$$\Rightarrow P(A_1 \cup \cdots \cup A_n) \leq P(A_1) + \cdots + P(A_n)$$

$\square$

\# Completed from Piyush's Notes. 16th January 2024

Δ Bonferroni's Inequality

$(\Omega, \mathcal{E}, P)$: Probability Space

$A_1, A_2, \ldots, A_n$ events in $\mathcal{E}$

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) \geq P(A_1) + P(A_2) + \cdots + P(A_n) - (n-1)$$

Proof: for $n=1$

$$P(A_1) \geq P(A_1) - 0$$

for $n=2$,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$\Rightarrow P(A_1 \cap A_2) = P(A_1) + P(A_2)$$

$\Rightarrow$ Now, $P(A_1 \cup A_2) \leq 1$ [Since $P: \mathcal{E} \to [0,1]$]

$$\Rightarrow - P(A_1 \cup A_2) \geq -1$$

$$\Rightarrow P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$$

Induction hypothesis: The claim holds for $m \in \mathbb{N}$

$$P(\underbrace{A_1 \cap \cdots \cap A_m}_{A} \cap \underbrace{A_{m+1}}_{B}) \geq P(A) + P(B) - 1$$

$$= P(A_1 \cap \cdots \cap A_m) + P(A_{m+1}) - 1$$

$$\geq P(A_1) + \cdots + P(A_m) - (m-1) + P(A_{m+1}) - 1$$

$$= P(A_1) + \cdots + P(A_{m+1}) - m$$

So the induction hypothesis holds for $m+1$ when it holds for m.

$\rightarrow$ The statement is true.

Δ Conditional Probability →

$(\Omega, \mathcal{E}, P)$: Probability Space

$B \in \mathcal{E}$ has already occured as the result of the random experiment.

→ If B has occured, our sample space is reduced to B.

$A \in \mathcal{E}$

- If $A \cap B = \phi$, then A hasn't occured
- If $A \cap B \neq \phi$, then the probability of this event is measured relative to the probability of B

Δ **Defn**: Let $B \in \mathcal{E}$ be such that $P(B) > 0$. Then, for $A \in \mathcal{E}$ we define,

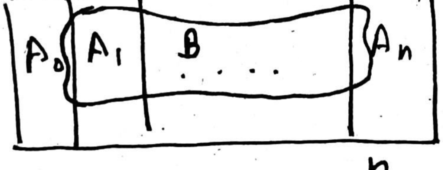
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

→ In particular, $P(B \mid B) = 1$

□ **Lemma**: (Total Probability) →

Let $(\Omega, \mathcal{E}, P)$ be a probability space and let $\{A_i\}_{i=0}^{\infty}$ be pairwise mutually exclusive exhaustive events such that $P(A_0) = 0$ and $P(A_i) > 0 \ \forall i \in \mathbb{N}$. Then, for any $B \in \mathcal{E}$, we have,

$$P(B) = \sum_{i=1}^{\infty} P(A_i) P(B \mid A_i)$$



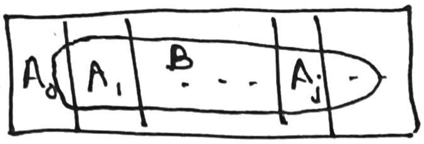
If $\bigcup_{i=0}^{\infty} A_i = \Omega$, then $B = \bigcup_{i=0}^{n} (B \cap A_i)$

$$\Rightarrow P(B) = \sum_{i=0}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(B \cap A_i)$$

$$\boxed{P(B) = \sum_{i=1}^{n} P(A_i)\, P(B \mid A_i)}$$

for infinite partition it just follows trivialy,



$$B = (B \cap A_0) \cup (B \cap A_1) \cup \dots \cup (B \cap A_j) \cup \dots$$

$$\Rightarrow P(B) = \underbrace{P(B \cap A_0)}_{=0} + P(B \cap A_1) + \dots + P(B \cap A_j) + \dots$$

$$= P(B \cap A_1) + \dots + P(B \cap A_j) + \dots$$

$$= \sum_{i=1}^{\infty} P(A_i)\, P(B \mid A_i)$$

Bayes' Theorem →

Let $(\Omega, \mathcal{E}, P)$ be a probability space, and let $\{A_i\}_{i=1}^{\infty}$ be pairewise mutually exclusive and exhaustive events with $P(A_i) > 0 \ \forall i \in \mathbb{N}$.

Then, for any $B \in \mathcal{E}$ with $P(B) > 0$ we have,

$$P(A_j \mid B) = \frac{P(A_j)\, P(B \mid A_j)}{\sum_{i=1}^{\infty} P(A_i)\, P(B \mid A_i)}$$

Proof: $P(A_j|B) = \frac{P(A_j \cap B)}{P(B)}$

$$\Rightarrow P(A_j|B) = \frac{P(A_j)P(B|A_j)}{\sum_{i=1}^{\infty} P(A_i)P(B|A_i)}$$

Ex: Let's roll a fair die till we get an outcome of 6.
Call A_n the even in which we stop at the n^{th} roll.
Let B be the event that all the outcomes preceding the last one are odd.

$$P(A_n) = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}$$

$$P(B|A_n) = \frac{P(B \cap A_n)}{P(A_n)} = \frac{\frac{1}{6}\left(\frac{3}{6}\right)^{n-1}}{\frac{1}{6}\left(\frac{5}{6}\right)^{n-1}}$$

$$\Rightarrow P(B|A_n) = \left(\frac{3}{5}\right)^{n-1}$$

(*) $$P(A_m|B) = \frac{P(A_m)P(B|A_m)}{\sum_{n=1}^{\infty} P(A_n)P(B|A_n)} = \frac{\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{m-1}\left(\frac{3}{5}\right)^{m-1}}{\sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}\left(\frac{3}{5}\right)^{n-1}}$$

$$\Rightarrow P(A_m|B) = \frac{\left(\frac{1}{2}\right)^{m-1}}{\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}} = \frac{1}{2^m}$$

Ex: Suppose you find someone interesting and you'd like to ask him/her out for a coffee. Let's assume there are three mutually exclusive, exhaustive, and equally likely ~~events~~ cases

A: He/She finds you interesting too

B: He/She fells indifferent towards you

C: He/She is repulsed by you.

$P(Y|A) = 0.9$, $P(Y|B) = 0.5$ $P(Y|C) = 0.1$

(i) Find the probability that ~~a~~ he/she accepts your invitation

(ii) Given that he/she accepts your invitation, find the ~~probab~~ probability that he/she finds you interesting too.

$$P(Y) = P(Y|A)P(A) + P(Y|B)P(B) + P(Y|C)P(C)$$

$P(A \cup B \cup C) = 1$ [Since A, B, C are exhaustive]

$\Rightarrow$ ~~P(P~~ $P(A) + P(B) + P(C) = 1$ [mutually exclusive]

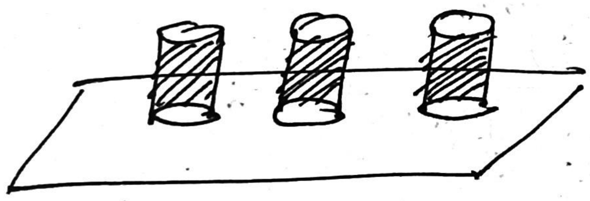
$\Rightarrow P(A) = P(B) = P(C) = \frac{1}{3}$ [Equally likely]

$$\Rightarrow P(Y) = \frac{1}{3}(0.9 + 0.5 + 0.1)$$

$$\Rightarrow P(Y) = 0.5$$

(ii) $P(A|Y) = \dfrac{P(Y|A)P(A)}{P(Y)} = \dfrac{(0.9)\left(\frac{1}{3}\right)}{(0.5)} = 0.6$

Ex: (Monty Hall Problem) $\rightarrow$



A: Your initial guess is right

~~B~~ A^c: Your initial guess is wrong

L: Your friend lifts an empty cup

We must compare,

$P(A|L)$ and $P(A^c|L)$

$P(L) = P(L|A) = 1$ [Since both are sure events]

$$\Rightarrow P(A|L) = \frac{P(A)\,P(L|A)}{P(L)} = \frac{(\frac{1}{3})(1)}{(1)} = \frac{1}{3}$$

$$P(A^c|L) = 1 - P(A|L) = \frac{2}{3}$$

Ex: Random Monty Hall problem.

☐ Independence: Let $(\Omega, \mathcal{E}, P)$ be a probability space. Let $A, B \in \mathcal{E}$ be two events such that the occurrence of any of them doesn't influence the other. In other words:

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

$$\Leftrightarrow P(A|B) = \frac{P(A\cap B)}{P(B)} = P(A)$$

$$\Leftrightarrow \boxed{P(A\cap B) = P(A)P(B)}$$

[Could be taken as formal definition of Independence]

○ Mutual vs. Pairwise independence –

If $S \subseteq \mathcal{E}$ s.t. $P(A\cap B) = P(A)P(B)\ \forall A, B \in S$, we say the events in S are pairwise independent, whereas if $\forall T \subseteq S$ we have

$$P\left(\bigcap_{A\in T} A\right) = \prod_{A\in T} P(A),$$ the events in S are called mutually independent.