Δ Random Variables: A real valued function on the sample space such that the pre-image of every interval is an event.

(*) For this course, by a r.v. we will always mean a real-valued r.v

Δ Probability distribution of a r.v -

Let $(\Omega, \mathcal{E}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable. Then X translates $(\Omega, \mathcal{E}, P)$ to $(\mathbb{R}, \mathcal{E}_X, P_X)$, where $\mathcal{E}_X = \{A \subseteq \mathbb{R} \mid X^{-1}(A) \in \mathcal{E}\}$

$$P_X(A) := P(X^{-1}(A))$$

Notation: Hence forth, ~~we'll~~ we'll use the notations $P_X(A)$ and $P(X \in A)$ interchangeably.

In particular, we will write $P_X((-\infty, a])$ as $P(X \leq a)$

Δ Random variables -

$$X : \Omega \to \mathbb{R}$$

$$\text{st } X^{-1}((-\infty, a]) \in \mathcal{E} \quad \forall a \in \mathbb{R}.$$

Eg: $(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - \frac{1}{n}]$

The fact that $\mathcal{E}$ is closed under countable unions and complementation, also carries over to $\mathcal{E}_X$

$(a, b) \Rightarrow \mathbb{R} \setminus ((-\infty, a] \cup [b, \infty))$

$$= \mathbb{R} \setminus ((-\infty, a] \cup (-\infty, b]^c)$$

Lemma: $\forall x \in \mathbb{R}$ we have $\{x\} \in \mathcal{E}_X$

Proof: $\{x\} = [x, x]$

$$= ((-\infty, x) \cup (x, \infty))^c$$

$$\underset{\mathcal{E}}{\cap\!\mid} \qquad \underset{\mathcal{E}}{\cap\!\mid}$$

and $\mathcal{E}$ is closed under countable unions and complementation.

So, $\{x\} \in \mathcal{E}_X$

Corollary: ~~Rf~~ $\forall$ r.v $X : \Omega \to \mathbb{R}$,

$P(X = x) = P_X(X^{-1}\{x\})$ is welldefined.

This is called PMF (Probability Mass function)

Δ CDF: Cumulative distribution function.

For $a \in \mathbb{R}$, we define $F_X(a) = P(X \leq a)$

⊛ The CDF is an increasing function, because for $a \leq b$

$$F_X(a) = P(X \leq a) = P(X^{-1}(-\infty, a]) \leq P(X^{-1}(-\infty, b]) = P(X \leq b) = F_X(b)$$