Δ Random Variables: A real valued function on the sample space such that the pre-image of every interval is an event.

(*) For this course, by a r.v. we will always mean a real-valued r.v

Δ Probability distribution of a r.v -

Let $(\Omega, \mathcal{E}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable. Then X translates $(\Omega, \mathcal{E}, P)$ to $(\mathbb{R}, \mathcal{E}_X, P_X)$, where $\mathcal{E}_X = \{A \subseteq \mathbb{R} \mid X^{-1}(A) \in \mathcal{E}\}$

$P_X(A) := P(X^{-1}(A))$

Notation: Hence forth, ~~we the~~ we'll use the notations $P_X(A)$ and $P(X \in A)$ interchangeably.

In particular, we will write $P_X((-\infty, a])$ as $P(X \leq a)$

Δ Random variables -

$$X : \Omega \to \mathbb{R}$$
$$\text{st } X^{-1}((-\infty, a]) \in \mathcal{E} \quad \forall a \in \mathbb{R}.$$

Eg: $(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - \frac{1}{n}]$

The fact that $\mathcal{E}$ is closed under countable unions and complementation, also carries over to $\mathcal{E}_X$

$$(a, b) \Rightarrow \mathbb{R} \setminus ((-\infty, a] \cup [b, \infty))$$
$$= \mathbb{R} \setminus ((-\infty, a] \cup [-\infty, b]^c)$$

Lemma: $\forall x \in \mathbb{R}$ we have $\{x\} \in \mathcal{E}_X$

Proof: $\{x\} = [x, x]$

$$= \Big(\underbrace{(-\infty, x)}_{\in\, \mathcal{E}} \cup \underbrace{(x, \infty)}_{\in\, \mathcal{E}}\Big)^c$$

and $\mathcal{E}$ is closed under countable unions and complementation.

So, $\{x\} \in \mathcal{E}_X$

Corollary: ~~RE~~ $\forall$ r.v $X : \Omega \to \mathbb{R}$,

$P(X = x) = P_X(X^{-1}\{x\})$ is well defined.

This is called PMF (Probability Mass function)

Δ CDF: Cumulative distribution function.

For $a \in \mathbb{R}$, we define ~~F_a~~ $F_X(a) = P(X \leq a)$

⊛ The CDF is an increasing function,

because for $a \leq b$

$$F_X(a) = P(X \leq a) = P(X^{-1}(-\infty, a]) \leq P(X^{-1}(-\infty, b]) = P(X \leq b) = F_X(b)$$