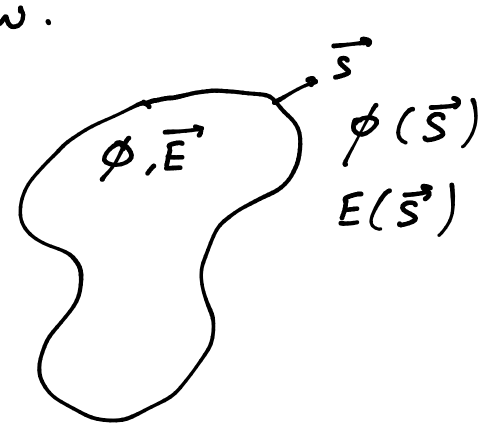
14th January 2025 (Tuesday) →

Boundary Value Problem:

Imp because we have converted electromagnetic (static) laws to boundary value problem – evident from Gauss' Law.

Poisson → $\nabla^2 \phi = \frac{\rho}{\epsilon_0}$



☐ Review: Green's Function

Green's function used for Poisson,
Laplace transform for Laplace.

→ Some incoherent rambling on types of boundaries, Green's theorem..

We will solve Laplace equation

We may reduce n dim to n-1 dim on Laplace/Poisson only on special conditions.

Laplace eqn in 2D →

1. Cartesian
2. Plane Polar.

$$\boxed{\nabla^2 \phi = 0}$$

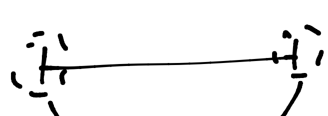
(*) Always choose coords based on given boundary.

Why? LE is not separable in all coord system.

Note: Some boundaries may extend to infinity – but they must be consistent.

→ He rambles on about infinite boundaries.

Why do we not use finite boundaries? Like a finite wire?



Singularities

We need to account for them. Before, we used to represent equipotential curves

$u(x, y) + i\,v(x, y)$

where we see CR conditions give us two Laplace eqn

$$\nabla^2 u = 0 = \nabla^2 v$$

I guess this was useful for plotting or something back before computers.

→ In Cartesian: $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$

→ In Plane Polar: $\dfrac{1}{r}\left[\dfrac{\partial}{\partial r}\left(r\dfrac{\partial\phi}{\partial r}\right) + \dfrac{1}{r}\dfrac{\partial^2\phi}{\partial\theta}\right] = 0$

We make separable ansatz, often.

$$\phi = X(x)\,Y(y)$$

$$\Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

with some constant of separation, α^2

$$\therefore \frac{d^2X}{dx^2} + \alpha^2 X = 0, \qquad \frac{d^2Y}{dy^2} - \alpha^2 Y = 0$$

$\downarrow$ $\sin\alpha x$, $\cos\alpha x$; $e^{i\alpha x}$, $e^{-i\alpha x}$

$\downarrow$ $\sinh y$, $\cosh y$; $e^{\pm\alpha y}$

↳ we select solution based on boundary.

⊙ Fourier Series become important here — any L^2 function can be expressed as as a sum of sin / cos by Fourier decomposition.

On some interval $[-L, L]$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)$$

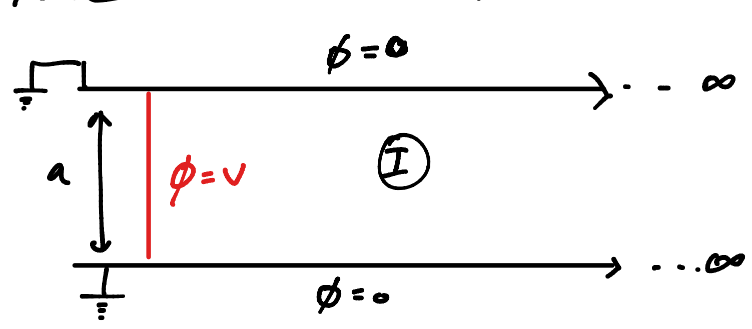
Where,

$$a_0 = \int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

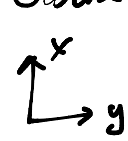
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

☐ Check the formulae please — this is no authorative source you know.

We now solve an example 2D problem.



We choose coords,



We solve in region (I) with boundary

$$\phi(x=0, y) = 0 = \phi(x=a, y)$$
$$\phi(x, y=0) = V$$

We assume it separable, and write,

$$\frac{d^2\phi_x}{dx^2} + \alpha^2 \phi_x = 0$$

$$\frac{d^2\phi_y}{dy^2} - \alpha^2\phi_y = 0$$

(*) Where do we choose to place the minus?

Hyperbolic → Infinities bound

Trigonometric → Finite bound

Here, $y \to$ infinite ⇒ y soln should by hyperbolic $(-\alpha^2)$

$x \to$ finite ⇒ x soln should be trigonometric $(+\alpha^2)$

$$\therefore \phi_x = A\sin\alpha x + B\cos\alpha x$$

Now, $\phi_x(0) = 0 \Rightarrow B = 0$

$\phi_x(a) = 0 \Rightarrow A\sin\alpha a = 0 \Rightarrow \alpha a = n\pi \Rightarrow \alpha = \frac{n\pi}{a}$

$$\therefore \phi_x(a) = A\sin\left(\frac{n\pi x}{a}\right)$$

$\therefore \alpha(n)$, $n \in \mathbb{N}$.

Now,

$$\frac{d^2\phi_y}{dy^2} - \left(\frac{n\pi}{a}\right)^2\phi_y = 0$$

$\phi_y = e^{-\frac{n\pi}{a}y}$, as $e^{+\alpha y}$ blows up as $y \to +\infty$

$$\therefore \phi = \sum_{n=1}^{\infty} A_n e^{\frac{-n\pi y}{a}} \sin\frac{n\pi x}{a}$$

We now need to figure out A_n's.

$$\phi(x, 0) = V$$

$\therefore \phi(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} = V$ → Fourier Series.

$$\therefore A_m = \frac{2}{a}\int_0^a V\sin\frac{m\pi x}{a}\,dx = \frac{2}{a}\left[\sum_n A_n \int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx\right]$$

$$\Rightarrow \boxed{A_n = \frac{4V}{a}}$$

☐ Redo properly ☐ Revise Fourier Series.

Tomorrow: Plane polar Laplace eqn

15th January 2025 (Wednesday) →

Again, Boundary Value Problem →

BVP means, ϕ → potential, is given on some boundary.

(or, $\vec{\nabla}\phi, \vec{E}$)

How do we compute surface charge density on a metal?

$\boxed{\begin{array}{c}\phi = \text{const} \\ E = 0\end{array}}$

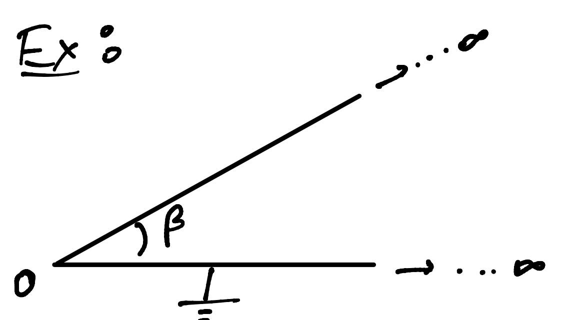
Using Gauss' Law, the surface charge density of a metal is,

$E_{\hat{x}}$ (component ∥ to surface) $= 0$

$E_{\hat{n}}$ (component ⊥ to surface) $= \frac{\sigma}{\epsilon_0}$

$[d\vec{s} = \hat{n}\, da]$

Ex:



plane polar coordinate is suitable for this problem.

$$\nabla^2 \psi = 0 \qquad \psi = \phi \text{, just notation change.}$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} = 0$$

$$\Rightarrow \rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial^2\psi}{\partial\phi^2} = 0$$

$\left.\begin{array}{l}\rho \to \text{Radial} \\ \phi \to \text{Angular.}\end{array}\right\}$

We assume separability,

$$\psi = \psi_\rho \psi_\phi$$

$$\therefore \rho\frac{\partial}{\partial\rho}\left(\rho\psi_\phi\frac{\partial\psi_\rho}{\partial\rho}\right) + \psi_\rho\frac{\partial^2\psi_\phi}{\partial\phi^2} = 0$$

$$\Rightarrow \rho\left(\psi_\phi\frac{\partial\psi_\rho}{\partial\rho} + \rho\psi_\phi\frac{\partial^2\psi_\rho}{\partial\rho^2}\right) + \psi_\rho\frac{\partial^2\psi_\phi}{\partial\phi^2} = 0$$

∴ The separated diffn. equations are →

$$\frac{\rho}{\Psi_\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\Psi_\rho}{\partial\rho}\right) = \alpha^2 \quad , \quad \frac{1}{\Psi_\phi}\frac{\partial^2\Psi_\phi}{\partial\phi^2} = -\alpha^2$$ (Alike to Helmholtz eqn, apparently)

How do we easily solve this? These are eigenvalue problem.

$$\therefore \rho\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\Psi_\rho}{\partial\rho}\right) = \alpha^2\Psi_\rho \longrightarrow$$ Euler differential equation

Put in, $\Psi_\rho = \rho^n$ (Trial soln)

$$\rho\frac{\partial}{\partial\rho}\left(n\rho^n\right) = n\rho\left(n\rho^{n-1}\right) = n^2\rho^n$$

$$\therefore n^2 = \alpha^2 \Rightarrow \boxed{n = \pm\alpha}$$

$$\therefore \Psi_\rho(\rho) \propto \rho^{\pm\alpha}$$

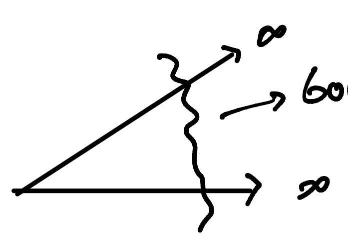
And the angular equation gives us usual trig (sin, cos) Solution.

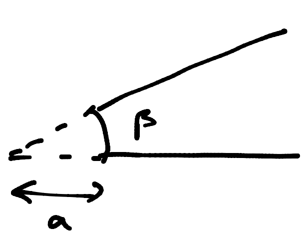
<u>Boundary conditions</u> →

$$\left.\begin{aligned}\Psi(\rho, \phi=0) &= 0\\ \Psi(\rho, \phi=\beta) &= 0\end{aligned}\right\} \to \Psi_\phi \propto \sin\left(\frac{n\pi}{\beta}\phi\right)$$

We have to reject both Ψ_ρ soln, as they both blow up.

What do we do? We need to either →

① 

② 

The plates never touch

Now, $\Psi_\rho = \rho^{\frac{n\pi}{\beta}}$

$$\therefore \Psi = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi\phi}{\beta}\right)\rho^{\frac{n\pi}{\beta}}$$

We look at $\lim_{\rho\to 0}$, then leading growth term is $n=1$

$\therefore$ In this limit,

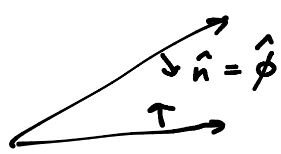
$$\Psi \simeq a_1 \sin\left(\frac{\pi\phi}{\beta}\right)\rho^{\pi/\beta}$$

Now, we want to find $\vec{E}$.

$$E_\rho = -\frac{\partial \Psi}{\partial \rho} = -a_1 \frac{\pi}{\beta} \rho^{\frac{\pi}{\beta}-1} \sin\left(\frac{\pi \phi}{\beta}\right)$$

$$E_\phi = -\frac{1}{\rho}\frac{\partial \Psi}{\partial \phi} = -a_1 \frac{\pi}{\beta} \rho^{\frac{\pi}{\beta}-1} \cos\left(\frac{\pi \phi}{\beta}\right)$$

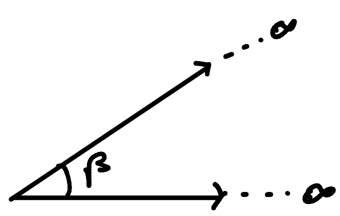
What is the surface charge density of these plates?



$$\Rightarrow \quad E_\phi \Big|_{\phi = 0, \beta} = -a_1 \frac{\pi}{\beta} \rho^{\frac{\pi}{\beta}-1} \quad \rightarrow \text{Surface charge density sensitive to angle.}$$

16th January 2025 (Thursday) →

We were solving the Laplace eqn in plane-polar coordinates.



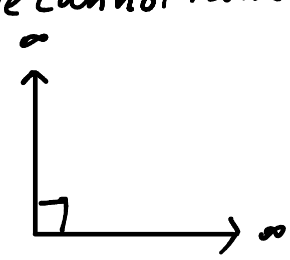
$$\Psi = a_1 \rho^{\pi/\beta} \sin\left(\frac{\pi}{\beta}\phi\right)$$

$$E_\rho = -\frac{\pi a_1}{\beta} \rho^{\pi/\beta - 1} \sin\left(\frac{\pi \phi}{\beta}\right)$$

$$E_\phi = -\frac{\pi}{\beta} a_1 \rho^{\frac{\pi}{\beta}-1} \cos\left(\frac{\pi}{\beta}\phi\right)$$

$\sigma = \frac{\pi}{\beta} a_1 \rho^{\frac{\pi}{\beta}-1}$ → $\rho \to \infty$ is a problem, infinite. Taking $\rho \to 0$ is also a problem, as polar has a singularity

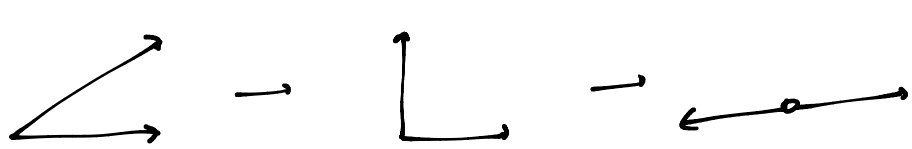
⊛ Note : We cannot solve for $\beta = \frac{\pi}{2}$,



using cartesian.
We need plane polar to so

☐ Numerically solve –
① The problem solved in Tutorial 2
② The two plate problem here
on cartesian (my job) and maybe even polar.
Plot $\vec{E}$, ϕ, and surface charge density too.

Extended discussion on,



He explains that at the hinge, we have singularities and thus we require complex analysis.

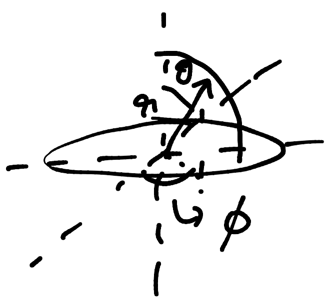
⊛ Spherical polar coordinate system →

Note, in Laplacian we cannot have trigonometric solution on all axes.

Why? Because the operator has a (+) on two dimension, and one (−) on the other dimension.

Something about the wave equation →

$$\nabla_x^2 + \nabla_y^2 - \nabla_t^2 = 0$$



$$\nabla^2 \psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2 \sin\theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = 0$$

$$\psi(r, \theta, \phi) = R(r)\, P(\theta)\, \Theta(\phi)$$

$$\Theta(\phi) = e^{im\phi}$$

$\rightarrow m \in \mathbb{Z}$

Why? Single valued quality of the function.