

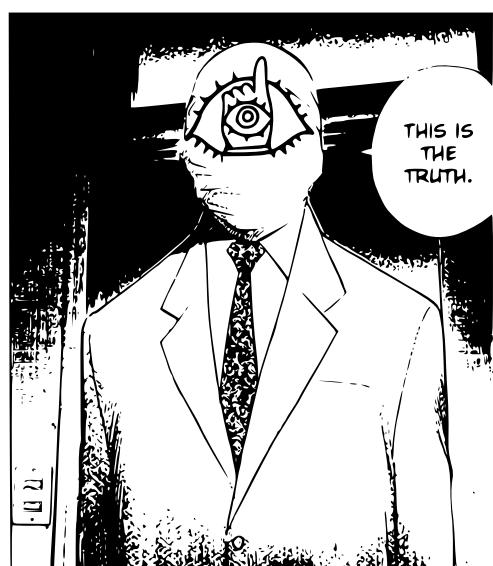
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This document summarizes and presents the topics for the Robotics 2 course for the Master's degree in Artificial Intelligence and Robotics at Sapienza University of Rome. The document is free for any use. If the reader notices any typos, they are kindly requested to report them to the author.



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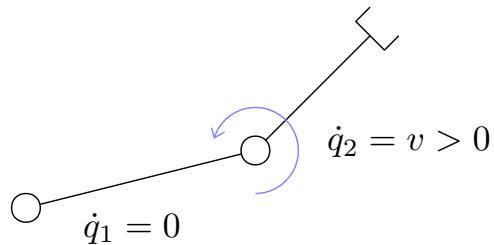
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CHAPTER

1

INTRODUCTION

Previously, in the Robotics 1 course, we described the kinematic behavior of a robotic manipulator, assuming that no force are involved, and that we can control directly the angular velocity of each joint, without any kind of resistance. We consider the following example

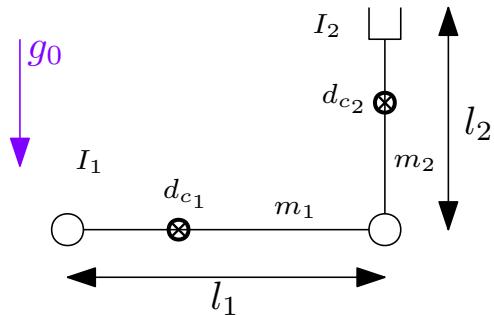


it is clear that, considering the positive angular velocity $v > 0$, in the next configuration the second link will rotate counter clock wise, while the first joint remain fixed. If we have to take in account the dynamics, we should also consider

- the center of mass of the first and the second link
- the mass of the first and the second link
- the inertia the first and the second link

and instead of describing the control trough angular velocities on the two joints, we define the torques τ_1, τ_2 applied on them. In this case (we ignore the effects of gravity), by applying a zero torque $\tau_1 = 0$ on the first joint, and a positive torque $\tau_2 > 0$ on the second joint, both joints will move, according with the laws of rigid body dynamics.

Let's consider the following example of a 2R planar robot, with the joint axis orthogonal respect to the direction of gravity.



where I_1, I_2 are the moment of inertia of the links around their center of mass, d_{c1}, d_{c2} is the distance along the links between the joint and the center of mass, m_1, m_2 are the mass of the links, and l_1, l_2 the lengths. The initial configuration at $t = 0$ is

$$\mathbf{q}_0 = \begin{pmatrix} 0 \\ \frac{\pi}{2} \end{pmatrix} \quad \dot{\mathbf{q}}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.1)$$

Consequently, also $\dot{\mathbf{p}} = \mathbf{0}$ (the velocity of the effector end). We want to know, what will be $\ddot{\mathbf{q}}$ (and consequently $\ddot{\mathbf{p}}$) in the initial configuration at $t = 0$, by considering the dynamics and the forces. The Jacobian matrix of this simple manipulator is

$$J = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix} \quad (1.2)$$

in the initial configuration $\mathbf{q}(0) = \mathbf{q}_0$ we have

$$J(\mathbf{q}_0) = \begin{pmatrix} -l_2 & -l_2 \\ l_1 & 0 \end{pmatrix} \quad (1.3)$$

from the differential relation

$$\ddot{\mathbf{p}} = J\ddot{\mathbf{q}} + \dot{J}\dot{\mathbf{q}} \quad (1.4)$$

since $\dot{\mathbf{q}}_0 = \mathbf{0}$

$$\ddot{\mathbf{p}}_0 = \begin{pmatrix} -l_2 & -l_2 \\ l_1 & 0 \end{pmatrix} \ddot{\mathbf{q}}_0 \quad (1.5)$$

we know that, by finding the initial joint acceleration $\ddot{\mathbf{q}}_0$ we can predict the next configuration and where the end effector will move. The full dynamic model is a system of differential equation and is the following

$$M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \mathbf{0} \quad (1.6)$$

where M is the 2×2 inertia matrix, and depends only on the current configuration, \mathbf{c} is a term representing the centrifugal and coriolis forces, and depends also on the velocities $\dot{\mathbf{q}}$. The last term \mathbf{g} is the gravity term, and depends on the current configuration. We are not explaining in details how the model is derived, since this is an introductory section, and all these topics will be described and expanded in the following chapters.

Since the robot is at rest, the \mathbf{c} vector is null, so

$$\ddot{\mathbf{q}} = M^{-1}(\mathbf{q})\mathbf{g}(\mathbf{q}) \quad (1.7)$$

the inertia matrix in the initial configuration (that for now we get for granted) is

$$M(\mathbf{q}_0) = \begin{pmatrix} I_1 + m_1 d_{c1}^2 + I_2 + m_2 d_{c2}^2 + m_2 l_1^2 & I_2 + m_2 d_{c2}^2 \\ I_2 + m_2 d_{c2}^2 & I_2 + m_2 d_{c2}^2 \end{pmatrix} \quad (1.8)$$

Theorem 1 (Steiner) *If I is the moment of inertia around an axis of a rigid body, and d is the distance between this axis and the center of mass, the inertia around the parallel axis passing through the center of mass will be $\bar{I} = I + md^2$, where m is the mass of the body.*

In the inertia matrix we see the terms $I_i + m_i d_{c_i}^2$, that is exactly the moment of inertia around the parallel axis passing through the center of mass, that we denote \bar{I}_i .

$$M(\mathbf{q}_0) = \begin{pmatrix} \bar{I}_1 + \bar{I}_2 + m_2 l_1^2 & \bar{I}_2 \\ \bar{I}_2 & \bar{I}_2 \end{pmatrix} \quad (1.9)$$

This inertia matrix is *always* positive definite, so it is invertible, in particular

$$M^{-1}(\mathbf{q}_0) = \frac{1}{\det M(\mathbf{q}_0)} \begin{pmatrix} \bar{I}_2 & -\bar{I}_2 \\ -\bar{I}_2 & \bar{I}_1 + \bar{I}_2 + m_2 l_1^2 \end{pmatrix} = \quad (1.10)$$

$$\frac{1}{(\bar{I}_1 + m_2 l_1^2)\bar{I}_2} \begin{pmatrix} \bar{I}_2 & -\bar{I}_2 \\ -\bar{I}_2 & \bar{I}_1 + \bar{I}_2 + m_2 l_1^2 \end{pmatrix} \quad (1.11)$$

the gravity term is the following

$$\mathbf{g}(\mathbf{q}_0) = \begin{pmatrix} m_1 d_{c1} g_0 + m_2 l_1 g_0 \\ 0 \end{pmatrix} = \begin{pmatrix} (m_1 d_{c1} + m_2 l_1) g_0 \\ 0 \end{pmatrix} \quad (1.12)$$

the first component $m_1 d_{c1} g_0 + m_2 l_1 g_0$ represents the total gravitational torque acting on the first joint.

- $m_1 d_{c_1} g_0$: This is the torque generated by the weight of the first link.
- $m_2 l_1 g_0$: This is the torque generated by the weight of the second link acting at the end of the first link.

due to the vertical alignment of the initial configuration, the gravity does not exert any torque on the second joint. By putting together the equations we get:

$$\ddot{\mathbf{q}}_0 = \frac{(m_1 d_{c_1} + m_2 l_1) g_0}{(\bar{I}_1 + m_2 l_1^2) \bar{I}_2} \begin{pmatrix} -\bar{I}_2 \\ \bar{I}_2 \end{pmatrix} = \begin{cases} \star < 0 \\ \star > 0 \end{cases} \quad (1.13)$$

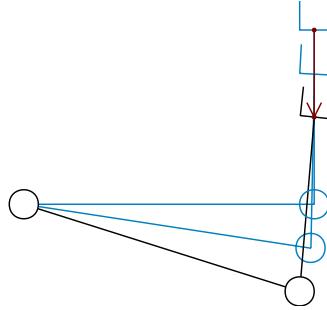
so the first joint will rotate clockwise while the second counter clockwise. We want to know if the end effector will fall to the left or to the right. We compute

$$\ddot{\mathbf{p}}_0 = \begin{pmatrix} -l_2 & -l_2 \\ l_1 & 0 \end{pmatrix} \ddot{\mathbf{q}}_0 = \quad (1.14)$$

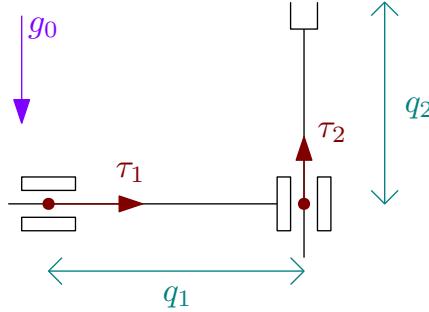
$$\ddot{\mathbf{p}}_0 = \begin{pmatrix} -l_2 & -l_2 \\ l_1 & 0 \end{pmatrix} \frac{(m_1 d_{c_1} + m_2 l_1) g_0}{(\bar{I}_1 + m_2 l_1^2) \bar{I}_2} \begin{pmatrix} -\bar{I}_2 \\ \bar{I}_2 \end{pmatrix} = \quad (1.15)$$

$$\ddot{\mathbf{p}}_0 = \begin{pmatrix} 0 \\ -\frac{l_1(m_1 d_{c_1} + m_2 l_1) g_0}{\bar{I}_1 + m_2 l_1^2} \end{pmatrix} \quad (1.16)$$

so the end effector will accelerate downward without moving on the x axis.



Now, we consider a simpler example from a dynamic point of view, but we will show the process to derive the dynamic model. We consider a planar 2P robot, where the joint's are orthogonal, in this simple case the Jacobian matrix is the identity. For this manipulator there will be no angular terms, since all the joint's are prismatic, we still refers with torques to the linear forces τ_1, τ_2 applied on the prismatic joints.



We consider two different methods to derive the dynamic model (both these methods we will further explained in details in their relative sections).

Newton-Euler

This methods perform a balance of forces acting on the system of rigid bodies. We start by the last link, backtracking towards the first. The forces acting on the second link are the torque τ_2 (the commands) and the gravity, so the dynamic equation for the second link (according to the Newton laws) is:

$$\tau_2 - m_2 g_0 = m_2 \ddot{q}_2 \quad (1.17)$$

on the first link, there is no force exerts from the gravity (since this link is constrained to move only along the x axis), so is affected only by the command torque τ_1 and the inertia of the second link (which opposes its motion):

$$\tau_1 - m_2\ddot{q}_1 = m_1\ddot{q}_1 \quad (1.18)$$

by putting the two equations together we obtain the following system

$$\begin{pmatrix} m_1m_2 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ m_2g_0 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \quad (1.19)$$

the first matrix M is the inertia matrix while the vector $\begin{pmatrix} 0 \\ m_2g_0 \end{pmatrix}$ is the gravity term.

Euler-Lagrange

This methods is based on the energy of the system. The kinetic energy T is given by the sum of the kinetic energies of the two links $T_1 + T_2$, the same applies for the potential energy $U = U_1 + U_2$ (that in this case, is given by the gravity field). The kinetic energy of the first link (that can move only along the x axis) is

$$T_1 = \frac{1}{2}m_1\dot{q}_1^2 \quad (1.20)$$

since the second link can move in both directions, and their velocity is given by $\mathbf{v}_{c_2} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$, the kinetic energy is

$$T_2 = \frac{1}{2}m_2\|\mathbf{v}_{c_2}\|^2 = \frac{1}{2}m_2(\dot{q}_1^2 + \dot{q}_2^2) \quad (1.21)$$

the potential energy of the first link is constant, since it can't move along the y axis, we denote this constant energy just U_1 . The potential energy of the second link, given by the gravity field, is $U_2 = m_2g_0q_2$.

The *Lagrangian* is a function describing the dynamics of the system (will be explained in the next chapters), is given by the difference between the kinetic and the potential energy

$$\mathcal{L} = T - U \quad (1.22)$$

depends on the current configuration \mathbf{q} and the velocity $\dot{\mathbf{q}}$. In this case we have

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + \dot{q}_2^2) - U_1 - m_2g_0q_2 \quad (1.23)$$

there is a simple principle in Physics, the trajectories of the system (the evolution over time of \mathbf{q} and $\dot{\mathbf{q}}$) are the one that *minimizes the action over time*, that is the following scalar function

$$\mathcal{S} = \int_0^t \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) dt \quad (1.24)$$

this is a problem of Calculus of Variations (the minimization of a functional over a set of functions) and is proved to be solved by the trajectories that satisfies the **Euler-Lagrange Equations**:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \quad 1 \leq i \leq n \quad (1.25)$$

for our 2P robotic system, we have the two following equations

$$\begin{cases} (m_1 + m_2)\ddot{q}_1 = \tau_1 \\ m_2\ddot{q}_2 + m_2g_0 = \tau_2 \end{cases} \quad (1.26)$$

That fully describes the model.