Art of Problem Solving

WOOT

Class Transcript 10/07 - Sequences & Series

MellowMelon 7:30:03 pm
WOOT 2013-14: Sequences & Series

MellowMelon 7:30:12 pm
Today, we will be looking at problems involving sequences and series. Let's start with a simple warm-up problem.

MellowMelon 7:30:29 pm
Let S be the set of positive integers of the form 2^a3^b, where a and b are nonnegative integers, so the first few elements of S are 1, 2, 3, 4, 6, 8, and so on. Find the sum of the reciprocals of the elements of S.

MellowMelon 7:30:54 pm
We write out the first few terms of the sum:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{2\cdot 3}+\frac{1}{2^3}+\cdots.$$

Does anyone see anything?

ProbaBillity 7:31:59 pm
product of two infinite series

TheStrangeCharm 7:31:59 pm
factor into two infinite geometric sequences

Arithmophobia 7:31:59 pm
Factor it!

lawrenceli 7:31:59 pm
factor in terms of 2^a

Bg1 7:31:59 pm
group terms

noodleeater 7:31:59 pm
(1+1/2+1/2^2+...)(1+1/3+1/3^2+...)

thkim1011 7:31:59 pm
(1 + 1/2 + 1/4 + 1/8 + ...)(1+ 1/3 + 1/9 + 1/27 + ...)

MellowMelon 7:32:03 pm
We can factor this sum as the product of two sums:

$$\begin{aligned}&1+\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{2\cdot 3}+\frac{1}{2^3}+\cdots\\&=\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots\right).\end{aligned}$$

Clearly, each term in the original sum is a product of a term from the first factor and a term in the second factor.

willwang123 7:33:08 pm
=2*3/2=3

TheStrangeCharm 7:33:08 pm
geometric series formula and we get 3

brian22 7:33:08 pm
2*3/2=3

Coly 7:33:08 pm
sum of geometric series

Arithmophobia 7:33:08 pm
geometric sequence sum formula

cerberus88 7:33:08 pm
sum of each geometric sequence

lawrenceli 7:33:08 pm
compute sum for each of the geometric series

ProbaBillity 7:33:08 pm
So the answer is $\frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} = \boxed{3}$.

vincenthuang75025 7:33:08 pm
First one is 2, second one is 3/2

chenjamin 7:33:08 pm
=(2)(3/2)=3

MellowMelon 7:33:10 pm
Then each factor is an infinite geometric series, so we get

$$\begin{aligned}
&1+\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{2\cdot 3}+\frac{1}{2^3}+\cdots \\
&=\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots\right) \\
&=\frac{1}{1-1/2}\cdot\frac{1}{1-1/3} \\
&=2\cdot\frac{3}{2} \\
&=3.
\end{aligned}$$

MellowMelon 7:33:19 pm
So although geometric series are relatively simple objects, they can turn up in unexpected ways.

MellowMelon 7:33:36 pm
Find

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}.$$

MellowMelon 7:33:50 pm
If we want to analyze a sequence, a good way to start is to look at the first few terms to see if we can find a pattern. How can we apply this technique to an infinite sum?

willwang123 7:34:50 pm
replace infinity with 2, 3, 4, etc.

steve314 7:34:50 pm
write out the first few sums

sirknightingfail 7:34:50 pm
for k=1,2,3 etc.

nuggetfan 7:34:50 pm
k =1,2,3,...

nilaisarda 7:34:50 pm
partial sums?

MellowMelon 7:34:52 pm
We can look at the sequence defined by the sum of the first few terms.

MellowMelon 7:34:57 pm
Let

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}.$$

(Given an infinite sum, these sums are called *partial sums*.)

MellowMelon 7:35:22 pm
We now compute the first few partial sums.

$$\begin{aligned}
S_1 &= \frac{1}{1\cdot 2\cdot 3} = \frac{1}{6}, \\
S_2 &= \frac{1}{6} + \frac{1}{2\cdot 3\cdot 4} = \frac{5}{24}, \\
S_3 &= \frac{5}{24} + \frac{1}{3\cdot 4\cdot 5} = \frac{9}{40}, \\
S_4 &= \frac{9}{40} + \frac{1}{4\cdot 5\cdot 6} = \frac{7}{30}, \\
S_5 &= \frac{7}{30} + \frac{1}{5\cdot 6\cdot 7} = \frac{5}{21}.
\end{aligned}$$

Unfortunately, there do not seem to be any apparent patterns.

MellowMelon 7:35:41 pm
The sequence of partial sums continues $\frac{27}{112}$, $\frac{35}{144}$, $\frac{11}{45}$, $\frac{27}{110}$, $\frac{65}{264}$, and so on. Do these numbers seem to be approaching any nice value?

zhuangzhuang 7:36:22 pm
approaching 1/4

ProbaBillity 7:36:22 pm
1/4

yfang88 7:36:22 pm
1/4?

willwang123 7:36:22 pm
1/4

RocketSingh 7:36:22 pm
1/4

sunny2000 7:36:22 pm
1/4?

TheStrangeCharm 7:36:22 pm
close to 1/4

cerberus88 7:36:22 pm
1/4

sujaykazi 7:36:22 pm
1/4

Coly 7:36:22 pm
1/4

lawrenceli 7:36:22 pm
1/4?

superpi83 7:36:22 pm
1/4

cothurn 7:36:22 pm
1/4

ssk9208 7:36:22 pm
1/4

Bg1 7:36:22 pm
1/4

MellowMelon 7:36:28 pm
They seem to be approaching $\frac{1}{4}$, so this gives us a guess as to what the infinite sum is.

MellowMelon 7:36:46 pm
Is there any way we can make use of this observation?

lawrenceli 7:38:13 pm
S_n = 1/4 + f(n)

mentalgenius 7:38:13 pm
consider 1/4 - partial sums

cerberus88 7:38:13 pm
difference from each term to 1/4?

MellowMelon 7:38:15 pm
We can also look at the differences between $\frac{1}{4}$ and the partial sums. If the infinite sum is $\frac{1}{4}$, then we expect the differences to go to 0:

$$\frac{1}{4} - S_1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12},$$
$$\frac{1}{4} - S_2 = \frac{1}{4} - \frac{5}{24} = \frac{1}{24},$$
$$\frac{1}{4} - S_3 = \frac{1}{4} - \frac{9}{40} = \frac{1}{40},$$
$$\frac{1}{4} - S_4 = \frac{1}{4} - \frac{7}{30} = \frac{1}{60},$$
$$\frac{1}{4} - S_5 = \frac{1}{4} - \frac{5}{21} = \frac{1}{84}$$

Now we see something interesting and useful. Each difference is the reciprocal of a positive integer, which gives us a new sequence to analyze.

MellowMelon 7:38:36 pm
We need to figure out the sequence $12, 24, 40, 60, 84,$ etc. What do all these integers have in common?

yfang88 7:39:07 pm
factor of 4

sirknightingfail 7:39:07 pm
multiples of 4

vincenthuang75025 7:39:07 pm
divisible by 4

cerberus88 7:39:07 pm
divisible by 4

werdnerd360 7:39:07 pm
multiples of 4

Piya31415 7:39:07 pm
multiples of 4

zhuangzhuang 7:39:07 pm
They are all divisible by 4

cothurn 7:39:07 pm
multiples of 4

MellowMelon 7:39:10 pm
All these integers are divisible by 4, so let's divide by 4 to get $3, 6, 10, 15, 21,$ etc.

brian22 7:39:42 pm
4* (3,6,10...triangle #s)!

ProbaBillity 7:39:42 pm
four times the triangular numbers

Piya31415 7:39:42 pm
triangle numbers * 4

noodleeater 7:39:42 pm
triangular numbers

vincenthuang75025 7:39:42 pm
triangular numbers

thkim1011 7:39:42 pm
triangular number

zhuangzhuang 7:39:42 pm
triangle numbers!

cerberus88 7:39:42 pm
triangular numbers?

willwang123 7:39:42 pm
triangular numbers

olado22 7:39:42 pm
Difference of 3,4,5,6

soy_un_chemisto 7:39:42 pm
triangular numbers

werdnerd360 7:39:42 pm
triangular numbers

nuggetfan 7:39:44 pm
triangle numbers!!!

Coly 7:39:46 pm
triangular numbers

MellowMelon 7:39:52 pm
These are the triangular numbers!

MellowMelon 7:39:55 pm
So it *appears* that

$$\frac{1}{4} - S_n = \frac{1}{4 \cdot (n+1)(n+2)/2} = \frac{1}{2(n+1)(n+2)}.$$

MellowMelon 7:40:12 pm
Let's prove that

$$S_n = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

zhuangzhuang 7:40:19 pm
Use induction!

sirknightingfail 7:40:19 pm
induction time?

MellowMelon 7:40:21 pm
Let's prove this two ways. First induction.

MellowMelon 7:40:33 pm
Let

$$T_n = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

We want to show that $S_n = T_n$.

MellowMelon 7:40:39 pm
We want T_n in terms of T_{n-1}, so what should we do?

noodleeater 7:42:02 pm
take out the T_(n-1) first, then add remaining stuff

sirknightingfail 7:42:02 pm
take the difference?

vincenthuang75025 7:42:02 pm
take $S_n - S_{n-1}$ and $T_n - T_{n-1}$

zhuangzhuang 7:42:02 pm
Add on the next term, and see if it is the right difference.

nuggetfan 7:42:02 pm
t_n+1 based on t_n formula

ProbaBillity 7:42:02 pm
lets subtract S_{n-1} = T_{n-1} from both sides

lazorpenguin27143 7:42:02 pm
Tn = Tn-1 +1/(2n(n+1)) - 1/(2(n+1)(n+2))

MellowMelon 7:42:05 pm
We can compute the difference of consecutive terms:

MellowMelon 7:42:12 pm
We have that

$$
\begin{aligned}
&T_n - T_{n-1} \\
&= \left[\frac{1}{4} - \frac{1}{2(n+1)(n+2)}\right] - \left[\frac{1}{4} - \frac{1}{2n(n+1)}\right] \\
&= \frac{1}{2n(n+1)} - \frac{1}{2(n+1)(n+2)} \\
&= \frac{n+2}{2n(n+1)(n+2)} - \frac{n}{2n(n+1)(n+2)} \\
&= \frac{2}{2n(n+1)(n+2)} \\
&= \frac{1}{n(n+1)(n+2)}.
\end{aligned}
$$

What do you notice about this expression?

ProbaBillity 7:43:08 pm
the summand

brian22 7:43:08 pm
its Sn

RocketSingh 7:43:08 pm
its what you add in S_n betwen terms

yfang88 7:43:08 pm
if n=k, then it is identical to our original sums

Coly 7:43:08 pm
its the same as original question

lawrenceli 7:43:08 pm
term in the sum

cerberus88 7:43:08 pm
that is T_n term

noodleeater 7:43:08 pm
it is the same as our original expression

cothurn 7:43:08 pm
the original sum

MellowMelon 7:43:09 pm
This is the n^{th} summand in our original sum!

MellowMelon 7:43:16 pm
Specifically,

$$
\begin{aligned}
T_n &= T_{n-1} + \frac{1}{n(n+1)(n+2)} \\
S_n &= S_{n-1} + \frac{1}{n(n+1)(n+2)}
\end{aligned}
$$

MellowMelon 7:43:24 pm
So is $T_n = S_n$?

RocketSingh 7:43:58 pm
We know the base case so we're done

lawrenceli 7:43:58 pm
we need to check T_1 and S_1

mentalgenius 7:43:58 pm
we have to check the base case

TheStrangeCharm 7:43:58 pm
They start out the same so yes

ssk9208 7:43:58 pm
No we have to prove the base

Arithmophobia 7:43:58 pm
Check base cases

brian22 7:43:58 pm
base case!

Piya31415 7:43:58 pm
If for any x, t_x = s_x, then all t_x=s_x

MellowMelon 7:44:00 pm
First we need to show that $T_1 = S_1$ and then we are done. However

$$T_1 = \frac{1}{4} - \frac{1}{2\cdot 2\cdot 3} = \frac{1}{6}$$

and

$$S_1 = \frac{1}{1\cdot 2\cdot 3} = \frac{1}{6}$$

so $T_n = S_n$.

MellowMelon 7:44:17 pm
To finish the problem, what happens when n gets large?

lawrenceli 7:45:15 pm
converges to 1/4!

cerberus88 7:45:15 pm
it goes to 1/4

Piya31415 7:45:15 pm
S_n approaches 1/4

zhuangzhuang 7:45:15 pm
The denomitator term goes to 0, so the answer is 1/4.

Coly 7:45:15 pm
the second part becomes 0, so 1/4

yfang88 7:45:15 pm
the second fractional part will diminish to 0

eyzhang 7:45:15 pm
gets close to 1/4

TheStrangeCharm 7:45:15 pm
1/4 - somethingclosetozero = 1/4

nuggetfan 7:45:15 pm
T_n gets small and so does the difference between 1/4 and T_n, so it approaches 1/4 and we're don

MellowMelon 7:45:16 pm
The $\frac{1}{n(n+1)(n+2)}$ bit in T_n goes away and we get $\frac{1}{4}$.

MellowMelon 7:45:30 pm
OK, but we could have done that even more cleanly. Let's go back to our expression for $T_n - T_{n-1}$ which tells us that

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2n(n+1)} - \frac{1}{2(n+1)(n+2)}.$$

MellowMelon 7:45:45 pm
What does that tell us about the sum $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}$?

sujaykazi 7:46:26 pm
It telescopes.

zhuangzhuang 7:46:26 pm
It telescopes.

Cosmynx 7:46:26 pm
it telescopes

chenjamin 7:46:26 pm
telescoping

Arithmophobia 7:46:26 pm
Telescopes

MellowMelon 7:46:31 pm
The sum telescopes, giving us:

$$\begin{aligned}&\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}\\ &= \sum_{k=1}^{n} \left[\frac{1}{2k(k+1)} - \frac{1}{2(k+1)(k+2)} \right]\\ &= \left(\frac{1}{2\cdot 1\cdot 2} - \frac{1}{2\cdot 2\cdot 3} \right)\\ &\quad + \left(\frac{1}{2\cdot 2\cdot 3} - \frac{1}{2\cdot 3\cdot 4} \right) + \cdots\\ &\quad + \left(\frac{1}{2n(n+1)} - \frac{1}{2(n+1)(n+2)} \right)\\ &= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.\end{aligned}$$

ProbaBillity 7:46:35 pm
it is 1/4

Coly 7:46:35 pm
1/4

sunny2000 7:46:35 pm
=1/4

cerberus88 7:46:35 pm
it converges to 1/4

MellowMelon 7:46:38 pm
Finally, letting n go to infinity, we see that the original infinite sum is $\frac{1}{4}$.

MellowMelon 7:47:20 pm
And then there's yet another method that can solve this problem, which makes these problems rather mechanical.

ProbaBillity 7:47:22 pm
partial fractions

thkim1011 7:47:22 pm
Partial Fractions?

lawrenceli 7:47:22 pm
partial fraction decomposition

RocketSingh 7:47:22 pm
partial fractions and telescope

brian22 7:47:22 pm
Partial fractions!!

zhuangzhuang 7:47:22 pm
Partial Fractions??

MellowMelon 7:47:38 pm
Partial fractions. I won't explain the details of it here; it's a good topic to post on the message board if you aren't familiar with it.

MellowMelon 7:48:01 pm
Suffice it to say, in this problem you can use partial fractions to find the identity

$$\frac{1}{k(k+1)(k+2)} = \frac{1/2}{k} - \frac{1}{k+1} + \frac{1/2}{k+2}.$$

Then we find that the sum telescopes using this identity, and the answer pops right out.

nsun48 7:48:41 pm
Much faster 🙂

MellowMelon 7:48:43 pm
Assuming you're efficient at computing the partial fractions (which is probably quicker than what we did).

superpi83 7:48:59 pm
1/(k(k+1)(k+2))=1/2(1/(k(k+1))-1/((k+1)(k+2))) is also sufficient

MellowMelon 7:49:07 pm
Oh, interesting. That's somewhat equivalent to our original method.

MellowMelon 7:49:13 pm
Now we look at problems involving linear recurrences.

MellowMelon 7:49:22 pm
Find the set of real numbers a_0 for which the infinite sequence (a_n) of real numbers defined by $a_{n+1} = 2^n - 3a_n$ for $n = 0, 1, 2, \ldots$ is strictly increasing, that is, $a_n < a_{n+1}$ for $n > 0$.

MellowMelon 7:49:49 pm
One way to attack the problem would be to solve for a_n and then analyze. How might we do that?

nsun48 7:50:45 pm
We need to get rid of 2^n term somehow

RocketSingh 7:50:45 pm
try to get rid of the 2^n term

MellowMelon 7:50:47 pm
Note that the equation $a_{n+1} = 2^n - 3a_n$ is not quite a linear recurrence - there is a term of 2^n. How can we deal with it?

noodleeater 7:51:47 pm
subtract a_(n+1) = 2^n - 3a_n and 2a_n = 2^n - 6a_(n-1) to get an equation in terms of a's only

Cosmynx 7:51:47 pm
shift the index and multiply by 2

nsun48 7:51:47 pm
multiply term a_n by 2

zhuangzhuang 7:51:47 pm
Subtract consecutive terms

Arithmophobia 7:51:47 pm
subtract 2a_n=2^n-6a_(n-1)

sujaykazi 7:51:47 pm
Substitute a(n) for an expression of a(n-1) and try to cancel.

sunny2000 7:51:47 pm
$a_{n+2} = 2^{n+1} - 3a_{n+1}$ use that and the equation multiplyed by two. then subtract

MellowMelon 7:51:50 pm
We can, for example, look at two consecutive terms:

$$\begin{aligned} a_{n+1} &= 2^n - 3a_n, \\ a_n &= 2^{n-1} - 3a_{n-1}. \end{aligned}$$

Now what?

noodleeater 7:52:34 pm
multiply the second by 2 and subtract

zhuangzhuang 7:52:34 pm
Subtract twice the 2nd eq form the 1st

RocketSingh 7:52:34 pm
subtract

sirknightingfail 7:52:34 pm
subtract twice the second equation from the first

superpi83 7:52:34 pm
multiply second equation by 2

lawrenceli 7:52:34 pm
multiply by 2 and subtract

MellowMelon 7:52:38 pm
Multiplying the second equation by 2, we get

$$2a_n = 2^n - 6a_{n-1}.$$

We can then subtract this equation from the first equation, to get

$$\begin{aligned} a_{n+1} - 2a_n &= -3a_n + 6a_{n-1} \\ \Rightarrow \quad a_{n+1} &= -a_n + 6a_{n-1}. \end{aligned}$$

This equation does describe a linear recurrence, which we know how to solve. What is the characteristic polynomial?

lawrenceli 7:53:15 pm
x^2 + x - 6 = 0

lazorpenguin27143 7:53:15 pm
x^2 + x - 6

RocketSingh 7:53:15 pm
x^2 + x - 6 = 0

thkim1011 7:53:15 pm
x^2 + x - 6 = 0

zhuangzhuang 7:53:15 pm
x^2+x-6

olado22 7:53:15 pm
x^2+x-6

sirknightingfail 7:53:15 pm
x^2+x-6

MellowMelon 7:53:16 pm
Right. What are the roots?

sirknightingfail 7:53:33 pm
-3 and 2

noodleeater 7:53:33 pm
-3,2

sujaykazi 7:53:33 pm
2,-3

lazorpenguin27143 7:53:33 pm
-3, 2

ProbaBillity 7:53:33 pm
Roots are 2 and -3

RocketSingh 7:53:33 pm
x = 2 , -3

lawrenceli 7:53:33 pm
-3, 2

Double_Double 7:53:33 pm
-3, 2

Coly 7:53:33 pm
-3, 2

MellowMelon 7:53:35 pm
The characteristic polynomial is $x^2 + x - 6$, which factors as $(x - 2)(x + 3)$.

MellowMelon 7:53:41 pm
Therefore, $a_n = c_1 2^n + c_2(-3)^n$ for some constants c_1 and c_2.

MellowMelon 7:53:48 pm
How do we solve for the coefficients c_1 and c_2?

noodleeater 7:54:43 pm
find a_0 and a_1 and solve

steve314 7:54:43 pm
`use the first 2 terms of the sequence

olado22 7:54:43 pm
Plug in n=0 and n=1

MellowMelon 7:54:45 pm
We solve for c_1 and c_2 by using the initial terms a_0 and a_1.

MellowMelon 7:54:50 pm
We leave a_0 as the variable we wish to solve for. What is a_1?

zhuangzhuang 7:56:05 pm
1-3a0

ProbaBillity 7:56:05 pm
$a_1 = 1 - 3a_0$

mentalgenius 7:56:05 pm
1 - $3a_0$

noodleeater 7:56:05 pm
a_1 = 1-3a_0

RocketSingh 7:56:05 pm
a_1 = 1 - 3a_0

chenjamin 7:56:05 pm
1 - 3a_0

Cosmynx 7:56:07 pm
1-3a_0

MellowMelon 7:56:08 pm
$a_1 = 2^0 - 3a_0 = 1 - 3a_0$.

MellowMelon 7:56:23 pm
We can also write a_1 as...

fprosk 7:56:26 pm
2c_1-3c_2

lazorpenguin27143 7:56:26 pm
2c_1 - 3c_2

MellowMelon 7:56:33 pm
And for a_0 ,

TheStrangeCharm 7:56:34 pm
c_1 + c_2 = a_0

MellowMelon 7:56:37 pm
Thus, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= a_0, \\ 2c_1 - 3c_2 &= 1 - 3a_0. \end{aligned}$$

Solving for c_1 and c_2, we find

$$c_1 = \frac{1}{5}, \quad c_2 = a_0 - \frac{1}{5}.$$

Therefore,

$$a_n = \frac{1}{5} \cdot 2^n + \left(a_0 - \frac{1}{5} \right) (-3)^n.$$

Now, we must find a_0 such that the sequence is increasing.

MellowMelon 7:57:16 pm
From the formula for a_n above, how does a_n behave as n grows large?

nuggetfan 7:58:12 pm
it oscillates

mentalgenius 7:58:12 pm
it is dominated by the second term

zhuangzhuang 7:58:12 pm
It makes a "zigzag" pattern

RocketSingh 7:58:12 pm
it oscillates

Cosmynx 7:58:12 pm
it basically becomes the second term

brian22 7:58:12 pm
fluctuates because of that (-3)^n

lazorpenguin27143 7:58:12 pm
a_n oscillates depending on the parity of n

MellowMelon 7:58:15 pm
As n grows large, the term $(-3)^n$ grows faster (in magnitude) than the term 2^n, which means for sufficiently high n, the terms of the sequence will alternate in sign. In such a case, the sequence cannot be increasing.

MellowMelon 7:58:40 pm
So then... is the sequence never strictly increasing? What now?

lazorpenguin27143 7:59:17 pm
so a_0 - 1/5 has to equal 0

delta1 7:59:17 pm
so a_0=1/5

chenjamin 7:59:17 pm
a_0 = 1/5 :)

nuggetfan 7:59:17 pm
a_0 = 1/5

superpi83 7:59:17 pm
unless the coefficient of (-3)^n is 0

zhuangzhuang 7:59:17 pm
What if a0 is 1/5 ??

sujaykazi 7:59:17 pm
a(0)=1/5

sirknightingfail 7:59:17 pm
if a_0=1/5

TheStrangeCharm 7:59:17 pm
we must have a_0 = 1/5

ssk9208 7:59:17 pm
$a_0 = 1/5$

AndrewKwon97 7:59:17 pm
just make the (-3)^n term vanish, so a_0=1/5

lawrenceli 7:59:17 pm
unless a_0 = 1/5

brian22 7:59:21 pm
a_0 = 1/5?

MellowMelon 7:59:23 pm
There is only one exception, and that is when the coefficient of $(-3)^n$ is 0. The sequence doesn't do any oscillation then.

MellowMelon 7:59:38 pm
(Note that $a_0 - 1/5 < 0$ flips the sign of $(-3)^n$, but we still get oscillation.)

MellowMelon 7:59:52 pm
We see that this occurs when $a_0 = \frac{1}{5}$. When $a_0 = \frac{1}{5}$, $a_n = \frac{2^n}{5}$, which is an increasing sequence. Therefore, the solution is $\frac{1}{5}$.

TheStrangeCharm 8:00:34 pm
cool problem

ssk9208 8:00:34 pm
YAY!

MellowMelon 8:00:35 pm
Next up:

MellowMelon 8:00:37 pm
Let F_n denote the n^{th} Fibonacci number. Show that $F_n - 2n3^n$ is divisible by 5 for all $n \geq 0$.

MellowMelon 8:00:51 pm
How can we start on this problem?

noodleeater 8:01:58 pm
list out small cases

lawrenceli 8:01:58 pm
small cases

mentalgenius 8:01:58 pm
write out a couple base cases

cerberus88 8:01:58 pm
try a few examples

nuggetfan 8:01:58 pm
compute some terms

MellowMelon 8:02:02 pm
We can try looking at some small terms.

MellowMelon 8:02:37 pm
But let's focus on the $2n3^n$ part first. Should we look at the whole number, or can we simplify our life a bit?

yangwy 8:03:16 pm
take it mod 5

mentalgenius 8:03:16 pm
just take it mod 5

eyzhang 8:03:16 pm
take them apart and look at the mod 5

steve314 8:03:16 pm
take that mod 5

MellowMelon 8:03:18 pm
Right, let's just think in mod 5. How does n behave modulo 5?

cerberus88 8:04:05 pm
0,1,2,3,4

brian22 8:04:05 pm
either0,1,2,3, or 4

nuggetfan 8:04:05 pm
1,2,3,4,0

superpi83 8:04:05 pm
0,1,2,3,4

Cpi2728 8:04:05 pm
1 2 3 4 0 1 2 3 4 0 1 2 3 4 0

ProbaBillity 8:04:05 pm
`0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 ...`

lazorpenguin27143 8:04:09 pm
0,1,2,3,4,0,1,2,3,4,...

MellowMelon 8:04:11 pm
The sequence n modulo 5 is periodic, with period 5 : $0, 1, 2, 3, 4, 0, 1, 2, 3, 4,$ etc.

thkim1011 8:04:41 pm
3 has order 4.

RocketSingh 8:04:41 pm
3^4 cong 1 mod 5

lawrenceli 8:04:41 pm
3^4 = 1 (mod 5)

MellowMelon 8:04:44 pm
In the case of 3^n , we can see that the first power of 3 that is 1 mod 5 is 3^4 (that is, the order of 3 mod 5 is 4). So then we get a period of 4 for 3^n .

MellowMelon 8:04:51 pm
The period is: $1, 3, 4, 2, 1, 3, 4, 2,$ etc.

MellowMelon 8:05:02 pm
So what does that say about $2n3^n$?

noodleeater 8:05:59 pm
0,1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1,...

zhuangzhuang 8:05:59 pm
1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1,0 then it repeats mod 5

chenjamin 8:05:59 pm
period of 20

superpi83 8:05:59 pm
it should have period 20

ProbaBillity 8:05:59 pm
periodic with period 20

thkim1011 8:05:59 pm
it repeats every 20?

fprosk 8:05:59 pm
it has period 20

delta1 8:05:59 pm
period of 20

mentalgenius 8:05:59 pm
has a period of 20

Arithmophobia 8:05:59 pm
It is periodic with period 20

Cosmynx 8:05:59 pm
it's periodic with period 4

TheStrangeCharm 8:05:59 pm
it has period lcm(5,4) = 20

nuggetfan 8:05:59 pm
period of 20

willwang123 8:05:59 pm
period 20?

MellowMelon 8:06:02 pm
It says that the sequence $2n3^n$ modulo 5 has period 20. We can easily compute the residues of the first twenty values.

n	$2n3^n$	n	$2n3^n$	n	$2n3^n$	n	$2n3^n$
0	0	5	0	10	0	15	0
1	1	6	3	11	4	16	2
2	1	7	3	12	4	17	2
3	2	8	1	13	3	18	4
4	3	9	4	14	2	19	1

What else do we need?

mathaops1123 8:06:49 pm
fibonacci mod 5

delta1 8:06:49 pm
F_n mod 5

thkim1011 8:06:49 pm
residues of first 20 fibonacci

Piya31415 8:06:49 pm
fibonacci mod 5

zhuangzhuang 8:06:49 pm
The residues of F_N

mathaops1123 8:06:49 pm
Fibonacci mod 5

noodleeater 8:06:49 pm
fibonacci numbers period

superpi83 8:06:49 pm
If F_n has period 20, we're done

mentalgenius 8:06:49 pm
the period of the Fibonacci numbers mod 5

RocketSingh 8:06:49 pm
the fibonacci period

cerberus88 8:06:49 pm
Fibonacci numbers residue mod 5

steve314 8:06:49 pm
check to see if the fibonacci numbers have a pattern mod 5

ProbaBillity 8:06:49 pm
the period of F_n mod 5

brian22 8:06:49 pm
show that fibonnaci repeats mod 5

MellowMelon 8:06:51 pm
Now we check that F_n modulo 5 has period 20, and that the residues are the same.

n	F_n	n	F_n	n	F_n	n	F_n
0	0	5	0	10	0	15	0
1	1	6	3	11	4	16	2
2	1	7	3	12	4	17	2
3	2	8	1	13	3	18	4
4	3	9	4	14	2	19	1

The residues agree, so $F_n - 2n3^n$ is divisible by 5 for all n.

MellowMelon 8:07:06 pm
So we've solved it. Again, that wasn't so bad, but there are other ways.

MellowMelon 8:07:12 pm
Here's one that's really simple:

thkim1011 8:07:27 pm
induction

sirknightingfail 8:07:27 pm
induction?

TheStrangeCharm 8:07:27 pm
strong induction

ProbaBillity 8:07:27 pm
induction

brian22 8:07:27 pm
its induction, so a base case?

mathcool2009 8:07:27 pm
induction?

MellowMelon 8:07:51 pm
Induct on the hypothesis that $F_n \equiv 2n3^n \pmod{5}$. You should be able to use the Fibonacci recurrence to show it continues to hold.

MellowMelon 8:08:12 pm
But the theory of linearly recurrent sequences offers yet another approach.

MellowMelon 8:08:18 pm
What is the characteristic polynomial of the Fibonacci sequence?

Piya31415 8:08:51 pm
Is there a way that is less brute force?

thkim1011 8:08:51 pm
x^2 - x + 1

lazorpenguin27143 8:08:51 pm
x^2 - x - 1 = 0

superpi83 8:08:51 pm
x^2-x-1

TheStrangeCharm 8:08:51 pm
x^2 - x - 1

sujaykazi 8:08:51 pm
x^2 -x -1

brian22 8:08:51 pm
x^2 - x- 1

delta1 8:08:51 pm
x^2-x-1

cerberus88 8:08:51 pm
x^2-x-1

MellowMelon 8:08:52 pm
The characteristic polynomial of the Fibonacci sequence is $x^2 - x - 1$.

MellowMelon 8:08:56 pm
The sequence $2n3^n$ also satisfies a linear recurrence. What is its characteristic polynomial?

zhuangzhuang 8:09:59 pm
x^2-6x+9

lawrenceli 8:09:59 pm
(x-3)^2 = 0?

superpi83 8:09:59 pm
(x-3)^2

MellowMelon 8:10:02 pm
The characteristic polynomial of the sequence $2n3^n$ is $(x-3)^2 = x^2 - 6x + 9$. (The coefficient of n in front of the 3^n requires us to take 3 as a double root.)

MellowMelon 8:10:25 pm
Meaning that if we define $G_n = 2n3^n$, what linear recurrence does the sequence (G_n) satisfy?

thkim1011 8:11:49 pm
g_n = 6g_(n-1) - 9g(n-2)

zhuangzhuang 8:11:49 pm
6a(n-1)-9a(n-2)

superpi83 8:11:49 pm
x_n=6x_(n-1)-9x_(n-2)

lawrenceli 8:11:49 pm
G_n = 6G_n-1 - 9G_n-2

noodleeater 8:12:03 pm
G_n = 6G_n-1 - 9G_n-2

MellowMelon 8:12:04 pm
The sequence (G_n) satisfies

$$G_n = 6G_{n-1} - 9G_{n-2}$$

for all $n \geq 2$. This is by using the coefficients of the characteristic polynomial.

MellowMelon 8:12:25 pm
So we have a recurrence for G_n and of course one for F_n . What do we notice about them?

sujaykazi 8:13:26 pm
They are congruent mod 5.

zhuangzhuang 8:13:26 pm
Their Difference is congruent to 0 modulo 5!

superpi83 8:13:26 pm
they're congruent mod 5

Cosmynx 8:13:26 pm
if we take mod 5, the coefficients are congruent

lawrenceli 8:13:26 pm
difference is a multiple of 5?

delta1 8:13:26 pm
they're congruent mod 5

nilaisarda 8:13:26 pm
They are the same mod 5

Piya31415 8:13:32 pm
The recurrence for F - the recursion for G is a multiple of 5 always

Coly 8:13:32 pm
when subtracted are mod 5

MellowMelon 8:13:34 pm
We see that in mod 5, we can write

$$G_n \equiv G_{n-1} + G_{n-2} \pmod 5$$

for all $n \geq 2$. This is the same recursion for the Fibonacci numbers, also modulo 5 !

MellowMelon 8:13:55 pm
What's the last step we need for this to solve the problem?

superpi83 8:14:47 pm
check that first two terms match

ProbaBillity 8:14:47 pm
To make sure that they start with the same two numbers

chenjamin 8:14:47 pm
base case

Cosmynx 8:14:47 pm
base cases

nilaisarda 8:14:47 pm
G_0 = F_0 mod 5, G_1 = F_1 mod 5

steve314 8:14:47 pm
base case

brian22 8:14:47 pm
basis step

noodleeater 8:14:47 pm
base cases

zhuangzhuang 8:14:47 pm
base cases!

nuggetfan 8:14:47 pm
base cases?

sophiazhi 8:14:47 pm
G_1 and G_2?

MellowMelon 8:14:49 pm
We need to check that $G_0 \equiv F_0$ and $G_1 \equiv F_1 \pmod 5$, which is easy. Then the identical recurrences will give us that $G_n \equiv F_n \pmod 5$ for all n .

MellowMelon 8:15:00 pm
So since the sequences (F_n) and (G_n) have the same initial terms modulo 5, and satisfy the same recurrence modulo 5, the terms are exactly the same modulo 5, i.e. $F_n \equiv G_n \pmod 5$ for all n. In other words, $F_n - G_n = F_n - 2n3^n$ is divisible by 5 for all n.

MellowMelon 8:15:31 pm
This idea of obtaining identical recurrences and then showing the base cases are the same is an easy way to show two sequences match identically.

MellowMelon 8:15:50 pm
Also, the use of characteristic polynomials here is a concept that was described in the handout. In dealing with linear recurrences, what usually happens is we are given the linear recurrence and initial conditions (like $G_n = 6G_{n-1} - 9G_{n-2}$), and then we solve the recurrence to obtain a formula (like $G_n = 2n3^n$.).

MellowMelon 8:16:22 pm
However, when we are given the formula, and we see that it should satisfy a linear recurrence (e.g. exponential, etc.), then we can turn the method around to derive the linear recurrence that the sequence satisfies. This gives us another way of analyzing the sequence.

MellowMelon 8:17:12 pm
So again, there were many ways to solve that problem. On a contest, induction is probably the most straightforward and quickest, but hopefully you learned something from our method with characteristic polynomials. On to the next one then...

MellowMelon 8:17:22 pm
Show that $F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n}$ for all $n \geq 1$.

MellowMelon 8:17:46 pm
What might you try first?

mentalgenius 8:18:09 pm
YES BINET'S FORMULA!!!

superpi83 8:18:09 pm
plugging in formula for F_n

MellowMelon 8:18:12 pm
One way to prove this identity is to use the (not simple) formula for F_n , known as Binet's formula, and expand it out. It's a little tedious, but purely mechanical. Given that we have a closed formula for F_n, it's a perfectly reasonable method.

RocketSingh 8:19:03 pm
that seems really really messy

brian22 8:19:03 pm
that looks yucky

MellowMelon 8:19:08 pm
I agree, but it will work.

thkim1011 8:19:21 pm
Induction

lazorpenguin27143 8:19:21 pm
induction

RocketSingh 8:19:21 pm
induction

Piya31415 8:19:21 pm
Induction

MellowMelon 8:19:25 pm
And induction should work too, but it might get a bit hectic as well.

ProbaBillity 8:19:30 pm
Show that the LHS and the RHS have the same recursion

ProbaBillity 8:19:30 pm
and show that they have the same base cases

MellowMelon 8:19:40 pm
This however, is quite close to the way I want to solve this. 🙂

MellowMelon 8:19:46 pm
Write

$$F_n = c_1\alpha^n + c_2\beta^n,$$

where $\alpha = (1+\sqrt{5})/2$ **and** $\beta = (1-\sqrt{5})/2$.

MellowMelon 8:20:15 pm
(This is Binet's formula when expanded out. We won't do everything in a flurry of algebra though; you'll see in a moment.)

MellowMelon 8:20:27 pm
Then

$$\begin{aligned} F_n^3 &= (c_1\alpha^n + c_2\beta^n)^3 \\ &= c_1^3\alpha^{3n} + 3c_1^2c_2\alpha^{2n}\beta^n + 3c_1c_2^2\alpha^n\beta^{2n} + c_2^3\beta^{3n}. \end{aligned}$$

We can also write

$$F_n^3 = c_1^3(\alpha^3)^n + 3c_1^2c_2(\alpha^2\beta)^n + 3c_1c_2^2(\alpha\beta^2)^n + c_2^3(\beta^3)^n.$$

MellowMelon 8:20:52 pm
So that's a big scary looking equation, but can we say anything about F_n^3 **from this?**

superpi83 8:22:10 pm
It's recursive

MellowMelon 8:22:24 pm
This is a sum of exponentials, so the sequence F_n^3 **satisfies a linear recurrence. What is its characteristic polynomial?**

superpi83 8:23:46 pm
(x-a^3)(x-a^2b)(x-ab^2)(x-b^3)

RocketSingh 8:23:46 pm
(x- a^3)(x- a^2*b)(x - ab^2)(x-b^3)

MellowMelon 8:23:53 pm
The characteristic polynomial is given by

$$(x-\alpha^3)(x-\beta^3)(x-\alpha^2\beta)(x-\alpha\beta^2).$$

The roots are just the bases for the exponentials.

MellowMelon 8:24:02 pm
Can we do any simplifications here?

superpi83 8:25:12 pm
ab=-1

Cosmynx 8:25:12 pm
a*b=-1

mentalgenius 8:25:12 pm
ab = -1

RocketSingh 8:25:12 pm
(x- a^3)(x+b)(x +a)(x-b^3)

eyzhang 8:25:12 pm
we know that a and b are from x^2-x-1. so plug in sum and product.

MellowMelon 8:25:59 pm
We know $\alpha\beta = -1$ so let's simplify $\alpha^2\beta = -\alpha$ and $\alpha\beta^2 = -\beta$ to get

$$(x - \alpha^3)(x - \beta^3)(x + \alpha)(x + \beta).$$

MellowMelon 8:26:35 pm
Okay... not sure what else we can do with that. What else is there to analyze in the problem?

lazorpenguin27143 8:27:39 pm
F_3n

nuggetfan 8:27:39 pm
f_3n

nilaisarda 8:27:39 pm
F_3n

noodleeater 8:27:39 pm
F_3n

RocketSingh 8:27:39 pm
what is the characteristic polynomial f_3n

chenjamin 8:27:39 pm
F_3n

olado22 8:27:39 pm
F_3n

MellowMelon 8:27:43 pm
We could go ahead and analyze F_{3n} , and we will, but let's get something else out of the way before we do...

cerberus88 8:27:48 pm
the other terms

lawrenceli 8:27:48 pm
find linear recurrence of the other terms

Cpi2728 8:27:48 pm
Plug this in for F3_n-1, F3_n, F3_n+1.

MellowMelon 8:28:01 pm
We have the characteristic polynomial of F_n^3 , but that's not the left hand side.

MellowMelon 8:28:13 pm
What can we say about F_{n-1}^3 , as a sequence?

MellowMelon 8:29:04 pm
Think about what we did for F_n^3 ...

RocketSingh 8:29:42 pm
The c1 and c2 values are different

cerberus88 8:29:42 pm
binet's formula

brian22 8:29:42 pm
lets take the fomrula for F_n and replace the ns with n-1?

MellowMelon 8:29:43 pm
Right, we could do the same thing, and although we would get c_1 and c_2 to be different...

MellowMelon 8:29:48 pm
... what wouldn't change?

lazorpenguin27143 8:30:25 pm
the characteristic equation

nilaisarda 8:30:25 pm
the characteristic polynomial

lawrenceli 8:30:25 pm
characteristic polynomial?

avery 8:30:25 pm
the characteristic polynomial

noodleeater 8:30:25 pm
characteristic polynomial

RocketSingh 8:30:25 pm
the characteristic equation is the same

MellowMelon 8:30:26 pm
This linear recurrence is the same as the linear recurrence of F_n^3, because the terms of F_{n-1}^3 are the same as the terms of F_n^3, just shifted by one. So we get the same characteristic polynomial.

MellowMelon 8:30:41 pm
By the same reasoning, the sequence F_{n+1}^3 also satisfies the same linear recurrence and has the same characteristic polynomial.

MellowMelon 8:30:58 pm
Finally, if we let $A_n = F_{n+1}^3 + F_n^3 - F_{n-1}^3$, do we know the characteristic polynomial of A_n ?

cerberus88 8:31:44 pm
yes

superpi83 8:31:44 pm
it's the same

RocketSingh 8:31:44 pm
its the same f^3_n

lazorpenguin27143 8:31:44 pm
(x-a^3)(x-b^3)(x+a)(x+b)

zhuangzhuang 8:31:44 pm
It is the function we have derived.

MellowMelon 8:31:46 pm
Yes, A_n will also satisfy the same recurrence. So the characteristic polynomial of the sequence (A_n) is

$$(x-\alpha^3)(x-\beta^3)(x+\alpha)(x+\beta).$$

MellowMelon 8:32:10 pm
Now I put something on hold...

superpi83 8:32:14 pm
So if we can show F_3n also satisfies this polynomial, then all that's left to do is show that a few initial terms match

MellowMelon 8:32:20 pm
We have F_{3n} to analyze now.

MellowMelon 8:32:34 pm
Let

$$B_n = F_{3n} = c_1\alpha^{3n} + c_2\beta^{3n}.$$

Can we get a characteristic polynomial out of this?

superpi83 8:33:44 pm
(x-a^3)(x-b^3)

lawrenceli 8:33:44 pm
(x-a^3)(x-b^3)

cerberus88 8:33:44 pm
yes, (x-a^3)(x-b^3)

RocketSingh 8:33:44 pm
(x-a^3)(x -b^3)

noodleeater 8:33:44 pm
(x-a^3)(x-b^3)

MellowMelon 8:33:47 pm
The characteristic polynomial for this sequence is

$$(x-\alpha^3)(x-\beta^3).$$

... that looks familiar, doesn't it?

brian22 8:35:01 pm
thats part of what we have for A_n

lawrenceli 8:35:01 pm
but additional (x+a)(x+b)

cerberus88 8:35:01 pm
part of A_n

Cpi2728 8:35:01 pm
But not equal to the LHS.

olado22 8:35:01 pm
Same thing except without (x+a)(x+b)

MellowMelon 8:35:06 pm
This polynomial is a factor of the characteristic polynomial for A_n , although it's not the same. What does it mean in terms of the linear recurrences that the characteristic polynomial for B_n divides the characteristic polynomial for A_n?

MellowMelon 8:36:44 pm
This is tricky. Try starting with the formula for B_n and throwing in some extra exponential terms with coefficient 0...

superpi83 8:38:38 pm
B_n also satisfies the recurrence of A_n

MellowMelon 8:38:43 pm
It says the sequence (B_n) satisfies the linear recurrence for the sequence (A_n) as well!

MellowMelon 8:39:01 pm
One way to see this: we can also write

$$B_n = c_1(\alpha^3)^n + c_2(\beta^3)^n + 0 \cdot (\alpha^2\beta)^n + 0 \cdot (\alpha\beta^2)^n.$$

MellowMelon 8:39:43 pm
With this as the formula, we get a characteristic polynomial for "B_n " that is the same as that for A_n , so both have the same linear recurrence.

MellowMelon 8:40:23 pm
Let me try to show you more concretely how this sleight of hand worked, since I think that was quite difficult. I'll do a different example.

MellowMelon 8:40:37 pm
Start with the characteristic polynomial of F_n : $x^2 - x - 1$. What do you get when you multiply it with $x + 1$?

mentalgenius 8:42:10 pm
$x^3 - 2x - 1$

ProbaBillity 8:42:10 pm
x^3 - 2x - 1

olado22 8:42:10 pm
x^3-2x-1

lazorpenguin27143 8:42:10 pm
x^3 - 2x - 1

avery 8:42:10 pm
x^3-2x-1

Coly 8:42:10 pm
x^3-2x-1

MellowMelon 8:42:13 pm
We get $x^3 - 2x - 1$. As a characteristic polynomial, this corresponds to a linear recurrence $a_n = 2a_{n-2} + a_{n-3}$.

MellowMelon 8:42:22 pm
Try putting the Fibonacci series into that recurrence. Does it work?

lazorpenguin27143 8:43:18 pm
Yes

eyzhang 8:43:18 pm
Yes

mentalgenius 8:43:18 pm
yes

noodleeater 8:43:18 pm
yes

Johnzh 8:43:18 pm
yes

lawrenceli 8:43:18 pm
yes

cerberus88 8:43:18 pm
yes

RocketSingh 8:43:18 pm
yes it good

zhuangzhuang 8:43:18 pm
Yes, it does; it "bunches up"

nilaisarda 8:43:18 pm
Yes

ProbaBillity 8:43:18 pm
Yes

MellowMelon 8:43:20 pm
Indeed. And we could have derived this recurrence directly by adding these two equations:

$$F_n = F_{n-1} + F_{n-2},$$

$$F_{n-1} = F_{n-2} + F_{n-3}$$

$$\Rightarrow \quad F_n = 2F_{n-2} + F_{n-3}.$$

MellowMelon 8:44:05 pm
Similarly, if I had taken two times the first equation and three times the second there, I would have also gotten a recurrence whose characteristic polynomial is $(2x+3)(x^2-x-1)$. Above, we would have gotten the equation $2F_n = -F_{n-1} + 5F_{n-2} + 3F_{n-3}$, and indeed $(2x+3)(x^2-x-1) = 2x^3 + x^2 - 5x - 3$.

MellowMelon 8:44:40 pm
The point here is that any linearly recurrent sequences satisfies lots of different linear recurrences. But usually we just look at the one with the smallest possible degree. Here, we just exploited the existence of other recurrences B_n satisfies to show that A_n and B_n in our problem satisfy the same recurrence, although it wasn't the one of smallest degree for B_n .

mentalgenius 8:44:47 pm
can this technique of multiplying characteristic functions by other polynomials prove non-trivial identities?

MellowMelon 8:44:57 pm
Well, we used the idea in reverse on this problem, didn't we? I can't think of a forward usage off the top of my head.

MellowMelon 8:45:30 pm
Alright, so let me bring things back. We just argued that A_n and B_n satisfy the same 4th degree linear recurrence. Let's use this to wrap up the problem.

MellowMelon 8:45:37 pm
We want to show that $A_n = B_n$ for all $n \geq 1$. We have shown that the sequences (A_n) and (B_n) satisfy the same linear recurrence. So what else do we have to do?

superpi83 8:47:04 pm
Show that the sequences' first four terms match

cerberus88 8:47:04 pm
show the initial terms are the same

noodleeater 8:47:04 pm
base cases

nilaisarda 8:47:04 pm
Prove base cases equal

chenjamin 8:47:04 pm
base cases

lazorpenguin27143 8:47:04 pm
Show they have the same first few terms

lawrenceli 8:47:04 pm
check base cases

ProbaBillity 8:47:04 pm
Show that they have the same base case

MellowMelon 8:47:08 pm
All we have to do now is show that the two sequences have the same four initial terms:

$$\begin{aligned}
A_1 &= F_2^3 + F_1^3 - F_0^3 = 2,\\
A_2 &= F_3^3 + F_2^3 - F_1^3 = 8,\\
A_3 &= F_4^3 + F_3^3 - F_2^3 = 34,\\
A_4 &= F_5^3 + F_4^3 - F_3^3 = 144,\\
B_1 &= F_3 = 2,\\
B_2 &= F_6 = 8,\\
B_3 &= F_9 = 34,\\
B_4 &= F_{12} = 144.
\end{aligned}$$

Thus, the identity holds.

MellowMelon 8:47:29 pm
So we reduced the computation and guessing of intermediate formulas a lot, but we did need a very good understanding of characteristic polynomials to put it off.

MellowMelon 8:47:46 pm
Linearly recurrent sequences are defined by the linear recurrence and the initial terms, so they are often most easily accessed via this information, rather than the closed form solution.

MellowMelon 8:53:16 pm
One practical way that linear recurrences actually arise is in the solution of certain combinatorial problems.

MellowMelon 8:53:23 pm
For $n \geq 1$, define a_n to be the number of sequences of n 0 s, 1 s, and 2 s such that no three consecutive numbers in the sequence are all different. Find a formula for a_n, and show that if $p \geq 3$ is a prime, then $a_p \equiv 3 \pmod{p}$.

MellowMelon 8:53:47 pm
What should we do?

sujaykazi 8:55:20 pm
Find a recursive formula for a(n).

Cpi2728 8:55:20 pm
Find N in terms of N-1. Induction/linear recurrence.

lawrenceli 8:55:20 pm
write a recurrence

MellowMelon 8:55:24 pm
We can use recursion by thinking about how to build sequences with n numbers from sequences with $n-1$ numbers.

MellowMelon 8:55:32 pm
For example, under what conditions can we add a 0 to the end of a sequence?

TheStrangeCharm 8:56:14 pm
as long as we didnt end with 12

brian22 8:56:14 pm
if the last two nunbers are not 1, 2 or 2,1

ProbaBillity 8:56:14 pm
if the previous two terms are anything but 1,2 or 2,1 then it's ok

sirknightingfail 8:56:14 pm
01,10,11,22,02,20,00

willwang123 8:56:14 pm
the previous terms are not 12 or 21

lazorpenguin27143 8:56:14 pm
if the last 2 terms were not 1,2

RocketSingh 8:56:14 pm
if it does not end in 12 or 21

MellowMelon 8:56:16 pm
We can add a 0 to the end of a sequence as long as the last two numbers are not 1 and 2, in some order.

MellowMelon 8:56:29 pm
More generally, we can add a number to the end of a sequence as long as the last two numbers are not the other two numbers, in some order.

MellowMelon 8:56:51 pm
How can we use this to create a recurrence? What should we look at?

lawrenceli 8:57:55 pm
we care about the last 2 terms in the sequence

MellowMelon 8:57:56 pm
The last two terms of the sequence are certainly important...

cerberus88 8:58:11 pm
the choices we have at each term?

MellowMelon 8:58:13 pm
And we can easily determine how many possible terms we can append with a little more information on the last two...

sirknightingfail 8:59:41 pm
Let a_n=number with last two digits the same, b_n be the number with different last two digits

willwang123 8:59:41 pm
whether or not the last two are the same

MellowMelon 8:59:46 pm
We define s_n to be the number of such sequences of length n where the last two numbers are the same, and d_n as the number of such sequences of length n where the last two numbers are different.

MellowMelon 8:59:53 pm
Suppose we have a sequence of length $n-1$, where the last two numbers are the same, say both numbers are x. What can we add to the end of the sequence?

cerberus88 9:00:42 pm
0,1,2

delta1 9:00:42 pm
anything

brian22 9:00:42 pm
either a 0, 1, or 2

Cpi2728 9:00:42 pm
Anything.

ProbaBillity 9:00:42 pm
we can add either x or something other than x

TheStrangeCharm 9:00:42 pm
any 3 possible lettesr

chenjamin 9:00:42 pm
any of 0, 1, 2

sirknightingfail 9:00:48 pm
any

MellowMelon 9:00:54 pm
We can either add x, y, or z at the end, where x, y, and z stand for the numbers 0, 1, and 2.

MellowMelon 9:01:10 pm
If we add x, then we obtain a sequence of length n where the last two numbers are the same. If we add y or z, then we obtain a sequence of length n where the last two numbers are different.

MellowMelon 9:01:17 pm
Now suppose we have a sequence of length $n-1$ where the last two numbers are different, say x and y. What numbers can we add at the end?

brian22 9:02:03 pm
x or y

Coly 9:02:03 pm
x, y

Johnzh 9:02:03 pm
x or y

noodleeater 9:02:03 pm
x, y

TheStrangeCharm 9:02:03 pm
only 2, we cant add the number we didn't use

chenjamin 9:02:03 pm
x or y

sujaykazi 9:02:03 pm
x or y

delta1 9:02:03 pm
x or y

fprosk 9:02:03 pm
x or y

sirknightingfail 9:02:03 pm
x or y

Piya31415 9:02:03 pm
not z

eyzhang 9:02:03 pm
x,y

MellowMelon 9:02:05 pm
We can only add the numbers x and y at the end, not z. Adding one of these results in a sequence of n numbers where the last two numbers are the same, and the other, different.

MellowMelon 9:02:22 pm
So what is s_n in terms of s_{n-1} and d_{n-1}?

sirknightingfail 9:04:30 pm
s_n=s_n-1+d_n-1

chenjamin 9:04:30 pm
s_n = s_n-1+d_n-1

nuggetfan 9:04:30 pm
s_n-1+d_n-1

TheStrangeCharm 9:04:30 pm
s_n = s_n-1 + d_n-1

Cpi2728 9:04:30 pm
s_n-1 plus d_n-1

MellowMelon 9:04:33 pm
We have $s_n = s_{n-1} + d_{n-1}$. We have one way to make the last two terms the same in both cases.

sirknightingfail 9:04:49 pm
d_n=2s_n-1+d_n-1

MellowMelon 9:04:51 pm
If we similarly analyze d_n , we get these two recurrences:

$$s_n = s_{n-1} + d_{n-1},$$
$$d_n = 2s_{n-1} + d_{n-1}.$$

Now what?

lawrenceli 9:05:54 pm
get rid of one of the terms

zhuangzhuang 9:05:54 pm
eliminate a variable?

16navidr 9:05:54 pm
We have to find s in terms of d or vice versa

eyzhang 9:05:57 pm
put it in characteristic form?

MellowMelon 9:06:00 pm
We only know how to solve linear recurrences in one variable, so we need to isolate a recursion that is only in terms of s_n, or only in terms of d_n. How can we do that?

noodleeater 9:08:11 pm
replace d_n with s_(n+1)-s_n and d_(n-1) with s_n - s_(n-1) in the second equation

MellowMelon 9:08:14 pm
First, let's isolate d_{n-1} in the first equation:

$$d_{n-1} = s_n - s_{n-1}.$$

sirknightingfail 9:08:35 pm
plug that into the second

MellowMelon 9:08:37 pm
Then if we substitute into the second equation, both for d_n and d_{n-1} , we get an equation with only terms of the s_i sequence. Namely,

$$s_{n+1} - s_n = 2s_{n-1} + s_n - s_{n-1}.$$

This simplifies as

$$s_{n+1} = 2s_n + s_{n-1},$$

which gives us a linear recurrence for the sequence (s_n).

zhuangzhuang 9:08:43 pm
s_n=2s_(n-1)+S_(n-2)

Johnzh 9:08:43 pm
plug d_(n-1) = s_n - s_(n-1) into the second equation to get s_(n+1) = 2s_n + s_(n-1)

MellowMelon 9:08:57 pm
So we have s_n . How about d_n ?

noodleeater 9:10:38 pm
same, but replaced the s's

Johnzh 9:10:38 pm
d_(n+1) = 2d_n + d_(n-1)

MellowMelon 9:10:40 pm
Since

$$d_{n-1} = s_n - s_{n-1},$$

the sequence (d_n) satisfies the same linear recurrence as (s_n).

MellowMelon 9:10:53 pm
Hence,

$$\begin{aligned} s_{n+1} &= 2s_n + s_{n-1}, \\ d_{n+1} &= 2d_n + d_{n-1} \end{aligned}$$

for all $n \geq 1$.

MellowMelon 9:11:05 pm
Finally, what is the original sequence a_n in terms of s_n and d_n?

sirknightingfail 9:12:22 pm
s_n+d_n

cerberus88 9:12:22 pm
s_n+d_n

Cpi2728 9:12:22 pm
sn+dn

brian22 9:12:22 pm
s_n + d_n (or is that too easy?)

nilaisarda 9:12:22 pm
a_n = s_n + d_n

MellowMelon 9:12:26 pm
$a_n = s_n + d_n$.

noodleeater 9:12:48 pm
a_n = 2a_(n-1) + a_(n-2)

sirknightingfail 9:12:48 pm
a_n=s_n+d_n=2(s_n-1+d_n-1)+s_n-2+d_n-2=2a_n-1+a_n-2

MellowMelon 9:12:49 pm
So we can add the recurrences for (s_n) and (d_n) to get

$$a_{n+1} = 2a_n + a_{n-1}.$$

MellowMelon 9:12:58 pm
We also can count that $a_1 = 3$ and $a_2 = 9$, so we can solve this linear recurrence.

MellowMelon 9:13:21 pm
Let's spare you some of the tedious computations. Solving for a_n , we find

$$a_n = \frac{3}{2}\left((1+\sqrt{2})^n + (1-\sqrt{2})^n\right).$$

Yay, one part done!

brian22 9:13:40 pm
i smell one of those characteristic polynomials

MellowMelon 9:13:51 pm
Indeed. If you don't think you could have done that computation, you may want to review the handout.

cerberus88 9:13:53 pm
now the proofy part

MellowMelon 9:13:59 pm
Now, let $p \geq 3$ be a prime. We must show that $a_p \equiv 3 \pmod{p}$.

MellowMelon 9:14:03 pm
How?

sirknightingfail 9:14:38 pm
plug n=p in

sirknightingfail 9:14:38 pm
and then try some stuff

MellowMelon 9:14:40 pm
That sounds about right.... What stuff though?

zhuangzhuang 9:15:22 pm
use binomial expansion

mathocean97 9:15:22 pm
Binomial Theorem!

superpi83 9:15:22 pm
expand with Binomial Theorem and reduce mod p

TheStrangeCharm 9:15:22 pm
binomial theorem?

MellowMelon 9:15:45 pm
We can expand a_p using the Binomial Theorem.

MellowMelon 9:15:50 pm
We get

$$\begin{aligned} a_p &= \frac{3}{2}\left[(1+\sqrt{2})^p + (1-\sqrt{2})^p\right] \\ &= \frac{3}{2}\left[1 + \binom{p}{1}\sqrt{2} + \binom{p}{2}(\sqrt{2})^2 - \binom{p}{3}(\sqrt{2})^3 + \cdots\right] \\ &\quad + \frac{3}{2}\left[1 - \binom{p}{1}\sqrt{2} + \binom{p}{2}(\sqrt{2})^2 - \binom{p}{3}(\sqrt{2})^3 + \cdots\right] \\ &= 3\left[1 + \binom{p}{2}2 + \binom{p}{4}2^2 + \cdots\right]. \end{aligned}$$

What happens to this sum modulo p?

cerberus88 9:16:54 pm
becomes 3

lawrenceli 9:16:54 pm
only 1 remains

ProbaBillity 9:16:54 pm
3*1 = 3

sirknightingfail 9:16:54 pm
all terms are divisible by p except for that 1

noodleeater 9:16:54 pm
the pCk ones are multiples of p, since p is prime

Piya31415 9:16:54 pm
p choose n is congruent to 0 mod p for n >0

RocketSingh 9:16:54 pm
only the 1 is not congruent to 0 mod p

Piya31415 9:16:54 pm
so the sum simplifies to 3(1)

superpi83 9:16:54 pm
everything vanishes except 3(1)=3

steve314 9:16:54 pm
every term within the brackets except for 1 has a factor of p, so the sum is 3 mod p

delta1 9:17:05 pm
it's just 3 since there isn't a p choose p term

soy_un_chemisto 9:17:05 pm
3 * 1. all other terms are 0 mod p

willwang123 9:17:05 pm
all the combinations are 0 mod p

eyzhang 9:17:05 pm
only 3 is left

MellowMelon 9:17:07 pm
Each binomial coefficient of the form $\binom{p}{2k}$ is divisible by p for $k \geq 1$, and p is odd, so there is no $\binom{p}{p}$ term. The expression reduces to

$$a_p \equiv 3 \pmod{p},$$

as desired.

brian22 9:17:38 pm
YAY!

cerberus88 9:17:38 pm
Q.E.D.

MellowMelon 9:18:52 pm
Now we turn to a final problem with sequences and series that can't necessarily be solved by the techniques that we have seen before. In these problems, our main weapons will be the ability to find patterns, and to notice unusual or odd features of the problem that will help us accomplish our goal.

MellowMelon 9:19:22 pm
The sequence (a_n) satisfies $a_1 = 1$ and

$$a_{n+1} = \frac{1 + 4a_n + \sqrt{1 + 24a_n}}{16}$$

for $n \geq 1$. Show that a_n is rational for all n .

MellowMelon 9:19:42 pm
What can we try first?

willwang123 9:20:23 pm
that looks like quadratic formula sort of

RocketSingh 9:20:23 pm
use quad formula bkwards

brian22 9:20:23 pm
that looks like the quadratic formula

MellowMelon 9:20:25 pm
That's interesting. I don't know if you can get the numbers to work out exactly right though. You might see if this can be finished on your own.

steve314 9:20:32 pm
small values of n

Piya31415 9:20:32 pm
find some small values?

Johnzh 9:20:32 pm
get the first few terms

sujaykazi 9:20:32 pm
small cases

soy_un_chemisto 9:20:32 pm
first few terms

MellowMelon 9:20:40 pm
The first thing we should try is computing the first few terms. This is a good way of easing into the problem, and may allow us to find a pattern.

$$\begin{aligned} a_1 &= 1, \\ a_2 &= \frac{1 + 4 \cdot 1 + \sqrt{1 + 24 \cdot 1}}{16} = \frac{5}{8}, \\ a_3 &= \frac{1 + 4 \cdot \frac{5}{8} + \sqrt{1 + 24 \cdot \frac{5}{8}}}{16} = \frac{15}{32}, \\ a_4 &= \frac{1 + 4 \cdot \frac{15}{32} + \sqrt{1 + 24 \cdot \frac{15}{32}}}{16} = \frac{51}{128}. \end{aligned}$$

So far, we only get rational numbers.

MellowMelon 9:20:45 pm
(I should hope so...)

MellowMelon 9:20:52 pm
What do you see?

brian22 9:21:27 pm
denominators are powers of 2

lawrenceli 9:21:27 pm
powers of 2 in the denominator

mentalgenius 9:21:27 pm
denominators always multiplying by 4

Arithmophobia 9:21:27 pm
powers of 2

TheStrangeCharm 9:21:27 pm
powers of 2 on bottom

superpi83 9:21:27 pm
the denominators are powers of 2

eyzhang 9:21:27 pm
denominator multiply by 4 each time

Cosmynx 9:21:27 pm
powers of 2 in the denominator

MellowMelon 9:21:28 pm
Already, one pattern is evident. In the above terms, starting with the second, the denominator is always a power of 2. In fact, the denominators appear to form a geometric sequence, with common ratio 4.

MellowMelon 9:22:01 pm
We might need that later. Now let's step back a bit. We want to show the terms are rational. What would stop a term from being rational?

brian22 9:22:37 pm
the square root

lawrenceli 9:22:37 pm
the square root

sirknightingfail 9:22:37 pm
if 1+24a_n-1 is not a squar

steve314 9:22:37 pm
the square root

zhuangzhuang 9:22:37 pm
1+24a is not a perfect square

werdnerd360 9:22:37 pm
sqrt

Piya31415 9:22:37 pm
1+24a_n is not a square number

TheStrangeCharm 9:22:37 pm
1 + 24a_n-1 is not a perfect square

delta1 9:22:37 pm
the square root isn't rational

superpi83 9:22:37 pm
the sqrt(1+24a_n) term

Cpi2728 9:22:37 pm
The square root term.

willwang123 9:22:37 pm
that square root

cerberus88 9:22:43 pm
that square root

MellowMelon 9:22:44 pm
If a_n is rational, then the only part of a_{n+1} that is potentially irrational is the square root. So let's take a closer look at this part.

MellowMelon 9:22:56 pm
Let $b_n = \sqrt{1 + 24a_n}$, or equivalently $a_n = \frac{1}{24}(b_n^2 - 1)$.

MellowMelon 9:23:39 pm
What can we do with that?

Johnzh 9:24:24 pm
plug it in?

lawrenceli 9:24:24 pm
substitute into the recurrence

sirknightingfail 9:24:24 pm
solve for b_n in terms of b_n-1

soy_un_chemisto 9:24:24 pm
plug into a_{n + 1}

ProbaBillity 9:24:31 pm
find the recursion

brian22 9:24:31 pm
plug it back into a_(n+1)

MellowMelon 9:24:34 pm
Then, we can rewrite the given recursion in terms of b_n :

$$\frac{b_{n+1}^2 - 1}{24} = \frac{1 + (b_n^2 - 1)/6 + b_n}{16} = \frac{b_n^2 + 6b_n + 5}{96}.$$

Isolating b_{n+1}^2, we find

$$b_{n+1}^2 = \frac{b_n^2 + 6b_n + 9}{4}.$$

Hence,

$$b_{n+1}^2 = \left(\frac{b_n + 3}{2}\right)^2.$$

What can we do with this equation?

noodleeater 9:25:41 pm
take sqrt

lawrenceli 9:25:41 pm
take the square root

brian22 9:25:41 pm
b_(n+1) = (B_n + 3)/2

Johnzh 9:25:41 pm
take the square root of both side since b_n > 0

sirknightingfail 9:25:41 pm
square root of both siden

zhuangzhuang 9:25:41 pm
square root

MellowMelon 9:25:53 pm
Since b_n is positive for all n, we can take the square root of both sides to get

$$b_{n+1} = \frac{b_n + 3}{2}.$$

MellowMelon 9:26:03 pm
Looks like we're pretty close. What's the last thing we should check?

cerberus88 9:26:42 pm
so if the first term is rational, then b_n is rational

ProbaBillity 9:26:42 pm
check that b_1 is rational, then we done

noodleeater 9:26:42 pm
base cases is rational

brian22 9:26:42 pm
base case

superpi83 9:26:42 pm
b_1 is rational

Cosmynx 9:26:42 pm
b_1 is rational

Johnzh 9:26:42 pm
base case

nilaisarda 9:26:42 pm
base cases rational?

zhuangzhuang 9:26:42 pm
base cases?

sujaykazi 9:26:44 pm
base case

MellowMelon 9:26:48 pm
We just check that b_1 is rational. We find that $b_1 = \sqrt{1+24a_1} = 5$. Then the recurrence shows all terms are rational.

MellowMelon 9:26:54 pm
What do we conclude?

lawrenceli 9:27:39 pm
a_n is rational

ProbaBillity 9:27:39 pm
Q.E.D.

RocketSingh 9:27:39 pm
its always rational

willwang123 9:27:39 pm
rational b_n means rational a_n

sirknightingfail 9:27:39 pm
all of a_n is rational

ProbaBillity 9:27:39 pm
we conclude that b_n is rational for all n , and thus so is a_n .

eyzhang 9:27:39 pm
(a_n) is always rational!

brian22 9:27:39 pm
The stuff under the square root will always turn out rational, so QED

steve314 9:27:39 pm
the square root part of a_n is always rational, so a_n is always rational

cerberus88 9:27:39 pm
that a_n is rational

nilaisarda 9:27:39 pm
(a_n) rational for n >= 1

MellowMelon 9:27:41 pm
Therefore, a_n is rational for all n as well, since its recurrence only involves rational numbers. Apparently we didn't need our small cases at all...

MellowMelon 9:27:53 pm
This is a classic problem solving technique. If there is something ugly about your expression, then focus your attention on the part that is ugly. Maybe give it a new name too.

MellowMelon 9:28:07 pm
I have a couple extra minutes; any questions about anything today?

brian22 9:28:59 pm
Can you re-explain the problem before break

MellowMelon 9:29:01 pm
Probably not in two minutes. ☹ Maybe ask on the message board at the end of the week?

MellowMelon 9:29:28 pm
SUMMARY

MellowMelon 9:29:33 pm
We saw how telescoping sums and linear recurrences are powerful tools for analyzing sequences and series. You may be surprised how often linear recurrences appear in sequence problems. We also saw how effective it was to work with the linear recurrence itself, rather than the closed form solution.

MellowMelon 9:29:41 pm
However, not every sum telescopes and not every sequence is linearly recurrent. In such a case, you can fall back on classic problem solving techniques. Generalize. Exploit symmetry. And especially look for a pattern. Write out the terms of the sequence, until you find one. Look carefully and be persistent, because sometimes it's the smallest detail that breaks open a problem.

MellowMelon 9:29:55 pm
That's it for today's class. See you in a couple weeks!