Individual Assignment 2 E

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1 Part 1

1.1 Expected running time of B

We know that algorithm A has two different runtimes where the first is 5 seconds and the second is 395 seconds. Both of them have the same probability of happening which is 0.5. We then have algorithm B that runs algorithm A and if it takes longer than 5 seconds it gets interrupted and restarts. This means that the only way for this algorithm to succeed is if algorithm A is able to execute in 5 seconds.

This type of problem is of the type of geometric distribution where each trial is independent. This means that the expected runtime can be calculated as $E[B] = \frac{1}{p} \times$ time per trial. This means the expected number of trials until the first success is:

$E[B] = \frac{1}{0.5} = 2 \times$ time per trial. Since each trial takes 5 seconds the expected runtime is $2 \times 5 = 10$ seconds

1.2 Probability that B runs for (strictly) more than $5k$ time units

Algorithm B runs for more than $5k$ time units for every integer where $k \geq 1$. This means that the algorithm will run k times. Because the probability of restarting the algorithm is 0.5 it means that the probability of Algorithm B running for more than $5k$ time units can be calculated by:

$$P(\text{B runs more than } 5k \text{ time units}) = 0.5^k$$

2 Part 2

2.1 Upper bound on the probability that A runs for more than 4,000 time units

To calculate an upper bound on the probability that A runs for more than 4,000 times units can be calculated using Markov's inequality [1]. Markov's inequality states that if X is a nonnegative random variable, then, for any $a > 0$:

$$\Pr[X \geq a \times E[X]] \leq \frac{1}{a}$$

Using this we know that $a \times E[A] = 4000$. We also know that the $E[A] = 200$ which makes that $a = 20$. This means that:

$$Pr[X \geq 4000] \leq \frac{1}{20}$$
$$Pr[X \geq 4000] \leq 0.05$$

2.2 Probability of B taking longer than 4,000 time units is at most 0.1%

To find a threshold value for a cutoff time for algorithm A so that the probability of it running more than 4,000 time units is below 0.001. This means that the probability of A completing within a threshold value T can be stated as $P(A < T)$.

Based on this it means that the probability that a single run is longer than the threshold value T is $1 - P(A < T)$. We then can calculate the max number of iterations that the algorithm can complete within the timeframe of 4,000 time units as $k = \frac{4000}{T}$. Therefore we have:

$$(1 - P(A < T))^{\frac{4000}{T}} < 0.001$$

Using Markov's inequality we know that $P(A \geq T) \leq \frac{E[A]}{T}$. This also means that $P(A < T) > 1 - \frac{E[A]}{T}$. We then want to find a threshold T that upholds this:

$$\left(1 - \left(1 - \frac{E[A]}{T}\right)\right)^{\frac{4000}{T}} < 0.001$$
$$\left(\frac{E[A]}{T}\right)^{\frac{4000}{T}} < 0.001$$
$$\left(\frac{200}{T}\right)^{\frac{4000}{T}} < 0.001$$

We then could use programs like Wolfram Alpha to find a value that upholds this or manually guess. We can see that values in the range of ≈ 400 to 800. This means that for example with a cutoff value of 500 time units (if algorithm A runs longer than 500 time units we interrupt it and start it again). Using this allows algorithm B to uphold the constraint that the probability that the algorithm runs more than 4,000 time units is at most 0.001.

References

[1] Ioana Bercea, “Lecture 6: Sampling and probability amplification,” October 2023.