10th January 2024

~~Ab~~ $\boxed{\,}$ Leibnitz rule →

Let $f, g : J \to \mathbb{R}$ be smooth. Then,

$$(fg)' = f'g + fg'$$

$$(fg)'' = f''g + 2f'g' + fg''$$

$$\vdots$$

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

$\square$ Prove using induction.

<u>Ex</u>: $f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

Then, f is a smooth function.

~~Ex~~ <u>Qn</u>: Construct a smooth function s.t $f(x) = 0$ $\forall x \in [a,b]^c$, where a, b are given.

(*) <u>Notation</u>: Let $J \subseteq \mathbb{R}$ be an interval

$$C(J) = \{ f : J \to \mathbb{R} : f \text{ is contd} \}$$

$$C^1(J) = \{ f : J \to \mathbb{R} : f' \in C(J) \}$$

$$C^2(J) = \{ f : J \to \mathbb{R} : f'' \in C(J) \}$$

$$\vdots$$

→ $C^k(J) \subseteq C^{k-1}(J) \subseteq \dots \subseteq C(J)$

(*) $C^\infty(J) = \{ f : J \to \mathbb{R} : f \in C^k(J) \; \forall \, k \geq 0 \}$

$\therefore$ $f, g \in C^\infty \Rightarrow fg \in C^\infty$

Back to question,

Soln: Construct $f_1 : \mathbb{R} \to \mathbb{R}$ s.t

$f_1 \in C^\infty(\mathbb{R})$ and $f_1(x) = 0 \ \forall \ x \in (-\infty, a)$

$$f_1 = \begin{cases} e^{-\frac{1}{x-a}} & , x > a \\ 0 & , x \leq a \end{cases}$$

Construct $f_2 : \mathbb{R} \to \mathbb{R}$ s.t

$f_2 \in C^\infty(\mathbb{R})$ and $f_2(x) = 0 \ \forall \ x \in (b, \infty)$

$$f_2 = \begin{cases} e^{\frac{1}{x-b}} & , x < b \\ 0 & , x \geq b \end{cases}$$

Then, take

$$f(x) = f_1(x) f_2(x)$$

$\Rightarrow f(x) \in C^\infty(\mathbb{R})$

$\Rightarrow f(x) = 0 \ \forall \ x \in [a, b]^c$ (All limit points)

Δ Support of a function — → Closure

$$\text{Supp}(f) =: \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$$

~~= largest closed set where f is non-zero~~ ← Not accurate

⊛ Bump function → $f \in C^\infty(J)$ and Supp(f) is compact.

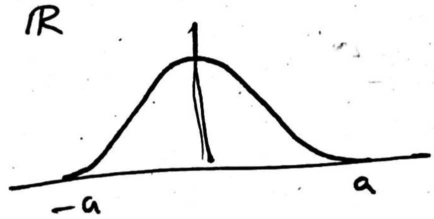
Ex: (Bump)

$$f(x) = \begin{cases} c e^{+\frac{1}{x^2 - a^2}} & , |x| < a \\ 0 & , |x| \geq a \end{cases}$$

Then, $f \in C^\infty(\mathbb{R})$, $f = 0 \ \forall x, |x| \geq a$

and c is such that

$$\int_{\mathbb{R}} f(x) = 1$$



Can we approximate the Dirac Delta with a sequence of functions?

we take the ~~taf~~ last function - $f(x) = \begin{cases} c e^{\frac{1}{x^2-1}}, & |x|<1 \\ 0, & |x| \geq 1 \end{cases}$

For $n \in \mathbb{N}$

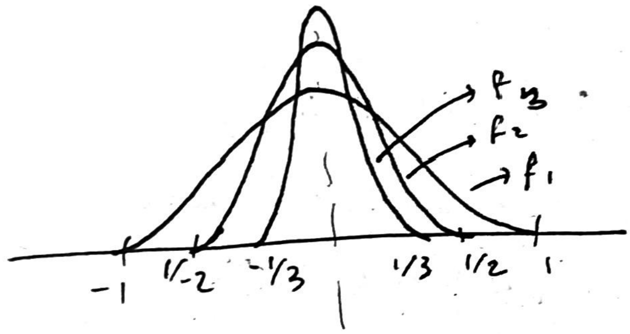
$$f_n(x) := n f(nx)$$

$$\int_{\mathbb{R}} f_n(x)\,dx = \int_{-\infty}^{\infty} n f(nx)\,dx$$

$$= \int_{-\infty}^{\infty} f(y)\,dy = 1$$

$\rightarrow \text{Supp}(f_n) = \left[\frac{-1}{n}, \frac{1}{n}\right]$ → That is why closure is req, so that by cantor intersection it converges to $\{0\}$

$\rightarrow \int f_n = 1$



$f_n(x) = 0 \quad \forall n \geq k$

In some sense, this converges to Dirac Delta.

$\boxed{\neg}$ Taylor's theorem →

Let $f: J \rightarrow \mathbb{R}$ be smooth. Let ~~x~~ $x_0 \in J$

Then, $\exists$ c lies between x and x_0 s.t.

$$f(x) = f(x_0) + (x-x_0)\, f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \underbrace{\frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)}_{\text{Error}}$$

<u>Ex</u>: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. Such that,

$$f^{(k)}(x) = 0 \quad \forall x \in \mathbb{R}$$

for some $k \in \mathbb{N}$. Then f is a polynomial of degree $(k-1)$

Proof : Define

$F: J \to \mathbb{R}$ by

$$F(t) = f(t) + \sum_{k=1}^{n} \frac{(x-t)^k}{k!} f^{(k)}(t) + M(x-t)^{n+1}$$

We choose M s.t

$$F(x) = F(x_0) \quad —— (1)$$

Note, $F(x) = f(x)$

Since F satisfies the hypothesis of Rolle's theorem

$\exists$ c lies b/w x and x_0 s.t

$$F'(c) = 0 \quad —— (2)$$

From (1) and (2), we get the desired result.

$\square$ Calculate.

⊗ Next two-three classes are problem solving.

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be a diff'ble function.

Let $c \in \mathbb{R}$ and $x_n < c < y_n$ s.t.

$x_n - y_n \to 0$. Then show that $\lim_{n\to\infty}$

$$\lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(c)$$

Can be made difficult by saying diff'ble at c:

$\square$ Solve both