21st January 2025 (Tuesday) →

(*) Coherent States will be included in the syllabus.

Reminder →

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

Complex, as a is not Hermitian.

We also know,

$$\langle H\rangle_{|\alpha\rangle} = \langle\alpha|H|\alpha\rangle = \hbar\omega\left[\langle\alpha|a^\dagger a + \tfrac{1}{2}|\alpha\rangle\right]$$

$$= \hbar\omega\left[\langle\alpha|a^\dagger a|\alpha\rangle + \tfrac{1}{2}\underbrace{\langle\alpha|\alpha\rangle}_{1}\right]$$

$$= \hbar\omega\left[\alpha^*\alpha\langle\alpha|\alpha\rangle + \tfrac{1}{2}\right]$$

$$= \hbar\omega\left[|\alpha|^2 + \tfrac{1}{2}\right]$$

(*) Lowest value of α is 0.

But since α is non-Hermitian, we have to show that its eigenstates form an orthonormal eigenbasis.

Phase shift operator: $U(\theta) = e^{-i\theta\hat{N}}$

Why? $\alpha = r e^{i\theta_\alpha}$ → So we require operators to give us freedom in r and θ.

$$\hat{U}(\theta)|\alpha\rangle = |e^{-i\theta}\alpha\rangle$$

$$= |\alpha'\rangle$$

$$\Rightarrow a|\alpha'\rangle = r e^{i(\theta_\alpha - \theta)}|\alpha'\rangle$$

→ So, this operator $U(\theta)$ allows us to move around in phase space.

When we normally require space translation,

$$T = e^{-i\hat{p}x_0}, \quad T|x\rangle = |x + x_0\rangle$$

Note: $\hat{p}$ and $\hat{x}$ are conjugates, i.e. $[\hat{p}, \hat{x}] \neq 0$ and $\hat{p}$ is the generator of translation on x.

∴ Looking at, $U(\theta)$,

$$[\hat{\theta}, \hat{N}] \neq 0$$, i.e, they are conjugates.

→ We can not have simultaneous eigenstates of $\hat{\theta}$ and $\hat{N}$.

We now try to act on coherent states with translation operator.

$$\hat{T}(x_0) = e^{-i\hat{p}x_0/\hbar}$$

⊛ Next day: Some basic Group Theory. (maybe not, on the extra class)

Some rules →

- $T(x_1)\, T(x_2) = T(x_1+x_2)$ [Group closure]

Why? $e^A e^B = e^{A+B}$ provided $[A,B]=0$

- Also, $\hat{T}$ is unitary.

$\therefore (\hat{T}^+)(\hat{T}) = \underbrace{e^{\frac{i}{\hbar}\hat{p}x_0}\; e^{-\frac{i}{\hbar}\hat{p}x_0}}_{=\mathbb{I}}$

$\Rightarrow \hat{T}^+(x_0) = \hat{T}^{-1}(x_0)$

Schroedinger: $\hat{T}|\psi\rangle$

Heisenberg: $\hat{T}^+\hat{x}\,\hat{T} = \hat{x} + x_0\mathbb{I}$ (BCH to prove)

Take a definition,

we will show this.
This is currently just a claim

$\hat{T}(x_0)\,|0\rangle = |\tilde{x}_0\rangle = |\alpha\rangle$

we deal with this now.

Dealing with it in coordinate, not energy, basis.

$\hat{T}(x_0)\;|0\rangle = |\tilde{x}_0\rangle$

Adjoint,

$\langle 0|\hat{T}^+(x_0) = \langle\tilde{x}_0| = \langle 0|\hat{T}(-x_0)$

$\Rightarrow \langle\tilde{x}_0|\tilde{x}_0\rangle = \langle 0|\hat{T}^+(x_0)\hat{T}(x_0)|0\rangle$

$= \langle 0|0\rangle$

$= 1$

Q All the symmetries in QM are represented by unitary and anti-unitary transformation.

Why? Because it preserves the inner product.

We now try to find the wave function.

$\langle x|\alpha\rangle = \psi_\alpha(x)$

Now,

$\langle x|\tilde{x}_0\rangle = \psi_{x_0}(x) = \langle x|\hat{T}(x_0)|0\rangle$

$= \langle x-x_0|0\rangle$

(*) Note: The (−1) minus sign might be being flipped here.

What it should be is,

$$\hat{T}(x_0)\,|x\rangle = |x - x_0\rangle$$

$$\langle x|\hat{T}(x_0) = \langle x + x_0|$$

$$\therefore\ \psi_{x_0}(x) = \langle x + x_0|0\rangle$$

$$= \psi_0(x + x_0)$$

So, $\langle\tilde{x}_0|\hat{x}|\tilde{x}_0\rangle = x_0$

whereas,

$\langle 0|\hat{x}|0\rangle = 0$

$\rightarrow$ The ground state wavefunction has been <u>shifted</u> by x_0

What is the expectation of momentum operator?

$$\langle\tilde{x}_0|\hat{p}|\tilde{x}_0\rangle = \langle 0|\hat{T}^{\dagger}(x_0)\,\hat{p}\,\hat{T}(x_0)|0\rangle$$

Commute

$$= \langle 0|\hat{p}|0\rangle$$

$$= 0$$

So, the expectation remain the same. So, we may conjecture that the wave function was just shifted.

<u>BUT</u>, this only holds if fluctuations have not increased.

☐ What does this even mean? I don't think that I know what fluctuations are.

☐ <u>Tutorial Exercise</u>: Work out $\hat{T}|x\rangle = |x - x_0\rangle$ (Taylor expand to linear)

22nd January 2025 (Wednesday) →

We clear up the algebra related to the translation operator.

$$e^{-\frac{i}{\hbar}\hat{p}x_0}|x\rangle = ?$$

$$\downarrow$$

$$\hat{x}|x\rangle = x|x\rangle$$

Now, $e^{-i/\hbar \hat{p} x_0} |x\rangle \simeq \left(1 - \frac{i}{\hbar}\Delta x \hat{p}\right)|x\rangle$

$\hat{p} = -i\hbar \frac{\partial}{\partial x}$

$$\therefore e^{-i/\hbar \hat{p} x_0} = |x\rangle - \Delta x \underbrace{\frac{\partial}{\partial x}|x\rangle}_{} \simeq \Delta |x\rangle$$

↳ We may define this as,

$$\lim_{\Delta x \to 0} \frac{|x+\Delta x\rangle - |x\rangle}{\Delta x} = \lim_{\Delta x \to 0} \frac{|x\rangle - |x-\Delta x\rangle}{\Delta x}$$

$$\Rightarrow e^{-i/\hbar \hat{p} x_0} = |x\rangle - \left(\lim_{\Delta x \to 0} |x+\Delta x\rangle - |x\rangle\right)$$

$$= \lim_{\Delta x \to 0} |x+\Delta x\rangle$$

⊛ Formally(?) →

$$\exp\left[-\frac{i}{\hbar}\hat{p}x_0\right]|x\rangle = |\psi\rangle$$

$$\Rightarrow \hat{x}|\psi\rangle = (x + x_0)|\psi\rangle$$

Identities used →

$\hat{C} = [\hat{A}, \hat{B}]$

$[\hat{A}, \hat{C}] = 0$

$[\hat{B}, \hat{C}] = 0$

So,

① $e^{-\hat{B}}\hat{A}e^{B} = (\hat{A} + \hat{C})$

② $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{\hat{C}/2} = e^{\hat{A}+\hat{B}+\hat{C}/2}$

③ $e^{-\hat{B}}f(\hat{A})e^{\hat{B}} = f(\hat{A}+\hat{C})$

④ $[\hat{A}, e^{-\hat{B}}] = \hat{C}e^{\hat{B}}$

Let us use, $[\hat{A}, e^{\hat{B}}] = \hat{C}\hat{B}$

$Ae^B - e^B A = Ce^B$

$\Rightarrow Ae^B =$ Something. Partially convinced that something is off here.

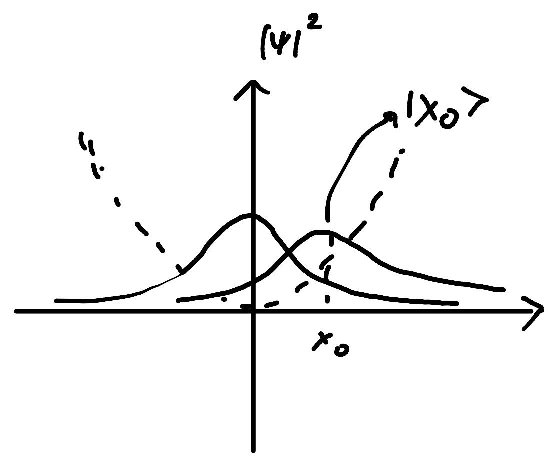
☐ Check the derivation for the action independently.

So,

$|x_0\rangle = \hat{T}(x_0)\,|0\rangle$

$\therefore \langle \tilde{x}_0 | \hat{x} | \tilde{x}_0 \rangle = x_0$

$\therefore \langle \tilde{x}_0 | \hat{p} | \tilde{x}_0 \rangle = 0$



Now, we calculate expectation.

☐ Tutorial HW: $T^{\dagger}(x_0)\,\hat{x}\,T(x_0) = \hat{x} + x_0 I$

$[\hat{T}(x_0), \hat{p}] = 0$

Now,

$\langle \tilde{x}_0 | \hat{H} | \tilde{x}_0 \rangle = \langle 0 | \hat{T}^{\dagger}(x_0)\,\hat{H}\,\hat{T}(x_0)\,|0\rangle$

Now,

$$\hat{T}^{\dagger}(x_0)\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)\hat{T}(x_0)$$

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(\hat{x}+x_0)^2$$

Now, $\langle \tilde{x}_0 | \hat{H} | \tilde{x}_0 \rangle$

$$= \langle 0 | \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(\hat{x}+x_0)^2 | 0 \rangle$$

$$= \langle 0 | \frac{\hat{p}^2}{2m} | 0 \rangle + \frac{1}{2}m\omega^2\Big[\langle 0 | \hat{x}^2 | 0 \rangle$$

$$+ 2x_0 \langle 0 | \hat{x} | 0 \rangle$$

$$+ x_0^2 \langle 0 | 0 \rangle\Big]$$

$$\Rightarrow \langle \hat{x}_0 | \hat{H} | x_0 \rangle = \underbrace{\langle 0 | \hat{H} | 0 \rangle}_{\frac{1}{2}\hbar\omega} + \underbrace{\frac{1}{2}m\omega^2 x_0^2}_{\frac{1}{2}kx_0^2}$$

(*) Classical energy cost.

→ All of this is assumed to be at time $t = 0$

☐ Tutorial Hw: Find the Heisenberg equation of motion for $\hat{x}(t)$

In Heisenberg,

$$|\hat{x}, t\rangle = e^{-\frac{iHt}{\hbar}} |\tilde{x}\rangle$$

Adjoint,

$$\langle \hat{x}, t| = \langle \hat{x}| e^{+i\frac{Ht}{\hbar}}$$

$$\therefore \langle \tilde{x}, t | \tilde{x}, t\rangle = \langle \tilde{x} | e^{i\frac{H}{\hbar}t} \hat{x} e^{-i\frac{H}{\hbar}t} |\tilde{x}\rangle$$

$$= \langle \tilde{x} | \hat{x}(t) | \tilde{x}\rangle$$

$$\hookrightarrow \dot{\hat{x}}(t) = e [\hat{x}, H]$$

Solving, we get,

$$\langle \tilde{x} | x(t) | \tilde{x}\rangle = \langle \tilde{x} | \hat{x} \cos\omega t + \frac{1}{m\omega} \hat{p} \sin\omega t | \tilde{x}\rangle$$

BREAK

$$\rightarrow \langle \tilde{x} | x(t) | \tilde{x}\rangle = x_0 \cos\omega t + 0$$

Now,

$$\langle \tilde{x}, t | \hat{p} | \tilde{x}, t\rangle = \langle \hat{x} | e^{i\frac{H}{\hbar}t} \hat{p} e^{-i\frac{H}{\hbar}t} |\tilde{x}\rangle$$

$$= \langle \hat{x} | (\hat{p} \cos\omega t - m\omega \hat{x} \sin\omega t | \tilde{x}\rangle$$

$$= - m\omega x \sin\omega t$$

$$\therefore \langle \hat{p}\rangle = m \frac{d}{dt} \langle x\rangle$$

We try to find fluctuations now.

$\langle \tilde{x}_0 | \hat{x}^2 | \tilde{x}_0\rangle$ and $\langle \tilde{x}_0 | \hat{p} | \tilde{x}_0\rangle$ are required.

$$|\tilde{x}\rangle = \hat{T}(x_0) |0\rangle$$

$$\therefore \langle \tilde{x} | \hat{x}^2 | \tilde{x}\rangle = \langle 0 | \hat{T}^\dagger(x_0) \hat{x}^2 \hat{T}(x_0) |0\rangle$$

$$= \langle 0 | \hat{T}^\dagger(x_0) \hat{x} \hat{T}(x_0) \hat{T}^\dagger(x_0) \hat{x} \hat{T}(x_0) |0\rangle$$

$$= \langle 0 | (\hat{x} + x_0)^2 | 0\rangle$$

$$= \langle 0 | \hat{x}^2 | 0\rangle + 2 x_0 \underbrace{\langle 0 | \hat{x} | 0\rangle}_{0} + x_0^2 \underbrace{\langle 0 | 0\rangle}_{1}$$

Now,

$$\langle 0 | \hat{x}^2 | 0\rangle = \frac{\hbar}{2m\omega} + x_0^2$$

Putting in ~~and~~ arranging,

$$\Delta x = \sqrt{\langle \tilde{x} | \hat{x}^2 | \tilde{x}\rangle - \underbrace{\langle \tilde{x} | \hat{x} | \tilde{x}\rangle^2}_{x_0^2}} = \sqrt{\frac{\hbar}{2m\omega}}$$

In the normal ground state,

$$\langle 0|\hat{x}^2|0\rangle = \frac{\hbar}{2m\omega} \quad , \langle 0|\hat{x}|0\rangle = 0$$

$$\Rightarrow \Delta X = \sqrt{\frac{\hbar}{2m\omega}}$$

We donot really need to compute, $\langle\tilde{x}|\hat{p}^2|\tilde{x}\rangle$, as $[\hat{T}(x_0), \hat{p}^2] = 0$

$\Rightarrow$ The fluctuation in $\hat{p}$ remains the same.

Now, we look at how uncertainty changes with time.

$$\Delta x(t) = \langle\tilde{x},t|\hat{x}^2|\tilde{x},t\rangle$$

$$= \langle\tilde{x}|e^{i\frac{H}{\hbar}t}\hat{x}^2 e^{-i\frac{H}{\hbar}t}|\tilde{x}\rangle$$

So, we pull old trick of,

$$e^{i\frac{H}{\hbar}t}\hat{x}^2 e^{-i\frac{H}{\hbar}t} = e^{i\frac{H}{\hbar}t}\hat{x}e^{-i\frac{H}{\hbar}t}e^{i\frac{H}{\hbar}t}\hat{x}e^{-i\frac{H}{\hbar}t}$$

$$\therefore \langle\tilde{x}|(\hat{x}\cos\omega t + \frac{1}{m\omega}\hat{p}\sin\omega t)(\hat{x}\cos\omega t + \frac{1}{m\omega}\hat{p}\sin\omega t)|\tilde{x}\rangle$$

$$= \langle\tilde{x}|\left[\cos^2\omega t\,\hat{x}^2 + \frac{1}{(m\omega)^2}\sin^2\omega t\,\hat{p}^2 + \frac{\sin\omega t\cos\omega t}{m\omega}\{\hat{x},\hat{p}\}\right.|\tilde{x}\rangle$$

$\square$ Tutorial HW: Show that $\langle\tilde{x},t|\{\hat{x},\hat{p}\}|\tilde{x},t\rangle$

$$= \cos^2\omega t\,\langle\tilde{x}|\hat{x}^2|\tilde{x}\rangle + \frac{1}{(m\omega^2)}\sin^2\omega t\,\langle\tilde{x}|\hat{p}^2|\tilde{x}\rangle$$

We have already computed these. So,

$$\Delta x(t) = \left[\langle\tilde{x}|\hat{x}^2(t)|\tilde{x}\rangle - \langle\tilde{x}|\hat{x}(t)|\tilde{x}\rangle\right]^{1/2}$$

$$= \left[\frac{\hbar}{2m\omega} + x_0^2\cos\omega t - x_0^2\cos\omega t\right]^{1/2}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}$$

So, the uncertainty remains pinned to the uncertainty of vaccuum state.

We can also show that $\Delta p(t)$ follows same behaviour.

We also see,

$$\Delta x\,\Delta p = \frac{\hbar}{2}$$

Next, we try to expand the coherent state in Energy eigenstates.

Since $|\tilde{x}\rangle = \hat{T}(x_0)|0\rangle$, we would try to express $\hat{T}(x_0)$ in terms of number operator/creation/annihilation operator.

We Know,

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(a^+ - a)$$

$$\therefore \hat{T}(x_0) = \exp\left[\frac{-i}{\hbar} x_0 \sqrt{\frac{m\omega\hbar}{2}} \cdot (a^+ - a)\right]$$

$$= \exp\left[\frac{-i}{\hbar} x_0 \beta (a^+ - a)\right]$$

$$= \exp\left[\frac{-i}{\hbar} x_0 \beta a^+\right] \exp\left[\underbrace{\frac{i}{\hbar} x_0 \beta a}_{\text{Ignore, annihilates.}}\right] \exp\left[\frac{-i}{\hbar} x_0 \beta' [a^+, a]\right]$$

$$= e^{-\beta_1} e^{\beta_2 a^+}$$

Now, $e^{\# a^+}|0\rangle = \left(1 + \# a^+ + \frac{\#^2}{2!} a^{+2} + \dots\right)|0\rangle$

$$= \left(|0\rangle + \#|1\rangle + \#|2\rangle + \dots\right)$$

So, the coherent states are linear combination <u>all</u> the energy eigenstates.

26th January 2025 (Sunday) → Extra Class

$\langle x + x_0 | \psi \rangle$ → Arbitrary wave function.

Now, the definition (reminder) is →

$$\hat{X}|x + x_0\rangle = (x + x_0)|x + x_0\rangle$$

$$\hat{T}(x_0)|x\rangle = |x + x_0\rangle \xleftrightarrow[\text{Adj}]{} \langle x| \hat{T}^+(x_0) = \langle x + x_0|$$

So what is $\langle x + x_0 | \psi \rangle = ?$

Is it $\psi(x)$, $\psi(x + x_0)$ or $\psi(x - x_0)$

Note,

$$\langle x + x_0 | \psi \rangle = \langle x | \hat{T}^+(x_0) | \psi \rangle$$

$$= \psi(x - x_0)$$ → Makes sense, the wave func shifts right.

Randomness →

If you have a lot of electrons, we may have something like

↑ ↓ ↑ ↑ ↑ ↓ . . . .

up, or down, are assigned randomly. Also, all of these outcomes are uncorrelated! This naturally leads us to Poisson distribution in large N limit.

If you have two energy levels,

$$\begin{array}{ll} |\uparrow\rangle \;\text{———} & \\ & \updownarrow \Delta E \\ |\downarrow\rangle \;\text{———} & \end{array}$$

What happens if we have have an infinite temperature state here?

$$P(E_n) \propto \frac{e^{-\beta E_n}}{\mathrm{Tr}\, e^{-\beta H}}$$

Now, if $T \to \infty$, $\quad P(\downarrow) = P(\uparrow) = \frac{1}{2}$, as $\Delta E << k_B T$

They become <u>degenerate</u>.

So, a system with N states at infinite temp will have probability of $\frac{1}{N}$ for each state.

So, at $T=0$,

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \cdots \quad \longrightarrow$ No flips

$T+=1$

$\uparrow \quad \downarrow \quad \uparrow \quad \uparrow \quad \cdots \quad \longrightarrow$ $^nC_1$ ways of 1 flip

$\vdots$

until we reach $\frac{n}{2}$ flips $\rightarrow$ we can no longer say anything about the system.

We know, for a coherent state $\rightarrow$ (of QSHO)

$$(\Delta x(t))^2 = \frac{\hbar}{2m\omega} \quad \rightarrow \text{Time independent.}$$

$$\psi_0(x) = \langle x|0\rangle \sim e^{-\frac{x^2}{2d^2}}, \quad d = \sqrt{\frac{\hbar}{m\omega}}$$

So when does $\psi_0(x)$ 'essentially' dies off?

On the order of d, of course – it is standard deviation.

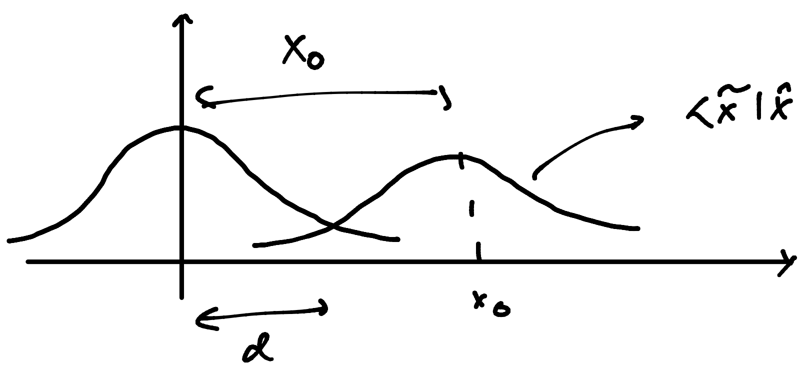
Note, $(\Delta x(t))^2 = \frac{\hbar}{2m\omega} = \frac{d^2}{2}$

$\hookrightarrow$ fundamentally, then, the wave function has fluctuations of this scale.

Also, we may represent this in fluctuations in momentum too.

$$(\Delta p(t))^2 = \frac{m\hbar\omega}{2} \Rightarrow \Delta p(t) = m\omega \frac{d}{2}$$

So, d is the fundamental scale here.
Is d small or large? We have nothing to compare to. So, we use translation to stretch state from $|0\rangle$ to some $|x_0\rangle$



Here,

$\frac{x_0}{d} < 1 \rightarrow$ Quantum

$\frac{x_0}{d} \gg \rightarrow$ Classical.

⊛ Sort of makes sense, if you stretch it far enough you can "discern" the state better than fluctuations.

Now,

$$|\tilde{x}\rangle = e^{-i/\hbar \; \hat{p} x_0} |0\rangle$$

$$= e^{\left(\frac{x_0}{\sqrt{2}d} (a^\dagger - a)\right)} |0\rangle$$

↓ Can be expressed as an algebraic function of a^+ and a

we know, $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$

$\therefore$ Let, $X = \frac{x_0}{\sqrt{2}d} a^+$, $Y = \frac{-x_0}{\sqrt{2}d} \hat{a}$

Now, $[X,Y] = -\frac{x_0^2}{2d^2} [a^+, a] = \frac{-x_0^2}{2d^2}$

↓ Scale parameter, as expected.

$$\therefore e^{-\frac{i}{\hbar}\hat{p} x_0} |0\rangle = e^{-\left(\frac{1}{4}\frac{x_0^2}{d^2}\right)} e^0 e^{\left(\frac{x_0}{\sqrt{2}d}\right) a^+} |0\rangle$$

$$= \exp\left[-\frac{1}{4}\frac{x_0^2}{d^2}\right] \cdot \underbrace{\left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x_0}{\sqrt{2}d}\right)^n (a^+)^n |0\rangle\right]}_{\text{Inf series of all energy eigenstates.}}$$

So, $\hat{T}(x_0)|0\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$

↳ we can identify these from the last expression.

We expect c_n's to be in general, complex.

Like if we expand,

$$\sigma_y = \#_1 |\uparrow\rangle_z + \#_2 |\downarrow\rangle_z$$

↳ Complex.

If we have an n-dimensional Hilbert space,

$$|\psi\rangle = \alpha_1 |1\rangle + \alpha_2 |2\rangle + \dots + \alpha_n |n\rangle$$

$$\langle i|j\rangle = \delta_{ij}$$

Say, we have,

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

Then, via normalization,

$$|\alpha|^2 + |\beta|^2 = 1 \Rightarrow x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$$

↳ 4-sphere

(*) Same stuff as in the very first lecture.

Onto, Bloch sphere.

(*) Note: Two dimensional Hilbert spaces can be thought of as points on a 3-sphere.

How?

Say, $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, a general state, with phase,

$$|\psi\rangle = \begin{pmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{pmatrix} \rightarrow \text{we draw out one phase.}$$

$$= e^{i\theta_1} \begin{pmatrix} r_1 \\ r_2 e^{i(\theta_2 - \theta_1)} \end{pmatrix}$$

What if we find density matrix of this?

Note, $\rho = |\psi\rangle\langle\psi| \rightarrow$ Is insensitive to phase (Note how the phase from bra and ket will cancel due to conjugation)

(*) HW!!

☐ Find density matrix of $|\psi\rangle$ as described here.

We will see that such a density matrix is parametrized by 3 values →
The sphere this creates is the Bloch Sphere.

So, in the energy eigenspace expansion of a coherent state, where we have only real coeffs, we get a lower dimensional sphere than expected.