6^{th} January 2025 (Monday) →

(*) Ankified

Probability theory →

A common example of a random event is, of course, a coin toss.

We know,

$$S = \{H, T\}$$

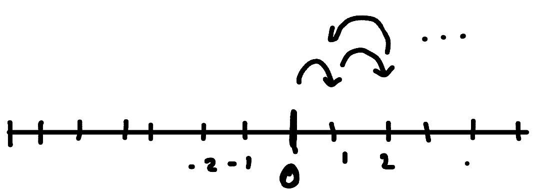
$$P_H = \frac{1}{2} = P_T \quad \text{(Equally likely)}$$

If we toss N times, we get a sequence,

H T H H T T T T H H . . .

$$\Rightarrow \langle \# \text{ of } H \rangle \approx \langle \# \text{ of } T \rangle \quad (\text{as } N \to \infty)$$

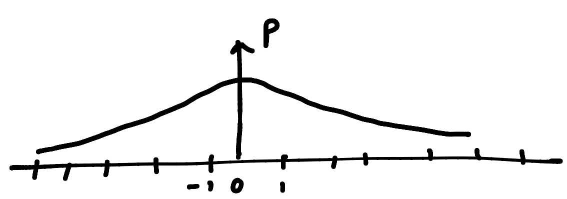
Let us now imagine a lattice.



If we get a heads, we go +1 unit

If we get tails, we go −1 unit.

The probability of existence at some lattice point looks like this →



for 2D (Lattice), and 1D Lattice, the probability of coming back to origin in finite non-zero value.

For 3D Lattice, it is zero. (Polya proved this)

Δ Random variable: An experiment outcome that is stochastic. (Random)

Δ Sample Space: Set of all possible outcomes

Ex: {H, T}, {1, 2, . . ., 6} (Roll of dice), may even be continuous.

Δ Event: Subset of sample space

Ex: {2, 4, 6} (Getting even number on roll of dice)

(*) Note: Here, all members of the sample space is equally likely.

⊟ Theorem: For an event E, the probability P(E) assigned to it is,

(I) $P(E) \geq 0 \quad , \forall E \subseteq \Omega$ (Sample Space)

(II) $P(\Omega) = 1$

(III) $E \subset F \subset \Omega \Rightarrow P(E) \leq P(F)$

(IV) $A \cap B = \phi \Rightarrow P(A \cup B) = P(A) + P(B)$

(V) $P(\bar{A}) = 1 - P(A)$ ($\bar{A}$ is complement of A)

There are two methods to determine the values of probability.

(1) All n events are equally likely $\Rightarrow$ p(one member of Ω) $= \frac{1}{n}$

(2) Empirical: We carry out experiment a lot of times.

$$\therefore P(E) = \lim_{N \to \infty} \frac{\#\text{ of E event occuring}}{N}$$

Δ CDF (Cumulative distribution function)

Gives us probability that a random variable takes a value $< x$, $x \in (-\infty, \infty)$

i.e.

$$P(x) = \text{prob}\left[E \subset (-\infty, x)\right]$$

$$P(-\infty) = 0$$

$$P(+\infty) = 1$$

(*) $P(x)$ is monotonically increasing.

We also define,

$$p(x) = \frac{dP(x)}{dx}$$

Why? Note that,

$$p(x)dx = dP(x) = \text{prob}\left[E \subset (x, x+dx)\right]$$

Thus,

$$\text{prob}(\Omega) = \int_{-\infty}^{\infty} p(x)dx = 1$$

- Dimensions: $[p(x)] = [x]^{-1}$
- $0 < p(x) < \infty$ (Unlike $P(x)$, which is bounded above by 1)
- How do we calculate average value of a function of the random variable x?

$$\langle F(x) \rangle = \int_{-\infty}^{\infty} F(x)\, p(x)\, dx$$

(VI) Change of variable:

We have a random variable x, and pdf $p(x)$

We transform,

$$\left.\begin{aligned} y &= y(x) = f(x) \\ x &= x(y) = \underbrace{f^{-1}(y)}_{g(y)\ (\text{say})} \end{aligned}\right\} f \text{ is invertible}$$

Now, what is the value of

$\tilde{p}(y) = ?$ (PDF is also transformed)

Let us say, A is an event, and $x \in A$

B is an event, and $y = y(x) \in B$

let x, y are continuous variables

Then,

$$\int_{y \in B} \tilde{p}(y)\,dy = \int_{x \in A} p(x)\,dx = \int_{y \in B} p(g(y)) \left| \frac{dg(y)}{dy} \right|$$

(*) Note, $f^{-1} = g : Y \to X$

Comparing,

$$\tilde{p}(y) = p(g(y)) \left| \frac{dg(y)}{dy} \right|$$

(Note: This is a Jacobian)

Example: $p(x) = 3x^2, \quad 0 < x < 1$

Now,

$y(x) = f(x) = x^2, \quad 0 < y < 1$

what is,

$\tilde{p}(y) = ?$

We use the formula derived above,

$x = g(y) = \sqrt{y}$

$\Rightarrow \frac{dg(y)}{dy} = \frac{1}{2} y^{-1/2}$

$\therefore \tilde{p}(y) = 3(\sqrt{y})^2 \cdot \frac{1}{2} y^{-1/2}$

$= \frac{3}{2}\sqrt{y}$

Another method $\to$

$\tilde{P}(y) = \text{prob}(B \le y) = \text{prob}(x^2 \le y) = \text{prob}[x \le \sqrt{y}]$

$= \text{prob}(A \le \sqrt{y})$

$= P(\sqrt{y})$

Now,

$$\int_0^{\sqrt{y}} 3t^2\,dt = y^{3/2}$$ (Simply computing CDF)

And,

$\tilde{p}(y) = \frac{d\tilde{P}(x)}{dx}$ (Definition of the pdf)

$= \frac{3}{2}\sqrt{y}.$

(*) Practise

8th January 2025 (Wednesday) →

⊛ To be Ankified

Moment:

$$M_n = \langle x^n \rangle = \int x^n \, p(x) dx \quad (n^{th} \text{ moment})$$

⊛ It is often much easier to use a MGF (Moment Generating Function). Also called characteristic function.

This minus is purely convention, plus is also allowed.

CF: $$\tilde{p}(k) = \int dx \; p(x) \, e^{-ikx}$$

Thus, we fourier transform the PDF.

Note, thus, as per definition of moment,

$$\tilde{p}(k) = \langle e^{-ikx} \rangle$$

Of course, we can find $p(x)$ again by a simple inverse FT.

Now, $$\langle e^{-ikx} \rangle = \left\langle \sum_{n=0}^{\infty} \frac{(-ik)^n x^n}{n!} \right\rangle$$

Since x is only R.V. here, we may draw the rest out.

$$\Rightarrow \tilde{p}(k) = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle$$

→ n^{th} moment!

So to recover $\langle x^n \rangle$,

$$\left. \frac{\partial^n \tilde{p}(k)}{\partial k^n} \right|_{k=0} = (-i)^n \langle x^n \rangle$$

So, we only need to →

(I) FT the PDF $(x \to k)$, we call this MGF

(II) $\frac{\partial^n}{\partial k^n}$ the MGF

(III) Evaluate at $k=0$, divide by $(-i)^n$ to get $\langle x^n \rangle$

This method does not gives us variance, Kurtosis, etc – cumulants

To get them, we use cumulative generating function. This is defined as,

$$\ln(\tilde{p}(k)) = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle_c$$

→ Note that this is $\neq \langle x^n \rangle$, it is something else.

(we assume this) → Again, note that this is a definition

⊛ We use a FT to ensure convergence of the MGF integral.

⊛ Cumulants are defined using $\ln(\tilde{p}(k))$ properties.

Again, $$\tilde{p}(k) = 1 + \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle = 1 + \epsilon$$

We know, $\ln(1+\epsilon) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\epsilon^m}{m}$

Using this here,

$$\ln(\tilde{p}(k)) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m} \left[\sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle \right]^m$$

How is this useful? We equate the powers of k with the defining power series of cumulative MGF.

So, $$\ln(\tilde{p}(k)) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m} \left[\left(\frac{-ik}{1!}\right) \langle x \rangle + \left(\frac{-ik}{2!}\right)^2 \langle x^2 \rangle + \cdots \right]^m$$

$$= (1)\left(\frac{1}{1}\right) \left[\left(\frac{-ik}{1!}\right) \langle x \rangle + \left(\frac{-ik}{2!}\right)^2 \langle x^2 \rangle + \cdots \right]^1$$

$$+ (-1)\left(\frac{1}{2}\right) \left[\left(\frac{-ik}{1!}\right) \langle x \rangle + \left(\frac{-ik}{2!}\right)^2 \langle x^2 \rangle + \cdots \right] \left[\left(\frac{-ik}{1!}\right) \langle x \rangle + \left(\frac{-ik}{2!}\right)^2 \langle x^2 \rangle + \cdots \right]$$

So, k^1 occurs only for $n=1, m=1$, k^2 for $n=1, m=2$ <u>or</u> $n=2, m=1$

$\therefore \langle x \rangle_c = \langle x \rangle$

$\therefore \langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2$

$\vdots$

<u>And so on</u> to get maps from moments to cumulants.

(*) Also,

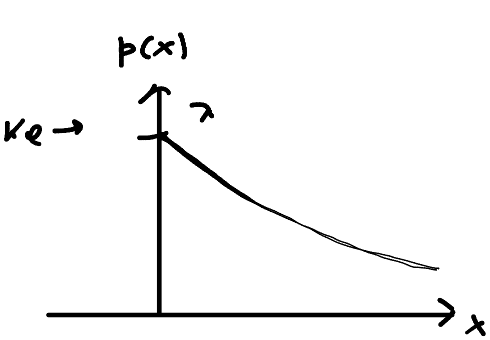
$$\boxed{\left.\frac{\partial^n \ln \tilde{p}(k)}{\partial k^n}\right|_{k=0} = (-i)^n \langle x^n \rangle_c}$$

Defn. of cumulant (explicit)

Now, we deal with some probability distributions (Prof: "Anything beyond this? Exam")

(I) Exponential →

PDF: $p(x) = \lambda e^{-\lambda x}$, $x \in [0, \infty)$, Looks like →



$$\Rightarrow \text{CDF: } P(x) = \int_0^x \lambda e^{-\lambda y} dy = 1 - e^{-\lambda x}$$

Now, we compute some moments of this.

$$\text{MGF: } \tilde{p}(k) = \int_0^{\infty} dx\, \lambda e^{-\lambda x} e^{-ikx} = \int_0^{\infty} dx\, \lambda e^{-(\lambda+ik)x}$$

$$\Rightarrow \tilde{p}(k) = \lambda \left[\frac{e^{-(\lambda+ik)x}}{-(\lambda+ik)} \right]_0^{\infty}$$

$$\Rightarrow \tilde{p}(k) = \lambda\left[0 + \frac{1}{(\lambda+ik)}\right]$$

$$\Rightarrow \boxed{\tilde{p}(k) = \frac{\lambda}{\lambda+ik}}$$

We use it to find moments now,

$$(-i)\langle x\rangle = \left.\frac{\partial \tilde{p}(k)}{\partial k}\right|_{k=0} = \lambda(-1)\cdot\frac{1}{(\lambda+ik)^2}\cdot i\Big|_{k=0} = \frac{-i\lambda}{\lambda^2} = \frac{-i}{\lambda}$$

$$\Rightarrow \boxed{\langle x\rangle = \frac{1}{\lambda}}$$

Similarly,

$$\langle x^2\rangle = \left.\frac{\partial^2 \tilde{p}(k)}{\partial k^2}\right|_{k=0} = (-\lambda)(-2)\cdot\frac{1}{(\lambda+ik)^3}(i)\Big|_{k=0} = \frac{-2i}{\lambda^2}$$

$\Rightarrow \langle x^2\rangle = \frac{2}{\lambda^2}$ (Note the pattern, $\langle x^n\rangle = \frac{n}{\lambda^n}$)

We may also find cumulants,

$$\langle x\rangle_c = \langle x\rangle = \text{mean}$$

$$\langle x^2\rangle_c = \langle x^2\rangle - \langle x\rangle^2 = \text{variance}$$

OR we can use the $\ln\tilde{p}(k)$ approach too.

Bernoulli Trial → Any stochastic process which has only two outcomes.

$\Rightarrow$ 'success' rate $= p$

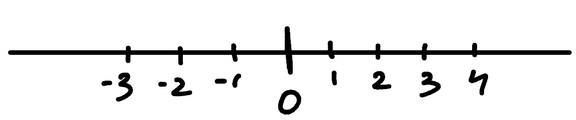
'failure' rate $= 1-p = q$

Now, in an unbiased coin, $p = q = \frac{1}{2}$ (Special case of Bernoulli trial)

We imagine a random walk on a 1D Lattice.

Total no. of steps in walk → $N = N_p + N_q$

N_p → Right-step (Say), N_q → Left-step



Using some combinatorics,

$$\text{prob}_N[N_p] = \frac{N!}{N_p!\,(N-N_p)!}\,p^{N_p}\,q^{N-N_p}$$

So how far from the origin has the random walker gone?

Note, we may also represent this distribution like this $\rightarrow$

$$\text{Prob}_N(m) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!} p^{\frac{N+m}{2}} (1-p)^{\frac{N-m}{2}}$$

$\text{Prob}_N(m) \downarrow$ Net disp from origin

as, $2N_p = N+m$

If we want to compute an average,

$$\langle m \rangle = 2\langle N_p \rangle - N$$

$$= 2Np - N \quad \text{(Why? Two right steps} \rightarrow p\cdot p,\ 3 \rightarrow p\cdot p\cdot p, \ldots)$$

$$= N(p-q)$$

As expected: for equally probable right and left steps, we are on average still at origin.

☐ Exercise: Compute variance, i.e. $\langle m^2 \rangle_c$

9th January 2025 (Thursday) $\rightarrow$

Binomial Expansion $\rightarrow$ $p+q=1$

$$(p+q)^N = \sum_{N_p=0}^{N} P_N(N_p)$$

$$\rightarrow P_N(N_p) = \frac{N!}{N_p!(N-N_p)!} p^{N_p} (1-p)^{N_p} \quad \text{(Binomial expansion)}$$

Characteristic function $\rightarrow$

$$\tilde{p}_N(k) = \int dx\, p(x) e^{-ikx}$$

discrete sum $\curvearrowright$
$$= \sum_{N_p=0}^{N} \frac{N!}{N_p!(N-N_p)!} p^{N_p} q^{N-N_p} e^{-ikN_p}$$

$$= \sum_{N_p=0}^{N} \frac{N!}{N_p!(N-N_p)!} (pe^{-ik})^{N_p} q^{N-N_p}$$

$$\equiv (pe^{-ik}+q)^N \quad \rightarrow \text{using Binomial expansion.}$$

Cumulant generating function $\rightarrow$

$$\ln \tilde{p}(k) = N \ln\left(pe^{-ik}+q\right) \equiv \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle_c$$

We know, $(-i)^n \langle x^n \rangle_c = \left.\frac{\partial^n \ln \tilde{p}(k)}{\partial k^n}\right|_{k=0}$

We know,

$$\langle N_p \rangle_c = \langle N_p \rangle_c$$

So,

$$(-i)^2 \langle N_p^2 \rangle_c = \frac{\partial^2}{\partial k^2}\left[N \ln\left(p e^{-ik} + q\right)\right]\Big|_{k=0}$$

$$= \frac{\partial}{\partial k}\left[N \cdot \frac{1}{pe^{-ik}+q} \cdot \left(-ipe^{-ik}\right)\right]\Big|_{k=0}$$

$$= N\left[(-1) \cdot \frac{1}{(pe^{-ik}+q)^2} \cdot \left(-ipe^{-ik}\right)\left(-ipe^{-ik}\right) + \frac{1}{pe^{-ik}+q} \cdot (-ip)(-i)e^{-ik}\right]\Big|_{k=0}$$

$$= N\left[(-1) \cdot \frac{1}{(p+q)^2} \cdot (-ip)(-ip) + \frac{1}{p+q}(-ip)(-i)\right]$$

$$= N\left[\frac{p^2}{(p+q)^2} - \frac{p}{(p+q)}\right]$$

$$= Np\,[p-1]$$

$$= Npq$$

So, we see that,

$$\boxed{\begin{matrix} \mu \sim N \\ \sigma \sim \sqrt{N} \end{matrix}}$$ for the random walk.

we calculated last lecture,

$$\langle m \rangle = N(p-q)$$

We may calculate,

$$\sigma_m^2 = 4Npq$$ HW: Derive this

So for $p=q$, the mean remains const with N, but the distribution spreads with N.

Consider a 2 state system.

1 2

Labelled states.

We may thus define,

$P_1(t) \rightarrow$ Prob of sys in state 1 at time t

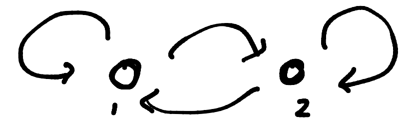
$P_2(t) \rightarrow$ " " " 2 " " "

Since there exist only 2 states,

$$P_1(t) + P_2(t) = 1 \quad \forall t$$

Now, $P_1(t+\Delta t) = ?$

Will it choose to stay, or hop to other state?



$\therefore P_1(t+\Delta t) = P_1(t) \cdot$ prob of staying in 1

$+ P_2(t) \cdot$ prob of jumping '2 → 1'

$= P_1(t)\,[1 - \omega(1\to 2)\,dt] + P_2(t)\,[w(2\to 1)]\,dt$

ω is like a 'rate' of transition.

(*) Note: We only need to know probabilities at t to compute them at $t+\Delta t$.

The process is Markovian or memoryless.

Now,

$$\frac{P_1(t+\Delta t) - P_1(t)}{\Delta t} = -\omega(1\to 2)\,P_1(t) + \omega(2\to 1)\,P_2(t)$$

$$= \frac{dP_1}{dt}$$

$$\Rightarrow \frac{dP_1}{dt} = -\omega(1\to 2)\,P_1(t) + \omega(2\to 1)\,P_2(t)$$

and similarly,

$$\frac{dP_2}{dt} = -\omega(2\to 1)\,P_2(t) + \omega(1\to 2)\,P_1(t)$$

Generally,

$$\frac{dP_i}{dt} = \sum_{j\neq i} \left[-\omega(i\to j)\,P_i(t) + \omega(j\to i)\,P_j(t)\right]$$

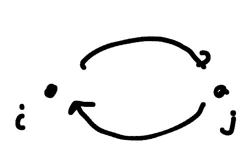
→ Master equation.

When we study steady state, we set $\frac{dP_i}{dt} = 0$, i.e. no more dynamics.

How does it go to zero?

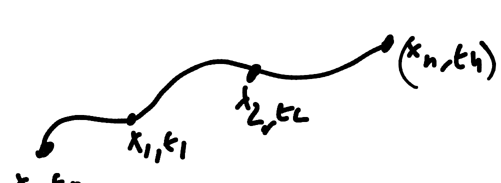
We may claim that the entire sum goes to zero term by term

$\therefore \omega(i\to j)\,P_i = \omega(j\to i)\,P_j$ → Detailed balance



This is an equilibrium system.

If we claim that entire sum goes to zero, not term by term, we get a non-equilibrium steady state → They may be currents/transitions but they cancel on whole.



System traversing coordinate space.

→ $\{x_1(t), x_2(t), \ldots, x_n(t)\}$

Set of R.V. describing path.

We make a Markovian assumption,

$$p(x_n, t_n | x_{n-1}, t_{n-1}, \dots, x_2, t_2, x_1, t_1) = p(x_n, t_n | x_{n-1}, t_{n-1})$$

So now, we set some intermediate state x', and compute,

$$p(x,t|x_0,t_0) = \int p(x,t|x',t')\, p(x',t'|x_0,t_0)\, dx'$$

Chapman-Kolmogorov equation.

☐ We can reduce the Chapman-Kolmogorov to the master equation.