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Schroedinger eq^n $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$

admits a conservation equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

where $\rho = \psi^* \psi$

$$\vec{J} = \frac{\hbar}{2mi}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$$

If $\int_V d^3x \, (\vec{\nabla}\cdot\vec{J}) = \int_S \vec{J}\cdot d\vec{s} = 0$ (for eg ψ vanishes at boundary)

then $\int d^3x \, \psi^*\psi$ is conserved.

⊛ Max Born (1926) → (Most accepted interpretation)

He gave a statistical interpretation.

$\rho = \psi^*(x,t)\psi(x,t)$ is the probability density of finding the particle at point x

⊛ Convention (as in statistics):

Normalization ⇒ Total probability = 1

In QM: Convention is to ~~no~~ normalize wave function as,

$$\int d^3x \, \psi^*\psi = 1$$

→ dimension 3+1

⊛ $\boxed{Q1/A4}$: Show that for the infinite square well potential (as being studied in class), the normalized energy eigenstates can be expressed as $\psi_1 = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$ (do it for time independent)

⊛ A will be fixed from this. — The normalization fixes it.

(*) Consider two solutions (say) ψ_1 and ψ_2 such that both satisfy Schroedinger equation: s.t

$$\boxed{i\hbar \frac{\partial \psi_1}{\partial t} = \hat{H}\psi_1} \text{ and } \boxed{i\hbar \frac{\partial \psi_2}{\partial t} = \hat{H}\psi_2}$$

are both true.

<u>Claim</u>: Any arbitrary linear combination of ψ_1 and ψ_2 ~~at~~ is also a solution of the Schroedinger equation.

<u>Proof</u>: Consider $\psi_3 = c_1\psi_1 + c_2\psi_2$

c_1 and c_2 are two Complex numbers.

<u>LHS</u> →

$$i\hbar \frac{\partial \psi_3}{\partial t}$$

$$= c_1 i\hbar \frac{\partial \psi_1}{\partial t} + c_2 i\hbar \frac{\partial \psi_2}{\partial t}$$

$$= c_1 \hat{H}\psi_1 + c_2 \hat{H}\psi_2$$

$$= \hat{H}(c_1\psi_1 + c_2\psi_2)$$

$$= \hat{H}\psi_3 \quad (\text{RHS})$$

→ This result is Known as the principle of linear superposition.

⇒ All solutions of the Schroedinger solution forms a linear vector space

(*) <u>Define</u>: Inner product or dot product on this vector space as,

$$(\psi, \psi) = \langle \psi | \psi \rangle = \int d^3x \, \psi^* \psi$$

$\boxed{E1}$ Compute the inner product between the two following energy eigenstates.

$$\psi_1 = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

$$\psi_2 = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right)$$

Soln: $(\psi_1, \psi_2) \equiv \psi_1 \cdot \psi_2 = \langle \psi_1 | \psi_2 \rangle$

$$= \int_0^a \psi_1 \cdot \psi_2 \, dx$$

$$= \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right) dx$$

$$= 0$$

They are orthogonal to each other. (w.r.t their given linear product)

* For different eigenvalues, the eigenstates are orthogonal.

* $\boxed{A4/Q2}$: Show that all energy eigenstates form an orthonormal set of vectors, i.e. $(\psi_m, \psi_n) = \delta_{m,n}$

Where $\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$

* If a linear vector space say V along with an inner product $\langle \cdot | \cdot \rangle$ is such that for all vectors ψ,

$$\langle \psi | \psi \rangle < \infty$$

i.e, $(\psi, \psi) \equiv \langle \psi | \psi \rangle = \int d^3x \, \psi^* \psi < \infty$

i.e, squared norm is $||\psi^2|| = (\psi, \psi)$ is finite.

i.e, ψ is square integrable

Then such vector space is called a Hilbert Space.