Graph theory

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1 Well-known results

We begin by collecting some basic facts which can be proved via "bare-hands" techniques.

1. The sum of all of the degrees is equal to twice the number of edges. Deduce that the number of odd-degree vertices is always an even number.

   Solution: By counting in two ways, we see that the sum of all degrees equals twice the number of edges.

2. A graph is called *bipartite* if it is possible to separate the vertices into two groups, such that all of the graph's edges only cross between the groups (no edge has both endpoints in the same group). Prove that this property holds if and only if the graph has no cycles of odd length.

   Solution: Separate into connected components. For each, choose a special vertex, and color based on parity of length of shortest path from that special vertex.

3. Every connected graph with all degrees even has an *Eulerian circuit*, i.e., a walk that traverses each edge exactly once.

   Solution: Start walking from a vertex v_1 without repeating any edges, and observe that by the parity condition, the walk can only get stuck at v_1, so we get one cycle. If we still have more edges left to hit, connectivity implies that some vertex v_2 on our current walk is adjacent to an unused edge, so start the process again from v_2. Splice the two walks together at v_2, and repeat until done.

4. Suppose that a graph has at least as many edges as vertices. Show that it contains a cycle.

   Solution: As long as there are vertices with degree exactly 1, delete both the vertex and its incident edge. Also delete all isolated vertices. These operations preserve $E \geq V$, but we can never delete everything because once $V = 1$, E must be 0, so we can never get down to only 1 vertex or less.

   Therefore we end up with a nonempty graph with all degrees ≥ 2, and by taking a walk around and eventually hitting itself, we get a cycle.

5. Suppose that the graph G has all degrees at most Δ. Prove that it is possible to color the vertices of G using $\leq \Delta + 1$ colors, such that no pair of adjacent vertices receives the same color.

   Solution: Consider the greedy algorithm for coloring vertices.

6. Let G_1, G_2, G_3 be three (possibly overlapping) graphs on the same vertex set, and suppose that G_1 can be properly colored with 2 colors, G_2 can be properly colored with 3 colors, and G_3 can be properly colored with 4 colors. Let G be the graph on the same vertex set, formed by taking the union of the edges appearing in G_1, G_2, G_3. Prove that G can be properly colored with 24 colors.

   Solution: Product coloring.

7. Let G be a graph. It is possible to partition the vertices into two groups such that for each vertex, at least half of its neighbors ended up in the other group.

   Solution: Take a max-cut: the bipartition which maximizes the number of crossing edges.

8. Let δ be the minimum degree of G, and suppose that $\delta \geq 2$. Then G contains a cycle of size $\geq \delta + 1$. In particular, it contains a path with $\geq \delta$ edges.

   Solution: Take a longest path. Let v be its last endpoint. By maximality, every one of v's $\geq \delta$ neighbors lie on the path. So path has length $\geq \delta + 1$.

9. (Dirac.) Let G be a graph on n vertices with all degrees at least $n/2$. Show that G has a Hamiltonian cycle.

   Solution: Suppose the longest path has t vertices $x_1, \ldots, x_t$. We will show there is a cycle of t vertices as well. Suppose not. All neighbors of x_1 and x_t must lie on the path or else it is not longest. Minimum degree condition implies that both have degree $\geq t/2$. But if $x_1 \sim x_k$, then $x_t \not\sim x_{k-1}$ or else we can re-route to get a cycle. So, each of x_1's $t/2$ neighbors on the path prohibit a potential neighbor of x_t. Yet x_t's neighbors come from indices $1 \ldots t-1$, so there is not enough space for x_t to have $t/2$ neighbors there, avoiding the prohibited ones...

   Now if this longest path is not the full n vertices, then we get a cycle C missing some vertex x. But min-degree $n/2$ implies that the graph is connected (smallest connected component is $n/2+1$), so there is a shortest path from x to C, and adding this to the cycle gives a longer path than t, contradiction.

1.1 Matching

Consider a bipartite graph $G = (V, E)$ with partition $V = A \cup B$. A *matching* is a collection of edges which have no endpoints in common. We say that A *has a perfect matching to* B if there is a matching which hits every vertex in A.

Theorem. *(Hall's Marriage Theorem) For any set $S \subset A$, let $N(S)$ denote the set of vertices (necessarily in B) which are adjacent to at least one vertex in S. Then, A has a perfect matching to B if and only if $|N(S)| \geq |S|$ for **every** $S \subset A$.*

This has traditionally been called the "marriage" theorem because of the possible interpretation of edges as "acceptable" pairings, with the objective of maximizing the number of pairings. In real life, however, perhaps there may be varying degrees of "acceptability." This may be formalized by giving each vertex (in both parts) an ordering of its incident edges. Then, a matching M is called *unstable* if there is an edge $e = ab \notin M$ for which both a and b both prefer the edge e to their current partner (according to M).

Theorem. *(Stable Marriage Theorem) A stable matching always exists, for every bipartite graph and every collection of preference orderings.*

1.2 Planarity

When we represent graphs by drawing them in the plane, we draw edges as curves, permitting intersections. If a graph has the property that it can be drawn in the plane without any intersecting edges, then it is called *planar*. Here is the tip of the iceberg. One of the most famous results on planar graphs is the Four-Color Theorem, which says that every planar graph can be properly colored using only four colors. But perhaps the most useful planarity theorem in Olympiad problems is the Euler Formula:

Theorem. *Every connected planar graph satisfies $V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces.*

Solution: Actually prove that $V - E + F = 1 + C$, where C is the number of connected components. Each connecting curve is piecewise-linear, and if we add vertices at the corners, this will keep $V - E$ invariant. Now we have a planar graph where all connecting curves are straight line segments.

Then induction on $E + V$. True when $E = 0$, because $F = 1$ and $V = C$. If there is a leaf (vertex of degree 1), delete both the vertex and its single incident edge, and $V - E$ remains invariant. If there are no leaves, then every edge is part of a cycle. Delete an arbitrary edge, and that will drop E by 1, but also drop F by 1 because the edge was part of a cycle boundary, and now that has merged two previously distinct faces.

Now, use the theorem to solve the following problems:

1. Prove that K_5 is not planar.

2. Prove that $K_{3,3}$ is not planar.

3. Prove that $K_{4,4}$ is not planar.

4. Prove that every planar graph can be properly colored using at most 6 colors.

The Euler criterion immediately implies that every connected graph has at least $E - (3V - 6)$ crossings. As it turns out, one can do much better:

Theorem. *(Ajtai, Chvátal, Newborn, Szemeredi; Leighton.) Every connected graph with $E \geq 4V$ has at least $\frac{E^3}{64V^2}$ crossings.*

1.3 Extremal graph theory

The classical starting point is Turán's theorem, which proves the extremality of the following graph: let $T_r(n)$ be the complete r-partite graph with its n vertices distributed among its r parts as evenly as possible (because rounding errors may occur).

Theorem. *(Turán.) For $r \geq 3$, the Turán graph $T_{r-1}(n)$ is the unique n-vertex graph with the maximum number of edges subject to having no K_r subgraphs.*

There are many proofs of Turán's theorem, and one particular approach using Zykov symmetrization appears in the free online textbook of R. Diestel. The following results are somewhat easier to establish, and the reader is encouraged to attempt them:

1. Every graph G with average degree d contains a subgraph H such that all vertices of H have degree at least $d/2$ (with respect to H).

   Solution: Condition on G is that the number of edges is at least $nd/2$. If there is a vertex with degree $< d/2$, then delete it, and it costs 1 vertex and $< d/2$ edges, so the condition is preserved. But it can't go on forever, because once there is 1 vertex left, average degree is 0.

2. (Approximation to Erdős-Sós conjecture.) Let T be a tree with t edges. Then every graph with average degree at least $2t$ contains T as a subgraph.

   Solution: Graph has subgraph with minimum degree at least t. Then embed greedily. Suppose we already put down v vertices. ($v < t + 1$ or else we are done.) Pick a current node to which to adjoin a new leaf. Degree is at least t, and $v - 1$ vertices are already down (so blocked for embedding), so $t - v + 1 > 0$ choices remain. Pick one of them for the new leaf, and continue.

3. We say that a graph G is *t-degenerate* if every subgraph has a vertex of degree $\leq t$. Show that G can be properly colored with $\leq t + 1$ colors.

   Solution: Iteratively peel off vertex of degree $\leq t$, and put these into an ordering. That is, v_1 is the first vertex pulled off, then v_2, etc. Now greedily color from v_n to v_1.

2 Ramsey theory

Complete disorder is impossible.
— T. S. Motzkin, on the theme of Ramsey Theory.

The *Ramsey Number* $R(s,t)$ is the minimum integer n for which **every** red-blue coloring of the edges of K_n contains a completely red K_s or a completely blue K_t. Ramsey's Theorem states that $R(s,t)$ is always finite, and we will prove this in the first exercise below. The interesting question in this field is to find upper and lower bounds for these numbers, as well as for quantities defined in a similar spirit.

1. Prove by induction that $R(s,t) \le \binom{s+t-2}{s-1}$. Note that in particular, $R(3,3) \le 6$.

   Solution: Observe that $R(s,t) \le R(s-1,t)+R(s,t-1)$, because if we have that many vertices, then if we select one vertex, then it cannot simultaneously have $< R(s-1,t)$ red neighbors and $< R(s,t-1)$ blue neighbors, so we can inductively build either a red K_s or a blue K_t. But

   $$\binom{(s-1)+t-2}{(s-2)} + \binom{s+(t-1)-2}{s-1} = \binom{s+t-2}{s-1},$$

   because in Pascal's Triangle the sum of two adjacent guys in a row equals the guy directly below them in the next row.

2. Show that $R(t,t) \le 2^{2t}$. Then show that $R(t,t) > 2^{t/2}$ for $t \ge 3$, i.e., there is a red-blue coloring of the edges of the complete graph on $2^{t/2}$, such that there are no monochromatic K_t.

   Solution: The first bound follows immediately from the Erdős-Szekeres bound. The second is an application of the probabilistic method. Let $n = 2^{t/2}$, and consider a random coloring of the edges of K_n, where each edge independently receives its color with equal probabilities. For each set S of t vertices, define the event E_S to be when all $\binom{t}{2}$ edges in S are the same color. It suffices to show that $\mathbb{P}\left[\text{some } E_S \text{ occurs}\right] < 1$. But by the union bound, the LHS is

   $$\begin{aligned} \binom{n}{t} \cdot \left(2 \cdot 2^{-\binom{t}{2}}\right) &\le \frac{n^t}{t!} \cdot 2 \cdot 2^{-\frac{t^2}{2}+\frac{t}{2}} \\ &= \frac{\left(2^{t/2}\right)^t}{t!} \cdot 2 \cdot 2^{-\frac{t^2}{2}+\frac{t}{2}} \\ &= 2 \cdot \frac{2^{t/2}}{t!}. \end{aligned}$$

   This final quantity is less than 1 for all $t \ge 3$.

3. (IMO 1964/4.) Seventeen people correspond by mail with one another—each one with all the rest. In their letters only 3 different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least 3 people who write to each other about the same topic.

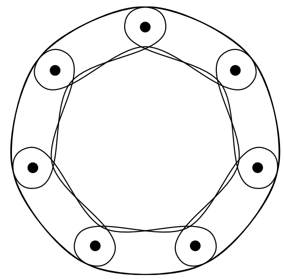
   Solution: This is asking us to prove that the 3-color Ramsey Number $R(3,3,3)$ is ≤ 17. By the same observation as in the previous problem, $R(a,b,c) \le R(a-1,b,c)+R(a,b-1,c)+R(a,b,c-1)-1$. Then using symmetry, $R(3,3,3) \le 3R(3,3,2)-1$. It suffices to show that $R(3,3,2) \le 6$. But this is immediate, because if we have 6 vertices, if we even use the 3rd color on a single edge, we already get a K_2. So we cannot use the 3rd color. But then from above, we know $R(3,3) \le 6$, so we are done.

3 Problems

1. Prove that every n-vertex graph with $n+1$ edges contains at least two (possibly overlapping) cycles. Does it always contain at least 3?

   Solution: Since the graph has at least n edges, there is a cycle. Delete one edge of that cycle. There are still at least n edges, so there is another cycle.

2. (Open.) Does every 3-uniform hypergraph with at least $1000n^2$ edges contain a (not necessarily spanning) *tight cycle*:



3. (USAMO 1986/2.) Five professors attended a lecture. Each fell asleep just twice. For each pair there was a moment when both were asleep. Show that there was a moment when three of them were asleep.

   Solution: Assume there is never a moment when three people were simultaneously asleep. The timeline then produces a permutation of the edges of K_5, where the first edge is the first pair of simultaneously asleep professors. This permutation must hit each vertex exactly 4 times, and those times can be spread over at most 2 intervals. It is relatively easy to show (via cases) that it is impossible for a single one of those intervals to hit a vertex 3 or more times, so we conclude that each interval hits every vertex exactly twice. This then corresponds to an Eulerian tour of K_5. However, the vertex at which the tour begins and ends is hit in 3 distinct intervals.

4. (IMO 1983 shortlist.) A country has 1983 cities. Every pair of cities is connected by a road. Each road is owned by one of 10 companies. Prove that there must be a way to travel in a circuit of odd length along a sequence of roads that are all owned by a single company.

   Solution: For each company, define a (not necessarily induced or spanning) subgraph of K_{1983} by taking every edge owned by the company. The condition implies that K_{1983} is the union of these 10 (possibly overlapping) subgraphs. Yet if each subgraph is bipartite, then the chromatic number of K_{1983} would be bounded by $2^10 = 1024$, contradiction.

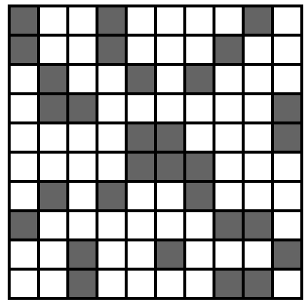
5. (From Ehud Friedgut.) Let T_1 and T_2 be two edge-disjoint spanning trees on the same vertex set. Prove that their union G is 4-colorable. Now let T_3 be a third spanning tree, also edge-disjoint from both T_1 and T_2. Prove that their union G is 8-colorable. Is it always 6-colorable?

   Solution: It is always 6-colorable due to 5-degeneracy. Every induced k-vertex subgraph spans at most $3(k-1)$ edges, with equality if and only if all three trees are connected on the subgraph. Therefore, there is always a vertex of degree at most 5.

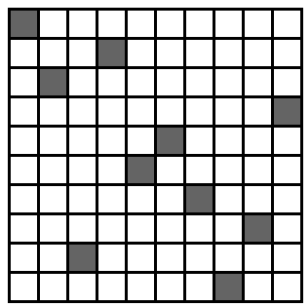
6. A **tournament** is a complete graph in which every edge has been oriented toward one of its endpoints. Prove that every tournament has a *Hamiltonian path*: one which passes through each vertex, respecting the orientations of the edges.

   Solution: Take a median order.

7. You are given a 10×10 grid, with the property that in every row, exactly 3 squares are shaded, and in every column, exactly 3 squares are shaded. An example is below.



Prove that there must always be a shaded *transversal*, i.e., a choice of 10 shaded squares such that no two selected squares are in the same row or column. An example is below.



Solution: Hall's theorem.

8. (MOP 2007/4/K2.) Let S be a set of 10^6 points in 3-dimensional space. Show that at least 79 distinct distances are formed between pairs of points of S.

   Solution: Zarankiewicz counting for the excluded $K_{3,3}$ in the unit distance graph. This upper-bounds the number of edges in each constant-distance graph, and therefore lower-bounds the number of distinct distances.

9. (MOP 2007/10/K4.) Let S be a set of $2n$ points in space, such that no 4 lie in the same plane. Pick any $n^2 + 1$ segments determined by the points. Show that they form at least n (possibly overlapping) triangles.

   Solution: In fact, every $2n$-vertex graph with at least $n^2 + 1$ edges already contains at least n triangles. No geometry is needed.

10. (Open.) Determine, as a function of n, the minimum number of crossings that appear in any plane drawing of K_n.

11. (Bondy 1.5.9.) There are n points in the plane such that every pair of points has distance ≥ 1. Show that there are at most $3n$ (unordered) pairs of points that span distance exactly 1 each.

    Solution: The unit distance graph is planar.

12. (St. Petersburg 1997/13.) The sides of a convex polyhedron are all triangles. At least 5 edges meet at each vertex, and no two vertices of degree 5 are connected by an edge. Prove that this polyhedron has a face whose vertices have degree 5, 6, 6, respectively.

    Solution: By Euler, $E \leq 3V - 6$, so in particular the sum of degrees is less than $6V$. We will use this for a contradiction. Suppose there are no 5,6,6 faces. We will count the number of edges which connect vertices of degree 5 to vertices of degree ≥ 7.

Let x_i be the number of vertices of degree i for each i. No 5,6,6 implies that each 5-vertex has at most 2 neighbors of degree 6, thus it contributes 3 edges which cross from degree 5 to degree ≥ 7. On the other hand, any vertex of degree d has at most $\lfloor d/2 \rfloor$ neighbors of degree 5 because no two degree-5 guys are adjacent. Thus, double-counting gives:

$$\begin{aligned} 3x_5 &\leq \sum_{d=7} x_d \cdot \left\lfloor \frac{d}{2} \right\rfloor \\ x_5 &\leq \sum_{d=7} x_d \cdot \frac{1}{3} \left\lfloor \frac{d}{2} \right\rfloor . \end{aligned}$$

Note that for $d \geq 7$, the cumbersome expression satisfies $\lfloor d/2 \rfloor / 3 \geq d - 6$. Adding to the LHS so that it becomes 6 times the number of vertices:

$$\begin{aligned} x_5 &\leq \sum_{d=7} x_d \cdot (d-6) \\ 6x_5 + 6x_6 + \sum_{d=7} 6x_d &\leq 5x_5 + 6x_6 + \sum_{d=7} x_d \cdot d. \end{aligned}$$

Recognize the LHS as $6V$ and the RHS as sum of degrees, and this contradicts our opening observation.