2nd January 2025 (Thursday) →

(*) Ankified

Pre-req →

1. Vector Calculus ($\mathbb{R}^3$)
   - → · and × product
   - → div / curl
2. Analysis III / II (Imp) → Inv F.T, → Im F.T, → Regular values.
3. Topology.
4. Linear Algebra
   - → Determinants
   - → Trace
   - → Eigen stuff.
   - → $SO(3)$, $O(3)$, etc.

Will deal with curves and Surfaces in $\mathbb{R}^3$

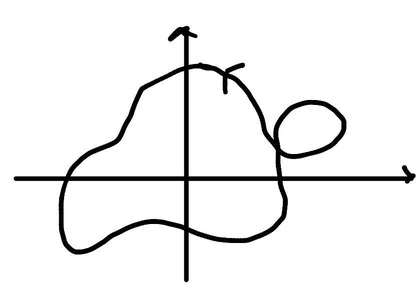
Surfaces: Locally 2D

Imp Theorems →

(I) Four vertex theorem: The curvature of a simple, closed, smooth, plane ($\mathbb{R}^2$) curve has atleast 2 local maxima and 2 local minima

Must have a period, like circle

Ex of smooth curve →



Periodic map from $\mathbb{R}$ to $\mathbb{R}^2$
OR map from circle to $\mathbb{R}^2$

Simple → No self intersection

Curvature is a real number assigned to each point $(x(t), y(t))$, i.e. is a function of t.

→ $(\text{Radius})^{-1}$ of circle of best fit touching curve tangentially at the point
(Assume that this exists right now)

Like, for a unit circle, at any point on the circle, the unit circle itself is best fit.

⇒ Curvature $= \frac{1}{1} = 1$ for all points on the unit circle.

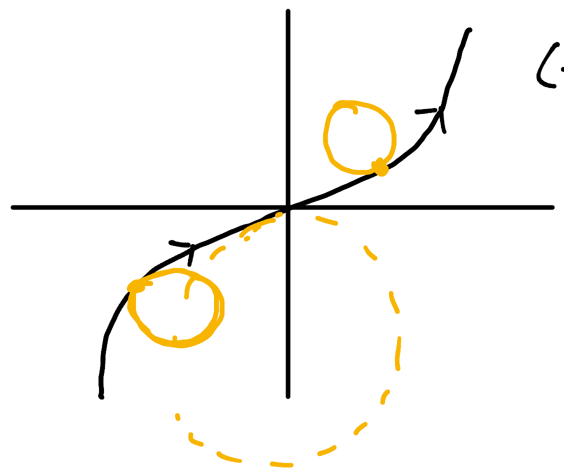
Or, for a straight line, the best fit is a circle with ∞ radius ⇒ Curvature $= 0$.

(*) Ellipse is where the four vertex theorem holds exactly – 2 local max, 2 local min.

(*) for a curve in $\mathbb{R}^3$, we have replace circle analogy with sphere.

(II) Fáry - Milnor Theorem :

first, note that curvature has a sign. →



$(t, t^3) = \gamma(t)$ (cubic)

If to the right, curvature is +ve
If to right, curvature is −ve } → Does not hold in $\mathbb{R}^3$ curves?!

No sign possible in $\mathbb{R}^3$!!

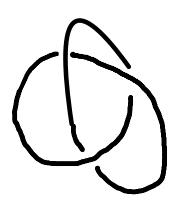
Also, in 2D, all simple closed smooth curves can be manipulated to a circle, but in 3D, Knots are possible

FM Theorem statement : If the total (absolute) curvature of a knot K (denoted by $\oint_K |K(s)| ds$) is at most 4π, then K is an unknot.

Δ Knot : Simple, closed, smooth space curve (i.e. in $\mathbb{R}^3$)

Ex: The unit circle is a Knot — it is a space curve on $\mathbb{R}^2$

or,



→ Trefoil (?) — Chiral, not the circle. → cannot be deformed to circle without cutting.

Δ Unknot : Something that may be "deformed" to a unit circle (i.e. without cuts or self intersection) — equivalent to unit circle on xy plane.

⊛ Absolute curvature for unit circle →

$$\int_0^{2\pi} 1 \cdot ds = 2\pi$$

So, there is nothing below 2π either, as total curvature.

⊛ The trefoil is a torus - knot — it can be drawn on a torus.

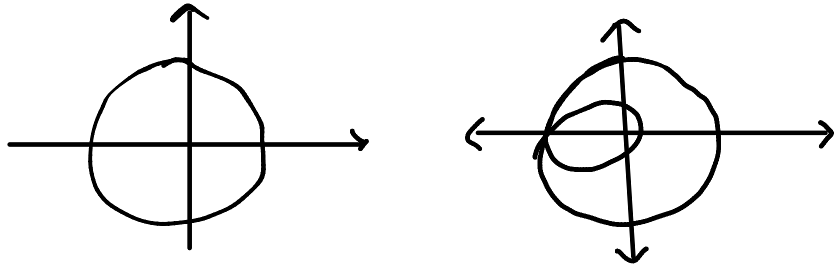
⊛ Simple is required because otherwise there is no unique curvature at points of intersection.

(III) Hirsch - Smale Theory : Any two immersed loops in $\mathbb{R}^2$ are isotopic iff these have the same winding number.

Δ Loop : Closed, smooth curve (No simple condition)

Δ Immersed : $\frac{d\gamma(t)}{dt} \neq 0$ at all points on curve (non zero

Ex : Immersed loops →



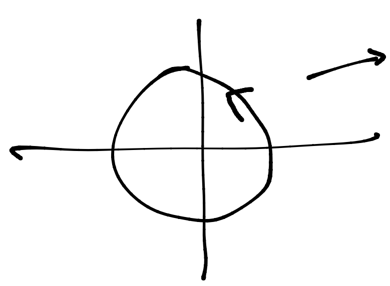
Δ Isotopic: All isotopic curves are parametrised by an interval, where at the ends they are not necessarily immersed but in the middle they are

☐ Clarify this.

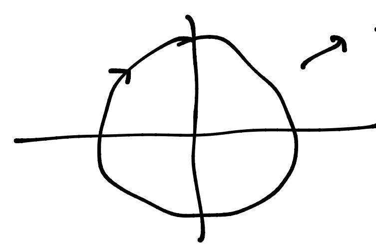
Δ Winding number: (Correct idea of equivalence here)

If we draw tangent along curve in one direction, and count the no. of times we complete a circle anticlockwise (-ve if clockwise) in the rotation of tangent vector drawn from origin

Like, unit circle →

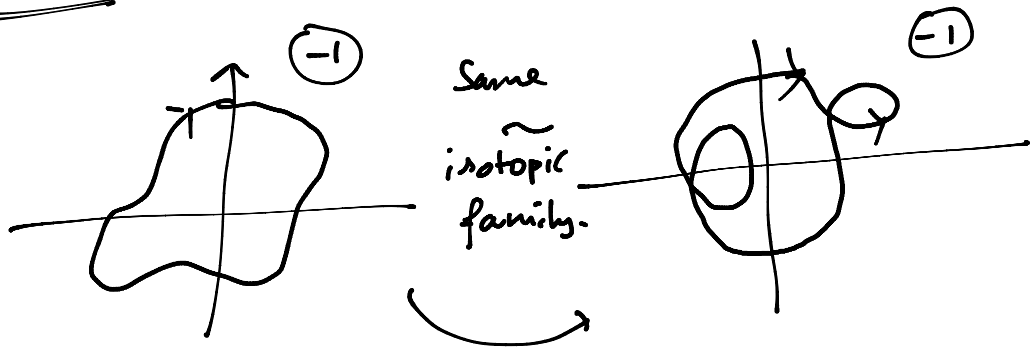


→ +1 winding no.



→ -1 winding no.

Exercise: Try to walk through an immersed family

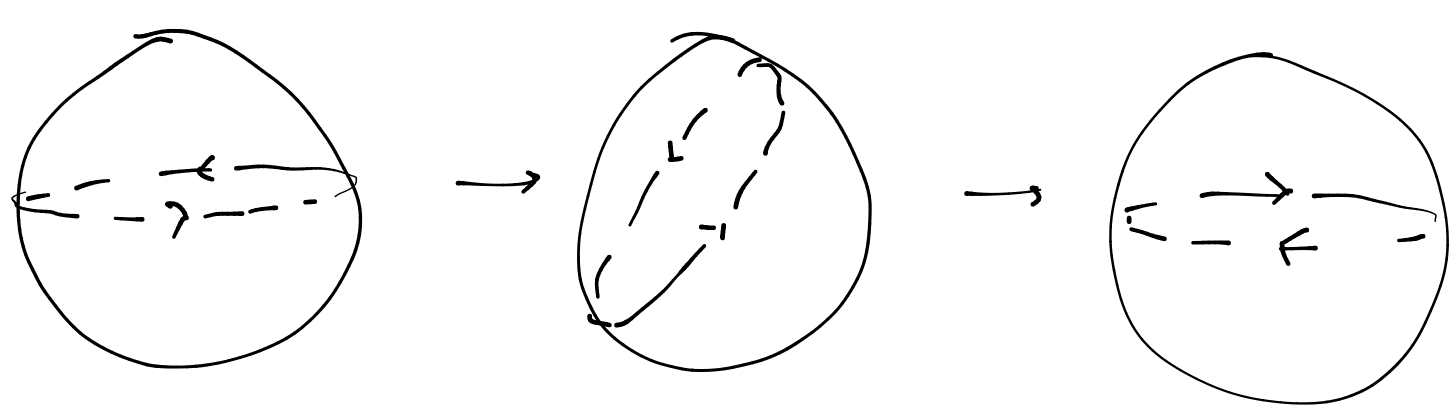


How?

Note:  → Not allowed! (T is not immersed)

(*) Now, in HS theory,
what if we replace $\mathbb{R}^2$ with S^2 (unit sphere)

Now → clockwise and anticlockwise unit circles are now isotopic



(*) We have to not consider winding no. here, but no. of time crossing complete circle of tangent mod 2

(IV) Gauss-Bonnet Theorem:

first, we are allowed only closed curves - no finite/infinite cylinders, etc, but we are allowed torus, etc.

No point where it should look, locally, as a half disc ◖ → Compact + no boundary

GB statement: Let Σ^2 be a closed 2 dimensional (Riemannian) manifold. Then,

Gaussian curvature

$$\int_{\Sigma^2} K \, dA = 2\pi \, \chi(\Sigma)$$

dA: infinitesimal area on surface

χ: Euler characteristic (Basically a func of the # of holes) → Can be defined purely topologically. (invariant topologically)