

Gli esercizi presentati in questo documento, esposti negli appunti di Optimization di Simone Bianco, sono tratti da esercizi del libro, e da esercizi che il professore ha presentato in classe durante il corso delle lezioni.

Consider the following linear program:

$$\begin{aligned} \max \quad & x_1 + 2x_2 - 3x_3 + 7x_5 \\ \text{s.t.} \quad & x_1 + 2x_2 + 2x_3 + x_4 = 3 \\ & x_1 + 2x_2 + 7x_3 + x_5 = 3 \\ & 2x_1 + 4x_2 + 7x_3 + x_6 = 6 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Determine and justify which of the following vectors is a basic feasible solution (BFS):

1. $a = [1 \ 1 \ 0 \ 0 \ 0 \ 0]$

2. $b = [1 \ 0 \ 0 \ 2 \ 2 \ 4]$

3. $c = [0 \ 0 \ 0 \ 3 \ 3 \ 6]$

La matrice dei vincoli e'

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & 4 & 7 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{Le colonne 4, 5, 6 sono lin. indep. quindi } C \text{ e' una BFS associata a tale base } B = \{4, 5, 6\}$$

Given a general linear program in equational form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

let \bar{x} and \bar{y} be two optimal solutions to the problem. Prove or disprove the following statements:

1. Any convex combination of \bar{x} and \bar{y} is also an optimal solution to the linear program

2. There exists an optimal solution \bar{z} on the line through \bar{x} and \bar{y} which is also basic

Una combinazione convessa di due punti \bar{x}, \bar{y} e' il segmento di linea.

$$c^T \bar{z} = c^T \bar{y} = \gamma \quad z = \alpha \bar{x} + (1-\alpha) \bar{y}$$
$$c^T z = c^T (\alpha \bar{x} + (1-\alpha) \bar{y}) = \alpha c^T \bar{x} + (1-\alpha) c^T \bar{y} = \alpha \gamma + (1-\alpha) \gamma = \gamma \Rightarrow z \text{ e' ottimale.}$$

La 2 e' falsa perche' \bar{x} e \bar{y} sono BFS, ossia vertici del poliedro, e ogni punto sul segmento di linea di 2 vertici non e' (per definizione) un vertice.

Consider the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ and let $v \in P$. Prove that v is a vertex of P if and only if there exists a vector c such that v is the unique optimal basic feasible solution to the linear program:

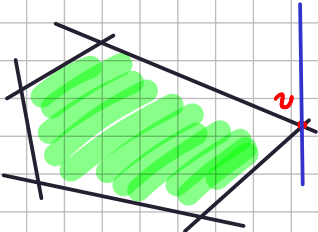
$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & x \in P \end{aligned}$$

$$[\Rightarrow]: v \text{ e' un vertice di } P \Rightarrow \exists \mathcal{H} = \{x : c^T x = \alpha\} \text{ t.c.}$$

$$c^T x < \alpha \quad \forall x \in P \setminus \{v\} \quad \wedge \quad c^T v = \alpha$$

ma allora v e' sol. ottimale per $\max_{x \in P} c^T x$.

$$[\Leftarrow]: v \text{ e' l'unica sol. di } \max_{x \in P} c^T x \Rightarrow \text{necessariamente } v \text{ e' una BFS}$$
$$\Rightarrow v \text{ e' un vertice di } P.$$



Solve the following linear program using the Simplex method:

$$\begin{aligned} \max \quad & 2x_1 + x_2 - x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

La prima base e' $B = \{4, 5\}$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 8 \\ -x_1 + x_2 - 2x_3 &= 4 \end{aligned} \Rightarrow \begin{aligned} x_4 &= 8 - x_1 - 2x_2 - x_3 \\ x_5 &= 4 + x_1 - x_2 + 2x_3 \end{aligned} \Rightarrow$$
$$z = 2x_1 + x_2 - x_3$$

$$B = \{1, 5\}$$
$$\begin{aligned} x_1 &= 8 - x_4 - 2x_2 - x_3 \\ x_5 &= 4 + x_1 - x_2 + 2x_3 \end{aligned} \Rightarrow \text{12 sol ott. e'}$$
$$z = 16 - 2x_4 - 3x_2 - 3x_3$$

$\begin{cases} x_1 = 8 \\ x_5 = 12 \end{cases} \Rightarrow \text{per il problema originale e' } x = [8 \ 0 \ 0]^T$

Consider the following linear program

$$\begin{aligned} \min \quad & 3x_1 + 2x_2 - x_3 \\ & x_1 - 2x_2 + x_3 - x_4 = 0 \\ & x_1 + x_2 + 2x_3 - x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Beginning with the basis $B = \{1, 2\}$, solve the linear program with the Simplex method

$$B = \{1, 2\} \quad \begin{cases} x_1 - 2x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 + 2x_3 - x_4 = 1 \end{cases} \quad \begin{cases} x_1 = -\frac{5}{3}x_3 + x_4 + \frac{2}{3} \\ x_2 = \frac{1}{3} - \frac{1}{3}x_3 \end{cases}$$

$$B = \{1, 2\}$$

$$\begin{aligned} x_1 &= -\frac{5}{3}x_3 + x_4 + \frac{2}{3} \\ x_2 &= \frac{1}{3} - \frac{1}{3}x_3 \end{aligned}$$

$$z = -\frac{8}{3} + \frac{20}{3}x_3 + 3x_4$$

$$B = \{2, 4\}$$

$$\begin{aligned} x_4 &= -x_1 - \frac{5}{3}x_3 + \frac{2}{3} \\ x_2 &= \frac{1}{3} - \frac{1}{3}x_3 \end{aligned}$$

$$z = -\frac{2}{3} + \frac{20}{3}x_3 - 3x_1 - 5x_3$$

$$B = \{3, 4\}$$

$$\begin{aligned} x_4 &= -x_1 - \frac{5}{3} + 5x_2 + \frac{2}{3} \\ x_3 &= 1 - 3x_2 \end{aligned}$$

$$z = 1 - 17x_2 - 3x_1$$

Finito, il massimo e' 1, il minimo del problema originale e' -1,

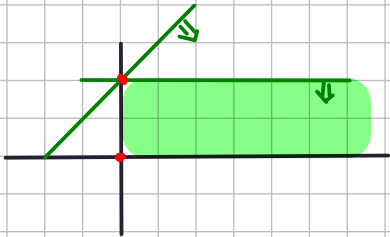
Give an explicit example of a linear program in the form of:

$$\begin{aligned} \max \quad & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

such that the problem is unbounded, it has a degenerate BFS and infinitely many optimal solutions

Si consideri il problema

$$\begin{aligned} \max \quad & -x_1 \\ \text{s.t.} \quad & x_2 - x_1 \leq 2 \\ & x_2 \leq 2 \end{aligned}$$



e' illimitato, le due BFS $[0 \ 0 \ 0 \ 0]$ e $[0 \ 2 \ 0 \ 0]$ sono ottimali;

in forma di eq:
$$\begin{aligned} x_2 - x_1 + x_3 &= 2 \\ x_2 + x_4 &= 2 \end{aligned}$$

Si consideri $B = \{1, 2\} \Rightarrow \begin{cases} x_1 = -x_4 + x_3 - 2 + 2 \\ x_2 = -x_4 + 2 \end{cases} \Rightarrow \text{Sol: } [0 \ 2 \ 0 \ 0]^T \Rightarrow \text{Una variabile di base e' } 0 \Rightarrow [0 \ 2 \ 0 \ 0]^T \text{ e' degenera.}$

Solve the following linear program using the Simplex method:

$$\begin{aligned} \min \quad & 12x_1 + 3x_2 + 4x_3 \\ & 4x_1 + 2x_2 + 3x_3 \geq 2 \\ & 8x_1 + x_2 + 2x_3 \geq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Trasformo il problema in forma canonica, e poi in forma di equazione.

$$\max \quad -12x_1 - 3x_2 - 4x_3$$

$$\max \quad -12x_1 - 3x_2 - 4x_3$$

la base $B = \{4, 5\}$ ha BFS

$$-4x_1 - 2x_2 - 3x_3 \leq -2$$

$$\Rightarrow -4x_1 - 2x_2 - 3x_3 + x_4 = -2$$

associata $[0 \ 0 \ 0 \ -2 \ -3]^T \Rightarrow$ non e'

$$-8x_1 - x_2 - 2x_3 \leq -3$$

$$-8x_1 - x_2 - 2x_3 + x_5 = -3$$

ammissibile.

$$B = \{1, 2\}$$

$$B = \{1, 3\}$$

$$\begin{aligned} x_1 &= -\frac{3}{20}x_3 + \frac{1}{20}x_4 + \frac{1}{10}x_5 + \frac{2}{5} \\ x_2 &= -\frac{4}{5}x_3 + \frac{2}{5}x_4 - \frac{1}{5}x_5 + \frac{1}{5} \\ z &= \frac{13}{5}x_3 - \frac{9}{5}x_4 - \frac{3}{5}x_5 - \frac{27}{5} \end{aligned}$$

$$\begin{aligned} x_1 &= \frac{35}{80}x_2 + \frac{3}{16}x_3 - \frac{1}{8}x_4 + \frac{5}{16} \\ x_3 &= -\frac{5}{4}x_2 + \frac{1}{2}x_4 - \frac{1}{4}x_5 + \frac{1}{4} \\ z &= -\frac{13}{4}x_2 - \frac{1}{2}x_4 - \frac{25}{20}x_5 - \frac{19}{4} \end{aligned}$$

Sol: $x^* = [\frac{5}{16}, 0, \frac{1}{4}]$
 \Rightarrow opt: $\frac{19}{4}$
 \uparrow
per l'LP originale

Consider the standard form polyhedron $P = \{x \mid Ax = b, x \geq 0\}$. Suppose that the matrix A of dimensions $m \times n$ has linearly independent rows and that all basic feasible solutions are non-degenerate.

Given $\bar{x} \in P$ such that \bar{x} has exactly m positive components:

1. Show that \bar{x} is a BFS
2. Show that the previous point is false if we remove the non-degeneracy assumption

Se \bar{x} ha m componenti positive, e non ci sono soluzioni degenerate, allora $B = \{i : \bar{x}_i > 0\}$ e' la base di \bar{x} , ed e' una BFS solo se A_B , che ha m colonne, e' non singolare, essendo che il rango delle righe di A e' m , lo e' anche quello di $A_B \Rightarrow$ le colonne di A_B sono lin. ind. $\Rightarrow A_B$ e' non singolare.

Se l'assunzione sulle sol. non degenerate e' falsa, non si puo' affermare che $B = \{i : \bar{x}_i > 0\}$ sia la base di \bar{x} .

Consider the following primal program:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

1. Form the dual program and state it as an equivalent minimization problem
2. Derive the conditions on A, b and c such that the dual is identical to the primal, meaning that the problem is *self-dual*
3. Give a concrete example of a primal program identical to its dual

Il duale e' $\begin{cases} \max & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \leq 0 \end{cases} \equiv \begin{cases} \min & -b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \leq 0 \end{cases}$, e' self-dual se

$$\begin{aligned} \{x : Ax \geq b\} &= \{x : A^T x \leq c\} \Rightarrow \begin{cases} b = -c \\ A = \bar{A}^T \end{cases} \\ \{x : Ax \geq b\} &= \{x : -A^T x \geq -c\} \implies \begin{cases} b = -c \\ A = \bar{A}^T \end{cases} \end{aligned}$$

Un'esempio e': $\max [1 \ -1] x$
 $\text{s.t.} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x \leq \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
 $x \geq 0$

Consider the following linear program:

$$\begin{aligned} \max \quad & 2x_1 + 16x_2 + 12x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 - x_3 \leq 3 \\ & -3x_1 + 8x_2 + 2x_3 \leq 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Considero il problema duale:

$$\begin{cases} \min & 3\gamma_1 + 12\gamma_2 \\ \text{s.t.} & 2\gamma_1 - 3\gamma_2 \geq 2 \\ & \gamma_1 + 8\gamma_2 \geq 16 \\ & -\gamma_1 + 2\gamma_2 \geq 12 \\ & \gamma_1, \gamma_2 \geq 0 \end{cases}$$

Determine if the feasible solution $x^* = [6 \ 0 \ 12]$ is also optimal.

Essendo $x_1^*, x_3^* > 0 \Rightarrow$

$$\begin{cases} 2\gamma_1 - 3\gamma_2 = 2 \\ \gamma_1 + 8\gamma_2 \geq 16 \\ -\gamma_1 + 2\gamma_2 = 12 \end{cases} \Rightarrow \begin{cases} 2\gamma_1 = 2 + 3\gamma_2 \\ 2\gamma_2 = 12 + \gamma_1 \end{cases} \Rightarrow \begin{cases} \gamma_1 = \frac{3}{2}\gamma_2 + 1 \\ 2\gamma_2 = 12 + \frac{3}{2}\gamma_2 + 1 \end{cases} \Rightarrow \begin{cases} \gamma_1 = 40 \\ \gamma_2 = 26 \end{cases} \Rightarrow \gamma^* = [40 \ 26]^T$$

γ^* dovrebbe essere la sol ottimale per il duale, verifichiamo

$$2x_1^* + 16x_2^* + 12x_3^* = 12 + 144 = 156$$

$$3\gamma_1 + 12\gamma_2 = 3 \cdot 40 + 12 \cdot 26 = 432, \text{ gli ottimi non coincidono } \Rightarrow x^* \text{ non e' ottimale.}$$

Consider the following linear program:

$$\max 2x_1 + \alpha x_2 + \beta x_3$$

$$2x_1 + x_2 - x_3 \leq 3$$

$$-3x_1 + 8x_2 + 2x_3 \leq 12$$

$$x_1, x_2, x_3 \geq 0$$

L'ottimo del duale deve coincidere con
l'ottimo del primario. Il duale e':

Find all the values α, β such that $x^* = \begin{bmatrix} 0 & 0 & 6 \end{bmatrix}$ is an optimal solution.

$$\begin{cases} \min 3\gamma_1 + 12\gamma_2 \\ 2\gamma_1 - 3\gamma_2 \geq 2 \\ \gamma_1 + 8\gamma_2 \leq \alpha \\ -\gamma_1 + 2\gamma_2 \geq \beta \\ \gamma_1, \gamma_2 \geq 0 \end{cases} \quad \begin{aligned} &\text{essendo } x_3^* > 0 \\ &\Rightarrow -\gamma_1 + 2\gamma_2 = \beta \\ &\Rightarrow \gamma^* = \begin{bmatrix} 0 & \frac{\beta}{2} \end{bmatrix} \\ &\text{Inoltre } 2x_1^* + x_2^* - x_3^* < 3 \\ &\Rightarrow \gamma_1^* = 0 \end{aligned}$$

Voglio che $\begin{cases} \gamma_1^* + 8\gamma_2^* \leq \alpha \\ 2\gamma_1^* - 3\gamma_2^* \geq 2 \end{cases} \Rightarrow \begin{cases} 4\beta \geq \alpha \\ -\frac{3}{2}\beta \geq 2 \end{cases} \Rightarrow \begin{cases} \beta \geq \frac{\alpha}{4} \\ \beta \geq -\frac{4}{3} \end{cases} \Leftarrow \text{banale perche } \beta \geq 0$

Voglio che gli ottimi siano identici:

$$2x_1^* + \alpha x_2^* + \beta x_3^* = 3\gamma_1^* + 12\gamma_2^* \Rightarrow 6\beta = 12\frac{\beta}{2} \Rightarrow \beta = 1, \quad x^* \text{ e' sol per } \beta = 1, \alpha \leq 4$$