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ECONOMETRICS OF FIRST-PRICE AUCTIONS

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In this paper we propose an estimation method for the empirical study of theoretical auction models. We focus on first-price sealed bid and descending auctions and we adopt the private value paradigm, where each bidder is assumed to have a different private value, only known to him, for the object that is auctioned.

Following McFadden (1989) and Pakes and Pollard (1989), our proposed method is based on simulations. Specifically, the method relies on a simulated nonlinear least squares objective function appropriately adjusted so as to obtain consistent estimates of the parameters of interest.

We illustrate the proposed method by studying a market of agricultural products, where descending auctions are used. Our analysis takes into account heterogeneity of the auctioned objects and the fact that only the winning bid is observed. We estimate the parameters that characterize the distribution of the unobserved private values for each auctioned object.

KEYWORDS: First-price auctions, descending auctions, private values, simulated nonlinear least squares.

1. INTRODUCTION

SINCE VICKREY'S (1961) WORK, the theory of auctions has considerably expanded with the development of appropriate game theoretical tools (see Milgrom (1985, 1987), McAfee and McMillan (1987), and Wilson (1992) for recent surveys). It has generated an impressive body of experimental work (see, e.g., Cox, Smith, and Walker (1985, 1988)) and a recent critique (Harrison (1989)) with comments and reply (see the *American Economic Review* (1992, pp. 1374–1443)).

In contrast, only a few empirical studies have attempted to validate theoretical auction models using real auction data. Examples include Hansen (1985, 1986), Hendricks, Porter, and Boudreau (1987), and Hendricks and Porter (1988). Most empirical studies have concentrated on testing some implications of the theory of auctions using reduced-form econometric models. Recent exceptions are Paarsch (1989, 1992) who estimates econometric models that are closely derived from theory. A possible reason for this gap between theoretical and empirical work arises from the computational difficulties due to the nonlinearity and numerical complexity associated with the estimation of structural econometric models.

In this paper we focus on one of the simplest theoretical auction models, the first-price sealed bid auction with independent private values. The symmetric

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Bayesian Nash equilibrium of the corresponding game of incomplete information was characterized by Riley and Samuelson (1981) among others. The equilibrium expresses the optimal bid as a function of the bidder's private value, the reservation price of the object, the number of bidders, and the distribution of private values.

In general, bids are observed while actual private values are not observed. The preceding theoretical model leads, however, to a closely related structural econometric model. Because optimal bids are functions of private values, which are theoretically random, then observed bids are also random with a distribution that is uniquely determined by the structural elements of the model. Unfortunately, the equilibrium function relating optimal bids to private values is typically untractable. As a consequence, only very specific distributions of private values have been considered in empirical work.

In this paper we develop an estimation method that can handle a broad class of distributions of private values. It is based on simulations as proposed by Lerman and Manski (1981) and studied recently by McFadden (1989), Pakes and Pollard (1989), Laroque and Salanié (1989, 1990), Gourieroux and Monfort (1990, 1993), Ruud (1991), and Hajivassiliou and McFadden (1993) among others. Our method relies on a simulated nonlinear least squares objective function appropriately adjusted so as to obtain consistent estimates of the parameters of interest. Then we establish the asymptotic properties of our estimator and discuss some extensions.

Next, we illustrate the estimation method by studying a market of agricultural products where descending auctions are used. The econometric formulation takes into account the heterogeneity of the auctioned objects and the fact that only the winning bids are observed. Estimation of the model provides estimates of the parameters that characterize the distribution of private values for each auctioned object.

The paper is organized as follows. In Section 2 we briefly present the theoretical first-price sealed bid auction model with independent private values. We indicate the difficulties associated with the estimation of the corresponding econometric model. In Section 3 we observe that the equilibrium solution of the auction model has a form that is naturally suitable to simulations. Then we propose a simulated nonlinear least squares estimator, which is shown to be consistent and asymptotically normal. In Section 4 we present the data and the market under study. The model is estimated and the empirical results are discussed. Section 5 concludes. Proofs are collected in Appendix A and some descriptive statistics are presented in Appendix B.

2. FIRST-PRICE AUCTIONS

2.1. *The Basic Theoretical Model*

A single and indivisible object is auctioned. All the bids are collected simultaneously. The object is sold to the highest bidder who pays his bid to the seller, provided the bid is at least as high as a reservation price. In such an institutional framework each bidder does not know the bids of the others when forming his bid. On the other hand, it is assumed that the reservation price p^o and the number of bidders I are known by all bidders in advance.

In the private value paradigm each bidder ($i=1,\dots,I$) is assumed to have a private value v^i for the object that is auctioned. When forming this bid, each bidder knows his private value but does not know others' private values. However, each bidder knows that all private values including his own have been drawn independently from a probability distribution. This probability distribution is identical across bidders and assumed to be common knowledge. Therefore all bidders are identical a priori. Because of the latter property, the game is said to be symmetric.

Let $F(\cdot)$ denote the cumulative distribution function of private values. It is assumed that $F(\cdot)$ is absolutely continuous with respect to the Lebesgue measure. The density and support are $f(\cdot)$ and $V\subset[0,+\infty)$ respectively.

Under the assumption that each buyer is risk neutral, the symmetric Bayesian Nash equilibrium of the preceding model gives, for every $i=1,\dots,I$, the optimal bid b^i of the ith bidder as

$$(1) \qquad b^i = e(v^i, I, p^o, F) \equiv v^i - \frac{1}{(F(v^i))^{I-1}} \int_{p^o}^{v^i} (F(x))^{I-1}\, dx$$

if $v^i \geqslant p^o$ (see, e.g., Riley and Samuelson (1981)). If $v^i < p^o$, then b^i can be any value strictly less than the reservation price p^o.

The equilibrium strategy (1) is the unique symmetric differentiable Bayesian Nash equilibrium (see Maskin and Riley (1984)). Also $b^i \leqslant v^i$ if $v^i \geqslant p^o$ and $b^i = p^o$ if $v^i = p^o$. Another important property is that b^i, as a function of v^i, is strictly increasing on $[p^o, +\infty)\cap V$. Hence, the equilibrium is revealing (provided $v^i \geqslant p^o$).

A bid of particular interest is the winning bid, denoted b^w. The winning bid is the highest bid $b_{(I)} = \max_i b^i$ if $b_{(I)} \geqslant p^o$. It is undefined otherwise. Let $v_{(I)} = \max_i v^i$. Since relation (1) is monotone increasing in v^i and starts at p^o when $v^i = p^o$, then we have

$$(2) \qquad b^w = e(v_{(I)}, I, p^o, F)$$

if $v_{(I)} \geqslant p^o$. Thus the winner of the auction is the individual with the highest private value provided $v_{(I)} \geqslant p^o$. (Ties occur with zero probability because $F(\cdot)$ is absolutely continuous with respect to the Lebesgue measure.) If $v_{(I)} < p^o$, the auctioned object is not sold and there is no winning bid.

A related quantity is the expected revenue for the seller. Let R denote revenue. Then R is equal to b^w if the object is sold and zero otherwise. From the point of view of the seller, $v_{(I)}$ and hence R are random. By integrating (2) on $[p^0, +\infty)$ with respect to the density of $v_{(I)}$, which is $If(\cdot)(F(\cdot))^{I-1}$, one obtains

$$(3) \qquad E[R] = I\int_{p^o}^{+\infty} \left(v(F(v))^{I-1} - \int_{p^o}^{v} (F(x))^{I-1}\, dx \right) f(v)\, dv.$$

The preceding equilibrium solution also applies to a descending or Dutch auction. In such an auction the selling price of the object starts from an arbitrary price, which is assumed to be sufficiently high so as to exceed all possible bid values. For instance, since b^i cannot be larger than v^i, the auction can start from the highest value in the support V. The selling price of the auctioned object decreases continuously. The auction stops and the object is sold to the first individual who signals that he will buy the object at the current posted price. The auction automatically ends, however, if the descending posted price reaches the reservation price p^o.

In such an auction b^i corresponds to the price at which the ith bidder will "stop the clock." Under the same informational assumptions as in the first-price sealed bid auction, the symmetric Bayesian Nash equilibrium bid for the ith individual is still given by (1) when $v^i \geqslant p^o$. If $v^i < p^o$, then b^i is indeterminate and can be any value strictly less than p^o as before. Indeed, descending and first-price auctions are strategically equivalent (see, e.g., Milgrom and Weber (1982)) and therefore have the same equilibrium. As a result, the winning bid in the descending auction, when it exists, is given by (2) and the expectation of the seller's gross revenue is given by (3). A difference, however, between a first-price sealed bid auction and a descending auction is that only the winning bid can be observed in the latter.

2.2. *The Structural Econometric Model*

We shall adopt a structural approach to the empirical analysis of auctions. This means that the econometric model for the observed bids is derived directly from the underlying theoretical auction model. Thus the basic ingredients that determine the econometric model for observed bids are the reservation price, the number of bidders, and the distribution of private values. This modelling strategy allows us to estimate the distribution of the unobserved private values, which is useful for policy analysis.

For example, consider the choice of an optimal price for the seller. Assume that the seller is risk neutral and that his value for the object is v^o. As it is well known (see, e.g., Laffont and Maskin (1980) or Riley and Samuelson (1981)), the optimal reservation price p^o_*, which maximizes $E[R]+v^o(F(p^o))^I$, where $E[R]$ is given by (3), solves

$$(4) \qquad p^o_* = v^o + \frac{1-F(p^o_*)}{f(p^o_*)}.$$

Hence the optimal reservation price crucially depends on the distribution of private values.[2]

[2] As suggested by a referee, (4) can be used to improve estimation efficiency. Specifically, assuming that the seller behaves optimally, (4) gives $p^o_* \equiv e_o(v^o, F)$, where v^o is assumed to be drawn from some distribution G (say). Then (4) can be combined with (2) to estimate the parameters of F and G. For reasons given in footnote 23, we shall exclusively focus upon the separate estimation of (2). Note, however, that an assumption, which is natural and consistent with the theoretical model of Section 2.1 and the optimal rule (4), is that v^o and $v^1, \dots, v^I$, and hence v^o and $v_{(I)}$, are mutually independent conditional upon the exogenous variables in F and G. Thus, provided all these exogenous variables are observed by the investigator (see also Section 3.3), $p^o = p^p_*$ is still weakly exogenous in (2) so that our estimation method for (2) proposed below remains valid.

In an econometric investigation, one frequently considers more than one auction. This is clearly necessary for statistical inference in a descending auction since, otherwise, one would have only one observation, namely, the winning bid. Hereafter we let L denote the number of auctions and we subscript by l all quantities relevant to the lth auction.

Considering more than one auction raises two difficulties. The first concerns the repetition of an auction. When successive auctions for various objects are allowed, bidders face a complex dynamic strategic situation and may want to conceal their private values at the beginning of the sequence of auctions. This may lead to underbidding during the first auctions.[3] Thus, to preserve the equilibrium strategy (1), we assume that bidders draw new independent private values at every auction. This assumption will be justified in the empirical application of Section 4.

The second difficulty arises from a likely heterogeneity across auctions due to changes in the nature of the auctioned objects or in the environment of the auctions. Formally, this means that the distribution of private values for the lth auction varies across auctions. We let $F_l(\cdot)$ denote such a distribution.

We shall adopt a parametric formulation. For every l, we assume

$$F_l(\cdot) = F(\cdot | z_l, \theta), \tag{5}$$

where θ is an unknown parameter vector in $\Theta \subset \boldsymbol{R}^k$ and z_l is a vector of variables affecting bidders' valuations through the distribution of private values. For instance, z_l may contain characteristics of the lth auctioned object as well as variables describing the relevant environment of the lth auction. In this paper, however, the reservation price p_l^o is not included in the vector of characteristics z_l. This corresponds to our empirical application, where buyers and sellers have the same information about the market conditions affecting $F_l(\cdot)$.[4]

To begin with, we assume that the vector z_l is fully observed and that the number of bidders is known to the investigator. We shall consider relaxing these assumptions in Section 3.3. To simplify the presentation, we assume that I is constant so that $I_l = I$ for every l. This is not necessary for the results of Sections 3.1 and 3.2 as long as I_l is observed.

We can now derive the structural econometric model corresponding to the theoretical auction model. We focus on the case where only the winning bid is available, as in the descending auction analyzed in Section 4. As noted earlier, however, the winning bid is undefined when the object is unsold, i.e., when $v_{(I)l} < p_l^o$. To avoid a discontinuity at $v_{(I)l} = p_l^o$, it is convenient to set $b_l^w = p_l^o$ when this occurs. Then

$$b_l^w = e(v_{(I)l}, I, p_l^o, F_l)\mathbf{1}(v_{(I)l} \geqslant p_l^o) + p_l^o \mathbf{1}(v_{(I)l} < p_l^o), \tag{6}$$

where $\mathbf{1}(\cdot)$ is the indicator of the event $(\cdot)$.

[3] Characterization of the equilibrium solutions in repeated auctions is, however, much more difficult (see, e.g., Weber (1983) and Bikhchandani (1988)).

[4] On the other hand, if the seller had private information about such conditions, the announcement of p_l^o would have a signalling value. Hence p_l^o would have to be included in the vector z_l. See the conclusion for a discussion of difficulties implied by this signalling model.

Let $H_l(\cdot)$ denote the distribution of b_l^w. Since b_l^w is a function of $v_{(I)l}$, which is distributed as $F_l(\cdot)^I$ according to the theoretical model, then $H_l(\cdot)$ is uniquely determined. Specifically, $H_l(\cdot)$ has a mass at p_l^o with probability $(F_l(p_l^o))^I$ and a density with respect to the Lebesgue measure given by

$$(7) \qquad h_l(b) = \frac{I}{I-1} \frac{\left(F_l(e_l^{-1}(b))\right)^I}{e_l^{-1}(b) - b}$$

for $b \geqslant p_l^o$, where $e_l^{-1}(\cdot) \equiv e^{-1}(\cdot, I, p_l^o, F_l)$ denotes the inverse of the equilibrium function defined in (1) with respect to its first argument.

Given independence across auctions and taking into account censoring of the winning bids by the reservation prices, the likelihood function of the model takes the familiar Tobit form. Then maximum likelihood (ML) estimates of the parameter vector θ can be obtained.

Unfortunately, ML estimation presents two major difficulties. First, it can be shown that the support of the distribution of b_l^w is $[p_l^o, E_l(\max(X, p_l^o))]$ where X is the largest order statistic in $I-1$ independent draws from $F_l(\cdot)$. Thus, for each l, the support of the distribution of b_l^w depends on θ, which violates the usual assumptions underlying ML estimation.[5] The second difficulty is computational. Indeed, except for some simple specifications such as those considered by Paarsch (1989, 1992), the equilibrium function (1) is nonlinear in its first argument v. As a result, the inverse function $e_l^{-1}(\cdot)$ cannot be obtained explicitly and must be evaluated numerically. Computational burdens then become overwhelming since $e_l^{-1}(b_l^w)$ must be determined for every l and every trial value for θ.

Alternatively, one can use other estimation methods such as nonlinear least squares (NLLS) or a generalized method of moments (GMM) (see Hansen (1982)). Exact computation of the moment of b_l^w for every l and trial value for θ becomes rapidly excessive, however, as the equilibrium solution (1) becomes untractable.

3. ECONOMETRIC METHODS

3.1. *A Simulated Nonlinear Least Square Estimator*

In view of these computational difficulties, we shall propose a relatively simple estimation method that can handle a broad class of valuation distributions. The method requires the exact computation of neither the equilibrium strategy (1) nor the moments of the winning bid. It is based on simulations following McFadden (1989) and Pakes and Pollard (1989) among others.

In this paper we focus mainly on the first moment of the winning bid. This moment is of special interest in view of the revenue equivalence theorem (see,

[5] Two recent papers by Donald and Paarsch (1992, 1993) address this issue. In particular, the first paper derives the asymptotic distribution of the ML estimator, which is not standard. Note also that the upper bound $E_l(\max(X, p_l^o))$ is typically a complex function of the parameters, which complicates the numerical implementation of the ML procedure.

e.g., Riley and Samuelson (1981) and Milgrom and Weber (1982)). The seller's expected revenue is $E[R]=E[b^w]+p^oF^I(p^o)$ using (6) given our convention $b^w=p^o$ when the object is unsold. Thus, by this theorem, a large class of auctions including first-price, descending, English, and second-price auctions can be analyzed within the private value paradigm by the estimation method explained below.

Let $E[b_l^w]\equiv m_l(\theta)\equiv m(x_l,\theta)$ denote the conditional expectation of b_l^w given I (assumed constant) and $x_l'\equiv(p_l^o,z_l')$. The usual NLLS estimator minimizes the objective function $Q_L(\theta)=(1/L)\Sigma_{l=1}^L(b_l^w-m_l(\theta))^2$ with respect to θ. Because $m_l(\theta)$ is not readily available, a natural idea is to replace $m_l(\theta)$ by an unbiased simulator $\overline{X}_l(\theta)$, i.e., a simulator such that $E[\overline{X}_l(\theta)]=m_l(\theta)$. We show later how such a simulator can be constructed for the theoretical auction model of Section 2.1.

As is well-known, however, minimizing the objective function

$$(8)\qquad Q_{S,L}(\theta)=\frac{1}{L}\sum_{l=1}^{L}\left(b_l^w-\overline{X}_l(\theta)\right)^2$$

with respect to θ produces an inconsistent estimator for any fixed number of simulations S as L increases to infinity. For, using a uniform Law of Large Numbers (see Appendix A) and the properties of the simulator $\overline{X}_l(\theta)$ mentioned below, we have that $Q_{S,L}(\theta)-[Q_L(\theta)+\Delta_L(\theta)]$ converges in probability to zero (uniformly in θ), for any fixed S, as $L\to\infty$, where

$$(9)\qquad \Delta_L(\theta)=\frac{1}{L}\sum_{l=1}^{L}E\left[\overline{X}_l^2(\theta)-m_l^2(\theta)\right]=\frac{1}{L}\sum_{l=1}^{L}E\left(\mathrm{Var}_x\left[\overline{X}_l(\theta)\right]\right)>0,$$

and $\mathrm{Var}_x[\cdot]$ denotes the conditional variance given x_l. Hence the θ that minimizes $Q_{S,L}(\theta)$ converges to the value that minimizes $\mathrm{plim}_{L\to\infty}Q_L(\theta)+\Delta_L(\theta)$, which is typically not the true value θ_o because the latter minimizes $\mathrm{plim}_{L\to\infty}Q_L(\theta)$.[6]

The preceding argument suggests a simple adjustment to the objective function (8) so as to obtain a consistent estimator for fixed S as $L\to\infty$. Using the usual unbiased estimator of $\mathrm{Var}_x[\overline{X}_l(\theta)]$, we expect that $\Delta_{S,L}(\theta)-\Delta_S(\theta)\xrightarrow{p}0$ (uniformly in θ) for any fixed S, as $L\to\infty$, where

$$(10)\qquad \Delta_{S,L}(\theta)\equiv\frac{1}{L}\sum_{l=1}^{L}\frac{1}{S(S-1)}\sum_{s=1}^{S}\left(X_{sl}(\theta)-\overline{X}_l(\theta)\right)^2.$$

Thus, subtracting $\Delta_{S,L}(\theta)$ from $Q_{S,L}(\theta)$, we define the "simulated" NLLS objective function

$$(11)\qquad Q_{S,L}^*(\theta)=\frac{1}{L}\sum_{l=1}^{L}\left[\left(b_l^w-\overline{X}_l(\theta)\right)^2-\frac{1}{S(S-1)}\sum_{s=1}^{S}\left(X_{sl}(\theta)-\overline{X}_l(\theta)\right)^2\right].$$

[6] An alternative reason is that $m_l^2(\theta)$ is not estimated unbiasedly by $\overline{X}_l^2(\theta)$. Such an argument based on Jensen's inequality is similar to the one used to show the inconsistency of simulated maximum likelihood (SML) estimation (see, e.g., Ruud (1991)).

We let $\hat{\theta}$ denote the SNLLS estimator obtained by minimizing $Q^*_{S,L}(\theta)$ with respect to θ.

We now turn to the construction of a simulator $\bar{X}_l(\theta)$ that does not require exact computation of the equilibrium solution (1). The next lemma underlies our construction by giving an alternative expression for (1). It can be proved by applying Theorem 14 in Milgrom and Weber (1982) to the private value paradigm with reservation prices and by noticing that the conditional distribution function of the second highest private value $v_{(I-1)}$ given the highest $v_{(I)}$ is $[F(\cdot)/F(v_{(I)})]^{I-1}$.[7]

LEMMA: *Given I, p^o, and $F(\cdot)$, then for $v \geqslant p^o$,*

$$(12) \qquad e(v, I, p^o, F) = E\left[\max\left(v_{(I-1)}, p^o\right) | v_{(I)} = v\right],$$

where $E[\cdot | v_{(I)}]$ denotes the conditional expectation given $v_{(I)}$.[8]

This lemma can be used in at least two important ways. First, given I, p^o, $F(\cdot)$, and $v \geqslant p^o$, (12) can be used to evaluate the optimal bid by averaging $\max(v_{(I-1)}, p^o)$ over many random draws from the conditional distribution of $v_{(I-1)}$ given $v_{(I)} = v$. Hence, by varying v, the equilibrium function (1) can be estimated unbiasedly.

Second, the lemma can be used to obtain an expression for the winning bid that is prone to simulations. Specifically, from (6) we obtain

$$(13) \qquad \begin{aligned} b_l^w &= E\left[\max\left(v_{(I-1)l}, p_l^o\right) | v_{(I)l}\right] \mathbf{1}\left(v_{(I)l} \geqslant p_l^o\right) + p_l^o \mathbf{1}\left(v_{(I)l} < p_l^o\right) \\ &= E\left[\max\left(v_{(I-1)l}, p_l^o\right) | v_{(I)l}\right] \end{aligned}$$

given I, p_l^o, and $F_l(\cdot)$, i.e., given I and $x_l = (p_l^o, z_l')$. Equation (13) can be used to simulate the moments of b_l^w.

Taking expectation of (13) with respect to $v_{(I)l}$ gives

$$(14) \qquad E\left[b_l^w\right] = E\left[\max\left(v_{(I-1)}, p_l^o\right)\right].$$

Equation (14) can be viewed as an integral with respect to the density of $v_{(I-1)l}$. Alternatively, $v_{(I-1)l}$ can be viewed as a function of $v_l^1, \dots, v_l^I$, which are independently drawn from the distribution $F_l(\cdot)$. Then (14) becomes

$$(15) \qquad E\left[b_l^w\right] = \int_{V_l} \cdots \int_{V_l} \max\left(u_{(I-1)}, p_l^o\right) f_l(u_1) \dots f_l(u_I)\, du_1 \dots du_I,$$

where $u_{(I-1)}$ is the second highest value in $u_1, \dots, u_I$. Equation (15) is the basis of our proposed simulator of $m_l(\theta)$.

There are many ways to simulate the (conditional) mean of b_l^w. We use importance sampling, one of the oldest techniques in Monte Carlo integration

[7] A direct and simple proof using (1) is also available from the authors.

[8] By convention, we take $v_{(I-1)} = 0$ if $I = 1$. Thus, when there is only one bidder and $v \geqslant p_o$, the optimal bid is p_o as required by (1).

(see, e.g., Hammersley and Handscomb (1964) and Rubinstein (1981)). This technique provides some flexibility in the choice of the importance function which can reduce the sampling variability of the random draws and hence improve the precision of estimators based on simulations.[9] In addition, as noted below, this technique leads here to a continuously differentiable statistical objective function. As a result, the derivation of asymptotic properties is greatly simplified relative to McFadden (1989) and Pakes and Pollard (1989).

For each l, let $g_l(\cdot)$ be a known density with a support V_{gl} at least including that of $f_l(\cdot)$, i.e., V_l. In practice, it is convenient to let the importance function $g_l(\cdot)$ depend on the observation index l. For instance, $g_l(\cdot)$ can be defined as $g_l(\cdot)=g(\cdot|z_l)$ for some known conditional density $g(\cdot|\cdot)$. This is assumed hereafter. Then (15) can be written as

$$E[b_l^w] = \int_{V_{gl}} \cdots \int_{V_{gl}} \max\left[u_{(I-1)}, p_l^o\right] \times \frac{f_l(u_1)\dots f_l(u_I)}{g_l(u_1)\dots g_l(u_I)} g_l(u_1)\dots g_l(u_I)\, du_1 \dots du_I. \tag{16}$$

Now, for each $l=1,\dots,L$, we draw S independent samples, each of size I, denoted $u_{1l}^s,\dots,u_{Il}^s$ where u_{il}^s is independently drawn from the distribution with density $g_l(\cdot)$ for $s=1,\dots,S$. Then, for every l, (16) suggests to estimate $E[b_l^w]$ by the sample mean

$$\bar{X}_l(\theta) = \frac{1}{S}\sum_{s=1}^{S} X_{sl}(\theta) \quad \text{where} \tag{17}$$

$$X_{sl}(\theta) = \max\left[u_{(I-1)l}^s, p_l^o\right] \frac{f_l(u_{1l}^s)\dots f_l(u_{Il}^s)}{g_l(u_{1l}^s)\dots g_l(u_{Il}^s)},$$

and we have emphasized the dependence of $\bar{X}_l(\theta)$ and $X_{sl}(\theta)$ on θ.

Then, by construction, $E[\bar{X}_l(\theta)]=m_l(\theta)$ given x_l. Second, note that the LSI random draws u_{il}^s are independent of θ and are drawn prior to estimation. Hence, for any given θ, the variables $\{X_{sl}(\theta);\ s=1,\dots,S\}$ and hence $\bar{X}_l(\theta)$ are conditionally independent of b_l^w given x_l. Third, because all the u_{il}^s are drawn from the importance functions $g_l(\cdot)$, which are independent of θ, and because θ appears only in the $f_l(\cdot)$'s, which are outside the "max" in (17), then $X_{sl}(\theta)$, $\bar{X}_l(\theta)$, and hence the objective function $Q_{S,L}^*(\theta)$ defined in (11) are twice continuously differentiable in θ if $f(\cdot|z_l,\theta)$ is. This is assumed hereafter.

3.2. *Asymptotic Properties*

Let θ_o be the true parameter value with $\theta_o \in \Theta$. From the form of $Q_{S,L}^*(\theta)$, the estimator $\hat{\theta}$ is an M-estimator (see, e.g., Amemiya (1985)). Properties of M-estimators have been derived under various assumptions. Below, we use

[9] The choice of an "optimal" importance function is an important issue. See Rubinstein (1981, Chapter 4) and Section 4.4 for a further discussion.

those of Gallant and White (1988). This allows dynamic specifications, where the weakly exogenous variables in x_l can follow heterogeneous and weakly dependent processes.

Consistency

The next result establishes formally the consistency of $\hat{\theta}$. It also provides a consistent estimator of the mean squared error $EQ_L(\theta_o)$.

PROPOSITION 1: *For any fixed S, $\hat{\theta}$ is a strongly consistent estimator of θ_o as $L \to \infty$. Moreover, $Q^*_{S,L}(\hat{\theta}) - EQ_L(\theta_o)$ converges almost surely to* 0.

Though found independently, our estimator is related to an estimator recently proposed by Bierings and Sneek (1989). These authors have only established the consistency of their estimator, which minimizes (11), where the second term in brackets is replaced by the exact variance of $\overline{X}_l(\theta)$. Relative to other simulated M-estimators such as SML or simulated pseudo maximum likelihood (SPML) estimators (see Laroque and Salanié (1989, 1990)), our SNLLS estimator has the advantage of being consistent for a fixed number of simulations as $L \to \infty$.

Another class of simulation-based estimators arises from the class of generalized method of moment (GMM) estimators. This gives the method of simulated moments (MSM) (see McFadden (1989) and Pakes and Pollard (1989)) and the method of simulated scores (MSS) (see Ruud (1991) and Hajivassiliou and McFadden (1993)). In particular, an estimator related to ours is the SNLLS estimator recently proposed by Gourieroux and Monfort (1993). These alternative estimators also possess the computationally convenient property of being consistent for a fixed number of simulations as L increases to infinity. They will be discussed below.

Asymptotic Distribution

Let $\mu_l^o \equiv \partial m_l(\theta_o)/\partial\theta$. Let $\sigma_o^2(x_l)$ denote the conditional variance of b_l^w given x_l. Following (17), define $\overline{Y}_l(\theta) \equiv (1/S)\Sigma_{s=1}^S Y_{sl}(\theta)$, where $Y_{sl}(\theta) \equiv \partial X_{sl}(\theta)/\partial\theta$. Let ω_l, Ω_l, and C_l denote respectively the variances of $X_{sl}(\theta_o), Y_{sl}(\theta_o)$, and their covariance conditional upon x_l.

The next result gives the asymptotic distribution of $\hat{\theta}$.

PROPOSITION 2: *For any fixed S, $\Sigma_{S,L}^{-1/2}\sqrt{L}(\hat{\theta} - \theta_o)$ converges in distribution to a normal $N(0, I_k)$ as $L \to \infty$, where $\Sigma_{S,L} = A_L^{-1}B_{S,L}A_L^{-1}$ with*

$$(18) \qquad A_L = \frac{1}{L}\sum_{l=1}^{L} E(\mu_l^o \mu_l^{o\prime})$$

$$(19) \qquad B_{S,L} = \frac{1}{L}\sum_{l=1}^{L} E\left[\sigma_o^2(x_l)\mu_l^o\mu_l^{o\prime} + \frac{1}{S}\left(\sigma_o^2(x_l)\Omega_l + \omega_l\mu_l^o\mu_l^{o\prime}\right) + \frac{1}{S(S-1)}\left(\omega_l\Omega_l + C_lC_l'\right)\right],$$

and $E[\cdot]$ is the expectation with respect to $x_l' = (p_l^o, z_l')$.

Our SNLLS estimator differs from the SNLLS estimator recently proposed by Gourieroux and Monfort (1993), which is also consistent for any fixed number of simulations. There, the basic idea is to simulate the NLLS first-order conditions by simulating independently $m_l(\theta)$ and $\partial m_l(\theta)/\partial\theta$ by $\bar{X}_l(\theta)$ and $\bar{Y}_l^*(\theta)$, respectively, where $\bar{Y}_l^*(\theta)$ is similar to $\bar{Y}_l(\theta)$ but uses an additional independent set of SI random draws u_{il}^{s*} from $g_l(\cdot)$. This gives the orthogonality conditions $E[\bar{Y}_l^*(\theta_o)(b_l^w - \bar{X}_l(\theta_o))] = 0$. Then the corresponding best GMM estimator $\tilde{\theta}$ (say) is Gourieroux-Monfort's SNLLS estimator.[10]

For a total of $2LSI$ random draws, the asymptotic covariance matrix of $\tilde{\theta}$ is $A_L^{-1}B_{S,L}^* A_L^{-1}$ where

$$B_{S,L}^* = B_{S,L} - \frac{1}{S(S-1)}\frac{1}{L}\sum_{l=1}^{L} E\left[\frac{1}{S}\omega_l\Omega_l + C_lC_l'\right]. \tag{20}$$

See Gourieroux and Monfort (1993). Thus, with half as many simulations, the asymptotic covariance matrix of our SNLLS estimator $\hat{\theta}$ differs from that of $\tilde{\theta}$ by a term of order $(1/S^2)$ and $(1/S^3)$ if C_l is close to zero.[11]

It is also interesting to compare our SNLLS estimator to the usual NLLS estimator, which requires the exact computation of the mean $m_l(\theta)$. The asymptotic covariance matrix of the NLLS estimator is $A_L^{-1}B_L A_L^{-1}$ where $B_L = (1/L)\Sigma_{l=1}^L E[\sigma_0^2(x_l)\mu_l^o\mu_l^{o\prime}]$. Thus the asymptotic covariance matrix of our SNLLS estimator is larger than that of the NLLS estimator by a term of order $(1/S)$. In particular, if the number of simulations S increases to infinity, then our SNLLS estimator $\hat{\theta}$ becomes as asymptotically efficient as the usual NLLS estimator.

When $\sigma_o^2(\cdot)$ is constant, the NLLS estimator is asymptotically efficient in the class of estimators that uses conditional moment restriction $E[b_l^w|x_l] = m_l(\theta_o)$ only.[12] When $\sigma_o^2(\cdot)$ is not constant, the NLLS estimator need not be asymptotically efficient in the sense of reaching Chamberlain's (1987) lower bound in this class of estimators. As proposed there, one can exploit the orthogonality conditions $E[W_l(b_l^w - m_l(\theta_o))] = 0$, where W_l is, for instance, a vector of powers of x_l, to construct a MSM estimator following McFadden (1989) and Pakes and Pollard (1989). For a fixed number of simulations, the resulting MSM estimator may be more or less efficient than our SNLLS estimator depending on the choice of the instruments W_l.

[10] An alternative estimator that uses only SI random draws can be obtained by adjusting appropriately the NLLS first-order conditions. Specifically, following the argument leading to (11), one can consider the best GMM estimator based on the orthogonality conditions

$$E\left[\bar{Y}_l(\theta_o)(b_l^w - \bar{X}_l(\theta_o)) - \frac{1}{S(S-1)}\sum_{s=1}^{S}(Y_{sl}(\theta_o) - \bar{Y}_l(\theta_o))(X_{sl}(\theta_o) - \bar{X}_l(\theta_o))\right] = 0.$$

[11] Of course, this gain requires the computation of the correction term (10). On the other hand, Gourieroux and Monfort's estimator requires a consistent estimate of the asymptotic covariance matrix of $\bar{Y}_l^*(\theta_o)(b_l^w - \bar{X}_l(\theta_o))$.

[12] See Section 3.3 for the use of higher conditional moments.

Consistent Estimation of $\Sigma_{S,L}$

To use Property 2 to conduct inferences on the parameter θ_o, it is necessary to estimate consistently the asymptotic covariance matrix $\Sigma_{S,L}$. This matrix depends on $\mu_l^o, \sigma_o^2(x_l)$, and the variances and covariances of $X_{sl}(\theta_o)$ and $Y_{sl}(\theta_o)$. The latter variances and covariances are easily estimated.

However, because $\mu_l^o = \partial m_l(\theta_o)/\partial\theta$, then μ_l^o cannot be determined explicitly for the same reason as $m_l(\theta_o)$ could not. Below we propose an estimator that relies on simulating μ_l^o through $Y_{sl}(\theta_o)$ and that is also consistent for a fixed number of simulations as L increases to infinity. Second, the winning bids are typically heteroscedastic across auctions so that the conditional variance $\sigma_o^2(x_l)$ of b_l^w depends on x_l. Because $\sigma_o^2(x_l)$ as a function of x_l and θ_o is typically untractable, we follow White (1980) and estimate $B_{S,L}$ and hence $\Sigma_{S,L}$ without relying upon the form of $\sigma_o^2(x_l)$.

We define the $k \times k$ matrices

$$(21)\qquad \hat{A}_{S,L} = \frac{1}{L}\sum_{l=1}^{L}\Bigg[\bar{Y}_l(\hat{\theta})\bar{Y}_l(\hat{\theta})' - \frac{1}{S(S-1)} \times \sum_{s=1}^{S}\big(Y_{sl}(\hat{\theta}) - \bar{Y}_l(\hat{\theta})\big)\big(Y_{sl}(\hat{\theta}) - \bar{Y}_l(\hat{\theta})\big)'\Bigg],$$

$$(22)\qquad \hat{B}_{S,L} = \frac{1}{L}\sum_{l=1}^{L} d_{S,l}(\hat{\theta})\, d_{S,l}(\hat{\theta})',$$

where $\hat{\theta}$ is our SNLLS estimator and $d_{S,l}(\hat{\theta})$ is the k-dimensional vector

$$(23)\qquad d_{S,l}(\theta) = \big(b_l^w - \bar{X}_l(\theta)\big)\bar{Y}_l(\theta) + \frac{1}{S(S-1)}\sum_{s=1}^{S}\big(X_{sl}(\theta) - \bar{X}_l(\theta)\big)Y_{sl}(\theta).$$

The next result provides a consistent estimator of $\Sigma_{S,L}$.

PROPOSITION 3: *For any fixed* S, $\hat{\Sigma}_{S,L} - \Sigma_{S,L}$ *converges almost surely to* 0 *as* $L \to \infty$, *where* $\hat{\Sigma}_{S,L} = \hat{A}_{S,L}^{-1}\hat{B}_{S,L}\hat{A}_{S,L}^{-1}$.

A practical advantage of the proposed estimator $\hat{\Sigma}$ is that it requires neither the number of simulations S to be large nor the exact computation of the conditional variances $\sigma_o^2(x_l)$. Moreover, as the proof shows, $\hat{A}_{S,L}$ differs from $(1/2)\,\partial^2 Q_{S,L}^*(\hat{\theta})/\partial\theta\,\partial\theta'$ by an $o_p(1)$, while $d_{S,l}(\hat{\theta})$ is the lth individual gradient of $(-1/2)Q_{S,L}^*(\theta)$ evaluated at $\hat{\theta}$. Hence $\hat{A}_{S,L}$ and $\hat{B}_{S,L}$ can typically be obtained from the last iteration of commonly-used minimization algorithms.

3.3. *Some Extensions*

In this section we discuss separately some important assumptions underlying our proposed estimation method. This leads to some extensions that can be useful in empirical work.

The Number of Bidders

The number of bidders I is an important determinant of the equilibrium strategy (1) and hence of our structural econometric model. Up to now we have assumed that this number is known to the investigator. In practice, however, this may not be the case.

Let I_o denote the true number of bidders. In agreement with our application, we assume that I_o is constant and belongs to $\mathscr{I}$, a bounded set of positive integers. Then we can treat I_o as an additional (discrete) parameter.[13]

The next result provides a method for estimating I_o. We subscript by I all preceding statistical quantities when the number of bidders is I.

PROPOSITION 4: *Let $\hat{I} \equiv \arg\min_{I \in \mathscr{I}} Q^*_{S,L,I}(\hat{\theta}_I)$. Then, for any fixed S, $\hat{I}$ and $\hat{\theta}_{\hat{I}}$ converge almost surely to I_o and θ_o, respectively, as $L \to \infty$.*

Proposition 4 suggests the following procedure. For every number $I \in \mathscr{I}$, compute the corresponding SNLLS estimator $\hat{\theta}_I$ and estimated mean square error $Q^*_{S,L,I}(\hat{\theta}_I)$. Then, the true number of bidders I_o is estimated consistently by the integer $\hat{I}$ that minimizes $Q^*_{S,L,I}(\hat{\theta}_I)$ with respect to I. The second part of Proposition 4 ensures that $\hat{\theta}_{\hat{I}}$ is still a consistent estimator of θ_o. Then, statistical inference about θ_o can be based on $\hat{\theta}_{\hat{I}}$ as if $I_o = \hat{I}$.[14]

Unobserved Heterogeneity

Throughout, we have assumed that the vector z_l of characteristics influencing the distribution $F_l(\cdot)$ of bidders valuations is fully observed by the investigator. According to the theoretical model of Section 2.1, z_l is common knowledge to all bidders. Thus the assumption that the investigator observes z_l simply means that the investigator has access to this common knowledge.

In some situations one may argue that the investigator is less informed than bidders. For instance, the investigator observes only a vector z_{1l} where $z_l = (z_{1l}, z_{2l})$. Then, applying SNLLS to the econometric specification (5) omitting z_{2l} leads, in general, to inconsistent estimates. Following Gourieroux and Monfort (1990), consistency can be recovered by modelling the unobserved heterogeneity implied by the omission of z_{2l} and by adapting our SNLLS estimation method.

Specially, let $\{h_l(\cdot) \equiv h(\cdot | p_l^o, z_{1l}, \gamma);\ \gamma \in \Gamma\}$ be a conditional parametric model for z_{2l} given (p_l^o, z_{1l}). For instance, this model can be derived from a model for

[13] As suggested by a referee, a more general framework would be to assume that I_l is random with some unknown discrete distribution $\Pi_l(\cdot)$ depending on the exogenous variables z_l. Then, our SNLLS estimator can be adapted by noting that the right-hand side of (15) must be further integrated (summed) with respect to $\Pi_l(\cdot)$. Such an extension, however, was unnecessary in our empirical application.

[14] As in Ryu (1992), the rate of convergence of $\hat{I}$ is much faster than that of $\hat{\theta}_I$ for every I. This is because $\hat{I}$ and I_o are both discrete. Thus $\sqrt{L}(\hat{I} - I_o)$ converges to zero in probability and hence is asymptotically independent of $\sqrt{L}(\hat{\theta}_I - \theta_o)$ for every I. See also the end of the proof of Proposition 4.

p_l^o given (z_{1l}, z_{2l}) such as (4) and a model for z_{2l} given z_{1l}. Let $\gamma_o \in \Gamma$ denote the true value of γ. The parameter vector is now $(\theta, \gamma) \in \Theta \times \Gamma$. Let $\mathbf{Z}_{2l}$ be the support of $h_l(\cdot)$. Then

$$(24)\qquad E[b_l^w | p_l^o, z_{1l}] = \int_{\mathbf{Z}_{2l}} E[b_l^w | p_l^o, z_{1l}, z_{2l}] h_l(z_{2l})\, dz_{2l}$$
$$= \int_{\mathbf{Z}_{2l}} \int_{V_l} \cdots \int_{V_l} \max(u_{(I-1)}, p_l^o) f_l(u_1) \ldots f_l(u_I) h_l(z_2)\, du_1 \ldots du_I\, dz_2,$$

where the second equality follows from (15). Thus, introducing importance functions as in (16), we can consider sample averages $\bar{X}_l(\theta, \gamma)$ of $X_{sl}(\theta, \gamma)$ now defined by

$$(25)\qquad X_{sl}(\theta, \gamma) = \frac{1}{S} \sum_{s=1}^{S} \max(u_{(I-1)l}^s, p_l^o) \frac{f_l(u_{1l}^s) \ldots f_l(u_{Il}^s)}{g_{sl}(u_{1l}^s) \ldots g_{sl}(u_{Il}^s)} \frac{h_l(z_{2l}^s)}{\psi_l(z_{2l}^s)},$$

where, for each $l = 1, \ldots, L$ and $s = 1, \ldots, S, (u_{1l}^s, \ldots, u_{Il}^s)$ are independent draws from the importance conditional density $g_{sl}(\cdot) \equiv g(\cdot | z_{1l}, z_{2l}^s)$ and z_{2l}^s is drawn from the importance density $\psi_l(\cdot) \equiv \psi(\cdot | p_l^o, z_{1l})$. Then Propositions 1–4 apply to $(\hat{\theta}, \hat{\gamma})$ with this modification provided there is no dynamic misspecification.[15]

Using Higher Moments

Up to now, we have focused on the first moment of the winning bid. As mentioned earlier, the first moment is of special interest in view of the revenue equivalence theorem. As a result our SNLLS method applies to the analysis of a large class of auctions including first-price, descending, English, and second-price auctions within the risk neutral private value paradigm.

When analyzing a particular form of auctions such as first-price or descending auctions, however, one may gain efficiency by using higher moments of the winning bid. Moreover, this may be necessary when θ_o is not identified from the first moment only, a condition that was assumed in Propositions 1–4. In this case, (12) can be used to simulate higher moments of b_l^w (see Laffont and Vuong (1993)). Then, a MSM can be constructed following McFadden (1989) and Pakes and Pollard (1989) to obtain a computationally convenient estimator of θ_o that is consistent and $\sqrt{L}$-asymptotically normal for a fixed number of simulations as $L \to \infty$.[16]

[15] That is when $E[b_l^w | x_{1l}] = E[b_l^w | x_{1l}, \mathcal{F}_1^{l-1}]$, where $\mathcal{F}_1^{l-1}$ denotes the past values of (b_l^w, x_{1l}), and $x_{1l} = (p_l^o, z_{1l})$. See Appendix A. This is the case if z_{2l} is independent of past (b_l^w, p_l^o, z_{1l}) given (p_l^o, z_{1l}), which holds when (b_l^w, p_l^o, z_l) are independent across l.

[16] Alternatively, using the first two conditional moments $m_l(\theta_o)$ and $\sigma_l^2(\theta_o)$ of b_l^w given x_l, one can correct for heteroscedasticity by minimizing (11), where the term in brackets is divided by $\sigma_l^2(\hat{\theta}_o)$, and $\hat{\theta}_o$ is a preliminary consistent estimate of θ_o such as our SNLLS estimator $\hat{\theta}$. This requires, however, the exact computation of $\sigma_l(\cdot)$, which is typically untractable.

4. APPLICATION TO AN AGRICULTURAL MARKET

In this section, we illustrate the preceding econometric methods by an empirical analysis of a descending auction. Our analysis should be viewed as a preliminary study using the theoretical benchmark model of Section 2.1. In particular, we assume that there is no unobserved heterogeneity. On the other hand, we assume that the number of bidders is not known to the investigator, as is typically the case in empirical work.

4.1. *The Descending Auction of Eggplants in Marmande*

Descending auctions are frequently used to sell agricultural products as they are very fast and well adapted to perishable commodities. The particular auction we study is a descending auction of greenhouse eggplants in Marmande, France. In this market the sellers are farmers and the buyers are resale trade firms. The buyers can be considered as agents of retail sellers who have placed orders at specific prices before the opening of the market. These prices are the valuations of the buyers in the auction. At each round of the auction, a case of eggplants—15 to 350 kilos—is displayed. It is also described in the general catalog of the day given to each buyer before the market. The seller announces a reservation unit price, i.e., a price for one kilogram.

The general principles of this auction are the same as in any descending auction. The auction starts with a very high unit price, for example, fifteen francs when the usual unit price of the commodity is about ten francs per kilo. From then on, the price drops very quickly, until one of the bidders makes a bid. At that instant the reverse clock is stopped. The first buyer who makes a bid before the reservation price is reached wins the auction. His unit payment equals his bid.

4.2. *Data*

We have data on daily sales during the summer of 1990 (from June to October). In our theoretical economic model it is important that each day a single case be placed on the market to avoid dynamic strategic considerations, which are outside the scope of this paper.[17] Thus we have selected days for which only a single case of eggplants of a given size category is offered. Moreover, eggplants deteriorate quickly enough so that the supply of day $t+1$ can be considered independent of day t's exchanges. We have data for 81 auctions.

With this data set, the theoretical model of Section 2.1 seems appropriate or at least a benchmark model for a first analysis. The private value paradigm is justified by the organization of the market. Buyers come to the market with the willingnesses to pay of the retailers, who sell in geographically distinct markets.

[17] If several cases are auctioned the same day, buyers may wish to understate their willingness to pay in the first auction to acquire information about other players' valuations.

Since each day the market is quite specific and since we use data for auctions of single cases, the independence assumptions, which enable us to model the equilibrium as a static Bayesian equilibrium, seem satisfied. For each auction, we have the winning bid, the reservation price (see Graph B1, Appendix B), and other relevant variables (see next subsection).

As for the number of bidders we will explore two hypotheses. The first hypothesis is to ignore the potential asymmetry of the bidders and to consider that the number of bidders is the 11 participants who always attend this market. The second hypothesis, which is motivated by the observation that one buyer wins almost half of the auctions, considers this particular buyer as the agent of a number (unknown to us) of retail traders.

4.3. *The Distribution of Private Values*

We assume that private values follow a log-normal distribution for each auction. Unlike the cases considered by Paarsch (1989, 1992), the log-normal distribution does not lead to a closed form bidding function (1). We let this distribution depend on various characteristics, which are drawn from our data file (see (5)). Specifically, we assume that the mean of the logarithm of valuations is a linear function of six exogenous variables:

$$E \log v_l^i = \mu_l = \theta_1 + \theta_2 \text{ seller}_l + \theta_3 \text{ size } 1_l + \theta_4 \text{ size } 2_l + \theta_5 \text{ period}_l + \theta_6 \text{ date}_l + \theta_7 \text{ supply}_l \qquad (l = 1, \ldots, 81). \tag{26}$$

The first three exogenous variables control for the heterogeneity of eggplant cases. There are two sellers. In general, buyers prefer one of them, who has a reputation for better quality. The dummy variable "seller" is assigned the value zero for the preferred seller. There are three official size categories for eggplants: Less than three hundred grams by eggplant, between three and four hundred, and above four hundred. They are introduced through two dummy variables. The variable "size 1" takes the value one for the middle category and zero otherwise, while the variable "size 2" takes the value one for the largest category and zero otherwise.

The last three exogenous variables are market variables. In this market each year prices drop from the beginning of August until the first week of September because of a large supply of substitute goods (other types of summer eggplants). There is accordingly a shock to demand that we accommodate with the dummy variable "period," which takes the value 0 during this period and 1 otherwise. A question of particular interest to the organizer of the auctions is whether eggplant prices increase over time. This is taken into account with the "date" variable, which records the time at which the auction takes place within the year 1990 normalized by 100. Hence June 25th, which is the opening of the season, corresponds to 1.76. The last variable "supply" measures the quantity of eggplants (of all categories) in tons supplied at Marmande that day. It is related to the general state of the market supply in France.

In this market the standard deviation of retail prices for a given auctioned object is roughly equal to 5% of their mean, i.e., two standard deviations represent one franc for a price of ten francs per kilo. Since buyers are agents of retailers and since the distribution of private values is specified to be log-normal, this gives $.05 = \text{std}(v)/E(v) = (e^{\sigma^2} - 1)^{1/2}$, where σ is the standard deviation of $\log v$. Solving for σ, we obtain $\sigma \approx 0.05$, which is maintained throughout our empirical study.[18]

Some statistics describing the data are provided in Appendix B.

4.4. *Estimation Results*

We estimate the structural model associated with the theoretical benchmark model of Section 2.1 by minimizing the criterion (11). We use 20 simulations per auction.[19] For the choice of the importance functions $g_l(\cdot)$, we use the following convenient procedure. For each auction, the importance function $g_l(\cdot)$ is a log-normal density with a mean given by (26) where θ is equal to some preliminary consistent estimate $\tilde{\theta}$ and a standard deviation equal to 0.05 as discussed above.[20]

Because $\tilde{\theta}$ should be close to θ_o, the importance function $g_l(\cdot)$ should be close to $f_l(\cdot)$. Hence from (17) the variance ω_l of the random draws $X_{sl}(\theta)$ at $\theta = \theta_o$ is approximately equal to the (true) variance of $\max[u_{(I-1)l}, p_l^o]$. Thus, because the latter variance is strictly positive, our importance function $g_l(\cdot)$ is not equal to the optimal but infeasible importance function (see Rubinstein (1981, Theorem 4.3.1)). Hence the precision of our estimates could be improved by applying a number of variance reduction techniques (see, e.g., Rubinstein (1981, Section 4.3), Hendry (1984), and Geweke (1988)).

A first step is to determine the number I of bidders, i.e., the actual size of the market. In our sample, there were 11 active buyers, each of whom regularly

[18] We decided to use information on the retail market due to convergence problems when estimating σ with our data. An explanation for this is the choice of the log-normal distribution with the parameterization (26) combined with the fact that reservation prices p_l^o are too low, as suggested by the fact that all auctioned lots are sold. For when private values are log-normally distributed with parameters μ and σ, then $E[b^w] = e^{\mu} r(\sigma, \log p^o - \mu)$, where $r(\cdot,\cdot)$ is a function that involves integrals (see footnote 20). Thus, when p^o is zero and μ is parameterized as in (26), then $E[b^w] = e^{\mu} r(\sigma, -\infty)$ so that the constant term in μ and σ are not separately identified. Alternatively, we could have used another parameterization for μ and σ such as $E[v] = e^{z'\beta}$ and $\text{Var}[v] = \text{constant}$.

[19] Results are analogous with 30 simulations. Laroque and Salanié (1989) used a similar number of simulations for a simulation method which is consistent when the number of simulations must also increase to infinity.

[20] Taking advantage of the log-normal specification and parameterization (26), our preliminary estimate $\tilde{\theta}$ minimizes (11), where $E[b_l^w]$ is simulated with standard normal random draws using $E[b_l^w] = e^{\mu_l} r(\sigma, \log p_l^o - \mu_l)$ with $\mu_l = z_l' \beta$,

$$r(\sigma, \alpha) = \int_R \cdots \int_R \exp\left[\max(\sigma u_{I-1)}, \alpha)\right] \phi(u_1) \ldots \phi(u_I)\, du_1 \ldots du_I,$$

and $\phi(\cdot)$ is the density of the univariate standard distribution. By Proposition 1, such an estimator is consistent.

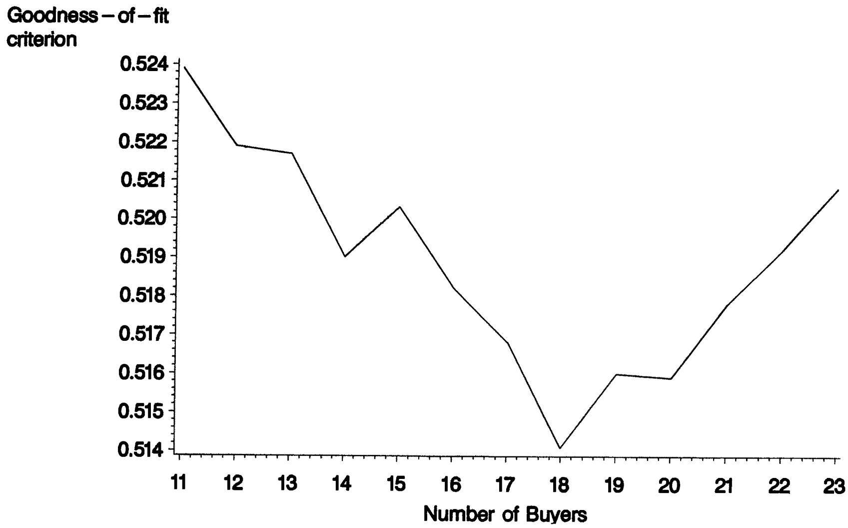

FIGURE 1.—Size of the market.

attended the auction. Thus, between $I = 11$ and $I = 23$, we search for the best value of I in terms of our lack-of-fit criterion $(1/L)Q^*_{S,L,I}(\hat{\theta}_I)$ (see Proposition 4). Figure 1 suggests that this value is obtained for $I = 18$.[21] This result lends support to our second hypothesis that the large buyer is an agent of a number of traders. Specifically $I = 18$ implies that this buyer represents 8 such traders, which is consistent with his share of the market (slightly less than one half). As long as this buyer is just an intermediary who bids according to the 8 traders' instructions determined independently, this does not affect the strategies of the other bidders. See also the conclusion.

Table I displays our empirical results for $I = 11$ and 18. Student statistics computed from Proposition 3 are shown in parentheses and the criterion value is the quantity $(1/L)Q^*_{S,L,I}(\hat{\theta}_I)$. Given the choice of the log-normal distribution and the parameterization (26), each parameter estimate of Table I can be interpreted as the percentage change of the expected value of the auctioned lot because $\theta_k = (1/E_l[V]\ \partial E_l[V]/\partial x_{kl}$. For instance, when $I = 18$, the midsize eggplants and the large size eggplants are respectively 24% and 12% more valuable then the small size eggplants. This agrees with the conventional wisdom that midsize eggplants are the highest quality.

Table I shows some slight differences between the coefficient estimates for $I = 11$ and 18. All parameters have the expected signs. The seller parameter identifies correctly the seller who is known to be of higher quality. The signs for the size dummies agree with conventional wisdom, as noted previously. The significantly positive coefficient for the period dummy agrees with a demand

[21] In principle, the significance of the difference between $I = 11$ and $I = 18$ can be tested but this requires the development of nonnested tests (following Vuong (1989)), which is outside the scope of this paper.

TABLE I

Variables	Parameters: First Model	Second Model
Number of Buyers (I)	11	18
Number of Simulations (S)	20	20
Number of Auctions (L)	81	81
Constant	0.1297	0.0286
	(0.02)	(0.06)
Seller	−0.0107	−0.0240
	(−0.17)	(−0.51)
Size 1	0.2402	0.2402
	(3.57)	(4.39)
Size 2	0.1373	0.1213
	(1.39)	(1.60)
Period	1.2404	1.1998
	(2.16)	(2.90)
Date	0.3115	0.3202
	(3.04)	(4.03)
Supply	−0.0340	−0.0357
	(−0.59)	(−0.81)
Criterion Value	0.52395	0.51401

shock due to the supply of a substitute good during the period coded zero. The trend coefficient is significantly positive. The supply coefficient is negative, although not significant. The lack of significance of the supply variable might be explained as follows. This variable is the local supply and not the global supply on this market. The final buyers buy on several markets so that the idiosyncratic local shocks in this market do not affect their willingness to pay. The global movement of supply is already taken into account by the period and date variables.

Concerning the goodness-of-fit of our estimated model for $I = 18$, our explanatory variables enable us to track closely the winning bids b_l^w by the simulated winning bids $\overline{X}_l(\hat{\theta})$ (see Figure 2). In view of Proposition 1, an R^2 measure can be computed as $1 - Q_{S,L}^*(\hat{\theta})/\hat{\text{var}}\, b^w$. This gives $R^2 = .895$.[22]

5. CONCLUSION

The major contribution of this paper is to describe a new research strategy for analyzing auction data sets. Using a simulated NLLS estimation method we have shown that a traditional structural econometric approach can be employed.

[22] In principle, the log-normality of the private value distribution can also be tested. For instance, we can nest the log-normal family in $\{f^*(\cdot|z, \theta, \mu) = (1-\mu)f(\cdot|z, \theta) + \mu h(\cdot|z)\colon \ \mu \in \boldsymbol{R}\}$, where $h(\cdot|z)$ is a fixed distribution. The condition $\int v h(v|z)\,dv = 0$ may be imposed so that the conditional mean of v given z remains as specified in (26). Then standard classical tests of $\mu = 0$ can be used.

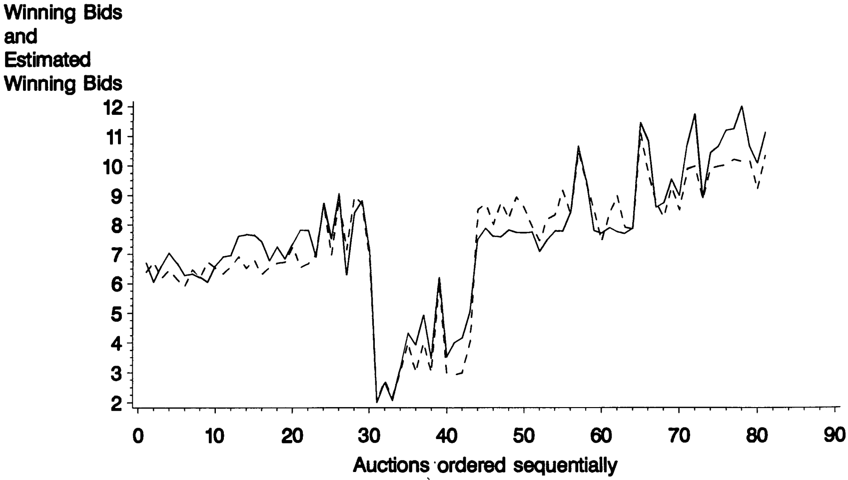

FIGURE 2.—Winning bids (continuous line) and estimated winning bids (dotted line).

We have developed the statistical tools necessary for the analysis of first-price or Dutch auctions with independent private values. Clearly similar methods can be developed for other types of auctions such as the English auction and the common value paradigm.

Many directions of research are possible with our statistical approach. They include:

—An analysis of the optimality of the observed reservation prices so as to provide a pricing strategy based on (4) that would enable the sellers to increase their expected income.[23]

—Tests of the gains provided by the game theoretic approach to auctions as opposed to nonstrategic approaches. For example the game theoretic approach can be confronted with a bounded rationality approach, where bid functions are restricted to be linear or quadratic. Such tests have been carried out in experimental economics where the distribution of characteristics is known.

—Tests of the strategic behavior of the large trader. This is explored in Laffont and Vuong (1994). In our case there is a natural potential coalition of traders. This is a modest step towards the more ambitious program of testing the existence of collusive behavior among buyers in general. In our view, this program still suffers from the weakness of economic theory in describing coalitional behavior under incomplete information. The step we propose raises a whole set of difficult issues associated with asymmetric auctions.[24]

[23] This is an important practical issue. Because our estimation method does not require sellers to behave according to (4), a Hausman test of the optimality of observed reservation prices can be derived by comparing our parameter estimate of θ with its parameter estimate obtained by simulation-based joint estimation of (2) and (4) (see footnote 1).

[24] These issues are not trivial because the asymmetry created by the large trader prevents the analytic solution for the optimal bidding strategy. Numerical integration of differential equations has to be part of the estimation procedure.

—Tests of the most relevant informational model as in practice many auctions seem to be somewhere between the private values paradigm and the common value paradigm (see also Paarsch (1992)).

—Comparison with other auctions (secret reservation price, auctions that maximize social welfare, etc.) and evaluation of the gain for the seller from a first-price auction with an optimal reservation price.

It may also be necessary to extend our analysis to take into account dynamic elements. First, auctions of similar objects are often repeated and this raises issues of reputation as well as strategic dynamic behavior because buyers may adopt strategies that do not reveal their private values during the first auctions. Within a given auction, there is also a dynamic feature worth exploring. The reservation price set by the seller (in conjunction with the organizer of the market) may be chosen on the basis of better information than buyers' information. Then a two-stage game in which the buyers try to infer the seller's information from the reservation price would be appropriate. The reservation price then enters in the structural model in two ways: by truncating the distribution of private valuations as above but also by modifying directly the distribution of private valuations. From Laffont and Maskin (1990) we know that such models exhibit a large number of Bayesian perfect equilibria. Then our approach could be used to select the most relevant equilibrium.

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APPENDIX A

Throughout, we maintain that $E[b_l^w - m(x_l, \theta_o)|x_l, \mathscr{F}^{l-1}] = 0$, where $\mathscr{F}^{l-1}$ denotes the past values of (b_l^w, x_l). Thus $E[b_l^w|x_l, \mathscr{F}^{l-1}] = E[b_l^w|x_l]$ and there is no dynamic misspecification. In addition, we assume that the conditions of Gallant and White (1988) hold.

PROOF OF PROPOSITION 1: Let $q_{S,l}^*(\theta)$ denote the term in brackets in (11). We have

$$Q_{S,L}^*(\theta) - \frac{1}{L}\sum_{l=1}^{L} E[q_{S,l}^*(\theta)] \xrightarrow{\text{a.s.}} 0$$

uniformly on Θ (see Gallant and White (1988, Theorem 3.18)). But, omitting the subscript l whenever there is no confusion,

$$\begin{aligned} E[q_S^*(\theta)] &= E[b^w - \bar{X}(\theta)]^2 - \frac{1}{S(S-1)}E\left[\sum_{s=1}^{S}\left(X_s(\theta) - \bar{X}(\theta)\right)^2\right] \\ &= E[b^w - m(x, \theta_o)]^2 + E[m(x, \theta_o) - \bar{X}(\theta)]^2 - \frac{1}{S}E\operatorname{Var}_x X_s(\theta) \\ &= E[b^w - m(x, \theta_o)]^2 + E[m(x, \theta_o) - m(x, \theta)]^2, \end{aligned}$$

where $\mathrm{Var}_x(\cdot)$ denotes the variance conditional on x. The second equality follows from $E[b^w|x]=m(x,\theta_o)$, the unbiased estimation of $\mathrm{Var}_x\,\bar{X}_s(\theta)$, and $E[b^w-m(x,\theta_o)][\bar{X}(\theta)-m(x,\theta_o)]=0$, which follows from the conditional independence of b_l^w and the simulations $u_l^s\equiv(u_{1l}^s,\dots,u_{Il}^s)$ given x_l. The third equality follows from $E[\bar{X}(\theta)|x]=m(x,\theta)$, which holds by construction. Hence

$$\text{(A.1)}\qquad Q_{S,L}^*(\theta)-\frac{1}{L}\sum_{l=1}^{L}\left(E[b_l^w-m(x_l,\theta_o)]^2+E[m(x_l,\theta_o)-m(x_l,\theta)]^2\right)\xrightarrow{\text{a.s.}}0$$

uniformly in θ. Thus $\theta=\theta_o$ minimizes the second term. Hence, provided θ_o is identifiably unique (see Gallant and White (1988, Definition 3.2)), then $\hat{\theta}$ is a strongly consistent estimator of θ_o as $L\to\infty$, for any fixed S.

The second term in (A.1) is nothing else than the limit of the usual NLLS objective function $QL(\theta)=(1/L)\Sigma_{l=1}^{L}(b_l^w-m_l(\theta))^2$. Hence θ_o is identifiably unique if it is identified from the first moment of b^w, i.e., if it is first-order identified. Moreover, it follows that $Q_{S,L}^*(\theta)-EQ_L(\theta)\xrightarrow{\text{a.s.}}0$ uniformly in θ. Hence $Q_{S,L}^*(\hat{\theta})-EQ_L(\theta_o)\xrightarrow{\text{a.s.}}0$ as $L\to\infty$, for any fixed S.

Proof of Proposition 2: First, we note that $\partial q_{S,l}^*(\theta)/\partial\theta=-2d_{S,l}(\theta)$ as defined in (23). From Gallant and White (1988, Theorem 5.7) it follows that $\Sigma_{S,L}^{-1/2}\sqrt{L}(\hat{\theta}-\theta_o)\xrightarrow{\text{d}}N(0,I_k)$ as $L\to\infty$, for any fixed S, where $\Sigma_{S,L}=\mathscr{A}_{S,L}^{-1}\mathscr{B}_{S,L}\mathscr{A}_{S,L}^{-1}$ and

$$\text{(A.2)}\qquad \mathscr{A}_{S,L}=-\frac{2}{L}\sum_{l=1}^{L}E\left[\frac{\partial d_{S,l}(\theta_o)}{\partial\theta'}\right],$$

$$\text{(A.3)}\qquad \mathscr{B}_{S,L}=\mathrm{Var}\left[\frac{2}{\sqrt{L}}\sum_{l=1}^{L}\left(d_{S,l}(\theta_o)-E[d_{S,l}(\theta_o)]\right)\right].$$

It remains to show that $\mathscr{A}_{S,L}=2A_L$ (independent of S) and $\mathscr{B}_{S,L}=4B_{S,L}$, where A_L and $B_{S,L}$ are defined by (18) and (19) respectively.

Consider $\mathscr{A}_{S,L}$. Differentiating $d_{S,l}(\theta)$ and using $Y_{sl}(\theta)=\partial X_{sl}(\theta)/\partial\theta$, we obtain $\mathscr{A}_{S,L}=-2(A_1+A_2+A_3+A_4)$, where

$$A_1=-\frac{1}{L}\sum_{l=1}^{L}E[\bar{Y}_l(\theta_o)\bar{Y}_l(\theta_o)'],$$

$$A_2=\frac{1}{L}\sum_{l=1}^{L}E\left[\left(b_l^w-\bar{X}_l(\theta_o)\right)\frac{\partial\bar{Y}_l(\theta_o)}{\partial\theta'}\right],$$

$$A_3=\frac{1}{L}\sum_{l=1}^{L}E\left[\frac{1}{S(S-1)}\sum_{s=1}^{S}Y_{sl}(\theta_o)\left(Y_{sl}(\theta_o)-\bar{Y}_l(\theta_o)\right)'\right],$$

$$A_4=\frac{1}{L}\sum_{l=1}^{L}E\left[\frac{1}{S(S-1)}\sum_{s=1}^{S}\left(X_{sl}(\theta_o)-\bar{X}_l(\theta_o)\right)\frac{\partial Y_{sl}(\theta_o)}{\partial\theta'}\right].$$

Now, for each l, conditionally upon x_l, $(X_{sl}(\theta_o),Y_{sl}(\theta_o))$, $s=1,\dots,S$, are (i) iid and (ii) independent of b_l^w. Thus, first taking conditional expectation given x_l and using $E[b_l^w|x_l]=E[X_{sl}(\theta_o)|x_l]=m(x_l,\theta_o)$, we obtain

$$\begin{aligned}A_1&=-\frac{1}{L}\sum_{l=1}^{L}\left(E[\mathrm{Var}_x\,\bar{Y}_l(\theta_o)]-E[E_x\bar{Y}_l(\theta_o)E_x\bar{Y}_l'(\theta_o)]\right)\\&=-\frac{1}{L}\sum_{l=1}^{L}\left(\frac{1}{S}E[\mathrm{Var}_x\,Y_{sl}(\theta_o)]-E[E_xY_{sl}(\theta_o)E_xY_{sl}'(\theta_o)]\right),\end{aligned}$$

$$A_2 = \frac{1}{L}\sum_{l=1}^{L} E\left[E_x\left(\left(m(x_l,\theta_o) - \bar{X}_l(\theta_o)\right)\frac{\partial \bar{Y}_l(\theta_o)}{\partial \theta'}\right)\right]$$
$$= -\frac{1}{L}\sum_{l=1}^{L}\frac{1}{S}E\left[\mathrm{Cov}_x\left(X_{sl}(\theta_o), \frac{\partial Y_{sl}(\theta_o)}{\partial\theta'}\right)\right],$$
$$A_3 = \frac{1}{L}\sum_{l=1}^{L}\frac{1}{S}E[\mathrm{Var}_x\, Y_{sl}(\theta_o)],$$
$$A_4 = \frac{1}{L}\sum_{l=1}^{L}\frac{1}{S}E\left[\mathrm{Cov}_x\left(X_{sl}(\theta_o), \frac{\partial Y_{sl}(\theta_o)}{\partial\theta'}\right)\right].$$

Cancelling terms, it follows that $\mathscr{A}_{S,L} = 2A_L$ as desired, using $E_x Y_{sl}(\theta_o) = \partial E_x X_{sl}(\theta_o)/\partial\theta = \mu_l(\theta_o)$ by interverting differentiation and expectation.

We now turn to $\mathscr{B}_{S,L}$. Dropping the l subscript, we have

$$\text{(A.4)}\qquad d_S(\theta_o) = (b^w - \mu_X^o)\bar{Y}^o - (\bar{X}^o - \mu_X^o)\mu_Y^o - \frac{S}{S-1}(\bar{X}^o - \mu_X^o)(\bar{Y}^o - \mu_Y^o)$$
$$+ \frac{1}{S(S-1)}\sum_{s=1}^{S}(X_s^o - \mu_X^o)(Y_s^o - \mu_Y^o)$$
$$= d_1 - d_2 - d_3 + d_4,$$

where $X_x^o \equiv X_s(\theta_o)$, $Y_s^o \equiv Y_s(\theta_o)$, $\mu_X^o \equiv m(x,\theta_o)$, and $\mu_Y^o \equiv \partial m(x,\theta_o)/\partial\theta (= \mu^o)$. Taking conditional expectation given x, we have $E_x[d_1] = E_x[d_2] = 0$ and

$$E_x[d_3] = \frac{S}{S-1}\mathrm{Cov}_x(\bar{X}^o, \bar{Y}^o) = \frac{1}{S-1}\mathrm{Cov}_x(X_s^o, Y_s^o),$$
$$E_x[d_4] = \frac{1}{S-1}\mathrm{Cov}_x(X_s^o, Y_s^o).$$

Hence $E_x[d_S(\theta_o)] = 0$. In fact, by construction of the random draws and because there is no dynamic misspecification, we have $E[d_{Sl}(\theta_o)|\mathscr{F}^{l-1}, \mathscr{U}^{l-1}] = 0$, where $\mathscr{U}^{l-1}$ denotes the past values of the random draws. Hence $\{d_{S,l}(\theta_o), (\mathscr{F}^{l-1}, \mathscr{U}^{l-1})\}$ is a martingale difference. Thus all the covariance terms in (A.3) disappear. Then, because $\mathrm{Var}[d_S(\theta_o)] = E[\mathrm{Var}_x\, d_S(\theta_o)]$, (A.3) reduces to

$$\text{(A.5)}\qquad \mathscr{B}_{S,L} = \frac{4}{L}\sum_{l=1}^{L}E\left[\mathrm{Var}_x\left(d_{S,l}(\theta_o)\right)\right].$$

It remains to compute $\mathrm{Var}_x[d_S(\theta_o)]$. Using (A.4), we have $\mathrm{Cov}_x(d_1, d_j) = E_x[(b^w - \mu_X^o)\bar{Y}^o d_j'] = 0$, for every $j = 2, 3, 4$, and

$$\mathrm{Cov}_x(d_2, d_3) = \frac{1}{S^2(S-1)}\mu_Y^o E_x\left[\left(\sum_{s=1}^{S}(X_s^o - \mu_X^o)^2 + 2\sum_{s<t}(X_s^o - \mu_X^o)(X_t^o - \mu_X^o)\right)\left(\sum_{s=1}^{S}(Y_s^o - \mu_Y^o)'\right)\right]$$
$$= \frac{1}{S(S-1)}\mu_Y^o E_x\left[(X_s^o - \mu_X^o)^2(Y_s^o - \mu_Y^o)'\right],$$
$$\mathrm{Cov}_x(d_2, d_4) = \frac{1}{S^2(S-1)}\mu_Y^o E_x\left[\left(\sum_{s=1}^{S}(X_x^o - \mu_X^o)\right)\left(\sum_{s=1}^{S}(X_s^o - \mu_X^o)(\bar{Y}^o - \mu_Y^o)'\right)\right]$$
$$= \frac{1}{S(S-1)}\mu_Y^o E_x\left[(X_s^o - \mu_X^o)^2(\bar{Y}^o - \mu_Y^o)'\right].$$

Thus $\text{Cov}_x(d_2, d_3 - d_4) = 0$. Hence

$$\text{(A.6)} \qquad \text{Var}_x\, d_S(\theta_o) = \text{Var}_x\, d_1 + \text{Var}_x\, d_2 + \text{Var}_x\,(d_3 - d_4).$$

It remains to compute the variance covariance matrices of d_1, d_2, and $(d_3 - d_4)$, conditional on x. Note that these three vectors have zero conditional expectations. Since $\sigma_o^2(x) = E_x[(b^w - \mu_X^o)^2]$, we obtain

$$\text{(A.7)} \qquad \text{Var}_x\, d_1 = \sigma_o^2(x)\left(\frac{1}{S}\text{Var}_x\, Y_s^o + \mu_Y^o \mu_Y^{o\prime}\right),$$

$$\text{(A.8)} \qquad \text{Var}_x\, d_2 = \frac{1}{S}\text{Var}_x\, X_s^o \mu_Y^o \mu_Y^{o\prime}.$$

For the third conditional variance covariance matrix, we have

$$\text{(A.9)} \qquad \text{Var}_x\,(d_3 - d_4) = E_x(d_3 d_3') + E_x(d_4 d_4') - E_x(d_3 d_4') - E_x(d_4 d_3').$$

But

$$\begin{aligned}
\text{(A.10)} \qquad & E_x(d_3 d_3') \\
&= \frac{1}{S^2(S-1)^2} E_x\left[\left(\sum_{s=1}^{S} (X_s^o - \mu_X^o)_s^2 + 2\sum_{s<t} (X_s^o - \mu_X^o)(X_t^o - \mu_X^o)\right)\right. \\
&\qquad \left. \times \left(\sum_{s=1}^{S} (Y_s^o - \mu_Y^o)(Y_s^o - \mu_Y^o)' + 2\sum_{s<t} (Y_s^o - \mu_Y^o)(Y_t^o - \mu_Y^o)'\right)\right] \\
&= \frac{1}{S^2(S-1)^2} E_x\left[\left(\sum_{s=1}^{S} (X_s^o - \mu_X^o)^2 \sum_{s=1}^{S} (Y_s^o - \mu_Y^o)(Y_s^o - \mu_Y^o)'\right)\right. \\
&\qquad \left. + 4\left(\sum_{s<t} (X_s^o - \mu_X^o)(X_t^o - \mu_X^o) \sum_{s<t} (Y_s^o - \mu_Y^o)(Y_t^o - \mu_Y^o)'\right)\right] \\
&= \frac{1}{S^2(S-1)^2} E_x\left[\sum_{s=1}^{S} (X_s^o - \mu_X^o)^2 (Y_s^o - \mu_Y^o)(Y_s^o - \mu_Y^o)'\right. \\
&\qquad + \sum_{s \neq t}^{S} (X_s^o - \mu_X^o)^2 (Y_t^o - \mu_Y^o)(Y_t^o - \mu_Y^o)' \\
&\qquad \left. + 4\sum_{s<t} (X_s^o - \mu_X^o)(X_t^o - \mu_X^o)(Y_s^o - \mu_Y^o)(Y_t^o - \mu_Y^o)'\right] \\
&= \frac{1}{S(S-1)^2} E_x\left[(X_s^o - \mu_X^o)^2 (Y_s^o - \mu_Y^o)(Y_s^o - \mu_Y^o)'\right] \\
&\qquad + \frac{1}{S(S-1)}\left[\text{Var}_x\, X_s^o\, \text{Var}_x\, Y_s^o + 2\,\text{Cov}_x\,(Y_s^o, Y_s^o)\,\text{Cov}_x\,(X_s^o, Y_s^o)'\right],
\end{aligned}$$

$$\begin{aligned}
\text{(A.11)} \qquad E_x(d_4 d_4') &= \frac{1}{S^2(S-1)^2} E_x\left[\sum_{s=1}^{S} (X_s^o - \mu_X^o)^2 (Y_s^o - \mu_Y^o)(Y_s^o - \mu_Y^o)'\right. \\
&\qquad \left. + \sum_{s \neq t} (X_s^o - \mu_X^o)(X_t^o - \mu_X^o)(Y_s^o - \mu_Y^o)(Y_t^o - \mu_Y^o)'\right] \\
&= \frac{1}{S(S-1)^2} E_x\left[(X_s^o - \mu_X^o)^2 (Y_s^o - \mu_Y^o)(Y_s^o - \mu_Y^o)'\right] \\
&\qquad + \frac{1}{S(S-1)}\text{Cov}_x\,(X_s^o, Y_s^o)\,\text{Cov}_x\,(X_s^o, Y_s^o)',
\end{aligned}$$

$$(\text{A.12})\qquad E_x(d_3 d_4') = \frac{1}{S^2(S-1)^2} E_x\Bigg[\Bigg(\sum_{s=1}^{S}(X_s^o-\mu_X^o)(Y_s^o-\mu_Y^o) + \sum_{s\neq t}(X_s^o-\mu_X^o)(Y_t^o-\mu_Y^o)\Bigg) \times\Bigg(\sum_{s=1}^{S}(X_s^o-\mu_X^o)(Y_s^o-\mu_Y^o)'\Bigg)\Bigg]$$

$$= \frac{1}{S^2(S-1)^2} E_x\Bigg[\Bigg(\sum_{s=1}^{S}(X_s^o-\mu_X^o)(Y_s^o-\mu_Y^o)\Bigg) \times\Bigg(\sum_{s=1}^{S}(X_s^o-\mu_X^o)(Y_s^o-\mu_Y^o)'\Bigg)\Bigg]$$

$$= E_x(d_4 d_4').$$

Using (A.10)–(A.12) in (A.9), we obtain $\operatorname{Var}_x(d_3-d_4) = E_x(d_3 d_3') - E_x(d_4 d_4')$, i.e.

$$(\text{A.13})\qquad \operatorname{Var}_x(d_3-d_4) = \frac{1}{S(S-1)}\left[\operatorname{Var}_x X_s^o \operatorname{Var}_x Y_s^o + \operatorname{Cov}_x(X_s^o, Y_s^o)\operatorname{Cov}_x(X_s^o, Y_s^o)'\right].$$

Thus, using (A.6)–(A.8) and (A.13), $\operatorname{Var}_x[d_S(\theta_o)]$ is equal to the term in brackets in (19). Hence, from (A.5), $\mathscr{B}_{S,L} = 4B_{S,L}$ as desired.

Proof of Proposition 3: It suffices to prove that $\hat{A}_{S,L} - A_L \xrightarrow{\text{a.s.}} 0$ and $\hat{B}_{S,L} - B_{S,L} \xrightarrow{\text{a.s.}} 0$. The second convergence is an immediate consequence of (A.5) and Theorem 6.3 in Gallant and White (1988) since $\{d_{S,l}(\theta_o), (\mathscr{F}^{l-1}, \mathscr{U}^{l-1})\}$ is a martingale difference and $E[d_{Sl}(\theta_o)] = 0$ as noted in the preceding proof.

To establish the first convergence, we have as $L \to \infty$, for any fixed S,

$$(\text{A.14})\qquad \hat{A}_{S,L} - \sum_{l=1}^{L} E\Bigg[\bar{Y}_l(\theta_o)\bar{Y}_l(\theta_o)' - \frac{1}{S(S-1)}\sum_{s=1}^{S}\big(Y_{sl}(\theta_o)-\bar{Y}_l(\theta_o)\big)\big(Y_{sl}(\theta_o)-\bar{Y}_l(\theta_o)\big)'\Bigg] \xrightarrow{\text{a.s.}} 0$$

using $\hat{\theta} \xrightarrow{\text{a.s.}} \theta_o$ and Theorem 3.18 in Gallant and White (1988). Then, taking expectation conditional upon x_l and recalling the formula for the usual unbiased estimator of $\operatorname{Var}_x(\bar{Y}_l(\theta_o))$, we obtain $\hat{A}_{S,L} - (1/L)\sum_{l=1}^{L} E[E_x(\bar{Y}_l(\theta_o))E_x(\bar{Y}_l'(\theta_o))] \xrightarrow{\text{a.s.}} 0$. The desired result follows from $\mu_l^o = E_x(\bar{Y}_l(\theta_o))$.

Proof of Proposition 4: We now assume that the conditions for applying Theorem 3.18 in Gallant and White (1988) hold for each I in $\mathscr{I}$. Hence, by the same reasoning leading to (A.1), for each S and $I \in \mathscr{I}$ we have as $L \to \infty$

$$(\text{A.15})\qquad Q_{S,L,I}^*(\theta) - Q_{L,I}(\theta) \xrightarrow{\text{a.s.}} 0,$$

uniformly in θ, where

$$Q_{L,I}(\theta) \equiv \frac{1}{L}\sum_{l=1}^{L}\Big(E[b_l^w - m(x_l, I_o, \theta_o)]^2 + E[m(x_l, I_o, \theta_o) - m(x_l, I, \theta)]^2\Big).$$

Here $E[b_l^w|x_l] = m(x_l, I_o, \theta_o)$ and $E[X_{I,sl}(\theta)|x_l] = m(x_l, I, \theta)$, where $X_{I,sl}(\theta)$ is defined by (17) when I is the number of bidders.

Since $\mathscr{I}$ is finite, then (A.15) holds uniformly in (I, θ). Moreover, (I_o, θ_o) clearly minimizes the second term of (A.15) with respect to (I, θ). On the other hand, $(\hat{I}, \hat{\theta}_{\hat{I}})$ as defined in the text minimizes $Q_{S,L,I}^*(\theta)$ with respect to (I, θ). Hence, provided (I_o, θ_o) is identified from the first moment of b^w, then $(\hat{I}, \hat{\theta}_{\hat{I}})$ converges almost surely to (I_o, θ_o) as $L \to \infty$, for any fixed S.

As a matter of fact, the proof of the later statement does not follow from standard argument because $Q_{L,I}(\theta)$ is not continuous in I. However, because $Q_{L,I}(\theta)$ is continuous in θ for every I, and because of the above uniform convergence, we have $Q^*_{S,L,I_o}(\hat{\theta}_{I_o}) - Q_{L,I_o}(\theta_o) \xrightarrow{\text{a.s.}} 0$ and $\min_{I \in \mathcal{I}-\{I_o\}} Q^*_{S,L,I}(\hat{\theta}_I) - \min_{I \in \mathcal{I}-\{I_o\}} \min_\theta Q_{L,I}(\theta) \xrightarrow{\text{a.s.}} 0$. Because (I_o, θ_o) is identifiably unique by assumption, then $Q_{L,I_o}(\theta_o) < \min_{I \in \mathcal{I}-\{I_o\}} \min_\theta Q_{L,I}(\theta)$ for L sufficiently large a.s. Hence, $\hat{I} = I_o$ and $\hat{\theta}_{\hat{I}} = \hat{\theta}_{I_o}$ for L sufficiently large a.s. The desired result follows.

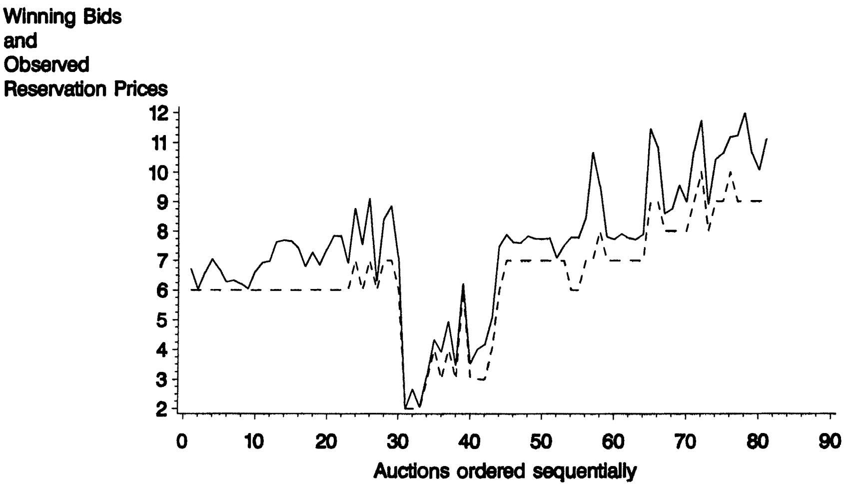

FIGURE 3.—Winning bids (continuous line) and reservation prices (dotted line).

APPENDIX B

Summary Statistics

Bids and reservation prices are measured in Francs per Kilo. The "supply" variable is measured in tons. The "date" variable and the qualitative variables are defined in Section 4.3.

NUMBER OF AUCTIONS: 81
Quantitative Variables

Variables	Mean	Standard Deviation	Maximum	Minimum
Bid	7.52	2.21	11.98	2.00
Reservation Price	6.43	1.81	10.00	2.00
Date	2.40	0.41	3.00	1.76
Supply	1.39	0.67	3.49	0.13

Qualitative Variables

Dummy	Number of Items Equal to 0	Equal to 1
Seller	62	19
Size 1	71	10
Size 2	47	34
Period	13	68

Correlation Matrix

	Bid	Reserv	Seller	Size 1	Size 2	Period	Date	Supply
Bid	1.000							
Reservation Price	0.962	1.000						
Seller	−0.393	−0.389	1.000					
Size 1	0.001	−0.090	0.324	1.000				
Size 2	0.259	0.267	−0.412	−0.319	1.000			
Period	0.736	0.773	−0.472	−0.245	0.167	1.000		
Date	0.479	0.481	0.072	−0.059	0.131	−0.011	1.000	
Supply	−0.015	0.041	−0.040	−0.376	0.231	−0.059	0.341	1.000

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