Exercise 1: consider the following linear program in standard equation form:

$$\max x_1 + 2x_2 - 3x_3 + 7x_5$$
$$x_1 + 2x_2 + 2x_3 + x_4 = 3$$
$$x_1 + 2x_2 + 7x_3 + x_5 = 3$$
$$2x_1 + 4x_2 + 7x_3 + x_6 = 6$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

a) For each of the following values of $x^T = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ say whether or not this value is a basic feasible solution of this linear program. Justify your answers.

i. $x^T = (1, 1, 0, 0, 0, 0)$
ii. $x^T = (1, 0, 0, 2, 2, 4)$
iii. $x^T = (0, 0, 0, 3, 3, 6)$

b) Solve the linear program via the simplex method

(ii) non e' una BFS perche' ha piu' di 3 componenti non nulle, (iii) e' una BFS perche' $A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ e' non singolare. (i) non puo' essere BFS perche' le colonne 1,2 di A sono linearmente dipendenti.

Risolvo col simplesso:

$$x_4 = -x_1 - 2x_2 - x_4 + 3$$
$$x_5 = -x_1 - 2x_2 - 7x_3 + 3$$
$$x_6 = -2x_1 - 4x_2 - 7x_3 + 6$$

$$z = x_1 + 2x_2 - 3x_3 + 7(-x_1 - 2x_2 - 7x_3 + 3)$$

$(0\ 0\ 0\ 3\ 3\ 6)^T$ e' ottimale.

$\Rightarrow$ l'ottimo e' 21.

Exercise 2: consider the polyhedron P equal to the convex hull of points $\{a_1, \dots, a_n\}$. Prove that if v is a vertex of P then $v = a_i$ for some index i.

Lets assume that v is not equal to a_i for some i, since the convex hull is the union of all convex combinations, v is a convex combination of $a_1, \dots, a_n$

$$v = \sum_{i=1}^{n} \beta_i a_i, \quad \sum_{i=1}^{n} \beta_i = 1, \quad \beta_i \geq 0\ \forall i$$

v is a vertex, so $\exists c, \alpha$ such that

$$c^T v = \alpha$$
$$c^T a_i < \alpha, \forall i$$

we can rewrite $c^T v$ as follows

$$c^T v = c^T \sum_{i=1}^{n} \beta_i a_i = \sum_{i=1}^{n} \beta_i c^T a_i$$

since $\sum \beta_i = 1$, we can rewrite α as follows

$$\alpha = \sum_{i=1}^{n} \beta_i \alpha$$

Since for each i, $c^T a_i < \alpha$, we get

$$\sum_{i=1}^{n} \beta_i c^T a_i < \sum_{i=1}^{n} \beta_i \alpha = \alpha$$

but $\alpha = c^T v$

$$\sum_{i=1}^{n} \beta_i c^T a_i < \sum_{i=1}^{n} \beta_i \alpha = c^T v$$

and $c^T v = \sum_{i=1}^{n} \beta_i c^T a_i$

$$c^T v = \sum_{i=1}^{n} \beta_i c^T a_i < \sum_{i=1}^{n} \beta_i \alpha = c^T v$$

we get

$$c^T v < c^T v$$

is a contraddiction, so it is impossible that v is a convex combination of the $a_i \Rightarrow v = a_i$ for some i. ■

Exercise 3: prove or disprove the following:

i. For a given linear program, if there exist two distinct bases B_1, B_2 such that the associated basic feasible solution x_1 and x_2 are optimal, then there exist infinitely many distinct optimal solutions.

ii. If x is an optimal solution to a given linear program in standard equality form, no more than m of its components can be positive, where m is the number of equality constraints.

i) FALSO, e' vero che se esistono 2 soluzioni ottimali, allora ne esistono infinite, non e' pero' detto che BFS associate a diverse basi siano diverse, dato che possono essere degeneri.

ii) Se esistono 2 BFS ottimali non degeneri, allora un qualsiasi punto sul segmento di linea e' ottimale e ha $K > m$ componenti maggiori di 0.

Exercise 4: consider the linear program

$$\begin{aligned} \min\ & 47x_1 + 93x_2 + 17x_3 - 93x_4 \\ & -x_1 - 6x_2 + x_3 + 3x_4 \leq -3 \\ & -x_1 - 2x_2 + 7x_3 + x_4 \leq 5 \\ & 3x_2 - 10x_3 - x_4 \leq -8 \\ & -6x_1 - 11x_2 - 2x_3 + 12x_4 \leq -7 \\ & x_1 + 6x_2 - x_3 - 3x_4 \leq 4 \end{aligned}$$

Prove that $(1, 1, 1, 1)$ is not an optimal solution.

Si assume che x^* e' ottimale, il duale del LP e'

$$\begin{cases} \min\ -3y_1 + 5y_2 - 8y_3 - 7y_4 + 4y_5 \\ -y_1 - y_2 - 6y_4 + y_5 = 47 \\ -6y_1 - 2y_2 + 3y_3 - 11y_4 + 6y_5 = 93 \\ y_1 + 7y_2 - 10y_3 - 2y_4 - y_5 = 17 \\ 3y_1 + y_2 - y_3 + 12y_4 + 3y_5 = -93 \\ y \geq 0 \end{cases}$$

il vincolo 5 e' slack per x^*
$\Rightarrow y$ e' ottim. se $y_5 = 0$

$$\begin{cases} -y_1 - y_2 - 6y_4 = 47 \\ -6y_1 - 2y_2 + 3y_3 - 11y_4 = 93 \\ y_1 + 7y_2 - 10y_3 - 2y_4 = 17 \\ 3y_1 + y_2 - y_3 + 12y_4 = -93 \end{cases}$$

e' un sistema di 4 equazioni in 4 incognite, ha quindi soluzione $\Rightarrow y^* = [-3 \ \ -2 \ \ -2 \ \ -7 \ \ 0]^T$

ma y^* ha componenti negative quindi non e' soluzione $\Rightarrow x^*$ non e' ottimale.