Algebra

1. Dividing both sides by abc, the inequality becomes

$$\sum \frac{bc-ab}{ab+ca} \geq 0 \iff \sum \frac{bc}{ab+ca} \geq \sum \frac{ab}{ab+ca},$$

which is true by the rearrangement inequality.■

2. It's equivalent to maximize the product

$$|x-x_1|\cdots|x-x_n|.$$

Note that $x_{r+s}-x_s \leq x_{r+t}-x_t$ if $s \leq t$ with equality iff $s=t$.

Assume for the sake of contradiction that the optimal point lies in the interval $[x_{i-1}, x_i)$ for some $2 \leq i \leq n-1$. Now consider $x' = x_{n-1} + (x_i - x) \in (x_{n-1}, x_n)$ (this is easier to visualize with n points the real line). Clearly we have $x'-x_k > x-x_k > 0$ for $1 \leq k \leq i-2$ and $x'-x_k \geq x_{n-1+i-k}-x \geq 0 \iff x_{n-1}-x_{n-1+i-k} \geq x_k - x_i$ for $i \leq k \leq n-1$. Furthermore,

$$\begin{aligned}(x_n-x')(x'-x_{i-1}) > (x_n-x)(x-x_{i-1}) \geq 0 &\iff x' - \frac{x_n+x_{i-1}}{2} \leq \frac{x_n+x_{i-1}}{2} - x\\ &\iff x_n - x_{n-1} \geq x_i - x_{i-1},\end{aligned}$$

so

$$\begin{aligned}&|(x'-x_1)\cdots(x'-x_{i-2})|\cdot|(x'-x_{n-1})\cdots(x'-x_i)|\cdot|(x'-x_{i-1})(x'-x_n)|\\ &>|(x-x_1)\cdots(x-x_{i-2})|\cdot|(x-x_i)\cdots(x-x_{n-1})|\cdot|(x-x_n)(x-x_{i-1})|,\end{aligned}$$

a contradiction.■

3. Let $P(x,y)$ denote the assertion that $f\left(\frac{x+f(x)}{2}+y\right) = 2x - f(x) + f(f(y))$.

First, suppose to the contrary that some $a<b$ exist such that $\ell = f(a) = f(b)$. Then $f(x) = \ell$ for all $a \leq x \leq b$, so for sufficiently small ϵ,

$$P(a,b), P(a+2\epsilon, b-\epsilon) \implies 2a - \ell + f(\ell) = f\left(\frac{a+\ell}{2}+b\right) = 2(a+2\epsilon) - \ell + f(\ell),$$

a contradiction.

Thus f is injective, so $a<b \implies f(a)<f(b)$ and $f(a) \geq f(b) \implies a \geq b$, whence

$$P(0,r) \implies f(f(r)) = f(0) + f\left(\frac{f(0)}{2}+r\right) \implies f(r) \geq r + \frac{f(0)}{2}\forall r.$$

Now assume for the sake of contradiction that $c = f(0)/2 > 0$. Then

$$\begin{aligned}P(f(r),r) \implies 2f(r) &= f\left(\frac{f(r)+f(f(r))}{2}+r\right)\\ &\geq \frac{r+c+r+2c}{2}+r+c\\ &= 2r+\frac{5c}{2} \implies f(r) \geq r + \frac{5c}{4}\end{aligned}$$

for all r, which is clearly impossible (since $(5/4)^n$ tends to infinity).

Hence $f(0)=0$, so

$$P(0,r) \implies f(f(r)) = f(r) \implies f(r) = r\forall r,$$

as desired.■

4.

5. If f is constant, then the desired result is trivial, so suppose that f is not constant. Then there exist integers a_i, d_i with $d_i > 0$ (by the non-decreasing condition) such that $f(p_i r + s_i) = a_i + d_i r$ for all $r \geq 1$. If $p_i/d_i \neq p_j/d_j$ for some i, j, then we easily get a bounding contradiction with the non-decreasing condition, so we have $d_i = kp_i$ for all i and some positive integer k. By CRT, there exists a positive integer x such that $x \equiv s_i \pmod{p_i}$ for all i. But then

$$f(x) = a_i + kp_i \left(\frac{x - s_i}{p_i} \right) \forall 1 \leq i \leq n \implies a_1 - ks_1 = \cdots = a_n - ks_n = c$$

for some integer c, i.e. $f(p_i r + s_i) = a_i + d_i r = c + k(p_i r + s_i)$ for all $r \geq 1$. Again by CRT, we can find a positive integer a such that $a + i \equiv s_i \pmod{p_i}$. Because $f(a+i) = c + k(a+i)$ for $1 \leq i \leq n$, we're done.■

Geometry

1. Obviously $ACBH_1$ is cyclic. Let M', N, N' be the midpoints of BH_1, BC, AH_1, respectively, and S be the foot from C_1 to BH_1. We'll show that M, M', N, N', P, Q, R, S lie on a circle centered at the center of rectangle $MNM'N'$, which clearly implies that $M_1 = M'$ (which lies on BH_1). By trivial angle chasing, $\angle NC_1C = \angle BCH_1 = \angle C_1AH_1 = \angle PC_1H_1$, so NC_1P is a line. Similarly, MC_1S, $N'C_1R$, and $M'C_1Q$ are lines. By right angles, M, M', Q, S lie on a circle with diameter MM' and N, N', R, P lie on a circle with diameter NN'. But the midpoints of MM' and NN' coincide at the center of rectangle $MNM'N'$ while $MM' = NN'$, so we're done.■

2. Consider either $\triangle ABC$ with maximal area or segment XY with maximal length. The rest is easy.■

3. Obviously $M_1M_3 \cap M_2M_4$, $I_1I_3 \perp I_2I_4$, and $K = I_1I_3 \cap I_2I_4$. Define $4\alpha = \angle AOB$, etc. such that $\alpha + \beta + \gamma + \delta = 90°$.

   First we show that $M_1M_3 \| I_1I_3$. Indeed, since I_1KI_3 bisects $\angle AKB = \angle CKD$,
   $$\angle AOI_1 - \angle DOI_3 = \angle AOI_1 + \angle COI_3 - 4\gamma = 2\alpha - 2\gamma = \angle AOM_1 - \angle COM_3,$$
   so $\angle I_1OM_1 = \angle I_3OM_3$. Similarly, $M_2M_4 \| I_2I_4$.

   Let $T = M_1M_3 \cap M_2M_4$, $R = M_1M_3 \cap I_2I_4$, and $S = M_2M_4 \cap I_1I_3$. Note that $M_1I_2 = M_1I_4$, $I_2R = I_4R$, etc. We have
   $$\frac{M_1R}{M_3R} = \frac{[M_1I_2I_4]}{[M_3I_2I_4]} = \frac{M_1I_4}{M_3I_4}\frac{M_1I_2}{M_3I_2}\frac{\sin 2\gamma}{\sin 2\alpha} = \frac{\tan\alpha}{\tan\gamma},\ \frac{M_2S}{M_4S} = \frac{\tan\beta}{\tan\delta}.$$
   Also,
   $$\frac{M_1T}{M_3T} = \frac{M_1M_2}{M_3M_2}\frac{M_1M_4}{M_3M_4} = \tan(\alpha+\beta)\tan(\alpha+\delta),\ \frac{M_2T}{M_4T} = \tan(\beta+\alpha)\tan(\beta+\gamma)$$
   and
   $$M_1M_3 = 2R\sin(\alpha+2\beta+\gamma),\ M_2M_4 = 2R\sin(\beta+2\gamma+\delta)$$
   while
   $$\frac{I_1K}{I_3K} = \frac{AB}{CD} = \frac{\sin 2\alpha}{\sin 2\gamma},\ \frac{I_2K}{I_4K} = \frac{\sin 2\beta}{\sin 2\delta}$$
   and
   $$I_1I_3 = \frac{2M_2M_4\sin\beta\sin\delta}{\sin(\beta+\delta)},\ I_2I_4 = \frac{2M_1M_3\sin\alpha\sin\gamma}{\sin(\alpha+\gamma)}.$$
   Now let $X = M_1I_1 \cap M_3I_3$ and $Y = M_2I_2 \cap M_4I_4$, so it suffices to show that $d(X, I_1I_3) = d(Y, I_1I_3)$ (so $d(X, I_2I_4) = d(Y, I_2I_4)$ and thus $X = Y$ follows by symmetry).

   All segments will be directed from now on. By easy ratios with similar triangles, we can compute
   $$\begin{aligned}
   d(X, I_1I_3) &= \frac{I_1I_3 \cdot ST}{M_1M_3 - I_1I_3} \\
   &= M_2M_4 \cdot \frac{\frac{M_2T}{M_2M_4} - \frac{M_2S}{M_2M_4}}{\frac{M_1M_3}{I_1I_3} - 1} \\
   &= M_2M_4 \cdot \frac{\frac{\tan(\beta+\alpha)\tan(\beta+\gamma)}{\tan(\beta+\alpha)\tan(\beta+\gamma)+1} - \frac{\tan\beta}{\tan\beta+\tan\delta}}{\frac{\sin(\alpha+2\beta+\gamma)\sin(\beta+\delta)}{2\sin\beta\sin\delta\sin(\beta+2\gamma+\delta)} - 1} \\
   &= M_2M_4 \frac{\sin(\alpha+\beta)\sin(\beta+\gamma)\sin(\beta+\delta) - \sin\beta\cos\delta\cos(\alpha-\gamma)}{\cos(\beta-\delta)\sin(\beta+\delta) - 2\sin\beta\sin\delta\cos(\alpha-\gamma)} \frac{2\sin\beta\sin\delta}{\sin(\beta+\delta)} \\
   &= M_2M_4 \frac{[2\sin(\alpha+\beta)\sin(\beta+\gamma)]\cos(\alpha+\gamma) - [2\sin\beta\sin(\alpha+\beta+\gamma)]\cos(\alpha-\gamma)}{\cos(\beta-\delta)\cos(\alpha+\gamma) - 2\sin\beta\sin\delta\cos(\alpha-\gamma)} \frac{\sin\beta\sin\delta}{\sin(\beta+\delta)} \\
   &= M_2M_4 \frac{-\cos(\alpha+2\beta+\gamma)\cos(\alpha+\gamma) + \cos(\alpha+2\beta+\gamma)\cos(\alpha-\gamma)}{\cos(\beta-\delta)\cos(\alpha+\gamma) - 2\sin\beta\sin\delta\cos(\alpha-\gamma)} \frac{\sin\beta\sin\delta}{\sin(\beta+\delta)} \\
   &= M_2M_4 \frac{-2\sin(\beta-\delta)\sin\alpha\sin\gamma}{\cos(\beta-\delta)\cos(\alpha+\gamma) - 2\sin\beta\sin\delta\cos(\alpha-\gamma)} \frac{\sin\beta\sin\delta}{\sin(\beta+\delta)}
   \end{aligned}$$

and

$$\begin{aligned}
\frac{d(Y,I_1I_3)-I_2K}{d(Y,I_1I_3)-M_2S} &= \frac{I_2I_4}{M_2M_4} \\
\implies d(Y,I_1I_3) &= \frac{M_2M_4\cdot I_2K - I_2I_4\cdot M_2S}{M_2M_4-I_2I_4} \\
&= M_2M_4\cdot\frac{\frac{I_2K}{I_2I_4}-\frac{M_2S}{M_2M_4}}{\frac{M_2M_4}{I_2I_4}-1} \\
&= M_2M_4\cdot\frac{\frac{\sin 2\beta}{\sin 2\beta+\sin 2\delta}-\frac{\tan\beta}{\tan\beta+\tan\delta}}{\frac{\sin(\beta+2\gamma+\delta)\sin(\alpha+\gamma)}{2\sin\alpha\sin\gamma\sin(\alpha+2\beta+\gamma)}-1} \\
&= M_2M_4\frac{\cos\beta-\cos\delta\cos(\beta-\delta)}{\cos(\alpha-\gamma)\sin(\alpha+\gamma)-2\sin\alpha\sin\gamma\cos(\beta-\delta)}\frac{2\sin\alpha\sin\beta\sin\gamma}{\sin(\beta+\delta)} \\
&= M_2M_4\frac{-\sin\delta\sin(\beta-\delta)}{\cos(\alpha-\gamma)\cos(\beta+\delta)-2\sin\alpha\sin\gamma\cos(\beta-\delta)}\frac{2\sin\alpha\sin\beta\sin\gamma}{\sin(\beta+\delta)}
\end{aligned}$$

Thus it remains to show that

$$\begin{gathered}
\cos(\beta-\delta)\cos(\alpha+\gamma)-2\sin\beta\sin\delta\cos(\alpha-\gamma)=\cos(\alpha-\gamma)\cos(\beta+\delta)-2\sin\alpha\sin\gamma\cos(\beta-\delta) \\
\iff \cos(\beta-\delta)\cos(\alpha-\gamma)=\cos(\alpha-\gamma)\cos(\beta-\delta),
\end{gathered}$$

as desired.■

We can also solve this problem with complex numbers. Note that $m_1^2=ab$, etc., and $abcd = -m_1m_2m_3m_4$, so there exist integers p,q,r,s such that

$$(a,b,c,d,m_1,m_2,m_3,m_4)=(p^2,q^2,r^2,s^2,-pq,qr,rs,sp).$$

Then $I_1 = AM_2\cap BM_4$, etc., so

$$i_1=\frac{pr(q^2+sp)-qs(p^2+qr)}{pr-qs},\ x_1=\frac{pr(q+s)-qs(p+r)}{p+r-q-s},$$

etc. where $\{M_i,X_i\}=M_iX_i\cap(ABCD)$ for all i. Then it just remains to show that $m_1x_1\cap m_3x_3 = m_2x_2\cap m_4x_4$, where to cycle through the variables (for ease of computation) we set

$$(p,q,r,s)\mapsto(-q,r,s,p)\mapsto(-r,s,p,-q)\mapsto(-s,p,-q,-r)\mapsto\cdots.■$$

4. By simple angle chasing $\angle AXY=\angle AYX=\angle BKX=\angle XKY=\angle CKY$, so A is the intersection of the tangents XX, YY of the circle (KXY). Set (KXY) as the unit circle in the complex plane, so $a=2xy/(x+y)$. From the three similarities $\triangle MXP\sim\triangle MKX$, $\triangle YXP\sim\triangle YBX$, and $\triangle XYP\sim\triangle XCY$ (all of these are with *opposite orientation*), we have

$$\begin{aligned}
\frac{m-x}{p-x} &= \overline{\left(\frac{m-k}{x-k}\right)} \implies p=\frac{k(x^2+y^2)-xy(x+y)}{k(x+y)-2xy} \\
\frac{y-b}{x-b} &= \overline{\left(\frac{y-x}{p-x}\right)} \implies b=k+\frac{(x-k)^2}{y-k} \\
\frac{x-c}{y-c} &= \overline{\left(\frac{x-y}{p-y}\right)} \implies c=k+\frac{(y-k)^2}{x-k}.
\end{aligned}$$

But Q is the spiral center taking XY to BC and XB to YC, so

$$(q-b)(q-y)=(q-c)(q-x)\implies q=\frac{by-cx}{b+y-c-x}=\frac{x(y-k)^2+y(x-k)^2}{(x-k)^2+(y-k)^2}.$$

Finally,

$$\frac{\frac{a-x}{q-x}}{\frac{p-k}{q-k}} = -\frac{x(x+y-2k)(k(x+y)-2xy)}{(x-k)^2(x+y)^2} \in \mathbb{R}$$

so $\measuredangle QXA = \measuredangle QKP$, as desired.■

5. Let $\alpha = \angle ABD = \angle ABC$. Then the tangent from A to the circumcircle of $\triangle AMP$ is parallel to BC iff $90^\circ - \angle DAM = \angle APM$, i.e. $\angle APM = \alpha$ or $MA = MP$. But $MA = MD = MB$, so it suffices to show that $ADPB$ is cyclic. By trivial angle chasing $\angle PAB = \angle PDB$, as desired.■

Number Theory

1. Suppose that $a_k = 2k$ and let p be the minimal prime p such that $p|k-1$, which exists since $k \geq 3$.. Clearly $a_{k+i} = 2k+i$ for $0 \leq i \leq p-2$ and $a_{k+p-1} = 2k+p-2$, so $a_{k+p-1} - a_{k+p-2} = p$. Since $a_{k+p-1} = 2(k+p-1)$, we can repeat this argument to get infinitely many primes in the sequence $\{a_n - a_{n-1}\}$.$\blacksquare$

2. Since x is relatively prime to n iff $n-x$ is, we have $f(n) = n(n+1-\phi(n))/2$.

   Case 1: p does not divide n. Then $n(n+1-\phi(n)) = (n+p)(n+p+1-\phi(n+p))$ and $\gcd(n, n+p) = 1$ force $n+p|n+1-\phi(n)$. But $0 < n+1-\phi(n) < n+p$, which is impossible.

   Case 2: $n = pk$ for some integer $k \geq 1$. Then

$$\begin{aligned} k(pk+1-\phi(pk)) &= (k+1)(pk+p+1-\phi(pk+p)) \\ \implies k(pk+1-\phi(pk)) - k[p(k+1)] &= (k+1)(pk+p+1-\phi(pk+p)) - (k+1)[pk] \\ \implies k(-p+1-\phi(pk)) &= (k+1)(p+1-\phi(pk+p)), \end{aligned}$$

   so for some integer m we have $\phi(pk)+p-1 = (k+1)m$ (1) and $\phi(pk+p)-p-1 = km$. (2)

   If $p=2$ and $k \geq 2$, then $\phi(pk), \phi(pk+p)$ are both even ($\phi(x)$ is even for $x \geq 3$) so $(k+1)m$ and km are both odd, which is impossible since $k, k+1$ have opposite parity. We can check that $(p,k) = (2,1)$ does not work, so now we can assume $p \geq 3$ is odd.

   From (1), we have $(k+1)m > 0 \implies m \geq 1$, and from (2), we have

$$km < pk + p - p - 1 = pk - 1 \implies m \leq p-1.$$

   Taking (1) and (2) $\pmod{p-1}$ and using the fact that

$$p - 1 = \phi(p)|\phi(pk), \phi(pk+p),$$

   we have $p-1|(k+1)m$ and $p-1|km+2$. Subtracting these two divisibility relations, we find that $m \equiv 2 \pmod{p-1}$.

   Since $1 \leq m \leq p-1$, we have $m=2$, so $\phi(pk) = 2k+3-p$ and $\phi(pk+p) = 2k+1+p$. If $p|k+1$, then $p|2k+1+p \implies p|1$, which is absurd, and if $p|k$, then $p|2k+3-p \implies p|3$, in which case $\phi(3k) = 2k$ and

$$2k+4 = \phi(3k+3) \leq \frac{2}{3}(3k+3) = 2k+2,$$

   another contradiction. (This also shows that $p=3$ does not work, so from now on assume that $p \geq 5$.)

   Thus $k, k+1$ are relatively prime to p, so $\phi(pk) = (p-1)\phi(k)$ and $\phi(pk+p) = (p-1)\phi(k+1)$, i.e. $\phi(k) = \frac{2k+3-p}{p-1}$ and $\phi(k+1) = \frac{2k+1+p}{p-1}$. But this means that $\phi(k+1) - \phi(k) = 2$. Since $2|\phi(x)$ for $x \geq 3$, we must have $k \geq 3$. But then $\phi(k+1), \phi(k)$ cannot be congruent $\pmod 4$, so because they're both even exactly one of them must be congruent to $2 \pmod 4$.

   First suppose that $\phi(k) \equiv 2 \pmod 4$, so that $k \in \{4, q^r, 2q^r\}$ for some odd prime q and integer $r \geq 1$. If $k=4$, then

$$2 = \phi(k) = \frac{2k+3-p}{p-1} = \frac{11-p}{p-1} \implies 3p = 13,$$

   which has no solutions. If $k = q^r$ or $k = 2q^r$, then

$$q^r - q^{r-1} = \phi(k) = \frac{2k+3-p}{p-1} \leq \frac{4q^r+3-p}{p-1} < \frac{4q^r}{p-1} \implies 1 < \frac{4}{p-1} + \frac{1}{q}.$$

   Because $q \geq 3$ and $p \geq 5$, we must have $p = 5$. Therefore $\phi(k) = \frac{k-1}{2}$, so k is odd, i.e. $k = q^r$ and $q^{r-1}(q-1) = \frac{q^r-1}{2}$, which is impossible.

Now suppose that $\phi(k+1) \equiv 2 \pmod 4$, so that instead $k \in \{3, q^r - 1, 2q^r - 1\}$ for some odd prime q and integer $r \geq 1$. If $k = 3$, then

$$2 = \phi(k+1) = \frac{2k+1+p}{p-1} = \frac{7+p}{p-1} \implies p = 9,$$

which has no solutions. If $k + 1 = q^r$ or $k + 1 = 2q^r$, then

$$q^r - q^{r-1} = \phi(k+1) = \frac{2k+1+p}{p-1} \leq \frac{4q^r - 1 + p}{p-1} = \frac{4q^r}{p-1} + 1 \implies 1 \leq \frac{4}{p-1} + \frac{1}{q^r} + \frac{1}{q}.$$

Because $q \geq 3$, $r \geq 1$, and $p \geq 5$, we must have $p \in \{5, 7, 11\}$. If $p = 5$, then $\phi(k+1) = \frac{k+3}{2}$, so k is odd and $q^{r-1}(q-1) = q^r + 1$, which is impossible. If $p = 7$, then

$$1 \leq \frac{4}{p-1} + \frac{2}{q} \implies q \leq 6 \implies q \in \{3, 5\},$$

and if $p = 11$, then $q \leq \frac{10}{3} \implies q = 3$. Furthermore, if $p = 7$ and $q = 5$, then $r \leq 1 \implies k \in \{4, 9\}$, if $p = 11$ and $q = 3$, then $r \leq 1 \implies k \in \{2, 5\}$, both of which are easy to check. If $p = 7$ and $q = 3$, then

$$3^r - 3^{r-1} = \frac{k+4}{3} \implies k = 3^{r+1} - 3^r - 4,$$

which yields no solutions.

Finally, we conclude that $f(n+p) = f(n)$ has no solutions, as desired.■

3. The condition that *no three points are collinear* is the key to this problem. Without it, the case in which $x_i = y_i = i$ for all $1 \leq i \leq n$ would clearly force $k \geq n - 1$ in general (even if we relax it to not all points being collinear, a similar construction gets $k \geq n - 2$), so we should expect an answer substantially less than n.

   We can easily see that $k_2 = 1$, $k_3 = 1$, and $k_4 = 2$, so $k_n = \lfloor n/2 \rfloor$ is a reasonable guess. The idea of taking $y = Q(x)$ for a polynomial in x as in the previous paragraph gets us $2k_n \geq n - 1$ in general when $\deg Q = 2$ (i.e. take $(x_i, y_i) = (i, Q(i))$).

   Now we just need to construct an example for $k_n = \lfloor n/2 \rfloor$. Since $\lfloor 2m/2 \rfloor = m = \lfloor (2m+1)/2 \rfloor$, it suffices to construct for $n = 2m + 1$ odd (just add an arbitrary triple $(x_{n+1}, y_{n+1}, c_{n+1})$ for $n = 2m$). We can set up an interpolation scheme (similar to Lagrange interpolation/CRT) by defining lines $A_iA_j \equiv L_{i,j}(x, y) = r_{i,j}x + s_{i,j}y + t_{i,j}$ so that $L_{i,j}(A_l) = 0$ iff $l \in \{i, j\}$. Taking all indices $\bmod 2m + 1$,

$$P(x, y) = \sum_{i=1}^{2m+1} c_i \prod_{j=1}^{m} \frac{L_{i+j,i+j+m}(x, y)}{L_{i+j,i+j+m}(x_i, y_i)}$$

   is clearly a working polynomial of degree m, as desired.■

4. By definition of r, WLOG $0 \leq r < p_1 < p_2 < \cdots < p_n$. Assume for the sake of contradiction that $r \geq n - 1$. Then $p_1 \geq r + 1 \geq n$, so

$$0 < \frac{1}{p_1} + \cdots + \frac{1}{p_n} - \frac{p_1}{p_1 \cdots p_n} < \frac{1}{p_1} + \cdots + \frac{1}{p_n} - \frac{r}{p_1 \cdots p_n} \leq n \cdot \frac{1}{n} - 0 = 1,$$

   contradicting the fact that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} - \frac{r}{p_1 \cdots p_n}$$

   is an integer.■

5. Fix a positive integer r and consider a prime $p > a_1^r + a_2^r + a_3^r + b_1^r + b_2^r + b_3^r$. Taking n such that $n \equiv r \pmod{p-1}$ and $n \equiv \frac{a_3^r - a_1^r}{a_1^r + a_2^r + a_3^r} \pmod p$ (which exists by CRT), we have

$$(n+1)a_1^n + na_2^n + (n-1)a_3^n \equiv n(a_1^r + a_2^r + a_3^r) - (a_3^r - a_1^r) \equiv 0 \pmod p.$$

Thus

$$0 \equiv (n+1)b_1^n + nb_2^n + (n-1)b_3^n \equiv n(b_1^r + b_2^r + b_3^r) - (b_3^r - b_1^r) \pmod{p},$$

so

$$\frac{b_3^r - b_1^r}{b_1^r + b_2^r + b_3^r} \equiv n \equiv \frac{a_3^r - a_1^r}{a_1^r + a_2^r + a_3^r} \pmod{p}$$

for all sufficiently large primes p, whence

$$(a_1^r + a_2^r + a_3^r)(b_3^r - b_1^r) - (b_1^r + b_2^r + b_3^r)(a_3^r - a_1^r) = 0 \implies 2a_3^r b_1^r + a_3^r b_2^r + a_2^r b_1^r = 2a_1^r b_3^r + a_2^r b_3^r + a_1^r b_2^r$$

for all positive integers r. The conclusion easily follows with some casework, since we must have the multisets $\{a_3b_1, a_3b_1, a_3b_2, a_2b_1\}$ and $\{a_1b_3, a_1b_3, a_2b_3, a_1b_2\}$ equal (clearly the largest elements must be equal or else one side will dominate the other as $r \to \infty$, etc.).$\blacksquare$

Combinatorics

1. Since 4 does not divide 10, some kid must have two colors of pencils. Then we can find the remaining two colors from two other children, so $n \leq 3$. If we distribute the pencils as

$$AAAA, AAAA, BBBB, BBBB, AACC, BBCC, CCDD, CCDD, DDDD,$$

we clearly need three kids, so $n \geq 3$, as desired.■

2. Suppose otherwise, and note that $a_{i+1} + a_{i+4} > a_{i+2} + a_{i+3} \iff a_{i+1} - a_{i+2} > a_{i+3} - a_{i+4}$ (indices taken $\pmod{100}$). Thus

$$a_1 - a_2 \leq a_3 - a_4 \leq \cdots \leq a_{99} - a_{100} \leq a_1 - a_2$$

and

$$a_2 - a_3 \leq a_4 - a_5 \leq \cdots \leq a_{100} - a_1 \leq a_2 - a_3,$$

so $\alpha = a_1 - a_2 = \cdots = a_{99} - a_{100}$ and $\beta = a_2 - a_3 = \cdots = a_{100} - a_1$ for some α, β. But $50\alpha + 50\beta = 0$, so $a_1 = a_3$, which is not allowed.■

3. It's natural to replace 17 by numbers $k \geq 2$ to find a pattern. We also expect d to be very large (i.e. close to 100^2), since otherwise the construction would be very difficult.

For $k = 2$, it's easy to get rid of all rows except for at most 1 odd row. In the odd row, it's easy to get rid of all squares but at most 1 odd square. The construction for $d_2 = 100^2 - 1$ is trivial.

In general, let the n rows mod$k$ have sums $a_1, \ldots, a_n$. Let the i^{th} partial sum be $s_i = a_1 + \cdots + a_i$. We'll prove by induction on $n \geq 0$ that we can color rows such that at most $k-1$ rows remain uncolored. If k divides any of the partial sums s_i then we can color rows 1 through i and then use the inductive hypothesis, so suppose that k divides none of the s_i. Now let $f(i)$ denote the largest index $j \in [i, n]$ such that $s_j \equiv s_i \pmod{k}$ (this exists since $s_i \equiv s_i$), and consider the sequence $b_1, \ldots, b_\ell$ such that $b_1 = 1$ and $b_m = f(b_{m-1} + 1)$ for $m \geq 1$. By the definition of f, we see that $s_{b_1}, s_{b_2}, \ldots, s_{b_\ell} \pmod{k}$ are pairwise distinct and $b_\ell = n$. Yet k does not divide any of $s_{b_1}, s_{b_2}, \ldots, s_{b_\ell}$ by assumption, so $\ell \leq k-1$. Thus we can color the ℓ sets of rows $(b_m, f(b_m)]$ for $1 \leq m \leq \ell$ to leave exactly $\ell \leq k-1$ rows uncolored, as desired.

Similarly, in each of the uncolored rows we can color the squares so that at most $k-1$ squares remain uncolored, so $d(n,k) \geq n^2 - (k-1)^2$. The construction is easy.■

4.

5.