June 27, 2003 - Elementary Number Theory

I. Divisibility and remainders

1. Prove that for any positive integer n, 3 does not divide n^2+1.

2. Find the remainder when 6^{100} is divided by 7.

3. Prove that 31 divides $30^{99}+61^{100}$.

4. Prove that
a) 66 divides $43^{101}+23^{101}$;
b) $(a+b)$ divides a^n+b^n for any odd n.

5. Prove that n divides $(1^n+2^n+\ldots+(n-1)^n)$ for any odd n.

6. There is a box with n balls in it. Each one of two players takes up to 7 balls. One who takes the last ball wins. For which n the first player can win?

7. Prove that there exists an infinite number of positive integers n that cannot be presented in the form $n=m^3+p^3+d^3$, where m,p and d are positive integers.

8. Prove that numbers of the type 10^{3n+1} cannot be presented as a sum of two squares.

9. Prove that among any 501 numbers there are two numbers a and b such that a^2 and b^2 have the same three last digits.

10. Let us call a number "convenient" if 10^6+1 divides n^2+1. Prove that there is an even number of convenient numbers among $1,2,\ldots,10^6$.

11. Is it possible to compose the square of a positive integer using the digits 2,3,7,8 only (probably some of them several times)?

12. Let k be the product of the first n primes. Prove that neither $k-1$ nor $k+1$ are squares.

13. For how many numbers among $1,2,\ldots,10000$ does 7 divide 2^n-n^2 ?

14. Does there exist such a positive integer that 2005 divides n^2+n+1?

15. Prove that 133 divides $11^{n+2}+12^{2n+1}$ for any positive integer n.

16. Suppose 24 divides $n+1$. Prove that 24 also divides the sum of all positive integer divisors of n.

17. Let $a_{n+2}=a_{n+1}a_n+1$. Prove that
a) if $a_1=a_2=1$ then 4 does not divide any a_n; b) for any $n>10$ the number a_n-22 is composite.

18. Find the remainder of the number $10^{10}+10^{100}+10^{1000}+\ldots+10^{10^{10}}$ when divided by 7.

II. Divisibility and the decimal representation of integers

19. n is a positive integer, the last digit of n^2 is 6. Prove that the digit before the last one in n^2 is odd.

20. Prove that 2^n cannot have 4 equal final digits, where n is a positive integer.

21. Is it possible to write the square of an integer using 10 times each one of the following digits:
a) 2,3,6 b) 1,2,3

22. n is a positive integer. Can the sum of the digits of n^2 be equal to 2003?

23. We found the sum of the digits of 2^{100}, then the sum of the digits of the sum and so on. Finally, we have a one-digit number. What is it?

24. A is the sum of the digits of the number 4444^{4444}, B is the sum of the digits of A. Find the sum of the digits of B.

25. 15 divides $\overline{a15b}$. Find a, b.

26. How many numbers $\overline{a97b}$ exist, such that 45 divides this number?

27. What is the smallest positive integer n, such that 36 divides n and the decimal representatio of N contains all 10 digits?

28. Is it possible to find two integers $n = 19m$, such that all the digits of n, m are 2,3,4 or 9?

29. Does there exist such a number $\overline{abc}$ that $\overline{abc} - \overline{cba}$ is the square of a positive integer?

30. Find digits a, b and an integer n, such that $\overline{aabb} = n^2$.

III. The Little Fermat's Theorem

Theorem. If p is prime and p does not divide A then p divides $A^{p-1} - 1$.

31. Prove that 1001 divides $300^{3000} - 1$.

32. Prove that 143 divides $7^{120} - 1$.

33. Prove that $30^{239} + 239^{30}$ is a composite number.

34. Prove that p divides $(a+b)^p - a^p - b^p$ for any integers a, b and for any prime number p.

35. Let p be a prime number that does not divide a. Prove that there exists such a positive integer b that p divides $ab - 1$.

36. (Wilson's Theorem) Let p be prime. Prove that p divides $(p-1)! + 1$.

37. a) Let $p \neq 3$ be prime. Prove that p does not divide $11\ldots1$ (p digits).
b) Let $p > 5$ be prime. Prove that p divides $11\ldots1$ ($p-1$ digits).