3rd Jan 2023

We begin by recounting the axioms of probability.

- Vitali sets (Example of non measurable sets)

Probability is an example of a measure on a set.

(a) $\Omega \in \mathcal{E}$

(b) If $A \in \mathcal{E}$, then $A^c \in \mathcal{E}$

(c) If $\{A_i\}^{\infty}$ are events, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{E}$

(d) $\exists\ P: \mathcal{E} \to [0,1]$ s.t

(1) $P(\Omega) = 1$

(2) If $\{A_i\}_{i=1}^{\infty}$ are pairwise disjoint

$(A_i \cap A_j = \phi \ \forall i \neq j)$

then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

If the sample space is infinite, it may cause a contradiction with (d)

A $(\Omega, \mathcal{E}, P)$: Probability space

→ If Ω is finite, we may always take $\mathcal{E} = 2^{\Omega}$.

Ex: Tossing a coin

$\Omega = \{H, T\}$

$\mathcal{E} = \{\phi, \{H\}, \{T\}, \{H,T\}\}$

first three axioms are satisfied.

$P(\phi) = 0$

$\boxed{P(A^c) = 1 - P(A)}$

Derivation → $P(\Omega) = 1$

$\Omega = A \cup A^c$

$\therefore A \cap A^c = \phi$

$\therefore P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$

$\Rightarrow P(A^c) = 1 - P(A)$

$P(H) = p$

$P(T) = 1 - p$

$P(\Omega) = 1$

2. Rolling a die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

3. Guessing the first letter of a stranger's name.

4. Guessing the no. of stars in the Universe (may be infinite) $\Omega = \{1, 2, 3, \ldots\}$

5. Guessing the temperature (uncountable infinite sample space as it is an interval)

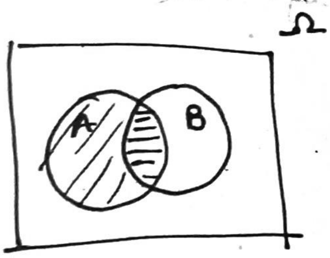
$\Omega = [55, 115]$

Mutually exclusive events —

$A, B \in \mathcal{E}$ are called mutually exclusive if

$A \cap B = \phi$

If $A \cap B = \phi$, $P(A \cup B) = P(A) + P(B)$



$A \cap B \neq \phi$

$A = (A \cap B^c) \cup (A \cap B)$

$\therefore P(A) = P(A \cap B^c) + P(A \cap B)$

$A \cup B = B \cup (A \cap B^c)$

$P(A \cup B) = P(B) + P(A \cap B^c)$

$\Rightarrow P(A \cap B^c) = P(A \cup B) - P(B)$

Putting this back,

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

By induction, this could be generalised to the principle of inclusion and exclusion. $\longrightarrow$ WORKS for all measures.

Δ Exhaustive set of events —

$S \subseteq \mathcal{E}$ is an exhaustive set of events (or the events in S are ~~so~~ said to be exhaustive)

if $\bigcup_{A \in S} A = \Omega$.

Ex $\rightarrow$ A, A^c are exhaustive events.

Let $\Omega = \{\omega_n\}_{n=1}^{\infty}$ be a countable sample space s.t.

$\mathcal{E} = 2^{\Omega}$

$$\Rightarrow \{\omega_n\} \in \mathcal{E} \ \forall\ \omega_n \in \Omega$$

$$\therefore P(\Omega) = P\left(\bigcup_{n=1}^{\infty} \{\omega_n\}\right) = \sum_{n=1}^{\infty} P(\{\omega_n\})$$

$$\Rightarrow \sum_{n=1}^{\infty} P(\{\omega_n\}) = 1$$

Ex: Let $p > 0$ be the probability of obtaining head if a coin is tossed. Show that if we keep on tossing the coin, then probability of obtaining a head eventually is 1.

$$\Omega = \overbrace{\{H, TH, TTH, TTTH, \ldots\}}^{A} \cup \overbrace{\{TTT \ldots \infty\}}^{B}$$

$P(H) = p$ $\qquad\qquad\qquad$ $\mathcal{E} = 2^{\Omega}$

$P(TH) = (1-p)p$

$P(TTH) = (1-p)^2 p$

$$P(\{H, TH, TTH, \ldots\})$$

$$= P(H) + P(TH) + P(TTH) + \cdots = P(A)$$

$$= p + (1-p)p + (1-p)^2 p + \cdots = \frac{p}{1-(1-p)} = 1$$

Roll a die ⊛ Outcomes define event.

$\Omega = \{1, 2, \ldots, 6\}$

$A = \{1, 3, 5\} \rightarrow$ odd outcomes (Event)

If you roll a die and if the roll results in an odd outcome, we say that event A has occurred.

$B = \{2, 3, 5\} \rightarrow$ prime outcomes (Event)

A and B can occur simultaneously.

Δ Equally likely –

Let $S \subseteq \mathcal{E}$. We say the events in S are equally likely if $P(A) = P(B) \;\; \forall A, B \in S$

More often, we say that the events A and B are equally likely if $P(A) = P(B)$

~~Δ Classical definition of probability –~~

Δ Random experiment –

An experiment of which we know the sample space but none of the ~~events except Ω occurs with~~ outcomes of the experiment occurs with certainty (probability 1).

Δ Non-random experiment –

Experiment for verification of physical, chemical biological / mathematical laws.

⊛ Segway to Riemann Hypothesis and number theoretic motivation ~~to~~ for probability.

△ Classical definition of probability —

☐ Make up your own definition — mutually exclusive, exhaustive, equally likely.