31st January 2024

⊛ Energy eigenvalues –

We know,

① $A_1 e^{-\kappa a} = B_1\left(e^{ila} + e^{-ila}\right) = 2B_1\cos(la)$

③ $\kappa A_1 e^{-\kappa a} = ilB_1\left(e^{-ila} - e^{ila}\right) = 2lB_1\sin(la)$

Taking their ratio,

$$\kappa = l\tan(la)$$

We know, $\kappa^2 = -\frac{2mE}{\hbar^2}$, $l^2 = \frac{2m(E+V_0)}{\hbar^2}$

$$\Rightarrow -\frac{2mE}{\hbar^2} = \frac{2m(E+V_0)}{\hbar^2}\tan^2\left(\sqrt{\frac{2ma^2(E+V_0)}{\hbar}}\right)$$

$$\Rightarrow E = -(E+V_0)\tan^2\left(\frac{\sqrt{2ma^2(E+V_0)}}{\hbar}\right)$$

$E = f(E)$ type equation

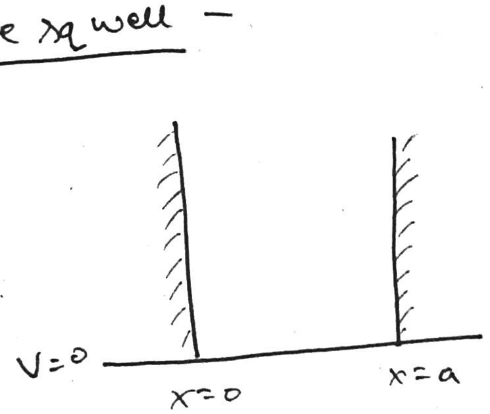
→ It is a transcendental equation for energy E.

It can be solved numerically using a computer.

⊛ Limit of finite sq well to finite sq well →

Infinite sq well –

(A)

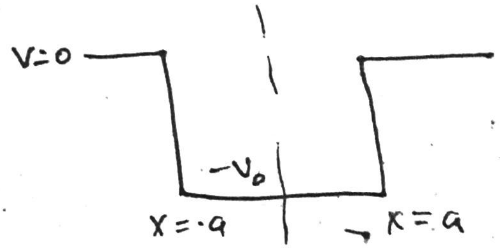


$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$n = 1, 2, 3,$

Finite sq well –

(B)



How do we go from configuration (B) to (A)

(i) Width: $a \rightarrow \frac{a}{2}$

(ii) Shift the origin of Energy : $(E+V_0) \rightarrow E$

(iii) Limit : $V_0 \rightarrow \infty$

$$E = -(E+V_0)\tan^2\left(\frac{\sqrt{2ma^2(E+V_0)}}{\hbar}\right)$$

(i) $a \rightarrow \frac{a}{2}$: $-V_0 + (V_0 + E) = -(E+V_0)\tan^2\left(\frac{\sqrt{ma^2(E+V_0)/2}}{\hbar}\right)$

(ii) $(E+V_0) \rightarrow E$

$$\frac{V_0}{E} - 1 = \tan^2\left(\frac{\sqrt{ma^2E/2}}{\hbar}\right)$$

(iii) $V_0 \rightarrow \infty \Rightarrow$

$$\tan\theta \rightarrow \infty$$

$$\Rightarrow \theta = \frac{n\pi}{2}, \quad n = 1, 3, 5, \dots$$

$$\therefore \frac{ma^2E}{2\hbar^2} = \frac{n^2\pi^2}{4} \Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$n = 1, 3, 5,$

The other half of the eigenvalues are in the case for $B_1 = -B_2$ (assymetrical)

⊛ Case $B_1 = -B_2$:

wave function, $\psi(x) = \begin{cases} A_1 e^{\kappa x} \\ 2i B_1 \sin(lx) \\ -A_1 e^{-\kappa x} \end{cases}$

$$\psi(-x) = -\psi(x)$$

⊛ $\boxed{A5/Q2}$: Repeat the steps for $B_1 = -B_2$ (as $B_1 = -B_2$) and show that the odd wave functions in the infinite square well limit lead to the energy eigenvalue

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}, \quad n = 2, 4, 6, \cdots$$

⊛ Class Test – 1 :

Feb 9, 2024 (Friday) at 2 P.M.

$$E = -(E + V_0) \tan^2\left(\frac{\sqrt{2ma^2(E+V_0)}}{\hbar}\right)$$

Case : Deep well (V_0 is large)

$$E_n \simeq -V_0 + \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

A Bound states : An quantum states with negative energy eigenvalue (i.e, $E_n < 0$)
(Assuming energy is zero at infinity)

for deep well : there are finite number of ~~bes~~ bound states.

Case : Shallow well (V_0 is small)

$V_0 \to 0$ so $(E + V_0) \to 0$ as $E < 0$

~~V_0~~ $\therefore$ E must be b/w V_0 and 0

(*) <u>NOT</u> a limit, it is just small

$\theta \to$ small, $\tan\theta \simeq \theta$

$$\therefore \; E = -\frac{2ma^2}{\hbar^2}(E+V_0)^2$$

$$\Rightarrow E^2 + \left(2V_0 + \frac{\hbar^2}{2ma^2}\right)E + V_0^2 = 0$$

Using quadratic formula,

$$E = -\left(V_0 + \frac{\hbar^2}{4ma^2}\right) \pm \sqrt{\left(V_0 + \frac{\hbar^2}{2ma^2}\right)^2 - V_0^2}$$

By construction,

$E + V_0 > 0$, only the +ve root survives.

$$E = -V_0 + \frac{\hbar^2}{4ma^2}\left[\sqrt{1 + \frac{8V_0ma^2}{\hbar^2}} - 1\right]$$

2nd Feb 2024

$$\sqrt{1 + \frac{8V_0ma^2}{\hbar^2}} = 1 + \frac{4V_0ma^2}{\hbar^2} - \frac{1}{8}\left(\frac{8V_0ma^2}{\hbar^2}\right)^2$$

So,

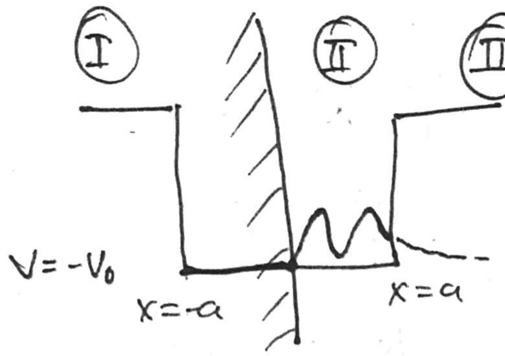
$$E = -V_0 + \frac{\hbar^2}{4ma^2}\left[\sqrt{1 + \frac{8V_0ma^2}{\hbar^2}} - 1\right]$$

$$= -\frac{\hbar^2}{4ma^2} \cdot \frac{1}{8}\left(\frac{8V_0ma^2}{\hbar^2}\right) < 0$$

$\Rightarrow$ It is a bound state.

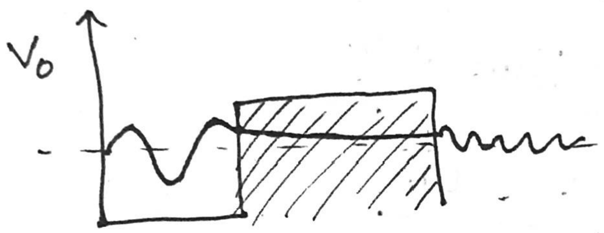
$\rightarrow$ There is atleast one bound state even for a shallow well.

* Suppose that the potential well has an infinite wall at $x=0$ on one side.



→ Solutions are the same as earlier, except with ~~new~~ the new boundary condition
$\psi(x=0) = 0$

→ picks up only the odd functions



• Something might "leak" out of a potential well.

* We have seen that trying to measure position causes error in momentum measurement, and vice-versa.

Let us denote :

$\hat{x} \rightarrow$ position operator

$\hat{p} \rightarrow$ momentum operator

$\psi(x) \rightarrow$ an arbitrary wavefunction.

$$\boxed{\hat{p}\,\psi(x) = \frac{\hbar}{i}\frac{d\psi(x)}{dx}}$$

$$\boxed{\hat{x}\,\psi(x) = x\,\psi(x)}$$

Now let us compute commutator,

$$\hat{x}\,\hat{p}\,\psi(x) - \hat{p}\,\hat{x}\,\psi(x) \qquad \alpha\, \hbar \text{ (we guess)}$$

LHS $\rightarrow$ $\hat{x}(\hat{p}\,\psi(x)) - \hat{p}\,(\hat{x}\,\psi(x))$

$$= \hat{x}\left(\frac{\hbar}{i}\frac{d\psi}{dx}\right) - \hat{p}\,(x\,\psi(x))$$

$$= \frac{\hbar}{i}\,x\,\frac{d\psi}{dx} - \frac{\hbar}{i}\frac{d}{dx}(x\,\psi(x))$$

$$= i\hbar\,\psi(x)$$

Now,

$$(\hat{x}\hat{p} - \hat{p}\hat{x})\psi = i\hbar\psi$$

$\hookrightarrow$ True $\forall\, \psi$

$$\Rightarrow \boxed{\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar}$$

(*) Let us define the commutator bracket between two operators, say $\hat{A}$ and $\hat{B}$ as,

$$\boxed{[\hat{A}, \hat{B}] \equiv \hat{A}.\hat{B} - \hat{B}\hat{A}}$$

(*) Canonical commutation relation (CCR) $\rightarrow$

$$\boxed{[\hat{x}, \hat{p}] = i\hbar} \rightarrow \text{First axiom of Quantum Mechanics}$$

Ex: Show that in position representation (x-representation) such that x operator acting on Ψ, i.e,

$$\boxed{\hat{x}\Psi = x\Psi(x)} \quad \text{(Def of x representation)}$$

The general form of the momentum operator is

$$\boxed{\hat{p} = \frac{\hbar}{i}\frac{d}{dx} + f(x)}$$

where f is an arbitrary function.

Proof: Consider,

$$LHS = [\hat{x},\hat{p}]\Psi = \hat{x}\hat{p}\,\Psi(x) - \hat{p}\hat{x}\,\Psi(x)$$

$$= \hat{x}\left(\frac{\hbar}{i}\frac{d\Psi(x)}{dx} + f(x)\Psi(x)\right) - \hat{p}\left(x\Psi(x)\right)$$

Some algebra $\rightarrow$ $= i\hbar\Psi = RHS$

$$\Rightarrow [\hat{x},\hat{p}] = i\hbar$$

$\longrightarrow$ $\hat{p} = \frac{\hbar}{i}\frac{d}{dx} + f(x)$ is a valid representation that satisfies the CCR.

$\boxed{A6/Q1}$ Show that in momentum representation i.e, $\boxed{\hat{p}\Psi(p) = p\Psi(p)}$, the position operator $\hat{x}$ can be expressed as

$$\boxed{\hat{x}\Psi(p) = -\frac{\hbar}{i}\frac{d}{dp}\Psi(p) + g(p)\Psi(p)}$$

Simple Harmonic Oscillator (SHO) $\rightarrow$

Newton's eqn $\quad m\ddot{x} = -Kx = -\frac{\partial}{\partial x}$

$$V = \frac{1}{2}Kx.$$