AWESOMELY FANTASTIC FUNCTIONAL EQUATION PROBLEMS

What do you try with a functional equation problem? Well, the main thing to do is to just PLUG IN VALUES! Really, don't shy away from this. Try to get as many concrete relations as possible and spot a pattern. Then, you can go on the offensive!

More concretely, here are a list of tips:

- Try to set expressions equal to 0, or things equal to each other. This can turn a very complicated relation into a relatively simple one.
- Plug in 0.
- Prove that f is surjective (meaning it takes every possible value in its image) or injective (meaning it takes every value at most once) or bijective (both are true!).
- Use what you've learned!! Seriously, this applies here more than any other type of problem. If you get useful relations that don't quite solve the problem, don't just forget about them. Like, if f is some function on the integers, and you've learned what $f(n)$ is for n even, remember that when solving for n odd.
- If f is a function on the reals, try first figuring out what f is on the integers. Then on the rationals. Then there are several ways to get to the reals. A common one is to prove that f is monotonic.

large**Problems**

(1) Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 + f(y)) = f(x)^2 + y$$

For all $x, y \in \mathbb{R}$.

(2) Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + f(y)^2) = f(x) + y^2$$

For all $x, y \in \mathbb{R}$.

(3) Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that for all real x, y, we have

$$f(x + y) + f(x - y) = 2f(x)\cos y$$

(4) Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^3) + f(y^3) = (x + y)\left(f(x^2) - f(xy) + f(y^2)\right)$$

for all $x, y \in \mathbb{R}$.

(5) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$2f(x) = f(x + y) + f(x + 2y)$$

for all $x, y \in \mathbb{R}$.

(6) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)^2 + f(y)) = xf(x) + y$, for all $x, y \in \mathbb{R}$.

(7) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(mn) = \mathrm{lcm}(m, n) \cdot \gcd(f(m), f(n))$$
for all positive integers m, n.

(8) Find all functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that for all $a, b, c, d \in \mathbb{R}$ satisfying $ab + bc + cd = 0$ we have
$$f(a - b) + f(c - d) = f(a) + f(b + c) + f(d).$$

(9) Find all pairs of functions $f, g : \mathbb{Q} \to \mathbb{Q}$ such that for all $x, y \in \mathbb{Q}$,
$$f\left(g\left(x\right) - g\left(y\right)\right) = f\left(g\left(x\right)\right) - y$$
$$g\left(f\left(x\right) - f\left(y\right)\right) = g\left(f\left(x\right)\right) - y$$
.

(10) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ we have
$$f(x + y) = f(x)f(y) - c \sin x \sin y.$$

(11) Find all functions $f : \mathbb{R} \to \mathbb{R}_{\geq}0$ such that for all $x, y \in \mathbb{R}$ we have
$$f(x + y) + f(x - y) - 2f(x) - 2y^2 = 0.$$

(12) Find all integer valued functions, f and g, defined on the integers, such that g is surjective and
$$f(g(x) + y) = g(f(y) + x)$$
for all integers x, y.

(13) Find all functions $f : \mathbb{R} \to \mathbb{Z}$ such that for all $x, y \in \mathbb{R}$ we have
- $f(x + y) < f(x) + f(y)$
- $f(f(x)) = \lfloor x \rfloor + 2.$

Past IMO functional equations

(1) *IMO 1987* Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n + 1987$ for all $n \in \mathbb{N}$.

(2) *IMO 2002*
Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy
$$f(x) + f(y))(f(t) + f(z)) = f(xt + yz) + f(xz - yt)$$
for all real x, y, z, t.

(3) *IMO 1999*
Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy
$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

(4) *IMO 1998*
Find the least possible value of $f(1998)$, where $f : \mathbb{N} \to \mathbb{N}$ satisfies
$$f(n^2 f(m)) = mf(n)^2$$

(5) *IMO shortlist 1996*
Show that there exists a bijective functions $f : \{0, 1, 2, \dots\} \to \{0, 1, 2, \dots\}$ Such that for all m, n, we have
$$f(3mn + m + n) = 4f(m)f(n) + f(n) + f(m) + 1$$