5th Feb 2024

Recall: Simple Harmonic Oscillator (SHO)

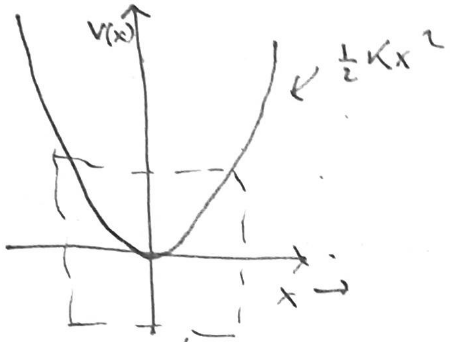
Newton's Law:

$$m\ddot{x} = -\frac{\partial V}{\partial x}, \quad V = \frac{1}{2}Kx^2$$

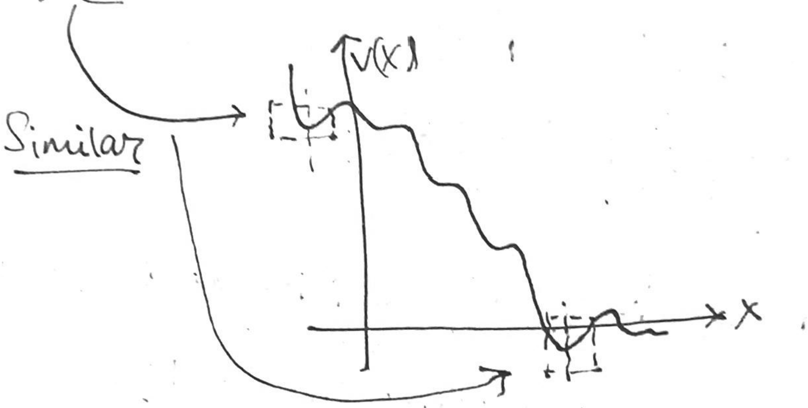
Energy or Hamiltonian of a SHO:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \quad \frac{K}{m} = \omega^2$$

(*) Why is SHO problem important physics?



All potential problems where the potential has a finite minima can be approximated as a SHO near its minima.



$$V(x) = V(x_0) + (x-x_0)V'(x_0) + \frac{1}{2!}(x-x_0)^2 V''(x_0) + \cdots$$

(Taylor exp near minima)

→ At $x = x_0$ (minima) $\Rightarrow V'(x_0) = 0$

→ $\bar{V}(x) = V(x) - V(x_0)$ define

$$= \frac{1}{2}K(x-x_0)^2$$

Define, $\bar{x} = x - x_0$

$$\Rightarrow \bar{V}(\bar{x}) = \frac{1}{2}K\bar{x}^2, \quad K = V''(x_0)$$

Quantum SHO problem →

* Time independent Schroedinger equation →

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x)$$

But we will not be solving the diff eqn in this method.

* Here we shall solve the eigen value problem using operator approach by using the CCR directly.

$$[\hat{x}, \hat{p}] = i\hbar$$

Classically,

$$H = \frac{1}{2}m\omega^2\left(x^2 + \frac{p^2}{m^2\omega^2}\right)$$

↳ Remember to always pull out coefficient of x^2

$$\Rightarrow H = \frac{1}{2}m\omega^2\left(x + \frac{ip}{m\omega}\right)\left(x - \frac{ip}{m\omega}\right)$$

$$\Rightarrow H = \frac{1}{2\hbar}m\omega\underbrace{\left(x + \frac{ip}{m\omega}\right)\left(x - \frac{ip}{m\omega}\right)}_{\text{Dimensionless}}\hbar\omega$$

↳ Dim of Energy

Dim of energy

Let us define two operators (dimensionless) —

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

$\hat{a}^{+}$ is the conjugate of $\hat{a}$.

(*) Let's compute the commutator –

$$[\hat{a}, \hat{a}^\dagger] = \left[\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right), \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\right]$$

Ex1: Show that $[c_1\hat{A}, c_2\hat{B}] = c_1c_2[\hat{A}, \hat{B}]$

where c_1 and c_2 are complex numbers (Commuting number – 'c' number)

LHS: $[c_1\hat{A}, c_2\hat{B}] = (c_1\hat{A})(c_2\hat{B}) - (c_2\hat{B})(c_1\hat{A})$

$$= c_1c_2\hat{A}\hat{B} - c_1c_2\hat{B}\hat{A}$$

$$= c_1c_2 (\hat{A}\hat{B} - \hat{B}\hat{C})$$

$$= c_1c_2[\hat{A}, \hat{B}] = RHS$$

Ex2: Show that $[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$

LHS: $[\hat{A}, \hat{B} + \hat{C}] = \hat{A}(\hat{B} + \hat{C}) - (\hat{B} + \hat{C})\hat{A}$

$$= \hat{A}\hat{B} + \hat{A}\hat{C} - \hat{B}\hat{A} - \hat{C}\hat{A}$$

$$= (\hat{A}\hat{B} - \hat{B}\hat{A}) + (\hat{A}\hat{C} - \hat{C}\hat{A})$$

$$= [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}] = RHS.$$

Using these,

$$[\hat{a}, \hat{a}^\dagger] = \left(\frac{m\omega}{2\hbar}\right)\left[\hat{x} + \frac{i\hat{p}}{m\omega}, \hat{x} - \frac{i\hat{p}}{m\omega}\right]$$

$$= \left(\frac{m\omega}{2\hbar}\right)\left([\hat{x}, \hat{x}] + \frac{i}{m\omega}[\hat{p}, \hat{x}] - \frac{i}{m\omega}[\hat{x}, \hat{p}] - \left(\frac{i}{m\omega}\right)^2[\hat{p}, \hat{p}]\right)$$

$$= \cancel{\left(\frac{m\omega}{2\hbar}\right)}$$

Ex3: Show that $[\hat{A},\hat{B}] = -[\hat{B},\hat{A}]$

$\underline{LHS}$: $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = -(\hat{B}\hat{A} - \hat{A}\hat{B})$

$= -[\hat{B},\hat{A}] = \underline{RHS}.$

Lemma: $[\hat{A},\hat{A}] = -[\hat{A},\hat{A}] = 0$

$$[\hat{a},\hat{a}^{+}] = \left(\frac{m\omega}{2\hbar}\right)\left[-\frac{2i}{m\omega}[\hat{x},\hat{p}]\right)$$

$$= -\frac{i}{\hbar}[\hat{x},\hat{p}]$$

$$= -\frac{i}{\hbar}(i\hbar) \qquad \text{(using CCR)}$$

$$= 1$$

$$\therefore \boxed{[\hat{a},\hat{a}] = 1}$$

(*) Consider the product operator $\hat{N} = \hat{a}^{+}\hat{a}$

$$\hat{N} = \hat{a}^{+}\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$= \left(\frac{m\omega}{2\hbar}\right)\left(\hat{x}^2 - \frac{i}{m\omega}\hat{p}\hat{x} + \frac{i}{m\omega}\hat{x}\hat{p} - \left(\frac{i}{m\omega}\right)^2\hat{p}^2\right)$$

$$= \frac{1}{\hbar\omega}\underbrace{\frac{1}{2}m\omega^2\left[\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right)\right.}_{\text{classical Hamiltonianesque}} + \frac{i}{m\omega}\underbrace{\left.\left(\hat{x}\hat{p} - \hat{p}\hat{x}\right)\right]}_{[\hat{x},\hat{p}] = i\hbar}$$

$$\hat{N} = \hat{a}^{+}\hat{a} = \frac{1}{\hbar\omega}\hat{H} + \frac{i}{2\hbar}[\hat{x},\hat{p}]$$

$$\Rightarrow \hat{N} = \hat{a}^{+}\hat{a} = \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}$$

The Hamiltonian operator $\hat{H} = \left(\hat{N} + \frac{1}{2}\right)\hbar\omega$

Recall : SHO

7th February 2024

Define: $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$

$$\hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

CCR : $[\hat{x}, \hat{p}] = i\hbar$

$$[\hat{a}, \hat{a}^{+}] = 1$$

$$\hat{N} \equiv \hat{a}^{+}\hat{a} = \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2} \Rightarrow \boxed{\hat{H} = \left(\hat{N} + \frac{1}{2}\right)\hbar\omega}$$

What is the physical meaning of $\hat{a}^{+}$ and $\hat{a}$?

Ex : Show that the ~~pro~~ $[\hat{A}\hat{B}, \hat{C}] = [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}]$

LHS : $[\hat{A}\hat{B}, \hat{C}] = (\hat{A}\hat{B})(\hat{C}) - (\hat{C})(\hat{A}\hat{B})$

$$= \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} + \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B}$$

$$= (\hat{A}\hat{C} - \hat{C}\hat{A})(\hat{B}) + \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B})$$

$$= [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}]$$

$= RHS.$

⊛ Compute the commutator bracket $[\hat{N}, \hat{a}]$

$$[\hat{N}, \hat{a}] = [\hat{a}^{+}\hat{a}, \hat{a}]$$

$$= [\hat{a}^{+}, \hat{a}]\hat{a} + \hat{a}^{+}[\hat{a}, \hat{a}]$$

$\cancel{= +\hat{a}}$

$$= -[\hat{a}, \hat{a}^{+}]\hat{a} + 0$$

$$= -1 \cdot \hat{a}$$

$$= -\hat{a}$$

$$\Rightarrow \boxed{[\hat{N}, \hat{a}] = -\hat{a}}$$

(*) Compute,

$$[\hat{N}, \hat{a}^{+}] = [\hat{a}^{+}\hat{a}, \hat{a}^{+}]$$
$$= \hat{a}^{+}[\hat{a}, \hat{a}^{+}] + [\hat{a}^{+}, \hat{a}^{+}]\hat{a}$$
$$= \hat{a}^{+} + 0$$
$$\Rightarrow \boxed{[\hat{N}, \hat{a}^{+}] = \hat{a}^{+}}$$

Let us consider the eigenstates of the operator $\hat{N}$.

$$\boxed{\hat{N}\Psi = n\Psi}$$

(*) We may ask, is the state, say $\phi = \hat{a}\Psi$ an eigenstate of the operator $\hat{N}$

$$\hat{N}\phi = \hat{a}^{+}\hat{a}(\hat{a}\Psi) \Rightarrow \hat{N}\phi = (\hat{N}\hat{a})\Psi$$
$$\Rightarrow \hat{N}\phi = (\hat{N}\hat{a} - \hat{a}\hat{N} + \hat{a}\hat{N})\Psi$$
$$\Rightarrow \hat{N}\phi = ([\hat{N}, \hat{a}] + \hat{a}\hat{N})\Psi$$
$$\Rightarrow \hat{N}\phi = (-\hat{a} + \hat{a}\hat{N})\Psi$$
$$\Rightarrow \hat{N}\phi = -\hat{a}\Psi + \hat{a}(\hat{N}\Psi)$$
$$\Rightarrow \hat{N}\phi = -\phi + n(\hat{a}\phi)$$
$$\Rightarrow \boxed{\hat{N}\phi = (n-1)\phi}$$

(*) Role of $\hat{a}$ is to reduce eigenvalue by 1.

$\Rightarrow$ $\phi = \hat{a}\Psi$ is also an eigenstate of $\hat{N}$ but with eigenvalue reduced exactly by 1.

$\hat{a} \longrightarrow$ Lowering operator. or annihilation operator.

(*) Compute same thing for $\chi = \hat{a}^{+}\psi$

$$\hat{N}\chi = \hat{N}(\hat{a}^{+}\psi) = (\hat{N}\hat{a}^{+})\psi$$
$$= (\hat{N}\hat{a}^{+} - \hat{a}^{+}\hat{N} + \hat{a}^{+}\hat{N})\psi$$
$$= ([\hat{N},\hat{a}^{+}] + \hat{a}^{+}\hat{N})\psi$$
$$= (\hat{a}^{+} + \hat{a}^{+}\hat{N})\psi$$
$$= \hat{a}\psi + \hat{a}^{+}(\hat{N}\psi)$$
$$= \chi + \hat{a}(n\psi)$$
$$= \chi + n\chi$$
$$= (n+1)\chi$$
$$\Rightarrow \boxed{\hat{N}\chi = (n+1)\chi}$$

→ χ is also an eigenstate but the eigenvalue is raised exactly by 1.

$\hat{a}^{+}$ → Raising operating operator or creation operator.

(*) What about the states like

$$(\hat{a}\hat{a}\psi) \text{ or } (\hat{a}^{+}\hat{a}^{+}\psi)\ ?$$

$$\hat{N}(\hat{a}\hat{a}\psi) = (n-2)(\hat{a}\hat{a}\psi)$$
$$\hat{N}(\hat{a}^{+}\hat{a}^{+}\psi) = \cancel{(N}\,(n+2)(\hat{a}^{+}\hat{a}^{+}\psi)$$

Inconvenient notation

Recall: Inner (Dot) product between two wave-functions $\psi(x)$ and $\phi(x)$:

$$(\phi,\psi) = \langle\phi|\psi\rangle = \int_{L_1}^{L_2} dx\, \phi^*(x)\,\psi(x)$$

$$\begin{array}{ccc} \langle \phi | & \cdot & | \psi \rangle \\ \downarrow & \downarrow & \downarrow \\ \langle \text{Bra} | & c | & \text{Ket} \rangle \end{array}$$

(???)

(Dirac notation)

wave function $\psi \rightarrow |\psi\rangle$ is called Ket vector

Conjugate (Dual) wave function,

$$\phi^* \rightarrow \langle \phi |$$

is called a Bra vector

Eg: Eigenvalue ?

$$\hat{O}\psi = \lambda \psi \Rightarrow \hat{O}|\psi\rangle = \lambda |\psi\rangle$$

Use the eigenvalue to denote the state $|\psi\rangle$

$$\hat{O}|\lambda\rangle = \lambda |\lambda\rangle$$

2 states with λ_1 and λ_2 eigenvalues,

$$\boxed{\begin{aligned} \hat{O}|\lambda_1\rangle &= \lambda_1 |\lambda_1\rangle \\ \hat{O}|\lambda_2\rangle &= \lambda_2 |\lambda_2\rangle \end{aligned}}$$

$$\hat{N}\psi = n\psi \Rightarrow \hat{N}|n\rangle = n|n\rangle$$

$$\therefore \hat{N}|n-1\rangle = (n-1)|n-1\rangle$$