22nd Jan 2024

Integration →

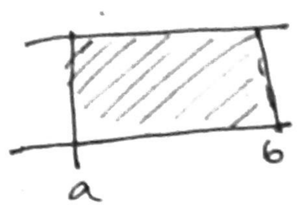
(*) Area under the curve → given by a function.

(*) Antiderivative

→ Known: Area of a square. → $f:[a,b] \to \mathbb{R}$

$f(x) = c$

$f:[a,b] \to \mathbb{R}$



Notation: area under the curve is denoted by,

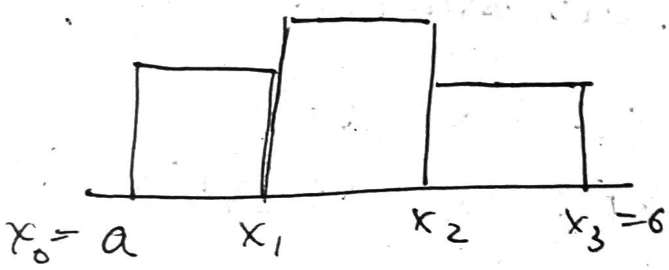
$\int_a^b f(x)dx$ / $\int_a^b f$ (dx has no meaning yet)

$$\int_a^b f = c(b-a)$$

Also, for piecewise constant,

$$a = x_0 < x_1 < x_2 < x_3 = b$$

$$f(x) = \begin{cases} c_1, & x \in [a, x_1) \\ c_2, & x \in [x_1, x_2) \\ c_3, & x \in [x_2, x_3] \end{cases}$$



$$\int_a^b f = (x_1 - x_0)c_1 + (x_2 - x_1)c_2 + (x_3 - x_2)c_3$$

A **Defn**: Let $[a,b]$ given A finite set
$P = \{x_0, x_1, \dots, x_n\}$ is called a partition of $[a,b]$ if

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

Defn: Let P and Q be two partitions of $[a,b]$. We say that Q is a refinement of P if

$$P \subseteq Q$$

Ex: $[0,1]$

$$P = \{0, \tfrac{1}{2}, \tfrac{1}{1.5}, 1\}$$

$$Q = \{0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{1.5}, 1\}$$

$$R = \{0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{1.5}, 1\}$$

$$P \subseteq Q, \quad P \subseteq R$$

But we cannot compare Q and R.

Assumption:

$f : [a,b] \to \mathbb{R}$ is bounded.

for all functions discussed in integration.

Notn: $m = \inf \{ f(x) : x \in [a,b] \}$

$M = \cancel{\max} \sup \{ f(x) : x \in [a,b] \}$.

They exist as f is bounded.

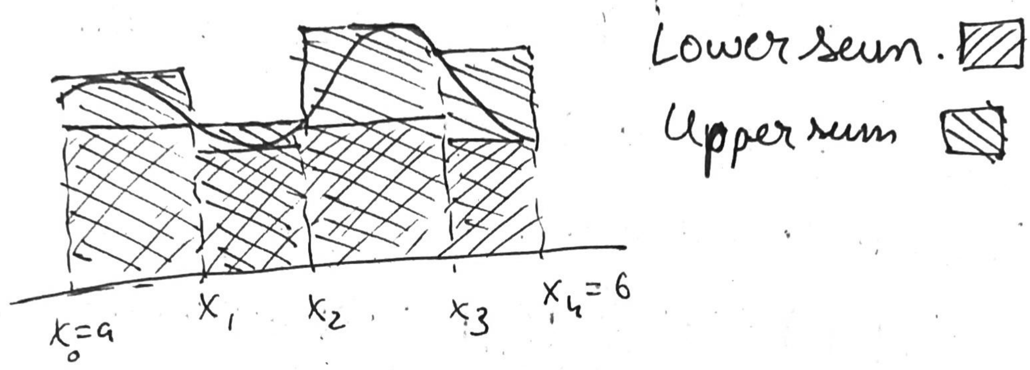
$P = \{x_0, x_1, \dots, x_n\}$, $a = x_0 < x_1 < \dots < x_n = b$

$i = 0, 1, \dots, n-1$

$$m_i(f) = \inf\{f(x) : x \in [x_i, x_{i+1})\}$$

$$M_i(f) = \sup\{f(x) : x \in [x_i, x_{i+1})\}$$

⊛ When a function is a piecewise, we know the area.



△ Lower sum: Let P be a given partition of $[a, b]$. The lower sum of f w.r.t P is denoted by,

$$L(f, P) = \sum_{i=0}^{n-1} m_i(f)(x_{i+1} - x_i)$$

△ Upper Sum: we denote the upper sum by $U(f, P)$ and defined by,

$$U(f, P) = \sum_{i=0}^{n-1} M_i(f)(x_{i+1} - x_i)$$

Now,

$$L(f, P) \le \int_a^b f \le U(f, P) \qquad \text{(Intuitively)}$$

Lemma : for any partition p of $[a,b]$,

~~$U(f,P)$~~

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$$

as, $m_i(f) \ge m$

$M_i(f) \le M$

Δ Def :

$$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$$

$L(f)$: Lower integral of f

(Due to boundedness of $L(f,P)$)

$$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

$U(f)$: ~~Lower~~ Upper integral of f

Δ Defn : A bdd fn., $f : [a,b] \to \mathbb{R}$ is said to be Darboux integrable if

$$L(f) = U(f)$$