7th January 2025 (Tuesday) →

⊗A

→ Definition of Hilbert Space (is $\mathbb{R}^2$ a Hilbert Space?)

Eigenstates of QSHO →

$$\psi(x) = C e^{-\alpha^2 x^2}$$ → we normally take this

$$\psi_\pm(x) = C' e^{\pm ikx}$$ ↪ Can also be complex

$\Rightarrow \psi_\pm(x) = \langle x | \psi_\pm \rangle$

→ Discussion of discrete / continuous normalization.

→ Discussion of infinite dimensional nature of free particle problem.

⊘ Spin 1/2 problem → (A 2 state problem)

→ Pauli matrices.

$$H = g\mu_B \vec{S}\cdot\vec{B}, \quad \text{where } \vec{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)$$

$$\Rightarrow H = g\mu_B \cdot \frac{\hbar}{2}\left(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z\right)$$

we orient our axes s.t $B_x\sigma_x = B_y\sigma_y = 0$

$$\therefore H = \underbrace{\left(g\mu_B \frac{\hbar}{2} B_z\right)}_{\text{Let us call this } \epsilon_0}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longrightarrow \text{Eigenstuff:}$$

$+\epsilon_0 \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\uparrow\rangle_z$

$-\epsilon_0 \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle_z$

These are clearly normalized.

Note,

$$\left.\begin{aligned} \langle\downarrow|\uparrow\rangle &= \langle\uparrow|\downarrow\rangle = 0 \\ \langle\downarrow|\downarrow\rangle &= \langle\uparrow|\uparrow\rangle = 1 \end{aligned}\right\} \text{Orthonormal basis}$$

A general state is thus → $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, $\alpha, \beta \in \mathbb{C}$ (generally)

Let $\alpha = x_1 + ix_2$, $\beta = x_3 + ix_4$

$$\therefore |\alpha|^2 + |\beta|^2 = 1 \Rightarrow \boxed{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1}$$

What does this mean? Every point on a 4-sphere is a valid state here.

H is a Hermitian matrix.

→ Is the set of Hermitian matrices vectorspace?

Over the field $\mathbb{R}$, yes. Why? $(iH)^{+} = -H \neq H \Rightarrow$ Scalar mult is not closed.

We thus get the idea that Hermitian operators like H may be represented as a linear combination of $\sigma_x, \sigma_y, \sigma_z$ AND $\mathbb{I}_{2\times 2}$

$$H = B_0 \mathbb{I}_{2\times 2} + \frac{g\mu_B \hbar}{2}\left(B_x \sigma_x + B_y \sigma_y + B_z \sigma_z\right)$$

$$= B_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \epsilon_0 & 0 \\ 0 & -\epsilon_0 \end{bmatrix}$$

How do define innerproduct rule here (To call it Hilbert Space)

$$\frac{1}{2} Tr[\sigma_i \sigma_j] = \delta_{ij} = \langle \sigma_i | \sigma_j \rangle$$ (SU(2) group structure)

Thus, $\{\mathbb{I}_{2\times 2}, \sigma_x, \sigma_y, \sigma_z\}$ is an orthonormal basis for 2 state Hilbert Space over $\mathbb{R}$.

Stern-Gerlach / HUP →

$$|\uparrow\rangle_z = \pm \frac{1}{\sqrt{2}} |\uparrow\rangle_x \pm \frac{1}{\sqrt{2}} |\downarrow\rangle_x$$ ⟶ Prepare this state.

Now we measure S_x on this set. What do we see? 50% times $|\uparrow\rangle_x$ and 50% $|\downarrow\rangle_x$

This may be used in cryptography.

Say we have binary information →

$$I = \begin{bmatrix} 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix}$$ → well defined.

Now, we carry out experiment of S_x measurement — we get a vector of ± 1 (completely random)

$$R_1 = \begin{bmatrix} +1 \\ -1 \\ +1 \\ +1 \\ \vdots \end{bmatrix}$$

Now, we form $R_1 + I$ and send it over to someone. That person has an entangled $|\uparrow\rangle_z$ state, and hence they can reproduce R_1

⇒ They may decode $I + R_1$ to get R_1, securely.

Claim → QM has ① wave-particle duality
② Entanglement
as core features.

→ 3 Lectures (Revision – Basics)

→ 1 Lecture (Revision – Time independent Perturbation)

→ Then Time dependent perturbation.

Next week we will NOT have classes. (He will be on a conference)
Will have weekend classes this weekend + next weekend.

Tomorrow : Why entanglement is an outcome of Linear Algebra.

8th January 2025 (wednesday) →

Say we have a system which has two $\pm\frac{1}{2}$ spins. How do we describe this system?

Each spin $\frac{1}{2}$ is described in terms of Hamiltonian $H \propto \vec{B}\cdot\vec{\sigma}$, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

We have $H_1 = A_1 \vec{B}\cdot\vec{\sigma}_1$, $H_2 = A_2 \vec{B}\cdot\vec{\sigma}_2$ → Each 2x2 matrices.

But if we define,

$$H = H_1 + H_2 \rightarrow \text{it is } 4\times 4 \text{ (4 basis is obvious)}$$

How do we promote these 2x2 to 4x4?
Direct products.

We say,
$$\left.\begin{aligned}\vec{\sigma}_1 &= (\sigma_x \otimes \mathbb{I}, \sigma_y \otimes \mathbb{I}, \sigma_z \otimes \mathbb{I}) \\ \vec{\sigma}_2 &= (\mathbb{I}\otimes\sigma_x, \mathbb{I}\otimes\sigma_y, \mathbb{I}\otimes\sigma_z)\end{aligned}\right\} \rightarrow \text{Promoted to } 4\times 4$$

☐ How??

↳ Why Identity! So that it does not touch 2nd spin.

We may of course add an interaction term here as well –

$$J\,\vec{S}_1\cdot\vec{S}_2 = \left(\frac{\hbar}{2}\right)^2(\vec{\sigma}_1\cdot\vec{\sigma}_2)$$

$$= \left(\frac{\hbar}{2}\right)^2\left(\sigma_{x_1}\otimes\sigma_{x_2} + \sigma_{y_1}\otimes\sigma_{y_2} + \sigma_{z_1}\otimes\sigma_{z_2}\right)$$

We may extend to n dimensions.

⊗ We have essentially composed a 4D Hilbert Space from two 2D Hilbert Space.

~~☐~~ Figure out direct product of Hilbert Spaces and How to define operators for it.
(Kroenecker product)

Now, you may write states →

$$|\psi\rangle_a = |\uparrow\rangle_z^1 \otimes |\uparrow\rangle_z^2$$
$$|\psi\rangle_b = |\downarrow\rangle_z^1 \otimes |\downarrow\rangle_z^2$$

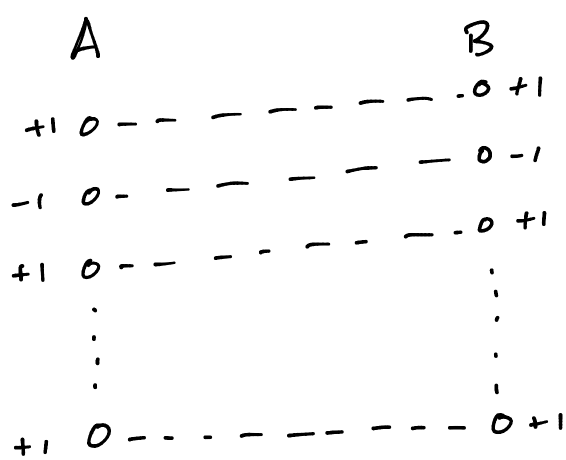
NOTE: These are distinguishable particles - we may label them 1 and 2.

A linear superposition of these is also a state →

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_z^1 \otimes |\uparrow\rangle_z^2 + |\downarrow\rangle_z^1 \otimes |\downarrow\rangle_z^2 \right]$$

In this state, the information is not classically well defined — it has a 50% chance of being in either state.

Say after preparing this state, we split up particle 1 and particle 2. 1 goes to Alice, 2 to Bob. Say we prepare 1000 (large) number of these states and do the same procedure — sending 1000 of particle 1 to Alice, the other 1000 particle 2 to Bob.



Now, say Alice measures the 1st one, gets ↑ ⇒ We immediately know Bob will get ↑ on the 1st one he has.

These are perfectly correlated.

NOTE: No FTL information transfer, as it is still stochastic on one side.

→ used in the cryptography example from last lecture.

☐ Tutorial 1 problem → Solve the eigen value problem of $\vec{S_1} \cdot \vec{S_2}$

Revision of Quantum Harmonic Oscillator →

$$H\psi = E\psi$$

$$\Rightarrow \left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2 \right)\psi = E\psi \quad \rightarrow \text{in } x \text{ basis.}$$

Rearranging a bit,

$$\rightarrow \frac{1}{2m}\Big[\underbrace{\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)}_{p}^2 + (m\omega \underbrace{x}_{\hat{x}})^2 \Big]\psi = E\psi$$

$$\Rightarrow \frac{1}{2m}\left[p^2 + \bar{x}^2\right]\psi = E\psi$$

<u>Note</u> : we may not write $[p^2+\bar{x}^2] = (\bar{x}+ip)(\bar{x}-ip)$ as $[\bar{x},p] \neq 0$

→ There will be another term.

Regardless, this inspires definition of,

$$a = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega\hat{x} + i\hat{p})$$

$$a^{+} = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega\hat{x} - i\hat{p})$$

$$\Rightarrow \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^{+})$$

$$\rightarrow \hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}(a - a^{+})$$

We may also use CCR to derive,

$$[a, a] = 0 = [a^{+}, a^{+}]$$

$$[a, a^{+}] = 1$$

And we can write,

$$H = \frac{1}{2}\hbar\omega(a^{+}a + aa^{+})$$

$$= \hbar\omega(a^{+}a + \tfrac{1}{2})$$

↳ $\hat{N}$ → Identical to something called a number operator in many-particle problems.

We derive the commutation relations,

$$[\hat{N}, \hat{a}^{+}] = a^{+}$$

$$[\hat{N}, a] = -a$$

We can thus diagonalise H by simply diagonalising $\hat{N}$ (everything else is $\mathbb{I}$)

$[H, \hat{N}] = 0$ (obvious)

So, we imagine eigenstates of $\hat{N}$,

$$\hat{N}|n\rangle = n|n\rangle$$

and these are also eigenstates of H ($[H,\hat{N}] = 0$, simultaneous eigenstate)

$$[\hat{N}, a^{+}]|n\rangle = a^{+}|n\rangle \Rightarrow [\hat{N}a^{+} - a^{+}\hat{N}]|n\rangle = a^{+}|n\rangle$$

$$\Rightarrow \hat{N}(a^{+}|n\rangle) = (n+1)(a^{+}|n\rangle)$$

$\Rightarrow$ a^+ increases $|n\rangle$ to state which is proportional to $|n+1\rangle$ (Creation)

Similarly, a decreases $|n\rangle$ to state which is proportional to $|n-1\rangle$ (Annihilation)

We thus know that $|0\rangle$ is the lowest eigenstate of $\hat{N}$. Thus,

$$a\,\psi_0 = 0 \rightarrow \text{Annihilated.}$$

Writing in position basis, we get a nice differential equation.

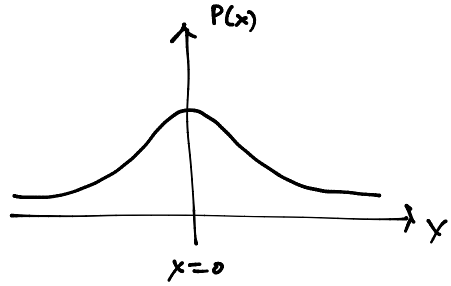
$$\frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\hat{x} + i\hat{p}\right)\psi_0(x) = 0$$

$$\Rightarrow \left(m\omega x + i\,\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi_0(x) = 0$$

Solving,

$$\psi_0(x) = N\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

We find N using square integrability + normalization to 1.



$$\langle\hat{x}\rangle = 0$$

as $\int_{-\infty} \underbrace{(x)}_{\text{odd}} \underbrace{P(x)}_{\text{Even}}\,dx = 0$

Now symmetry. If H commutes with some symmetry operator — are all H eigenstates also symmetric?

<u>Yes</u> $\rightarrow$ IF, there is no degeneracy. If there is degeneracy, we may not have linear combination of the H eigenstates which are not symmetric

BREAK

Now, we discuss the uncertainties associated with these states.

$$\langle\hat{x}\rangle = \langle n|\hat{x}|n\rangle \propto \langle n|a + a^+|n\rangle$$

$$\propto \langle n|\underbrace{a|n\rangle}_{|n-1\rangle} + \langle n|\underbrace{a^+|n\rangle}_{|n+1\rangle}$$

$$= 0$$ (Eigenstates of Hermitian operators are orthogonal)

Again,

$\langle\hat{p}\rangle = 0$ (Same method as for $\langle\hat{x}\rangle$)

Now, second moment.

$$\langle \hat{x}^2 \rangle = \frac{\hbar}{2m\omega} \langle n | a^2 + aa^+ + a^+a + a^{+2} | n \rangle$$

$$= \frac{\hbar}{2m\omega}\left[\underbrace{\langle n | a^2 | n\rangle}_{0} + \underbrace{\langle n | aa^+ | n \rangle + \langle n | a^+a | n\rangle}_{\neq 0} + \underbrace{\langle n | a^{+2} | n\rangle}_{0}\right]$$

$$= \frac{\hbar}{2m\omega}\left[\langle n | 2\hat{N} + 1 | n\rangle\right]$$

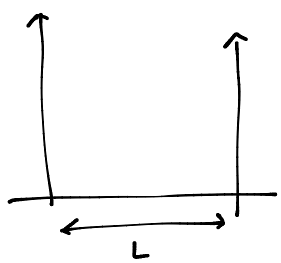
$$= x_0^2 \left(n + \frac{1}{2}\right)$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$ → Has both m and ω

This also appears in the exponential of $\psi_0(x)$

This, thus, tells us how 'spread' out the QSHO is.

How do we determine the minimum energy of particle in a box?



Note, $\Delta x \leq L$ (upper bound)

Using HUP,

$$\Delta p \geq \frac{\hbar}{2\Delta x}$$

$$\frac{\Delta p^2}{2m} \geq \frac{\hbar^2}{8m\Delta x^2} = \frac{\hbar^2}{8mL^2}$$

☐ <u>Tutorial</u>: Solve the eigenvalue problem on a ring
(Ring boundary condition)

So, the variance now,

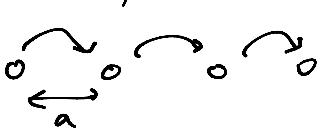
$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= x_0^2\left(n + \frac{1}{2}\right)$$

$$\Rightarrow \Delta x = x_0\sqrt{n+\frac{1}{2}}, \quad \Delta p = \left|\frac{\hbar}{x_0}\right|\sqrt{n+\frac{1}{2}}$$

How do we guess this?

On a lattice,



The lattice spacing bounds from above the maximum possible momenta.

In fact, the bound is $\sim \frac{\pi}{a}$

$\therefore$ Here, $\Delta x \Delta p = \hbar(n+\frac{1}{2})$

What is the dimension of the QSHO Hilbert Space?

It is a countable infinity — but why? Clearly, $|x\rangle$ is also a basis. And it is an uncountable infinity.

What then causes this issue — we cannot go from uncountable to countability by an unitary transform.

Ans: We need square integrability for a physical description.

But note,

$$\langle x'|x\rangle = \delta(x'-x) \rightarrow \underline{\text{NOT}} \text{ square normalizable}$$

$\therefore L^2$ norm is not followed.

$\rightarrow$ It cannot <u>really</u> be called a basis of the Hilbert Space, like $|n\rangle$ basis can,

due to $\langle n'|n\rangle = \delta_{nn'} < \infty \rightarrow$ Obeys L^2 norm.

Let us form a representation,

$$|n=0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad |n=1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \text{ and so on,}$$

Clearly, it is orthonormal.

$$\langle 2|2\rangle = 0+0+(1)(1)+0+\cdots = 1$$

$$\langle 1|2\rangle = 0+0+\cdots = 0$$

Clearly,

$$\langle n|m\rangle = \delta_{nm} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \\ \vdots & & \ddots \end{bmatrix}$$

We figure out matrix of a^+.

$$\langle k|a^+|n\rangle = \sqrt{n+1}\,\langle k|n+1\rangle = \sqrt{n+1}\,\delta_{k,(n+1)}$$

It is <u>not</u> diagonal, but all values lie on the diagonal right beside the principal diagonal.

$$\therefore a^{\dagger} = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \\ \vdots & \vdots & & \ddots \\ \vdots & \vdots & & & \ddots \end{bmatrix}$$

Similarly for a, with other (upper) adjacent diagonal.

$\hat{N}$ has a simple representation here in $|n\rangle$ basis.

$$\langle n|\hat{N}|m\rangle = m\,\delta_{mn}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \\ 0 & 0 & 2 & \\ \vdots & & & \ddots \\ \vdots & & & & \ddots \end{bmatrix}$$

Next day: Coherent states

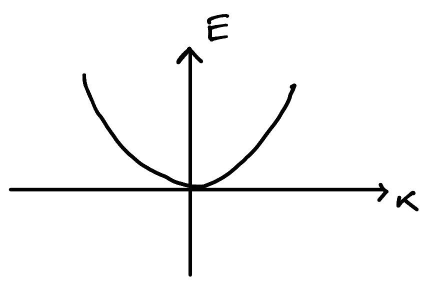
⊗ Will announce office hours after next week.

10th January 2025 (Friday) →

Apparently, a tutorial.

For a free particle →

$$H = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} = E$$



Potential →

① Periodic   Where?

② Weak   No idea.

↳ Say $V(K) = V_0 \exp(iKx) \ldots$

$\downarrow$

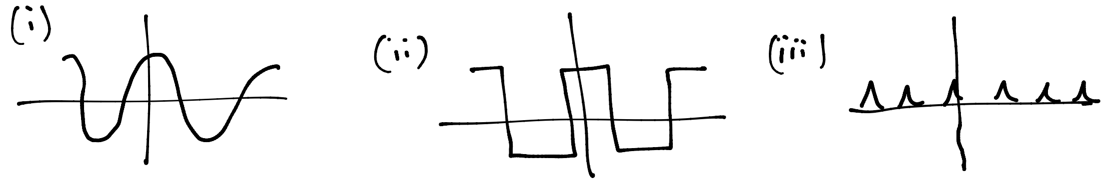
$V_0 << E \Rightarrow$ Weak potential.

So, we know, $\psi(x) = \frac{1}{\sqrt{L}} e^{\pm ikx}$

$$\langle k|k'\rangle = \delta_{kk'} = \frac{1}{L}\int e^{-ikx} e^{ik'x}\,dx$$

$$= \frac{1}{L}\cdot L\ \delta(k,k')$$

So now for a potential like mentioned above, we try to solve generally.



$$V(x+a) = V(x)$$

$$H(x) = \frac{p^2}{2m} + V(x)$$

Using periodic property, we expand into fourier series.

$V(x) =$ usual fourier sum.

$V_n = \int$ usual fourier coeff finding integral.

Note: The free particle is degenerate. Why? particles having $-k$ and $+k$ have same energy.

$\Rightarrow \langle k | V(x) | k' \rangle \neq 0$

$\rightarrow$ Degenerate perturbation theory.

Now,

$$E(k) = E_0(k) + \langle k | V | k \rangle + \sum_{k' \neq k} \frac{|\langle k | V | k' \rangle|}{E_k - E_{k'}}$$

$\rightarrow$ Nonzero. $\hookrightarrow 0$

Let us claim a state,

$$|\psi\rangle = \alpha |k\rangle + \beta |k'\rangle$$

Now,

$$H|\psi\rangle = E|\psi\rangle$$

$$\Rightarrow \langle \psi | H | \psi \rangle = \langle \psi | E | \psi \rangle$$

$$\Rightarrow \left(\alpha^* \langle k | + \beta^* \langle k' | \right) H \left(\alpha |k\rangle + \beta |k'\rangle \right) = \langle \psi | E | \psi \rangle$$

$$\Rightarrow (\alpha^* \ \beta^*) \begin{pmatrix} \langle k | H | k \rangle & \langle k | H | k' \rangle \\ \langle k' | H | k \rangle & \langle k' | H | k' \rangle \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (\alpha^* \beta^*) \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Now,

$$|k\rangle = \frac{1}{\sqrt{L}} \exp(\pm ikx)$$

$$H = \frac{\hbar^2 k^2}{2m} + V(x) = \frac{\hbar^2 k^2}{2m} + \sum_{n \in \mathbb{Z}} V_n \exp\left(\frac{2\pi i n x}{a} \right)$$

$$= E_0 + \langle k | \sum_n v_n \exp\left(\frac{2\pi i n x}{a}\right) | k' \rangle$$

Now,

$$\langle k|H|k'\rangle = E_0 + \frac{1}{L}\int \sum_n v_n e^{-ikx} e^{+ikx} e^{\frac{2\pi i n x}{a}}\, dx$$

$$\Rightarrow \langle k|H|k'\rangle = E_0 + \sum_n v_n \underbrace{\exp\left(i\left(k'-k+\frac{2\pi n}{a}\right)x\right)}_{\delta_{k,\,k'+\frac{2\pi n}{a}}}$$

I am not verifying these — Take for granted your own think.

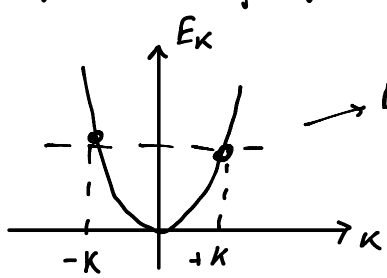
Sorry, but this is not very coherent

☐ Read about solutions to the Hamiltonian on a Lattice/periodic boundary using degenerate/non-degenerate perturbation theory.

→ Get the Notes from the TA.

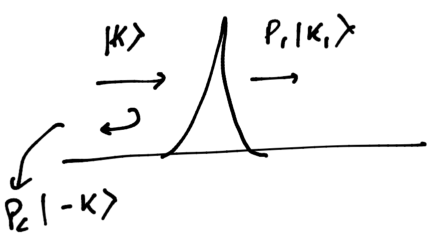
Extra Class — 10th January 2025 (Friday) →

A simple example of degeneracy in the free particle →



→ Degeneracy.

What happens when you encounter, say, Dirac Delta potential?



$P_1 + P_2 = 1$

⊛ If you start with momentum k, the only other momentum that you can go to conserving energy is $-k$, due to the degeneracy.

⇒ Energy is conserved.

So if we break translation symmetry by introducing barrier — the time-reversal symmetry constrain the other possibility to $|-k\rangle$ to conserve energy.

When we have a lattice/periodic boundary, there is some sort of translation symmetry that is retained.

☐ Do the problem of the lattice — something about magic momenta (that cause strong back scattering) and analogues to diffraction grating.

(*) Leads to band gaps, etc.

→ Discussion of Tutorial over.

III Coherent States →

Δ Defn: States with "optimized" uncertainty.
(i.e., no inequality, an equality in HUP)

This generally relates to states that are eigenstates of phase operators, not number operators (Superconductivity, Quantum Optics, etc)

Say, we have a state,

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha \in \mathbb{C}$$

Assume as normalized state

$$\therefore \alpha = |\alpha| e^{i\theta}$$

Now,

$$\langle\alpha|H|\alpha\rangle = \hbar\omega\left\{\langle\alpha|a^+a + \tfrac{1}{2}|\alpha\rangle\right\}$$

$$= \hbar\omega\left\{\langle\alpha|a^+a|\alpha\rangle + \tfrac{1}{2}\langle\alpha|\alpha\rangle\right\}$$

$$= \hbar\omega\left\{\langle\alpha|\alpha^*\alpha|\alpha\rangle + \tfrac{1}{2}\right\}$$

$$= \hbar\omega\left\{|\alpha|^2 + \tfrac{1}{2}\right\}$$

Strange things about non-hermitian operators →

$$\hat{O}|\lambda\rangle = \lambda|\lambda\rangle$$ → λ, in general, need not be real.

non-hermitian

Also, the adjoint,

$$\langle\lambda|\hat{O}^+ = \langle\lambda|\lambda^*$$

If $\hat{O}$ was Hermitian, $\lambda = \lambda^*$

Since this is not true, we notice that a non-Hermitian has a left and a right eigenvalues.

□ Exercise: We may claim – $\left(|\alpha\rangle_R\right)^+ \neq \left(\langle\alpha|_L\right)$

eigenstates of Non-Hermitian operators are distinct.

Coming back to eigenstates of a,

Note that, in QSHO, we may obtain all states by acting on $|0\rangle$ with a^+.

$$\text{All states} = (a^+)^n |0\rangle$$

We may posit that $|0\rangle$ is a part of our coherent states.

We label states by phases now, i.e. the eigenstates of a have phase as label.

$\theta \in [0, 2\pi]$

$$\therefore |\alpha\rangle = \{|\alpha| e^{i\theta}\rangle$$

We now want an operator (like a, a^+) that allows us to go b/w the phases – a phase shift operator.

It turns out that this operator is,

$$U(\theta) = e^{-i\theta\hat{N}}$$

This is what you would expect, as all translation operators look like this –

$$T(x) \sim e^{-ix\hat{p}}$$

↳ These are conjugate.

So, note, the translation operator for some coordinate is

$$\exp\left(-i\,(\text{that coordinate})(\text{conjugate of that operator})\right)$$

Like $[\hat{x}, \hat{p}] = i\hbar$, there exists some commutation relation for $[\hat{\theta}, \hat{N}]$

Using BCH, we can state –

$$U^+(\theta)\, a\, U(\theta) = a e^{-i\theta}$$

Now, we ask,

$$\hat{U} a |\alpha\rangle = ?$$

Since $\hat{U}$ is unitary, it is not an antilinear operator.

$$\therefore \hat{U}(a|\alpha\rangle) = \hat{U}(\alpha|\alpha\rangle) = \alpha(\hat{U}|\alpha\rangle)$$

← goes through due to linearity.

$$\therefore \hat{U} a \hat{U}^+ (\hat{U} a |\alpha)$$

$$= a e^{i\theta} U|\alpha\rangle$$

Compare.

$$\therefore a\hat{U}|\alpha\rangle = \alpha e^{-i\theta} U|\alpha\rangle$$

↳ Note: Phase of state has suffered a translation to $-\theta$.

⊛ We may have some algebra issue here – should've started with U^+ instead.

☐ Do this derivation yourself.

So, $u(\theta) = e^{-i\theta\hat{N}}$ is identified as phase shift operator.

Now, we require an operator to reduce $|\alpha|$ too.

We define,

$$D(\alpha) = e^{\alpha a^{+} - \alpha^{*} a}$$

Say we separate exponential,

$$e^{A+B} = e^{A} e^{B} e^{\frac{1}{2}[A,B]}$$ ONLY true if $[A,B]$ is a c number.

Why? In BCH, all higher order commutators go to zero.

$$[A \text{ or } B\ [A,B]] = 0$$

$$\therefore D(\alpha) = e^{\alpha a^{+}} e^{-\alpha^{*} a} e^{-\frac{1}{2}|\alpha|^2}$$

Note, $H' = \alpha a^{+} - \alpha^{*} a$

$$\Rightarrow (H')^{+} = \alpha^{*} a - \alpha a^{+}$$
$$= -H'$$

(Anti-Hermitian)

$$\therefore D^{+}(\alpha) = D^{-1}(\alpha)$$ → It is unitary ☐ Check

So,
$$D^{+}(\alpha)\, a\, D(\alpha) = (\alpha + a)$$

☐ Exercise: Try to switch $D^{+}(\alpha)$ and a,

i.e. $D^{+}(\alpha) a = a D^{+}(\alpha) +$ something.

Then, $D^{+}(\alpha) D(\alpha) = \mathbb{I}$

We start with, again,

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

We wish to find out what happens to states like $a u^{+} |\alpha\rangle = \alpha e^{-i\theta} u^{+}|\alpha\rangle$ here.

⊛ Class tomorrow at 3 P.M at G02