Enumeration Techniques

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MOP 2010 (Blue)

Often you want to find the number of objects of some type; find an upper or lower bound for this number; find its value modulo n for some n; or compare the number of objects of one type with the number of objects of another type. There are a lot of methods for doing all of these things.

I'm going to focus on methods rather than on knowing formulas, but I've attached a short list of useful formulas at the end. As an exercise, you can try to prove whichever ones you don't already know.

If you're looking for reading or reference materials, a general-purpose source for a lot of enumeration techniques is Graham, Knuth, and Patashnik's *Concrete Mathematics*. The bible of the subject (but much more advanced) is Stanley's *Enumerative Combinatorics*. Andreescu and Feng's book *A Path to Combinatorics for Undergraduates* is a more accessible, problem-solving-oriented treatment.

1 Counting techniques

A typical counting problem is as follows: you're given the definition of a quagga of order n, and told what it means for a quagga to be blue. How many blue quaggas of order n are there?

Here are some general-purpose techniques to approach such a problem:

- Write down a recurrence relation
- Count the non-blue quaggas
- Find a bijection with something you know how to count
  - If you only need a lower or upper bound, find a surjection or injection to something you know how to count
- Put all quaggas into groups of size n, such that there's one blue quagga in each group
- Count incarnations of blue quaggas, then show that each quagga has n incarnations

- To find out the number of quaggas mod n, find a way to put most of the blue quaggas into groups of size n and see how many are left over
- Use generating functions
- Use inclusion-exclusion
- Attach variables to parts of quaggas, then use algebra to count quaggas

A classic example of many of these techniques is Catalan numbers. The nth Catalan number is defined to be $C_n = \binom{2n}{n}/(n+1)$. This is always an integer, and moreover, there are lots of things it counts:

- ways to triangulate a regular $(n+2)$-gon by drawing $n-1$ diagonals
- Dyck paths of length $2n$ — that is, paths from $(0,0)$ to $(2n,0)$ via steps $(1,1)$ and $(1,-1)$ that never go below the x-axis
- ways to parenthesize the expression $1+1+\cdots+1$, with $n+1$ 1's
- ways to connect $2n$ points on a circle with n nonintersecting chords
- rooted, ordered binary trees with $n+1$ leaves

2 Problems

1. Given are positive integers n and m. Put $S = \{1, 2, \ldots, n\}$. How many ordered sequences are there of m subsets $T_1, \ldots, T_m$ of S, such that $T_1 \cup T_2 \cup \cdots \cup T_m = S$?
2. How many subsets $S \subseteq \{1, 2, \ldots, n\}$ are there that do not contain two consecutive integers?
3. [UMUMC] On an 8×8 grid, a *cross* consists of one square and its four diagonal neighbors. Let N be the number of ways of coloring the squares of the grid in red and blue so that there do not exist five red squares forming a cross. Prove that $\sqrt{N}$ is an integer.
4. Let $S(m,n)$ be the set of integer points $\{(i,j) \mid 1 \le i \le m,\ 1 \le j \le n\}$. A subset T of $S(m,n)$ is *safe* if, whenever $i \le i'$, $j \le j'$ and $(i',j') \in T$, we also have $(i,j) \in T$. How many safe subsets are there?
5. [Putnam, 1990] How many ordered pairs (A,B) are there, where A,B are subsets of $\{1,2,\ldots,n\}$ such that every element of A is larger than $|B|$ and every element of B is larger than $|A|$?

6. [Colombia, 1997] We are given an $n \times m$ grid of squares. In how many ways can we color the *edges* of the squares in yellow, red, and blue, so that each square has two sides of one color and two sides of another color?

7. [Putnam, 2002] A nonempty subset $S \subseteq \{1, 2, \ldots, n\}$ is *decent* if the average of its elements is an integer. Prove that the number of decent subsets has the same parity as n.

8. [USAMO, 1996] An n-term sequence in which every term is either 0 or 1 is called a "binary sequence" of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to $0, 1, 0$ in that order. Let b_n be the number of binary sequences of length n containing no four consecutive terms equal to $0, 0, 1, 1$ or $1, 1, 0, 0$ in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

9. An alphabet consists of $k > 1$ letters. A *word* is a finite sequence of letters. Let S be a finite set of words with the property that no word in S consists of the first n letters of any other word in S, for any n. If $|S_n|$ denotes the number of words in S of length n, prove that

$$\frac{|S_1|}{k} + \frac{|S_2|}{k^2} + \cdots + \frac{|S_n|}{k^n} \leq 1.$$

10. [ELMO, 2008] In how many ways can the numbers $1, \ldots, n$ be arranged in an order $a_1, \ldots, a_n$ so that $a_1 = 1$ and $a_{i+1} \leq a_i + 2$ for each $i = 1, \ldots, n-1$?

11. [Turkey, 1998] Let $A = \{1, 2, \ldots, n\}$. Let $P(A)$ be the set of nonempty subsets of A. Find the number of functions $f : P(A) \to A$ such that $f(B) \in B$ for all B, and $f(B \cup C)$ equals $f(B)$ or $f(C)$ for all B and C.

12. Find the number of strings of n letters, each equal to A, B, or C, such that the same letter never occurs three times in succession.

13. Prove that the number of partitions of a positive integer n into distinct parts equals the number of partitions into odd parts.

14. [Putnam, 2003] In a Dyck path, a "return" is a sequence of consecutive downsteps that is preceded by an upstep and ends on the x-axis. How many Dyck paths of length $2n$ contain no return of even length?

15. [Iran, 1999] In a deck of $n > 1$ cards, each card has some of the numbers $1, 2, \ldots, 8$ written on it. Each card contains at least one number; no number appears more than once on the same card; and no two cards have the same set of numbers. For every set containing between 1 and 7 numbers, the number of cards showing at least one of those numbers is even. Determine n, with proof.

16. [China, 2006] d and n are positive integers such that $d \mid n$. Consider the ordered n-tuples of integers $(x_1, \ldots, x_n)$ such that $0 \le x_1 \le \cdots \le x_n \le n$, and $x_1 + \cdots + x_n$ is divisible by d. Prove that exactly half of these n-tuples satisfy $x_n = n$.

17. Let $E(n)$ be the number of partitions of the natural number n into an even number of parts, and let $O(n)$ be the number of partitions of n into an odd number of parts. Prove that $|E(n) - O(n)|$ equals the number of partitions of n into distinct odd parts.

18. [Putnam, 2005] For positive integers m, n, let $f(m, n)$ be the number of n-tuples of integers $(x_1, \ldots, x_n)$ such that $|x_1| + \cdots + |x_n| \le m$. Prove that $f(m, n) = f(n, m)$.

19. Find the number of permutations σ of $\{1, 2, \ldots, n\}$ such that $\sigma(i) \ne i$ for all i.

    (a) Give a direct (non-inductive) solution.

    (b) Use a bijection to generate a recurrence that leads to the solution.

20. [TST, 2009] Let m, n be positive integers. The region R is a $2^m \times 2^n$ rectangle with one 1×1 corner square removed. You would like to choose $m + n$ rectangles, of integer side lengths and areas $2^0, 2^1, \ldots, 2^{m+n-1}$, and use them to tile R. Prove that this can be done in at most $(m + n)!$ ways.

21. [IMO, 1989] A permutation π of $\{1, 2, \ldots, 2n\}$ has property P if $|\pi(i) - \pi(i+1)| = n$ for some i. For any given $n \ge 1$, prove that there are more permutations with property P than without it.

22. [IMO Shortlist, 2008] Let $S = \{x_1, x_2, \ldots, x_{k+l}\}$ be a $(k + l)$-element set of real numbers contained in the interval $[0, 1]$, where k and l are positive integers. A k-element subset $A \subseteq S$ is called *nice* if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{l} \sum_{x_j \in S \setminus A} x_j \right| \le \frac{k + l}{2kl}.$$

    Prove that the number of nice subsets is at least $\frac{2}{k+l}\binom{k+l}{l}$.

23. [MOP, 1998] Let $a_1, a_2, \ldots$ be a sequence of integers such that for each n,

$$\sum_{d \mid n} a_d = 2^n.$$

    Prove that a_n is divisible by n for each n.

24. [TST, 2010] Let T be a finite set of positive integers greater than 1. A subset S of T is called *good* if, for every $t \in T$, there exists some $s \in S$ with $\gcd(s, t) > 1$. Prove that the number of good subsets of T is odd.

25. Given n vertices labeled $1, \ldots, n$, how many trees are there on these vertices?

Useful Counting Facts
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- Number of subsets of an n-element set: 2^n
- Number of permutations of n objects: $n!$
- Number of k-element subsets of an n-element set: $\binom{n}{k} = n!/k!(n-k)! \quad (0 \le k \le n)$
- Binomial coefficient identities:
  - $\binom{n}{k} = \binom{n}{n-k}$
  - $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
  - $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$
  - $\sum_{m=k-1}^{n} \binom{m}{k-1} = \binom{n+1}{k}$
  - $k\binom{n}{k} = n\binom{n-1}{k-1}$
  - $\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$ (Vandermonde convolution)
  - $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$
  - $\sum_{k=0}^{n} k\binom{n}{k} = 2^{n-1}n$
  - $\sum_{m=0}^{n} \binom{m}{j}\binom{n-m}{k} = \binom{n+1}{j+k+1}$
  - $\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$ for $n > 0$
  - more generally $\sum_{i=0}^{n}(-1)^i\binom{n}{i}P(x+i) = 0$ if P is a polynomial of degree $< n$

  All of these, except maybe the last statement, can be checked by direct counting arguments. They can also be proven algebraically.
- Number of functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$: m^n
- Number of choices of k elements of $\{1, 2, \ldots, n\}$, without regard to ordering and with repetitions allowed: $\binom{n+k-1}{k}$
- Number of paths from $(0, 0)$ to (m, n) using steps $(1, 0)$ and $(0, 1)$: $\binom{n+m}{m}$
- Number of ordered r-tuples of positive integers with sum n: $\binom{n-1}{r-1}$
- Number of ways of dividing $\{1, 2, \ldots, kn\}$ into k subsets of size n: $(kn)!/(n!)^k k!$
- Number of Dyck paths of length $2n$ or ways of triangulating a regular $(n + 2)$-gon by diagonals (see main handout for more): $C_n = \binom{2n}{n}/(n+1)$ (nth Catalan number)

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