8th January 2024

○ Mean Value Theorem:

⊢ Let $f : [a,b]$ be such that

(i) f is contd. on $[a,b]$

(ii) f is diff'ble on (a,b)

Then, $\exists\ c \in (a,b)$ s.t

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Another example of an application of MVT is,

⊢ Let $f : [a,b] \to \mathbb{R}$ be diff s.t $\exists M > 0$

$$|f'(x)| \leq M \quad \forall x \in [a,b]$$

Then f satisfies

$$|f(x) - f(y)| \leq M|x-y|$$
$$\forall x, y \in [a,b]$$

$\Rightarrow$ f is ~~Lipschitz~~ Lipschitz. $\Rightarrow$ f is uniformly contd.

Recall:

Lipschitz: A function $f : [a,b] \to \mathbb{R}$ is lipschitz if $\exists$ ~~$M \geq 0$~~ $M > 0$ s.t

$$|f(x) - f(y)| \leq M|x-y|$$
$$\forall x, y \in [a,b]$$

Recall: Lipschitz $\Rightarrow$ Uniform continuous

Ex) Find a function $f : [0,1] \to \mathbb{R}$ which is diff'ble but $f' : [0,1] \to \mathbb{R}$ is not contd.

⊢ **Darboux Theorem:** Let $f : [a,b] \to \mathbb{R}$ be diff. Let $f'(a) < \lambda < f'(b)$. Then $\exists\ c \in (a,b)$ s.t.

$$f'(c) = \lambda$$

Proof (of MVT Lipschitz) $\rightarrow$

If $x = y$, then it is trivial.

If $x \neq y$, then by MVT,

$$\frac{f(x)-f(y)}{x-y} = f'(c)$$

(*) c depends on x and y.

for some c

But $\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \leq m$ (By hypothesis)

$\Rightarrow |f(x)-f(y)| \leq m|x-y| \quad \forall x, y$

Proof of Darboux's theorem $\rightarrow$

Let us define,

$$g : [a,b] \rightarrow \mathbb{R}$$

by

$$g(x) = f(x) - \lambda x$$

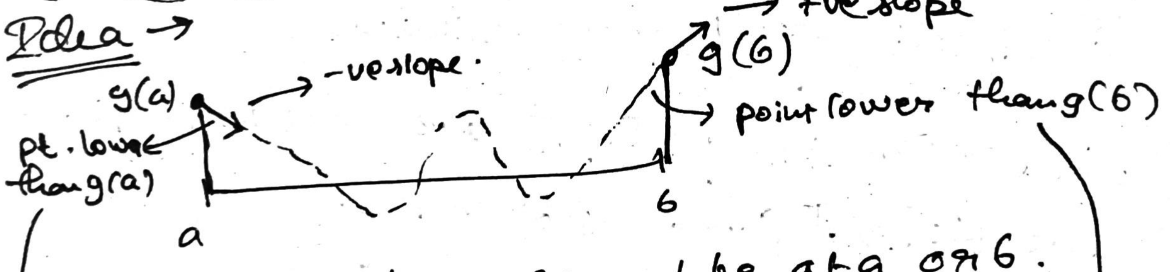
We have to find $c \in (a,b)$ such that $g'(c) = 0$

$\Rightarrow$ i.e, then $f'(c) = \lambda$

Note that $g'(a) = f'(a) - \lambda < 0$ (as $f'(a) < \lambda$)

Now, $g'(b) = f'(b) - \lambda > 0$ (as $f'(b) > \lambda$)

Idea $\rightarrow$



Minima or maxima cannot be at a or b.

Let $x_0 \in [a,b]$ s.t

$$g(x_0) = \min\{g(x) : x \in [a,b]\}$$

$\rightarrow x^0$ cannot be a or b. $\Rightarrow x_0 \in (a,b)$

Since $g : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, $\exists\, x_0 \in [a,b]$

s.t $g(x_0) = \min\{ g(x) : x \in [a,b] \}$

Claim ①: $x_0 \neq a$

Claim ②: $x_0 \neq b$

Pf. Claim ① $\to$ Since $g'(a) < 0$ $\exists\, x \in (a,b)$ s.t $g(a) > g(x)$

Therefore $g(a) > g(x_0)$

$\Rightarrow x_0 \neq a$

Similarly,

$x_0 \neq b$

$\Rightarrow x_0 \in (a,b)$

$\Rightarrow x_0$ is a local minima of g

$\Rightarrow g'(x_0) = 0$

(*) This is necessary because ~~g~~ x_0 is a global minima, we need to show it to be local minima to claim $g'(x_0) = 0$

7 Inverse function theorem $\to$

Let $f : [a,b] \to \mathbb{R}$ be diff'ble and $f'(x) \neq 0$ $\forall x \in (a,b)$, $I = (a,b)$

Then,

(i) $f : (a,b) \to \mathbb{R}$ is strictly monotone.

(ii) $f(I) = J$ is an interval

~~and $f^{-1}(J$ $f^{-1} : J \to \mathbb{R}$ exists and is continuous.~~

(iii) $(f^{-1}) : J \to \mathbb{R}$ is diff'ble and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

(As, $f^{-1}(f(x)) = x$, then differentiate)

Pf: either $f'(x) > 0 \ \forall x$

or

$f'(x) < 0 \ \forall x$

if this is not true, then $\exists\, x_1, x_2 \in (a,b)$ st

$f'(x_1) < 0, f'(x_2) > 0$

Then by Darboux's theorem,

$\exists\, c$ lies b/w x_1 and x_2 s.t

$f'(c) = 0$

$\Rightarrow\Leftarrow$ $f'(x) \neq 0 \ \forall x \in (a,b)$

$\Rightarrow$ WLOG we assume $f'(x) > 0 \ \forall x \in (a,b)$

Now this implies f is strictly increasing

* A cont.d. function takes an interval to an interval, if it is monotone.

$\Rightarrow$ proves (ii)

for (iii), define $f(c) = d$, $f(x) = y$

$$\lim_{y \to d} \frac{f^{-1}(y) - f^{-1}(d)}{y - d} = \lim_{x \to c} \frac{x - c}{f(x) - f(c)}$$

$$= \lim_{x \to c} \frac{1}{\frac{f(x) - f(c)}{x - c}}$$

$$= \frac{1}{f'(c)}$$