WINTER CAMP 2005
INEQUALITIES

A BRIEF SUMMARY OF BASIC INEQUALITIES.

1. **The triangle inequality**
   If a, b, c are real numbers, then $\left||a-c|-|b-c|\right| \le |a-b| \le \left||a-c|+|b-c|\right|$.

2. **The harmonic-geometric-arithmetic-quadratic means inequality**
   If $x_1, x_2, x_3, \dots, x_n$ are positive numbers, then
   $$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\dots+\frac{1}{x_n}} \le \sqrt[n]{x_1 x_2 x_3 \dots x_n} \le \frac{x_1+x_2+x_3+\dots+x_n}{n} \le \sqrt{\frac{x_1^2+x_2^2+x_3^2+\dots+x_n^2}{n}}$$
   with equality if and only if $x_1 = x_2 = x_3 = \dots = x_n$.

3. **The general means inequality**
   Let $x_1, x_2, x_3, \dots, x_n$ be positive numbers.
   We define $M_r = \left(\dfrac{x_1^r+x_2^r+x_3^r+\dots+x_n^r}{n}\right)^{1/r}$ for $r \ne 0$ and $M_0 = \sqrt[n]{x_1 x_2 x_3 \dots x_n}$.
   If $r > s$ then $M_r \ge M_s$, with equality if and only if $x_1 = x_2 = x_3 = \dots = x_n$.

4. **The general weighted means inequality**
   Let $x_1, x_2, x_3, \dots, x_n, w_1, w_2, w_3, \dots, w_n$ be positive numbers with $w_1 + w_2 + w_3 + \dots + w_n = 1$.
   We define $WM_r = \left(w_1 x_1^r + w_2 x_2^r + w_3 x_3^r + \dots + w_n x_n^r\right)^{1/r}$ for $r \ne 0$ and $WM_0 = x_1^{w_1} x_2^{w_2} x_3^{w_3} \dots x_n^{w_n}$.
   If $r > s$ then $WM_r \ge WM_s$, with equality if and only if $x_1 = x_2 = x_3 = \dots = x_n$.

5. **The Minkowski inequality**
   If $x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_n$ are all ≥ 0 and $p \ge 1$, then
   $$\left(\sum_{k=1}^{n}(x_k+y_k)^p\right)^{1/p} \le \left(\sum_{k=1}^{n}x_k^p\right)^{1/p} + \left(\sum_{k=1}^{n}y_k^p\right)^{1/p}$$
   with equality if and only if there exists λ such that $y_k = \lambda x_k$ for $k = 1, 2, 3, \dots, n$.
   The inequality is reversed if $0 < p < 1$.

6. **The Cauchy-Schwarz inequality**
   If $v_1, v_2, v_3, \dots, v_n$ and $w_1, w_2, w_3, \dots, w_n$ are real numbers, then
   $$|v_1w_1 + v_2w_2 + v_3w_3 + \dots + v_nw_n| \le \sqrt{v_1^2+v_2^2+v_3^2+\dots+v_n^2}\,\sqrt{w_1^2+w_2^2+w_3^2+\dots+w_n^2},$$
   with equality if and only if there exists λ such that $w_k = \lambda v_k$ for $k = 1, 2, 3, \dots, n$.

7. **The Hölder inequality**
   If $x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_n, p, q$ are all ≥ 0 and $p+q=1$, then
   $$\sum_{i=1}^{n} x_i^p y_i^q \le \left(\sum_{i=1}^{n} x_i\right)^p \left(\sum_{i=1}^{n} y_i\right)^q$$
   with equality if and only if there exists λ such that $y_k = \lambda x_k$ for $k = 1, 2, 3, \dots, n$.

8. The rearrangement inequality

Suppose that $x_1 \le x_2 \le x_3 \le \ldots \le x_n$ and $y_1 \le y_2 \le y_3 \le \ldots \le y_n$, and let $z_1, z_2, z_3, \ldots, z_n$ be any permutation of the numbers $y_1, y_2, y_3, \ldots, y_n$, then

$$\sum_{i=1}^{n} x_i y_{n+1-i} \le \sum_{i=1}^{n} x_i z_i \le \sum_{i=1}^{n} x_i y_i .$$

9. The Chebyshev inequality

Suppose that $0 \le x_1 \le x_2 \le x_3 \le \ldots \le x_n$ and $0 \le y_1 \le y_2 \le y_3 \le \ldots \le y_n$, then

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \le n \sum_{i=1}^{n} x_i y_i .$$

EXERCISES.

1. Prove that for any positive a, b and c, $(a+b)(b+c)(a+c) \ge 8abc$.
2. Prove that for any positive a, b and c, if $(1+a)(1+b)(1+c) = 8$ then $abc \le 1$.
3. Prove that for any positive a, b and c, if $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = 1$ then $(a-1)(b-1)(c-1) \ge 8$.
4. Prove that for $a, b, c > 0$, if $(a \sin\theta)^2 + (b\cos\theta)^2 < c^2$ then $a\sin^2\theta + b\cos^2\theta < c$.
5. If a, b and c are positive numbers, what is the minimum possible value of the expression

$$\frac{1+a+2b+3c}{\left(1+\sqrt[3]{a}+2\sqrt[3]{b}+3\sqrt[3]{c}\right)^3} \ ?$$

What are the values of a, b and c for which the minimum value is reached ?

6. What is the maximum possible value of the expression $\dfrac{1+a+2b+3c}{\sqrt{1+2(a^2+b^2+c^2)}}$?

$$\frac{1+\frac{1}{2}+2+\frac{9}{2}}{\sqrt{1+2\left(\frac{1}{4}+1+\frac{9}{4}\right)}}$$

What are the values of a, b and c for which the maximum value is reached ?

7. Find the volume of the largest rectangular box that fits inside the ellipsoid $x^2 + 3y^2 + 9z^2 = 9$, with faces parallel to the coordinate planes.

$9 = x^2+3y^2+9z^2 \ge 3\sqrt[3]{27x^2y^2z^2} = 9(xyz)^{2/3}$

$xyz \le 1$

$x = \sqrt{3}y = 3z$

$z = \frac{1}{\sqrt{3}}$

$y = 1$

$x = \sqrt{3}$

8. Prove that for $a, b, c, d > 0$, $\dfrac{(a^2+b^2+c^2+d^2)^3}{(abc+abd+acd+bcd)^2} \ge 4$.
9. Prove each of the following inequalities.

a) If $0 \le x \le \pi/2$ then $2x \le \pi\sin x \le \pi x$. (Jordan)

b) If $x > -1$ and $0 < r < 1$, then $(1+x)^r \le 1 + rx$. (Bernoulli)

c) If a, b, p, q are all positive and $p + q = 1$, then $ab \le p a^{1/p} + q b^{1/q}$. (Young)

d) If a, b, c are all positive, then $\dfrac{a}{b+c} + \dfrac{b}{a+c} + \dfrac{c}{a+b} \ge \dfrac{3}{2}$. (Nesbitt)

10. Suppose that there is a triangle whose sides have lengths a, b and c. Prove that there is a triangle whose sides have lengths $\dfrac{a^2+ab+ac+bc}{2a+b+c}$, $\dfrac{ab+ac+b^2+bc}{a+2b+c}$ and $\dfrac{ab+ac+bc+c^2}{a+b+2c}$.

11. Prove the rearrangement inequality.

12. Prove the Chebyshev inequality.

13. Let $n>3$ be an integer and let $x_1, x_2, x_3, \ldots, x_n$ be positive numbers such that $x_1^2+x_2^2+\ldots+x_n^2=1$.

Prove that $\dfrac{x_1}{1+x_2^2}+\dfrac{x_2}{1+x_3^2}+\ldots+\dfrac{x_n}{1+x_1^2} \geq \dfrac{4}{5}\left(x_1\sqrt{x_1}+x_2\sqrt{x_2}+\ldots+x_n\sqrt{x_n}\right)^2$.

14. Let $x_1, x_2, x_3, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2}+\frac{x_2}{1+x_1^2+x_2^2}+\ldots+\frac{x_n}{1+x_1^2+\ldots+x_n^2}<\sqrt{n}.$$

15. Find all positive integers n such that $3^n+4^n+\ldots+(n+2)^n=(n+3)^n$.

16. Find a solution to the system $\begin{cases} a+b+c+d+e=8 \\ a^2+b^2+c^2+d^2+e^2=16 \end{cases}$ for which the value of e is the maximum possible.

17. Let $x_i>0$, $x_1+x_2+x_3+\ldots+x_n=1$ and let s be the greatest of the numbers

$$\frac{x_1}{1+x_1}, \frac{x_2}{1+x_1+x_2}, \frac{x_3}{1+x_1+x_2+x_3}, \ldots, \frac{x_n}{1+x_1+x_2+\ldots+x_n}$$

Find the smallest value for s. Find the values of $x_1, x_2, x_3, \ldots, x_n$ for which s reaches its minimum.

18. **IMO 1975. A1.**

Let $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, y_3, \ldots, y_n$ be real numbers such that $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$. Prove that, if $z_1, z_2, z_3, \ldots, z_n$ is any permutation of $y_1, y_2; y_3, \ldots, y_n$, then

$$\sum_{i=1}^{n}(x_i-y_i)^2 \leq \sum_{i=1}^{n}(x_i-z_i)^2.$$

19. **IMO 1978. B2**

Let $a_1, a_2, a_3, \ldots, a_n$ be a sequence of distinct positive integers. Prove that, for all natural numbers n,

$$\sum_{k=1}^{n}\frac{a_k}{k^2} \geq \sum_{k=1}^{n}\frac{1}{k}.$$

20. **IMO 1984. A1**

Prove that $0 \leq xy+yz+zx-2xyz \leq 7/27$, where x, y, z are non-negative real numbers such that $x+y+z=1$.

SOME RECENT IMO PROBLEMS.

21. IMO 2004. B1.

Let $n \geq 3$ be an integer. Let $t_1, t_2, t_3, \dots, t_n$ be positive real numbers such that

$$n^2+1>(t_1+t_2+\dots+t_n)\left(\frac{1}{t_1}+\frac{1}{t_2}+\dots\frac{1}{t_n}\right)$$

Show that t_i, t_j and t_k are side lengths of a triangle for all i, j and k with $1 \leq i < j < k \leq n$.

22. IMO 2003. B2.

Let $n > 2$ be a positive integer and let $x_1, x_2, \dots, x_n$ be real numbers with $x_1 \leq x_2 \leq \dots \leq x_n$.

a) Show that $\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left|x_i-x_j\right|\right)^2 \leq \frac{2}{3}(n^2-1)\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^2$.

b) Show that equality holds if and only if $x_1, x_2, \dots, x_n$ is an arithmetic progression.

23. IMO 2001. A2.

Let a, b and c be positive real numbers. Prove that $\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1$.

24. IMO 2000. A2.

Let a, b and c be positive real numbers such that $abc = 1$.

Prove that $(a-1+1/b)(b-1+1/c)(c-1+1/a) \leq 1$.

25. IMO 1999. A2.

Let $n \geq 2$ be a fixed integer.

a) Determine the least constant C such that the inequality $\sum_{1\leq i<j\leq n} x_i x_j (x_i^2+x_j^2) \leq C\left(\sum_{1\leq i\leq n} x_i\right)^4$ holds for all real numbers $x_1, x_2, \dots, x_n \geq 0$.

b) For this constant C, determine when the equality holds.

26. IMO 1997. A3.

Let $x_1, x_2, \dots, x_n$ be real numbers satisfying the conditions $|x_1+x_2+\dots+x_n|=1$ and $|x_i| \leq \frac{n+1}{2}$ for $i = 1, 2, \dots, n$. Show that there exists a permutation $y_1, y_2, \dots, y_n$ of $x_1, x_2, \dots, x_n$ such that

$$|y_1+2y_2+\dots+ny_n| \leq \frac{n+1}{2}.$$