

Consider the orientation obtained by a (partial) Euler sequence with a rotation of an angle  $\alpha$  around  $z$ , followed by a rotation of an angle  $\beta$  around the current  $y$ . Find three angles  $\phi$ ,  $\chi$ , and  $\psi$  such that the product  $R_x(\phi)R_y(\chi)R_z(\psi)$  returns the same final orientation. Give the procedure for solving this problem in general, determine the singular cases, and provide then a numerical value of the sought triple of angles when  $\alpha = \pi/4$ ,  $\beta = -\pi/3$  [rad]. Check the result.

\* i use it for  $R_{x,y,z}$

$$R_z(\alpha)R_y(\beta) = \begin{bmatrix} c\alpha c\beta & -s\alpha & s\beta c\alpha \\ s\alpha c\beta & c\alpha & s\alpha s\beta \\ -s\beta & 0 & c\beta \end{bmatrix} \quad \text{i have to solve for } \phi, \chi, \psi \text{ the equation of the two matrices}$$

$$R_x(\phi)R_y(\chi)R_z(\psi) = \begin{bmatrix} c\chi & s\chi s\psi & s\chi c\psi \\ s\phi s\chi & -s\phi s\psi c\chi + c\phi c\psi & -s\phi c\chi c\psi - s\psi c\phi \\ -s\chi c\phi & s\phi c\psi + s\psi c\phi c\chi & -s\phi s\psi + c\phi c\chi c\psi \end{bmatrix}$$

$$c\chi = c\alpha c\beta \Rightarrow \pm s\chi = \pm \sqrt{1 - c^2 \alpha c^2 \beta} \Rightarrow \chi = \arctan 2 \left\{ \pm \sqrt{1 - c^2 \alpha c^2 \beta}, c\alpha, c\beta \right\}$$

$$\begin{cases} s\chi s\psi = -s\alpha \\ s\chi c\psi = s\beta c\alpha \end{cases} \Rightarrow \begin{cases} s\psi = \pm \frac{s\alpha}{s\chi} \\ c\psi = \pm \frac{s\beta c\alpha}{s\chi} \end{cases} \Rightarrow \psi = \arctan 2 \left\{ \pm \frac{s\alpha}{s\chi}, \pm \frac{s\beta c\alpha}{s\chi} \right\} \begin{array}{l} \arctan 2 \left\{ \frac{s\alpha}{s\chi}, \frac{s\beta c\alpha}{s\chi} \right\} \text{ if } +s\chi \\ \arctan 2 \left\{ \frac{s\alpha}{s\chi}, -\frac{s\beta c\alpha}{s\chi} \right\} \text{ if } -s\chi \end{array}$$

$$\begin{cases} s\phi s\chi = s\alpha c\beta \\ -s\chi c\phi = -s\beta \end{cases} \Rightarrow \begin{cases} s\phi = \pm \frac{s\alpha s\beta}{s\chi} \\ c\phi = \pm \frac{s\beta}{s\chi} \end{cases} \Rightarrow \begin{array}{l} \text{if } +s\chi \Rightarrow \phi = \arctan 2 \left\{ \frac{s\alpha s\beta}{s\chi}, \frac{s\beta}{s\chi} \right\} \\ \text{if } -s\chi \Rightarrow \phi = \arctan 2 \left\{ -\frac{s\alpha s\beta}{s\chi}, -\frac{s\beta}{s\chi} \right\} \end{array}$$

$$\text{let } \alpha = \frac{\pi}{4}, \beta = -\frac{\pi}{3} : c\chi = \frac{\sqrt{2}}{4} \Rightarrow \pm s\chi = \pm \frac{\sqrt{14}}{4} \Rightarrow \chi = \pm 1.2094$$

$$\psi = \arctan 2 \left\{ \frac{\frac{2\sqrt{2}}{7}}{\frac{\sqrt{2}}{4}}, -\frac{\frac{\sqrt{21}}{7}}{\frac{\sqrt{2}}{4}} \right\} = -2.28 \quad (\text{if } \chi = 1.2094)$$

$$\psi = \arctan 2 \left\{ \frac{\frac{2\sqrt{2}}{7}}{\frac{\sqrt{2}}{4}}, \frac{\frac{\sqrt{21}}{7}}{\frac{\sqrt{2}}{4}} \right\} = 0.85 \quad (\text{if } \chi = -1.2094)$$

$$\text{if } +s\chi \Rightarrow \phi = \arctan 2 \left\{ \frac{s\alpha s\beta}{s\chi}, \frac{s\beta}{s\chi} \right\} = \arctan 2 \left\{ -\frac{\sqrt{21}}{7}, -\frac{\sqrt{14}}{7} \right\} = -2.52$$

$$\text{if } -s\chi \Rightarrow \phi = \arctan 2 \left\{ -\frac{s\alpha s\beta}{s\chi}, -\frac{s\beta}{s\chi} \right\} : \arctan 2 \left\{ \frac{\sqrt{21}}{7}, \frac{\sqrt{14}}{7} \right\} = 0.615$$

$$\Rightarrow \begin{bmatrix} \phi \\ \chi \\ \psi \end{bmatrix} = \begin{bmatrix} -2.52 \\ 1.2094 \\ -2.28 \end{bmatrix} \vee \begin{bmatrix} 0.615 \\ -1.2094 \\ 0.85 \end{bmatrix}$$

**Exercise 2** [10 points]

Let a first rotation be defined by an angle  $\gamma$  around  $\mathbf{x}$ , followed by a rotation of an angle  $\delta$  around the unit vector  $\mathbf{v} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$  expressed in the original frame. Determine the resulting rotation matrix  $\mathbf{R}(\gamma, \delta)$  in symbolic form. For a numerical case with  $\gamma = -\pi/2$ ,  $\delta = \pi/3$  [rad], extract the invariant axis  $\mathbf{r}$  of the total rotation and the corresponding angle  $\theta$ . Check the result.

$$\mathbf{R}(\delta, \mathbf{v}) = \mathbf{v}\mathbf{v}^T + (\mathbf{I} - \mathbf{v}\mathbf{v}^T) \cos \delta + \mathbf{s}(\mathbf{v}) \sin \delta$$

$$\mathbf{R}_x = \mathbf{R}_{\infty}(\delta) \Rightarrow \mathbf{R}(\delta, \mathbf{v}) = \mathbf{R}(\delta, \mathbf{v})\mathbf{R}(\infty)$$

$$\begin{vmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{vmatrix} \quad \left| \begin{array}{c} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{array} \right. \theta$$

$$\mathbf{R}(\delta, \mathbf{v}) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cos \delta + \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \sin \delta = \begin{bmatrix} \frac{1}{2}(1+\cos \delta) & \frac{1}{2}(1+\cos \delta) & -\frac{1}{2}\sin \delta \\ \frac{1}{2}(1+\cos \delta) & \frac{1}{2}(1+\cos \delta) & -\frac{1}{2}\sin \delta \\ \frac{1}{\sqrt{2}}\sin \delta & \frac{1}{\sqrt{2}}\sin \delta & \cos \delta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{2}(1+\cos \delta) & \frac{1}{2}(1+\cos \delta) & -\frac{1}{\sqrt{2}}\sin \delta \\ \frac{1}{2}(1+\cos \delta) & \frac{1}{2}(1+\cos \delta) & -\frac{1}{\sqrt{2}}\sin \delta \\ \frac{1}{\sqrt{2}}\sin \delta & \frac{1}{\sqrt{2}}\sin \delta & \cos \delta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \delta & -\sin \delta \\ 0 & \sin \delta & \cos \delta \end{bmatrix} =$$

$$\mathbf{R} = \begin{bmatrix} \frac{1}{2}(1+\cos \delta) & \frac{1}{2}(1+\cos \delta)\cos \delta - \frac{1}{\sqrt{2}}\sin \delta \sin \delta & -\frac{1}{2}(1+\cos \delta)\sin \delta - \frac{1}{\sqrt{2}}\sin \delta \cos \delta \\ \frac{1}{2}(1+\cos \delta) & \frac{1}{2}(1+\cos \delta)\cos \delta - \frac{1}{\sqrt{2}}\sin \delta \sin \delta & -\frac{1}{2}(1+\cos \delta)\sin \delta - \frac{1}{\sqrt{2}}\sin \delta \cos \delta \\ \frac{1}{\sqrt{2}}\sin \delta & \frac{1}{\sqrt{2}}\sin \delta \cos \delta + \cos \delta \sin \delta & \cos \delta \cos \delta - \frac{1}{\sqrt{2}}\sin \delta \sin \delta \end{bmatrix}$$

$$\text{For } \delta : \begin{cases} \frac{\pi}{3} \\ -\frac{\pi}{2} \end{cases}$$

We have:

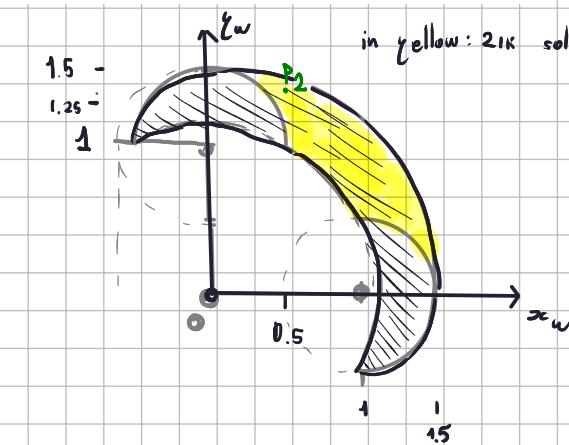
$$\begin{bmatrix} 0.75 & 0.61 & -0.25 \\ -0.25 & 0.61 & 0.75 \\ 0.61 & -0.5 & 0.61 \end{bmatrix}$$

To find  $r, \theta$  such that  $\mathbf{R} = \mathbf{R}(\theta, \mathbf{r})$ ; I know that  
 $1 + 2\cos \theta = \text{trace } \mathbf{R} \Rightarrow \cos \theta = 0.485 \Rightarrow \sin \theta = \pm 0.874$

$$\Rightarrow \mathbf{r} \cdot \frac{1}{2\sin \theta} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix} = \pm 0.572 \begin{bmatrix} -1.25 \\ -0.86 \\ -0.86 \end{bmatrix} \Rightarrow \theta = 1.064, \mathbf{r} = \begin{bmatrix} -0.715 \\ -0.431 \\ -0.491 \end{bmatrix} \text{ or } \theta = -1.064, \mathbf{r} = \begin{bmatrix} 0.715 \\ 0.431 \\ 0.491 \end{bmatrix}$$

### Exercise 3 [10 points]

Consider the 2R planar robot in Fig. 1, with  $L_1 = 1$ ,  $L_2 = 0.5$  [m]. The joint variables have a limited range:  $\theta_1 \in [0, \pi/2]$ ,  $\theta_2 \in [-\pi/2, \pi/2]$  [rad].  
 • Sketch the primary workspace of this robot, localizing the relevant points on its boundary.  
 • Indicate the region of the workspace where two inverse kinematics solutions exist.  
 • For each of the following five points, specify whether there are 0, 1, 2, or  $\infty$  inverse kinematics solutions:  
 $P_1 = (0.1, 1.5)$ ,  $P_2 = (0.5, 1.3)$ ,  $P_3 = (-0.4, 1.1)$ ,  $P_4 = (1.0, 1.0)$ ,  $P_5 = (1.0, -0.3)$  [m].



P1) Since  $\|P_1\| > L_1 + L_2 \Rightarrow P_1 \notin WS_1$

P2)  $\|P_2\| = 1.35$  and  $P_2 \in WS_1$

P3) is for sure inside.  $P_3 \in WS_1$

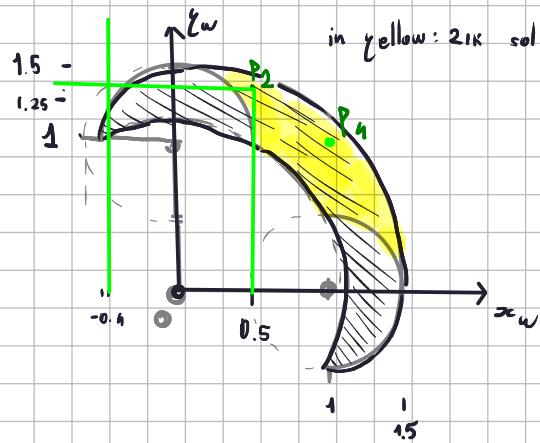
P4) is nob, since the only one point on the line  $x = 1$  in the WS have  $y = -0.5$ .

For  $P_5$  i can't see, i have to solve the IK

$$\Rightarrow \cos q_2 = 0.12 \Rightarrow \sin q_2 = \pm 0.992$$

$$\Rightarrow q_2 = \pm \tan^{-1}\left(\frac{\pm 0.992}{0.12}\right) = \pm 1.4504 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

I look for  $q_1$ .



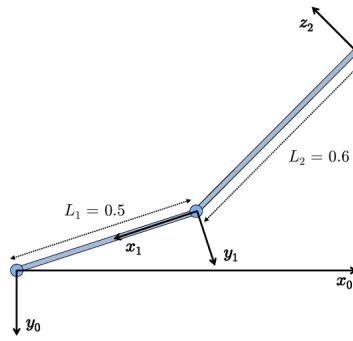
$$\pm q_1 = \text{atan} 2 \left\{ 1.1, -0.4 \right\} - \text{atan} 2 \left\{ \pm \frac{1}{2} \cdot 0.992, 1 + \frac{1}{2} \cdot 0.12 \right\} =$$

$$\therefore 1.31 - 0.43 = 1.48 \in [0, \pi/2] \Rightarrow P_5 \in WS_1 \text{ with only one sol: } q = \begin{pmatrix} 1.48 \\ 1.45 \end{pmatrix}$$

$$1.31 + 0.43 = 2.34 \notin [0, \pi/2]$$

### Exercise 4 [10 points]

Figure 2 shows an unusual but feasible choice of Denavit-Hartenberg (D-H) frames for a 2R planar robot. Provide the corresponding D-H table of parameters and the direct kinematics of this robot as an homogeneous transformation matrix  ${}^0T_2(q)$ . Evaluate then this matrix in numerical form at  $q^* = (\pi/2, -\pi/2)$  [rad] and draw the robot in this configuration.



$\alpha_i$	$z_i$	$d_i$	$\theta_i$
1	$-\pi$	$-L_1$	$0$
2	$-\pi$	$L_2$	$q_2$

$${}^0T_1 = \begin{bmatrix} c_1 & s_1 & 0 & -L_1 c_1 \\ s_1 & -c_1 & 0 & -L_1 s_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1T_2 = \begin{bmatrix} c_2 & s_2 & 0 & L_2 c_1 \\ s_2 & -c_2 & 0 & L_2 s_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_2(q) = \begin{bmatrix} c_1 c_2 + s_1 s_2 & c_1 s_2 - s_1 c_2 & 0 & L_2 c_1 c_2 + L_2 s_1 s_2 - L_1 c_1 \\ s_1 c_2 - c_1 s_2 & s_1 s_2 + c_1 c_2 & 0 & L_2 s_1 c_2 - L_2 c_1 s_2 - L_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_2(q^*) = \begin{bmatrix} -1 & 0 & 0 & -L_2 \\ 0 & -1 & 0 & -L_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

### Exercise 5 [10 points]

The differential equations of a DC motor are given in slide #14 of the block 03\_CompsActuators.pdf. With the motor unloaded and starting from rest, if we apply a constant armature voltage  $\bar{v}_a$ , the motor will start rotating and then reach a steady-state condition, with a constant angular velocity  $\bar{\omega}$  and a constant produced torque  $\bar{\tau}$ . What are the expressions of  $\bar{\omega}$  and  $\bar{\tau}$  in terms of the system parameters and  $\bar{v}_a$ ? If we attach a load with inertia  $I_L > 0$  to the motor shaft through a transmission with reduction ratio  $n_r > 1$  and assume no dissipative terms on the load side, will the steady-state velocity of the motor change? And what will be the velocity  $\omega_L$  of the load at steady state?

$$\dot{\theta} = \omega$$

$\omega \text{ const} \Rightarrow \ddot{\theta} = 0$

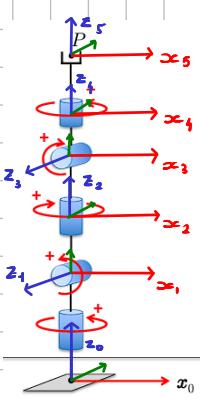
$$\begin{cases} V = RI + L \frac{di}{dt} + K_V \omega \\ \tau = I_m \frac{d\omega}{dt} + F_m \omega + \tau_{load} \end{cases} \Rightarrow \begin{cases} V = RI + L \frac{di}{dt} + K_V \omega \\ \tau = I_m \frac{d\omega}{dt} + F_m \omega + \tau_{load} \end{cases} \begin{cases} V = RI + K_V \omega \\ \tau = F_m \omega = K_T i \end{cases}$$

$\Downarrow \frac{di}{dt} = 0$

$$\begin{cases} V = RI + K_V \omega \\ \tau = F_m \omega = K_T i \end{cases} \Rightarrow \begin{cases} \omega = \frac{V - RI}{K_V} \\ \tau = \left(\frac{V - RI}{K_V}\right) F_m \end{cases}$$

### Exercise 6 [20 points]

The 5R robot in Fig. 3 is shown in its zero configuration (i.e., for  $q = 0$ ), with indication of the positive joint rotations. Assign the D-H frames consistently with these specifications and fill the corresponding table of parameters (specifying also the signs of the non-zero constant parameters). The origin of the last D-H frame should be at point  $P$ . Evaluate then numerically the position and the orientation of the last frame at  $q = 0$ , when all the non-zero kinematic lengths of the links are unitary.



	$\alpha_i$	$z_i$	$d_i$	$\theta_i$
1	$\pi/2$	0	$d_1 > 0$	$q_1$
2	$-\pi/2$	0	$d_2 > 0$	$q_2$
3	$\pi/2$	0	$d_3 > 0$	$q_3$
4	$-\pi/2$	0	$d_4 > 0$	$q_4$
5	0	0	$d_5 > 0$	$q_5$

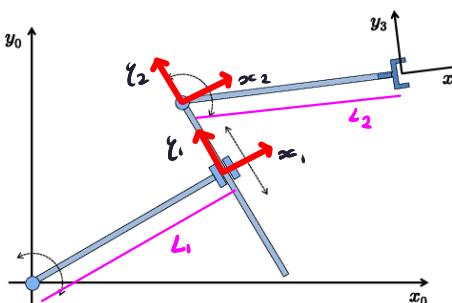
if  $d_i = 1$ , then

$$S_r(0) = (0 \ 0 \ 5)^T$$

### Exercise 7 [30 points]

Consider the planar RPR robot in Fig. 4, with the first and third joint revolute and the second prismatic.

- Determine the task kinematics  $r = f_r(q)$  for  $r = (p_x, p_y) \in \mathbb{R}^2$  the position of the end-effector and  $\phi \in (-\pi, \pi]$  its orientation angle with respect to  $x_0$ . [Hint: Use D-H joint variables.]
- Solve analytically the inverse kinematics problem for  $r_d = (p_{dx}, p_{dy}, \phi_d)$  in the regular case only.



$${}^0 T_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1 T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2 T_3 = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0 T_3 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_3 = \begin{bmatrix} C_1 & -S_1 & 0 & L_1 C_1 - q_2 S_1 \\ S_1 & C_1 & 0 & L_1 S_1 + q_2 C_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 & -S_3 & 0 & L_2 C_2 \\ S_3 & C_2 & 0 & L_2 S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} C_1 C_3 - S_1 S_3 & -(C_1 S_3 + S_1 C_3) & 0 & L_2 C_1 C_3 - L_2 S_1 S_3 + L_1 C_1 - q_2 S_1 \\ S_1 C_3 + C_1 S_3 & C_1 C_3 - S_1 S_3 & 0 & L_2 S_1 C_3 + L_2 C_1 S_3 + L_1 S_1 + q_2 C_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} C_{13} & -S_{13} & 0 & L_2 C_{13} + L_1 C_1 - q_2 S_1 \\ S_{13} & C_{13} & 0 & L_2 S_{13} + L_1 S_1 + q_2 C_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow S_r(q) = \begin{cases} P_x = L_2 C_{13} + L_1 C_1 - q_2 S_1, & I \text{ solve it for } q_1, q_2, q_3 \\ P_y = L_2 S_{13} + L_1 S_1 + q_2 C_1 \\ \phi = q_1 + q_3 \end{cases}$$

$$\phi = q_1 + q_3 \Rightarrow \begin{cases} P_x = L_2 \cos \phi + L_1 \cos q_1 - q_2 \sin q_1, & I \text{ square and sum} \\ P_y = L_2 \sin \phi + L_1 \sin q_1 + q_2 \cos q_1, & \Rightarrow \end{cases}$$

$$P_x^2 + P_y^2 = L_2^2 \cos^2 \phi + L_1^2 \cos^2 q_1 + q_2^2 \sin^2 q_1 + L_1 L_2 \cos q_1 \cos \phi - q_2 L_2 \cos \phi \sin q_1 - q_2 L_1 \cos q_1 \sin q_1 + L_2^2 \sin^2 \phi + L_1^2 \sin^2 q_1 + q_2^2 \cos^2 q_1 + L_1 L_2 \sin q_1 \sin \phi + q_2 L_2 \sin q_1 \cos q_1 + q_2 L_1 \cos q_1 \sin q_1 \Rightarrow$$

$$\Rightarrow P_x^2 + P_y^2 = L_2^2 + L_1^2 + q_2^2 + L_1 L_2 \cos q_3 + q_2 L_2 \sin q_3$$