of the sides of triangle ABC on which they are located.

3. On sides CB and CD of square $ABCD$ are chosen points M and K so that the perimeter of triangle CMK equals double the side of the square. Find $\angle MAK$.

4. Thc centre O of the circmcircle of triangle ABC lies inside the triangle. Perpendiculars are dropped from O to the sides. When produced beyond the sides they thought they meet the circuimcircle at points K, M and P. Prove that $OK + OM + OP = OI$, where I is the centre of the inscribed, circle of ABC.

5. $ABCDEF$ is a convex hexagon with AB parallel to CF, CD parallel to BE and EF parallel to AD. Prove that the areas of triangles ACE and BDF are equal.

6. M is a point inside a convex quadrilateral with perimeter of length P and diagonals of lengths D_1 and D_2. Prove that the sum of the distances from M to the vertices of the quadrilateral is not greater than $P + D_1 + D_2$.

7. In triangle ABC, AH is an altitude with H on BC, and BE is the bisector of $\angle ABC$, with E on AC. If $\angle BEA = 45°$, prove that $\angle EHC = 45°$.

8. $ABCD$ is a parallelogram which is not a rhombus. The bisector of $\angle BAD$ intersects the lines BC and CD at K and L respectively. Prove that the circumcentre of triangle CKL lies on the circumcircle of triangle BCD.

9. $ABCD$ is a quadrilateral with an incircle of radius R. AD is parallel to BC and perpendicular to AB. M is the point of intersection of AC and BD. Determine the area of triangle DCM in terms of R.

10. Perpendiculars are drawn from an interior point M of the equilateral triangle to its sides, intersecting them at D, E and F respectively. Find the locus of all points M such that DEF is a right triangle.

11. $ABCD$ is a square of side 1. Two lines are drawn from A, one intersecting BC and the other intersecting CD. The angle between these lines is θ. Perpendiculars from B and D are dropped to each of the two lines. Find the area of the quadrilateral determined by the four feet of perpendiculars in terms of θ.

12. From a point inside a triangle, perpendiculars are dropped to each altitude. Each of the segments on the altitudes from the vertices to the feet of these perpendiculars are of equal length. Prove that this common length is equal to the diameter of the incircle of the triangle.

13. A line cuts the diagonals, lateral sides and extended bases of a trapezoid in six points, forming five equal segments. Determine the ratio of the lengths of the bases.

14. M is a point inside a rectangle $ABCD$. Prove that the area of the rectangle is less than or equal to $AM \cdot CM + BM \cdot DM$.

15. For an interior point M of triangle ABC, $\angle BMC = 90° + \frac{1}{2}\angle BAC$. If the circumcentre of triangle BMC lies on the line AM, prove that M is the incentre of triangle ABC.

16. Choose a point A inside a circle of radius R. Construct a pair of perpendicular lines through A. Then rotate these lines through the same angle $\theta < \frac{\pi}{2}$ about A. The figure formed inside the circle, as the lines move from their initial to their final positions, is in the form of a cross with its centre at A. Find the area of this cross.

17. Let $ABCD$ be a trapezium with $AC = BC$. Let H be the midpoint of the base AB and let ℓ be a line passing through H. Let ℓ meet AD at P and BD at Q. Prove that $\angle ACP$ and $\angle QCB$ are either equal or have a sum of $180°$.

18. ABC is an equilateral triangle. P is the midpoint of arc AC of its circumcircle, and M is an arbitrary point on the arc. N is the midpoint of BM and K is the foot of perpendicular from P to MC. Prove that ANK is an equilateral triangle.

19. ABC is a triangle with $AC = BC$. D is a point on AB such that the inradius of triangle ACD is equal to the radius of the circle tangent to the segment BD and the extensions of CB and CD. Prove that this radius equals one fourth of the two equal altitudes of triangle ABC.

20. The chord MN on a circle is fixed. For every diameter AB of the circle, consider the point of intersection C of the lines AM and BN, and the line passing through C perpendicular to AB. Prove that all these lines pass through a fixed point.

21. $ABCD$ is a cyclic quadrilateral with $BC = CD$. Prove that its area is $\frac{1}{2}AC^2 \sin A$.

22. M is the centroid of triangle ABC. A $120°$ counterclockwise rotation about M takes B into the point P. A $240°$ counterclockwise rotation about M takes C into the point Q. Prove that either APQ is an equilateral triangle, or the points A, P and Q coincide.

23. The bisector of $\angle A$ of triangle ABC intersects its circumcircle at the point D. P is the point symmetric to the incentre of ABC with respect to the midpoint of the side BC, and M is second point of intersection of PD with the circumcircle. Prove that one of the distances AM, BM and CM is equal to the sum of two other distances.

24. In trigngle ABC, $BC = a$ and $CA = b$. M and N are the midpoints of BC and CA respectively. O is the centre of the square constructed outside triangle ABC, with AB as one of its sides. Determine in terms of a and b the maximum value of $OM + ON$.

25. From a point C outside a circle, two tangents are drawn touching the circle at A and B respectively. Consider the curved "triangle" bounded by the minor arc AB and the segments CA and CB. Prove that the length of any segment inside this triangle is not greater than the length of $CA = CB$.

26. $ABCDEF$ is a convex hexagon where ACE and BDF are equilateral triangles, not necessarily congruent to each other. $B'D'F'$ is obtained from BDF by a parallel shift. If $AB'CD'EF'$ is still convex, prove that its area is equal to that of $ABCDEF$.

27. D is a point on the side BC of triangle ABC. The third common tangent, besides AD and BC, of the incircles of triangles BAD and CAD cuts AD at K. Prove that the length AK is independent of the position of D.

28. The external bisectors of angles A and C of a convex quadrilateral $ABCD$ meet at K, and those of angles B and D at L. The extensions of AB and DC meet at P, and those of AD and BC at Q. The external bisectors of angles P and Q meet at M. Prove that K, L and M are collinear.

29. The median AD of triangle ABC intersects its incircle at the points X and Y. O is the incentre of ABC. If $AC = AB + AD$, determine $\angle XOY$.

30. $ABCD$ is a trapezium with AD parallel to BC. The point G of intersection of the diagonals AC and BD is outside the circles with diameters AB and CD respectively. Prove that all four tangents to the two circles from G are equal in length.

31. P is a point on the plane of a convex quadrilateral $ABCD$. The bisectors of $\angle APB, \angle BPC$, $\angle CPD$ and $\angle DPA$ meet AB, BC, CD and DA at K, L, M and N respectively.

    (a) Find a point P such that $KLMN$ is a parallelogram.

    (b) Find the locus of all such points P.

32. A captain finds his way to Treasure Island, which is circular in shape. He knows that there is treasure buried at the midpoint T of the segment joining the orthocentres of triangles ABC and DEF, where A, B, C, D, E and F are six palm trees on the shore of the island, not necessarily in this cyclic order. He finds the trees all right, but does not know which tree is denoted by which letter. What is the maximum number of points at which the captain has to dig in order to recover the treasure?

33. In triangle $ABC, AB = AC$ and $\angle BAC = \alpha$. D is the point on AB such that $AD = \frac{1}{n}AB$. $P_1, P_2, \ldots, P_{n-1}$ are points on BC which divide it into n equal segments. Find

$$\angle AP_1D + \angle AP_2D + \cdots + \angle AP_{n-1}D$$

    (a) if $n = 3$;

    (b) for any integer $n > 2$.

34. Let A', B', C', D', E' and F' be the midpoints of the sides AB, BC, CD, DE, EF and FA of an arbitrary convex hexagon $ABCDEF$ respectively. Express its area in terms of the areas of the triangles ABC', BCD', CDE', DEF', EFA' and FAB'.

35. D is the point on BC and E is the point on CA such that AD and BE are the bisectors of $\angle A$ and $\angle B$ of triangle ABC. If DE is the bisector of $\angle ADC$, find $\angle A$.

36. CM and BN are medians of triangle ABC. Points P and Q are on sides AB and AC respectively, such that the bisector of $\angle ACB$ also bisects $\angle MCP$, and the bisector of $\angle ABC$ also bisects $\angle NBQ$. If $AP = AQ$, does it follow that ABC is isosceles?

37. $ABCD$ is a convex quadrilateral. M is a point inside $ABCD$ such that $AM = MB, CM = MD$ and $\angle AMB = \angle CMD = 120°$. Prove that there exists a point N such that both BNC and DNA are equilateral triangles.

38. The segment AB is parallel to the line joining the centres of two equal circles. It intersects them, and all the points of intersection lie between A and B. From A and B, tangents are drawn to the circles nearer to them, respectively. The quadrilateral formed by the four tangents contains both circles. Prove that this quadrilateral has an incircle.

39. O is the centre of a circle passing through the vertices of a convex quadrangle $ABCD$. The circumcircles of triangles ABO and CDO intersect each other again at F. Prove that the circumcircle of triangle AFD passes through the point of intersection of AC and BD.

40. The excircle of triangle ABC opposite B touches CA at K, and the excircle opposite A touches CB at L. Prove that the line pasing through the midpoints of KL and AB

    (a) bisects the perimeter of triangle ABC;

    (b) is parallel to the bisecter of $\angle BCA$.

Inversion

On the Euclidean plane is a circle ω with centre o and radius r. We define a transformation as follow. Let P be any point other than O. Then the image of P is the point P' such that P', P and O are collinear, O is not between P' and P, and $OP \cdot OP' = r^2$. Note that points on ω are the only fixed points. The image of a point inside ω is a point outside, and vice versa. This is why we call this transformation an inversion with respect to ω.

In order for inversion to be a genuine transformation, we must provide the point O with an image. To do so, we extend the Eucliean plane by adding an ideal point I, which lies on every straight line. Then O and I are images of each other under the inversion. This new structure is called the inversive plane. Note that straight lines here are closed curves, just like circles.

Theorem 1.
Under inversion with respect to ω,

(a) a straight line passing through O turns into a striaght line passing through O;

(b) a straight line not passing through O turns into a circle passing through O;

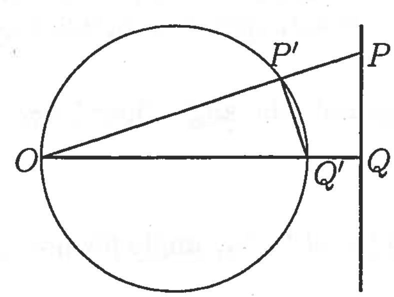
(c) a circle passing through O turns into a straight line not passing through O;

(d) a circle not passing through O turns into a circle not passing through O.

Proof:

(a) Since a point P and its image P' are collinear with O, the image of a straight line passing through O is the line itself.

(b) Let Q be the foot of perpendicular from O to a line ℓ not passing through O, and let Q' be the image of Q under the inversion. We claim that the image of ℓ is the circle ℓ' with OQ' as diameter. For any point P on ℓ, let P' be its image under inversion. Then $OP \cdot OP' = r^2 = OQ \cdot OQ'$ so that $\frac{OP}{OQ} = \frac{OQ'}{OP'}$. Since $\angle POQ = \angle Q'OP'$, triangles POQ and $Q'OP'$ are similar, so that $\angle OP'Q' = \angle OQP = 90°$. It follows that P' lies on ℓ'. Conversely, if P' lies on ℓ', we can prove in the same way that P must lie on ℓ.



(c) This is already proved in (b).

(d) Let γ be a circle which does not pass through O and let P be any point on γ. Let the line OP cut γ again at Q. If OP is tangent to γ, then Q coincides with P. Let P' be the image of P under the inversion. Then $OP \cdot OP' = r^2$. Also, $OP \cdot OQ$ is some constant k, independent of the position of P on γ. It follows that O, P' and Q are collinear, O is not between P' and Q, and $\frac{OP'}{OQ} = \frac{r^2}{k}$ is constant. Hence as Q traces out γ, P' will trace out a circle γ' which is the image of γ. Since γ does not pass through O, neither does γ'.

We use the term stircle to stand for a straight line or a circle. A straight line is considered to be a tangent to itself.

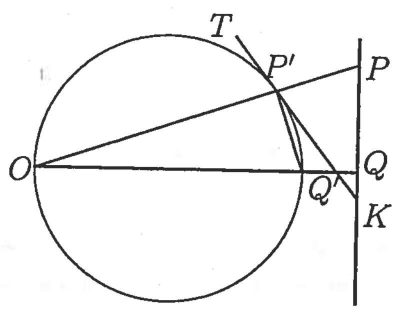
Lemma 2.

Suppose γ is a stircle whose image is γ' under an inversion. Let P be any point on γ and P' on γ' be its image. Then the tangent to γ at P is the reflection of the tangent to γ' at P' about the perpendicular bisector of PP'.

Proof:

(a) If γ is a straight line passing through O, then both γ and γ' coincide with PP' and the result is trivial.

(b) Suppose γ is a straight line not passing through O. Then γ' is a circle passing through O. Let the tangent to γ' at P' intersect γ at K and let T be any point such that P' is between K and T. Now $\angle KPP' = \angle OQ'P' = \angle TP'O = \angle KP'P$. The desired result follows.



(c) Again, this is already proved in (b).

(d) Using the notation and result of Theorem 1(d), the tangent to γ at Q is parallel to the tangent to γ' at P'. The desired result follows since the tangent to γ at Q is the reflection of the tangent to γ at P about the perpendicular bisector of PQ.

When two stircle intersect at a point, the angle they form is defined to be the angle formed by their tangents at that point.

Theorem 3.

The angle formed by two stircles is equal to the angle formed by their images under an inversion.

Proof:

Let γ_1 and γ_2 be two stircles intersecting at a point P. Let their tangents at P be ℓ_1 and ℓ_2. Then their images γ_1' and γ_2' intersect at P', with tangents ℓ_1' and ℓ_2' there. By Lemma 2, ℓ_1 and ℓ_1' are reflections of each other about the perpendicular bisector of PP', as are ℓ_2 and ℓ_2'. Hence ℓ_1 and ℓ_2 form the same angle as ℓ_1' and ℓ_2'. It follows that γ_1 and γ_2 form the same angle as γ_1' and γ_2'.