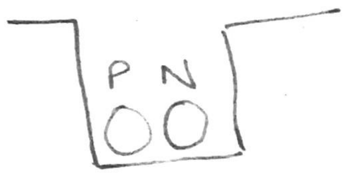
This phenomena of finding the particle even in classically prohibited regions is called ~~tunne~~ tunnelling.

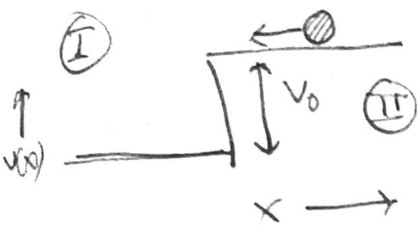
Ex: Spontaneous decay of atomic nucleus (radioactive)



Nucleus

Not explainable using CM, but explained by tunnelling. (Spontaneous and random)

13th March 2024



QM predicts that the particle can be scattered backwards too.

Case B: $E > V_0$: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$

Say $l^2 = \frac{2m(E-V_0)}{\hbar} > 0$

$\psi_{II} = Ce^{ilx} + De^{-ilx}$

Time dependent →

$$\Psi_I(t,x) = e^{-i\omega t}\left(Ae^{ikx} + Be^{-ikx}\right)$$

$$\Psi_{II}(t,x) = e^{-i\omega t}\left(Ce^{ilx} + De^{-ilx}\right)$$

Is it possible to have only right to left moving waves in both regions ~~I an~~ I and II

↳ Can we set both A and C to zero?

Continuity → $\psi_I(0) = \psi_{II}(0) \Rightarrow \boxed{A+B = C+D}$ — (I)

$\psi_I'(0) = \psi_{II}'(0) \Rightarrow \boxed{k(A-B) = l(C-D)}$ — (II)

$l \times (1) + (2) \Rightarrow C = \frac{A}{2}\left(1+\frac{k}{l}\right) + \frac{B}{2}\left(1-\frac{k}{l}\right)$

$l \times (1) - (2) \Rightarrow D = \frac{A}{2}\left(1-\frac{k}{l}\right) + \frac{B}{2}\left(1+\frac{k}{l}\right)$

If $A = 0 = C$
$\Rightarrow B = 0 = D$
(Trivial)

For non-trivial solution of ψ,

$R = L \Rightarrow \boxed{V_0 = 0}$

In QM, if there is difference in potential there will be scattering.

O Quantum mechanics in 3D →

3D ⇒ (3 space + 1 time)

In 1D:

Canonical Commutation relation (CCR)

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$$

$\hat{x}$ → position operator

$\hat{p}$ → momentum operator.

In 3D:

Position vector (Classical) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ ← Not operator

Use index notation: $\begin{cases} r_1 = x, \ r_2 = y, \ r_3 = z \\ \hat{e}_1 = \hat{i}, \ \hat{e}_2 = \hat{j}, \ \hat{e}_3 = \hat{k} \end{cases}$

$$\vec{r} = r_1\hat{e}_1 + r_2\hat{e}_2 + r_3\hat{e}_3$$

$$\Rightarrow \boxed{\vec{r} = \sum_{i=1}^{3} r_i\hat{e}_i}$$

Momentum vector:

$$\boxed{\vec{p} = \sum_{i=1}^{3} p_i\hat{e}_i}$$

$p_1 = p_x$

$p_2 = p_y$

$p_3 = p_z$

⊗ CCR in 3D →

$\boxed{[\hat{x}, \hat{p}_x] = i\hbar}$, $\boxed{[\hat{y}, \hat{p}_y] = i\hbar}$

$\boxed{[\hat{z}, \hat{p}_z] = i\hbar}$

Photon argument

Also, independence of directions,

$$[\hat{x}, \hat{P}_y] = 0 = [\hat{x}, \hat{P}_z]$$

and so on.

Same argument.

Also,

$$[\hat{P}_x, \hat{P}_y] = 0 = [\hat{P}_y, \hat{P}_z]$$

(Some thing about Space time stretching, idk)

and

$$[\hat{x}, \hat{y}] = 0 = [\hat{y}, \hat{z}]$$

In index notation,

$$\boxed{[\hat{r}_i, \hat{P}_j] = i\hbar\delta_{ij}} \quad \boxed{[\hat{r}_i, \hat{r}_j] = 0 = [\hat{P}_i, \hat{P}_j]}$$

CCR in 3D

* Angular momentum →

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$ Classical

In index notation,

$$\boxed{L_i = \sum_{j,k=1}^{3} \epsilon_{ijk} r_j P_k} \quad \boxed{\begin{matrix} L_1 = L_x \\ L_2 = L_y \\ L_3 = L_z \end{matrix}}$$

* ϵ_{ijk} : Levi-Civita symbol:

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$$

$$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$$

$$\epsilon_{ijk} = 0 \text{ if } i = j \text{ or } j = k \text{ or } k = i$$

Cyclically permute to get increasing or decreasing seq.

$$L_x = L_1 = \sum_{j,k=1}^{3} \epsilon_{ijk} r_j P_k = \epsilon_{123} r_2 P_3 + \epsilon_{132} r_3 P_2$$

$$\boxed{L_x = y P_z - z P_x}$$

(*) $\boxed{A9/Q1}$ Using Levi-civita symbol show that $L_y = zp_x - xp_z$ and $L_z = xp_y - yp_x$

Filling in notes. (Piyush's Notes)

Say, we define the form of angular momentum operator as

$$\hat{L}_x = \hat{y}\hat{P}_z - \hat{z}\hat{P}_y$$

Quantum operator corresponding to the x- component of $\vec{L}$ [classical]

(*) Hermitian Conjugate :

$$\hat{L}_x^{+} = (\hat{y}\hat{P}_z)^{+} - (\hat{z}\hat{P}_y)^{+}$$

$$= \hat{P}_z^{+}\hat{y}^{+} - \hat{P}_y^{+}\hat{z}^{+}$$

Since, $\hat{x}$, $\hat{P}_x$ are self adjoint [Hermitian]

$$\Rightarrow \hat{P}_z\hat{y} - \hat{P}_y\hat{z}$$

~~Since~~ $\cancel{\Rightarrow [\hat{P}_z\hat{y} - \hat{P}_y\hat{z}]}$

Since $[\hat{P}_z, \hat{y}] = 0 \Rightarrow \hat{P}_z$ and $\hat{y}$ commute.

$$\Rightarrow \hat{y}\hat{P}_z - \hat{z}\hat{P}_y$$

$$\Rightarrow \boxed{\hat{L}_x^{+} = \hat{L}_x} \rightarrow \text{Self adjoint [Hermitian]}$$

Eigenvalues of angular momentum operators $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$ are reals.

Ex: Compute the commutator bracket

$$[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{P}_z - \hat{z}\hat{P}_y, \hat{z}\hat{P}_x - \hat{x}\hat{P}_z]$$

$$= [\hat{y}\hat{P}_z, \hat{z}\hat{P}_x] - [\hat{z}\hat{P}_y, \hat{z}\hat{P}_x] -$$

$$[\hat{y}\hat{P}_z, \hat{x}\hat{P}_z] + [\hat{z}\hat{P}_y, \hat{x}\hat{P}_z]$$

$$= [(\hat{y}\hat{p}_z)(\hat{z}\hat{p}_x) - (\hat{z}\hat{p}_x)(\hat{y}\hat{p}_z)] - [(\hat{z}\hat{p}_y)(\hat{z}\hat{p}_x) - (\hat{z}\hat{p}_x)(\hat{z}\hat{p}_y)]$$

$$- [(\hat{y}\hat{p}_z)(\hat{x}\hat{p}_z) - (\hat{x}\hat{p}_z)(\hat{y}\hat{p}_z)]$$

$$= [\hat{y}(\hat{p}_z\hat{z})\hat{p}_x - \hat{y}(\hat{z}\hat{p}_z)\hat{p}_x] - \hat{z}[\hat{p}_y\hat{p}_x - \hat{p}_x\hat{p}_y]\hat{z}$$

$$- \hat{p}_z[\hat{y}\hat{x} - \hat{x}\hat{y}]\hat{p}_z + [\hat{x}(\hat{z}\hat{p}_z)\hat{p}_y - \hat{x}(\hat{p}_z\hat{z})\hat{p}_y]$$

$$= \hat{y}\underbrace{[\hat{p}_z, \hat{z}]}_{-i\hbar}\hat{p}_x - \hat{z}\underbrace{[\hat{p}_y, \hat{p}_x]}_{0}\hat{z} - \hat{p}_z\underbrace{[\hat{y}, \hat{x}]}_{0}\hat{p}_z$$

$$+ \hat{x}\underbrace{[\hat{z}, \hat{p}_z]}_{i\hbar}\hat{p}_y$$

$$= i\hbar\{\hat{x}\hat{p}_y - \hat{y}\hat{p}_x\} = i\hbar\hat{L}_z$$

$$\Rightarrow [\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$$

$\boxed{\text{A10/Q1}}$ Show that $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$

$\boxed{\text{A10/Q2}}$ Show that $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$

<u>Using Index Notation</u> →

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$$

→ Fundamental commutation relations of angular momentum operator.

$$[\hat{L}_x, \hat{L}_y] \neq 0$$

$$\Rightarrow \hat{L}_x\hat{L}_y \neq \hat{L}_y\hat{L}_x$$

Demo:

$$R_x\left(\theta=\frac{\pi}{2}\right) R_y\left(\theta=\frac{\pi}{2}\right) \neq R_y\left(\theta=\frac{\pi}{2}\right) R_x\left(\theta=\frac{\pi}{2}\right)$$

∴ angular momentum operator are the generator of rotation.

(*) Wave function Ψ, translated

$$\Psi(x+\Delta x) = \Psi(x) + \Delta x \frac{d}{dx}\Psi(x)$$

$$= \left(\mathbb{I} + \Delta x \frac{d}{dx}\right)\Psi(x)$$

$$= \left(\mathbb{I} + \frac{i}{\hbar}\Delta x \frac{\hbar}{i}\frac{d}{dx}\right)\Psi(x)$$

$$\Rightarrow \boxed{\Psi(x+\Delta x) = \left(\mathbb{I} + \frac{i\Delta x}{\hbar}\hat{P}_x\right)\Psi(x)}$$

$$\Rightarrow \boxed{\Psi(x+\Delta x) = \mathcal{T}(\Delta x)\Psi(x)}$$

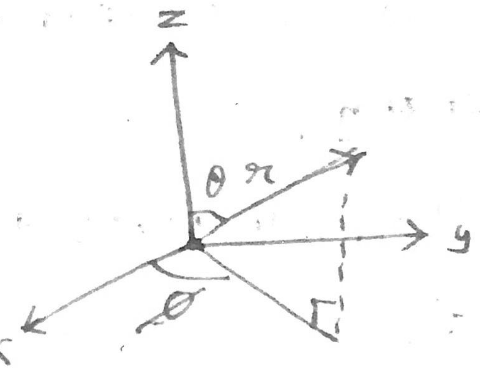
Where we have, $\boxed{\mathcal{T}(\Delta x) = \mathbb{I} + i\left(\frac{\Delta x}{\hbar}\right)\hat{P}_x}$

$\mathcal{T}(\Delta x)$: is a translation operator that takes a wave function at x (i.e $\Psi(x)$) and returns Ψ at $x+\Delta x$ (i.e $\Psi(x+\Delta x)$)

$\hat{P}_x \rightarrow$ generator of the translation.

In 3D:

$$\Psi(x,y,z) = \Psi(r,\theta,\phi)$$



* $\Psi(r,\theta,\phi+\Delta\phi)$

$$= \Psi(r,\theta,\phi) + \Delta\phi \frac{\partial}{\partial\phi}\Psi(r,\theta,\phi)$$

$$= \left[\mathbb{I} + \frac{i\,\Delta\phi}{\hbar}\left(\frac{\hbar}{i}\frac{\partial}{\partial\phi}\right)\right]\Psi(r,\theta,\phi)$$

$\therefore$ $\boxed{\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}}$ in spherical polar coordinates.

$\hat{L}_z$ — is the generator of the rotation around z-axis.

* Square magnitude of the angular momentum $\vec{L}$

$$L^2 = \vec{L}\cdot\vec{L} = L_x^2 + L_y^2 + L_z^2$$

$$\Rightarrow \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

* $[\hat{L}^2, \hat{L}_x] = [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_x]$

$$= [\hat{L}_x^2, \hat{L}_x] + [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x]$$

$$= \hat{L}_y[\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x]\hat{L}_y + \hat{L}_z[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x]\hat{L}_z$$

$$\Rightarrow [\hat{L}^2, \hat{L}_x] = \hat{L}_y(i\hbar\hat{L}_z) + (-i\hbar\hat{L}_z)\hat{L}_y + \hat{L}_z(i\hbar\hat{L}_y) + (i\hbar\hat{L}_y)\hat{L}_z$$

$$= i\hbar[\hat{L}_z\hat{L}_y - \hat{L}_z\hat{L}_y + \hat{L}_y\hat{L}_z - \hat{L}_y\hat{L}_z]$$

$$\Rightarrow \boxed{[\hat{L}^2, \hat{L}_x] = 0}$$

A10/Q3 Show that $[\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z^2] = 0$

$$\rightarrow \boxed{[\hat{L}^2, \vec{\hat{L}}] = 0 \quad ; \quad \vec{\hat{L}} = \hat{i}\,\hat{L}_x + \hat{j}\,\hat{L}_y + \hat{k}\,\hat{L}_z}$$

Claim: If $|\Psi\rangle$ is an eigenstate of an operator $\hat{A}$ then $|\Psi\rangle$ is also an eigenstate of the operator $\hat{B}$ if $[\hat{A}, \hat{B}] = 0$

Proof: Say $\boxed{\hat{A}|\Psi\rangle = \lambda_A |\Psi\rangle}$

$$|\Psi\rangle \xrightarrow{\hat{B}} |\Phi\rangle = \hat{B}|\Psi\rangle$$

$$A|\Phi\rangle = \hat{A}(\hat{B}|\Psi\rangle) = (\hat{A}\hat{B})|\Psi\rangle = (\hat{B}\hat{A})|\Psi\rangle$$

$$= \hat{B}(\hat{A}|\Psi\rangle) = \lambda_A \hat{B}|\Psi\rangle$$

$$= \lambda_A |\Phi\rangle$$

$$\Rightarrow \boxed{\hat{A}|\Phi\rangle = \lambda_A |\Phi\rangle}$$

$\Rightarrow |\Phi\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue λ_A

$$\Rightarrow |\phi\rangle \propto |\Psi\rangle$$

[True only if A has no degeneracy]

$\Rightarrow |\phi\rangle = \lambda_B |\Psi\rangle$ for some $\lambda_B \in \mathbb{R}$

$$\Rightarrow |\phi\rangle = \hat{B}|\Psi\rangle$$

$$\Rightarrow \boxed{\hat{B}|\Psi\rangle = \lambda_B |\Psi\rangle}$$

$\Rightarrow$ If $[\hat{A}, \hat{B}] = 0$ then eigenstates of $\hat{A}$ are also the eigenstates of $\hat{B}$

Say Eigenstate of $\hat{L}^2$ are $|\psi\rangle$

$$\hat{L}^2 |\psi\rangle = \lambda |\psi\rangle$$

$$\hat{L}_z |\psi\rangle = \mu |\psi\rangle$$

as $[\hat{L}^2, \hat{L}_z] = 0$

What are μ and λ?

~~wen~~ We employ the operator method (as in SHO, $\hat{a}$, $\hat{a}^{\dagger}$)

Define,

$$\hat{L}_+ \equiv \hat{L}_x + i\hat{L}_y$$

$$\hat{L}_- \equiv \hat{L}_x - i\hat{L}_y$$

$$[\hat{L}_+, \hat{L}_-] = [\hat{L}_x + i\hat{L}_y, \hat{L}_x - i\hat{L}_y]$$

$$= i[\hat{L}_y, \hat{L}_x] - i[\hat{L}_x, \hat{L}_y]$$

$$= -i\left[[\hat{L}_x, \hat{L}_y] - [\hat{L}_y, \hat{L}_x]\right]$$

$$= 2\hbar \hat{L}_z$$

$$[\hat{L}_z, \hat{L}_+] = [\hat{L}_z, \hat{L}_x] + i[\hat{L}_z, \hat{L}_y]$$
$$= i\hbar \hat{L}_y + \hbar \hat{L}_x$$
$$= \hbar(\hat{L}_x + i\hat{L}_y)$$
$$= \hbar \hat{L}_+$$
$$\Rightarrow \boxed{[\hat{L}_z, \hat{L}_+] = \hbar \hat{L}_+}$$

$$[\hat{L}_z, \hat{L}_-] = [\hat{L}_z, \hat{L}_x] - i[\hat{L}_z, \hat{L}_y]$$
$$= i\hbar \hat{L}_y - \hbar \hat{L}_x$$
$$= -\hbar(\hat{L}_x - i\hat{L}_y)$$
$$= -\hbar \hat{L}_-$$
$$\Rightarrow \boxed{[\hat{L}_z, \hat{L}_\pm] = \pm\hbar \hat{L}_\pm}$$

$$\boxed{[\hat{L}^2, \hat{L}_\pm] = 0}$$

We have,

$$\hat{L}^2 |\psi\rangle = \lambda |\psi\rangle \; ; \; \hat{L}_z |\psi\rangle = \mu |\psi\rangle$$

$$\left.\begin{array}{l} [\lambda] = [\hbar^2] \\ [\mu] = [\hbar] \end{array}\right\} \Rightarrow \boxed{\begin{array}{c} \lambda = q\hbar^2 \\ \mu = m\hbar \end{array}}$$

q, m are dimensionless

By Dirac notation,

$$|\psi\rangle \Rightarrow |q, m\rangle$$

$$\hat{L}^2 |q, m\rangle = q\hbar^2 |q, m]$$

$$\hat{L}_z |q, m\rangle = m\hbar |q, m\rangle$$

- Action of $\hat{L}_\pm$ on $|q, m\rangle \longrightarrow$

Define,

$$|\phi\rangle := \hat{L}_\pm |q, m\rangle$$

$$\hat{L}_z |\phi\rangle = \hat{L}_z \hat{L}_\pm |q, m\rangle$$

$$= (\pm\hbar \hat{L}_\pm + \hat{L}_\pm \hat{L}_z) |q, m\rangle$$

$$= \hat{L}_\pm (\pm\hbar |q, m\rangle + m\hbar |q, m\rangle)$$

$$= (m \pm 1)\hbar (\hat{L}_\pm |q, m\rangle)$$

$$= (m \pm 1)\hbar |\phi\rangle$$

$\Rightarrow |\phi\rangle$ is an eigenvector of $\hat{L}_z$ with eigenvalue

$(m \pm 1)\hbar$

$$\Rightarrow |\phi\rangle = |q, m'\rangle$$

$$\boxed{m' = (m \pm 1)}$$

$$\boxed{\hat{L}_+ |q,m\rangle = c_1 |q, m+1\rangle}$$ Raising operator

$$\boxed{\hat{L}_- |q,m\rangle = c_2 |q, m-1\rangle}$$ Lowering operator

(*) $\hat{L}^2 - \hat{L}_z^2 = \hat{L}_x^2 + \hat{L}_y^2$

$$\langle q,m| (\hat{L}^2 - \hat{L}_z^2) |q,m\rangle = \langle q,m| (\hat{L}_x^2 + \hat{L}_y^2) |q,m\rangle$$

$$(q - m^2)\hbar^2 = \langle q,m| \hat{L}_x^2 |q,m\rangle + \langle q,m| \hat{L}_y^2 |q,m\rangle$$

Define,

$$|x\rangle = \hat{L}_x |q,m\rangle$$

$$\hat{L}_x^2 = \hat{L}_x \hat{L}_x$$

$$= \hat{L}_x^{\dagger} \hat{L}_x$$

$$\Rightarrow \langle x|x\rangle = \langle q,m| \hat{L}_x^{\dagger} \hat{L}_x |q,m\rangle = \langle q,m| \hat{L}_x^2 |q,m\rangle$$

$$\Rightarrow (q - m^2)\hbar^2 \geq 0$$

$$\Rightarrow \boxed{q \geq m^2}$$

For a given q, there must be a pair (m_{max}, m_{min}) such that,

$$\hat{L}_+ |q, m_{max}\rangle = 0$$

$$\hat{L}_- |q, m_{min}\rangle = 0$$

$$\hat{L}_+ \hat{L}_- = (\hat{L}_x + i\hat{L}_y)(\hat{L}_x - i\hat{L}_y)$$

$$= \hat{L}_x + \hat{L}_y^2 + i[\hat{L}_y , \hat{L}_x]$$

$$= (\hat{L}^2 - \hat{L}_z^2) + i(-i\hbar \hat{L}_z)$$

$$= \hat{L}^2 - \hat{L}_z^2 + \hbar \hat{L}_z$$

$$\Rightarrow \boxed{\hat{L}^2 = \hat{L}_+ \hat{L}_- + \hat{L}_z^2 - \hbar \hat{L}_z}$$

$$\hat{L}^2 |q, m_{min}\rangle = (\hat{L}_+ \hat{L}_- + \hat{L}_z - \hbar \hat{L}_z) |q, m_{min}\rangle$$

$$\Rightarrow q\hbar^2 |q, m_{min}\rangle = (m_{min}^2 \hbar^2 - m_{min}\hbar^2) |q, m_{min}\rangle$$

$$\Rightarrow \boxed{q = (m_{min} - 1)\, m_{min}} \longrightarrow \text{I}$$

Similarly,

$$\boxed{\hat{L}^2 = \hat{L}_- \hat{L}_+ + \hat{L}_z^2 + \hbar \hat{L}_z}$$

$$\hat{L}^2 |q, m_{max}\rangle = (\hat{L}_- \hat{L}_+ + \hat{L}_z^2 + \hbar \hat{L}_z) |q, m_{max}\rangle$$

$$\Rightarrow \boxed{q = m_{max}\,(m_{max} + 1)} \longrightarrow \text{II}$$

II $-$ I $\Rightarrow m_{max}^2 - m_{min}^2 + m_{max} + m_{min} = 0$

$$\Rightarrow (m_{max} + m_{min})(m_{max} - m_{min} + 1) = 0$$

We Know,

$$m_{max} \geq m_{min}$$

$$\Rightarrow m_{max} = -m_{min} =: l$$

$$\Rightarrow \boxed{q = l(l+1)} \; ; \; m_{max} = l$$

$$m_{min} = -l$$

* m changes in steps of integers.

$$\Rightarrow m_{min}, m_{min}+1, m_{min}+2, \ldots, m_{max}-2, m_{max}-1, m_{max}$$

$\Rightarrow m_{max} = m_{min} + N$ for some integer N.

$$\Rightarrow 2l = N$$

$$\Rightarrow \boxed{m_{max} = l = \frac{N}{2}}$$

$$|q, m\rangle \rightarrow |l, m\rangle$$

$$\boxed{\hat{L}^2 |l,m\rangle = l(l+1)\hbar^2 |l,m\rangle}$$

$$\boxed{\hat{L}_z |l,m\rangle = m\hbar |l,m\rangle}$$

$$l = \frac{N}{2}, \quad N \in \mathbb{Z}_{\geq 0}$$

$$\Rightarrow l \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$$

$\Rightarrow m \in \{-l, -l+1, \ldots, l-1, l\}$

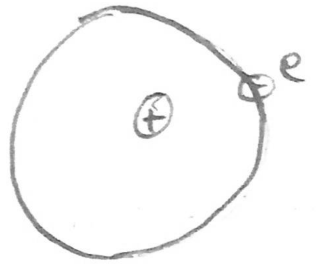
from $l(l+1) \geq m^2$

<u>Cases</u> $l = 1/2$, $m = -\frac{1}{2}, \frac{1}{2}$

$l = 1$, $m = -1, 0, 1$

$l = \frac{3}{2}$, $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

Hydrogen Atom in QM



(*) Hamiltonian

$H = KE + PE$

(*) We shall assume :

$m_p \gg m_e$

(*) Potential energy (of the electron) :

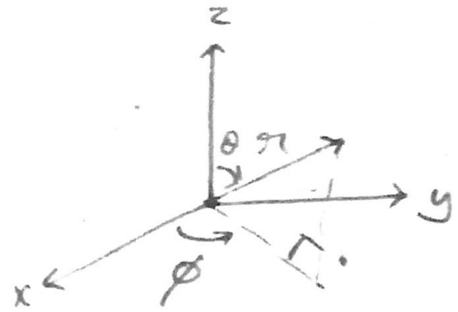
$$\boxed{V = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{e^2}{r}}$$

(*) Kinetic Energy $= \frac{\vec{p} \cdot \vec{p}}{2m}$

where $\vec{p} = m\frac{d\vec{r}}{dt}$ <u>in 3D</u>

(A) Given V depends only on r, its convenient to use spherical polar coordinate

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$



Position vector :

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= r\hat{r}$$

$$\hat{r} = \frac{\vec{r}}{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

Momentum : $\vec{p} = m\dfrac{d\vec{r}}{dt}$

Cartesian : $$m\left(\frac{dx}{dt}\hat{i} + x\overset{=0}{\frac{d\hat{i}}{dt}} + \frac{dy}{dt}\hat{j} + y\overset{=0}{\frac{d\hat{j}}{dt}} + \frac{dz}{dt}\hat{k} + z\overset{=0}{\frac{d\hat{k}}{dt}}\right)$$

$$\vec{p} = m\frac{d\vec{r}}{dt}$$
$$= m\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right)$$

Spherical polar : $$\vec{p} = m\frac{d\vec{r}}{dt} = m\left(\frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}\right)$$
$$= m\frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$
$$= p_r\hat{r} + r\frac{d\hat{r}}{dt}$$

$$\boxed{\frac{\vec{p}\cdot\vec{p}}{2m} = \frac{p_r^2}{2m} + \frac{1}{2}mr^2\left(\frac{d\hat{r}}{dt}\cdot\frac{d\hat{r}}{dt}\right)} + (\quad)\overset{=0}{\left(\hat{r}\cdot\frac{d\hat{r}}{dt}\right)}$$

Angular Momentum:

$$\vec{L} = \vec{r} \times \vec{p} = r\hat{r} \times \left(p_r \hat{r} + m r \frac{d\hat{r}}{dt} \right)$$

$$\Rightarrow \boxed{\vec{L} = m r^2 \left(\hat{r} \times \frac{d\hat{r}}{dt} \right)}$$

Recall:

Cartesian:

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

$$\hat{i} \times \hat{j} = \hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

Spherical polar:

$$\hat{r} \cdot \hat{\theta} = \hat{\theta} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{r} = 0$$

$$\hat{r} \times \hat{\theta} = \hat{\phi}$$

$$\hat{\theta} \times \hat{\phi} = \hat{r}$$

$$\hat{\phi} \times \hat{r} = \hat{\theta}$$

Since $\hat{r} \cdot \frac{d\hat{r}}{dt} = 0$

$$\Rightarrow \frac{d\hat{r}}{dt} = a\hat{\theta} + b\hat{\phi}$$

$$\hat{r} \times \frac{d\hat{r}}{dt} = a\hat{\phi} - b\hat{\theta}$$

$$L^2 = \vec{L}\cdot\vec{L}$$

$$= m^2 r^4 \left(\hat{r}\times\frac{d\hat{r}}{dt}\right)\cdot\left(\hat{r}\times\frac{d\hat{r}}{dt}\right)$$

$$= m^2 r^2 \left(a^2 + 6^2\right)$$

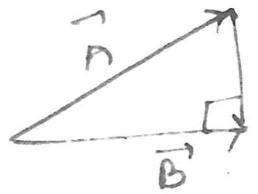
$$= m^2 r^2 \left(\frac{d\hat{r}}{dt}\right)\cdot\left(\frac{d\hat{r}}{dt}\right)$$

$$\frac{P^2}{2m} = \frac{P_r^2}{2m} + \frac{1}{2} m r^2 \left(\frac{d\hat{r}}{dt}\cdot\frac{d\hat{r}}{dt}\right)$$

$$= \frac{P_r^2}{2m} + \frac{1}{2mr^2} L^2$$

$$\Rightarrow \boxed{KE = \frac{P^2}{2m} = \frac{1}{2m}\left(P_r^2 + \frac{L^2}{r^2}\right)}$$

Let $\vec{A}$ be a given vector and $\hat{B}$ be unit vector such that $\hat{A} \neq \hat{B}$



$$\vec{A} = \vec{B} + \vec{C}$$

$$\Rightarrow \vec{A}\cdot\vec{A} = \vec{B}\cdot\vec{B} + \vec{C}\cdot\vec{C}$$

$$\Rightarrow \vec{A}\cdot\vec{A} = \cancel{\vec{B}}\left(\hat{B}\cdot\vec{A}\right)^2 + \left(\hat{B}\times\vec{A}\right)^2$$

Set $\hat{B} = \hat{r}$ and $\vec{A} = \vec{P}$

$$\Rightarrow \vec{P}\cdot\vec{P} = \left(\hat{r}\cdot\vec{P}\right)^2 + \cancel{\vec{r}}\left(\hat{r}\times\vec{P}\right)^2$$

$$\Rightarrow P^2 = P_r^2 + \left(\frac{\vec{r}\times\vec{P}}{r}\right)^2$$

$$\Rightarrow \boxed{P^2 = P_r^2 + \frac{L^2}{r^2}}$$

○ Time-Independent Schrodinger Eqn:

$$\hat{H}\Psi = E\Psi \quad \Big| \quad \Psi = \Psi(r, \theta, \phi)$$

Classically,

$$H = KE + V$$

$$\Rightarrow \hat{H}\Psi = \left[\frac{\hat{p}_r^2}{2m} + \frac{1}{2mr^2}\hat{L}^2 + V(r)\right]\Psi$$

$$= E.\Psi$$

(*) What are these $|l, m\rangle$ ket states in (θ, ϕ) representation?

Recall SHO.

Ground state: $|0\rangle$

We have seen $\psi_0(x) = \langle x|0\rangle$

$$= N e^{-\alpha x^2}$$

$$\psi_1(x) = \langle x|1\rangle = N x e^{-\alpha x^2}$$

In general,

$$\boxed{\psi_n(x) = \langle x|n\rangle = N H_n(\beta x) e^{-\alpha x^2}}$$

α, β are appropriate const.

where $H_n(x) \rightarrow$ Hermite polynomials.

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_n(x) = e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}$

⊛ Similarly,

$$\boxed{\psi_{l,m}(\theta,\phi) = \langle\theta,\phi|l,m\rangle \equiv Y_{lm}(\theta,\phi)}$$

↳ Spherical Harmonics.

$$\Rightarrow \begin{cases} \hat{L}^2 |l,m\rangle = l(l+1)\hbar^2 |l,m\rangle \\ \hat{L}_z |l,m\rangle = m\hbar |l,m\rangle \end{cases}$$

$$\hat{L}^2 Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 Y_{lm}(\theta,\phi)$$

$$\rightarrow \hat{H}\Psi = \left[\frac{\hat{P}_r^2}{2m} + \frac{\hat{L}^2}{2mr^2} + \hat{V}(r)\right]\Psi = E\Psi$$

Here, $\Psi = \Psi(r,\theta,\phi) = R(r)\,Y_{lm}(\theta,\phi)$ (Ansatz)

$\Rightarrow$ <u>Radial wave function</u> $(R = R(r))$

$$\boxed{\hat{P}_r^2 R(r) + \left[2m\left(\hat{V}(r) - E\right) + \frac{l(l+1)\hbar^2}{r^2}\right]R(r) = 0}$$

⊛ What is the operator $\hat{P}_r$?

Classically, $P_r = \frac{1}{r}(\vec{r}\cdot\vec{p})$

$= \frac{1}{r}(xp_x + yp_y + zp_z)$

Say, $\hat{P}_r = \frac{1}{r}(\hat{x}\hat{P}_x + \hat{y}\hat{P}_y + \hat{z}\hat{P}_z)$

$\hat{P}_r^\dagger \neq \hat{P}_r$ as $\vec{r}$ and $\vec{p}$ are non-commuting operators.

(*) How do we make $\hat{p}_r$ Hermitian?

We can choose

$$\boxed{\hat{P}_r = \frac{1}{2}\left(\frac{\vec{r}}{r}\cdot\vec{p} + \hat{\vec{p}}\cdot\frac{\vec{r}}{r}\right)}$$

(*) Operator form of $\frac{\vec{r}}{r}\cdot\vec{p}$ [in polar coordinates]

$$\vec{p} = \frac{\hbar}{i}\vec{\nabla} = \frac{\hbar}{i}\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

Say, $\frac{\vec{r}}{r}\cdot\hat{\vec{p}}\,\psi = \frac{\hbar}{i}\frac{1}{r}\left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y} + z\frac{\partial\psi}{\partial z}\right)$

Since $\psi = \psi(r)$

$$\frac{\partial\psi}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial\psi}{\partial r} = \left(\frac{x}{r}\right)\frac{\partial\psi}{\partial r}$$

$$\frac{\vec{r}}{r}\cdot\hat{\vec{p}}\,\psi = \frac{\hbar}{i}\left(\frac{x^2}{r^2}\frac{\partial\psi}{\partial r} + \frac{y^2}{r^2}\frac{\partial\psi}{\partial r} + \frac{z^2}{r^2}\frac{\partial\psi}{\partial r}\right)$$

$$\Rightarrow \boxed{\frac{\vec{r}}{r}\cdot\hat{\vec{p}}\,\psi = \left(\frac{\hbar}{i}\frac{\partial}{\partial r}\right)\psi}$$

$$\Rightarrow \boxed{\frac{\vec{r}}{r}\cdot\hat{\vec{p}} = \hat{r}\cdot\vec{p} = \frac{\hbar}{i}\frac{\partial}{\partial r}}$$

$$\hat{P}_r \psi = \frac{1}{2}\frac{\hbar}{i}\left[\frac{\vec{r}}{r}\cdot\vec{\nabla}\psi + \vec{\nabla}\left(\frac{\vec{r}}{r}\psi\right)\right]$$

$$= \frac{1}{2}\frac{\hbar}{i}\left[\frac{\vec{r}}{r}\vec{\nabla}\psi + \psi\vec{\nabla}\left(\frac{\vec{r}}{r}\right) + \frac{\vec{r}}{r}\vec{\nabla}\psi\right]$$

$$= \frac{1}{2}\frac{\hbar}{i}\left[2\frac{\hat{\vec{r}}}{r}\vec{\nabla}\psi + \psi\vec{\nabla}\left(\frac{\hat{\vec{r}}}{r}\right)\right]$$

$$\vec{\nabla}\cdot\left(\frac{\hat{\vec{r}}}{r}\right) = \frac{1}{r}\left(\vec{\nabla}\cdot\vec{r}\right) + \vec{r}\left(\vec{\nabla}\cdot\left(\frac{1}{r}\right)\right)$$

$$= \frac{3}{r} + \vec{r}\cdot\left\{\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\hat{k}\right\}$$

$$= \frac{3}{r} + \left[-\frac{\vec{r}}{r^2}\cdot\left\{\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right\}\right]$$

$$= \frac{3}{r} + \left(-\frac{(x^2+y^2+z^2)}{r^3}\right)$$

$$= \frac{2}{r}$$

$$\Rightarrow \hat{P}_r\psi = \frac{\hbar}{i}\left[\frac{\vec{r}}{r}\vec{\nabla}\cdot\psi + \frac{1}{r}\psi\right]$$

$$= \frac{\hbar}{i}\left[\frac{\partial}{\partial r}\psi + \frac{1}{r}\psi\right] \Rightarrow \frac{\hbar}{i}\left[\frac{1}{r} + \frac{\partial}{\partial r}\right]\psi$$

$$= \frac{\hbar}{i}\left[\frac{1}{r}\frac{\partial}{\partial r}r\right]\psi$$

(*) Radial wave function $R(r)$

$$\hat{P}_r R(r) = \left[2m\left(V(r) - E\right) + \frac{\hbar^2 (l)(l+1)}{r^2}\right]R(r) = 0$$

→ we define $u(r) = rR(r)$

$$\hat{P}_r R(r) = \frac{\hbar}{i}\frac{1}{r}\frac{d}{dr}\left(rR(r)\right) = \frac{\hbar}{i}\frac{1}{r}\frac{d}{dr}u(r)$$

$$\hat{P}_r^2 R(r) = \hat{P}_r\hat{P}_r R(r)$$

$$= \frac{\hbar}{i}\frac{1}{r}\frac{d}{dr}\left[\cancel{r}\frac{\hbar}{i}\frac{1}{\cancel{r}}\frac{d}{dr}u(r)\right]$$

$$= -\frac{\hbar^2}{r}\frac{d^2}{dr^2}u(r)$$

$$\Rightarrow \boxed{-\frac{\hbar^2}{r}\frac{d^2}{dr^2}u(r) + \left[2m\left(V(r)-E\right) + \frac{\hbar^2 l(l+1)}{r^2}\right]u(r) = 0}$$

— ($*$)

Say $k^2 = -\frac{2mE}{\hbar^2}$ | $E < 0$

$$\frac{d^2}{dr^2}u - \left[-\frac{2m}{\hbar^2}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + k^2 + \frac{l(l+1)}{r^2}\right]u(r) = 0$$

($*$) We define $\boxed{\rho = kr}$ $\quad [k] = \frac{1}{[L]}$

Now divide by k^2 in eqn.

$$\frac{d^2u}{d\rho^2} - \left[1 - \frac{me^2}{2\pi\varepsilon_0\hbar^2 k}\frac{1}{\rho} + \frac{l(l+1)}{\rho^2}\right]u = 0$$

↳ ρ_0 (define)

ρ is unitless

$$\boxed{\frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u = 0}$$ — ($*$)($*$)

→ A ~~not~~ neat form of the Schroedinger eqn for radial wave function.

($*$) Asymptotics Solutions.

[I] for large $r \Rightarrow$ large ρ

for $\rho >> 1$

$$\boxed{\frac{d^2u}{d\rho^2} - u = 0} \Rightarrow u = Ae^{-\rho} + Be^{-\rho}$$

for $r \to \infty$, $R(r) < \infty \Rightarrow B = 0$

$$\Rightarrow \boxed{R(r) = \frac{1}{r} A e^{-Rr}}$$

II For small r (i.e. small ρ)

$$\Rightarrow \frac{d^2u}{d\rho^2} - \frac{l(l+1)}{\rho^2} u = 0$$

__Claim__ : The general solution : $u = C\rho^{l+1} + D\rho^{-l}$

__Proof__ : $\frac{du}{d\rho} = (l+1) C\rho^{l} - Dl\rho^{-(l+1)}$

$$\frac{d^2u}{d\rho^2} = l(l+1) C\rho^{l-1} + Dl(l+1)\rho^{-(l+2)}$$

$$\Rightarrow l(l+1) C\rho^{l-1} + l(l+1)\rho^{-(l+2)}$$

$$= l(l+1) \left[C\rho^{l-1} + D\rho^{-(l+2)}\right]$$

$$= \frac{l(l+1)}{\rho^2} \left[C\rho^{l+1} + D\rho^{-l}\right]$$

$$= \frac{l(l+1)}{\rho^2} u$$

$$= \frac{d^2u}{d\rho^2} - \frac{l(l+1)}{\rho^2} u = 0$$

$$R(r) = \frac{1}{r} \left(C (Rr)^{l+1} + D (Rr)^{-l}\right)$$

for $R(r) < \infty$ for $r \to 0 \Rightarrow D = 0$

$$\boxed{R(r) \sim r^{l}}$$

In summary:

$$\rho \to \infty; \quad u = Ae^{-\rho}$$

$$\rho \to 0; \quad u = C\rho^{l+1}$$

Ansatz: $\boxed{u = \rho^{l+1} e^{-\rho} v(\rho)}$

$$\frac{du}{d\rho} = (l+1)\rho^{l} e^{-\rho} v - \rho^{l+1} e^{-\rho} v + \rho^{l+1} e^{-\rho} \frac{dv}{d\rho}.$$

$$\frac{d^2u}{d\rho^2} = l(l+1)\rho^{l-1} e^{-\rho} v - (l+1)\rho^{l} e^{-\rho} v$$
$$+ (l+1)\rho^{l} e^{-\rho} \frac{dv}{d\rho}$$
$$- (l+1)\rho^{l} e^{-\rho} v + \rho^{l+1} e^{-\rho} v -$$
$$\rho^{l+1} e^{-\rho} \frac{dv}{d\rho}$$
$$+ (l+1)\rho^{l} e^{-\rho} \frac{dv}{d\rho} - \rho^{l+1} e^{-\rho} \frac{dv}{d\rho}$$
$$+ \rho^{l+1} e^{-\rho} \frac{d^2v}{d\rho^2}$$

$$\Rightarrow \left[1 - \frac{2(l+1)}{\rho} + \frac{l(l+1)}{\rho^2}\right] \rho^{l+1} e^{-\rho} v$$
$$- 2\left[1 - \frac{(l+1)}{\rho}\right] \rho^{l+1} e^{-\rho} \frac{dv}{d\rho}$$
$$+ \rho^{l+1} e^{-\rho} \frac{d^2v}{d\rho^2}$$

We know,

$$\frac{d^2u}{d\rho^2} - \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u = 0$$

$$\Rightarrow \boxed{\frac{d^2v}{d\rho^2} - 2\left[1 - \frac{(l+1)}{\rho}\right] \frac{dv}{d\rho} - \left[\frac{2(l+1) - \rho_0}{\rho}\right] v = 0}$$

Let's define,

$$x = 2\rho$$

$$y(x) = v(\rho)$$

$$\alpha = 2l + 1 \; ; \; \beta = \frac{1}{2}\left[\rho_0 - 2(l+1)\right]$$

$$\Rightarrow 4\frac{d^2y}{dx^2} - 4\left(1 - \frac{\alpha+1}{x}\right)\frac{dy}{dx} + \frac{4\beta}{x}y = 0$$

$$\Rightarrow \boxed{x y'' + (\alpha + 1 - x)y' + \beta y = 0}$$

↳ Associated Laguerre differential eqn

⊛ It has exact solution which lead to normalizable radial wave function ($rR(r)$) provided β is a non-negative integer

[i.e, $\beta = N = 0, 1, 2, 3, \dots$]

These solution are known as the associated Laguerre Polynomial and denoted as,

$$L_N^\alpha(x) = y(x) = v(\rho)$$

$$L_0^\alpha(x) := 1$$

$$L_1^\alpha(x) := 1 + \alpha - x$$

Condition of ρ and l

$$\beta = \frac{1}{2}\left[\rho_0 - 2(l+1)\right] = N \geq 0$$

$$\Rightarrow 2(l+1) \leq \rho_0 \quad \text{as } N \geq 0$$

$$\Rightarrow \boxed{\rho_0 = 2(N + l + 1)}$$

We know,

$$l = \frac{N}{2}$$

$$l \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\right\}$$

Case (I) : $l = 0, 1, 2, \cdots$

$$\rho_0 = 2[N + l + 1] = 2n$$

where $\boxed{n = N + l + 1 = 1, 2, 3, \cdots}$

$$\Rightarrow l + 1 \leq \frac{\rho_0}{2} = n$$

$$\Rightarrow l \leq n - 1$$

$$\Rightarrow \boxed{l = 0, 1, \ldots, n-1}$$

$$\rho_0 = \frac{me^2}{2\epsilon_0 \hbar^2 \pi R}$$

* Energy eigen values :

$$E = -\frac{\hbar^2}{2m} R^2 \Rightarrow E = -\frac{\hbar^2}{2m} \frac{m^2 e^4}{4\epsilon_0^2 \pi^2 \hbar^4 \rho_0^2}$$

$$\Rightarrow E = \frac{-me^4}{8\epsilon_0^2 \pi^2 \hbar^2} \cdot \frac{1}{2^2 n^2}$$

$$\Rightarrow \boxed{E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{n^2}}$$

Bohr formula for energy levels of H atom.

Where are $l = \frac{1}{2}, \frac{3}{2}, \cdots$ eigen values or

Why $l = 0, 1, 2, \cdots$ only ?

$$\hat{L}_z |l, m\rangle = m\hbar |l, m\rangle$$

$$\langle \phi | \hat{L}_z | l, m\rangle = m\hbar \langle \phi | l, m\rangle$$

$$= \frac{\hbar}{i} \frac{\partial}{\partial \phi} \langle \phi | l, m\rangle$$

Define $\langle \phi | l, m\rangle = \psi_{lm}(\phi)$

$$\therefore m\hbar \psi_{lm}(\phi) = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \psi_{lm}(\phi)$$

$$\Rightarrow \boxed{\psi_{lm}(\phi) = e^{im\phi} \cdot f_{lm}(\theta)}$$

$$\psi_{lm}(\phi) = \psi_{lm}(\phi + 2\pi)$$

If we demand that the wave function be single valued function of the coordinate ϕ.

$$e^{im\phi} = e^{im(\phi + 2\pi)}$$

$$\Rightarrow e^{i2m\pi} = 1 \Rightarrow \boxed{m \in \mathbb{Z}}$$

Since $m \in \{-l, -l+1, \cdots, l-1, l\}$

l should be integer for orbital angular momentum (Not half integer).

$$\boxed{\hat{L}^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle} \text{ with } l = 0, 1, 2, \cdots$$

orbital angular momentum operator.

* $l = \frac{1}{2}, \frac{3}{2}, \ldots$ → corresponds to spin angular momentum.

○ Radial wave function

$$\boxed{R(r) = \frac{1}{r} u = \frac{1}{r} \rho^{l+1} e^{-\rho} L_{\beta}^{\alpha}}$$

$$\beta = \frac{1}{2}\left[\rho_0 - 2(l+1)\right]$$

$$\alpha = 2l+1$$

$l \sim n$

$\rho_0 \sim n$ $\boxed{\alpha \sim n}$

$\boxed{\beta \sim n}$

$$\boxed{R_{nl}(r)}$$

* Energy eigenkets:

$$\boxed{\Psi = \Psi(r, \theta, \phi) = R_{nl}(r)\, Y_{lm}(\theta, \phi) = \Psi_{nlm}(r, \theta, \phi)}$$

In Dirac notation,

$$\boxed{\hat{H} |n, l, m\rangle = E_n |n, l, m\rangle}$$

* Ground state $(n = 1)$

$l = 0, 1, \ldots, n-1 \Rightarrow \boxed{l = 0} \Rightarrow \boxed{m = 0}$

$$\hat{H} |1, 0, 0\rangle = E_1 |1, 0, 0\rangle \Rightarrow \cancel{\hat{H}\, \Psi_{lm}}$$

$$\Rightarrow \hat{H}\, \Psi_{100} = E_1 \Psi_{100}$$

(*) First excited state $(n=2) \rightarrow l = 0, 1$

(i) $l = 0, \; m = 0$

$$\hat{H}\,|2,0,0\rangle = E_2\,|2,0,0\rangle$$

(ii) $l = 1 \Rightarrow m = -1, 0, 1$

(a) $\hat{H}\,|2,1,-1\rangle = E_2\,|2,1,-1\rangle$

(b) $\hat{H}\,|2,1,0\rangle = E_2\,|2,1,0\rangle$

(c) $\hat{H}\,|2,1,1\rangle = E_2\,|2,1,1\rangle$

→ There is '1' state with energy E_1

→ There are '4' state with energy E_2

→ First excited eigenstate has 4 fold degeneracy {Same eigenvalue but different states}

n: Principal quantum no.

l: Azimuthal quantum no.

m: magnetic quantum no.