Extra Problems

Selections from Various Sources

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These problems are in *random* order, not sorted by difficulty. This is deliberate.

§1 First month

Problem 1.1. Let P be a chessboard polygon, and assume P may be tiled with S-tetrominoes. Prove that if P is tiled with S-tetrominoes and Z-tetrominoes, then the number of Z-tetrominoes used is even.

Problem 1.2. Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Problem 1.3. Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at E and F. The circumcircle of $\triangle DEF$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq F$. Prove that line AT passes through either S_1 or S_2.

Problem 1.4. Let n be a positive integer. Consider a triangular array of nonnegative integers as follows:

$$\begin{array}{rccccccccc}
\text{Row } 1: & & & & & a_{0,1} & & & & \\
\text{Row } 2: & & & & a_{0,2} & & a_{1,2} & & & \\
 & & & \vdots & & \vdots & & \vdots & & \\
\text{Row } n-1: & & a_{0,n-1} & & a_{1,n-1} & & \cdots & & a_{n-2,n-1} & \\
\text{Row } n: & a_{0,n} & & a_{1,n} & & a_{2,n} & & \cdots & & a_{n-1,n}.
\end{array}$$

Call such a triangular array *stable* if for every $0 \le i < j < k \le n$ we have

$$a_{i,j} + a_{j,k} \le a_{i,k} \le a_{i,j} + a_{j,k} + 1.$$

For $s_1, \ldots, s_n$ any nondecreasing sequence of nonnegative integers, prove that there exists a unique stable triangular array such that the sum of all of the entries in row k is equal to s_k.

Problem 1.5. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that whenever $a > b > c > d > 0$ and $ad = bc$,

$$f(a+d) + f(b-c) = f(a-d) + f(b+c).$$

§2 Second month

Problem 2.1. Let ABC be a triangle with incentre I and circumcircle ω. Let D and E be the second intersection points of ω with AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and B meet at a point K. Prove that the three lines AE, BD and KP are either parallel or concurrent.

Problem 2.2. Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

Problem 2.3. Let BC be a diameter of circle ω with center O. Let A be a point of circle ω such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of arc AB not containing C. Line ℓ passes through O and is parallel to line AD. Line ℓ intersects line AC at J. The perpendicular bisector of segment OA intersects circle ω at E and F. Prove that J is the incenter of triangle CEF.

Problem 2.4. Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geq n+2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$. These points split I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.

Problem 2.5. Each cell of an $m \times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be *adjacent* if their cells share a common side. (Note that two numbers in cells that share only a corner are not adjacent). The filling is called a *garden* if it satisfies the following two conditions:

(i) The difference between any two adjacent numbers is either 0 or 1.

(ii) If a number is less than or equal to all of its adjacent numbers, then it is equal to 0.

Determine the number of distinct gardens in terms of m and n.

§3 Third month

Problem 3.1. Let $ABCD$ be a cyclic quadrilateral, and let diagonals AC and BD intersect at X.Let C_1, D_1 and M be the midpoints of segments CX, DX and CD, respecctively. Lines AD_1 and BC_1 intersect at Y, and line MY intersects diagonals AC and BD at different points E and F, respectively. Prove that line XY is tangent to the circle through E, F and X.

Problem 3.2. Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled $1, 2, \ldots, 2n$ be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off. Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

Determine $\frac{N}{M}$.

Problem 3.3. Find all functions $f : \mathbb{R} \to [0, \infty)$ such that for any real numbers a, b, c, d with $ab + bc + cd = 0$ we have

$$f(a - b) + f(c - d) = f(a) + f(b + c) + f(d).$$

Problem 3.4. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$. (Here we always assume that an angle bisector is a ray.)

Problem 3.5. Let m be a positive integer. Consider a $4m \times 4m$ array of square unit cells. Two different cells are *related* to each other if they are in either the same row or in the same column. No cell is related to itself. Some cells are coloured blue, such that every cell is related to at lest two blue cells. Determine the minimum number of blue cells.