

# Algebraic Techniques in Combinatorics

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Yufei Zhao

yufreiz@mit.edu

## Linear algebra

### Useful facts in linear algebra

- Any set of  $n + 1$  vectors in an  $n$ -dimensional vector space is linearly dependent. That is, we can find scalars  $a_1, \dots, a_{n+1}$ , not all zero, such that  $a_1 \mathbf{v}_1 + \dots + a_{n+1} \mathbf{v}_{n+1} = 0$ .
- Almost all linear algebra results (especially the ones related to rank) are true over any field. For instance, the field  $\mathbb{F}_2$  is often useful for working with parity or incidence. However, note that we might not be able to use eigenvalues.
- Suppose that  $A$  is an  $m \times n$  matrix, then the subspace spanned by its columns has the same dimension as the subspace spanned by its rows. This common dimension number is called the rank of  $A$ . In particular, we have  $\text{rank } A \leq \min(m, n)$ .
- Rank-nullity theorem:  $\text{rank } A + \text{nullity } A = n$ .
- $\text{rank}(AB) \leq \min(\text{rank } A, \text{rank } B)$
- $\text{rank}(A + B) \leq \text{rank } A + \text{rank } B$
- $A$  is invertible if and only if  $m = n = \text{rank } A$ . If  $A$  is a square matrix, then it is invertible if and only if there does not exist a nonzero vector  $\mathbf{v}$  such that  $A\mathbf{v} = 0$ . Or equivalently, none of the eigenvalues of  $A$  is zero.
- The determinant of a square matrix  $A$  can be evaluated by taking a sum over all permutations of  $\{1, 2, \dots, n\}$

$$\det A = \sum_{\sigma \in S_n} (-1)^{\text{sgn } \sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

A matrix is invertible if and only if its determinant is nonzero. In  $\mathbb{F}_2$ , we no longer have to worry about the sign in front of the product.

- If  $W$  is a subspace of  $V$ , denote  $W^\perp = \{\mathbf{v} \in V \mid \mathbf{v}^t \mathbf{w} = 0 \ \forall \mathbf{w} \in W\}$ , then

$$\dim W + \dim W^\perp = \dim V.$$

However, it's not necessarily true that  $W \oplus W^\perp = V$ ! E.g., take  $W = \{(0, 0), (1, 1)\}$  in  $\mathbb{F}_2^2$ , then  $W^\perp = W$ .

- Suppose that we have a set of maps  $f_i : V \rightarrow \mathbb{R}$ , and points  $v_i \in V$ ,  $1 \leq i \leq n$ . If  $f_i(v_i) \neq 0$  and  $f_i(v_j) = 0$  whenever  $i \neq j$ , then  $f_1, f_2, \dots, f_n$  are linearly independent.

**Problem:** (St. Petersburg) Students in a school go for ice cream in groups of at least two. After  $k > 1$  groups have gone, every two students have gone together exactly once. Prove that the number of students in the school is at most  $k$ .

*Solution.* There is a combinatorial solution which is somewhat long and non-intuitive. However, a much quicker (and more intuitive) solution is available using the tools of linear algebra.

Let there be  $n$  students. Note if some student went for ice cream only once, then everyone else has to have gone with that student, due to the constraint that every pair of student have gone together exactly once. Furthermore, since each group consists of two students, no other groups can be formed. However,  $k > 1$ , so this situation cannot occur. Therefore, every student went for ice cream at least twice.

Let us construct **incidence vectors**  $\mathbf{v}_i \in \mathbb{R}^k$ ,  $1 \leq i \leq n$ , representing the students. That is, the  $j$ -th component of  $\mathbf{v}_i$  is 1 if student  $i$  went with the  $j$ -th group, and 0 otherwise.

The condition that every two students have gone together exactly once translates into the dot product  $\mathbf{v}_i \cdot \mathbf{v}_j = 1$  for  $i \neq j$ . The condition that every student went for ice cream at least twice translates into  $|\mathbf{v}_i|^2 \geq 2$ .

We want to prove that  $n \leq k$ . Suppose otherwise. Then  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  consists of at least  $k + 1$  vectors in  $\mathbb{R}^k$ , so they must be linearly dependent. It follows that there are real numbers  $\alpha_1, \dots, \alpha_n$ , not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = 0.$$

Let us square the above expression (i.e., taking the dot product with itself. We get

$$0 = \sum_{i=1}^n \alpha_i^2 |\mathbf{v}_i|^2 + 2 \sum_{i < j} \alpha_i \alpha_j \mathbf{v}_i \cdot \mathbf{v}_j = \sum_{i=1}^n \alpha_i^2 |\mathbf{v}_i|^2 + 2 \sum_{i < j} \alpha_i \alpha_j = \sum_{i=1}^n \alpha_i^2 (|\mathbf{v}_i|^2 - 1) + \left( \sum_{i=1}^n \alpha_i \right)^2.$$

However, the RHS expression is positive, since  $|\mathbf{v}_i|^2 \geq 2$  and some  $\alpha_i$  is nonzero. Contradiction.

**Alternatively**, we can use an **incidence matrix** to obtain a somewhat faster (albeit more technically involved) solution. The incidence matrix  $M$  is a  $k \times n$  matrix, where  $M_{ij}$  is 1 if the  $i$ -th group includes student  $j$ . Using the notation in the previous solution, we can write this as

$$M = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & \cdots & | \end{pmatrix}$$

Consider the product  $M^t M$ . Since  $\mathbf{v}_i \cdot \mathbf{v}_j = 1$  for  $i \neq j$  and  $|\mathbf{v}_i|^2 \geq 2$  for all  $i$ , we see that  $M^t M = J + A$ , where  $J$  is the  $n \times n$  matrix with all entries 1, and  $A$  is a diagonal matrix with positive diagonal entries. Note that  $J$  is positive semidefinite (meaning that  $\mathbf{x}^t J \mathbf{x} \geq 0$  for all vectors  $\mathbf{x}$ ) and  $M$  is positive definite (meaning that  $\mathbf{x}^t J \mathbf{x} > 0$  for all  $\mathbf{x} \neq 0$ ). Therefore, their sum  $J + A = M^t M$  is positive definite. In particular, this means that  $M^t M$  is invertible, so its rank is  $n$ . Since  $M$  is a  $k \times n$  matrix, we conclude that  $k \geq n$ .  $\square$

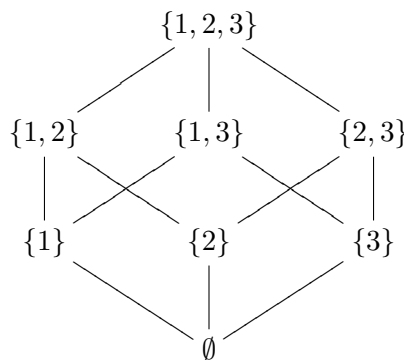
## Partially ordered sets (posets)

**Definition.** A *partially ordered set* (or *poset* for short)  $P$  is a set, also denoted  $P$ , together with a binary relation denoted  $\leq$  satisfying the following axioms:

- (reflexivity)  $x \leq x$  for all  $x \in P$
- (antisymmetry) If  $x \leq y$  and  $y \leq x$ , then  $x = y$ .
- (transitivity) If  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ .

An example of a poset is the set of all subsets of  $\{1, 2, \dots, n\}$  under the relation  $\subset$ . This poset is sometimes called the *Boolean algebra of rank  $n$* , and denoted  $B_n$ .

The **Hasse diagram** is a simple way of representing (small) posets. We say that  $x$  *covers*  $y$  if  $x > y$  (i.e.,  $x \geq y$  and  $x \neq y$ ) and there is no  $z \in P$  such that  $x > z > y$ . If  $x$  covers  $y$ , then we draw  $x$  above  $y$  and connect them using a line segment. Note that in general,  $x > y$  if and only if  $x$  is above  $y$  and we trace a downward path from  $x$  to  $y$  in the Hasse diagram. The Hasse diagram for  $B_3$  is depicted below.



There is one result about posets that has proven useful for olympiad problems. Before we state this result, let us go over some more terminology.

Two elements  $x, y$  of a poset are called *comparable* if  $x \geq y$  or  $x \leq y$ , otherwise they are called *incomparable*. A **chain** is a sequence of elements  $a_1 < a_2 < \dots < a_k$ , and an **antichain** is a set of pairwise incomparable elements. The *length* of a chain or antichain is one less the number of elements contained in it.

Now we are ready to state **Dilworth's Theorem**.

**Theorem 1.** (Dilworth) *Let  $P$  be a finite poset. Then the smallest set of chains whose union is  $P$  has the same cardinality as the longest antichain.*

There is also a dual version of this theorem that's much easier to prove.

**Theorem 2.** *Let  $P$  be a finite poset. Then the smallest set of antichains whose union is  $P$  has the same cardinality as the longest chain.*

The number of elements in the longest chain is *height* of the poset, and the number of elements in the longest antichain is *width* of the poset. Another way of stating the two theorems is that if a poset  $P$  has height  $h$  and width  $w$ , then  $P$  can be covered with  $h$  antichains and  $w$  chains. One simple consequence is that  $|P| \leq h \cdot w$ .

Instead of proving the the above two theorems, which you can do yourself<sup>1</sup>, let's see how we can apply them.

**Problem:** (Romania TST 2005) Let  $n$  be a positive integer and  $S$  a set of  $n^2 + 1$  positive integers with the property that every  $(n+1)$ -element subset of  $S$  contains two numbers one of which is divisible by the other. Show that  $S$  contains  $n + 1$  different numbers  $a_1, a_2, \dots, a_{n+1}$  such that  $a_i \mid a_{i+1}$  for each  $i = 1, 2, \dots, n$ .

*Solution.* Use the divisibility relation to obtain a poset on  $S$  (that is,  $x \leq y$  iff  $x \mid y$ . Check that this makes a poset). The condition that there does not exist an  $n + 1$  element subset of  $S$  that no element divides another translates into the condition that there does not exist an antichain with  $n + 1$  elements in  $S$ . So the longest antichain in  $S$  has at most  $n$  elements, and thus by Dilworth's theorem,  $S$  can be written as the union of at most  $n$  chains. Since  $S$  has  $n^2 + 1$  elements, this implies that one of these chains has a least  $n + 1$  elements. This implies the result.  $\square$

A very similar result is the **Erdős–Szekeres Theorem**, which states that within any sequence of  $ab + 1$  real numbers, there is either a nondecreasing subsequence of  $a + 1$  terms, or a nonincreasing subsequence of  $b + 1$  terms. This result is also a simple consequence of Dilworth's Theorem.

More examples are given in the problems section.

## Problems related to linear algebra

- (Nonuniform Fisher inequality) Let  $A_1, \dots, A_m$  be distinct subsets of  $\{1, 2, \dots, n\}$ . Suppose that there is an integer  $1 \leq \lambda < n$  such that  $|A_i \cap A_j| = \lambda$  for all  $i \neq j$ . Prove that  $m \leq n$ .
- (a) (China West 2002) Let  $A_1, A_2, \dots, A_{n+1}$  be non-empty subsets of  $\{1, 2, \dots, n\}$ . Prove that there exists nonempty disjoint subsets  $I, J \subset \{1, 2, \dots, n + 1\}$  such that

$$\bigcup_{k \in I} A_k = \bigcup_{k \in J} A_k.$$

- (b) (Lindstrom) Let  $A_1, A_2, \dots, A_{n+2}$  be non-empty subsets of  $\{1, 2, \dots, n\}$ . Prove that there exists nonempty disjoint subsets  $I, J \subset \{1, 2, \dots, n + 2\}$  such that

$$\bigcup_{k \in I} A_k = \bigcup_{k \in J} A_k, \quad \text{and} \quad \bigcap_{k \in I} A_k = \bigcap_{k \in J} A_k.$$

- (Russia 2001) A contest with  $n$  question was taken by  $m$  contestants. Each question was worth a certain (positive) number of points, and no partial credits were given. After all the papers have been graded, it was noticed that by reassigning the scores of the questions, any desired ranking of the contestants could be achieved. What is the largest possible value of  $m$ ?
- Oddtown and Eventown.* In a certain town with  $n$  citizens, a number of clubs are set up. No two clubs have exactly the set of members. Determine the maximum number of clubs that can be formed under each of the following constraints:

- The size of every club is odd, and every pair of clubs share an even number of members.

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<sup>1</sup>if you are really stuck, then you may consult the proof at, for example, [http://ocw.mit.edu/NR/rdonlyres/Mathematics/18-997Spring2004/FC143A49-2C1F-4653-85C4-C08A74990438/0/co\\_lec6.pdf](http://ocw.mit.edu/NR/rdonlyres/Mathematics/18-997Spring2004/FC143A49-2C1F-4653-85C4-C08A74990438/0/co_lec6.pdf)

- (b) The size of every club is even, and every pair of clubs share an odd number of members.
  - (c) The size of every club is even, and every pair of clubs share an even number of members.
  - (d) The size of every club is odd and every pair of clubs share an odd number of members.
5. (Moldova TST 2005) Does there exist a configuration of 22 distinct circles and 22 distinct points on the plane, such that every circle contains at least 7 points and every point belongs at least to 7 circles?
6. (Iran TST 1996, Germany TST 2004) Let  $G$  be a finite simple graph, and there is a light bulb at each vertex of  $G$ . Initially, all the lights are off. Each step we are allowed to chose a vertex and toggle the light at that vertex as well as all its neighbors'. Show that we can get all the lights to be on at the same time.
7. Let  $a_1, a_2, \dots, a_n$  be integers. Show that

$$\prod_{1 \leq i < j \leq n} \frac{a_i - a_j}{i - j}$$

is an integer. (Hint: use the Vandermonde determinant.)

8. (a) Let  $G$  be a graph with  $v$  vertices. Let  $f(n)$  denote the number of closed walks in  $G$  of length  $n$ . Show that there exists complex numbers  $\lambda_1, \dots, \lambda_v$  such that

$$f(n) = \lambda_1^n + \lambda_2^n + \dots + \lambda_v^n$$

for all positive integers  $n$ .

- (b) Let  $g(n, m)$  denote the number of sequences  $(x_1, x_2, \dots, x_n)$ , with terms from  $\{1, 2, \dots, m\}$ , such that  $x_1 = 1$ ,  $x_n \neq 1$ , and  $x_i \neq x_{i+1}$  for any  $i$ . Show that

$$g(n, m) = \frac{1}{m} ((m-1)^n + (m-1)(-1)^n).$$

9. (Crux 3037) There are 2007 senators in a senate. Each senator has enemies within the senate. Prove that there is a non-empty subset  $K$  of senators such that for every senator in the senate, the number of enemies of that senator in the set  $K$  is an even number.
10. (Classical) Let  $a_1, a_2, \dots, a_{2n+1}$  be real numbers, such that for any  $1 \leq i \leq 2n+1$ , we can remove  $a_i$  and separate the remaining  $2n$  numbers into two groups with equal sums. Show that  $a_1 = a_2 = \dots = a_{2n+1}$ .
11. (Russia 1998) Each square of a  $(2^n - 1) \times (2^n - 1)$  board contains either  $+1$  or  $-1$ . Such an arrangement is called successful if each number is the product of its neighbors (squares sharing a common side with the given square). Find the number of successful arrangements.
12. (Graham–Pollak) Show that the complete graph with  $n$  vertices,  $K_n$ , cannot be covered by fewer than  $n - 1$  complete bipartite graphs so that each edge of  $K_n$  is covered exactly once.
13. (Iran 2006) Let  $B$  be a set of  $n$ -tuples of integers such that for every two distinct members  $(a_1, \dots, a_n)$  and  $(b_1, \dots, b_n)$  of  $B$ , there exist  $1 \leq i \leq n$  such that  $a_i \equiv b_i + 1 \pmod{3}$ . Prove that  $|B| \leq 2^n$ .

14. (a) (Frankl–Wilson) Let  $A_1, A_2, \dots, A_m$  be distinct subset of  $\{1, 2, \dots, n\}$ . Let  $L$  be the set of numbers that occur as  $|A_i \cap A_j|$  for some  $i \neq j$ , and suppose that  $|L| = s$ . Show that

$$m \leq \binom{n}{s} + \binom{n}{s-1} + \dots + \binom{n}{0}$$

- (b) (Ray-Chaudhuri–Wilson) Let  $0 < k \leq n$  be positive integers, and let  $A_1, A_2, \dots, A_m$  be distinct  $k$ -element subsets of  $\{1, 2, \dots, m\}$ . Let  $L$  be the set of numbers that occur as  $|A_i \cap A_j|$  for some  $i \neq j$ , and suppose that  $|L| = s$ . Show that  $m \leq \binom{n}{s}$ .

## Problems related to posets

1. (Sperner) Let  $A_1, \dots, A_k$  be subsets of  $\{1, 2, \dots, n\}$  so that no  $A_i$  contains another  $A_j$ . Show that

$$\sum_{i=1}^k \frac{1}{\binom{n}{|A_i|}} \leq 1.$$

Conclude that  $k \leq \binom{n}{\lfloor n/2 \rfloor}$ .

2. (Romanian TST 2006) Let  $m$  and  $n$  be positive integers and  $S$  be a subset of  $\{1, 2, 3, \dots, 2^m n\}$  with  $(2^m - 1)n + 1$  elements. Prove that  $S$  contains  $m + 1$  distinct numbers  $a_0, a_1, \dots, a_m$  such that  $a_{k-1} \mid a_k$  for all  $k = 1, 2, \dots, m$ .
3. (Iran 2006) Let  $k$  be a positive integer, and let  $S$  be a finite collection of intervals on the real line. Suppose that among any  $k + 1$  of these intervals, there are two with a non-empty intersection. Prove that there exists a set of  $k$  points on the real line that intersects with every interval in  $S$ .
4. (Slovak competition 2004) Given 1001 rectangles with lengths and widths chosen from the set  $\{1, 2, \dots, 1000\}$ , prove that we can choose three of these rectangles,  $A, B, C$ , such that  $A$  fits into  $B$  and  $B$  fits into  $C$ .
5. Let  $G$  be a simple graph, and let  $\chi(G)$  be its chromatic number, i.e., the smallest number of colors needed to color its vertices so that no edge connects two vertices of the same color. Show that there is a path in  $G$  of length  $\chi(G)$  such that all  $\chi(G)$  vertices are of different colors.
6. Suppose that  $A$  and  $B$  are two distinct lattice points in  $\mathbb{R}^n$  with non-negative integer coordinates. We say that  $A$  *dominates*  $B$  (denote by  $A > B$ ) if all the components of  $A - B$  are nonnegative, and  $A \neq B$ .
- (a) Suppose that  $S$  is an infinite sequence of lattice points in  $\mathbb{R}^n$  with non-negative coordinates. Show that there exists an infinite subsequence satisfying  $S_{i_1} < S_{i_2} < S_{i_3} < \dots$ .
- (b) Suppose that  $T$  is a set of lattice points in the box  $[0, t_1] \times [0, t_2] \times \dots \times [0, t_n]$  (where  $t_1, t_2, \dots, t_n$  are fixed nonnegative integers). It is known that no element in  $T$  dominates another element. What is the maximum value of  $|T|$ ?