How to use AM-GM like a boss

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Introduction

AM-GM inequality is one of the most frequently used inequalities in olympiads. Not only in inequalities, but sometimes it is also used in solving (or disproving) equations. In this note we will actually exploit weighted AM-GM with rational weights, but in a more intuitive way where we don't have to deal with scary notations.

In this note we will try to shed some light on how some people seem to miraculously solve inequalities with *just* AM-GM. They actually exploit weighted AM-GM with rational weights, which you probably think is another name for black magic. Let's break that misconception, shall we? This note explores the application of weighted AM-GM, but in a more intuitive way where we don't have to deal with scary notations.

Here is the normal formulation of AM-GM.

For positive reals $x_1, x_2, \ldots, x_n$

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$

Which can be more conveniently written as:

For positive reals $a_1, b_1, \ldots, a_n$

$$a_1^n + a_2^n + \ldots + a_n^n \geq n a_1 a_2 \ldots a_n$$

Some problems

Problem 1. *For positive reals a, b, c prove that*

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

Solution: Using AM-GM for two variables, we get,

$$a^2 + b^2 \geq 2ab$$
$$b^2 + c^2 \geq 2bc$$
$$c^2 + a^2 \geq 2ca$$

Summing up and dividing by 2, we get,

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

Which was what we wanted. □

AM-GM for two variables is beautiful, isn't it? Elegant application, like in the previous problem, is the true power of 2-variable AM-GM. Not only algebra, but also combinatorial and geometric olympiad level problems exist that require clever usage of 2-variable AM-GM. But just like in romance, in inequalities beauty can often be a trap. A lot of people get trapped into trying only AM-GM for two or three variables in a problem, and thus missing quite easy stuff. This especially applies for our next few problems:

Problem 2. *For positive reals a, b, c prove that*

$$a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab$$

Solution: Using AM-GM for four variables, we get,

$$a^4 + a^4 + b^4 + c^4 \geq 4a^2bc$$
$$b^4 + b^4 + c^4 + a^4 \geq 4b^2ca$$
$$c^4 + c^4 + a^4 + b^4 \geq 4c^2ab$$

Summing up and dividing by 4, we get the desired inequality. □

The next problem has a very neat statement. Surely it must have an obvious solution? Inequalities as a subject has so much complicated algebra and notation that such nice-looking statements immediately catch our eye. But as you will see, that's another big trap in inequalities. Try to become comfortable with the complicated algebra by working out the steps and reading solutions properly - the confidence you gain will help you avoid the traps.

Problem 3. *For positive reals a, b, c so that $abc = 1$, prove that*

$$a^2 + b^2 + c^2 \geq a + b + c$$

Solution: Using AM-GM on 6 variables, we get

$$a^2 + a^2 + a^2 + a^2 + b^2 + c^2 \geq 6\sqrt[6]{a^8b^2c^2}$$

But $a^8b^2c^2 = a^6 \times a^2b^2c^2 = a^6$ Therefore,

$$a^2 + a^2 + a^2 + a^2 + b^2 + c^2 \geq 6a$$

Similarly,

$$b^2 + b^2 + b^2 + b^2 + c^2 + a^2 \geq 6b$$
$$c^2 + c^2 + c^2 + c^2 + a^2 + b^2 \geq 6c$$

Adding up and dividing by 6, we get the desired inequality. □

Note that in this problem(and also in problem 2), the positive condition is not necessary(Why?).

Are you confused by why AM-GM for 4 and 6 variables respectively worked in the above two problems, where the lower variable versions failed? Don't worry, we will discuss it in depth soon enough. In the meantime, please think about it on your own. We chose to use AM-GM in a certain way in all of the example problems, in order to get just one thing. What was it?

And if you're still a bit scared of AM-GM taking on so many variables, have patience young one ;). As you go into the next problem (and when you read its solution) remember the golden rule of inequalities: "Be Brave!"

Problem 4. *For positive reals a, b, c, d prove that*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{a} \geq a + b + c + d$$

Solution: We use AM-GM with 15 variables.

$$\frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{d^2}{a} \geq 15a$$

Similarly,

$$\frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{a^2}{b} \geq 15b$$

$$\frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{c^2}{d} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{b^2}{c} \geq 15c$$

$$\frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{d^2}{a} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{a^2}{b} + \frac{b^2}{c} + \frac{b^2}{c} + \frac{c^2}{d} \geq 15d$$

Summing up and dividing by 15, we get the desired inequality. □

Now I don't want you to hate me and inequalities for the rest of your life, so here's a simpler solution to the last problem:

We use AM-GM with 2(!) variables.

$$\frac{a^2}{b} + b \geq 2a$$

$$\frac{b^2}{c} + c \geq 2b$$

$$\frac{c^2}{d} + d \geq 2c$$

$$\frac{d^2}{a} + a \geq 2d$$

Summing up and after a bit of algebra, we get the desired inequality.

Now why would anyone post that abomination of a solution first when there's such an elegant solution? It might seem strange to you, but the first solution is actually well-motivated. Of course, the second solution also offers a nice bit of intuition (remove pesky denominators with AM-GM!). But the reason why this note focuses on the former is that it's much stronger than the latter. Don't believe me? Then be sure to see **Exercise 6** at the end, trying out both of the above approaches.

Cyclic sum

At this point, you are probably fed up with me for showing you overhead AM-GM solutions with no explanation whatsoever. Don't worry, I am now going to provide you with the motivation for these approaches. But before that, we need to get into some notation.

Wait, isn't the mission of this sheet to replace scary notation with intuition? Well cyclic sums are remarkable in that the notation is extremely intuitive. It might feel weird at first, but as you use it it'll become more natural and save a lot of time.

The cyclic sum of polynomial $P(a, b, c)$ will be defined as

$$P(a, b, c) + P(b, c, a) + P(c, a, b)$$

And will be denoted by

$$\sum_{cyc} P(a, b, c)$$

Confused? Maybe looking at some examples will help:

$$P(a,b,c) = a^2 \Longrightarrow \sum_{cyc} a^2 = a^2 + b^2 + c^2$$

$$P(a,b,c) = a^2bc \Longrightarrow \sum_{cyc} a^2bc = a^2bc + b^2ca + c^2ab$$

$$P(a,b,c) = abc \Longrightarrow \sum_{cyc} abc = abc + bca + cab = 3abc$$

$$P(a,b,c,d) = a^3b^2c \Longrightarrow \sum_{cyc} a^3b^2c = a^3b^2c + b^3c^2d + c^3d^2a + d^3a^2b$$

Play around with this notation until you get the hang of it.

Explanation

Did you see a pattern behind the solutions? Note that we create an inequality looking at just one term of the right side, and then cyclically sum them up to get the full RHS. So, lets assume we take the first term n_1 times, the second term n_2 times and so on. If we don't use the ith term, we take $n_i = 0$. Recall something from your textbook:

$$(a^m)^n = a^{mn}$$

That is the principle we will use when finding the desired inequality. Now, we certainly remember the definition of power, don't we? So, lets start with the first problem.

For positive reals a, b, c prove that

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

For practice, try to rewrite the proof using cyclic sums. It should go something like:

Using AM-GM for two variables, we get,

$$a^2 + b^2 \geq 2ab$$

And so

$$\sum_{cyc} a^2 + b^2 \geq \sum_{cyc} 2ab$$

Which is the same as:

$$\frac{1}{2}\sum_{cyc} a^2 + b^2 \geq \frac{1}{2}\sum_{cyc} 2ab$$

This is the same as the problem's statement (check to see if this is true!)

Now let's try to find a general technique to reach the solution. Lets assume there are l a^2's, m b^2's, n c^2's. We want to make them greater than ab:

$$la^2 + mb^2 + nc^2 \geq ab$$

So the product would be $a^{2l}b^{2m}c^{2n}$. If this equals ab, then we deduce $2l = 1$, $2m = 1$. So, $l = \frac{1}{2}$, $m = \frac{1}{2}$. It is also clear that $n = 0$. However $l : m : n$ is what truly matters. That's why we scale them by 2 and get $(l, m, n) = (1, 1, 0)$. So, we have to take 1 a^2 and 1 b^2. And that is precisely what we did.

Now, for the second problem. Setting up equations like the previous one lets us assume,

$$4l = 2,\ 4m = 1,\ 4n = 1$$

Therefore $l = \frac{1}{2}$, $m = \frac{1}{4}$, $n = \frac{1}{4}$. Scaling up by 4, we get $l = 2, m = 1, n = 1$. So, the desired inequality would be

$$a^4 + a^4 + b^4 + c^4 \geq 4a^2bc$$

Why does this work out? That is for you to discover.

Now, we move on to P3, which is a bit trickier because of the $abc = 1$ condition. However, for those acquainted with the homogenization technique, this should not be difficult.

Here, like before, let's take l, m, n and set up equations. Here we get: $l = m = n = 1$. But that doesn't work!!!

What we have to do here is to use the $abc = 1$ condition. We're trying to get a product which equals a. Therefore, the bs and cs must cancel out to 1. Then we have $m = n$. Also note that, since $abc = 1$, $a^m b^m c^m$=1. So

$$a^l b^m c^m = a^{l-m} a^m b^m c^m = a^{l-m}$$

From these we get $2l - 2m = 1$. Here we evoke the $l + m + n = 1$ condition(Why?). So $l + 2m = 1$. Solving the equations, we get $l = \frac{2}{3}$, and $m = n = \frac{1}{6}$. Scaling up by 6, We get $l = 4, m = n = 1$. From there we get the desired inequality.

For P4, let us set up equations with p, q, r, s. Since $\frac{a^2}{b} = a^2 b^{-1}$, we get,

$$2p - s = 1, 2q - p = 0, 2r - q = 0.2s - r = 0$$

Solving these and scaling yield $p = 8, q = 4, r = 2, s = 1$.

Now, why we can we take $l + m + n = 1$? Did you see that, before scaling, this was true in every problem? The reason is we directly took the products equal to equate powers. However, we neglected the factor in front of the product. So, we just took it as 1 for simplicity. However, in the real expression it will be the number of terms. So, we are taking the number of terms equal to 1(!).

For example, in problem 2, we wanted:

$$la^4 + mb^4 + nc^4 \geq a^2bc$$

AM-GM would give us an inequality for the LHS, like this:

$$la^4 + mb^4 + nc^4 \geq (l + m + n) \sqrt[l+m+n]{a^{4l} b^{4m} c^{4n}}$$

That's such an ugly RHS! D: But we can easily fix it if we consider $l + m + n = 1$

That is also the reason we get fraction numbers. Note that, I am not giving any rigorous proof of the technique because you don't need to show these in your actual rigorous solutions. These are simply rough work. However, you are encouraged to try and prove.

Now we present a bit more difficult problem.

Problem 5. *Let a, b, c be positive reals with $a + b + c = 1$. Prove that,*

$$3(a^5 + b^5 + c^5) \geq a^2b^2 + b^2c^2 + c^2a^2$$

Solution: We note that the degree of RHS is 4 whereas the degree of LHS is 5. However, if we multiply RHS with $a + b + c = 1$, then the degrees would be equal. So,

$$3a^5 + 3b^5 + 3c^5 \geq (a + b + c)(a^2b^2 + b^2c^2 + c^2a^2)$$

is an equivalent statement. Now, expanding the RHS, we get,

$$\sum_{cyc} a^3b^2 + \sum_{cyc} a^2b^3 + \sum_{cyc} a^2b^2c$$

Clearly,

$$a^5 + a^5 + a^5 + b^5 + b^5 \geq 5a^3b^2$$
$$a^5 + a^5 + b^5 + b^5 + b^5 \geq 5a^2b^3$$
$$a^5 + a^5 + b^5 + b^5 + c^5 \geq 5a^2b^2c$$

Summing each of them cyclically and adding them up and dividing by 5, we get the desired inequaltiy. $\square$

So, yeah. If the RHS seems too unsymmetric, then divide it into symmetric parts and then apply AM-GM like a boss.

However, one question remains. Why did we multiply $a + b + c$ to the LHS? It's simply to make the application of AM-GM more smooth. Note that in both formulations of AM-GM, the degree in both sides are the same. So, keeping the degree equal on both sides gives us a clear line of attack for using AM-GM.

Another important observation is, if you keep the degree equal in both sides, then if the inequality holds for (a, b, c), it will also hold for (ka, kb, kc) where k is a positive real. This helps us to disregard the weird equality conditions like $a + b + c = 1$, $abc = 1$, $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and etc. Actually, we could have done the same thing in problem 3. Let us recall the formulation.

For positive reals a, b, c such that $abc = 1$, prove that

$$a^2 + b^2 + c^2 \geq a + b + c$$

The condition $abc = 1$ is equivalent to $\sqrt[3]{abc} = 1$, and substituting(substituting is very important, use it whenever it seems like it won't harm the problem) $a = x^3, b = y^3$ and $c = z^3$. After that, we multiply $\sqrt[3]{abc} = xyz = 1$ to the RHS. So we get,

$$x^6 + y^6 + z^6 \geq xyz(x^3 + y^3 + z^3)$$

Or, equivalently,

$$x^6 + y^6 + z^6 \geq x^4yz + y^4zx + z^4xy$$

Which can be again solved similarly.

Practice problems

Exercise 1. *Create as many new inequality problems as you can.*

Exercise 2. *For positive reals a, b, c, prove that*

$$3(a^3 + b^3 + c^3) \geq (a + b + c)(a^2 + b^2 + c^2)$$

Exercise 3. *For positive reals a, b, c, prove that*

$$a^3 + b^3 + c^3 + ab^2 + bc^2 + ca^2 \geq 2(a^2b + b^2c + c^2a)$$

Exercise 4. *Rewrite the previous inequalities using cyclic sum notation.*

Exercise 5. *For positive reals a, b, c prove the following chain of inequalities.*

$$\sum_{cyc} a^5 \geq \sum_{cyc} a^4b \geq \sum_{cyc} a^3b^2 \geq \sum_{cyc} a^3bc \geq \sum_{cyc} a^2b^2c$$

Exercise 6. *For positive reals a, b, c prove that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Nesbitt's inequality

Exercise 7. *For positive reals a, b, c prove that*

$$(a+b+c)^4 \geq 27(a^2bc + b^2ca + c^2ab)$$

Exercise 8. *For positive reals a, b, c prove that*

$$(a+b+c)(a^7+b^7+c^7) \geq (a^4+b^4+c^4)^2$$

Exercise 9. *For positive reals x, y, z prove that*

$$(x+y+z)^2(x^2+y^2+z^2)^3 \geq 27(x^2yz + y^2zx + z^2xy)^2$$

Hint: <I:YO:U YEP FOGGRTRMY OMRWD (replace each letter with the one left to it on the qwerty keyboard)

Exercise 10. *For positive reals x, y, z prove that*

$$(x^2+y^2+z^2)(x^3+y^3+z^3) \geq (x^2yz + y^2zx + z^2xy)(xy+yz+zx)$$

Exercise 11. *For positive reals a, b, c with $abc = 1$, Prove that,*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c$$

Exercise 12. *For positive reals a, b, c with $a + b + c = 1$, prove that*

$$a^3 + b^3 + c^3 \geq \frac{a^2+b^2+c^2}{3}$$

Exercise 13. *Let a, b, c be positive reals with $a + b + c = 3$. Find the maximum possible value of*

$$ab^2 + bc^2 + ca^2$$

Exercise 14. *Let x_1, x_2, x_3, x_4 be positive reals such that $x_1x_2x_3x_4 = 1$. Prove that:*

$$\sum_{i=1}^{4} x_i^3 \geq \max\{\sum_{i=1}^{4} x_i, \sum_{i=1}^{4} \frac{1}{x_i}\}$$

Iran MO 1997

Exercise 15. *Let a, b, c be positive reals with $a + b + c = 3$. Prove that*

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$$

Russia 2004

Exercise 16. *Let a, b, c be positive reals with $abc = 1$. Prove that*

$$\sum_{cyc} \frac{a^2}{b+c} \geq \frac{3}{2}$$

Exercise 17. *Let a, b, c be positive reals with $abc = 1$. Prove that*

$$\sum_{cyc} \frac{1}{a^3(b+c)} \geq \frac{3}{2}$$

IMO 1995

Hint: DINDYOYIYR OMBRTDRD (replace each letter with the one left to it on the qwerty keyboard)

Appendix A

Here we provide a proof of AM-GM inequality

Statement: For positive reals $a_1, a_2, \dots a_n$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

Proof: We use cauchy induction, also known as forward backward induction.

The statement is true when $n = 2$ because

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \Leftrightarrow (a_1 + a_2)^2 \geq 4a_1 a_2 \Leftrightarrow (a_1 - a_2)^2 \geq 0$$

Next we show that if AM-GM holds for n variables, it also holds for $2n$ variables:

$$\frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = \frac{\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n}}{2}$$

$$\frac{\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n}}{2} \geq \frac{\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{a_{n+1} a_{n+2} \cdots a_{2n}}}{2}$$

$$\frac{\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{a_{n+1} a_{n+2} \cdots a_{2n}}}{2} \geq \sqrt{\sqrt[n]{a_1 a_2 \cdots a_n} \sqrt[n]{a_{n+1} a_{n+2} \cdots a_{2n}}}$$

$$\sqrt{\sqrt[n]{a_1 a_2 \cdots a_n} \sqrt[n]{a_{n+1} a_{n+2} \cdots a_{2n}}} = \sqrt[2n]{a_1 a_2 \cdots a_{2n}}$$

The first inequality follows from n-variable AM-GM, which is true by assumption, and the second inequality follows from 2-variable AM-GM, which is proven above.

Finally we show that if AM-GM holds for n variables, it also holds for $n - 1$ variables. By n-variable AM-GM, $\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$

Let $a_n = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}$

Then we have

$$\frac{a_1 + a_2 + \cdots + a_{n-1} + \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}}{n} = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n - 1}$$

So,

$$\frac{a_1 + a_2 + \cdots + a_{n-1}}{n - 1} \geq \sqrt[n]{a_1 a_2 \cdots a_{n-1} \cdot \frac{a_1 + a_2 + \cdots + a_{n-1}}{n - 1}}$$

$$\Rightarrow \left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n - 1}\right)^n \geq a_1 a_2 \cdots a_{n-1} \cdot \frac{a_1 + a_2 + \cdots + a_{n-1}}{n - 1}$$

$$\Rightarrow \left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n - 1}\right)^{n-1} \geq a_1 a_2 \cdots a_{n-1}$$

$$\Rightarrow \frac{a_1 + a_2 + \cdots + a_{n-1}}{n - 1} \geq \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}$$

By Cauchy Induction, this proves the AM-GM inequality for n variables.

Appendix B

For further reading, these are highly recommended.

- brilliant.org articles . These articles have a simple style and LOTS of good practice problms for beginners - check them out! Two of the coolest:
  - 'Cauchy-Scwarz Inequality'
  - 'Titu's Lemma'
- Basic of Olympiad Inequalities by Samin Riasat Nayel
- Olympiad Inequalities by Thomas J. Mildorf
- Secrets in Inequalities by Pham Kin Han
- Problem Solving Strategies by Arthur Engel

And finally, solve lots of problems! Because the only way to learn to solve problems is to solve problems.

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