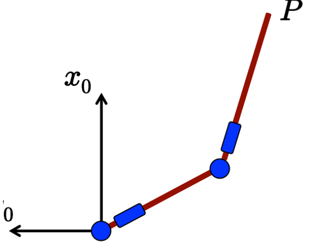
Exercise 1

Consider the 4-dof planar RPRP robot in Fig. 1 and assume that every joint has an unlimited range.

- Assign the link frames according to the Denavit-Hartenberg convention. Place the origin of the last frame coincident with point P. Make the free choices that are available so as to eliminate (i.e., "zeroing out") as many unnecessary constant parameters as possible. Draw the chosen frames directly on the robot in Fig. 1.
- Provide the Denavit-Hartenberg table of parameters associated to the frames that have been assigned. Draw the robot in the configuration $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}^T$.
- A task requires to place the end-effector frame at a desired position $\boldsymbol{p}_d = \begin{pmatrix} p_{dx} & p_{dy} \end{pmatrix}^T$, with a given orientation α_d of axis $\boldsymbol{x}_4$ w.r.t. axis $\boldsymbol{x}_0$ of the base. For the RPRP robot, define the analytic Jacobian associated to this three-dimensional task and determine all its singular configurations.



The DK is $f(q) = \begin{cases} q_2 c_1 + q_4 c_{13} \\ q_2 s_1 + q_4 s_{13} \end{cases}$ so the Jacobian is:

$$\frac{df}{dq} = \begin{pmatrix} -q_2 s_1 - q_4 s_{13} & c_1 & -q_4 s_{13} & c_{13} \\ q_2 c_1 + q_4 c_{14} & s_1 & q_4 c_{13} & s_{13} \end{pmatrix} \Rightarrow \text{the max rank is two.}$$

Is singular iff $q_2 = q_4 = 0$: $J_s = \begin{pmatrix} 0 & c_1 & 0 & c_{13} \\ 0 & s_1 & 0 & s_{13} \end{pmatrix}$ in that case moving q_1 or q_3 produces no end effector velocity:

$$\begin{pmatrix} 0 & c_1 & 0 & c_{13} \\ 0 & s_1 & 0 & s_{13} \end{pmatrix} \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \forall a, b \in \mathbb{R}.$$

Exercise 2

A planar 2R robot, with link lengths $\ell_1 = 1$ and $\ell_2 = 0.5$ [m], has its end-effector placed in the Cartesian position $\boldsymbol{p}_0 = \begin{pmatrix} 0.7 & 0.7 \end{pmatrix}^T$ [m] and is at rest at time $t = 0$. Using separation in space and time, plan a Cartesian trajectory for the robot end-effector in order to pick an object in the position $\boldsymbol{p}_d = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ [m] at a given time $T > 0$ (to be treated symbolically in this problem). The object is on a conveyor belt, moving with a constant velocity $\boldsymbol{v}_d = V \cdot \begin{pmatrix} -1 & 0 \end{pmatrix}^T$, where $V = 1$ [m/s] is the speed. The robot end effector should match this velocity at the final position. Moreover, the motion task should be executed with joint velocities $\dot{\boldsymbol{q}}(t)$ that are continuous for all $t \in [0, T]$.

- Provide the parametric expression $\boldsymbol{p}(s)$ of the chosen Cartesian path, and of its first and second derivative with respect to the path parameter s.
- Provide the expression of a timing law $s(t)$ that satisfies the required conditions.
- Assuming a motion time $T = 1.6$ [s], compute the joint velocity $\dot{\boldsymbol{q}}_{mid} = \dot{\boldsymbol{q}}(T/2)$ at $t = T/2$, when the robot is executing the planned Cartesian trajectory. How many solutions are there?

let $h(t)$ to be the moving object:

$$h(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} t, \quad \text{in } T \text{ we have: } h(T) = \begin{pmatrix} -T \\ 1 \end{pmatrix}$$

So for our end effector position p we want that:

$$p(0) = 0.7 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad p(T) = \begin{pmatrix} -T \\ 1 \end{pmatrix} \qquad \dot{p}(0) = 0 \qquad \dot{p}(T) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

I want $p'(T) = \begin{pmatrix} -1 & 0 \end{pmatrix}$ so

$$p_x(s) = as^2 + bs + c$$

$$\begin{cases} p_x(0) = c = 0.7 \\ p_x(T) = aT^2 + bT + 0.7 = -T \\ \dot{p}_x(T) = 2aT + b = -1 \end{cases} \Rightarrow \begin{cases} aT^2 + T(-1 - 2aT) + 0.7 = -T \\ b = -1 - 2aT \end{cases} \Rightarrow \begin{cases} a = \frac{0.7}{T^2} \\ b = -1 - \frac{1.4}{T} \end{cases}$$

$$p_y(s) = as^2 + bs + c$$

$$a := -\frac{0.3}{T^2}$$

$$\begin{cases} p_y(0) = c = 0.7 \\ p_y(T) = aT^2 + bT + 0.7 = 1 \\ \dot{p}_y(T) = 2aT + b = 0 \end{cases} \Rightarrow \begin{cases} aT^2 - 2aT^2 = 0.3 \\ b = -2aT \end{cases} \quad \begin{cases} a = -\frac{0.3}{T^2} \\ b = \frac{0.6}{T} \end{cases}$$

Since $\dot{p} = p' \dot{s}$ and $p'(T) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ i need $\dot{s}(T) = 1$.

$\Rightarrow s(t) = at^3 + bt^2 + ct + d$ so:

$$\begin{cases} s(0) = 0 \\ s(T) = 1 \\ \dot{s}(0) = 0 \\ \dot{s}(T) = 1 \end{cases} \Rightarrow \begin{cases} d = 0 \\ c = 0 \\ aT^3 + bT^2 = 1 \\ 3aT^2 + 2bT = 1 \end{cases} \Rightarrow \begin{cases} a = \frac{1 - bT^2}{T^3} \\ 3\frac{1 - bT^2}{T} + 2bT = 1 \end{cases} \Rightarrow \begin{cases} a = \frac{T-2}{T^3} \\ b = -\frac{1}{T} + \frac{3}{T^2} \end{cases}$$

IF $T = 1.6$: $s(t) = -\frac{25}{256}t^3 + \frac{35}{64}t^2$, $\dot{s}(t) = -\frac{75}{256}t^2 + \frac{70}{64}t$

$$p(s) = \begin{cases} \frac{35}{128}s^2 - \frac{15}{8}s + 0.7 \\ -\frac{15}{128}s^2 + \frac{3}{8}s + 0.7 \end{cases}, \quad p'(s) = \begin{cases} \frac{70}{128}s - \frac{15}{8} \\ -\frac{30}{128}s + \frac{3}{8} \end{cases}$$

$$s(\tfrac{T}{2}) = s(0.8) = \frac{3}{10} \Rightarrow p'(s(\tfrac{T}{2})) = p'(\tfrac{3}{10}) = \begin{cases} -\frac{219}{128} \\ \frac{39}{128} \end{cases} \Rightarrow \dot{p}(\tfrac{T}{2}) = \frac{11}{16}\begin{pmatrix} -\frac{219}{128} \\ \frac{39}{128} \end{pmatrix} = \begin{pmatrix} -\frac{2409}{2048} \\ \frac{429}{2048} \end{pmatrix} = \frac{1}{2048}\begin{pmatrix} -2409 \\ 429 \end{pmatrix}$$

$$\dot{s}(\tfrac{T}{2}) = \dot{s}(0.8) = \frac{11}{16}$$

$p(\tfrac{T}{2}) = \begin{pmatrix} \frac{83}{512} \\ \frac{1531}{2560} \end{pmatrix} \Rightarrow$ Since $S(q) = \begin{cases} c_1 + \frac{1}{2}c_{12} \\ s_1 + \frac{1}{2}s_{12} \end{cases}$ i solve the IK to find $q(\tfrac{T}{2})$

$$\begin{cases} c_1 + \frac{1}{2}c_{12} = \frac{83}{512} \\ s_1 + \frac{1}{2}s_{12} = \frac{1531}{2560} \end{cases} \Rightarrow c_2 = -0.866 \Rightarrow s_2 = \frac{1}{2} \Rightarrow q_2 = 2.618$$ so:

$$\begin{cases} c_1 + \frac{1}{2}(c_1c_2 - s_1s_2) = \frac{83}{512} \\ s_1 + \frac{1}{2}(s_1c_2 + c_1s_2) = \frac{1531}{2560} \end{cases} \begin{cases} \frac{567}{1000}c_1 - \frac{1}{4}s_1 = \frac{83}{512} \\ \frac{1}{4}c_1 + \frac{567}{1000}s_1 = \frac{1531}{2560} \end{cases} \Rightarrow \begin{cases} c_1 = 0.6287 \\ s_1 = 0.7775 \end{cases} \Rightarrow q_1 = 0.89$$

Since $J(q) = \begin{pmatrix} -s_1 - \frac{1}{2}s_{12} & -\frac{1}{2}s_{12} \\ c_1 + \frac{1}{2}c_{12} & \frac{1}{2}c_{12} \end{pmatrix} \Rightarrow J(q(\tfrac{T}{2})) = \begin{pmatrix} -\frac{1531}{2560} & \frac{2297}{12800} \\ \frac{83}{512} & -0.4673 \end{pmatrix}$, $\dot{q}(\tfrac{T}{2}) = J^{-1}(q(\tfrac{T}{2}))\dot{p}(\tfrac{T}{2}) = \begin{pmatrix} 2.045 \\ 0.261 \end{pmatrix}$