2003 Winter Camp

Open Contest

Algebra.

1. Determine the minimum value of

$$\sum_{i=1}^{10}\sum_{j=1}^{10}\sum_{k=1}^{10}|k(x+y-10i)(3x+6y-36j)(19x+95y-95k)|$$

where x and y range over all real numbers.

2. Let $n \geq 2$ be an integer. Let $x_1,\ x_2,\ \ldots,\ x_n$ be real numbers such that

$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n-1} x_i x_{i+1} = 1.$$

For any fixed k, $1 \leq k \leq n$, determine the maximum value of $|x_k|$.

3. Let n be a positive integer. Take $x_0 = 0$. For $1 \leq i \leq n$, let x_i be a positive real number, where $x_1 + x_2 + \cdots + x_n = 1$. Prove that

$$1 \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{1+x_0+x_1+\cdots+x_{i-1}}\sqrt{x_i+x_{i+1}+\cdots+x_n}} < \frac{\pi}{2}.$$

4. Let $\{a_1, a_2, \ldots\}$ be a sequence of non-negative numbers such that $a_{n+m} \leq a_n + a_m$ for all n and m. Prove that $a_n \leq ma_1 + (\frac{n}{m} - 1)a_m$ for all $n \geq m$.

5. Let $x_1, x_2, \ldots, x_{1997}$ be real numbers such that $-\frac{1}{\sqrt{3}} \leq x_i \leq \sqrt{3}$ for $1 \leq i \leq 1997$ and $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$. Determine the maximum value of $x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}$.

6. Let $n \geq 3$ be an integer and let $a_1,\ a_2,\ \ldots,\ a_n,\ b_1,\ b_2, \ldots,\ b_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$, $0 < a_1 = a_2$, $0 < b_1 \leq b_2$ and for $1 \leq i \leq n-2$, $a_i + a_{i+1} \leq a_{i+2}$ and $b_i + b_{i+1} < b_{i+2}$. Prove that $a_{n-1} + a_n \leq b_{n-1} + b_n$.

7. Let f be a function from the set of positive integers to itself such that $f(1) = 1$ and, for each positive integer n, $f(2n) < 6f(n)$ and $3f(n)f(2n+1) = f(2n)(1+3f(n))$. Determine all pairs (k, ℓ) such that $f(k) + f(\ell) = 293$ and $k < \ell$.

8. Let $f(x) = x^3 + ax^2 + bx + c$ be any cubic polynomial.

   (a) Determine the maximum value of λ if $f(x) \geq \lambda(x-a)^3$ whenever $f(x)$ has three non-negative roots.

   (b) Determine when $f(x) = \lambda(x-a)^3$ for the maximum value of λ.

9. Determine all functions $f : [1, \infty) \to [1, \infty)$ such that for all $x \geq 1$, $f(x) \leq 2(x+1)$ and $f(x+1) = \frac{1}{x}((f(x))^2 - 1)$.

10. A function f from the set of real numbers to itself satisfies

$$f(x^3 + y^3) = (x+y)((f(x))^2 - f(x)f(y) + (f(y))^2),$$

where x and y are arbitrary real numbers. Prove that $f(1996x) = 1996f(x)$ for any real number x.

11. Let a be a fixed real number. A sequence of polynomials $\{f_n(x)\}$ is defined by $f_0(x) = 1$ and for $n = 0, 1, 2, \ldots$,

$$f_{n+1}(x) = xf_n(x) + f_n(ax).$$

   (a) Prove that $f_n(x) = x^n f(\frac{1}{x})$ for $n = 0, 1, 2, \ldots$.

   (b) Find an explicit expression for $f_n(x)$.

12. The coefficient of the n-th degree polynomial

$$f(z) = c_0 z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_{n-1} z + c_n$$

are complex numbers. Prove that there exists a complex number z_0 such that $|z_0| \leq 1$ while $|f(z_0)| \geq |c_0| + |c_n|$.

Number Theory

13. Let $a,\ b,\ c,\ b+c-a,\ c+a-b,\ a+b-c$ and $a+b+c$ be seven distinct prime numbers such that $a+b=800$. Determine the maximum value of the difference between the largest and the smallest of these seven numbers.

14. Let $a_1,\ a_2,\ \ldots,\ a_{10}$ be ten distinct positive integers whose sum is 1995. Determine the minimum value of

$$a_1a_2 + a_2a_3 + \cdots + a_9a_{10} + a_{10}a_1.$$

15. Let n be an integer greater than 1. Do there always exist $2n$ distinct positive integers $a_1,\ a_2,\ \ldots,\ a_n,\ b_1,\ b_2,\ \ldots,\ b_n$ such that $a_1 + a_1 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$ and

$$n - 1 > \sum_{i=1}^{n} \frac{a_i - b_i}{a_i + b_i} > n - 1 - \frac{1}{1998}?$$

16. Let m and n be positive integers such that $m < 4002$, $n^2 - m^2 + 2mn \leq 4002(n-m)$ and $4002m - m^2 - n^2$ is divisible by $2n$.

   (a) Determine the minimum value of $\frac{4002m - m^2 - mn}{n}$.

   (b) Determine the maximum value of $\frac{4002m - m^2 - mn}{n}$.

17. Find all positive integers n for which there exist k integers $n_1,\ n_2,\ \ldots,\ n_k$, each greater than 3, such that

$$n = n_1 n_2 \cdots n_k = \sqrt[2^k]{2^{(n_1-1)(n_2-1)\cdots(n_k-1)}} - 1.$$

18. For any integer m, prove that $2m$ can be expressed in the form $a^{19} + b^{99} + k \cdot 2^{1999}$, where a and b are odd integers and k is a non-negative integer.

19. Let $S = \{1, 2, \ldots, 50\}$. Determine the smallest positive integer k such that for any k-element subset of S, there are two different elements a and b for which $a+b$ divides ab.

20. Let $S = \{1, 2, \ldots, 98\}$. Determine the smallest positive integer n for which any subset of S of size n contains 10 elements such that no matter how they are divided into two subsets of size 5, one subset contains an element relatively prime to each of the other four, while the other subset contains an element not relatively prime to any of the other four.

21. Determine the smallest positive integer m such that every subset of $\{1, 2, \ldots, 2001\}$ of size m contains two elements, not necessarily distinct, such that their sum is a power of 2.

22. Prove that there exist infintely many positive integers n for which the integers $1, 2, \ldots, 3n$ can be arranged in a $3 \times n$ array such that all rows have the same sum, all columns have the same sum, and both sums are divisible by 6.

23. Let $n > 1$ be an odd integer. Suppose

$$X_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) = (1, 0, 0, \ldots, 0, 1).$$

For $1 \leq k \leq n$, let

$$x_i^{(k)} = \begin{cases} 0 & \text{if } x_i^{(k-1)} = x_{i+1}^{(k-1)}; \\ 1 & \text{if } x_i^{(k-1)} \neq x_{i+1}^{(k-1)}. \end{cases}$$

We take $x_{n+1}^{(k-1)} = x_1^{(k-1)}$. Let

$$X_k = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}).$$

If the positive integer m satisfies $X_m = X_0$, prove that m is a multiple of n.

24. Let $A = \{0, 1, \ldots, 16\}$. For any mapping $f : A \longrightarrow A$, define $f^{(1)}(x) = f(x)$ and for any $n \geq 1$, $f^{(n+1)}(x) = f(f^{(n)}(x))$. Interpret $f^{(n)}(17)$ as $f^{(n)}(0)$. Suppose that for a bijection $f : A \longrightarrow A$, there exists a positive integer M such that

$$f^{(M)}(i+1) - f^{(M)}(i) \equiv \pm 1 \pmod{17}$$

for $0 \leq i \leq 16$ and for $m < M$, we have

$$f^{(m)}(i+1) - f^{(m)}(i) \not\equiv \pm 1 \pmod{17}$$

for $0 \leq i \leq 16$. Determine the maximum value of M taken over all bijections $f : A \longrightarrow A$ with the above properties.

Combinatorics

25. Let $\langle a_1, a_2, \ldots, a_n \rangle$ be any permutation of 1, 2, ..., n. Define $b_k = \max\{a_i : 1 \leq i \leq k\}$ for $k = 1, 2, \ldots, n$. Determine the average value of the first term a_1 of all permutations for which the sequence $\{b_1, b_2, \ldots, b_n\}$ takes on exactly two distinct values.

26. Determine all integers $n > 3$ such that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$ divides 2^{2000}.

27. Prove that $\displaystyle\sum_{k=0}^{n} 2^k \binom{n}{k} \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor} = \binom{2n+1}{n}$ for any positive integer n.

28. The sequence $\{a_n\}$ is defined by $a_1 = 0$, $a_2 = 1$ and $a_n = \frac{n}{2} a_{n-1} + \frac{n(n-1)}{2} a_{n-2} + (-1)^n (1 - \frac{n}{2})$ for $n \geq 3$. Simplify

$$a_n + 2\binom{n}{1} a_{n-1} + 3\binom{n}{2} a_{n-2} + \cdots + (n-1)\binom{n}{n-2} a_2 + n\binom{n}{n-1} a_1.$$

29. There are at least 4 smarties randomly distributed among at least 4 boxes. In each move, remove 1 smarty from each of 2 boxes and put both of them into a third box. Is it always possible to have all the smarties in 1 box?

30. Eight singers take part in a festival. The organizer wants to plan a number of concerts with four singers performing in each. The number of concerts in which a pair of singers performs together is the same for every pair. Determine the minimum number of concerts.

31. A multiple-choice examination has 5 questions, each with 4 choices. Each of 2000 students picks exactly 1 choice for each question. Among any n students for some positive integer n, there exist 4 such that any 2 of them give the same answers to at most 3 questions. Determine the minimun value of n.

32. A space city consists of 99 space stations. Every two stations are connected by a space highway. All highways are one-way except for 99 which are two-way. Design such a space city so that the number of *groups* is as large as possible, where a group is defined as a set of four stations such that we can travel from any one to any other of the four along the highways.

33. In a table-tennis tournament, all games are between pairs of participants. Each participant is a member of at most two pairs. No participant ever plays against another if the two form a pair. Two pairs play exactly once against each other as long as the preceding rule is not violated. A set $\{a_1, a_2, \ldots, a_k\}$ is given, where k is a positive integer and $0 < a_1 < a_2 < \cdots < a_k$ are multiples of 6. What is the minimum number of participants so that at the end of the tournament, the number of games played by each participant is a_i for some i, and for each i, at least one participant has played exactly a_i games?

34. On the circumference of a circle are 24 points which divide it into 24 arcs of length 1. In how many ways can we choose 8 of these points such that neither arc determined by any two chosen points has length 3 or 8.

35. On each vertex of a regular n-gon is a blue jay. They fly away and then return, again one blue jay on each vertex, but not necessarily to their original positions. Prove that there exist three blue jays such that the triangle determined by their earlier positions and the triangle determined by their later positions are of the same type, that is, both acute, both right or both obtuse.

36. Determine the number of ways of constructing a $4 \times 4 \times 4$ block from 64 unit cubes, exactly 16 of which are red, so that there is exactly one red cube within each $1 \times 1 \times 4$ subblock in any orientation.

Geometry

37. Let p be a prime. Determine the number of right triangles such that the incentre is (0,0), the vertex of the right angle is $(1994p, 7 \cdot 1994p)$, and the other two vertices have integer coordinates.

38. Let ABC be a non-obtuse triangle with circumcentre O and incentre I. Determine $\sin A$ if $AB > AC$, $\angle B = 45°$ and $\sqrt{2}OI = AB - AC$.

39. In triangle ABC, $\angle C = 90°$, $\angle A = 30°$ and $BC = 1$. Determine the minimum length of the longest sides of all triangles whose vertices lie respectively on the three sides of triangle ABC.

40. In triangle ABC, $a \le b \le c$ where $a = BC$, $b = CA$ and $c = AB$. The circumradius is R and the inradius is r. What can be said about $\angle C$ if $a + b - 2R - 2r$ is

    (a) positive;

    (b) zero;

    (c) negative?

41. Let D be a point inside an acute triangle ABC. Characterize geometrically the set of possible locations of the point D if it satisfies

$$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA = AB \cdot BC \cdot CA.$$

42. In an acute triangle ABC, $\angle C > \angle B$. D is a point on BC such that $\angle ADB$ is obtuse. H is the orthocentre of triangle BAD. F is a point inside triangle ABC and on the circumcircle of triangle BAD. Prove that F is the orthocentre of triangle ABC if and only if CF is parallel to HD and H is on the circumcircle of triangle ABC.

43. Let H be the orthocentre of an acute triangle ABC. From A draw two tangent lines AP and AQ to the circle whose diameter is BC, the points of tangency being P and Q respectively. Prove that P, H and Q are collinear.

44. $ABCD$ is a quadrilateral inscribed in a circle. The extensions of AB and DC meet at P, and the extensions of AD and BC meet at Q. The tangents from Q to the circle touch it at E and F. Prove that P, E and F are collinear.

45. Let $ABCD$ be a quadrilateral with AB parallel to DC. Let E be a point on AB and F a point on CD. The segments AF and DE intersect at G, while the segments BF and CE intersect at H.

    (a) Prove that the area of $EGFH$ is at most one-quarter that of $ABCD$.

    (b) Is this conclusion still valid if $ABCD$ is an arbitrary convex quadrilateral?

46. The quadrilateral $ABCD$ is inscribed in the unit circle such that it contains the centre of the circle and the length of its shortest side is a, where $\sqrt{2} < a < 2$. Let $A'B'C'D'$ be the quadrilateral determined by the tangents to the circle at A, B, C and D.

    (a) Determine the minimim value of the ratio of the area of $A'B'C'D'$ to that of $ABCD$.

    (b) Determine the maximum value of the ratio of the area of $A'B'C'D'$ to that of $ABCD$.

47. $A_1B_1C_1D_1$ is any convex quadrilateral. P is a point inside such that any line joining P to a vertex forms an acute angle with each of the two sides meeting at that vertex. Suppose A_{k-1}, B_{k-1}, C_{k-1} and D_{k-1} have been defined. Let A_k, B_k, C_k and D_k be the respective reflecitons of P across $A_{k-1}B_{k-1}$, $B_{k-1}C_{k-1}$, $C_{k-1}D_{k-1}$ and $D_{k-1}A_{k-1}$.

    (a) Which of $A_iB_iC_iD_i$, $1 \le i \le 12$, is necessarily similar to $A_{1997}B_{1997}C_{1997}D_{1997}$?

    (b) Which of $A_iB_iC_iD_i$, $1 \le i \le 12$, is necessarily cyclic if $A_{1997}B_{1997}C_{1997}D_{1997}$ is?

48. The radii of four spheres are 2, 2, 3 and 3 respectively. Each is externally tangent to the three others. If a smaller sphere is tangent to each of these 4 spheres, determine the radius of the smaller sphere.