: a graph with 2n vertices and every degree is at least n
& (there is a cycle go thru all vertices)
Hamiltonian graph is graph which has a cycle go thru all vertices MA

Hint: For #1 choose v st f(v) is minimized.

Suppose u is not adjacent to v and deg u + deg v ≥ 2n Then G is Hamiltonian ⟺ G + uv is Hamiltonian.

IMO Training Problem Set

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aft,

1. Let n be a positive integer. There are $2n$ people in a room and each person is friends with at least n other people. Prove that the $2n$ people can sit at a circular table such that each person is sitting beside two of his/her friends. Issac

2. A math competition has six problems where no participant solved all six problems. For every pair of problems, there are strictly more than $\frac{2}{5}$ of the participants that solved both. Prove that at least two participants solved five problems.

3. There are n people at a party. Suppose that the following conditions are satisfied:

   (i) No three people are mutually friends.

   (ii) For every pair of strangers, there is exactly one person that are friends with both.

   (iii) There does not exist a person who is friends with everyone else.

   Prove that each person at the party has the same number friends.

4. There are n people at a party. Suppose the following conditions are satisfied:

   (i) No three people are mutually friends.

   (ii) For every pair of strangers, there are exactly two people who is friends with both.

   (iii) n is not a multiple of 11 and $5 < n < 30$.

   Find all possible values of n.

5. There are 2007 people in a room such that for any four people, there is one person that knows the other three. What is the minimum number of people in the room that can know 2006 people. Yan

6. Five students are sitting in a class where each fell asleep on exactly two separate occasions. For each pair of students, there was a moment where both were asleep. Prove that there was a moment where three students were asleep.

7. There are n people where k pairs of people are friends and no three people are mutually friends. Prove that there exists a person P such that amongst the non-friends of P, there are at most $k(1-\frac{4k}{n^2})$ pairs of friends.

8. Let $n > 1$ be a positive integer. Prove that a complete graph on n vertices can be decomposed into $n-1$ paths $P_1, P_2, \cdots, P_{n-1}$ such that P_i contains i edges.

9. A **tournament** is a complete finite graph where each edge contains an arrow going from one endpoint of the edge to the other. Each edge is coloured red or blue. Prove that there exists a vertex v such that for every other vertex u, there exists a mono-colour directed path from v to u.

10. There are n hungry students and n distinct food items on a table. Each student has a list of food items that he/she is willing to eat. For any positive integer k, $1 \leq k \leq n$, the following statement is true.

    For any group of k students, the set of food items that at least one student in the group is willing to eat is at least k.

    Prove that each student can be assigned a unique food item that he/she is willing to eat.

11. We say that a group of n friends is *complicated* if there exists four people A, B, C, D such that A loves B, B loves C, C loves D and D loves A. (We will assume that the relation "love" is symmetric. i.e. if A loves B then B loves A.) What is the maximum pairs of people amongst n friends that can be in love without the group being complicated?

12. In a competition there are n contestants and k judges, where $k \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose m is a number such that for any two judges their ratings coincide for at most m contestants. Prove that

$$\frac{m}{n} \geq \frac{k-1}{2k}$$