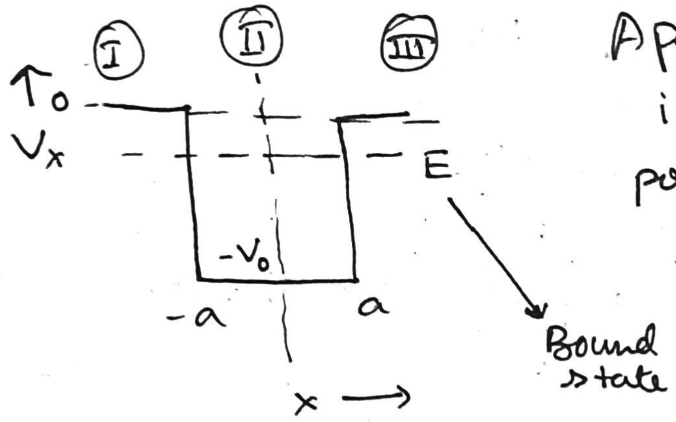
29th January 2024

Finite Square well →



A particle of mass m is moving in a potential $V(x)$ as

$$V(x) = \begin{cases} -V_0, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

(I) : $x < -a$: $V(x) = 0$

Time-independent Schrodinger equation →

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi$$

$E < 0$ (bound state)

$$\Rightarrow \frac{d^2\psi}{dx^2} - \kappa^2\psi = 0 \quad , \quad \kappa^2 = \frac{-2mE}{\hbar^2}$$

General solution : (Ansatz, $\psi \sim e^{\kappa x}$)

$$\psi_I(x) = A_1 e^{\kappa x} + A_2 e^{-\kappa x}$$

Similarly for region (III) : $x > a$:

General solution :

$$\psi_{III} = C_1 e^{\kappa x} + C_2 e^{-\kappa x}$$

for region (II) → $-a < x < a$:

$$V(x) = -V_0$$

Time independent Schrodinger eqn →

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + \frac{2m(E+V_0)}{\hbar^2}\psi = 0$$

$E + V_0 \rightarrow +ve$

$\therefore$ Say $l^2 = \frac{2m(E+V_0)}{\hbar^2} > 0$

$\therefore \frac{d^2\psi}{dx^2} + l^2\psi = 0$ → 2nd order ODE with const coefficient.

~~As~~ Ansatz: $\psi \sim e^{rx}$

Aux: $r^2 = -l^2 \rightarrow r = \pm il$

General Solution:

$$\psi_{II} = B_1 e^{ilx} + B_2 e^{-ilx}$$

⇒ We have 6 unknown constants – $A_1, A_2, B_1, B_2, c_1, c_2$

(*) In QM: we require the wave-function to be square integrable. i.e. $\int \psi^*\psi\, dx < \infty$

$$\Rightarrow \int_{-\infty}^{-a} \psi_I^*\psi_I\, dx + \int_{-a}^{a} \psi_{II}^*\psi_{II}\, dx + \int_{-a}^{\infty} \psi_{III}^*\psi_{III}\, dx < \infty$$

Given $\psi^*\psi > 0$ for each, ⇒ Each term is individually finite.

(III): $\int_a^\infty \psi_{III}^*\psi_{III}\, dx = \cancel{\int_a^\infty [c_1 e^{Rx} + c_2 e^{-Rx}]^2 dx}$

$$\cancel{= \int_a^\infty [c_1^2 e^{2Rx} + c_2 e^{-2Rx} + 2c_1c_2]\, dx}$$

$$= \int_a^\infty [c_1^* e^{Rx} + c_2^* e^{-Rx}][c_1 e^{Rx} + c_2 e^{-Rx}]\, dx$$

$$= |c_2|^2 \frac{e^{-2\mathcal{R}a}}{2\mathcal{R}} + \left[\frac{|c_1|^2 e^{2\mathcal{R}x}}{2\mathcal{R}}\right]_0^\infty \longrightarrow \text{Must be zero}$$

$$+ (c_1 c_2^* + c_2 c_1^*) x \Big|_0^\infty$$

We choose $c_1 = 0 \Rightarrow$ Physically allowed solution,

$$\psi_{III} = c_2 e^{-\mathcal{R}x} , \quad \psi_I = A_1 e^{\mathcal{R}x}$$

$\boxed{A5/Q1}$: Show that in the region (I), the square integrability of a wave function implies

$$\psi_I = A_1 e^{\mathcal{R}x}$$

Continuity of $\psi(x)$: (Must match at (I)-(II)-(III) boundaries)

(i) $x = -a$:

$$\lim_{h \to 0} \left[\psi_I(-a-h) = \psi_{II}(-a+h)\right]$$

$$\Rightarrow A_1 e^{-\mathcal{R}a} = B_1 e^{-iba} + B_2 e^{iLa} \quad \text{—(I)}$$

(ii) $x = a$:

$$\lim_{h \to 0} \psi_{II}(a-h) = \lim_{h \to 0} \psi_{III}(a+h)$$

$$\Rightarrow B_1 e^{+iLa} + B_2 e^{-iLa} = c_2 e^{-\mathcal{R}a} \quad \text{—(II)}$$

Continuity of $\psi'(x)$:

(i) $x = -a$:

$$-A\mathcal{R} e^{-\mathcal{R}a} = -iLB_1 e^{-iLa} + iLB_2 e^{iLa}$$

$$\Rightarrow A\mathcal{R} e^{-\mathcal{R}a} = iL\left[B_1 e^{+iLa} \mp B_2 e^{-iLa}\right]$$

—(III)

Similarly for

$x = a$:

$$iL\left(B_1 e^{ila} - B_2 e^{-ila}\right) = -\mathcal{R}\, C_2 e^{-\mathcal{R}a} \quad \text{— (IV)}$$

(I)/(II) $\Rightarrow$

$$\frac{A_1}{C_2} = \frac{B_1 e^{-ila} + B_2 e^{ila}}{B_1 e^{ila} + B_2 e^{-ila}} \quad \text{— (V)}$$

(III)/(IV) $\Rightarrow$

$$\frac{A_1}{C_2} = -\frac{B_1 e^{-ila} - B_2 e^{ila}}{B_1 e^{ila} - B_2 e^{-ila}} \quad \text{(VI)}$$

Equating (V) and (VI),

$$\left(B_1 e^{-ila} + B_2 e^{ila}\right)\left(B_1 e^{ila} - B_2 e^{-ila}\right) = -\left(B_1 e^{-ila} - B_2 e^{ila}\right)\left(B_1 e^{ila} + B_2 e^{-ila}\right)$$

$$\Rightarrow B_1^2 = B_2^2$$

$$\Rightarrow \boxed{B_1 = \pm B_2}$$

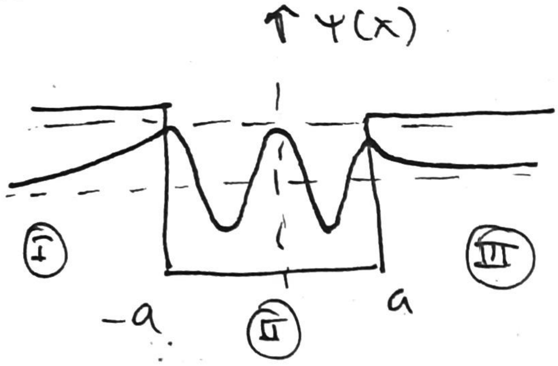
Case $B_1 = B_2$:

$$\psi(x) = \begin{cases} A_1 e^{\mathcal{R}x} \\ 2B_1 \cos(Lx) \\ C_2 e^{-\mathcal{R}x} \end{cases}$$

Now, $\boxed{A_1 = C_2}$ from (V) or (VI),

$$\psi(x) = \begin{cases} A_1 e^{\mathcal{R}x} \\ 2B_1 \cos(Lx) \\ A_1 e^{-\mathcal{R}x} \end{cases}$$

Plotting,



① for $B_1 = B_2$, $\Rightarrow$ $\psi(-x) = \psi(x) \rightarrow$ even function

② For $E < 0$, regions I and III are classically inacessible.

In QM, both ~~ψ_I and~~ $\psi_I^* \psi_I > 0$ and $\psi^* \psi > 0$

$\Rightarrow$ Non-zero probability of finding the particle.