7th January 2025 (Tuesday) →

(*) To be Ankified

Brief overview of electrostatics →

(Emphasis on Jackson, right out the gate)

We begin by discussing Coulomb's Law

→ It came from experiment.

Mathematical representation →

Let $\mathcal{O}$ be an inertial frame, with point charge q_1 at $\vec{x}_1$ and another point-charge q_2 at $\vec{x}_2$. The force is given by,

$$\vec{F} = Kq_1q_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} \quad , K = \frac{1}{4\pi\varepsilon_0} \quad (SI)$$

- **Electric field:** A thing, that only interacts with charge.

The electric field due to a charge Q at $\vec{x}$, at another point $\vec{x}'$,

$$\vec{E}(\vec{x}) = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}$$

Practically measured using an arbitrarily small charge q (test charge)

$$\vec{E}(\vec{x}) = \lim_{q \to 0} \frac{Qq}{4\pi\varepsilon_0} \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}$$

- **Principle of Linear superposition →**

Force due to different charges on a single charge can be vectorially added.

$\Rightarrow$ $\vec{E}(\vec{x})$ is also linearly added, as a consequence.

- **Discrete and Continuous Charge Distribution →**

We may have a smooth charge distribution, $\rho(\vec{x})$, considered as an amount,

$$\Delta q(x) = \rho(\vec{x})\, \Delta x\, \Delta y\, \Delta z \quad , \text{as } dv = dx\,dy\,dz \ \text{(cartesian)}$$

$$\rightarrow \vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}|^3}\, d^3x'$$

Similarly, a point charge can be viewed as a Dirac Delta,

$$\rho(\vec{x}) = \delta(\vec{x} - \vec{x}')$$

(*) Ensure you know the properties of the $\delta(\vec{x})$.

- **Quick Application →**

A sphere of radius a is uniformly charged with a total charge Q, what is electric field at an arbitrary point inside (Obviously, $\vec{0}$, but let us do it)

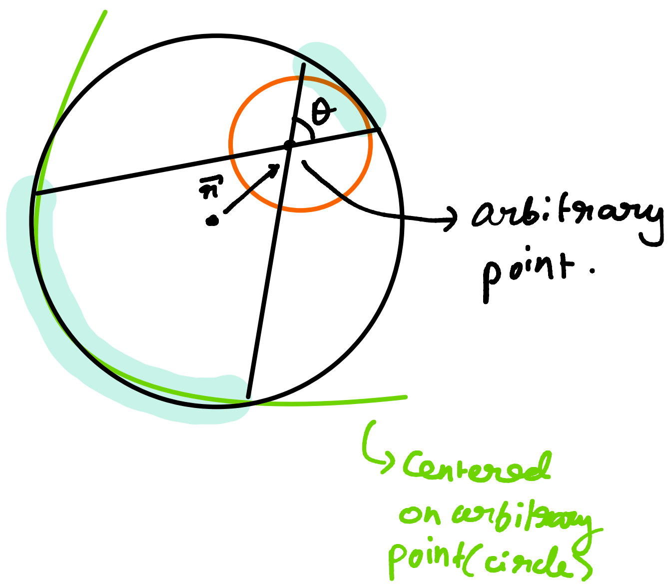
Let $\sigma(\theta, \phi) = \sigma$ be <u>surface</u> charge density

We take an infinitesimal surface element on this sphere.

$$d\Omega = a^2 \sin\theta \, d\theta \, d\phi \quad \text{(But this is from origin !)}$$

So we ignore this.

We take a solid angle instead.



→ Something about $\frac{1}{r}$ cancellation — check slides.

It does <u>NOT</u> work in 2D, we need 3D.

<u>A Steradian</u>: $\Omega = \dfrac{A}{r^2}$

A : Area of sphere centered on origin in the solid angle.

r : The radius of that sphere.

→ A sphere subtends a total solid angle of 4π

But also, surface area in vector → $\vec{A} = A\hat{n}$ ↳ Outward from surface, locally.

We return to the sphere problem.

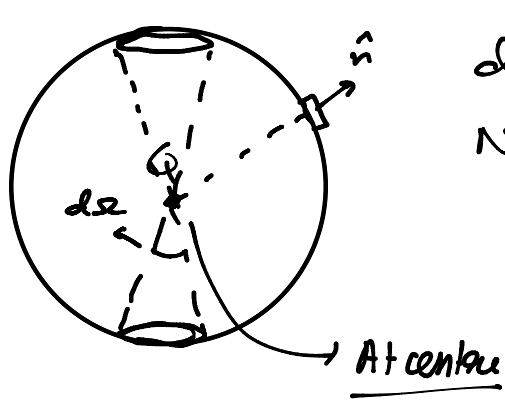
Now,

$$\Omega = \frac{A_1}{r_1^2} = \frac{A_2}{r_2^2}$$

→ The larger area is just far away enough to cancel effect of the closer, but smaller area.

We may take any such point, subtend an infinitesimal solid angle $d\Omega$, and integrate over sphere to get 0 total force (As it is zero for each $d\Omega$)

Now, let us place a charge Q, and place a sphere around it. (Reverse situation)



$d\vec{s} = da\,\hat{n}$

Now, we integrate over sphere,

$$\int \vec{E} \cdot d\vec{s} = da \cdot \hat{n} = \left(\text{We know, } \frac{Q}{\epsilon_0}, \text{ but we try to derive it} \right)$$

We subtend $d\Omega$ from origin here.

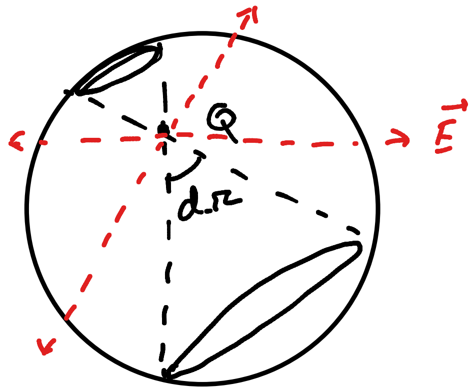
Now, $d\vec{s} = r^2 d\Omega \, \hat{r}$ (Area element on sphere)

Also,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

We see, $\displaystyle\int \vec{E} \cdot d\vec{s} = \frac{Q}{4\pi\epsilon_0} \int d\Omega$

Let us now place Q off centre at some arbitrary point in the sphere.

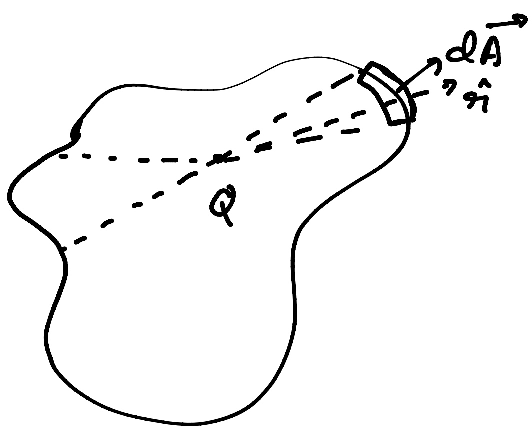


Let us draw two spheres, both centred on Q, and touching / coinciding with original sphere at the two surface elements subtended on original sphere by $d\Omega$ and drawn.

Here, $d\Omega = \frac{\vec{S}_1}{r_1^2} = \frac{\vec{S}_2}{r_2^2}$

Dividing sphere into such $d\Omega$, and integrating, we get $\frac{4\pi}{4\pi\varepsilon_0} \cdot q = \frac{Q}{\varepsilon_0}$

This generalises to Gauss' Law → we generalise to arbitrary closed surface.



If the manifold is not a sphere, $d\vec{A} \parallel \hat{r}$ may not be true

→ $\hat{r} \cdot d\vec{A} = dA \cos\theta$

Now, $\int_{\text{manifold}} \vec{E} \cdot d\vec{s} = \frac{Q}{4\pi\varepsilon_0} \int_{\text{manifold}} \frac{1}{r^2} \cos\theta \, dA$

If we take $|\vec{E}| \sin\theta$ and integrate over surface, we will get zero (what theorem ensures this? Hairy Ball?)

(The argument basically just justifies why we care about $\vec{E} \cdot d\vec{s}$ and not $|\vec{E} \times d\vec{s}|$ in such surface integrals)

☐ Gauss' Law → $\oint_{\text{surface}} \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$ → enclosed charge

(He will probably stress geometry a lot here – check Jackson)

8th January 2025 (Wednesday) →

⊛ To be Ankified

Gauss' Law →

For an electric field $\vec{E}$, enclosed by a surface $\vec{S}$ we have,

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\sum_{n=1}^{\infty} Q_n$$

↳ Sum of all charges enclosed by surface S.

Or, if the charge distribution is continuous,

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}\int_V \rho(x)\, dx^3$$

We then use the divergence theorem, $\oint_S \vec{A}\cdot d\vec{s} = \int_V \vec{\nabla}\cdot\vec{A}\cdot dV$

with the surface arbitrary (why? Generally if integrals are equal, we may not equate integrand — but if surface is arbitrary, we take lim → 0 and claim local equality of integrands)

$$\therefore \oint_S \vec{E}\cdot d\vec{s} = \int_V \vec{\nabla}\cdot\vec{E}\cdot dV = \frac{1}{\epsilon}\int_V \rho(\vec{x})\,dx^3$$

Equating, $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ → Total enclosed charge.

Differential form of Gauss' Law

This differential equation has a boundary condition cooked in — the bounding surface.
Still a vector eqn, so we try to convert to scalar.

We also know, $\vec{\nabla}\times\vec{E} = \vec{0}$

→ Mathematically, how many diff eqns are required to derive $\vec{E}$?

From Vector Analysis, we know,

$$\vec{A} = -\vec{\nabla}\cdot\phi + \vec{\nabla}\times\vec{B}$$ → Helmholtz Theorem

Where,

$$\phi = \frac{1}{4\pi}\int \frac{\vec{\nabla}\cdot\vec{A}}{|\vec{x}-\vec{x}'|}dx'^3 \ , \ \vec{B} = \frac{1}{4\pi}\int\frac{\vec{\nabla}\times\vec{A}}{|\vec{x}-\vec{x}'|}dx'^3$$

So, for $\vec{E}$ and diff form of Gauss' Law → We may show $\vec{\nabla}\times\vec{E} = \vec{0}$

☐ Do it (maybe also read Helmholtz from a book)

So whenever you require a vector field which describes some physics, we ask —

(i) Div of the field
(ii) Curl of the field. $\Big\} \rightarrow$ Use Helmholtz theorem to describe field now.

So, for no $\vec{B}$ in vaccum, we have →

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla}\times\vec{E} = 0 \longrightarrow \vec{E} = -\vec{\nabla}\cdot\phi$$

↳ Allows us to express.

Plug this in

Poisson's Equation $\Rightarrow$ $\boxed{-\nabla^2\phi = \frac{\rho}{\epsilon_0}}$ Solve on boundary Gaussian surface.

No longer a vector eqn!

We recover $\vec{E}$ by simply,

$$\vec{\nabla}\cdot\phi = \vec{E}$$

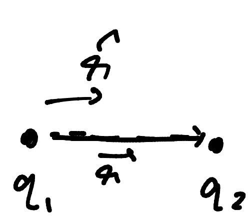
Also, $\rho = 0$ (may be the case)

$\Rightarrow$ $\boxed{\nabla^2\phi = 0}$ Laplace's equation

Alternatively, we may also do,

$$\phi = \int_V \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,dx'^3 \quad \text{(Difficult to solve)}$$

How do we now re-derive Coulomb's Law?



$$\vec{E} = \frac{q_1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}$$

In the case of point charge, $\rho(\vec{x}) = \delta(\vec{x}-\vec{x}')$ (Charge at $\vec{x}'$)

He rambles on about finding potential at the charge (incoherently)

Take away: $|\vec{E}| \propto \frac{1}{r^2}$, $\phi \propto \frac{1}{r}$

What are the boundary conditions?

Boundary: $|\vec{x}| \to \infty$, $\phi(|\vec{x}|) \to 0$ as $|\vec{x}| \to \infty$

He goes into a very incoherent discussion of unknown boundary condition — what are electric field + charge on a given boundary.

(*) Mentions that he might skip Green's functions.

(*) Tutorial tomorrow @ 8:00

9th January 2025 (Thursday) →

$\vec{E}$ is derived from a potential ϕ (as we saw, in vacuum)

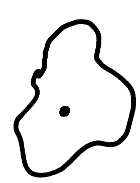
ϕ satisfies either →

① $\nabla^2 \phi = 0$

② $\nabla^2 \phi = \frac{\rho}{\epsilon_0}$

What is the role of boundary condition?

Isolated Charge :



as $r \to \infty$ we have,

$\phi \to 0$

$E \to 0$

Yet another incoherent discussion on boundaries follow. — maybe safely ignored.

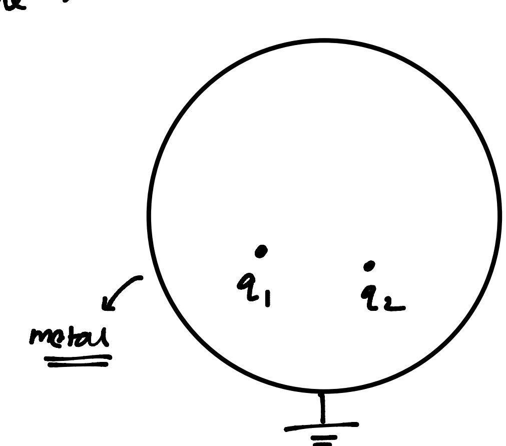
What is a metal?

for our purposes, it consists of free charges which can flow.

What does this mean for $\vec{E}$ inside metal? $\vec{E} = \vec{0}$ everywhere, as if there was a field, the charges would simply move to cancel it.

Now, grounded metal. → The ground is an infinite source or sink of charge, fixed at some potential.

Imagine →

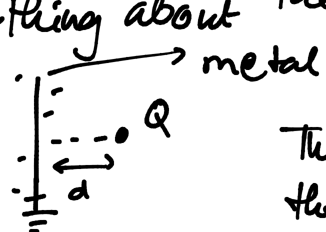


→ Here, the potential goes to zero at the metal boundary.

→ A simple, useful boundary.

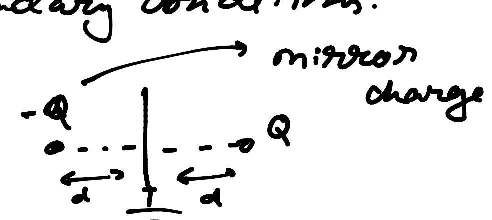
Yet more incoherence.

Something about the method of images used in boundary conditions.



There is surface charge on the metal now, due to pulling charge from the ground.

$\Rightarrow$



□ Practise mirror charge.

Something about solving $\nabla^2 \phi = \delta(x - x')$ with mirror charge is getting a Green's function.

Now, solving,

$$\nabla^2 \phi = 0 \quad \text{with boundary at infinity} \rightarrow G(x, x') = \frac{1}{|x - x'|}$$

We now solve Laplace's equation in several coordinate systems.

⊘ Cartesian → $\nabla^2 \phi = 0$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \phi_1(x)\, \phi_2(y) = 0$$

$$\Rightarrow \phi_2(y) \frac{\partial^2 \phi_1(x)}{\partial x^2} + \phi_1(x) \frac{\partial^2 \phi_2(y)}{\partial y^2} = 0$$

$$\therefore \frac{1}{\phi_1(x)} \frac{\partial^2 \phi_1(x)}{\partial x^2} = \frac{-1}{\phi_2(y)} \frac{\partial^2 \phi_2(y)}{\partial y^2}$$

$$\therefore \frac{\partial^2 \phi(x)}{\partial x^2} = a\, \phi_1(x)$$

So on,

for plane polar, spherical polar.

⊛ Revise Fourier Series before next class.