ECON-GA 1025 Macroeconomic Theory I

Lecture 3

John Stachurski

Fall Semester 2018

Today's Lecture

- Neumann series theorem
- Applications to finite state asset pricing
- Metric spaces
- Contractions and Banach's theorem
- Back to asset pricing

The Neumann Series Theorem

Let $A \in \mathcal{M}(n \times n)$ and let I be the $n \times n$ identity

The **Neumann series theorem** states that if $r(A) < 1$, then $I - A$ is nonsingular and

$$(I - A)^{-1} = \sum_{i=0}^{\infty} A^i \tag{1}$$

Example. If $r(A) < 1$, then $x = Ax + b$ has the unique solution

$$x^* = \sum_{i=0}^{\infty} A^i b$$

Full proof of the NSL: See the course notes

To show that (1) holds we can prove that $(I - A)\sum_{i=0}^{\infty} A^i = I$

This is true, since

$$\left\|(I-A)\sum_{i=0}^{\infty} A^i - I\right\| = \left\|(I-A)\lim_{n\to\infty}\sum_{i=0}^{n} A^i - I\right\|$$

$$= \lim_{n\to\infty}\left\|(I-A)\sum_{i=0}^{n} A^i - I\right\|$$

$$= \lim_{n\to\infty}\left\|A^{n+1}\right\| = 0$$

Application: Finite State Asset Pricing

An asset is a claim to anticipated future economic benefit

Example. Stocks, bonds, housing

Example. A friend asks if he can borrow \$100

If you agree, then you are purchasing an asset

Risk Neutral Prices

What is the time t price of a stochastic payoff G_{t+1} ?

The **risk neutral price** is

$$p_t = \beta \mathbb{E}_t G_{t+1}$$

More generally, the price of G_{t+n} at $t+n$ is

$$p_t = \beta^n \mathbb{E}_t G_{t+n}$$

Example. European call option that expires in n periods with strike price K has price

$$p_t = \beta^n \mathbb{E}_t \max\{S_{t+n} - K, 0\}$$

Pricing Dividend Streams

Now let's price the dividend stream $\{d_t\}$

We will price an **ex dividend** claim

- a purchase at time t is a claim to $d_{t+1}, d_{t+2}, \ldots$
- we seek p_t given β and these payoffs

The **risk-neutral price** satisfies

$$p_t = \beta \, \mathbb{E}_t \left(d_{t+1} + p_{t+1} \right)$$

That is, cost $=$ expected benefit, discounted to present value

A recursive expression with no natural termination point...

To solve

$$p_t = \beta \, \mathbb{E}_t \left(d_{t+1} + p_{t+1} \right)$$

let's assume that

- $d_t = d(x_t)$ for some nonnegative function d
- $\{x_t\}$ is a **Markov chain** on some **finite** set X with $|\mathsf{X}| = n$
- $\Pi(x, y) := \mathbb{P}\{x_{t+1} = y \,|\, x_t = x\}$

We **guess** there is a solution of the form $p_t = p(x_t)$ for some function p

Thus, our aim is to find a p satisfying

$$p(x_t) = \beta \, \mathbb{E}_t \left[d(x_{t+1}) + p(x_{t+1}) \right]$$

Equivalent: we seek a p with

$$p(x) = \beta \, \mathbb{E}_t \left[d(x_{t+1}) + p(x_{t+1}) \,|\, x_t = x \right]$$

for all $x \in \mathsf{X}$

Equivalent: for all $x \in \mathsf{X}$,

$$p(x) = \beta \sum_y \left[d(y) + p(y) \right] \Pi(x, y)$$

This is a **functional equation** in p

But also a **vector equation** in p, since X is finite!

Let's stack these equations:

$$p(x_1) = \beta \sum_y [d(y) + p(y)] \, \Pi(x_1, y)$$

$$\vdots$$

$$p(x_n) = \beta \sum_y [d(y) + p(y)] \, \Pi(x_n, y)$$

Treating $p = (p(x_1), \ldots, p(x_n))$ and $d = (d(x_1), \ldots, d(x_n))$ as column vectors, this is equivalent to

$$p = \beta \Pi d + \beta \Pi p$$

Does this have a unique solution and, if so, how can we find it?

Since Π a stochastic matrix we have $r(\Pi) = 1$

Hence $r(\beta\Pi) = \beta < 1$

Neumann series theorem implies that $p = \beta\Pi d + \beta\Pi p$ has the unique solution

$$p^* = (I - \beta\Pi)^{-1}\beta\Pi d = \sum_{i=1}^{\infty}(\beta\Pi)^i d$$

In particular, $p_t = p^*(x_t)$ is the risk-neutral price of the asset

Ex. Let u be a one period utility function and let lifetime value of consumption stream $\{c_t\}$ be defined recursively by

$$v_t = u(c_t) + \beta \mathbb{E}_t v_{t+1}$$

Assume that $\beta \in (0, 1)$ and, in addition

- $c_t = c(x_t)$ for some nonnegative function d
- $\{x_t\}$ is a Markov chain on finite set X with $|\mathsf{X}| = n$
- $\Pi(x, y) := \mathbb{P}\{x_{t+1} = y \,|\, x_t = x\}$

Guess there is a solution of the form $v_t = v(x_t)$ for some function v

Derive an expression for v using Neumann series theory

An Uncountable State Space

Now let's try to solve

$$p_t = \beta \, \mathbb{E}_t \left(d_{t+1} + p_{t+1} \right)$$

again but with

- $d_t = d(x_t)$ for some nonnegative function d
- x_t takes values in $\mathbb{R}$ with $x_{t+1} = F(x_t, \xi_{t+1})$
- $\{\xi_t\}$ is IID with common distribution φ

Example. $x_{t+1} = a\, x_t + b + \sigma \xi_{t+1}$ with $\{\xi_t\} \overset{\text{IID}}{\sim} N(0, 1)$

We guess a solution of the form $p_t = p(x_t)$ for some function p

Now the unknown p is a function on $\mathbb{R}$

It solves the **functional equation**

$$p(x) = \beta \int [d(F(x,z)) + p(F(x,z))] \, \varphi(dz) \qquad (x \in \mathbb{R})$$

Can we prove existence of a solution?

Uniqueness?

If so, how to compute the solution?

We cannot use any previous results because p is not a finite vector

Need a more general approach...

The approach in a nutshell

1. Introduce metric spaces
2. Introduce operators, fixed points and contractions
3. Show that contractive operators have unique fixed points
   - Banach's contraction mapping theorem
4. Frame the asset pricing functional equation as a fixed point problem
   - Solutions to functional eq = fixed points of a **pricing operator**
5. Show the contraction property of the pricing operator
6. Conclude existence of unique solution

Metric Space

Let M be any nonempty set

A function $\rho\colon M \times M \to \mathbb{R}$ is called a **metric** on M if, for any $u, v, w \in M$,

1. $\rho(u,v) \geqslant 0$ with $\rho(u,v) = 0 \iff u = v$
2. $\rho(u,v) = \rho(v,u)$
3. $\rho(u,v) \leqslant \rho(u,w) + \rho(w,v)$

Together, the pair (M, ρ) is called a **metric space**

Example. $(\mathbb{R}^d, \rho)$ with $\rho(u,v) := \|u - v\|$ is a metric space

Let X be any set and let $b\mathsf{X}$ be all bounded functions in $\mathbb{R}^{\mathsf{X}}$

For all f, g in $b\mathsf{X}$, the pair $(b\mathsf{X}, d_\infty)$ is a metric space when

$$\|f\|_\infty := \sup_{x \in \mathsf{X}} |f(x)| \quad \text{and} \quad d_\infty(f,g) := \|f - g\|_\infty$$

Triangle inequality: given f, g, h in $b\mathsf{X}$, we have

$$\begin{aligned} |f(x) - g(x)| &= |f(x) - h(x) + h(x) - g(x)| \\ &\leqslant |f(x) - h(x)| + |h(x) - g(x)| \\ &\leqslant d_\infty(f,h) + d_\infty(h,g) \end{aligned}$$

$$\therefore \quad d_\infty(f,g) \leqslant d_\infty(f,h) + d_\infty(h,g)$$

Let X be any countable set, fix $p \geqslant 1$ and define

$$\|h\|_p := \left\{ \sum_{x \in \mathsf{X}} |h(x)|^p \right\}^{1/p} \quad \text{and} \quad d_p(g,h) = \|g - h\|_p$$

Now set

$$\ell_p(\mathsf{X}) := \left\{ h \in \mathbb{R}^{\mathsf{X}} \,:\, \|h\|_p < \infty \right\}$$

The pair $(\ell_p(\mathsf{X}), d_p)$ is a metric space

The triangle inequality (in this case, the **Minkowski inequality**) follows from the **Hölder inequality**

$$\|fg\|_1 \leqslant \|f\|_p \, \|g\|_q \quad \text{whenever } p, q \in [1, \infty] \text{ with } 1/p + 1/q = 1$$

Example. If $\mathsf{X} = \{x_1, \dots, x_d\}$ and $p = 2$, then

$$\|h\|_p := \left\{ \sum_{x \in \mathsf{X}} |h(x)|^p \right\}^{1/p}$$

$$= \left\{ \sum_{i=1}^{d} |h(x_i)|^2 \right\}^{1/2}$$

$$= \text{Euclidean norm of } h$$

(Remember that h is identified with the vector $(h(x_1), \dots, h(x_d))$)

In particular, $(\ell_2(\mathsf{X}), d_2)$ "is" regular Euclidean space for such X

The case $p = +\infty$ is also admitted, with

$$\|h\|_\infty := \sup_{x \in \mathsf{X}} |h(x)|$$

Then $\ell_\infty(\mathsf{X}) = \{h \in \mathbb{R}^{\mathsf{X}} : \|h\|_\infty < \infty\}$

This space $\ell_\infty(\mathsf{X})$ coincides with $b\mathsf{X}$ when X is countable

For any $h \in \ell_\infty(\mathsf{X})$ with X finite we have

$$\|h\|_\infty = \lim_{p \to \infty} \|h\|_p$$

Let (M, ρ) be any metric space

Given any point $u \in M$, the **ϵ-ball** around u is the set

$$B_\epsilon(u) := \{v \in M : \rho(u, v) < \epsilon\}$$

A point $u \in G \subset M$ is called **interior** to G if there exists an ϵ-ball $B_\epsilon(u)$ such that $B_\epsilon(u) \subset G$

A set G in M is called **open** if all of its points are interior to G

A set F in M is called **closed** if F^c is open

A sequence $\{u_n\} \subset M$ is said to **converge to $u \in M$** if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geqslant N \implies u_n \in B_\epsilon(u)$$

Completeness

A sequence $\{u_n\} \subset M$ is called **Cauchy** if, given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $n, m \geqslant N$ implies $\rho(u_n, u_m) < \epsilon$

Ex. Show that if $M = \mathbb{R}$, $\rho(u, v) = |u - v|$ and $u_n = 1/n$, then $\{u_n\}$ is Cauchy.

A metric space (M, ρ) is called **complete** if every Cauchy sequence in M converges to some point in M

Under completeness, sequences that "look convergent" do in fact converge to some point in the space

Examples.

- Ordinary Euclidean space $(\mathbb{R}^d, \|\cdot\|)$ is complete
- $(b\mathsf{X}, d_\infty)$ is complete for any choice of X
- $(\ell_p(\mathsf{X}), d_p)$ is complete for any discrete X
- If $M = (0, 1]$ and $\rho(u, y) = |u - y|$, then (M, ρ) is *not* complete

Let (M, ρ) be any metric space

Fact. If $F \subset M$ is closed in M, then (F, ρ) is complete

Example. Let X be a metric space and let $bc\mathsf{X} :=$ all continuous functions in $(b\mathsf{X}, d_\infty)$

This set is closed because uniform limits of continuous functions are continuous

Hence $(bc\mathsf{X}, d_\infty)$ is complete

Fixed Points and Contractions

Let (M, ρ) be a metric space

A map T from M to itself is called a **self-mapping** on M

A point $x \in M$ is called a **fixed point** of T if $Tx = x$

There can be none, one or many...

Examples.

- If $f \colon \mathbb{R} \to \mathbb{R}$ is the identity $f(x) = x$, then every $x \in \mathbb{R}$ is a fixed point
- If $f \colon \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x + 1$, then no $x \in \mathbb{R}$ is a fixed point

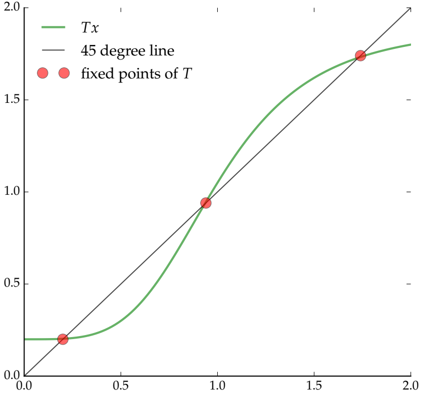
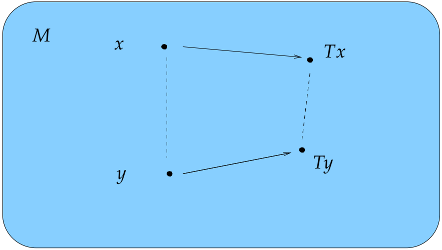


Figure: Fixed points in one dimension

Contractions

Self-mapping T on (M, ρ) is called a **contraction mapping with modulus λ** if

$$\exists\, \lambda < 1 \quad \text{s.t.} \quad \rho(Tx, Ty) \leqslant \lambda \rho(x, y) \quad \text{for all} \quad x, y \in M$$



Example. The nicest case: $Tx = ax + b$ on $\mathbb{R}$ where a and b are parameters

For any $x, y \in \mathbb{R}$ we have

$$\begin{aligned} |Tx - Ty| &= |ax + b - ay - b| \\ &= |ax - ay| \\ &= |a(x - y)| \\ &= |a||x - y| \end{aligned}$$

Hence $|a| < 1 \implies T$ is a contraction mapping on $\mathbb{R}$

Banach Contraction Mapping Theorem

Fact. If M is complete and T is a contraction mapping on M then

1. T has a unique fixed point $\bar{x} \in M$
2. $T^n x \to \bar{x}$ as $n \to \infty$ for any $x \in M$

Proof of uniqueness: Suppose that $x, y \in M$ with

$$Tx = x \quad \text{and} \quad Ty = y$$

Then

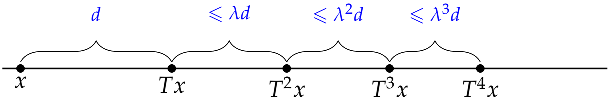
$$\rho(x, y) = \rho(Tx, Ty) \leqslant \lambda \rho(x, y)$$

Since $\lambda < 1$, it must be that $\rho(x, y) = 0$, and hence $x = y$

Sketch of existence proof: Fix $x \in M$ and let

$$d := \rho(Tx, x)$$

It can be shown that $\rho(T^{n+1}x, T^n x) \leqslant \lambda^n d$ for all n



One can then show that $\{x_n\} := \{T^n x\}$ is Cauchy

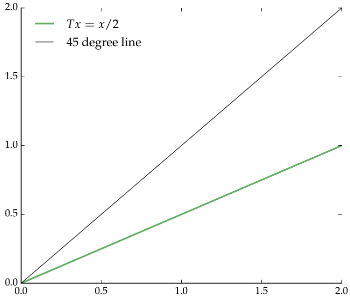
The Cauchy property implies convergence to some $\bar{x} \in M$

It can then be shown that $\bar{x}$ is a fixed point

By the way, why does M need to be complete?

An example of failure when M is not complete:

$$Tx = x/2 \quad \text{and} \quad M = (0, \infty)$$



Back to Asset Pricing

Recall that we wanted to solve for $\{p_t\}$ in

$$p_t = \beta \, \mathbb{E}_t \left(d_{t+1} + p_{t+1} \right)$$

Here $\beta \in (0, 1)$,

- $d_t = d(x_t)$ for some nonnegative function d
- $x_{t+1} = F(x_t, \xi_{t+1})$ in $\mathbb{R}$ with $\{\xi_t\} \overset{\text{IID}}{\sim} \varphi$

Guess a solution of the form $p_t = p(x_t)$

<u>Assumption</u>: d is bounded and d and F are both continuous

Reduces to the functional equation

$$p(x) = \beta \int \left[d(F(x,z)) + p(F(x,z)) \right] \varphi(dz) \qquad (x \in \mathbb{R}) \tag{2}$$

We seek a solution in $bc\mathbb{R}$ — paired with metric d_∞

Consider the operator T on $bc\mathbb{R}$ defined by

$$Tp(x) = \beta \int \left[d(F(x,z)) + p(F(x,z)) \right] \varphi(dz) \qquad (x \in \mathbb{R})$$

Important: $p \in bc\mathbb{R}$ solves (2) **iff** p is a fixed point of T

T is called the **equilibrium price operator**

Steps:

1. Show that T is a self-mapping on $bc\mathbb{R}$
2. Show that T is a contraction mapping on $bc\mathbb{R}$ of modulus β
3. Conclude that T has a unique fixed point in $bc\mathbb{R}$
4. Hence the pricing equation has a unique solution p^* in $bc\mathbb{R}$

Additional remarks

- $T^n p \to p^*$ as $n \to \infty$ for all $p \in bc\mathbb{R}$
- So we have a method to compute the solution

Step 1: T is a self-mapping on $bc\mathbb{R}$

Proof: For $p \in bc\mathbb{R}$ and $x \in \mathbb{R}$ we have

$$
\begin{aligned}
|Tp(x)| &= \left| \beta \int [d(F(x,z)) + p(F(x,z))] \, \varphi(dz) \right| \\
&\leqslant \beta \int |d(F(x,z)) + p(F(x,z))| \, \varphi(dz) \\
&\leqslant \beta \int |d(F(x,z))| \varphi(dz) + \beta \int |p(F(x,z))| \varphi(dz)
\end{aligned}
$$

Hence $|Tp(x)| \leqslant \beta(\|d\|_\infty + \|p\|_\infty)$

In particular, Tp is bounded on $\mathbb{R}$

Step 1 continued: T is a self-mapping on $bc\mathbb{R}$

Proof: For $p \in bc\mathbb{R}$, $x \in \mathbb{R}$ and $x_n \to x$, we have

$$\begin{aligned}\lim_{n\to\infty} Tp(x_n) &= \beta \lim_{n\to\infty} \int [d(F(x_n, z)) + p(F(x_n, z))] \, \varphi(dz) \\ &= \beta \int \left[\lim_{n\to\infty} d(F(x_n, z)) + \lim_{n\to\infty} p(F(x_n, z))\right] \varphi(dz) \\ &= \beta \int [d(F(x, z)) + p(F(x, z))] \, \varphi(dz)\end{aligned}$$

Hence $\lim_{n\to\infty} Tp(x_n) = Tp(x)$

In particular, Tp is continuous on $\mathbb{R}$

Step 2: T is a contraction on $bc\mathbb{R}$ of modulus β

Proof: For $p, q \in bc\mathbb{R}$ and $x \in \mathbb{R}$ we have

$$\begin{aligned}
|Tp(x) - Tq(x)| &= \left| \beta \int [p(F(x,z)) - q(F(x,z))] \varphi(dz) \right| \\
&\leqslant \beta \int |p(F(x,z)) - q(F(x,z))| \varphi(dz) \\
&\leqslant \beta \int \|p - q\|_\infty \varphi(dz) = \beta \|p - q\|_\infty
\end{aligned}$$

Taking the supremum over $x \in \mathbb{R}$ gives

$$\|Tp - Tq\|_\infty \leqslant \beta \|p - q\|_\infty$$

Step 3: From Banach's CMT we see that T has a unique fixed point in $bc\mathbb{R}$

Step 4: Hence the pricing equation has a unique solution in $bc\mathbb{R}$

We are done...

Question: Why did we use $bc\mathbb{R}$ as our space rather than $b\mathbb{R}$?

Extension: Lucas 1978

In Lucas (1978), the price process obeys

$$p_t = \beta \, \mathbb{E}_t \frac{u'(c_{t+1})}{u'(c_t)} \, (d_{t+1} + p_{t+1})$$

where c_t is consumption and u is utility

In equilibrium, $c_t = d_t = d(x_t)$ for all t

Taking $q_t := p_t \, u'(c_t)$ and $\kappa(x) := u'(d(x))d(x)$, we get

$$q_t = \beta \, \mathbb{E}_t \left[\kappa(x_{t+1}) + q_{t+1} \right]$$

Lucas adopts the following assumptions

- $x_{t+1} = F(x_t, \xi_{t+1})$ in $\mathbb{R}$ with $\{\xi_t\} \overset{\text{IID}}{\sim} \varphi$
- d and F are both continuous, $d \geqslant 0$
- u is continuously differentiable, strictly increasing, bounded and concave with $u(0) = 0$

Proposition: The function $\kappa(x) := u'(d(x))d(x)$ is bounded on $\mathbb{R}$

Proof: this is immediate if $u'(t)t$ is bounded over $t \geqslant 0$

Ex. Show that $\exists\, M < \infty$ with $|u'(t)t| \leqslant M$ for all $t \geqslant 0$

Proposition: The map $\kappa(x) := u'(d(x))d(x)$ is continuous on $\mathbb{R}$

Why?

Now we go back to

$$q_t = \beta \, \mathbb{E}_t \left[\kappa(x_{t+1}) + q_{t+1} \right]$$

and guess that $q_t = q(x_t)$ for some function q on $\mathbb{R}$

This leads to the **equilibrium pricing equation**

$$q(x) = \beta \int \left[\kappa(F(x,z)) + q(F(x,z)) \right] \varphi(\mathrm{d}z)$$

Proposition: There exists a function q in $bc\mathbb{R}$ that solves the equilibrium pricing equation

Ex. Check the details