Group 2: Assignment 1.1

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1 D-Level Problem (1.1)

We define a graph $G = (V, E)$. Let $V = \{x_1, ..., x_n\}$ be a set of all variables and let the edge set be of the form $E = \{(x_i, x_j) | x_i \neq x_j \in I\}$ with $I = \{E_1, ..., E_m\}$. $E_1, ..., E_m$ are the inequalities as defined in the task description.

Additionally, we define a function $K : V \to \{null\} \cup \{1..k\}$ which returns the assigned value for the variable or *null* if no value was assigned yet. $K_i = \{v \in V | K(v) = i\} \subseteq V$ is a subset of all variables that have been assigned a specific value.

Algorithm 1: D-level problem

Function `Main`(V, E, K)**:**
- **for** $x \in V$ **do**
  - $k_x = \text{argmin}_{i \in \{1..k\}} |E(x, y)|$ with $K(y) = i$
  - $K(x) := k_x$
- **return** K

For our proof we define three more sets:

$$I_t = \{(x, y) \in I | K(x) \neq K(y) \land K(x), K(y) \neq null\}$$

$$I_f = \{(x, y) \in I | K(x) = K(y) \land K(x), K(y) \neq null\}$$

I_t is a set of equations that are already satisfied by the algorithm. I_f is a set of equations that are not satisfied by the algorithm.

Lemma 1.1. *At every stage of the algorithm* $|I_t| \geq (k-1)\,|I_f|$.

Proof. Basecase: $I_t = I_f = \emptyset$
Inductionstep: Assume $|I_t| \geq (k-1)\,|I_f|$ when we process x. We assign value k_x to x, this gives us

$$|I'_t| = |I_t| + \sum_{\substack{i \neq k_x \\ i=1}}^{k} |E(x, K_i)| \geq |I_t| + (k-1)|E(x, K_{k_x})| \geq (k-1)\,|I_f| + (k-1)\,|E(x, K_{k_x})| = (k-1)\,|I'_f|$$

Where $|I'_t|$ and $|I'_f|$ are the sets after assignment of the value to x. □

Since at the end of the algorithm all vertices have a value $|I_t| + |I_f| = |I|$. The solution of our algorithm satisfies $|I_t|$ equations and therefore $Alg = |I_t|$. Lemma 1.1 implies

$$Alg \geq (1 - \frac{1}{k})|I| \geq (1 - \frac{1}{k})Opt$$

References