12th Jan 2024

Assignment on Diff'ble functions will be discussed

① Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Show that $f'(0)$ exists. Find $f'(0)$

Let us start with ③

③ Let $f : \mathbb{R} \to \mathbb{R}$ be continuously diff'ble at c. Let $x_n < c < y_n$ be such that ~~x_n~~ $y_n - x_n \to 0$. Show that

$$\lim_{n \to \infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(c)$$

Looks like we can use mean value theorem.

By MVT, $\dfrac{f(y_n) - f(x_n)}{y_n - x_n} = f'(c_n)$ where $x_n < c_n < y_n$

Since $x_n < c < y_n$

$\Rightarrow |c_n - c| \leq |y_n - x_n|$

Since $y_n - x_n \to 0 \Rightarrow c_n \to c$

Since f' is continuous,

$$\lim_{n \to \infty} f'(c_n) = f'(c)$$

Now, let us remove a condition. Continuously diff'ble $\to$ diff'ble at c

~~Prove~~, ~~Lim~~

Problem: $f : \mathbb{R} \to \mathbb{R}$ be diff'ble at c. Assume $x_n < c < y_n$ and $y_n - x_n \to 0$

Prove, $\lim_{n\to\infty} \dfrac{f(y_n) - f(x_n)}{y_n - x_n} = f'(c)$

$$f(x) = f(c) + f_1(x)(x-c)$$

And $f'(c) = f_1(c)$

$$\therefore\ f(y_n) = f(c) + f_1(y_n)(y_n - c)$$

$$f(x_n) = f(c) + f_1(x_n)(x_n - c)$$

$$f(y_n) - f(x_n) = f_1(y_n)(y_n - c) - f_1(x_n)(x_n - c)$$

$$\Rightarrow \cancel{f(y_n) - f(x_n) = f_1(y_n)\, y_n - f_1(x_n)\, x_n - c\,(f_1(y_n) + f(x_n))}$$

$$\Rightarrow \cancel{\frac{f(y_n) - f(x_n)}{y_n - x_n} = \frac{f_1(y_n)\, y_n - f_1(x_n)\, x_n - c\,(f_1(y_n) - f_1(x_n))}{y_n - x_n}}$$

$$\Rightarrow f(y_n) - f(x_n) = f_1(y_n)\, y_n - c f_1(y_n) - f_1(x_n)\, x_n + f_1(x_n)\, c$$

$$\Rightarrow f(y_n) - f(x_n) = f_1(y_n)\, y_n - f_1(x_n)\, x_n - c\,(f_1(y_n) - f_1(x_n))$$

The calculation is off somehow.

$$\frac{f(y_n)-f(x_n)}{y_n-x_n} = \frac{(f(y_n)-f(c))-(f(x_n)-f(c))}{(y_n-x_n)}$$

$$= \frac{\frac{f(y_n)-f(c)}{y_n-c}(y_n-c) + \frac{f(c)-f(x_n)}{c-x_n}(c-x_n)}{y_n-x_n}$$

$$= \alpha_n \frac{y_n-c}{y_n-x_n} + \beta_n \frac{c-x_n}{y_n-x_n}$$ $\to$ Convex comb of α and β

Why? $\frac{y_n-c}{y_n-x_n} + \frac{c-x_n}{y_n-x_n} = 1$

$\alpha_n \to f'(c)$

$\beta_n \to f'(c)$

$\alpha, \beta \in \mathbb{R}$ for $t \in [0,1]$, $\alpha t + (1-t)\beta$

Note, $\min(\alpha,\beta) \leq \alpha t + (1-t)\beta \leq \max(\alpha,\beta)$

Then,

$$\min\{\alpha_n, \beta_n\} \leq \frac{f(x_n)-f(y_n)}{x_n-y_n} \leq \max\{\alpha_n,\beta_n\}$$

Since $\alpha_n, \beta_n \to f'(c)$, By sandwitch

$\underline{\underline{Pb}}$: If $f:\mathbb{R}\to\mathbb{R}$ be diff'ble at c, then

$$\lim_{h\to 0} \frac{f(c+h)-f(c-h)}{2h} = f'(c)$$

(*) In general Does not mean that f is diff'ble at c