Δ **Def:** $\int dR\, f(R)\, \delta(R-R') = f(R')$

$$\delta(R-R') = 0 \text{ if } R \neq R'$$

<u>11th March 2024</u>

<u>Free particle in QM</u> →

<u>Recall:</u> $\hat{H}|k\rangle = E_K |k\rangle$

with $E_K = \dfrac{\hbar^2 k^2}{2m}$

$\hat{H}|n\rangle = E_n|n\rangle$

$\langle m|n\rangle = \delta_{mn}$

$\sum |n\rangle\langle n| = \mathbb{I}$

But for free particles,

$\psi^*\psi$ is not normalizable

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

<u>Analogy:</u> $|n\rangle$ is an abstract vector

$\vec{A}$ (random vector)

If we project this in all directions, and then recombine the set of all components (complete set of bases), we should be able to reconstruct $\vec{A}$.

$$\sum |n\rangle\langle n| = \mathbb{I} \implies \int dx\, |x\rangle\langle x| = \mathbb{I} \text{ or } 1$$

$$|\psi\rangle = \mathbb{I}\,|\psi\rangle$$

$$\Rightarrow \sum |n\rangle\langle n|\psi\rangle = \sum c_n|n\rangle$$

$\vec{n}$ is the direction of the random vector and the summation of $\sum |n\rangle\langle n|$ is over countably infinite elements (infinite dimensions) x-rep of $|k\rangle$

$$\psi_k(x) = \langle x|k\rangle = e^{-ikx}$$

$$\langle k'|k\rangle = \int dx\, e^{-i(k-k')x}$$

$$= \delta(k-k') = \delta(k,k')$$

Dirac Delta.

① $\delta(k-k') = 0$ if $k \neq k'$

② $\int dk\, f(k)\, \delta(k-k') = f(k')\ \forall f(k)$

for $f(k) = 1$

$$\Rightarrow \int_{-\infty}^{\infty} dk\, \delta(k-k') = 1 \quad \Big\}\ \text{also valid over } k+\epsilon \text{ and } k-\epsilon$$

(*) Note : It is a distribution, not a function.

As for the free particle, we do not gain anymore information from this notation, taking $(-\infty, \infty)$ as limits.

However, we can solve the free particle by claiming that the universe is not finite:

⟶ Particle in a 'very' large box.

$$\langle k|k'\rangle = \int_{-L}^{L} dx\, e^{-i(k-k')x}$$

$$= -2L\, \delta_{k,k'}$$

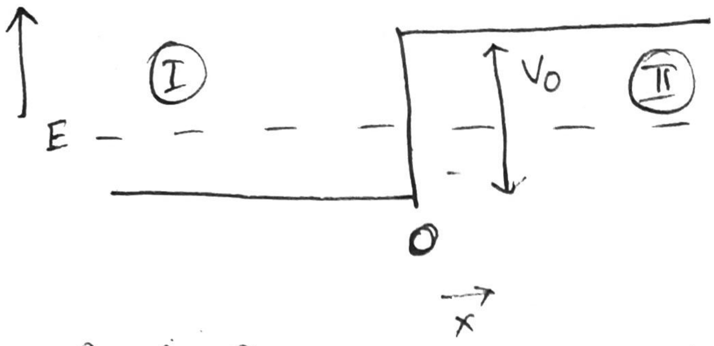
$$\boxed{\psi_k(L) = \psi_k(-L) = 0}$$

$$\therefore \langle x|k\rangle = \frac{1}{\sqrt{2L}} e^{-ikx}$$

CT2: Mar 18, 2024 (Monday)

Coming back to our discussion on Newtonian QM, we now study,

⊛ Scattering and Tunnelling phenomena →



$$V = \begin{cases} 0, & x < 0 \\ V_0, & x \geq 0 \end{cases}$$

Region I:
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi, \quad \boxed{E > 0}$$

$$\Rightarrow \psi = Ae^{iKx} + Be^{-iKx}, \quad K^2 = \frac{2mE}{\hbar^2}$$

Region II →

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi, \quad E > 0$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V_0)\psi$$

↳ $E < V_0$ Bound state problem.

Case A:
$$\frac{d^2\psi}{dx^2} + \frac{2m(E-V_0)}{\hbar^2}\psi = 0$$

$$\Rightarrow \boxed{l^2 = \frac{2m(V_0 - E)}{\hbar^2} > 0}$$

$$\psi_{II}(x) = Ce^{lx} + De^{-lx}$$

If we demand normalizability, $C = 0$

$$\psi_{II}(x) = De^{-lx}$$

$$|\psi_{II}|^2 > 0 \quad \forall x$$

This phenomena of finding the particle even in classically prohibited regions is called ~~tunn~~ tunnelling.

Ex: Spontaneous decay of atomic nucleus (radioactive)

Nucleus

Not explainable using CM,
but explained by tunnelling.
(Spontaneous and random)