⊛ Extracted from Piyush's Class Notes

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## L'Hôpital's Rule →

Let $J$ be an interval. Let $a \in J$ or $a$ be a boundary pt. of $J$.

Assume that

(i) $f, g : J \setminus \{a\} \to \mathbb{R}$ be diff'ble

(ii) $g'(x) \neq 0$ and $g(x) \neq 0 \ \forall \ x \in J \setminus \{a\}$

(iii) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) =: A$ and $A$ is either $0$ or $\infty$

If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\boxed{\frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{0}{0} \Big/ \frac{\infty}{\infty} \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

**Proof:** We will take $a$ to be an int pt. of $J$

**Case I:** $\frac{0}{0}$

assume that $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$

Note $f, g$ are contd. of $J \setminus \{a\}$ as it is differentiable on $J \setminus \{a\}$

Define

$$\tilde{f}(x) = \begin{cases} f(x), & x \in J \setminus \{a\} \\ 0, & x = a \end{cases}$$

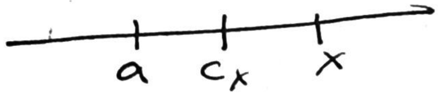
$$\tilde{g}(x) = \begin{cases} g(x), & x \in J \setminus \{a\} \\ 0, & x = a \end{cases}$$

then $\tilde{f}, \tilde{g} : J \to \mathbb{R}$ is contd.

Now,

$$\frac{f(x)}{g(x)} = \frac{\tilde{f}(x)}{\tilde{g}(x)} = \frac{\tilde{f}(x) - \tilde{f}(a)}{\tilde{g}(x) - \tilde{g}(a)}$$

By CMVT, $\exists\, c_x \in (a, x)$




$$\Rightarrow \frac{f(x)}{g(x)} = \frac{\tilde{f}'(c_x)}{\tilde{g}'(c_x)} = \frac{f'(c_x)}{g'(c_x)}$$

$$\Rightarrow \boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c_x \to a} \frac{f'(c_x)}{g'(c_x)}}$$

Now we will show that,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Let $x_n > a$ s.t $x_n \to a$

Need to prove that,

$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{n \to \infty} \frac{f'(x_n)}{g'(x_n)}$$

Now, $$\frac{f(x_n)}{g(x_n)} = \frac{\tilde{f}(x_n)}{\tilde{g}(x_n)} = \frac{\tilde{f}(x_n) - \tilde{f}(a)}{\tilde{g}(x_n) - \tilde{g}(a)}$$

as $\tilde{f}(a) = \tilde{g}(a) = 0$

According to CMVT, $\exists\, c_n \in (a, x_n)$ s.t

$$\frac{f(x_n)}{g(x_n)} = \frac{\tilde{f}'(c_n)}{\tilde{g}'(c_n)} = \frac{f'(c_n)}{g'(c_n)}$$

$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = \lim_{n \to \infty} \frac{f'(c_n)}{g'(c_n)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Since $\{x_n\}$ is arbitrary

Thus, $$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Similarly,

$$\lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

$$\Rightarrow \boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

<u>Case 2</u>: $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$

$$\frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f(x)}{g(x)} \cdot \frac{\left(1-\frac{f(c)}{f(x)}\right)}{\left(1-\frac{g(c)}{g(x)}\right)}$$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{f(x)-f(c)}{g(x)-g(c)} \frac{\left(1-\frac{g(c)}{g(x)}\right)}{\left(1-\frac{f(c)}{f(x)}\right)}$$

$\lim_{x\to a^+} g(x) = \infty$

$\Rightarrow$ for each $R>0$ $\exists$ $\delta>0$ s.t

$x\in(a,a+\delta) \Rightarrow g(x) > R$

Let $c\in J$ and $c>a$

for $R = |g(c)|$ $\exists$ $\delta$ s.t.

$x\in(a,a+\delta)$, $g(x) > |g(c)| \geq g(c)$

□ <u>Higher order derivative</u> $\rightarrow$

A diff'ble func. $f: J\to\mathbb{R}$ is twice diff'ble at $c\in J$ if

$f': J\to\mathbb{R}$ is diff'ble at $c$

$$\frac{d^2f}{dx^2}(c) = f''(c) = (f')'(c)$$

Similarly, we define $n^{th}$ order differentiability.

## Δ Infinitely diff'ble func →

A function $f : J \to \mathbb{R}$ is infinitely diff'ble if $n^{th}$ derivative $f^{(n)}$ exists for all $n \in \mathbb{N}$

## Δ Smooth function →

An infinitely diff'ble function is also called a smooth function

Eg: Any polynomial, $\sin(x)$, $e^x$

Ex: $f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0 \end{cases} \Rightarrow$ No convergent power series of this function.

Then $f$ is smooth.

Note, $f(0) = 0 = \lim_{x \to 0^+} f(x)$

$\to f$ is contd.

It is obv that $f$ is diff'ble for $x \in \mathbb{R} \setminus \{0\}$

Now,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x}$$

$$\Rightarrow \lim_{x \to 0^-} \frac{f(x)}{x} = 0$$

Now, $\lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{e^{-1/x}}{x} = \lim_{y \to \infty} \frac{e^{-y}}{1/y}$

$$= \lim_{y \to \infty} \frac{y}{e^y} = \lim_{y \to \infty} \frac{\frac{d}{dy}(y)}{\frac{d}{dy}(e^y)} = \lim_{y \to \infty} \frac{1}{e^y} = 0$$