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DETERMINING OPTIMAL PRODUCTION-INVENTORY CONTROL POLICIES FOR AN INVENTORY SYSTEM WITH PARTIAL BACKLOGGING

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Scope and Purpose—Most research studies in inventory control have been concerned with cases in which the demand during the stockout period is either completely backlogged or completely lost. However, in many real situations such as demand for spare parts, only a fraction of the unsatisfied demand can be backlogged because some customers may decide to buy elsewhere. This leads to a situation of partial backlogging. This paper attempts to present simple expressions to assist management in formulating production-inventory control policies when the unsatisfied demand is only partially backlogged and the items are replenished at a uniform rate. The theoretical results obtained are illustrated by means of a numerical example.

Abstract—In this paper, a mathematical model is developed for an inventory system in which the demand during the stockout period is partially backlogged and the items are replenished at a uniform rate. It is shown that by defining a time proportional backordered cost and a penalty cost per unit lost, a convex cost function can be obtained if a suitable set of decision variables is used in the analysis. The determination of optimal production-inventory control policies is thus greatly simplified since it is not necessary to use any *ad hoc* optimization procedure in the analysis. A numerical example is used to illustrate the theory. Computational results indicate that optimal policies are sensitive to the nature of the demand during the stockout period, and that erroneous assumption of the fraction of the unsatisfied demand that can be backlogged leads to a higher annual operating cost.

INTRODUCTION

The formulation of optimal policies for controlling inventory is one of the most widely researched topics in the literature. Aggarwal [1] reported a comprehensive survey on a wide variety of models developed on the basis of different assumptions, operational environments and solution procedures. However, in most of these models, one common assumption is that the demand during the stockout period is either entirely backlogged or entirely lost, though in many cases, such as the retail business, the typical situation is one where only some of the customers agree to wait until the stock is replenished, while the others decide to buy elsewhere, namely, a situation of partial backlogging. The formulation of inventory replenishment policies for such a situation has received relatively little attention. Fabrycky and Banks [2] and Jelen [3] described inventory models which considered a mixture of backorders and lost sales but no detail solution procedure was reported. Montgomery *et al.* [4] developed an infinite-time-horizon lot-size model in which the demand during the stockout period is partially backlogged. In view of the nonconvexity of the cost function obtained, a complicated two stage minimization procedure using non-singular transformation of the cost function was suggested to determine the optimal order quantity. Rosenberg [5] introduced the concept of fictitious demand rate and reformulated the model to determine the optimal solution using decomposition by projection in the stepwise manner specified. In this paper, the mathematical model studied by Montgomery *et al.* [4] and Rosenberg [5] is modified to incorporate a uniform replenishment rate. A time proportional backorder cost and a penalty cost per unit lost are also defined. It is shown that if a suitable set of decision variables is selected, the cost function derived is convex. Hence, the formulation of optimal policies is greatly simplified since it is not necessary to employ any complicated *ad hoc* optimization procedure [4, 5] in the analysis. The optimal batch size

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and the optimal cumulative shortage allowed per cycle are determined for such an inventory system. A numerical example is used to illustrate the theoretical results obtained.

MATHEMATICAL MODELLING AND ANALYSIS

The mathematical model of the inventory system considered in this paper is developed on the basis of the following assumptions:

(1) A single item is considered and the demand rate is known and constant.
(2) The item is replenished at a uniform rate which is known and constant. The replenishment rate is greater than the demand rate.
(3) The scheduling period is constant.
(4) The inventory level and demand can be treated as continuous variables.
(5) The setup time for a production run is assumed to be zero.
(6) Only a fraction of the demand during the stockout period is made up in the next inventory cycle. The remaining unfilled orders are lost.

The following symbols are used:

p = production rate, units/unit time
λ = demand rate, units/unit time
T = scheduling period
t = stockout period
α = a constant which indicates the proportion of the demand during the stockout period that can be backlogged
Q = production batch size
S = total demand during the stockout period in a cycle
ϕ = maximum stock level in a cycle
C_1 = fixed setup cost per production run, \$ per setup
C_2 = inventory carrying cost, \$ per unit per unit time
C_3 = shortage cost per unit backlogged per unit time, \$ per unit per unit time
C_4 = penalty cost of a lost sale including the lost profit, \$ per unit.

The behaviour of the inventory level of the system at any time during a given cycle is depicted in Fig. 1. It can be seen that the quantity backlogged is αS units and the quantity lost is $(1-\alpha)S$ units. Inventory depletion occurs at the constant rate λ and the next production run starts when the total demand during the stockout period reaches S units. Thus, the scheduling period T depends on the amount of unsatisfied demand which can be backlogged. Indeed, it can be shown that the total

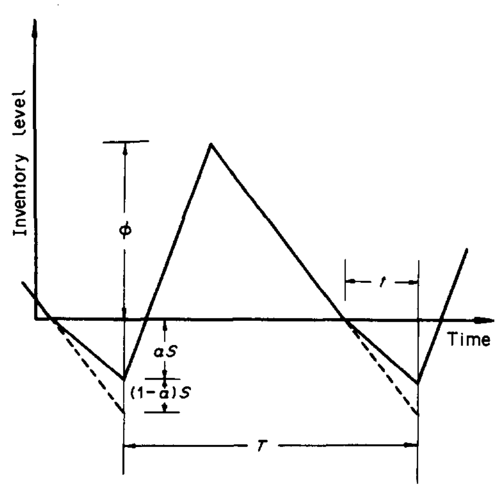

Fig. 1. Inventory system with partial backlogging.

operating cost per cycle is

$$C_1 + \frac{C_2}{2}\left(\frac{p}{\lambda}\right)\left(\frac{1}{p-\lambda}\right)\phi^2 + \frac{C_3}{2}\alpha\left(\frac{p-(1-\alpha)\lambda}{\lambda(p-\lambda)}\right)S^2 + C_4(1-\alpha)S \tag{1}$$

in which the first term represents the setup cost per production run, the second term the inventory carrying cost, the third term the shortage cost for units backlogged and the last term the penalty cost due to lost sales. It can also be shown that the maximum inventory level ϕ and the cumulative shortage S satisfy the respective equations

$$\phi = \frac{\lambda}{p}\{(p-\lambda)T - (p-(1-\alpha)\lambda)t\} \tag{2}$$

and

$$S = \lambda t. \tag{3}$$

The total cost per unit time therefore satisfies the equation

$$\begin{aligned}\mathrm{TC}(T,t) = {} & \frac{C_1}{T} + \frac{C_2}{2T}\left(\frac{\lambda}{p}\right)\left(\frac{1}{p-\lambda}\right)\{(p-\lambda)T-(p-(1-\alpha)\lambda)t\}^2 \\ & + \frac{C_3}{2T}\alpha\left(\frac{\lambda}{p-\lambda}\right)(p-(1-\alpha)\lambda)t^2 + \frac{C_4}{T}(1-\alpha)\lambda t.\end{aligned} \tag{4}$$

As the cost function given by equation (4) is convex (see Appendix), the optimal values of T and t can be obtained directly by solving the equations $\partial\mathrm{TC}(T,t)/\partial T = 0$ and $\partial\mathrm{TC}(T,t)/\partial t = 0$ which after simplication indicate that

$$\begin{aligned}C_1 + \frac{C_2}{2}\left(\frac{\lambda}{p}\right)\left(\frac{1}{p-\lambda}\right)(p-(1-\alpha)\lambda)^2t^2 + \frac{C_3}{2}\alpha\lambda\left(\frac{1}{p-\lambda}\right)(p-(1-\alpha)\lambda)t^2 + C_4(1-\alpha)\lambda t \\ = \frac{C_2}{2}\left(\frac{\lambda}{p}\right)(p-\lambda)T^2\end{aligned} \tag{5}$$

and that

$$\begin{aligned}C_2\left(\frac{\lambda}{p}\right)\left(\frac{1}{p-\lambda}\right)(p-(1-\alpha)\lambda)^2t + C_3\alpha\lambda\left(\frac{1}{p-\lambda}\right)(p-(1-\alpha)\lambda)t + C_4(1-\alpha)\lambda \\ = C_2\left(\frac{\lambda}{p}\right)(p-(1-\alpha)\lambda)T.\end{aligned} \tag{6}$$

Hence, it can easily be deduced by eliminating T from equations (5) and (6) that the optimal value of t is given by

$$\begin{aligned}t^* = {} & \left\{-C_4(1-\alpha)\lambda + \left(\frac{C_2(p-(1-\alpha)\lambda)}{C_3\alpha p}\left(2C_1C_3\alpha\lambda\frac{(p-(1-\alpha)\lambda)}{p-\lambda} + 2C_1C_2\lambda\frac{(p-(1-\alpha)\lambda)^2}{p(p-\lambda)}\right.\right.\right. \\ & \left.\left.\left. -(C_4(1-\alpha)\lambda)^2\right)\right)^{1/2}\right\} \Bigg/ \left\{\frac{\lambda(p-(1-\alpha)\lambda)}{p-\lambda}\left(\alpha C_3 + C_2\frac{1}{p}(p-(1-\alpha)\lambda)\right)\right\}.\end{aligned} \tag{7}$$

The optimal value of T can then be obtained by substituting t^* into equation (6). Indeed, it can be shown that

$$T^* = \left\{\frac{2C_1\left(\dfrac{C_3\alpha\lambda(p-(1-\alpha)\lambda)}{p-\lambda} + \dfrac{C_2\lambda(p-(1-\alpha)\lambda)^2}{p(p-\lambda)}\right) - (C_4(1-\alpha)\lambda)^2)p}{C_2C_3\alpha\lambda^2(p-(1-\alpha)\lambda)}\right\}^{\frac{1}{2}}. \tag{8}$$

However, it should be noted that t^* and T^* given by equations (7) and (8) are optimal provided that $t^* \geqslant 0$, otherwise no shortage should be allowed (i.e. $t^* = 0$) and the classical formula [6].

$$T^* = \left\{\frac{2C_1 p}{C_2\lambda(p-\lambda)}\right\}^{1/2} \tag{9}$$

is employed to determine the optimal scheduling period. This requires that the quantity inside the square root in equation (7) should not be less than $(C_4(1-\alpha)\lambda)^2$. Hence, it can be shown that

$$\left\{\frac{2C_1 p}{C_2\lambda(p-\lambda)}\right\}^{1/2} \geqslant \frac{C_4(1-\alpha)p}{C_2(p-(1-\alpha)\lambda)}. \tag{10}$$

The optimal batch size and the optimal cumulative shortage allowed in each cycle are given by the respective equations

$$Q^* = \lambda(T^* - (1-\alpha)t^*) \tag{11}$$

and

$$S^* = \lambda t^*. \tag{12}$$

The left-hand side of condition (10) is the classical formula for the optimal scheduling period when shortage is not allowed. It is therefore evident that whenever the penalty for not inventorying demanded units is relatively small such that the optimal scheduling period computed by using the classical formula exceeds the test ratio given by the right-hand side of condition (10), partial backlogging is a viable operating doctrine. Otherwise, $S^* = 0$ and the classical EBQ formula should be employed to determine the optimal batch size Q^*. In addition, if $\alpha = 1$, it follows from equations (7), (8), (11) and (12) that

$$Q^* = \left\{\frac{C_2 + C_3}{C_3\left(1-\frac{\lambda}{p}\right)}\right\}^{1/2}\left\{\frac{2C_1\lambda}{C_2}\right\}^{1/2} \tag{13}$$

and

$$S^* = \left\{\frac{2C_1C_2\lambda}{C_3(C_2+C_3)}\left(1-\frac{\lambda}{p}\right)\right\}^{1/2} \tag{14}$$

which give the optimal values of Q and S when the demand during the stockout period is completely backlogged.

NUMERICAL EXAMPLE

The preceding theory can be conveniently elucidated by considering a particular item which has the following characteristics:

$\lambda = 1100$ units per year,	$C_2 = \$2.0$ per unit per year,
$p = 9200$ units per year,	$C_3 = \$3.2$ per unit per year,
$C_1 = \$275$ per setup,	$C_4 = \$4.0$ per unit.

It follows from equations (4), (7), (8), (11) and (12) that

$$Q^* = 598 \text{ units}, \qquad S^* = 14 \text{ units},$$
$$T^* = 0.547 \text{ years}, \qquad t^* = 0.013 \text{ years},$$
$$\mathrm{TC}(T^*, t^*) = \$1031.7 \text{ per year},$$

when $\alpha = 0.75$, and that

$$Q^* = 747 \text{ units}, \qquad S^* = 253 \text{ units},$$
$$T^* = 0.679 \text{ years}, \qquad t^* = 0.230 \text{ years},$$
$$\mathrm{TC}(T^*, t^*) = \$809.7 \text{ per year}$$

when $\alpha = 1.0$. In both cases, the optimal policies synthesized on the basis of the model developed in this paper can be used to control the units since condition (10) is satisfied. Hence, partial backlogging represents an optimal operating doctrine. However, if C_4 is changed to \$7 per unit, it can easily be shown that condition (10) is not satisfied. Consequently, the classical production-lot-size doctrine is optimal. The optimal batch size is then equal to 586 units and no shortage should be allowed. The corresponding optimal annual total cost is \$1032.2. Table 1 shows the optimal replenishment policies for different values of α. It can be seen that as the value of α increases, Q^* and S^* increase, and the optimal annual operating cost decreases. Thus, the optimal replenishment policies are sensitive to the nature of the demand during the stockout period.

Furthermore, if complete backlogging ($\alpha = 1$) is assumed when in fact the demand is only partially captive with $\alpha = 0.75$, the substitution of $T = 0.679$ and $t = 0.230$ into equation (4) yields that $\mathrm{TC}(T, t) = \$1149.2$ per year. Thus, failure to use the appropriate model leads to an increase in the annual operating cost by \$117.5. The amount of cost savings that can be obtained from using the appropriate model has been calculated for different values of α. The results are presented in Fig. 2. It can be noted that erroneous assumption of α leads to a higher annual operating cost.

Table 1. Effects of α on the optimal replenishment policies

α	0.75	0.8	0.85	0.9	0.95	1.0
Q^*	598	654	693	720	737	747
S^*	14	86	142	187	223	253
$\mathrm{TC}(Q^*, S^*)$	1031.7	1014.1	978.4	930.7	873.9	809.7

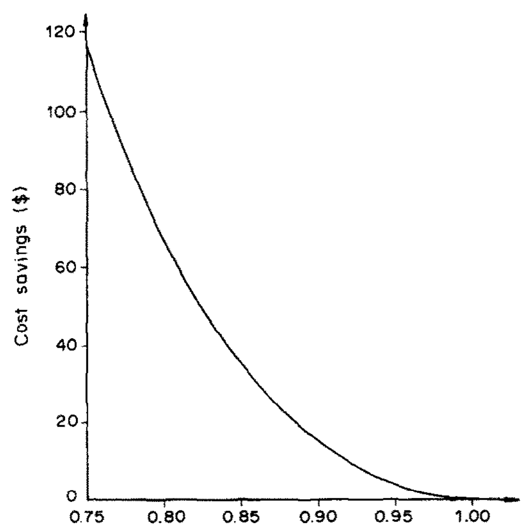

Fig. 2. Effects of erroneous assumption of α.

CONCLUSION

In this paper, a mathematical model has been developed for an inventory system in which the demand during the stockout period is partially backlogged and the items are replenished at a uniform rate. It has been shown that by defining a time proportional backordered cost and a penalty cost per unit lost, the cost function obtained is convex if a suitable set of decision variables is used in the analysis. Hence, the determination of optimal production-inventory control policies is greatly simplified since it is not necessary to employ any *ad hoc* optimization procedure [4, 5] in the analysis. Simple expressions have been obtained for the optimal production batch size and the optimal cumulative shortage allowed per cycle for such an inventory system. It has also been shown that partial backlogging is a viable operating doctrine if the penalty for not inventorying demanded items is relatively smaller than the production setup cost and the stock carrying cost. A numerical example has been used to illustrate the theory. Computational results have indicated that the optimal control policies are sensitive to the nature of the demand during the stockout period, and that making the assumption of complete backlogging when in fact the demand during the stockout period is only partially backlogged results in a higher annual operating cost. Furthermore, it has been demonstrated that the results obtained in this paper can be reduced to the well known results of the simple production-lot-size model with complete backlogging.

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APPENDIX

Convexity of TC(T, t)

Assuming that TC(T, t) is twice continuously differentiable, it can be shown from equation (4) that

$$\frac{\partial^2 \mathrm{TC}(T,t)}{\partial T^2} = \left\{2C_1 + \frac{C_2\lambda}{p(p-\lambda)}(p-(1-\alpha)\lambda)^2 t^2 + \frac{C_3\alpha\lambda}{p-\lambda}(p-(1-\alpha)\lambda)t^2 + 2C_4(1-\alpha)\lambda t\right\}\frac{1}{T^3},$$

$$\frac{\partial^2 \mathrm{TC}(T,t)}{\partial t^2} = \left\{\frac{C_2\lambda}{p(p-\lambda)}(p-(1-\alpha)\lambda)^2 + \frac{C_3\alpha\lambda}{p-\lambda}(p-(1-\alpha)\lambda)\right\}\frac{1}{T},$$

and

$$\frac{\partial^2 \mathrm{TC}(T,t)}{\partial T\,\partial t} = -\left\{\frac{C_2\lambda}{p(p-\lambda)}(p-(1-\alpha)\lambda)^2 t + \frac{C_3\alpha\lambda}{p-\lambda}(p-(1-\alpha)\lambda)t + C_4(1-\alpha)\lambda\right\}\frac{1}{T^2}.$$

The determinant of the hessian matrix H of TC(T,t) is

$$\begin{aligned}|H| &= \frac{\partial^2 \mathrm{TC}(T,t)}{\partial T^2}\frac{\partial^2 \mathrm{TC}(T,t)}{\partial t^2} - \left(\frac{\partial^2 \mathrm{TC}(T,t)}{\partial T\,\partial t}\right)^2 \\ &= \left\{2C_1C_2\frac{\lambda(p-(1-\alpha)\lambda)^2}{p(p-\lambda)} - (C_4(1-\alpha)\lambda)^2 + 2C_1C_3\alpha\frac{\lambda(p-(1-\alpha)\lambda)}{(p-\lambda)}\right\}\frac{1}{T^4} \\ &\geqslant 0\end{aligned}$$

in view of condition (10). Hence, the cost function TC(T, t) is convex.