Gli esercizi presentati in questo documento, esposti negli appunti di Optimization di Simone Bianco, sono tratti da esercizi del libro, e da esercizi che il professore ha presentato in classe durante il corso delle lezioni.

Consider the following linear program:

$$\max\ x_1 + 2x_2 - 3x_3 + 7x_5$$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= 3 \\ x_1 + 2x_2 + 7x_3 + x_5 &= 3 \\ 2x_1 + 4x_2 + 7x_3 + x_6 &= 6 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

Determine and justify which of the following vectors is a basic feasible solution (BFS):

1. $a = [1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0]$
2. $b = [1\ \ 0\ \ 0\ \ 2\ \ 2\ \ 4]$
3. $c = [0\ \ 0\ \ 0\ \ 3\ \ 3\ \ 6]$

La matrice dei vincoli e'

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & 4 & 7 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

Le colonne 4, 5, 6 sono lin. indip. quindi c e' una BFS associata a tale base $\mathcal{B} = \{4, 5, 6\}$

Given a general linear program in equational form

$$\begin{aligned} \max\ & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

let $\overline{x}$ and $\overline{y}$ be two optimal solutions to the problem. Prove or disprove the following statements:

1. Any convex combination of $\overline{x}$ and $\overline{y}$ is also an optimal solution to the linear program
2. There exists an optimal solution $\overline{z}$ on the line through $\overline{x}$ and $\overline{y}$ which is also basic

Una combinazione convessa di due punti $\overline{x}, \overline{y}$ e' il segmento di linea.

$c^T \overline{x} = c^T \overline{y} = \gamma \qquad z = \alpha \overline{x} + (1-\alpha)\overline{y}$

$c^T z = c^T(\alpha x + (1-\alpha) y) = \alpha c^T \overline{x} + (1-\alpha) c^T \overline{y} = \alpha\gamma + (1-\alpha)\gamma = \gamma \Rightarrow z$ e' ottimale.

La 2 e' falsa perche' $\overline{x}$ e $\overline{y}$ sono BFS, ossia vertici del poliedro, e ogni punto sul segmento di linea di 2 vertici non e' (per definizione) un vertice.

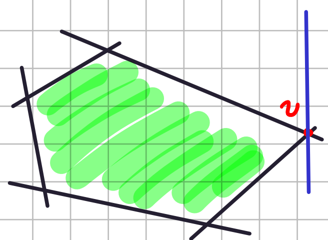
Consider the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ and let $v \in P$. Prove that v is a vertex of P if and only if there exists a vector c such that v is the unique optimal basic feasible solution to the linear program:

$$\begin{aligned} \max\ & c^T x \\ & x \in P \end{aligned}$$

[$\Rightarrow$]: v e' un vertice di $P \Rightarrow \exists\ \mathcal{H} = \{x : c^T x = \alpha\}$ t.c.

$c^T x < \alpha \quad \forall x \in P \setminus \{v\} \quad \wedge \quad c^T v = \alpha$

ma allora v e' sol. ottimale per $\max_{x \in P} c^T x$.



[$\Leftarrow$]: v e' l'unica sol. di $\max_{x \in P} c^T x \Rightarrow$ necessariamente v e' una BFS

$\Rightarrow v$ e' un vertice di P.

Solve the following linear program using the Simplex method:

$$\max\ 2x_1 + x_2 - x_3$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 8 \\ -x_1 + x_2 - 2x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Il problema in forma matriciale e'

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & -2 \end{bmatrix} \quad b = [8\ \ 4]^T$$

$\Rightarrow$ in forma di eq: $A' = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ -1 & 1 & -2 & 0 & 1 \end{bmatrix}$

La prima base e' $\mathcal{B} = \{4, 5\}$

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 8 \\ -x_1 + x_2 - 2x_3 + x_5 &= 4 \\ z &= 2x_1 + x_2 - x_3 \end{aligned} \Rightarrow$$

$\mathcal{B} = \{4, 5\}$

$$\begin{aligned} x_4 &= 8 - x_1 - 2x_2 - x_3 \\ x_5 &= 4 + x_1 - x_2 + 2x_3 \\ \hline z &= 2x_1 + x_2 - x_3 \end{aligned} \Rightarrow$$

$\mathcal{B} = \{1, 5\}$

$$\begin{aligned} x_1 &= 8 - x_4 - 2x_2 - x_3 \\ x_5 &= 4 + x_1 - x_2 + 2x_3 \\ \hline z &= 16 - 2x_4 - 3x_2 - 3x_3 \end{aligned}$$

$\Rightarrow$ la sol ott. e' $\begin{cases} x_1 = 8 \\ x_5 = 12 \end{cases} \Rightarrow$ per il problema originale e' $x = [8\ \ 0\ \ 0]^T$

Consider the following linear program

$$\min\ 3x_1 + 2x_2 - x_3$$
$$x_1 - 2x_2 + x_3 - x_4 = 0$$
$$x_1 + x_2 + 2x_3 - x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Beginning with the basis $\mathcal{B} = \{1, 2\}$, solve the linear program with the Simplex method

Riscrivo il problema in Forma max e parto con il primo tableu

$\mathcal{B} = \{1,2\}$

$$\begin{cases} x_1 - 2x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 + 2x_3 - x_4 = 1 \end{cases} \quad \begin{cases} x_1 = -\frac{5}{3}x_3 + x_4 + \frac{2}{3} \\ x_2 = \frac{1}{3} - \frac{1}{3}x_3 \end{cases}$$

$\mathcal{B} = \{1,2\}$

$$x_1 = -\frac{5}{3}x_3 + x_4 + \frac{2}{3}$$
$$x_2 = \frac{1}{3} - \frac{1}{3}x_3$$
$$z = -\frac{8}{3} + \frac{20}{3}x_3 + 3x_4$$

$\Rightarrow$

$\mathcal{B} = \{2,4\}$

$$x_4 = -x_1 - \frac{5}{3}x_3 + \frac{2}{3}$$
$$x_2 = 1/3 - 1/3\, x_3$$
$$z = -\frac{2}{3} + \frac{20}{3}x_3 - 3x_1 - 5x_3$$

$\Rightarrow$

$\mathcal{B} = \{3,4\}$

$$x_4 = -x_1 - \frac{5}{3} + 5x_2 + \frac{2}{3}$$
$$x_3 = 1 - 3x_2$$
$$z = 1 - 17x_2 - 3x_1$$

Finito, il massimo e' 1, il minimo del problema originale e' -1,

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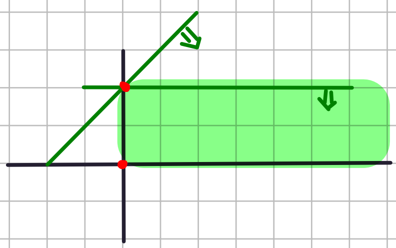
Give an explicit example of a linear program in the form of:

$$\max c^T x$$
$$Ax = b$$
$$x \geq 0$$

such that the problem is unbounded, it has a degenerate BFS and infinitely many optimal solutions

Si consideri il problema

$$\max\ -x_1$$
$$\text{s.t. } x_2 - x_1 \leq 2$$
$$x_2 \leq 2$$



e' illimitato, le due BFS $[0\ 0\ 0\ 0]$ e $[0\ 2\ 0\ 0]$ sono ottimali

in Forma di eq: $x_2 - x_1 + x_3 = 2$
$x_2 + x_4 = 2$

Si consideri $\mathcal{B} = \{1,2\} \Rightarrow \begin{cases} x_1 = -x_4 + x_3 - 2 + 2 \\ x_2 = -x_4 + 2 \end{cases} \Rightarrow$ Sol : $[0\ 2\ 0\ 0]^T \Rightarrow$ Una variabile di base e' $0 \Rightarrow [0\ 2\ 0\ 0]^T$ e' degenere.

Solve the following linear program using the Simplex method:

$$\min\ 12x_1 + 3x_2 + 4x_3$$
$$4x_1 + 2x_2 + 3x_3 \geq 2$$
$$8x_1 + x_2 + 2x_3 \geq 3$$
$$x_1, x_2, x_3 \geq 0$$

Trasformo il problema in forma canonica, e poi in forma di equazione.

$$\max\ -12x_1 - 3x_2 - 4x_3$$
$$-4x_1 - 2x_2 - 3x_3 \leq -2$$
$$-8x_1 - x_2 - 2x_3 \leq -3$$

$\Rightarrow$

$$\max\ -12x_1 - 3x_2 - 4x_3$$
$$-4x_1 - 2x_2 - 3x_3 + x_4 = -2$$
$$-8x_1 - x_2 - 2x_3 + x_5 = -3$$

la base $\mathcal{B} = \{4,5\}$ ha BFS associata $[0\ 0\ 0\ -2\ -3]^T \Rightarrow$ non e' ammissibile.

$\mathcal{B} = \{1,2\}$

$$x_1 = \frac{3}{4} + \frac{1}{4}x_3 - \frac{1}{4}x_4 + \frac{1}{4}x_5$$
$$x_2 = -\frac{1}{2} - 2x_3 + x_4 - \frac{1}{2}x_5$$
$$z = -\frac{15}{2} + 3x_3 - \frac{3}{2}x_5$$

$3 \curvearrowright 2$

$\Rightarrow$

$\mathcal{B} = \{1,3\}$

$$x_1 = \frac{3}{4} + \frac{1}{4}x_3 - \frac{1}{4}x_4 + \frac{1}{4}x_5$$
$$x_3 = -\frac{1}{4} - \frac{1}{2}x_2 + \frac{1}{2}x_4 - \frac{1}{4}x_5$$