Art of Problem Solving

WOOT 2010–11

Practice AIME 2

Instructions

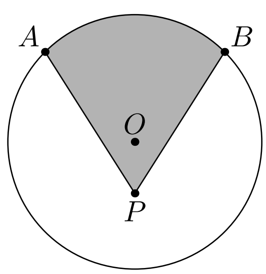
- The time limit is 3 hours. These 3 hours should be continuous, e.g. do not work for 2 hours on one day, and then 1 hour on another day.
- All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor a penalty for wrong answers.
- No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted**.
- Enter your answers at the following link, before **midnight ET/9 PM PT** on **Wednesday, February 2, 2011**:

  `http://www.artofproblemsolving.com/School/WOOT/aime.php`

- Do not discuss the problems before Wednesday, February 2. The answers will be posted on Thursday, February 3.

1. The ratio of the surface area of a cube with side length 2011 to the perimeter of one of its faces can be written in the form m/n, where m and n are relatively prime positive integers. Find $m - 2700n$.

2. Points A and B lie on a circle of radius 42, centered at O, such that $\angle AOB = 90^\circ$. Point P is located inside the circle such that $PA = PB$, $OP = 17$, and O lies inside triangle ABP. The area of the shaded region can be expressed in the form $m\pi + n\sqrt{p}$, where m, n, and p are positive integers, and p is not divisible by the square of any prime. Find $m + n + p$.



3. Evaluate
$$\frac{21^3 + 22^3 + 23^3 + \cdots + 39^3}{21 + 22 + 23 + \cdots + 39}.$$

4. Let n be a positive integer. When the fraction
$$\frac{n}{n+42}$$
is reduced, the result is the fraction $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the sum of all possible values of $q - p$.

5. Let N be the number of ways to divide 5050 students into 100 teams, so that the first team has one student, the second team has two students, and so on. (The order of the students on a team is irrelevant.) Compute the number of zeros at the end of the decimal representation of N.

6. Let ABC be a triangle, and let D be the point on BC such that AD bisects $\angle BAC$. Let T be the intersection of line BC with the perpendicular bisector of AD. If $BD = 20$ and $CD = 30$, then find TD.

7. Gary rolls three standard, six-sided dice and discards the die with the lowest number. The expected value of the sum of the remaining two dice can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

8. In triangle ABC, let D, E, and F be points on sides BC, AC, and AB, respectively such that AD, BE, and CF are the three angle bisectors. If $AF = 7$, $BC + CE = 21$, and $BC + BF = 27$, then the length of AE can be expressed in the form m/n, where m and n are relatively prime positive integers. Find mn.

9. Find the number of ten-digit numbers that satisfy the following properties:
   - All the digits 0, 1, 2, ..., 9 appear exactly once (and 0 cannot be the first digit).
   - The digits 1, 2, 3, 4, 5, 6, and 7 are in order, reading from left to right.
   - The 8 does *not* appear anywhere after the 7, reading from left to right.

   For example, the number 1823405967 satisfies these conditions, but the number 1239456708 does not.

10. In triangle ABC, $AB = 13$, $BC = 14$, and $AC = 15$. Let the bisector of $\angle BAC$ intersect BC at D and the circumcircle of triangle ABC at E. Let ω_1 be the circle that passes through D and E and is tangent to AB; let G be the point of tangency. Similarly, let ω_2 be the circle that passes through D and E and is tangent to AC; let H be the point of tangency. Determine the area of triangle AGH.

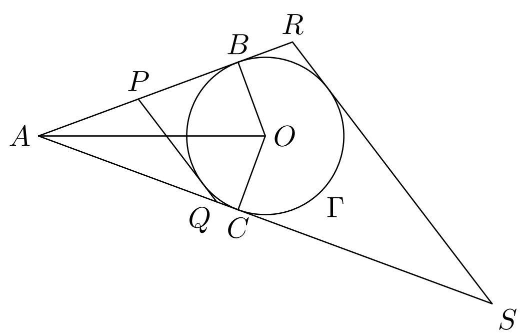
11. Let z_1, z_2, ..., z_9 be distinct complex numbers that form a regular nonagon in the complex plane. If $z_1 + z_2 + \cdots + z_9 = 9$ and $z_1 z_2 \cdots z_9 = 512$, then find $(z_1 - 2)(z_2 - 2)\cdots(z_9 - 2)$.

12. The sum
$$\sum_{k=0}^{12} \frac{(-1)^k \binom{12}{k}}{k^2 - 29k + 210}$$
can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

13. Let $N = 10203\ldots979899$ be the positive integer formed by concatenating all the positive integers from 1 to 99 inclusive, with the single-digit numbers using two digits (so 4 is represented as 04, for example). Let r be the remainder when 100^{100} is divided by N. Find the last three digits of r.

14. Let Γ be a circle with center O and radius $\sqrt{3}$, and let A be a point such that $AO = 15$. Let AB and AC be the tangents from A to Γ, and let P be a point on AB, other than A or B. Let Q be the point on AC such that PQ is tangent to Γ. Let R and S be points on AB and AC, respectively, such that RS is parallel to PQ and tangent to Γ (and does not coincide with PQ). Find $[ARS] \cdot [APQ]$.

    Note: $[XYZ]$ denotes the area of triangle XYZ.



15. Let a_0, a_1, ... be the sequence sequence defined by $a_0 = 1$ and $a_n = 2^{a_{n-1}}$ for all $n \geq 1$. Find the remainder when a_{2011} is divided by 1000.

Note: All problems are based on proposals submitted by WOOT students.