Linear Programs

$$\begin{aligned}\min_{\mathbf{x}}\ & \mathbf{f}^T\mathbf{x}\\ \text{subject to:}\ & \mathbf{A}\mathbf{x} \le \mathbf{b}\\ & \mathbf{A}_{\text{eq}}\mathbf{x} = \mathbf{b}_{\text{eq}}\\ & \mathbf{l}_{\text{b}} \le \mathbf{x} \le \mathbf{u}_{\text{b}}\end{aligned}$$

$$\min_{\mathbf{x}}\ \mathbf{f}^T\mathbf{x} + objbias$$

Ex1 LP

$$\begin{aligned}\min_{\mathbf{x}}\ & -6x_1 - 5x_2\\ \text{subject to:}\ & x_1 + 4x_2 \le 16\\ & 6x_1 + 4x_2 \le 28\\ & 2x_1 - 5x_2 \le 6\\ & 0 \le \mathbf{x} \le 10\end{aligned}$$

$$\begin{aligned}\min_{\mathbf{x}}\ & \begin{bmatrix}-6\\-5\end{bmatrix}^T \mathbf{x}\\ \text{subject to:}\ & \begin{bmatrix}1 & 4\\ 6 & 4\\ 2 & -5\end{bmatrix}\mathbf{x} \le \begin{bmatrix}16\\28\\6\end{bmatrix}\\ & 0 \le \mathbf{x} \le 10\end{aligned}$$

Ex2 LP

$$\begin{aligned}\min_{\mathbf{x}}\ & -x_1 - 2x_2 - 3x_3\\ \text{subject to:}\ & -x_1 + x_2 + x_3 \le 20\\ & x_1 - 3x_2 + x_3 \le 30\\ & x_1 + x_2 + x_3 = 40\\ & 0 \le x_1 \le 40\\ & 0 \le x_2\\ & 0 \le x_3\end{aligned}$$

MI Linear Programs

$$\begin{aligned}\min_{\mathbf{x}}\ & \mathbf{f}^T\mathbf{x}\\ \text{subject to: } & \mathbf{A}\mathbf{x}\leq \mathbf{b}\\ & \mathbf{A}_{\text{eq}}\mathbf{x}=\mathbf{b}_{\text{eq}}\\ & \mathbf{l}_{\text{b}}\leq \mathbf{x}\leq \mathbf{u}_{\text{b}}\\ & x_i\in\mathbb{Z}\\ & x_j\in\{0,1\}\end{aligned}$$

Ex1 MILP

$$\begin{aligned}\min_{\mathbf{x}}\ & -6x_1-5x_2\\ \text{subject to: } & x_1+4x_2\leq 16\\ & 6x_1+4x_2\leq 28\\ & 2x_1-5x_2\leq 6\\ & 0\leq \mathbf{x}\leq 10\\ & \mathbf{x}\in\mathbb{Z}\end{aligned}$$

Ex2 MILP

$$\begin{aligned}\min_{\mathbf{x}}\ & -x_1-x_2-3x_3-2x_2-2x_5\\ \text{subject to: } & -x_1-x_2+x_3+x_4\leq 30\\ & x_1+x_3-3x_4\leq 30\\ & 0\leq x_1\leq 40\\ & 0\leq x_2\leq 1\\ & 0\leq x_3\\ & 0\leq x_4\\ & 0\leq x_5\leq 1\end{aligned}$$

Quadratic Programs

$$\begin{aligned}\min_{\mathbf{x}}\ & \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x}+\mathbf{f}^T\mathbf{x}\\ \text{subject to: } & \mathbf{A}\mathbf{x}\leq \mathbf{b}\\ & \mathbf{A}_{\text{eq}}\mathbf{x}=\mathbf{b}_{\text{eq}}\\ & \mathbf{l}_{\text{b}}\leq \mathbf{x}\leq \mathbf{u}_{\text{b}}\end{aligned}$$

$$
\begin{aligned}
\min_{\mathbf{x}}\ 0.5x_1^2 + x_2^2 - x_1x_2 - 2x_1 - 6x_2 \\
\text{subject to:} \qquad x_1 + x_2 \leq 2 \\
-x_1 + 2x_2 \leq 2 \\
2x_1 + x_2 \leq 3 \\
\mathbf{0} \leq \mathbf{x}
\end{aligned}
$$

$$
\begin{aligned}
\min_{\mathbf{x}}\ & -2x_1x_2 \\
\text{subject to:}\ & -\mathbf{0.5} \leq \mathbf{x} \leq \mathbf{1}
\end{aligned}
$$

MI Quadratic Programs

$$
\begin{aligned}
\min_{\mathbf{x}}\ & \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{f}^T\mathbf{x} \\
\text{subject to:}\ & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\
& \mathbf{A}_{\text{eq}}\mathbf{x} = \mathbf{b}_{\text{eq}} \\
& \mathbf{l}_{\text{b}} \leq \mathbf{x} \leq \mathbf{u}_{\text{b}} \\
& x_i \in \mathbb{Z} \\
& x_j \in \{0, 1\}
\end{aligned}
$$

$$
\begin{aligned}
\min_{\mathbf{x}}\ 0.5x_1^2 + x_2^2 - x_1x_2 - 2x_1 - 6x_2 \\
\text{subject to:} \qquad x_1 + x_2 \leq 2 \\
-x_1 + 2x_2 \leq 2 \\
2x_1 + x_2 \leq 3 \\
\mathbf{0} \leq \mathbf{x} \\
x_1 \in \mathbb{Z}
\end{aligned}
$$

QC Quadratic Programs

$$
\begin{aligned}
\min_{\mathbf{x}}\ & \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{f}^T\mathbf{x} \\
\text{subject to:}\ & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\
& \mathbf{A}_{\text{eq}}\mathbf{x} = \mathbf{b}_{\text{eq}} \\
& \mathbf{l}_{\text{b}} \leq \mathbf{x} \leq \mathbf{u}_{\text{b}} \\
& \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{l}^T\mathbf{x} \leq \mathbf{r}
\end{aligned}
$$

QC Row

$$\mathbf{q}_{\text{rl}} \leq \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{l}^T\mathbf{x} \leq \mathbf{q}_{\text{ru}}$$

$$\begin{aligned} \min_{\mathbf{x}}\ & 0.5x_1^2 + 0.5x_2^2 - 2x_1 - 2x_2 \\ \text{subject to:} \quad & -x_1 + x_2 \leq 2 \\ & x_1 + 3x_2 \leq 5 \\ & x_1^2 + x_2^2 - 2x_2 \leq 1 \\ & \mathbf{0} \leq \mathbf{x} \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{x}}\ & 0.5x_1^2 + 0.5x_2^2 - 2x_1 - 2x_2 \\ \text{subject to:} \quad & -x_1 + x_2 \leq 2 \\ & x_1 + 3x_2 \leq 5 \\ & x_1^2 + x_2^2 - 2x_2 \leq 1 \\ & x_1^2 + x_2^2 - x_1 + 2x_2 \leq 1.2 \\ & \mathbf{0} \leq \mathbf{x} \end{aligned}$$

MIQC Quadratic Programs

$$\begin{aligned} \min_{\mathbf{x}}\ & \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x} + \mathbf{f}^T\mathbf{x} \\ \text{subject to:}\ & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{A}_{\text{eq}}\mathbf{x} = \mathbf{b}_{\text{eq}} \\ & \mathbf{l}_{\text{b}} \leq \mathbf{x} \leq \mathbf{u}_{\text{b}} \\ & \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{l}^T\mathbf{x} \leq \mathbf{r} \\ & x_i \in \mathbb{Z} \\ & x_j \in \{0, 1\} \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{x}}\ & 0.5x_1^2 + 0.5x_2^2 - 2x_1 - 2x_2 \\ \text{subject to:} \quad & -x_1 + x_2 \leq 2 \\ & x_1 + 3x_2 \leq 5 \\ & x_1^2 + x_2^2 - 2x_2 \leq 1 \\ & \mathbf{0} \leq \mathbf{x} \\ & x_1 \in \mathbb{Z} \end{aligned}$$

SDP

$$
\begin{aligned}
\min_{\mathbf{x}} \ & \mathbf{f}^T\mathbf{x} \\
\text{subject to: } & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\
& \mathbf{l}_\mathrm{b} \leq \mathbf{x} \leq \mathbf{u}_\mathrm{b} \\
& \mathbf{X} = \sum_{i=1}^{n} x_i \mathbf{F}_i - \mathbf{F}_0 \\
& \mathbf{X} \succeq \mathbf{0} \text{ [Positive Semidefinite]}
\end{aligned}
$$

Ex1 SDP

$$
\begin{aligned}
\min_{\mathrm{x}} \ & x \\
\text{subject to: } & \begin{bmatrix} x & \sqrt{2} \\ \sqrt{2} & x \end{bmatrix} \succeq \mathbf{0}
\end{aligned}
$$

Ex1 SDP Equation

$$
x \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{F}_1} - \underbrace{\begin{bmatrix} 0 & -\sqrt{2} \\ -\sqrt{2} & 0 \end{bmatrix}}_{\mathbf{F}_0} \succeq \mathbf{0}
$$

Ex2 SDP

$$
\begin{aligned}
\min_{\mathbf{x}} \ & x_1 + x_2 \\
\text{subject to: } & \begin{bmatrix} x_1 & 2 \\ 2 & x_2 \end{bmatrix} \succeq \mathbf{0} \\
& \mathbf{0} \leq \mathbf{x} \leq \mathbf{10}
\end{aligned}
$$

Ex2 SDP Equation

$$
x_1 \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{A}_1} + x_2 \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{A}_2} - \underbrace{\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}}_{\mathbf{C}} \succeq \mathbf{0}
$$

Ex3 SDP

$$\begin{aligned} & \min_{\mathbf{x}} \; x_1 \\ \text{subject to:} \; & \begin{bmatrix} x_2 & x_3 \\ x_3 & x_4 \end{bmatrix} \preceq \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} \\ & \begin{bmatrix} x_2 & x_3 \\ x_3 & x_4 \end{bmatrix} \succeq \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix} \end{aligned}$$

Nonlinear Least Squares

$$\begin{aligned} & \min_{\mathbf{x}} \; \left\| \mathbf{F}\left(\mathbf{x}\right) - \mathbf{ydata} \right\|_2^2 \\ \text{subject to:} \; & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{A}_{\mathrm{eq}}\mathbf{x} = \mathbf{b}_{\mathrm{eq}} \\ & \mathbf{l}_{\mathrm{b}} \leq \mathbf{x} \leq \mathbf{u}_{\mathrm{b}} \end{aligned}$$

$$\min_{\mathbf{x}} \; \sum_i \left(\mathrm{F}_i\left(\mathbf{x}, \mathbf{xdata}\right) - ydata_i\right)^2$$

Ex1

$$\min_{\mathbf{x}} \; \left\| \begin{matrix} 100\left(x_2 - x_1^2\right) \\ 1 - x_1 \end{matrix} \right\|_2^2$$

Ex2

$$\begin{aligned} & \min_{\mathbf{x}} \; \left\| \begin{matrix} 100\left(x_2 - x_1^2\right) \\ 1 - x_1 \end{matrix} \right\|_2^2 \\ \text{subject to:} \; & -2 \leq \mathbf{x} \leq 0.5 \end{aligned}$$

Ex3

$$\mathbf{F}\left(\mathbf{x}, \mathbf{xdata}\right) = x_1 e^{x_2 \mathbf{xdata}}$$

Dynamic Nonlinear Least Squares

$$\dot{\mathbf{z}}(t) = \mathbf{f}\left(t, \mathbf{z}(t), \mathbf{p}\right), \qquad \mathbf{z}(t_0) = \mathbf{z}_0$$

$$\theta := \begin{bmatrix} \mathbf{p} \\ \mathbf{z}_0 \end{bmatrix}$$

$$\dot{\mathbf{S}} = \mathbf{J}\mathbf{S} + \begin{bmatrix} \mathbf{J}_\mathbf{p} & \mathbf{0} \end{bmatrix}$$

where

$$\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{z}} = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \cdots & \frac{\partial f_2}{\partial z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1} & \frac{\partial f_n}{\partial z_2} & \cdots & \frac{\partial f_n}{\partial z_n} \end{bmatrix}$$

$$\mathbf{J}_\mathbf{p} = \frac{\partial \mathbf{f}}{\partial \mathbf{p}} = \begin{bmatrix} \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial p_2} & \cdots & \frac{\partial f_1}{\partial p_m} \\ \frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial p_2} & \cdots & \frac{\partial f_2}{\partial p_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial p_1} & \frac{\partial f_n}{\partial p_2} & \cdots & \frac{\partial f_n}{\partial p_m} \end{bmatrix}$$

$$\begin{aligned} \dot{z_1} &= -p_1 z_1 + 4 \\ \dot{z_2} &= 2z_1 - p_1 z_2 + 5 \\ \dot{z_3} &= -4z_1 - 2z_2 z_3 - p_2 \end{aligned}$$

$$\dot{z_1} = p_1 z_1^2 - z_1^3$$

System of Nonlinear Equations

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}$$

$$\begin{aligned} 2x_1 - x_2 - e^{-x_1} &= 0 \\ -x_1 + 2x_2 - e^{-x_2} &= 0 \end{aligned}$$

$$
\begin{aligned}
10\left(x_2 - x_1^2\right) &= 0\\
\sqrt{90}\left(x_4 - x_3^2\right) &= 0\\
\sqrt{10}\left(x_2 + x_4 - 2\right) &= 0\\
\frac{1}{\sqrt{10}}\left(x_2 - x_4\right) &= 0
\end{aligned}
$$

System of Constrained Nonlinear Equations

$$
\begin{aligned}
&\mathbf{F}\left(\mathbf{x}\right) = \mathbf{0}\\
\text{subject to: } &\mathbf{A}\mathbf{x} \leq \mathbf{b}\\
&\mathbf{A}_{\text{eq}}\mathbf{x} = \mathbf{b}_{\text{eq}}\\
&\mathbf{l}_{\text{b}} \leq \mathbf{x} \leq \mathbf{u}_{\text{b}}
\end{aligned}
$$

Ex1

$$
\begin{aligned}
2x_1 - x_2 - e^{-x_1} &= 0\\
-x_1 + 2x_2 - e^{-x_2} &= 0\\
\text{subject to: } 0.6 \leq x_1 &\leq 1\\
0 \leq x_2 &\leq 1
\end{aligned}
$$

Nonlinear Programs

$$
\begin{aligned}
\min_{\mathbf{x}} \; &f\left(\mathbf{x}\right)\\
\text{subject to: } &\mathbf{A}\mathbf{x} \leq \mathbf{b}\\
&\mathbf{A}_{\text{eq}}\mathbf{x} = \mathbf{b}_{\text{eq}}\\
&\mathbf{l}_{\text{b}} \leq \mathbf{x} \leq \mathbf{u}_{\text{b}}\\
&\mathbf{c}\left(\mathbf{x}\right) \leq \mathbf{d}\\
&\mathbf{c}_{\text{eq}}\left(\mathbf{x}\right) = \mathbf{d}_{\text{eq}}
\end{aligned}
$$

Ex2 NLP

$$
\begin{aligned}
\min_{\mathbf{x}} \; &100(x_2 - x_1^2)^2 + (1 - x_1)^2\\
\text{subject to: } &-5 \leq \mathbf{x} \leq 5
\end{aligned}
$$

Ex2 NLP

$$\begin{aligned} \min_{\mathbf{x}} \ & \log(1+x_1^2) - x_2 \\ \text{subject to: } & \left(1+x_1^2\right)^2 + x_2^2 = 4 \end{aligned}$$

Ex2 NLP

$$\begin{aligned} \min_{\mathbf{x}} \ & x_1 x_4 \left(x_1 + x_2 + x_3\right) + x_3 \\ \text{subject to: } & x_1 x_2 x_3 x_4 \geq 25 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40 \\ & 1 \leq \mathbf{x} \leq 5 \end{aligned}$$

Ex2 LinearCon NLP

$$\begin{aligned} \min_{\mathbf{x}} \ & (x_1 - x_2)^2 + (x_2 - x_3 - 2)^2 + (x_4 - 1)^2 + (x_5 - 1)^2 \\ \text{subject to: } & x_1 + 3x_3 = 4 \\ & x_3 + x_4 - 2x_5 = 0 \\ & x_2 - x_5 = 0 \end{aligned}$$

Ex 3

$$\min_{\mathbf{x}} \ 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

MI Nonlinear Programs

$$\begin{aligned} \min_{\mathbf{x}} \ & f\left(\mathbf{x}\right) \\ \text{subject to: } & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{A}_{\mathrm{eq}}\mathbf{x} = \mathbf{b}_{\mathrm{eq}} \\ & \mathbf{l}_{\mathrm{b}} \leq \mathbf{x} \leq \mathbf{u}_{\mathrm{b}} \\ & \mathbf{c}\left(\mathbf{x}\right) \leq \mathbf{d} \\ & \mathbf{c}_{\mathrm{eq}}\left(\mathbf{x}\right) = \mathbf{d}_{\mathrm{eq}} \\ & x_i \in \mathbb{Z} \\ & x_j \in \{0, 1\} \end{aligned}$$

EX1 MINLP

$$
\begin{aligned}
\min_{\mathbf{x}} \; & -x_1 - x_2 - x_3 \\
\text{subject to: } & (x_2 - 0.5)^2 + (x_3 - 0.5)^2 \leq 0.25 \\
& x_1 - x_2 \leq 0 \\
& x_1 + x_3 + x_4 \leq 2 \\
& x_1 \leq 1 \\
& x_4 \leq 5 \\
& \mathbf{0} \leq \mathbf{x} \\
& x_1 \in \{0, 1\} \\
& x_4 \in \mathbb{Z}
\end{aligned}
$$

EX2 MINLP

$$
\begin{aligned}
\min_{\mathbf{x}} \; & 20 + x_1^2 + x_2^2 - 10(\cos 2\pi x_1 + \cos 2\pi x_2) \\
\text{subject to: } & 5\pi \leq x_1 \leq 20\pi \\
& -20\pi \leq x_2 \leq -4\pi \\
& x_1 \in \mathbb{Z}
\end{aligned}
$$

Constraint Stuff

$$\mathbf{r_l} \leq \mathbf{A}\mathbf{x} \leq \mathbf{r_u}$$

1st Derivatives
Gradient

$$\nabla f = \frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Jacobian

$$\nabla \mathbf{F} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Ex1

$$f(\mathbf{x}) = x_1 x_4 (x_1 + x_2 + x_3) + x_3$$

2nd Derivatives

$$\nabla^2 f = \frac{\partial^2 f}{\partial \mathbf{x}^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Hess Lagrangian

$$\nabla^2 \mathbf{L} = \sigma \nabla^2 f + \sum_i \lambda_i \nabla^2 c_i$$

Adv Options
Tolr

$$\frac{|f_i - f_{i-1}|}{|f_i|} \leq \text{tol}_{rel}$$

Tola

$$|f_i - f_{i-1}| \leq \text{tol}_{abs}$$

Toli

$$|x_r - x_z| \leq \text{tol}_{int}$$

Integer Tricks
Constrained Set

$$x \in \{3, 5, 8, 20\}$$

$$\begin{aligned} \min_x \; & x \\ \text{subject to: } & l \leq x \leq 20 \\ & x \in \{3, 5, 8, 20\} \end{aligned}$$

Many nlcon example

$$\begin{bmatrix} x_{(1)}x_{(2N+1)} - r_{(1)}x_{(1)}x_{(N+1)} + (1 - r_{(1)})x_{(N+1)}x_{(2N+1)} \\ x_{(2)}x_{(2N+2)} - r_{(2)}x_{(2)}x_{(N+2)} + (1 - r_{(2)})x_{(N+2)}x_{(2N+2)} \\ \vdots \\ x_{(N)}x_{(2N+N)} - r_{(N)}x_{(N)}x_{(N+N)} + (1 - r_{(N)})x_{(N+N)}x_{(2N+N)} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix}$$

$$
\begin{aligned}
\ddot{\theta}_e &= \left(\frac{K_f l_a}{J_e}\right)(V_f + V_b)\cos(\theta_p) - \frac{T_g}{J_e} \\
\ddot{\theta}_p &= \left(\frac{K_f l_h}{J_p}\right)(V_f - V_b) \\
\ddot{\theta}_r &= -\left(\frac{F_g l_a}{J_t}\right)\sin(\theta_p)
\end{aligned}
\tag{1}
$$