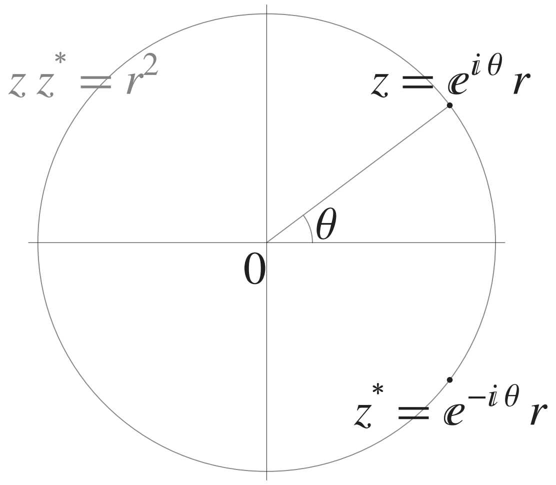
Complex numbers

We assume that you are familiar with complex numbers in algebra, and delve immediately into the use of complex numbers in geometry. We are able to use complex numbers in two-dimensional geometry because the Euclidean plane, $\mathbb{R}^2$, is isomorphic to the complex plane, $\mathbb{C}$. We represent a point with Cartesian coordinates (x, y) with the complex number $z = x + i\, y$, where i is the imaginary unit. Complex numbers are superior to two-dimensional vectors in that rotations are easy to define, as we shall see shortly.

Basic properties of complex numbers

Complex numbers can be added, subtracted and multiplied by real numbers in precisely the same way that vectors can. They have a *magnitude* and *argument*, which correspond to the length and direction of a complex number. Also, for a complex number $z = x + i\, y$, we define its *complex conjugate* to be $z^* = x - i\, y$. As i and $-i$ have definitions analogous to 'left' and 'right', we can interchange all instances of i with $-i$ in an algebraic equation without affecting anything. Hence, $(z\, w)^* = z^*\, w^*$ and $(z + w)^* = z^* + w^*$.



In the above diagram, $\arg(z) = \theta$ is the *argument* of z, and $|z| = r$ is the *magnitude* (or *modulus*) of z. An important identity is that $z\, z^* = r^2$, which enables the (squared) modulus of a complex number to be calculated. Using vector subtraction, this gives us $A\,B^2 = (a - b)\,(a^* - b^*)$ for the squared distance between two points.

If we multiply a complex number with polar form $\langle r_1, \theta_1 \rangle$ with another complex number $\langle r_2, \theta_2 \rangle$, it is easy to verify that we obtain the complex number $\langle r_1\, r_2, \theta_1 + \theta_2 \rangle$. Hence, $\arg(z_1\, z_2) = \arg(z_1) + \arg(z_2)$ and $|z_1\, z_2| = |z_1|\ |z_2|$.

We can define the *inner product* in terms of complex numbers as $z \cdot w = \frac{1}{2}\,(z\, w^* + w\, z^*)$, which is analogous to the dot product of vectors. Similarly, we define the *exterior product* as $z \times w = \frac{1}{2}\, i (w\, z^* - z\, w^*)$, which resembles the cross product.

1. Prove that the area of $A\,O\,B$ is given by $[A\,O\,B] = \frac{1}{2}\,(a \times b) = \frac{1}{4}\, i (b\, a^* - a\, b^*)$.

2. Hence prove that the area of triangle $A\,B\,C$ is given by

$$[A\,B\,C] = \frac{1}{2}\,(b \times a + c \times b + a \times c) = \frac{1}{4}\, i \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^* & b^* & c^* \end{pmatrix}.$$ **[Area of a triangle]**

3. If we have a triangle and erect equilateral triangles on its sides, prove that the centres of those equilateral triangles themselves form an equilateral triangle. **[Napoleon's theorem]**

Angles, circles and concyclicity

4. Show that the directed angle $\measuredangle A B C = \arg(a-b) - \arg(c-b) = \arg\left(\frac{a-b}{c-b}\right)$.

5. Hence deduce that $\measuredangle A B C \equiv \measuredangle A D C \pmod{\pi}$ if and only if $\frac{(a-b)(c-d)}{(b-c)(d-a)}$ is real. **[Real cross ratio ⇔ concyclicity]**

More specifically, if this value is equal to -1, then $A B C D$ is known as a *harmonic* quadrilateral. Harmonic quadrilaterals are covered in the chapter on projective geometry.

6. Show that the equation of a circle with centre P and radius r has the equation $z z^* - p z^* - p^* z + p p^* - r^2 = 0$. **[General form of a circle]**

7. Prove that four points, $A B C D$, are mutually concyclic (or collinear) if and only if $(a-b)(b^*-c^*)(c-d)(d^*-a^*) = (a^*-b^*)(b-c)(c^*-d^*)(d-a)$.

8. Prove that four points, $A B C D$, are mutually concyclic (or collinear) if and only if
$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^* & b^* & c^* & d^* \\ a a^* & b b^* & c c^* & d d^* \end{pmatrix} = 0.$$

The observant amongst you may notice that the previous two questions are equivalent quartic expressions. This demonstrates the equivalence between the 'angles in the same segment' and 'equidistant from a common point' conditions for concyclicity. Any quadratic function, which vanishes only on the circumference of a circle, must necessarily be proportional to the power of a point with respect to that circle. This gives us a more general result, which I believe has yet to be published elsewhere:

- For any four points, $A B C D$, no three of which are collinear, we have
$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^* & b^* & c^* & d^* \\ a a^* & b b^* & c c^* & d d^* \end{pmatrix} = -4\, i\, [A B C]\ \text{Power}(P,\ A B C).$$ **[Goucher's theorem]**

Interestingly, it was noted that at almost the same time a problem particularly vulnerable to this theorem was proposed at an International Mathematical Olympiad. There are ordinary Euclidean methods of proving this, but they are less inspired and do not explain *why* this result should hold.

9. Suppose we have a non-cyclic quadrilateral, $P_1 P_2 P_3 P_4$. Let O_1 and R_1 be the centre and radius, respectively, of the circumcircle of $P_2 P_3 P_4$, and define O_2, O_3, O_4 and R_2, R_3, R_4 similarly. Show that $\frac{1}{O_1 P_1{}^2 - R_1{}^2} + \frac{1}{O_2 P_2{}^2 - R_2{}^2} + \frac{1}{O_3 P_3{}^2 - R_3{}^2} + \frac{1}{O_4 P_4{}^2 - R_4{}^2} = 0$. **[IMO 2011 shortlist, Question G2]**

Reflections and rotations

A useful property of complex numbers is the ability to express rotations and reflections rather simply. We have concise expressions for reflection about the real axis (complex conjugation), rotation about the origin (multiplication by a unit complex number) and translation (addition of a complex number). Here are the three 'elementary' operations:

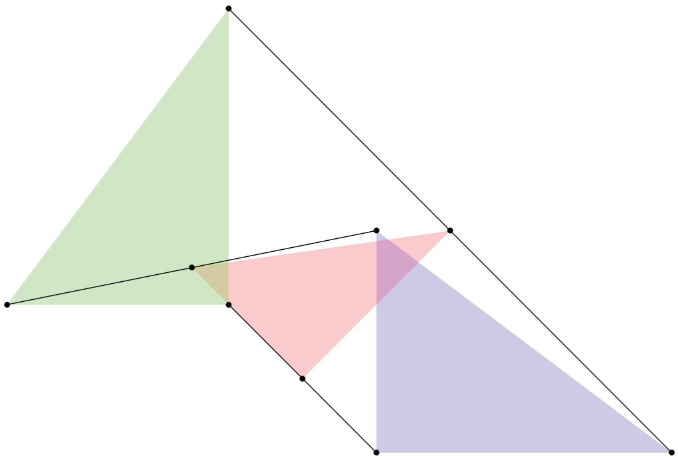
- A translation parallel to the vector $\overrightarrow{OA}$ is represented by $z \to z + a$. **[Translation]**
- An anticlockwise rotation about the origin by the angle θ is represented by $z \to z\, e^{i\theta}$. **[Rotation about the origin]**
- A reflection in the real line is represented by $z \to z^*$. **[Reflection in the real axis]**

These become more useful when one realises that they can be composed to yield any Euclidean transformation.

10. Show that an anticlockwise rotation by θ about the point A is represented by $z \to z\, e^{i\theta} + a\left(1 - e^{i\theta}\right)$.

11. Hence demonstrate that an arbitrary direct congruence is a transformation of the form $z \to a\,z + b$, where $a\,a^* = 1$.

If we relax the condition that $a\,a^* = 1$, we obtain the result that two directly similar figures can be related by a transformation of the form $z \to a\,z + b$. As this is a linear function, we can linearly interpolate between any two directly similar figures to obtain a third directly similar figure. Specifically, if triangles ABC and $A'B'C'$ are directly similar, then the (optionally weighted) midpoints of AA', BB' and CC' form a third similar triangle.

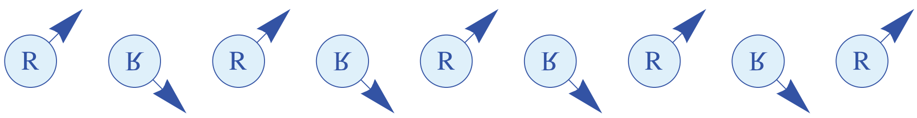


This is known as the *fundamental theorem of directly similar figures*. In the diagram above, the red triangle is the 'arithmetic mean' of the blue and green triangles. If we have five directly similar figures in general position, then any directly similar figure can be expressed as a 'weighted mean' of those five.

12. Show that a reflection in the line $z\, e^{-i\theta} \in \mathbb{R}$ is represented by $z \to z^*\, e^{2i\theta}$.

13. Hence demonstrate that an arbitrary indirect congruence is a transformation of the form $z \to a\,z^* + b$, where $a\,a^* = 1$.

In two dimensions, direct congruences can be either translations or rotations. Indirect congruences can be either reflections or *glide-reflections*. A glide-reflection is a composition of a reflection in a line and a translation parallel to the line.



14. Show that a glide-reflection has no fixed points.

15. Hence demonstrate that a reflection in the line BC is represented by $z \to \frac{(b-c)(z^*-b^*)}{b^*-c^*} + b$.

16. If $b\,b^* = c\,c^* = R^2$, show that a reflection in the line BC is represented by $z \to b + c - \frac{b\,c\,z^*}{R^2}$.

The comparative complexities of the previous two expressions show that the calculations become simpler when we assume that the circumcentre of a triangle ABC is the origin. This is explored more thoroughly in a later section of this chapter.

17. Let ABC be a triangle, and P be a point in the plane. Let the reflections of P in BC, CA and AB be D, E and F, respectively. Prove that $\frac{[DEF]}{[ABC]} = \frac{R^2 - OP^2}{R^2}$. **[Euler's formula for pedal triangles]**

As a special case of the above, we have the Simson line property:

- Let ABC be a triangle, and P be a point in the plane. Then the reflections of P in BC, CA and AB are collinear if and only if P lies on the circumcircle of ABC. Moreover, the orthocentre H lies on this line. **[Dilated Simson line]**

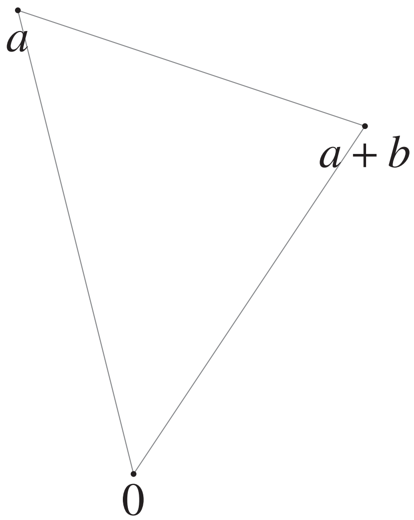
Usually, these results are quoted when D, E and F are the feet of the perpendiculars from P, rather than reflections of P. However, Euler's formula is more elegant with this version, and the standard Simson line in general does not contain H.

Triangle inequality

One of the more rudimentary inequalities governing vectors (and thus complex numbers) is the triangle inequality.

- If a and b are nonzero complex numbers, then $|a + b| \le |a| + |b|$, with equality if and only if a is a **positive real multiple** of b. **[Triangle inequality]**

This follows immediately from the following configuration, together with the notion that the shortest path between two points is a straight line.



18. Show that $(a-b)(c-d) + (a-d)(b-c) = (a-c)(b-d)$, where $a, b, c, d \in \mathbb{C}$.

19. Hence prove that $|a-b|\,|c-d| + |a-d|\,|b-c| \ge |a-c|\,|b-d|$, with equality if and only if $\frac{(a-b)(c-d)}{(b-c)(d-a)}$ is a negative real.

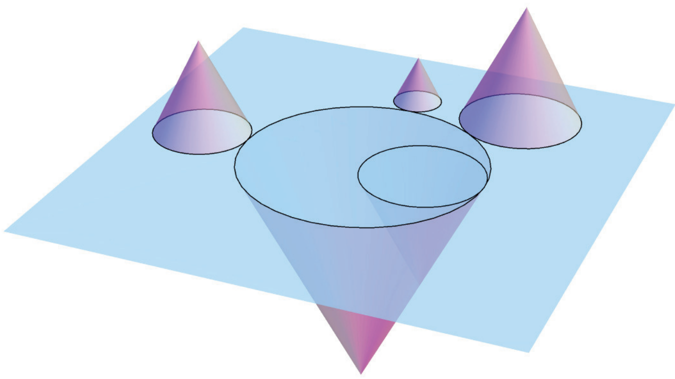
20. Let $ABCD$ be a quadrilateral. Show that $AB \cdot CD + BC \cdot DA \geq AC \cdot BD$, with equality if and only if $ABCD$ is a convex cyclic quadrilateral. **[Ptolemy's inequality]**

Casey's theorem

The equality case of Ptolemy's inequality can be regarded as a special case of Casey's theorem.

- Let Γ be a circle, and φ_1, φ_2, φ_3 and φ_4 be four circles tangent to Γ at P_1, P_2, P_3 and P_4, respectively. The chords P_1P_3 and P_2P_4 intersect inside Γ. For each pair of circles φ_i and φ_j, we let $d(i, j)$ denote the length of the common outer tangents if φ_i and φ_j are both on the same side of Γ, or the length of the common inner tangents if they lie on opposite sides of Γ. Then we have $d(1, 3) \cdot d(2, 4) = d(1, 2) \cdot d(3, 4) + d(2, 3) \cdot d(4, 1)$. **[Casey's theorem]**

The following exercise demonstrates how the latter can be inferred from the former using some basic trigonometry. Firstly, we consider a circle tangent externally to Γ to have positive radius, and a circle tangent internally to Γ to have negative radius. Γ itself is considered to have negative radius.



Erecting cones and 'anticones' on the circles with positive and negative radii, respectively, gives the diagram shown above.

21. If φ_1 and φ_2 are two circles with radii r_1 and r_2 and centres O_1 and O_2, respectively, then show that $d^2 = O_1O_2{}^2 - (r_1 - r_2)^2$, where d is the length of the common outer tangents.

The value of d is dependent only on the positions of the centres and difference between the radii. This means we can fix the centres and uniformly increase the radii (using the sign convention described above) of all five circles in the problem by the same amount, without changing the values of $d(i, j)$ or affecting the tangency of the circles. (In the three-dimensional diagram, this is equivalent to moving the horizontal reference plane upwards or downwards.) So, we can assume without loss of generality that Γ is a single point (circle of zero radius) through which each φ_i passes. This greatly simplifies the analysis.

- Let P be a point, and φ_1, φ_2, φ_3 and φ_4 be four circles passing through P with centres O_1, O_2, O_3 and O_4, respectively. The V-shaped line O_1PO_3 separates the plane into two regions; O_2 and O_4 lie in opposite regions. For each pair of circles φ_i and φ_j, we let $d(i, j)$ denote the length of the common outer tangents. Then we have $d(1, 3) \cdot d(2, 4) = d(1, 2) \cdot d(3, 4) + d(2, 3) \cdot d(4, 1)$. **[Simplified Casey's theorem]**

By proving the simplification of Casey's theorem, we will therefore implicitly prove the original theorem.

22. In the above problem, let Φ_1 and Φ_2 have radii r_1 and r_2, respectively. Show that $d(1, 2)^2 = 2\, r_1\, r_2(1 - \cos\theta)$, where $\theta = O_1 \hat{P} O_2$.

23. Hence show that $d(1, 3)\cdot d(2, 4)$ and the other terms in the simplified Casey's theorem are unaffected when r_1, r_2, r_3 and r_4 are simultaneously replaced with their geometric mean.

24. Hence prove the simplified Casey's theorem.

Casey's theorem, like Ptolemy's theorem, has a converse. If we have four (directed) circles and $d(1, 3)\cdot d(2, 4) = d(1, 2)\cdot d(3, 4) + d(2, 3)\cdot d(4, 1)$, then there exists a fifth circle tangent to all four circles. This is the basis of the shortest known proof of Feuerbach's theorem, demonstrating that there is a circle (the nine-point circle) tangent to the incircle and three excircles of a generic triangle.

Solutions

1. Suppose $\angle A\,O\,B = \theta$. Then we have $[A\,O\,B] = \frac{1}{2}\left|a\right|\,\left|b\right|\,\sin\theta = \frac{1}{2}\,(a\times b)$.

2. $[A\,B\,C] = [B\,O\,A] + [C\,O\,B] + [A\,O\,C] = \frac{1}{2}\,(b\times a + c\times b + a\times c)$. Using the formula for cross product, this equals $\frac{i}{4}\,(a\,b^* - b\,a^* + b\,c^* - c\,b^* + c\,a^* - a\,c^*) = \frac{1}{4}\,i\det\begin{pmatrix}1 & 1 & 1\\ a & b & c\\ a^* & b^* & c^*\end{pmatrix}$.

3. Assume, without loss of generality, that the triangle is labelled anticlockwise. Let x, y and z be the centres of the equilateral triangles erected on $B\,C$, $C\,A$ and $A\,B$, respectively. We have $x = \frac{1}{3}\left(b + c - \omega\,c - \omega^2\,b\right)$, where $\omega = e^{\frac{2}{3}\,i\,\pi}$ is a cube root of unity. This gives us $x - y = \frac{1}{3}\left(a(\omega - 1) + b\left(1 - \omega^2\right) + c(\omega - 1)\right)$. The symmetrical expression means that $(x - y) = \omega(z - x)$, which is a sufficient condition for the triangle to be equilateral.

4. We translate the configuration so that B is the origin, and A and C are represented by complex numbers $a - b$ and $c - b$, respectively. Hence, we have $\angle A\,B\,C = \arg(a - b) - \arg(c - b)$, as required. The final part of the proof, namely that this also equals $\arg\left(\frac{a-b}{c-b}\right)$, follows from the prosthaphaeretic property of the arg() function.

5. $\angle A\,B\,C \equiv \angle A\,D\,C \pmod{\pi} \Leftrightarrow \arg\left(\frac{a-b}{c-b}\right) \equiv \arg\left(\frac{a-d}{c-d}\right) \pmod{\pi} \Leftrightarrow \frac{(a-b)\,(c-d)}{(b-c)\,(d-a)} \in \mathbb{R}$.

6. $|\,z - p\,| = r \Leftrightarrow (z - p)\,(z^* - p^*) = r^2 \Leftrightarrow z\,z^* - p\,z^* - p^*\,z + p\,p^* - r^2 = 0$.

7. This is a consequence of the ‘angle in the same segment’ theorem for concyclicity and the result of Question 5. As $\frac{(a-b)\,(c-d)}{(b-c)\,(d-a)} \in \mathbb{R}$, it must be equal to its complex conjugate $\frac{(a^*-b^*)\,(c^*-d^*)}{(b^*-c^*)\,(d^*-a^*)}$. We then multiply throughout by the common denominator, giving the quartic equation
$(a - b)\,(b^* - c^*)\,(c - d)\,(d^* - a^*) = (a^* - b^*)\,(b - c)\,(c^* - d^*)\,(d - a)$.

8. Suppose D is variable and A, B and C are constants. The equation multiplies out to the form in Question 6, so is the condition that D lies on some circle. As the determinant vanishes whenever two columns are equal, $D = A$, $D = B$ and $D = C$ all satisfy this equation. Hence, it must be the circumcircle of $A\,B\,C$, and we are done.

9. Multiply through by $\frac{1}{4}\,i\det\begin{pmatrix}1 & 1 & 1 & 1\\ a & b & c & d\\ a^* & b^* & c^* & d^*\\ a\,a^* & b\,b^* & c\,c^* & d\,d^*\end{pmatrix}$ and use Goucher’s theorem. This leads to the equivalent statement about signed areas $[A\,B\,C] + [C\,D\,A] = [B\,C\,D] + [D\,A\,B]$.

10. To obtain this transformation, we compose a translation by $\overrightarrow{A\,O}$, rotation by θ anticlockwise about O, and a translation by $\overrightarrow{O\,A}$. They have the formulae $z \to z - a$, $z \to z\,e^{i\,\theta}$ and $z \to z + a$, respectively. Composing these in order yields $z \to (z - a)\,e^{i\,\theta} + a = z\,e^{i\,\theta} + a\left(1 - e^{i\,\theta}\right)$.

11. As $z \to a\,z + b$ is a linear transformation, it is closed under composition. Translations and rotations are of this form, ergo every rigid transformation is.

12. Again, we compose a rotation by θ clockwise ($z \to z\, e^{-i\theta}$), a reflection in the real axis ($z \to z^*$) and a rotation by θ anticlockwise ($z \to z\, e^{i\theta}$). The composite transformation has rule $z \to z^*\, e^{2i\theta}$.

13. Composing a function of the form $z \to a\, z^* + b$ with any linear function results in another function of the form $z \to a\, z^* + b$. All indirect congruences can be built from a reflection in the real axis and a rigid transformation, so must also be of this form.

14. Composing a glide-reflection with itself results in a translation, which clearly has no fixed points.

15. The reflection must have the form $z \to p\, z^* + q$, where $p\, p^* = 1$, as we demonstrated earlier. Having B and C as fixed points (as the transformation does indeed) proves that it is not a glide reflection, and must be the reflection in the line $B\,C$.

16. The reasoning is identical to the previous question.

17. Without loss of generality, assume $a\,a^* = b\,b^* = c\,c^* = R^2$. Then $d = b + c - \frac{b\,c\,p^*}{R^2}$, and e and f have similar forms. The determinant (proportional to the area) is given by $\sum_{\text{cyc}} (d\,e^* - e\,d^*)$, where $\sum_{\text{cyc}}$ denotes the cyclic sum interchanging a, b and c. This evaluates to
$\det\begin{pmatrix} 1 & 1 & 1 \\ d & e & f \\ d^* & e^* & f^* \end{pmatrix} = \sum_{\text{cyc}} \left(a\,b^* - b\,a^* + \frac{b\,a^*\,p\,p^*}{R^2} - \frac{a\,b^*\,p\,p^*}{R^2}\right) = \sum_{\text{cyc}} (a\,b^* - b\,a^*)\left(1 - \frac{p\,p^*}{R^2}\right) = \det\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^* & b^* & c^* \end{pmatrix}\left(1 - \frac{O\,P^2}{R^2}\right)$. Or, in other words, $[D\,E\,F] = [A\,B\,C]\left(1 - \frac{O\,P^2}{R^2}\right)$.

18. Both sides of the equation expand to $a\,b - b\,c + c\,d - d\,a$.

19. As the modulus function is multiplicative, this is equivalent to $|(a-b)(c-d)| + |(a-d)(b-c)| \geq |(a-c)(b-d)|$. This is the triangle inequality, so equality only holds when $\frac{(a-b)(c-d)}{(a-d)(b-c)}$ is a positive real, or $\frac{(a-b)(c-d)}{(b-c)(d-a)}$ is a negative real.

20. The real cross-ratio condition forces A, B, C and D to be concyclic. If A and C lie on the same side of $B\,D$, the cross-ratio would be positive; hence, chords $A\,C$ and $B\,D$ must intersect.

21. Assume, without loss of generality, that $r_1 \geq r_2$. Let one of the common outer tangents meet Γ_1 at A and Γ_2 at B. Further, let C be the point on the radius $O_1\,A$ such that $A\,C = r_2$ and $O_1\,C = r_1 - r_2$. As $C\,O_2\,B\,A$ is a rectangle, we have $d = A\,B = C\,O_2$ (not carbon dioxide!). By applying Pythagoras' theorem to the triangle $O_1\,O_2\,C$, we obtain $C\,O_2{}^2 = O_1\,O_2{}^2 - C\,O_1{}^2$, each term of which is equal to $d^2 = O_1\,O_2{}^2 - (r_1 - r_2)^2$.

22. Using the formula from the previous question, we have $d(1, 2)^2 = O_1\,O_2{}^2 - (r_1 - r_2)^2$. The cosine rule gives us $O_1\,O_2{}^2 = r_1{}^2 + r_2{}^2 - 2\,r_1\,r_2 \cos\theta$, and multiplying out yields $(r_1 - r_2)^2 = r_1{}^2 + r_2{}^2 - 2\,r_1\,r_2$. The difference between these expressions is $2\,r_1\,r_2(1 - \cos\theta)$, as required.

23. The previous question results in $(d(1, 3)\cdot d(2, 4))^2 = 4\,r_1\,r_2\,r_3\,r_4(1 - \cos\theta)(1 - \cos\phi)$, where θ and ϕ are defined in the obvious way. This is unaffected when we replace each of r_1, r_2, r_3 and r_4 with $r = \sqrt[4]{r_1\,r_2\,r_3\,r_4}$, as the product remains equal to r^4. Hence, the value of $d(1, 3)\cdot d(2, 4)$ also remains invariant. By symmetry, so do the other terms.

24. We can assume without loss of generality that $r_1 = r_2 = r_3 = r_4 = r$. The distance $d(1, 2) = O_1\,O_2$, *et cetera*. Since O_1, O_2, O_3 and O_4 lie on a circle by Ptolemy's theorem, we are done.