1 Mock CMO

1. *(Lithuania team contest 2000, #3)*
   Solve the equation

   $$\sqrt{-3x^2+18x+37}+\sqrt{-5x^2+30x-41}=\sqrt{x^2-6x+109}.$$

   Solution:
   After setting $t=(x-3)^2$, the equation reduces to $\sqrt{64-3t}+\sqrt{4-5t}=\sqrt{t+100}$. $t=0$ is one solution, and the left-hand side is decreasing in t while the right-hand side is increasing in t. Therefore, $t=0$ (and hence $x=3$) is the unique solution.

2. *(Ukraine 10th grade 2008, #2)*
   Let $ABCD$ be a parallelogram with $\angle BCD > 90°$. Extend line BC to K so that $DK=DC$, and extend line DC to L so that $BL=BC$. Let the bisectors of $\angle CDK$ and $\angle CBL$ meet at point Q. If $\angle BQD=\alpha$ and $KL=a$, express the circumradius of triangle AKL in terms of a and α.

   Solution:
   $ADLB$ and $ADKB$ are isosceles trapezoids, so K, L lie on the circumcircle of $\triangle ABD$. $\angle BAD = \angle LCK = 180° - \angle BQD$ since BQ and QD are perpendicular to LC and CK. Therefore, $ABLQKD$ is a cyclic hexagon. Now, $\angle LAK = \angle LAQ + \angle KAQ = \angle LDQ + \angle KBQ = 180° - 2\alpha$. Applying the extended sine law, we have the radius is $\frac{a}{2\sin 2\alpha}$.

3. *(Japan junior math olympiad 2008, #10)*
   Consider a board consisting of 9 squares of length 1, arranged in 3 rows and 3 columns as in the figure below. Let us number squares as indicated in the figure.

1	2	3
8		4
7	6	5

   Let A, B, C, D be points chosen from the interior (not on the boundary) of the square 1, 3, 5, 7, respectively. Denote by X the sum of the areas of the intersection of the quadrilateral $ABCD$ with the squares 1, 3, 5, and 7. Denote also by Y the sum of the areas of the intersections of the quadrilateral $ABCD$ with the squares 2, 4, 6, and 8. Prove that $X < Y$.

   Solution:
   Suppose that $ABCD$ intersects the boundary of square i and square $i+1$ at point X_i, and that X_i is distance x_i from the central square. Then, the area of $ABCD$ within square $2i$ is exactly $\frac{x_{2i-1}+x_{2i}}{2}$, so $Y=\frac{1}{2}\sum_{i=1}^{8} x_i$. Now let Y_i and Z_i denote the outer and inner corners of square i for $i=1,3,5,7$. Then, the area of $ABCD$ within square $2i-1$ is less than the area of quadrilateral $X_{2i-2}Y_{2i-1}X_{2i-1}Z_{2i-1}$, which is exactly $1-\frac{1-x_{2i-2}}{2}-\frac{1-x_{2i-1}}{2}=\frac{x_{2i-2}+x_{2i-1}}{2}$. Therefore, $X < \frac{1}{2}\sum_{i=1}^{8} x_i = Y$.

4. *(Lithuania team contest 2000, #10)*
   A positive integer n is said to be *reducible* if there exist positive integers m and d such that

   $$n=\frac{m+1}{d+1}+\frac{m+2}{d+2}.$$

   How many reducible numbers are there from the set $\{1,2,3,\ldots,2000\}$?

   Solution:
   $\frac{m+1}{d+1}+\frac{m+2}{d+2}=\frac{(m+1)(d+2)+(m+2)(d+1)}{(d+1)(d+2)}$. For this to be an integer, we must have $(m+2)(d+1)\equiv 0 \pmod{d+2} \implies m+2\equiv 0 \pmod{d+2}$ and $(m+1)(d+2)\equiv 0 \pmod{d+1} \implies m+1\equiv$

0 (mod $d+1$). Equivalently $m \equiv d \pmod{\operatorname{lcm}(d+1, d+2)}$. Since $d+1$ and $d+2$ are relatively prime, it follows that $m = C(d+1)(d+2) + d$, which gives $n = C(2d+3) + 2$.

Clearly, $n = 2$ can be expressed in this form but $n = 1$ cannot. For $n > 2$, the condition is equivalent to requiring that $n-2$ have an odd divisor greater than 3. This fails only for $n = 2 + 2^k$ and $n = 2 + 3 \cdot 2^k$. Altogether, that gives only $1 + 11 + 10$ numbers that are not reducible, so the answer is 1978.

5. *(Japan math olympiad final 2008, #5)*
   Does there exist a positive integer n satisfying the following condition?

   Condition: For an arbitrary rational number r, there exists an integer b and nonzero integers $a_1, a_2, \ldots, a_n$ such that $r = b + \frac{1}{a_1} + \ldots + \frac{1}{a_n}$.

   Solution:

   No. We prove by induction on n that for each integer n and rational number q, there exists a rational number $q' < q$ such that no number in the interval (q', q) can be expressed in this form. For $n = 1$, the claim is trivial.

   Now suppose the claim has been proven for $n-1$. Fix an arbitrary q, and let f denote the fractional part of q (except if q is an integer, we take $f = 1$). Then, if we are to express some number $y \in (q - \frac{f}{2}, q)$ in the required form, we must have some $\frac{1}{a_i} \geq \frac{f}{2} \implies a_i \leq \frac{2}{f}$. This leaves only a finite number of options for a_i. Once a_i is fixed to some value u, we have reduced the problem to the $n-1$ case, so by the inductive hypothesis, there exists q'_u such that no number in (q'_u, q) can be achieved, given that $a_i = u$. The inductive step follows from taking $q' = \max(q - \frac{f}{2}, \max_u q'_u)$.

2 Mock IMO Day 1

1. *(Ukraine 11th grade 2008, #3)*
A point O is placed inside triangle ABC so that $\angle BOC = 90°$ and $\angle BAO = \angle BCO$. If M and N are the midpoints of the segments AC and BC respectively, prove that $\angle OMN = 90°$.

Solution:

Let P be the midpoint of OC. N is the circumcenter of right triangle BOC since it is the midpoint of the hypoteneuse, so $\angle BCO = \angle NCP = \angle NOP$. Since MP is parallel to AO and MN is parallel to AB, we have $\angle NMP = \angle BAO = \angle BCO = \angle NOP$. Therefore, $OMPN$ is a cyclic quadrilateral and $\angle OMN = \angle OPN = \angle BOC = 90°$.

2. *(Japan math olympiad final 2008, #4)*
Determine all real-valued functions f defined on the real line, which satisfy

$$f(x+y)f(f(x)-y) = xf(x) - yf(y).$$

for all real numbers x and y.

Solution:

Taking $y = 0$ gives $f(x)f(f(x)) = xf(x)$, so $f(x) = 0$ or $f(f(x)) = x$. Taking $y = f(x)$ gives $f(x+f(x))f(0) = xf(x) - f(x)f(f(x)) = 0$. Taking $y = -x$ gives $0 = f(0)f(f(x)+x) = xf(x) + xf(-x)$, which implies $f(-x) = -f(x)$ for $x \neq 0$.

Now suppose $f(x) = 0$ and $y, f(y) \neq 0$. Then, $f(x+y)f(-y) = -yf(y) = yf(-y) \implies f(x+y) = y$. In particular, $f(x+y) \neq 0$ so $f(f(x+y)) = x+y$, and $f(y) = x+y$ (*). If $x \neq 0$, then $f(-x) = 0$ also and the same argument implies $f(y) = -x+y$, which is impossible. Therefore, if $f(x) = 0$ for some $x \neq 0$, then $f(x) = 0$ for all $x \neq 0$. It is easy to check any such function is a solution.

Otherwise, $f(x) \neq 0$ for all $x = 0$. Then $f(0)f(f(0)) = 0$ implies $f(0) = 0$, and (*) implies that in this case, $f(y) = y$ for all y. This, in turn, is easily checked to be a valid solution.

3. *(Bulgarian math olympiad team selection test 2008, #1)*
The number -1 is written at k of the vertices of a regular 2009-gon and the number 1 is written at the remaining $2009 - k$ vertices. A vertex is said to be *good* if, starting from this vertex and running around the polygon in either direction, every partial sum is positive. Find the largest number k such that there exists a good vertex for any arrangement of the 1's and -1's.

Solution:

Let p be the number of positive vertices, and n the number of negative vertices. If $p \leq 2n$, put all positive vertices together. Then, any positive vertex can reach the n negative vertices after going through at most n positive vertices, so it is bad. Conversely, if $p > 2n$, then for each positive vertex, there must exist a minimal interval with it as an endpoint and sum 0. The opposite endpoint is negative, and each negative point is endpoints to two such intervals. Therefore, some vertex is good. $k = 669$.

3 Mock IMO Day 2

4. *(IMO short list 2008, N1)* Let n be a positive integer and let p be a prime number. Prove that if a, b, c are integers (not necessarily positive) satisfying the following equations

$$a^n + pb = b^n + pc = c^n + pa,$$

then $a = b = c$.

Solution:

If two values are equal, the claim is trivial. Otherwise, multiply the equations $a^n - b^n = p(c-b)$ to get $\frac{a^n-b^n}{a-b} \cdot \frac{b^n-c^n}{b-c} \cdot \frac{c^n-a^n}{c-a} = -p^3$. If n is odd, $a^n - b^n$ has the same sign as $a - b$, and the left-hand side is positive, which is impossible. Otherwise, two of a, b, c are the same parity, and in this case, the corresponding $\frac{a^n-b^n}{a-b}$ term is even, which implies $p = 2$.

The original equation now implies a, b, c must all be the same parity, and hence $\frac{a^n-b^n}{a-b}$ is even (as above). Since the product of these terms is -2^3, each such term is ± 2. Comparing with the original equation, we have $a - b = \pm(b - c)$. If $a - b, b - c, c - a$ are all the same sign, they are all equal, but their sum is 0 so then $a = b = c$. Otherwise, $a - b = c - b$ or some shift thereof, and two values are equal. Either way, we have a contradiction.

5. *(Ukraine 11th grade 2008, #7)*
Prove that the inequality

$$\frac{x}{\sqrt{x^2+y+z}} + \frac{y}{\sqrt{x+y^2+z}} + \frac{z}{\sqrt{x+y+z^2}} \le \sqrt{3}$$

holds for any non-negative real numbers x, y, z satisfying $x^2 + y^2 + z^2 = 3$.

Solution:

The given condition implies $x+y+z \le x^2+y^2+z^2 = 3$, and hence $\frac{x(1+y+z)+y(1+z+x)+z(1+x+y)}{(x+y+z)^2} \le 1$. By Cauchy-Schwarz, $\frac{1}{x^2+y+z} \le \frac{1+y+z}{(x+y+z)^2}$, so therefore, $\frac{x}{x^2+y+z} + \frac{y}{x+y^2+z} + \frac{z}{x+y+z^2} \le 1$. The result now follows from $x + y + z \le 3$ and Cauchy-Schwarz.

6. *(Romanian master in mathematics 2008, #4)*
Prove that from among any $(n + 1)^2$ points inside a square of side length positive integer n, one can pick three determining a triangle with area at most $\frac{1}{2}$.

Solution:

If three of the $N = (n+1)^2$ points are collinear, the problem is trivial. Otherwise, let H denote the convex hull of the points, and let k denote the number of points on the boundary of H. Then, H has area at most n^2 and perimeter at most $4n$. We first triangulate H. This uses exactly $2(N - 1) - k$ triangles, so one triangle has area at most $\frac{n^2}{2(N-1)-k}$. Next, there must exist consecutive sides of H with lengths a, b such that $\frac{a+b}{2} \le \frac{4n}{k}$. The area of the triangle with these two sides is at most $\frac{1}{2} \cdot ab \le \frac{1}{2} \cdot \left(\frac{a+b}{2}\right)^2 \le \frac{8n^2}{k^2}$.

Therefore, some triangle has area at most $X = \min\left(\frac{n^2}{2(N-1)-k}, \frac{8n^2}{k^2}\right)$. The first term is increasing in k, and the second is decreasing, so X is maximized when $\frac{n^2}{2(N-1)-k} = \frac{8n^2}{k^2}$, or equivalently $k = 4n$. In this case, $X = \frac{1}{2}$, and the result follows.