Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise 1

Consider the spatial 3R robot in Fig. 1.

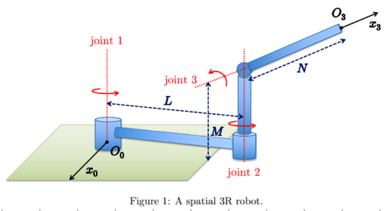
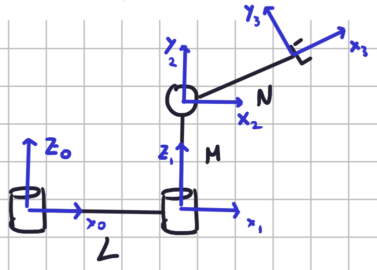


Figure 1: A spatial 3R robot.

- Provide the 3×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ of the robot in
$$\boldsymbol{v} = \dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$
and determine all the kinematic singularities, each with the associated rank of $\boldsymbol{J}(\boldsymbol{q})$.
- In a singularity $\boldsymbol{q}_s$ where rank $\boldsymbol{J}(\boldsymbol{q}_s) = 1$, find an admissible end-effector velocity $\boldsymbol{v}_s \in \mathbb{R}^3$ and a joint velocity $\dot{\boldsymbol{q}}_s \in \mathbb{R}^3$ such that $\boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}}_s = \boldsymbol{v}_s \neq \boldsymbol{0}$. Is such $\dot{\boldsymbol{q}}_s$ unique for a given admissible end-effector velocity $\boldsymbol{v}_s$?



Joint	α_i	a_i	d_i	θ_i
1	0	L	0	q_1
2	$\frac{\pi}{2}$	0	M	q_2
3	0	N	0	q_3

$${}^0T_1 = \begin{pmatrix} C_1 & -S_1 & 0 & LC_1 \\ S_1 & C_1 & 0 & LS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1T_2 = \begin{pmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & M \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^2T_3 = \begin{pmatrix} C_3 & -S_3 & 0 & NC_3 \\ S_3 & C_3 & 0 & NS_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0T_2 = \begin{pmatrix} C_1 & -S_1 & 0 & LC_1 \\ S_1 & C_1 & 0 & LS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & M \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C_{12} & 0 & S_{12} & LC_1 \\ S_{12} & 0 & -C_{12} & LS_1 \\ 0 & 1 & 0 & M \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0T_3 = \begin{pmatrix} C_{12} & 0 & S_{12} & LC_1 \\ S_{12} & 0 & -C_{12} & LS_1 \\ 0 & 1 & 0 & M \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} C_3 & -S_3 & 0 & NC_3 \\ S_3 & C_3 & 0 & NS_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C_{12}C_3 & -C_{12}S_3 & S_{12} & NC_{12}C_3+LC_1 \\ S_{12}C_3 & -S_{12}S_3 & -C_{12} & NS_{12}C_3+LS_1 \\ S_3 & C_3 & 0 & NS_3+M \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So $f_r(q) = \begin{cases} NC_{12}C_3+LC_1 \\ NS_{12}C_3+LS_1 \\ NS_3+M \end{cases}$

then $J = \begin{pmatrix} -NS_{12}C_3-LS_1 & -NS_{12}C_3 & -NC_{12}S_3 \\ NC_{12}C_3+LC_1 & NC_{12}C_3 & -NS_{12}S_3 \\ 0 & 0 & NC_3 \end{pmatrix}$

$$\det J = N^2C_{12}C_3^2(-NS_{12}C_3-LS_1) + N^2C_3^2S_{12}(NC_{12}C_3+LC_1) = -N^3C_3^3C_{12}S_{12} - N^2LC_{12}C_3^2S_1 + N^3C_3^3S_{12}C_{12} + N^2LC_3^2S_{12}C_1 =$$

$$= N^2LC_3^2S_{12}C_1 - N^2LC_{12}C_3^2S_1 = N^2LC_3^2(C_1S_{12}-S_1C_{12}) = N^2LC_3^2S_2 = 0 \iff \begin{cases} C_3 = 0 \\ S_2 = 0 \end{cases} \iff \begin{cases} q_3 = \pm\frac{\pi}{2} \\ \vee \\ q_2 \in \{0, \pi\} \end{cases}$$

IF $q_3 = \pm\frac{\pi}{2}$ then:

$J = \begin{pmatrix} -LS_1 & 0 & \mp NC_{12} \\ LC_1 & 0 & \mp NS_{12} \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank} J = 1$

if $q_2 \in \{0, \pi\}$ then:

$J = \begin{pmatrix} -S_1(NC_3+L) & -NS_1C_3 & -NC_1S_3 \\ C_1(NC_3+L) & NC_1C_3 & -NS_1S_3 \\ 0 & 0 & NC_3 \end{pmatrix} \Rightarrow \text{rank } J = 2$

let $q_s = (0, \frac{\pi}{2}, 0)$:

$J(q_s) = \begin{pmatrix} 0 & 0 & 0 \\ L & 0 & N \\ 0 & 0 & 0 \end{pmatrix}$, $\dot{q} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow v = \begin{pmatrix} 0 \\ N \\ 0 \end{pmatrix}$. is not unique since with $\dot{q} = (\frac{N}{2L}, 0, \frac{1}{2})$ $\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ L & 0 & N \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} N/2L \\ 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ N \\ 0 \end{pmatrix}$

Exercise 2

A planar RP robot is commanded at the acceleration level. Its end-effector position is given by
$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}. \tag{1}$$

If the robot is in a generic nonsingular configuration $\boldsymbol{q}$ and with non-zero velocities for both joints, determine the explicit expression of a command $\ddot{\boldsymbol{q}}$ such that the end-effector acceleration is instantaneously $\ddot{\boldsymbol{p}} = \boldsymbol{0}$. Is this command unique?

Since $\ddot{p} = J\ddot{q} + \dot{J}\dot{q} \Rightarrow \ddot{q} = J^{-1}(\ddot{p} - \dot{J}\dot{q})$ For $\ddot{p} = 0 \Rightarrow \ddot{q} = -J^{-1}\dot{J}\dot{q}$

$J = \begin{pmatrix} -q_2S_1 & C_1 \\ q_2C_1 & S_1 \end{pmatrix}$ $\dot{J} = \begin{pmatrix} (-S_1 - q_2C_1)\dot{q}_2 & -S_1\dot{q}_1 \\ (C_1 - q_2S_1)\dot{q}_2 & C_1\dot{q}_1 \end{pmatrix} \Rightarrow \ddot{q} = \begin{pmatrix} -q_2S_1 & C_1 \\ q_2C_1 & S_1 \end{pmatrix}^{-1}\begin{pmatrix} (-S_1 - q_2C_1)\dot{q}_2 & -S_1\dot{q}_1 \\ (C_1 - q_2S_1)\dot{q}_2 & C_1\dot{q}_1 \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$

Exercise 3

Consider again the same RP robot of Exercise #2. Suppose that the generalized forces $\boldsymbol{\tau} \in \mathbb{R}^2$ that the robot actuators can provide at the two joints are bounded componentwise as

$$|\tau_1| \leq \tau_{max,1} = 10\,[\mathrm{Nm}], \qquad |\tau_2| \leq \tau_{max,2} = 5\,[\mathrm{N}].$$

In the configuration $\boldsymbol{q} = (\pi/3, 1.5)$ [rad,m], find the set of feasible Cartesian forces $\boldsymbol{F} = (F_x, F_y) \in \mathbb{R}^2$ (expressed in [N]) which can applied to the end effector and that the robot can sustain while in static equilibrium.

In that conf. the Jacobian is: $J = \begin{pmatrix} -q_2 s_1 & c_1 \\ q_2 c_1 & s_1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4}\sqrt{3} & 1/2 \\ 3/4 & \sqrt{3}/2 \end{pmatrix}$ and $J^T = \begin{pmatrix} -\frac{3}{4}\sqrt{3} & 3/4 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$

Since $J^T F = \tau$ i have to solve $|(J^T F)_1| \leq 10$, $|(J^T F)_2| \leq 5$.

$$\begin{pmatrix} -\frac{3}{4}\sqrt{3} & 1/2 \\ 3/4 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} F_x \\ F_y \end{pmatrix} \Rightarrow \begin{cases} \left|-\frac{3}{4}\sqrt{3}\cdot F_x + \frac{1}{2}F_y\right| \leq 10 \\ \left|\frac{3}{4}F_x + \frac{\sqrt{3}}{2}F_y\right| \leq 5 \end{cases} \Rightarrow \begin{cases} -10 \leq -\frac{3}{4}\sqrt{3}\cdot F_x + \frac{1}{2}F_y \leq 10 \\ -5 \leq \frac{3}{4}F_x + \frac{\sqrt{3}}{2}F_y \leq 5 \end{cases}$$

Exercise 4

The end-effector of a 2R planar robot with unitary link lengths has to track a linear path with constant speed $v_d = 0.5$ [m/s] between $\boldsymbol{P}_1 = (1, 0.5)$ and $\boldsymbol{P}_2 = (1, 1.5)$ [m]. However, at the initial time $t = 0$, the end effector is positioned in $\boldsymbol{P}_0 = (0.5, 0.5)$ [m]. The robot is commanded by a joint velocity $\dot{\boldsymbol{q}}$ that is limited componentwise as

$$|\dot{q}_1| \leq V_{max,1} = 3\,[\mathrm{rad/s}], \qquad |\dot{q}_2| \leq V_{max,2} = 2\,[\mathrm{rad/s}].$$

Design a kinematic control law that is able to achieve the fastest exponential convergence to zero of the trajectory tracking error, uniformly in all Cartesian directions, while being still feasible in terms of robot commands at time $t = 0$ for the given task. Provide some discussion on where/how fast the return to the original trajectory will be achieved.

let $p = S(q)$ and p_d to be the desired trajectory: $p_d = P_1 + vt(P_2 - P_1) = \begin{pmatrix} 1 \\ 1 + \frac{1}{4}t \end{pmatrix}$ $t \in [0,2]$

$e = p_d - p$ is the error, i want that $\dot{e} = -Ke$ with K a 2×2 diagonal matrix. positive

$\dot{e} = \dot{p}_d - \dot{p} = \dot{p}_d - J\dot{q}$ so i choose $\dot{q} = J^{-1}(\dot{p}_d + Ke)$ so: $\dot{e} = \dot{p}_d - J\dot{q} = \dot{p}_d - JJ^{-1}(\dot{p}_d + Ke) = -Ke$

$$\Rightarrow \begin{cases} \dot{e}_x = -K_1 e \\ \dot{e}_y = -K_2 e \end{cases} \Rightarrow \begin{cases} e_x = e_{0x}\cdot exp(-K_1 t) \\ e_y = e_{0y}\cdot exp(-K_2 t) \end{cases} \text{ where } \begin{cases} e_{0x} = 0.5 \\ e_{0y} = 0 \end{cases} \quad \text{since } J = \begin{pmatrix} -s_1 - s_{12} & -s_{12} \\ c_1 + c_{12} & c_{12} \end{pmatrix}$$

$$\dot{q} = \begin{pmatrix} -s_1 - s_{12} & -s_{12} \\ c_1 + c_{12} & c_{12} \end{pmatrix}^{-1} \cdot \left[\begin{pmatrix} 0 \\ 1/4 \end{pmatrix} + \begin{pmatrix} K_1 \frac{1}{2} exp(-K_1 t) \\ 0 \end{pmatrix}\right] = \begin{pmatrix} -s_1 - s_{12} & -s_{12} \\ c_1 + c_{12} & c_{12} \end{pmatrix}^{-1} \begin{pmatrix} \frac{K_1}{2} exp(-K_1 t) \\ 1/4 \end{pmatrix} = \frac{1}{s_2}\begin{pmatrix} c_{12} & s_{12} \\ -c_1 - c_{12} & -s_1 - s_{12} \end{pmatrix}\begin{pmatrix} \frac{K_1}{2} exp(-K_1 t) \\ 1/4 \end{pmatrix} =$$

$$= \begin{cases} \frac{1}{4}s_{12} + \frac{K_1}{2}c_{12}\cdot exp(-K_1 t) \\ -\frac{K_1}{2} exp(-K_1 t)(c_1 + c_{12}) - \frac{1}{4}(s_1 + s_{12}) \end{cases} \quad \text{i have to solve} \quad \begin{cases} \left|\frac{1}{4}s_{12} + \frac{K_1}{2}c_{12}\cdot exp(-K_1 t)\right| \leq 3 \\ \left|-\frac{K_1}{2} exp(-K_1 t)(c_1 + c_{12}) - \frac{1}{4}(s_1 + s_{12})\right| \leq 2 \end{cases}$$

the maximum value of $exp(-Kt)$ is reached in $t = 0$ so:

$$\begin{cases} \left|\frac{1}{4}s_{12} + \frac{K_1}{2}c_{12}\right| \leq \left|\frac{1}{4} + \frac{K_1}{2}\right| \leq 3 \\ \left|-\frac{K_1}{2}(c_1 + c_{12}) - \frac{1}{4}(s_1 + s_{12})\right| \leq \underbrace{\left|-K_1 - \frac{1}{2}\right|}_{K_1 > 0} \leq 2 \end{cases} \Rightarrow \begin{cases} \frac{1}{4} + \frac{K_1}{2} \leq 3 \Rightarrow K_1 \leq 5.5 \\ K_1 - \frac{1}{2} \leq 2 \Rightarrow K_1 \leq 2.5 \end{cases} \Rightarrow K = \begin{pmatrix} 2.5 & 0 \\ 0 & 0 \end{pmatrix}$$