Summer 2005 I MO Camp

Problems:

1. How many arrangements of the word MATHEMATICS are there with the vowels occurring in alphabetical order?
2. How many integers solutions are there to $x_1 + x_2 + x_3 + x_4 = 2$ where $x_i \geq -4$?
3. For how many integers from 1 to 9999 is the sum of their digits equal to 11?
4. Show by combinatorial argument that
$$k\binom{n}{k} = n\binom{n-1}{k-1}$$
5. Show that
$$n2^{n-1} = \sum_{i=1}^{n} i\binom{n}{i}$$
6. Prove that
$$\sum_{j=0}^{k}\binom{n}{j}\binom{m}{k-j} = \binom{m+n}{k}$$
7. Given $n > 4$ points in the plane such that no three are collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.
8. In how many ways can 5 indistinguishable rooks be placed on an 8×8 chess board so that no rook can attack another and neither the first row or the first column are empty?
9. A king is placed on an $n \times n$ chess board. For a given k, we would like to construct a route for the king to take so that he visits every square exactly k times. Find all values of n and k for which he can do this?
10. First we define two squares of a chess board to be touching if they have at least one vertex in common. Determine if a king on a standard chess board can visit every square exactly once in such a way that each move except the first is to a square which is touching an even number of squares already visited.

11. A king is placed on an $n \times n$ chess board and is permitted to make the following moves: one space left, one space down or one space diagonally up and to the right.

    a) Can the king visit all squares exactly once and return to his original square when $n = 8$?
    b) For what values of n can the king visit all squares exactly once and return to his original square?

12. Adrian and Paul play a game with 3 piles of beans. On each player's turn, they can either remove 1 or 2 beans from pile A, 1,2 or 3 beans from pile B OR 1,2,3 or 4 beans from pile C. Whoever takes the last bean wins the game.

    a) Show that if Adrian plays first and all three piles have 3 beans then Paul has a winning strategy.
    b) Find all n such that when Adrian plays first and all three piles start with n beans then Paul wins the game.

13. On an island there are two kinds of people: Knights who always tell the truth and crooks who always lie. On a given day, 2005 people from the island take seats randomly around a table and they all declare "Both of my neighbours are crooks". The following day, they meet again, but there is one member who is sick so there are only 2004 who attend. They once more take seats around the table and they all declare "Both of my neighbours belong to a category which is not mine". Is the sick member a crook or a knight?

14. A group of 10 pirates discover treasure consisting of 10 equally sized gold bars. To divide this bounty in a fair manner, they decide a democratic method would be best. The pirates order themselves from 1 to 10. Then, pirate 10 suggests a method of dividing up the gold. Then, all pirates vote on this suggestion. If at least 50% vote yes, then the gold is divided up and we are done. Otherwise, the pirate making the suggestion is killed and we repeat the process with one fewer pirate. We must make the following assumptions: 1. The pirates are infinitely logical and know that all other pirates are as well. 2. The pirates are infinitely greedy, so they will always vote no if they can determine they can get more gold in a later suggestion. 3. The pirates are bloodthirsty,

so they will vote no if they can determine that they can get the same amount of gold in a later suggestion.

a) Is there any suggestion that pirate 10 can make that will be accepted? If so, what is the maximum amount of gold he can get for himself?
b) Assume there are k bars of gold. Find all values of n such that if we start with n pirates, the first pirate to make a suggestion will not be killed.
c) Repeat parts a) and b) with the proviso that a proposal needs $> 50\%$ of the votes to be accepted.

15. A total of 119 residents live in a building with 120 apartments. We call an apartment overpopulated if there are at least 15 people living there. Every day, the inhabitants of an overpopulated apartment have a quarrel and each goes off to a different apartment in the building. Is it true that this process will eventually end?

16. In a given lottery, each ticket has the integers 1 through 50 written in some order. When the draw is made, a permutation of the first 50 integers is announced. If for any particular position the number on the ticket matches the number announced, then you win a prize. What is the minimum number of tickets that need to be purchased to guarantee that at least one will win a prize?

17. In a soccer tournament, each team plays another team exactly once and receives 3 points for a win, 1 point for a draw and 0 points for a loss. After the tournament, it is observed that there is a team which has earned the most points, but has won the fewest games. Find the smallest number of teams in the tournament for which this is possible.

18. Choose any 5 lattice points in the plane. Prove that for some pair of these points, the midpoint of the line segment they define is also a lattice point.

19. Prove that there is an integer whose digits are composed entirely of 0s and 1s which is divisible by 2005.

20. Prove that there is a multiple of $\sqrt{2}$ that is within $\frac{1}{100000}$ of an integer.