Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise #2

The absolute initial orientation of the end effector of a 6R robot with a spherical wrist is specified by the YXY sequence of Euler angles $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (45°, -45°, 120°)$. A different orientation is expressed instead by the rotation matrix

$$ {}^0\boldsymbol{R}_f = \begin{pmatrix} 0 & \sin\phi & \cos\phi \\ 0 & \cos\phi & -\sin\phi \\ -1 & 0 & 0 \end{pmatrix}, \qquad \text{with } \phi = \frac{\pi}{3}.$$

Find an axis-angle representation $(\boldsymbol{r}, \theta)$ of the relative rotation between these two end-effector orientations. Further, if a motion is imposed to the end effector with constant angular velocity $\boldsymbol{\omega} = 1.1 \cdot \boldsymbol{r}$ [rad/s], what will be the time T_ω needed to accomplish this change of orientation?

I denote the first orientation ${}^0R_i$. Since

$$ {}^0R_i = R_y(45°)R_x(-45°)R_y(120°) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} -1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & -1/2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -0.866 & 0 & -0.5 \\ 0.5 & 0 & -0.866 \end{pmatrix}$$

Now i have to find r, θ s.t. $R(\theta, r) = {}^iR_f = {}^0R_i^T\,{}^0R_f$

$$ {}^iR_f = \begin{pmatrix} -0.5 & -0.433 & 0.75 \\ 0 & -0.866 & -0.5 \\ 0.866 & -0.25 & 0.433 \end{pmatrix} \Rightarrow 1+2\cos\theta = \text{trace}({}^iR_f) \Rightarrow 1+2\cos\theta = -0.933 \Rightarrow \cos\theta = -0.9665 \Rightarrow \theta = 2.882$$

then:

$$ r = \frac{1}{2\sin\theta}\begin{pmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{pmatrix} \Rightarrow r = 1.948\begin{pmatrix} 0.25 \\ -0.116 \\ 0.433 \end{pmatrix} = \begin{pmatrix} 0.487 \\ -0.226 \\ 0.843 \end{pmatrix}$$

If i consider $R(t) = {}^0R_i\,R(\theta, r)$ with $\theta(t) \in [0, 2.882]$ i have a parametrization of the orientation. the angular speed is $\omega = r\dot{\theta}$ so $\dot{\theta} = 1.1 \Rightarrow$

$\theta(t) = 1.1t = 2.882 \Rightarrow T_\omega = 2.62$ [sec].

Exercise #3

Assume that the motion of a 3R planar robot having equal links of unitary length is commanded by the joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^3$. With reference to Fig. 2, the robot end effector should follow a desired smooth trajectory $\boldsymbol{p}_d(t) = \begin{pmatrix} p_{x,d}(t) & p_{y,d}(t) \end{pmatrix}^T \in \mathbb{R}^2$ in position, while keeping constant its angular speed at some value $\omega_{z,d} \in \mathbb{R}$ (perhaps, after an initial transient).

i. Provide the general form of the command $\ddot{\boldsymbol{q}}$ that executes the full task in nominal conditions.

ii. Study the singularities that may be encountered during the execution of the task.

the desired trajectory is $r_d = (p_{x,d}, p_{y,d}, \omega_{z,d}t)^T$.

Since $\ddot{r} = J(q)\ddot{q} + \dot{J}(q)\dot{q}$ i impose $\ddot{q} = J^{-1}(q)(\ddot{r}_d - \dot{J}\dot{q})$, this might not be realizable if J is not invertible. Since:

$$ J = \begin{pmatrix} -S_1 - S_{12} - S_{123} & -S_{12} - S_{123} & -S_{123} \\ C_1 + C_{12} + C_{123} & C_{12} + C_{123} & C_{123} \\ 1 & 1 & 1 \end{pmatrix} \qquad \dot{J} = \begin{pmatrix} -(C_1 + C_{12} + C_{123}) & -(C_{12} + C_{123}) & -C_{123} \\ -S_1 - S_{12} - S_{123} & -S_{12} - S_{123} & -S_{123} \\ 0 & 0 & 0 \end{pmatrix}$$

iii. Compute the numerical value of $\ddot{\boldsymbol{q}}$ when the robot is in the nominal state $\boldsymbol{x}_d = (\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d) \in \mathbb{R}^6$ and for a desired $\ddot{\boldsymbol{p}}_d \in \mathbb{R}^2$, as given by

$$ \boldsymbol{q}_d = \begin{pmatrix} \pi/4 \\ \pi/3 \\ -\pi/2 \end{pmatrix} \text{ [rad]}, \qquad \dot{\boldsymbol{q}}_d = \begin{pmatrix} -0.8 \\ 1 \\ 0.2 \end{pmatrix} \text{ [rad/s]}, \qquad \ddot{\boldsymbol{p}}_d = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ [m/s}^2\text{]}.$$

What are the values of $\boldsymbol{p}_d$, $\dot{\boldsymbol{p}}_d$, and of $\omega_{z,d}$ in this nominal robot state?

For $q = (\frac{\pi}{4}, \frac{\pi}{3}, -\frac{\pi}{2})$:

$$ J = \begin{pmatrix} -1.931 & -1.224 & -0.259 \\ \sqrt{2} & 0.707 & 0.966 \\ 1 & 1 & 1 \end{pmatrix} \qquad \dot{J} = \begin{pmatrix} -\sqrt{2} & -0.707 & -0.966 \\ -1.931 & -1.224 & -0.259 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \ddot{q} = J^{-1}(q)(\ddot{r}_d - \dot{J}\dot{q}) = (0.585 \;\; -1.81 \;\; 1.225)^T$$

iv. If at some time $t \geq 0$, there is a position and/or a velocity error in the execution of the desired end-effector trajectory $\boldsymbol{p}_d(t)$, how would you modify the commanded acceleration $\ddot{\boldsymbol{q}}(t)$ so as to recover exponentially[1] the error to zero, both in position and velocity? And what if also the angular velocity $\omega_z(t)$ is not the desired one?

let's consider a pos. error given by $e = r_d - r$ and i want that to go to zero. Since $\dot e = \dot r_d - \dot r = \dot r_d - J\dot q$ and

$$\ddot e = \ddot r_d - \ddot r = \ddot r_d - J(q)\ddot q - \dot J(q)\dot q =$$

$$\ddot r_d - J(q)\left[J^{-1}(q)(\ddot r_d - \dot J \dot q) + C\right] - \dot J(q)\dot q$$

where C is an unknown command that i need to realize: $\ddot e = -K\dot e$ with $K = \mathrm{diag}\{a, a, a\}$ and $a \in \mathbb{R}^+$.

$$\ddot r_d - J(q)\left[J^{-1}(q)(\ddot r_d - \dot J \dot q) + C\right] - \dot J(q)\dot q = -K\dot e$$

i solve for C

$$\ddot r_d - \ddot r_d + \dot J \dot q - J(q)C - \dot J(q)\dot q = -K\dot e \Rightarrow -J(q)C = -K\dot e \Rightarrow C = J(q)K\dot e$$

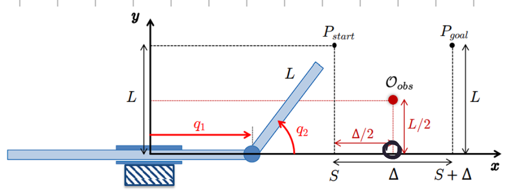
so

$\Rightarrow$ new comand is $\ddot q = J^{-1}(q)(\ddot r_d - \dot J \dot q) + J(q)K\dot e$ so

$\ddot e = -\dot e \Rightarrow$ the sol. for e is an exponential of the type $e(t) = Ae^{-t}$

Exercise #4

Consider the situation in Fig. 3, with all data defined therein in symbolic form. The PR robot starts at rest with its end effector placed in $P_{start} = (S, L)$ and should move the end effector to $P_{goal} = (S + \Delta, L)$ in a given time T and stop there, without colliding with the obstacle $\mathcal{O}_{obs}$ located at $(S + (\Delta/2), L/2)$. Design a joint trajectory $\boldsymbol{q}_d(t) \in \mathbb{R}^2$, $t \in [0, T]$, that realizes the task with continuous acceleration $\ddot{\boldsymbol{q}}_d(t)$ and no instant of zero velocity in the open interval $(0, T)$. The solution should be parametric with respect to $L > 0$ (length of the second link of the robot), $S > 0$ (x-coordinate of P_{start}), $\Delta > L/2$ (distance of the two Cartesian points in the x-direction), and T (motion time). Provide then a numerical example, sketching the plot of $\boldsymbol{q}_d(t)$.



I want the following conditions:

$q_1(0) = S \qquad q_1(\frac{T}{2}) = S + \frac{\Delta}{2} - L \qquad q_1(T) = S + \Delta \qquad \dot q_1(0) = 0$

$q_2(0) = \frac{\pi}{2} \qquad q_2(\frac{T}{2}) = 0 \qquad q_2(T) = \frac{\pi}{2} \qquad \dot q_2(0) = 0$

For both i use a 3-degree polynomial.

$q_1(t) = at^3 + bt^2 + ct + d \qquad q_2(t) = at^3 + bt^2 + ct + d$

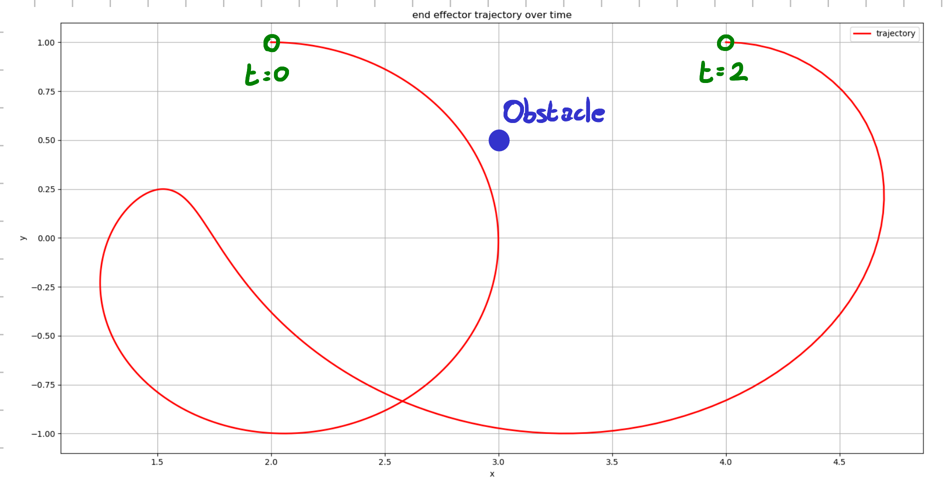
$$\begin{cases} q_1(0) = d = S \\ \dot q_1(0) = c = 0 \\ q_1(T/2) = \frac{a}{8}T^3 + \frac{b}{4}T^2 = \frac{\Delta}{2} - L \\ q_1(T) = aT^3 + bT^2 = \Delta \end{cases} \qquad \begin{cases} q_2(0) = d = \frac{\pi}{2} \\ \dot q_2(0) = c = 0 \\ q_2(T/2) = \frac{a}{8}T^3 + \frac{b}{4}T^2 + \frac{\pi}{2} = 0 \\ q_2(T) = aT^3 + bT^2 = 0 \end{cases}$$

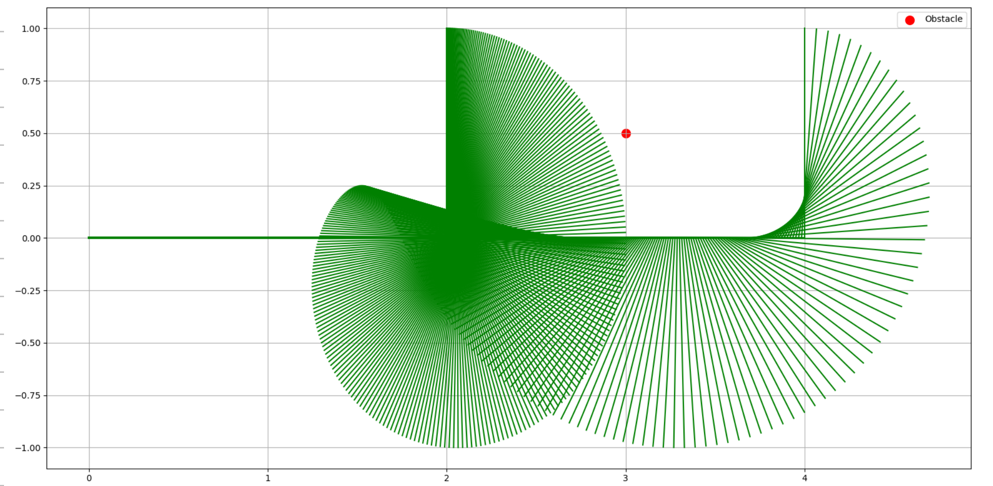
I choos the values $L = 1$, $\Delta = 2$, $S = 2$, $T = 2$

$$\begin{cases} q_1(T/2) = \frac{a}{8}T^3 + \frac{b}{4}T^2 = \frac{\Delta}{2} - L \\ q_1(T) = aT^3 + bT^2 = \Delta \end{cases} \quad \begin{cases} \frac{1}{8}a + \frac{1}{4}b = 0 \\ 8a + 4b = 2 \end{cases} \Rightarrow q_1(t) = \frac{1}{3}t^3 - \frac{1}{6}t^2 + 2$$

$$\begin{cases} q_2(T/2) = \frac{a}{8}T^3 + \frac{b}{4}T^2 + \frac{\pi}{2} = 0 \\ q_2(T) = aT^3 + bT^2 = 0 \end{cases} \quad \begin{cases} \frac{1}{8}a + \frac{1}{4}b = -\frac{\pi}{2} \\ 8a + 4b = 0 \end{cases} \Rightarrow q_2(t) = \frac{4}{3}\pi t^3 - \frac{8}{3}\pi t^2 + \frac{\pi}{2}$$

$$q(t) = \begin{cases} q_1(t) = \frac{1}{3}t^3 - \frac{1}{6}t^2 + 2 \\ q_2(t) = \frac{4}{3}\pi t^3 - \frac{8}{3}\pi t^2 + \frac{\pi}{2} \end{cases} \qquad O_{obs} = (3, 1/2)$$





in this stroboscopic view is clear that the two arms of the manipulator never touch the obstacle.