Copied from Sobarno's notes

1st March 2024

- Time independent →

$$\hat{H}\Psi_n = E_n \Psi_n$$

- Time-dependent →

$$i\hbar \frac{dT}{dt} = E_n T(t)$$

$$\Rightarrow \int \frac{dT}{T} = \frac{-iE_n}{\hbar} \int dt$$

$$\Rightarrow T(t) = T(0) e^{-i\frac{E_n t}{\hbar}}$$

full solution → (nth eigenvalue)

$$\Psi_n(t;x) = N e^{-\frac{iE_n}{\hbar}t} \Psi_n(x)$$

↳ Normalization const.

$$\Rightarrow |n;t\rangle = e^{-\frac{iE_n t}{\hbar}} |n\rangle$$

General Solution →

$$|\Psi;t\rangle = \sum_{n=0}^{\infty} c_n |n;t\rangle = \sum_{n=0}^{\infty} c_n e^{-\frac{iE_n t}{\hbar}} |n\rangle$$

Norm of $|\Phi, t\rangle$ →

$$\langle \Psi;t|\Psi;t\rangle = \left(\sum_n c_n^* e^{i\frac{E_n t}{\hbar}} \langle m|\right) \left(\sum_m c_m e^{-\frac{iE_m t}{\hbar}} |n\rangle\right)$$

$$= \sum_{m,n} c_n^* c_m e^{i\left(\frac{E_m - E_n}{\hbar} t\right)} \langle m|n\rangle$$

$$\Rightarrow \langle \psi;t|\psi;t\rangle = \sum_{m,n} c_n^* c_m e^{i\left(\frac{E_m-E_n}{\hbar}\right)t} \delta_{mn}$$

$$= \sum_n |c_n|^2 = \langle \psi,0|\psi,t=0\rangle$$

$\therefore$ Time evolution keeps total probability conserved || unitary evolution $\rightarrow$ Inner prod. invariant

$$|\psi\rangle \rightarrow A|\psi\rangle$$

$$|\phi\rangle = A|\phi\rangle$$

$$\therefore \langle A\phi|A\psi\rangle = \langle \phi|A^* A|\psi\rangle$$

$$= \langle \phi, \psi\rangle$$

Ex 1: Compute the expectation value

$$\langle \psi;t|\hat{x}|\phi;t\rangle \text{ where}$$

$$|\psi;t\rangle = |1;t\rangle$$

Proof: $|1;t\rangle = e^{-i\frac{E_n t}{\hbar}}$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^{+})$$

$$\therefore \langle 1;t|\hat{x}|1;t\rangle = \langle 1|(a+a^{+})|1\rangle = 0$$

Thus, this $|1,t\rangle$ is a purely quantum mechanical state.

Ex: Compute $\langle \hat{x} \rangle_\psi$ where

$$|\psi; x\rangle = \frac{1}{\sqrt{2}}\left(|0;t\rangle + |1;t\rangle\right)$$

Proof: $\langle \psi; x | \hat{x} | \psi; x \rangle$

$$|0;t\rangle = e^{-\frac{iE_0 t}{\hbar}} |0\rangle$$

$$|1;t\rangle = e^{-\frac{iE_0 t}{\hbar}} |1\rangle$$

$$\hat{x}|\psi;x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(e^{-\frac{iE_0 t}{\hbar}}(a+a^\dagger)|0\rangle + e^{-\frac{iE_1 t}{\hbar}}|1\rangle\right)$$

$$\langle \psi; t | \hat{x} | \psi; x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{\sqrt{2}}$$

$$\left(e^{-\frac{iE_0 t}{\hbar}}(a+a^\dagger)|0\rangle + \langle 0| e^{+i\frac{E_0 t}{\hbar}} + \langle 1| e^{\frac{iE_1 t}{\hbar}}\right)\left((a+a^\dagger)|0\rangle e^{-i\frac{E_0 t}{\hbar}} + (a+a^\dagger)|1\rangle e^{-\frac{iE_1 t}{\hbar}}\right)$$

$$= \underbrace{\langle 0|(a+a^\dagger)|0\rangle}_{=0} + \underbrace{\langle 1|(a+a^\dagger)|1\rangle}_{=0}$$

$$+ e^{i\frac{(E_0 - E_1)t}{\hbar}}\langle 0|a+a^\dagger|1\rangle$$

$$+ e^{i\frac{(E_1 - E_0)t}{\hbar}}\langle 1|a+a^\dagger|0\rangle$$

$$= \lambda_1\left(\langle 0|a|1\rangle + \langle 0|a^\dagger|1\rangle\right)$$

$$+ \lambda_2\left(\langle 1|a|0\rangle + \langle 1|a^\dagger|0\rangle\right)$$

$$= \lambda_1\left(\langle 0|0\rangle + \langle 0|a^\dagger|1\rangle\right)$$

$$+ \lambda_2\left(\langle 1|0\rangle + \langle 1|1\rangle\right)$$

$$= \lambda_1 + \lambda_2$$

$$= \left(\sqrt{\frac{\hbar}{2m\omega}}\right)\left(\frac{1}{2}\right)\left[e^{i\frac{(E_0-E_1)t}{\hbar}} + e^{i\frac{(E_1-E_0)t}{\hbar}}\right.$$

We know, $\cancel{E_0 - E_1 = \hbar\omega}$

$$E_1 - E_0 = \hbar\omega$$

$$\therefore \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{2}\right)\left[e^{i\omega t} + e^{-i\omega t}\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{2}\right)\cos(\omega t)$$ * Clarify these notes

* $\boxed{A8/Q1}$ Show $\langle \psi; t | \hat{p} | \psi; t \rangle$

$$= \sqrt{\frac{m\omega\hbar}{2}} \sin(\omega t)$$

$$p = m\frac{dx}{dt}$$

$$\langle \hat{p} \rangle = -m\frac{d}{dt}\langle \hat{x} \rangle$$

6th March 2024

Recall: For a given $|\psi\rangle = \frac{1}{\sqrt{2}}(|0;t\rangle + |1;t\rangle)$

$$\langle \psi | \hat{x} | \psi \rangle = \langle \hat{x} \rangle_\psi = \sqrt{\frac{\hbar}{2m\omega}} \cos\omega t$$

and $\langle \hat{p} \rangle_\psi = \langle \psi | \hat{p} | \psi \rangle = -\sqrt{\frac{m\omega\hbar}{2}} \sin\omega t.$

We know classically that,

$$\boxed{p = m\frac{dx}{dt}}$$

* We note that,

$$\boxed{m\frac{d}{dt}\langle \hat{x} \rangle_\psi = \langle \hat{p} \rangle_\psi}$$

We also know that,

$$\frac{dp}{dt} = F = -Kx = -m\omega^2 x$$

We can also see that, Similar

$$\frac{d}{dt}\langle\hat{p}\rangle_\psi = -m\omega^2\langle\hat{x}\rangle$$

There are states in Q Mech in which expectation values of $\hat{x}$, $\hat{p}$ shows similar properties as classical mechanics.

→ Quantum to classical correspondence

→ Ehrenfest theorem

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0;t\rangle + |1;t\rangle\right)$$

$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{iE_0 t}{\hbar}}|0\rangle + e^{-\frac{iE_1 t}{\hbar}}|1\rangle\right)$$

↳ This is an example of a so-called wave packet. → a linear combination of two or more basis states

You can have a linear combination of as many states as possible as long as

$$\langle\psi|\psi\rangle < \infty$$

$$\Rightarrow \sum_n |c_n|^2 < \infty$$

Schroedinger eqn :

Energy eigenfunc

$$\boxed{\hat{H}\,\Psi_n(x) = E_n\,\Psi_n(x)}$$

Operator method

How to relate them?

$$\boxed{\hat{H}\,|n\rangle = E_n\,|n\rangle}$$

Kets

(*) Relation b/w $|n\rangle$ and $\Psi_n(x)$:

$\hat{A} \rightarrow$ vector.

What is the 'x' component of the vector A?

$$A_x = (\hat{e}_x, \hat{A})$$

We take the dot

$$= \langle \hat{e}_x, \hat{A} \rangle$$

Similarly

$$\boxed{\Psi_n(x) = \langle x | n \rangle}$$

(*) Action of $\hat{x}$ and $\hat{p}$ on position basis ket $|x\rangle$

So in this basis, by def, ~~sho~~ $\hat{x}$ should act as eigenvalue expression

$$\hat{x}\,|x\rangle = x\,|x\rangle$$

$$\hat{p}\,|x\rangle = \frac{\hbar}{i}\frac{d}{dx}|x\rangle$$

We find duals of this,

$$\langle x|\hat{x} = \langle x|\,x$$

$$\langle x|\hat{p} = \langle x|\,\frac{\hbar}{i}\frac{d}{dx} = \frac{\hbar}{i}\frac{d}{dx}\langle x|$$

as $\hat{x}$ and $\hat{p}$ are Hermitian.

Ground State :

$$|0\rangle \rightarrow \psi_0(x) = \langle x|0\rangle$$

We know that, $\hat{a}|0\rangle = 0$ (Null)

$$\Rightarrow \langle x|\hat{a}|0\rangle = 0$$

$$\Rightarrow \langle x|\hat{x} + \frac{i}{m\omega}\hat{p}|0\rangle = 0$$ as they are c-num

$$\Rightarrow x\langle x|0\rangle + \frac{i}{m\omega}\frac{\hbar}{i}\frac{d}{dx}\langle x|0\rangle = 0$$

$$\Rightarrow x\,\psi_0(x) + \frac{\hbar}{m\omega}\frac{d}{dx}\psi_0(x) = 0$$

$$\Rightarrow \frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}x\cdot dx$$

$$\Rightarrow \ln\psi_0 = -\frac{m\omega}{\hbar}x^2 + c$$

$$\rightarrow \boxed{\psi_0(x) = N e^{-\frac{m\omega}{2\hbar}x^2}}$$

Gaussian

Does this satisfy Schroedinger eqn?

$$\frac{-\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + \frac{1}{2}m\omega^2x^2\psi_0 = E\psi_0$$

$$\psi_0' = N e^{-\frac{m\omega}{2\hbar}x^2}(2x)\left(-\frac{m\omega}{2\hbar}\right)$$

$$\psi_0'' = N e^{-\frac{m\omega}{2\hbar}x^2}\left(-\frac{m\omega}{2\hbar}\right) + N(2x)\left(-\frac{m\omega}{2\hbar}\right)e^{-\frac{m\omega}{2\hbar}x^2}(2x)\left(-\frac{m\omega}{2\hbar}\right)$$

$$\Rightarrow \psi_0'' = N e^{-\frac{m\omega}{2\hbar}x^2}\left(-\left(\frac{m\omega}{\hbar}\right) + \left(\frac{m\omega}{\hbar}\right)^2 x^2\right)$$

$$\Rightarrow E = \frac{\hbar\omega}{2} - \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}\hbar\omega$$

$\Rightarrow \psi_0(x)$ is a solution of Schroedinger equation with $E = \frac{1}{2}\hbar\omega$

$\psi_0(x) \rightarrow$ is the ground state.

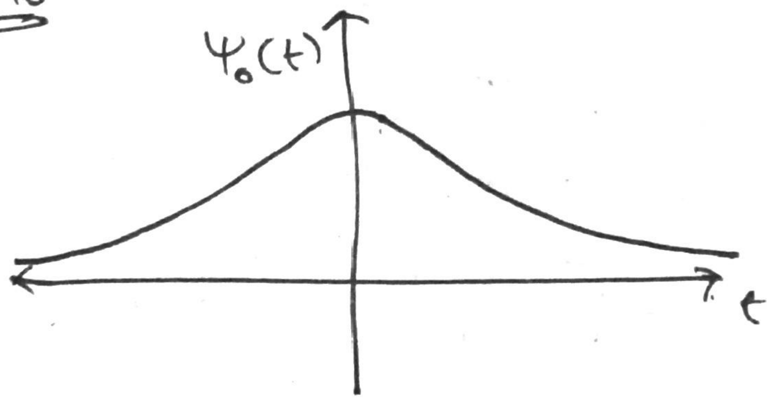
*$\boxed{\text{A8 | Q2}}$ Show that the normalization constant N for the state

$\psi_0(x) = N$ for the state

$$\psi_0(x) = N e^{-\frac{m\omega}{2\hbar}x^2}$$

is $N = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

<u>Plotting ψ_0:</u>



→ No node
→ 0 at $-\infty$ and $+\infty$
→ Ground state.

⊛ What is the position representation (x-repr) of the ket $|1\rangle$?

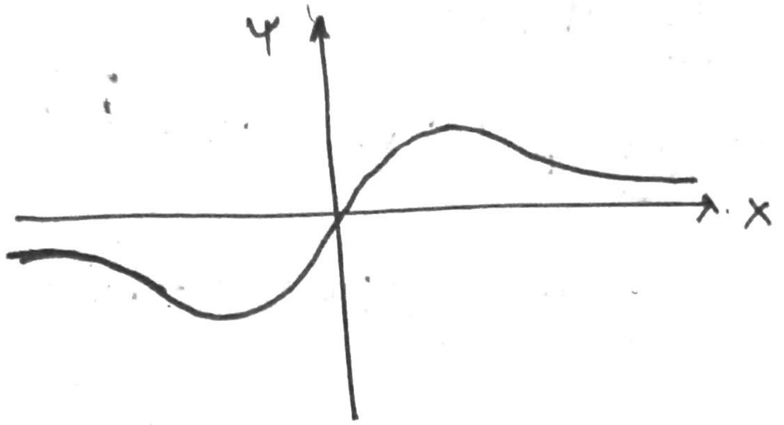
$$|1\rangle = \hat{a}^{+}|0\rangle$$

$$\psi_1(x) = \langle x|1\rangle = \langle x|\hat{a}^{+}|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x|\hat{x} - \frac{i}{m\omega}\hat{p}|0\rangle$$

$$\Rightarrow \psi_1(x) = \sqrt{\frac{m\omega}{2\hbar}}\Big[x\underbrace{\langle x|0\rangle}_{\psi_0} - \frac{\hbar}{m\omega}\frac{d}{dx}\underbrace{\langle x|0\rangle}_{\psi_0}\Big)$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left[x\psi_0 - \frac{\hbar}{m\omega}\frac{d\psi_0}{dx}\right]$$

$$\Rightarrow \psi_1(x) = \mathcal{N}\, x\, e^{-\frac{m\omega}{2\hbar}x^2}$$



⊛ $\boxed{A8/Q3}$ Using Schroedinger equation verify that $\psi_1(x)$ is a solution with energy eigenvalue $E = \frac{3}{2}\hbar\omega$

⊛ 1 class test before Spring break.

8th March 2024

Recall: $\boxed{\hat{H}|n\rangle = E_n|n\rangle}$

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega$$

$$|n;t\rangle = e^{-i\frac{E_n t}{\hbar}}|n\rangle$$

→ c-number.

$$\hat{H}|n;t\rangle = e^{-i\frac{E_n t}{\hbar}}\hat{H}|n\rangle$$

$$= E_n\left(e^{-iE_n t/\hbar}|n\rangle\right)$$

$$= E_n|n;t\rangle$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0;t\rangle + |1;t\rangle\right)$$

$$\hat{H}|\Psi\rangle = \frac{1}{\sqrt{2}}\left(\hat{H}|0;t\rangle + \hat{H}|1;t\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(E_0|0;t\rangle + E_1|1;t\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\frac{\hbar\omega}{2}\left(|0;t\rangle + 3|1;t\rangle\right) \neq \frac{1}{\sqrt{2}}\frac{\hbar\omega}{2}|\Psi\rangle$$

Not an eigenfunction.

(*) Destroys the original state — So how do we find the energy of wavepacket? (Not eigenval anymore)

Ex: Compute the expectation value of the Hamiltonian op $\hat{H}$ in the state $|\Psi\rangle$

$$\langle \hat{H} \rangle_\psi = \langle \psi | \hat{H} | \psi \rangle$$

$$= \frac{1}{\sqrt{2}} \left(\langle 0; t| + \langle 1, t| \right) \cdot \frac{1}{\sqrt{2}} \left(E_0 |0, t\rangle + E_1 |1, t\rangle \right)$$

$$= \frac{1}{2}(E_0 + E_1) = \hbar\omega$$

(*) As the eigenstates are normalized and are orthogonal to each other at all times.

Here we end SHO.

$$P \xrightarrow{\text{Do not comp}} \hat{P}$$

$\hat{P} \downarrow |\psi\rangle \rightarrow$ Initial values encoded here.

$P \leftrightarrow$ Comparable $\leftrightarrow \langle \hat{P} \rangle_\psi$ expectation w.r.t state

o Free particles in QM $\rightarrow$ (1+1 dim)

NLM 2: $\boxed{m \frac{d^2x}{dt^2} = F}$

If force is absent, we call it free particle.

i.e, if $F = 0$, then the particle is called a free particle.

$$F = -\frac{\partial V}{\partial x} = 0 \Rightarrow V = V_0$$

$\therefore$ In QM we still have a free parameter of V.

By convention, we choose it to be zero, i.e,

$V_0 = 0$

(*) Hamiltonian :

$$\boxed{H = \frac{p^2}{2m}}$$

Solving NLM, we have,

$v = \text{const}.$

$x = v_0 t + x_0$

So what is the probability of finding the particle at some point in space?

It can be anywhere, ~~ax~~ as t varies and x_0 could be anything

This causes a problem (??)

(*) QM : Time - independent Schroedinger eqn

$$\hat{H} \psi(x) = E \psi(x)$$

$$\hat{p} = \frac{\hbar}{i} \frac{d}{dx}$$

$$\therefore \ -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi \Rightarrow \frac{d^2\psi}{dx^2} + k^2\psi = 0$$

$$\Rightarrow k^2 = \frac{2mE}{\hbar^2}$$

Ansatz : $\psi \sim e^{rx}$

$$\Rightarrow r^2 + k^2 = 0 \Rightarrow \boxed{r = \pm ik}$$

Gen. Solution (time indep) :

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad , E = \frac{\hbar^2 k^2}{2m}$$

(*) Time - dependent part of Schr. equ:

$$i\hbar \frac{dT}{dt} = ET$$

$$\Rightarrow \Psi(x,t) = T(t)\psi(x) \quad \text{(using this)}$$

$$\therefore T(t) = e^{-\frac{iE}{\hbar}t}\, T(0)$$

Define, $\frac{E}{\hbar} = \omega$

$$\therefore \boxed{T(t) = e^{-i\omega t}\, T(0)}$$

Combining to get full solution,

full solution of SE →

$$\psi(t,x) = T_0 e^{-i\omega t}\left(Ae^{ikx} + Be^{-ikx}\right)$$

$$\Rightarrow \boxed{\psi(t,x) = A_0 e^{-i(\omega t - kx)} + Be^{-i(\omega t + kx)}}$$

↳ waves (or particle ?)

$e^{-i(\omega t - kx)}$ → A moving wave from left to right as t increases.

$e^{-i(\omega t + kx)}$ → A moving wave from right to left as t increases.

(*) How do we determine A and B?

Sq. Norm of $\psi(t,x)$ →

$$\|\psi(t,x)\|^2 = \int_{-\infty}^{\infty} \psi^* \psi \, dx$$

$$\Rightarrow \|\Psi(t,x)\|^2 = \int_{-\infty}^{\infty} (|A|^2 + |B|^2)\,dx + \int_{-\infty}^{\infty} dx\,[AB^* e^{2ikx} + A^*B e^{-2ikx}]$$

$$\Rightarrow \|\Psi(t,x)\|^2 = (|A|^2 + |B|^2)\int_{-\infty}^{\infty} dx + (AB^* + A^*B)\int_{-\infty}^{\infty} dx\, e^{i2kx}$$

Oscillating

Goes to ∞
But cannot be killed
as $A = B = 0$ required

$\hookrightarrow < \infty$

$\Rightarrow$ There is no solution of free particle in SE in Hilbert Space.

$$\Rightarrow \boxed{\|\Psi(t,x)\|^2 \not< \infty}$$

$\hookrightarrow$ wave function Ψ is not normalizable. i.e. it is not part of Hilbert Space.

(*) Free particle is a pathological ~~inte~~ system for QM.

For SHO, we had,

$$\hat{H}|n\rangle = E_n|n\rangle$$

$$\langle n|m\rangle = \delta_{nm} \quad \text{(Kronecker Delta)}$$

$$\sum |n\rangle\langle n| = \mathbb{I} \quad \text{(Completeness relation)}$$

$$\langle x|n\rangle = \Psi_n(x)$$

$\hat{H}\,\psi(x) = E_k \psi(x)$

$E_k = \dfrac{\hbar^2 k^2}{2m}$ These are analogies to SHO.

$\hat{H}\,|k\rangle = E_k\,|k\rangle$

$|k\rangle$ ket in x-repr.

Now, what is the analogy of the ~~completeness~~

$\langle n|m\rangle = \delta_{nm}$?

Note that the continuous ext of the complete-ness relation (Sturm-Louiville) holds.

$\sum |n\rangle\langle n| = \mathbb{I}$ and $\int_{-\infty}^{\infty} dx\,|x\rangle\langle x| = \mathbb{I}$

$$\langle k'|k\rangle = \langle k'|\mathbb{I}|k\rangle$$

$$= \int_{-\infty}^{\infty} dx\, \underbrace{\langle k'|x\rangle}_{\psi^*_{k'}} \underbrace{\langle x|k\rangle}_{\psi_k}$$

$$= \int_{-\infty}^{\infty} dx\, \psi^*_{k'}(x)\,\psi_k(x)$$

$$= \int_{-\infty}^{\infty} dx\, e^{-i(k-k')x}$$

$$= \delta(k-k')$$

(*) <u>Dirac delta</u> →
- It is a distribution.
- It is not a function.

Δ <u>Def</u>: $\int d\mathbf{r}\, f(\mathbf{r})\, \delta(\mathbf{r}-\mathbf{r}') = f(\mathbf{r}')$

$$\delta(\mathbf{r}-\mathbf{r}') = 0 \text{ if } \mathbf{r} \neq \mathbf{r}'$$