Lesson 11

- Dynamic Programming (Chapter 8)
  - Fibonacci numbers
  - Robot Coin Collecting
- Backtracking (Chapter 12)
  - n Queen problem
- Branch and Bound (Chapter 12)
  - Assignment problem

Dynamic Programming

- Dynamic Programming is a general algorithm design technique for solving optimization problems
- Invented by American mathematician Richard Bellman in the1950s
- “Programming” here means “planning”

Fibonacci numbers

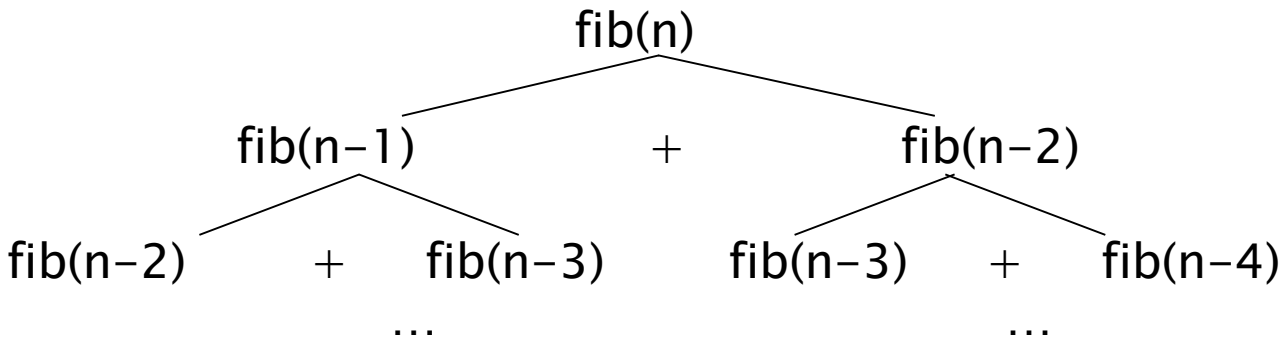
- Fibonacci numbers:

  0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...
  where each number is the sum of the preceding two.

  fib(0) = 0
  fib(1) = 1
  fib(n) = fib(n−1) + fib(n−2)

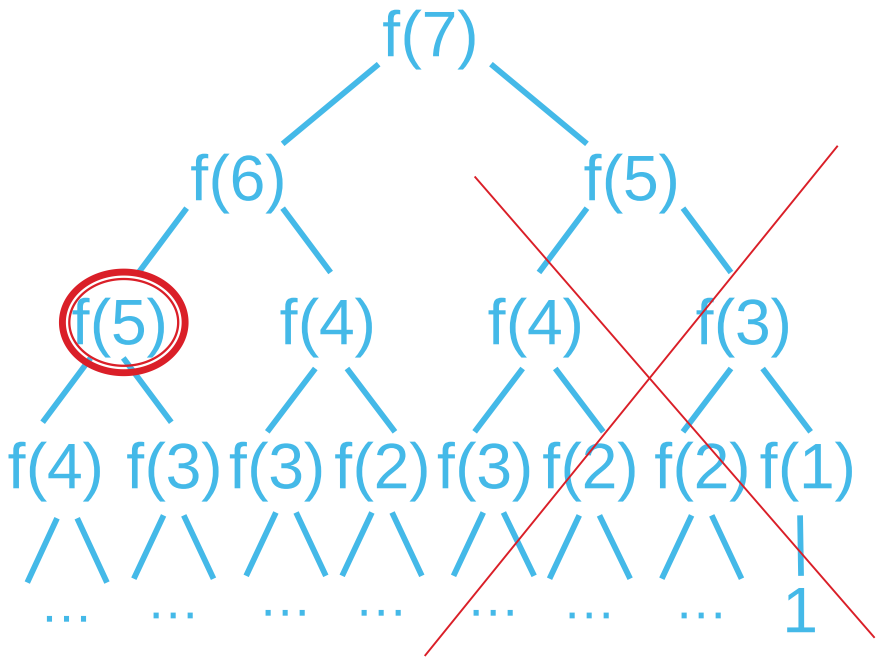
Fibonacci numbers (Divide & Conquer)

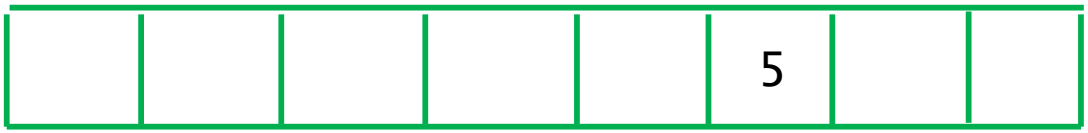
```
fib (n) {
   if n < 2
      f = n;
   else
      f=fib(n-1) + fib(n-2)
   return f
}
```



F(n) takes exponential time to compute.

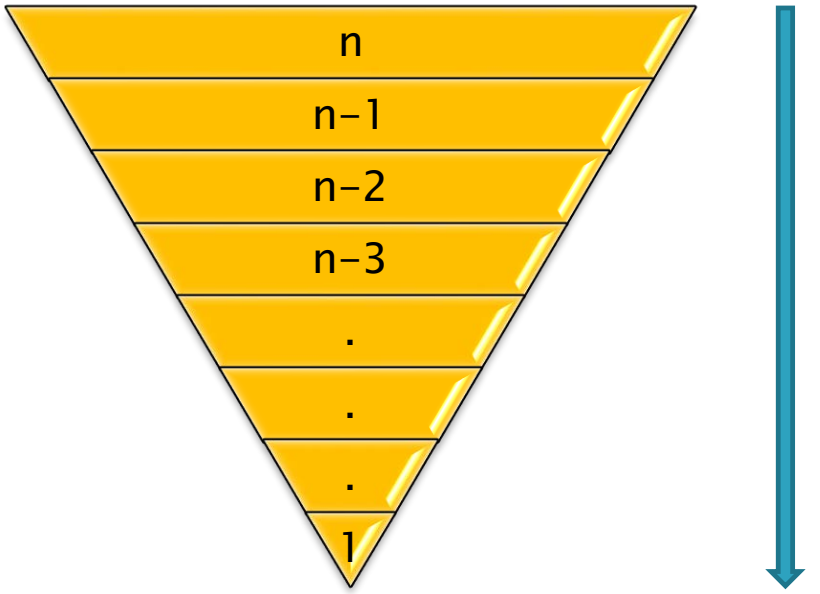
Fibonacci numbers





DP,(top- down)

```
fib (n) {
   If n is in memo, return  memo[n];
   if n < 2
      f = n;
   else
      f=fib(n-1) + fib(n-2)
   memo[n] = f;
   return f
}
```



top-down (Recursive)

memo

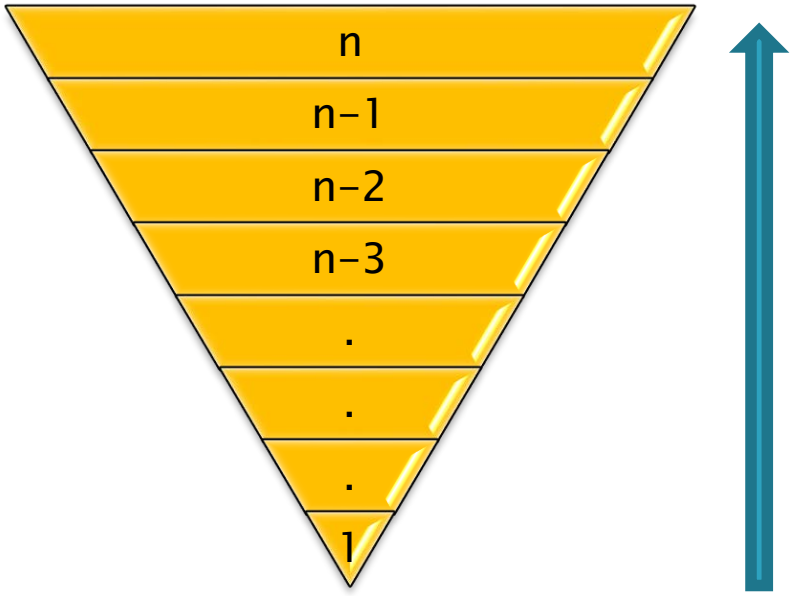
0	1	1	. . .	*fib*(*n*-2)	*fib*(*n*-1)	*fib*(*n*)

Efficiency:

- time: O(n)
- space: Needs an extra array

DP,(Bottom –up)

```
fib (n) {
    memo[0]= 0;
    memo[1]= 1;
    for i← 0 to n do
        memo [i] = memo[i-1]+ memo[i-2]
    return memo[n]
}
```



bottom-up

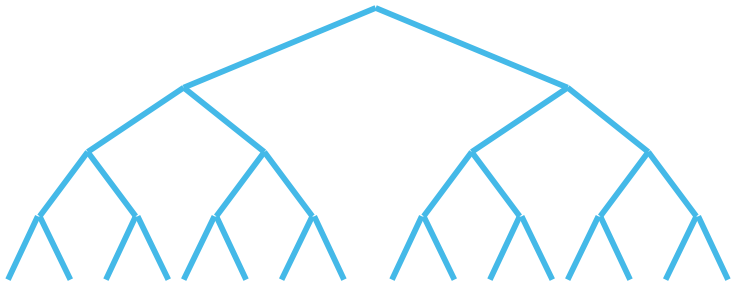
memo	0	1	1	. . .	*fib*(*n*-2)	*fib*(*n*-1)	*fib*(*n*)

Efficiency:

- time: O(n)
- space: Needs an extra array

Dynamic programming

- Exactly the same as divide-and-conquer … but store the solutions to sub-problems for possible reuse.
- A good idea if many of the sub-problems are the same as one another.



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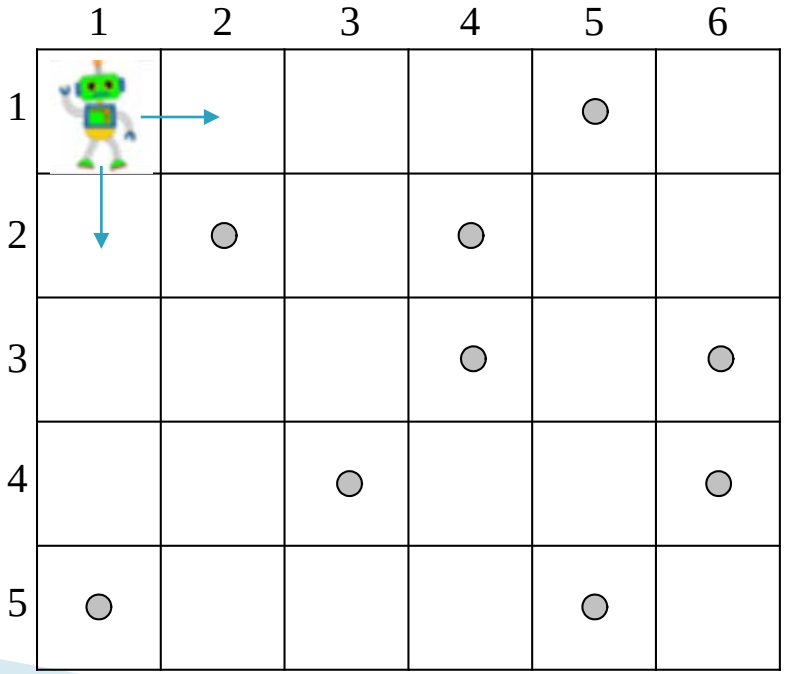
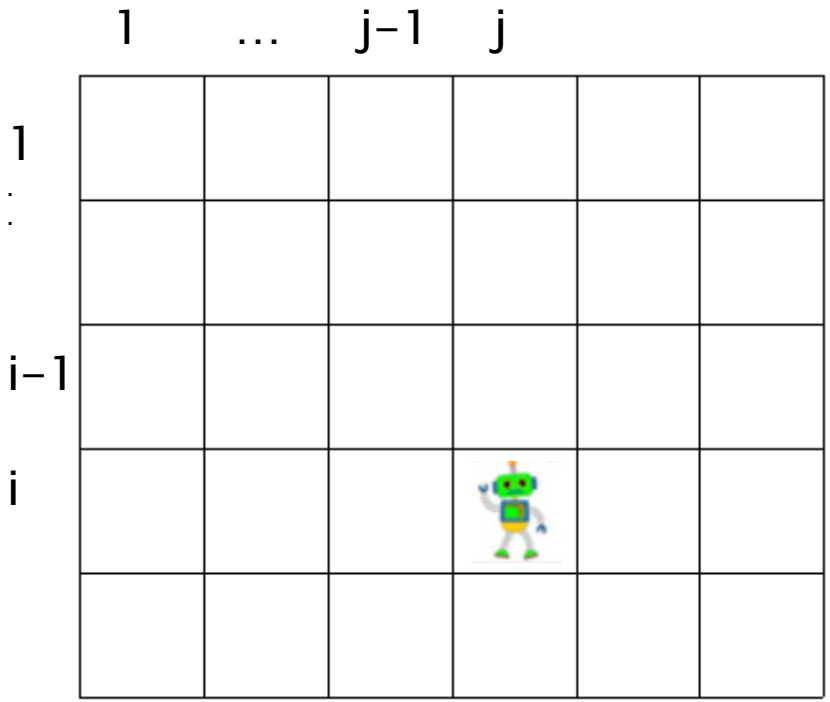
Robot Coin-collecting

Several coins are placed in cells of an $n \times m$ board. A robot, located in the upper left cell of the board, needs to collect as many of the coins as possible and bring them to the bottom right cell. On each step, the robot can move either one cell to the right or one cell down from its current location.



Solution

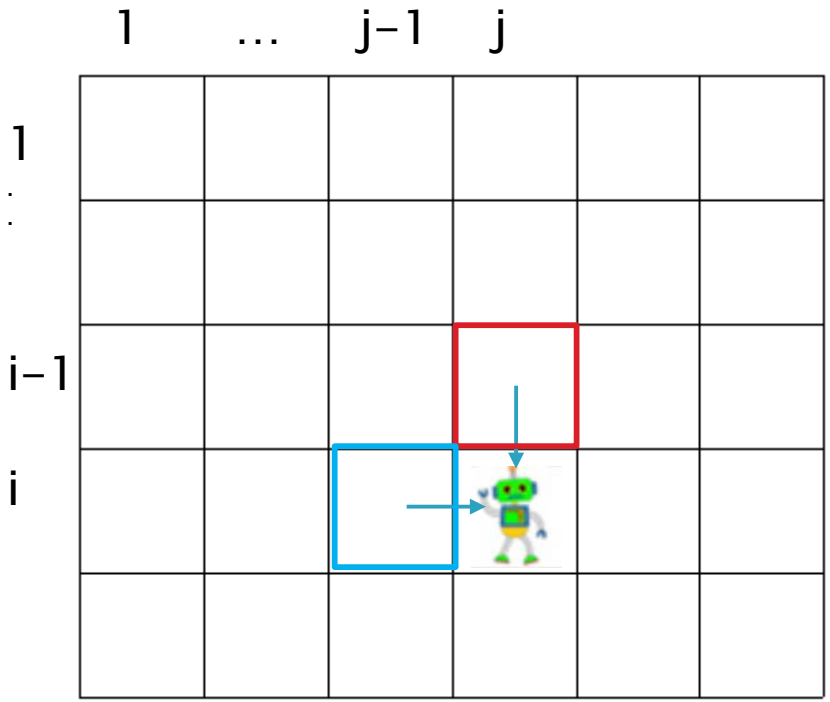
- Let F(i,j) be the largest number of coins the robot can collect and bring to cell (i,j) in the *ith* row and *j*th column.



Solution

The largest number of coins that can be brought to cell (*i*,*j*):

from the left neighbor ? F(i, j−1)
from the neighbor above? F(i−1, j)

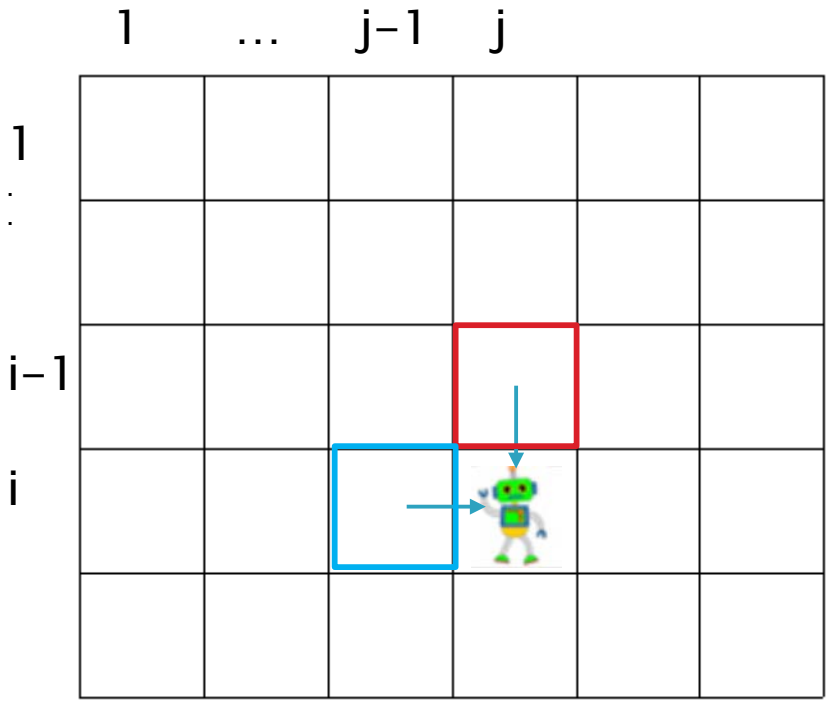


Solution

The recurrence:

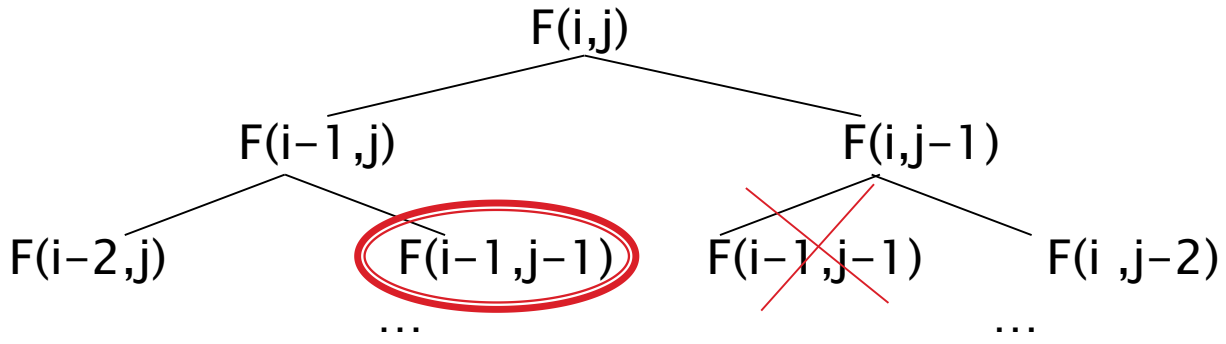
$F(i, j) = \max\{F(i-1, j),\ F(i, j-1)\} + c_{ij}$ for $1 \le i \le n$, $1 \le j \le m$

where $c_{ij} = 1$ if there is a coin in cell (i,j), and $c_{ij} = 0$ otherwise



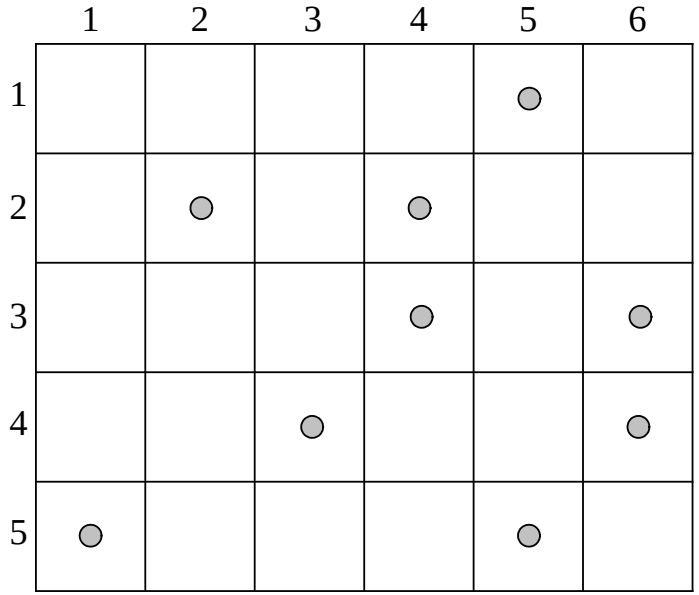
Solution (cont.)

$F(i, j) = \max\{F(i-1, j),\ F(i, j-1)\} + c_{ij}$
$F(0, j) = 0$ for $1 \le j \le m$ and $F(i, 0) = 0$ for $1 \le i \le n$.



Solution (cont.)

$$F(i, j) = \max\{F(i-1, j),\ F(i, j-1)\} + c_{ij} \quad \text{for } 1 \le i \le n,\ 1 \le j \le m$$



C

	1	2	3	4	5	6
1	0	0	0	0	1	1
2	0	1	1	2	2	2
3	0	1	1	3	3	4
4	0	1	2	3	3	5
5	1	1	2	3	4	**5**

F

Robot Coin Collection

```
ALGORITHM RobotCoinCollection(C[1..n, 1..m])
  // Robot coin collection using dynamic programming
  // Input: Matrix C[1..n, 1..m] with elements equal to 1 and 0 for
  //        cells with and without coins, respectively.
  // Output: Returns the maximum collectible number of coins
  F[1, 1] ← C[1, 1]
  for j ← 2 to m do
    F[1, j] ← F[1, j - 1] + C[1, j]
  for i ← 2 to n do
    F[i, 1] ← F[i - 1, 1] + C[i, 1]
    for j ← 2 to m do
      F[i, j] ← max(F[i - 1, j], F[i, j - 1]) + C[i, j]
  return F[n, m]
```

Complexity? $\Theta(nm)$ time, $\Theta(nm)$ space

DP Algos: General Principles

- **Step 1:**
  - Decompose problem into simpler sub-problems
- **Step 2:**
  - Express solution in terms of sub-problems
- **Step 3:**
  - Use table to compute optimal value bottom-up
- **Step 4:**
  - Find optimal solution based on steps 1-3

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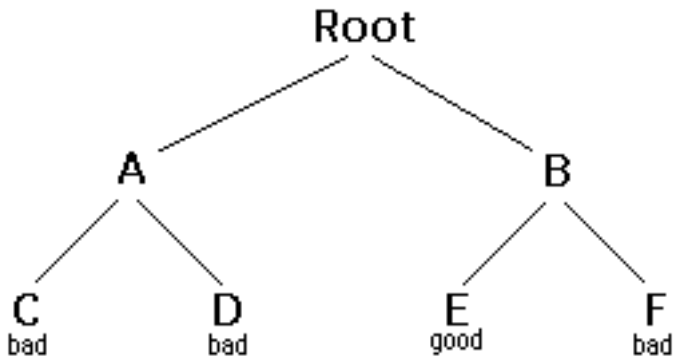
Backtracking

- Suppose you have to make a series of *decisions*, among various *choices*, where
  - You don't have enough information to know what to choose
  - Each decision leads to a new set of choices
  - Some sequence of choices (possibly more than one) may be a solution to your problem
- Backtracking is a methodical way of trying out various sequences of decisions, until you find one that "works"

Backtracking

- Think of the solutions as being organized in a tree
  - The root represents initial state before the search begins
  - Nodes at first level represent first choice
  - Second… second choice..etc

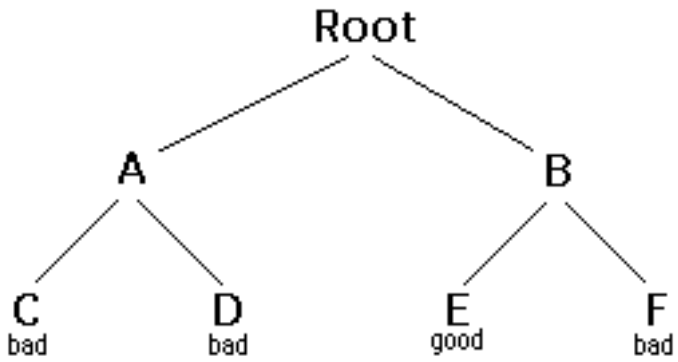


Backtracking in words

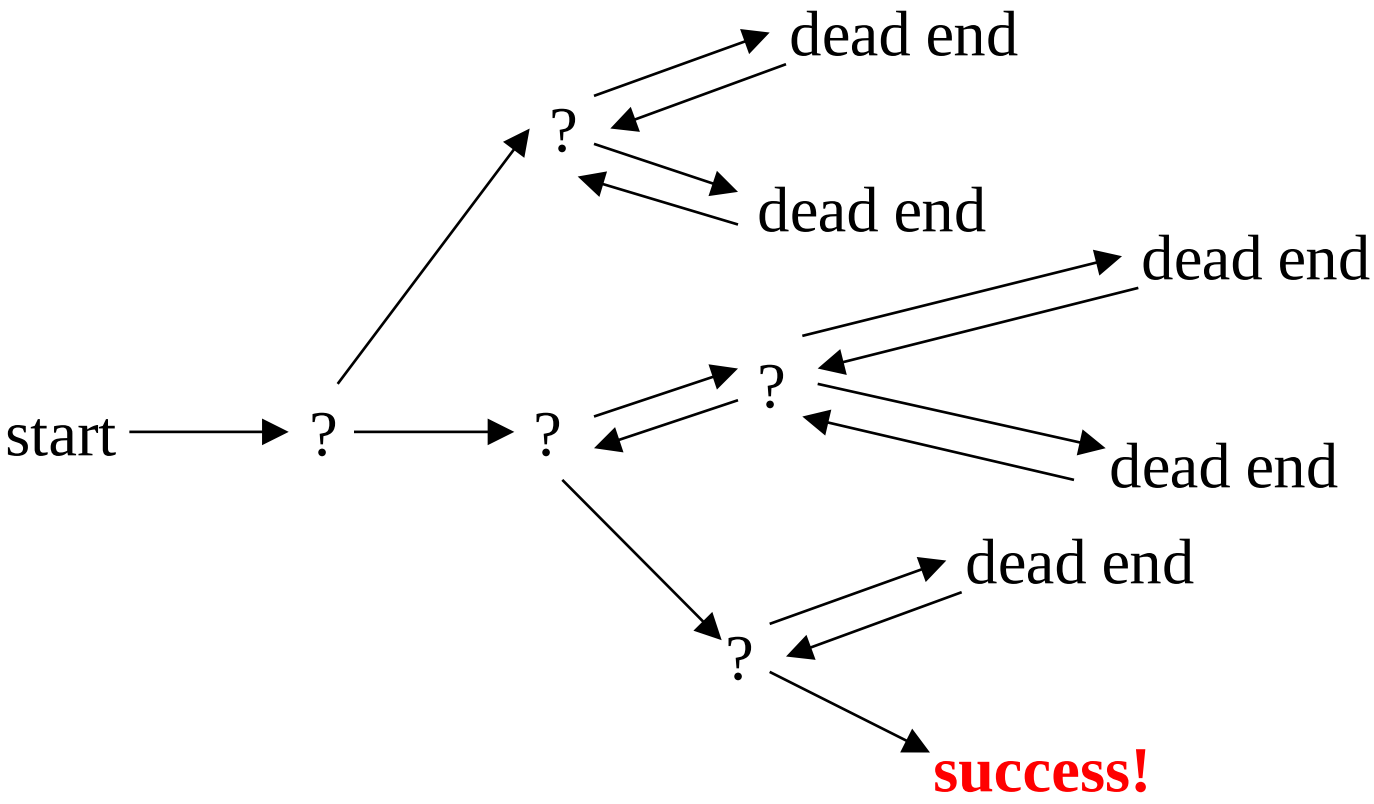
- IDEA:
  - Construct solutions one component at a time
  - If a partial solution can be developed further without violating constraints:
    - Choose first legitimate option for the next component
  - If there is *no option* for the next component
    - Backtrack to replace the last component of partial solution

Backtracking – Abstract Example

- Starting at Root, your options are A and B. You choose A.
- At A, your options are C and D. You choose C.
- C is bad. Go back to A.
- At A, you have already tried C, and it failed. Try D.
- D is bad. Go back to A.
- At A, you have no options left to try. Go back to Root.
- At Root, you have already tried A. Try B.
- At B, your options are E and F. Try E.
- E is good. Congratulations!



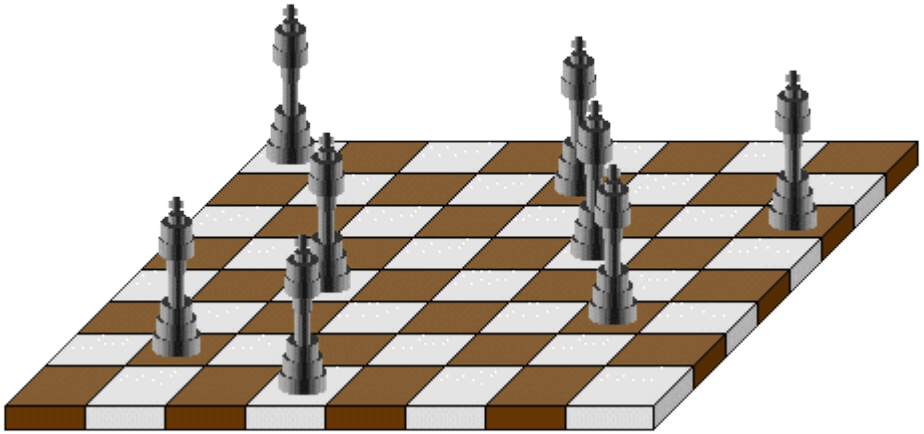
Backtracking (animation)



The tree used to build solutions is called the *state-space tree*
The nodes are *partial solutions*
The edges are *choices*

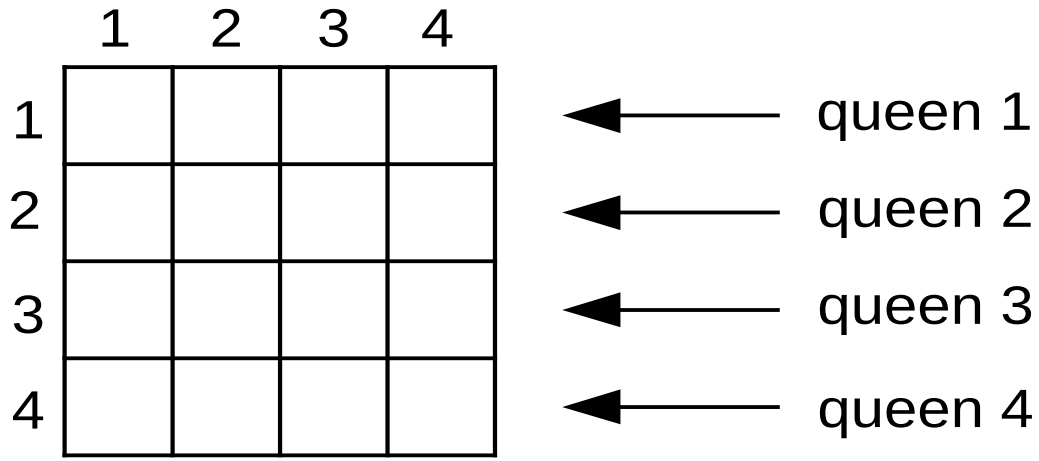
Example: n-Queens Problem

- Place *n* queens on an *n*-by-*n* chess board so that no pair of them are in the same row, column or diagonal



Example: 4-Queens

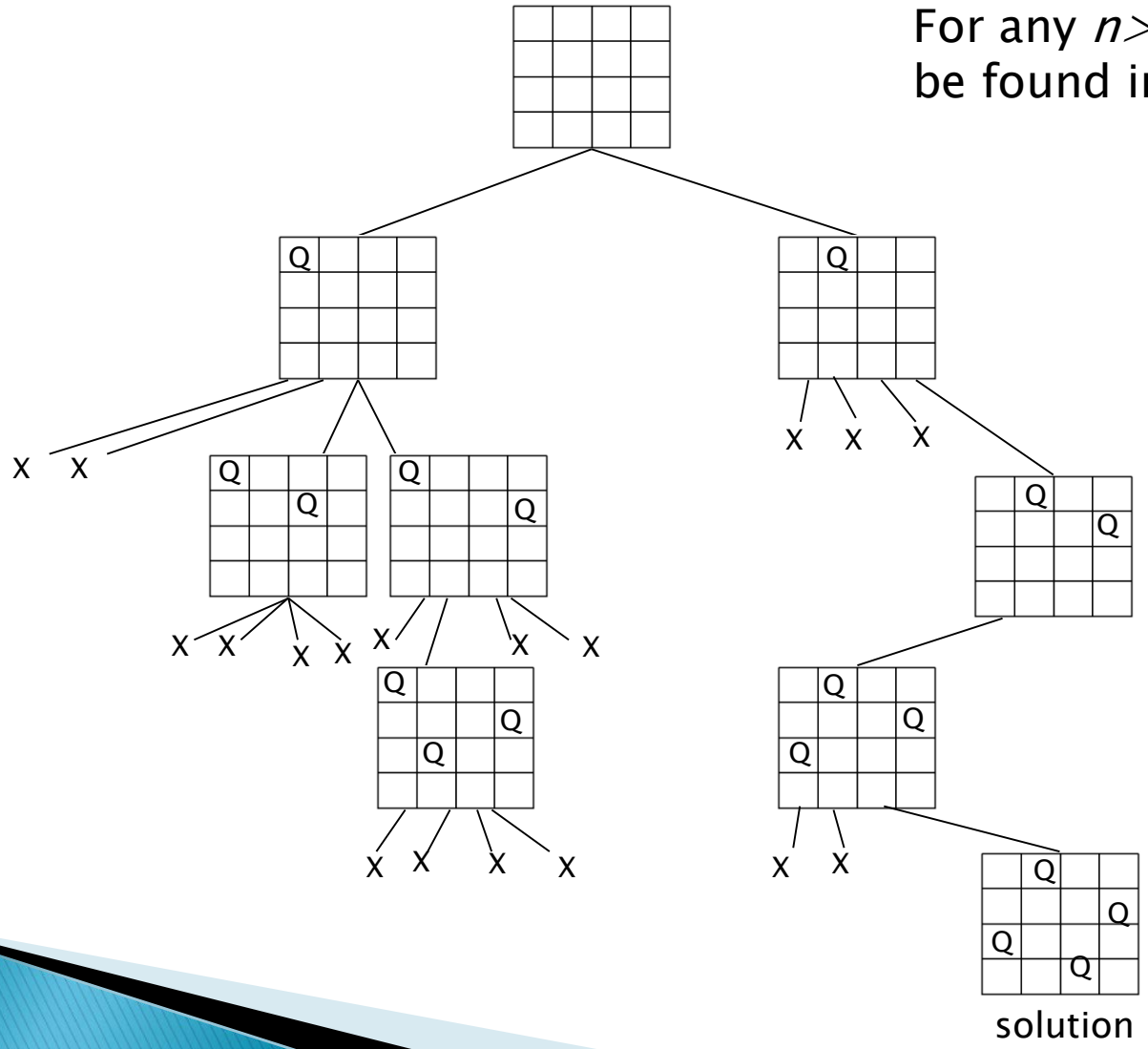
- n=4



- We can solve it by backtracking
  - Root is empty board
  - At level i… put a queen in row i

State-Space Tree of 4-Queens

For any $n>3$, a solution can be found in linear time



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Branch and Bound

- The idea:

  Set up a **bounding function**, which is used to compute a **bound** (for the value of the objective function) **at a node** on a state-space tree and determine **if it is promising**
  - **Promising** (if the bound is better than the value of the best solution so far): expand beyond the node.
  - **Non-promising** (if the bound is no better than the value of the best solution so far): not expand beyond the node (pruning the state-space tree).

Assignment problem

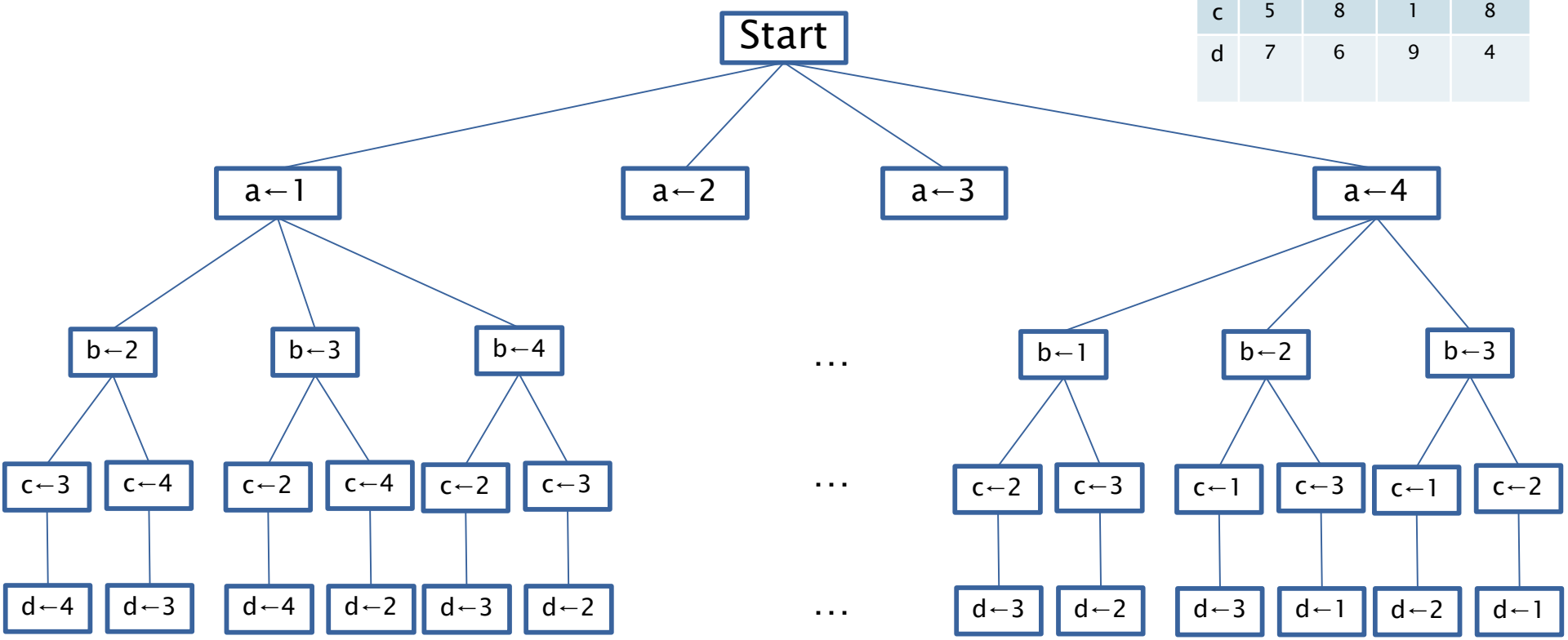
Select one element in each row of the cost matrix C so that:

- no two selected elements are in the same column
- the sum is minimized

	Job 1	Job 2	Job 3	Job 4
Person a	9	2	7	8
Person b	6	4	3	7
Person c	5	8	1	8
Person d	7	6	9	4

Assignment Problem (Brute Force)

	Job 1	Job 2	Job 3	Job 4
a	9	2	7	8
b	6	4	3	7
c	5	8	1	8
d	7	6	9	4



Assignment Problem (Branch & Bound)

	Job 1	Job 2	Job 3	Job 4
a	9	2	7	8
b	6	4	3	7
c	5	8	1	8
d	7	6	9	4

Lower bound: Any solution to this problem will have total cost at least 10

$lb = 2+3+1+4 = 10$

Start

Assignment Problem (Branch & Bound)

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a	9	2	7	8
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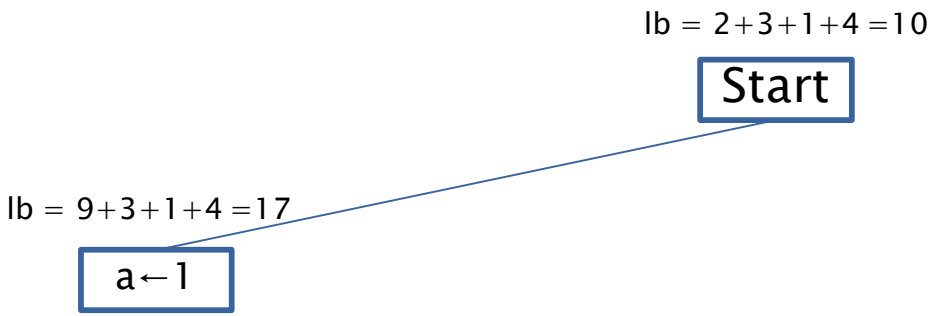

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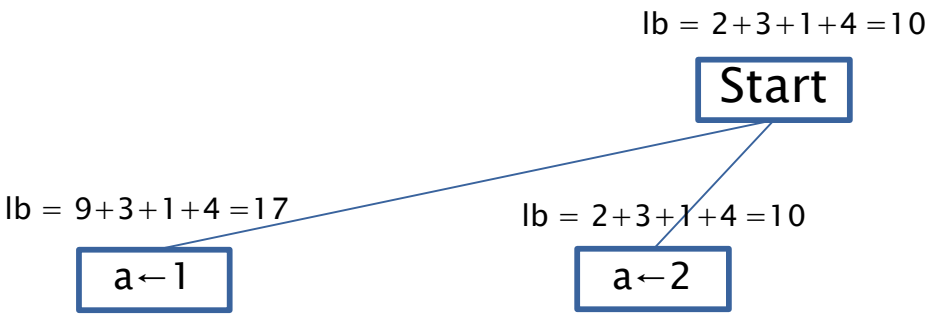

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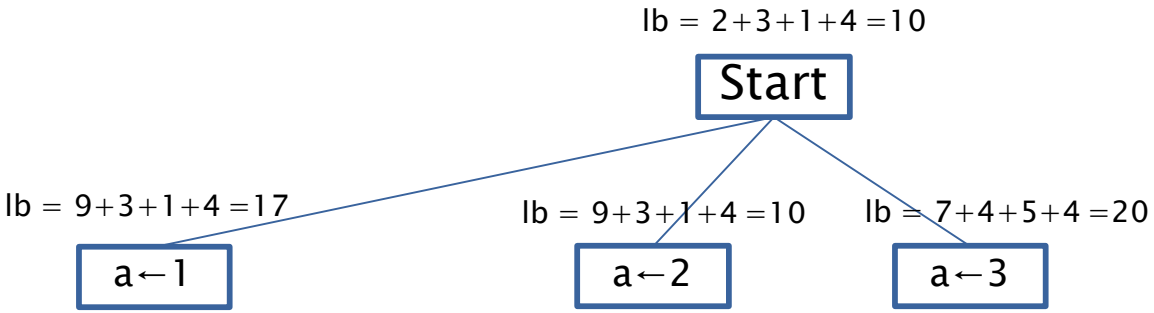

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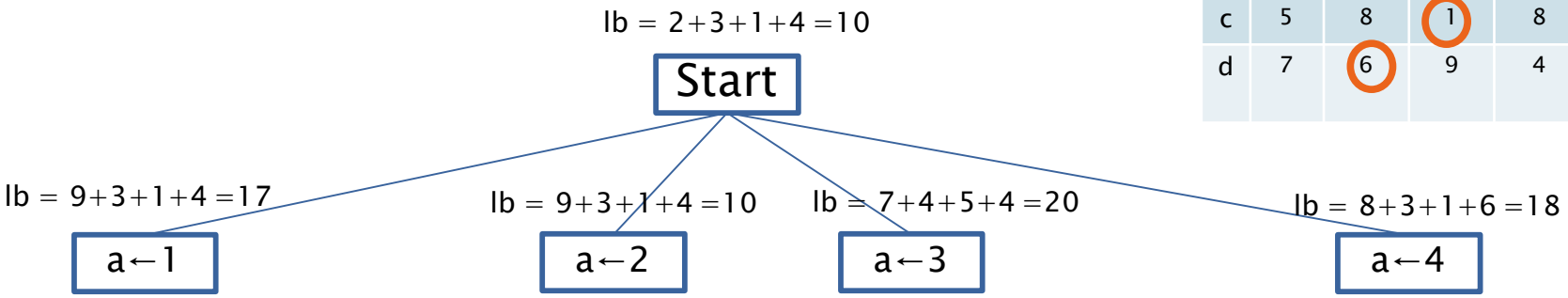

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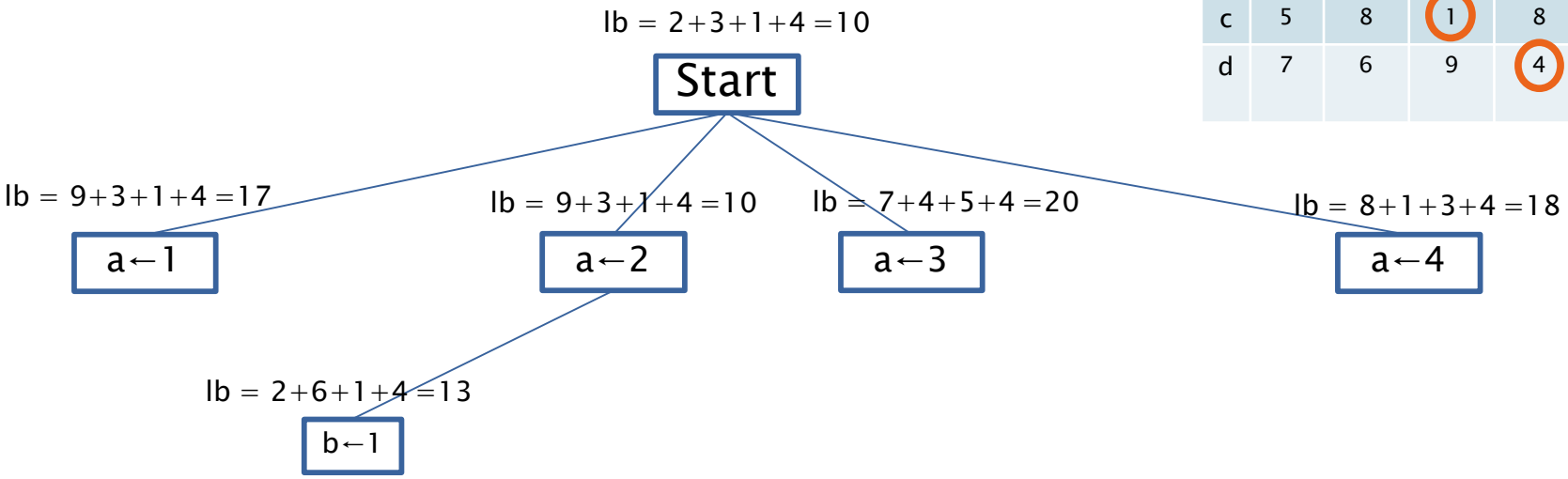

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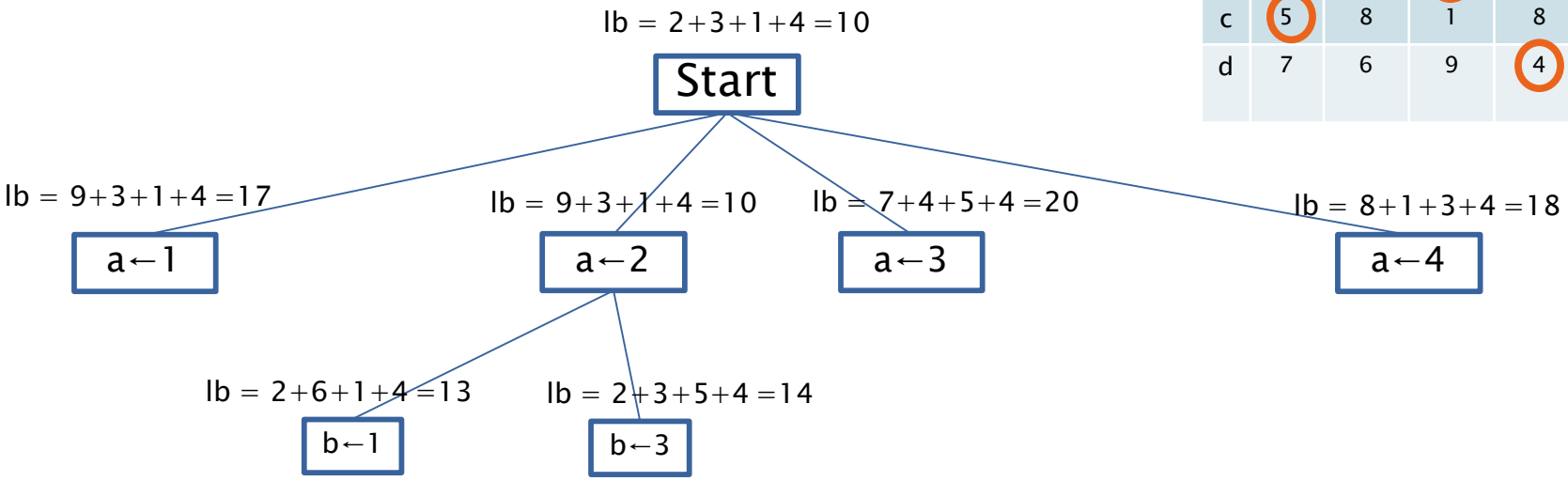

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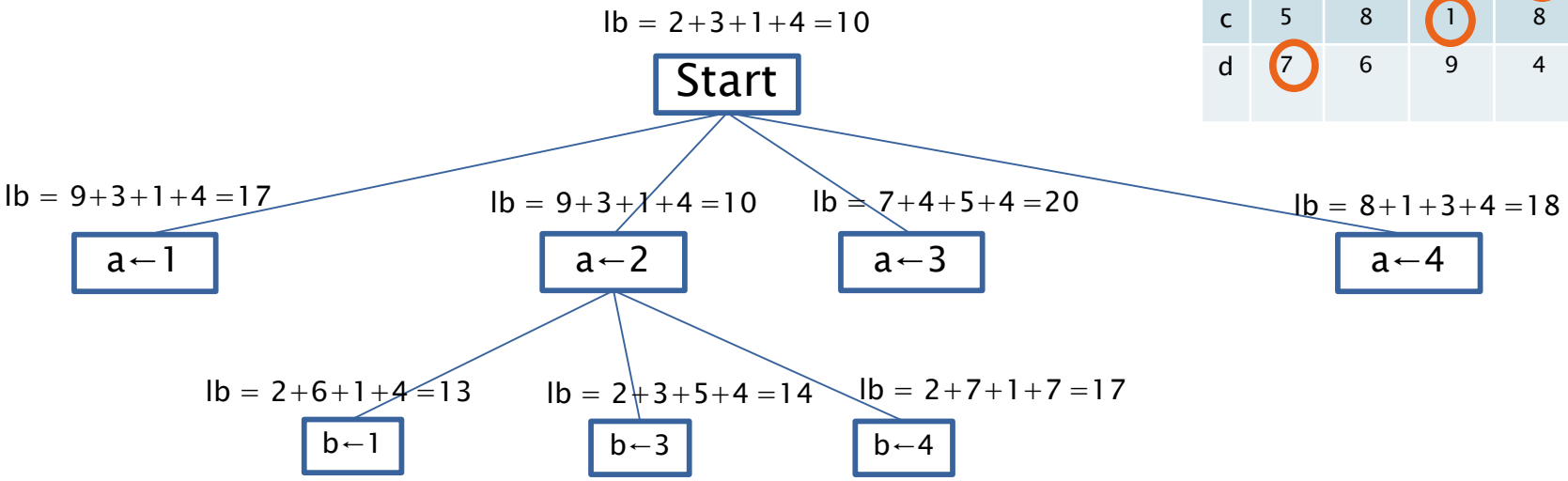

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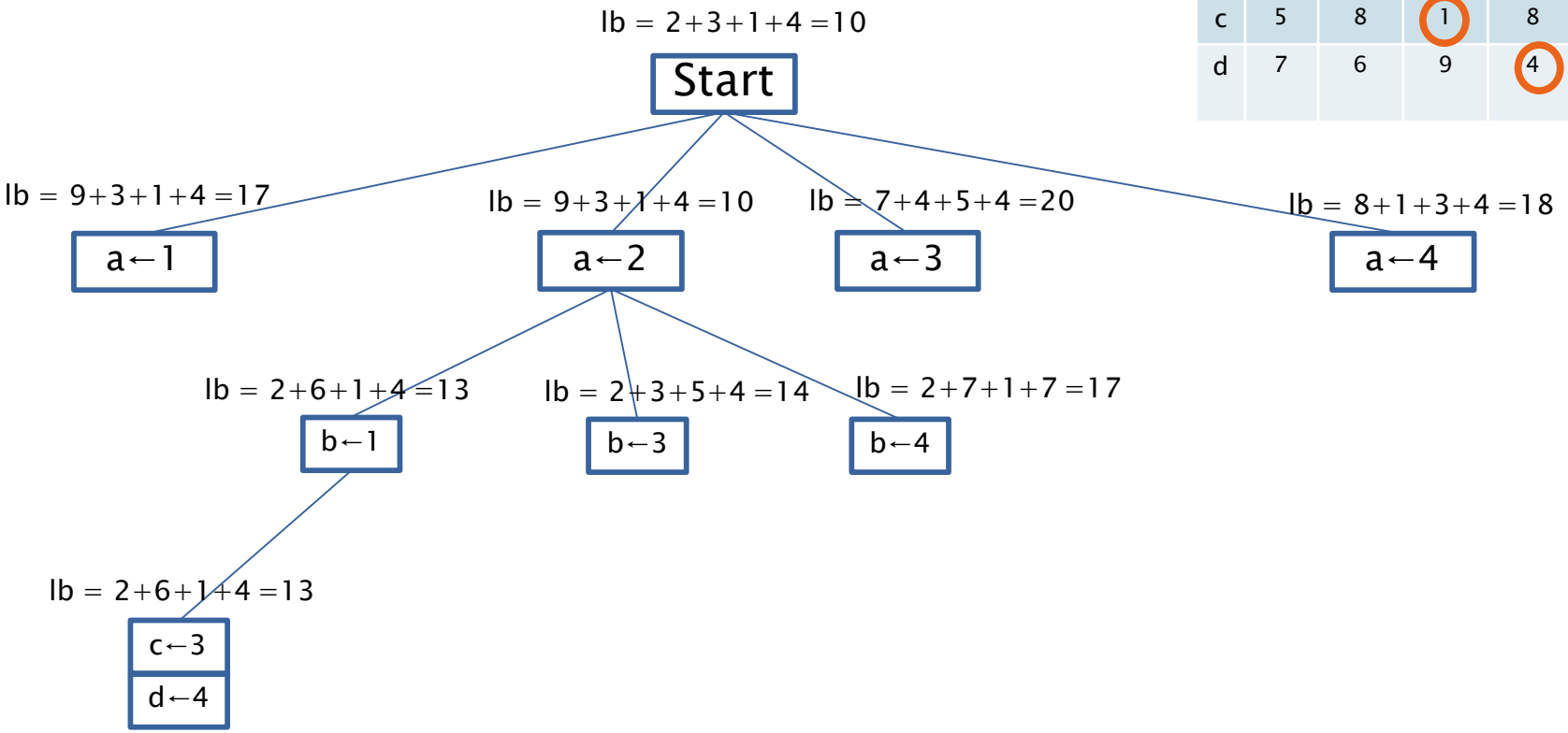

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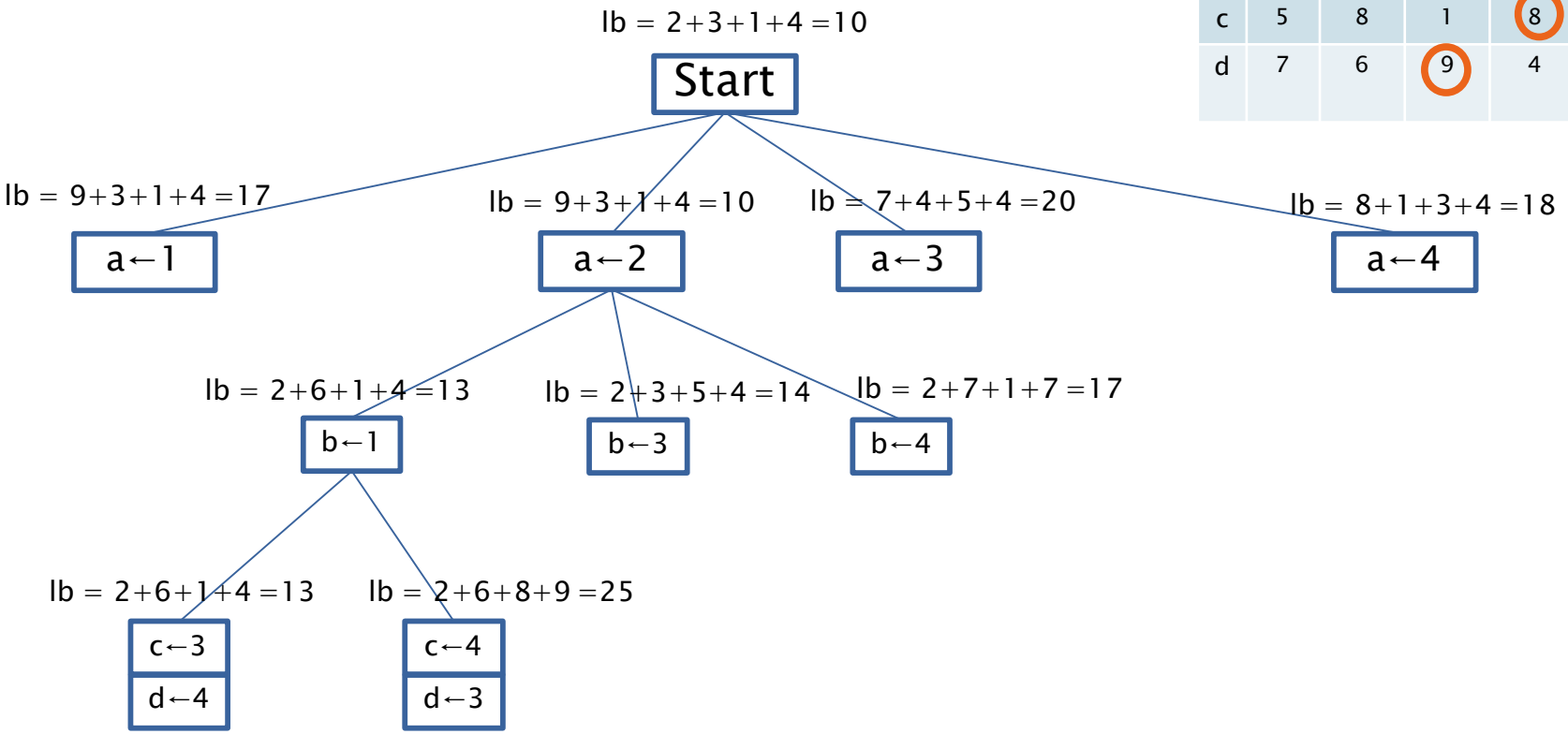

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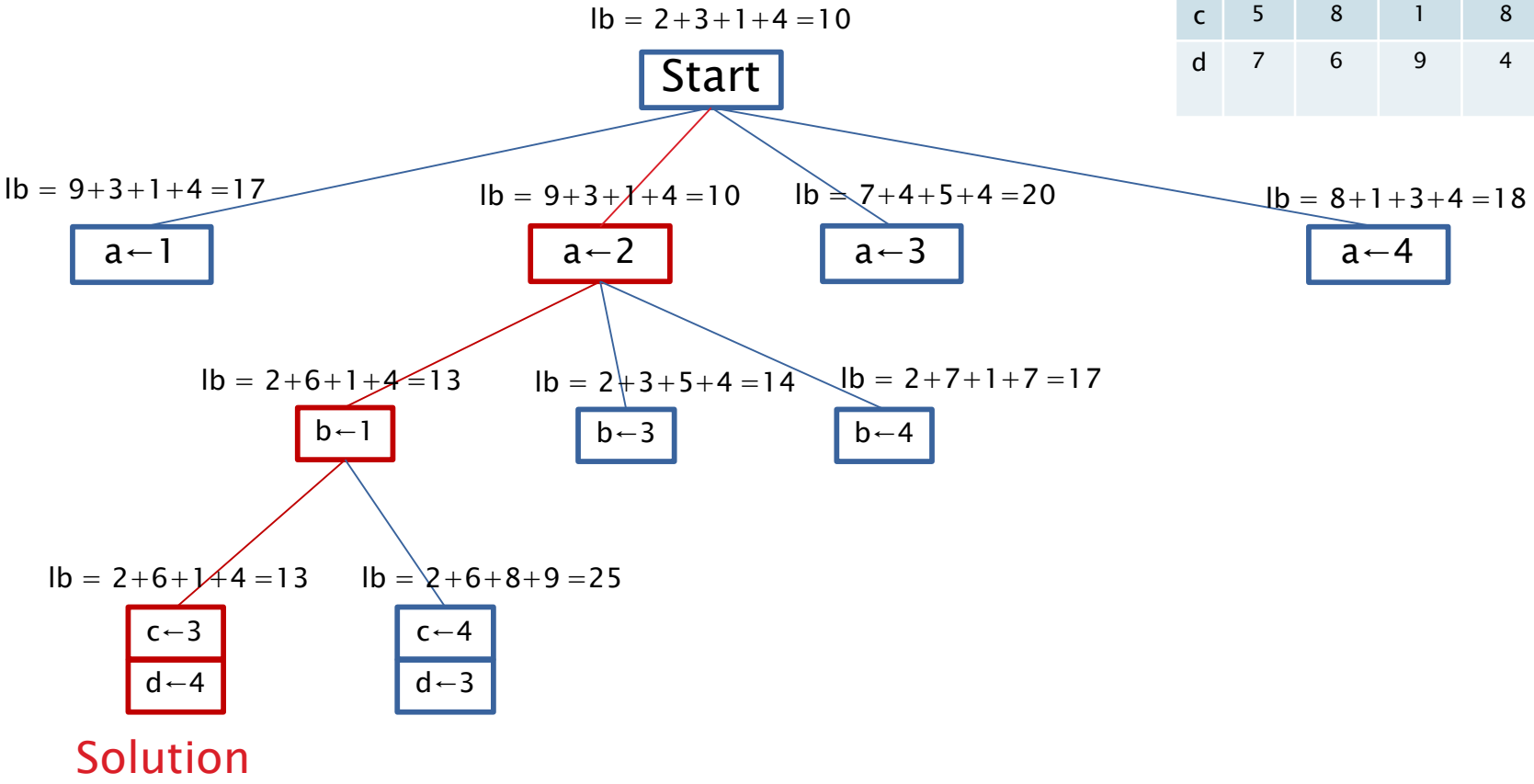
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