Individual Assignment

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1 E-Level Problem (Problem 3)

Let X_i be an indicator of the random variable for the i-th student assigned to a given house (e.g. house 1).

$$\begin{cases} 1 & \text{if the } i\text{-th student is assigned to a specific house (e.g. house 1)} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Let X be the number of students assigned to a specific house (e.g. house 1). For this solution. This means that we can represent X as the sum of the X_i for the given house.

$$X = \sum_{i=1}^{n} X_i$$

Let n be the total number of students. Based on this the expected number of students in a specific house (e.g. house 1) is:

$$E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \frac{n}{4}$$

The Hat claims that, with a probability of at least 0.99, the number of students in each house X is between $0.23n$ and $0.27n$. This can be stated as $P(0.23n < X < 0.27n) \geq 0.99$. Thus, to be able to either prove or disprove the hat's claim, the probability we are interested in is given by $P(X \leq 0.23n) + P(X \geq 0.27n)$. If the sum of these probabilities is less than 0.01, then the hat's claim holds true otherwise it does not.

Because we are dealing with independent Bernoulli trials with a large number of trials we can use Chernoff Bound to define a lower and upper bound to solve the problem. Chernoff Bound is defined as [1]:

$$\Pr\left[X \geq (1+\delta)\mu_{\max}\right] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu_{\max}}$$
$$\Pr\left[X \leq (1-\delta)\mu_{\min}\right] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu_{\min}}$$

We can than solve for δ for both the lower and upper:

$$\begin{aligned}\text{For the upper bound we have } E[X] \times (1+\delta) &= 0.27n \\ 1+\delta &= 0.27 \times 4 \\ \delta &= 0.08\end{aligned}$$

$$\begin{aligned}\text{For the lower bound we have } E[X] \times (1-\delta) &= 0.23n \\ 1-\delta &= 0.23 \times 4 \\ \delta &= 0.08\end{aligned}$$

Using the Chernoff Bound for the upper bound:

$$\Pr\left[X \geq (1+\delta)\mu_{\max}\right] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu_{\max}}$$

By plugging in the values with $n = 10000$:

$$\Pr\left[X \geq 1.08 \times \frac{10000}{4}\right] \leq \left(\frac{e^{0.08}}{1.08^{1.08}}\right)^{\frac{10000}{4}}$$

$$\Pr\left[X \geq 2700\right] \leq \left(\frac{e^{0.08}}{1.08^{1.08}}\right)^{2500}$$

Using the Chernoff Bound for the lower bound:

$$\Pr\left[X \leq (1-\delta)\mu_{\min}\right] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu_{\min}}$$

By plugging in the values with $n = 10000$:

$$\Pr\left[X \leq 0.92 \times \frac{10000}{4}\right] \leq \left(\frac{e^{-0.08}}{0.92^{0.92}}\right)^{\frac{10000}{4}}$$

$$\Pr\left[X \leq 2300\right] \leq \left(\frac{e^{-0.08}}{0.92^{0.92}}\right)^{2500}$$

With the Chernoff Bounds we can add them together to see if they add up to more or less than what the hat stated:

$$\begin{aligned}\left(\frac{e^{0.08}}{1.08^{1.08}}\right)^{2500} + \left(\frac{e^{-0.08}}{0.92^{0.92}}\right)^{2500} &\leq 0.01 \\ 0.00068 &\leq 0.01\end{aligned}$$

Under the assumptions of independence and randomness, the calculations support the Hat's claim: with a probability over 0.99, all houses would receive between 23% and 27% of the 10000 or more students.

References

[1] N. Harvey. (2014) CPSC 536N: Randomized Algorithms 2014-15 Term 2 Lecture 3. University of British Columbia. [Online]. Available: https://www.cs.ubc.ca/~nickhar/W15/Lecture3Notes.pdf