2004 Winter Camp

1 Cauchy's Equation

Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function satisfying **Cauchy's equation**, namely that

$$f(x + y) = f(x) + f(y) \tag{1}$$

for all real x and y.

1. Show that (1) implies

   $$f(x_1 + x_2 + \ldots + x_n) = f(x_1) + f(x_2) + \ldots f(x_n) \tag{2}$$

   for all $x_1, \ldots, x_n \in \mathbf{R}$.

2. Show

   $$f(n\,x) = n\,f(x) \tag{3}$$

   for all positive integers n and for all real x.

3. Prove that

   $$f(q\,t) = q\,f(t) \tag{4}$$

   for all real t and all positive rationals q.

4. Prove that

   $$f(q\,t) = q\,f(t) \tag{5}$$

   for all real values of t and all rational values of q.

5. Show that there exists a real number a such that $f(q) = a\,q$ for all rational numbers q.

6. We would like to extend the last result yet further so that we can conclude that $f(x) = a\,x$ for all real x. Unfortunately, this results does not follow from Cauchy's equation. We need to add an extra assumption. **Henceforth we shall assume that** there exists some interval $[c, d]$ of real numbers, where $c < d$, such that f is bounded below on $[c, d]$. In other words, there exists a real number A such that $f(x) \geq A$ for all $c \leq x \leq d$.

7. Define $p = d - c$. Show that f is bounded below on the interval $[0, p]$. (However, it need not be bounded below by the same constant A as on the interval $[c, d]$.)

8. Define the function
$$g(x) = f(x) - \frac{f(p)}{p}\,x$$
Prove that g is also bounded below on the interval $[0, p]$ and satisfies Cauchy's equation.

9. Show that g is periodic with period p in the sense that $g(x + p) = g(x)$ for all real x. Conclude from this, and the fact that g is bounded below on the interval $[0, p]$ that g is bounded below on the entire real line $(-\infty, +\infty)$.

10. Suppose that there exists some x_0 for which $g(x_0) \neq 0$. Prove a contradiction, by showing that the sequence of values $g(n\,x_0)$, $n = \pm 1, \pm 2, \pm 3, \ldots$ is not bounded below.

11. **Conclude** that $g(x) = 0$ for all real x, and **therefore** that $f(x) = a\,x$ for all real x, where $a = f(p)/p$.

2 Jensen's Equation

Jensen's equation has the form
$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2} \tag{6}$$
for all real x and y. It can be thought of as a version of Cauchy's equation using averages. Once again, the function f is assumed to be bounded below on some interval.

1. Find all solutions to Jensen's equation.

3 Pexider's Equation

Pexider's equation has the form
$$f(x + y) = g(x) + h(y) \tag{7}$$
for all x and y. The function f is assumed to be bounded below on some interval.

1. Find all functions f, g, h satisfying (7) for all real values of x and y.

4 Two Simultaneous Functional Equations

We can also mix together an additive version of Cauchy's equation with a multiplicative version. Suppose f satisfies

$$f(x+y) = f(x) + f(y) \qquad f(x\,y) = f(x)\,f(y)$$

simultaneously for all x and y. We do **not** assume anything about the boundedness of f. *(Why not?)*

1. Find all solutions for f.

5 Euler's Equation

Let k be any real number. For given k, the equation

$$f(t\,x, t\,y) = t^k\,f(x,y) \tag{8}$$

for all positive x, y and t, is called *Euler's equation.* A function f satisfying Euler's equation is said to be a *homogeneous function of degree k.* For example, the function

$$f(x,y) = \frac{x+y}{2}$$

is homogeneous of degree one, the function

$$f(x,y) = \frac{x}{y}$$

has degree zero, and the function

$$f(x,y) = x^2 + 2\,y^2 + 7\,x\,y$$

has degree two.

How can we find the class of all functions which satisfy Euler's equation for a given choice of k? This looks difficult, but is really quite easy. Let us substitute $t = x^{-1}$ into (8). It then becomes

$$f(x,y) = x^k\,f(1, x^{-1}\,y) \tag{9}$$

Suppose that we let $g(z) = f(1,z)$. We can write

$$f(x,y) = x^k\,g(x^{-1}\,y) \tag{10}$$

to obtain the solution to Euler's equation. It can be seen that any function g defined on the positive real axis determines a homogeneous function of degree k, and vice versa.

The simplicity of (10) suggests that it might be reasonable to generalize Euler's equation to

$$f(t\,x, t\,y) = h(t)\, f(x, y) \tag{11}$$

where the function $h(t)$ is given. However, it turns out that the generality of this modified Euler equation is a bit of an illusion here.

1. Prove that if h is bounded below on an interval then h can be written in the form $h(t) = t^k$, provided that f is not the function which is zero everywhere.

6 Additional Problems

Here are some additional problems. You may assume that the functions are bounded on some interval if that will help the proof.

1. Find all solutions in continuous functions $f, g, h : \mathbf{R} \to \mathbf{R}$ to the functional equation
$$f(x + y) = g(x)\, h(y)$$

2. Consider continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that
$$f(x_1) + f(x_2) + f(x_3) = f(y_1) + f(y_2) + f(y_3)$$
whenever $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0$. Prove that there exist real numbers a and b such that $f(x) = a\,x + b$ for all real x.

3. Suppose $f : \mathbf{R} \to \mathbf{R}$ has the property that there exists a constant K such that
$$|f(x) - f(y)| \leq K\,(x - y)^2$$
for all x and y. Prove that f is a constant function.

4. Find all $f : \mathbf{R} \to \mathbf{R}$ such that $f(x) \leq x$ and $f(x + y) \leq f(x) + f(y)$ for all $x, y \in \mathbf{R}$.

5. Determine which functions $f(x)$ mapping the rationals to the rationals satisfy the functional equation

$$f[x + f(y)] = f(x)\, f(y)$$

6. Find all $f : \mathbf{R} \to \mathbf{R}$ such that

$$f\left(\sqrt{x^2 + y^2}\right) = f(x)\, f(y)$$

for all real x and y.

7. Find all f such that for all real x and y

$$f(x + y)\, f(x - y) = [f(x)\, f(y)]^2$$

8. For any real numbers x and y, we define the *quasiarithmetic mean* of x and y with generating function f to be a function of the form

$$m(x, y) = f^{-1}\left[\frac{f(x) + f(y)}{2}\right]$$

where f is a continuous strictly increasing or strictly decreasing function. For example, the ordinary arithmetic mean is obtained by setting $f(x) = x$ and the geometric mean by setting $f(x) = \log x$. Other means that can be constructed include the harmonic mean ($f(x) = x^{-1}$) and the root-mean-square ($f(x) = x^2$). These last three definitions are usually defined only for x and y positive.

(a) A question of immediate interest is when distinct functions f and g give distinct quasiarithmetic means. Suppose that

$$f^{-1}\left[\frac{f(x) + f(y)}{2}\right] = g^{-1}\left[\frac{g(x) + g(y)}{2}\right]$$

prove that there exist constants $a \neq 0$ and b such that

$$g(x) = a\, f(x) + b$$

In addition, prove the converse. Thus prove that two quasiarithmetic means are equivalent if and only if their generating functions are linearly related.

(b) A quasiarithmetic mean is said to be *translative* if it has the property that

$$m(x+t, y+t) = m(x,y) + t$$

for all real values x, y and t. Obviously, the arithmetic mean ($f(x) = x$) is translative. Is it the only one? It turns out that there is another family of translative quasiarithmetic means. Solve

$$f^{-1}\left[\frac{f(x+t)+f(y+t)}{2}\right] = f^{-1}\left[\frac{f(x)+f(y)}{2}\right] + t$$

to find all possible generating functions f.

(c) A quasiarithmetic mean is said to be *homogeneous* if it has the property that

$$m(t\,x, t\,y) = t\,m(x,y)$$

for all positive values x, y and t. Obviously, the geometric mean ($f(x) = \log x$) is homogeneous. Find all others.

9. For each positive integer n, let f_n be a real-valued symmetric function of n real variables. Suppose that for all n and for all real numbers $x_1, \ldots x_n$, y, it is true that

(a) $f_n(x_1 + y, \ldots, x_n + y) = f_n(x_1, \ldots, x_n) + y$

(b) $f_n(-x_1, \ldots, -x_n) = -f_n(x_1, \ldots, x_n)$

(c) $f_{n+1}(f_n(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n), x_{n+1}) = f_{n+1}(x_1, \ldots, x_{n+1})$

Prove that

$$f_n(x_1, \ldots, x_n) = \frac{x_1 + \ldots + x_n}{n}$$

(Note that a function f is said to be symmetric if its value is independent of the order of its variables: $f(x,y) = f(y,x)$ for two variables, and so on.)