Max Flow / Min Cut

Problem: Network Connectivity

- You have two computers that are indirectly connected through a network of other computer systems. How many internal network disconnections is your connection resilient to?

Problem: School Dance

- Boys and girls need to be paired up for the school dance, but the kids only want to be paired with someone that they know. Is such a pairing possible? And if so, what's the pairing?

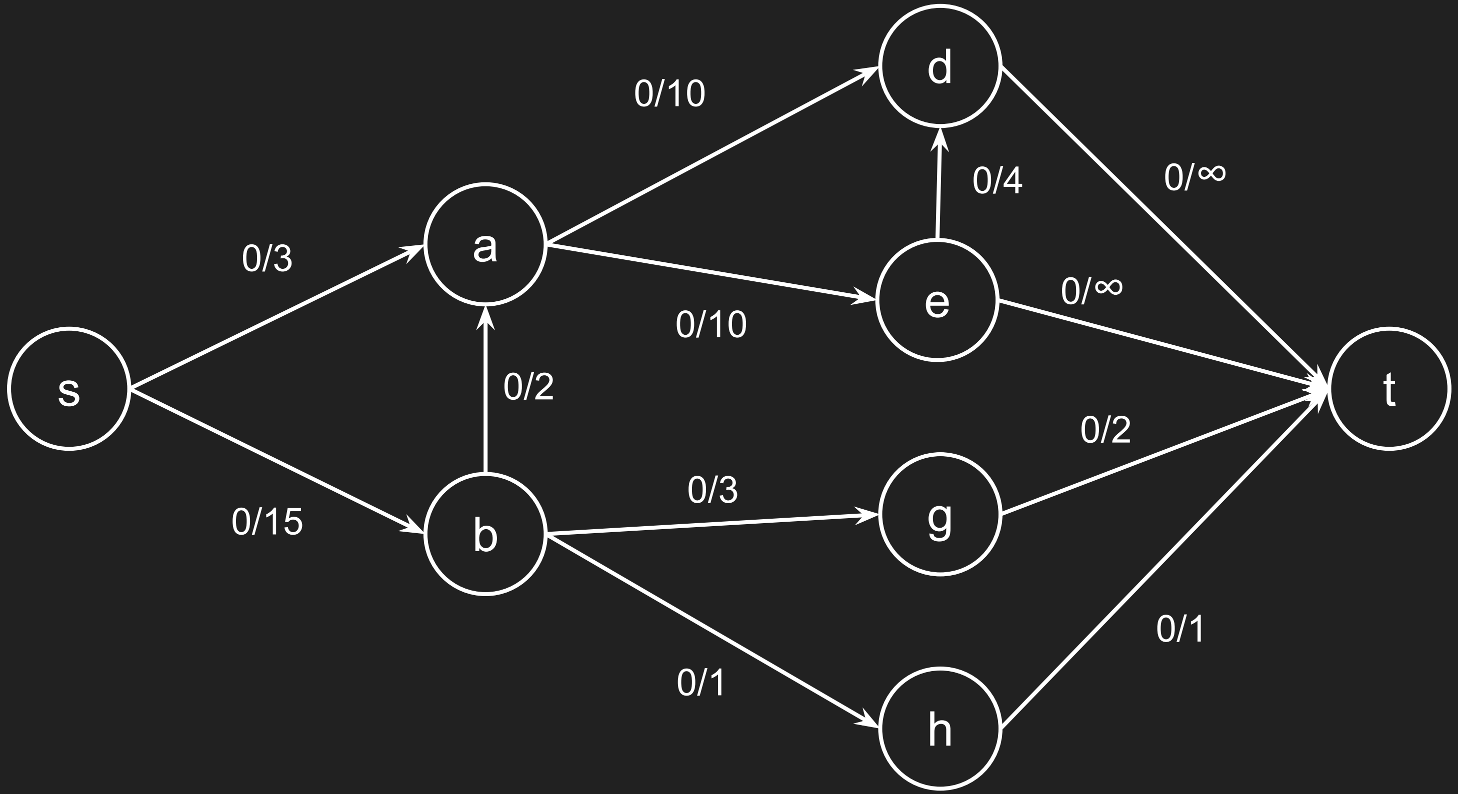
Problem: Project Selection

- You have a set of projects p_i which will each net a revenue of $r(p_i)$. Each project will require purchasing one or more machines q_j each of which costs $c(q_j)$. Machines can be shared by multiple projects. The goal is to maximize profit.

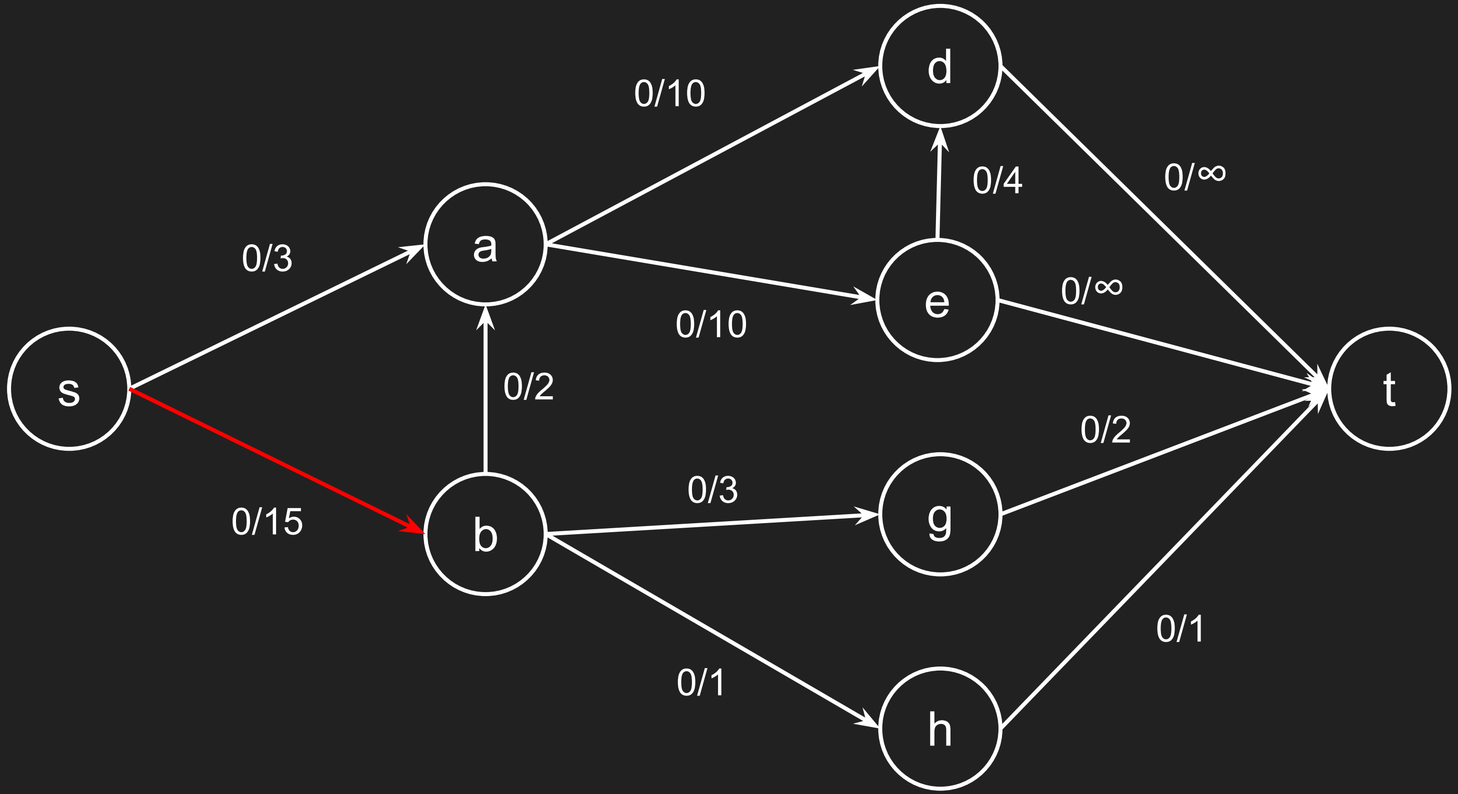
Flow Networks

- Given a directed graph *G* with a special source node *s*, a special sink node *t*
    - *s* has no inbound edges, and *t* has no outbound edges
- For each edge *e* in the graph, *c(e)* is the given “capacity” of the edge
    - The capacity must be greater than 0
    - For simplicity, assume the capacity is an integer or ∞
- Define *f(e)* to be the “flow” along an edge
    - The flow must be non-negative
    - The flow must also no greater than the capacity for a given edge
- For any given node, the sum of flows of inbound edges must equal the sum of flows of outbound edges (“conservation of flow”)
    - Exceptions: *s* may have any amount of outbound flow, and *t* may have any amount of inbound flow
- Question: what is the maximum amount of flow that can be sent from *s* to *t*?

Max Flow



Max Flow



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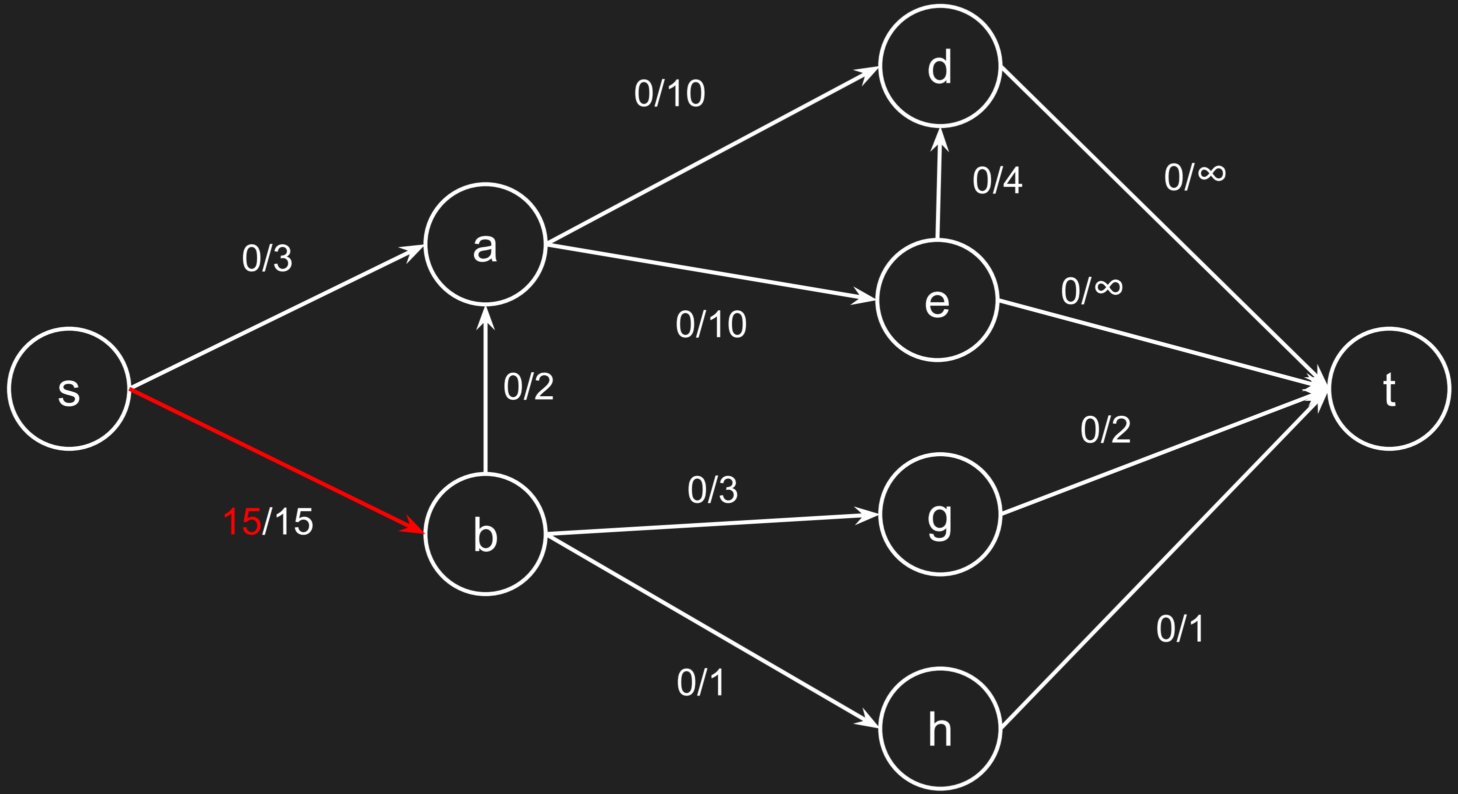

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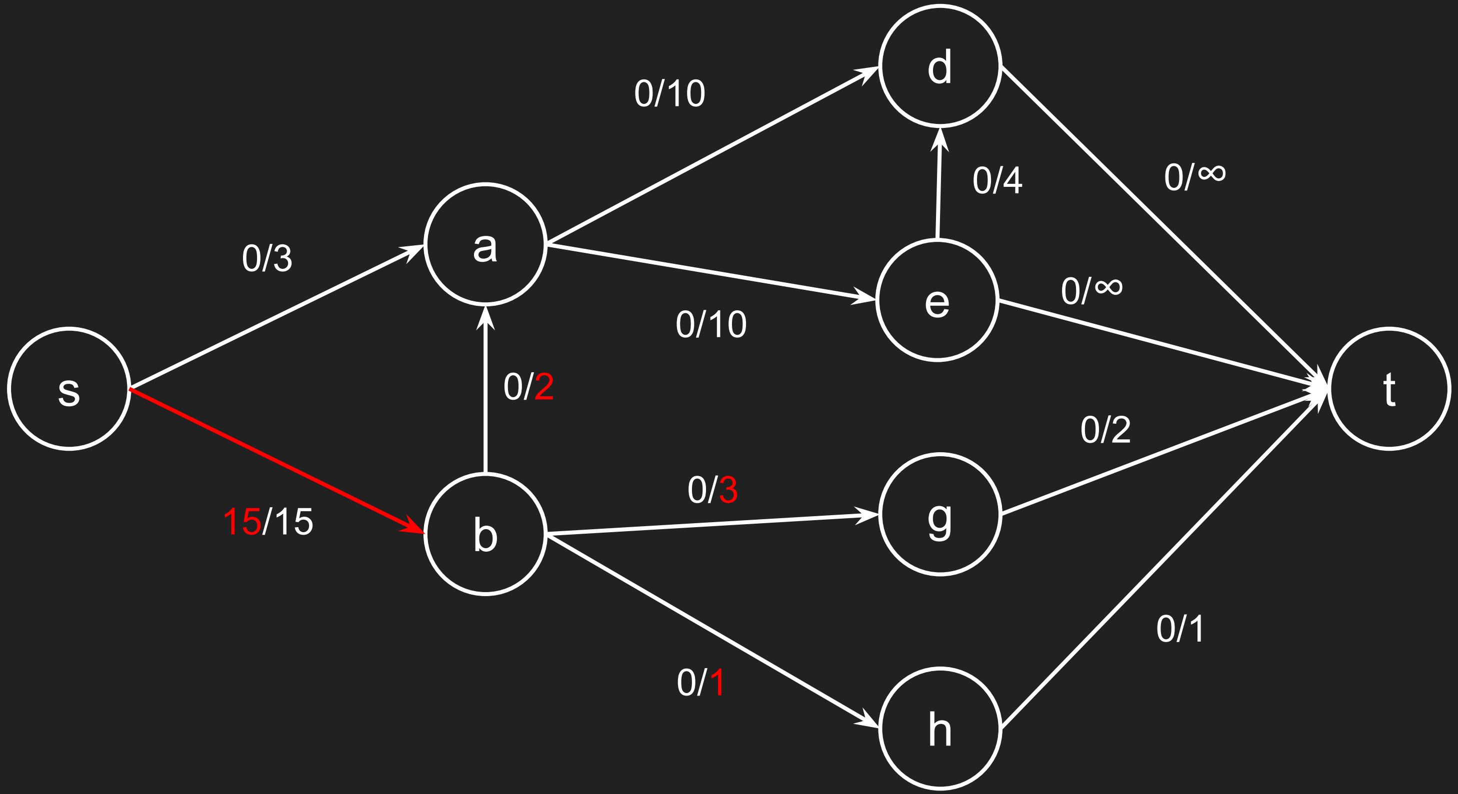

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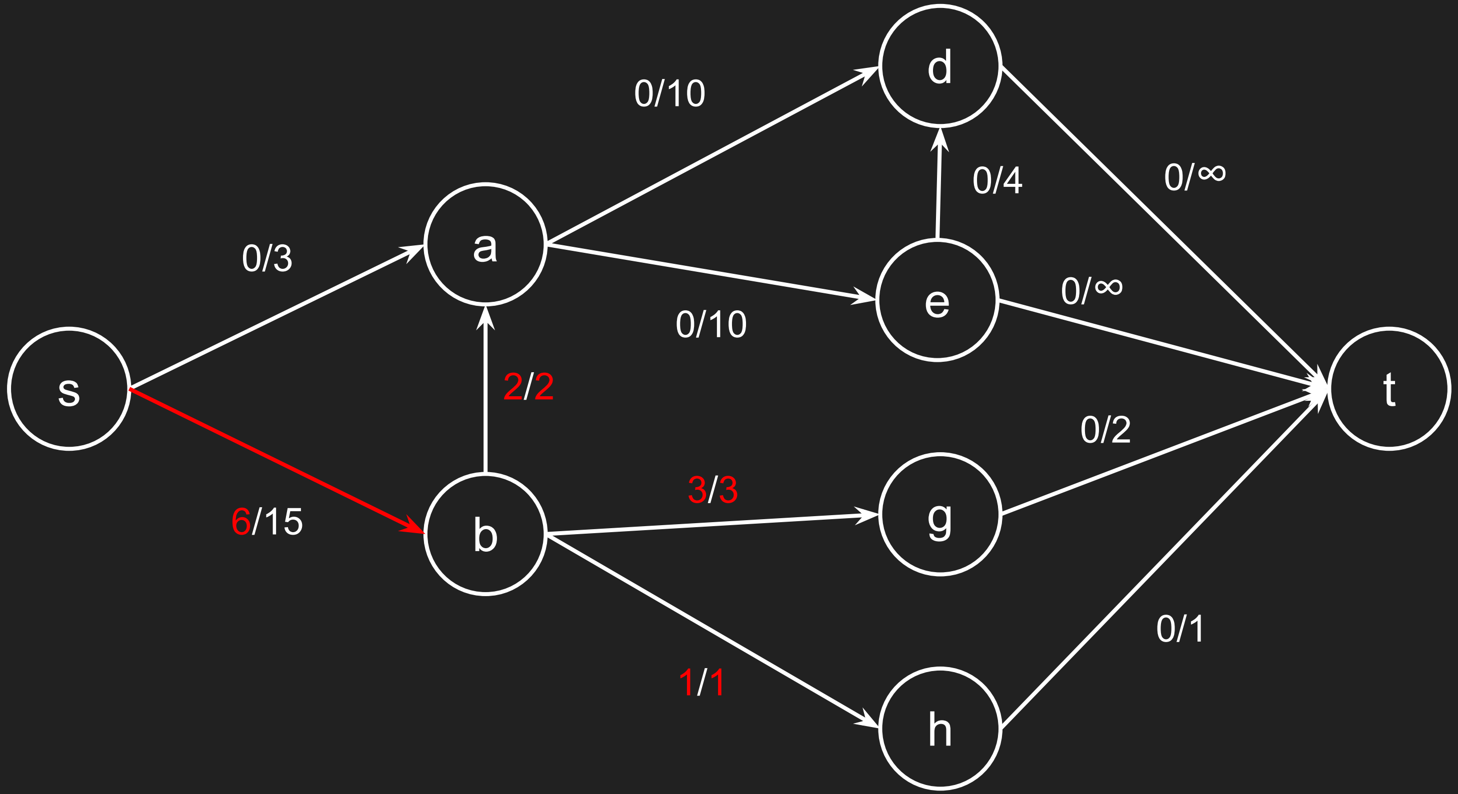

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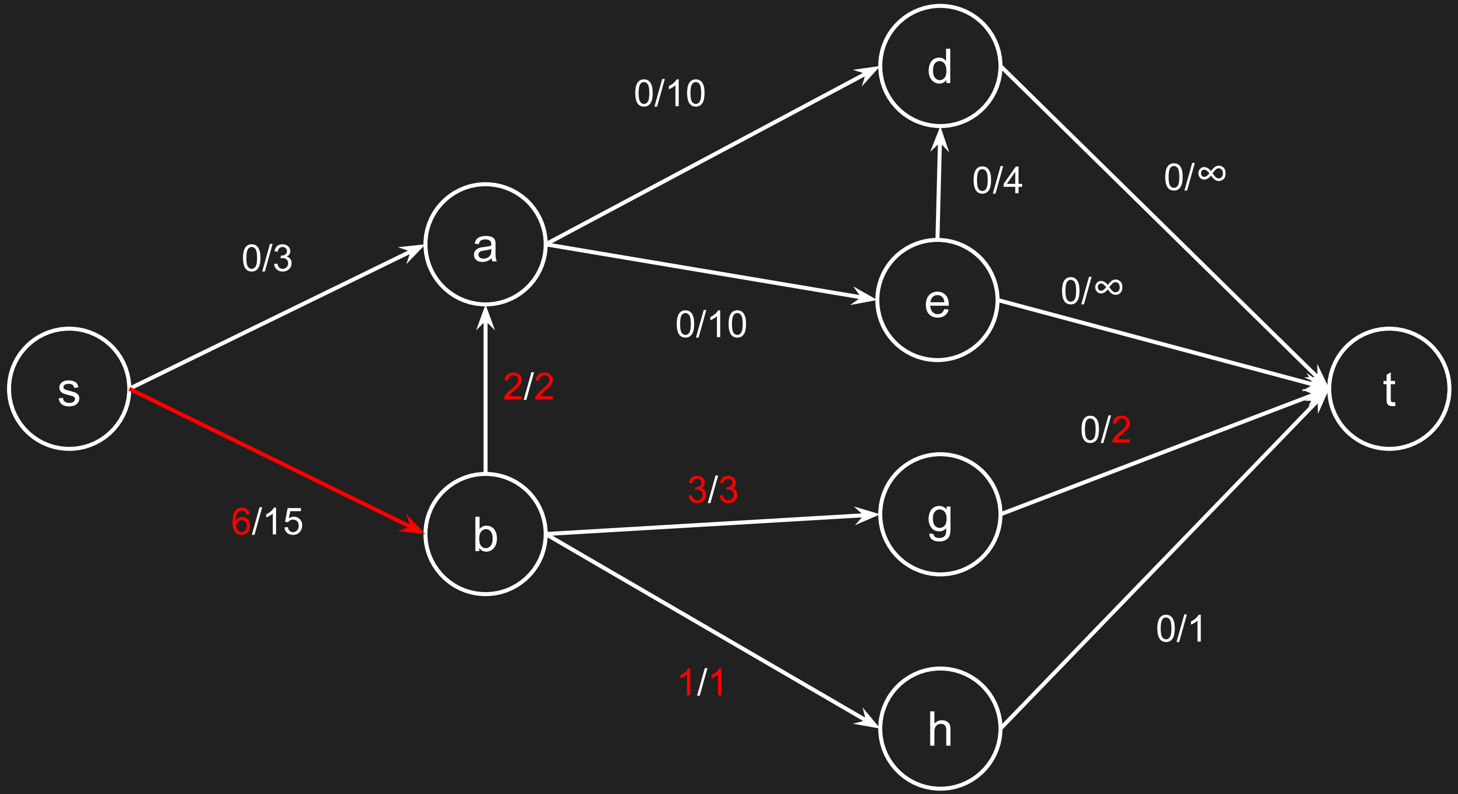

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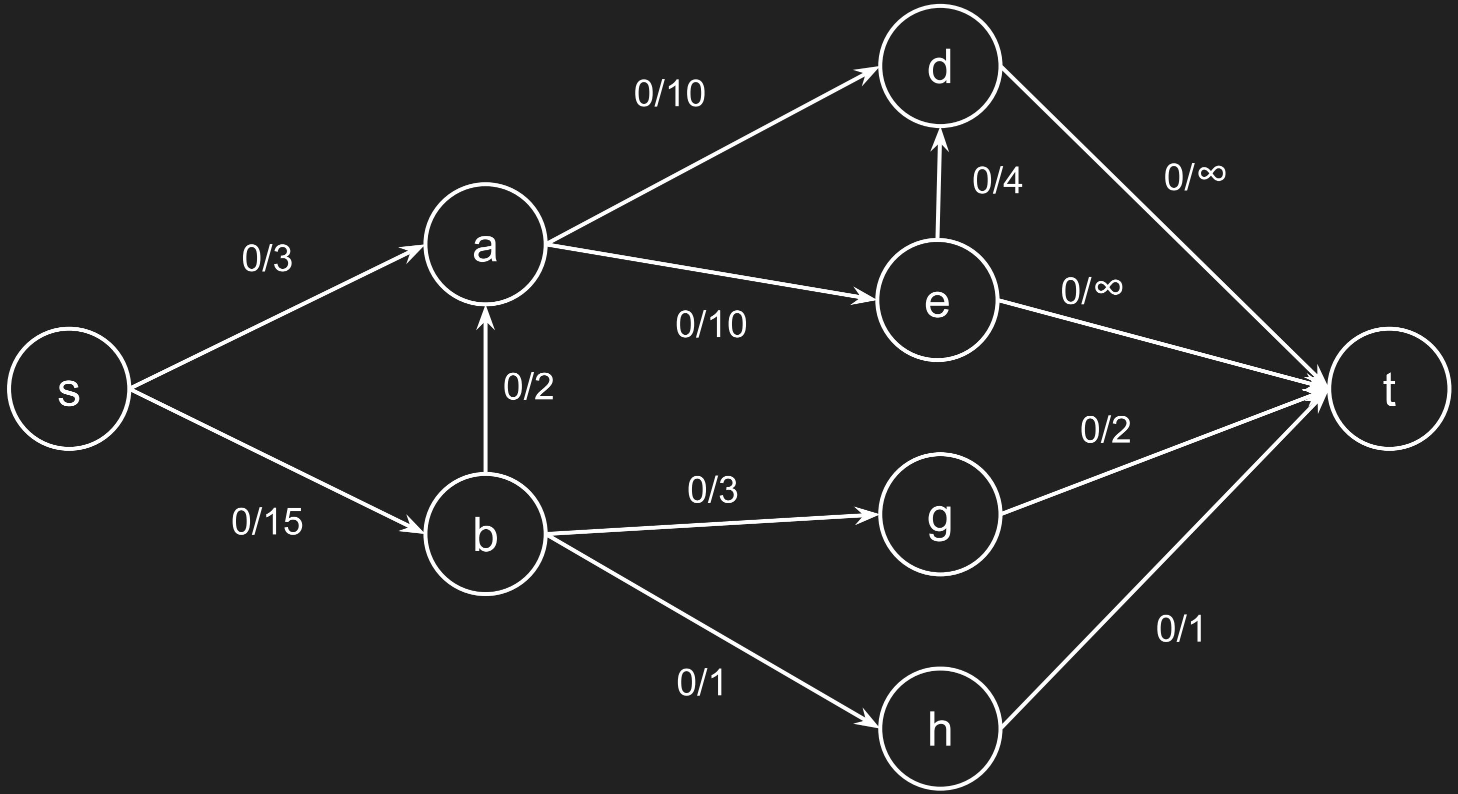

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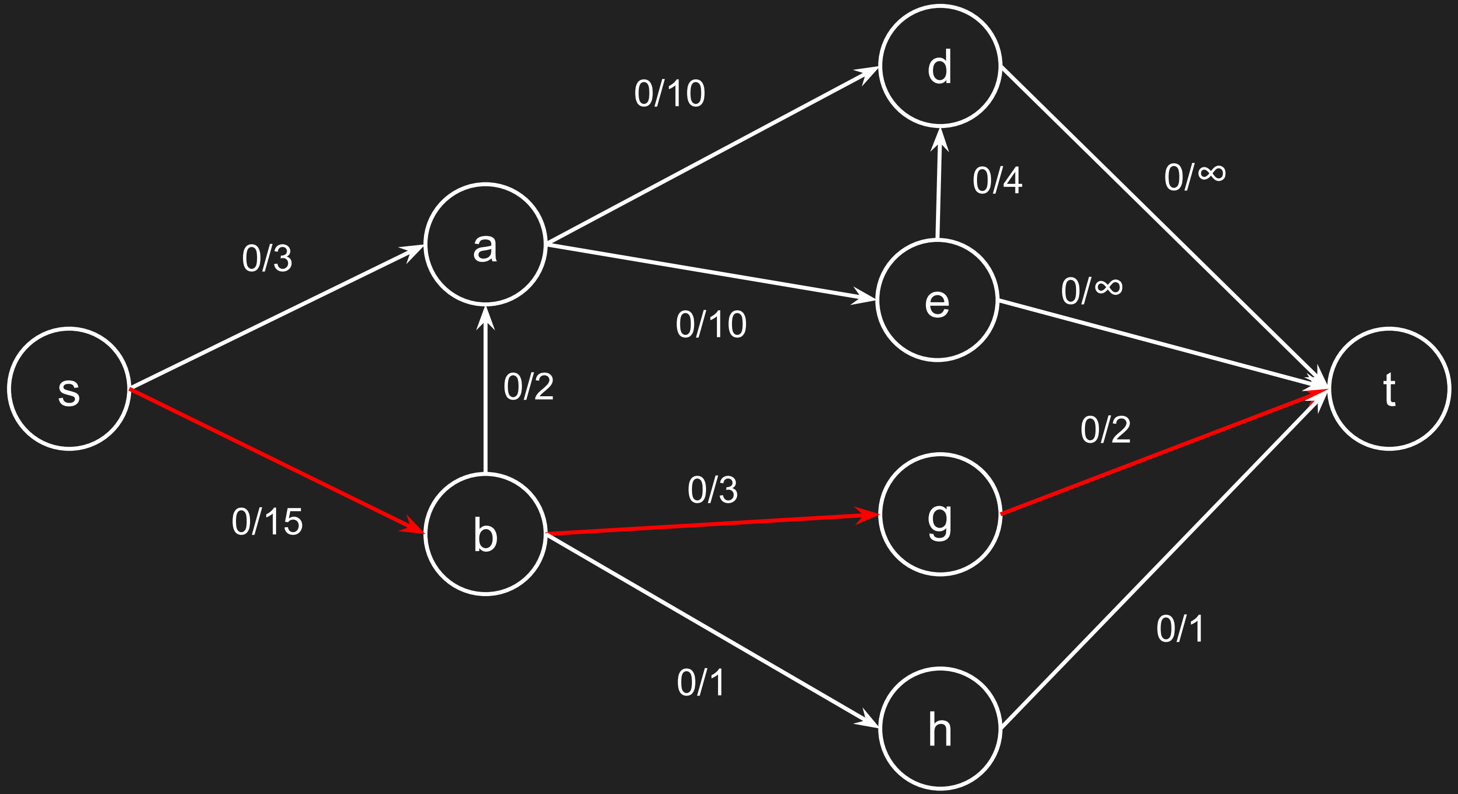

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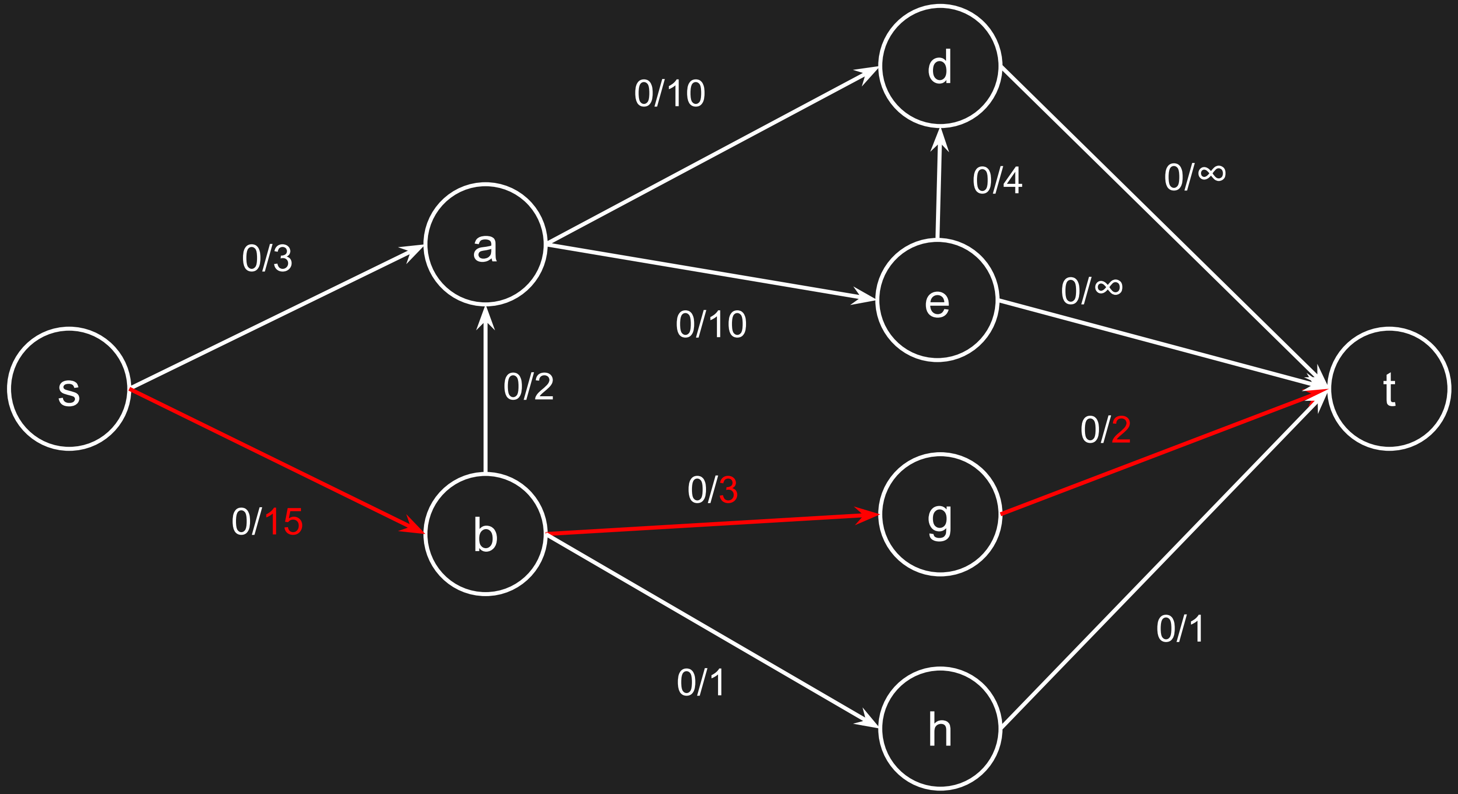

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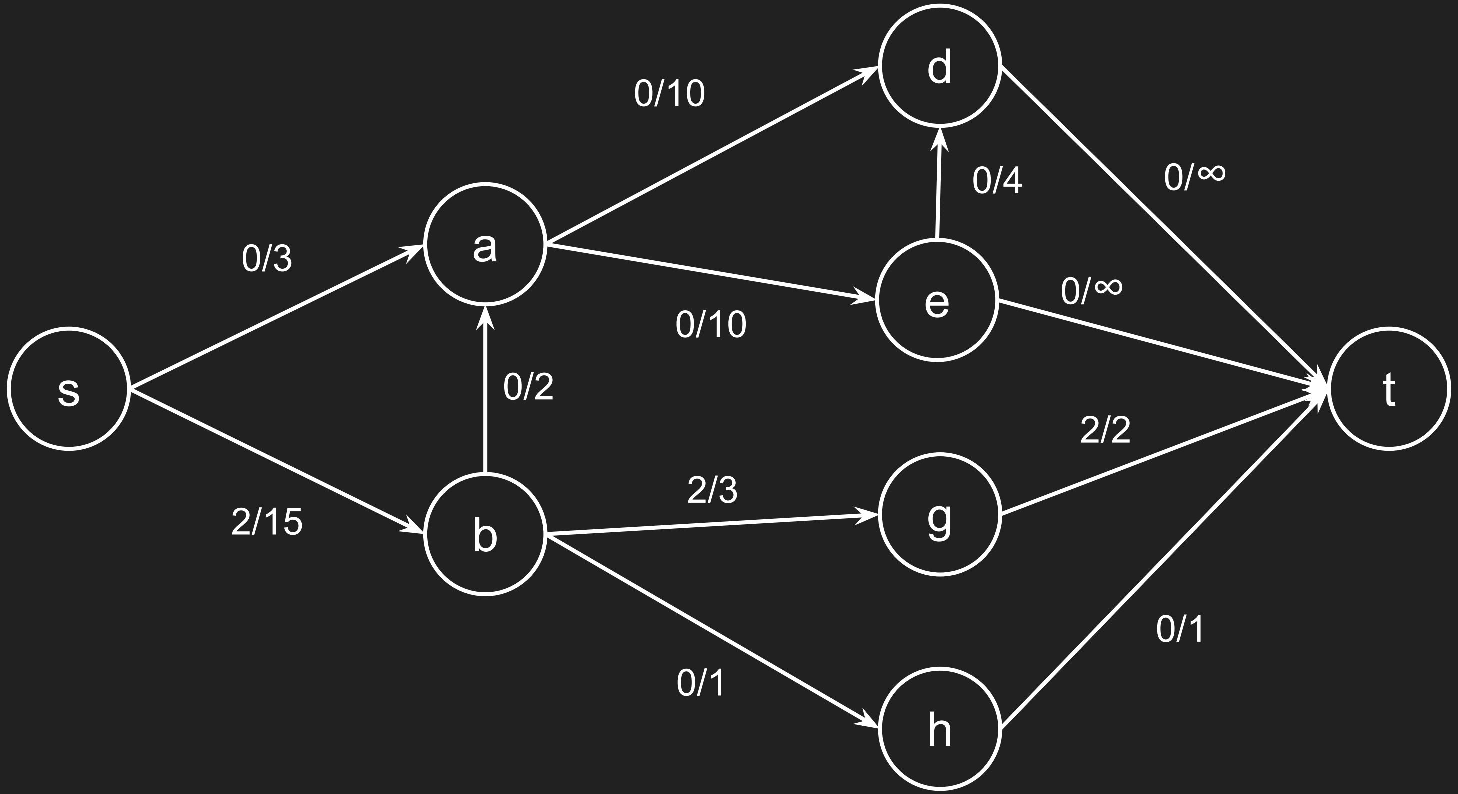

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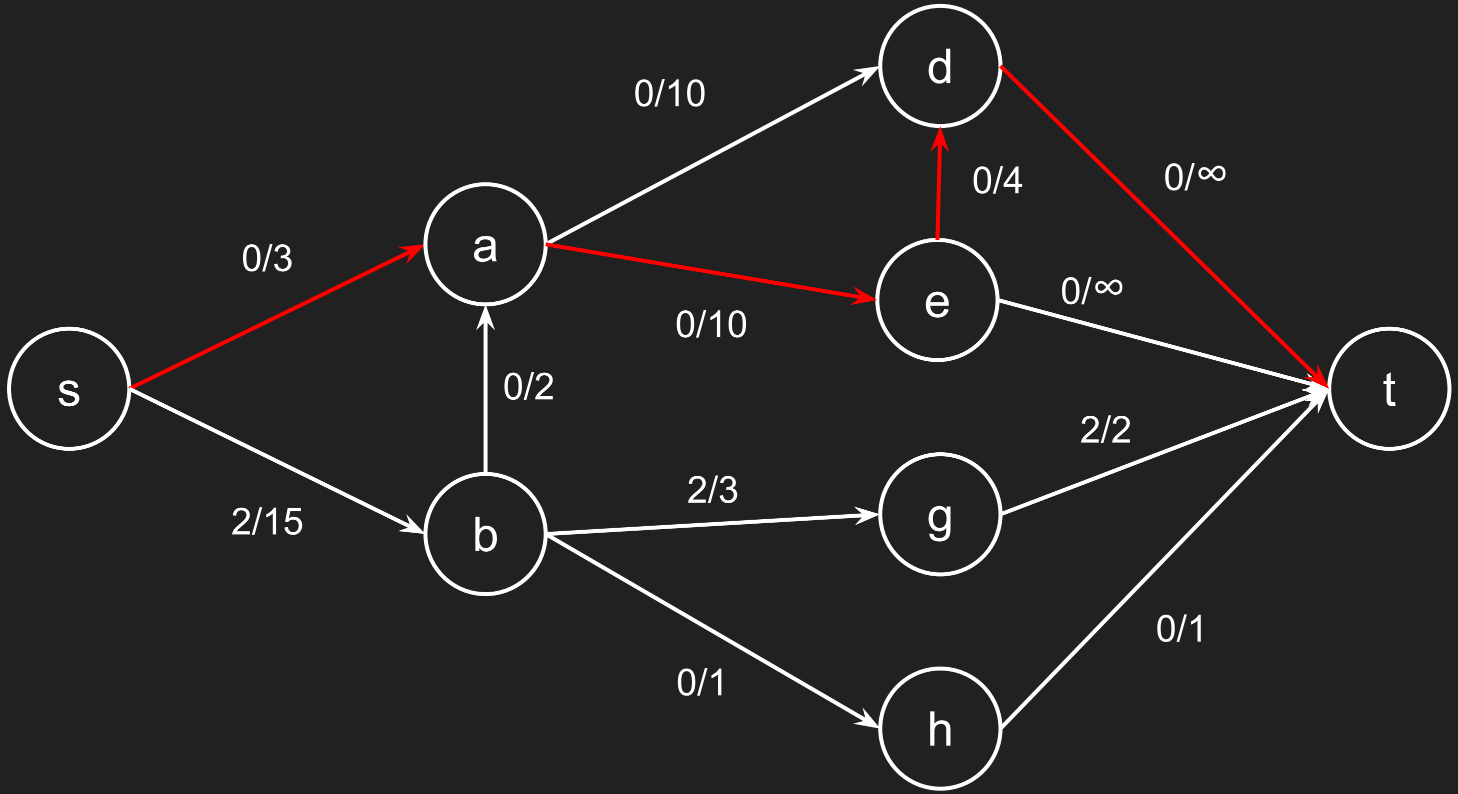

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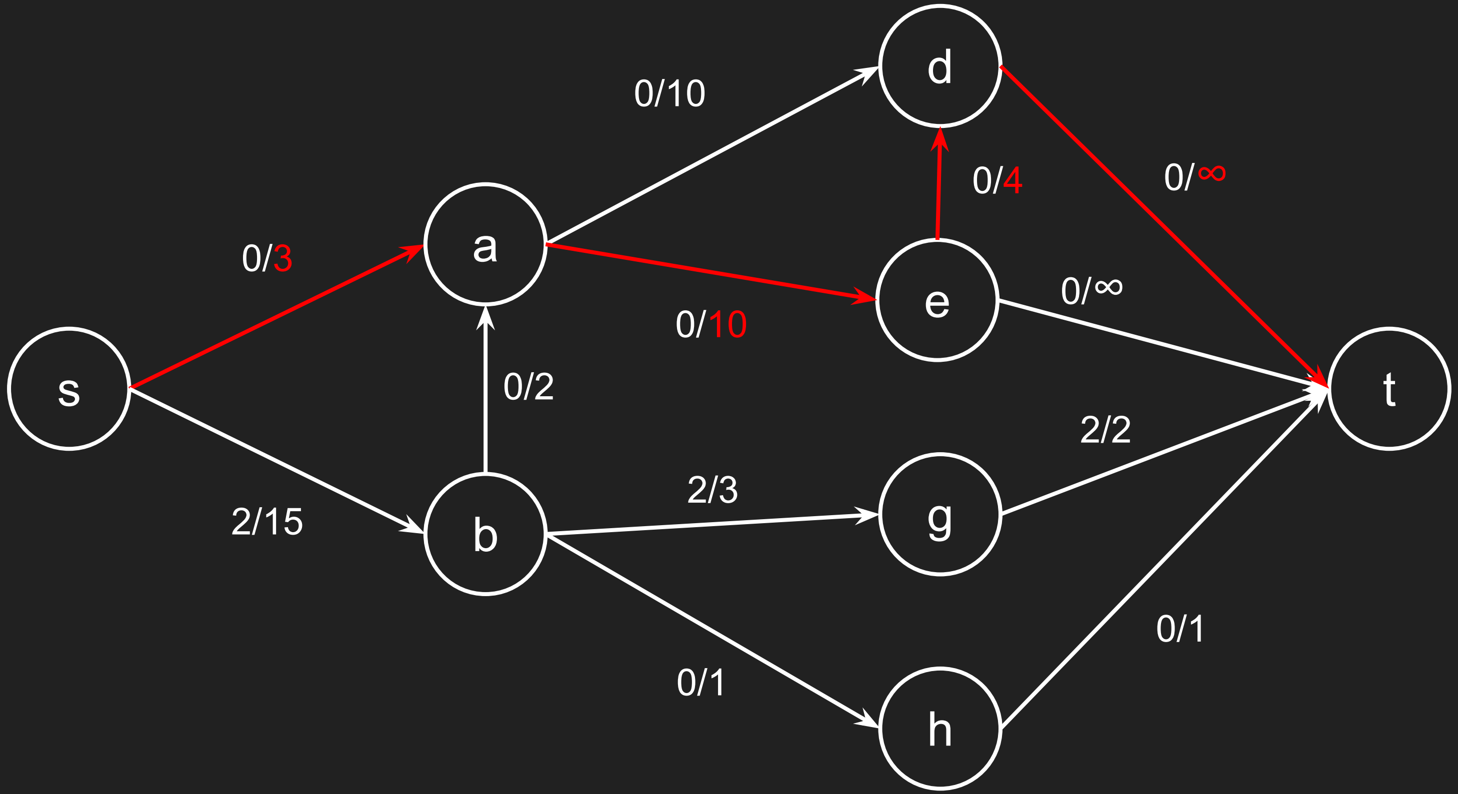

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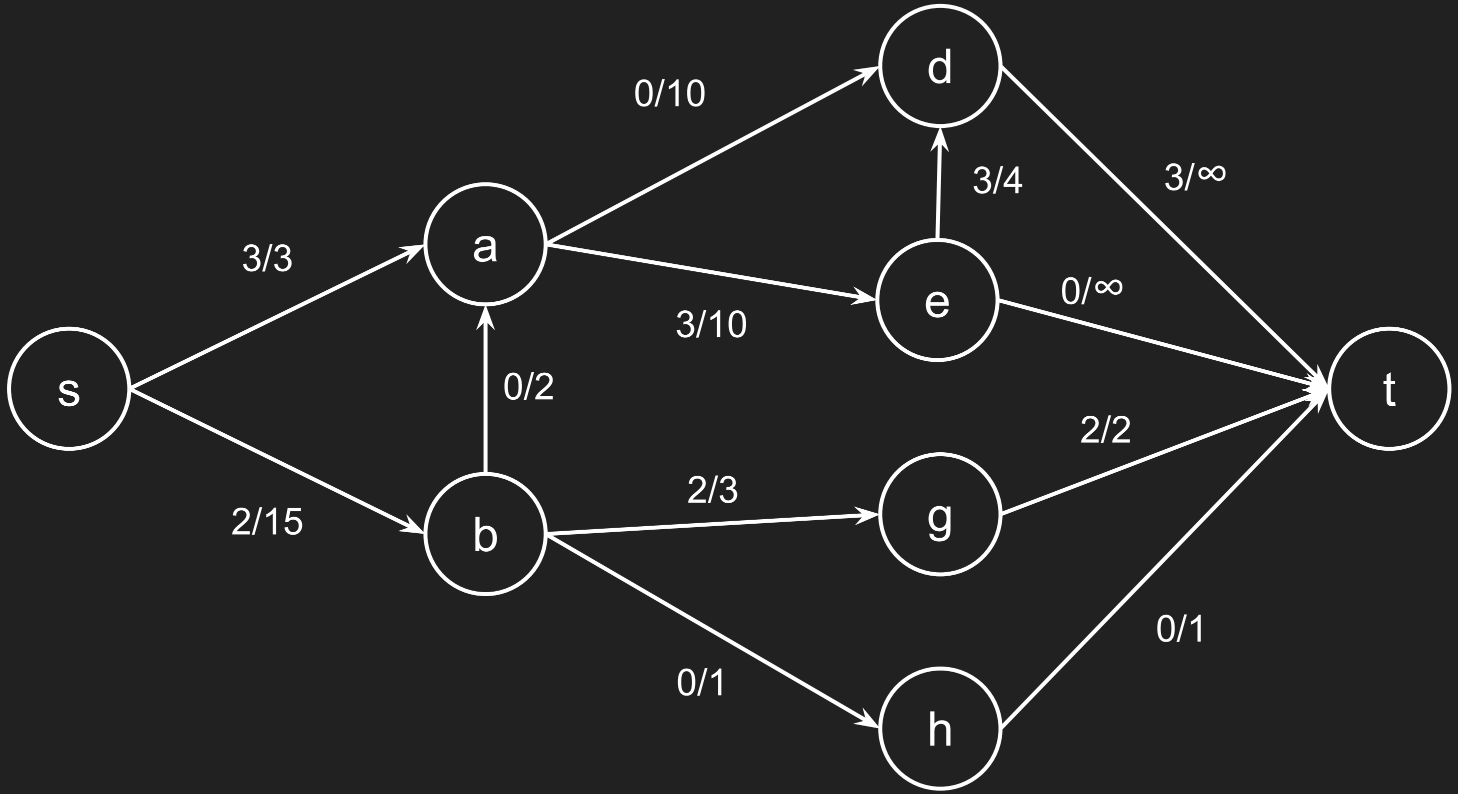

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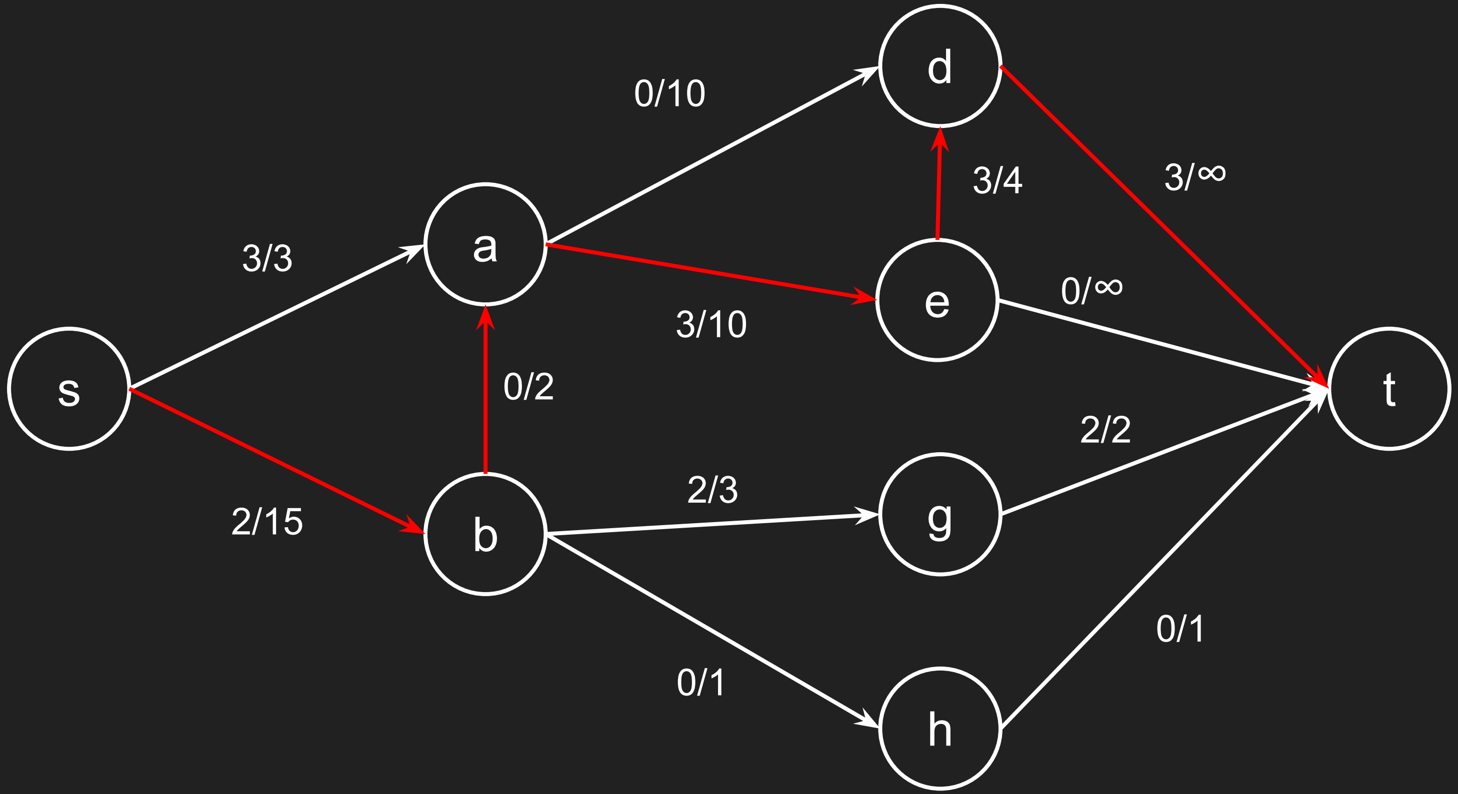

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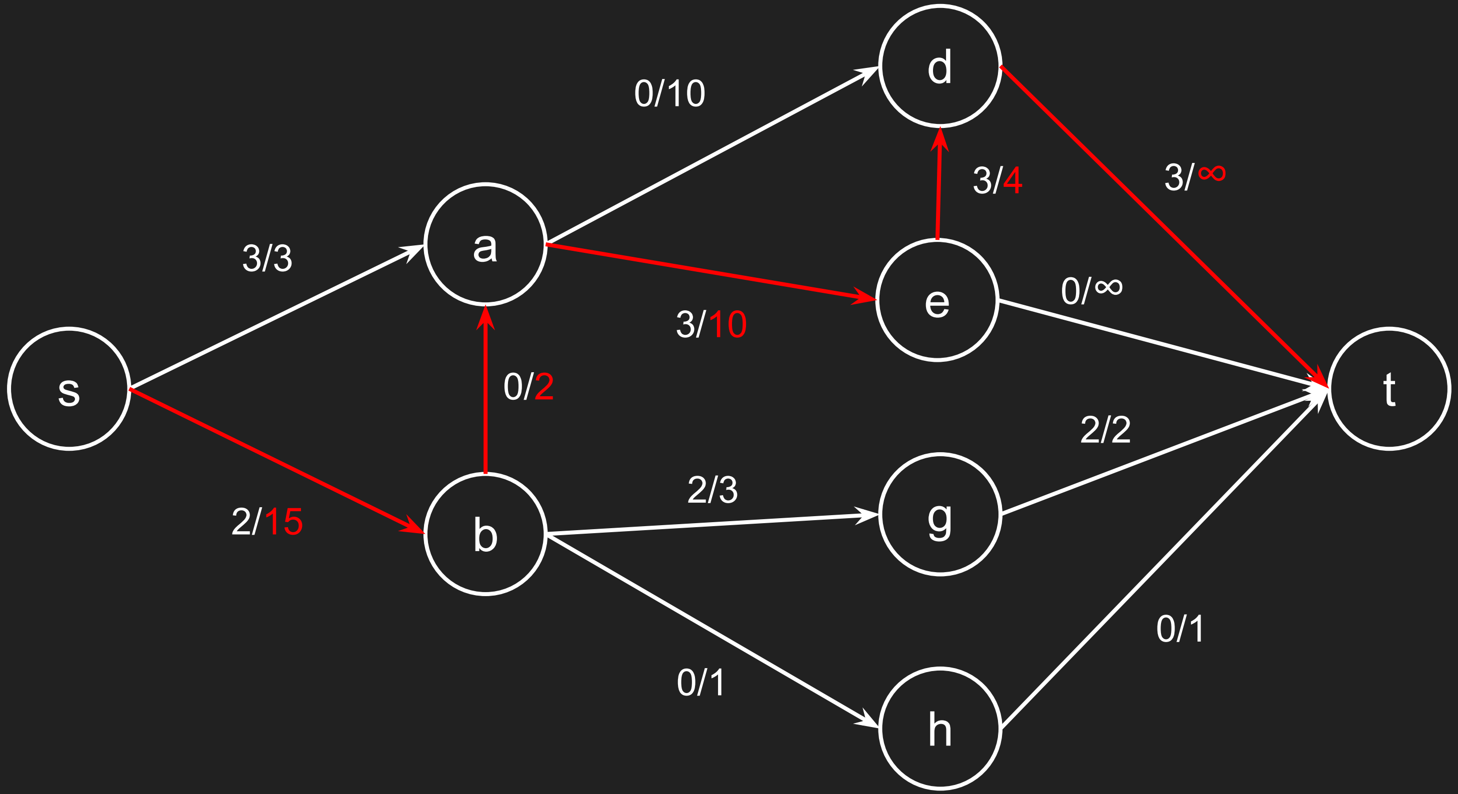

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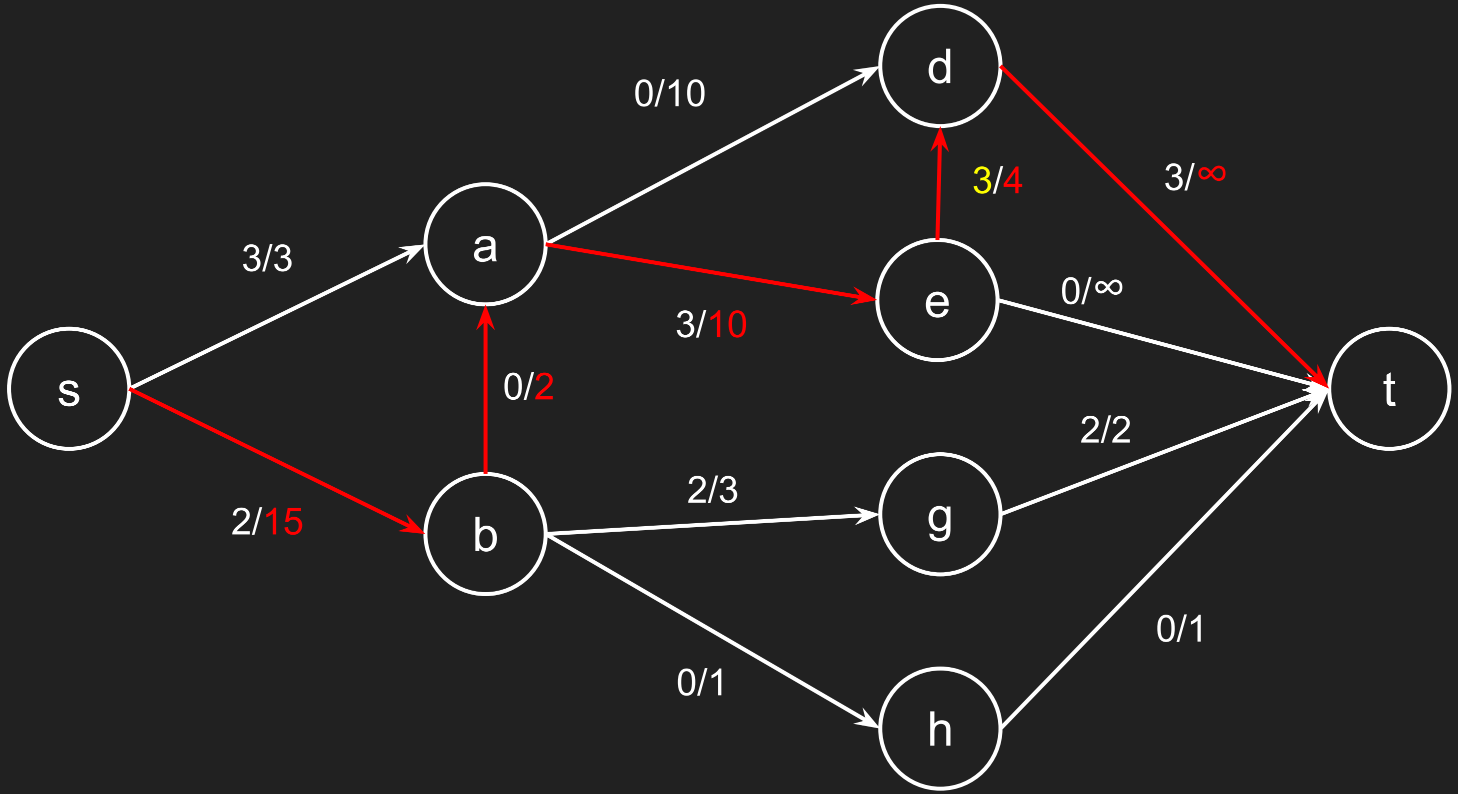

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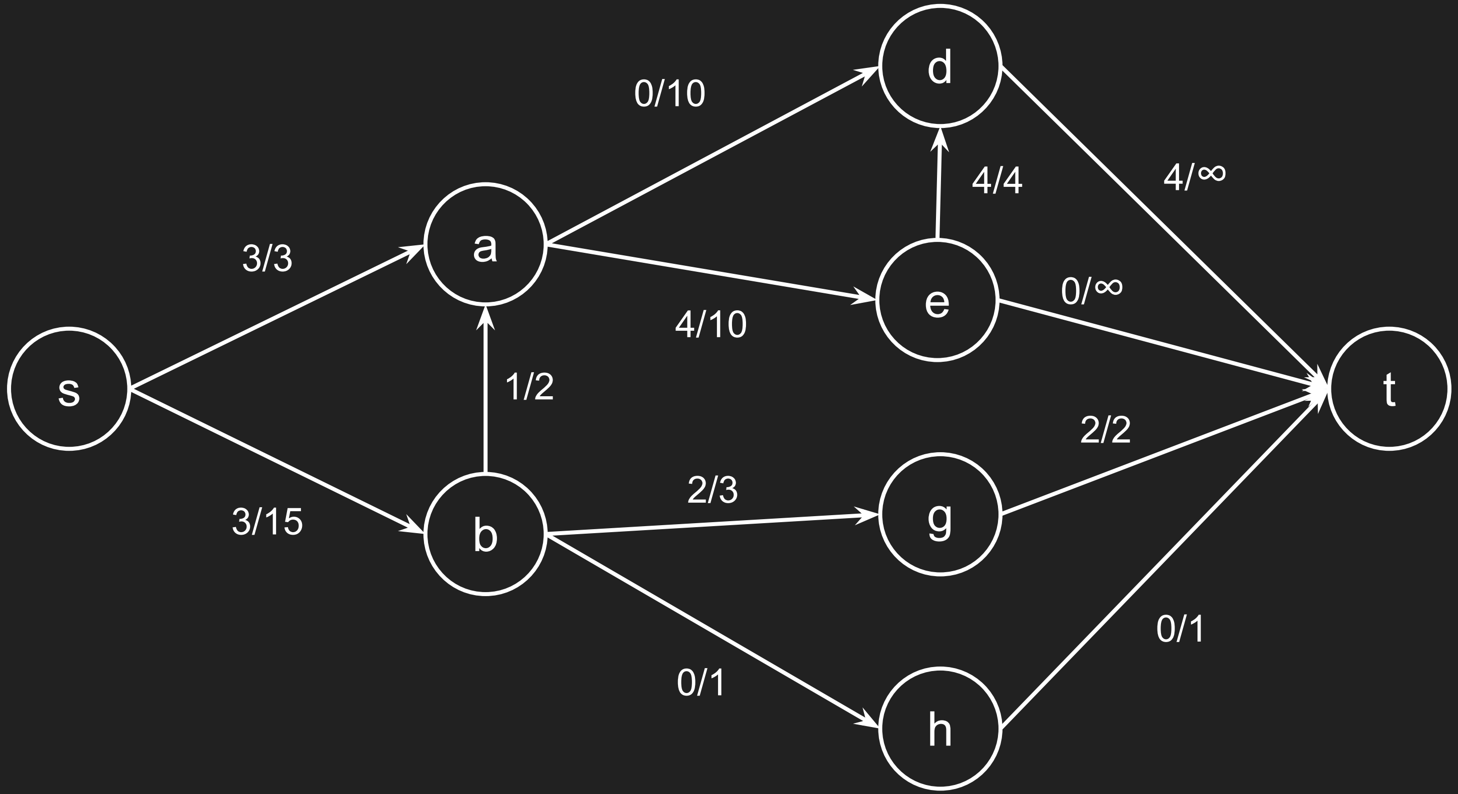

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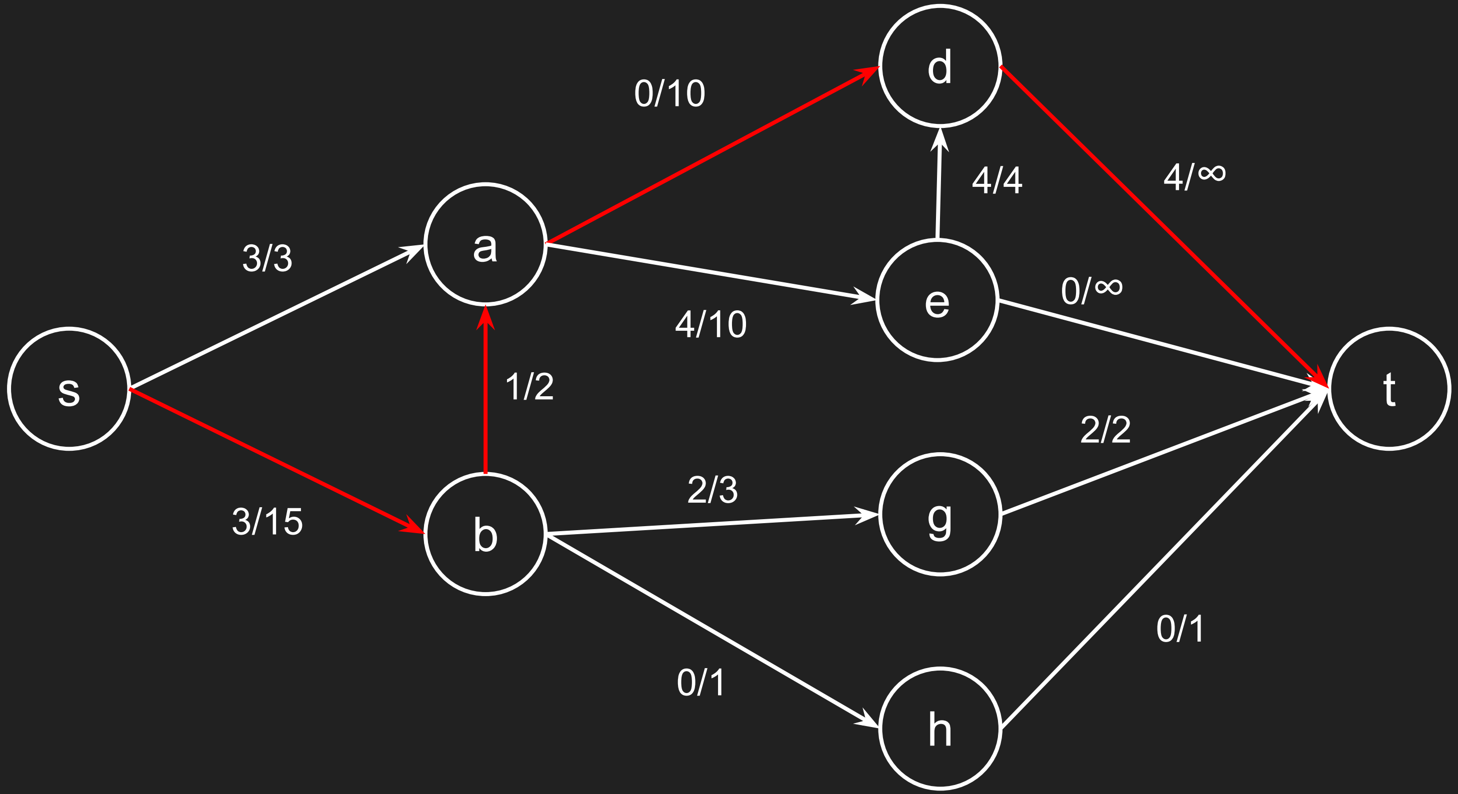

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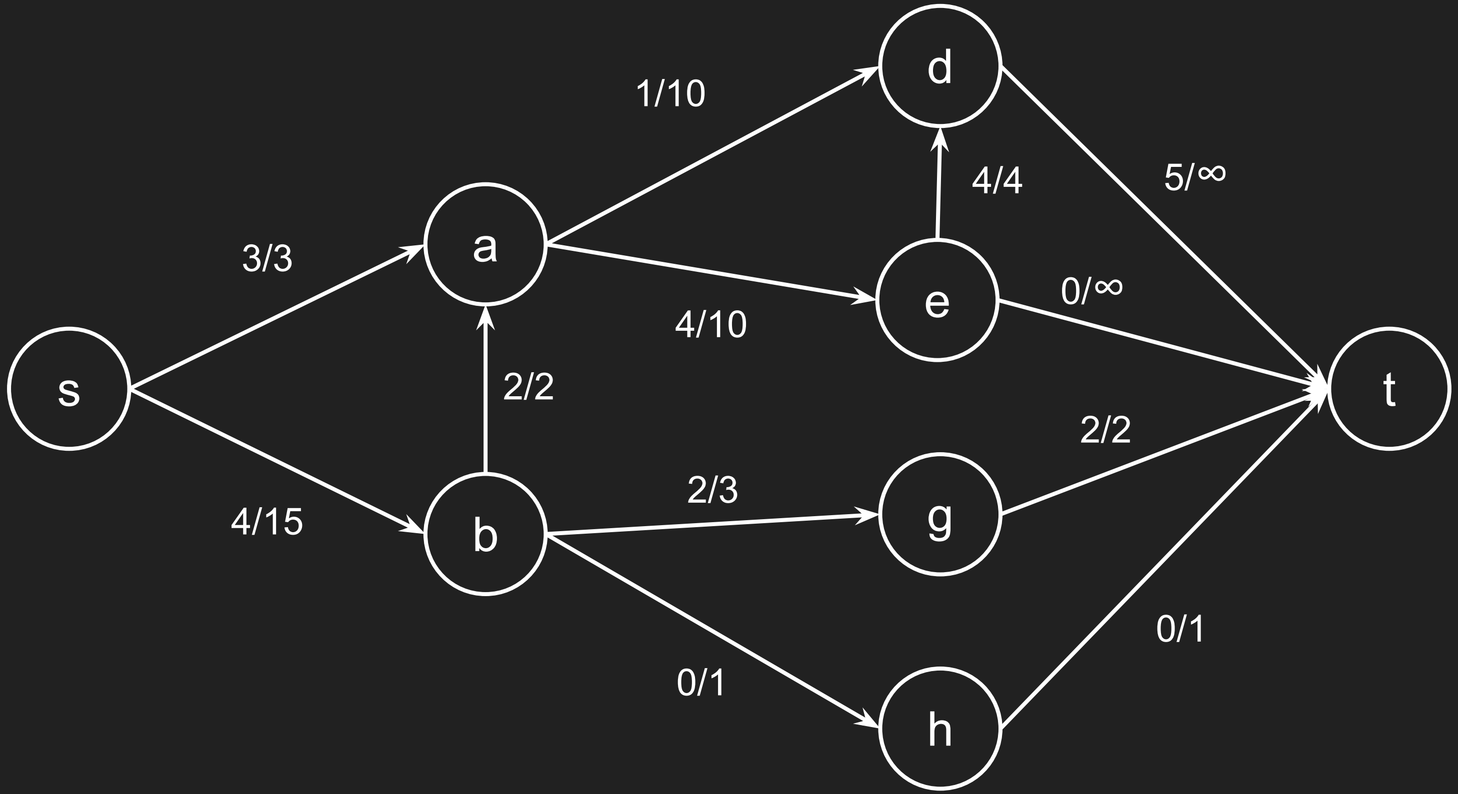

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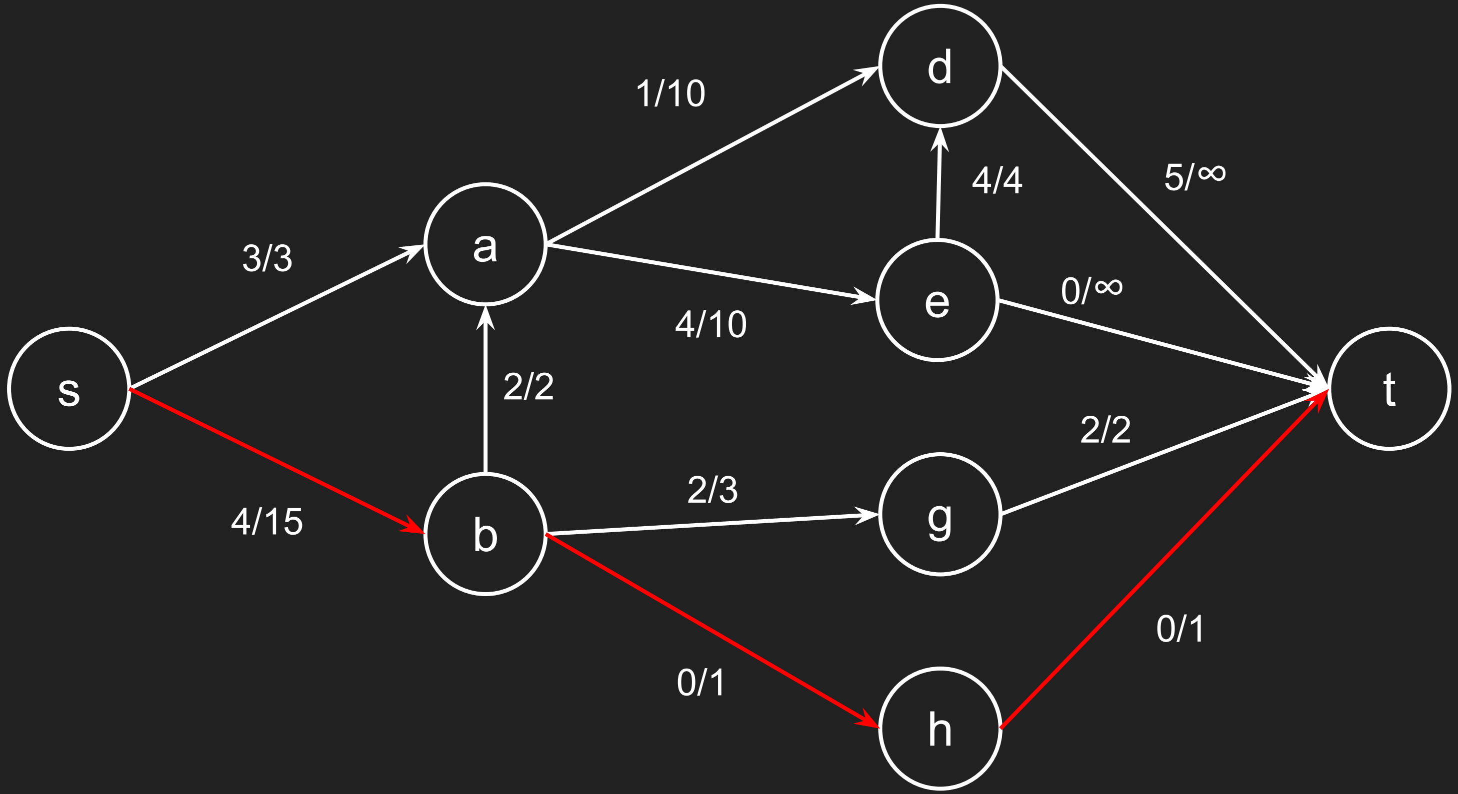

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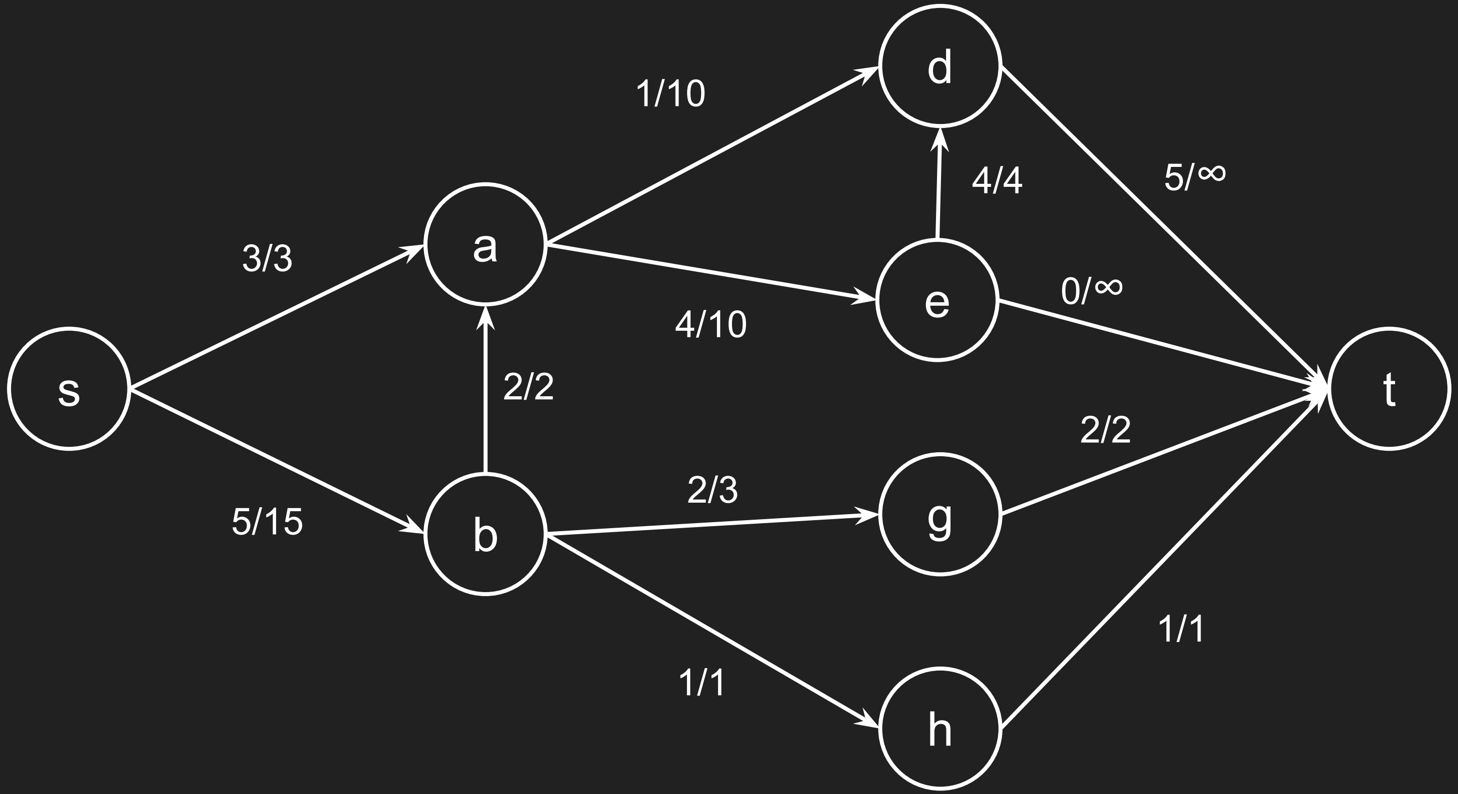

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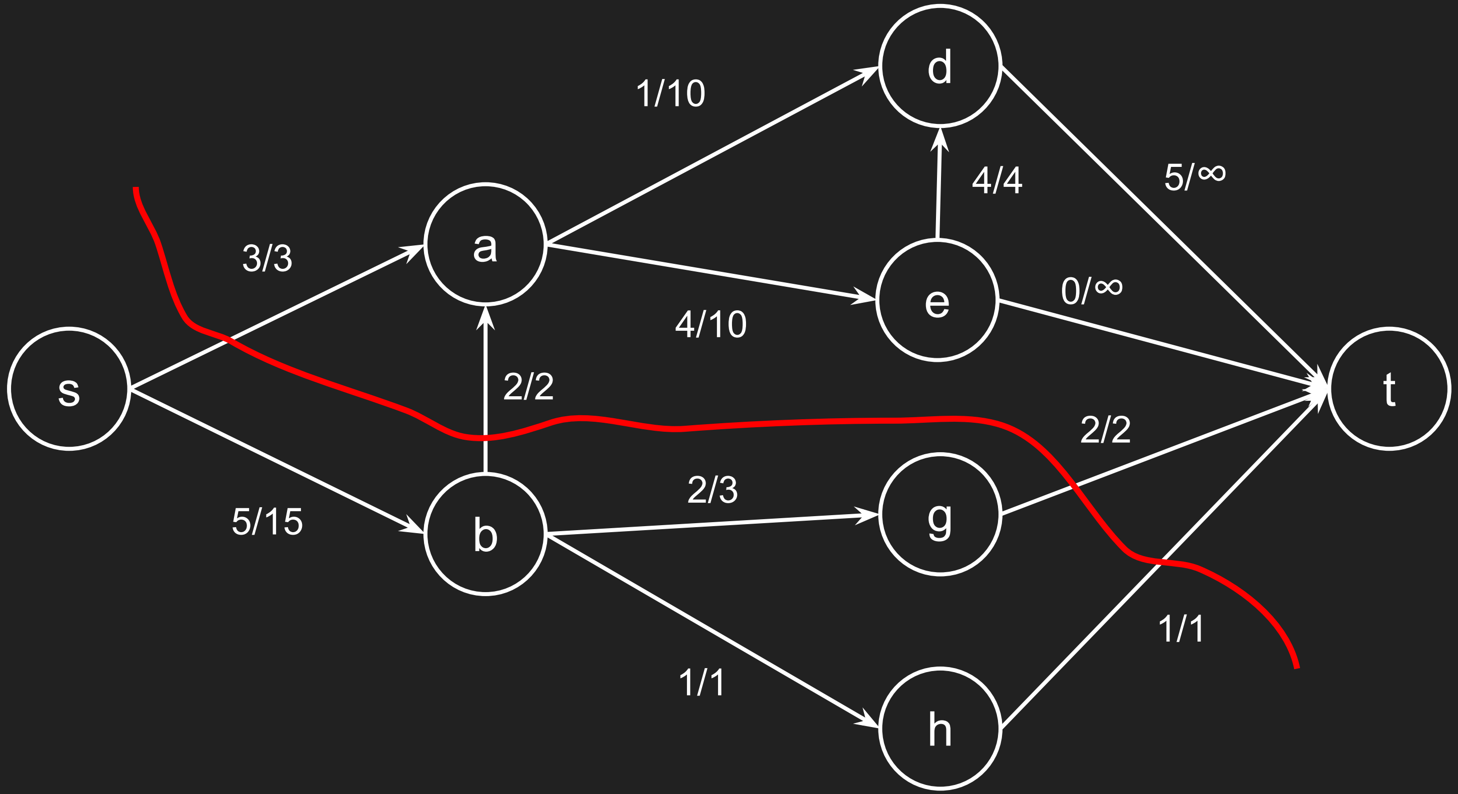

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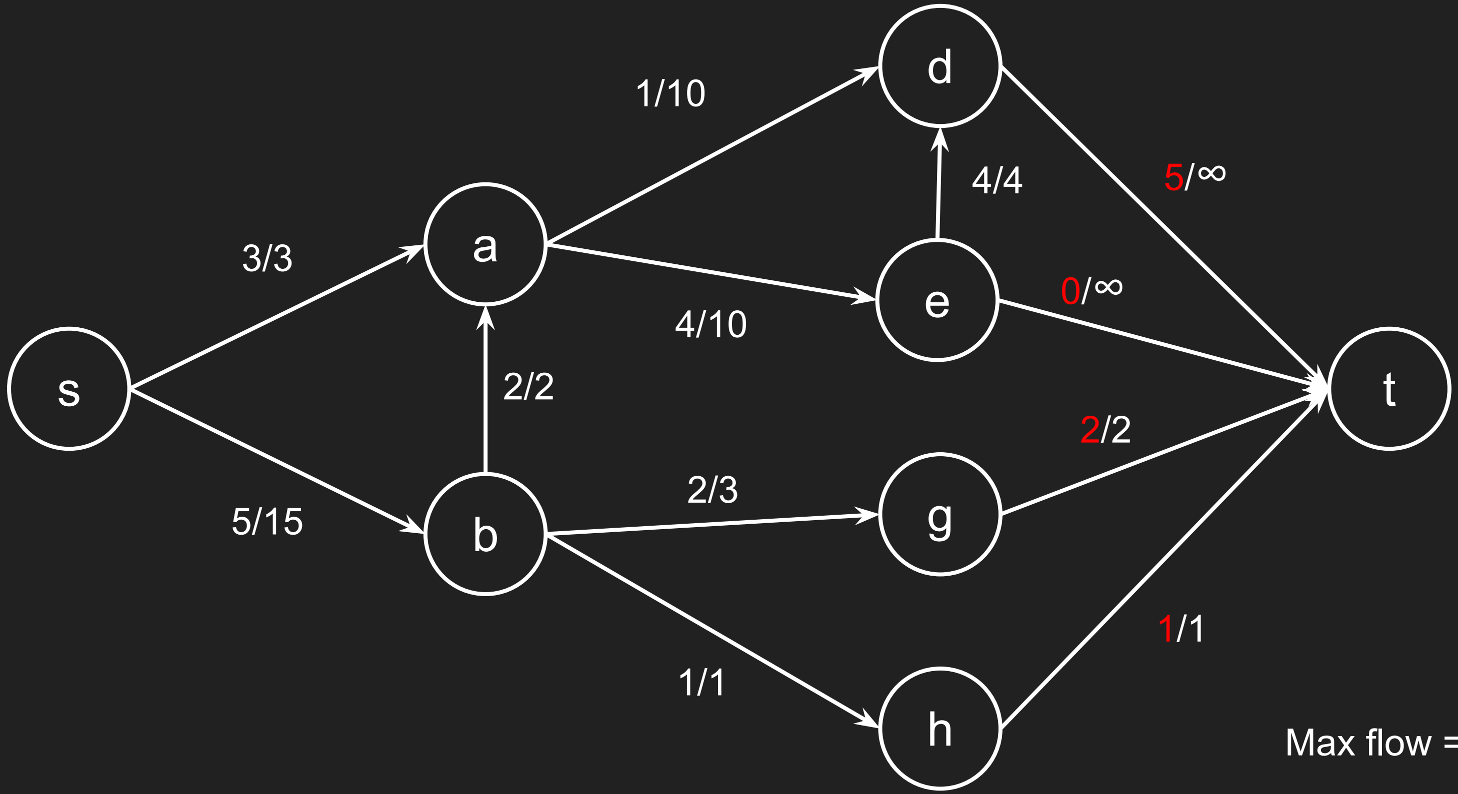
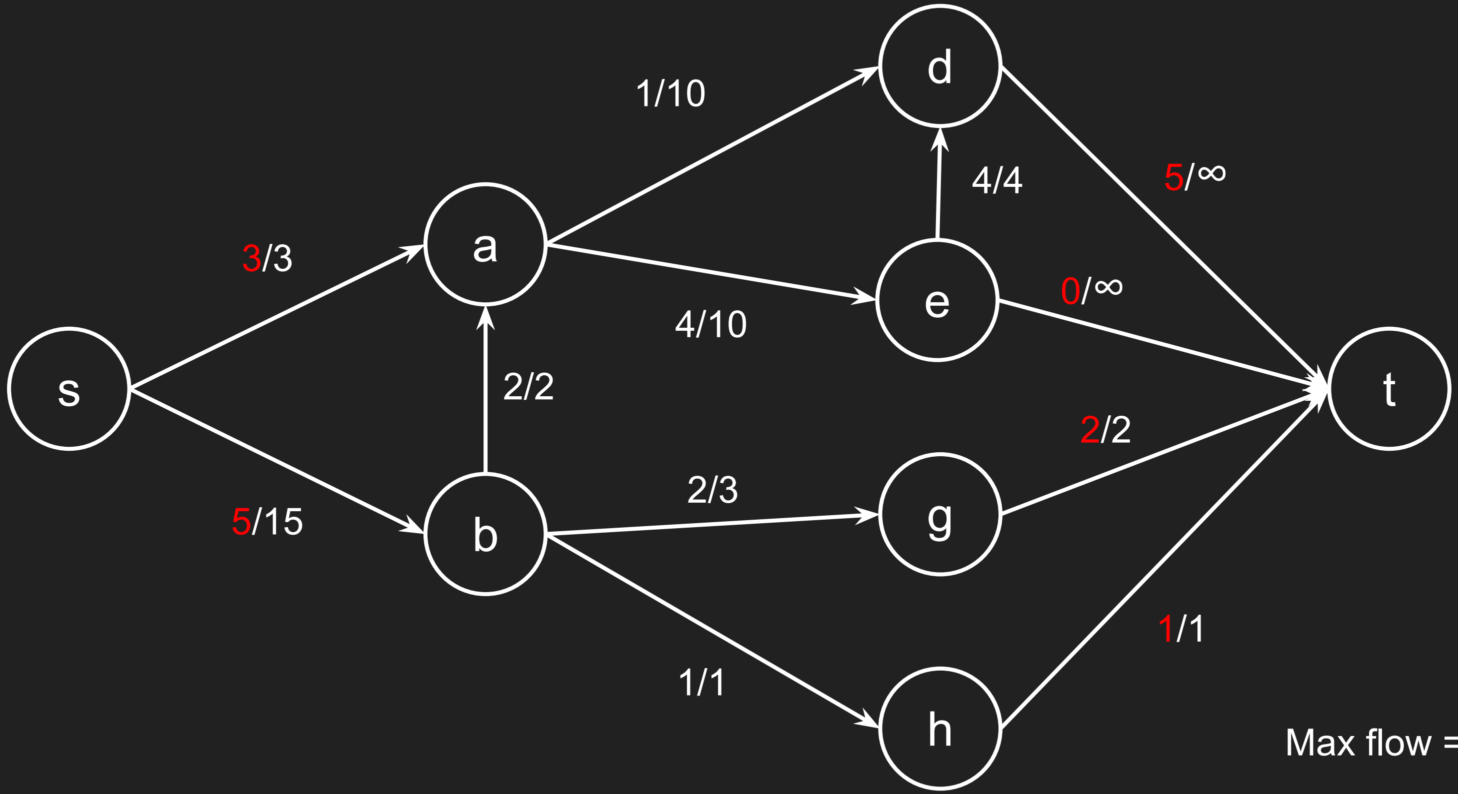


Max flow = 8

Max Flow



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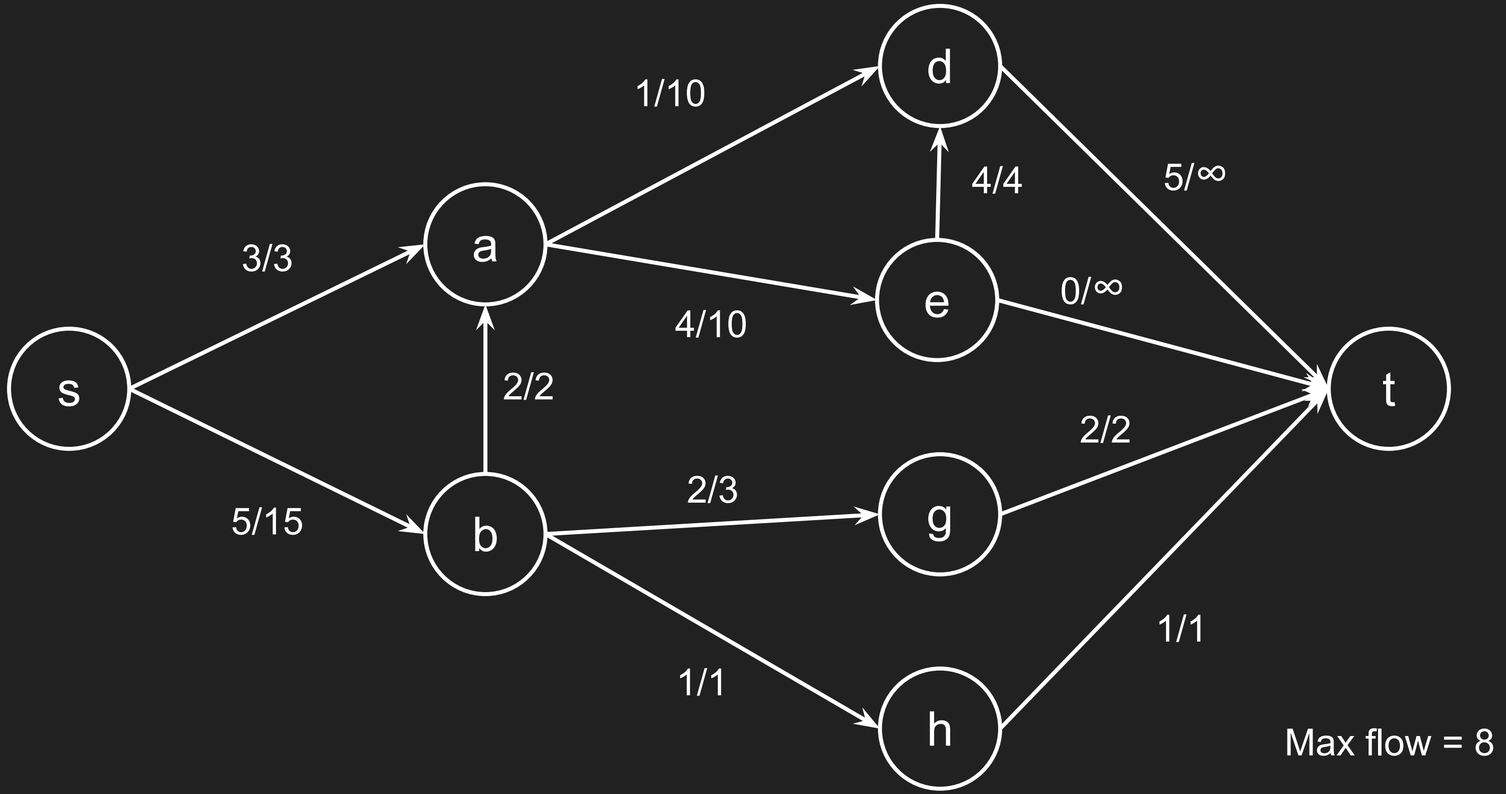

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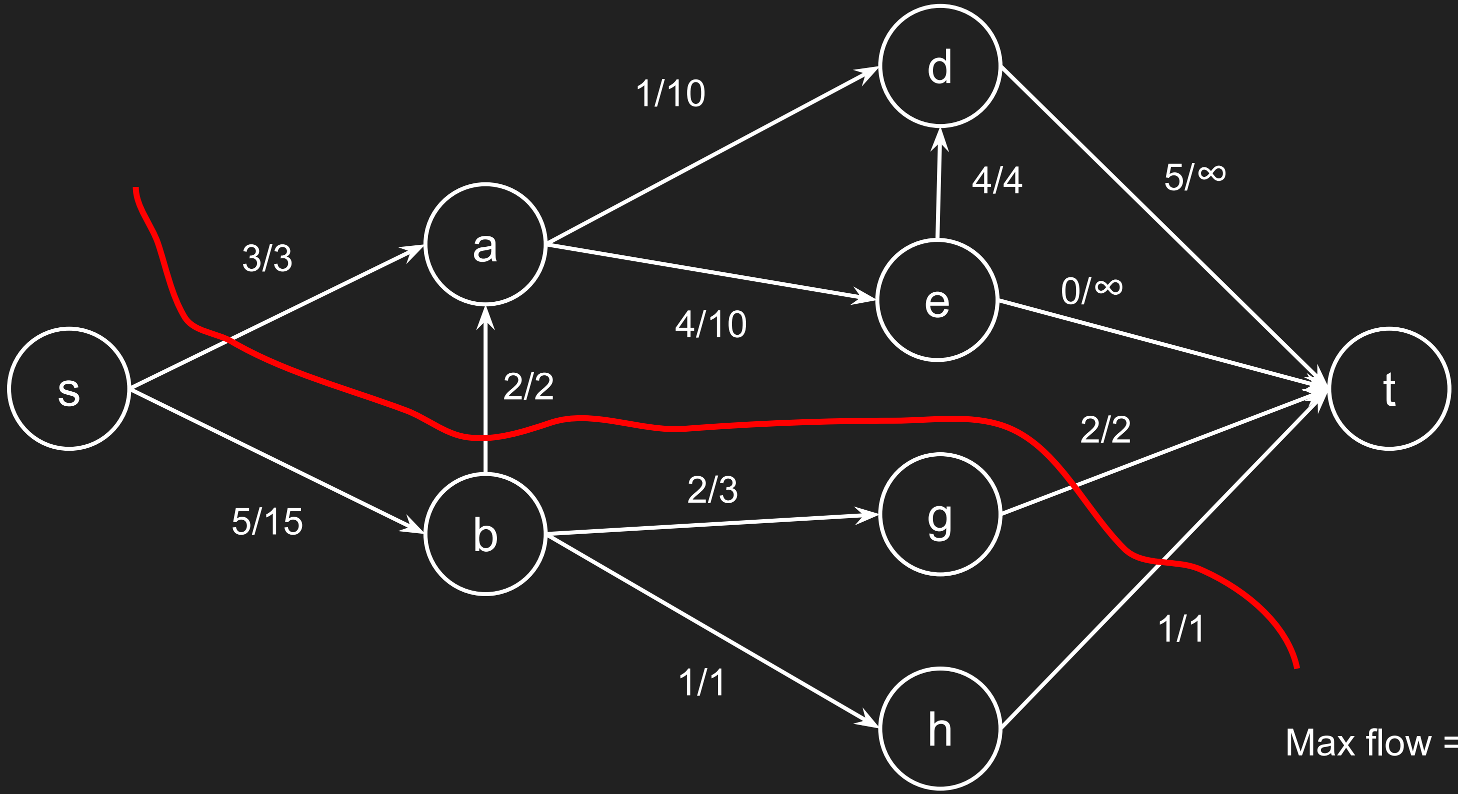

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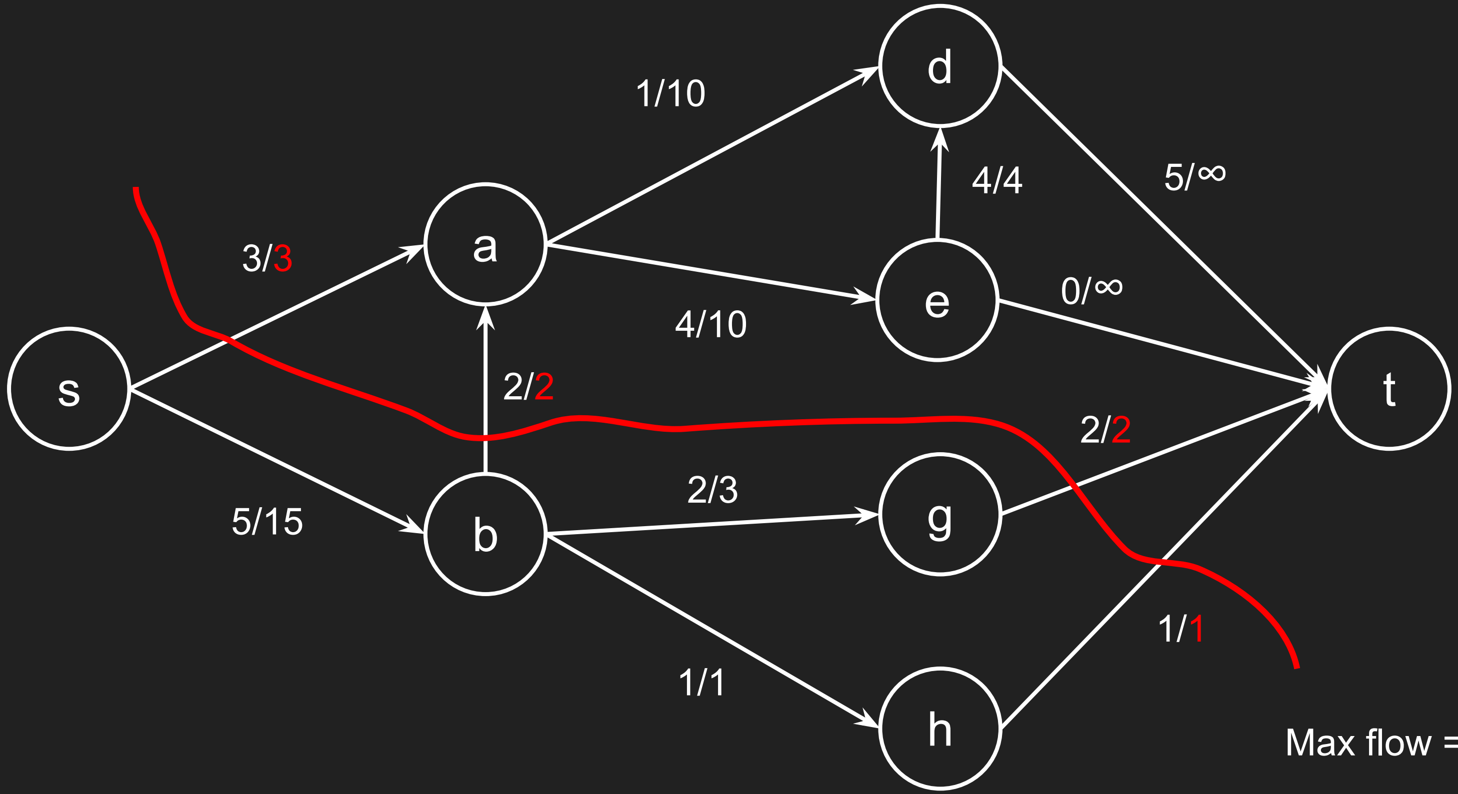


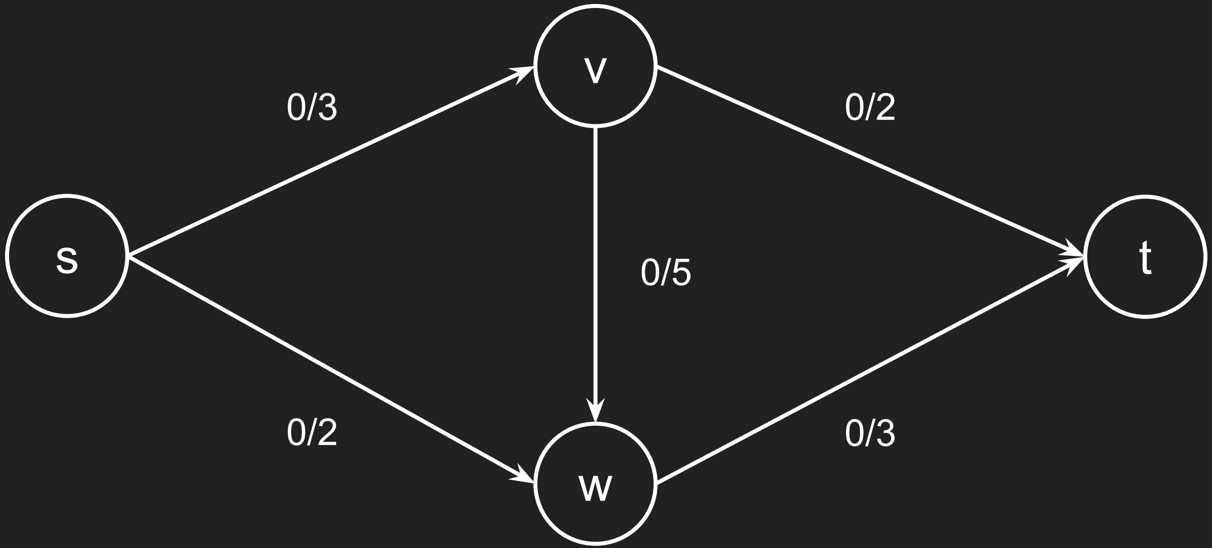
Max flow = 8

Max Flow / Min Cut

- Define a "cut" to be a partition of the nodes into two sets. An *s*-*t* cut is one where one set contains s and the other contains t
- Define the capacity of an *s*-*t* cut to be the sum of the capacities of the edges that "cross the partition boundary" from the s set to the t set
  - In other words, if the nodes are partitioned into sets A and B with $s \in A$ and $t \in B$, the edges from u to v where $u \in A$ and $v \in B$
- The Min Cut problem is to find the *s*-*t* cut with the minimum capacity
- The Max Flow / Min Cut Theorem says the answers are the same!
  - Essentially formalizes the notion of "bottlenecks"

- ...but does our bottleneck finding/saturating algorithm always work?

Max Flow



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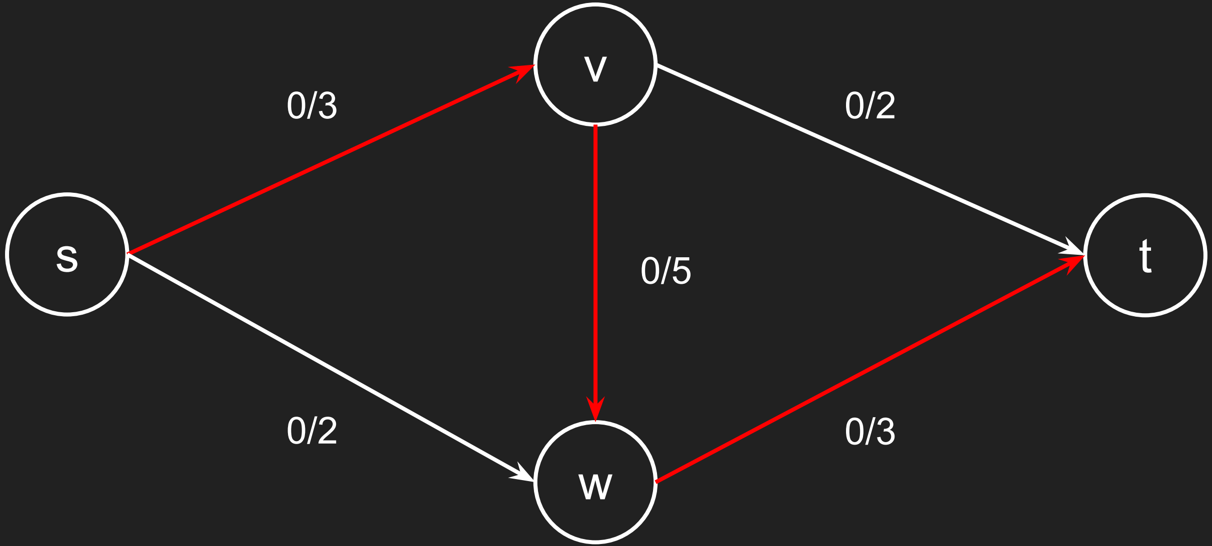

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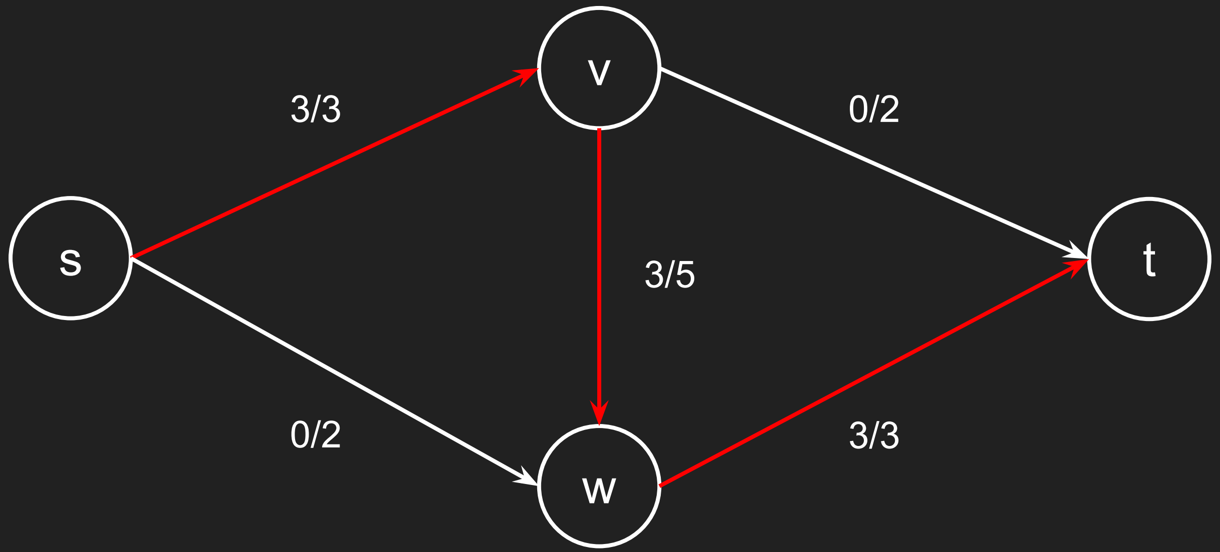

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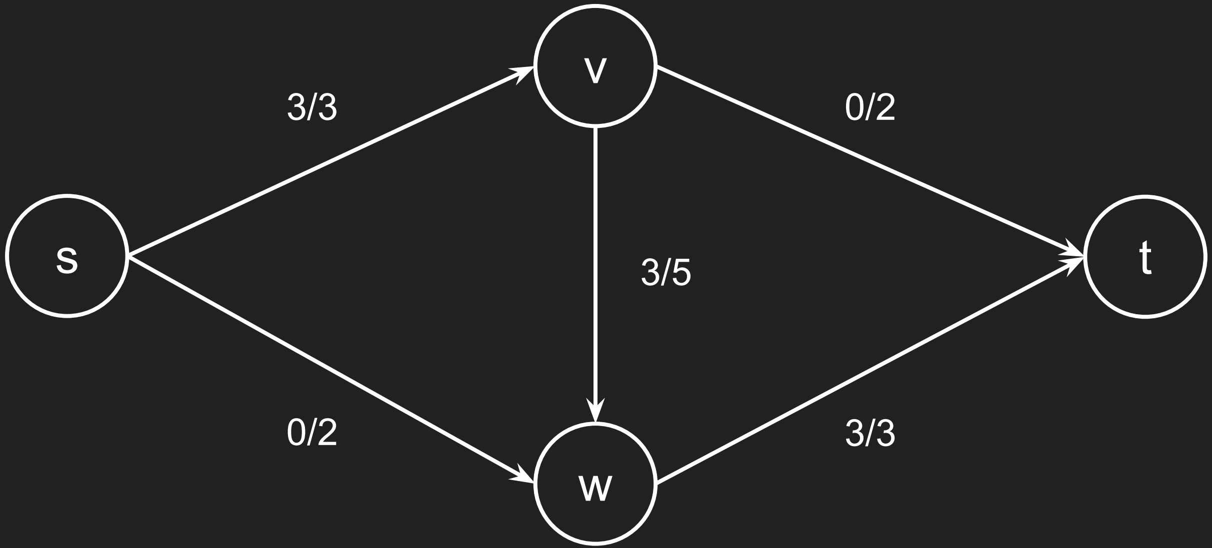
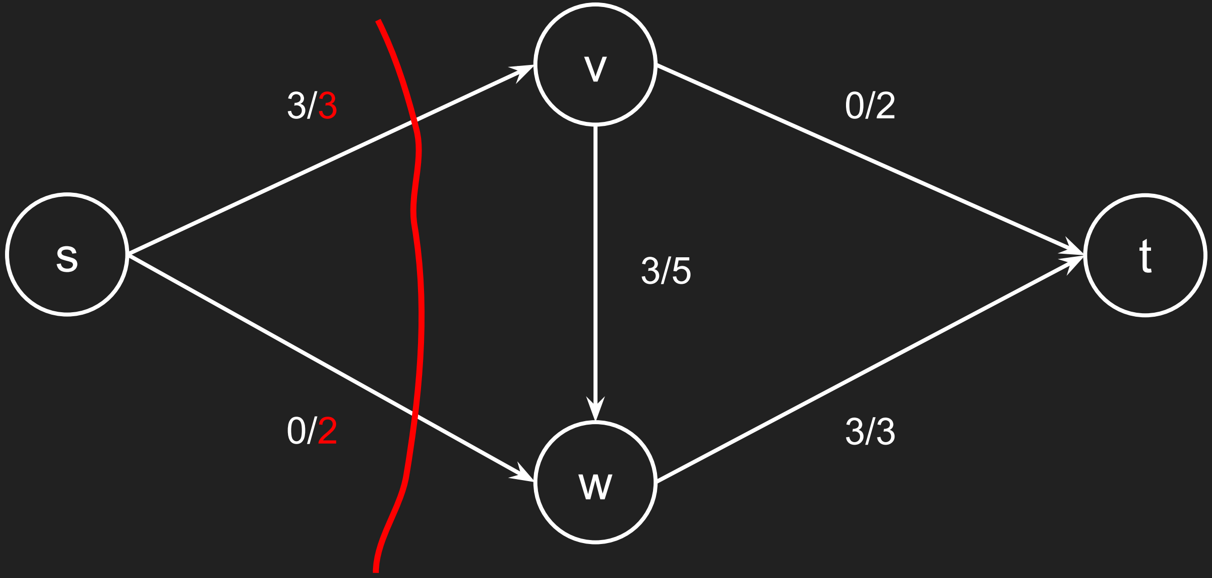


Max flow = 3 ?

Max Flow



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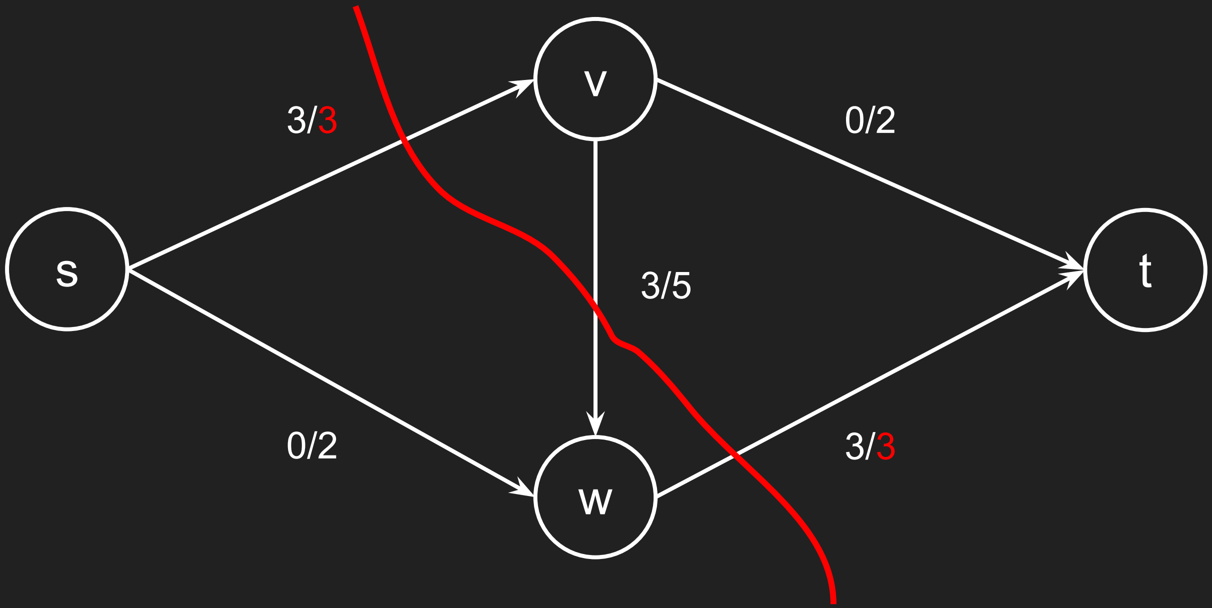

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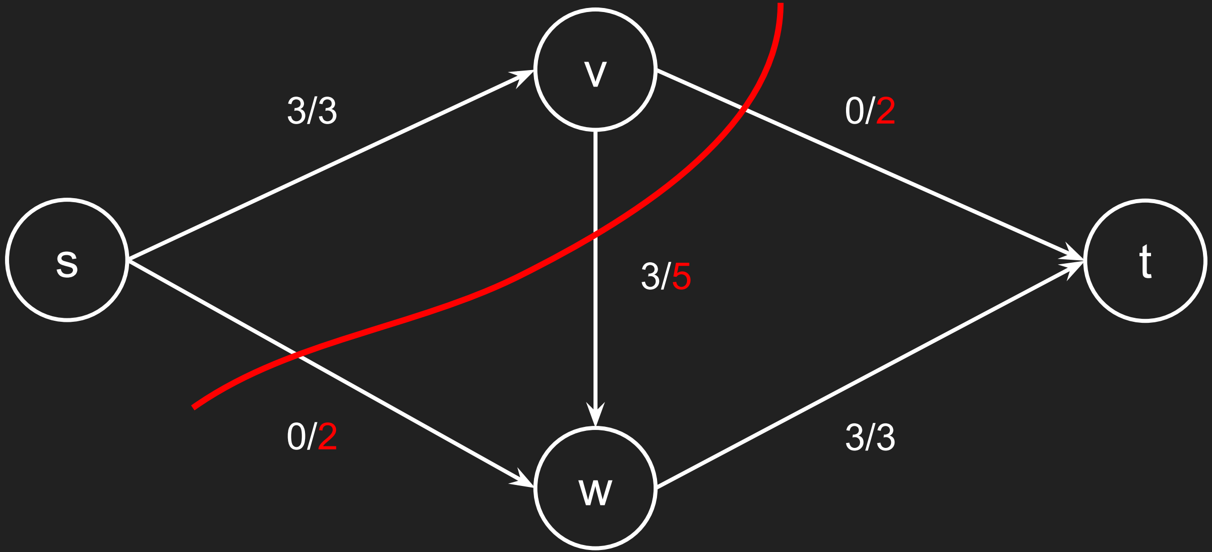

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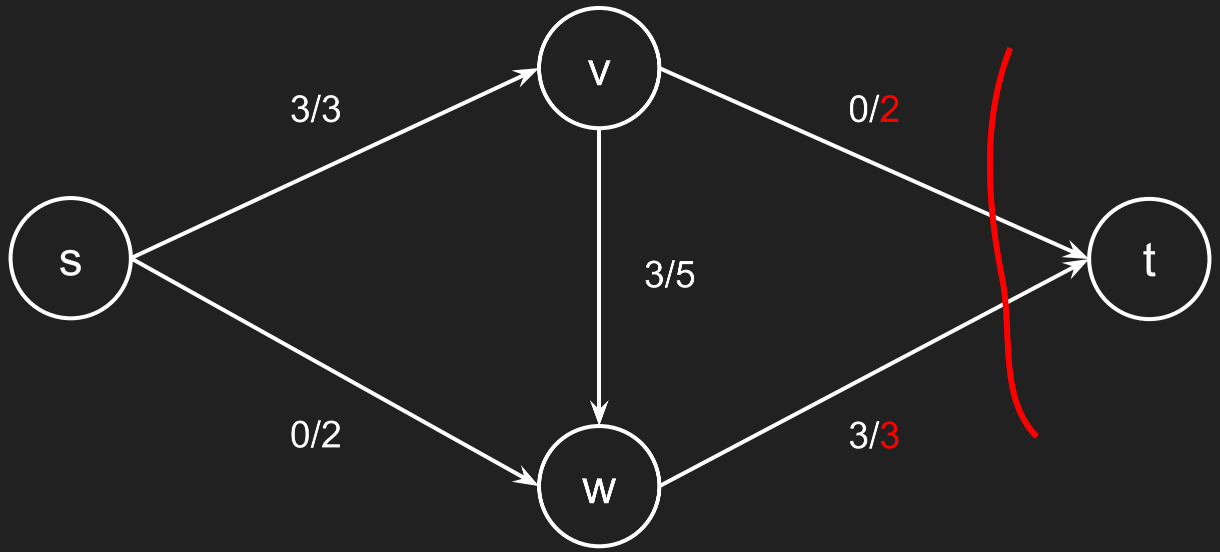

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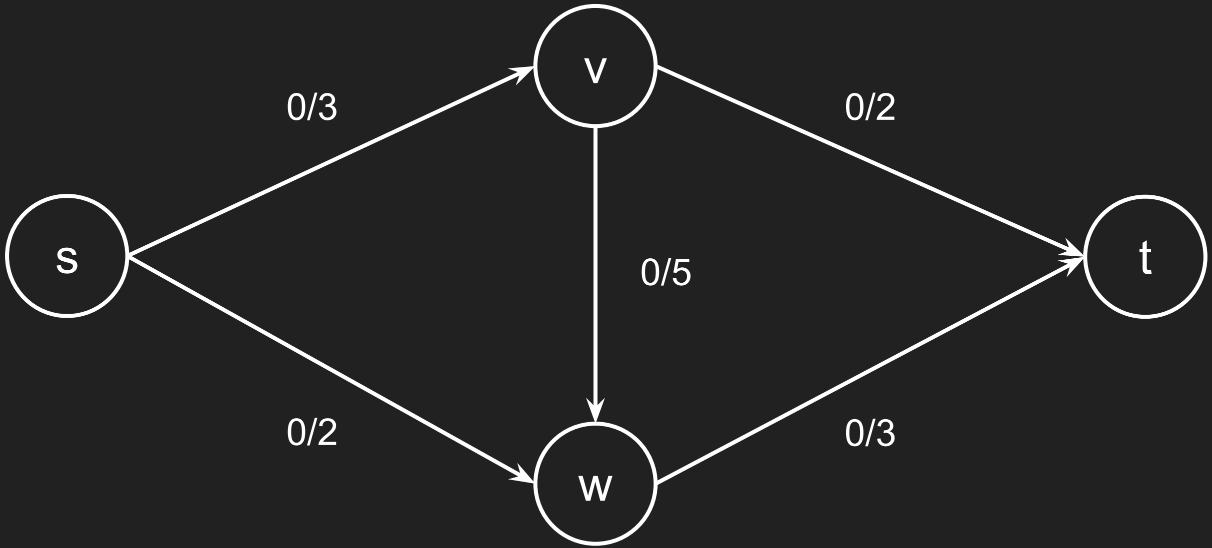

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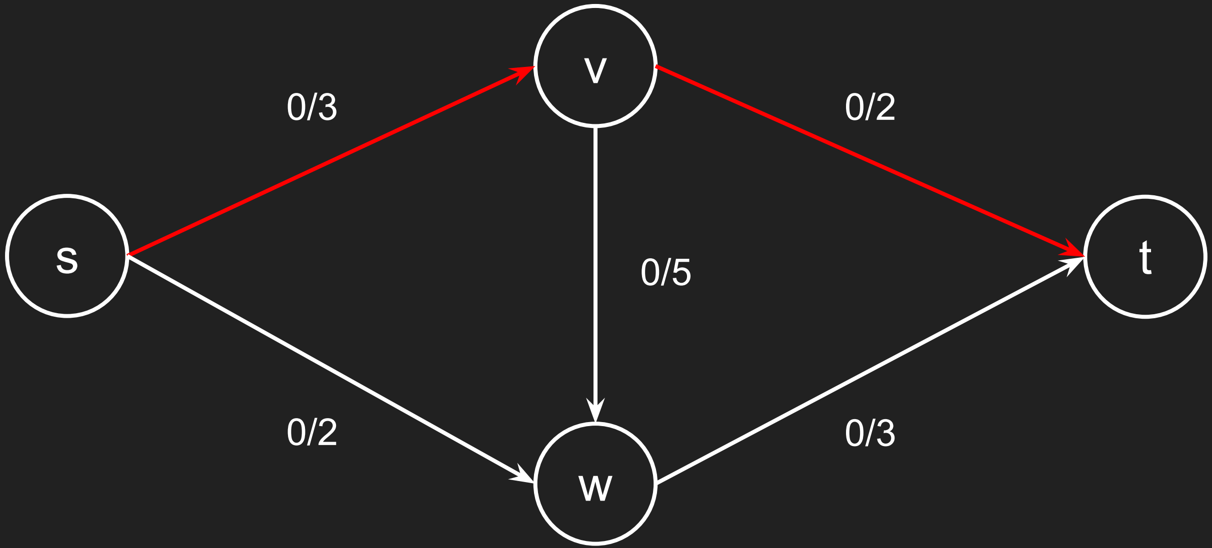

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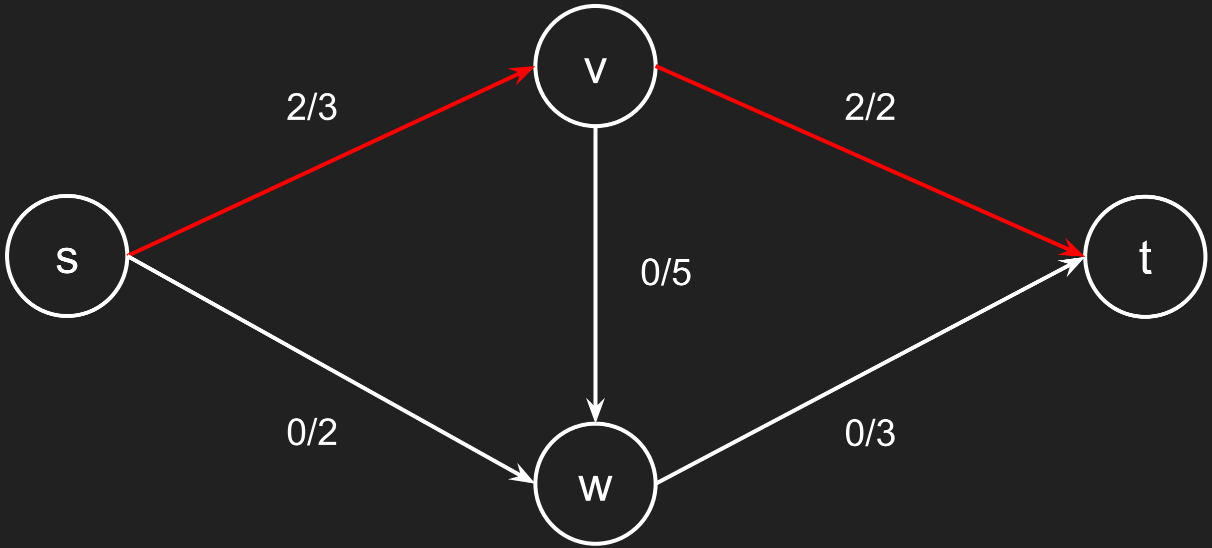

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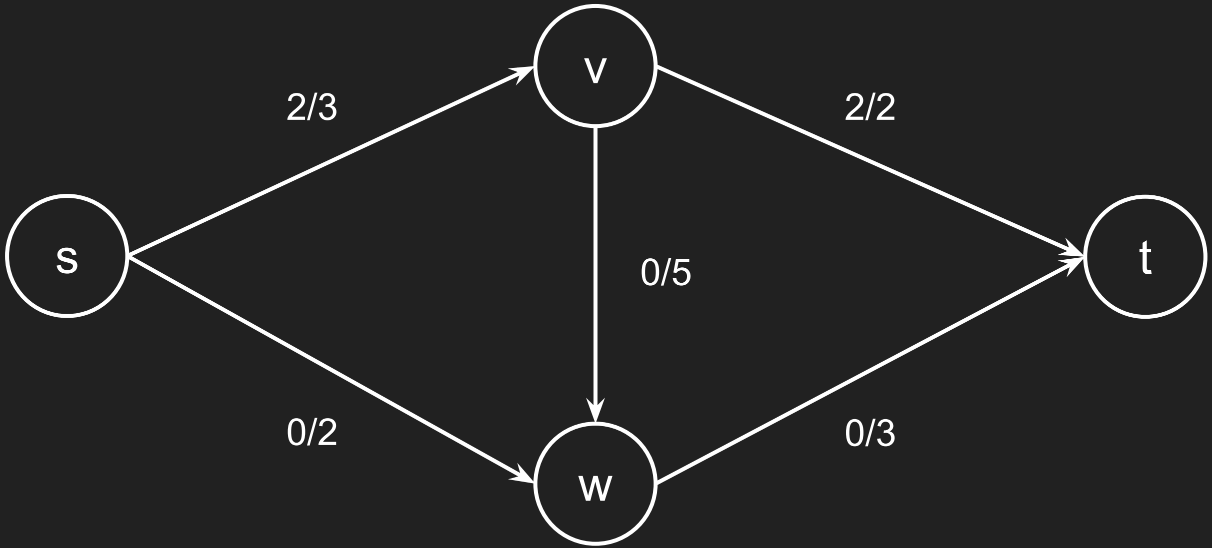

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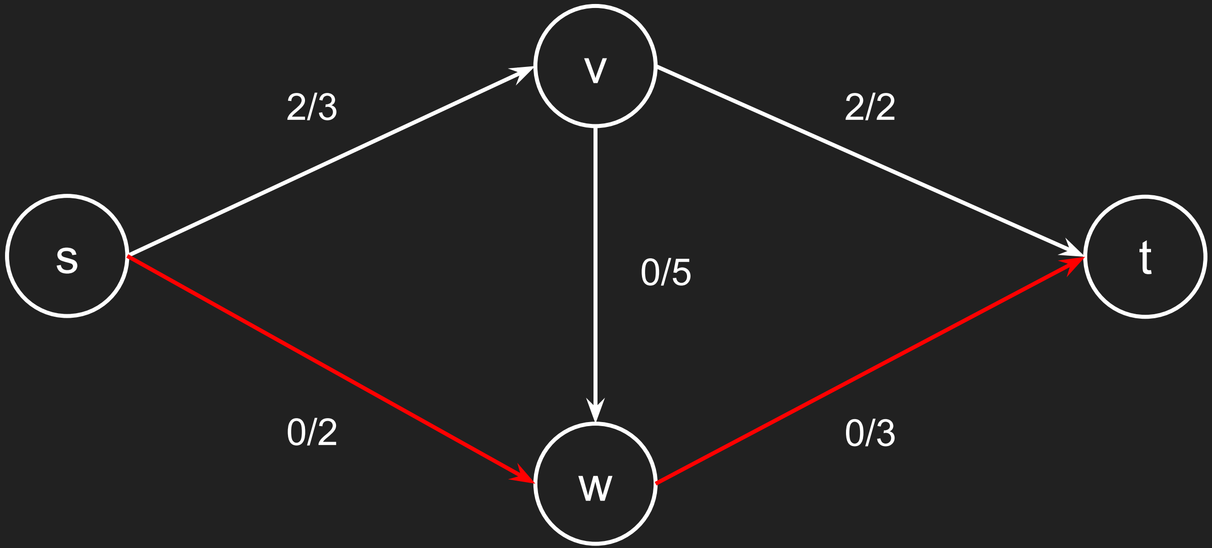

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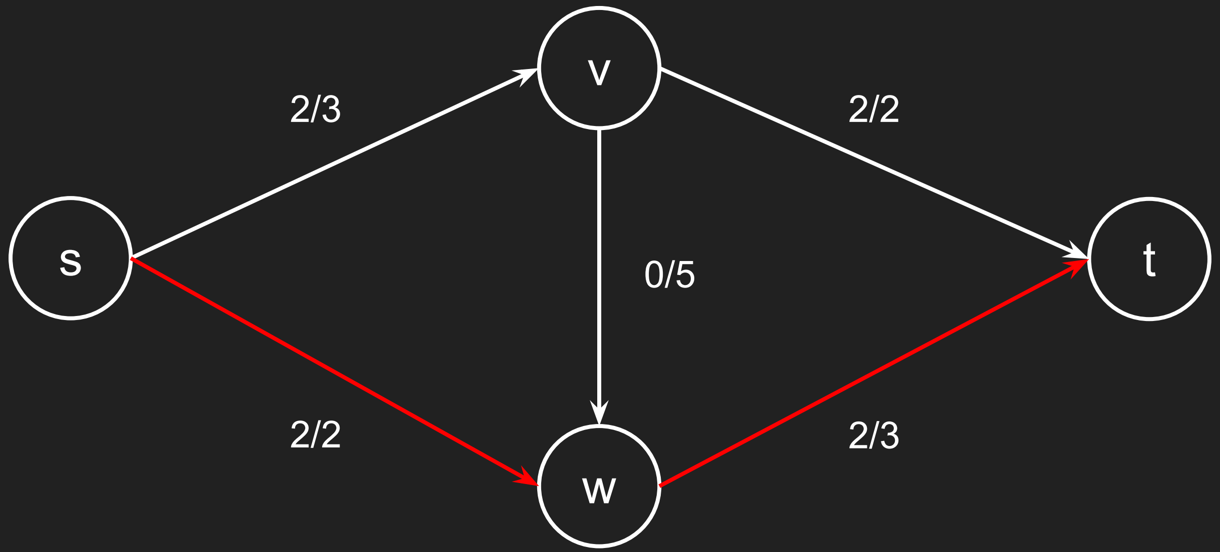

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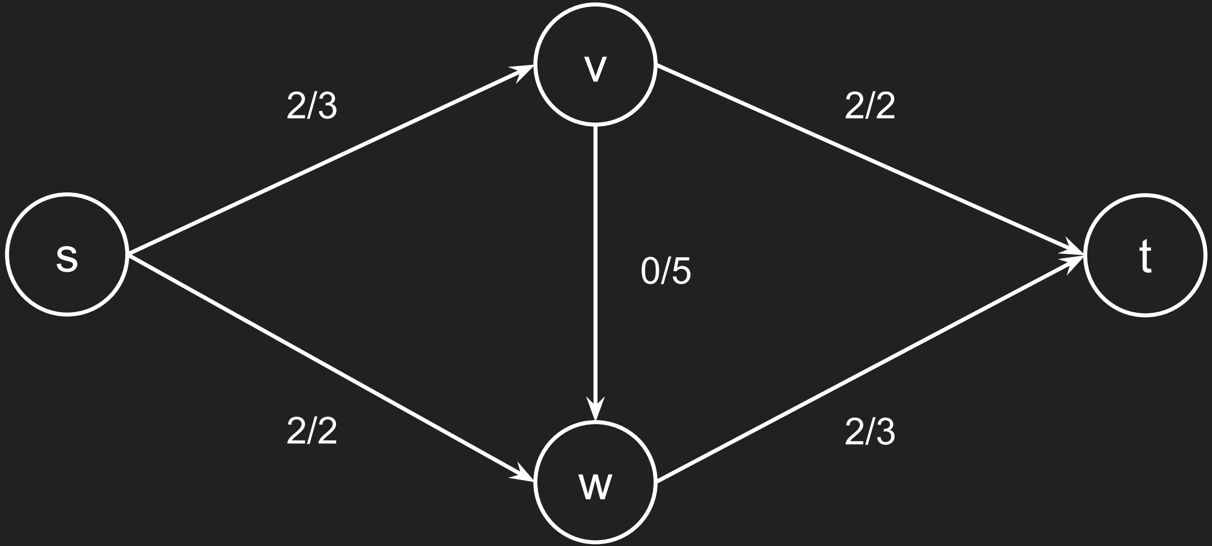

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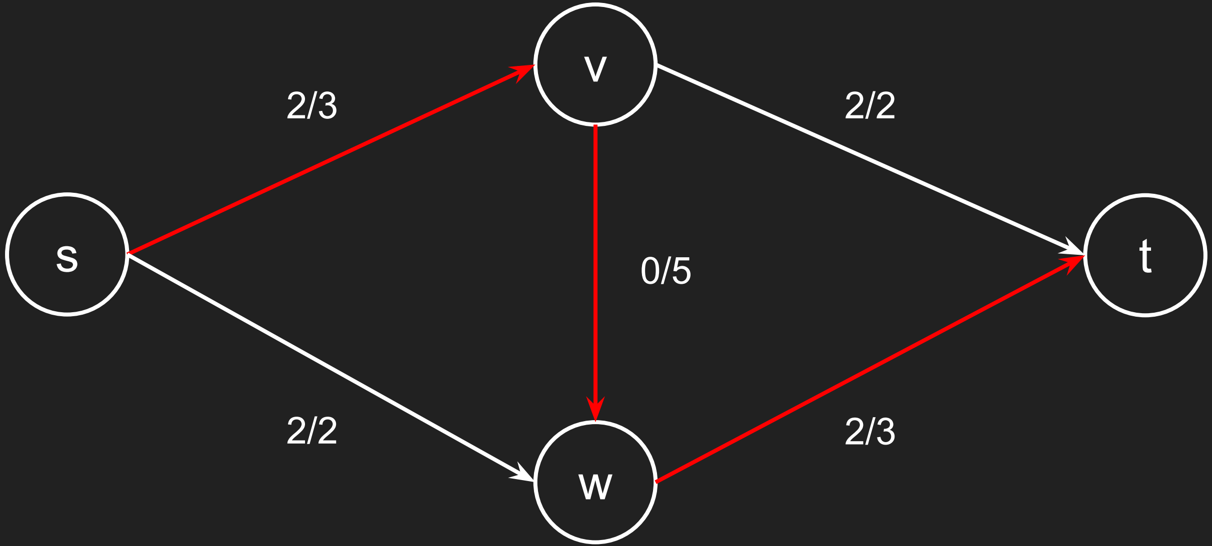

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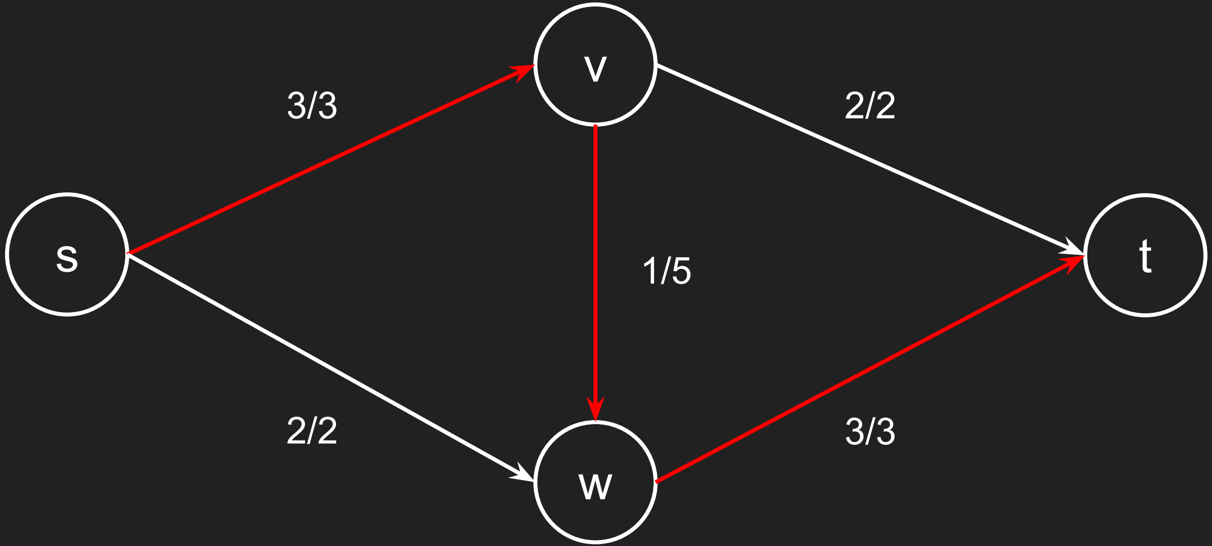
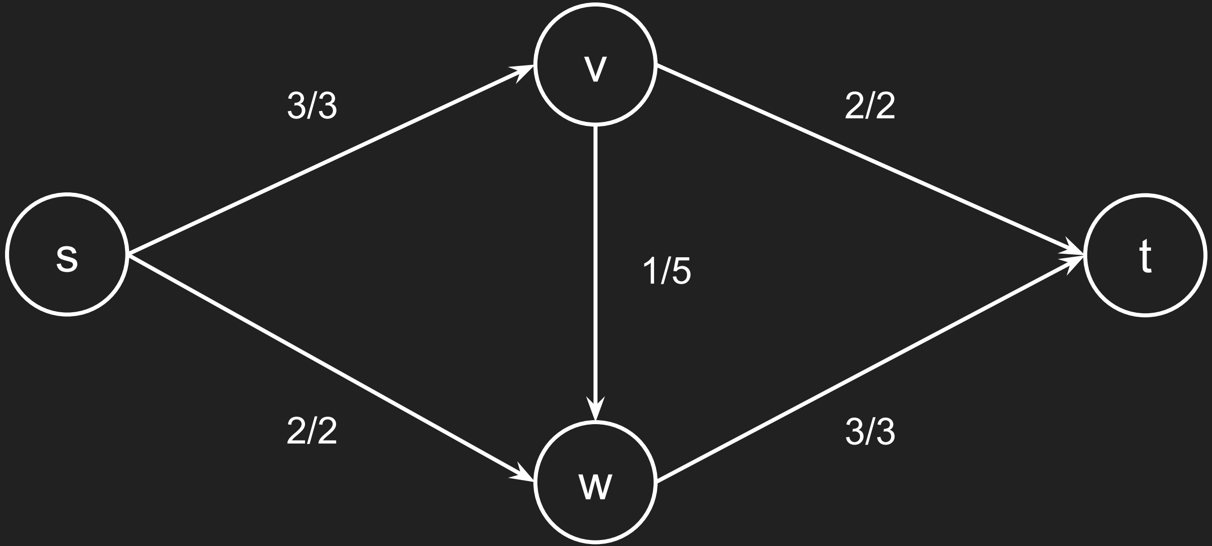


Max Flow

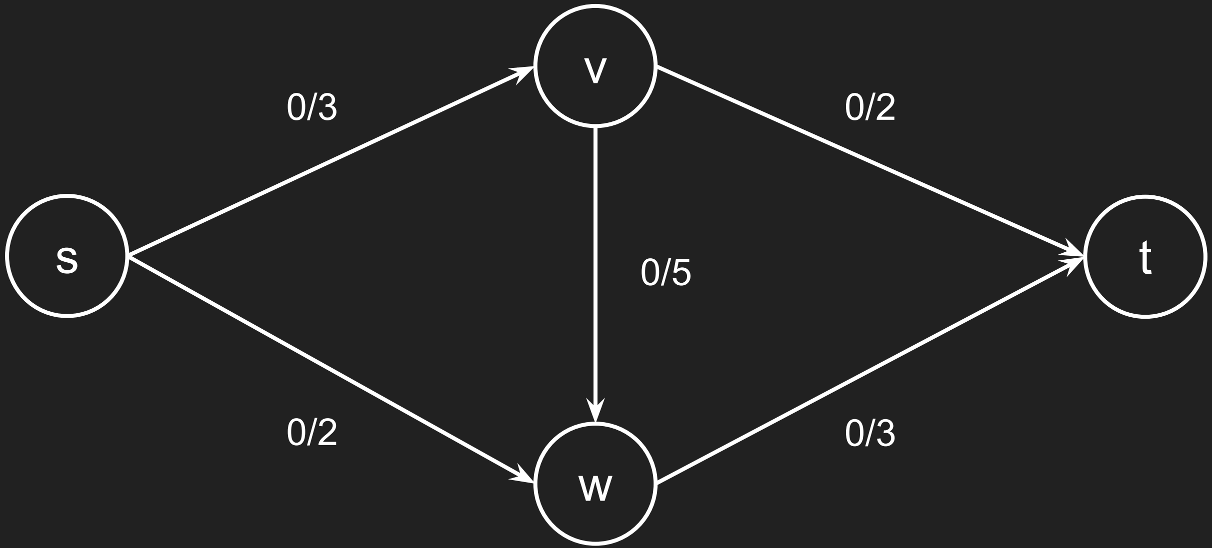


Max flow = 5 !

Max Flow

- Being greedy isn't quite good enough
  - At least, the order you pick the paths seems to matter
- Key idea: just because we *can* saturate a path doesn't necessarily mean we *should*
  - We want to be able to "undo" choices if it turns out they boxed us in a corner
- Solution: "residual graphs"
  - Augment the graph with information that allows algorithm to "undo" or "push flow back"

Max Flow



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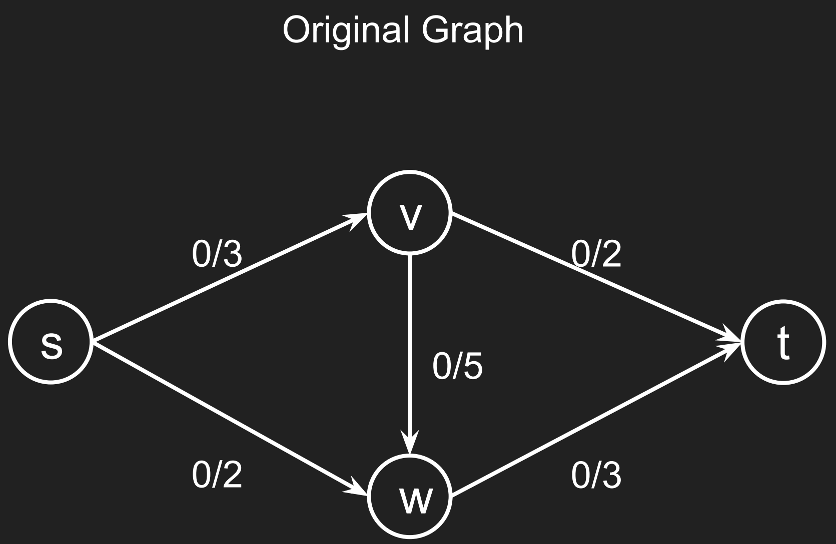
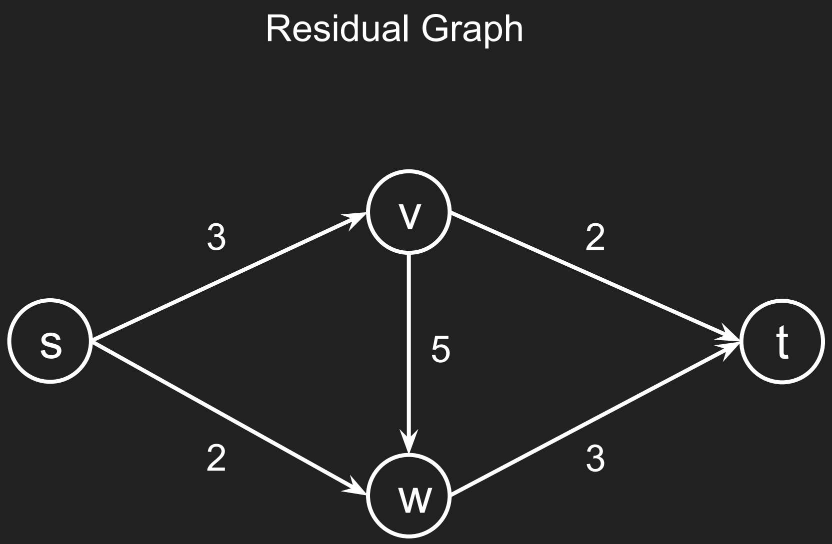
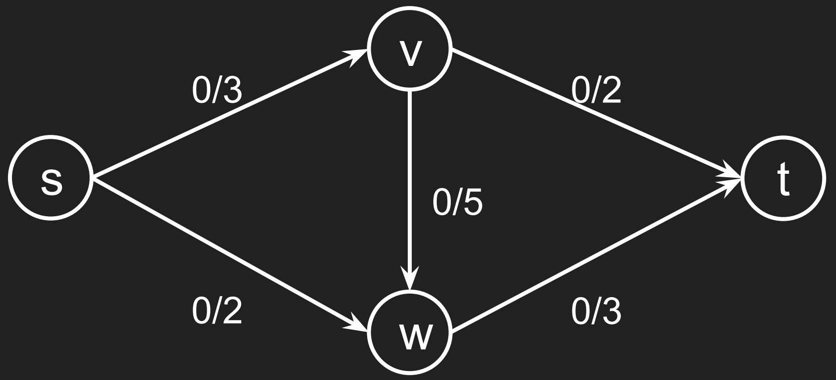


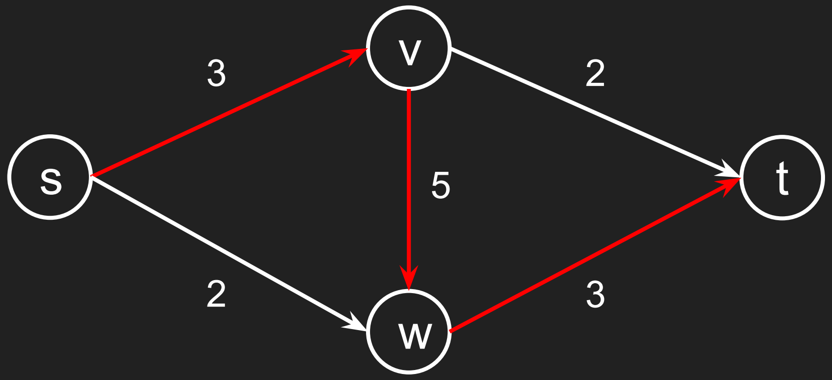


Max Flow

Original Graph



Residual Graph



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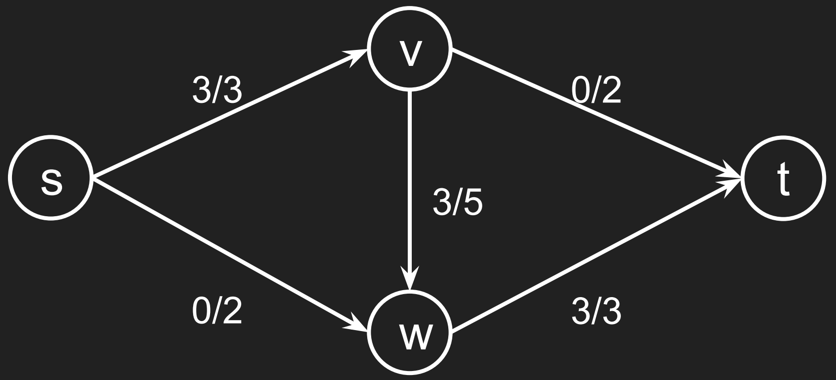

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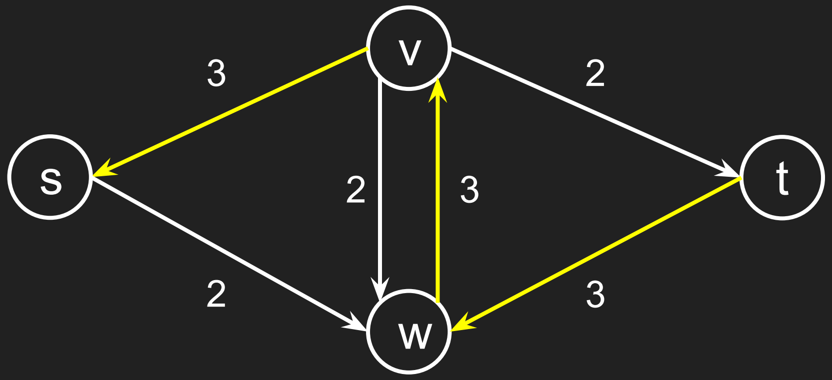

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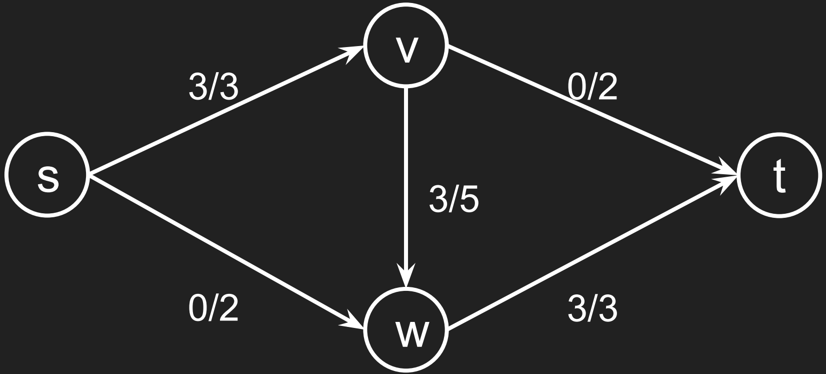

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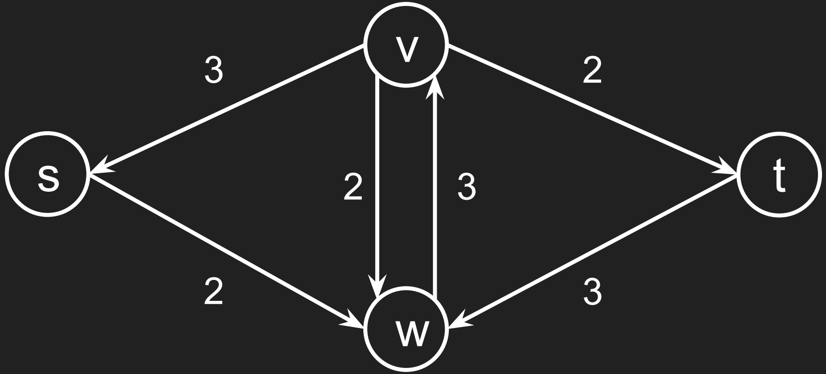

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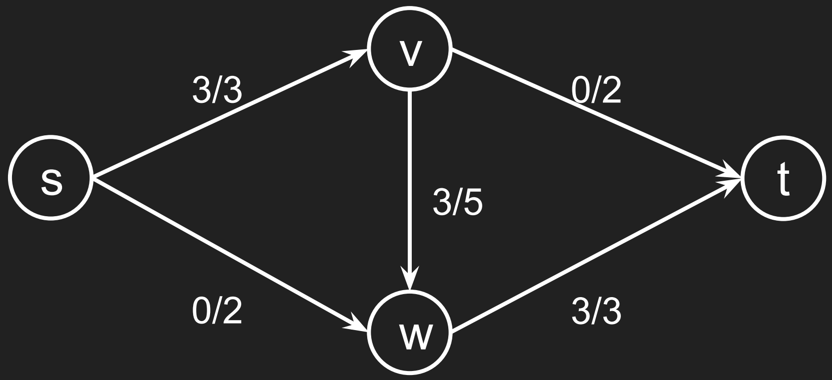

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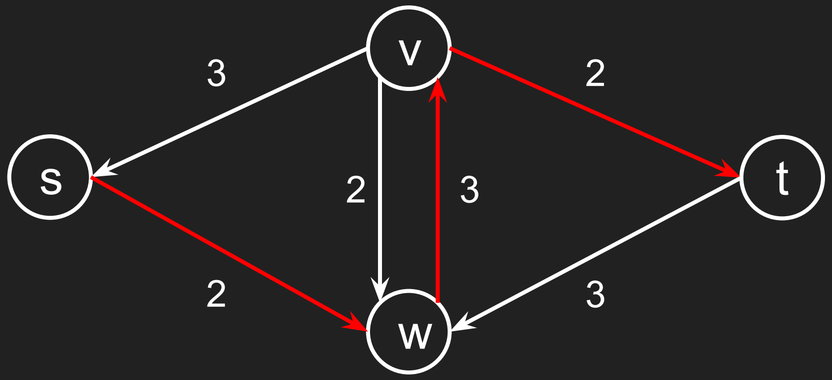

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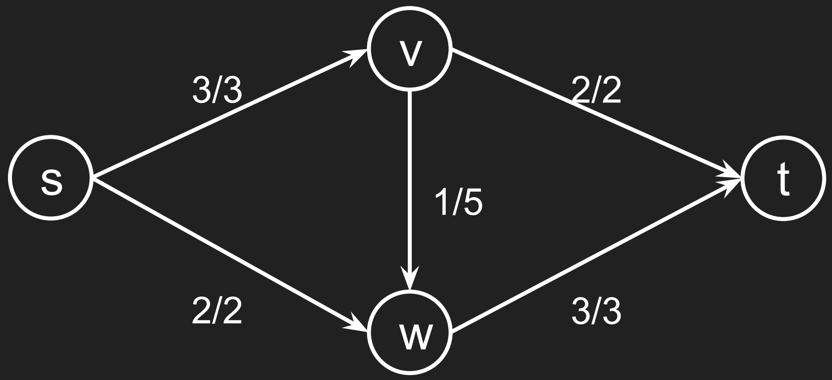

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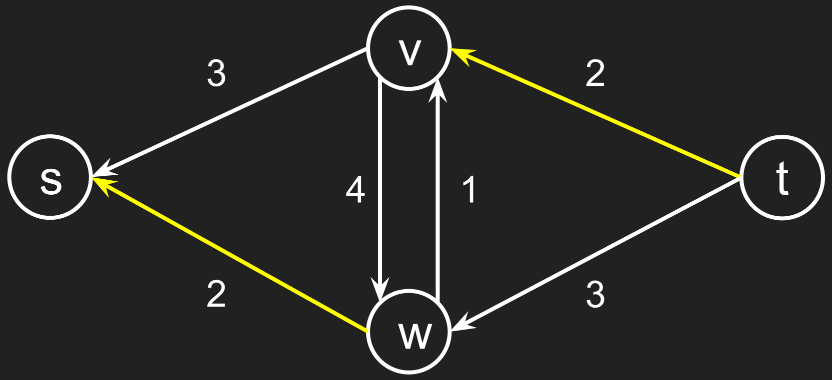

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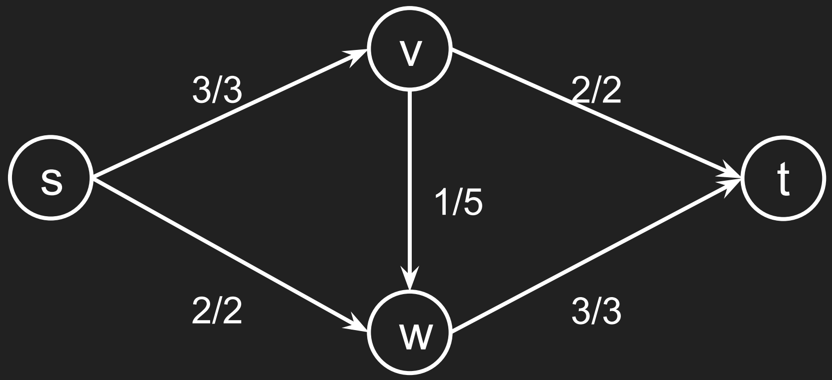

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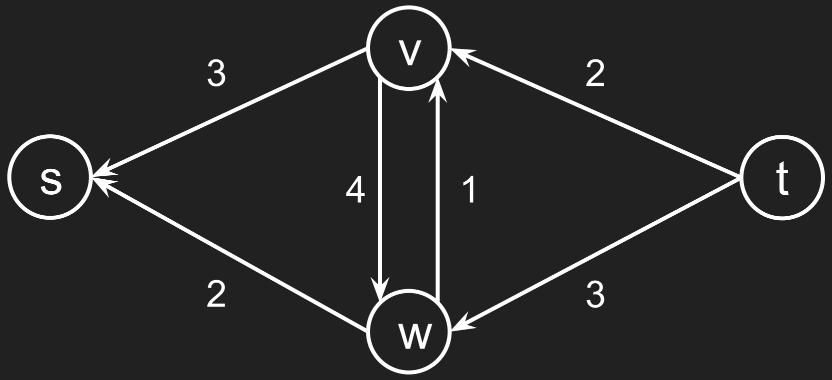


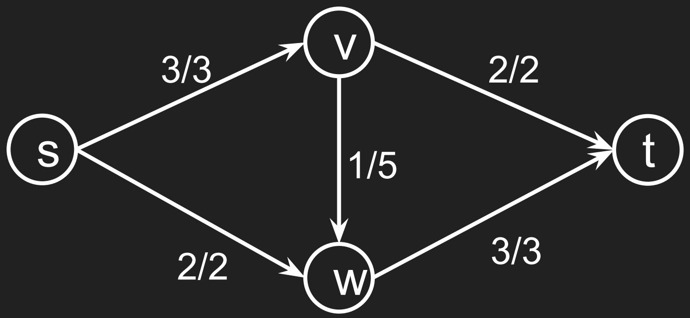
Max Flow: Ford-Fulkerson Method

- Given graph G, define G_R (the residual graph) to be the same graph but with only capacity (no flow) and capacities denoted by $c_R(e)$.
- Maintain that for any edge (u, v) in G that $c_R(u, v) + c_R(v, u) = c(u, v)$
  - Called “skew symmetry”
- While there’s a path P from s to t in G_R:
  - Find the minimum capacity c_R among all edges in P, call it m
  - For each edge (u, v) in P:
    - If (u, v) in G, update $f(u, v)$ += m
    - Otherwise, (v, u) is in G so update $f(v, u)$ -= m
    - Update $c_R(u, v)$ and $c_R(v, u)$ accordingly to match remaining capacity in G and maintain skew symmetry
- Called a “method” because the path finding mechanism is not explicitly defined
  - If you use Breadth-First search, it’s called the Edmonds-Karp algorithm

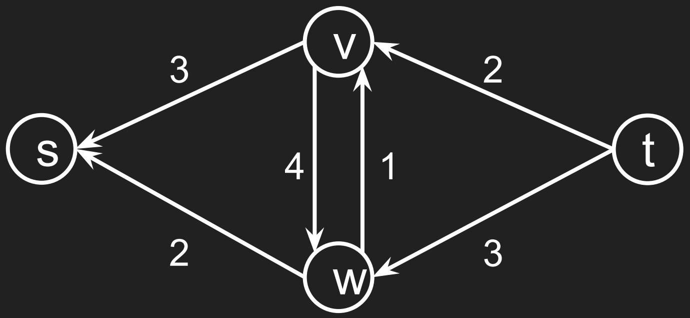
Min Cut

- The "*s*-side" of the partition is simply all nodes still reachable from *s* (including itself) in the residual graph at the end of Ford-Fulkerson. All other nodes are on the "*t*-side" of the partition.
    - Note, we know *s* and *t* will be in different partitions, since if *t* were reachable from *s* then Ford-Fulkerson could still find another path to augment the flow.

Original Graph



Residual Graph



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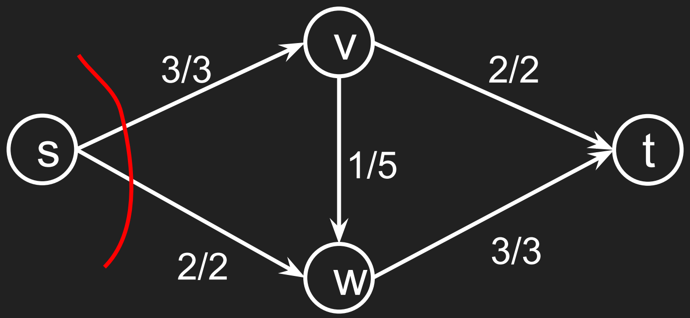
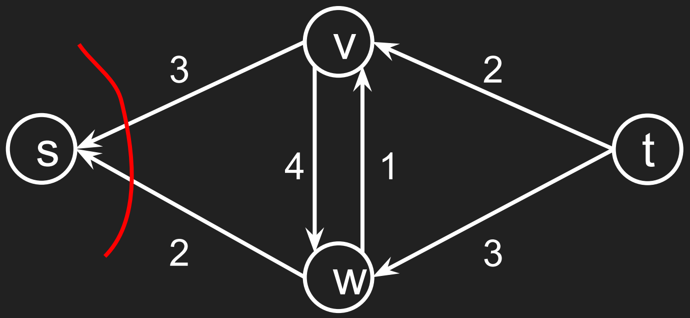
  - Note, we know *s* and *t* will be in different partitions, since if *t* were reachable from *s* then Ford-Fulkerson could still find another path to augment the flow.

Original Graph



Residual Graph

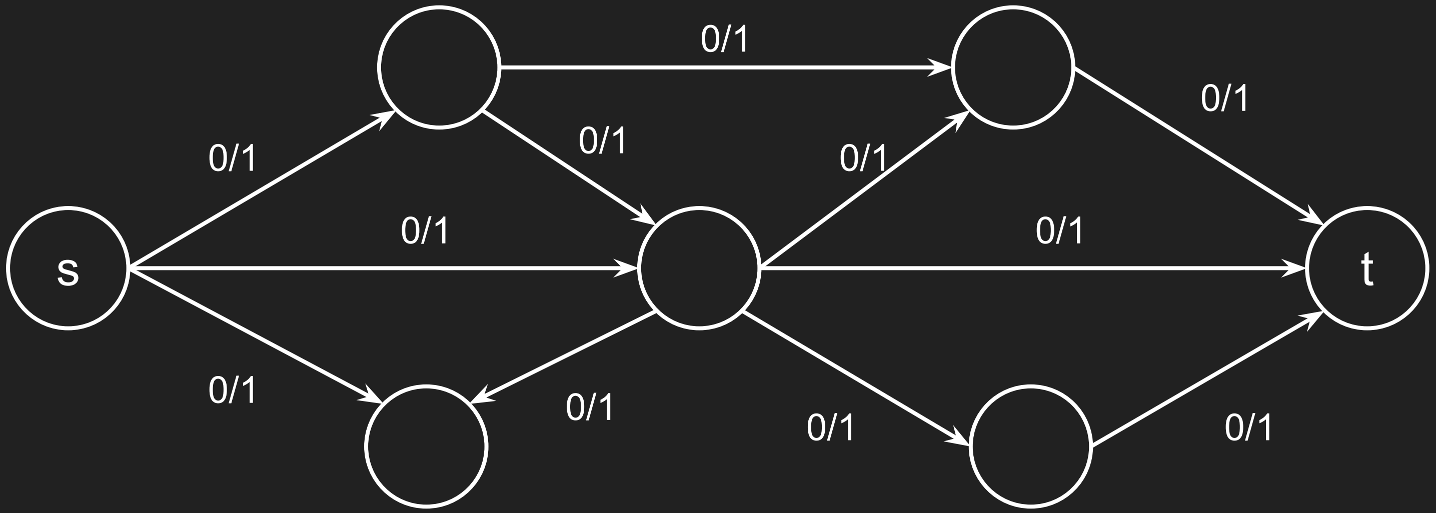


Max Flow: Applications

- Many seemingly unrelated problems map nicely into a network flow equivalent
- Useful fact: If all edge capacities are integers, the max flow will also be an integer (and the flow along any given edge will also be an integer)
  - Known as the Integral Flow Theorem

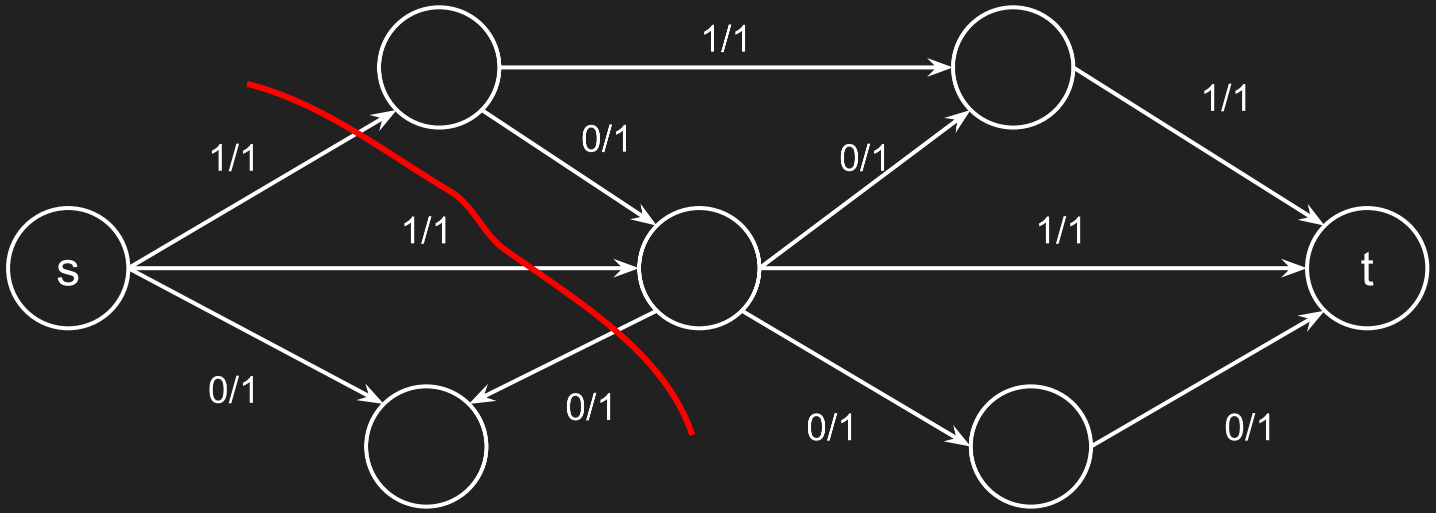
Max Flow: Network Connectivity

- You have two computers that are indirectly connected through a network of other computer systems. How many internal network disconnections is your connection resilient to?



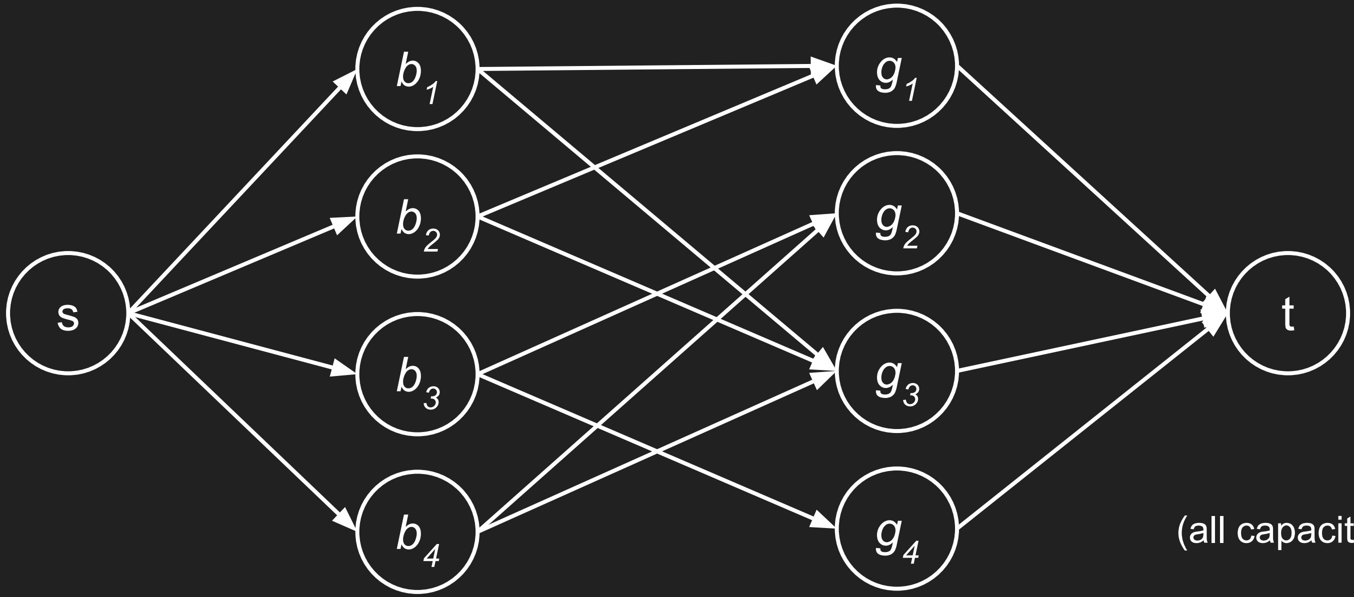
Max Flow: Network Connectivity

- You have two computers that are indirectly connected through a network of other computer systems. How many internal network disconnections is your connection resilient to?



Max Flow: School Dance

- Boys and girls need to be paired up for the school dance, but the kids only want to be paired with someone that they know. Is such a pairing possible? And if so, what's the pairing?



(all capacities = 1)

Min Cut: Project Selection

- You have a set of projects p_i which will each net a revenue of $r(p_i)$. Each project will require purchasing one or more machines q_j each of which costs $c(q_j)$. Machines can be shared by multiple projects. The goal is to maximize profit.

Let P be the set of projects NOT taken, and Q be the set of machines purchased. Then we want:

$$\max \left[\sum_i r(p_i) - \sum_{p_i \in P} r(p_i) - \sum_{q_j \in Q} c(q_j) \right]$$

Which can be reformulated as:

$$\sum_i r(p_i) - \min \left[\sum_{p_i \in P} r(p_i) + \sum_{q_j \in Q} c(q_j) \right]$$

Min Cut: Project Selection

$$\sum_i r(p_i) - \min \left[\sum_{p_i \in \bar{P}} r(p_i) + \sum_{q_j \in Q} c(q_j) \right]$$