Exercise 1

Consider the rotation matrix

$$R_d = \frac{1}{3}\begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ -1 & -2 & -2 \end{pmatrix}.$$

Find, if possible, all angle-axis pairs $(\theta, \boldsymbol{r})$ that provide the desired orientation R_d. At the end, check your results by verifying that $R(\theta, \boldsymbol{r}) = R_d$.

By considering the explicit representation of $R(\theta, r)$, we know that:

$$\text{trace}\, R_d = \frac{1}{3}(-2+1-2) = -1 = 1 + 2\cos\theta \Rightarrow 2\cos\theta = -2 \Rightarrow \cos\theta = -1 \Rightarrow \sin\theta = 0$$

$\Rightarrow \theta = \pm\pi$. We have to solve $R_d = 2rr^T - I$ for r:

$$\frac{1}{3}\begin{bmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ -1 & -2 & -2 \end{bmatrix} = 2\begin{bmatrix} r_x^2 - 1 & r_x r_y & r_x r_z \\ r_x r_y & r_y^2 - 1 & r_y r_z \\ r_x r_z & r_y r_z & r_z^2 - 1 \end{bmatrix} \Rightarrow$$

$$2r_y^2 - 2 = \frac{1}{3} \Rightarrow 2r_y^2 = \frac{1}{3} + 2 \Rightarrow r_y^2 = \frac{1}{6} + 1 \Rightarrow r_y = \pm\sqrt{7/6} = \pm\sqrt{42}/6 \approx \pm 1.08$$

Case $r_y = \frac{\sqrt{42}}{6}$:

Since $2r_y r_z = -\frac{2}{3} \Rightarrow r_z = -\frac{1}{3r_y} = -0.3$

Since $2r_y r_x = \frac{2}{3} \Rightarrow r_x = \frac{1}{3r_y} = \sqrt{42}/21 = 0.3$

$\Rightarrow$ 2 solutions Found

- $r = (0.3,\ 1.08,\ -0.3)$ $\theta = \pi$
- $r = (0.3,\ 1.08,\ -0.3)$ $\theta = -\pi$

$\Uparrow$

4 solutions in total

$\Downarrow$

Case $r_y = -\frac{\sqrt{42}}{6}$:

Since $2r_y r_z = -\frac{2}{3} \Rightarrow r_z = -\frac{2}{3 \cdot 2r_y} = 0.3$

Since $2r_y r_x = \frac{2}{3} \Rightarrow r_x = -0.3$

$\Rightarrow$ 2 solutions Found

- $r = (-0.3,\ 1.08,\ 0.3)$ $\theta = \pi$
- $r = (-0.3,\ 1.08,\ 0.3)$ $\theta = -\pi$

Exercise 2

The end-effector of a robot undergoes a change of orientation between an initial R_i and a final R_f, as specified by

$$R_i = \begin{pmatrix} 0 & 0.5 & -\frac{\sqrt{3}}{2} \\ -1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0.5 \end{pmatrix}, \qquad R_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Provide a minimal representation of the relative rotation between the initial and the final orientation using YXY Euler angles $(\alpha_1, \alpha_2, \alpha_3)$. At the end, check your solutions by performing the direct computation.

Since we have ${}^0R_i$ and ${}^0R_f$, the relative relation is ${}^iR_f = {}^iR_0\, {}^0R_f = {}^0R_i^T R_f =$

$${}^iR_f = \begin{bmatrix} 0 & 0 & -1 \\ 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \end{bmatrix}$$

the symbolic matrix For the euler angles YXY is

$$R_\alpha = \begin{bmatrix} c\alpha_1 & 0 & s\alpha_1 \\ 0 & 1 & 0 \\ -s\alpha_1 & 0 & c\alpha_1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_2 & -s\alpha_2 \\ 0 & s\alpha_2 & c\alpha_2 \end{bmatrix}\begin{bmatrix} c\alpha_3 & 0 & s\alpha_3 \\ 0 & 1 & 0 \\ -s\alpha_3 & 0 & c\alpha_3 \end{bmatrix}$$

$$\Rightarrow R_\alpha = \begin{bmatrix} c\alpha_1 & s\alpha_1 s\alpha_2 & s\alpha_1 c\alpha_2 \\ 0 & c\alpha_2 & -s\alpha_2 \\ -s\alpha_1 & c\alpha_1 s\alpha_2 & c\alpha_1 c\alpha_2 \end{bmatrix}\begin{bmatrix} c\alpha_3 & 0 & s\alpha_3 \\ 0 & 1 & 0 \\ -s\alpha_3 & 0 & c\alpha_3 \end{bmatrix}$$

I have to solve

$$\begin{bmatrix} & S_1S_2 & \\ S_2S_3 & C_2 & -S_2C_3 \\ & C_1S_2 & \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \end{bmatrix}$$

$$\cos\alpha_2 = -\sqrt{3}/2 \Rightarrow \sin\alpha_2 = \pm 1/2$$

$$\Rightarrow \begin{cases} \sin\alpha_1 \sin\alpha_2 = 0 \\ \cos\alpha_1 \sin\alpha_2 = -1/2 \end{cases} \Rightarrow \begin{cases} \sin\alpha_1 = 0 \\ \cos\alpha_1 \left(\pm \frac{1}{2}\right) = -1/2 \end{cases} \Rightarrow \begin{cases} \sin\alpha_2 = 1/2 \Rightarrow \cos\alpha_1 = -1 \Rightarrow \\ \sin\alpha_2 = -1/2 \Rightarrow \cos\alpha_1 = 1 \Rightarrow \end{cases}$$

For now, we have 2 cases:

$$\begin{cases} \cos\alpha_2 = -\sqrt{3}/2 \\ \sin\alpha_2 = 1/2 \\ \cos\alpha_1 = -1 \\ \sin\alpha_1 = 0 \end{cases} \text{ AND } \begin{cases} \cos\alpha_2 = -\sqrt{3}/2 \\ \sin\alpha_2 = -1/2 \\ \cos\alpha_1 = 1 \\ \sin\alpha_1 = 0 \end{cases}$$

Since $-\sin\alpha_2 \cos\alpha_3 = 0 \Rightarrow$

$$\cos\alpha_3 = 0$$

$$\Rightarrow \sin\alpha_2 = 1/2 \Rightarrow \frac{1}{2}\sin\alpha_3 = \frac{1}{2} \Rightarrow \sin\alpha_3 = 1$$

$$\Rightarrow \sin\alpha_2 = -1/2 \Rightarrow -\frac{1}{2}\sin\alpha_3 = \frac{1}{2} \Rightarrow \sin\alpha_3 = -1$$

$$\Rightarrow \begin{cases} \cos\alpha_2 = -\sqrt{3}/2 \\ \sin\alpha_2 = 1/2 \\ \cos\alpha_1 = -1 \\ \sin\alpha_1 = 0 \\ \cos\alpha_3 = 0 \\ \sin\alpha_3 = 1 \end{cases} \text{ AND } \begin{cases} \cos\alpha_2 = -\sqrt{3}/2 \\ \sin\alpha_2 = -1/2 \\ \cos\alpha_1 = 1 \\ \sin\alpha_1 = 0 \\ \cos\alpha_3 = 0 \\ \sin\alpha_3 = -1 \end{cases}$$

$$\Rightarrow \alpha = \left(-\pi/2, \frac{5\pi}{6}, \pi/2\right) \text{ or } \alpha = \left(\pi/2, -\frac{5\pi}{6}, -\pi/2\right)$$

Exercise 3

A DC motor is used to move a link of length $L = 0.7$ [m], as shown in Fig. 1. The motor mounts on its axis an absolute encoder and uses as transmission elements an Harmonic Drive having a flexspline with $N_{FS} = 160$ teeth and a gear with two toothed wheels of radius $r_1 = 2$ and $r_2 = 4$ [cm], respectively.

- Compute the reduction ratio $n_r > 1$ of the transmission system. Which is the direction of rotation of the link when the motor angular position θ_m is turning counterclockwise?
- Determine the resolution of the absolute encoder that allows distinguishing two link tip positions that are $\Delta r = 0.1$ [mm] away. What should be the minimum number of tracks N_t of the encoder?
- If the link has an angular range $\Delta\theta_{max} = 180°$, how many turns of the motor are needed to cover the entire range? With a multi-turn absolute encoder, what is the minimum number of bits for counting all these turns?
- If the motor inertia is $J_m = 1.2 \cdot 10^{-4}$ [kgm^2], determine the optimal value of the link inertia J_l around the axis at its base which minimizes the motor torque τ_m needed for a desired link acceleration $\ddot{\theta}$. What is then the value of τ_m (in [Nm]) for $\ddot{\theta} = 7$ [rad/s^2]?

the red. ratio of the harmonic drive is $\frac{N_{FS}}{2} = 80$. The red. ratio of the gears is $\frac{r_2}{r_1} = 2 \Rightarrow n_r = 160$, every 160 revolution of the motor, we have one revolution of the link. $\theta_m = 160\,\theta_L$, $\theta_L = \frac{1}{160}\theta_m$

If two link tip positions are at distance $\Delta r = 0.1$ mm,

$$\frac{2\pi L}{360} = \frac{\Delta r}{x} \Rightarrow \frac{2\pi \cdot 0.7}{360} = \frac{(1/10000)}{\Delta\theta_L} \Rightarrow \Delta\theta_L = 8.18 \cdot 10^{-3}$$ degrees the encoder, to detect a change of $\Delta\theta_L$ on the link, should detect a change of $\Delta\theta_m = 160\,\Delta\theta_L = 0.13$ degrees. Number of tracks is $N_t = \left\lceil \log_2\left(\frac{360}{0.13}\right)\right\rceil = 12$

the optimal ratio is $n_r = \sqrt{J_L / J_m}$ so the optimal J_L is

$160 = \sqrt{J_L / 1.2 \cdot 10^{-4}} \Rightarrow J_L = 3.072 \ Kgm^2$ the torque τ_m needed

is $\tau_m = \left(1.2 \cdot 10^{-4} \cdot 160 + \frac{1}{160} \cdot 3.072\right) 7 = 0.2688$ Nm

To perform 180° on the link the motor should perform $160 \cdot 180$ degrees = 80 turns.

Exercise 4

A large 6R robot manipulator is mounted on the ceiling of an industrial cell and holds firmly a cylindric object in its jaw gripper. The world frame RF_w of the cell is placed on the floor, at about the cell center. The robot base frame RF_0 is defined by ${}^w\boldsymbol{T}_0$, while its end-effector frame RF_e has the origin O_e at the center of the grasped object. The robot direct kinematics is expressed in symbolic form by ${}^0\boldsymbol{T}_e(\boldsymbol{q})$, in terms of the joint variables $\boldsymbol{q}$. A camera is placed in the cell and its frame RF_c, having the origin O_c at the center of the image plane and the $\boldsymbol{z}_c$ unit vector along the focal axis of the camera, is defined by ${}^w\boldsymbol{T}_c$.

the relation between the ee and the camera is expressed by the following matrix: ${}^eT_c = \begin{bmatrix} A & B & \begin{matrix} 0 \\ 0 \\ -1 \end{matrix} & \begin{matrix} 0 \\ 0 \\ d \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$ where $A, B \in \mathbb{R}^3$ such that $\begin{bmatrix} A & B & \begin{matrix} 0 \\ 0 \\ -1 \end{matrix} \end{bmatrix} \in SO(3)$ and $\det \begin{bmatrix} A & B & \begin{matrix} 0 \\ 0 \\ -1 \end{matrix} \end{bmatrix} = 1$.

A and B are free but constrained vectors because it's irrelevant the angle between x_c and x_e from z_c.

We know that ${}^eT_c = {}^0T_e^{-1}(q) \, {}^wT_0^{-1} \, {}^wT_c$, this is the task kinematics equation.

The task is redundant: we have 6 dof but we don't want to control the full pose, just the position and the allignment to the camera, in the regular case we have ∞ solutions.

Exercise 5

For the spatial RPR robot of Fig. 3, complete the assignment of Denavit-Hartenberg (DH) frames and fill in the associated table of parameters. The origin of the last frame should be placed at the point P. Moreover, the frame assignment should be such that all constant DH parameters are *non-negative* and the value of the joint variables q_i, $i = 1, 2, 3$, are *strictly positive* in the shown configuration. Compute then the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ for the position of point P.

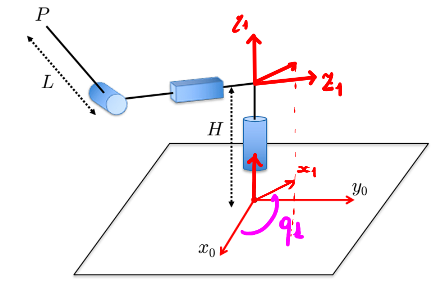
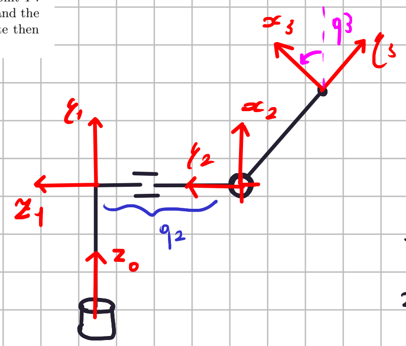


Figure 3: A spatial RPR robot.



	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	H	q_1
2	$\pi/2$	0	q_2	$\pi/2$
3	0	L	0	q_3

$$\begin{bmatrix} cq_1 & 0 & sq_1 & 0 \\ sq_1 & 0 & -cq_1 & 0 \\ 0 & 1 & 0 & H \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_3 & -sq_3 & 0 & L\,cq_3 \\ sq_3 & cq_3 & 0 & L\,sq_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & sq_1 & cq_1 & q_2 sq_1 \\ 0 & -cq_1 & sq_1 & -q_2 cq_1 \\ 1 & 0 & 0 & H \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_3 & -sq_3 & 0 & L\,cq_3 \\ sq_3 & cq_3 & 0 & L\,sq_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} s_1 s_3 & s_1 c_3 & c_1 & L s_1 s_3 + q_2 s_1 \\ -c_1 s_3 & -c_1 c_3 & s_1 & -L c_1 s_3 - q_2 c_1 \\ c_3 & -s_3 & 0 & L c_3 + H \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow f(q) = p : \begin{cases} L \sin q_1 \sin q_3 + q_2 \sin q_1 & x \\ -L \cos q_1 \sin q_3 - q_2 \cos q_1 & y \\ L \cos q_3 + H & z \end{cases}$$

Exercise 6

For the spatial RPR robot of Fig. 3, provide the closed-form expression of the inverse kinematics for the position $\boldsymbol{p}$ of point P. Assuming for simplicity that the joints have unlimited ranges, how many inverse kinematics solutions are there in the regular case? Compute the numerical values of all inverse solutions $\boldsymbol{q}$ when $\boldsymbol{p} = (3, 4, 1.5)$ [m] and the geometric parameters of the robot are $H = L = 1$ [m]. Check the solutions!

note: s3 is not sin(q3) but just a symbol to replace the square root of 1 minus...ecc

$$\begin{cases} p_x = L\sin q_1 \sin q_3 + q_2 \sin q_1 \\ p_y = -L\cos q_1 \sin q_3 - q_2 \cos q_1 \\ p_z = L\cos q_3 + H \end{cases}$$

We can find q_3:

$$p_z = L\cos q_3 + H \Rightarrow \cos q_3 = \frac{p_z - H}{L} \Rightarrow \sin q_3 = \pm \underbrace{\sqrt{1 - \frac{p_z - H}{L}}}_{S_3}$$

$$\Rightarrow q_3 = \text{atan2}\{\pm S_3, \cos q_3\}$$

$\Rightarrow$ to find q_2 we square and sum the first two

$$p_x^2 + p_y^2 = L^2 S_3^2 + q_2^2 + 2L S_3 q_2 \sin^2 q_1 + 2L S_3 q_2 \cos^2 q_1 =$$

$$\Rightarrow q_2^2 + (2LS_3)q_2 + (L^2 S_3^2 - p_x^2 - p_y^2) = 0 \Leftarrow \text{Solvable for } q_2$$

$$\Delta = 4L^2S_3^2 - 4(L^2S_3^2 - p_x^2 - p_y^2) = 4L^2S_3^2 - 4L^2S_3^2 + 4p_x^2 + 4p_y^2 = 4p_x^2 + 4p_y^2$$

$$q_2 = \frac{-2LS_3 \pm \sqrt{\Delta}}{2} \Rightarrow \begin{matrix} \nearrow q_2^+ \\ \searrow q_2^- \end{matrix}$$ for each value of $S_3 \Rightarrow 4$ solutions in total.

$\Rightarrow$ the remaining q_1 can be found by solving the 2×2 linear system for $\cos q_1$, $\sin q_1$.

Let $H = L = 1$ and $p = (3, 4, 1.5)$. $\cos q_3 = \frac{1.5 - 1}{1} = \frac{1}{2} \Rightarrow \sin q_3 = \pm\sqrt{1/2} = \pm\frac{\sqrt{2}}{2}$

$\Rightarrow q_3 = 0.955$ or $q_3 = -0.955$

Case $q_3 = 0.955$

$$\Delta = 4\frac{\sqrt{2}}{2} - 4\left(\left(\frac{\sqrt{2}}{2}\right)^2 - 3^2 - 4^2\right) =$$

$$2\sqrt{2} - 4\left(\frac{1}{2} - 9 - 16\right) = 100.82$$

$$\Rightarrow q_2 = \frac{-\sqrt{2} + \sqrt{\Delta}}{2}, \quad q_2 = \frac{-\sqrt{2} - \sqrt{\Delta}}{2}$$

$\Rightarrow q_2 = 4.31$ or $q_2 = -5.727$

$$\begin{cases} 3 = \sin q_1 \frac{\sqrt{2}}{2} + 4.31 \sin q_1 \\ 4 = -\cos q_1 \frac{\sqrt{2}}{2} - 4.31 \sin q_1 \end{cases}$$

$$\Rightarrow \begin{cases} 3 = \sin q_1 \frac{\sqrt{2}}{2} - 5.72 \sin q_1 \\ 4 = -\cos q_1 \frac{\sqrt{2}}{2} + 5.72 \cos q_1 \end{cases}$$