Numerical Analysis homework 1

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I. The bisection method interval

I-a

If we take the width of the bisection interval as $3.5 - 1.5 = 2$ at the first step, then get the following answer:

$$\text{Width}_n = 2^{2-n}$$

I-b

The $\sup dist_n$ is half the width of the interval at the nth step, so the answer is:

$$\sup dist_n = 2^{1-n}$$

II. Proof of the choice of n

Before the start of the process, we consider the real error at this moment. To calculate the error from root r, we use the midpoint x_0 actually, and we can get the relative error $e_0 = \frac{|x_0 - r|}{r} = \frac{|(a_0+b_0)/2-r|}{r}$. Based on this and Problem I, we have the max distance from real point to the root gotten after n steps:

$$\max\{x_n - r\} = a_0 + \frac{b_0 - a_0}{2^{n+1}}$$

And the realtive error is

$$\begin{aligned} e_n = & \frac{|x_n - r|}{r} \\ & \leq \frac{|(b_0 - a_0)/2^{n+1}|}{r} \\ & \leq \frac{|(b_0 - a_0)/2^{n+1}|}{a_0} \leq \epsilon \end{aligned}$$

After taking logarithms, we can easily get

$$\log(b_0 - a_0) - \log \epsilon - \log a_0 \leq (n+1)\log 2$$

So the proposition is proved.

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III. Perform four iterations of Newton's method

$$
\begin{aligned}
&p(x) = 4x^3 - 2x^2 + 3 \\
&p'(x) = 12x^2 - 4x \\
&x_1 = x_0 - \frac{p(x_0)}{p'(x_0)} = -\frac{13}{16} \approx -0.8125 \\
&x_2 = x_1 - \frac{p(x_1)}{p'(x_1)} \approx -0.7708 \\
&x_3 = x_2 - \frac{p(x_2)}{p'(x_2)} \approx -0.7688 \\
&x_4 = x_3 - \frac{p(x_3)}{p'(x_3)} \approx -0.7689
\end{aligned}
$$

IV. Find C & s

Expanding $f(x)$ about x_n and letting $x = \alpha$, α is the root of $f(x)$

$$
\begin{aligned}
f(\alpha) &= 0 \\
&= f(x_n) + (\alpha - x_n)f'(x_n) + o(\alpha - x_n)^2 \\
&= f(x_n) + e_n f'(x_n) + o(e_n^2) \\
&\rightarrow f(x_n) + e_n f'(x_n)
\end{aligned}
$$

We know that $x_{n+1} = x_n + \frac{f(x_n)}{f'(x_0)}$, so we can get

$$
\begin{aligned}
f'(x_0)(x_n - x_{n+1}) + e_n f'(x_0) = 0 \\
f'(x_0)(e_n - e_{n+1}) + e_n f'(x_0) = 0 \\
(f'(x_n) + f'(x_0))e_n = f'(x_0)e_{n+1}
\end{aligned}
$$

i.e. $s = 1, C = \frac{f'(x_n)+f'(x_0)}{f'(x_0)}$

V. Convergence of iteration

$f = tan^{-1}$, when $x \in \mathbb{R}, f \in (-\frac{\pi}{2}, \frac{\pi}{2})$. And according to condition, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence, we get a continuous compressed mapping, which means we can find a fixed point to ensure the convergence of iteration of f.

VI. Prove that the sequence of values converges.

we can let $x = \lim_{n\to\infty} x_n$, and the recursive relationship can be written as:

$$x_0 = 0, \quad x_1 = \frac{1}{p + x_0}, \quad x_2 = \frac{1}{p + x_1}, \quad x_3 = \frac{1}{p + x_2}, \ldots$$

i.e we have iteration

$$x_n = \frac{1}{p + x_{n-1}}$$

To solve it with fixed point method, we get:

$$
\begin{aligned}
\frac{1}{p + x} &= x \\
(p + x)x - 1 &= 0 \\
x_r &= \frac{-p \pm \sqrt{p^2 + 4}}{2}
\end{aligned}
$$

since $p > 1$, we select the positive root:

$$x_r = \frac{-p + \sqrt{p^2 + 4}}{2}$$

Using the fixed-point theorem, the function $f(x) = \frac{1}{p+x}$ is continuous and decreasing. Its graph intersects the line $y = x$ at a unique point, meaning the sequence x_n converges to x_r.

VII. Derive an inequality similar to that in II

Based on the analysis and solution in Problem II, we get:

$$\max\{x_n - r\} = a_0 + \frac{b_0 - a_0}{2^{n+1}}$$

We take absolute error instead here, and we have:

$$\begin{aligned} e_n &= |x_n - r| \\ &\leq |\frac{b_0 - a_0}{2^{n+1}}| \leq \epsilon \end{aligned}$$

After taking logarithms, get the inequality:

$$n \geq \frac{\log(b_0 - a_0) - \log \epsilon}{\log 2} - 1$$

If α is very close to zero, the value of $|\alpha|$ becomes very small. In this situation, the denominator in the relative error expression $|\alpha|$ approaches zero, which causes the relative error to become extremely large. That's why we don't take relative error here.

VIII. Newton's method on a root of multiplicity of k

We know the root α is a zero of multiplicity k of the function f, and we can set f as $f(x) = (x-\alpha)^k h(x)$ iff $h(\alpha) \neq 0$. With Newton's method, we have:

$$g(x) = x - \frac{f(x)}{f'(x)}; \tag{1}$$

$$f'(x) = k(x-\alpha)^{k-1}h(x) + (x-\alpha)^k h'(x); \tag{2}$$

so

$$g(x) = x - \frac{(x-\alpha)h(x)}{kh(x) + (x-\alpha)h'(x)}; \tag{3}$$

$$g'(x) = 1 - \frac{h(x)}{kh(x) + (x-\alpha)h'(x)} - (x-\alpha)\frac{d}{dx}\left[\frac{h(x)}{kh(x) + (x-\alpha)h'(x)}\right]; \tag{4}$$

$$g'(\alpha) = 1 - \frac{1}{k} \tag{5}$$

From above analysis, we can easily get the answers:
VIII-a when x is close to α, in equation(3), $(x-\alpha)h'(x)$ can be neglected, so we have $g(x) = x - \frac{(x-\alpha)h(x)}{kh(x)} = x - \frac{x-\alpha}{k}$. Further, we can get:

$$x_{n+1} - \alpha = \frac{1}{k}(x_n - \alpha)$$

It's obvious that the error decreases linearly (rather than quadratically), which means for a multiple zero with multiplicity k, the convergence of Newton's method is generally slower.

VIII-b

$$\begin{aligned} r - x_{n+1} &= g(r) - g(x_n) \\ &= -g'(r)(x_n - r) - \frac{1}{2}(x_n - r)^2 g''(\xi_n) \\ &= -\frac{1}{2}(x_n - r)^2 g''(\xi_n) \end{aligned}$$

With $\xi_n$between x_n and r. Thus the modification effectively cancels the multiplicity effect and restores the quadratic convergence of the Newton's method.

Acknowledgement

References

[1] Kendall Atkinson. *An introduction to numerical analysis*. John wiley & sons, 1991.

[2] OpenAI. *GPT-4*. Accessed: 2024-09-23. 2023. URL: `https://openai.com/index/hello-gpt-4o/`.