2011 Mathematical Olympiad Summer Program Homework

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Chapter 1

Red and Green groups homework

1.1 algebra

1. The real numbers a, b, c, and d satisfy the following equations:

$$abc - d = 1, \quad bcd - a = 2, \quad cda - b = 3, \quad dab - c = -6.$$

   Prove that $a + b + c + d \neq 0$.

2. Let F denote the set of functions $f : \mathbb{N} \to \mathbb{N}$ with the following property: for all $a, b \in \mathbb{N}$, where $a \geq b$, we have $f(a^2 - b^2) = f(a)^2 - f(b)^2$. Find the cardinality of F.

3. Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers such that

$$\sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k} = a_n^2$$

   for all n. Prove that $(a_n)_{n\geq 0}$ is a geometric sequence.

4. Let a, b, and c be positive real numbers. Prove that

$$\frac{a^2 b(b-c)}{a+b} + \frac{b^2 c(c-a)}{b+c} + \frac{c^2 b(a-b)}{c+a} \geq 0.$$

5. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Let $a_n = n! + n$. Suppose $\frac{P(a_n)}{Q(a_n)}$ is an integer for all n. Prove that for every integer n such that $Q(n) \neq 0$, we have $\frac{P(n)}{Q(n)}$ is an integer.

1.2 geometry

1. Let A, B, and P be three points on a circle. Let a and b denote the distances from P to the tangents at A and B. Let c denote the distance from P to AB. Prove that $c^2 = ab$.

2. Let $ABCD$ be a rhombus. A tangent of its incircle intersects with sides BC and CD. Let R and S denote the intersections of the tangent with lines AB and AD. Prove that the value of the product $|BR| \cdot |DS|$ is independent of the choice of the tangent.

3. Let AL and BK be angle bisectors in the non-isosceles triangle ABC (L lies on the side BC, K lies on the side AC). The perpendicular bisector of BK intersects the line AL at point M. Point N lies on the line BK such that LN is parallel to MK. Prove that $LN = NA$.

4. Let $ABCDE$ be a convex pentagon such that $AB + CD + BC + DE$. Let k be a circle with center on side AE that touches sides AB, BC, CD and DE at points P, Q, R and S (different from vertices of the pentagon) respectively. Prove that lines PS and AE are parallel.

5. Let ABC be an acute triangle with orthocenter H, and let M be the midpoint of AC. Point C_1 lies on AB such that CC_1 is an altitude of triangle ABC. Let H_1 be the reflection of H in AB. The orthogonal projections of C_1 onto the lines AH_1, AC and BC are P, Q and R, respectively. Let M_1 be the point such that the circumcenter of triangle PQR is the midpoint of MM_1. Prove that M_1 lies on segment BH_1.

1.3 Number Theory

1. Let a, b, c, and d be positive integers. Let $p = a + b + c + d$. Prove that if p is not a prime, then p does not divide $ab - cd$.

2. For any integer $n \geq 2$, let A_n denote the set of solutions of the equation
$$x = \lfloor \frac{x}{2} \rfloor + \lfloor \frac{x}{3} \rfloor + \cdots + \lfloor \frac{x}{n} \rfloor.$$
Prove that the set $A = \cup_{n \geq 2} A_n$ is finite and find maximal element of A.

3. Find all integers $n \geq 1$ such that $n \cdot 2^{n+1} + 1$ is a perfect square.

4. Find all triples of nonnegative integers (a, b, c) such that $2^a 3^b + 9 = c^2$

5. Let p be a prime. Determine the number of triples (a, b, c), where $a, b, c, \in \{1, 2, 3, \ldots, 2p^2\}$, which satisfies
$$\frac{[a, c] + [b, c]}{a + b} = \frac{p^2 + 1}{p^2 + 2} \cdot c,$$
where $[x, y]$ denote the least common multiple of x and y.

1.4 Combinatorics

1. Let S be a subset of $\{1, 2, \ldots, 2010\}$ with 673 elements. Prove that there exist two distinct elements of S for which their sum is divisible by 6.

2. Each cell of a 50 by 50 grid can be colored in either red or blue. Initially all squares are red. At each step, one may choose a row or a column and change all the colors of its cells. Is it possible that after a sequence of steps, exactly 2010 cell are blue?

3. A hundred points are given in the plane with no three collinear. The points are arranged in ten groups; each contains at least three points. A segment is drawn between any pair of points in the same group.
a) Determine which of the possible arrangements into ten such groups has the minimal number of triangles.
b) Prove that there exists an arrangement in which each segment can be colored with one of three given colors and no triangle has all edges of the same color.

4. A 9×7 rectangle is tiled with tiles of the two types: L-shaped tiles composed of three unit squares (can be rotated repeatedly with $90°$) and square tiles composed of four unit squares. Let n be the number of the 2×2 tiles which can be used in such a tiling. Find all the values of n.

5. Each one of 2009 distinct points in the plane is colored in blue or red, so that on every blue-centered unit circle there are exactly two red points. Find the greatest possible number of blue points.

6. For all natural n, an n-staircase is a figure consisting of unit squares, with one square in the first row, two squares in the second row, and so on, up to n squares in the n^{th} row, such that all the left-most squares in each row are aligned vertically. Let $f(n)$ denote the minimum number of square tiles requires to tile the n-staircase, where the side lengths of the square tiles must be natural number. For example, $f(2) = 3$ and $f(4) = 7$.
(a) Find all n such that $f(n) = n$.
(b) Find all n such that $f(n) = n + 1$.

7. Three speed skaters have a friendly "race" on a skating oval. They all start from the same point and skate in the same direction, but with different speeds that they maintain throughout the race. The slowest skater does 1 lap per minute, the fastest one does 3.14 laps per minute, and the middle one does L laps a minute for some $1 < L < 3.14$. The race ends at the moment when all three skaters again come together to the same point on the oval (which may differ from the starting point.) Determine the number of different choices for L such that exactly 117 passings occur before the end of the race.

8. Each vertex of a finite graph can be colored either black or white. Initially all vertices are black. At each step, we are allowed to pick a vertex P and change the color of P and all of its neighbors. Is it possible to change the color of every vertex from black to white by a sequence of steps?

9. A teacher has 24 pencils of 4 colors (there are 6 pencils of each color). He distributes them to 6 kids so that each kid has 4 pencils. Find the least value of n such that the class leader can always find a group of n kids having altogether the pencils of all 4 colors.

10. A hundred pairwise distinct real numbers are arranged in a circle. Prove that there exist four successive numbers among them such that the sum of the two middle ones is strictly less than the sum of the other two.

Chapter 2

Black and blue groups homework

2.1 algebra

1. Let a, b, and c be positive real numbers. Prove that

$$\frac{a^2b(b-c)}{a+b} + \frac{b^2c(c-a)}{b+c} + \frac{c^2b(a-b)}{c+a} \geq 0.$$

2. A polynomial $P(x)$ of degree $n \geq 3$ has n real roots $x_1 < x_2 < \cdots < x_n$ such that $x_2 - x_1 < x_3 - x_2 < \cdots < x_n - x_{n-1}$. Prove that the maximum of the function $y = |P(x)|$ on the segment $[x_1, x_n]$ is attained on the segment $[x_{n-1}, x_n]$.

3. Find all non-decreasing functions $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that for each $x, y \in \mathbb{R}^+ \cup \{0\}$, we have

$$f\left(\frac{x + f(x)}{2} + y\right) = 2x - f(x) + f(f(y)).$$

4. Let a, b, and c be complex numbers such that, for any complex number z with $|z| \leq 1$, we have $|az^2 + bz + c| \leq 1$. Find the maximum value of $|bc|$.

5. Let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function and let n be an arbitrary natural number. Suppose that there are prime numbers $p_1, p_2, \ldots, p_n$ and natural numbers $s_1, s_2, \ldots, s_n$ such that for each $1 \leq i \leq n$ the set $\{f(p_i r + s_i) | r = 1, 2, \ldots\}$ is an infinite arithmetic progression. Prove that there is a natural number a such that

$$f(a+1), f(a+2), \ldots, f(a+n)$$

form an arithmetic progression.

2.2 geometry

1. Let ABC be an acute triangle with orthocenter H, and let M be the midpoint of AC. Point C_1 lie on AB such that CC_1 is an altitude of triangle ABC. Let H_1 be the reflection of H in AB. The orthogonal projections of C_1 onto the lines AH_1, AC and BC are P, Q and R, respectively. Let M_1 be the point such that the circumcenter of triangle PQR is the midpoint of the segment MM_1. Prove that M_1 lies on BH_1.

2. A strip of width w is the set of all points on or between two parallel lines of distance w apart. Let S be a set of n $(n \geq 3)$ points on the plane such that any three different points of S can be covered by a strip of width 1. Prove that S can be covered by a strip of width 2.

3. Quadrilateral $ABCD$ is inscribed into circle ω. Diagonals AC and BD intersect at K. Points M_1, M_2, M_3, and M_4 are midpoints of arcs AB, BC, CD, and DA, respectively. Points I_1, I_2, I_3, and I_4 are incenters of triangles ABK, BCK, CDK, and DAK respectively. Prove that lines M_1I_1, M_2I_2, M_3I_3, and M_4I_4 all concur.

4. Circles W_1 and W_2 intersect at points P and K. Let XY denote the common tangent of two circles closer to P, where X is on W_1 and Y is on W_2. Segment XP intersects W_2 for the second time in C and YP intersects W_1 in B. Let A be intersection point of BX and CY. Let Q denote the second intersection point of circumcircles of ABC and AXY. Prove that $\angle QXA = \angle QKP$.

5. Let ABC be an isosceles triangle with $BC > AB = AC$. Let D and M denote the midpoints of BC and AB. Let X denote the point for which $BX \perp AC$ and $XD||AB$. Let BX and AD meet at H. Let P be the intersection point of DX and circumcircle of AHX (other than X). Prove that the tangent from A to the circumcircle of triangle AMP is parallel to BC

2.3 Number Theory

1. Let k be an integer greater than or equal to 3. A sequence $\{a_n\}$ is defined as follows: $a_k = 2k$, and for all $n > k$,
$$a_n = \begin{cases} a_{n-1} + 1 & \text{if } (a_{n-1}, n) = 1 \\ 2n & \text{if } (a_{n-1}, n) > 1 \end{cases}.$$
Prove that there are infinitely many primes in the sequence $\{a_n - a_{n-1}\}$.

2. For each integer n $(n \geq 2)$, let $f(n)$ denote the sum of all positive integers less than or equal to n that are not relatively prime to n. Prove that $f(n+p) \neq f(n)$ for all primes p.

3. Given a positive integer $n \geq 3$. Find the minimal integer k that satisfies the following property.
For any n points $A_i = (x_i, y_i)$ on the plane with no three collinear, and for any set of real numbers $\{c_i : 1 \leq i \leq n\}$, there exists a polynomial $P(x, y)$ with degree at most k such that $P(x_i, y_i) = c_i$ for all $i = 1, \ldots, n$.

4. Let $p_1, p_2, ..., p_n$ $(n \geq 3)$ be pairwise different prime numbers. Let r be an integer. Suppose for any $k \in \{1, 2, ..., n\}$, residue of $\prod_{i \neq k} p_i$ divided by p_k is equal to r. Prove that $r \leq n - 2$.

5. Let a_1, a_2, a_3, b_1, b_2, and b_3 be pairwise distinct positive integers such that
$$(n+1)a_1^n + na_2^n + (n-1)a_3^n | (n+1)b_1^n + nb_2^n + (n-1)b_3^n$$
holds for all positive integer n. Prove that there exists a positive integer k such that $b_i = ka_i$ for $i = 1, 2, 3$.

2.4 Combinatorics

1. A teacher has 40 pencils of 4 colors (there are 10 pencils of each color). He distributes them to 10 kids so that each kid has 4 pencils. Find the least value of n such that the class leader can always find a group of n kids having altogether the pencils of all 4 colors.

2. A hundred pairwise distinct real numbers are arranged in a circle. Prove that there exist four successive numbers among them such that the sum of the two middle ones is strictly less than the sum of the other two.

3. Each cell of a 100 by 100 grid contains a natural number. A grid rectangle is called good if the sum of the numbers in all its cells is divisible by 17. In each move, one is allowed to color all the cells of some good rectangle. It is not allowed to color a certain cell twice. Find the maximal d such that for every arrangement of numbers, it is possible to color at least d cells in some sequence of moves.

4. In a certain country some pairs of cities are linked by bi-directional direct airlines so that the system of airlines is connected (that is, from each city one can reach any other city by some flights). It appears that for each cyclic route containing an odd number of airlines, canceling all airlines of this cyclic route makes the system of airlines not connected. Prove that one can distribute all the cities into 4 regions so that each airline links two cities from different regions. (A region may contain no cities.)

5. In a boarding school, 512 students learn 9 disciplines. These students live in 256 double rooms. It is known that for every two students the sets of disciplines in which they are interested are distinct (in particular, exactly one student is interested in nothing). Prove that all the students can be arranged in a circle so that (i) every student is next to his or her roommate, and (ii) for each two adjacent students who are not roommates, one of them is interested in all the disciplines in which the other is interested, and the first one is interested in exactly one additional discipline.

Chapter 3

Additional homework for the Black groups

3.1 Additional homework for the Black group

1. Let $a > 2$ be given, and starting $a_0 = 1, a_1 = a$ define recursively:

$$a_{n+1} = \left(\frac{a_n^2}{a_{n-1}^2} - 2\right) \cdot a_n.$$

Show that for all integers $k > 0$, we have: $\sum_{i=0}^{k} \frac{1}{a_i} < \frac{1}{2} \cdot (2 + a - \sqrt{a^2 - 4})$.

2. Let f be a function from the set of real numbers $\mathbb{R}$ into itself such for all $x \in \mathbb{R}$, we have $|f(x)| \leq 1$ and

$$f\left(x + \frac{13}{42}\right) + f(x) + f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right).$$

Prove that f is a periodic function (that is, there exists a non-zero real number c such $f(x + c) = f(x)$ for all $x \in \mathbb{R}$).

3. Let k, m, n be integers such that $1 < n \leq m - 1 \leq k$. Determine the maximum size of a subset S of the set $\{1, 2, 3, \ldots, k - 1, k\}$ such that no n distinct elements of S add up to m.

4. Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, \cdots, x_n)$ be an $(n + 1)$-tuple of integers satisfying the following conditions :

(a) $x_0 + x_n = 0$, and

(b) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

5. Let the sides of two rectangles be $\{a, b\}$ and $\{c, d\}$, respectively, with $a < c \le d < b$ and $ab < cd$. Prove that the first rectangle can be placed within the second one if and only if

$$\left(b^2 - a^2\right)^2 \le (bc - ad)^2 + (bd - ac)^2 .$$

6. In the plane, consider a point X and a polygon $\mathcal{F}$ (which is not necessarily convex). Let p denote the perimeter of $\mathcal{F}$, let d be the sum of the distances from the point X to the vertices of $\mathcal{F}$, and let h be the sum of the distances from the point X to the sidelines of $\mathcal{F}$. Prove that $d^2 - h^2 \ge \frac{p^2}{4}$.

7. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

8. A finite sequence of integers $a_0, a_1, \ldots, a_n$ is called quadratic if for each i in the set $\{1, 2 \ldots, n\}$ we have the equality $|a_i - a_{i-1}| = i^2$.

   a.) Prove that any two integers b and c, there exists a natural number n and a quadratic sequence with $a_0 = b$ and $a_n = c$.

   b.) Find the smallest natural number n for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.

9. Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1 + x_1^2} + \frac{x_2}{1 + x_1^2 + x_2^2} + \cdots + \frac{x_n}{1 + x_1^2 + \cdots + x_n^2} < \sqrt{n}.$$

10. Find all positive integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{99}{100} = \frac{a_0}{a_1} + \frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n},$$

    where $a_0 = 1$ and $(a_{k+1} - 1)a_{k-1} \ge a_k^2(a_k - 1)$ for $k = 1, 2, \ldots, n - 1$.

11. Define a k-[i]clique[/i] to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

12. A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called *historic* if $\{z - y, y - x\} = \{1776, 2001\}$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.

13. Let $p \ge 5$ be a prime number. Prove that there exists an integer a with $1 \le a \le p - 2$ such that neither $a^{p-1} - 1$ nor $(a + 1)^{p-1} - 1$ is divisible by p^2.

14. The sequence $c_0, c_1, ..., c_n, ...$ is defined by $c_0 = 1, c_1 = 0$, and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set S of ordered pairs (x, y) for which there is a finite set J of positive integers such that $x = \sum_{j \in J} c_j$, $y = \sum_{j \in J} c_{j-1}$. Prove that there exist real numbers α, β, and M with the following property: An ordered pair of nonnegative integers (x, y) satisfies the inequality $m < \alpha x + \beta y < M$ if and only if $(x, y) \in S$.

    Remark: A sum over the elements of the empty set is assumed to be 0.

15. A cake has the form of an n x n square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement A. Let B be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement B than of arrangement A.

    Prove that arrangement B can be obtained from A by performing a number of switches, defined as follows: A switch consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

16. Consider a convex polyheadron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it. Call a pair of points of the polyheadron [i]antipodal[/i] if there exist two parallel planes passing through these points and such that the polyheadron is contained between these planes. Let A be the number of antipodal pairs of vertices, and let B be the number of antipodal pairs of midpoint edges. Determine the difference $A - B$ in terms of the numbers of vertices, edges, and faces.

17. In a triangle ABC, let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively, and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not containing the vertices A, B, C, respectively. For $i \in \{a, b, c\}$, let w_i be the circle with M_iT_i as diameter. Let p_i be the common external common tangent to the circles w_j and w_k (for all $\{i, j, k\} = \{a, b, c\}$) such that w_i lies on the opposite side of p_i than w_j and w_k do. Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.

18. Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that $\angle PAB + \angle PDC \leq 90°$ and $\angle PBA + \angle PCD \leq 90°$. Prove that $AB + CD \geq BC + AD$.

19. Prove that the equation $\frac{x^7-1}{x-1} = y^5 - 1$ doesn't have integer solutions.

20. For all positive integers n, show that there exists a positive integer m such that n divides $2^m + m$.