Derivation of the LCS, MLCS and MLCS2k2 templates

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1 LCS (Light Curve Shapes)

We can start from the derivation of LCS introduced by RPK 1995, [1]. Here they stick to V-band data only. They describe the vector of measured apparent magnitudes, $\mathbf{y}$, as

$$\mathbf{y} = \mathbf{s} + \mathbf{L}\mathbf{q} + \mathbf{n}, \tag{1}$$

the vector $\mathbf{s}$ is the absolute magnitude, here playing the role of the "signal". $\mathbf{n}$ carries unmodeled and observational noise. $\mathbf{L}$ represent the corrections applied to the signal to accommodate the variations seen in Ia SNe light curves. And lastly, $\mathbf{q}$ represents the coefficients, controlling the amount of correction added to each SNe. In this description:

$$\mathbf{L} = \begin{pmatrix} 1 & R(t_1) \\ 1 & R(t_2) \\ \vdots & \vdots \\ 1 & R(t_M) \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix}. \tag{2}$$

The χ^2 of the fit between data and model is

$$\chi^2 = (\mathbf{y} - \mathbf{s} - \mathbf{L}\mathbf{q})^T \mathbf{C}^{-1} (\mathbf{y} - \mathbf{s} - \mathbf{L}\mathbf{q}). \tag{3}$$

The template can then be found if we consider $\mathbf{q}$ known and minimize the above expression w.r.t the template itself, $R(t)$. We have a contribution for each Ia SNe from the training set, these contributions are not correlated. Also, at each time, t, we have a contribution that is correlated to the remaining values of time, for each SN Ia. We have $\chi^2 = \sum_i \sum_t \chi_i^2(t)$, leading to

$$\chi^2 = \sum_i \sum_t \sum_{t'} (y_i(t) - s(t) - \mu_{0,i} - \mu_{1,i} R(t)) C^{-1}(t, t') (y_i(t') - s(t') - \mu_{0,i} - \mu_{1,i} R(t')). \tag{4}$$

The covariance matrix does not carry a Ia SN index since its assumed to be the same for every object at this moment, though when performing light-curve fitting, each SN will contribute with their own measurements errors. For a specific value of t^*, $R(t^*)$ is a constant parameter and we can minimize the total χ^2 w.r.t to it.

$$
\begin{aligned}
\frac{\partial \chi^2}{\partial R(t^*)} &= \sum_i \sum_t \sum_{t'} (-\mu_{1,i}) \delta(t-t^*) C^{-1}(t,t')(y_i(t') - s(t') - \mu_{0,i} - \mu_{1,i} R(t'))+ && (5)\\
&+ \sum_i \sum_t \sum_{t'} (y_i(t) - s(t) - \mu_{0,i} - \mu_{1,i} R(t)) C^{-1}(t,t')(-\mu_{1,i}) \delta(t'-t^*) && (6)\\
&= \sum_i \sum_{t'} (-\mu_{1,i}) C^{-1}(t^*,t')(y_i(t') - s(t') - \mu_{0,i} - \mu_{1,i} R(t'))+ && (7)\\
&+ \sum_i \sum_t (y_i(t) - s(t) - \mu_{0,i} - \mu_{1,i} R(t)) C^{-1}(t,t^*)(-\mu_{1,i}) && (8)\\
&= \sum_i \sum_t (-\mu_{1,i})(y_i(t) - s(t) - \mu_{0,i} - \mu_{1,i} R(t))(C^{-1}(t^*,t) + C^{-1}(t,t^*)). && (9)
\end{aligned}
$$

This equation is valid for every t^* and its equal to 0 to satisfy the minimum criteria, rearranging

$$
\sum_t (C^{-1}(t^*,t) + C^{-1}(t,t^*)) \sum_i (-\mu_{1,i})(y_i(t) - s(t) - \mu_{0,i} - \mu_{1,i} R(t)) = 0. \tag{10}
$$

For this equation to be satisfied we should have then

$$
\sum_i (-\mu_{1,i})(y_i(t) - s(t) - \mu_{0,i} - \mu_{1,i} R(t)) = 0. \tag{11}
$$

Leading us to

$$
\begin{aligned}
\sum_i \mu_{1,i}(y_i(t) - s(t) - \mu_{0,i}) - \sum_i \mu_{1,i}^2 R(t) = 0, && (12)\\
\sum_i \mu_{1,i}(y_i(t) - s(t) - \mu_{0,i}) = R(t) \sum_i \mu_{1,i}^2, && (13)
\end{aligned}
$$

and finding

$$
\boxed{R(t) = \frac{\langle \mu_1 (y(t) - s(t) - \mu_0) \rangle}{\langle \mu_1^2 \rangle}} \tag{14}
$$

This equation is equivalent to eq. 4 from [1]. This method does not account for extinction due to dust in host galaxies.

2 MLCS (Multicolor Light-Curve Shapes)

In this next paper, [2], they introduce MLCS. Here, they decided to add corrections not only to the V-band magnitudes but also to some color curves, then including an extra parameter describing the extinction due to dust in the host galaxy in V band, A_V. They propose

$$
\begin{aligned}
\mathbf{m}_V &= \mathbf{M}_V + \mu_V + \mathbf{R}_V \Delta + \mathbf{n}_V, && (15)\\
\mathbf{m}_B - \mathbf{m}_V &= (\mathbf{B} - \mathbf{V})_0 + E_{B-V} + \mathbf{R}_{B-V} \Delta + \mathbf{n}_{B-V}, && (16)\\
\mathbf{m}_V - \mathbf{m}_R &= (\mathbf{V} - \mathbf{R})_0 + E_{V-R} + \mathbf{R}_{V-R} \Delta + \mathbf{n}_{V-R}, && (17)\\
\mathbf{m}_V - \mathbf{m}_I &= (\mathbf{V} - \mathbf{I})_0 + E_{V-I} + \mathbf{R}_{V-I} \Delta + \mathbf{n}_{V-I}, && (18)
\end{aligned}
$$

where $\mathbf{X} \equiv \mathbf{M}_X$ for each filter X={B,V,R,I}. The correction templates are $\mathbf{R}_V$, $\mathbf{R}_{B-V}$, $\mathbf{R}_{V-R}$, and $\mathbf{R}_{V-I}$. The fit parameters are μ_V, A_V and Δ, where the color excesses are defined in terms of ratios of A_V as $\frac{A_V}{E_{B-V}}$, $\frac{A_V}{E_{V-R}}$ and $\frac{A_V}{E_{V-I}}$, which are equal to the reddening parametrizations usually employed. The templates are found following the same technique of analytical minimization as previously. However, now $\mathbf{L}$ and $\mathbf{q}$ are

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & R_V(t_1) \\ \vdots & \vdots & \vdots \\ 1 & 0 & R_V(t_N) \\ 0 & \frac{1}{3.1} & R_{B-V}(t_1) \\ \vdots & \vdots & \vdots \\ 0 & \frac{1}{3.1} & R_{B-V}(t_N) \\ 0 & \frac{1}{3.9} & R_{V-R}(t_1) \\ \vdots & \vdots & \vdots \\ 0 & \frac{1}{3.9} & R_{V-R}(t_N) \\ 0 & \frac{1}{1.9} & R_{V-I}(t_1) \\ \vdots & \vdots & \vdots \\ 0 & \frac{1}{1.9} & R_{V-I}(t_N) \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \mu_V \\ A_V \\ \Delta \end{pmatrix}. \tag{19}$$

With

$$\chi^2 = (\mathbf{m}_{obs} - \mathbf{M} - \mathbf{Lq})^T \mathbf{C}^{-1} (\mathbf{m}_{obs} - \mathbf{M} - \mathbf{Lq}), \tag{20}$$

we can write

$$\chi^2 = \sum_i \sum_t \sum_{t'} (m_{obsV,i}(t) - M_V(t) - \mu_{V,i} - \Delta_i R_V(t)) C_V^{-1}(t,t') \times \tag{21}$$

$$\times (m_{obsV,i}(t') - M_V(t') - \mu_{V,i} - \Delta_i R_V(t')) + \tag{22}$$

$$+ \sum_i \sum_t \sum_{t'} (m_{obsB,i}(t) - m_{obsV,i}(t) - (B-V)_0(t) - \frac{A_V}{3.1} - \Delta_i R_{B-V}(t)) C_{B-V}^{-1}(t,t') \times \tag{23}$$

$$\times (m_{obsB,i}(t') - m_{obsV,i}(t') - (B-V)_0(t') - \frac{A_V}{3.1} - \Delta_i R_{B-V}(t)') + \tag{24}$$

$$+ \sum_i \sum_t \sum_{t'} (m_{obsV,i}(t) - m_{obsR,i}(t) - (V-R)_0(t) - \frac{A_V}{3.9} - \Delta_i R_{V-R}(t)) C_{V-R}^{-1}(t,t') \times \tag{25}$$

$$\times (m_{obsV,i}(t') - m_{obsR,i}(t') - (V-R)_0(t') - \frac{A_V}{3.9} - \Delta_i R_{V-R}(t)') + \tag{26}$$

$$+ \sum_i \sum_t \sum_{t'} (m_{obsR,i}(t) - m_{obsI,i}(t) - (R-I)_0(t) - \frac{A_V}{1.9} - \Delta_i R_{R-I}(t)) C_{V-I}^{-1}(t,t') \times \tag{27}$$

$$\times (m_{obsR,i}(t') - m_{obsI,i}(t') - (R-I)_0(t') - \frac{A_V}{1.9} - \Delta_i R_{V-I}(t)'). \tag{28}$$

The covariance matrix again does not carry a Ia SN index since its assumed to be the same for every object at this moment. For this derivation I believe they did not consider any correlation terms between the different curves, i.e. $\mathbf{C}$ is a block diagonal matrix, otherwise the templates would not be independent of $\mathbf{C}$ as stated in the paper. For this scenario we have

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_V & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{B-V} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{V-R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}_{V-I} \end{pmatrix}, \tag{29}$$

where each block can be represented for $Y = \{V, B-V, V-R, V-I\}$ as

$$\mathbf{C}_Y = \begin{pmatrix} C_Y(t_1,t_1) & \dots & C_Y(t_1,t_N) \\ \vdots & \ddots & \vdots \\ C_Y(t_N,t_1) & \dots & C_Y(t_N,t_N) \end{pmatrix}. \tag{30}$$

This off diagonal terms were estimated afterwards, only being included in the fits. When taking the derivative w.r.t each of the templates at a fixed time t^*, all remaining terms are constant and we reobtain the same structure as seen for LCS. Thus, repeating the same process as previously, we obtain

$$\boxed{R_V(t) = \frac{\langle \Delta(m_{obsV}(t) - M_V(t) - \mu_V)\rangle}{\langle \Delta^2 \rangle}}, \tag{31}$$

$$\boxed{R_{B-V}(t) = \frac{\langle \Delta(m_{obsB}(t) - m_{obsV}(t) - (B-V)_0(t) - \frac{A_V}{3.1})\rangle}{\langle \Delta^2 \rangle}}, \tag{32}$$

$$\boxed{R_{V-R}(t) = \frac{\langle \Delta(m_{obsV}(t) - m_{obsR}(t) - (V-R)_0(t) - \frac{A_V}{3.9})\rangle}{\langle \Delta^2 \rangle}}, \tag{33}$$

$$\boxed{R_{V-I}(t) = \frac{\langle \Delta(m_{obsV}(t) - m_{obsI}(t) - (V-I)_0(t) - \frac{A_V}{1.9})\rangle}{\langle \Delta^2 \rangle}}. \tag{34}$$

If we considered in this step a non-null covariance matrix between the different curves or if we considered different correlation matrix for each SN, the templates would have other contributions and would not be independent of $\mathbf{C}$, as they state. Our eq. 31-34 are consistent with eq. 11 and eq. 12 from [2].

They argue that their model is able to distinguish between the effects of distance, dust reddening and intrinsic variation on Ia SN luminosity. This separation is possible due to the inclusion of color curves with the purpose of describing the relation between luminosity and intrinsic color. For each observed color curve they subtract the color excess (which is only an offset) and the remaining behaviour is considered to be due to intrinsic color variations and is corrected via the other templates.

They reaffirm that the assumption Ia SNe have uniform color at maximum is wrong and leads to incorrect predictions. This assumption in the past was responsible for increasing the dispersion in Hubble diagrams and also for nonphysical dust effects, e.g. dust making spectra bluer. They also state that the effect of color variations and dust reddening should not be corrected together since they do not cause exactly the same changes. (I guess there is no current consensus, since SNEMO, [3], follows this procedure and this same idea is somehow build into SALT2, [4, 5] when not questioning the origin of their c parameter.)

They find the best fit parameters also through a analytical minimization of the previous described χ^2,

$$\mathbf{q}_{best} = [\mu_V, A_V, \Delta]^T_{best} = (\mathbf{L}^T\mathbf{C}^{-1}\mathbf{L})^{-1}\mathbf{L}^T\mathbf{C}^{-1}[\mathbf{y}-\mathbf{s}], \tag{35}$$

with errors given by the diagonal elements of the covariance matrix $(\mathbf{L}^T\mathbf{C}^{-1}\mathbf{L})^{-1}$. It is important to note the distance modulus depends on the measurements of every epoch, not only data at maximum light in B-band.

And lastly, they estimate afterwards the covariance matrix in a more minimalist fashion. They are not able to estimate the off-diagonal elements of $\mathbf{C}_B$, $\mathbf{C}_V$, $\mathbf{C}_R$ e $\mathbf{C}_I$. Instead, they augment each block matrix multiplying by a specific value, which reflects the fact some bands are better at predicting the parameters than others. Later, a rotation is applied and the covariance matrix is estimated for $\mathbf{C}_{B-V}$, $\mathbf{C}_{V-R}$, $\mathbf{C}_{V-I}$ e $\mathbf{C}_V$, now exhibiting some off-diagonal elements.

3 MLCS2k2 (Multicolor Light-Curve Shapes 2002)

The third version of MLCS is presented in [6], this version was originally introduced in Saurabh Jha's 2002 PhD Thesis, hence the name.

They describe the model as

$$\mathbf{m}_X(t-t_0) = \mathbf{M}^0_X + \mu_0 + \boldsymbol{\zeta}_X\left(\alpha_X + \frac{\beta_X}{R_V}\right)A^0_V + \mathbf{P}_X\Delta + \mathbf{Q}_X\Delta^2, \tag{36}$$

as previously, they solve for the optimal vectors given the training sample. Afterwards they estimate the covariance matrix, which enters only when performing the light curve fitting.

In analogy with the previous descriptions we can rewrite $\mathbf{L}$ and $\mathbf{q}$ as

$$\mathbf{L} = \begin{pmatrix}
1 & \zeta_U(t_1)\left(\alpha_U + \frac{\beta_U}{R_V}\right) & P_U(t_1) & Q_U(t_1) \\
\vdots & \vdots & \vdots & \vdots \\
1 & \zeta_U(t_N)\left(\alpha_U + \frac{\beta_U}{R_V}\right) & P_U(t_N) & Q_U(t_N) \\
1 & \zeta_V(t_1)\left(\alpha_V + \frac{\beta_V}{R_V}\right) & P_V(t_1) & Q_V(t_1) \\
\vdots & \vdots & \vdots & \vdots \\
1 & \zeta_V(t_N)\left(\alpha_V + \frac{\beta_V}{R_V}\right) & P_V(t_N) & Q_V(t_N) \\
1 & \zeta_R(t_1)\left(\alpha_R + \frac{\beta_R}{R_V}\right) & P_R(t_1) & Q_R(t_1) \\
\vdots & \vdots & \vdots & \vdots \\
1 & \zeta_R(t_N)\left(\alpha_R + \frac{\beta_R}{R_V}\right) & P_R(t_N) & Q_R(t_N) \\
1 & \zeta_I(t_1)\left(\alpha_I + \frac{\beta_I}{R_V}\right) & P_I(t_1) & Q_I(t_1) \\
\vdots & \vdots & \vdots & \vdots \\
1 & \zeta_I(t_N)\left(\alpha_I + \frac{\beta_I}{R_V}\right) & P_I(t_N) & Q_I(t_N)
\end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \mu_0 \\ A^0_V \\ \Delta \\ \Delta^2 \end{pmatrix}. \tag{37}$$

With

$$\chi^2 = (\mathbf{m}_{obs} - \mathbf{M} - \mathbf{Lq})^T \mathbf{C}^{-1} (\mathbf{m}_{obs} - \mathbf{M} - \mathbf{Lq}), \tag{38}$$

we can write

$$\chi^2 = \sum_i \sum_t \sum_{t'} (m_{obsU,i}(t) - M_U^0(t) - \mu_{0,i} - \zeta_U(t) \left(\alpha_U + \frac{\beta_U}{R_V} \right) A_{V,i}^0 - \Delta_i P_U(t) - \Delta_i^2 Q_U(t)) \times \tag{39}$$

$$\times C_U^{-1}(t,t')(m_{obsU,i}(t') - M_U^0(t') - \mu_{0,i} - \zeta_U(t') \left(\alpha_U + \frac{\beta_U}{R_V} \right) A_{V,i}^0 - \Delta_i P_U(t') - \Delta_i^2 Q_U(t')) + \tag{40}$$

$$+ \sum_i \sum_t \sum_{t'} (m_{obsB,i}(t) - M_B^0(t) - \mu_{0,i} - \zeta_B(t) \left(\alpha_B + \frac{\beta_B}{R_V} \right) A_{V,i}^0 - \Delta_i P_B(t) - \Delta_i^2 Q_B(t)) \times \tag{41}$$

$$\times C_B^{-1}(t,t')(m_{obsB,i}(t') - M_B^0(t') - \mu_{0,i} - \zeta_B(t') \left(\alpha_B + \frac{\beta_B}{R_V} \right) A_{V,i}^0 - \Delta_i P_B(t') - \Delta_i^2 Q_B(t')) + \tag{42}$$

$$+ \sum_i \sum_t \sum_{t'} (m_{obsV,i}(t) - M_V^0(t) - \mu_{0,i} - \zeta_V(t) \left(\alpha_V + \frac{\beta_V}{R_V} \right) A_{V,i}^0 - \Delta_i P_V(t) - \Delta_i^2 Q_V(t)) \times \tag{43}$$

$$\times C_V^{-1}(t,t')(m_{obsV,i}(t') - M_V^0(t') - \mu_{0,i} - \zeta_V(t') \left(\alpha_V + \frac{\beta_V}{R_V} \right) A_{V,i}^0 - \Delta_i P_V(t') - \Delta_i^2 Q_V(t')) + \tag{44}$$

$$+ \sum_i \sum_t \sum_{t'} (m_{obsR,i}(t) - M_R^0(t) - \mu_{0,i} - \zeta_R(t) \left(\alpha_R + \frac{\beta_R}{R_V} \right) A_{V,i}^0 - \Delta_i P_R(t) - \Delta_i^2 Q_R(t)) \times \tag{45}$$

$$\times C_R^{-1}(t,t')(m_{obsR,i}(t') - M_R^0(t') - \mu_{0,i} - \zeta_R(t') \left(\alpha_R + \frac{\beta_R}{R_V} \right) A_{V,i}^0 - \Delta_i P_R(t') - \Delta_i^2 Q_R(t')) + \tag{46}$$

$$+ \sum_i \sum_t \sum_{t'} (m_{obsI,i}(t) - M_I^0(t) - \mu_{0,i} - \zeta_I(t) \left(\alpha_I + \frac{\beta_I}{R_V} \right) A_{V,i}^0 - \Delta_i P_I(t) - \Delta_i^2 Q_I(t)) \times \tag{47}$$

$$\times C_I^{-1}(t,t')(m_{obsI,i}(t') - M_I^0(t') - \mu_{0,i} - \zeta_I(t') \left(\alpha_I + \frac{\beta_I}{R_V} \right) A_{V,i}^0 - \Delta_i P_I(t') - \Delta_i^2 Q_I(t')). \tag{48}$$

As before, in this step the covariance matrix is block diagonal, not including the contributions of correlations between different filters. We see here they are not using the color curves anymore, they only use light curves, spanning 5 filters. On top of that we have a contribution in second order of the Δ parameter, meaning the parameter can not be anymore recognized as the luminosity correction.

Following the same procedure of taking the derivative w.r.t each template at a fixed time t^*, we obtain a slightly different structure for the $P_X(t)$ templates

$$\frac{\partial \chi^2}{\partial P_X(t^*)} = \sum_i \sum_{t'} (-\Delta_i) C_X^{-1}(t^*, t') \Big[m_{obsX,i}(t') - M_X^0(t') - \mu_{0,i} + \tag{49}$$

$$- \zeta_X(t') \left(\alpha_X + \frac{\beta_X}{R_V} \right) A_{V,i}^0 - \Delta_i P_X(t') - \Delta_i^2 Q_X(t') \Big] + \tag{50}$$

$$+ \sum_i \sum_t (-\Delta_i) C_X^{-1}(t, t^*) \Big[m_{obsX,i}(t) - M_X^0(t) - \mu_{0,i} + \tag{51}$$

$$- \zeta_X(t) \left(\alpha_X + \frac{\beta_X}{R_V} \right) A_{V,i}^0 - \Delta_i P_X(t) - \Delta_i^2 Q_X(t) \Big] \tag{52}$$

$$= \sum_t (C_X^{-1}(t^*, t) + C_X^{-1}(t, t^*)) \sum_i (-\Delta_i) \Big[m_{obsX,i}(t) - M_X^0(t) + \tag{53}$$

$$- \mu_{0,i} - \zeta_X(t) \left(\alpha_X + \frac{\beta_X}{R_V} \right) A_{V,i}^0 - \Delta_i P_X(t) - \Delta_i^2 Q_X(t) \Big]. \tag{54}$$

We then obtain

$$\sum_i \Delta_i \left[m_{obsX,i}(t) - M_X^0(t) - \mu_{0,i} - \zeta_X(t) \left(\alpha_X + \frac{\beta_X}{R_V} \right) A_{V,i}^0 - \Delta_i P_X(t) - \Delta_i^2 Q_X(t) \right] = 0. \tag{55}$$

From this equation above we can isolate whether $P_X(t)$ or $Q_X(t)$, finding

$$P_X(t) = \frac{\left\langle \Delta \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t) \left(\alpha_X + \frac{\beta_X}{R_V} \right) A_V^0 - \Delta^2 Q_X(t) \right] \right\rangle}{\langle \Delta^2 \rangle}, \tag{56}$$

and

$$Q_X(t) = \frac{\left\langle \Delta \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t) \left(\alpha_X + \frac{\beta_X}{R_V} \right) A_V^0 - \Delta P_X(t) \right] \right\rangle}{\langle \Delta^3 \rangle}. \tag{57}$$

If instead the derivative is taken w.r.t $Q_X(t^*)$ we obtain

$$\sum_i \Delta_i^2 \left[m_{obsX,i}(t) - M_X^0(t) - \mu_{0,i} - \zeta_X(t) \left(\alpha_X + \frac{\beta_X}{R_V} \right) A_{V,i}^0 - \Delta_i P_X(t) - \Delta_i^2 Q_X(t) \right] = 0. \tag{58}$$

Leading us to two other equations for $P_X(t)$ and $Q_X(t)$,

$$P_X(t) = \frac{\left\langle \Delta^2 \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t) \left(\alpha_X + \frac{\beta_X}{R_V} \right) A_V^0 - \Delta^2 Q_X(t) \right] \right\rangle}{\langle \Delta^3 \rangle}, \tag{59}$$

and

$$Q_X(t) = \frac{\left\langle \Delta^2 \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t) \left(\alpha_X + \frac{\beta_X}{R_V} \right) A_V^0 - \Delta P_X(t) \right] \right\rangle}{\langle \Delta^4 \rangle}. \tag{60}$$

Equating the equation for $P_X(t)$ we have for $Q_X(t)$

$$\frac{\left\langle \Delta \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t)\left(\alpha_X + \frac{\beta_X}{R_V}\right)A_V^0\right]\right\rangle}{\langle\Delta^2\rangle} - \frac{\langle\Delta^3\rangle}{\langle\Delta^2\rangle}Q_X(t) = \tag{61}$$

$$= \frac{\left\langle \Delta^2 \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t)\left(\alpha_X + \frac{\beta_X}{R_V}\right)A_V^0\right]\right\rangle}{\langle\Delta^3\rangle} - \frac{\langle\Delta^4\rangle}{\langle\Delta^3\rangle}Q_X(t), \tag{62}$$

this leads to

$$\boxed{\begin{aligned} Q_X(t) = {} & \frac{\left\langle \Delta^2 \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t)\left(\alpha_X + \frac{\beta_X}{R_V}\right)A_V^0\right]\right\rangle}{\langle\Delta^4\rangle\langle\Delta^2\rangle - \langle\Delta^3\rangle^2}\langle\Delta^2\rangle + \quad (63)\\ & - \frac{\left\langle \Delta \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t)\left(\alpha_X + \frac{\beta_X}{R_V}\right)A_V^0\right]\right\rangle}{\langle\Delta^4\rangle\langle\Delta^2\rangle - \langle\Delta^3\rangle^2}\langle\Delta^3\rangle. \quad (64)\end{aligned}}$$

Equating the equation for $Q_X(t)$ we have for $P_X(t)$

$$\frac{\left\langle \Delta \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t)\left(\alpha_X + \frac{\beta_X}{R_V}\right)A_V^0\right]\right\rangle}{\langle\Delta^3\rangle} - \frac{\langle\Delta^2\rangle}{\langle\Delta^3\rangle}P_X(t) = \tag{65}$$

$$\frac{\left\langle \Delta^2 \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t)\left(\alpha_X + \frac{\beta_X}{R_V}\right)A_V^0\right]\right\rangle}{\langle\Delta^4\rangle} - \frac{\langle\Delta^3\rangle}{\langle\Delta^4\rangle}P_X(t), \tag{66}$$

this leads to

$$\boxed{\begin{aligned} P_X(t) = {} & \frac{\left\langle \Delta^2 \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t)\left(\alpha_X + \frac{\beta_X}{R_V}\right)A_V^0\right]\right\rangle}{\langle\Delta^3\rangle^2 - \langle\Delta^4\rangle\langle\Delta^2\rangle}\langle\Delta^3\rangle + \quad (67)\\ & - \frac{\left\langle \Delta \left[m_{obsX}(t) - M_X^0(t) - \mu_0 - \zeta_X(t)\left(\alpha_X + \frac{\beta_X}{R_V}\right)A_V^0\right]\right\rangle}{\langle\Delta^3\rangle^2 - \langle\Delta^4\rangle\langle\Delta^2\rangle}\langle\Delta^4\rangle. \quad (68)\end{aligned}}$$

So following the same procedure as the previous models we obtain sightly bigger but similar equations for the correction templates. This average is also taken over the training sample, as for the previous examples. We can check this expression comparing the curves we obtain with the ones provided by [6].

Below on Fig. 1 and Fig. 2 we see the original model vectors from [6] in solid orange curves. In solid blue we see the curves obtained for the same training sample as the original version, using the expressions from eq. 64 and eq. 68 with parameters estimates given by Table 4 of [6]. To construct Fig. 1 model vectors we perform a Gaussian Process Regression (GPR) onto the type Ia supernovae data after corrections for Milk Way dust extinction and redshift (i.e. correcting for the rest-frame). An example of the Gaussian Process Regression (GPR) which is performed with the Python package *GPy* is shown

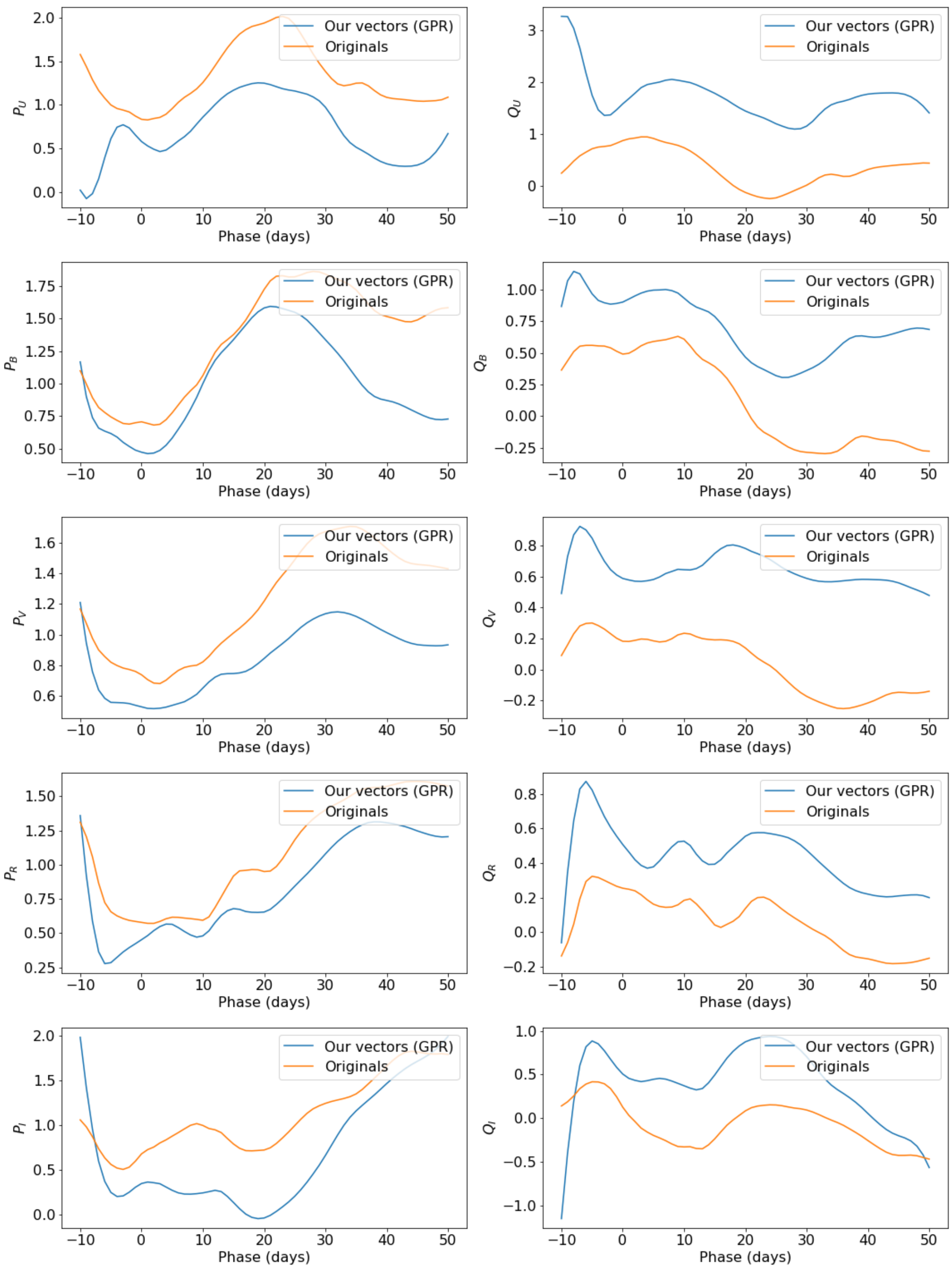

Figure 1: Comparison of original model vectors and the model vectors obtained using eq. 64 and eq. 68 after applying GPR on the training sample data.

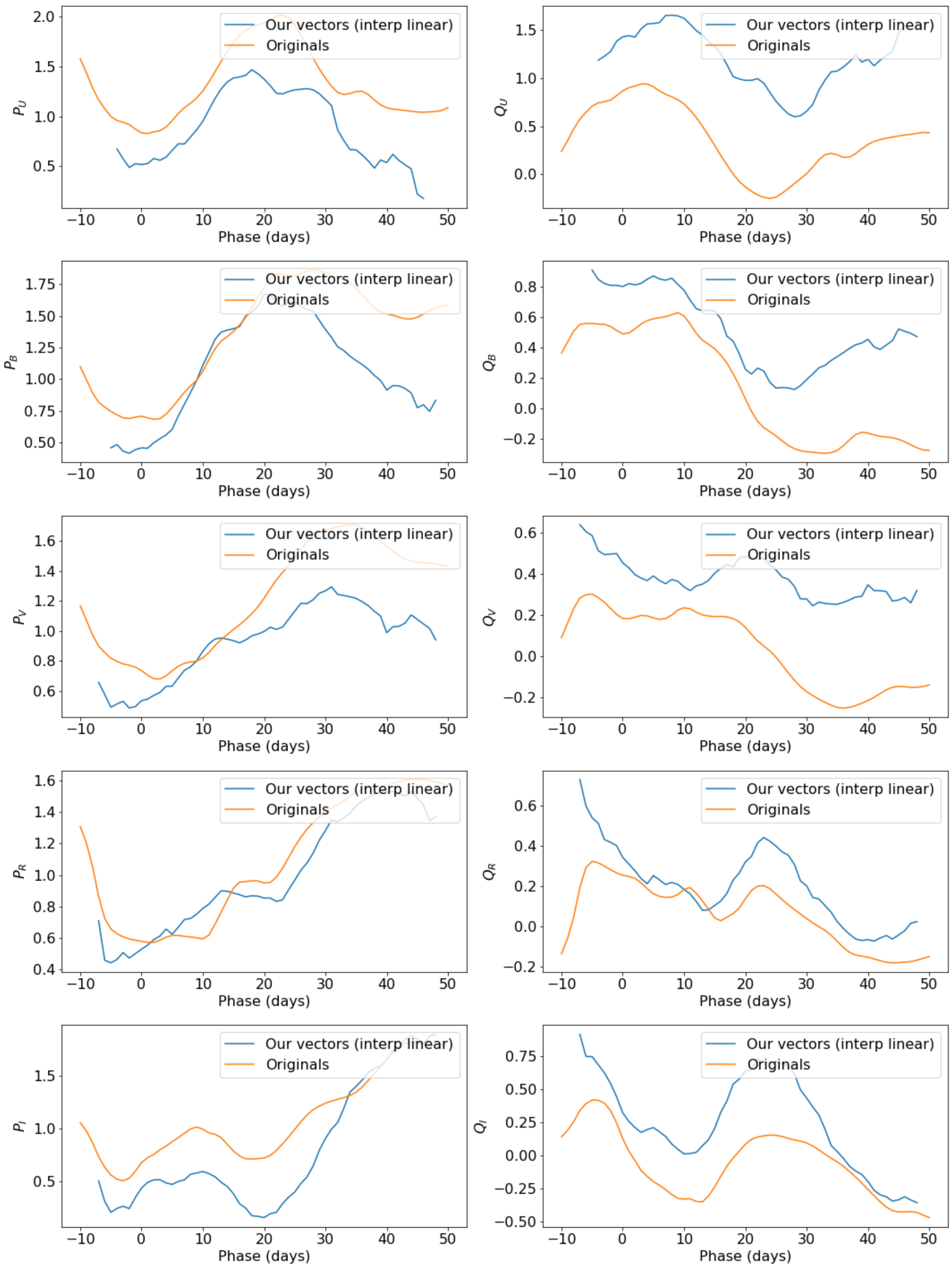

Figure 2: Comparison of original model vectors and the model vectors obtained using equations eq. 64 and eq. 68 after applying linear interpolation on the training sample data.

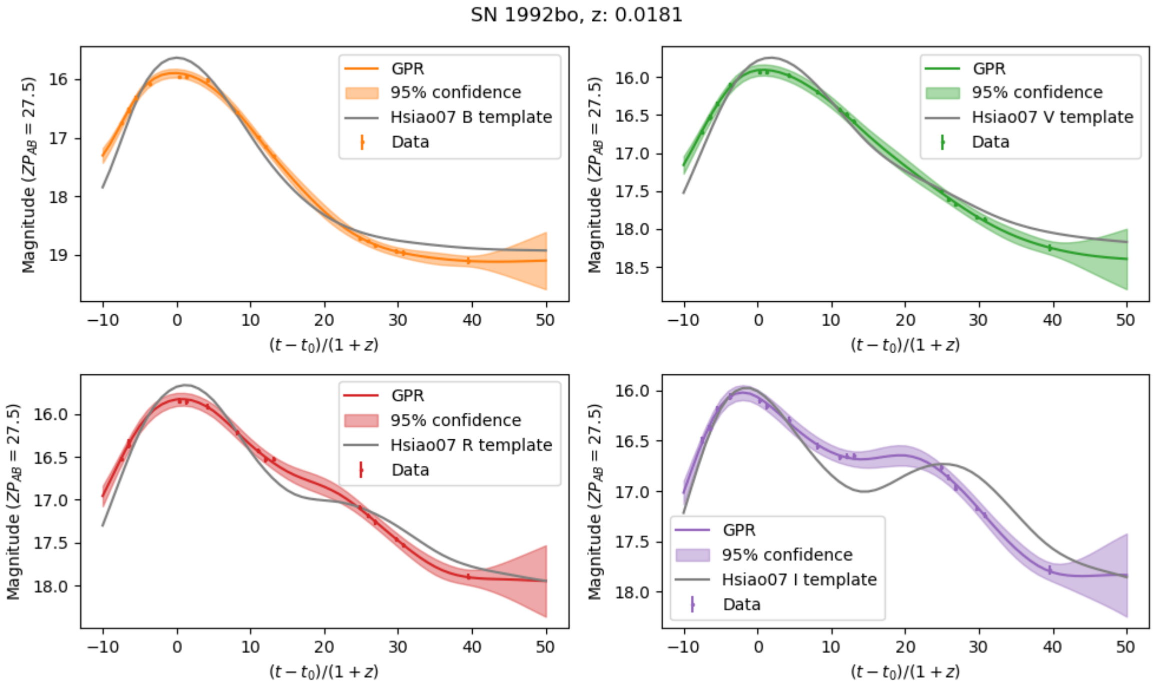

Figure 3: Example of the Gaussian Process Regression obtained for the training sample data.

on Fig. 3. Fig. 2 shows the same procedure but performing a simple linear interpolation without extrapolation.

We observe many similar behaviours on Fig. 1 and Fig. 2 for the original model vectors and the ones we obtain. The main differences are believed to be due to the data interpolation and extrapolation and the fact that the paramaters estimate used to construct this model vectors were in fact obtained in a posterior fit of [6] including the complete covariance matrix, which is estimated after this step.

References

[1] Adam G Riess, William H Press, and Robert P Kirshner. Using type Ia supernova light curve shapes to measure the hubble constant. *Astrophysical Journal, Part 2-Letters (ISSN 0004-637X), vol. 438, no. 1, p. L17-L20*, 438:L17–L20, 1995.

[2] Adam G Riess, William H Press, and Robert P Kirshner. A precise distance indicator: Type Ia supernova multicolor light-curve shapes. *The Astrophysical Journal*, 473(1):88, 1996.

[3] Claire Saunders et al. SNEMO: Improved empirical models for type Ia supernovae. *The Astrophysical Journal*, 869(2):167, 2018.

[4] Julien Guy et al. SALT2: using distant supernovae to improve the use of type Ia supernovae as distance indicators. *Astronomy & Astrophysics*, 466:11–21, 4 2007.

[5] Julien Guy et al. The supernova legacy survey 3-year sample: Type Ia supernovae photometric distances and cosmological constraints. *Astronomy & Astrophysics*, 523:A7, 11 2010.

[6] Saurabh Jha, Adam G Riess, and Robert P Kirshner. Improved distances to type Ia supernovae with multicolor light-curve shapes: MLCS2k2. *The Astrophysical Journal*, 659(1):122, 2007.