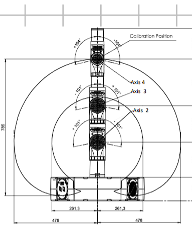
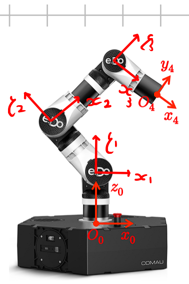
Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise 1

The Italian robot manufacturer Comau has recently put on the market two educational manipulators of small size and weight called *e.Do*. The version with four actuated revolute joints is shown in Fig. 1.



i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				



all constant DH parameters should be ≥ 0

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	202	q_1
2	0	210.5	0	q_2
3	0	268	0	q_3
4	0	174.5	0	q_4

Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated table of parameters so that all constant parameters are *non-negative*. Specify also their numerical values. Draw the frames and fill in the table directly on the extra sheet #1 provided separately. The two DH frames 0 and 4 are already assigned and should not be modified. Finally, write the DH homogeneous transformation matrices. [Please, make clean drawings and return the completed sheet with your name written on it.]

$$^0T_1 = \begin{pmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 202 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1T_2 = \begin{pmatrix} C_2 & -S_2 & 0 & 210.5C_2 \\ S_2 & C_2 & 0 & 210.5S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^2T_3 = \begin{pmatrix} C_3 & -S_3 & 0 & 268\,C_3 \\ S_3 & C_3 & 0 & 268\,S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3T_4 = \begin{pmatrix} C_4 & -S_4 & 0 & 174.5C_4 \\ S_4 & C_4 & 0 & 174.5S_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^0T_2 = \begin{pmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 202 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_2 & -S_2 & 0 & 210.5C_2 \\ S_2 & C_2 & 0 & 210.5S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C_1C_2 & -C_1S_2 & S_1 & 210.5\,C_1C_2 \\ S_1C_2 & -S_1S_2 & -C_1 & 210.5\,S_1C_2 \\ S_2 & C_2 & 0 & 210.5\,S_2+202 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^0T_3 = \begin{pmatrix} C_1C_2 & -C_1S_2 & S_1 & 210.5\,C_1C_2 \\ S_1C_2 & -S_1S_2 & -C_1 & 210.5\,S_1C_2 \\ S_2 & C_2 & 0 & 210.5\,S_2+202 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_3 & -S_3 & 0 & 268\,C_3 \\ S_3 & C_3 & 0 & 268\,S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C_1C_{23} & -C_1S_{23} & S_1 & C_1(268\,C_{23}+210\,C_2) \\ S_1C_{23} & -S_1S_{23} & -C_1 & S_1(268\,C_{23}+210\,C_2) \\ S_{23} & C_{23} & 0 & 268\,S_{23}+210\,S_2+202 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

the DK for the pos. is given by the 4-th column of $^0T_4$:

$$f = \begin{pmatrix} C_1C_{23} & -C_1S_{23} & S_1 & C_1(268\,C_{23}+210\,C_2) \\ S_1C_{23} & -S_1S_{23} & -C_1 & S_1(268\,C_{23}+210\,C_2) \\ S_{23} & C_{23} & 0 & 268\,S_{23}+210\,S_2+202 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 174.5C_4 \\ 174.5S_4 \\ 0 \\ 1 \end{pmatrix} = \begin{cases} C_1(174.5\,C_{234}+268\,C_{23}+210\,C_2) \\ S_1(174.5\,C_{234}+268\,C_{23}+210\,C_2) \\ 174\,S_{234}+268\,S_{23}+210\,S_2+202 \end{cases}$$

Exercise 3

Determine the symbolic expression of the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ for the robot in Fig. 1 (do not enter numerical values). Partition this matrix in blocks as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix}, \qquad \boldsymbol{v} = \boldsymbol{J}_L(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{1}$$

- Find all configurations $\boldsymbol{q}_L^*$, if any, where $\boldsymbol{J}_L(\boldsymbol{q})$ loses rank.
- Determine the range space of all feasible angular velocities $\boldsymbol{\omega} \in \mathbb{R}^3$.
- Find all singular configurations $\boldsymbol{q}^*$ of $\boldsymbol{J}(\boldsymbol{q})$, if any.

Choose next a configuration $\boldsymbol{q}_0$ where $\boldsymbol{J}_L$ is full rank, and substitute all the available numerical data in this matrix. Sketch this configuration and compute then a non-zero joint velocity $\dot{\boldsymbol{q}}_0 \in \mathbb{R}^4$ such that the resulting linear velocity $\boldsymbol{v}$ of the robot end-effector at $\boldsymbol{q}_0$ is identically zero.

the linear part is given by $\frac{\partial f}{\partial q}$:

$$J_L(q) = \begin{bmatrix} -S_1(174.5\,C_{234}+268\,C_{23}+210\,C_2) & -C_1(174.5\,S_{234}+268\,S_{23}+210\,S_2) & -C_1(174.5\,S_{234}+268\,S_{23}) & -C_1 174.5\,S_{234} \\ C_1(174.5\,C_{234}+268\,C_{23}+210\,C_2) & -S_1(174.5\,S_{234}+268\,S_{23}+210\,S_2) & -S_1(174.5\,S_{234}+268\,S_{23}) & -S_1 174.5\,S_{234} \\ 0 & 174\,C_{234}+268\,C_{23}+210\,C_2 & 174\,C_{234}+268\,C_{23} & 174\,C_{234} \end{bmatrix}$$

$$J_A(q) = \begin{bmatrix} z_0 & z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 0 & S_1 & S_1 & S_1 \\ 0 & -C_1 & -C_1 & -C_1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$J_A\dot{q} = \begin{bmatrix} 0 & S_1 & S_1 & S_1 \\ 0 & -C_1 & -C_1 & -C_1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{pmatrix} S_1(\dot{q}_2+\dot{q}_3+\dot{q}_4) \\ -C_1(\dot{q}_2+\dot{q}_3+\dot{q}_4) \\ \dot{q}_1 \end{pmatrix} \Rightarrow \mathcal{R}(J_A) = \text{Span}\left\{ \begin{bmatrix} S_1 \\ -C_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The rank of J_L in the regular case is 3.

$$J_L(q) = \begin{bmatrix} -s_1(174.5\,c_{234}+268\,c_{23}+210\,c_2) & -c_1(174.5\,s_{234}+268\,s_{23}+210\,s_2) & -c_1(174.5\,s_{234}+268\,s_{23}) & -c_1 174.5\,s_{234} \\ c_1(174.5\,c_{234}+268\,c_{23}+210\,c_2) & -s_1(174.5\,s_{234}+268\,s_{23}+210\,s_2) & -s_1(174.5\,s_{234}+268\,s_{23}) & -s_1 174.5\,s_{234} \\ 0 & 174\,c_{234}+268\,c_{23}+210\,c_2 & 174\,c_{234}+268\,c_{23} & 174\,c_{234} \end{bmatrix}$$

if $c_2 = c_{23} = c_{234} = 0$, this happens if $q_2 = \pm\frac{\pi}{2}$ and $q_3, q_4 \in \{0, \pi\}$ (or vice versa):

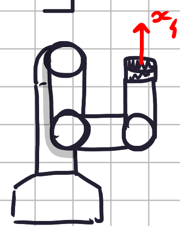
$$J_L(q) = \begin{bmatrix} 0 & -c_1(174.5\,s_{234}+268\,s_{23}+210\,s_2) & -c_1(174.5\,s_{234}+268\,s_{23}) & -c_1 174.5\,s_{234} \\ 0 & -s_1(174.5\,s_{234}+268\,s_{23}+210\,s_2) & -s_1(174.5\,s_{234}+268\,s_{23}) & -s_1 174.5\,s_{234} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank} = 1$$

since the 3 columns are linearly dependent.

if $s_2 = s_{23} = s_{234} = 0$, this happens if $q_2 = 0$ and $q_3, q_4 \in \{0, \pi\}$ (or vice versa):

$$J_L(q) = \begin{bmatrix} -s_1(174.5\,c_{234}+268\,c_{23}+210\,c_2) & 0 & 0 & 0 \\ c_1(174.5\,c_{234}+268\,c_{23}+210\,c_2) & 0 & 0 & 0 \\ 0 & 174\,c_{234}+268\,c_{23}+210\,c_2 & 174\,c_{234}+268\,c_{23} & 174\,c_{234} \end{bmatrix}$$

$\Rightarrow$ in this case, rank $= 2$.



let's consider $q_0 = (0, \pi, \frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow J_L(q_0) = \begin{pmatrix} 0 & 268 & 268 & 0 \\ -35.5 & 0 & 0 & 0 \\ 0 & -35.5 & 174.5 & 174.5 \end{pmatrix}$

$$\begin{pmatrix} 0 & 268 & 268 & 0 \\ -35.5 & 0 & 0 & 0 \\ 0 & -35.5 & 174.5 & 174.5 \end{pmatrix} \begin{pmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \\ \dot q_4 \end{pmatrix} = \begin{cases} 268(\dot q_2 + \dot q_3) = 0 \\ -35.5\,\dot q_1 = 0 \\ -35.5\,\dot q_2 + 174.5(\dot q_3 + \dot q_4) = 0 \end{cases} \Rightarrow \begin{cases} \dot q_2 = -\dot q_3 \\ \dot q_1 = 0 \\ \dot q_3 = -\frac{349}{366}\dot q_4 \end{cases} \Rightarrow \dot q^* = \begin{pmatrix} 0 \\ 1 \\ -1 \\ \frac{366}{349} \end{pmatrix} \neq 0 \text{ but } J_L \dot q^* = 0.$$

Exercise 4

Plan a cubic spline trajectory $q(t)$ that interpolates the following data at given time instants

$$t_1 = 1,\ q(t_1) = 45°, \quad t_2 = 2,\ q(t_2) = 90°, \quad t_3 = 2.5,\ q(t_3) = -45°, \quad t_4 = 4,\ q(t_4) = 45°, \tag{2}$$

starting with $\dot q(t_1) = 0$ and arriving with $\dot q(t_4) = 0$.

- Give an expression and the associated numerical values of the coefficients of each cubic polynomial.
- Find the maximum (absolute) values attained by the velocity $\dot q(t)$ and the acceleration $\ddot q(t)$ over the whole motion interval $[t_1, t_4]$, as well as the time instants at which these occur.
- Check if the following bounds are satisfied throughout the motion,

$$|\dot q(t)| \le V_{max} = 250°/s, \qquad |\ddot q(t)| \le A_{max} = 1000°/s^2, \tag{3}$$

  and, if needed, determine the minimum uniform scaling factor for the trajectory so that feasibility is recovered.
- Provide the total motion time of the feasible trajectory and sketch as accurately as possible the profiles of the resulting velocity and acceleration.

We have 4 knots, so we need 3 cubic.

$q_1 = 45 \quad q_2 = 90 \quad q_3 = -45 \quad q_4 = 45$

$t_1 = 1 \quad t_2 = 2 \quad t_3 = 2.5 \quad t_4 = 4 \Rightarrow$

$h_1 = 1, \quad h_2 = \frac{1}{2}, \quad h_3 = \frac{3}{2}$ since $h_k = t_{k+1} - t_k$ I construct the matrix:

$$A = \begin{pmatrix} 2(h_1+h_2) & h_1 \\ h_3 & 2(h_3+h_2) \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3/2 & 4 \end{pmatrix}$$

I construct the vector:

$$b = \begin{pmatrix} \frac{3}{h_1 h_2}(h_1^2(\bar{q}_3 - \bar{q}_2) + h_2^2(\bar{q}_2 - \bar{q}_1)) - h_2 v_1 \\ \frac{3}{h_2 h_3}(h_2^2(\bar{q}_4 - \bar{q}_3) + h_3^2(\bar{q}_3 - \bar{q}_2)) \end{pmatrix} = \begin{pmatrix} -\frac{1485}{2} \\ -\frac{10935}{16} \end{pmatrix}$$

$* \, v_1 = v_4 = 0$

$\Rightarrow$ i solve $Av = b$:

$$\begin{pmatrix} 3 & 1 \\ 3/2 & 4 \end{pmatrix}\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -\frac{1485}{2} \\ -\frac{10935}{16} \end{pmatrix} \Rightarrow \begin{cases} 3v_2 + v_3 = -\frac{1485}{2} \\ \frac{3}{2}v_2 + 4v_3 = -\frac{10935}{16} \end{cases} \Rightarrow \begin{cases} v_2 = -282.765 \\ v_3 = 105.795 \end{cases}$$

For each $k = 1,2,3$ let $a_{k0} = q_k$ and $a_{k1} = v_k$ and

in $[t_k, t_{k+1}]$: $q(t) = \theta_k(\tau) = \sum_{i=1}^{3} a_{ki}\tau^i$ with $\tau = t - t_k \in [0, h_k]$

For each $k = 1,2,3$ i solve: $\begin{pmatrix} h_k^2 & h_k^3 \\ 2h_k & 3h_k^2 \end{pmatrix}\begin{pmatrix} a_{k2} \\ a_{k3} \end{pmatrix} = \begin{pmatrix} q_{k+1} - q_k - v_k h_k \\ v_{k+1} - v_k \end{pmatrix}$

$$k=1) \quad \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{13} \end{pmatrix} = \begin{pmatrix} 45 \\ -282.765 \end{pmatrix} \Rightarrow \begin{cases} a_{12} = 417.765 \\ a_{13} = -372.765 \end{cases}$$

$$k=2) \quad \begin{pmatrix} 1/4 & 1/8 \\ 1 & 3/8 \end{pmatrix}\begin{pmatrix} a_{22} \\ a_{23} \end{pmatrix} = \begin{pmatrix} 6.3825 \\ 388.56 \end{pmatrix} \Rightarrow \begin{cases} a_{22} = 1477.65 \\ a_{23} = -2904.24 \end{cases}$$

$$k=3) \quad \begin{pmatrix} 9/4 & 27/8 \\ 3 & 81/8 \end{pmatrix}\begin{pmatrix} a_{32} \\ a_{33} \end{pmatrix} = \begin{pmatrix} -158.6925 \\ -105.795 \end{pmatrix} \Rightarrow \begin{cases} a_{32} = -98.742 \\ a_{33} = 18.808 \end{cases}$$

$$\Rightarrow \quad q(t) = \begin{cases} \theta_1(\tau) = 45 + 417.765\tau^2 - 372.765\tau^3 & \tau = t-1 \quad \text{if } t \in [1,2] \\ \theta_2(\tau) = 90 - 282.765\tau + 1447.65\tau^2 - 2904.24\tau^3 & \tau = t-2 \quad \text{if } t \in [2, 2.5] \\ \theta_3(\tau) = -45 + 105.795\tau - 98.742\tau^2 + 18.808\tau^3 & \tau = t-2.5 \quad \text{if } t \in [2.5, 4] \end{cases}$$

$$\dot{q}(t) = \begin{cases} \dot{\theta}_1(\tau) = 835.53\tau - 1118.295\tau^2 & \tau = t-1 \quad \text{if } t \in [1,2] \\ \dot{\theta}_2(\tau) = 282.765 + 2895.3\tau - 8712.72\tau^2 & \tau = t-2 \quad \text{if } t \in [2,2.5] \\ \dot{\theta}_3(\tau) = 105.795 - 197.484\tau + 56.424\tau^2 & \tau = t-2.5 \quad \text{if } t \in [2.5, 4] \end{cases}$$

$\max|\dot{\theta}_1| = 156.066 \quad \max|\theta_2| = 523.30 \quad \max|\theta_3| = 67.0035 \Rightarrow \max|\dot{q}| = 523.3 > \overset{250}{\overset{\parallel}{V_{max}}}$

$$\ddot{q}(t) = \begin{cases} \ddot{\theta}_1(\tau) = 835.53 - 2236.59\tau & \tau = t-1 \quad \text{if } t \in [1,2] \\ \ddot{\theta}_2(\tau) = 2895.3 - 17425.44\tau & \tau = t-2 \quad \text{if } t \in [2,2.5] \\ \ddot{\theta}_3(\tau) = -197.484 + 112.848\tau & \tau = t-2.5 \quad \text{if } t \in [2.5, 4] \end{cases}$$

$\Rightarrow \max|\ddot{q}| = 66806.46 \Rightarrow$

scaling factor $= \max_{j=1,\dots,n}\left\{\frac{v_{j,peak}}{v_{j,max}}, \sqrt{\frac{a_{j,peak}}{a_{j,max}}}\right\} = \max\left\{\frac{523.3}{250}, \sqrt{\frac{6806.46}{1000}}\right\} = 2.609 \Rightarrow T_{min} = 10.436$