Analysis I

Cheat Sheet · v1.0 · 2020

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1 Algebra

1.1 Exponential Properties

(i) $x^0 = 1$
(ii) $x^n x^m = x^{n+m}$
(iii) $\frac{x^n}{x^m} = x^{n-m} = \frac{1}{x^{m-n}}$
(iv) $(x^n)^m = x^{nm}$
(v) $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$
(vi) $x^{-n} = \frac{1}{x^n}$
(vii) $\frac{1}{x^{-n}} = x^n$
(viii) $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n = \frac{y^n}{x^n}$
(ix) $x^{\frac{n}{m}} = \left(x^{\frac{1}{m}}\right)^n = (x^n)^{\frac{1}{m}} = \sqrt[m]{x^n}$

1.2 Logarithm Properties

(i) $\log_n(0) = Undefined$
(ii) $\log_n(1) = 0$
(iii) $\log_n(n) = 1$
(iv) $\log_n(n^x) = x$
(v) $n^{\log_n(x)} = x$
(vi) $\log_n(x^r) = r\log_n(x) \neq \log_n^r(x) = (\log_n(x))^r$
(vii) $\log_n(xy) = \log_n(x) + \log_n(y)$
(viii) $\log_n\left(\frac{x}{y}\right) = \log_n(x) - \log_n(y)$
(ix) $-\log_n(x) = \log_n\left(\frac{1}{x}\right)$
(x) $\frac{\log(x)}{\log(n)} = \log_n(x)$

1.3 Radical Properties

(i) $\sqrt[n]{x} = x^{\frac{1}{n}}$
(ii) $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$
(iii) $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$
(iv) $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$
(v) $\sqrt[n]{x^n} = x, \quad if\ n\ is\ odd$
(vi) $\sqrt[n]{x^n} = |x|, \ if\ n\ is\ even$

1.4 Absolute Value Properties

(i) $|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$
(ii) $|x| \geq 0$
(iii) $|-x| = |x|$
(iv) $|ca| = c|a|, \quad if\ c > 0$
(v) $|xy| = |x||y|$
(vi) $|x^2| = x^2$
(vii) $|x^n| = |x|^n$
(viii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
(ix) $|a-b| = b-a$, if $a \leq b$
(x) $|a+b| \leq |a| + |b|$
(xi) $|a| - |b| \leq |a-b|$

1.5 Factorization

(i) $x^2 - a^2 = (x+a)(x-a)$
(ii) $x^2 + 2ax + a^2 = (x+a)^2$
(iii) $x^2 - 2ax + a^2 = (x-a)^2$
(iv) $x^2 + (a+b)x + ab = (x+a)(x+b)$
(v) $x^3 + 3ax^2 + 3a^2x + a^3 = (x+a)^3$
(vi) $x^3 - 3ax^2 + 3a^2x - a^3 = (x-a)^3$
(vii) $x^3 + a^3 = (x+a)(x^2 - ax + a^2)$
(viii) $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$
(ix) $x^{2n} - a^{2n} = (x^n - a^n)(x^n + a^n)$

1.6 Complete The Square

$$\boldsymbol{ax^2 + bx + c = 0 \quad \Rightarrow \quad a(x+d)^2 + e = 0}$$

- $d = \frac{b}{2a}$
- $e = c - \frac{b^2}{4a}$

1.7 Quadratic Formula

$$\boldsymbol{ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

- *If* $b^2 - 4ac > 0 \Rightarrow$ *Two real unequal solutions.*
- *If* $b^2 - 4ac = 0 \Rightarrow$ *Two repeated real solutions.*
- *If* $b^2 - 4ac < 0 \Rightarrow$ *Two complex solutions.*

2 Functions

2.1 Domain

- **Fractions** denominator $\neq 0$.
- **Logarithms** if the base is a number, the argument must be > 0, if the base depends on a variable, the base must be $> 0 \wedge \neq 1$.
- **Roots** with even index, the argument must be ≥ 0, for roots with odd index the domain is $\mathbb{R}$.
- **Arccos/Arcsin** the agrument must be $\in [-1, 1]$. For other trig functions we use trig properties to change them to cos and sin.
- **Exponential** base > 0.

2.2 Parity

We consider the partiy of the function only if $Dom(f)$ is mirrored on the origin: $(Dom(f) = [-2, 2] \vee (-\infty, \infty) \vee (-\infty, -1] \cup [1, \infty])$.

- **Even function** (with respect to the y axis) if: $f(-x) = f(x)$.
- **Odd function** (with respect to the origin) if: $f(-x) = -f(x)$.
- In every other case the function is neither even nor odd.

2.3 Axis Intercept

- **X intercept** can be many; is calculated by solving $f(x) = 0$. If $f(x) = \frac{g(x)}{h(x)}$ we solve just $g(x) = 0$. The points are then $(x_i, 0)$.
- **Y intercept** can be just one; is calculated by setting $x = 0$, the point is then $(0, f(0))$. If $x = 0 \notin Dom(f)$ there is no Y intercept.

2.4 Sign

The sign can only change when there is a x intercept (if the function is continous), thus if we solve $f(x) \geq 0$ we get both the X intercepts and where the function is positive.

2.5 Asymptotes/Holes

- **Hole** at point $(x_0, f_{semplified}(x_0))$ if plugging the critical point x_0 in the numerator of f gives $\frac{0}{0}$.
- **Vertical** asymptote at a critical point x_0 if:
  $\lim_{x \to x_0^-} f(x) = \pm\infty$ (left at $x = x_0$)
  $\lim_{x \to x_0^+} f(x) = \pm\infty$ (right at $x = x_0$).
- **Horizontal** aysmptote (if domain is unlimited at $\pm\infty$) if:
  $\lim_{x \to +\infty} f(x) = k$ (right $y = k$)
  $\lim_{x \to -\infty} f(x) = h$ (left $y = h$).
- **Oblique** aysmptote (if domain is unlimited at $\pm\infty$) if:
  $\lim_{x \to +\infty} \frac{f(x)}{x} = m \wedge \lim_{x \to +\infty}[f(x) - mx] = q$ (right at $y = mx + q$)
  $\lim_{x \to -\infty} \frac{f(x)}{x} = m \wedge \lim_{x \to -\infty}[f(x) - mx] = q$ (left at $y = mx + q$).

2.6 Monotonicity

A funciton f is:

- **Monotonically increasing** if: $\forall x, y : x \leq y \Rightarrow f(x) \leq f(y)$
- **Monotonically decreasing** if: $\forall x, y : x \leq y \Rightarrow f(x) \geq f(y)$
- **Strictly increasing** if: $\forall x, y : x < y \Rightarrow f(x) < f(y)$
- **Strictly decreasing** if: $\forall x, y : x < y \Rightarrow f(x) > f(y)$

2.7 Max, Min

Calculate $f'(x) = 0$, then all the solutions x_i are our candidates, where for a small $\epsilon > 0$:

- **Max** if: $f'(x_i - \epsilon) > 0 \wedge f'(x_i + \epsilon) < 0$.
- **Min** if: $f'(x_i - \epsilon) < 0 \wedge f'(x_i + \epsilon) > 0$.
- **Inflection** if (use sign table):
  $f'(x_i - \epsilon) < 0 \wedge f'(x_i + \epsilon) < 0$, or
  $f'(x_i - \epsilon) > 0 \wedge f'(x_i + \epsilon) > 0$

If $f'(x) > 0$, then f is strictly increasing.
If $f'(x) < 0$, then f is strictly decreasing.
If $f'(x) = 0$ f is constant.

2.8 Convexity

- **Convex** ($\cup$) if: $f''(x) > 0$
- **Concave** ($\cap$) if: $f''(x) < 0$

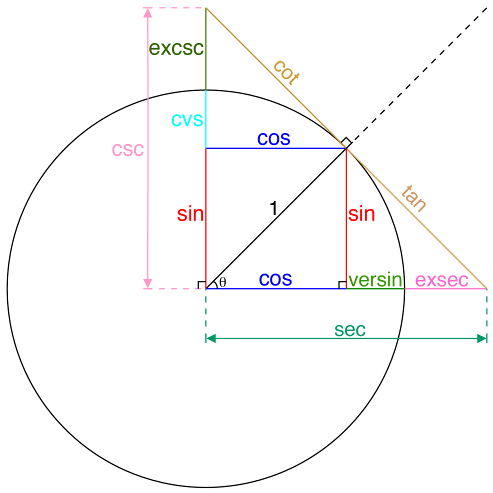
2.9 Inflection Points

Calculate $f''(x) = 0$, then all the solutions x_i are our candidates (except where $f(x)$ is not defined), where for a small $\epsilon > 0$:

- **Increasing Inflection** if: $f''(x_i - \epsilon) < 0 \wedge f''(x_i + \epsilon) > 0$
- **Decreasing Inflection** if: $f''(x_i - \epsilon) > 0 \wedge f''(x_i + \epsilon) < 0$
- Otherwise nothing happens on x_i.

3 Trigonometry

3.1 Unit Circle



3.2 Domain and Range

- $\sin : \mathbb{R} \longrightarrow [-1, 1]$
- $\cos : \mathbb{R} \longrightarrow [-1, 1]$
- $\tan : \{x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + k\pi\} \longrightarrow \mathbb{R}$
- $\cot : \{x \in \mathbb{R} \mid x \neq k\pi\} \longrightarrow \mathbb{R}$
- $\csc : \{x \in \mathbb{R} \mid x \neq k\pi\} \longrightarrow \mathbb{R} \setminus (-1, 1)$
- $\sec : \{x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + k\pi\} \longrightarrow \mathbb{R} \setminus (-1, 1)$
- $\sin^{-1} : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\cos^{-1} : [-1, 1] \longrightarrow [0, \pi]$
- $\tan^{-1} : \mathbb{R} \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

3.3 Pythagorean Identities

(i) $\sin^2(x) + \cos^2(x) = 1$
(ii) $\tan^2(x) + 1 = \sec^2(x)$
(iii) $1 + \cot^2(x) = \csc^2(x)$

3.4 Periodicity Identities

(i) $\sin(x \pm 2\pi) = \sin(x)$
(ii) $\cos(x \pm 2\pi) = \cos(x)$
(iii) $\tan(x \pm \pi) = \tan(x)$
(iv) $\cot(x \pm \pi) = \cot(x)$
(v) $\csc(x \pm 2\pi) = \csc(x)$
(vi) $\sec(x \pm 2\pi) = \sec(x)$

3.5 Reciprocal Identities

(i) $\cot(x) = \frac{1}{\tan(x)}$
(ii) $\csc(x) = \frac{1}{\sin(x)}$
(iii) $\sec(x) = \frac{1}{\cos(x)}$

3.6 Quotient Identities

(i) $\tan(x) = \frac{\sin(x)}{\cos(x)}$
(ii) $\cot(x) = \frac{\cos(x)}{\sin(x)}$

3.7 Sum Identities

(i) $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
(ii) $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
(iii) $\tan(x + y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$

3.8 Difference Identities

(i) $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
(ii) $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
(iii) $\tan(x - y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

3.9 Double Angle Identities

(i) $\sin(2x) = 2\sin(x)\cos(x)$
(ii) $\cos(2x) = \cos^2(x) - \sin^2(x)$
(iii) $\cos(2x) = 2\cos^2(x) - 1 \Rightarrow \cos^2(x) = \frac{\cos(2x)+1}{2}$
(iv) $\cos(2x) = 1 - 2\sin^2(x) \Rightarrow \sin^2(x) = \frac{1-\cos(2x)}{2}$
(v) $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

3.10 Co-Function Identities

(i) $\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$
(ii) $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$
(iii) $\tan\left(\frac{\pi}{2} - x\right) = \cot(x)$
(iv) $\cot\left(\frac{\pi}{2} - x\right) = \tan(x)$
(v) $\csc\left(\frac{\pi}{2} - x\right) = \sec(x)$
(vi) $\sec\left(\frac{\pi}{2} - x\right) = \csc(x)$

3.11 Even-Odd Identities

(i) $\sin(-x) = -\sin(x)$
(ii) $\cos(-x) = \cos(x)$
(iii) $\tan(-x) = -\tan(x)$
(iv) $\cot(-x) = -\cot(x)$
(v) $\csc(-x) = -\csc(x)$
(vi) $\sec(-x) = \sec(x)$

3.12 Half-Angle Identities

(i) $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$
(ii) $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}}$
(iii) $\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$
(iv) $\tan\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{\sin(x)}$
(v) $\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1+\cos(x)}$

3.13 Sum-to-Product Formulas

(i) $\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
(ii) $\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$
(iii) $\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
(iv) $\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

3.14 Product-to-Sum Formulas

(i) $\sin(x)\sin(y) = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$
(ii) $\cos(x)\cos(y) = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$
(iii) $\sin(x)\cos(y) = \frac{1}{2}[\sin(x + y) + \sin(x - y)]$
(iv) $\cos(x)\sin(y) = \frac{1}{2}[\sin(x + y) - \sin(x - y)]$

3.15 Tangent expression

If $u = \tan(\frac{x}{2})$: $\left[dx = \frac{2}{1+u^2}du\right]$

(i) $\cos(x) = \frac{1-u^2}{1+u^2}$
(ii) $\sin(x) = \frac{2u}{1+u^2}$
(iii) $\tan(x) = \frac{2u}{1-u^2}$

3.16 Hyperbolic Functions

(i) $\sinh(x) = \frac{e^x - e^{-x}}{2}$
(ii) $\cosh(x) = \frac{e^x + e^{-x}}{2}$
(iii) $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

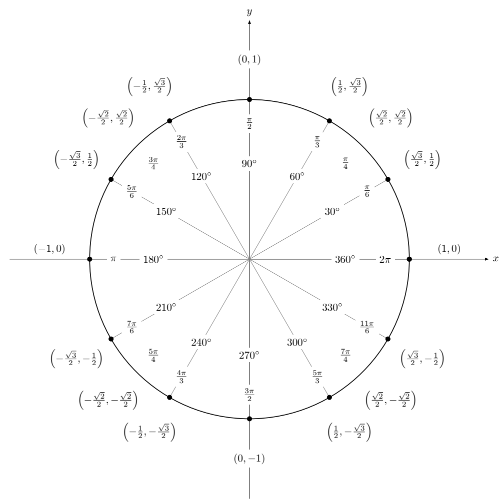
3.17 Laws of Sines

(i) $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$

3.18 Laws of Cosines

(i) $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$
(ii) $b^2 = a^2 + c^2 - 2ac\cos(\beta)$
(iii) $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$

3.19 Degrees



θ		sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
Rad	Deg						
0 / 2π	0	0	1	0	Undef	1	Undef
π/6	30	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$	$\sqrt{3}$
π/4	45	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}$	$\sqrt{2}$	1
π/3	60	$\sqrt{3}/2$	1/2	$\sqrt{3}$	$2\sqrt{3}/3$	2	$\sqrt{3}/3$
π/2	90	1	0	Undef	1	Undef	0
2π/3	120	$\sqrt{3}/2$	− 1/2	$-\sqrt{3}$	$2\sqrt{3}/3$	-2	$-\sqrt{3}/3$
3π/4	135	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
5π/6	150	1/2	$-\sqrt{3}/2$	$-\sqrt{3}/3$	2	$-2\sqrt{3}/3$	$-\sqrt{3}$
π	180	0	-1	0	Undef	-1	Undef
7π/6	210	− 1/2	$-\sqrt{3}/2$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$	$\sqrt{3}$
5π/4	225	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
4π/3	240	$-\sqrt{3}/2$	− 1/2	$\sqrt{3}$	$-2\sqrt{3}/3$	-2	$\sqrt{3}/3$
3π/2	270	-1	0	Undef	-1	Undef	0
5π/3	300	$-\sqrt{3}/2$	1/2	$-\sqrt{3}$	$-2\sqrt{3}/3$	2	$-\sqrt{3}/3$
7π/4	315	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
11π/6	330	− 1/2	$\sqrt{3}/2$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$	$-\sqrt{3}$

4 Limits, Sup and Inf

Definition 4.1 (Limits). *Let $f(x)$ be a function defined on an open interval that contains $x = x_0$, except possibly $x = x_0$, then the limit $\lim_{x\to x_0} f(x) = L$ exists if and only if for all ϵ there is a δ such that:*

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon \ \forall x$$

Sequence Definition:
$\lim\limits_{x\to x_0} f(x) = L \Leftrightarrow \forall (x_n)$ where $\lim\limits_{n\to\infty} x_n = x_0$, then $\lim\limits_{n\to\infty} f(x_n) = L$

4.1 Limit Properties

Assume that $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exists ant that $c \in \mathbb{R}$, then:

(i) $\lim\limits_{x\to x_0} [cf(x)] = c \lim\limits_{x\to x_0} f(x)$

(ii) $\lim\limits_{x\to x_0} [f(x) \pm g(x)] = \lim\limits_{x\to x_0} f(x) \pm \lim\limits_{x\to x_0} g(x)$

(iii) $\lim\limits_{x\to x_0} [f(x)g(x)] = \lim\limits_{x\to x_0} f(x) \lim\limits_{x\to x_0} g(x)$

(iv) $\lim\limits_{x\to x_0} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}, \lim\limits_{x\to x_0} g(x) \neq 0$

(v) $\lim\limits_{x\to x_0} [f(x)]^n = \left[\lim\limits_{x\to x_0} f(x)\right]^n$

(vi) $\lim\limits_{x\to x_0} \left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim\limits_{x\to x_0} f(x)}$

(vii) $\lim\limits_{x\to x_0} x = x_0$

4.2 Chain Rule

Let f and g be continous, and given $\lim_{x\to x_0} f(g(x))$ of composed function we can solve $\lim_{x\to x_0} g(x) = y_0$, then:

$$\lim_{x\to x_0} f(g(x)) = \lim_{y\to y_0} f(y)$$

4.3 Exponential Rule

Let f and g be continous, where $\lim_{x\to x_0} f(x) = f(x_0) > 0$ and $\lim_{x\to x_0} g(x) = g(x_0)$ (where both limits exists), then:

$$\lim_{x\to x_0} f(x)^{g(x)} = f(x_0)^{g(x_0)}$$

4.4 Root Trick

$$\lim_{x\to x_0} \sqrt{f} - g = \lim_{x\to x_0} \sqrt{f} - g \cdot \frac{\sqrt{f} + g}{\sqrt{f} + g}$$

4.5 E-Log Trick

$$\lim_{x\to x_0} f^g = \lim_{x\to x_0} e^{g\ln(f)}$$

Theorem 1: L'Hospital's Rule

If by plugging x_0 in $\frac{f(x)}{g(x)}$ we get $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L \Leftrightarrow L \neq \pm\infty$$

Theorem 2: Limit Squeeze Theorem

Let $\lim\limits_{x\to x_0} f(x)$, if $g(x) \leq f(x) \leq h(x), \forall x$, and $\lim\limits_{x\to x_0} g(x) = \lim\limits_{x\to x_0} h(x) = L$, then:

$$\lim_{x\to x_0} f(x) = L$$

4.6 Important Limits

- $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$
- $\lim\limits_{n\to 0} \frac{a^n - 1}{n} = \ln(a)$
- $\lim\limits_{n\to\infty} \ln(n) = \infty$
- $\lim\limits_{n\to\infty} \frac{\log_a(1+n)}{n} = \frac{1}{\ln(a)}$
- $\lim\limits_{n\to 0} \frac{\log_a(1+n)}{n} = \frac{1}{\ln(a)}$
- $\lim\limits_{n\to 0} \frac{\sin(n)}{n} = 1$
- $\lim\limits_{n\to 0} \frac{1 - \cos(n)}{n} = 0$
- $\lim\limits_{n\to 0} \frac{1 - \cos(n)}{n^2} = \frac{1}{2}$
- $\lim\limits_{n\to 0} \frac{\tan(n)}{n} = 1$
- $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0$
- $\lim\limits_{n\to 0} \frac{e^n - 1}{n} = 1$
- $\lim\limits_{n\to\infty} \sqrt[n]{n!} = \infty$
- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

4.7 Strategy

Given a limit $\lim\limits_{x\to x_0} f(x)$:

1. Is $f(x_0)$ solvable normally (polynomials and radicals) ?
2. Try to decompose the limit with the properties and go back to step 1 for each piece.
3. If it contains a **radical expession** try with the root trick, pay attention that if it's not a square root you can try with the third root factorization, but for bigger roots it's probably another method. If the root contains the entire limit it can be put out (PR6).
4. If conatins a **trigonometric function** try with the Squeeze Theorem, if the trig function contains another function go with the composed function decomposition. If it simplifies well with the series definition of $\cos, \sin$, or tan try to simplify the sum and solve each piece.
5. If it's a **composed function** try the chain rule.
6. If it's raised to an **unusual power** try the E-Log trick.
7. If you get $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ use l'Hopital.
8. If you get $\pm\infty \cdot 0$ or $0 \cdot \pm\infty$ tranform the function into a fraction so that you get $\frac{0}{0}$ or or $\frac{\pm\infty}{\pm\infty}$ then use l'Hopital.

4.8 Supremum and Infimium

Definition 4.2.

- *The* ***Supremum*** *of a set S denoted $sup(S) = u$ is a number u that satisfies the condition that u is an upper bound of S and for any upper bound v of S, $u \leq v$.*
- *The* ***Infimium*** *of a set S denoted $inf(S) = u$ is a number u that satisfies the condition that u is an lower bound of S and for any lower bound v of S, $u \geq v$.*

- If the supremum doesn't exists we can write: $sup(S) = \infty$.
- If the infimium doesn't exist we can write: $inf(S) = -\infty$.
- To prove that the minimum doesn't exist: $\forall\epsilon\exists n_0 \in \mathbb{N} : f(x) \leq inf(a) + \epsilon \ \forall x \geq n_0$.
- To prove that the maximum doesn't exist: $\forall\epsilon\exists n_0 \in \mathbb{N} : f(x) \leq sup(a) - \epsilon \ \forall x \geq n_0$.

5 Continuity

Definition 5.1 (Pointwise Continous). *A function $f : [a, b] \to \mathbb{R}$ is* ***pointwise continous*** *at $x_0 \in [a, b]$ if $\lim_{x\to x_0} f(x) = f(x_0)$, or:*

$$\forall x_0, \epsilon\exists\delta\ \forall x : (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$$

Definition 5.2 (Uniformly Continous). *A function $f : [a, b] \to \mathbb{R}$ is* ***uniformly continous*** *if it's continous at every point in it's domain $\forall x_0 \in [a, b] : \lim_{x\to x_0} f(x) = f(x_0)$, or:*

$$\forall\epsilon\exists\delta\ \forall x_0, x : (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)$$

Definition 5.3 (Lipschitz Continous). *A function $f : [a, b] \to \mathbb{R}$ is* ***Lipschitz continous*** *if:*

$$\exists L \forall x, x_0 : |f(x) - f(x_0)| \leq L|x - x_0|$$

5.1 Properties

Let f and g be continous, then also $f \pm g$, $f \cdot g$, $\frac{f}{g} \Leftrightarrow g \neq 0$ and $f \circ g$ are continous.

(i) **Polynomials:** All polinomials $P(x)$ are uniformly continous in their entire domain.

(ii) **Bijective:** If $f : [a, b] \to \mathbb{R}$ is continous and monotone, then it's bijective and f^{-1} is also continous.

Theorem 3: Intermediate Value

Let f be a continous function on $[a, b]$ and let s be a number with $f(a) < s < f(b)$, then there exists at least one solution to $f(x) = s$.

Theorem 4: Extreme Value

Let $f : I \to \mathbb{R}$ be a continous function on $I = [a, b]$ then there exist two numbers $c \in I$ and $d \in I$ such that:

$$\forall x \in I : \ m = f(c) \leq f(x) \leq f(d) = M$$

Where m is a lower bound and M an upper bound.

6 Derivatives

Definition 6.1 (Derivative)**.** *The **derivative** of $f(x)$ with respect to x is:*

$$\frac{df}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{x_0\to x}\frac{f(x_0)-f(x)}{x_0-x}$$

6.1 Properties

(i) $\frac{d}{dx}(c) = 0$
(ii) $(cf)' = cf'(x)$
(iii) $(f\pm g)' = f'(x) + g'(x)$
(iv) $(fg)' = f'g + fg'$
(v) $\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$
(vi) $\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1}f'(x)$
(vii) $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$
(viii) $[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))}$

6.2 Common Derivatives

· $\frac{d}{dx}(x) = 1$
· $\frac{d}{dx}(|x|) = sign(x)$
· $\frac{d}{dx}(e^x) = e^x$
· $\frac{d}{dx}(a^x) = a^x\ln(a)$
· $\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}$
· $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$
· $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)} = \frac{1}{x}$ *if* $f(x) = x$
· $\frac{d}{dx}(\ln|x|) = \frac{1}{x},\ x\neq 0$
· $\frac{d}{dx}(\log_a(x)) = \frac{1}{xln(a)},\ x>0$
· $\frac{d}{dx}(\sin(x)) = \cos(x)$
· $\frac{d}{dx}(\cos(x)) = -\sin(x)$
· $\frac{d}{dx}(\tan(x)) = \sec^2(x) = \tan^2(x)+1$
· $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
· $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
· $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$
· $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$
· $\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$
· $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$
· $\frac{d}{dx}(\sinh(x)) = \cosh(x)$
· $\frac{d}{dx}(\cosh(x)) = \sinh(x)$
· $\frac{d}{dx}(\tanh(x)) = \frac{1}{\cosh(x)} = 1-\tanh^2(x)$
· $\frac{d}{dx}(\sinh^{-1}(x)) = \frac{1}{\sqrt{x^2+1}}$
· $\frac{d}{dx}(\cosh^{-1}(x)) = \frac{1}{\sqrt{x^2-1}}$
· $\frac{d}{dx}(\tanh^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

6.3 Differentiable

Theorem 5: Differentiable

A function f is differentiable at a point x_0 iff:

$$\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}$$

(i) **Tangent Line:** The function f has a tangent point at a if and only if f is differentiable at a. The equation of the tangent line is:

$$y = f'(a)(x-a) + f(a)$$

(ii) **Continous:** If $f(x)$ is differentiable at a, then f is continous at a. The converse is not true (e.g. $f(x) = |x|,\ a = 0$).

(iii) **Classes:** If $f:[a,b]\to\mathbb{R}$ is differentiable k times we say that $f\in C^k([a,b])$ where C is called *classification function.* If f is differentiable infinite times we say that f is *smooth* ($f\in C^\infty([a,b])$).

Theorem 6: Inverse Function Theorem

Let $f:[a,b]\longrightarrow\mathbb{R}$ be continous, differentiable and stricly increasing where $\forall x\in[a,b]:$ $f'(x)>0$ and

$$c = \inf_{a<x<b} f(x) < \sup_{a<x<b} f(x) = d$$

then:

· $f:]a,b[\to]c,d[$ is bijective.
· $f^{-1}:]c,d[\to]a,b[$ is differentiable with $[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))}$

Theorem 7: Mean Value Theorem

Let $f:[a,b]\longrightarrow\mathbb{R}$ be continous and differentiable on $]a,b[$, then exists $c\in]a,b[$ with: $f(b) = f(a)+f'(c)(b-a)$, or:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

7 Integrals

Definition 7.1 (Riemann-Integral). *Given:*

- *A continous function* $f(x) : [a,b] \longrightarrow \mathbb{R}$
- *A partition* $P := \{a = x_0, ..., x_{n-1}, x_n = b\}$ *where* $I_i = [x_{i-1}, x_i]$
- *A set of points* $\xi := \{\xi_1, ..., \xi_n\}$ *where* $\xi_i \in I_i = [x_{i-1}, x_i]$.

Then the Riemann-Sum is defined as:

$$S(f, P, \xi) := \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$$

Where the ***Riemann-Integral*** *is:*

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$$

(i) **Over Sum:**
$$\overline{S}(f,P) := \lim_{n\to\infty} \sum_{i=1}^{n} \sup_{x\in I_i} f(x) \cdot (x_i - x_{i-1})$$
Infimum of the over sum:
$$\inf_P \overline{S}(f,P) := (b-a) \cdot \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sup_{x\in I_i} f(x)$$

(ii) **Under Sum:**
$$\underline{S}(f,P) := \lim_{n\to\infty} \sum_{i=1}^{n} \inf_{x\in I_i} f(x) \cdot (x_i - x_{i-1})$$
Supremum of the under sum:
$$\sup_P \underline{S}(f,P) := (b-a) \cdot \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \inf_{x\in I_i} f(x)$$
$$I_i = \left[a + \frac{(i-1)(b-a)}{n}, a + \frac{i(b-a)}{n}\right]$$

(iii) **Inequality:**
$$\sup_{p\in P(I)} \underline{S}(f,P) \leq \inf_{p\in P(I)} \overline{S}(f,P)$$

(iv) **Monotone:** A monotone function $f : I \to \mathbb{R}$ is Riemann-Integrable over I.

(v) **Continous:** A continous function $f : I \to \mathbb{R}$ is Riemann-Integrable over I.

Theorem 8: Riemann-Integrable

A function f is Riemann-Integrable iff:

$$\sup_{P_1} \underline{S}(f,P) = \inf_{P_2} \overline{S}(f,P)$$

More formally:

$$\forall \epsilon \exists P : |\underline{S}(f,P) - \overline{S}(f,P)| < \epsilon$$

7.1 Properties

(i) $\int_a^a f(x)dx = 0$

(ii) $\int_a^b cf(x) = c\int_a^b f(x)$

(iii) $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

(iv) $\int_a^b f(x)dx = -\int_a^b f(x)dx$

(v) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(vi) $\int_a^b cdx = c(b-a)$

(vii) If $f(x) \geq g(x)$, then: $\int_a^b f(x) \geq \int_a^b g(x)$

(viii) If $m \leq f(x) \leq M$, then: $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

(ix) If $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

7.2 Common Integrals ($+C$)

Basic

- $\int kdx = kx$
- $\int x^n dx = \frac{x^{n+1}}{n+1},\ n \neq -1$
- $\int \frac{1}{x^n} = \frac{-1}{(n-1)x^{n-1}}$
- $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x|$
- $\int a^x dx = \frac{a^x}{ln(a)}$
- $\int e^x dx = e^x$
- $\int \log_a(x)dx = x\log_a(x) - x\log_a(e)$

Trigonometirc

- $\int \sin(x)dx = -\cos(x)$
- $\int \cos(x)dx = \sin(x)$
- $\int \tan(x)dx = -\ln|\cos(x)| = \ln|\sec(x)|$
- $\int \cot(x)dx = \ln|\sin(x)|$
- $\int \sec(x)dx = \ln|\sec(x) + \tan(x)|$
- $\int \csc(x)dx = -\ln|\csc(x) + \cot(x)|$
- $\int \sin^{-1}(x)dx = x\sin^{-1}(x) + \sqrt{1-x^2}$
- $\int \cos^{-1}(x)dx = x\cos^{-1}(x) - \sqrt{1-x^2}$
- $\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \sqrt{1}2\ln(1+x^2)$
- $\int \cot^{-1}(x)dx = x\cot^{-1}(x) + \sqrt{1}2\ln(1+x^2)$
- $\int \sin^2(x)dx = \frac{1}{2}(x - \sin(x)\cos(x)$
- $\int \cos^2(x)dx = \frac{1}{2}(x + \sin(x)\cos(x)$
- $\int \tan^2(x)dx = \tan(x) - x$
- $\int \cot^2(x)dx = -\cot(x) - x$
- $\int \sec^2(x)dx = \tan(x)$
- $\int \csc^2(x)dx = -\cot(x)$
- $\int \csc(x)\cot(x)dx = -\csc(x)$
- $\int \frac{1}{\sin(x)}dx = \ln\left|\frac{1-\cos(x)}{\sin(x)}\right|$
- $\int \frac{1}{\cos(x)}dx = \ln\left|\frac{1+\sin(x)}{\cos(x)}\right|$
- $\int \frac{1}{\sin^2(x)}dx = -\cot(x)$
- $\int \frac{1}{\cos^2(x)}dx = \tan(x)$
- $\int \frac{1}{1+\sin(x)}dx = \frac{-\cos(x)}{1+\sin(x)}$
- $\int \frac{1}{1+\cos(x)}dx = \frac{\sin(x)}{1+\cos(x)}$
- $\int \frac{1}{1-\sin(x)}dx = \frac{\cos(x)}{1-\sin(x)}$
- $\int \frac{1}{1-\cos(x)}dx = \frac{-\sin(x)}{1-\cos(x)}$
- $\int \sin(ax)dx = -\frac{1}{a}\cos(ax)$
- $\int \cos(ax)dx = \frac{1}{a}\sin(ax)$
- $\int \tan(ax)dx = -\frac{1}{a}\ln(\cos(ax))$
- $\int x\sin(ax)dx = -\frac{1}{a}x\cos(ax) + \frac{1}{a^2}\sin(ax)$
- $\int x\cos(ax)dx = \frac{1}{a}x\sin(ax) + \frac{1}{a^2}\cos(ax)$
- $\int \sinh(x)dx = \cosh(x)$
- $\int \cosh(x)dx = \sinh(x)$
- $\int \tanh(x)dx = \ln(\cosh(x))$
- $\int \coth(x)dx = \ln|\sinh(x)|$
- $\int \sinh^{-1}(x)dx = x\sinh^{-1}(x) - \sqrt{x^2+1}$
- $\int \cosh^{-1}(x)dx = x\cosh^{-1}(x) - \sqrt{x^2-1}$
- $\int \tanh^{-1}(x)dx = x\tanh^{-1}(x) + \frac{1}{2}\ln(1-x^2)$
- $\int \coth^{-1}(x)dx = x\coth^{-1}(x) + \frac{1}{2}\ln(x^2-1)$

Logarithmic

- $\int \ln(ax)dx = x\ln(ax) - x$
- $\int x\ln(ax)dx = \frac{x^2}{4}(2\ln(ax) - 1)$
- $\int \frac{\ln(ax)}{x}dx = \frac{1}{2}(\ln(ax)^2$

Exponential

- $\int e^{ax}dx = \frac{1}{a}e^{ax}$
- $\int xe^x dx = (x-1)e^x$
- $\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$

Rational Functions

- $\int \frac{1}{\sqrt{x}} = 2\sqrt{x}$
- $\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1},\ n \neq -1$
- $\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)}$
- $\int \frac{ax+b}{cx+d}dx = \frac{ax}{c} - \frac{ad-bc}{c^2}\ln|cx+d|$
- $\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a}$
- $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$
- $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
- $\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)$
- $\int \frac{1}{(x-a)(x-b)}dx = \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right|$
- $\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln|a^2+x^2|$
- $\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\left(\frac{x}{a}\right)$
- $\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2|$
- $\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x} + \ln|a+x|$
- $\int \frac{x}{ax^2+bx+c} = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)$

Square Roots

- $\int \sqrt{x-a}dx = \frac{2}{3}(x-a)^{\frac{3}{2}}$
- $\int \sqrt{ax+b}dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b}$
- $\int \sqrt{x^2+a}dx = \frac{1}{2}x\sqrt{x^2+a} + \frac{a}{2}\ln|x+\sqrt{x^2+a}|$
- $\int \sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$
- $\int x\sqrt{x-a}dx = \frac{2}{3}a(x-a)^{\frac{3}{2}} + \frac{2}{5}(x-a)^{\frac{5}{2}}$
- $\int x\sqrt{x^2 \pm a^2}dx = \frac{1}{3}(x^2 \pm a^2)^{\frac{3}{2}}$
- $\int (ax+b)^{\frac{3}{2}}dx = \frac{2}{5a}(ax+b)^{\frac{5}{2}}$
- $\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$
- $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)$
- $\int \frac{1}{\sqrt{x \pm a}}dx = 2\sqrt{x \pm a}$
- $\int \frac{x}{\sqrt{x^2 \pm a^2}}dx = \sqrt{x^2 \pm a^2}$

Other

- $\int x\sin(ax)dx = -\frac{1}{a}x\cos(ax) + \frac{1}{a^2}\sin(ax)$
- $\int x\cos(ax)dx = \frac{1}{a}x\sin(ax) + \frac{1}{a^2}\cos(ax)$
- $\int e^{bx}\sin(ax)dx = \frac{1}{a^2+b^2}e^{bx}(b\sin(ax) - a\cos(ax))$
- $\int e^{bx}\cos(ax)dx = \frac{1}{a^2+b^2}e^{bx}(a\sin(ax) + b\cos(ax))$

7.3 U-Substitution

The substitution, $u = g(x)$, $du = g'(x)dx$ is:
$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

7.4 Integration By Parts

$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$

- $u = f(x)$, $\quad v = g(x)$
- $du = f'(x)dx$, $\quad dv = g'(x)dx$

$[\int udv = uv - \int vdu]$. As a rule of thumb use the following order, u should be the function that comes first beween: Logarthmic $\leftrightarrow$ Inverse trig. $\rightarrow$ Algebraic (Ax^n) $\rightarrow$ Trigonimetric $\rightarrow$ Exponential (k^x).

7.5 Trig-Function Trick

For $\int \sin^n(x)\cos^m(x)dx$ evaluate the following:

(i) **Deg(n) odd:** strip one sin out and convert the rest to cos with $\sin^2(x) = 1-\cos^2(x)$, then use substitution on $u = \cos(x)$.

(ii) **Deg(m) odd:** strip one cos out and convert the rest to sin with $\cos^2(x) = 1-\sin^2(x)$, then use substitution on $u = \sin(x)$.

(iii) **Deg(n) and Deg(m) both odd:** use either (i) or (ii).

(iv) **Deg(n) and Deg(m) both even:** use double angle and/or half angle trig identities to reduce the integral.

For $\int \tan^n(x)\sec^m(x)dx$ evaluate the following:

(i) **Deg(n) odd:** strip one tan and one sec out, and convert the rest to sec using $\tan^2(x) = \sec^2(x) - 1$, then use substitution on $u = \sec(x)$.

(ii) **Deg(m) even:** strip 2 sec out and convert the rest to tan with $\sec^2(x) = 1+\tan^2(x)$, then use substitution on $u = \tan(x)$.

(iii) **Deg(n) odd and Deg(m) even:** use either (i) or (ii).

(iv) **Deg(n) even and Deg(m) odd:** Deal with each integral differently.

7.6 Root-Trig Substitution Trick

If the integrals is one of the following roots use the given substitution and formula to convert it to an integral involving trig functions.

(i) $\sqrt{a^2 - b^2x^2} \implies x = \frac{a}{b}\sin(u)$, with property $\cos^2(x) = 1 - \sin^2(x)$.

(ii) $\sqrt{b^2x^2 - a^2} \implies x = \frac{a}{b}\sec(u)$, with property $\tan^2(x) = 1 - \sec^2(x)$.

(iii) $\sqrt{a^2 + b^2x^2} \implies x = \frac{a}{b}\tan(u)$, with property $\sec^2(x) = 1 + \tan^2(x)$.

7.7 Rational Functions

Given an integral $\int \frac{P(x)}{Q(x)}dx$:

- For $deg(P(x)) \geq deg(Q(x))$, then apply a polynomial division so that we get an equivalent integral $\int A(x) + \frac{R(x)}{Q(x)}dx$ where $\int \frac{R(x)}{Q(x)}dx$ is easier to solve.
- For $deg(P(x)) < deg(Q(x))$, then factor $Q(x)$ as completely as possible and find the partial fraction decomposition (P.F.D) of the rational expression.

1. $Q(x) = (ax+b)(cx^2+dx+e)$, then the P.F.D. is:
$\frac{A}{ax+b} + \frac{B}{cx^2+dx+e}$
2. $Q(x) = (ax+b)^n$, then the P.F.D. is:
$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$

7.8 Improper Integrals

Convergent, if $\lim = k$ with k finite.
Divergent, if $\lim = \pm\infty \vee D.N.E.$

Infinite Limit:

(i) $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$

(ii) $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$

(iii) $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$ provided that both integrals are convergent.

Discontinous Integrand:

(i) Discontinuity at a:
$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$

(ii) Discontinuity at b:
$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$

(iii) Discontinuity at a and b $(a < c < b)$:
$\int_a^c f(x)dx + \int_c^b f(x)dx$, if both convergent.

Convergence Tests:

- **Comparison Test:** If $f(x) \geq g(x) \geq 0$ on $[a, \infty[$, then:
If $\int_a^\infty f(x)dx$ converges $\Rightarrow \int_a^\infty g(x)dx$ converges.
If $\int_a^\infty g(x)dx$ diverges $\Rightarrow \int_a^\infty f(x)dx$ diverges.
Useful: If $a > 0 \Rightarrow \int_a^\infty \frac{1}{x^p}dx$ converges if $p > 1$ and diverges if $p \leq 1$.
- **Limit Comparison Test:** If f, g are continous on $[a, \infty[$ with $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq \infty$, then:
$\int_a^\infty |f(x)|dx$ converges $\Leftrightarrow \int_a^\infty |g(x)|dx$ converges
- **Absolute Converpgence:**
$\int_a^\infty |f(x)|dx$ converges $\Rightarrow \int_a^\infty f(x)$ converges

Definition 7.2 (Antiderivative)**.**
Let $f : [a, b] \to \mathbb{R}$ be a function where

$$f(x) = F'(x)\ \forall x \in [a, b]$$

*then F is called the **antiderivative** of f.*

Theorem 9: Mean Value Theorem

(Integration) Let f be continous on $[a, b]$, then there exists a c such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

$(f(c) = f_{avg})$

Theorem 10: Fundamental Theorem

Part 1: Suppose that f is continous on $[a, b]$ and F is defined as: $F(x) := \int_a^x f(t)dt$, then F is differentiable on $]a, b[$ and for all $x \in]a, b[$:

$$F'(x) = f(x)$$

Part 2: Suppose that f is continous on $[a, b]$ and F is the antiderivative of f, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

7.9 Derivative of Integrals

If we have to evaluate the derivative of an integral where $F(x) = \int_a^{g(x)} f(t)dt$, then by the first part of the Fundamental Theorem of Calculus (and the Chain Rule) we have: $F'(x) = f(g(x)) \cdot g'(x)$.
If $F(x) = \int_{h(x)}^{g(x)} f(t)dt = \int_a^{g(x)} f(t)dt - \int_a^{h(x)} f(t)dt$, then $F'(x) = f(g(x))g'(x) - f(h(x))h'(x)$.

8 Sequences

Definition 8.1 (Sequence)**.** *A sequence is set of numbers in a specific order, more formally: $(a_n)_{n=1}^{\infty}$ is a function $f : \mathbb{N} \to \mathbb{R}$ where $f(n) = a_n$.*

Definition 8.2 (Convergence)**.** *A sequence (a_n) is **convergent** to a value L if $\lim_{n\to\infty} a_n = L$, or:*

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} : (n > N \Rightarrow |a_n - L| < \epsilon)$$

*If the limit doesn't exist ($\pm\infty$ or doesn't converge) we say that (a_n) is **divergent**.*

Inutition: If for any small number ϵ there is, we can find a number $N(\epsilon)$ and L such that all points of a_n after N are at most at distance ϵ from L, the series converges.

8.1 Convergence Criteria

(i) **Linearity:** If (a_n) converges to a, (b_n) converges to b and $k \in \mathbb{N}$, then $(ka_n + b_n)$ converges to $ka + b$.

(ii) **Multiplication:** If (a_n) converges to a and (b_n) converges to b then $(a_n \cdot b_n)$ converges to $a \cdot b$.

(iii) **Division:** If (a_n) converges to a and (b_n) converges to $b \neq 0$ then $(\frac{a_n}{b_n})$ converges to $\frac{a}{b}$.

(iv) **Uniqueness:** If (a_n) is convergent to a, then: $\lim_{n\to\infty} a_n = a$ is unique.

(v) **Subsequence:** If (a_n) converges to a, then: any subsequence (a_{nk}) is also convergent to a.

(vi) **Squeeze Theorem:** If we have 3 convergent sequences $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, and $\lim_{n\to\infty} b_n = b$ where $a_n \leq b_n \leq c_n$, then $b = L$.

(vii) **Absolute:** If (a_n) is convergent to a, then: $|a_n|$ also converges and $\lim_{n\to\infty} |a_n| = |a|$.

(viii) **Ratio Test:** Let (a_n) be a sequence where $\forall n \in \mathbb{N} : a_n > 0$, then if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = a$ and $a < 1$, $\lim_{n\to\infty} a_n = 0$.

(ix) **Boundedness:** If (a_n) converges, then (a_n) is bounded.

(x) **Monotone Convergence:** (a_n) is monotone then it's convergent $\Leftrightarrow$ (a_n) is bounded.
If (a_n) is increasing and bounded $\Rightarrow$ $\lim_{n\to\infty} a_n = sup\{a_n : n \in \mathbb{N}\}$
If (a_n) is decreasing and bounded $\Rightarrow$ $\lim_{n\to\infty} a_n = inf\{a_n : n \in \mathbb{N}\}$

8.2 Divergence Criteria

(i) (a_n) is divergent if it has two subsequence that converge to different limits.

(ii) (a_n) is divergent if it has a divergent subsequence.

(iii) (a_n) is divergent if it's unbounded.

8.3 Monotonicity

Definition 8.3.

- *A sequence is **increasing** if:* $\forall n : a_n < a_{n+1}$
- *A sequence is **decreasing** if:* $\forall n : a_n > a_{n+1}$
- *A sequence is **monotonic** if it's either increasing or decreasing.*

Lemma: Every sequence has a monotonic subsequence.

8.4 Boundedness

Definition 8.4.

- *A sequence is **bounded above** if:* $\exists M > 0 \forall n \in \mathbb{N} : a_n \leq M$
- *A sequence is **bounded below** if:* $\exists m > 0 \forall n \in \mathbb{N} : m \leq a_n$
- *A sequence is **bounded** if it's either bounded above or below.*

8.5 Cauchy Sequence

Definition 8.5.
*A sequence (a_n) is **Cauchy** if:*

$$\forall \epsilon > 0 \exists N \in \mathbb{N} : \forall m, n \geq N \Rightarrow |a_n - a_m| < \epsilon$$

Inutition: If for any small number ϵ there is, we can find a number N such that all points of a_n after N are at most at distance ϵ from each other, the series is Cauchy.

(i) **Cauchy Convergence Criterion:** (a_n) is convergent $\Leftrightarrow$ it's Cauchy.

(ii) **Cauchy Bounded:** If (a_n) is cauchy, then it's also bounded.

(iii) **Linearity:** If (a_n) is Cauchy, (b_n) is Cauchy and $k \in \mathbb{N}$, then $(ka_n + b_n)$ is also Cauchy.

The advantage is that we don't have to find a limit L to prove that the sequence converges.

8.6 Accumulation Points

Definition 8.6.
*A number a is an **accumulation point** of (a_n) if there exists a subsequence (a_{n_k}) that converges to a, or:*

$$\forall \epsilon > 0 \exists K \in \mathbb{N} : (k \geq K \Rightarrow |a_{n_k} - a| < \epsilon)$$

(i) **Convergence:** If (a_n) converges to L, then L is the only accumulation point of (a_n).

(ii) **Boundedness**: If (a_n) is bounded, then it has at least one accumulation point.

(iii) **Divercence:** If a_n diverges, then it has no accumulation point.

8.7 Strategy

- **Convergence:** Treat (a_n) like a function and calculate the limit, if it exists it's convergent. If it's a recursive sequence use the Monotone Convergence Criteria by first proving that it's both monotonic increasing/decreasing and then that it's bounded above if increasing and bounded below if decreasing. To find the limit let $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$ and solve $L = a_\infty$ by plugging L inside of a_n.
- **Monotonicity:** To prove that the sequence is monotonic pick a candidate between increasing/decreasing and solve the inequality with a_n, a_{n+1} to prove your candidate. If the sequence is recursive prove your candidate by induction.
- **Boundedness:** Try to change n in a_n to make the sequence as small as possible to find a lower bound m, and similarly as big as possible to find an upper bound M. Give the result in terms of $m \leq a_n \leq M$. If it's a recursive sequence pick a candidate of upper/lower bound and prove it by induction.

9 Sequences of Functions

Definition 9.1. *A **sequence of a function** (f_n) is a list of functions $(f_1, f_2, ...)$ such that each f_n maps a given subset of $\mathbb{R}$ into $\mathbb{R}$:*

$$(f_n)_{n\in\mathbb{N}}, \ f_n : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$$

9.1 Convergence

Definition 9.2. *A sequence of a function (f_n) can converge to a function $f(x)$ in two different ways:*

- ***Pointwise** if $\forall x \in I$:*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

- ***Uniformly** if $\forall x \in I$:*

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0, \ \textit{or:}$$

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} : \ n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon$$

(i) **Convergence:** If (f_n) converges uniformly, then it also converges pointwise.

(ii) **Continuity:** If (f_n) converges uniformly, then f is continous.

(iii) **Differentiability:** If (f_n) converges pointwise to f, and f_n' converges uniformly to the function g on $]a, b[$, then f is differentiable on $]a, b[$ and $f' = g$, or: $\lim_{n\to\infty} f_n' = (\lim_{n\to\infty} f_n)' = f'$

(iv) **Integrability:** If a sequence of integrable function f_n converges uniformly to f on $[a, b]$, then f is integrable and: $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} (f_n(x))\, dx = \int_a^b f(x)dx$

10 Series

Definition 10.1.

- *A **partial sum** is the sum of the first n numbers of $(a_n)_{n=1}^{\infty}$, or: $s_n := \sum_{i=1}^{n} a_i$*
- *An **infinite series** is the sum of all terms of an infinite sequence $(a_n)_{n=1}^{\infty}$, or:*

$$\lim_{n\to\infty} s_n = \sum_{i=1}^{\infty} a_i := \lim_{N\to\infty} \sum_{i=1}^{N} a_i$$

10.1 Convergence

Definition 10.2 (Convergence)**.**
*An infinite series is called **convergent** if:*

$$\sum_{k=1}^{\infty} a_k \text{ converges} \Leftrightarrow \lim_{n\to\infty} \sum_{k=1}^{n} a_k \text{ exists}$$
$$\Leftrightarrow (s_n) \text{ converges}$$

(i) **Linearity:** If $\sum_{i=0}^{\infty} a_i = a$, $\sum_{i=0}^{\infty} b_i = b$, and $c \in \mathbb{R}$, then: $\sum_{i=0}^{\infty}(ca_i + b_i) = ca + b$

(ii) **Comparison:** If $\sum_{i=0}^{\infty} a_i = a$, $\sum_{i=0}^{\infty} b_i = b$, and $\forall n \in \mathbb{N}$ $a_n \leq b_n$, then $a \leq b$.

(iii) **Start Convergence:** $\sum_{i=0}^{\infty} a_i$ is convergent $\Leftrightarrow \sum_{i=N}^{\infty} a_i$ $\forall N \in \mathbb{N}$ is convergent.

(iv) **Bounded Convergence:** If (a_n) is ultimately positive and (s_n) is abounded above, then $\sum_{i=0}^{\infty} a_i$ converges. Otherwise the series diverges to infinity.

(v) **Unbounded Divergence:** If (a_n) is unbounded and $\lim_{n\to\infty} a_n = L \neq 0$, then if $L > 0$ the series diverges to $+\infty$ and if $L < 0$ the series diverges to $-\infty$.

10.2 Absolute Convergence

Definition 10.3 (Absolute Convergence)**.**
*An **absolute convergent** series $\sum_{i=0}^{\infty} a_n$ is a convergent series where also:*

$$\sum_{i=0}^{\infty} |a_n| \text{ converges}$$

*If $\sum a_n$ is convergent but $\sum |a_n|$ is divergent, it's called **conditionally convergent**.*

(i) **Theorem:** If $\sum_{i=0}^{\infty} |a_n|$ converges so does $\sum_{i=0}^{\infty} a_n$.

(ii) **Inequality:** $\left|\sum_{n=0}^{\infty} a_n\right| \leq \sum_{n=0}^{\infty} |a_n|$

(iii) **Unsorted Property:** If $\sum_{i=0}^{\infty} a_n$ converges absolutely, so does $\sum_{i=0}^{\infty} b_n$ where b_n is a bijection of the elements in a_n.

(iv) **Sum Property:** $\sum_{i=0}^{\infty}(a_n + b_n)$ converges absolutely if both $\sum_{i=0}^{\infty} a_n$ and $\sum_{i=0}^{\infty} b_n$ are absolute convergent.

10.3 Common Series

(i) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(ii) $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

(iii) $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

(iv) $\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

(v) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

(vi) $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

10.4 Common Sums

(i) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

(ii) $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{2n^3+3n^2+n}{6}$

(iii) $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$

Definition 10.4 (Power Series)**.**
*A **power series** f is a series of the form:*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

- ***Convergence:** the series converges absolutely for $0 \leq |x-c| < R$, and diverges otherwise. To calculate the radius of convergence we use the ratio (or root) test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$ then*

$$R = \frac{1}{L} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{\lim_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

.

(i) **Continuity:** A Power Series $f(x)$ is continous on $\{x : |x-c| < R\}$.

(ii) **Differentiability:** A Power Series $f(x)$ is differentiable in its radius of convergence R and:

$$f'(x) = \sum_{n=0}^{\infty} n \cdot a_n (x-c)^{n-1}$$

Definition 10.5 (Geometric Series)**.**
*A **geometric series** is a type of power series of the form:*

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1}$$

- ***Convergence:** converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges otherwise.*
- ***Partial Sum:** the n^{th} partial sum of a geometric series is $s_n = \frac{a(1-r^n)}{(1-r)}$.*

10.5 Convergence Tests

(i) **Divergence Test:** Let $\sum_{n=1}^{\infty} a_n$ be a series with $\lim_{n\to\infty} a_n \neq 0$ or undefined, then the series diverges.

(ii) **P-Test:** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(iii) **Comparison Test:** Let $(a_n), (b_n)$ be ultimately positive such that $\exists N \in \mathbb{N}$ $\forall n \geq N$: $0 \leq a_n \leq b_n$, then:
If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is also convergent.
If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ is also divergent.

(iv) **Limit Comparison Test:** Let $(a_n), (b_n)$ be positive sequences and assume $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, then: If $0 < L < \infty$: $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{n=1}^{\infty} b_n$ converges.
If $L = 0$: $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.
If $L = \infty$: $\sum_{n=1}^{\infty} b_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

(v) **Root Test:** Let $\sum_{n=1}^{\infty} a_n$ be a series with (a_n) ultimately and $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L \geq 0$, then:
If $0 \leq L < 1$ the series converges absolutely.
If $1 < L \leq \infty$ the series diverges.
If $L = 1$, this test is inconclusive.

(vi) **Ratio Test:** Let $\sum_{n=1}^{\infty} a_n$ be a series with (a_n) ultimately positive and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then:
If $L < 1$ the series converges absolutely.
If $1 < L \leq \infty$ the series diverges.
If $L = 1$ the test is inconclusive.

(vii) **Integral Test:** If $f(n) = a_n$ with $f(x)$ continous,eventually positive and decreasing, then: $\int_k^{\infty} f(x)dx$ converges $\Leftrightarrow \sum_{n=k}^{\infty} a_n$ converges.

(viii) **Alternating Series Test:** Let $\sum_{n=1}^{\infty} a_n$ be a series where $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$, then:
If $\lim_{n\to\infty} b_n = 0$ and b_n is decreasing $\Rightarrow$ the series converges.

10.6 Convergence Strategy

1. **Divergence Test:** If it's easy to see that the limit is not 0.
2. **P-Test/Geometric Series:** If it's of the form $\sum \frac{1}{n^p}$, $\sum ar^n$, or $\sum ar^{n+1}$.
3. **Comparison Test:** If it's similar to a p-series or geometric series.
4. **Limit Comparison Test:** If it's a rational expression with polynomials with positive terms.
5. **Root Test:** If can be written as $a_n = (b_n)^n$.
6. **Ratio Test:** If it contains factorials or c^n.
7. **Alternating Series Test:** If can be written as $a_n = (-1)^{n+c} b_n$, if $c \notin \{0, 1\}$ we have to manipulate it to make it 0 or 1 (e.g.: $(-1)^{n+2} = (-1)^n(-1)^2 = (-1)^n$).
8. **Integral Test:** If $f(n) = a_n$ is easy to integrate and f is positive and decreasing (ev. use derivative).

10.7 Value Calculation

To calculate the value of a series there are two ways:

1. Find the series representation as a Geometric Power Series, and calcualte its convergence value. Some tricks are: multiply the series by a number, strip out the first terms (how many are necessary), subtract the starting series to the obtained one to balance the multiplied term. By repeating this process we might be able to get to a geometric series.
2. If the series converges absolutley we can rearrange the sum such that they cancel each other, to do so we have to find the partial fraction decomposition of the series so that there are subtracting terms. Subsequently we will evaluate enough terms to find a repeating pattern (factoring a constant out might help) such that they cancel out indefinitely. Then we will rewrite the series as a partial sum $\sum_{i=0}^{N}$ with all the terms that do not cancel (at the beginning and end of the infinite series) and evaluate the limit to find its value.

11 Other

11.1 Length of a curve

Given a parametric curve where $x = f(t)$ and $y = g(t)$ defined on an interval $t \in [a, b]$ then the length of the curve is evaluated as follows:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

We assume that the curve is traced exactly once as t increases from a to b, and that the curve is traced out from left to right as t increases.

11.2 Bisection Method

The Bisection method is used to approximate solution to $f(x) = 0$ in an interval $[a, b]$ where $f(a) \cdot f(b) < 0$ (x_a is positive and x_b negative or vice-versa).

1. Calculate the midpoint $c \leftarrow \frac{b-a}{2}$ and evaluate $f(c)$.
2. If $|f(x)|$ is small enough, stop and return c.
3. If $f(a) \cdot f(c) > 0$ let $a \leftarrow c$ otherwise let $b \leftarrow c$ and restart from step 1.

This method works by keeping two points a and b with opposed sign and always shrinking the distance between them and the solution of $f(x) = 0$ which must exist by the Intermediate Value Theorem.

11.3 Newton method

The Newton method is used to approximate solutions to $f(x) = 0$, pay attention, not always this method converges, and it could also converge to a wrong value.

1. If an interval $I = [a, b]$ is given we start by making a random guess for the approximation by taking $\frac{b-a}{2}$ as our x_0.
2. We evaluate the next $(n+1)^{st}$ guess with the following formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ provided that $f'(x_n)$ exists.
3. To get n decimal places of precision we repeat point 2. until the last n digits are unchanged for two consecutive cycles.

The Newton method works by recursively finding the intersection between the original function f and the tangent line where the current guess lies (g). Subsequently it uses this line's intercept with the x axis, to find the next guess.

$$g(x) = \underbrace{f(x_n)}_{y_n} + \underbrace{f'(x_n)}_{slope}(x - \underbrace{x_n}_{x_n}) \Rightarrow g(x) = 0$$

11.4 Taylor Approximation

Definition 11.1 (Taylor Series).
*A **Taylor Series** is the rappresentation of a function as an infinte power series where f is differentiable any times at a point x_0 ($f \in C^\infty(x_0)$) of the form:*

$$T_\infty(f)(x; x_0) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(x_0)}{n!}}_{a_n} \cdot (x - x_0)^n$$

Where a_n is the Taylor coefficient.

We can use the first n terms of a Taylor Series to approximate the value of a function $f(x)$ around x_0 with $T_n(f)(x; x_0)$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = T_1(f)(x; x_0)$$

This is a rough approximation ($n = 1$) of $f(x)$ at the point x_0 with a polinomial of $deg = 1$, the value and derivative will be the same. If we derivate using the power rule, the first term will cancel leaving just the derivative, to get a better approximation we add more terms so that also higher derivatives will get the same values, the factorial/exponent are used to get the correct derivative when the power rule is applied multiple times, and $(x - x_0)$ will just shift the function if x_0 is not centered at 0.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x - x_0)^2}{2!} + \cdots + \frac{f^{(n)}(x - x_0)^n}{n!} = T_n(f)(x; x_0)$$

Remainder:

$$f(x) = T_n(f)(x; x_0) + R_n(f)(x; x_0)$$
$$R_n(f)(x; x_0) := |f(x) - T_n(f)(x; x_0)|$$

The remainder R_n quantifies how good is the estimate of the Taylor Series with respect to the actual value of the function $f(x)$.

Theorem 11: Taylor's Theorem

If $f : I \to \mathbb{R}$ is differentiable $n + 1$ times $f \in C^{(n+1)}(I)$ in an interval I containing the center $x_0 \in I$, then for each $x \in I$ there exists a $\xi \in]x, x_0[$ such that:

$$R_n(f)(x; x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

Lagrange Error Bound:
$|R_n(f)(x; x_0)| \leq \sup_{x_0 < \xi < a} \left|f^{(n+1)}(\xi)\right| \frac{(x - x_0)^{n+1}}{(n+1)!}$,

11.5 Integral Series Approximation

Given a convergent infinite series $\sum_{n=1}^{\infty} f(n)$ it's usually hard to find it's value. With this method we can approximate the sum if $f(n)$ is continous, positive and decreasing.

$$S = \sum_{n=1}^{\infty} f(n) = \underbrace{\sum_{n=1}^{k} f(n)}_{S_k \textit{ Partial Sum}} + \underbrace{\sum_{n=k+1}^{\infty} f(n)}_{R_k \textit{ Remainder}}$$

Since we can calculate an approximation $S_k \approx S$, R_k will tell us the difference from the actual value of S ($R_k = S - S_k$). Using integrals we can find upper and lower bound for R_k:

$$R_k \geq \int_{k+1}^{\infty} f(x)dx, \qquad R_k \leq \int_{k}^{\infty} f(x)dx$$

Then the value of the infinte series will be:

$$S_k + \int_{k+1}^{\infty} f(x)dx \leq S \leq S_k + \int_{k}^{\infty} f(x)dx$$

Thus to calculate the approximation:

1. Choose a value for k, the higher the better the approximation since we evaluate more terms where the intergal would find an unprecise bound.
2. Evaluate $S_k = \sum_{n=1}^{k} f(n)$.
3. Evaluate both improper intergrals $L = \int_{k+1}^{\infty} f(x)dx$ and $U = \int_{k}^{\infty} f(x)dx$.
4. Then S will be between $L + S_k$ and $U + S_k$, evaluate the mean to get an average term $S_{approx} = \frac{2S_k + L + U}{2}$.

11.6 Prove Bijectivity

To prove that a function $f : X \to Y$ is bijective we have to prove that it's both:

- **Surjective:** ($\forall x \in X \; \exists y \in Y : F(x) = y$) we have to prove that f is continous and either one of the following is true:
  1. $\lim_{x \to inf(X)} f(x) = inf(Y)$ and $\lim_{x \to sup(X)} f(x) = sup(Y)$
  2. $\lim_{x \to inf(X)} f(x) = sup(Y)$ and $\lim_{x \to sup(X)} f(x) = inf(Y)$

  Then by the Intermediate Value Theorem $f(x)$ covers the entire domain and thus it's surjective.
- **Injective:** ($F(x) = F(y) \Rightarrow x = y$) we have to show that $f(x)$ is strictly increasing if when we proved surjectivity we used 1) or strictly decreasing if we used 2) which can be proved with the first derivative (> 0 *or* < 0).

11.7 Approximating Definite Integrals

$$\int_a^b f(x)dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a + \frac{i(b-a)}{n}\right)$$

11.8 Theorems

Theorem 12: Polynomial Roots

A polynomial P_n of degree d has:

- From 0 to n distinct real roots if d is even.
- From 1 to n distinct real roots if d is odd.
- Always n complex roots (Fundamental Theorem of Algebra).

Theorem 13: Archimedean Property

$\forall x \in \mathbb{R} \; \exists n_x \in \mathbb{N} : x < n_x$

Theorem 14: Density Theorem

$\forall x, y \in \mathbb{R} \; \exists z \in \mathbb{Q} : x < y \Rightarrow x < z < y$

Theorem 15: Function Implication

Given a function f, the following implications hold:

diff. $\Rightarrow$ *continous* $\Rightarrow$ *r. integrable* $\Rightarrow$ *bounded*

None of the properties on the right implies one of the properties on the left.

11.9 Extra

Arithmetic Geometric Series

$$\sum_{n=1}^{\infty} nq^{n-1} = 1 + 2q + 3q^2 + \cdots + nq^{n-1} = \frac{1 - (n+1)q^n + nq^{n+1}}{(1-q)^2}$$

Continous Piecewise Function

$$f(x) = \begin{cases} x^2 - ax + b & x \leq -1 \\ (a+b)x & -1 < x < 1 \\ x^2 + ax - b & x \geq 1 \end{cases}$$

Then to have continuity both must be true:

$$\begin{aligned} f(-1) = 1 + a + b &\stackrel{!}{=} \lim_{x \to -1^+} f(x) \\ &= \lim_{x \to -1^+} (a+b)x = -(a+b) \\ f(1) = 1 + a - b &\stackrel{!}{=} \lim_{x \to 1^-} f(x) \\ &= \lim_{x \to 1^-} (a+b)x = a + b \end{aligned}$$

Function Length If f is differentiable on $[a, b]$, then the graph of the function has a length:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$