1998 IMO Camp

APPENDIX I Selected Definitions, Postulates, and Theorems

1 If two angles are vertical angles then the two angles are congruent.
2 Two triangles are congruent if two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle. (S.A.S.)
3 Two triangles are congruent if two angles and the included side of the first triangle are congruent to the corresponding parts of the second triangle. (A.S.A.)
4 Two triangles are congruent if the sides of the first triangle are congruent to the corresponding sides of the second triangle. (S.S.S.)
5 If a triangle has two congruent sides, then the triangle has two congruent angles opposite those sides. Also converse.
6 An equilateral triangle is equiangular. Also converse.
7 If a pair of corresponding angles formed by a transversal of two lines are congruent, then the two lines are parallel. Also converse.
8 If a pair of alternate interior angles formed by a transversal of two lines are congruent, then the lines are parallel. Also converse.
9 Two lines are parallel if they are perpendicular to the same line.
10 If a line is perpendicular to one of two parallel lines, then it is also perpendicular to the other.
11 If a pair of consecutive interior angles formed by a transversal of two lines are supplementary, then the lines are parallel. Also converse.
12 The measure of an exterior angle of a triangle equals the sum of the measures of the two non-adjacent interior angles.
13 The sum of the measures of the three angles of a triangle is 180, a constant.
14 The acute angles of a right triangle are complementary.

15 The sum of the measures of the four interior angles of a convex quadrilateral is 360, a constant.

16 Two triangles are congruent if two angles and a non-included side of the first triangle are congruent to the corresponding parts of the second triangle.

17 Two right triangles are congruent if the hypotenuse and a leg of one triangle are congruent to the corresponding parts of the other triangle.

18 Any point on the perpendicular bisector of a line segment is equidistant from the endpoints of the line segment. Two points equidistant from the endpoints of a line segment, determine the perpendicular bisector of the line segment.

19 Any point on the bisector of an angle is equidistant from the sides of the angle.

20 Parallel lines are everywhere equidistant.

21a The opposite sides of a parallelogram are parallel. Also converse.

21b The opposite sides of a parallelogram are congruent. Also converse.

21c The opposite angles of a parallelogram are congruent. Also converse.

21d Pairs of consecutive angles of a parallelogram are supplementary. Also converse.

21e A diagonal of a parallelogram divides the parallelogram into two congruent triangles.

21f The diagonals of a parallelogram bisect each other. Also converse.

21g A rectangle is a special parallelogram; therefore 21a through 21f hold true for the rectangle.

21h A rectangle is a parallelogram with congruent diagonals. Also converse.

21i A rectangle is a parallelogram with four congruent angles, right angles. Also converse.

21j A rhombus is a special parallelogram; therefore 21a through 21f hold true for the rhombus.

21k A rhombus is a parallelogram with perpendicular diagonals. Also converse.

21l A rhombus is a quadrilateral with four congruent sides. Also converse.

21m The diagonals of a rhombus bisect the angles of the rhombus.

21n A square has all the properties of both a rectangle and a rhombus; hence 21a through 21m hold true for a square.

22 A quadrilateral is a parallelogram if a pair of opposite sides are both congruent and parallel.

23 The base angles of an isosceles trapezoid are congruent. Also converse.

24 If a line segment is divided into congruent (or proportional) segments by three or more parallel lines, then any other transversal will similarly contain congruent (or proportional) segments determined by these parallel lines.

25 If a line contains the midpoint of one side of a triangle and is parallel to a second side of the triangle, then it will bisect the third side of the triangle.

26 The line segment whose endpoints are the midpoints of two sides of a triangle is parallel to the third side of the triangle and has a measure equal to one-half of the measure of the third side.

27 The measure of the median on the hypotenuse of a right triangle is one-half the measure of the hypotenuse.

28 The median of a trapezoid, the segment joining the midpoints of the non-parallel sides, is parallel to each of the parallel sides, and has a measure equal to one-half of the sum of their measures.

29 The three medians of a triangle meet in a point, the centroid, which is situated on each median so that the measure of the segment from the vertex to the centroid is two-thirds the measure of the median.

30 A line perpendicular to a chord of a circle and containing the center of the circle, bisects the chord and its major and minor arcs.

31 The perpendicular bisector of a chord of a circle contains the center of the circle.

32a If a line is tangent to a circle, it is perpendicular to a radius at the point of tangency.

32b A line perpendicular to a radius at a point on the circle is tangent to the circle at that point.

32c A line perpendicular to a tangent line at the point of tangency with a circle, contains the center of the circle.

32d The radius of a circle is only perpendicular to a tangent line at the point of tangency.

33 If a tangent line (or chord) is parallel to a secant (or chord) the arcs intercepted between these two lines are congruent.

34 Two tangent segments to a circle from an external point are congruent.

35 The measure of a central angle is equal to the measure of its intercepted arc.

36 The measure of an inscribed angle equals one-half the measure of its intercepted arc.

36a A quadrilateral is cyclic (i.e. may be inscribed in a circle) if one side subtends congruent angles at the two opposite vertices.

37 The opposite angles of a cyclic (inscribed) quadrilateral are supplementary. Also converse.

38 The measure of an angle whose vertex is on the circle and whose sides are formed by a chord and a tangent line, is equal to one-half the measure of the intercepted arc.

39 The measure of an angle formed by two chords intersecting inside the circle, is equal to half the sum of the measures of its intercepted arc and of the arc of its vertical angle.

40 The measure of an angle formed by two secants, or a secant and a tangent line, or two tangent lines intersecting outside the circle, equals one-half the difference of the measures of the intercepted arcs.

41 The sum of the measures of two sides of a non-degenerate triangle is greater than the measure of the third side of the triangle.

42 If the measures of two sides of a triangle are not equal, then the measures of the angles opposite these sides are also unequal, the angle with the greater measure being opposite the side with the greater measure. Also converse.

43 The measure of an exterior angle of a triangle is greater than the measure of either non-adjacent interior angle.

44 The circumcenter (the center of the circumscribed circle) of a triangle is determined by the common intersection of the perpendicular bisectors of the sides of the triangle.

45 The incenter (the center of the inscribed circle) of a triangle is determined by the common intersection of the interior angle bisectors of the triangle.

46 If a line is parallel to one side of a triangle it divides the other two sides of the triangle proportionally. Also converse.

47 The bisector of an angle of a triangle divides the opposite side into segments whose measures are proportional to the measures of the other two sides of the triangle. Also converse.

48 If two angles of one triangle are congruent to two corresponding angles of a second triangle, the triangles are similar. (A.A.)

49 If a line is parallel to one side of a triangle intersecting the other two sides, it determines (with segments of these two sides) a triangle similar to the original triangle.

50 Two triangles are similar if an angle of one triangle is congruent to an angle of the other triangle, and if the measures of the sides that include the angle are proportional.

51a The measure of the altitude on the hypotenuse of a right triangle is the mean proportional between the measures of the segments of the hypotenuse.

51b The measure of either leg of a right triangle is the mean proportional between the measure of the hypotenuse and the segment, of the hypotenuse, which shares one endpoint with the leg considered, and whose other endpoint is the foot of the altitude on the hypotenuse.

52 If two chords of a circle intersect, the product of the measures of the segments of one chord equals the product of the segments of the other chord.

53 If a tangent segment and a secant intersect outside the circle, the measure of the tangent segment is the mean proportional between the measure of the secant and the measure of its external segment.

54 If two secants intersect outside the circle, the product of the measures of one secant and its external segment equals the product of the measures of the other secant and its external segment.

55 (The Pythagorean Theorem) In a right triangle the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse. Also converse.

55a In an isosceles right triangle (45–45–90 triangle), the measure of the hypotenuse is equal to $\sqrt{2}$ times the measure of either leg.

55b In an isosceles right triangle (45–45–90 triangle), the measure of either leg equals one-half the measure of the hypotenuse times $\sqrt{2}$.

55c In a 30–60–90 triangle the measure of the side opposite the 30 angle is one-half the measure of the hypotenuse.

55d In a 30–60–90 triangle, the measure of the side opposite the 60 angle equals one-half the measure of the hypotenuse times $\sqrt{3}$.

55e In a triangle with sides of measures 13, 14, and 15, the altitude to the side of measure 14 has measure 12.

56 The median of a triangle divides the triangle into two triangles of equal area. An extension of this theorem follows. A line segment joining a vertex of a triangle with a point on the opposite side, divides the triangle into two triangles, the ratio of whose areas equals the ratio of the measures of the segments of this "opposite" side.

APPENDIX II Selected Formulas

1 The sum of the measures of the interior angles of an n-sided convex polygon $= (n - 2)180$.

2 The sum of the measures of the exterior angles of any convex polygon is constant, 360.

3 The area of a rectangle:
$K = bh$.

4a The area of a square:
$K = s^2$.

4b The area of a square:
$K = \frac{1}{2}d^2$.

5a The area of any triangle:
$K = \frac{1}{2}bh$, where b is the base and h is the altitude.

5b The area of any triangle:
$K = \frac{1}{2}ab \sin C$.

5c The area of any triangle:
$K = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a + b + c)$.

5d The area of a right triangle:
$K = \frac{1}{2}l_1 l_2$, where l is a leg.

5e The area of an equilateral triangle:
$K = \frac{s^2\sqrt{3}}{4}$, where s is any side.

5f The area of an equilateral triangle:
$K = \frac{h^2\sqrt{3}}{3}$, where h is the altitude.

6a The area of a parallelogram:
$K = bh$.

8. Menelaus and Ceva: Collinearity and Concurrency

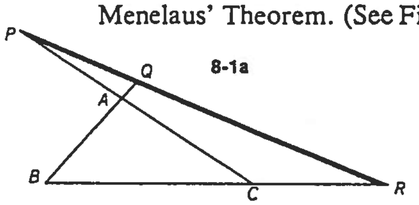
Proofs of theorems dealing with collinearity and concurrency are ordinarily clumsy, lengthy, and, as a result, unpopular. With the aid of two famous theorems, they may be shortened.

The first theorem is credited to Menelaus of Alexandria (about 100 A.D.). In 1678, Giovanni Ceva, an Italian mathematician, published Menelaus' Theorem and a second one of his own, related to it. The problems in this section concern either Menelaus' Theorem, Ceva's Theorem, or both. Among the applications investigated are theorems of Gerard Desargues, Blaise Pascal, and Pappus of Alexandria. A rule of thumb for these problems is: try to use Menelaus' Theorem for collinearity and Ceva's Theorem for concurrency.

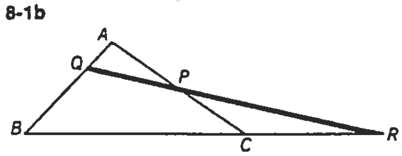
8-1 Points P, Q, and R are taken on sides $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$ (extended if necessary) of $\triangle ABC$. Prove that if these points are collinear,

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1.$$

This theorem, together with its converse, which is given in the Challenge that follows, constitute the classic theorem known as Menelaus' Theorem. (See Fig. 8-1a and Fig. 8-1b.)



8-1a



8-1b

Challenge In $\triangle ABC$ points P, Q, and R are situated respectively on sides $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$ (extended when necessary). Prove that if

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1,$$

then P, Q, and R are collinear. This is part of Menelaus' Theorem.

8-2 Prove that three lines drawn from the vertices A, B, and C of $\triangle ABC$ meeting the opposite sides in points L, M, and N, respectively, are concurrent if and only if $\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1$.

6b The area of a parallelogram:
$K = ab \sin C$.

7 The area of a rhombus:
$K = \frac{1}{2} d_1 d_2$.

8 The area of a trapezoid:
$K = \frac{1}{2} h(b_1 + b_2)$.

9 The area of a regular polygon:
$K = \frac{1}{2} ap$, where a is the apothem and p is the perimeter.

10 The area of a circle:
$K = \pi r^2 = \frac{\pi d^2}{4}$, where d is the diameter.

11 The area of a sector of a circle:
$K = \frac{n}{360} \pi r^2$, where n is the measure of the central angle.

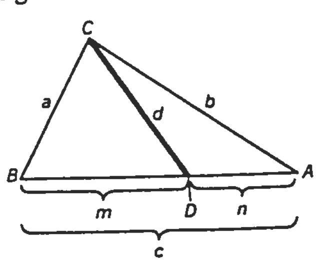
12 The circumference of a circle:
$C = 2\pi r$.

13 The length of an arc of a circle:
$L = \frac{n}{360} 2\pi r$, where n is the measure of the central angle of the arc.

10. The Theorem of Stewart

The geometry student usually feels at ease with medians, angle bisectors, and altitudes of triangles. What about 'internal line segments' (segments with endpoints on a vertex and its opposite side) that are neither medians, angle bisectors, nor altitudes? As the problems in this section show, much can be learned about such segments thanks to Stewart's Theorem. Named after Matthew Stewart who published it in 1745, this theorem describes the relationship between an 'internal line segment', the side to which it is drawn, the two parts of this side, and the other sides of the triangle.

10-1



10-1 A classic theorem known as Stewart's Theorem, is very useful as a means of finding the measure of any line segment from the vertex of a triangle to the opposite side. Using the letter designations in Fig. 10-1, the theorem states the following relationship: $a^2n + b^2m = c(d^2 + mn)$. Prove the validity of the theorem.