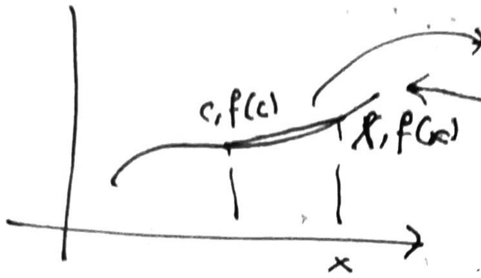
1st Jan 2023

Differentiability of a function at a point →

Defn: Let J be an interval and $c \in J$. Let $f : J \to \mathbb{R}$. We say that f is differentiable at c if

$$\boxed{\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists.}}$$

Slope of chord



The curve

This limit is denoted by $f'(c)$ / $\left.\frac{df}{dx}\right|_{x=c}$

Geometrically, $f'(c)$ denotes the ~~tangent~~ slope of the tangent at $(c, f(c))$ to the curve $\{(x, f(x)) : x \in J\}$

~~A ϵ-δ defn~~:

(Alter) f is diff at c if $\exists\, \alpha \in \mathbb{R}$ s.t

$$\lim_{x \to c} \frac{f(x) - f(c) - \alpha(x - c)}{x - c} = 0.$$

$\exists\, \alpha \in \mathbb{R}$ s.t

A $\epsilon - \delta$ defn: f is diff'ble at c if for $\epsilon > 0$ $\exists\, \delta > 0$ s.t

$$|f(x) - f(c) - \alpha(x - c)| < \epsilon |x - c|$$

$\forall x \in J$, $0 < |x - c| < \delta$

1 Let J be an interval and ~~cist~~ $c \in \mathbb{R}$.

A func $f: J \to \mathbb{R}$ is diff'ble at $c \in \mathbb{R}$ iff $\exists f_1 : J \to \mathbb{R}$ satisfying:

(i) $f(x) = f(c) + f_1(x)(x-c)$, $x \in J$

(ii) f_1 is continuous at c.

(*) Imp for higher dimensions.

→ $f_1(x)$ we know at all points but c

Proof ($\Rightarrow$) Assume f is diff at c.

Define,

$$f_1(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c}, & x \neq c \\ f'(c), & x = c \end{cases}$$

↳ Because f is diff'ble.

Then f and f_1 satisfy (i)

(ii) is also obviously satisfied, from definition.

($\Leftarrow$) Suppose f_1 exists

Since f_1 is cont at c,

$$\lim_{x \to c} f_1(x) = f_1(c)$$

$$\Rightarrow \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f_1(c)$$

$\therefore \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists $\Rightarrow f$ is diff'ble.

$$\therefore \boxed{f'(c) = f_1(c)}$$

(*) Moreover

(*) f_1 independant on c

○ Corollary: If f is diff'ble at c, then f is cont'd. at c.

(follows from (ii) of prev ~~st~~ theorem).

Remark: Let $f: J \to \mathbb{R}$ be differentiable at $c \in J$

Let $\delta > 0$ be s.t. $(c-\delta, c+\delta) \subset J$

Now consider the function $f|_{(c-\delta, c+\delta)}$

$f|_{(c-\delta, c+\delta)}$ is also diff'ble at c $\to$ Don't req info on f over entire interval.

Also, derivative of $f|_{(c-\delta, c+\delta)}$ and f at c are the same.

Derivative at a point is a local property - only depends on the vicinity.

○ Algebra of diff'ble

Let $f, g: J \to \mathbb{R}$, $c \in J$. Assume that f, g are diff at c. Then,

(i) $(f+g): J \to \mathbb{R}$ is differentiable at c and
$$(f+g)'(c) = f'(c) + g'(c)$$

(ii) for $\alpha \in \mathbb{R}$, (αf) is diff'ble at c and
$$(\alpha f)'(c) = \alpha f'(c)$$

(iii) fg is also diff'ble at c and
$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$
(Product rule)

(iv) Let ~~$f(c)$~~ $f(c) \neq 0$. Then $\left(\frac{1}{f}\right)$ is diff'ble at c (Note, no domain mentioned)

and $\left(\frac{1}{f}\right)'(c) = \frac{-f'(c)}{(f(c))^2}$

□ Prove using last theorem.

How? $g(x) = g(c) + g_1(x)(x-c)$
and add. — simple.

(*) We have to make sure that $\frac{1}{f}$ is def^n in open neighbourhood of c.

$\Rightarrow$ ~~f is~~ $\frac{1}{f}$ is diff'ble $\Rightarrow$ It is cont. at c

Since $f(c) \neq 0$, Then $\exists \delta > 0$ s.t $f(x) \neq 0$ $\forall x \in (c-\delta, c+\delta)$ (Neighbourhood property)

Now, we can study,

$$\frac{1}{f} : (c-\delta, c+\delta) \to \mathbb{R} \text{ is well defined.}$$

Since f is differentiable, $\exists$ $f_1 : J \to \mathbb{R}$ s.t

$f(x) = f(c) + f_1(x)(x-c)$

and f_1 is cont. at c and $f_1(c) = f'(c)$

We want to write,

$$\left(\frac{1}{f}\right)(x) = \left(\frac{1}{f}\right)(c) + \tilde{f}(x)(x-c)$$

for some $\tilde{f}$

So, $$\frac{1}{f(x)} = \frac{1}{f(c) + (x-c)f_1(x)}$$

$$\Rightarrow \frac{1}{f(x)} - \frac{1}{f(c)} = \frac{1}{f(c) + f_1(x)(x-c)} - \frac{1}{f(c)}$$

$$\Rightarrow \frac{1}{f(x)} - \frac{1}{f(c)} = \frac{-f_1(x)(x-c)}{f(c)\left(f(c) + f_1(x)(x-c)\right)}$$

$x \in (c-\delta, c+\delta)$

Here, $$\tilde{f}(x) = \frac{-f_1(x)}{f(c)\left(f(c) + f_1(x)(x-c)\right)} = \frac{-f_1(x)}{f(c)f(x)}$$

f_1 is cont. at c,
f is cont at c $\Rightarrow$ $\tilde{f}(x)$ is cont. at c

Now, $\left(\frac{1}{f}\right)'(c) = \tilde{f}(c)$

$$= -\frac{f_1(c)}{(f(c))^2} = \frac{-f'(c)}{(f(c))^2}$$

$\boxed{\text{T}}$ Let J be an interval

Let $c \in J$ is a point.

Let $f: J \to \mathbb{R}$ be diff'ble at c

Assume $f(J) \subseteq J_1$

Let $g: J_1 \to \mathbb{R}$ be diff'ble at $f(c)$.

Then ~~\$~~ $g \circ f : J \to \mathbb{R}$ is diff'ble at c and,

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

(Chain rule) (Derivative for composition)

☐ Prove using the theorem.