Chicago Mercantile Exchange Eurodollar Contract Implied Quotes *

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Abstract

Using ad hoc modifications to standard linear programming techniques, we develop a highly efficient system to compute implied quotes on Chicago Mercantile Exchange Eurodollar quotes using combinations of traded contracts. The optimization techniques are implemented in `MATLAB` and applied to market tick data over the trading day Wednesday, April 2, 2008 and the resulting optimal implied quotes are compared to market quotes for a series of contracts. The feasibility of implementing such a system in a real time trading environment is discussed along with the impact of transaction costs on the system.

1 Introduction

Traded on the Chicago Mercantile Exchange (CME), Eurodollar (ED) futures contracts allow investors to take positions on the future value of three month London Interbank Offered Rate (LIBOR). This link between ED futures prices and LIBOR rates allows investors to use these contracts to hedge interest rate risk. Investors and academics are frequently finding new applications for trading ED futures. In recent studies, Lind Waldock (2008) discuss using ED contracts to hedge against mortgage rates exposure and Martinez and Tse (2008) develop trading algorithms based on ED volatility models. As a result of their many uses, ED contracts are the most liquid interest rate product traded on the CME.

The exchange lists quarterly maturing contracts for the next decade, 40 contracts in total, and contracts for the four front months not covered by the quarterly contracts. Hereafter, these instruments will be referred to as base contracts. In general the short term contracts are far more liquid than the long term contracts indicating that demand for ED futures is primarily in hedging short term interest rate risk. To hedge increasingly more complex interest rate positions combinations of base contracts are frequently required. To mitigate legging risk in entering these complex positions and to increase the liquidity of the market, the CME lists several *derived* contracts constructed from integral combinations of long and short positions in the 44 base contracts. [CME Group (2007a)] Simple derived contracts such as calendar spreads and butterflies are frequently more liquid in the short term than the underlying base contracts from which they are constructed. Larger structures like packs and bundles are more common longer term instruments. The derived contracts traded on the exchange, and their constituent base contracts, are discussed further in Section 2.

While increasing market liquidity and decreasing legging risk, the introduction of derived contracts allows for redundancy in the market that can produce pricing inefficiencies. For example, a calendar spread that is long a short term base contract and short a longer term base contract can cause arbitrage if priced outside of a range determined by the underlying base contract market quotes. In practice these arbitrage opportunities seldom occur due to the large tick size on CME ED contracts and considerable number of market players. However, it is frequently possible to provide better market quotes and almost always possible to improve market liquidity by considering combinations of base and derived contracts when executing a desired market order. To this end, the exchange computes and lists *first generation* implied quotes using combinations of base contracts and a single derived contract. [CME Group (2007b)] The exchange further computes *second generation* implied quotes by combining first generation quotes with base and derived contracts but the prices produced by these combinations are not disseminated and only execute when sufficient volume is not available in the market and first generation quotes to fill the order, even when they offer price improvement.

Even with the first generation implied quotes computed by the exchange there is often price improvement possible over the listed quotes by using more complex contract combinations. The purpose of this study is to develop techniques to optimize implied quotes using combinations of market and first generation implied quotes listed on the exchange. A rigorous mathematical description of the optimization problem and an efficient algorithm to compute optimal implied quotes is presented in Section 3. The techniques developed here are applied to historical CME ED tick data to study the frequency and magnitude of price improvement in Section 4 to determine where these methods are most effective. The feasibility and value of implementing this system in a live trading environment is discussed in Section 5.

2 Chicago Mercantile Exchange Eurodollar Contracts

2.1 Overview

In this section the mechanics of the CME ED contract market are explained. Contract codes are enumerated and their relationship to specific maturities defined. The settlement values of the base contracts are explained in terms of forward starting three month LIBOR rates on specific fixing dates. The market tick size, which changes over the term of a contract, is explained and translated into dollar values. The derived contracts traded on the exchange are enumerated and the translation of their contract codes into an equivalent collection of base contracts explained. Market mechanics and implied prices are explained and traded volume is explored.

2.2 Base Contracts

Eurodollar futures settle two business days before the third Wednesday of their expiry months. Base ED futures are positions on three month LIBOR rate fixed at the future contract settlement date. At settlement time s, the value of the contract, P, is computed from LIBOR rate, L_s, using the simple 30/360 interest discount factor on one million dollars notional:

$$P = \$1\,000\,000 \times (1 - L_s) \times \frac{90}{360} = \$250\,000 \times (1 - L_s)\,. \tag{1}$$

Market quotes, Q, in terms of market expectation of forward LIBOR rate, L, are

$$Q = 100 \times (1 - L) \tag{2}$$

Combining equations (1) and (2), at settlement the contract value is

$$P = \$2\,500 \times Q \tag{3}$$

Outright base contracts are labeled via four character codes. The first two characters denote the quote location – quotes may be listed on either the CME Globex electronic trading platform, denoted `GE`, or via open outcry on the trading floor. As the methods discussed herein are only amenable to electronic platforms, the study will be limited to Globex contracts. The third character denotes the month of contract expiry, the character codes are related to month in Table 1. The fourth character is a digit corresponding to the last digit in the year of contract expiry. Contracts are available up to ten years forward so that there is no ambiguity in this system. For example, on December 8, 2008 the `GE:Z8` contract refers to the December 15, 2008 expiry contract listed on the Globex platform. On December 17, 2008 the `GE:Z8` contract will refer to the December 17, 2018 expiry contract, a contract that is not offered until settlement of December 2008 contracts. As only Globex contracts are considered in this discussion, the `GE` prefix will be dropped herein.

With the exception of the front month base contract, which have tick size 0.0025, all ED contract prices trade with tick size 0.005. Applying equation (3), the standard tick has dollar value \$12.50 for base contracts and value \$6.25 for the front month base contract.

Month	Code
January	F
February	G
March	**H**
April	J
May	K
June	**M**
July	N
August	Q
September	**U**
October	V
November	X
December	**Z**

Tab. 1: CME contract expiry codes by month with quarterly contracts highlighted.

2.3 Derived Contracts

In addition to the 44 base ED contracts that trade on the CME, derived contracts comprised of combinations of base contracts are traded on the exchange. Derived contracts are frequently more liquid than outright contracts as they enable investors to enter more complicated positions without assuming legging risk – the risk that there is a price or liquidity change in some of the constituent contracts of a desired position before execution is complete. As each derived contract can be reconstructed from combinations of integer multiples of base contracts there are pricing implications in the composition of derived contracts related to their respective base contracts and other derived contracts sharing the same base contracts. To examine these effects the derived contracts must first be decomposed into combinations of the base contracts. The 44 base contracts form a spanning set of all possible derived contracts; each derived contract can be expressed as a linear combination of elements in the base contrast basis. Each contract is specified by a 44 element vector $\mathbf{d}^{\text{code}}$ with elements corresponding to number of each base contract held. The elements are listed in chronological order of expiry, so that the augmented matrix constructed by combining the base contracts in order of expiry is the 44×44 identity. On December 8, 2008,

$$\left(\mathbf{d}^{\mathtt{Z8}} \,\middle|\, \mathbf{d}^{\mathtt{F9}} \,\middle|\, \mathbf{d}^{\mathtt{G9}} \quad \ldots \quad \mathbf{d}^{\mathtt{H8}} \,\middle|\, \mathbf{d}^{\mathtt{M8}} \,\middle|\, \mathbf{d}^{\mathtt{U8}} \right) = \mathbf{I} \tag{4}$$

Ten types of derived ED contracts currently trade on the CME: calendar spreads, butterflies, condors, packs, month packs, pack butterflies, double butterflies, pack spreads, bundles, and bundle spreads. [CME Group (2007a)] These derived contracts are explained in the next sections and examples for each contract are plotted in Figures 2 and 3. For each of the contract examples it is assumed that the current date is December 8, 2008. A time series of the daily traded volume from April 2007 to November 2008 for the most liquid derived contracts is plotted in Figure 1 to illustrate the relative volume of the contracts.

2.3.1 Calendar Spread

Composed of a long position in a short-dated contract and a short position in a long-dated contract, calendar spreads are the simplest derived contract traded on the exchange. As an example, the spread listed as `H9-M9` consists of a long position with a March 2009 expiry and a short position with June 2009 expiry. The trading of spreads with different durations allow investors to take positions on Eurodollar rates at different maturities. While seemingly very simple, the spread is a structure frequently repeated in collections of other derived contracts. Calendar spreads are the most liquid of the derived contracts, comprising about a third of the daily traded volume.

2.3.2 Butterfly

Butterflies are constructed from opposing positions in two overlapping spreads, resulting in a $1 : -2 : 1$ ratio of base contracts. They are effectively an overlapping spread of calendar spreads. A butterfly listed as `BF:U9-Z9-H0` can be deconstructed into a long position in a `U9-Z9` spread and a short position in a `Z9-H0` spread with the net effect of being long the short-dated contract `U9`, short two of the middle dated contract `Z9` and long the long-dated contract `H0`. It should be noted that a butterfly can span a number of durations, but the time between short and middle contracts and middle and long contracts is the same for any listed contract. Butterflies comprise about a tenth of the daily traded volume and are becoming increasingly more utilized in the market.

2.3.3 Condor

Condors are also constructed from opposing positions in two spreads, but in this case they are non-overlapping. Condors are comprised of four consecutive quarterly contracts in a ratio 1:-1:-1:1. Similar to butterflies, the holder of a condor is long the near spread and short the long spread. As an example, `CF:H9-M9-U9-Z9` would hold long positions in the short and long-dated contracts, March 2009 and December 2009, and would be short the two middle contracts, June 2009 and September 2009.

2.3.4 Pack

The pack is constructed from long positions in four equally spaced concurrent base contracts. The most common packs consist of four quarterly contracts. For example `PK:01Y U9` is long a contract expiring four consecutive quarters beginning in November 2009. In this case `01Y` refers to the duration of the pack contract and the month code `U9` indicates expiry of the first

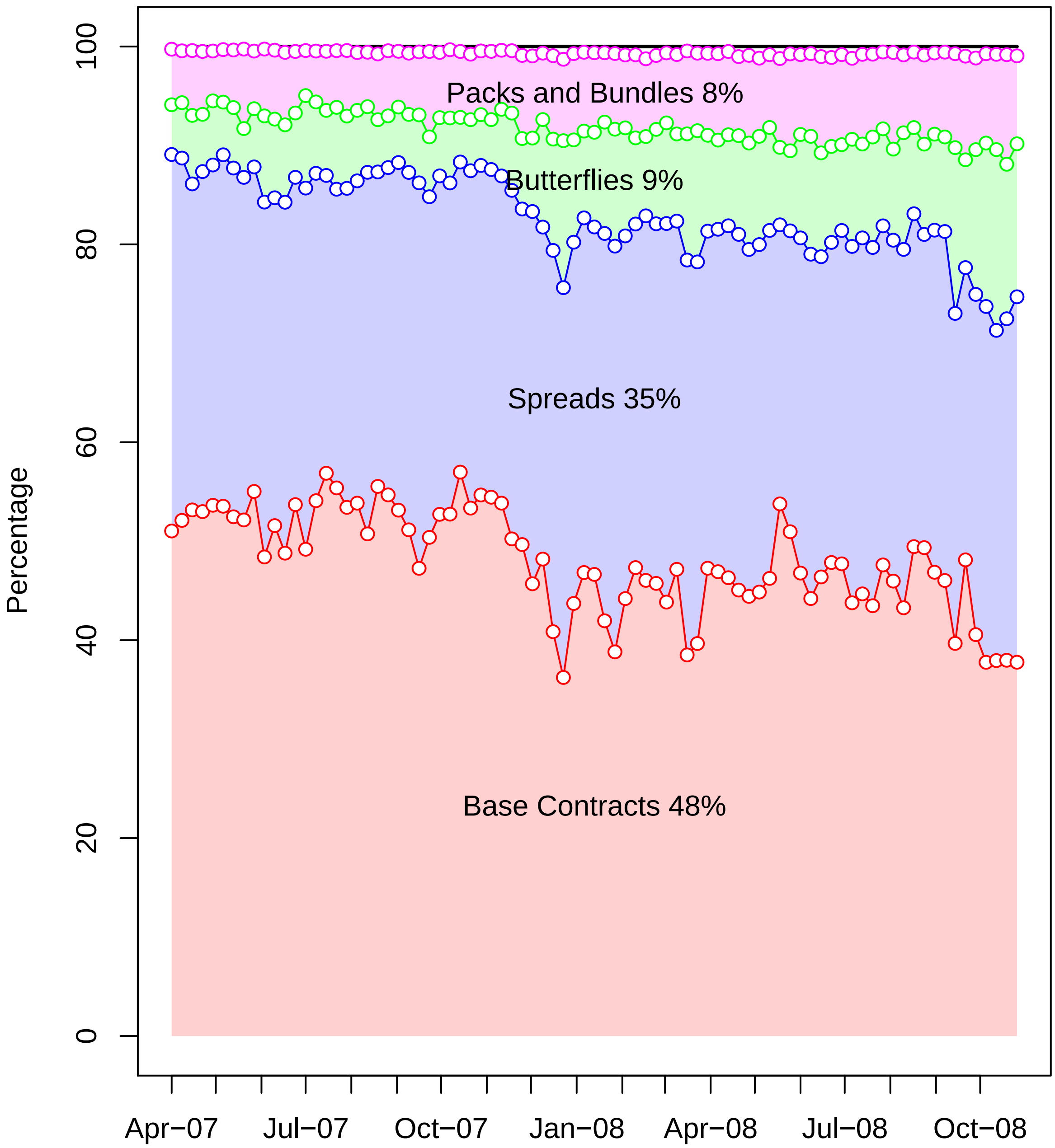


Fig. 1: Time series of relative volume of contracts traded by contract type from April 2007 through November 2008. Average relative volume over the period is specified for the four contract types.

contract. Packs and bundles allow investors to hedge short-term risk over longer positions and are more common at later maturities.

2.3.5 Month Pack

Month packs are constructed from a position long four outright contracts in a short-dated expiry combined with a short position in a long-dated pack. As an example, `MP:Z9 1YH0` would be long four contracts expiring in December 2009 and short a contract expiring in each consecutive quarter beginning in March 2010. `Z9` refers to the four contract long position, `1Y` again refers to the duration of the pack, and `H0` refers to the expiry of the first leg of the pack.

2.3.6 Pack Butterfly

The pack butterfly is a butterfly spread of pack legs. Holding a pack butterfly corresponds to being long the short-dated pack, short two of the next closest maturity and long the furthest maturity pack. As with the butterfly described above it results in a ratio of $1 : -2 : 1$. As an example, buying `PB:Z9-Z0-Z1` corresponds to buying a December 2009 pack, selling two December 2010 packs and buying a December 2001 pack. Each of these packs would have quarterly contracts for the year beginning with the listed date code.

2.3.7 Double Butterfly

A double butterfly can be thought of a spread of partially overlapping butterflies. Holding this derived contract is the same as being long a short-dated butterfly and short a long-dated butterfly, such that the last two legs of the short-dated contract coincide with the first two legs of the long-dated contract. The net effect of holding the double butterfly `DF:Z9-H0-M0-U0` is long one December 2009, short three contracts expiring in March 2010, long three June 2010 and short one contract in September 2010.

2.3.8 Pack Spread

The pack spread is a calendar spread with each leg being a pack of different maturities. Buying a pack spread is equivalent to buying a pack of closer maturity and selling a pack of later maturity. As an example, `PS:Z9-M0` would buy a pack spread with maturity December 2009 and sell a pack spread with maturity June 2010. When the investor uses spreads of such close maturity it has a net effect of canceling out many of the positions.

2.3.9 Bundle

The bundle consists of between eight and 40 contracts with consecutive quarterly maturity months. A two-year bundle consists of contracts expiring in eight consecutive quarters, five-year bundles have 20 consecutive quarters and ten-year have 40. The simplest example is `FB:02Y M9`, where `02Y` indicates the duration of the bundle and `M9` is the month code for the

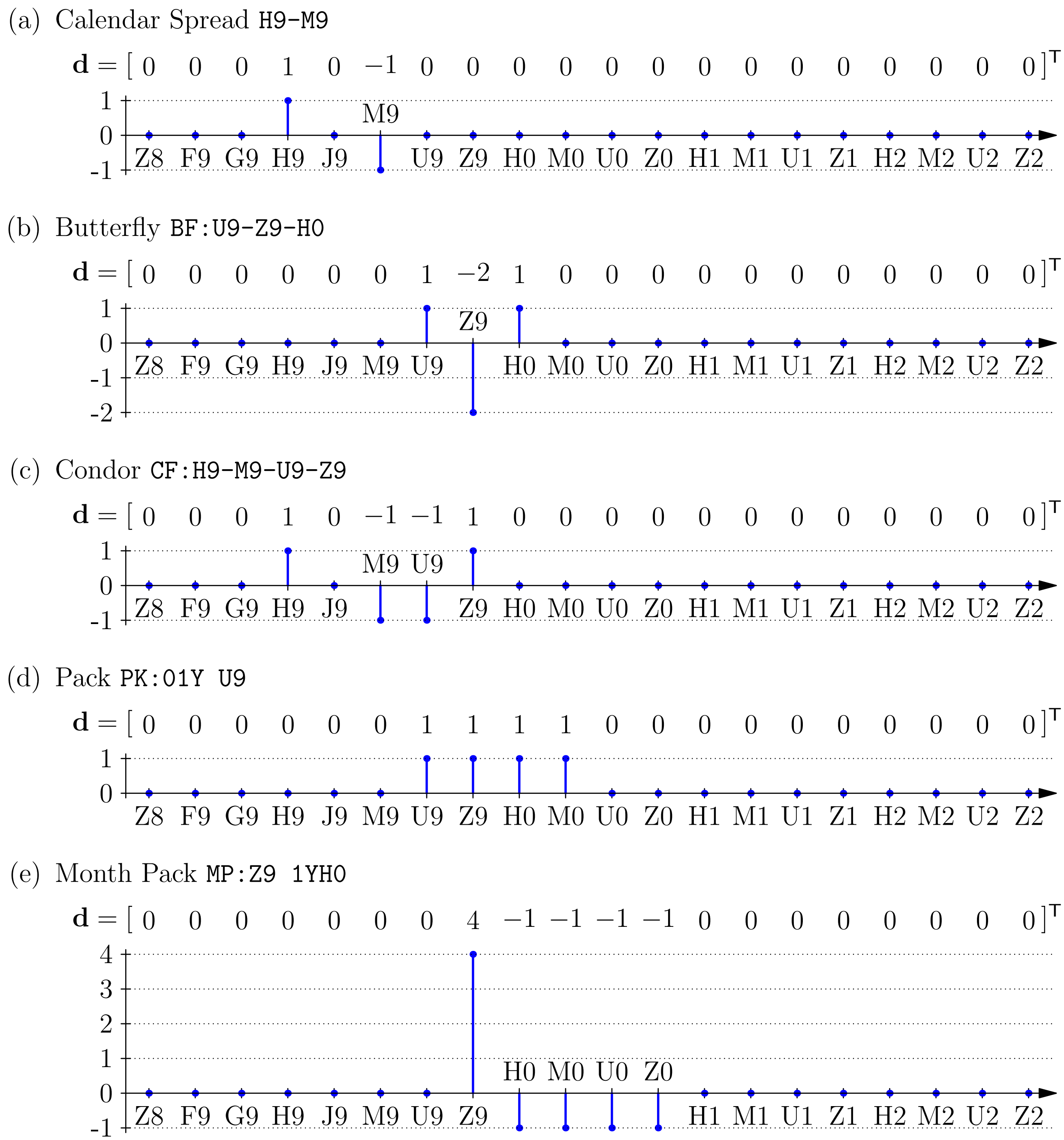
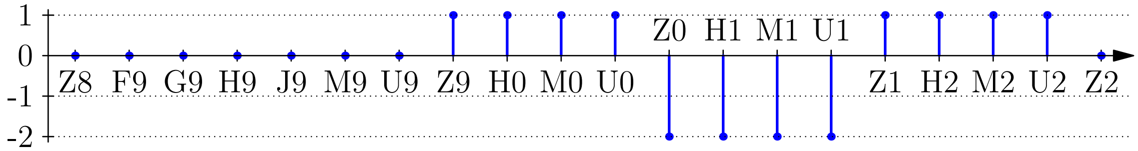


Fig. 2: Examples of decomposition of derived contracts into the corresponding base contracts for: (a) calendar spread, (b) butterfly, (c) condor, (d) pack, and (e) month pack.

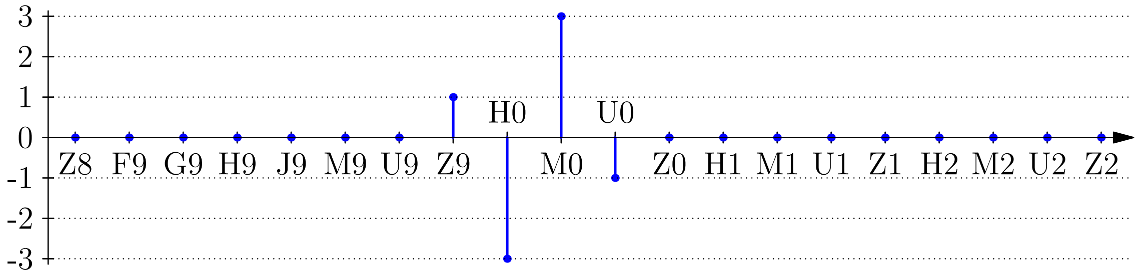
(f) Pack Butterfly `PB:Z9-Z0-Z1`

$$\mathbf{d} = [\,0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ {-2}\ \ {-2}\ \ {-2}\ \ {-2}\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\,]^{\mathsf{T}}$$



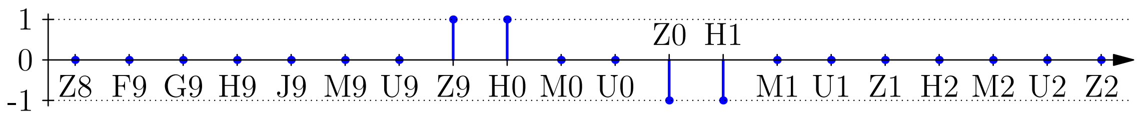
(g) Double Butterfly `DF:Z9-H0-M0-U0`

$$\mathbf{d} = [\,0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ {-3}\ \ 3\ \ {-1}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\,]^{\mathsf{T}}$$



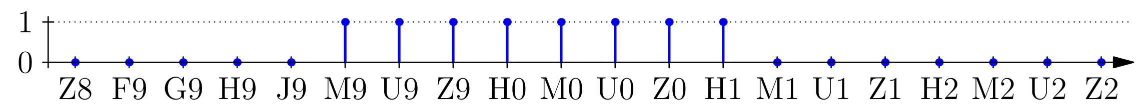
(h) Pack Spread `PS:Z9-M0`

$$\mathbf{d} = [\,0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0\ \ {-1}\ \ {-1}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\,]^{\mathsf{T}}$$



(i) Bundle `FB:02Y M9`

$$\mathbf{d} = [\,0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\,]^{\mathsf{T}}$$



(j) Bundle Spread `BS:2YU9 2YU0`

$$\mathbf{d} = [\,0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ {-1}\ \ {-1}\ \ {-1}\ \ {-1}\ \ 0\ \ 0\,]^{\mathsf{T}}$$

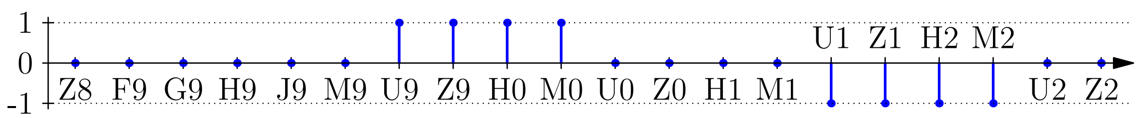


Fig. 3: Examples of decomposition of derived contracts into the corresponding base contracts for: (f) pack butterfly, (g) double butterfly, (h) pack spread, (i) bundle, and (j) bundle spread.

first contract of the bundle. This particular example consists of the eight consecutive quarterly contracts between June 2009 and March 2011. Although they comprise only about a tenth of the traded volume, due to the large number of base contracts in each pack and bundle, these contracts represent a significant amount of the total ED positions.

2.3.10 Bundle Spread

The bundle spread is a calendar spread with each leg corresponding to a bundle. Buying a bundle spread corresponds to buying the shorter maturity bundle and selling the longer-dated bundle. Bundle spreads must have an equal number of legs in each bundle, so only equal duration bundles can be paired together to form a bundle spread. For example, `BS:2YU9 2YU0` is a bundle spread created by buying a two year bundle for September 2009 and selling a two year bundle for September 2010. In this example, four of the long and short positions would cancel each other, leaving the investor long the first four maturities of the first bundle and short the last four maturities of the last bundle.

2.4 Market Mechanics

The CME publishes its ED market book five levels deep in both the bid and ask. [CME Group (2008b)] At each of the five levels the available quantity is disseminated. For each of the listed contract codes, with corresponding decomposition vector $\mathbf{d}^{\text{code}}$, the bid price for each of the five levels is denoted $b^{\text{code}}_{1,2,3,4,5}$ where $b_1 > b_2 > b_3 > b_4 > b_5$ and the ask price at each level is $a^{\text{code}}_{1,2,3,4,5}$ where $a_1 < a_2 < a_3 < a_4 < a_5$. To ensure that markets remain uncrossed, it is generally true that $b_1 < a_1$. The corresponding quantity of contracts available at each bid level is $\beta^{\text{code}}_{1,2,3,4,5}$ and at each ask level is $\alpha^{\text{code}}_{1,2,3,4,5}$. For the most active contracts the bid and ask levels are each separated by a tick and the volume available is significant, for less liquid contracts spreads can be much wider and volume can be intermittent. For example, on April 2, 2008 at 12:00:00 the market quotes for the highly liquid `Z9` and relatively illiquid `BF:H1-H2-H3` contracts are listed in Table 2.

In addition to outright prices, the CME will also consider first generation implied prices. These prices are computed by constructing the desired contract from any two listed contracts. For example, one could calculate an implied price for `H0` by recognizing that $\mathbf{d}^{\texttt{H0}} = \mathbf{d}^{\texttt{H0-H1}} + \mathbf{d}^{\texttt{H1}}$. When a market order is placed, the CME fills it at the best of the market and first generation prices. When there is insufficient liquidity in the market and first generation listed quotes, the system searches through second generation implied quotes to complete the order. [CME Group (2008a)] Second generation prices are derived from the combination of a first generation implied contract and an outright contract. It should be noted that the CME does not disseminate the second generation prices, nor will it use these prices to fill orders except in the case of insufficient liquidity even when the second generation prices are optimal.

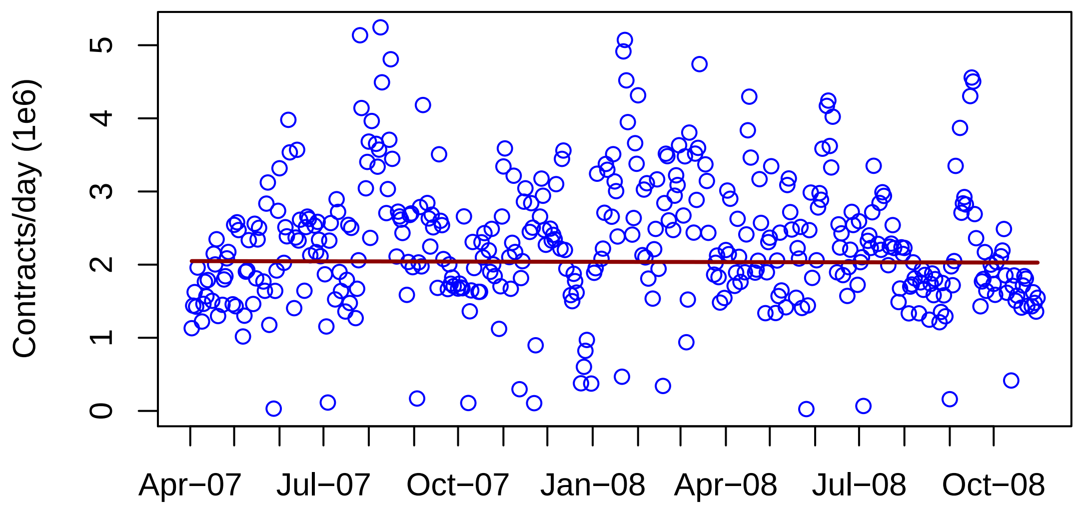
		Z8		BF:H1-H2-H3	
a_5	α_5	97.58	2052	–.–	0
a_4	α_4	97.575	192	–.–	0
a_3	α_3	97.57	680	0.125	200
a_2	α_2	97.565	583	0.12	100
a_1	α_1	97.56	136	0.115	99
b_1	β_1	97.555	599	0.85	41
b_2	β_2	97.55	543	0.8	505
b_3	β_3	97.545	41	–.–	0
b_4	β_4	97.54	290	–.–	0
b_5	β_5	97.535	161	–.–	0

Tab. 2: CME ED market snapshot for Z8 and BF:H1-H2-H3 contracts on April 2, 2008 at 12:00:00.

2.5 Order Size

Figure 4 plots a time series of the total daily traded volume and the average trade size of ED contracts on the CME from April 2007 to November 2008. The nearly flat fit in part (a) of the figure shows that total traded daily volume over the period changed very little. However, there is a clear decreasing trend in the average trade size in part (b) of the figure. In April 2007 a typical average trade size was over twenty contracts, by November 2008 this value had fallen to well below ten. This suggests that not only are order sizes getting smaller, but the number of orders is increasing significantly. These trends are symptomatic of a significant increase in high frequency and algorithmic trading on the exchange. As the market state changes increasingly quickly, the techniques developed here become more and more important as the frequency of possible price improvements increases and the ability to manually watch the market becomes exceedingly difficult.

(a) Daily volume of ED contracts traded on the CME



(b) Daily average of contracts exchanged per trade on the CME

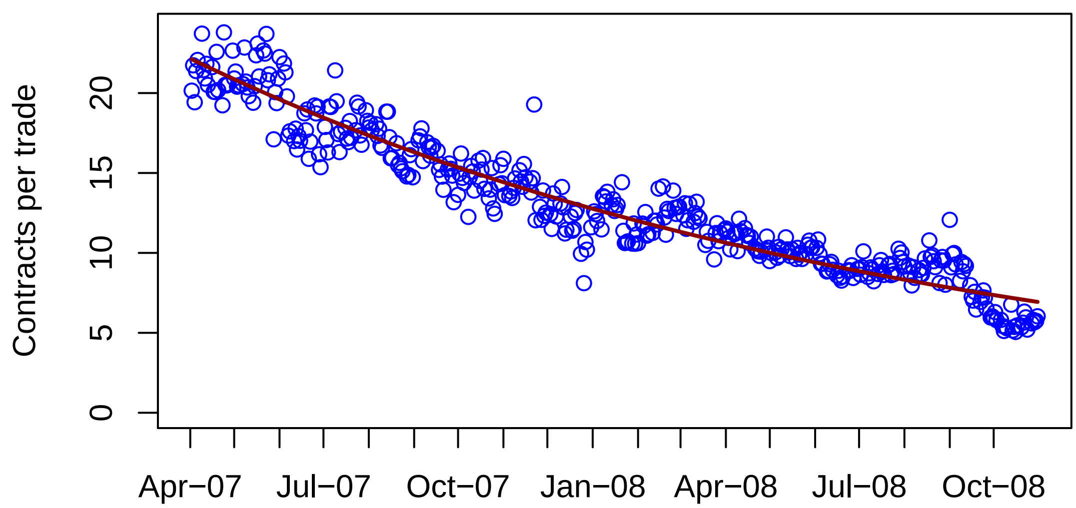


Fig. 4: Daily volume of CME ED contracts and average number of contracts per trade between April 2007 and November 2008 with exponential trend lines.

3 Price Optimization

3.1 Overview

In this section a formal framework for constructing the best market quote for a desired contract from the displayed market state is developed. The formal framework initially laid out by Almgren (2008) is extended to include a practical technique to implement the price optimization considering multiple market levels and integer constraints. Techniques for incorporating transaction costs are discussed and an algorithm to perform the pricing are presented.

3.2 Integer Programming

As discussed in the previous section, CME contracts trade only as full integer multiples – fractional contracts cannot be traded. As a result, when computing optimal contract combinations only integral solutions are viable. These restrictions are satisfied by the standard integer programming specification.

At any instant in time the observable ED market is completely specified by: the matrix of contract components, $\mathbf{D}$; the corresponding vector of quantities available on the bid, $\boldsymbol{\beta}_{1,2,3,4,5}$, and ask, $\boldsymbol{\alpha}_{1,2,3,4,5}$; and the corresponding bid, $\mathbf{b}_{1,2,3,4,5}$, and ask, $\mathbf{a}_{1,2,3,4,5}$, prices. Matrix $\mathbf{D}$ of contract components is constructed by the augmentation of column vectors describing each of the contracts in terms of the base components, as discussed in section 2:

$$\mathbf{D} = \left(\; \mathbf{d}_1^{\text{code}_1} \;\middle|\; \mathbf{d}^{\text{code}_2} \;\middle|\; \mathbf{d}^{\text{code}_3} \quad \cdots \quad \mathbf{d}^{\text{code}_{m-2}} \;\middle|\; \mathbf{d}^{\text{code}_{m-1}} \;\middle|\; \mathbf{d}^{\text{code}_m} \; \right) \tag{5}$$

where there are m unique contract types traded. This is the matrix of ask positions at the first level. The corresponding bid positions are $-\mathbf{D}$. As the market is quoted five levels deep in the bid and ask, the total market is described by the augmented matrix

$$\mathbf{M} = \left(\; \mathbf{D} \mid -\mathbf{D} \mid \mathbf{D} \mid -\mathbf{D} \mid \mathbf{D} \mid -\mathbf{D} \mid \mathbf{D} \mid -\mathbf{D} \mid \mathbf{D} \mid -\mathbf{D} \; \right) \tag{6}$$

with corresponding available quantity vector

$$\mathbf{q} = \left(\; \boldsymbol{\alpha}_1 \mid \boldsymbol{\beta}_1 \mid \boldsymbol{\alpha}_2 \mid \boldsymbol{\beta}_2 \mid \boldsymbol{\alpha}_3 \mid \boldsymbol{\beta}_3 \mid \boldsymbol{\alpha}_4 \mid \boldsymbol{\beta}_4 \mid \boldsymbol{\alpha}_5 \mid \boldsymbol{\beta}_5 \; \right) \tag{7}$$

and position cost vector

$$\mathbf{c} = \left(\; \mathbf{a}_1 \mid -\mathbf{b}_1 \mid \mathbf{a}_2 \mid -\mathbf{b}_2 \mid \mathbf{a}_3 \mid -\mathbf{b}_3 \mid \mathbf{a}_4 \mid -\mathbf{b}_4 \mid \mathbf{a}_5 \mid -\mathbf{b}_5 \; \right). \tag{8}$$

Denote the target position in the 44 base contracts as vector $\mathbf{t}$. The goal is to achieve this position using any combination of the m listed contracts while minimizing the cost of entry. The position is achieved by buying and selling integral amounts of the m underlings, constrained by the vector $\mathbf{q}$ of quantities. Decision vector $\mathbf{x}$, containing $10m$ elements, denotes the required position in each of the underlying contracts to construct desired position $\mathbf{t}$. By construction, position $\mathbf{t}$ is reproduced when

$$\mathbf{M}\mathbf{x} = \mathbf{t}. \tag{9}$$

As only integral positions are allowed, it is required that

$$\mathbf{x} \in \mathbb{I}^{10m}. \tag{10}$$

Further, as contracts must be bought at the ask and sold at the bid, components of $\mathbf{x}$ must be non-negative:

$$\mathbf{I}\mathbf{x} \geq \mathbf{0}. \tag{11}$$

The price of entering position $\mathbf{x}$ is then

$$\mathrm{P}\left(\mathbf{x}\right) = \mathbf{c}^{\mathsf{T}}\mathbf{x}. \tag{12}$$

Given the market state completely specified by $\mathbf{M}$, $\mathbf{q}$, and $\mathbf{c}$, market participants with desired position $\mathbf{t}$ wish to minimize the cost of entering the position. This is done by finding a decision vector $\mathbf{x}$ that satisfies

$$\mathbf{x} = \arg\min_{\mathbf{y}} \mathrm{P}\left(\mathbf{y}\right) \tag{13}$$

subject to the conditions specified in equations (9) $-$ (11). This is a well posed integer programming problem.

3.3 Transaction Costs

Execution of ED market orders requires clearing through the CME via a clearing house. There are fees associated with the order clearing that become significant when considering price optimization. Typically, transaction fees are on a per contract per base contract basis and range from about \$1 to \$3 depending on the relationship with the exchange, which translates to 0.04 bp to 0.12 bp. Brokerage houses that execute a large number of transactions operate at the lower end of the range while infrequent traders face higher costs. As an example, the transaction fee on three butterfly contracts, which are each comprised of four base contracts, would range from \$12 to \$36. To accommodate transaction costs in the model the bid and ask levels presented to the optimizer are universally widened based on the number of base contracts in the derived contract. Using the \$1 transaction cost level, the bid on a butterfly would be decreased by 0.16 and the ask increased by the same amount at each level in the book. For large contracts like the bundle spread `GE:BS 2YU9 2YU2` that contains 18 base contracts, the bid and ask would each move 0.72 resulting in a bid-ask spread nearly *three* ticks wider. As a result, it quickly becomes expensive to use derived contracts containing many legs and optimal solutions rarely contain these type of contracts.

3.4 Problem Reduction to Linear Programming

While the integer programming problem discussed in the previous section is conceptually well posed, algorithms do not exist to solve integer programming problems with guaranteed optimality. In practice most integer programming algorithms are at the core linear programming algorithms, which can guarantee optimality of their solutions, with additional techniques added to achieve the integer constraints. Due to the integral nature of the derived contracts it is possible to develop a very efficient ad hoc integer constraint system that directly drives a linear programming algorithm. In the current study a the linear programming simplex method technique, as described in Arthur (1961) and Cormen et al. (2002), is used. In addition to the increased accuracy of the technique described herein, there is a significant speed improvement as the levels of the order book are considered sequentially – that is only one bid and ask price is used at each level – rather then the much larger problem of considering all order levels simultaneously.

In developing the linear programming approach a technique to separate the parallel five order level problem into a sequential single order level problem with minimal reductions in price optimality is described. As the typical goal is improvement of quotes for contracts listed on the exchange, the target vectors typically are structurally similar to the derived contracts considered in Section 2. As such, large orders can be factored into a desired market *direction* vector and scalar quantity multiplier. For example, if a market participant wished to obtain 15 of a specific butterfly contract the target vector would look like $\mathbf{t} = \begin{pmatrix} \dots & 0 & 15 & -30 & 15 & 0 & \dots \end{pmatrix}$ from which it is possible to extract the desired quantity, 15, and the desired integer direction vector by dividing by the greatest common divisor: $\mathbf{t} = 15 \begin{pmatrix} \dots & 0 & 1 & -2 & 1 & 0 & \dots \end{pmatrix}$. Then the integer program discussed in the previous section is applied using only the first level of the order book to find the optimal market direction solution $\mathbf{x}$. The quantity of available contracts for each of the nonzero components in the solution is then checked for sufficient quantity to fill the target quantity. If sufficient volume is not available the target quantity is sequentially reduced until sufficient volume is available. The market is then updated to remove the contracts used in the solution. If a liquidity level is completely used up in the process, the next level becomes the market level. The process is continued with the new market until the desired order quantity is completely filled.

A similar technique is used to fulfill the integer constraints. As the target vector must have integral components, all viable decision vectors contain only rational values. Furthermore, as the derived contracts contain base contracts in *only* $1:1$, $2:1$, $3:1$, and $4:1$ ratios, it is always possible to construct an optimal decision vector with only $1/2$, $1/3$, and $1/4$ fractional components. In practice, it is typically only the $1/2$ factors introduced by the very liquid butterfly contracts that appear with great frequency. Proceeding as in the market level case, an optimal solution direction is found using the linear programming techniques. The largest multiplying factor that satisfies both the integer constraints and volume constraints is then found and the desired contracts removed from the market. The resulting market them must have a contract with available quantity 1, 2, or 3 that introduces fractional components in the solution vector. If only one of these derived contracts exist it is removed from the system. If multiple contracts of this form exist they are sequentially removed and the optimization reapplied and then restored, until the *binding* contract is found at which point it is removed. Using this technique it is possible that an optimal price is mistakenly removed from the system for a few contracts, however this is only an issue when large quantities are required and will only affect a small portion of the order. As such, the overall fractional price effect is minimal.

To summarize, at each step of the algorithm the reduced linear programming problem is considered:

$$\arg\min_{\mathbf{x}} \left(\mathbf{a}_1^\mathsf{T} \;\middle|\; -\mathbf{b}_1^\mathsf{T} \right) \mathbf{x} \tag{14}$$

subject to constraints

$$\left(\mathbf{D} \;\middle|\; -\mathbf{D} \right) \mathbf{x} = \mathbf{t} \tag{15}$$

and

$$\mathbf{I}\mathbf{x} \geq \mathbf{0} \tag{16}$$

where $\mathbf{x} \in \mathbb{R}^{2m}$. This problem is easily solved using the standard simplex method. The solution vector $\mathbf{x}$ is then checked for liquidity and integer constraints, the system updated, and the process repeated until the desired order quantity is reached.

4 Market Applications

4.1 Method

The techniques developed in Section 3 are implemented in `MATLAB` and applied to CME ED tick data. Using this code, the optimization was performed on ED tick data and the resulting performance analyzed. Due to the very high frequency of the data, in the one and a half years of data spanning April 2007 to November 2008 there are over seven trillion market updates, and the limited computation power available, analysis of the entire data set was not feasible. As such, the analysis focuses solely on the one day period from the market open at `6:00 pm` on Tuesday, April 1, 2008 through to the market close at `5:00 pm` on a Wednesday, April 2, 2008. The period was specifically chosen because it was a relatively quiet day – there was no significant news that day. In order to complete the analysis, samples of sequential tick data were selected from over the day stratified according to intraday tick volume by hour in such a way that none of the sample periods were overlapping.

The price optimization routine was run at each time sampled time increment seeking both bid and ask quotes for various target contracts, both base and derived, chosen based on average volume over the day. A broad cross section of contracts were chosen to capture the dynamics of the optimization in different liquidity regimes. The target contracts chosen for optimization are listed in Table 3. High volume indicates contracts were had the highest or second highest volume of the contract type over the analysis period. Medium and Low indicates volume approximately 50% and 10% respectively of the high volume contracts. The condor contract had significantly lower volume, with fewer than 400 trades over the day.

Due to a lack in availability for some of the price quote data some contracts had to be omitted in the analysis. This included Packs, Bundles and all derivations there of. While these products represent a significant volume of the total contracts traded, their large transaction costs usually preclude them from inclusion in optimal pricing.

Tick data was supplied by the CME and stored in a MySQL database housed on a high performance multi disk server. Simulations were conduced on 25 computers, each with quad core 2.4 GHz processors. Fifty instances of the optimization algorithm were run simultaneously, optimizing over the 13 contracts listed in Table 3, for four hours of real time – 200 hours of CPU time. In this period more than 5 million optimizations were performed, solving for the 13 target contracts on about 4% of the total periods ticks. Combining these values, it is clear that the grid can compute solutions in parallel for a single target contract at approximately triple real time.

4.2 Shortcomings

The optimization routine was fairly stable but in some instances crashed. This was due to two factors: sparsely populated sample areas and imperfections in the optimization routine. The former is easier to understand, if there are too few order than no solution can be found. The

Contract	Type	Relative Liquidity
Z8	Base Contract	High
M9		High
H0		Medium
H1		Low
U8-Z8	Spread	High
Z8-H9		High
Z9-H0		Medium
H9-Z9		Low
BF:M8-U8-Z8	Butterfly	High
BF:U8-Z8-H9		High
BF:U9-Z9-H0		Medium
BF:H1-H2-H3		Low
CF:M8-U8-Z8-H9	Condor	Very low

Tab. 3: Relative liquidity of several CME ED contracts on April 2, 2008.

latter is a result of using the simplex method implemented in MATLAB to solve the problem. In some instances the simplex routine returned a vector solution which was not easily convertible to an integer value. This problem began arising when we added the more complicated derived contracts such as double butterflies. Although it was attempted adjust the optimizer to account for this problem this was not successful. Further analysis of the simplex method would likely remedy these issues.

4.3 Analysis

Overall, when the optimization routine was successful, price improvement was possible approximately 19% of the time. The selected contracts were optimized with a quantity equal to the minimum size of the implied and the direct quotes and 10. In Figure 5 the success rate is divided into the respective contracts and it evident the optimization success occurs more frequently when there is both an open implied and direct quote.

The success rate plotted in Figure 6 by hour of day, it does not appear that there is any point in the day where the optimization is more successful, it is known however that the optimization algorithm converges on a much more frequently basis when there is more liquidity.

Breaking up the success rate by contract in Figure 5, it appears less liquid contracts are more prone to price improvement than more liquid contracts. With base contracts, spreads, and butterflies price improvement occurs more frequently than with the low liquidity contracts versus the high or medium contracts. The condor, however, has very low volume for the day

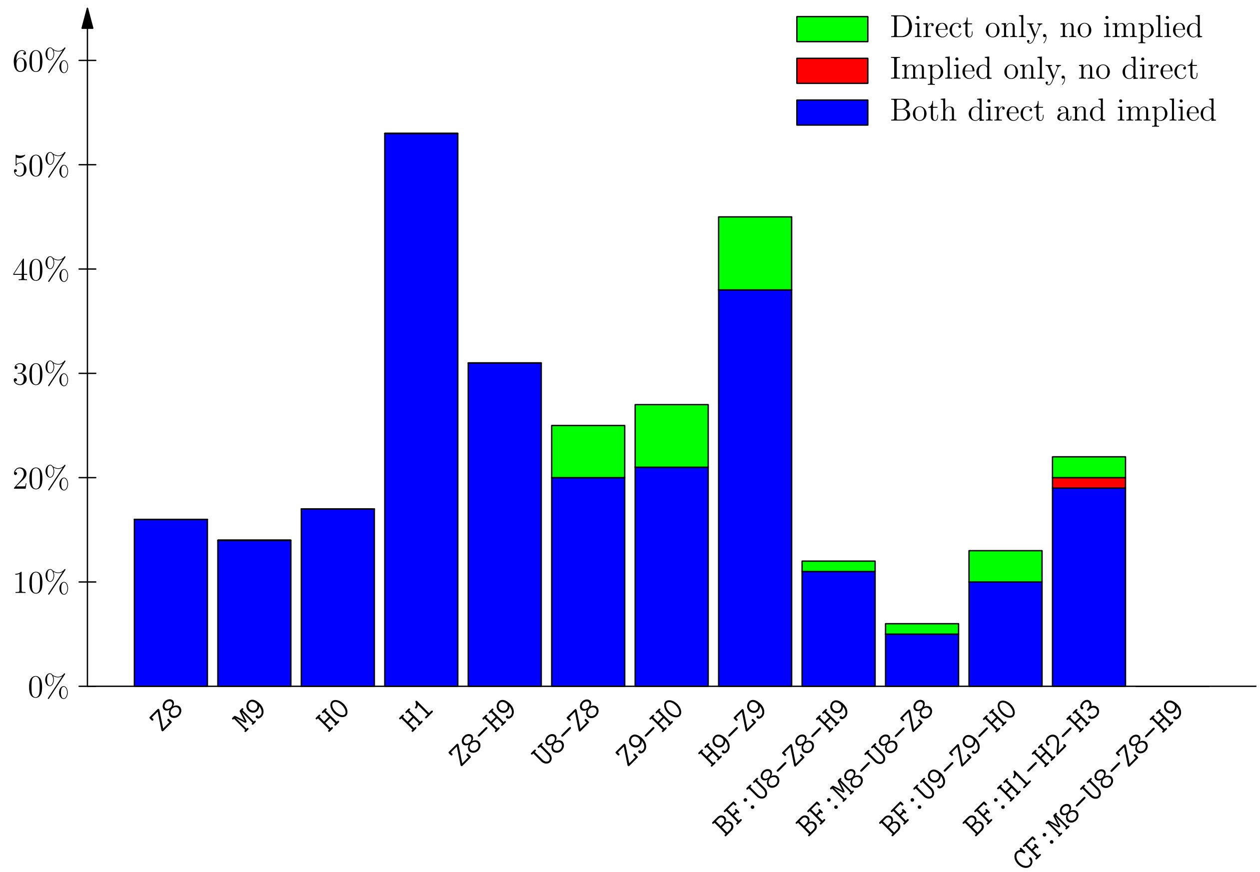


Fig. 5: Overall optimization success rate over the optimization period: All Contracts

and does not show any price improvement for the day.

In an analysis comparing the base contracts with the highest and lowest liquidity in 7 and 8, there is no strong relationship between what time of day either contracts is more successful, the price improvement appears to be independent of what time of day it is with the except of the late night hours when liquidity is very low.

Finally in an analysis of the spreads on both contracts and their optimizations, there does not appear to be a strong relationship between these variables. Spreads can be seen in Figures 9 and 10.

From these graphs no strong conclusions can be drawn as the sample period is only one day of a much larger set of data. Further data and enhancement of the optimization algorithm to improve convergence would also be required.

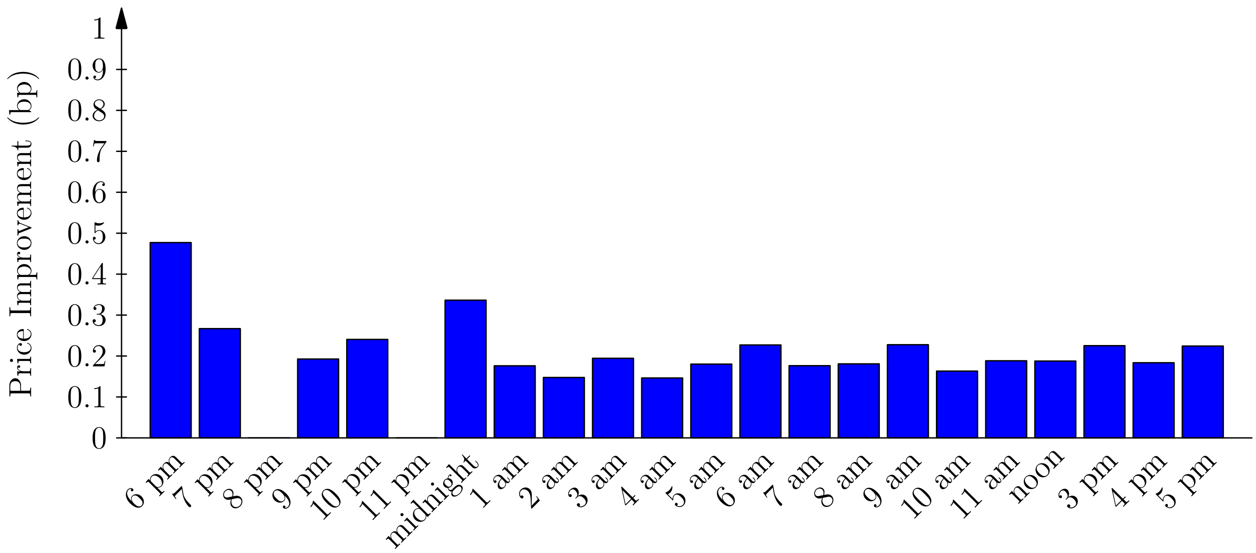


Fig. 6: Hourly optimization success rate: All Contracts

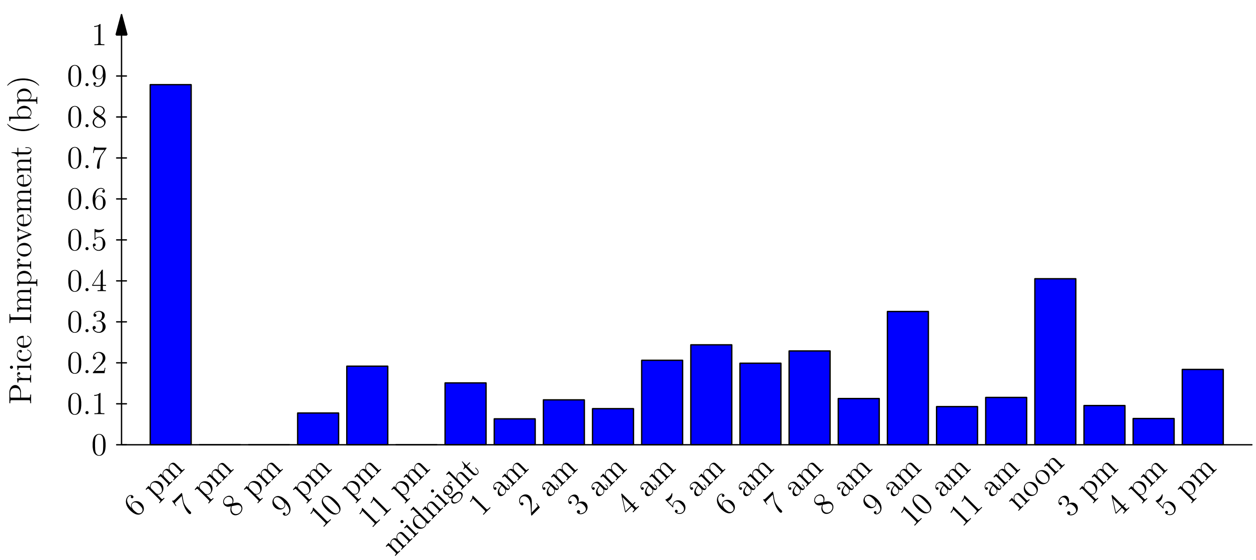


Fig. 7: Hourly optimization success rate: Contract Z8

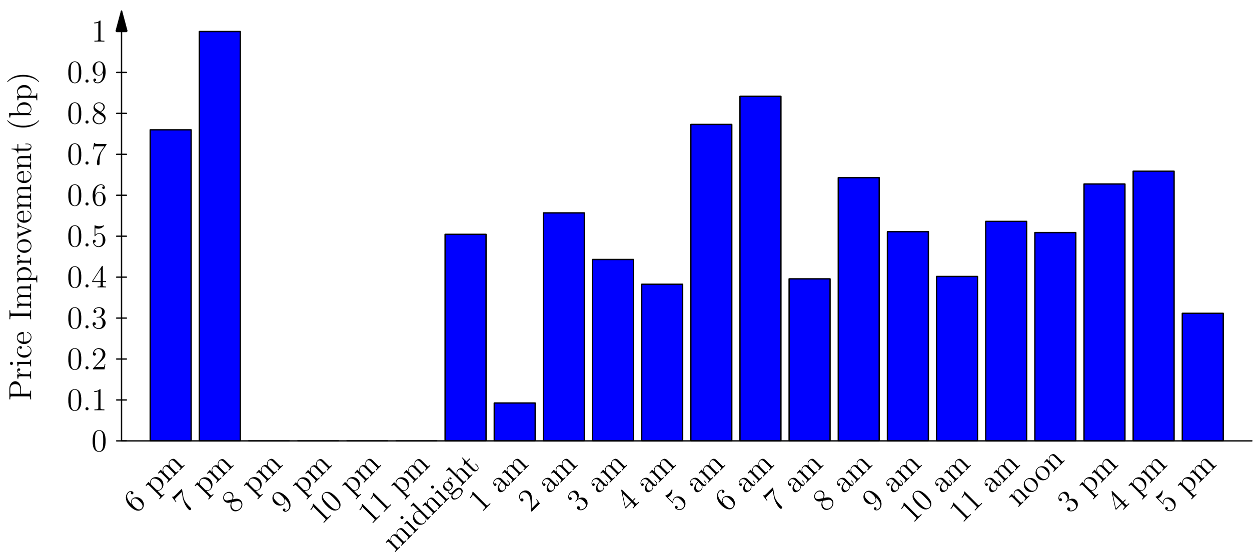


Fig. 8: Hourly optimization success rate: Contract H1

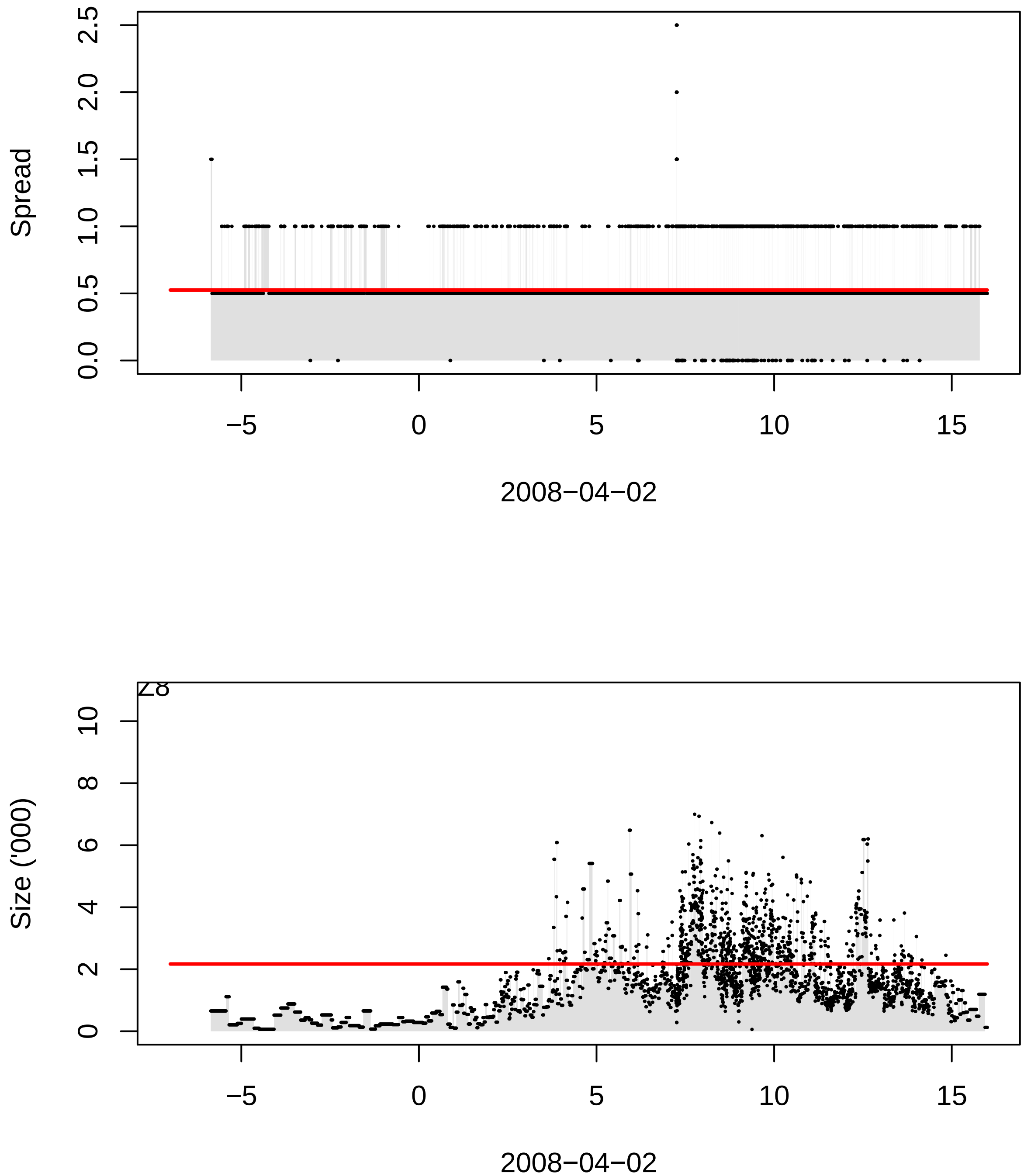
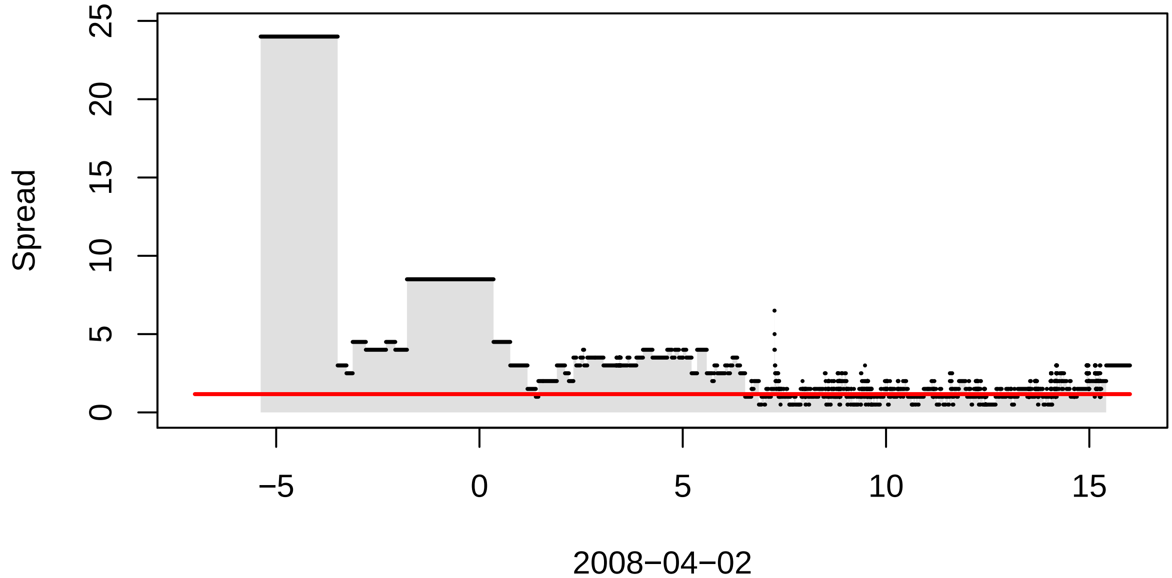


Fig. 9: Time series of Spread and Volume over the optimization period: Contract Z8



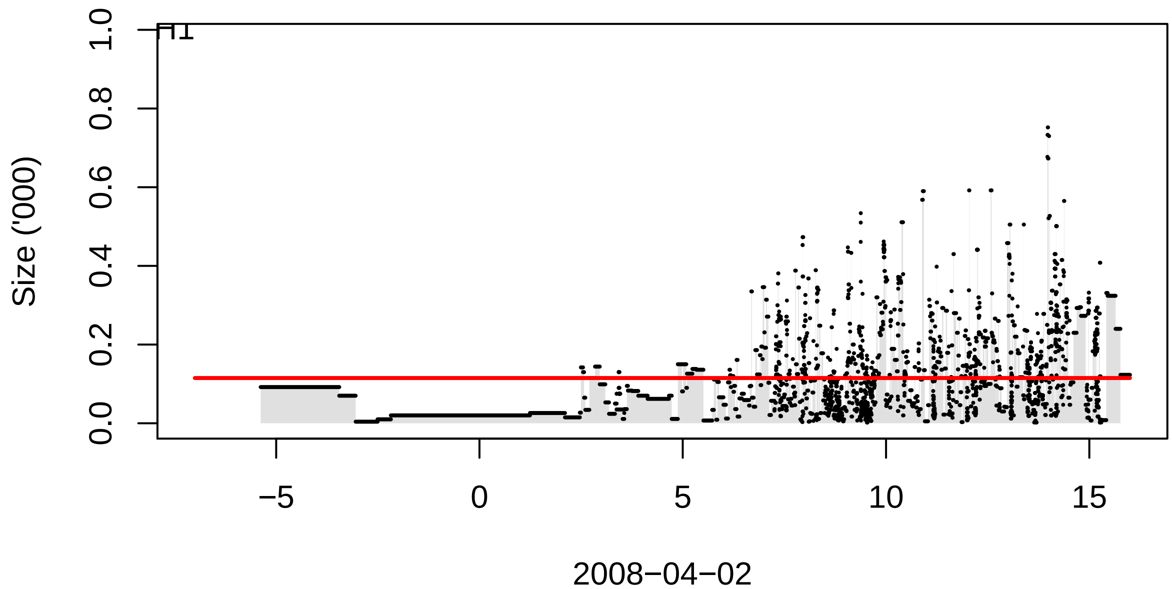


Fig. 10: Time Series of Spread and Volume over the optimization period: Contract `H1`

5 Conclusion

Due to the multiple applications in hedging interest rate risk, Eurodollar futures contracts are one of the most frequently traded products on the CME. By exploiting inefficiencies in the market, it is frequently possible to beat the outright or first order implied prices listed by the exchange. We developed a technique to offer clients the best available price for a given contract using any combination of listed contracts. Using the redundancies generated by derived contracts, an ad hoc procedure was created to find the best available price given the current market state.

Noting that contracts only trade in integral multiples, the problem is initially defined using an integer programming framework. With the realization that fractional values only occur when considering derived contracts with positions of greater than one, a modified linear programming technique was developed to handle spreads, butterflies, and double butterflies. The method has the benefit of being faster than typical integer programming technique however our technique may not be as robust. As the market state updates on the order of every $10ms$, with an increase in computational efficiency the technique could potentially be applied in a real-time setting, given appropriate hardware. Our method also accounts for transaction costs on a per contract per base contract basis and volume constraints imposed by available liquidity in the market. The transaction cost constraint has the effect of limiting the number of contracts included in the optimal construction and limiting the inclusion of the more complicated derived contracts. In turn, this reduces the frequency of modified solutions that result from forcing the integer constraints upon the optimal solution found from linear programming.

We then tested our technique on historical market data. With access to a year and a half of CME data and thus over seven trillion ticks, it was infeasible to undertake a complete study on frequency of price improvement. Instead, we focused on a market tick examination of one day of trading data. The examined date was selected to avoid anomalous behavior caused by significant news announcement. A tick analysis of the market on April 2, 2008 demonstrates that price improvements can be found for approximately 19% of listed contracts throughout the day. These price improvements tend to occur more frequently in less liquid contracts, perhaps due to increased market inefficiencies resulting from limited trading. Price improvements range as high as two ticks and average 0.2 ticks, a substantial improvement considering ticks have real value $12.50. The incorporation of transaction cost constraints results in optimal solutions generally avoiding many of the less liquid contracts.

Going forward, our major concern is increasing efficiency to achieve real-time computations. Using advanced linear programming techniques the serial nature of the data can be exploited. Rather than recomputing the complete solution with each market update, it is possible to examine the size of market changes and using precomputed values determine quickly if the optimization needs to be completely recomputed. Coupling more advanced linear programing techniques, as well faster hardware, we believe significant speed gains are possible. Further analysis on the feasibility of using graphics processing units may also be useful. The framework outlined is robust enough to be applicable and produces substantial price improvements for less frequently traded contracts. For investors looking to take larger positions, this procedure could result in substantial savings over listed CME prices.

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