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Earthquake Modelling with Differential Equations

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Abstract

The effect of earthquakes on multi-storey structures was modelled using systems of linear differential equations. The method of equivalent static lateral force analysis was applied to structures based on a spring-mass damper system model. Using a harmonic forcing function, the lateral oscillations of each storey of some regular two-dimensional structures undergoing seismic vibrations, was explored. Maple Software and MATLAB was used to compute numerical approximations when the structure underwent earthquake forces of varying frequencies. The case of resonance was modelled by setting the frequency of the earthquake to be close to or equal to the building's natural frequency. To model more realistic situations, non-linear damping cases were also simulated.

1 Introduction

This report presents the research conducted during the summer vacation of 2018-2019 under the guidance of my supervisor. To the best of my knowledge and belief, the report does not contain material previously published or written by another person, except where due reference has been made in the text. I am the sole author of this report.

Modelling is an important tool used in the mathematical realm to describe physical processes and situations. In the case of earthquake modelling, systems of differential equations can be used to describe oscillations of buildings undergoing seismic forces. By considering the magnitude of the applied external force on the building, the lateral movement of each storey with respect to its equilibrium position, can be simulated.

1.1 Seismic Waves

Seismic waves travel fastest through materials with high densities and elasticities, such as steel, which is a key structural element of modern infrastructure. As such, earthquake-resistant technology has been employed in recent years to reduce damage when buildings undergo violent vibrations. Base Isolation Devices are used to isolate buildings from the ground so that seismic waves are not transmitted directly upwards through them. These devices consist of flexible pads which are placed between the ground and the foundation, and which act to resist lateral movement and absorb the earthquake force. By introducing flexibility in the structure, the isolators also add damping to the system. Seismic Dampers are often introduced into buildings in place of structural elements, such as braces. Viscous, friction and yielding dampers all act as shock absorbers. By absorbing part of the earthquake energy, they work to damp the motion of the building **[2]**. Given such technology, it is important to consider damping when modelling earthquake-building interactions.

1.2 Equivalent Static Lateral Force Analysis

Realistically, earthquake forces are comprised of a multitude of frequency components, and each component can interact uniquely with different areas of the building's base. However, for simplicity in the modelling process, the method of equivalent static lateral force analysis is often used to simulate earthquake effects. The dynamic loading caused by an earthquake is approximated by laterally distributing an equal static force on the structure. The force is applied only to the base of the structure and induces oscillations to the first mode of the building only. Since no force is applied to any subsequent floor, the behaviour of the first mode determines the behaviour of all other modes of the building **[3]**. This method is a simple way to realistically model the ground vibrations produced during an earthquake, and how they affect certain structures.

There are some significant assumptions, as well as limitations, associated with this analysis method. Firstly, the structure is assumed to be rigid, and perfect fixity is assumed between the ground and foundation. As such, the effect of earthquake-resistant technology, such as Base Isolation Devices, cannot be accurately accounted for. Secondly, it is assumed that during ground motion, an equal seismic force is applied to every node of the building's base such that every point on the base experiences the same accelerations **[4]**. Thirdly, to accurately analyse the lateral movements of the building purely due to the earthquake force, any movement due to torsion needs to be minimised **[5]**. Torsion is the twisting of an object due to an applied torque, a phenomenon that is common in irregular structures **[6]**. As such, the structures considered in the project needed to be low-rise and relatively symmetric.

The main limitation of this method is that the model cannot accurately account for irregular structures, such as those with significant variations in floor-to-floor mass, or those with their centre of mass displaced from the fixture centre of the building. As such, the models used in the project were based on idealised structures, where the displacement of each floor was simulated based on a point mass located at the floor's centre of mass. To consider irregular structures, the nature of the forcing term and the way it is applied to the building, would need to be appropriately adjusted.

2 Numerical Results and Analysis

Building-earthquake interactions can be modelled using a spring-mass damper system. The following model is based around a two-dimensional building structure consisting of a series of floors with distinct mass. To account for the elasticity in real-world structures, elastic connectors are assumed to exist between successive floors of the building. The connectors mimic the functionality of a spring, and act to supply a restoring force for each level, in accordance with Hooke's Law:

$$F = -kx \tag{1}$$

The proportionality constant relating the restoring force and the floor's displacement is the stiffness parameter, k. For a two-storey structure, the spring-mass model is as follows:

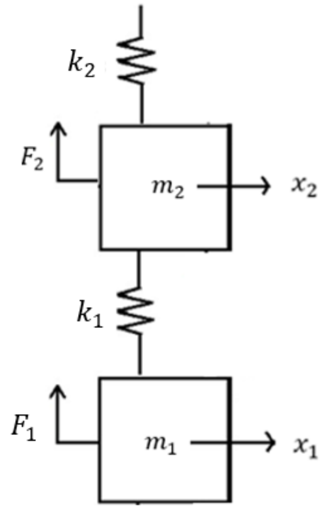

Figure 1: Spring-Mass Damper System

When an earthquake force is applied, each storey will experience a horizontal acceleration, in accordance with Newton's Second Law:

$$F = ma = mx'' \tag{2}$$

Combining Hooke's and Newton's Laws, an expression for the total force on a floor can be derived:

$$mx'' - kx = F_{external} \tag{3}$$

$$x'' = \frac{k}{m}x + \frac{F_{external}}{m} \tag{4}$$

Here, $F_{external}$ represents the magnitude of the earthquake force.

When considering a multi-storey structure, this differential equation is extended to a differential system by defining a mass matrix, M, and stiffness matrix, K.
The differential system is thus defined as:

$$X'' = M^{-1}KX + F_{external} \tag{5}$$

Here, $X = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ and $F_{external} = \begin{pmatrix} F_1(t) \\ \vdots \\ F_n(t) \end{pmatrix}$

By solving this system of differential equations, the lateral oscillations for each storey of the structure can be modelled.

By computing the eigenvalues for this system, the natural frequencies and periods of the building's oscillation can be determined. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues for a building of n storeys, then the natural frequencies $\omega_1, \dots, \omega_n$ are given by:

$$\omega_j = \sqrt{-\lambda_j} \quad for\ j = 1, \dots, n \tag{6}$$

Additionally, the natural periods $T_1, \dots, T_n$ are given by:

$$T_j = \frac{1}{f_j} = \frac{2\pi}{\omega_j} \quad for\ j = 1, \dots, n \tag{7}$$

During an earthquake, vibrations of the ground cause a large horizontal force to be applied to the first floor of the structure. If this is oscillatory in nature, then large displacements may develop in the building. The magnitude of this displacement is dependent on the frequency of the earthquake relative to the natural frequencies of the structure **[7]**.

2.1 Second Order Linear Differential Equations

The horizontal earthquake oscillation was assumed to be harmonic and was defined with the following function:

$$F(t) = F_0 \cos(\omega t) \tag{8}$$

Here, ω represents the frequency of the earthquake, and F_0 represents the amplitude of the earthquake oscillation. The acceleration of this earthquake oscillation is given by:

$$F''(t) = -\omega^2 F_0 \cos(\omega t) \tag{9}$$

By Newton's Second Law, as defined in (2), the earthquake force is thus:

$$F_{external} = ma = mF''(t) = -m\omega^2 F_0 \cos(\omega t) \tag{10}$$

As per (4), the net acceleration of a floor, displacement $x(t)$, is given by:

$$x'' = \frac{k}{m}x - \omega^2 F_0 \cos(\omega t) \tag{11}$$

Based on the spring-mass model, the displacement of a spring can be given by the following function:

$$x = A\cos(\omega t - \varphi) \tag{12}$$

Here, A is the amplitude, φ is the phase angle, and ω is the angular frequency. Substituting into the homogenous equation for (11) yields:

$$-\omega^2 A\cos(\omega t - \varphi) - \frac{k}{m}\big(A\cos(\omega t - \varphi)\big) = 0 \tag{13}$$

$$A\cos(\omega t - \varphi)\left(-\omega^2 - \frac{k}{m}\right) = 0 \tag{14}$$

$$-\omega^2 - \frac{k}{m} = 0 \tag{15}$$

$$\omega \cong i\sqrt{\frac{k}{m}} \tag{16}$$

For $\omega \cong i\sqrt{\frac{k}{m}}$, the amplitude of $x(t)$ is large compared to the amplitude of the earthquake force, $F_{external}$.

Extending this to a regular rectangular structure consisting of five stories, yields the system given in (5), with the mass, stiffness and force matrices defined as follows:

$$M = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{pmatrix} \tag{17}$$

$$K = \begin{pmatrix} -(k_1+k_2) & k_2 & 0 & 0 & 0 \\ k_2 & -(k_2+k_3) & k_3 & 0 & 0 \\ 0 & k_3 & -(k_3+k_4) & k_4 & 0 \\ 0 & 0 & k_4 & -(k_4+k_5) & k_5 \\ 0 & 0 & 0 & k_5 & -k_5 \end{pmatrix} \tag{18}$$

$$F_{external} = \begin{pmatrix} -\omega^2 F_0 \cos(\omega t) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{19}$$

The homogenous equation for the system is given as:

$$X'' - M^{-1}KX = 0 \tag{20}$$

The characteristic equation was found by seeking solutions of the form $\boldsymbol{x(t)} = \boldsymbol{v}e^{\lambda t},\ \boldsymbol{v} \neq 0$. Substituting into (20) yielded:

$$\lambda^2 \boldsymbol{v}e^{rt}I - M^{-1}K(ve^{rt}) = 0 \tag{21}$$

$$\boldsymbol{v}e^{rt}(\lambda^2 I - M^{-1}K) = 0 \tag{22}$$

$$(\lambda^2 I - M^{-1}K)\boldsymbol{v} = 0 \tag{23}$$

$$\det(\lambda^2 I - M^{-1}K) = 0 \ since\ \boldsymbol{v} \neq 0 \tag{24}$$

The solution to (24) gives the eigenvalues for the system, which can in turn be used to compute the natural frequencies of the structure.

2.2 Linear Damping Models

Most modern infrastructure includes friction damper systems which are used to control structural vibrations due to phenomena such as earthquakes. Such devices act to cause a decay in the amplitude of oscillations of the structure with time. To accurately simulate the motion of real-world structure with an applied external force, this damping factor was taken into consideration.

For the linear cases, the damping term was relative to the velocity of the floor's oscillation. As such, the system was defined as follows:

$$X'' = M^{-1}KX + cM^{-1}X' + F_{external} \tag{25}$$

The spring-mass damper system for this structure is represented in Figure 2 below. The geometrical representation of the five-storey rectangular structure is represented in Figure 3. The displacement of each storey is simulated by modelling the movement of the point masses located at the top of each floor.

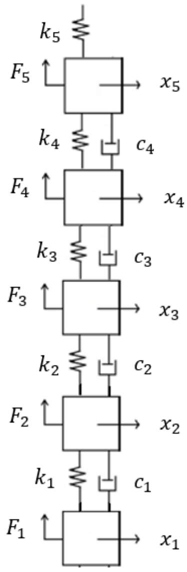

Figure 2: Spring-Mass Damper System for Five-Storey Structure

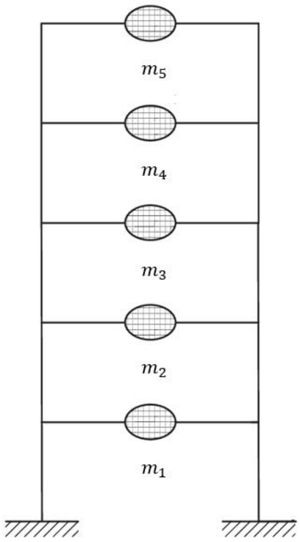

Figure 3: Geometric Structure with Point Masses

In the first case, the damping was introduced as a constant in the system. The values for the parameters were assigned as follows:

$$m_1 = m_2 = \cdots = m_5 = 1000\ units,$$
$$k_1 = k_2 = \cdots = k_5 = 10000\ units,$$
$$c = c_1 = c_2 = \cdots = c_5 = 500\ units \qquad (25)$$

The initial conditions were set as:

$$X(0) = 0, \ \ X'(0) = 0 \qquad (26)$$

To make the amplitudes of oscillations of each level sufficiently large, the earthquake frequency was set to be approximately equal to $\sqrt{\frac{k}{m}}$. That is, $\omega = 3.2$. Additionally, the amplitude of the earthquake oscillation was set to $F_0 = 0.1$, and the earthquake duration was set to 2.0 seconds.

Using Maple Software, the following plots representing the displacement of each floor with respect to their equilibrium position, was derived:

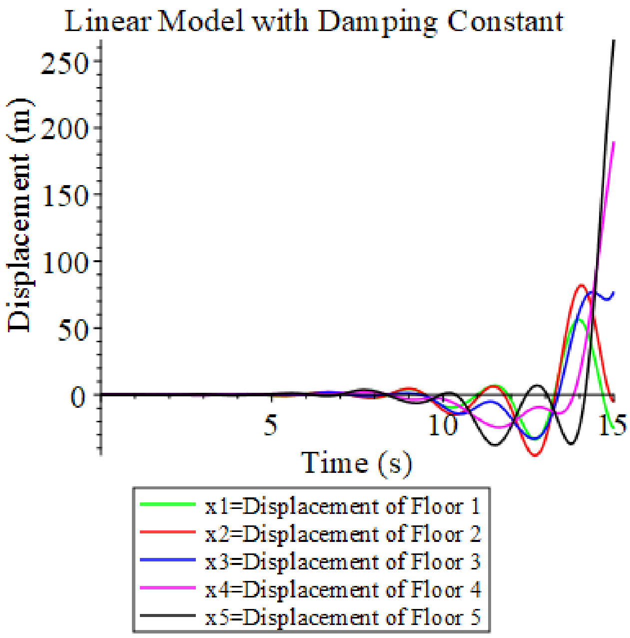

Figure 4: Linear Model with Damping Constant

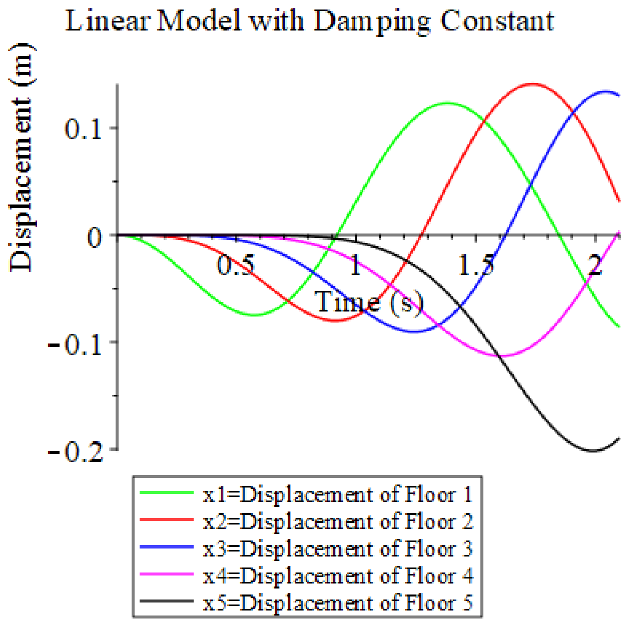

Figure 5: Linear Model for $t \leq 2.1$ $secs$

By looking at floor displacement profiles from a report for the 14th World Conference on Earthquake Engineering 2008, the trend of normalized maximum floor displacement in multi-storey structures was analysed **[8]**. It was concluded that the amplitudes of oscillations should initially be largest in the first floor of the building, as this is where the force was applied. With time, the amplitude of oscillations of each floor should be seen to increase for higher levels of the building, and maximum displacement should be observed for the highest floor of the building.

The model demonstrated this trend initially, particularly in the first 2.0 seconds when the earthquake force was being applied, as per Figure 5. The oscillations were seen to begin in the first floor and slowly to move to subsequent floors. At the same time however, defining the damping factor as a constant meant that the decay rate of each individual storey was not appropriately accounted for. As such, it was found that the amplitudes of oscillations of each level increased drastically with time, rather than decaying, as per Figure 4. Not only did the increase occur significantly quickly, but the displacements the levels' reached were much larger than expected. The nature of these solution curves over time were not indicative of the behaviour of a real-world structure undergoing damping.

To account for these inconsistencies, the next method involved defining the damping coefficients in a matrix, instead of as a constant. For this second model, the system was defined as follows:

$$X'' = M^{-1}KX + M^{-1}CX' + F_{external} \tag{27}$$

The damping matrix for the five-storey case was:

$$C = \begin{pmatrix} -c_1 & c_1 & 0 & 0 & 0 \\ c_1 & -(c_1+c_2) & c_2 & 0 & 0 \\ 0 & c_2 & -(c_2+c_3) & c_3 & 0 \\ 0 & 0 & c_3 & -(c_3+c_4) & c_4 \\ 0 & 0 & 0 & c_4 & -c_4 \end{pmatrix} \tag{28}$$

By seeking solutions of the form $\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{v}e^{\lambda t}, \ \boldsymbol{v} \neq 0$ for the homogenous case, the following equation was derived:

$$\det(\lambda^2 I - M^{-1}C\lambda - M^{-1}K) = 0 \tag{29}$$

By solving (29) in MATLAB, the following eigenvalues were calculated:

$$\lambda_1 = -0.01 + 0.90i, \ \ \lambda_2 = -0.12 + 2.63i, \ \ \lambda_3 = -0.34 + 4.13i,$$
$$\lambda_4 = -0.63 + 5.28i, \ \ \lambda_5 = -0.89 + 6.00i \tag{30}$$

Using (6), the following natural frequencies for this five-storey structure were computed:

$$\omega_1 = \sqrt{-(-0.01 + 0.90i)}, \ \ \omega_2 = \sqrt{-(-0.12 + 2.63i)},$$
$$\omega_3 = \sqrt{-(-0.34 + 4.13i)}, \ \ \omega_4 = \sqrt{-(-0.63 + 5.28i)},$$
$$\omega_5 = \sqrt{-(-0.89 + 6.00i)} \tag{31}$$

Using Maple, the solution for (27) was computed and the following numerical plot was derived:

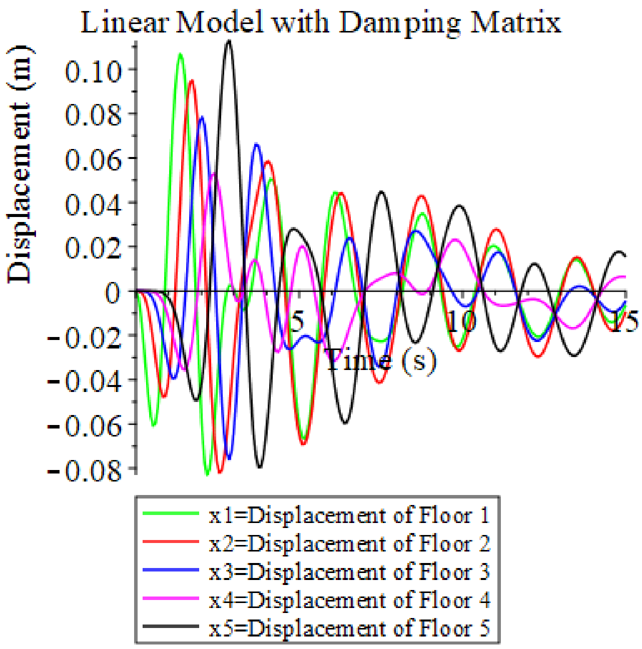

Figure 6: Linear Model with Damping Matrix

Here, the displacement of the first floor is largest in the first 2.0 seconds, when the earthquake force is present. During this time, an increased number of oscillations is also seen to occur. After this time, as expected, the amplitude of the first floor was seen to decay. Damping was present in each floor, and the largest oscillations occurred for the most part in the highest floor of the building. Importantly, by defining the damping constants in a matrix, the system not only considered the damping of each individual floor, but also the effect of each floor's damping on its surrounding floors. As such, a much more accurate display of a damping situation was observed.

2.3 Resonance

A significant aspect of the project involved exploring different earthquake forces, specifically forces of different frequencies. The effect of increasing the frequency of the earthquake was that the number of oscillations and amplitude of oscillations were seen to increase drastically in the period where the earthquake force was present, particularly in the first floor. Additionally, the amplitudes of oscillation were much higher for subsequent floors, and decay occurred more slowly. When the earthquake frequency became close to one of the natural frequencies $\omega_1, \ldots, \omega_5$, or close to a linear combination of these frequencies, the amplitude of the solution became very large. When close enough, the displacement became too large to preserve the structural integrity of the floor.

To model this, the earthquake frequency was set to $\omega = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 = -6.9344 + 6.3067i$. Using ode45 in MATLAB, numerical approximations for this case were derived:

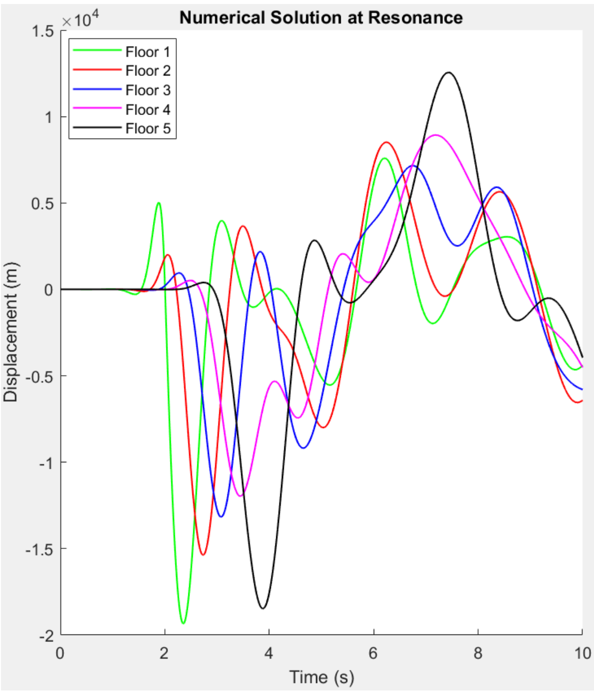

Figure 7: Resonance Due to Earthquake Frequency

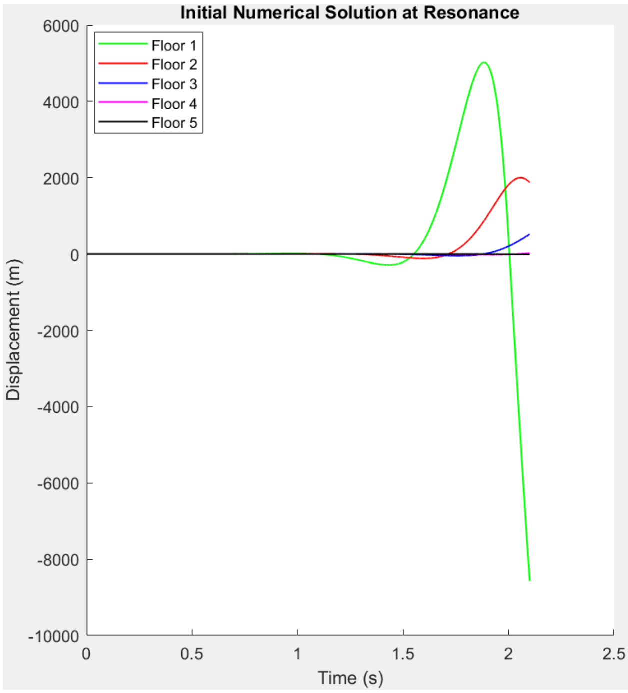

Figure 8: Resonance for $t \leq 2.1$ $secs$

Although Figure 7 indicates that the oscillations continue (and decay) with time, the initial displacements of the floors are in the thousands of metres, meaning that the structure would collapse after only a few oscillations. As evident in Figure 8, the displacements of the floors (particularly the first floor) in the period where the earthquake force is present, became extremely large. The proximity of the earthquake frequency to a linear combination of the levels' natural frequencies caused the amplitudes of oscillation to exceed the values at which the structure could maintain its integrity. This phenomenon is referred to as resonance.

From analysis of a variety of situations, it was found that the likelihood of an earthquake destroying a building depended more so on its frequency than its duration or amplitude. In fact, a very small amplitude of oscillation was still adequate to source floor oscillations. The amplitude of the earthquake force in no way affected the shape of the solution curves, only the magnitude of their displacements. The primary effect of changing the duration of the earthquake was in the amplitudes of oscillation of the solution curves. Since the earthquake force was being applied for more time, the decay of the solution curves was seen to occur over a longer period. As the earthquake's duration was increased, the floors were seen to oscillate with slightly higher displacements. Interestingly however, this increase became limited as the earthquake force reached longer durations. When this occurred, the primary factor influencing the floors' displacements was instead the earthquake's frequency.

Considering these results, it was determined that the solution curves were most significantly influenced by the frequency of the earthquake force, particularly its relativity to the natural frequencies of the building.

2.4 Non-Linear Damping Models

When structures undergo intense vibrations, resistive forces can occur. Because the surfaces between floors are not perfectly smooth, friction can occur when the structure moves. Frictional forces can contribute to damping in the system. In the case of phenomena such as friction and yielding, the damping term can become non-linear. To model more realistic earthquake-building situations, some non-linear damping systems were considered.

In two different cases, the effect of defining the damping term relative to the velocity as well as the displacement of the oscillations, was explored. In the non-linear cases, differences in the results were only clear when the damping was of significant magnitude, and so the damping parameters were assigned as follows:

$$c_1 = c_2 = \cdots = c_5 = 100000\ units \tag{32}$$

For the first case, the damping term was defined in terms of the magnitude of the velocity as well as the displacement of the storey, as per the following system:

$$X'' = M^{-1}KX + M^{-1}C|X'|X + F_{external} \tag{33}$$

Using Maple, the following solution was derived:

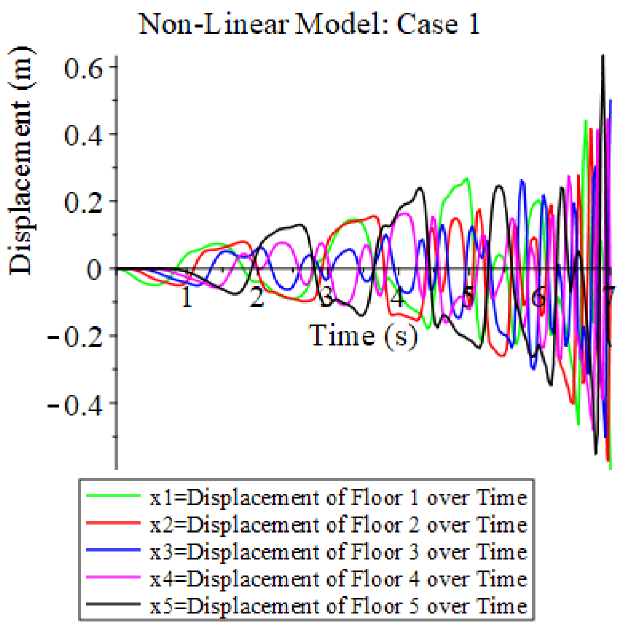

Figure 9: First Non-Linear Case

In comparison to the previous linear models, the predominant effect of adjusting the damping term in the first case was that the amplitude of the oscillation for each floor was increased. More specifically, the amplitudes of oscillation of each floor were seen to increase drastically with time, as per Figure 9. As expected, the oscillations begun in the first floor and the displacements of each floor were seen to increase for higher floors of the building. However, the damping condition was not displayed as expected. It was predicted that this was a result of the way the damping term had been defined.

Oscillation of floors surrounding a specific storey could influence the oscillations of that storey. As such, the damping for each level needed to not only account for the decay of the specific storey, but also for the decay of any surrounding storeys. This was achieved by redefining the damping term in terms of the modulus of the velocity of the entire building. The resulting system was:

$$X'' = M^{-1}KX + M^{-1}CX\sqrt{\sum_{j=1}^{5}\left(x_j'\right)^2} + F_{external} \tag{34}$$

In Maple, the following solution was derived:

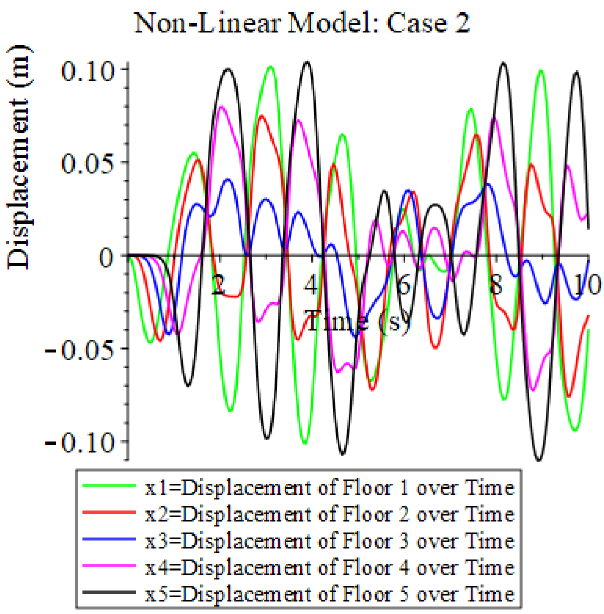

Figure 10: Second Non-Linear Case

By redefining the damping term, decay in the floors' oscillations with time was observed. Interestingly, the motion was seen to decay and then increase multiple times before coming to a stop. This trend mimicked that of the 'stick-slip' phenomenon. The non-linearity of the damping term meant that damping due to phenomena such as friction was also considered. The occurrence of the stick-slip implied that the frictional force oscillated as a function of distance or time, rather than remaining constant. The force built during the 'stick' phase, and energy was released causing the frictional force to decrease in the 'slip' phase. As such, the oscillations were seen to reach a maximum, rapidly decay, and repeat **[9]**. Such a trend can be used to describe the nature of the oscillations in Figure 10. By defining the system as per (34), a much more realistic damping situation was modelled.

3 Summary

Through analysis of the motion of buildings undergoing seismic forces, it was found that the nature of the results was highly dependent on the frequency of the earthquake and the way the damping term had been defined. It was important to consider the natural frequencies and periods at which the buildings oscillated to understand how certain earthquake frequencies would affect the building's motion. The proximity of the earthquake frequency to the natural frequency of the building greatly affected the amplitudes of oscillation for the levels. When the earthquake frequency was close enough to the natural frequency, resonance was seen to occur. The duration of the earthquake effected the magnitude of the displacement as well as the decay time for the system, however the key factor effecting change was the frequency.

Linear damping situations were effectively modelled by considering a damping matrix. It was found that more realistic damping conditions could be represented using non-linear models, as these accounted for mechanisms such as friction. However, to accurately model damping situations, the oscillations of certain floors and their effect on subsequent floors, needed to be appropriately considered.

Ultimately, the project focussed on using an adjusted model of the spring-mass damper system to represent the motion of a multi-storey structure experiencing an earthquake force. By considering different definitions for the damping in the models, and by analysing the relationship between natural frequencies and seismic wave frequencies, earthquake-building interactions were effectively modelled using differential systems.

4 Extensions

There are a few extensions that could be made to this project to allow more realistic situations to be modelled. One of the most significant extensions involves exploring different forcing functions. For the models investigated, a harmonic forcing term was used and only a singular frequency component for the earthquake force was considered. Realistically, however, earthquakes involve multiple seismic waves, each composed of many frequency components. Additionally, the cosine forcing function used offers only a general approximation of the ground motion produced during an earthquake disturbance. To more effectively simulate solutions, it would be prudent to consider delta functions or piecewise continuous functions, as these may be used to account for multiple seismic wave frequencies **[10]**.

The project could additionally be extended to explore more complex structures. By readjusting the definition of the stiffness matrix, structures consisting of multiple adjacent buildings connected with diagonal braces, can be considered. The difficulty of modelling such structures is that the displacement of the floors would need to be simulated relative to multiple point masses, rather than just one. Additionally, irregularity factors may need to be accounted for. For more complex structures, non-symmetry – where the centre of mass of the building is displaced from the building's fixture centre – needs to be considered. The method of equivalent static lateral force analysis fails to account for such situations, and so significant adjustments to the modelling method would need to be made. Finally, it would be interesting to explore the effect of elastic versus inelastic designs. For example, the behaviour of concrete reinforced buildings could be compared to more ductile building structures, such as those with steel reinforcement bars. However, to do so, the spring-mass damper system model would need to be re-evaluated.

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