Art of Problem Solving

WOOT 2010–11

Practice Olympiad 6

Instructions

- You should take the test under "olympiad conditions," meaning that the test should be completed in one sitting, with handwritten solutions (just like on an actual olympiad exam). Take the test using only the resources that would be available to you on an actual olympiad, meaning you should use scrap paper, a ruler, a compass, etc., but no calculators and no reference materials.
- You should allot 4.5 hours to take the test.
- Completely fill out the cover sheet and make sure it is the first page of your solutions.
- On the WOOT Home Page, there is a **WOOT Practice Olympiad Answer Sheet**. Print out (or copy) several blank copies of the answer sheets, and write all of your work on these sheets. Use only black pen or very dark pencil. Make sure that the top of every answer sheet page is completely filled out. Each problem's solution should start on a new page, along with new page numbering.
- Do not discuss the problems on or before the due date of Wednesday, April 13, 2011.

How to submit your solutions

You can submit solutions by upload, email, or by fax. DO NOT MAIL YOUR SOLUTIONS!

By Upload: Scan your solutions as a single PDF file. **Check to make sure that the file is legible before submitting it!** In your "My Classes" area, follow the link that says "Submit."

By email: Scan your solutions as a single PDF file. **Check to make sure that the file is legible before emailing it!** Email to `woot@artofproblemsolving.com`. Put "WOOT Practice Olympiad 6" in the subject line, attach your solutions as a single PDF file, and write something in the message body - if you leave the message body blank, it will get blocked by our spam filters.

By fax: Fax to (619)659-8146.

Solutions are due by Wednesday, April 13, 2011

Late submissions will not be accepted except under extraordinary circumstances

WOOT Practice Olympiad Cover Sheet

Username: ____

Class ID: __________

User ID: ____________________
(Your Class ID and User ID can be found in the "My Classes" section of the website)

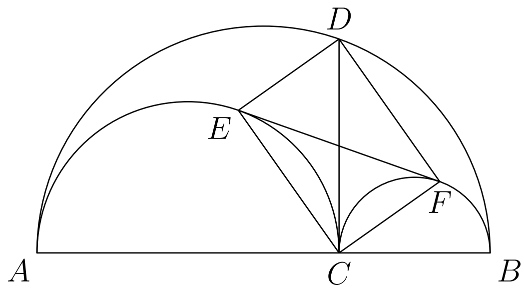
Practice Olympiad Number: 6

Beginning	
Intermediate	
Advanced	

Number of pages (including cover sheet): ____

1. Let x and y be positive real numbers such that x^3, y^3, and $x+y$ are all rational. Show that both x and y are also rational.

2. Let C be a point on line segment AB. We construct semicircles with diameters AB, AC, and BC, all on the same side of AB. Let D be the point on the semicircle with diameter AB such that AB and CD are perpendicular. Let E be a point on the semicircle with diameter AC, and let F be a point on the semicircle with diameter BC, such that EF is the common external tangent to these semicircles. Prove that quadrilateral $CEDF$ is a rectangle.



3. There are n tennis players in a tournament, where $n \geq 4$, and certain pairs of players played games against each other. In every group of four players, there exist three players, say A, B, and C, such that A and B played a game, A and C played a game, and B and C played a game. What is the minimum number of games that could have been played in the tournament?

4. Let a_1, a_2, $\ldots$, a_n be distinct real numbers. Show that

$$\min_{1 \leq i,j \leq n} (a_i - a_j)^2 \leq \frac{12}{n(n^2-1)}(a_1^2 + a_2^2 + \cdots + a_n^2).$$

5. Let $\mathcal{P}$ be the set of all primes, and let M be a subset of $\mathcal{P}$ containing at least three elements. For any proper subset A of M, all of the prime factors of the number

$$-1 + \prod_{p \in A} p$$

are in M. Prove that $M = \mathcal{P}$.

6. Let x_1, x_2, $\ldots$, x_n be real numbers. Prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i + x_j| \geq n \sum_{i=1}^{n} |x_i|.$$

7. Let I be the incenter of triangle ABC. Let A_1 and A_2 be points on side BC such that $\angle BIA_1 = \angle CIA_2 = 90°$, let B_1 and B_2 be points on side AC such that $CIB_1 = \angle AIB_2 = 90°$, and let C_1 and C_2 be points on side AB such that $\angle AIC_1 = \angle BIC_2 = 90°$.

Let A', B', and C' be the midpoints of arcs BC, AC, and AB on the circumcircle of triangle ABC. Let $A'A_1$ intersect AC at A_1', let $A'A_2$ intersect AB at A_2', let $B'B_1$ intersect AB at B_1', let $B'B_2$ intersect BC at B_2', let $C'C_1$ intersect BC at C_1', and let $C'C_2$ intersect AC at C_2'. Prove that $A_1'A_2'$, $B_1'B_2'$, and $C_1'C_2'$ are concurrent.

Practice Olympiad 6

Remember . . .

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- Complete the WOOT Practice Olympiad Cover Sheet (included in this document), and make it the first page of your solutions. We will not accept your solutions without this cover sheet.
- Do not mail your solutions. Use only upload, e-mail, or fax. If you use e-mail, send your solutions to

  woot@artofproblemsolving.com,

  NOT **classes**@artofproblemsolving.com. If you use e-mail, we will only accept solutions in PDF format. In particular, solutions in JPG or Word format will not be accepted.