ECON-GA 1025 Macroeconomic Theory I

Lecture 4

John Stachurski

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Today's Lecture

- Dynamical systems
- Order
- Monotone dynamical systems
- From monotonicity to stability

We previously studied dynamics through 45 degree diagrams

Informal discussions of

- stability
- steady states
- convergence
- etc.

Let's formalize ideas and state some general results

A **dynamical system** is a pair (M, g), where

1. M is a metric space and
2. g is a self-mapping on M

In this context, M is called the **state space**

Example. In the Solow–Swan model we saw that

$$k_{t+1} = g(k_t) \quad \text{where} \quad g(k) := sf(k) + (1 - \delta)k$$

Since g maps $\mathbb{R}_+$ to itself, the pair $(\mathbb{R}_+, g)$ is a dynamical system when $\mathbb{R}_+$ has its usual topology

If $g\colon u \mapsto 2u$, then $([0,1], g)$ is **not** a dynamical system because?

Let (M, g) be a dynamical system and consider

$$u_{t+1} = g(u_t), \quad \text{where } u_0 = \text{ some given point in } M$$

For this sequence we have

$$u_2 = g(u_1) = g(g(u_0)) =: g^2(u_0)$$

and, more generally,

$$u_t = g^t(u_0) \quad \text{where} \quad g^t = \underbrace{g \circ g \circ \cdots \circ g}_{t \text{ compositions of } g}$$

The sequence $\{g^t(u_0)\}_{t \geqslant 0}$ is called the **trajectory** of $u_0 \in M$

We will also call it a **time series**

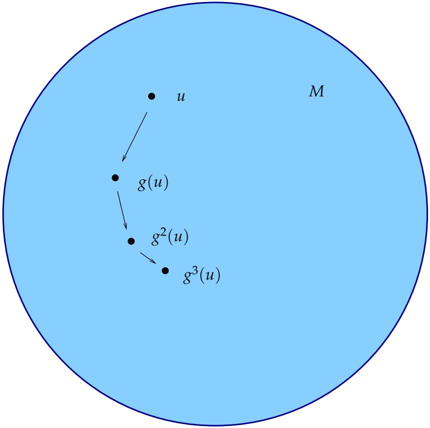


Figure: The trajectory of u under g

Fact. If g is increasing on M and $M \subset \mathbb{R}$, then every trajectory is monotone (either increasing or decreasing)

Proof: Pick any $u \in M$

Either $u \leqslant g(u)$ or $g(u) \leqslant u$ — let's treat the first case

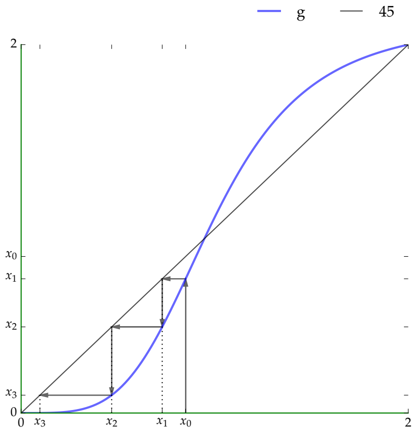
Since g is increasing and $u \leqslant g(u)$ we have $g(u) \leqslant g^2(u)$

Putting these inequalities together gives

$$u \leqslant g(u) \leqslant g^2(u)$$

Continuing in this way gives

$$u \leqslant g(u) \leqslant g^2(u) \leqslant g^3(u) \leqslant \cdots$$



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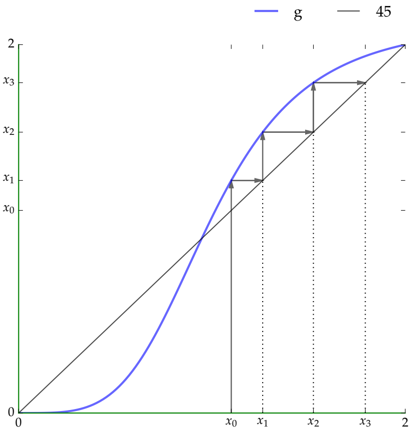
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Hence, in 1D, increasing functions generate simple dynamics

If g is not increasing then the dynamics can be far more erratic

Example. Let $M := [0, 1]$ and g be the **quadratic map**

$$g(x) = 4x(1 - x) \tag{1}$$

Almost all starting points generate "complicated" trajectories

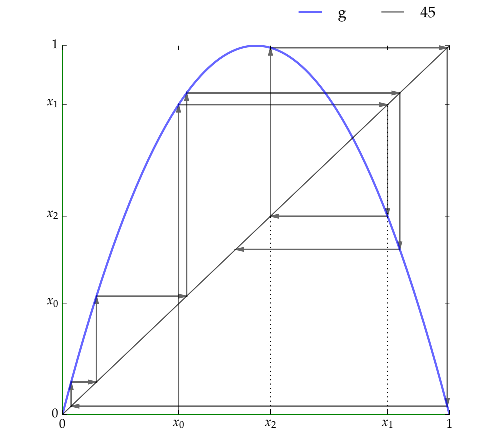

Figure: Logistic map $g(x) = 4x(1-x)$ with $x_0 = 0.3$

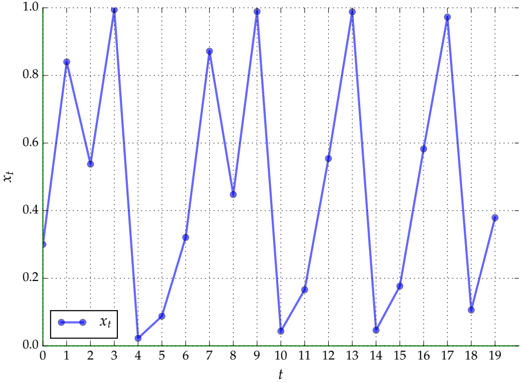

Figure: The corresponding time series

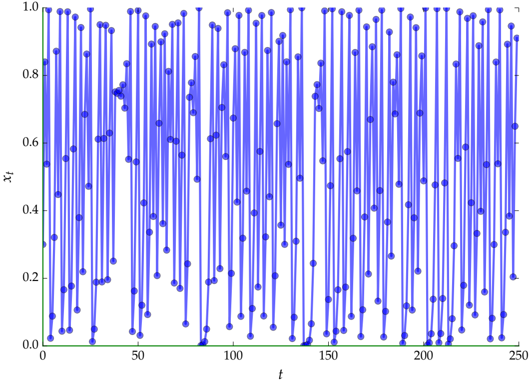

Figure: A longer time series

Steady States

Let (M, g) be a dynamical system

Suppose that u^* is a fixed point of g, so that

$$g(u^*) = u^*$$

Then, for any trajectory $\{u_t\}$ generated by g,

$$u_t = u^* \quad \implies \quad u_{t+1} = g(u_t) = g(u^*) = u^*$$

In other words, if we ever get to u^* we stay there

Hence, a fixed point of g in M is also called a **steady state**

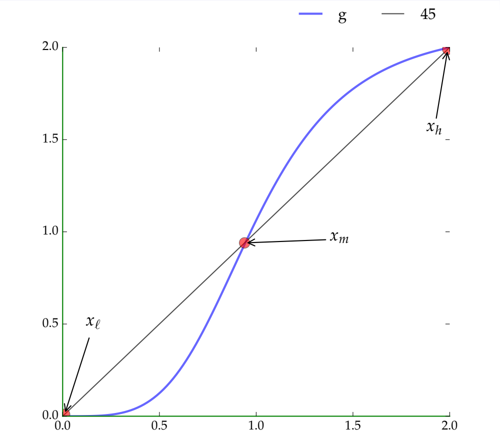

Figure: Steady states of $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$

Let (M, g) be a dynamical system

Fact. If $g^t(u) \to u^*$ for some $u, u^* \in M$ and g is continuous at u^*, then u^* is a fixed point of g

Proof: Assume the hypotheses, let $u_t := g^t(u)$

By continuity and $u_t \to u^*$ we have $g(u_t) \to g(u^*)$

But $\{g(u_t)\}$ is just $\{u_t\}$ without the first element and $u_t \to u^*$

Hence $g(u_t) \to u^*$

We now have

$$g(u_t) \to g(u^*) \quad \text{and} \quad g(u_t) \to u^*$$

Limits are unique, so $u^* = g(u^*)$

Local Stability

Let u^* be a steady state of (M, g)

The **stable set** of u^* is

$$\mathscr{O}(u^*) := \{u \in M : g^t(u) \to u^* \text{ as } t \to \infty\}$$

This set is nonempty (why?)

The steady state u^* is called **locally stable** or an **attractor** if there exists an $\epsilon > 0$ such that

$$B_\epsilon(u^*) \subset \mathscr{O}(u^*)$$

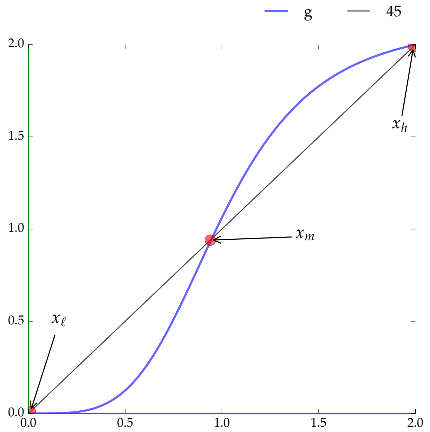

Figure: Steady states of $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$

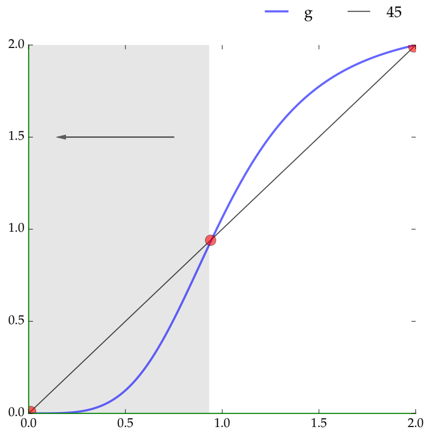

Figure: $\mathscr{O}(x_\ell)$

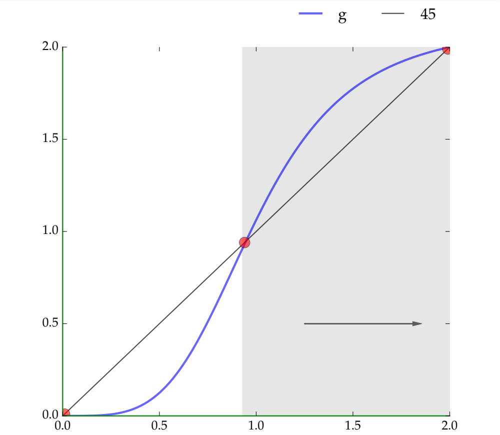

Figure: $\mathscr{O}(x_h)$

Global Stability

Dynamical system (M, g) is called **globally stable** if

1. g has a fixed point u^* in M
2. u^* is the only fixed point of g in M
3. $g^t(u) \to u^*$ as $t \to \infty$ for all $u \in M$

Equivalent: g has a fixed point u^* in M and $\mathscr{O}(u^*) = M$

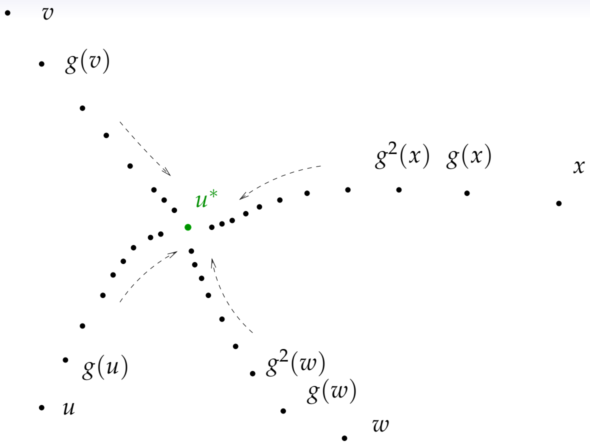

Figure: Visualizing global stability in $\mathbb{R}^2$

Example. Recall the Solow-Swan growth model where

$$g(k) = sAk^{\alpha} + (1 - \delta)k$$

with

1. $M = (0, \infty)$
2. $A > 0$ and $0 < s, \alpha, \delta < 1$

The system (M, g) is globally stable with unique fixed point

$$k^* := \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)}$$

Proof: Simple algebra shows that for $k > 0$ we have

$$k = sAk^{\alpha} + (1-\delta)k \quad \iff \quad k = \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)}$$

Hence (M, g) has unique steady state k^*

It remains to show that $g^t(k) \to k^*$ for every $k \in M := (0, \infty)$

Let's show this for any $k \leqslant k^*$, leaving $k^* \leqslant k$ as an exercise

Since calculating $g^t(k)$ directly is messy, let's try another strategy

Claim: If $0 < k \leqslant k^*$, then $\{g^t(k)\}$ is increasing and bounded

Proof increasing: Since g increasing $\{g^t(k)\}$ is monotone

From $k \leqslant k^*$ and some algebra (exercise) we get

$$k \leqslant k^* \implies g(k) \geqslant k \implies \{g^t(k)\} \text{ increasing}$$

Proof bounded: From $k \leqslant k^*$ and the fact that g is increasing,

$$g(k) \leqslant g(k^*) = k^*$$

Applying g to both sides gives $g^2(k) \leqslant k^*$ and so on

Hence both bounded and increasing

Hence $g^t(k) \to \hat{k}$ for some $\hat{k} \in M$

Because g is continuous, $\hat{k}$ is a fixed point

But k^* is the only fixed point of g on M, as discussed above

Hence $\hat{k} = k^*$

In other words, $g^t(k) \to k^*$ as claimed

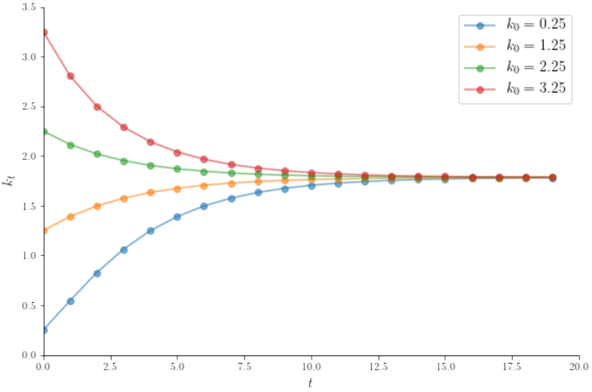

Figure: Global stability in the Solow–Swan model

Example. Consider again the Solow-Swan growth model

$$g(k) = sAk^{\alpha} + (1 - \delta)k$$

where parameters are as before

If $M = [0, \infty)$ then (M, g) is **not** globally stable

- We showed above that g has a fixed point k^* in $(0, \infty)$
- However, 0 is also a fixed point of g on $[0, \infty)$
- Hence (M, g) has two steady states in $M = [0, \infty)$

Moral: The state space matters for dynamic properties

Global Stability of Powers

The next result will be used in our study of Markov chains

Fact. Let (M, g) be a dynamical system

If

1. (M, g^i) is globally stable for some $i \in \mathbb{N}$ and
2. g is continuous at the steady state u^* of g^i,

then (M, g) is globally stable with unique steady state u^*

Proof: See course notes

Closed Invariant Sets

Let (M, g) be a globally stable dynamical system with fixed point u^* and let F be a closed subset of M

We say that g is **invariant** on F if $u \in F$ implies $g(u) \in F$

Fact. If F is nonempty and g is invariant on F, then $u^* \in F$

Ex. Check it

We use this many times in what follows

Examples.

- Concavity of the value function in savings problems
- Monotonicity of reservation wages, etc.

Application: Asset Pricing

When are asset prices increasing in x?

Recall that the equilibrium risk neutral price function satisfies

$$p^*(x) = \beta \int \left[d(F(x,z)) + p^*(F(x,z))\right] \varphi(dz) \qquad (x \in \mathbb{R})$$

Under what conditions does p^* increase in x?

Additional assumptions:

- d is increasing on $\mathbb{R}$
- $F(x,z)$ is increasing in x for each z

Does this sound like enough?

Recall that p is the unique fixed point in $bc\mathsf{X}$ of

$$Tp(x) = \beta \int [d(F(x,z)) + p(F(x,z))] \, \varphi(dz)$$

The pair $(bc\mathbb{R}, T)$ forms a dynamical system!

Let $ibc\mathbb{R}$ be the increasing functions in $bc\mathbb{R}$

Ex. Show that this set is closed in $(bc\mathbb{R}, d_\infty)$

Hence, **if** T is invariant on $ibc\mathbb{R}$, then

- its fixed point lies in $ibc\mathbb{R}$
- in particular, p^* is increasing

Under the stated assumptions, T is invariant on $ibc\mathbb{R}$

Proof: Pick any p in $ibc\mathbb{R}$ and fix x, x' in $\mathbb{R}$ with $x \leqslant x'$

For any z,

$$d(F(x,z)) \leqslant d(F(x',z)) \text{ and } p(F(x',z)) \leqslant p(F(x',z))$$

$$\begin{aligned} \therefore \quad Tp(x) &= \beta \int [d(F(x,z)) + p(F(x,z))] \, \varphi(dz) \\ &\leqslant \beta \int \left[d(F(x',z)) + p(F(x',z))\right] \varphi(dz) \\ &= Tp(x') \end{aligned}$$

In particular, $Tp \in ibc\mathbb{R}$

Sufficient Conditions for Global Stability

When is dynamical system (M, g) globally stable?

One sufficient condition is the Banach CMT

Requires that

- M is complete
- g is a contraction map on M

But this theorem doesn't always apply...

Example. Consider $g(k) = sf(k) + (1 - \delta)k$ with general f

- Typically **not** a contraction mapping...
- Moreover, the state $(0, \infty)$ is **not** complete

We require some alternative fixed point / stability results

Some of them use order theory

These results will be useful for many other problems so let's state them in an **abstract order-theoretic setting**

Order Structure

To study order in an abstract setting we introduce abstract notions of

- (partial) order
- suprema and infima
- lattices and sublattices
- isotonicity (increasing functions)

A **partial order** on nonempty set M is a relation $\preceq$ on $M \times M$ satisfying, for any u, v, w in M,

1. $u \preceq u$,
2. $u \preceq v$ and $v \preceq u$ implies $u = v$ and
3. $u \preceq v$ and $v \preceq w$ implies $u \preceq w$

Paired with $\preceq$, the set M is called a **partially ordered set**

Example. A subset M of $\mathbb{R}^d$ with the pointwise order $\leqslant$

Example. Let X be any set and let $\wp(\mathsf{X})$ be the set of all subsets

Then $\subset$ is a partial order on $\wp(\mathsf{X})$, since

1. $A \subset A$
2. $A \subset B$ and $B \subset A$ implies $A = B$
3. $A \subset B$ and $B \subset C$ implies $A \subset C$

Example. Let X be **any** set and, given $f, g \in \mathbb{R}^{\mathsf{X}}$, write

$$f \leqslant g \text{ if } f(x) \leqslant g(x) \text{ for all } x \in \mathsf{X}$$

This is the **pointwise partial order** on $\mathbb{R}^{\mathsf{X}}$

Ex. Check it satisfies the definition of a partial order

Given a subset E of a partially ordered set M, we call $u \in M$ an **upper bound** of E in M if $e \preceq u$ whenever $e \in E$

If there exists an $s \in M$ such that

1. s is an upper bound of E and
2. $s \preceq u$ whenever u is an upper bound of E,

then s is called the **supremum** of E in M

Note: Equivalent to the traditional definition when $M \subset \mathbb{R}$

Ex. Show that a subset E of M can have at most one supremum

Given a subset E of a partially ordered set M, we call $\ell \in M$ a **lower bound** of E in M if $\ell \preceq e$ for all $e \in E$

If there exists an $i \in M$ such that

1. i is a lower bound of E and
2. $\ell \preceq i$ whenever ℓ is a lower bound of E,

then i is called the **infimum** of E in M

Note: Equivalent to the traditional definition when $M \subset \mathbb{R}$

Ex. Show that a subset E of M can have at most one infimum

Example. Let $\leqslant$ be the pointwise partial order on $\mathbb{R}^{\mathsf{X}}$

Fix $K \in \mathbb{R}_+$ and let $E \subset B_K(0) = \{f \in b\mathsf{X} : \|f\|_\infty \leqslant K\}$

Fact. The supremum of E exists in $(b\mathsf{X}, \leqslant)$ and is given by

$$\hat{g}(x) := \sup_{g \in E} g(x) \qquad (x \in \mathsf{X}) \tag{2}$$

Proof: Sups of bounded sets in $\mathbb{R}$ exist, so $\hat{g}$ exists in $b\mathsf{X}$

Moreover,

1. $\hat{g} \geqslant g$ for all $g \in E$
2. $h \geqslant g$ for all $g \in E$ implies $h \geqslant \hat{g}$

Similarly, $\check{g}(x) := \inf_{g \in E} g(x)$ is the infimum of E

Given u and v in M, the supremum of $\{u,v\}$, when it exists, is also called the **join** of u and v, and is written $u \vee v$

The infimum of $\{u,v\}$, when it exists, is also called the **meet** of u and v, and is written $u \wedge v$

This is consistent with our earlier notation for vectors...

Suprema and infima do not necessarily exist

Example. Consider $M = \mathbb{R}$ with the usual order, where $E = \mathbb{R}_+$ has no upper bounds in M and hence no supremum

If $(M, \preceq)$ has the property that every **finite** subset of M has both a supremum and in infimum then $(M, \preceq)$ is called a **lattice**

Example. Given metric space X, the set bcX is a lattice when endowed with the pointwise partial order $\leqslant$

Proof: If f and g are continuous and bounded on X, then

- $f \wedge g$ is continuous and bounded
- $f \vee g$ is continuous and bounded

Example. The set of continuously differentiable functions on $[-1, 1]$ is **not** a lattice under the pointwise partial order $\leqslant$

For example, the supremum of $\{x \mapsto x, x \mapsto -x\}$ is $x \mapsto |x|$

A subset L of a lattice M is called a **sublattice** of M if

$$u, v \in L \implies u \wedge v \in L \text{ and } u \vee v \in L$$

Examples. Given metric space X,

- bcX is a sublattice of the lattice bX
- The set of nonnegative functions in bcX is a sublattice of bcX
- The set strictly positive functions in bcX is a sublattice of bcX

Suppose we have a metric space (M, ρ) and $\preceq$ is a partial order on M

- Often we want outcomes to replicate what we see in $\mathbb{R}^d$
- In Euclidean space, weak orders are preserved under limits

For this reason, we often require that $\preceq$ is **closed** with respect to ρ

This means that

$$u_n \to u, \; v_n \to v \;\text{ and } u_n \preceq v_n \;\text{ for all } n \in \mathbb{N} \implies u \preceq v$$

Example. The pointwise partial order $\leqslant$ is closed on $(b\mathsf{X}, d_\infty)$

Proof: Suppose that

- $f_n \to f$ and $g_n \to g$ in d_∞
- $f_n \leqslant g_n$ for all n

For any fixed $x \in \mathsf{X}$,

- $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ in $\mathbb{R}$ (why?)
- $f_n(x) \leqslant g_n(x)$ for all n

Since orders are preserved by limits in $\mathbb{R}$, we have $f(x) \leqslant g(x)$

Since x was arbitrary, we have $f \leqslant g$ in $(b\mathsf{X}, \leqslant)$

Given two partially ordered sets $(M, \preceq)$ and $(L, \trianglelefteq)$, a function g from M to L is called **isotone** if

$$u \preceq v \quad \Longrightarrow \quad g(u) \trianglelefteq g(v) \tag{3}$$

If $M = L = \mathbb{R}$ and $\preceq$ and $\trianglelefteq$ are both equal to $\leqslant$, the standard order on $\mathbb{R}$, then isotone means increasing (i.e., nondecreasing)

Other terms for isotone

- monotone increasing
- monotone
- order-preserving

Example. Recall the equilibrium price operator T on $bc\mathbb{R}$ defined by

$$Tp(x) = \beta \int [d(F(x,z)) + p(F(x,z))] \, \varphi(dz) \qquad (x \in \mathbb{R})$$

Endow $bc\mathbb{R}$ with the pointwise partial order $\leqslant$

For p, q in $bc\mathbb{R}$ with $p \leqslant q$ and arbitrary $x \in \mathbb{R}$, we have

$$\begin{aligned} Tp(x) &= \beta \int [d(F(x,z)) + p(F(x,z))] \, \varphi(dz) \\ &\leqslant \beta \int [d(F(x,z)) + q(F(x,z))] \, \varphi(dz) \\ &= Tq(x) \end{aligned}$$

Hence $Tp \leqslant Tq$ and T is isotone

Let S and T be isotone self-mappings on partially ordered set M

Ex. Show that $S \circ T$ is also an isotone self-mapping on M

Ex. Show that if u is a point in M with $u \preceq Tu$, then the sequence defined by $u_n := T^n u$ is monotone increasing

(Meaning: $u_n \preceq u_{n+1}$ for all n)

Parametric Monotonicity

A major concern in economic modeling is whether or not endogenous objects are shifted up (or down) by a change in some underlying parameter

Examples.

- Does a given policy intervention decrease steady state inflation?
- Does faster productivity growth increase firm profits?
- Does higher unemployment compensation increase average unemployment duration?

Let's see what we can say about such parametric monotonicity when the endogenous objects are **fixed points**

Let $\preceq$ be a closed partial order on metric space M

Given two self-maps g and h on M, we write

$$g \preceq h \quad \text{if} \quad g(u) \preceq h(u) \text{ for every } u \in M$$

- Sometimes h is said to **dominate** the function g

Domination is related to ordering of fixed points but does not guarantee it

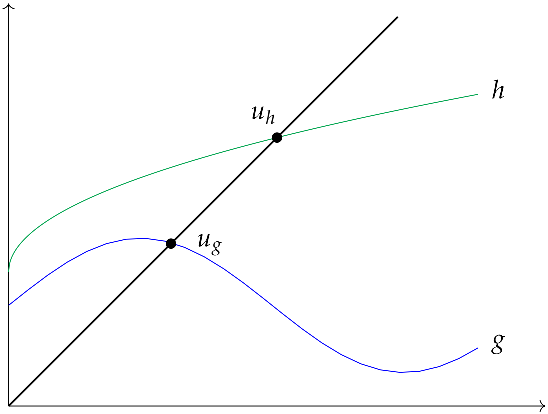

Figure: The dominating function has a higher fixed point

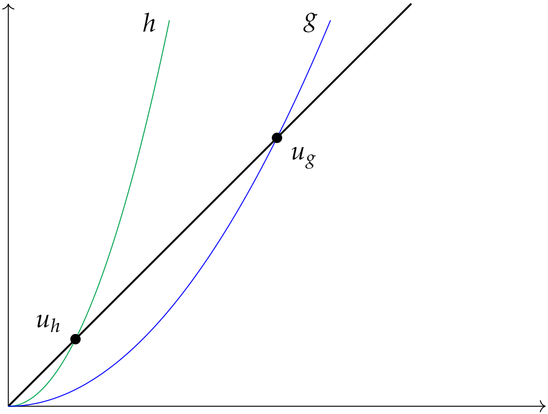

Figure: The dominating function has a lower fixed point

Fact. If (M, g) and (M, h) are dynamical systems such that

1. h is isotone and dominates g on M
2. (M, h) is globally stable with unique fixed point u_h,

then $u_g \preceq u_h$ for every fixed point u_g of g

Proof: Since $g \preceq h$, we have $u_g = g(u_g) \preceq h(u_g)$

Hence (by what laws?)

$$h(u_g) \preceq h^2(u_g) \text{ and therefore } u_g \preceq h^2(u_g)$$

Continuing in this fashion yields $u_g \preceq h^t(u_g)$ for all t

Taking the limit in t gives $u_g \preceq u_h$

Ex. Let $g(k) = sAk^\alpha + (1-\delta)k$ where

- all parameters are strictly positive
- $\alpha \in (0,1)$ and $\delta \leqslant 1$

Let $k^*(s, A, \alpha, \delta)$ be the unique fixed point of g in $(0, \infty)$

Without using the expression we derived for k^* previously, show that

1. $k^*(s, A, \alpha, \delta)$ is increasing in s and A
2. $k^*(s, A, \alpha, \delta)$ is decreasing in δ

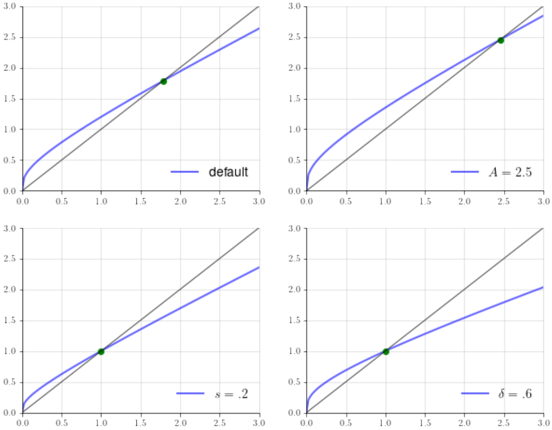

Figure: Deviations from the default $A = 2.0$, $s = \alpha = 0.3$ and $\delta = 0.4$

Application: Patience and Asset Prices

Let's go back to the equilibrium risk neutral price function

$$p(x) = \beta \int [d(F(x,z)) + p(F(x,z))] \, \varphi(dz) \qquad (x \in \mathbb{R})$$

How does it vary with parameters?

Consider two discount values β_1 and β_2

Let p_1 and p_2 be the corresponding equilibrium price functions

If $\beta_1 \leqslant \beta_2$, is it true that $p_1 \leqslant p_2$?

In other words, to we get higher prices for the asset in **all** states?

The functions p_1 and p_2 are fixed points of the operators

$$T_1 p(x) = \beta_1 \int [d(F(x,z)) + p(F(x,z))] \, \varphi(dz)$$

and

$$T_2 p(x) = \beta_2 \int [d(F(x,z)) + p(F(x,z))] \, \varphi(dz)$$

If $\beta_1 \leqslant \beta_2$, then the following **equivalent** statements are true

- $T_1 p(x) \leqslant T_2 p(x)$ for all $p \in bc\mathsf{X}$, $x \in \mathsf{X}$
- $T_1 p \leqslant T_2 p$ in the pointwise partial order for all $p \in bc\mathsf{X}$
- T_1 is dominated by T_2 on $(bc\mathsf{X}, \leqslant)$

Summarizing what we know,

1. T_1 is dominated by T_2 on $(bc\mathsf{X}, \leqslant)$
2. T_2 is isotone on $(bc\mathsf{X}, \leqslant)$
3. $(bc\mathsf{X}, T_2)$ is globally stable

Hence $p_1 \leqslant p_2$ in $(bc\mathsf{X}, \leqslant)$

In particular, $p_1(x_t) \leqslant p_2(x_t)$ for all realizations of x_t

Thus, p_2 yields higher prices in all states

From Order to Stability

Monotonicity is also connected to fixed points and stability

To illustrate, let's think again about the Solow–Swan growth model

$$k_{t+1} = g(k_t) := sf(k_t) + (1 - \delta)k_t$$

So far, we have proved stability in the case of

- Cobb–Douglas production $f(k) = Ak^{\alpha}$
- some suitable parameter restrictions

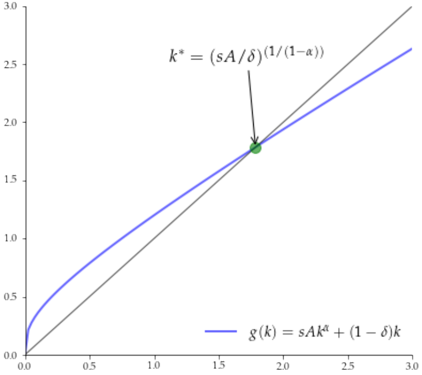

Figure: 45 degree diagram for the Solow–Swan model

It seems that global stability will hold more generally

All we really need is a similar shape for f

Example.

- f is strictly increasing and concave
- $f'(0) = \infty$ and $f'(\infty) = 0$

Then the 45 degree diagram will be similar too

But what proof technique can we use?

Not the Banach CMT, since

- g is **not** a contraction
- The set $(0, \infty)$ is **not** complete

Clearly we need another fixed point theorem

There aren't many that give

1. existence
2. uniqueness
3. global convergence of successive approximations

But we need one...

Our plan is to exploit

1. order structure (e.g., the Solow map is **increasing**)
2. algebraic structure (e.g., the Solow map is **concave**)
3. topological structure (e.g., small points are mapped up **strictly** and large points are mapped down **strictly**)

Order Structure in $\mathbb{R}^d$: Reminders

We use the standard pointwise partial order $\leqslant$ in $\mathbb{R}^d$ discussed earlier: for $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ in $\mathbb{R}^d$,

$$u \leqslant v \iff u_i \leqslant v_i \text{ for all } i$$

In addition,

- if $u_i \leqslant v_i$ for all i and $u \neq v$ then we write $u < v$
- if $u_i < v_i$ for all i then we write $u \ll v$

As usual,

- $u \wedge v := (u_1 \wedge v_1, \ldots, u_d \wedge v_d)$
- $u \vee v := (u_1 \vee v_1, \ldots, u_d \vee v_d)$

Recall: A subset L of $\mathbb{R}^d$ is called a sublattice of $\mathbb{R}^d$ if, given u, v in $\mathbb{R}^d$, we have

$$u, v \in L \quad \Longrightarrow \quad u \wedge v \in L \text{ and } u \vee v \in L$$

Examples.

- The **positive cone**

$$C := \mathbb{R}^d_+ := \{u \in \mathbb{R}^d : u \geqslant 0\}$$

  is a sublattice of $\mathbb{R}^d$

- The interior of the positive cone is a sublattice of $\mathbb{R}^d$
- The unit ball is **not** a sublattice of $\mathbb{R}^d$

Recall that a map T from $M \subset \mathbb{R}^d$ to itself is called isotone if

$$u, v \in M \text{ and } u \leqslant v \quad \Longrightarrow \quad Tu \leqslant Tv$$

Example. If $A = A(x, y)$ is a nonnegative matrix, then $v \mapsto Av$ is isotone, since

$$u \leqslant v \implies \sum_y A(x,y)u(y) \leqslant \sum_y A(x,y)v(y)$$

Hence $Au \leqslant Av$ pointwise on $\mathbb{R}^d$

Concavity and Convexity in $\mathbb{R}^d$

A subset C of $\mathbb{R}^d$ is called **convex** if

$$u, v \in C \text{ and } 0 \leqslant \lambda \leqslant 1 \implies \lambda u + (1-\lambda)v \in C$$

An self-map T on C is called **convex** if, for any $u, v \in C$ and $\lambda \in [0,1]$,

$$T(\lambda u + (1-\lambda)v) \leqslant \lambda Tu + (1-\lambda)Tv$$

An self-map T on C is called **concave** if, for any $u, v \in C$ and $\lambda \in [0,1]$,

$$T(\lambda u + (1-\lambda)v) \geqslant \lambda Tu + (1-\lambda)Tv$$

Let C be a sublattice of $\mathbb{R}^d$

Theorem FPT2 (finite dimensional case): Let T be an isotone self-mapping on C such that

1. $\forall\, u \in C$, there exists a point $a \in C$ with $a \leqslant u$ and $Ta \gg a$
2. $\forall\, u \in C$, there exists a point $b \in C$ with $b \geqslant u$ and $Tb \ll b$

If, in addition, T is either concave or convex, then (C, T) is globally stable

Proof: See the course notes (appendix)

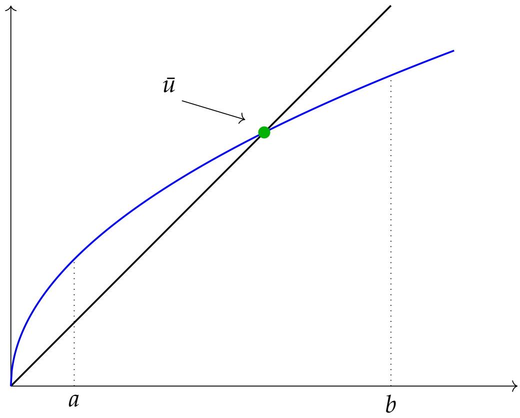

Figure: Global stability for an increasing concave functions

Corollary: Let g be a function from $(0, \infty)$ to itself with the following two properties:

1. For each $x > 0$, there is an $a \leqslant x$ such that $g(a) > a$.
2. For each $x > 0$, there is a $b \geqslant x$ such that $g(b) < b$.

If g is also increasing and concave, then

- g has a unique fixed point $\bar{x}$ in $(0, \infty)$ and
- $g^n(x) \to \bar{x}$ for every $x \in (0, \infty)$

Corollary of corollary: If

$$g(k) = sf(k) + (1-\delta)k$$

where $0 < s, \delta < 1$ and f is a increasing concave function on $(0, \infty)$ satisfying

1. $f'(k) \to \infty$ as $k \to 0$ and
2. $f'(k) \to 0$ as $k \to \infty$,

then g has a unique fixed point k^* in $(0, \infty)$ and $g^n(x) \to \bar{x}$ for every $x \in (0, \infty)$

Ex. Check the details