21st January 2025 (Tuesday) →

Laplacian in spherical polar coordinate →

$$\{r, \theta, \phi\}$$

$$r \in [0, \infty) \;,\; \theta \in [0, \pi] \;,\; \phi \in [0, 2\pi]$$

The Laplacian looks like →

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0$$

We may seperate,

$$r\sin\theta\frac{\partial^2}{\partial r^2}(r\psi) + \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \underbrace{\frac{\partial^2\psi}{\partial\phi^2}}_{-m^2} = 0$$

we replace,

$$r\sin\theta\frac{\partial^2}{\partial r^2}(r\psi) + \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - m^2 = 0$$

$$\Rightarrow \underbrace{r\frac{\partial^2}{\partial r^2}(r\psi)}_{f(r)} + \underbrace{\underbrace{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)}_{f(\theta)} - \frac{m^2}{\sin\theta}}_{-\alpha^2} = 0$$

$$\alpha^2 = L(L+1)$$

Looking at the radial part with $\psi = R(r)P(\theta)$ ...

$$\frac{r}{R}\frac{\partial^2}{\partial r^2}(rR) = \alpha^2$$

$$\Rightarrow \frac{\partial^2}{\partial r^2}(rR) = \frac{\alpha^2}{r}R$$

Now, this is an eigenvalue problem as usual.

$$\therefore R = r^L$$

$$\therefore r\frac{\partial^2}{\partial r^2}(r^{L+1}) = (L+1)(L)\,r^L$$

$\therefore$ $(L+1)L$ is the eigenvalue.

After quite a bit of scrambling, we find that the solutions are,

$$r^L \to L(L+1)$$

$$r^{-L-1} \to L(L+1)$$

$$\therefore \alpha^2 = L(L+1)$$

Now, we have,

$$\frac{1}{\sin\theta}\cdot\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P=0$$

We do a coordinate transform to make it recognizable.

$$x=\cos\theta \Rightarrow dx=-\sin\theta\, d\theta \Rightarrow \sin^2\theta=1-x^2$$

$$\therefore \frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[l(l+1)-\frac{m^2}{(1-x^2)}\right]P=0$$ → Associated Legendre diff'ble reaction.

Standard form of this diff'ble equation,

$$\frac{d^2y}{dx^2}+\frac{Q(x)}{P(x)}\frac{dy}{dx}+\frac{R(x)}{P(x)}=0$$

We may use power series to solve this. → $\sum_{n=0}^{\infty} a_n x^n$

→ $P(x)$, $Q(x)$ is a regular point $x$

Singular point → regular singular
→ irregular singular

→ $p(x)$ has pole of order $-1$ (Simple pole)

→ $q(x)$ has pole of order $-2$

Now, we have,

$$y=\sum_{n=0}^{\infty} b_n x^{n+s}$$ → Indicial eqn

☐ Review power series soln. of Associated Legendre diffn. eqn.

Note that in our case, $x\in\{-1,1\}$ are singularities.

Moreover, they are regular singular pts.

⊛ Something about plates? (Charged ones, of course)

If $\phi$ is symmetric, we have azimuthal/axial symmetry.

Setting $m=0$, we have,

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l}{dx}\right]+l(l+1)P_l(x)=0$$

☐ Just look up the solution and do it, the class is not really going on very coherently.

There is a one-one correspondence b/w Legendre polynomials and powers of x.

$$1 \longrightarrow P_0$$
$$x \rightarrow P_1$$
$$x^2 \rightarrow P_2 \dots$$

So,

Any $L^2$ function in the interval $[-1, 1]$ can be expressed as,

$$f(x) = \sum_n a_n P_n(x)$$

☐ Look up orthogonality of $P_l(x)$.

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22nd January 2025 (Wednesday) →

Legendre's differential equation →

Shows up when dealing with Spherical Laplacian.

⊛ Azimuthal symmetry: when potential does not depend on azimuth.

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l}{dx}\right] + l(l+1)P_l = 0$$

Where the solution, $P_l(x)$ is a polynomial in $x$

Note: $x \in [-1, 1]$ here, as it stands in for $\theta$ in spherical coord.

The differential equation looks like an eigenvalue eqn, thus suggesting that $P_l(x)$'s are orthogonal eigenfunctions — allows us to represent any $L^2$ function on $[-1, 1]$.

$$\Rightarrow \int_{-1}^{+1} P_l(x) P_m(x)\, dx \propto \delta_{lm}$$

Also,

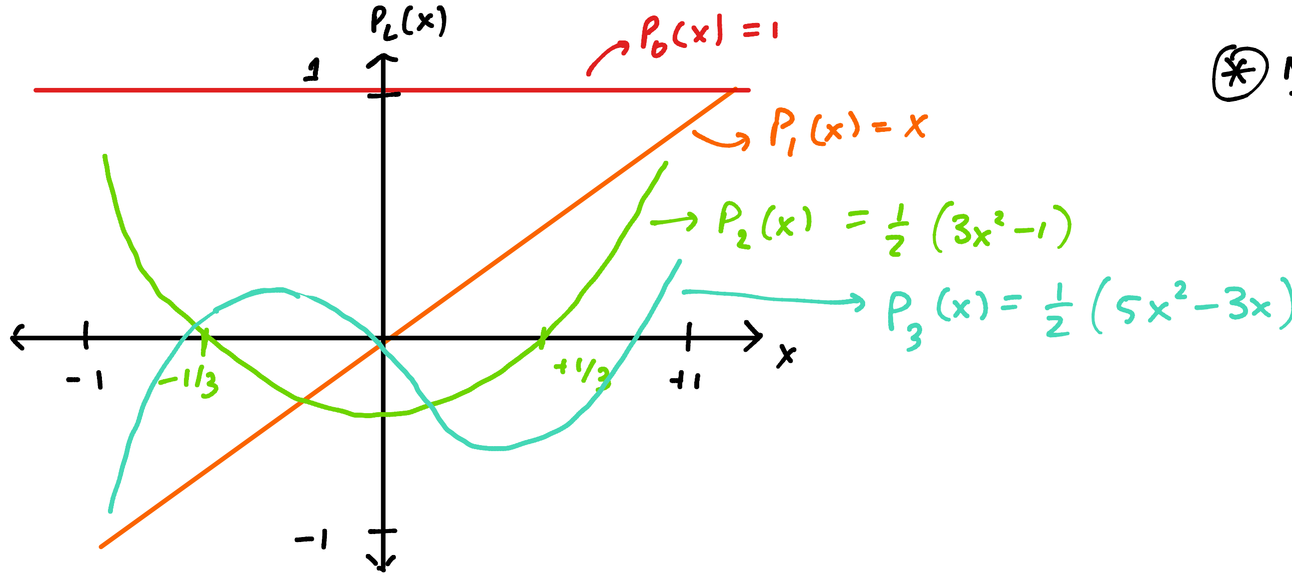
$$\int_{-1}^{1} [P_l(x)]^2 dx = N_l = \frac{2l+1}{2}$$

⊛ Rodrigues' formula →

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}\left[(x^2-1)^l\right]$$

We look at plots of $P_l(x)$

(*) **Note:** $P_l(\pm 1) = (\pm 1)^l$




Sort of like sin/cos x

"If you switch off your rational brain, it is like sin and cos"

— RKN

◉ **Fourier Legendre Series** →

If a function $f(x)$ is $\mathbb{L}^2$ in interval $[-1, 1]$. Then it can be expressed as,

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x)$$

Where, $A_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x)\, dx$ → Probably is $\frac{2}{2l+1}$

☐ Pls check all equations.

Now,

$$f(x) = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$$\frac{2}{2l+1} A_l = \int_{-1}^{+1} f(x) P_l(x)\, dx$$

$$= -\int_{-1}^{0} P_l(x)\, dx + \int_{0}^{1} P_l(x)\, dx$$

☐ Can be integrated using Rodrigues formula.

$$\rightarrow \quad A_l = \left(\frac{-1}{2}\right)^{\frac{l-1}{2}} \frac{(2l+1)(l-2)!!}{2\left(\frac{l+1}{2}\right)!}$$

$A_1 = \frac{3}{2}$, $A_2 = \frac{7}{8}$, $A_3 = \frac{11}{16}$, …

☐ HW: Decompose step function $P_l$ in interval $[-1, 1]$

→ Establish relation between $x^l$ and $P_l(x)$ for $l = 1, 3, 5, \dots$

→ Write step function as $\sum_n a_n x^n$ accurate upto 5th order.

□ Actual assignment — due next Monday. (10:00 A.M)

→ Also, plot with pen and paper.

If there is azimuthal symmetry, we just ignore z axis — we take projection on some x-y plane.

BVP with azimuthal symmetry →

$$\nabla^2 \phi = 0$$

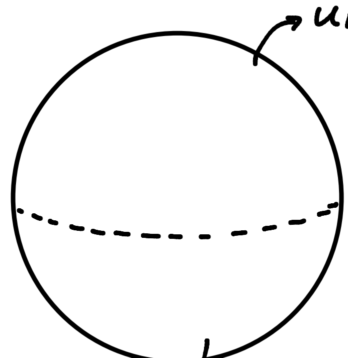
Something about "Object oriented"? Nevermind.

$$\phi(r,\theta) = \sum_{l=0} \left[ A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta)$$

Again something about $r \to 0$, $r \to \infty$.

Problem →

Upper hemisphere $= +V$



Lower hemisphere $= -V$

We solve Laplace inside this sphere.

$$\nabla^2 \phi = 0$$

$$\Rightarrow \phi(r,\theta) = \sum A_l \left(\frac{r}{a}\right)^l P_l(\cos\theta)$$

We know, as $r \to a$,

$$\phi(r,\theta) = \begin{cases} +V, & 0 \leq \theta < \frac{\pi}{2} \\ -V, & \pi \geq \theta > \frac{\pi}{2} \end{cases}$$

$$\Rightarrow \phi(r=a,\theta) = \sum_l A_l P_l(\cos\theta)$$

$$A_1 = \frac{3}{2}, \quad A_2 = \frac{7}{8}, \quad A_3 = \frac{11}{16}, \dots$$

(from Legendre Series of Step function)

Associated Legendre →

$$\frac{d}{dx}\left[ (1-x^2) \frac{dP_l^m}{dx} \right] + \left[ (l+1)l - \frac{m^2}{(1-x^2)} \right] P_l^m(x) = 0$$

$$l = 0, 1, \dots \quad , m = -l, \dots, +l$$

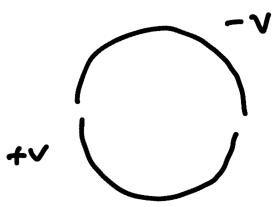
Very incoherent rambling on polar plots of associated Legendre polynomials.

23rd January 2025 (Thursday) →

- Slides: Representing functions on circles and spheres →

Plotting $\sin(x)$ vs. $x$ is simple.

But in problems like →




→ Plotting potential on this boundary is problematic, more so because it is on 2D polar.