Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Question #5 [all students]

The direct kinematics and the initial configuration of a planar RP robot are given by

$$p = f(q) = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \qquad q^{\{0\}} = \begin{pmatrix} \pi/4 \\ \varepsilon \end{pmatrix},$$

where $0 < \epsilon \ll 1$ is a very small number. Given the following desired end-effector positions,

$$p_{d,\mathrm{I}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad p_{d,\mathrm{II}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

compute the first iteration (i.e., $q^{\{1\}}$) of a Newton method and of a Gradient method for solving the two inverse kinematics problems. Discuss what happens in each of the four cases when $\varepsilon \to 0$.

$$J: \begin{pmatrix} -q_2 s_1 & c_1 \\ q_2 c_1 & s_1 \end{pmatrix} \Rightarrow J(q^0): \begin{pmatrix} -\varepsilon\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \varepsilon\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\hookrightarrow J^{-1}(q^0) = \begin{pmatrix} -\frac{1}{\varepsilon\sqrt{2}} & -\frac{1}{\varepsilon\sqrt{2}} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

the newton method use the Following rule:

$q^{k+1} = q^k + J^{-1}(q^k)(p_d - f_r(q^k))$ so, For $p_{d,\mathrm{I}}$:

$$f_r(q^0) = \varepsilon\frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$q^1_{\mathrm{I}} = \begin{pmatrix} \pi/4 \\ \varepsilon \end{pmatrix} + \begin{pmatrix} -\frac{1}{\varepsilon\sqrt{2}} & -\frac{1}{\varepsilon\sqrt{2}} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} -1-\varepsilon\frac{\sqrt{2}}{2} \\ 1-\varepsilon\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{3\sqrt{2}} + \frac{\varepsilon\sqrt{2}}{2\varepsilon\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{\varepsilon\sqrt{2}}{2\varepsilon\sqrt{2}} + \pi/4 \\ 2/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{\pi}{4} + 1 \\ 2/\sqrt{2} \end{pmatrix}$$

For $p_{d,\mathrm{II}} \Rightarrow$

$$q^1_{\mathrm{II}} = \begin{pmatrix} \pi/4 \\ \varepsilon \end{pmatrix} + \begin{pmatrix} -\frac{1}{\varepsilon\sqrt{2}} & -\frac{1}{\varepsilon\sqrt{2}} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1-\varepsilon\frac{\sqrt{2}}{2} \\ 1-\varepsilon\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{2}{\varepsilon\sqrt{2}} + 1 + \frac{\pi}{4} \\ 2/\sqrt{2} \end{pmatrix} \Rightarrow \lim_{\varepsilon \to 0} q^1_{\mathrm{II}} = \begin{pmatrix} \infty \\ 2/\sqrt{2} \end{pmatrix}$$ undefined $f(q)$

the gradient have the rule:

$$q^{k+1} = q^k + \alpha J^T(q^k)(r_d - f_r(q^k))$$

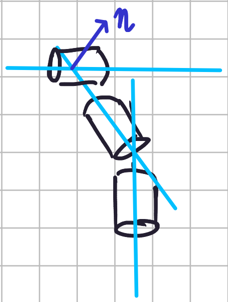
$$q^1_{\mathrm{I}} = \begin{pmatrix} \pi/2 \\ \varepsilon \end{pmatrix} + \alpha\begin{pmatrix} -\varepsilon\frac{\sqrt{2}}{2} & \varepsilon\frac{\sqrt{2}}{2} \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} -1-\varepsilon\frac{\sqrt{2}}{2} \\ 1-\varepsilon\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \pi/2 \\ \varepsilon \end{pmatrix} + \alpha\begin{pmatrix} \varepsilon\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \pi/2 + \alpha\varepsilon\sqrt{2} \\ \varepsilon \end{pmatrix} \quad \lim_{\varepsilon \to 0} q^1_{\mathrm{I}} = \begin{pmatrix} \pi/2 \\ 0 \end{pmatrix} \Rightarrow f(q) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

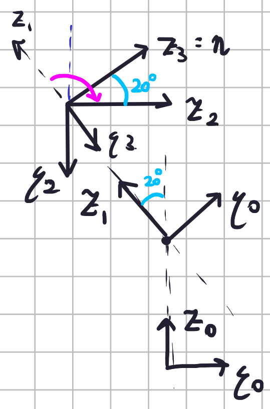
$$q^1_{\mathrm{II}} = \begin{pmatrix} \pi/2 \\ \varepsilon \end{pmatrix} + \alpha\begin{pmatrix} -\varepsilon\frac{\sqrt{2}}{2} & \varepsilon\frac{\sqrt{2}}{2} \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 1-\varepsilon\frac{\sqrt{2}}{2} \\ 1-\varepsilon\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \pi/2 \\ \varepsilon \end{pmatrix} + \alpha\begin{pmatrix} 0 \\ \sqrt{2}-\varepsilon \end{pmatrix} = \begin{pmatrix} \pi/2 \\ \alpha\sqrt{2} \end{pmatrix}$$

Question #6 [all students]

The 3R robotic device in Fig. 3 has joint axes that intersect two by two. The second joint axis is inclined by an angle $\delta \approx 20°$. This structure is mainly intended for pointing the final axis $\boldsymbol{n}$ at a moving target in 3D. Provide the explicit expression of the square angular part $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian of this robot. Find the singularities, if any, of the mapping $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$. Compute the relation between $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ and the time derivative $\dot{\boldsymbol{n}}$ of the pointing axis.

We notice that, since we are interested only in the angular part, we can ignore the link lengths.





Joints	α_i	a_i	d_i	θ_i
1	$20°$	0	0	q_1
2	$-110°$	0	0	q_2
3	$20°$	0	0	q_3

$$\begin{cases} \cos(20°) = 0.939 \\ \sin(20°) = 0.342 \\ \cos(-110°) = -0.342 \\ \sin(-110°) = -0.939 \end{cases}$$

$${}^0R_1 = \begin{pmatrix} c_1 & -0.939 s_1 & 0.342 s_1 \\ s_1 & 0.939 c_1 & -0.342 c_1 \\ 0 & 0.342 & 0.939 \end{pmatrix} \qquad {}^1R_2 = \begin{pmatrix} c_2 & 0.342 s_2 & -0.939 s_2 \\ s_2 & -0.342 c_2 & 0.939 c_2 \\ 0 & -0.939 & -0.342 \end{pmatrix} \Rightarrow$$

$${}^2R_3 = \begin{pmatrix} c_3 & -0.939 s_3 & 0.342 s_3 \\ s_3 & 0.939 c_3 & -0.342 c_3 \\ 0 & 0.342 & 0.939 \end{pmatrix}$$

$${}^0R_2 = \begin{pmatrix} c_1 & -0.939 s_1 & 0.342 s_1 \\ s_1 & 0.939 c_1 & -0.342 c_1 \\ 0 & 0.342 & 0.939 \end{pmatrix} \begin{pmatrix} c_2 & 0.342 s_2 & -0.939 s_2 \\ s_2 & -0.342 c_2 & 0.939 c_2 \\ 0 & -0.939 & -0.342 \end{pmatrix}$$

I denote $a = 0.939$, $b = 0.342$

$$\begin{pmatrix} c_1c_2 - a s_1 s_2 & b c_1 s_2 + d s_1 c_2 - ab s_1 & -a c_1 s_2 - a^2 s_1 c_2 - b^2 s_1 \\ s_1 c_2 + a c_1 s_2 & b s_1 s_2 - d c_1 c_2 + d c_1 & -a s_1 s_2 + a^2 c_1 c_2 + b^2 c_1 \\ b s_2 & -b^2 c_2 - a^2 & d c_2 - d \end{pmatrix}$$

$d = ab = 0.321138$

z_1

${}^0R_3 = {}^0R_2\, {}^2R_3$; take only the last column

$$z_2 = \begin{pmatrix} c_1c_2 - a s_1 s_2 & b c_1 s_2 + d s_1 c_2 - ab s_1 & -a c_1 s_2 - a^2 s_1 c_2 - b^2 s_1 \\ s_1 c_2 + a c_1 s_2 & b s_1 s_2 - d c_1 c_2 + d c_1 & -a s_1 s_2 + a^2 c_1 c_2 + b^2 c_1 \\ b s_2 & -b^2 c_2 - a^2 & d c_2 - d \end{pmatrix} \begin{pmatrix} 0.342 s_3 \\ -0.342 c_3 \\ 0.939 \end{pmatrix}$$

$$\begin{pmatrix} b s_3 (c_1c_2 - a s_1 s_2) - b c_3 (b c_1 s_2 + d s_1 c_2 - ab s_1) + a(-a c_1 s_2 - a^2 s_1 c_2 - b^2 s_1) \\ b s_3 (s_1 c_2 + a c_1 s_2) - b c_3 (b s_1 s_2 - d c_1 c_2 + d c_1) + a(-a s_1 s_2 + a^2 c_1 c_2 + b^2 c_1) \\ b^2 s_2 s_3 - b c_3 (-b^2 c_2 - a^2) + a d (c_2 - 1) \end{pmatrix} \Rightarrow J_A = \begin{pmatrix} \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} & z_1 & z_2 \end{pmatrix}$$

Question #8 [all students]

The planar 3R robot with unitary link lengths shown in Fig. 4 is initially in the configuration $\boldsymbol{q}_{in} = (-\pi/9,\ 11\pi/18,\ -\pi/4)$. Commanded by a joint velocity $\dot{\boldsymbol{q}}(t)$ that uses feedback from the current $\boldsymbol{q}(t)$, the robot should perform a self-motion so as to reach asymptotically the final value $q_{3,fin} = -\pi/2$ for the third joint, while keeping the position of its end-effector always at the same initial point P_{in}. Verify first that such task is feasible. Design then a control scheme that completes the task in a robust way, i.e., by rejecting also possible transient errors and without encountering any singular situation in which the control law is ill conditioned. *Hint: Use an approach based on joint space decomposition.*

First, I need P_{in}. $q_1+q_2=\frac{\pi}{2}$ $\quad q_1+q_2+q_3=\frac{\pi}{4}$

$$P_{in} = f_r(q_{in}) = \begin{pmatrix} 1.6467 \\ 1.365 \\ \pi/4 \end{pmatrix}.$$ I check if exists q^* s.t. $q_3^* = -\pi/2$ and $f(q^*) = P_{in}$

$$\begin{cases} c_1 + c_{12} + \cos\left(q_1+q_2-\frac{\pi}{2}\right) = 1.6467 \\ s_1 + s_{12} + \sin\left(q_1+q_2-\frac{\pi}{2}\right) = 1.365 \\ q_1+q_2-\frac{\pi}{2} = \frac{\pi}{4} \end{cases} \Rightarrow \begin{cases} q_1+q_2+q_3 = \frac{\pi}{4} \\ q_1+q_2 = q_1+\frac{3}{4}\pi - q_1 = \frac{3}{4}\pi \\ q_2 = \frac{3}{4}\pi - q_1 \end{cases}$$

$$q_2 = 2.385$$

$$\Rightarrow \begin{cases} c_1 = 1.6467 \\ s_1 + \sqrt{2} = 1.365 \end{cases} \quad \begin{cases} c_1 = 1.6467 \\ s_1 = -0.049 \end{cases} \Rightarrow q_1 = \text{atan2}\{-0.049,\ 1.6467\} = -0.029$$

$\rightarrow$ it's possible

the error on the joint space is $e_3 = -\frac{\pi}{2} - q_3$

the error on the task space is $e_p = P_{in} - p$.

$$\Rightarrow \dot{q} = \alpha\begin{pmatrix} 0 \\ 0 \\ q_3+\pi/2 \end{pmatrix}$$

On the task space:

$$\dot{p} = -Ke_p \qquad \text{with } e_p = p - P_{in}$$

$$J\dot{q} = -Ke_p \Rightarrow \dot{q} = -J^{\#}Ke_p$$

$$\dot{e}_p = \dot{p} = J\dot{q} = -JJ^{\#}Ke_p = -Ke_p \Rightarrow \lim_{t\to\infty} e_p = 0$$

$$\Rightarrow \dot{q} = \beta\left(-J^{\#}Ke_p\right) + \alpha\begin{pmatrix} 0 \\ 0 \\ q_3+\pi/2 \end{pmatrix} \qquad \alpha+\beta=1$$

$$\Rightarrow \dot{q} = -\beta J^{\#}e_p + \alpha\begin{pmatrix} 0 \\ 0 \\ q_3+\pi/2 \end{pmatrix} = -\beta J^{\#}(p-P_{in}) + \alpha\begin{pmatrix} 0 \\ 0 \\ q_3+\pi/2 \end{pmatrix}$$