CHAPTER 46

2-4 TREES AND B-TREES

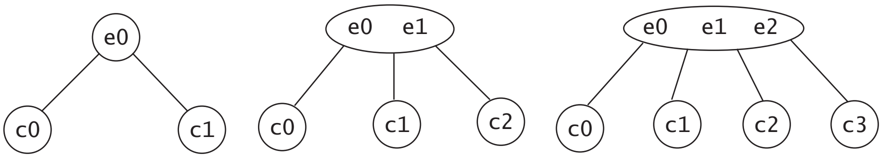
Objectives

- To know what a 2-4 tree is (§46.1).
- To design the **`Tree24`** class that implements the **`Tree`** interface (§46.2).
- To search an element in a 2-4 tree (§46.3).
- To insert an element in a 2-4 tree and know how to split a node (§46.4).
- To delete an element from a 2-4 tree and know how to perform transfer and fusion operations (§46.5).
- To traverse elements in a 2-4 tree (§46.6).
- To implement and test the **`Tree24`** class (§§46.7–46.8).
- To analyze the complexity of the 2-4 tree (§46.9).
- To use B-trees for indexing large amount of data (§46.10).

46.1 Introduction

completely balanced tree
2-node
3-node
4-node

A *2-4 tree*, also known as a *2-3-4 tree*, is a *completely balanced* search tree with all leaf nodes appearing on the same level. In a 2-4 tree, a node may have one, two, or three elements. An interior *2-node* contains one element and two children. An interior *3-node* contains two elements and three children. An interior *4-node* contains three elements and four children, as shown in Figure 46.1.



(a) 2-node (b) 3-node (c) 4-node

FIGURE 46.1 An interior node of a 2-4 tree has two, three, or four children.

ordered

Each child is a sub 2-4 tree, possibly empty. The root node has no parent, and leaf nodes have no children. The elements in the tree are distinct. The elements in a node are ordered such that

$$E(c_0) < e_0 < E(c_1) < e_1 < E(c_2) < e_2 < E(c_3)$$

$E(c_k)$
left subtree
right subtree

where $E(c_k)$ denote the elements in c_k. Figure 46.2 shows an example of a 2-4 tree. c_k is called the *left subtree* of e_k and c_{k+1} is called the *right subtree* of e_k.

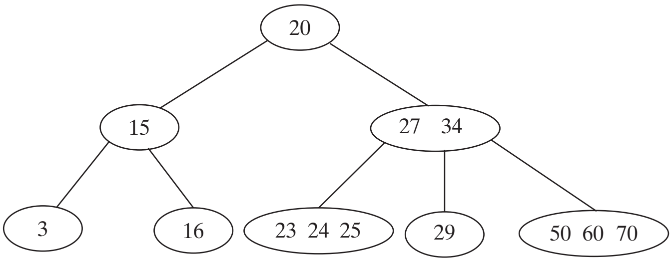


FIGURE 46.2 A 2-4 tree is a full complete search tree.

binary vs. 2-4

a binary tree, each node contains one element. A 2-4 tree tends to be shorter than a corresponding binary search tree, since a 2-4 tree node may contain two or three elements.

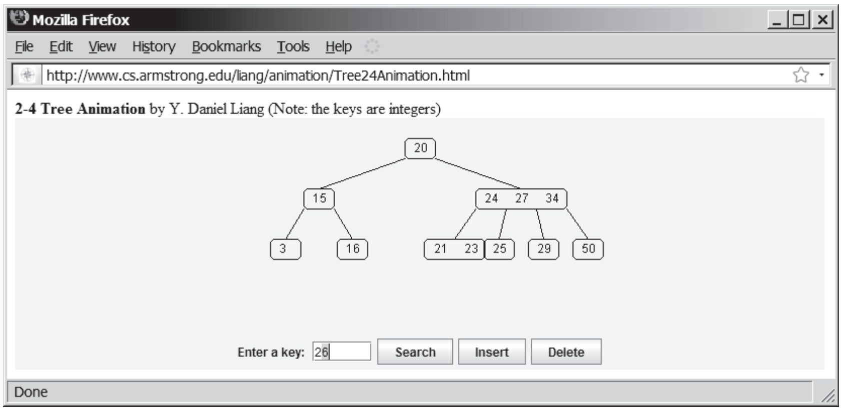
2-4 tree animation

Pedagogical Note

Run from www.cs.armstrong.edu/liang/animation/Tree24Animation.html to see how a 2-4 tree works, as shown in Figure 46.3.

46.2 Designing Classes for 2-4 Trees

The `Tree24` class can be designed by implementing the `Tree` interface, as shown in Figure 46.4. The `Tree` interface was defined in Listing 26.3 Tree.java. The `Tree24Node` class defines tree nodes. The elements in the node are stored in a list named `elements` and the links to the child nodes are stored in a list named `child`, as shown in Figure 46.5.

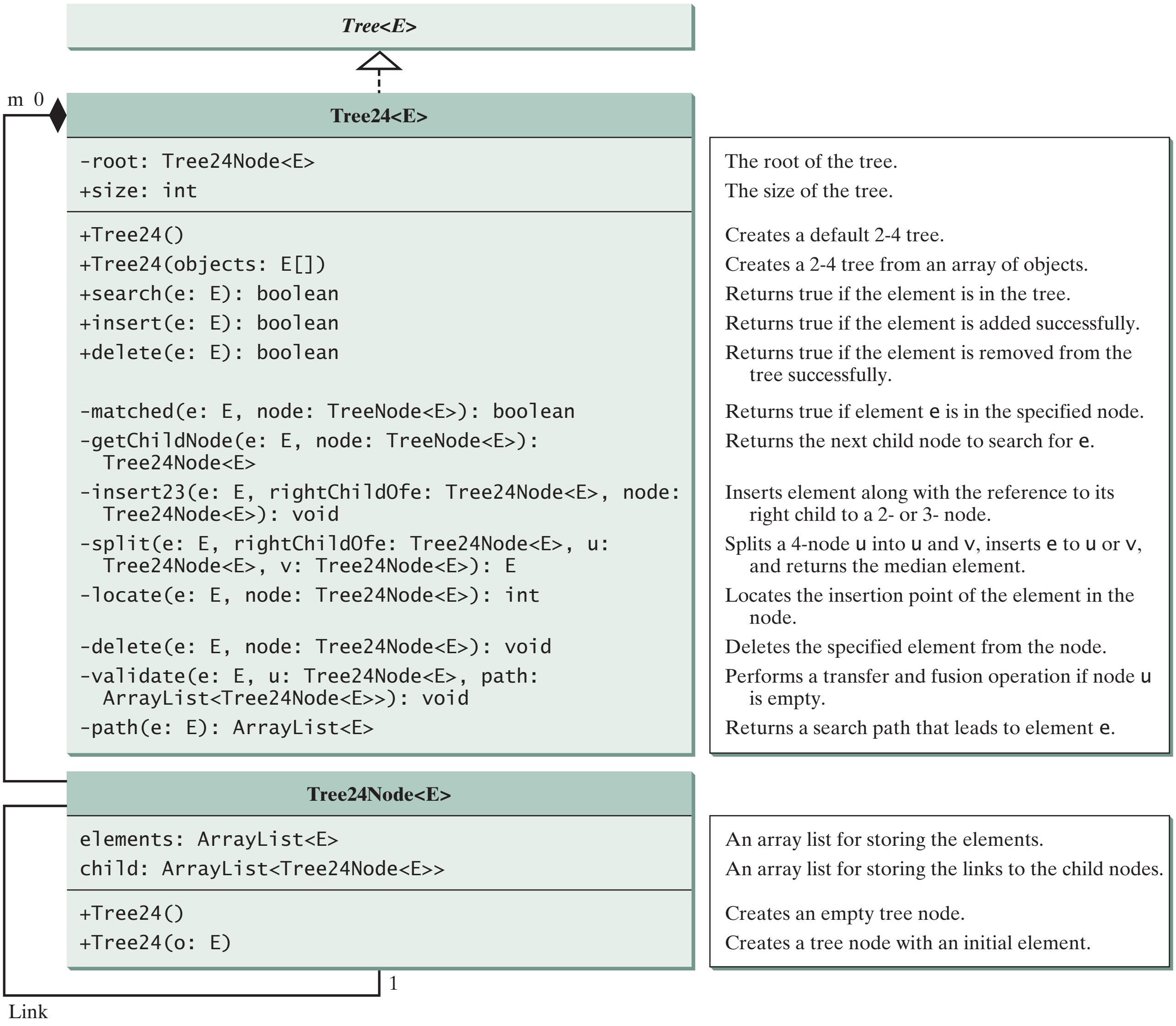


http://www.cs.armstrong.edu/liang/animation/Tree24Animation.html

2-4 Tree Animation by Y. Daniel Liang (Note: the keys are integers)

Enter a key: 26

FIGURE 46.3 The animation tool enables you to insert, delete, and search elements in a 2-4 tree visually.



`-root: Tree24Node<E>`	The root of the tree.
`+size: int`	The size of the tree.
`+Tree24()`	Creates a default 2-4 tree.
`+Tree24(objects: E[])`	Creates a 2-4 tree from an array of objects.
`+search(e: E): boolean`	Returns true if the element is in the tree.
`+insert(e: E): boolean`	Returns true if the element is added successfully.
`+delete(e: E): boolean`	Returns true if the element is removed from the tree successfully.
`-matched(e: E, node: TreeNode<E>): boolean`	Returns true if element `e` is in the specified node.
`-getChildNode(e: E, node: TreeNode<E>): Tree24Node<E>`	Returns the next child node to search for `e`.
`-insert23(e: E, rightChildOfe: Tree24Node<E>, node: Tree24Node<E>): void`	Inserts element along with the reference to its right child to a 2- or 3- node.
`-split(e: E, rightChildOfe: Tree24Node<E>, u: Tree24Node<E>, v: Tree24Node<E>): E`	Splits a 4-node `u` into `u` and `v`, inserts `e` to `u` or `v`, and returns the median element.
`-locate(e: E, node: Tree24Node<E>): int`	Locates the insertion point of the element in the node.
`-delete(e: E, node: Tree24Node<E>): void`	Deletes the specified element from the node.
`-validate(e: E, u: Tree24Node<E>, path: ArrayList<Tree24Node<E>>): void`	Performs a transfer and fusion operation if node `u` is empty.
`-path(e: E): ArrayList<E>`	Returns a search path that leads to element `e`.

`elements: ArrayList<E>`	An array list for storing the elements.
`child: ArrayList<Tree24Node<E>>`	An array list for storing the links to the child nodes.
`+Tree24()`	Creates an empty tree node.
`+Tree24(o: E)`	Creates a tree node with an initial element.

FIGURE 46.4 The `Tree24` class implements `Tree`.

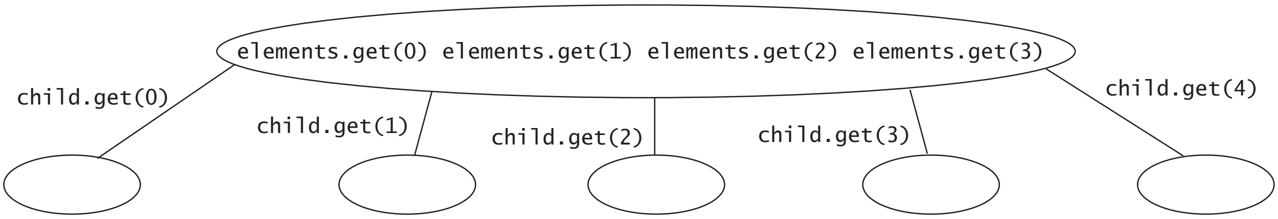


FIGURE 46.5 A 2-4 tree node stores the elements and the links to the child nodes in array lists.

46.3 Searching an Element

Searching an element in a 2-4 tree is similar to searching an element in a binary tree. The difference is that you have to search an element within a node in addition to searching elements along the path. To search an element in a 2-4 tree, you start from the root and scan down. If an element is not in the node, move to an appropriate subtree. Repeat the process until a match is found or you arrive at an empty subtree. The algorithm is described in Listing 46.1.

LISTING 46.1 Searching an Element in a 2-4 Tree

start from root

found

search a subtree

not found

```
 1 boolean search(E e) {
 2   current = root; // Start from the root
 3
 4   while (current != null) {
 5     if (match(e, current)) { // Element is in the node
 6       return true; // Element is found
 7     }
 8     else {
 9       current = getChildNode(e, current); // Search in a subtree
10     }
11   }
12
13   return false; // Element is not in the tree
14 }
```

The `match(e, current)` method checks whether element `e` is in the current node. The `getChildNode(e, current)` method returns the root of the subtree for further search. Initially, let `current` point to the root (line 2). Repeat searching the element in the current node until `current` is `null` (line 4) or the element matches an element in the current node.

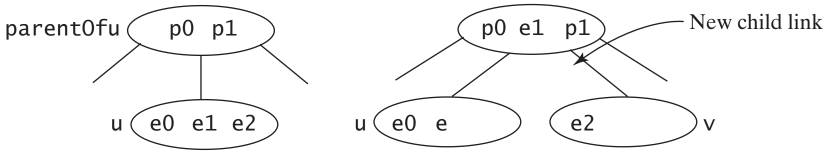
46.4 Inserting an Element into a 2-4 Tree

overflow

split

To insert an element `e` to a 2-4 tree, locate a leaf node in which the element will be inserted. If the leaf node is a 2-node or 3-node, simply insert the element into the node. If the node is a 4-node, inserting a new element would cause an *overflow*. To resolve overflow, perform a *split* operation as follows:

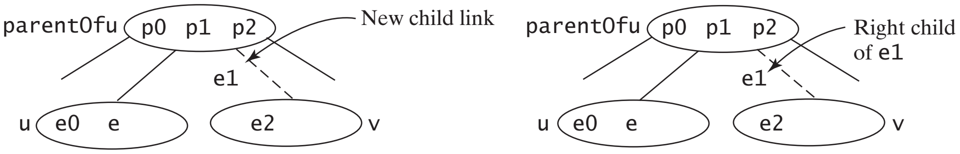
- Let `u` be the *leaf* 4-node in which the element will be inserted and `parentOfu` be the parent of `u`, as shown in Figure 46.6(a).
- Create a new node named `v`; move `e2` to `v`.
- If `e` < `e1`, insert `e` to `u`; otherwise insert `e` to `v`. Assume that `e0` < `e` < `e1`, `e` is inserted into `u`, as shown in Figure 46.6(b).
- Insert `e1` along with its right child (i.e., `v`) to the parent node, as shown in Figure 46.6(b).



(a) Before inserting `e` (b) After inserting `e`

FIGURE 46.6 The splitting operation creates a new node and inserts the median element to its parent.

The parent node is a 3-node in Figure 46.6. So, there is room to insert **`e`** to the parent node. What happens if it is a 4-node, as shown in Figure 46.7? This requires that the parent node be split. The process is the same as splitting a leaf 4-node, except that you must also insert the element along with its right child.

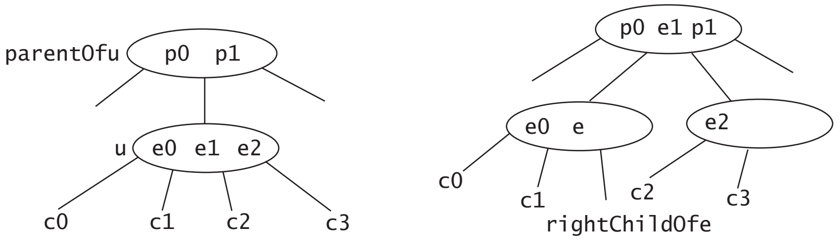


(a) The parent is a 4-node (b) Inserting `e1` into the parent

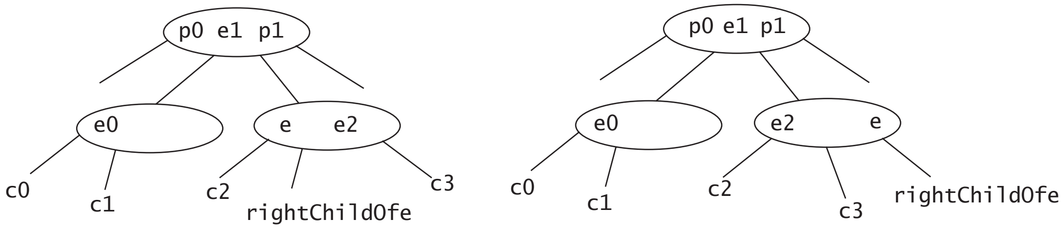
FIGURE 46.7 Insertion process continues if the parent node is a 4-node.

The algorithm can be modified as follows:

- Let **`u`** be the 4-node (*leaf or nonleaf*) in which the element will be inserted and **`parentOfu`** be the parent of **`u`**, as shown in Figure 46.8(a).
- Create a new node named **`v`**, move **`e2`** and its children **`c2`** and **`c3`** to **`v`**.
- If **`e`** < **`e1`**, insert **`e`** along with its right child link to **`u`**; otherwise insert **`e`** along with its right child link to **`v`**, as shown in Figure 46.6(b), (c), (d) for the cases **`e0`** < **`e`** < **`e1`**, **`e1`** < **`e`** < **`e2`**, and **`e2`** < **`e`**, respectively.
- Insert **`e1`** along with its right child (i.e., **`v`**) to the parent node, recursively.



(a) Before inserting `e` (b) After inserting `e` (`e0` < `e` < `e1`)



(c) After inserting `e2` < `e` (`e1` < `e` < `e2`) (d) After inserting `e` (`e2` < `e`)

FIGURE 46.8 An interior node may be split to resolve overflow.

Listing 46.2 gives an algorithm for inserting an element.

LISTING 46.2 Inserting an Element to a 2-4 Tree

create a new node
search **e**
insert **e**
one element added
element added
insert to a node
a 2- or 3-node
split 4-node
new root
insert median to parent

```
 1 public boolean insert(E e) {
 2   if (root == null)
 3     root = new Tree24Node<E>(e); // Create a new root for element
 4   else {
 5     Locate leafNode for inserting e
 6     insert(e, null, leafNode); // The right child of e is null
 7   }
 8
 9   size++; // Increase size
10   return true; // Element inserted
11 }
12
13 private void insert(E e, Tree24Node<E> rightChildOfe,
14     Tree24Node<E> u) {
15   if (u is a 2- or 3- node) { // u is a 2- or 3-node
16     insert23(e, rightChildOfe, u); // Insert e to node u
17   }
18   else { // Split a 4-node u
19     Tree24Node<E> v = new Tree24Node<E>(); // Create a new node
20     E median = split(e, rightChildOfe, u, v); // Split u
21
22     if (u == root) { // u is the root
23       root = new Tree24Node<E>(median); // New root
24       root.child.add(u); // u is the left child of median
25       root.child.add(v); // v is the right child of median
26     }
27     else {
28       Get the parent of u, parentOfu;
29       insert(median, v, parentOfu); // Inserting median to parent
30     }
31   }
32 }
```

The `insert(E e, Tree24Node<E> rightChildOfe, Tree24Node<E> u)` method inserts element `e` along with its right child to node `u`. When inserting `e` to a leaf node, the right child of `e` is `null` (line 6). If the node is a 2- or 3-node, simply insert the element to the node (lines 15–17). If the node is a 4-node, invoke the `split` method to split the node (line 20). The `split` method returns the median element. Recursively invoke the `insert` method to insert the median element to the parent node (line 29). Figure 46.9 shows the steps of inserting elements `34`, `3`, `50`, `20`, `15`, `16`, `25`, `27`, `29`, and `24` into a 2-4 tree.

46.5 Deleting an Element from a 2-4 Tree

To delete an element from a 2-4 tree, first search the element in the tree to locate the node that contains the element. If the element is not in the tree, the method returns false. Let `u` be the node that contains the element and `parentOfu` be the parent of `u`. Consider three cases:

Case 1: `u` is a leaf 3-node or 4-node. Delete `e` from `u`.

underflow

Case 2: `u` is a leaf 2-node. Delete `e` from `u`. Now `u` is empty. This situation is known as *underflow*. To remedy an underflow, consider two subcases:

transfer

Case 2.1: `u`'s immediate left or right sibling is a 3- or 4-node. Let the node be `w`, as shown in Figure 46.10(a) (assume that `w` is a left sibling of `u`). Perform a *transfer* operation that

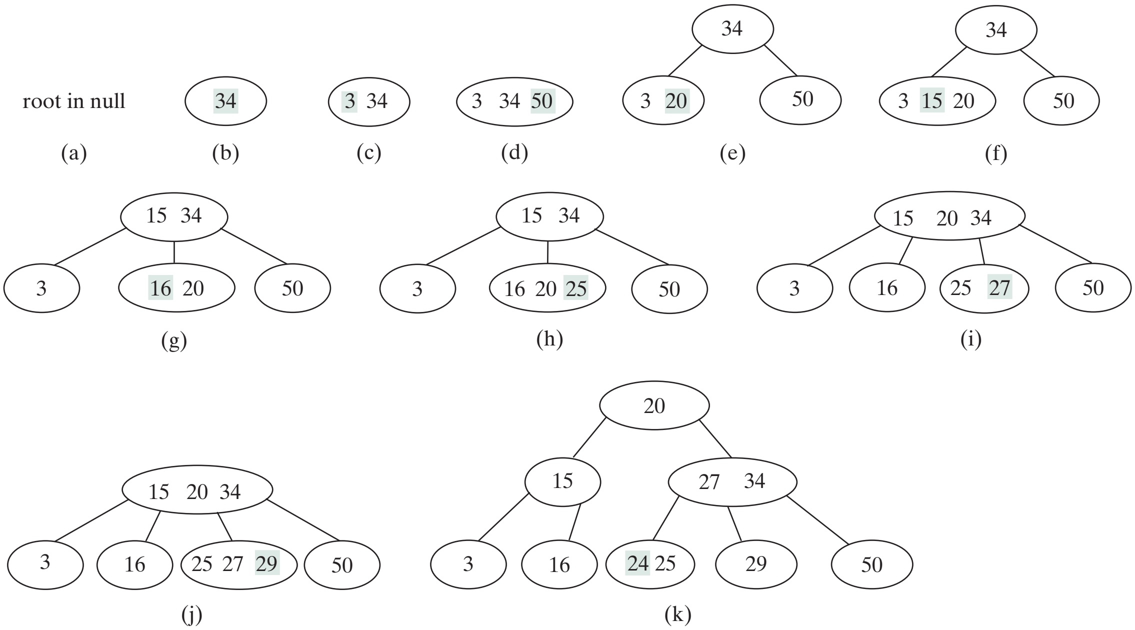
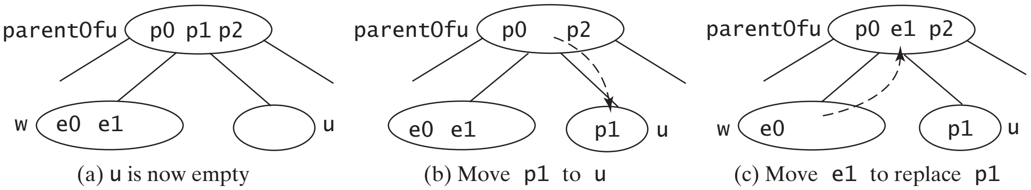


FIGURE 46.9 The tree changes after `34`, `3`, `50`, `20`, `15`, `16`, `25`, `27`, `29`, and `24` are added into an empty tree.



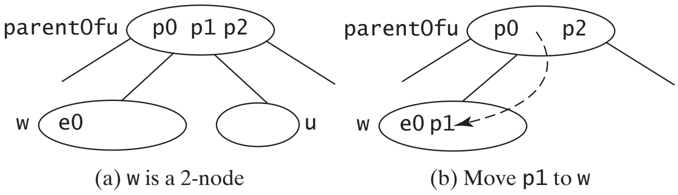
(a) `u` is now empty (b) Move `p1` to `u` (c) Move `e1` to replace `p1`

FIGURE 46.10 The transfer operation fills the empty node `u`.

moves an element from `parentOfu` to `u`, as shown in Figure 46.10(b), and move an element from `w` to replace the moved element in `parentOfu`, as shown in Figure 46.10(c).

Case 2.2: Both `u`'s immediate left and right sibling are 2-node if they exist (`u` may have only one sibling). Let the node be `w`, as shown in Figure 46.11(a) (assume that `w` is a left sibling of `u`). Perform a *fusion* operation that discards `u` and moves an element from `parentOfu` to `w`, as shown in Figure 46.11(b). If `parentOfu` becomes empty, repeat Case 2 recursively to perform a transfer or a fusion on `parentOfu`.

fusion

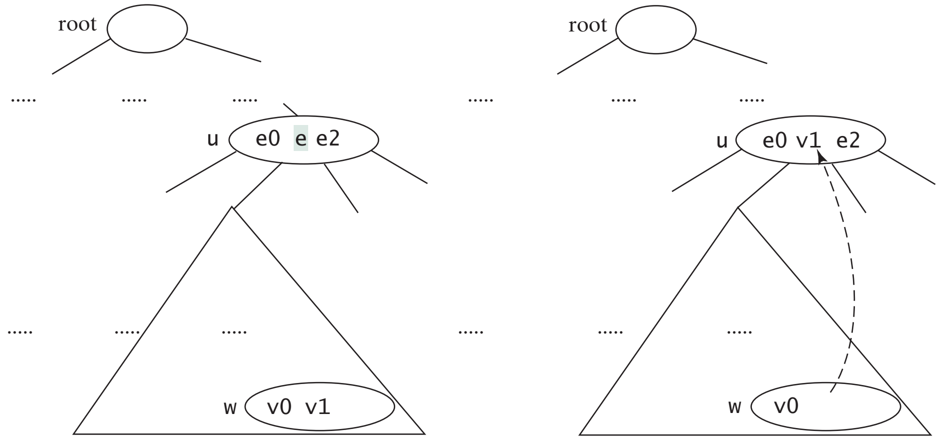


(a) `w` is a 2-node (b) Move `p1` to `w`

FIGURE 46.11 The fusion operation discards the empty node `u`.

internal node

Case 3: **u** is a nonleaf node. Find the rightmost leaf node in the left subtree of **e**. Let this node be **w**, as shown in Figure 46.12(a). Move the last element in **w** to replace **e** in **u**, as shown in Figure 46.12(b). If **w** becomes empty, apply a transfer or fusion operation on **w**.



(a) e is in u

(b) Replace e with v1

FIGURE 46.12 An element in the internal node is replaced by an element in a leaf node.

Listing 46.3 describes the algorithm for deleting an element.

LISTING 46.3 Deleting an Element from a 2-4 Tree

```
/** Delete the specified element from the tree */
public boolean delete(E e) {
  Locate the node that contains the element e
  if (the node is found) {
     delete(e, node); // Delete element e from the node
    size--; // After one element deleted
    return true; // Element deleted successfully
  }

  return false; // Element not in the tree
}

/** Delete the specified element from the node */
private void delete(E e, Tree24Node<E> node) {
  if (e is in a leaf node) {
    // Get the path that leads to e from the root
    ArrayList<Tree24Node<E>> path = path(e);

    Remove e from the node;

    // Check node for underflow along the path and fix it
    validate(e, node, path); // Check underflow node
  }
  else { // e is in an internal node
    Locate the rightmost node in the left subtree of node u;
    Get the rightmost element from the rightmost node;

    // Get the path that leads to e from the root
    ArrayList<Tree24Node<E>> path = path(rightmostElement);

```

locate the node

delete **e**

element not found

delete **e**

delete **e**

check and fix underflow

locate rightmost element

```
31     Replace the element in the node with the rightmost element
32
33     // Check node for underflow along the path and fix it
34     validate(rightmostElement, rightmostNode, path);
35   }
36 }
37
38 /** Perform a transfer or fusion operation if necessary */
39 private void validate(E e, Tree24Node<E> u,
40     ArrayList<Tree24Node<E>> path) {
41   for (int i = path.size() - 1; i >= 0; i--) {
42     if (u is not empty)
43       return; // Done, no need to perform transfer or fusion
44
45     Tree24Node<E> parentOfu = path.get(i - 1); // Get parent of u
46
47     // Check two siblings
48     if (left sibling of u has more than one element) {
49       Perform a transfer on u with its left sibling
50     }
51     else if (right sibling of u has more than one element) {
52       Perform a transfer on u with its right sibling
53     }
54     else if (u has left sibling) { // Fusion with a left sibling
55       Perform a fusion on u with its left sibling
56       u = parentOfu; // Back to the loop to check the parent node
57     }
58     else { // Fusion with right sibling (right sibling must exist)
59       Perform a fusion on u with its right sibling
60       u = parentOfu; // Back to the loop to check the parent node
61     }
62   }
63 }
```

check and fix underflow

check and fix underflow

The **`delete(E e)`** method locates the node that contains the element **`e`** and invokes the **`delete(E e, Tree24Node<E> node)`** method (line 5) to delete the element from the node.

If the node is a leaf node, get the path that leads to **`e`** from the root (line 17), delete **`e`** from the node (line 19), and invoke **`validate`** to check and fix the empty node (line 22). The **`validate(E e, Tree24Node<E> u, ArrayList<Tree24Node<E>> path)`** method performs a transfer or fusion operation if the node is empty. Since these operations may cause the parent of node **`u`** to become empty, a path is obtained in order to obtain the parents along the path from the root to node **`u`**, as shown in Figure 46.13.

If the node is a nonleaf node, locate the rightmost element in the left subtree of the node (lines 25–26), get the path that leads to the rightmost element from the root (line 29), replace **`e`** in the node with the rightmost element (line 31), and invoke **`validate`** to fix the rightmost node if it is empty (line 34).

The **`validate(E e, Tree24Node<E> u, ArrayList<Tree24Node<E>> path)`** checks whether **`u`** is empty and performs a transfer or fusion operation to fix the empty node. The **`validate`** method exits when node is not empty (line 43). Otherwise, consider one of the following cases:

1. If **`u`** has a left sibling with more than one element, perform a transfer on **`u`** with its left sibling (line 49).
2. Otherwise, if **`u`** has a right sibling with more than one element, perform a transfer on **`u`** with its right sibling (line 52).

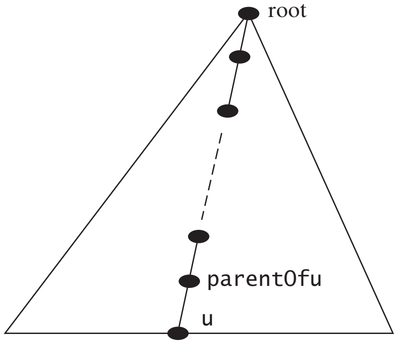
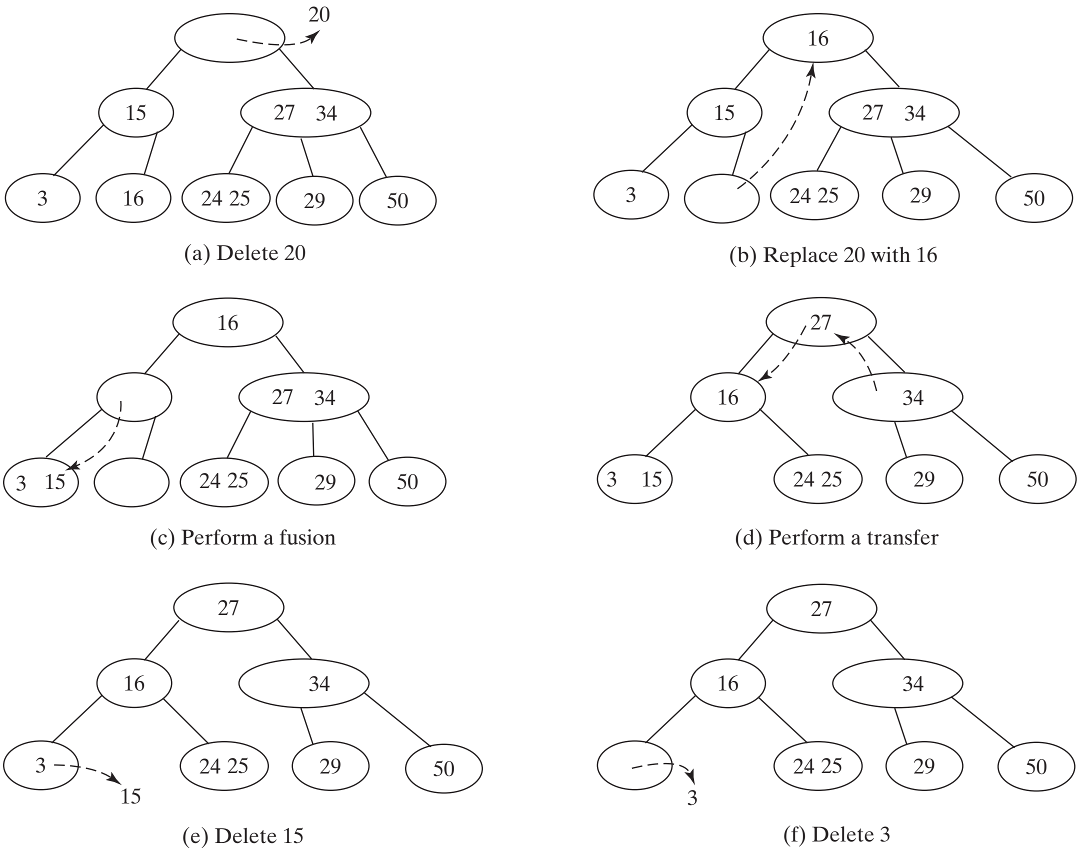


FIGURE 46.13 The nodes along the path may become empty as result of a transfer and fusion operation.

3. Otherwise, if **u** has a left sibling, perform a fusion on **u** with its left sibling (line 55) and reset **u** to **parentOfu** (line 56).
4. Otherwise, **u** must have a right sibling. Perform a fusion on **u** with its right sibling (line 59) and reset **u** to **parentOfu** (line 60).

Only one of the preceding cases is executed. Afterward, a new iteration starts to perform a transfer or fusion operation on a new node **u** if needed. Figure 46.14 shows the steps of deleting elements **20**, **15**, **3**, **6**, and **34** are deleted from a 2-4 tree in Figure 46.9(k).



(a) Delete 20

(b) Replace 20 with 16

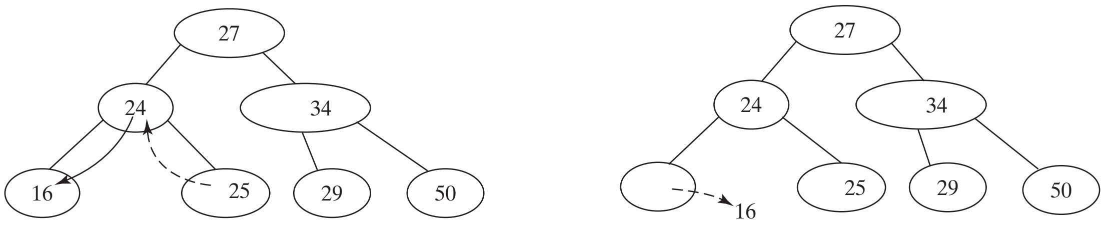
(c) Perform a fusion

(d) Perform a transfer

(e) Delete 15

(f) Delete 3

FIGURE 46.14 The tree changes after **20**, **15**, **3**, **6**, and **34** are deleted from a 2-4 tree.

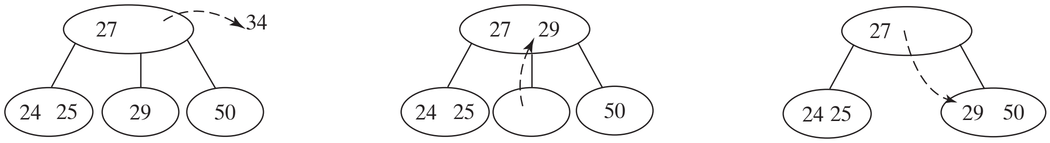


(g) Perform a transfer

(h) Delete 16

(i) Perform a fusion

(j) Perform a fusion



(k) Delete 34

(l) Replace 34 with 16

(m) Perform a fusion

Figure 46.14 *continued*

46.6 Traversing Elements in a 2-4 Tree

Inorder, preorder, postorder traversals are useful for 2-4 trees. Inorder traversal visits the elements in increasing order. Preorder traversal visits the elements in the root, then recursively visits the subtrees from the left to right. Postorder traversal visits the subtrees from the left to right recursively, and then the elements in the root.

For example, in the 2-4 tree in Figure 46.9(k), the inorder traversal is

```
3 15 16 20 24 25 27 29 34 50
```

The preorder traversal is

```
20 15 3 16 27 34 24 25 29 50
```

The postorder traversal is

```
3 16 1 24 25 29 50 27 34 20
```

46.7 Implementing the **Tree24** Class

Listing 46.4 gives the complete source code for the `Tree24` class.

Listing 46.4 Tree24.java

```
import java.util.ArrayList;

public class Tree24<E extends Comparable<E>> implements Tree<E> {
  private Tree24Node root;
  private int size;

  /** Create a default 2-4 tree */
  public Tree24() {
 }

```

root

size

no-arg constructor

constructor
search
found?
next subtree
find a match
matched?
next subtree
leaf node?
insertion point
insert to tree
empty tree?
find leaf node
insert to node

```java
/** Create a 2-4 tree from an array of objects */
public Tree24(E[] elements) {
  for (int i = 0; i < elements.length; i++)
    insert(elements[i]);
}

/** Search an element in the tree */
public boolean search(E e) {
  Tree24Node<E> current = root; // Start from the root

  while (current != null) {
    if (matched(e, current)) { // Element is in the node
      return true; // Element found
    }
    else {
      current = getChildNode(e, current); // Search in a subtree
    }
  }

  return false; // Element is not in the tree
}

/** Return true if the element is found in this node */
private boolean matched(E e, Tree24Node<E> node) {
  for (int i = 0; i < node.elements.size(); i++)
    if (node.elements.get(i).equals(e))
      return true; // Element found

  return false; // No match in this node
}

/** Locate a child node to search element e */
private Tree24Node<E> getChildNode(E e, Tree24Node<E> node) {
  if (node.child.size() == 0)
    return null; // node is a leaf

  int i = locate(e, node); // Locate the insertion point for e
  return node.child.get(i); // Return the child node
}

/** Insert element e into the tree
 *  Return true if the element is inserted successfully
 */
public boolean insert(E e) {
  if (root == null)
    root = new Tree24Node<E>(e); // Create a new root for element
  else {
    // Locate the leaf node for inserting e
    Tree24Node<E> leafNode = null;
    Tree24Node<E> current = root;
    while (current != null)
      if (matched(e, current)) {
        return false; // Duplicate element found, nothing inserted
      }
      else {
        leafNode = current;
        current = getChildNode(e, current);
       }

    // Insert the element e into the leaf node
    insert(e, null, leafNode); // The right child of e is null
```

```
    }

    size++; // Increase size
    return true; // Element inserted
  }

  /** Insert element e into node u */
  private void insert(E e, Tree24Node<E> rightChildOfe,
      Tree24Node<E> u) {
    // Get the search path that leads to element e
    ArrayList<Tree24Node<E>> path = path(e);

    for (int i = path.size() - 1; i >= 0; i--) {
      if (u.elements.size() < 3) { // u is a 2-node or 3-node
        insert23(e, rightChildOfe, u); // Insert e to node u
        break; // No further insertion to u's parent needed
      }
      else {
        Tree24Node<E> v = new Tree24Node<E>(); // Create a new node
        E median = split(e, rightChildOfe, u, v); // Split u

        if (u == root) {
          root = new Tree24Node<E>(median); // New root
          root.child.add(u); // u is the left child of median
          root.child.add(v); // v is the right child of median
          break; // No further insertion to u's parent needed
        }
        else {
          // Use new values for the next iteration in the for loop
          e = median; // Element to be inserted to parent
          rightChildOfe = v; // Right child of the element
          u = path.get(i - 1); // New node to insert element
        }
      }
    }
  }

  /** Insert element to a 2- or 3- and return the insertion point */
  private void insert23(E e, Tree24Node<E> rightChildOfe,
      Tree24Node<E> node) {
    int i = this.locate(e, node); // Locate where to insert
    node.elements.add(i, e); // Insert the element into the node
    if (rightChildOfe != null)
      node.child.add(i + 1, rightChildOfe); // Insert the child link
  }

  /** Split a 4-node u into u and v and insert e to u or v */
  private E split(E e, Tree24Node<E> rightChildOfe,
      Tree24Node<E> u, Tree24Node<E> v) {
    // Move the last element in node u to node v
    v.elements.add(u.elements.remove(2));
    E median = u.elements.remove(1);

    // Split children for a nonleaf node
    // Move the last two children in node u to node v
    if (u.child.size() > 0) {
      v.child.add(u.child.remove(2));
      v.child.add(u.child.remove(2));
    }

```

insert to node

no overflow

overflow

split

u is root?

insert to **parentOfu**

insert to node

insertion point

split

get median

insert **e**

insert **rightChildOfe**

return median

get path

add node searched

return path

delete from tree

locate the node

found?

delete from node

delete from node

leaf node?

delete **e**

node is root?

validate tree

```
    // Insert e into a 2- or 3- node u or v.
    if (e.compareTo(median) < 0)
      insert23(e, rightChildOfe, u);
    else
      insert23(e, rightChildOfe, v);

    return median; // Return the median element
  }

  /** Return a search path that leads to element e */
  private ArrayList<Tree24Node<E>> path(E e) {
    ArrayList<Tree24Node<E>> list = new ArrayList<Tree24Node<E>>();
    Tree24Node<E> current = root; // Start from the root

    while (current != null) {
      list.add(current); // Add the node to the list
      if (matched(e, current)) {
        break; // Element found
      }
      else {
        current = getChildNode(e, current);
      }
    }

    return list; // Return an array of nodes
  }

  /** Delete the specified element from the tree */
  public boolean delete(E e) {
    // Locate the node that contains the element e
    Tree24Node<E> node = root;
    while (node != null)
      if (matched(e, node)) {
        delete(e, node); // Delete element e from node
        size--; // After one element deleted
        return true; // Element deleted successfully
      }
      else {
        node = getChildNode(e, node);
      }

    return false; // Element not in the tree
  }

  /** Delete the specified element from the node */
  private void delete(E e, Tree24Node<E> node) {
    if (node.child.size() == 0) { // e is in a leaf node
      // Get the path that leads to e from the root
      ArrayList<Tree24Node<E>> path = path(e);

      node.elements.remove(e); // Remove element e

      if (node == root) { // Special case
        if (node.elements.size() == 0)
          root = null; // Empty tree
        return; // Done
      }

      validate(e, node, path); // Check underflow node
    }
```

```
    else { // e is in an internal node
      // Locate the rightmost node in the left subtree of the node
      int index = locate(e, node); // Index of e in node
      Tree24Node<E> current = node.child.get(index);
      while (current.child.size() > 0) {
        current = current.child.get(current.child.size() - 1);
      }
      E rightmostElement =
        current.elements.get(current.elements.size() - 1);

      // Get the path that leads to e from the root
      ArrayList<Tree24Node<E>> path = path(rightmostElement);

      // Replace the deleted element with the rightmost element
      node.elements.set(index, current.elements.remove(
        current.elements.size() - 1));

      validate(rightmostElement, current, path); // Check underflow
    }
  }

  /** Perform transfer and confusion operations if necessary */
  private void validate(E e, Tree24Node<E> u,
      ArrayList<Tree24Node<E>> path) {
    for (int i = path.size() - 1; u.elements.size() == 0; i--) {
      Tree24Node<E> parentOfu = path.get(i - 1); // Get parent of u
      int k = locate(e, parentOfu); // Index of e in the parent node

      // Check two siblings
      if (k > 0 && parentOfu.child.get(k - 1).elements.size() > 1) {
        leftSiblingTransfer(k, u, parentOfu);
      }
      else if (k + 1 < parentOfu.child.size() &&
        parentOfu.child.get(k + 1).elements.size() > 1) {
        rightSiblingTransfer(k, u, parentOfu);
      }
      else if (k - 1 >= 0) { // Fusion with a left sibling
        // Get left sibling of node u
        Tree24Node<E> leftNode = parentOfu.child.get(k - 1);

        // Perform a fusion with left sibling on node u
        leftSiblingFusion(k, leftNode, u, parentOfu);

        // Done when root becomes empty
        if (parentOfu == root && parentOfu.elements.size() == 0) {
          root = leftNode;
          break;
        }

        u = parentOfu; // Back to the loop to check the parent node
      }
      else { // Fusion with right sibling (right sibling must exist)
        // Get left sibling of node u
        Tree24Node<E> rightNode = parentOfu.child.get(k + 1);

        // Perform a fusion with right sibling on node u
        rightSiblingFusion(k, rightNode, u, parentOfu);

        // Done when root becomes empty
        if (parentOfu == root && parentOfu.elements.size() == 0) {
```

nonleaf node

rightmost element

replace element

validate tree

validate tree

transfer with left sibling

transfer with right sibling

fusion with left sibling

fusion with right sibling

```
          root = rightNode;
          break;
        }

        u = parentOfu; // Back to the loop to check the parent node
      }
    }
 }

  /** Locate the insertion point of the element in the node */
```

locate insertion point

```
  private int locate(E o, Tree24Node<E> node) {
    for (int i = 0; i < node.elements.size(); i++) {
      if (o.compareTo(node.elements.get(i)) <= 0) {
        return i;
      }
    }

    return node.elements.size();
  }

  /** Perform a transfer with a left sibling */
```

transfer with left sibling

```
  private void leftSiblingTransfer(int k,
      Tree24Node<E> u, Tree24Node<E> parentOfu) {
    // Move an element from the parent to u
    u.elements.add(0, parentOfu.elements.get(k - 1));

    // Move an element from the left node to the parent
    Tree24Node<E> leftNode = parentOfu.child.get(k - 1);
    parentOfu.elements.set(k - 1,
      leftNode.elements.remove(leftNode.elements.size() - 1));

    // Move the child link from left sibling to the node
    if (leftNode.child.size() > 0)
      u.child.add(0, leftNode.child.remove(
        leftNode.child.size() - 1));
  }

  /** Perform a transfer with a right sibling */
```

transfer with right sibling

```
  private void rightSiblingTransfer(int k,
      Tree24Node<E> u, Tree24Node<E> parentOfu) {
    // Transfer an element from the parent to u
    u.elements.add(parentOfu.elements.get(k));

    // Transfer an element from the right node to the parent
    Tree24Node<E> rightNode = parentOfu.child.get(k + 1);
    parentOfu.elements.set(k, rightNode.elements.remove(0));

    // Move the child link from right sibling to the node
    if (rightNode.child.size() > 0)
      u.child.add(rightNode.child.remove(0));
  }

  /** Perform a fusion with a left sibling */
```

fusion with left sibling

```
  private void leftSiblingFusion(int k, Tree24Node<E> leftNode,
      Tree24Node<E> u, Tree24Node<E> parentOfu) {
    // Transfer an element from the parent to the left sibling
    leftNode.elements.add(parentOfu.elements.remove(k - 1));

    // Remove the link to the empty node
    parentOfu.child.remove(k);

```

```
    // Adjust child links for nonleaf node
    if (u.child.size() > 0)
      leftNode.child.add(u.child.remove(0));
  }

```

fusion with right sibling

```
  /** Perform a fusion with a right sibling */
  private void rightSiblingFusion(int k, Tree24Node<E> rightNode,
      Tree24Node<E> u, Tree24Node<E> parentOfu) {
    // Transfer an element from the parent to the right sibling
    rightNode.elements.add(0, parentOfu.elements.remove(k));

    // Remove the link to the empty node
    parentOfu.child.remove(k);

    // Adjust child links for nonleaf node
    if (u.child.size() > 0)
      rightNode.child.add(0, u.child.remove(0));
  }

  /** Get the number of nodes in the tree */
  public int getSize() {
    return size;
  }

```

preorder

```
  /** Preorder traversal from the root */
  public void preorder() {
    preorder(root);
  }

```

recursive preorder

```
  /** Preorder traversal from a subtree */
  private void preorder(Tree24Node<E> root) {
    if (root == null)return;
    for (int i = 0; i < root.elements.size(); i++)
      System.out.print(root.elements.get(i) + " ");

    for (int i = 0; i < root.child.size(); i++)
      preorder(root.child.get(i));
  }

  /** Inorder traversal from the root*/
  public void inorder() {
    // Left as exercise
  }

  /** Postorder traversal from the root */
  public void postorder() {
    // Left as exercise
  }

  /** Return true if the tree is empty */
  public boolean isEmpty() {
    return root == null;
  }

  /** Return an iterator to traverse elements in the tree */
  public java.util.Iterator iterator() {
    // Left as exercise
    return null;
  }

```

inner **Tree24Node** class

element list

child list

```
373   /** Define a 2-4 tree node */
374   protected static class Tree24Node<E extends Comparable<E>> {
375     // elements has maximum three values
376     ArrayList<E> elements = new ArrayList<E>(3);
377     // Each has maximum four childres
378     ArrayList<Tree24Node<E>> child
379       = new ArrayList<Tree24Node<E>>(4);
380
381     /** Create an empty Tree24 node */
382     Tree24Node() {
383     }
384
385     /** Create a Tree24 node with an initial element */
386     Tree24Node(E o) {
387       elements.add(o);
388     }
389   }
390 }
```

root
size

The `Tree24` class contains the data fields `root` and `size` (lines 4–5). `root` references the root node and `size` stores the number of elements in the tree.

constructors

The `Tree24` class has two constructors: a no-arg constructor (lines 8–9) that constructs an empty tree and a constructor that creates an initial `Tree24` from an array of elements (lines 12–15).

search

The `search` method (lines 18–31) searches an element in the tree. It returns `true` (line 23) if the element is in the tree and returns `false` if the search arrives at an empty subtree (line 32).

matched

The `matched(e, node)` method (lines 34–40) checks where the element `e` is in the node.

getChildNode

The `getChildNode(e, node)` method (lines 43–49) returns the root of a subtree where `e` should be searched.

insert(e)

The `insert(E e)` method inserts an element in a tree (lines 54–76). If the tree is empty, a new root is created (line 56). The method locates a leaf node in which the element will be inserted and invokes `insert(e, null, leafNode)` to insert the element (line 71).

insert(e, rightChildOfe, u)

The `insert(e, rightChildOfe, u)` method inserts an element into node `u` (lines 79–107). The method first invokes `path(e)` (line 82) to obtain a search path from the root to node `u`. Each iteration of the `for` loop considers `u` and its parent `parentOfu` (lines 84–106). If `u` is a 2-node or 3-node, invoke `insert23(e, rightChildOfe, u)` to insert `e` and its child link `rightChildOfe` into `u` (line 86). No split is needed (line 87). Otherwise, create a new node `v` (line 90) and invoke `split(e, rightChildOfe, u, v)` (line 91) to split `u` into `u` and `v`. The `split` method inserts `e` into either `u` and `v` and returns the median in the original `u`. If `u` is the root, create a new root to hold median, and set `u` and `v` as the left and right children for median (lines 95–96). If `u` is not the root, insert median to `parentOfu` in the next iteration (lines 101–103).

insert23

The `insert23(e, rightChildOfe, node)` method inserts `e` along with the reference to its right child into the node (lines 110–116). The method first invokes `locate(e, node)` (line 112) to locate an insertion point, then insert `e` into the node (line 113). If `rightChildOfe` is not `null`, it is inserted into the child list of the node (line 115).

split

The `split(e, rightChildOfe, u, v)` method splits a 4-node `u` (lines 119–139). This is accomplished as follows: (1) move the last element from `u` to `v` and remove the median element from `u` (lines 122–123); (2) move the last two child links from `u` to `v` (lines 127–130) if `u` is a nonleaf node; (3) if `e < median`, insert `e` into `u`; otherwise, insert `e` into `v` (lines 133–136); (4) return median (line 138).

path

The `path(e)` method returns an `ArrayList` of nodes searched from the root in order to locate `e` (lines 142–157). If `e` is in the tree, the last node in the path contains `e`. Otherwise the last node is where `e` should be inserted.

delete(e)

The `delete(E e)` method deletes an element from the tree (lines 160–174). The method first locates the node that contains `e` and invokes `delete(e, node)` to delete `e` from the node (line 165). If the element is not in the tree, return `false` (line 173).

delete(e, node)

The `delete(e, node)` method deletes an element from node `u` (lines 177–211). If the node is a leaf node, obtain the path that leads to `e` (line 180), delete `e` (line 182), set root to `null` if the tree becomes empty (lines 184–188), and invoke `validate` to apply transfer and fusion operation on empty nodes (line 190). If the node is a nonleaf node, locate the rightmost element (lines 194–200), obtain the path that leads to `e` (line 203), replace `e` with the rightmost element (lines 206–207), and invoke `validate` to apply transfer and fusion operations on empty nodes (line 209).

validate

The `validate(e, u, path)` method ensures that the tree is a valid 2-4 tree (lines 214–259). The `for` loop terminates when `u` is not empty (line 216). The loop body is executed to fix the empty node `u` by performing a transfer or fusion operation. If a left sibling with more than one element exists, perform a transfer on `u` with the left sibling (line 222). Otherwise, if a right sibling with more than one element exists, perform a transfer on `u` with the left sibling (line 226). Otherwise, if a left sibling exists, perform a fusion on `u` with the left sibling (lines 230–239), and validate `parentOfu` in the next loop iteration (line 241). Otherwise, perform a fusion on `u` with the right sibling.

locate

The `locate(e, node)` method locates the index of `e` in the node (lines 262–270).

transfer

The `leftSiblingTransfer(k, u, parentOfu)` method performs a transfer on `u` with its left sibling (lines 273–287). The `rightSiblingTransfer(k, u, parentOfu)` method performs a transfer on `u` with its right sibling (lines 290–302). The `leftSiblingFusion(k, leftNode, u, parentOfu)` method performs a fusion on `u` with its left sibling `leftNode` (lines 305–316). The `rightSiblingFusion(k, rightNode, u, parentOfu)` method performs a fusion on `u` with its right sibling `rightNode` (lines 319–330).

fusion

preorder

The `preorder()` method displays all the elements in the tree in preorder (lines 338–350).

Tree24Node

The inner class `Tree24Node` defines a class for a node in the tree (lines 374–389).

46.8 Testing the **Tree24** Class

Listing 46.5 gives a test program. The program creates a 2-4 tree and inserts elements in lines 6–20, and deletes elements in lines 22–56.

Listing 46.5 TestTree24.java

```
 1 public class TestTree24 {
 2   public static void main(String[] args) {
 3     // Create a 2-4 tree
 4     Tree24<Integer> tree = new Tree24<Integer>();
 5
 6     tree.insert(34);
 7     tree.insert(3);
 8     tree.insert(50);
 9     tree.insert(20);
10     tree.insert(15);
11     tree.insert(16);
12     tree.insert(25);
13     tree.insert(27);
14     tree.insert(29);
15     tree.insert(24);
16     System.out.print("\nAfter inserting 24:");
17     printTree(tree);
18     tree.insert(23);
19     tree.insert(22);
20     tree.insert(60);
21     tree.insert(70);
```

create a **Tree24**

insert 34

insert 3

insert 50

insert 24

insert 70

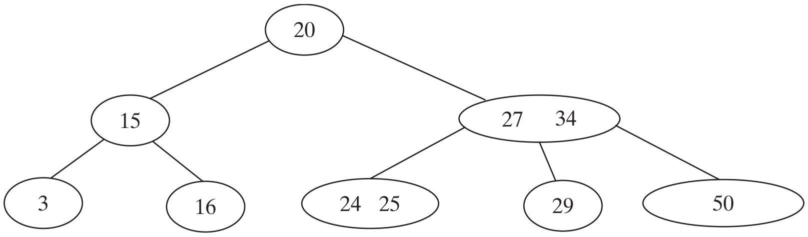
delete 34

```
    System.out.print("\nAfter inserting 70:");
    printTree(tree);

    tree.delete(34);
    System.out.print("\nAfter deleting 34:");
    printTree(tree);

    tree.delete(25);
    System.out.print("\nAfter deleting 25:");
    printTree(tree);

    tree.delete(50);
    System.out.print("\nAfter deleting 50:");
    printTree(tree);

    tree.delete(16);
    System.out.print("\nAfter deleting 16:");
    printTree(tree);

    tree.delete(3);
    System.out.print("\nAfter deleting 3:");
    printTree(tree);

    tree.delete(15);
    System.out.print("\nAfter deleting 15:");
    printTree(tree);
  }

  public static void printTree(Tree tree) {
    // Traverse tree
    System.out.print("\nPreorder: ");
    tree.preorder();
    System.out.print("\nThe number of nodes is " + tree.getSize());
    System.out.println();
  }
}
```

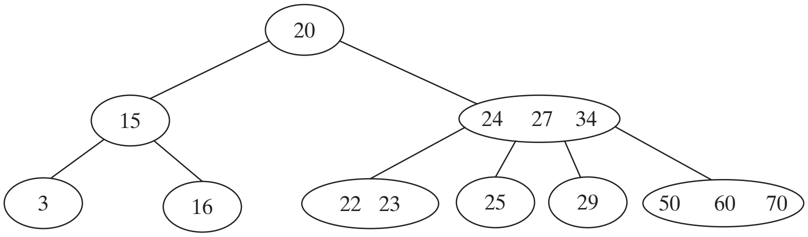
```
After inserting 24:
Preorder: 20 15 3 16 27 34 24 25 29 50
The number of nodes is 10

After inserting 70:
Preorder: 20 15 3 16 24 27 34 22 23 25 29 50 60 70
The number of nodes is 14

After deleting 34:
Preorder: 20 15 3 16 24 27 50 22 23 25 29 60 70
The number of nodes is 13

After deleting 25:
Preorder: 20 15 3 16 23 27 50 22 24 29 60 70
The number of nodes is 12

After deleting 50:
Preorder: 20 15 3 16 23 27 60 22 24 29 70
The number of nodes is 11

After deleting 16:
Preorder: 23 20 3 15 22 27 60 24 29 70
The number of nodes is 10
```

```
After deleting 3:
Preorder: 23 20 15 22 27 60 24 29 70
The number of nodes is 9

After deleting 15:
Preorder: 27 23 20 22 24 60 29 70
The number of nodes is 8
```

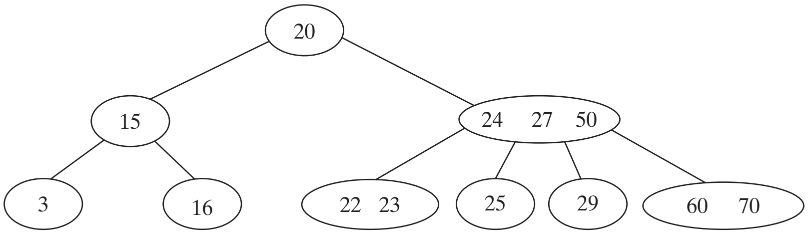
Figure 46.15 shows how the tree evolves as elements are added. After **34**, **3**, **50**, **20**, **15**, **16**, **25**, **27**, **29**, and **24** are added to the tree, it is as shown in Figure 46.15(a). After inserting **23**, **22**, **60**, and **70**, the tree is as shown in Figure 46.15(b). After inserting **23**, **22**, **60**, and **70**, the tree is as shown in Figure 46.15(b). After deleting **34**, the tree is as shown in Figure 46.15(c). After deleting **25**, the tree is as shown in Figure 46.15(d). After deleting **50**, the tree is as shown in Figure 46.15(e). After deleting **16**, the tree is as shown in Figure 46.15(f). After



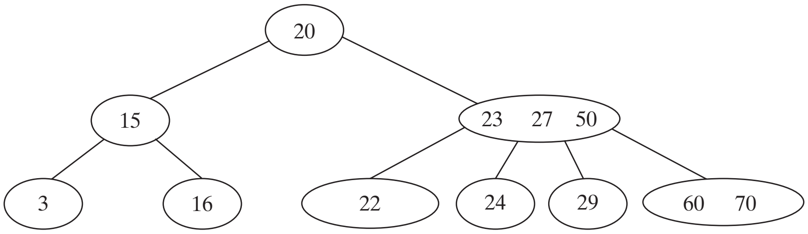
(a) After inserting 34, 3, 50, 20, 15, 16, 25, 27, 29, and 24, in this order



(b) After inserting 23, 22, 60 and 70

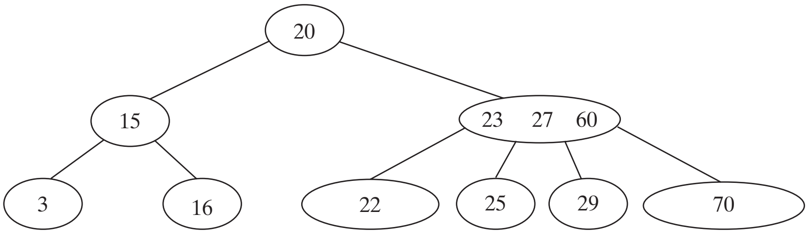


(c) After deleting 34

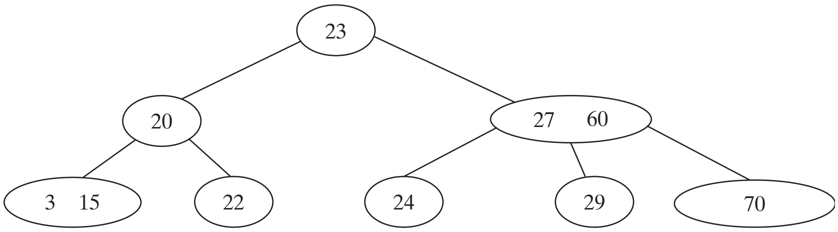


(d) After deleting 25

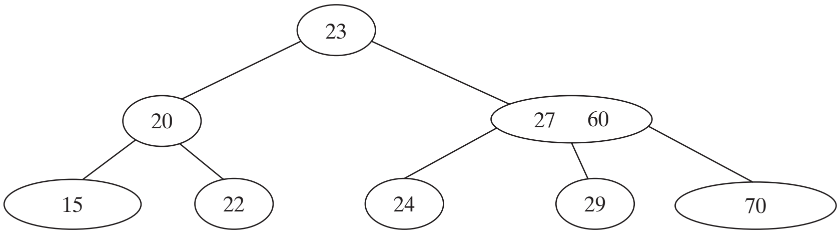
FIGURE 46.15 The tree evolves as elements are inserted and deleted.



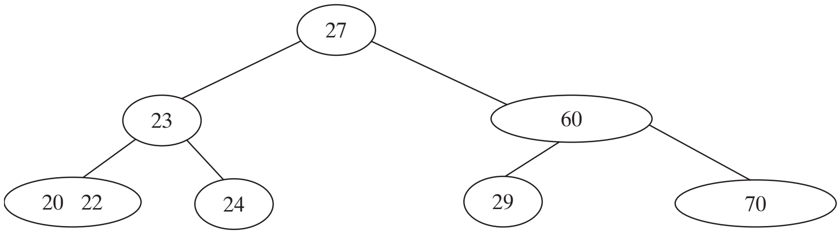
(e) After deleting 50



(f) After deleting 16



(g) After deleting 3



(h) After deleting 15

Figure 46.15 *continued*

deleting `3`, the tree is as shown in Figure 46.15(g). After deleting `15`, the tree is as shown in Figure 46.15(h).

46.9 Time-Complexity Analysis

Since a 2-4 tree is a completely balanced binary tree, its height is at most $O(\log n)$. The `search`, `insert`, and `delete` methods operate on the nodes along a path in the tree. It takes a constant time to search an element within a node. So, the `search` method takes $O(\log n)$ time. For the `insert` method, the time for splitting a node takes a constant time. So, the `insert` method takes $O(\log n)$ time. For the `delete` method, it takes a constant time to perform a transfer and fusion operation. So, the `delete` method takes $O(\log n)$ time.

46.10 B-Tree

So far we assume that the entire data set is stored in main memory. What if the data set is too large and cannot fit in the main memory, as in the case with most databases where data is stored on disks? Suppose you use an AVL tree to organize a million records in a database table. To find a record, the average number of nodes traversed is $\log_2 1{,}000{,}000 \approx 20$. This is fine if all nodes are stored in main memory. However, for nodes stored on a disk, this means 20 disk reads. Disk I/O is expensive, and it is thousands of times slower than memory access. To improve performance, we need to reduce the number of disk I/Os. An efficient data structure for performing search, insertion, and deletion for data stored on secondary storage such as hard disks is the B-tree, which is a generalization of the 2-4 tree.

A B-tree of order *d* is defined as follows:

1. Each node except the root contains between $\lceil d/2 \rceil - 1$ and $d - 1$ keys.
2. The root may contain up to $d - 1$ keys.
3. A nonleaf node with k keys has $k + 1$ children.
4. All leaf nodes have the same depth.

Figure 46.16 shows a B-tree of order 6. For simplicity, we use integers to represent keys. Each key is associated with a pointer that points to the actual record in the database. For simplicity, the pointers to the records in the database are omitted in the figure.

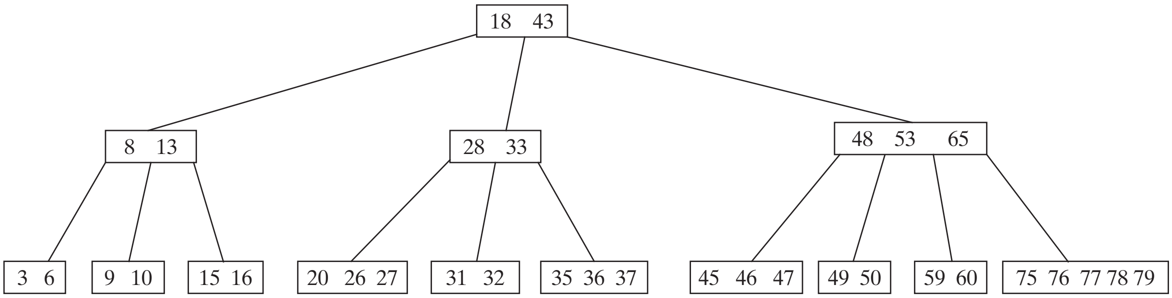


FIGURE 46.16 In a B-tree of order 6, each node except the root may contain between 2 and 5 keys.

Note that a B-tree is a search tree. The keys in each node are placed in increasing order. Each key in an interior node has a left subtree and a right subtree, as shown in Figure 46.17. All keys in the left subtree are less than the key in the parent node, and all keys in the right subtree are greater than the key in the parent node.

one block per node

The basic unit of the IO operations on a disk is a block. When you read data from a disk, the whole block that contains the data is read. You should choose an appropriate order *d* so that a node can fit in a single disk block. This will minimize the number of disk IOs.

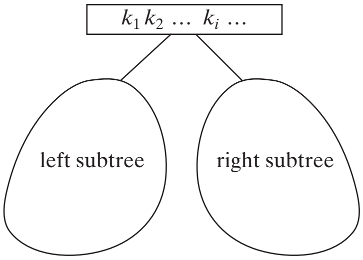


FIGURE 46.17 The keys in the left (right) subtree of key k_i are less than (greater than) k_i.

A 2-4 tree is actually a B-tree of order 4. The techniques for insertion and deletion in a 2-4 tree can be easily generalized for a B-tree.

insertion

Inserting a key to a B-tree is similar to what was done for a 2-4 tree. First locate the leaf node in which the key will be inserted. Insert the key to the node. After the insertion, if the leaf node has d keys, an overflow occurs. To resolve overflow, perform a *split* operation similar to the one used in a 2-4 tree, as follows:

Let u denote the node needed to be split and let `m` denote the median key in the node. Create a new node and move all keys greater than `m` to this new node. Insert `m` to the parent node of `u`. Now `u` becomes the left child of `m` and `v` becomes the right child of `m`, as shown in Figure 46.18. If inserting `m` into the parent node of `u` causes an overflow, repeat the same split process on the parent node.

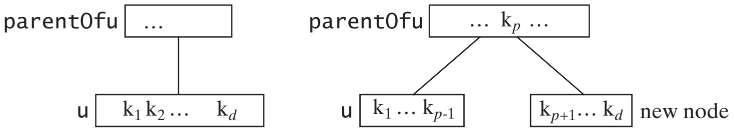
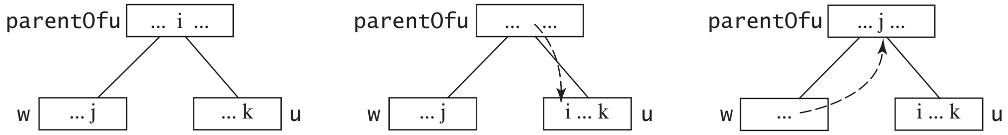


FIGURE 46.18 (a) After inserting a new key to node `u`. (b) The median key k_p is inserted to `parentOfu`.

deletion

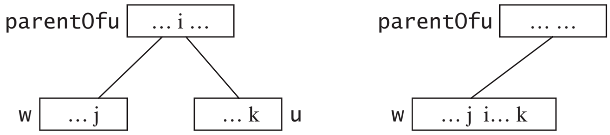
A key k can be deleted from a B-tree in the same way as in a 2-4 tree. First locate the node u that contains the key. Consider two cases:

Case 1: If `u` is a leaf node, remove the key from `u`. After the removal, if `u` has less than $\lceil d/2 \rceil - 1$ keys, an underflow occurs. To remedy an underflow, perform a transfer with a sibling `w` of `u` that has more than $\lceil d/2 \rceil - 1$ keys if such sibling exists, as shown in Figure 46.19. Otherwise perform a fusion with a sibling `w` of `u`, as shown in Figure 46.20.



(a) Before a transfer is performed (b) Key i moved to node `u` (c) Key j moved to `parentOfu`

FIGURE 46.19 The transfer operation transfers a key from the `parentOfu` to `u` and transfers a key from `u`'s sibling `parentOfu`.



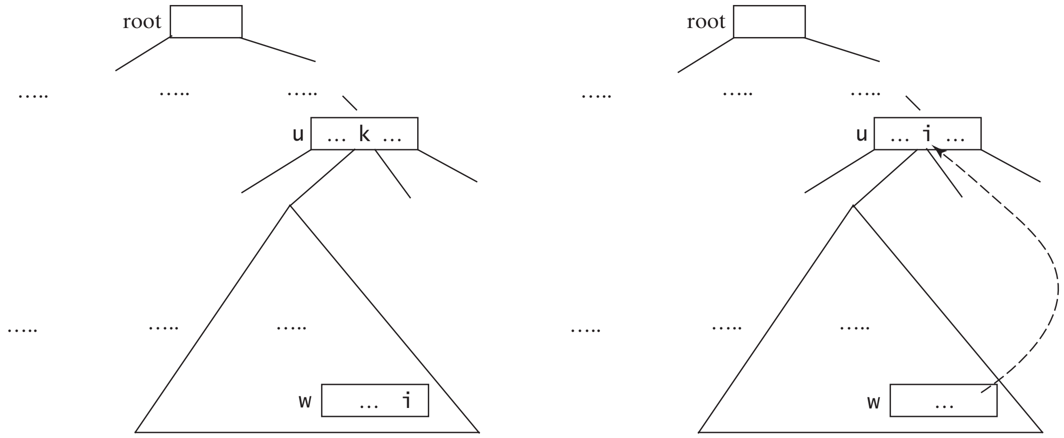
(a) Before a fusion is performed (b) After a fusion is performed

FIGURE 46.20 The fusion operation moves key i from the `parentOfu` to w and moves all keys in `u` to `w`.

Case 2: `u` is a nonleaf node. Find the rightmost leaf node in the left subtree of `k`. Let this node be `w`, as shown in Figure 46.21(a). Move the last key in `w` to replace `k` in `u`, as shown in Figure 46.21(b). If `w` becomes underflow, apply a transfer or fusion operation on `w`.

B-tree performance

The performance of a *B-tree* depends on the number of disk IOs (i.e., the number of nodes accessed). The number of nodes accessed for search, insertion, and deletion operations depends on the height of the tree. In the worst case, each node contains $\lceil d/2 \rceil - 1$ keys. So, the height of the tree is $\log_{\lceil d/2 \rceil} n$, where n is the number of keys. In the best case, each node



(a) Key is in u (b) Replace key *k* with key *i*

FIGURE 46.21 A key in the internal node is replaced by an element in a leaf node.

contains $d - 1$ keys. So, the height of the tree is $\log_d n$. Consider a B-tree of order 12 for ten million keys. The height of the tree is between $\log_6 1{,}000{,}000 \approx 7$ and $\log_{12} 10{,}000{,}000 \approx 9$. So, for search, insertion, and deletion operations, the maximum number of nodes visited is 46. If you use an AVL tree, the maximum number of nodes visited is $\log_2 10{,}000{,}000 \approx 24$.

KEY TERMS

2-3-4 tree	46–2
2-4 tree	46–2
2-node	46–2
3-node	46–2
4-node	46–2
B-tree	46–24
fusion operation	46–7
split operation	46–4
transfer operation	46–7

CHAPTER SUMMARY

1. A 2-4 tree is a completely balanced search tree. In a 2-4 tree, a node may have one, two, or three elements.
2. Searching an element in a 2-4 tree is similar to searching an element in a binary tree. The difference is that you have searched an element within a node.
3. To insert an element to a 2-4 tree, locate a leaf node in which the element will be inserted. If the leaf node is a 2- or 3-node, simply insert the element into the node. If the node is a 4-node, split the node.
4. The process of deleting an element from a 2-4 tree is similar to that of deleting an element from a binary tree. The difference is that you have to perform transfer or fusion operations for empty nodes.
5. The height of a 2-4 tree is $O(\log n)$. So, the time complexities for the search, insert, and delete methods are $O(\log n)$.
6. A B-tree is a generalization of the 2-4 tree. Each node in a B-tree of order d can have between $\lceil d/2 \rceil - 1$ and $d - 1$ keys except the root. 2-4 trees are flatter than AVL trees and B-trees are flatter than 2-4 trees. B-trees are efficient for creating indexes for data in database systems where large amounts of data are stored on disks.

REVIEW QUESTIONS

Sections 46.1–46.2

46.1 What is a 2-4 tree? What are a 2-node, 3-node, and 4-node?

46.2 Describe the data fields in the `Tree24` class and those in the `Tree24Node` class.

46.3 What is the minimum number of elements in a 2-4 tree of height 5? What is the maximum number of elements in a 2-4 tree of height 5?

Sections 46.3–46.5

46.4 How do you search an element in a 2-4 tree?

46.5 How do you insert an element into a 2-4 tree?

46.6 How do you delete an element from a 2-4 tree?

46.7 Show the change of a 2-4 tree when inserting `1`, `2`, `3`, `4`, `10`, `9`, `7`, `5`, `8`, `6` into it, in this order.

46.8 For the tree built in the preceding question, show the change of the tree after deleting `1`, `2`, `3`, `4`, `10`, `9`, `7`, `5`, `8`, `6` from it in this order.

46.9 Show the change of a B-tree of order 6 when inserting `1`, `2`, `3`, `4`, `10`, `9`, `7`, `5`, `8`, `6`, `17`, `25`, `18`, `26`, `14`, `52`, `63`, `74`, `80`, `19`, `27` into it, in this order.

46.10 For the tree built in the preceding question, show the change of the tree after deleting `1`, `2`, `3`, `4`, `10`, `9`, `7`, `5`, `8`, `6` from it, in this order.

PROGRAMMING EXERCISES

46.1* (*Implementing `inorder`*) The `inorder` method in `Tree24` is left as an exercise. Implement it.

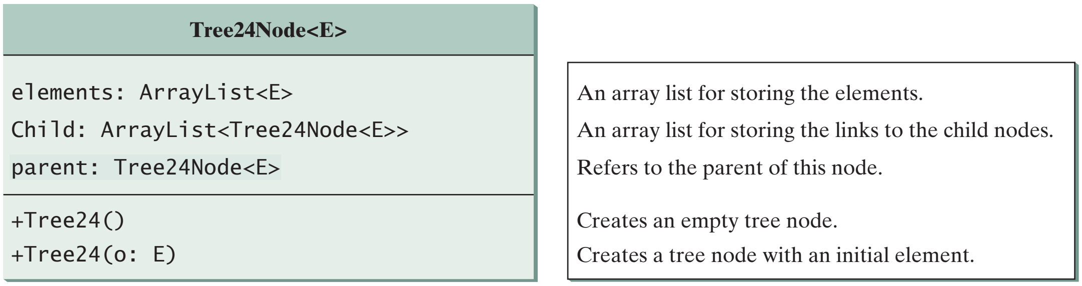
46.2 (*Implementing `postorder`*) The `postorder` method in `Tree24` is left as an exercise. Implement it.

46.3 (*Implementing `iterator`*) The `iterator` method in `Tree24` is left as an exercise. Implement it to iterate the elements using inorder.

46.4* (*Displaying a 2-4 tree graphically*) Write an applet that displays a 2-4 tree.

46.5*** (*2-4 tree animation*) Write a Java applet that animates the 2-4 tree `insert`, `delete`, and `search` methods, as shown in Figure 46.4.

46.6** (*Parent reference for `Tree24`*) Redefine `Tree24Node` to add a reference to a node's parent, as shown below:



Tree24Node<E>	
`elements: ArrayList<E>`	An array list for storing the elements.
`Child: ArrayList<Tree24Node<E>>`	An array list for storing the links to the child nodes.
`parent: Tree24Node<E>`	Refers to the parent of this node.
`+Tree24()`	Creates an empty tree node.
`+Tree24(o: E)`	Creates a tree node with an initial element.

Add the following two new methods in `Tree24`:

```
public Tree24Node<E> getParent(Tree24Node<E> node)
```

Returns the parent for the specified node.

```
public ArrayList<Tree24Node<E>> getPath(Tree24Node<E> node)
```

Returns the path from the specified node to the root in an array list.

Write a test program that adds numbers `1`, `2`, . . . , `100` to the tree and displays the paths for all leaf nodes.

46.7*** (*The* `BTree` *class*) Design and implement a class for B-trees.