5th Jan 2023

△ **Defn:** Let J be an interval and let $f: J \to \mathbb{R}$ be a function.

A point $c \in J$ is said to be a local maximum

(i) c is an interior point, i.e
$\exists \delta > 0$ s.t $(c-\delta, c+\delta) \subseteq J$.

(ii) $f(x) \le f(c) \quad \forall x \in (c-\delta, c+\delta)$

△ **Defn:** Let $f: J \to \mathbb{R}$ be a function. A point $c \in J$ is said to be global maximum if

$$f(x) \le f(c) \quad \forall x \in J$$

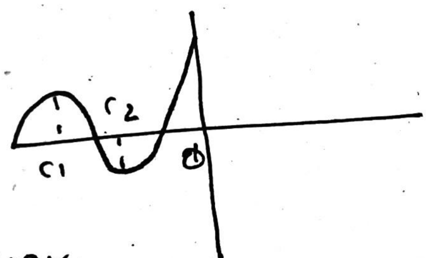
Ex: $f(x) = \sin c = \begin{cases} \frac{\sin x}{x}, & x = 0 \\ 1, & x = 0 \end{cases}$

$f(x) \le 1$, $f(0) = 1$

Local max, $M = \{c : \sin c = 1\}$

$c \in M \Rightarrow c$ is a local maxima.

Ex2:



$c_1 \to$ local max
0 is global max
0 is not a ~~global~~ local max

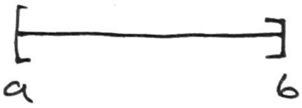
Ex3: $f(x) = x$, $x \in [0, 1]$

1 is not a local max (Not interior pt)

1 is global max.

$\boxed{}$ $f: J \to \mathbb{R}$ and c is a local max.

$\Rightarrow f'(c) = 0$



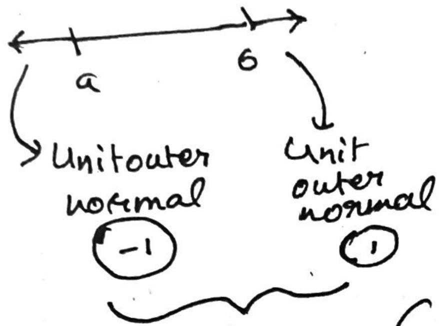
$a \to$ global max

$f'(a) \le 0$

$b \to$ global max

$f'(b) \ge 0$

Not going to be defined $\to$



Normal derivative

$f'(a)\nu_a \ge 0$

$$\nu_{a/b} = \begin{cases} -1, & a \text{ is left end} \\ +1, & a \text{ is right end.} \end{cases}$$

$\boxed{}$ Let $f: J \to \mathbb{R}$ be diff'ble

~~But~~ Let $c \in J$ be a local maxima of J.

Then $f'(c) = 0$

Proof: Since $c \in J$ is a local maxima, $\exists\, \delta > 0$ s.t

(i) $(c-\delta, c+\delta) \subseteq J$

and (ii) $f(x) \le f(c)$ $\forall x \in (c-\delta, c+\delta)$.

Now, $f'(c) = \lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$

$\Rightarrow \lim_{h \to 0^+} \dfrac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \dfrac{f(c+h) - f(c)}{h}$

for $h > 0$, $f(c+h) \le f(c)$

~~for h~~ then $\lim_{h \to 0^+} \dfrac{f(c+h) - f(c)}{h} \le 0$

for $h < 0$, $f(c+h) \leq f(c)$

then, $\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$

$\Rightarrow f'(c) = 0$

$\boxed{7}$ <u>Rolle's Theorem:</u>

Let $f : [a,b] \to \mathbb{R}$ ~~be~~ be s.t

(i) f is cont. on $[a,b]$

(ii) f' is cont. on (a,b)

(iii) $f(a) = f(b)$

$\exists\ c \in (a,b)$ s.t

$$f'(c) = 0$$

<u>Proof</u> → If I can show that f has a local maxima or minima, I am done.

If a global maxima or minima is an interior pt, it must be a local maxima or minima.

Since $f : [a,b] \to \mathbb{R}$ is cont. ~~there are~~ $\exists\ x_1, x_2$ ~~∈~~ $\in [a,b]$
↳ compact

s.t

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a,b]$$

☐ Look this up

(i.e) x_1 is global min and x_2 is a global max.

Consider two cases —

① $f(x_1) = f(x_2)$

Then $f(x) = f(x_1)$ (const func)

$\Rightarrow f'(x) = 0 \quad \forall x \in (a,b)$

② $f(x_1) \neq f(x_2)$

Now by assumptions, if x_1 and x_2 are boundary pt,

$f(x_1) = f(x_2)$ (By (iii))

∴ Either x_1 or x_2 is an interior pt of $[a,b]$

~~→ Either $f'(x_1)$~~

~~⇒ If $x_1 \notin [a,b]$ then f'~~

→ If $x_1 \notin \{a,b\}$, then $f'(x_1)=0$

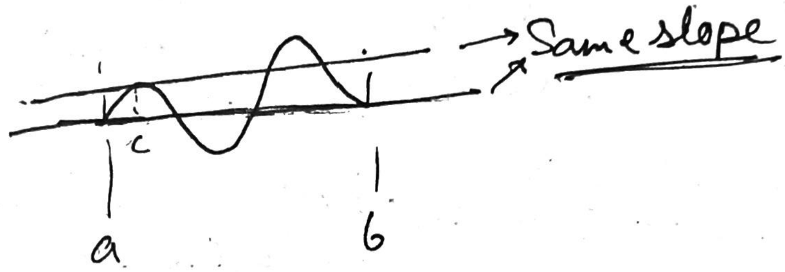
Now, application of Rolle's theorem.

$\boxed{-}$ <u>Mean Value Theorem (MVT)</u> →

Let $f:[a,b] \to \mathbb{R}$ be cont. on $[a,b]$ and diff in (a,b)

Then $\exists\, c \in (a,b)$ s.t

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$



<u>Proof</u>: Recall (using auxilliary function)

$\boxed{-}$ <u>Cauchy's Mean Value Theorem (CMVT)</u> →

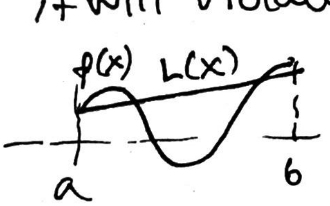
Let $f, g : [a,b] \to \mathbb{R}$ be diff'ble

Assume that $g'(x) \neq 0 \;\forall x \in (a,b)$

$\exists\, c \in (a,b)$ s.t

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$g'(x) \neq 0 \;\forall x \in (a,b)$ so $g(b) \neq g(a)$ as otherwise it will violate Rolle's Theorem.



Take $f(x) - L(x) = g(x)$

$g(x)$ satisfies rolle's, so find the rest where $g'(x) = 0$

for CMVT, choose λ such that

$$G(x) = f(x) - \lambda g(x)$$

s.t. $G(a) = G(b)$

Then apply Rolle's.

Applications of MVT :

① Let $f: J \to \mathbb{R}$ be diff and J. Assume that

$$f'(x) = 0 \ \forall x \in J$$

then

$$f(x) = \text{constant} \ \forall x \in J$$

$$f(x) = \begin{cases} 1, & x \in (1,2) \\ 2, & x \in (3,4) \end{cases}$$

Important that the domain is an interval.

$f'(x) = 0$

but $f(x)$ is not const $\Rightarrow$ As $(1,2) \cup (3,4)$ is not an interval.

② Let $f'(x) \geq 0 \ \forall x \in J$

then f is increasing

③ Let f' bdd $\Rightarrow$ f is Lipschitz continuous.