Harmonic Division and its Applications

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Let d be a line and A, C, B, and D four points which lie in this order on it. The four-point $(ACBD)$ is called a *harmonic division*, or simply *harmonic*, if

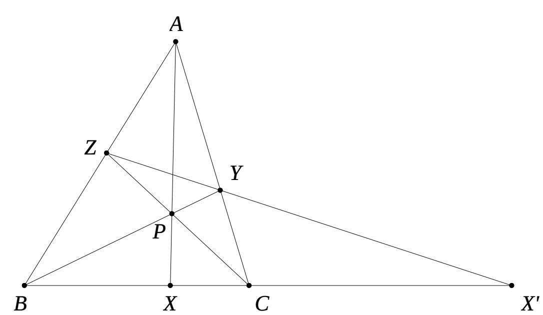
$$\frac{CA}{CB} = \frac{DA}{DB}.$$

If X is a point not lying on d, then we say that pencil $X(ACBD)$ (which consists of the four lines XA, XB, XC, XD) is harmonic if $(ACBD)$ is harmonic.

In this note, we show how to use harmonic division as a tool in solving some difficult Euclidean geometry problems.

We begin by stating two very useful lemmas without proof. The first lemma shows one of the simplest geometric characterizations of harmonic divisions, based on the theorems of Menelaus and Ceva.

Lemma 1. *In a triangle ABC consider three points X, Y, Z on the sides BC, CA, respective AB. If X' is the point of intersection of YZ with the extended side BC, then the four-point $(BXCX')$ forms and harmonic division if and only if the cevians AX, BY and CZ are concurrent.*



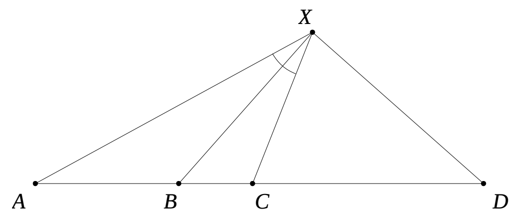
The second lemma is a consequence of the Appollonius circle property. It can be found in [1] followed by several interpretations.

Lemma 2. *Let four points A, B, C and D, in this order, lying on d. Then, if two of the following three propositions are true, then the third is also true:*

(1) The division $(ABCD)$ is harmonic.

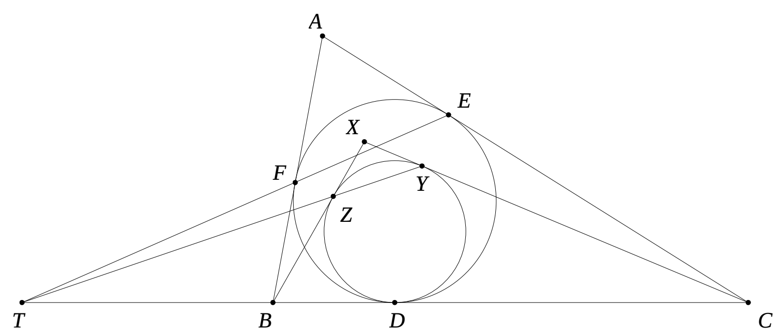
(2) XB is the internal angle bisector of $\angle AXC$.

(3) $XB \perp XD$.



We begin our journey with a problem from the IMO 1995 Shortlist.

Problem 1. Let ABC be a triangle, and let D, E, F be the points of tangency of the incircle of triangle ABC with the sides BC, CA and AB respectively. Let X be in the interior of ABC such that the incircle of XBC touches XB, XC and BC in Z, Y and D respectively. Prove that $EFZY$ is cyclic.

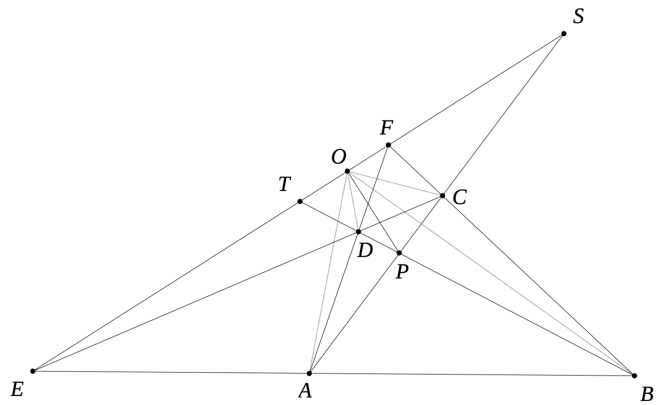


Solution. Denote $T = BC \cap EF$. Because of the concurency of the lines AD, BE, CF in the Gergonne point of triangle ABC, we deduce that the division $(TBDC)$ is harmonic. Similarly, the lines XD, BY and CZ are concurrent in the Gergonne point of triangle XBC, so $T \in YZ$ as a consequence of Lemma 1.

Now expressing the power of point T with respect to the incircle of triangle ABC and the incircle of triangle XBC we have that $TD^2 = TE \cdot TF$ and $TD^2 = TZ \cdot TY$. So $TE \cdot TF = TZ \cdot TY$, therefore the quadrilateral $EFZY$ is cyclic. $\square$

For our next application, we present a problem given at the Chinese IMO Team Selection Test in 2002.

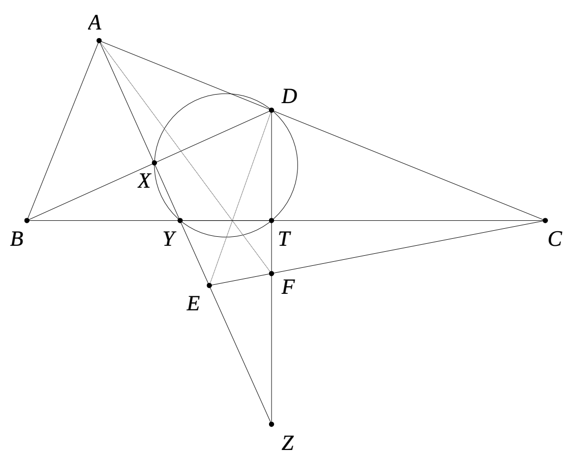
Problem 2. Let $ABCD$ be a convex quadrilateral. Let $E = AB \cap CD$, $F = AD \cap BC$, $P = AC \cap BD$, and let O the foot of the perpendicular from P to the line EF. Prove that $\angle BOC = \angle AOD$.



Solution. Denote $S = AC \cap EF$ and $T = BD \cap EF$. As from Lemma 1, we deduce that the division $(ETFS)$ is harmonic. Furthermore, the division $(APCS)$ is also harmonic, due to the pencil $B(ETFS)$. But now, the pencil $E(APCS)$ is harmonic, so by intersecting it with the line BD, it follows that the four-point $(BPDT)$ is harmonic. Therefore, the pencil $O(APCS)$ is harmonic and $OP \perp OS$, thus by Lemma 2, $\angle POA = \angle POC$. Similarly, the pencil $O(BPDT)$ is harmonic and $OP \perp OT$, thus again by Lemma 2, $\angle POB = \angle POD$. It follows that $\angle AOD = \angle BOC$. □

We continue with an interesting problem proposed by Dinu Serbanescu at the Romanian Junior Balkan MO 2007, Team Selection Test.

Problem 3. Let ABC be a right triangle with $\angle A = 90°$ and let D be a point on side AC. Denote by E the reflection of A across the line BD and F the intersection point of CE with the perpendicular to BC at D . Prove that AF, DE and BC are concurrent.



Solution. Denote the points $X = AE \cap BD$, $Y = AE \cap BC$, $Z = AE \cap DF$ and $T = DF \cap BC$. From Lemma 1, applied to triangle AEC and for the cevians AF

and ED, we observe that the lines AF, DE and BC are concurrent if and only if the division $(AYEZ)$ is harmonic.

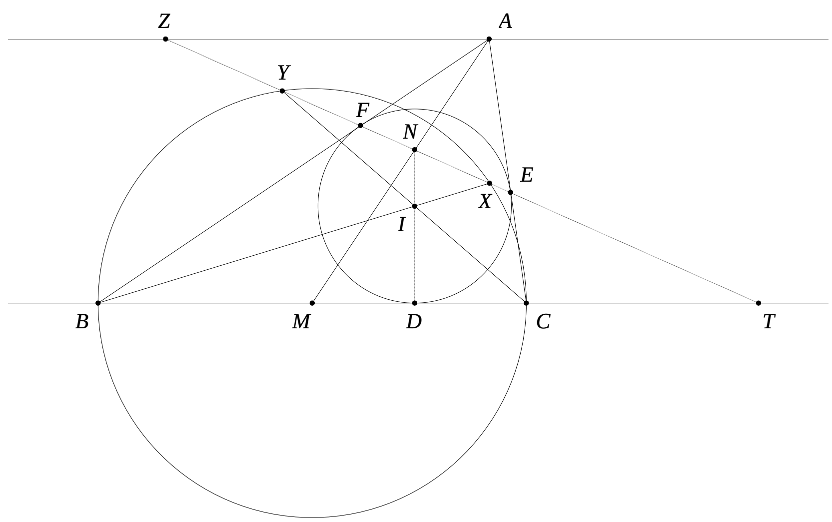
Since the quadrilateral $XYTD$ is cyclic, $\tan XYB = \tan XDZ$, which is equivalent to $XB/XY = XZ/XD$. So $XB \cdot XD = XY \cdot XZ$.

Since triangles XAB and XDA are similar, we have that $XA^2 = XB \cdot XD$, so $XA^2 = XY \cdot XZ$. Using $XA = XE$, we obtain that $\frac{YA}{YE} = \frac{ZA}{ZE}$, and thus the division $(AYEZ)$ is harmonic. $\square$

The next problem was proposed by the author and given at the Romanian IMO Team Selection Test in 2007.

Problem 4. Let ABC be a triangle, let E, F be the tangency points of the incircle $\Gamma(I)$ to the sides AC, respectively AB, and let M be the midpoint of the side BC. Let $N = AM \cap EF$, let $\gamma(M)$ be the circle of diameter BC, and let lines BI and CI meet γ again at X and Y, respectively. Prove that

$$\frac{NX}{NY} = \frac{AC}{AB}.$$



Solution. We will assume $AB \geq AC$, so the solution matches a possible drawing. Let $T = EF \cap BC$ (for $AB = AC$, $T = \infty$), and D the tangency point of Γ to BC.

Claim 1. In the configuration described above, for $X' = BI \cap EF$, one has $BX' \perp CX'$.

Proof. The fact that BI effectively intersects EF follows from $\angle DFE = \frac{1}{2}(\angle ABC + \angle BAC) = \frac{1}{2}\pi - \frac{1}{2}\angle ACB < \frac{1}{2}\pi$, and $BI \perp DF$ (similarly, CI effectively intersects EF).

The division $(TBDC)$ is harmonic, and triangles BFX' and BDX' are congruent, therefore $\angle TX'B = \angle DX'B$, which is equivalent to $BX' \perp CX'$ (similarly, for $Y' = CI \cap EF$, one has $CY' \perp BY'$). □

Claim 2. In the configuration described above, one has $N = DI \cap EF$.

Proof. It is enough to prove that $NI \perp BC$. Let d be the line through A, parallel to BC. Since the pencil $A(BMC\infty)$ is harmonic, it follows the division $(FNEZ)$ is harmonic, where $Z = d \cap EF$. Therefore N lies on the polar of Z relative to circle Γ, and as $N \in EF$ (the polar of A), it follows that AZ is the polar of N relative to circle Γ, hence $NI \perp d$, so $NI \perp BC$. In conclusion, since $DI \perp BC$, one has $N \in DI$. □

It follows, according to Claim 1, that $X = X'$ and $Y = Y'$, therefore $X, Y \in EF$. Since the division $(TBDC)$ is harmonic, it follows that D lies on the polar p of T relative to circle γ. But $TM \perp p$, so $BC \perp p$, and since $DI \perp BC$, it follows that p is, in fact, DI.

Now, according to Claim 2, it follows that D, I, N are collinear. Since DN is the polar, it means the division $(TYNX)$ is harmonic, thus the pencil $D(TYNX)$ is harmonic. But $DT \perp DN$, so DN is the angle bisector of $\angle XDY$, hence

$$\frac{NX}{NY} = \frac{DX}{DY} = \frac{\sin\angle DYX}{\sin\angle DXY}.$$

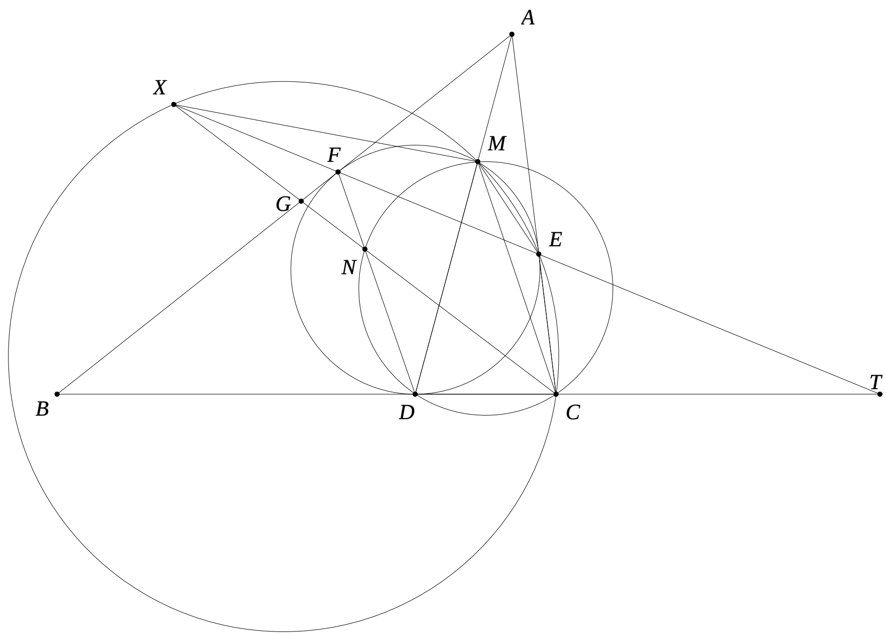
As quadrilaterals $BDIY$ and $CDIX$ are cyclic (since pairs of opposing angles are right angles), it follows that $\frac{1}{2}\angle ABC = \angle DBI = \angle DYI = \frac{1}{2}\angle DYX$ (triangles CDY and CEY are congruent), so $\angle DYX = \angle ABC$. Similarly, $\angle DXY = \angle ACB$. Therefore

$$\frac{NX}{NY} = \frac{DX}{DY} = \frac{\sin\angle DYX}{\sin\angle DXY} = \frac{\sin\angle ABC}{\sin\angle ACB} = \frac{AC}{AB}.$$

□

The following problem was posted on the MathLinks forum [2]:

Problem 5. Let ABC be a triangle and $\rho(I)$ its incircle. D, E and F are the points of tangency of $\rho(I)$ with BC, CA and AB respectively. Denote $M = \rho(I) \cap AD$, N the intersection of the circumcircle of CDM with DF and $G = CN \cap AB$. Prove that $CD = 3FG$.



Solution. Denote $X = EF \cap CG$ and $T = EF \cap BC$. Now because the four-point $(TBDC)$ forms an harmonic division, so does the pencil $F(TBDC)$ and now by intersecting it with the line CG, we obtain that the division $(XGNC)$ is harmonic.

According to the Menelaus theorem applied to BCG for the transversal DNF, we find that $CD = 3GF$ is equivalent to $CN = 3NG$.

Since $(XGNC)$ is harmonic, $\frac{NC}{NG} = \frac{XC}{XG}$, so it suffices to show that N is the midpoint of CX.

Observe that $\angle MEX = \angle MDF = \angle MCX$, therefore the quadrilateral $MECX$ is cyclic, which implies that $\angle MXC = \angle MEA = \angle ADE$ and $\angle MCX = \angle ADF$.

Also, $\angle CMN = \angle FDB$ and $\angle XMN = \angle XMC - \angle CMN = \angle CEF - \angle FDB = \angle EDC$.

Using the above angle relations and the equation

$$\frac{NX}{NC} = \frac{\sin \angle MCX}{\sin \angle MXC} \cdot \frac{\sin \angle XMN}{\sin \angle CMN},$$

we obtain that $NC = NX$, so

$$\frac{\sin \angle FDA}{\sin \angle EDA} = \frac{\sin \angle BDF}{\sin \angle CDE}.$$

On other hand, DA coincides with a symmedian of triangle DEF, so

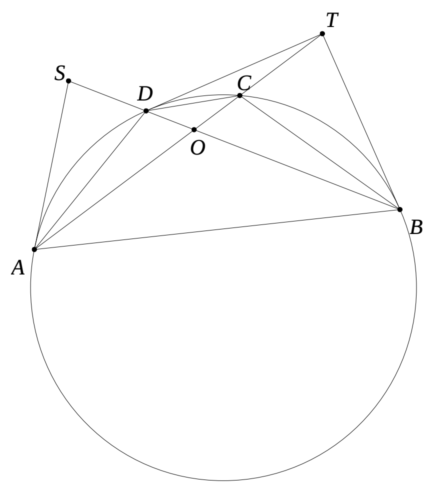
$$\frac{\sin \angle FDA}{\sin \angle EDA} = \frac{FD}{ED} = \frac{\sin \angle DEF}{\sin \angle DFE} = \frac{\sin \angle BDF}{\sin \angle CDE}.$$

Therefore, N is the midpoint of CX. □

Let $ABCD$ be a cyclic quadrilateral and X a point on the circle. Then, the $ABCD$ is called *harmonic* if the pencil $X(ABCD)$ is harmonic. For a list of properties regarding the harmonic quadrilateral, interested readers may can consult [1] and [3].

The following problem was given at an IMO Team Preparation Contest, held in Bacau, Romania, in 2006.

Problem 6. Let $ABCD$ be a convex quadrilateral, for which denote $O = AC \cap BD$. If BO is a symmedian of triangle ABC and DO is a symmedian of triangle ADC, prove that AO is a symmedian of triangle ABD.



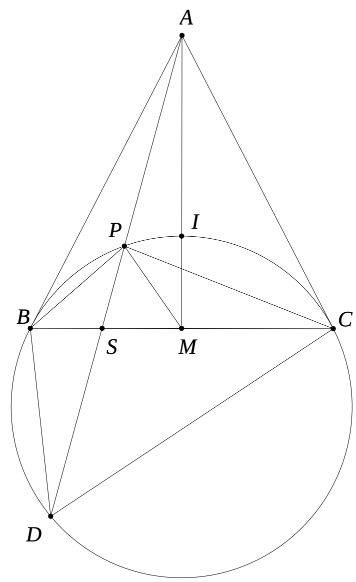
Solution. Denote $T_1 = DD \cap AC$, $T_2 = BB \cap AC$, $T = BB \cap DD$, where DD, respective BB represents the tangent in D to the circumcircle of ADC and the tangent in B to the circumcircle of ABC.

Since BO is a symmedian of triangle ABC and DO is a symmedian of triangle ADC, the divisions $(AOCT_1)$ and $(AOCT_2)$ are harmonic, so $T_1 = T_2 = T$.

Hence, BD is the polar of T_1 with respect to the circumcircle of ADC and also the polar of T_2 with respect to the circumcircle of ABC. But because $T_1 = T_2$, we deduce that the circles ABC and ADC coincide, i.e. the quadrilateral $ABCD$ is cyclic, and since the division $(AOCT)$ is harmonic, the pencil $D(AOCT)$ is, and by intersecting it by the circle $ABCD$, it follows that the quadrilateral $ABCD$ is also harmonic. Then, the pencil $A(ABCD)$ is harmonic. By intersecting it with the line BD, we see that the division $(BODS)$ is harmonic, where $S = AA \cap BD$. It follows that AO is a symmedian of triangle BAD. $\square$

The next problem was also given in an IMO Team Preparation Test, at the IMAR Contest, held in Bucharest in 2006.

Problem 7. Let ABC be an isosceles triangle with $AB = AC$, and M the midpoint of BC. Find the locus of the point P interior to the triangle for which $\angle BPM + \angle CPA = \pi$.



Solution. Denote the point D as the intersection of the line AP with the circumcircle of BPC and $S = DP \cap BC$.

Since $\angle SPC = 180 - \angle CPA$, it follows that $\angle BPS = \angle CPM$.

From the Steiner theorem applied in to triangle BPC for the isogonals PS and PM,

$$\frac{SB}{SC} = \frac{PB^2}{PC^2}.$$

On other hand, using Sine Law, we obtain

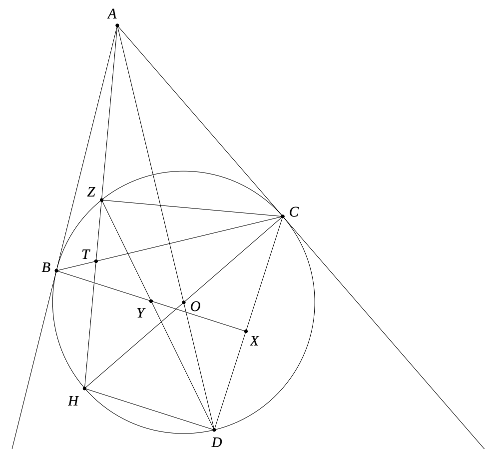
$$\frac{SB}{SC} = \frac{DB}{DC} \cdot \frac{\sin \angle SDB}{\sin \angle SDC} = \frac{DB}{DC} \cdot \frac{\sin \angle PCB}{\sin \angle PBC} = \frac{DB}{DC} \cdot \frac{PB}{PC}.$$

Thus by the above relations, it follows that $\frac{DB}{DC} = \frac{PB}{PC}$, i.e. the quadrilateral $PBDC$ is harmonic, therefore the point $A' = BB \cap CC$ lies on the line PD.

If $A' = A$, then lines AB and AC are always tangent to the circle BPC, and so the locus of P is the circle BIC, where I is the incircle of ABC. Otherwise, if $A' \neq A$, then $A' = AM \cap PS \cap BB \cap CC$, due to the fact that $A' \in PD$ and and $A = PS \cap AM$, therefore by maintaining the condition that $A' \neq A$, we obtain that $PS = AM$, therefore P lies on AM. $\square$

The next problem was selected in the Senior BMO 2007 Shortlist, proposed by the author.

Problem 8. Let $\rho(O)$ be a circle and A a point outside it. Denote by B, C the points where the tangents from A with respect to $\rho(O)$ meet the circle, D the point on $\rho(O)$, for which $O \in AD$, X the foot of the perpendicular from B to CD, Y the midpoint of the line segment BX and by Z the second intersection of DY with $\rho(O)$. Prove that $ZA \perp ZC$.

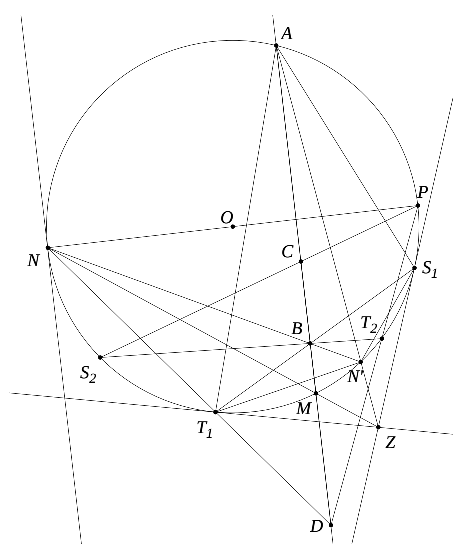


Solution. Let us call $H = CO \cap \rho(O)$. Thus $DC \perp DH$, so $DH \| BX$.

Because Y is the midpoint of BX, we deduce that the division $(BYX\infty)$ is harmonic, so also is the pencil $D(BYXH)$ and by intersecting it with $\rho(O)$, it follows that the quadrilateral $HBZC$ is harmonic. Then, the pencil $C(HBZC)$ is harmonic, so by intersecting it with the line HZ, it follows that the division $(A'ZTH)$ is harmonic, where $A' = HZ \cap CC$ and $T = HZ \cap BC$.

So, the line CH is the polar of A' with respect to $\rho(O)$, but $CH = BC$ is the polar of A as well, so $A = A'$, hence the points H, Z, A are collinear, therefore $ZA \perp ZC$. $\square$

The last problem is a generalization of a problem by Virgil Nicula [4]. The solution covers all concepts and methods presented throughout this paper.



Problem 9. Let d be a line and A, C, B, D four points in this order on it such that the division $(ACBD)$ is harmonic. Denote by M the midpoint of the line segment CD. Let ω be a circle passing through A and M. Let NP be the diameter of ω perpendicular to AM. Let lines NC, ND, PC, PD meet ω again at S_1, T_1, S_2, T_2, respectively. Prove that $B = S_1T_1 \cap S_2T_2$.

Solution. Since the four-point $(ACBD)$ is harmonic, so is the pencil $N(ACBD)$ and by intersecting it with ω, it follows that the quadrilateral $AS_1N'T_1$ is harmonic, hence the lines S_1S_1, T_1T_1 and AN' are concurrent, where $N' = NB \cap \omega$.

Because the tangent in N to ω is parallel with the line AM and since M is the midpoint of CD, the division $(CMD\infty)$ is harmonic, therefore the pencil $N(NDMC)$ also is, and by intersecting it with ω, it follows that the quadrilateral NT_1MS_1 is harmonic, hence the lines S_1S_1, T_1T_1 and MN are concurrent.

From the above two observations, we deduce that the lines S_1S_1, T_1T_1, MN, AN' are concurrent at a point Z.

On the other hand, since the pencils $B(AS_1N'T_1)$ and $B(NT_1MS_1)$ are harmonic, by intersecting them with ω, it follows that the quadrilaterals NT_3MS_3 and $AS_3N'T_3$ harmonic, where $S_3 = BS_1 \cap \omega$ and $T_3 = BT_1 \cap \omega$.

Similarly, we deduce that the lines S_3S_3, T_3T_3, MN and AN' are concurrent in the same point Z.

Therefore, S_3T_3 is the polar of Z with respect to ω, but so is S_1T_1, thus $S_1T_1 = S_3T_3$, so $S_1 = S_3$ and $T_1 = T_3$, therefore the points S_1, B, T_1 are collinear.

Similarly, the points S_2, B, T_2 are collinear, from which it follows that $B = S_1T_1 \cap S_2T_2$. □

References

[1] Virgil Nicula, Cosmin Pohoata. *Diviziunea armonica.* GIL, 2007.

[2] `http://www.mathlinks.ro/viewtopic.php?t=151320`

[3] `http://www.mathlinks.ro/viewtopic.php?t=70184`

[4] `http://www.mathlinks.ro/viewtopic.php?t=155875`

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