Group Assignment 2.1

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1 E-Level Problem (2.1)

We want to find an estimate $\hat{p}$ for the probability that the coin shows heads. From the task, we know that the true probability is p. Let us define a set of random variables $Y_i = \{0, 1\}$ where Y_i represents the i-th coin flip and $Y_i = 1$ if the coin shows heads. Let further $\hat{p} = \frac{1}{k}\sum_i Y_i$ be the average of these random variables. The Y_i are by definition uniform and independent. By this definition, $\hat{p}$ is an estimator of p. In the next section, we define an algorithm that calculates $\hat{p}$ such that $P(|\hat{p} - p| \leq \epsilon) \geq 1 - \delta$. The algorithm is based on the fact, that by flipping a coin often enough $\hat{p}$ converges to p. By defining the number of flips k dependent on ϵ and δ we can guarantee the requested probability bounds. We assume $\delta \in (0, 1)$ and $\epsilon > 0$.

1.1 Algorithm

Algorithm 1: Estimate Coin Bias

Function `CoinBias`(ϵ, δ):
- $k = \lceil -\frac{\ln(\delta/2)}{2\epsilon^2} \rceil$
- $numberOfHeads = 0$
- **for** $i \leftarrow 1$ **to** k **do**
  - $isHead = FlipCoin()$;
  - **if** $isHead$ **then**
    - $numberOfHeads = numberOfHeads + 1$
- **return** $numberOfHeads/k$

1.2 Correctness

First, we can convince ourselves that $\mathbb{E}[Y_i] = p$ since p is the true probability that the coin shows heads. Since $\hat{p}$ is the average of all Y_i by linearity of expectation we get $\mathbb{E}[\hat{p}] = p$. Since all Y_i are independent and uniformly at random Chernoff bounds can be applied. This gives the following two probabilities:

$$P(\hat{p} \leq p - \epsilon) \leq e^{-2k\epsilon^2}$$

$$P(\hat{p} \geq p + \epsilon) \leq e^{-2k\epsilon^2}$$

We define a bad Event $B =$"$\hat{p} \leq p - \epsilon$ or $\hat{p} \geq p + \epsilon$" and can calculate its probability using union bound

$$P(B) \leq P(\hat{p} \leq p - \epsilon) + P(\hat{p} \geq p + \epsilon) \leq 2e^{-2k\epsilon^2}$$

From the problem statement and the now established formula, we see that we can now show $P(|\hat{p} - p| \leq \epsilon) \geq 1 - P(B) \geq 1 - \delta$. This is equivalent to $P(B) \leq \delta$. Since we have an upper bound on $P(B)$ we can use this to calculate k to fulfil this requirement by rewriting the formula

$$\begin{aligned} P(B) \leq 2e^{-2k\epsilon^2} &\leq \delta \\ \Leftrightarrow k &\geq -\frac{1}{2\epsilon^2} \ln \frac{\delta}{2} \end{aligned}$$

This is exactly the k we defined in our algorithm. Therefore this proves that our algorithm fulfils the requirements and calculates a good enough estimator $\hat{p}$.