Gli esercizi presentati in questo documento, esposti negli appunti di Optimization di Simone Bianco, sono tratti da esercizi del libro, e da esercizi che il professore ha presentato in classe durante il corso delle lezioni.

Consider the following linear program:

$$\max\ x_1 + 2x_2 - 3x_3 + 7x_5$$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= 3 \\ x_1 + 2x_2 + 7x_3 + x_5 &= 3 \\ 2x_1 + 4x_2 + 7x_3 + x_6 &= 6 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

Determine and justify which of the following vectors is a basic feasible solution (BFS):

1. $a = [1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0]$
2. $b = [1\ \ 0\ \ 0\ \ 2\ \ 2\ \ 4]$
3. $c = [0\ \ 0\ \ 0\ \ 3\ \ 3\ \ 6]$

La matrice dei vincoli e'

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & 4 & 7 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

Le colonne 4, 5, 6 sono lin. indip. quindi c e' una BFS associata a tale base $\mathcal{B} = \{4, 5, 6\}$

Given a general linear program in equational form

$$\max\ c^T x \qquad Ax = b \qquad x \geq 0$$

let $\overline{x}$ and $\overline{y}$ be two optimal solutions to the problem. Prove or disprove the following statements:

1. Any convex combination of $\overline{x}$ and $\overline{y}$ is also an optimal solution to the linear program
2. There exists an optimal solution $\overline{z}$ on the line through $\overline{x}$ and $\overline{y}$ which is also basic

Una combinazione convessa di due punti $\overline{x}, \overline{y}$ e' il segmento di linea.

$c^T\overline{x} = c^T\overline{y} = \gamma \qquad z = \alpha\overline{x} + (1-\alpha)\overline{y}$

$c^T z = c^T(\alpha x + (1-\alpha)y) = \alpha c^T\overline{x} + (1-\alpha)c^T\overline{y} = \alpha\gamma + (1-\alpha)\gamma = \gamma \Rightarrow z$ e' ottimale.

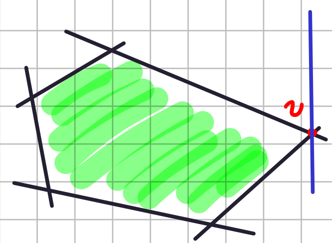
La 2 e' falsa perche' $\overline{x}$ e $\overline{y}$ sono BFS, ossia vertici del poliedro, e ogni punto sul segmento di linea di 2 vertici non e' (per definizione) un vertice.

Consider the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ and let $v \in P$. Prove that v is a vertex of P if and only if there exists a vector c such that v is the unique optimal basic feasible solution to the linear program:

$$\max\ c^T x \qquad x \in P$$

$[\Rightarrow]$: v e' un vertice di $P \Rightarrow \exists\ \mathcal{H} = \{x : c^T x = \alpha\}$ t.c.

$c^T x < \alpha \quad \forall x \in P \setminus \{v\} \quad \wedge \quad c^T v = \alpha$



ma allora v e' sol. ottimale per $\max\limits_{x \in P} c^T x$.

$[\Leftarrow]$: v e' l'unica sol. di $\max\limits_{x \in P} c^T x \Rightarrow$ necessariamente v e' una BFS $\Rightarrow v$ e' un vertice di P.

Solve the following linear program using the Simplex method:

$$\max\ \ 2x_1 + x_2 - x_3$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 8 \\ -x_1 + x_2 - 2x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Il problema in Forma matriciale e'

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & -2 \end{bmatrix} \qquad b = [8\ \ 4]^T$$

$\Rightarrow$ in Forma di eq: $A' = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ -1 & 1 & -2 & 0 & 1 \end{bmatrix}$

La prima base e' $\mathcal{B} = \{4, 5\}$

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 8 \\ -x_1 + x_2 - 2x_3 + x_5 &= 4 \end{aligned} \qquad z = 2x_1 + x_2 - x_3 \qquad \Rightarrow$$

$\mathcal{B} = \{4, 5\}$

$$\begin{aligned} x_4 &= 8 - x_1 - 2x_2 - x_3 \\ x_5 &= 4 + x_1 - x_2 + 2x_3 \\ \hline z &= 2x_1 + x_2 - x_3 \end{aligned} \Rightarrow$$

$\mathcal{B} = \{1, 5\}$

$$\begin{aligned} x_1 &= 8 - x_4 - 2x_2 - x_3 \\ x_5 &= 4 + x_1 - x_2 + 2x_3 \\ \hline z &= 16 - 2x_4 - 3x_2 - 3x_3 \end{aligned}$$

$\Rightarrow$ la sol ott. e' $\begin{cases} x_1 = 8 \\ x_5 = 12 \end{cases} \Rightarrow$ per il problema originale e' $x = [8\ \ 0\ \ 0]^T$

Consider the following linear program

$$\begin{aligned} \min\ & 3x_1 + 2x_2 - x_3 \\ & x_1 - 2x_2 + x_3 - x_4 = 0 \\ & x_1 + x_2 + 2x_3 - x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Beginning with the basis $\mathcal{B} = \{1,2\}$, solve the linear program with the Simplex method

Riscrivo il problema in forma max e parto con il primo tableu

$\mathcal{B} = \{1,2\}$

$$\begin{cases} x_1 - 2x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 + 2x_3 - x_4 = 1 \end{cases} \qquad \begin{cases} x_1 = -\frac{5}{3}x_3 + x_4 + \frac{2}{3} \\ x_2 = \frac{1}{3} - \frac{1}{3}x_3 \end{cases}$$

$\mathcal{B} = \{1,2\}$

$$x_1 = -\frac{5}{3}x_3 + x_4 + \frac{2}{3}$$
$$x_2 = \frac{1}{3} - \frac{1}{3}x_3$$
$$z = -\frac{8}{3} + \frac{20}{3}x_3 + 3x_4$$

$\Rightarrow$

$\mathcal{B} = \{2,4\}$

$$x_4 = -x_1 - \frac{5}{3}x_3 + \frac{2}{3}$$
$$x_2 = \frac{1}{3} - \frac{1}{3}x_3$$
$$z = -\frac{2}{3} + \frac{20}{3}x_3 - 3x_1 - 5x_3$$

$\Rightarrow$

$\mathcal{B} = \{3,4\}$

$$x_4 = -x_1 - \frac{5}{3} + 5x_2 + \frac{2}{3}$$
$$x_3 = 1 - 3x_2$$
$$z = 1 - 17x_2 - 3x_1$$

Finito, il massimo e' 1, il minimo del problema originale e' -1,

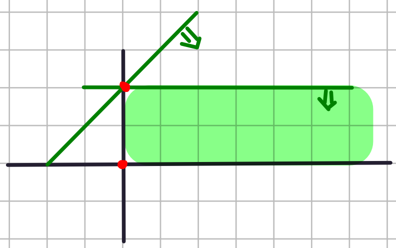
Give an explicit example of a linear program in the form of:

$$\begin{aligned} & \max c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

such that the problem is unbounded, it has a degenerate BFS and infinitely many optimal solutions

Si consideri il problema

$$\begin{aligned} & \max\ -x_1 \\ \text{s.t. } & x_2 - x_1 \leq 2 \\ & x_2 \leq 2 \end{aligned}$$



e' illimitato, le due BFS $[0\ 0\ 0\ 0]$ e $[0\ 2\ 0\ 0]$ sono ottimali

in forma di eq: $x_2 - x_1 + x_3 = 2$
$x_2 + x_4 = 2$

Si consideri $\mathcal{B} = \{1,2\} \Rightarrow \begin{cases} x_1 = -x_4 + x_3 - 2 + 2 \\ x_2 = -x_4 + 2 \end{cases} \Rightarrow$ Sol: $[0\ 2\ 0\ 0]^T \Rightarrow$ Una variabile di base e' $0 \Rightarrow [0\ 2\ 0\ 0]^T$ e' degenere.

Solve the following linear program using the Simplex method:

$$\begin{aligned} \min\ & 12x_1 + 3x_2 + 4x_3 \\ & 4x_1 + 2x_2 + 3x_3 \geq 2 \\ & 8x_1 + x_2 + 2x_3 \geq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Trasformo il problema in forma canonica, e poi in forma di equazione.

$$\begin{aligned} \max\ & -12x_1 - 3x_2 - 4x_3 \\ & -4x_1 - 2x_2 - 3x_3 \leq -2 \\ & -8x_1 - x_2 - 2x_3 \leq -3 \end{aligned} \Rightarrow \begin{aligned} \max\ & -12x_1 - 3x_2 - 4x_3 \\ & -4x_1 - 2x_2 - 3x_3 + x_4 = -2 \\ & -8x_1 - x_2 - 2x_3 + x_5 = -3 \end{aligned}$$

la base $\mathcal{B} = \{4,5\}$ ha BFS associata $[0\ 0\ 0\ -2\ -3]^T \Rightarrow$ non e' ammissibile.

$\mathcal{B} = \{1,2\}$

$$x_1 = -\frac{7}{20}x_3 + \frac{1}{20}x_4 + \frac{1}{10}x_5 + \frac{2}{5}$$
$$x_2 = -\frac{4}{5}x_3 + \frac{2}{5}x_4 - \frac{1}{5}x_5 + \frac{1}{5}$$
$$z = \frac{13}{5}x_3 - \frac{9}{5}x_4 - \frac{3}{5}x_5 - \frac{27}{5}$$

$3 \leftrightarrow 2$ $\Rightarrow$

$\mathcal{B} = \{1,3\}$

$$x_1 = \frac{35}{80}x_2 + \frac{3}{16}x_3 - \frac{1}{8}x_4 + \frac{5}{16}$$
$$x_3 = -\frac{5}{4}x_2 + \frac{1}{2}x_4 - \frac{1}{4}x_5 + \frac{1}{4}$$
$$z = -\frac{13}{4}x_2 - \frac{1}{2}x_4 - \frac{25}{20}x_5 - \frac{19}{4}$$

$\Rightarrow$ Sol: $x^* = \left[\frac{5}{16}, 0, \frac{1}{4}\right]$

opt: $\frac{19}{4}$

per l'LP originale

Consider the standard form polyhedron $P = \{x \mid Ax = b, x \geq 0\}$. Suppose that the matrix A of dimensions $m \times n$ has linearly independent rows and that all basic feasible solutions are non-degenerate.

Given $\overline{x} \in P$ such that $\overline{x}$ has exactly m positive components:

1. Show that $\overline{x}$ is a BFS
2. Show that the previous point is false if we remove the non-degeneracy assumption

Se $\bar{x}$ ha m componenti positive, e non ci sono soluzioni degenere, allora $B=\{i : \bar{x}_i > 0\}$ e' la base di $\bar{x}$, ed e' una BFS solo se A_B, che ha m colonne, e' non singolare, essendo che il rango delle righe di A e' m, lo e' anche quello di $A_B \Rightarrow$ le colonne di A_B sono lin. ind. $\Rightarrow A_B$ e' non singolare.

Se l'assunzione sulle sol. non degenere e' Falsa, non si puo' affermare che $B=\{i : \bar{x}_i > 0\}$ sia la base di $\bar{x}$.

Consider the following primal program:

$$\begin{aligned} \min\ & c^T x \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

1. Form the dual program and state it as an equivalent minimization problem
2. Derive the conditions on A, b and c such that the dual is identical to the primal, meaning that the problem is *self-dual*
3. Give a concrete example of a primal program identical to its dual

Il duale e' $\begin{cases} \max\ b^T y \\ A^T y \leq c \\ y \leq 0 \end{cases} \equiv \begin{cases} \min\ -b^T y \\ A^T y \leq c \\ y \leq 0 \end{cases}$, e' self-dual se

$\{x : Ax \geq b\} = \{x : A^T x \leq c\} \Rightarrow$

$\{x : Ax \geq b\} = \{x : -A^T x \geq -c\} \Longrightarrow \begin{cases} b = -c \\ A = -A^T \end{cases}$

Un'esempio e':

$$\begin{aligned} \max\ & [1\ -1]\, x \\ \text{s.t.}\ & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x \leq \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ & x \geq 0 \end{aligned}$$

Consider the following linear program:

$$\begin{aligned} \max\ & 2x_1 + 16x_2 + 12x_3 \\ & 2x_1 + x_2 - x_3 \leq 3 \\ & -3x_1 + 8x_2 + 2x_3 \leq 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Determine if the feasible solution $x^* = [6 \ \ 0 \ \ 12]$ is also optimal.

Considero il problema duale:

$$\begin{cases} \min\ 3y_1 + 12y_2 \\ 2y_1 - 3y_2 \geq 2 \\ y_1 + 8y_2 \geq 16 \\ -y_1 + 2y_2 \geq 12 \\ y_1, y_2 \geq 0 \end{cases}$$

Essendo $x_1^*, x_3^* > 0 \Rightarrow$

$$\begin{cases} 2y_1 - 3y_2 = 2 \\ y_1 + 8y_2 \geq 16 \\ -y_1 + 2y_2 = 12 \end{cases} \Rightarrow \begin{cases} 2y_1 = 2 + 3y_2 \\ 2y_2 = 12 + y_1 \end{cases} \Rightarrow \begin{cases} y_1 = \frac{3}{2}y_2 + 1 \\ 2y_2 = 12 + \frac{3}{2}y_2 + 1 \end{cases} \Rightarrow \begin{cases} y_1 = 40 \\ y_2 = 26 \end{cases} \Rightarrow y^* = [40 \ \ 26]^T$$

y^* dovrebbe essere la sol ottimale per il duale, verifichiamo

$2x_1^* + 16x_2^* + 12x_3^* = 12 + 144 = 156$

$3y_1^* + 12y_2^* = 3 \cdot 40 + 12 \cdot 26 = 432$, gli ottimi non coincidono $\Rightarrow x^*$ non e' ottimale.

Consider the following linear program:

$$\max\ 2x_1 + \alpha x_2 + \beta x_3$$

$$\begin{aligned} 2x_1 + x_2 - x_3 &\le 3 \\ -3x_1 + 8x_2 + 2x_3 &\le 12 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Find all the values α, β such that $x^* = \begin{bmatrix} 0 & 0 & 6 \end{bmatrix}$ is an optimal solution.

L'ottimo del duale deve coincidere con l'ottimo del primario. Il duale e':

$$\begin{cases} \min\ 3y_1 + 12y_2 \\ 2y_1 - 3y_2 \ge 2 \\ y_1 + 8y_2 \ge \alpha \\ -y_1 + 2y_2 \ge \beta \\ y_1, y_2 \ge 0 \end{cases}$$

essendo $x_3^* > 0$
$\Rightarrow -y_1 + 2y_2 = \beta.$ $\Rightarrow y^* = \begin{bmatrix} 0 & \frac{\beta}{2} \end{bmatrix}$

Inoltre $2x_1^* + x_2^* - x_3^* < 3$
$\Rightarrow y_1^* = 0$

Voglio che $\begin{cases} y_1^* + 8y_2^* \ge \alpha \\ 2y_1^* - 3y_2^* \ge 2 \end{cases}$ $\begin{cases} 4\beta \ge \alpha \\ -\frac{3}{2}\beta \ge 2 \end{cases}$ $\begin{cases} \beta \ge \frac{\alpha}{4} \\ \beta \ge -\frac{4}{3} \Leftarrow \text{banale perche' } \beta \ge 0 \end{cases}$

Voglio che gli ottimi siano identici:

$$2x_1^* + \alpha x_2^* + \beta x_3^* = 3y_1^* + 12y_2^* \Rightarrow 6\beta = 12\frac{\beta}{2} \Rightarrow \beta = 1,\ x^* \text{ e' sol per } \beta = 1,\ \alpha \le 4$$

Consider the following linear program:

$$\max\ 5x_1 + \alpha x_2 + 8x_3$$

$$\begin{aligned} x_1 + x_2 + 9x_3 &\le 5 \\ 3x_1 + 4x_2 + 13x_3 &\le 12 \\ 5x_1 + x_2 + 17x_3 &\le \beta \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Find all the values of α, β such that $x^* = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$ is an optimal solution.

Il problema duale e':

$$\begin{cases} \min\ 5y_1 + 12y_2 + \beta y_3 \\ y_1 + y_2 + 5y_3 \ge 5 \\ y_1 + 4y_2 + y_3 \ge \alpha \\ 9y_1 + 13y_2 + 17y_3 \ge 8 \\ y_1, y_2, y_3 \ge 0 \end{cases}$$

Sia y^* la sol. ottimale per il duale, se x^* fosse ottimale, dato che $x_1^*, x_2^* > 0$, si avrebbe:

$$\begin{cases} y_1^* + y_2^* + 5y_3^* = 5 \\ y_1^* + 4y_2^* + y_3^* = \alpha \end{cases}$$

inoltre, essendo che $x_1^* + x_2^* + 9x_3^* < 5$ e $3x_1^* + 4x_2^* + 13x_3^* < 12$ $\Rightarrow y_1^*, y_2^* = 0$

$$\Rightarrow \begin{cases} y_1^* + y_2^* + 5y_3^* = 5 \\ y_1^* + 4y_2^* + y_3^* = \alpha \\ 9y_1^* + 13y_2^* + 17y_3^* \ge 8 \end{cases} \Rightarrow \begin{cases} 5y_3^* = 5 \\ y_3^* = \alpha \\ 17y_3^* \ge 8 \end{cases} \Rightarrow \begin{array}{l} y^* = \begin{bmatrix} 0 & 0 & \alpha \end{bmatrix} \\ \alpha = 1 \end{array}$$

allora si vuole che

$$5y_1^* + 12y_2^* + \beta y_3^* = 5x_1^* + \alpha x_2^* + 8x_3^* \Rightarrow$$

$\beta\alpha = 5 + 2\alpha$ ma $\alpha = 1 \Rightarrow$

$\beta = 5 + 2 \Rightarrow (\alpha, \beta) = (1, 7)$

Consider the following linear program:

$$\begin{aligned} \min\ 2x_1 + 4x_2 + 2x_3 & \\ x_1 + x_2 + 3x_3 &\le 1 \\ -x_1 + 2x_2 + x_3 &\ge 1 \\ 3x_2 - 6x_3 &= 0 \\ x_1, x_3 &\ge 0 \\ x_2 &\in \mathbb{R} \end{aligned}$$

Determine if $x^* = \begin{bmatrix} 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$ is an optimal solution.

trasformo il terzo vincolo in due vincoli e scrivo il duale:

$$(P): \begin{cases} \min\ 2x_1 + 4x_2 + 2x_3 \\ x_1 + x_2 + 3x_3 \le 1 \\ -x_1 + 2x_2 + x_3 \ge 1 \\ 3x_2 - 6x_3 \ge 0 \\ 3x_2 - 6x_3 \le 0 \\ x_1, x_3 \ge 0,\ x_2 \in \mathbb{R} \end{cases} \qquad (D): \begin{cases} \max\ y_1 + y_2 \\ y_1 - y_2 \ge 2 \\ y_1 + 2y_2 + 3y_3 + 3y_4 = 4 \\ 3y_1 + y_2 - 6y_3 - 6y_4 \ge 2 \\ y_1, y_4 \ge 0 \\ y_2, y_3 \le 0 \end{cases}$$

Sia y^* la sol. ott. di (D), essendo $x_3^* > 0 \Rightarrow$

$$3y_1^* + y_2^* - 6y_3^* - 6y_4^* = 2$$

Essendo i vincoli di (P) binding per x^*, non si puo' costruire un sistema di K eq. in K incognite. Risolvo (D) col simplesso.

$$\begin{cases} \max\ y_1 - y_2 \\ y_1 + y_2 + y_5 = 2 \\ y_1 - 2y_2 - 3y_3 + 3y_4 = 4 \\ 3y_1 - y_2 + 6y_3 - 6y_4 = 2 \\ y \geq 0 \end{cases}$$

Serve una base ammissibile, provo con $\mathcal{B} = \{1,2,3\}$

$A_\mathcal{B}$ e´ non singolare.

$\mathcal{B} = \{1,2,3\}$

$$\begin{array}{|l} y_1 = -\frac{6}{5} y_4 + \frac{13}{30} y_5 + 2 \\ y_2 = \frac{6}{5} y_4 - \frac{1}{10} y_5 \\ y_3 = -\frac{1}{5} y_4 - \frac{7}{30} y_5 - \frac{2}{3} \\ \hline z = -\frac{12}{5} y_4 + \frac{8}{15} y_5 + 2 \end{array} \Rightarrow$$

$\mathcal{B} = \{1,3,5\}$

$$\begin{array}{|l} y_1 = -\frac{6}{5} y_4 + \frac{13}{30} y_5 + 2 \\ y_5 = -10 y_2 + 12 y_4 \\ y_3 = -\frac{1}{5} y_4 - \frac{7}{30} y_5 - \frac{2}{3} \\ \hline z = 4 y_4 - \frac{16}{3} y_2 + 2 \end{array} \Rightarrow$$

per far aumentare z serve $\mathcal{B} = \{3,4,5\}$ ma le colonne 3,4 di A sono lin. dip.

$y^* = [2 \ 0 \ -\frac{2}{3} \ 0 \ 0] \Rightarrow$

l'ottimo di (D) e´ uguale all'ottimo di (P)

$\Rightarrow y_1^* - y_2^* = 2 = 2x_1^* + 4x_2^* + 2x_3^* = 4\frac{2}{5} + 2\frac{1}{5} = 2 \Rightarrow x^*$ e´ ottimale.

The following tableau corresponds to an optimal basis for a linear program where x_1, x_2 and x_3 are the original primal variables and s_1, s_2 are the slack variables corresponding to the two constraints.

$$\begin{array}{rcl} x_1 & = & \frac{5}{2} + \frac{1}{2}x_2 + \frac{1}{6}s_1 - \frac{1}{3}s_2 \\ x_3 & = & \frac{5}{2} - \frac{1}{4}x_2 - \frac{1}{2}s_1 \\ \hline z & = & \frac{65}{2} - 2x_2 - 3s_1 - 2s_2 \end{array}$$

1. Write the original linear program
2. Write the dual of the original program
3. Solve the dual problem via Simplex method