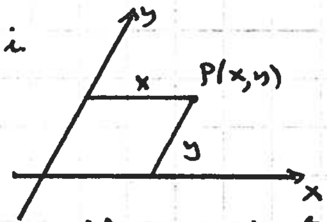
1998 IMO Camp

AREAL CO-ORDINATES [The definitive version].

I. Theory

1. Some Background

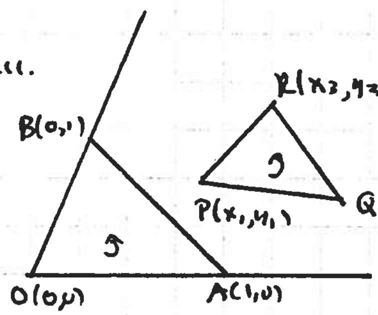
i.



Affine coordinates of P

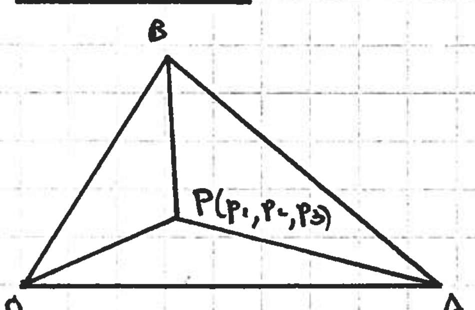
ii. Using affine coordinates, equations of lines are still $Ax + By + C = 0$.

iii.



$$\frac{(PQR)}{(OAB)} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

2. Definition



Point P has Areal coordinates (p_1, p_2, p_3) wrt $\triangle OAB$

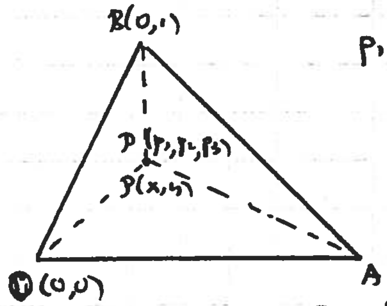
if $p_1 = \dfrac{(PBO)}{(OAB)}$

$p_2 = \dfrac{(POA)}{(OAB)}$

$p_3 = \dfrac{(PAB)}{(OAB)}$

N.B. $p_1 + p_2 + p_3 = 1$

3. Relation between Affine & Areal Coordinates



$$p_1 = \frac{(PBO)}{(OAB)} = \begin{vmatrix} x & y & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = x$$

$$p_2 = \frac{(POA)}{(OAB)} = \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = y$$

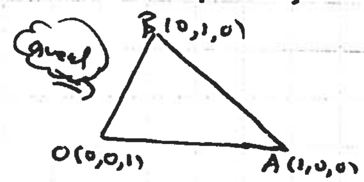
$$p_3 = \frac{(PAB)}{(OAB)} = \begin{vmatrix} x & y & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1-x$$

So $p_1 = x$ ← affine

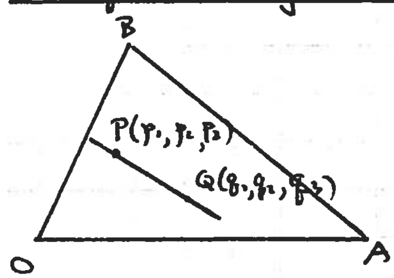
$p_2 = y$

$p_3 = 1 - x - y$

areal



4. Equations of Lines in Areal Coordinates

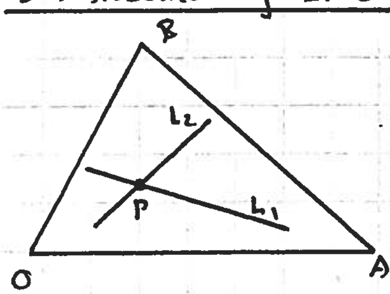


The Line through P & Q is given by

$$\begin{vmatrix} x & y & z \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = 0$$

[Reason: the determinant is linear in x, y, z and vanishes for $(x,y,z) = P$ or Q.

5. Intersections of Lines



If L_1 is $A_1x + B_1y + C_1z = 0$

L_2 is $A_2x + B_2y + C_2z = 0$

then find P by calculating

$$\begin{vmatrix} x & y & z \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix}$$

This determinant $= p_1x + p_2y + p_3z$, then $P = (p_1, p_2, p_3)$

(will need to be normalised by dividing by $p_1 + p_2 + p_3$)

6. Homogeneity of Areal Coordinates

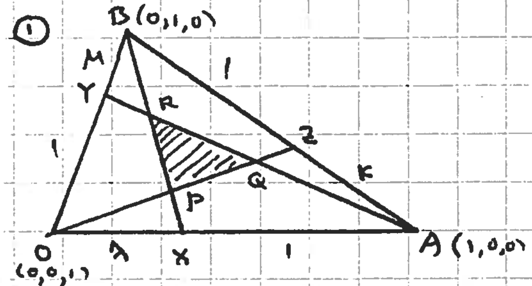
$(p_1, p_2, p_3) \leftrightarrow (kp_1, kp_2, kp_3)$ — but may

[illegible]

AREAL COORDINATES

III. Some Problems

①



Routh's Theorem: If the sides of $\triangle OAB$ are divided at X, Z and Y in the ratios $\lambda:1$, $\kappa:1$ and $\mu:1$ respectively, then

$$\frac{(PQR)}{(OAB)} = \frac{(\lambda\kappa\mu-1)^2}{(\lambda\kappa+\lambda+1)(\kappa\mu+\kappa+1)(\mu\lambda+\mu+1)}$$

Pf: $X=(\lambda,0,1)$, $Y=(0,1,\mu)$, $Z=(1,\kappa,0)$

$$BX:\ \begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{vmatrix} = 0 \Rightarrow \boxed{x-\lambda z=0}$$

$$AY:\ \begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & 1 & \mu \end{vmatrix} = 0 \Rightarrow \boxed{\mu y - z = 0}$$

$$OZ:\ \begin{vmatrix} x & y & z \\ 0 & 0 & 1 \\ 1 & \kappa & 0 \end{vmatrix} = 0 \Rightarrow \boxed{\kappa x - y = 0}$$

Now find P, Q, R

$$P:\ \begin{vmatrix} x & y & z \\ \kappa & -1 & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda x + \lambda\kappa y + z$$

$\therefore P = (\lambda, \lambda\kappa, 1)$

$$Q:\ \begin{vmatrix} x & y & z \\ \kappa & -1 & 0 \\ 0 & \mu & -1 \end{vmatrix} = x + \kappa y + \kappa\mu z$$

$\therefore Q = (1, \kappa, \kappa\mu)$

$$R:\ \begin{vmatrix} x & y & z \\ 1 & 0 & -\lambda \\ 0 & \mu & -1 \end{vmatrix} = \lambda\mu x + y + \mu z$$

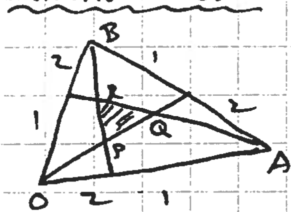
$R = (\lambda\mu, 1, \mu)$

$$\therefore \frac{(PQR)}{(OAB)} = \frac{\begin{vmatrix} \lambda & \lambda\kappa & 1 \\ 1 & \kappa & \kappa\mu \\ \lambda\mu & 1 & \mu \end{vmatrix}}{(\lambda\kappa+\lambda+1)(\kappa\mu+\kappa+1)(\mu\lambda+\mu+1)}$$

normalize here

$$= \frac{(\lambda\kappa\mu-1)^2}{(\lambda\kappa+\lambda+1)(\kappa\mu+\kappa+1)(\mu\lambda+\mu+1)}$$

Familiar Problem: When $\lambda=\kappa=\mu=2$



$$\frac{(PQR)}{(OAB)} = \frac{(2^3-1)^2}{7^3} = \frac{7^2}{7^3} = \frac{1}{7}.$$

Consequence 1 When 3 lines are concurrent

$\Rightarrow (PQR)=0 \Rightarrow (\lambda\kappa\mu-1)^2=0$

$\Rightarrow \boxed{\lambda\kappa\mu=1}$ Ceva's Theorem

Consequence 2 When X, Y, Z are collinear

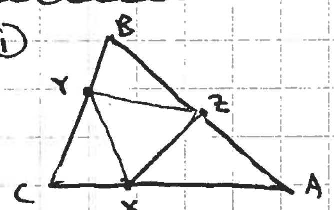
$$\Rightarrow \frac{(XYZ)}{(OAB)} = 0 \Rightarrow \begin{vmatrix} \lambda & 0 & 1 \\ 1 & \kappa & 0 \\ 0 & 1 & \mu \end{vmatrix} = 0$$

$\Rightarrow \lambda\kappa\mu+1=0 \Rightarrow \boxed{\lambda\kappa\mu=-1}$

Menelaus's Theorem

TWO PROBLEMS:

①

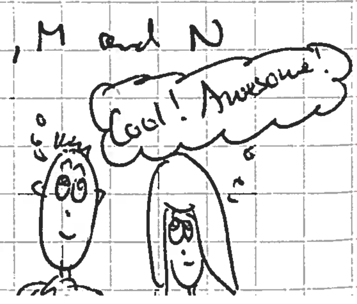


Let XYZ be a triangle whose vertices are on AC, CB, BA respectively of $\triangle ABC$. Prove that among the triangles AXZ, BZY, CYZ and XYZ, triangle XYZ cannot have smallest area except when X, Y, Z are midpoints.

② The diagonals AC and CE of the regular hexagon ABCDEF are divided by the inner points respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

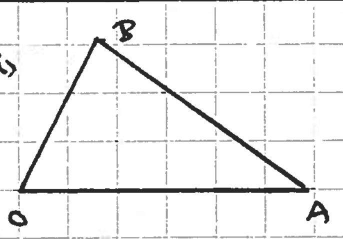
Determine r if B, M and N are collinear



Areal Coordinates

II. Special Points

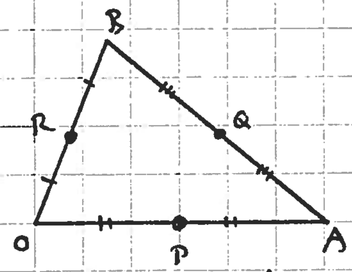
(i)



O : (0,0,1)
A : (1,0,0)
B : (0,1,0)

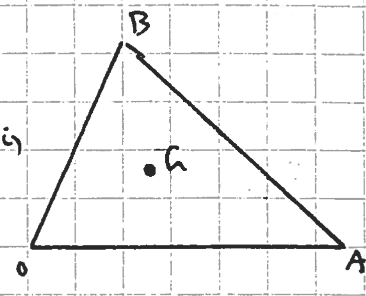
OA: $y=0$
OB: $x=0$
OC: $z=0$

(ii)



$P = (1,0,1)\ \left[\left(\tfrac{1}{2},0,\tfrac{1}{2}\right)\right]$
$Q = (1,1,0)\ \left[\left(\tfrac{1}{2},\tfrac{1}{2},0\right)\right]$
$R = (0,1,1)\ \left[\left(0,\tfrac{1}{2},\tfrac{1}{2}\right)\right]$

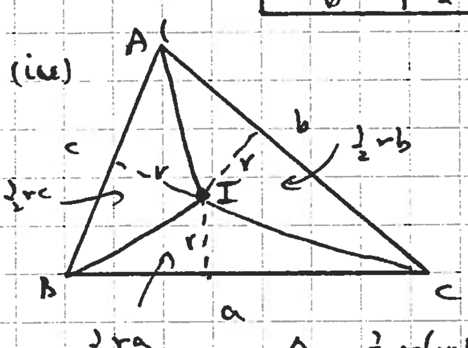
(iii)



$G = (1,1,1)$
$\left[\left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\right)\right]$

$$\boxed{\text{If the areal coordinates of } P \text{ are } (p_1,p_2,p_3) \text{ where } p_1+p_2+p_3=1, \text{ then the vector form of } P \text{ is } P = p_1A+p_2B+p_3O}$$

(iv)



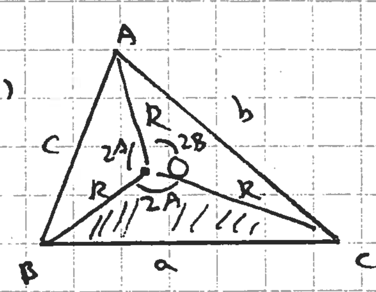
So areal coord. of I are

$$\left(\frac{a}{a+b+c},\ \frac{b}{a+b+c},\ \frac{c}{a+b+c}\right)$$

$\Delta = \frac{1}{2}r(a+b+c)$

$\Rightarrow$ In vector form $I = \dfrac{aA+bB+cC}{a+b+c}$

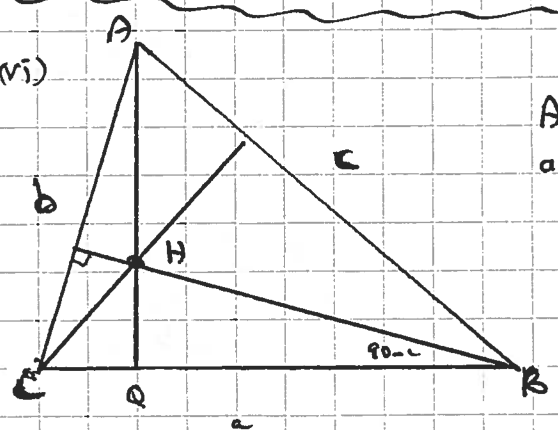
(v)



$(OBC) = \frac{R^2}{2}\sin(2A)$

So areal coordinates of $O = (\sin 2A, \sin 2B, \sin 2C)$ (normalize)

(vi)



Areal coord. of H are $(\tan A, \tan B, \tan C)$

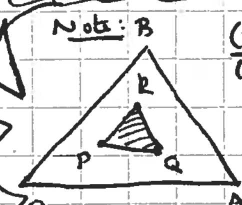
For $AQ = c\sin B$
$BQ = c\cos B$

so

$\cot C = \tan(90-C) = \dfrac{HQ}{c\cdot\cos B}$

This is important!

Note:



$$\frac{(PQR)}{(OAB)} = \begin{vmatrix} p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \\ r_1 & r_2 & 1 \end{vmatrix} = \begin{vmatrix} p_1 & p_2 & p_1+p_2+p_3 \\ q_1 & q_2 & q_1+q_2+q_3 \\ r_1 & r_2 & r_1+r_2+r_3 \end{vmatrix} = \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$$

So $HQ = \dfrac{c\cdot\cos B\cos C}{\sin C} = 2R\cos B\cos C$

Hence $\dfrac{HQ}{AQ} = \dfrac{(HCB)}{(ACB)} = \dfrac{2R\cos B\cos C}{2R\sin B\sin C} = \cot B\cot C$.

So one form of Areal coordinates of H is $(\cot B\cot C, \cot C\cot A, \cot A\cot B)$

Note: From $\tan A+\tan B+\tan C = \tan A\tan B\tan C$ we note that $\dfrac{\tan A}{\cot B\cot C} = \dfrac{\tan B}{\cot C\cot A} = \dfrac{\tan C}{\cot A\cot B} = \tan A+\tan B+\tan C$

$\Rightarrow$ A second way to express Ar. coor. for H is $(\tan A, \tan B, \tan C)$