Fourier Transformation
Signal and Image Processing 2014

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Monday, not to be handed-in

1. **Complex numbers**: The following questions are intended as a repetition of basic algebra of complex numbers. They need not an essay answer, but please write the steps used in order to reach the result.

   (a) Using $i = \sqrt{-1}$, $a_1, a_2, b_1, b_2, d_1, d_2 \in \mathbb{R}$, and $a, b \in \mathbb{C}, a = (a_1 + ia_2), b = (b_1 + ib_2)$, reduce each of the following to the form $d = (d_1 + id_2)$:

      i. $d = a + b$
      ii. $d = a - b$
      iii. $d = a * b$
      iv. $d = \frac{a}{b}$

   (b) Given a vector $\vec{v} = [x, y]^T$, represent this vector as a single complex number, and use the complex conjugate and some elementary operations (addition, subtraction, multiplication, division) to show how to calculate the squared length of $\vec{v}$.

   (c) Rewrite $d = \sqrt{-3}$ to the form $d = (d_1 + id_2)$.

   (d) Using $i = \sqrt{-1}$, $a_r, a_\theta, b_r, b_\theta, d_r, d_\theta \in \mathbb{R}$, and $a, b \in \mathbb{C}, a = a_r e^{ia_\theta}, b = b_r e^{ib_\theta}$ (polar form), reduce each of the following to the form $d = d_r e^{id_\theta}$:

      i. $d = a * b$
      ii. $d = \frac{a}{b}$

   (e) Write the complex conjugate of $a_r e^{ia_\theta}$ on polar form.

   (f) Given a and b as complex numbers on polar form, use Euler's formula $e^{ix} = \cos(x) + i\sin(x)$, simplify the following to the form $d = (d_1 + id_2)$

      i. $d = a + b$
      ii. $d = a - b$

   (g) Given $a = (a_1 + ia_2)$, rewrite it to polar form $d = d_r e^{id_\theta}$.

Wednesday, to be handed-in

1. **Fourier Transform – Theory**: The following investigate theoretical properties of Fourier series and transform. Remember to include crucial steps in derivations, and a short comment to each answer.

   (a) What is the difference between a Fourier series and the Fourier Transform?

   (b) Prove that the continuous Fourier transform of a real and even function is real and even.

   (c) Derive the continuous Fourier transform of $\delta(x - d) + \delta(x + d)$ for some constant d.

   (d) Consider the box function

   $$b_a(x) = \begin{cases} 1/a & \text{if } |x| \leq \frac{a}{2} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

   show that

i. $\int_{-\infty}^{\infty} b_a(x)\,dx = 1$

ii. the continuous Fourier transform of b using (5.10) is $B(k) = \frac{1}{ak\pi} \sin \frac{ak}{2}$. Rewrite $B(k)$ using the $\mathrm{sinc}(x) = \frac{\sin x}{x}$.

iii. $\lim_{a \to 0} B(k) = \frac{1}{2\pi}$ (Hint: $\lim_{x \to 0} \frac{\sin x}{x} = 1$). Does this prove an entry in Table 5.2?

iv. The filter b has compact support in space (only uses a small set of neighbouring pixels in x). Is the same true in the frequency domain, k? Explain your answer.

2. **Fourier Transform – Practice**: Each of theses answers should include examples of the input and output, possibly crucial Matlab code snippets, and definitely a description of which problems were solved, how, and an evaluation of the results.

   (a) Use Matlab to calculate the power spectrum of `lena.tif`. Apply the function `fftshift` and interpret the result representation of the image.

   (b) Write two programs: 1 that implements convolution as a nested for loop of the spatial representation of the kernel and image, 2 that implements the same convolution using Fast Fourier Transformation. Compare the two both in terms of the result and the computation time for a number of kernel sizes and image sizes.

   (c) Write a program that adds the function $a_0 cos(v_0 x + w_0 y)$ to `lena.tif`, and evaluate and describe the power spectrum of the result. Design a filter, which removes any such planar waves given v_0 and w_0.

   (d) Write the function `scale`, which implements convolution with a isotropic Gaussian kernel, parametrized with the its standard deviation - the scale. Apply it to `lena.tif` for a range of scales.

   (e) Spatial derivatives may be written as the multiplication of a kernel in the Fourier Domain. Derive the exact relation and discuss its practicality.

   (f) Implement a function which takes 2 derivative orders and a 2-dimensional image, and returns the derivative of the image using FFT.