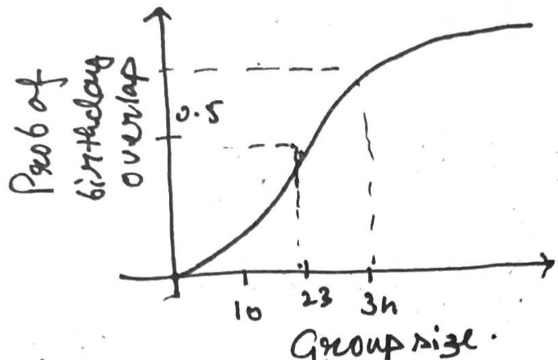
2nd Jan 2024

We begin by recounting the definition of random experiment.

Δ Experiment : An act which can be repeated under similar circumstances.

Classical "notion" $\rightarrow$ The proportion of favorable outcomes among the set of all ~~favorable~~ outcomes.

Birth day curve $\rightarrow$



So in the gform there was a very high probability of overlap with the 34 replies.

Now we prove the probabilistic PHP inequality.

$\boxed{\dashv}$ Given $\rightarrow$ $m = \#$ pigeonholes.

$p \in [0, 1)$

$n = \#$ pigeons.

If $n > \frac{1}{2} + \sqrt{2m \log\left(\frac{1}{1-p}\right) + \frac{1}{4}}$

then the probability of overlap $> p$.

Proof : Total no. of ways in which n pigeons could be put in m pigeonholes $= m^n$

~~Collect notes — too sleepy.~~

The no. of ways in which n pigeons could be put in n pigeonhole without any overlap $= {}^mP_n$ (Why not C ? Pigeons are identical)

$$= \frac{m!}{(m-n)!} = m(m-1)(m-2)\cdots(m-n+1)$$

$\therefore$ Probability of overlap $= \frac{{}^mP_n}{m^n} = \frac{m(m-1)\cdots(m-n+1)}{m^n}$

$$= \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\left(1-\frac{n-1}{m}\right)$$

$$= \prod_{i=1}^{n-1}\left(1-\frac{i}{m}\right)$$

Prob of overlap $> p$

$\Leftrightarrow$ Prob of no overlap $< 1-p$

$\therefore$ Required,

$$\prod_{i=1}^{n-1}\left(1-\frac{i}{m}\right) < 1-p$$

$$\Rightarrow \sum_{i=1}^{n-1}\log\left(1-\frac{i}{m}\right) < \log(1-p)$$

$$\Rightarrow -\sum_{i=1}^{n-1}\log\left(1-\frac{i}{m}\right) > \log\left(\frac{1}{1-p}\right)$$

Now, $\log(1-x) = -\sum_{l=1}^{\infty}\frac{x^l}{l}$, for $|x|<1$.

$$\Rightarrow \sum_{i=1}^{n-1}\sum_{l=1}^{\infty}\frac{i^l}{l\,m^l} > \sum_{i=1}^{n-1}\frac{i}{m} = \frac{1}{m}\sum_{i=1}^{n-1} i$$

$$= \frac{n(n-1)}{2m}$$ $\rightarrow$ Insert this b/w the bounds of the inequality.

So, $n > \frac{1}{2} + \sqrt{2m\log\left(\frac{1}{1-p}\right) + \frac{1}{4}}$

$$\Rightarrow \left(n-\frac{1}{2}\right)^2 > 2m\log\left(\frac{1}{1-p}\right) + \frac{1}{4}$$

$$\Rightarrow \frac{n(n-1)}{2m} > \log\left(\frac{1}{1-p}\right)$$

$$\therefore -\sum_{j=1}^{n-1}\log\left(1-\frac{j}{m}\right) < \frac{n(n-1)}{2m} < \log\left(\frac{1}{1-p}\right)$$

(*) Can be solved in alternate way ~~as~~ using the fact that $e^{-x} > 1-x$

Ex: (Probabilistic lie detector) →

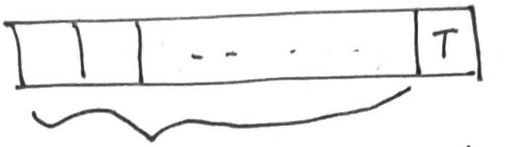
Find out the probability of obtaining no two heads if a fair coin is tossed 7 times.

Ans: F_n : The no. of cases s.t there are no two consecutive heads if a fair coin is tossed n-times.

$$F_1 = |\{T, H\}| = 2$$

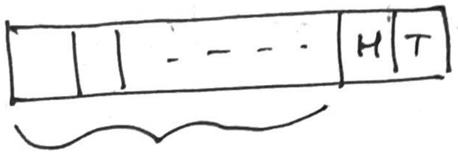
$$F_2 = |\{TT, HT, TH\}| = 3 \ldots$$

F_n :



→ There can be two ending cases

F_{n-1} → No two cons. heads.



F_{n-2}

$$\therefore \boxed{F_n = F_{n-1} + F_{n-2}} \text{ (Fibonacci recurrence)}$$

$F_7 = 34$

$$\therefore \text{Req. prob} = \frac{34}{2^7} = \frac{17}{64}$$

(*) Segways into deriving Binet's closed form expression for the n^{th} term of Fibonacci seq using LA

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$$

Eigenvalues of this are ϕ and $\bar{\phi}$, we diagonalise then raise to power to find form.

Axiom of Probability →

Δ Experiment → An act that can be repeated under similar circumstances.

Δ Sample Space → Set of all outcomes of an experiment : Ω

Δ Set of events $(\mathcal{E})$: A subset of the power set of Ω s.t

1. $\Omega \in \mathcal{E}$
2. If $A \in \mathcal{E}$ then $A^c := \Omega \setminus A$
3. $\mathcal{E}$ is closed under countable union
4. There is a func. $P : \mathcal{E} \to [0, 1]$ s.t

   * → $P(\Omega) = 1$

   * → If $A_1, A_2, \ldots, \in \mathcal{E}$
     s.t $A_i \cap A_j = \phi \ \forall \ i \neq j$
     then $P(A_1 \cup A_2 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$