Exercise 1

The end-effector of a robot manipulator has an initial orientation specified by the ZXY Euler angles $(\alpha, \beta, \gamma) = (\pi/2, \pi/4, -\pi/4)$ [rad] and should reach a final orientation specified by an axis-angle pair $(\boldsymbol{r}, \theta)$, with $\boldsymbol{r} = (0, -\sqrt{2}/2, \sqrt{2}/2)$ and $\theta = \pi/6$ rad. What is the required rotation matrix $\boldsymbol{R}_{if}$ between these two orientations? Represent $\boldsymbol{R}_{if}$ by the RPY-type angles (ϕ, χ, ψ) around the fixed-axes sequence YXY.

The First matrix ^0R_i is $R_z(\frac{\pi}{2})R_x(\frac{\pi}{4})R_y(-\frac{\pi}{4}) = \begin{bmatrix} 0.5 & -0.7 & 0.5 \\ 0.7 & 0 & -0.7 \\ 0.5 & 0.7 & 0.5 \end{bmatrix}$

the second is

$$^0R_f \quad \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}}_{rr^T} + (I - rr^T)\cos\frac{\pi}{6} + \begin{bmatrix} 0 & -\sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 & 0 \\ \sqrt{2}/2 & 0 & 0 \end{bmatrix} \sin\frac{\pi}{6} = \begin{bmatrix} 0.86 & -0.35 & -0.35 \\ 0.35 & 0.93 & -0.06 \\ 0.35 & -0.06 & 0.93 \end{bmatrix}$$

$$\Rightarrow \quad R_{if} = {}^iR_0 \, {}^0R_f = {}^0R_i^T \, {}^0R_f = \begin{bmatrix} 0.85 & 0.44 & 0.24 \\ -0.36 & 0.2 & 0.9 \\ 0.35 & -0.87 & 0.33 \end{bmatrix}$$

The RPY matrix for axes YXY in angles (ϕ, χ, ψ) is :

$$R_y(\psi)R_x(\chi)R_y(\phi) = \begin{pmatrix} -s\phi s\psi c\chi + c\phi c\psi & s\chi s\psi & s\phi c\psi + s\psi c\phi c\chi \\ s\phi s\chi & c\chi & -s\chi c\phi \\ -s\phi c\chi c\psi - s\psi c\phi & s\chi c\psi & -s\phi s\psi + c\phi c\chi c\psi \end{pmatrix}$$

I have to solve $R_{yxy}(\psi, \chi, \phi) = R_{if}$.

Since $\cos\chi = 0.2 \Rightarrow \sin\chi = \pm 0.979 \Rightarrow \chi = \pm 1.369$

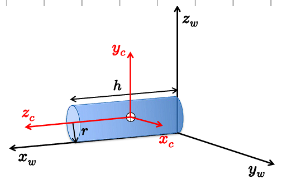
$$\begin{cases} s\phi s\chi = -0.36 \Rightarrow \\ -c\phi s\chi = 0.9 \Rightarrow \end{cases} \begin{cases} \pm 0.979 \cdot \sin\phi = -0.36 \Rightarrow \mp 0.367 \\ \mp 0.979 \cdot \cos\phi = 0.9 \Rightarrow \mp 0.919 \end{cases} \Rightarrow \begin{cases} \chi = 1.369 \Rightarrow \phi = \operatorname{atan2}\{-0.367, -0.919\} = -2.76 \\ \chi = -1.369 \Rightarrow \phi = \operatorname{atan2}\{0.367, 0.919\} = 0.37 \end{cases}$$

$$\begin{cases} s\chi s\psi = 0.44 \\ s\chi c\psi = -0.87 \end{cases} \Rightarrow \begin{cases} \pm 0.979 \sin\psi = 0.44 \Rightarrow \pm 0.449 \\ \pm 0.979 \cos\psi = -0.87 \Rightarrow \mp 0.88 \end{cases} \Rightarrow \begin{cases} \chi = 1.369 \Rightarrow \psi = \operatorname{atan2}\{0.449, -0.88\} = 2.66 \\ \chi = -1.369 \Rightarrow \psi = \operatorname{atan2}\{-0.449, 0.88\} = -0.47 \end{cases}$$

$$\Rightarrow (\phi, \chi, \psi) = (-2.76, 1.369, 2.66) \text{ or } = (0.37, -1.369, -0.47)$$

Exercise 2

A cylinder of height h and radius r lies on the plane (x_w, y_w) in the initial pose shown in Fig. 1, with a frame $RF_c = (x_c, y_c, z_c)$ attached to the geometric center of its body. The cylinder rolls without slipping by a ground distance $d > 0$ in the y_w-direction, and rotates then by an angle ϑ around the original z_w-axis. Finally, a rotation φ is performed around the current direction of the z_c-axis. Determine the expression of the elements of the homogeneous transformation matrix ${}^w\boldsymbol{T}_c(h, r, d, \vartheta, \varphi)$ that characterizes the final pose of the cylinder. Evaluate then ${}^w\boldsymbol{T}_c$ for $h = 0.5$, $r = 0.1$, $d = 1.5$ [m] and $\vartheta = \pi/3$, $\varphi = -\pi/2$ [rad]. *Hint: Check your intermediate results with simpler data.*



let RF_{cin} to be the Frame of the cylinder at the beginning.

The transformation from RF_w to RF_{cin} is :

2π

$2\pi r$

$$ {}^wT_{cin} = \begin{bmatrix} 0 & 0 & 1 & h/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

then, if the cylinder rolls towards $y_w = x_c$,

he performs a rotation around z_c of d/r radiants.

$$ {}^{cin}T_d = \begin{bmatrix} \cos\frac{d}{r} & -\sin d/r & 0 & d \\ \sin d/r & \cos d/r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

the rotation

$$ T_\theta = \begin{bmatrix} R_z(\theta) & 0 \\ 0 & 1 \end{bmatrix} $$

must be the first in the product since is around the original z_w axis.

The final rotation is $T_\varphi = \begin{bmatrix} R_z(\varphi) & 0 \\ 0 & 1 \end{bmatrix}$. The full transformation is:

$$ \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & h/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos d/r & -\sin d/r & 0 & d \\ \sin d/r & \cos d/r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & -\cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

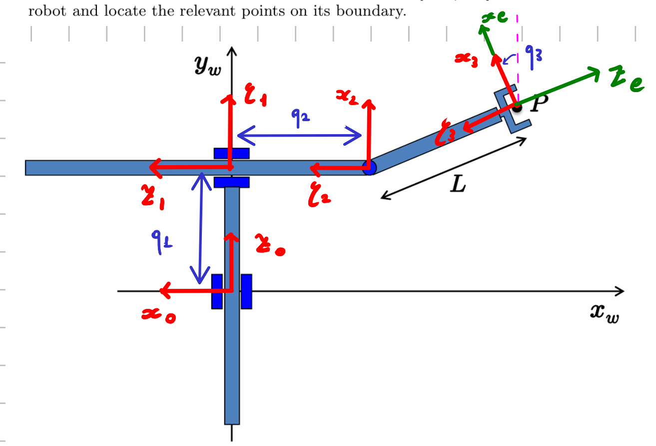
For $h = 0.5$, $r = 0.1$, $d = 1.5$, $\theta = \pi/3$ and $\varphi = -\pi/2$ we have:

$$ {}^wT_c = \begin{bmatrix} -0.56 & 0.65 & 0.5 & -1.17 \\ 0.32 & -0.37 & 0.86 & 0.96 \\ 0.75 & 0.65 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Exercise 3

Consider the PPR planar robot with a 2-jaw gripper in Fig. 2, shown together with the world frame RF_w.

- Assign the link frames and fill in the associated table of parameters according to the Denavit–Hartenberg (DH) convention (use the extra sheet). The origin of the last DH frame should be placed at the gripper's center (point P). Choose the frames so that there is **no** axis pointing inside the sheet.
- Determine the homogeneous transformation matrices ${}^w\boldsymbol{T}_0$ and ${}^3\boldsymbol{T}_e$, respectively between the world frame RF_w and the zero-th DH frame RF_0 and between the last DH frame RF_3 and the end-effector frame RF_e placed at the gripper, with the usual convention ($\boldsymbol{z}_e$ in the approach direction and $\boldsymbol{y}_e$ in the open/close slide direction of the jaws).
- Provide the direct kinematics for the end-effector position ${}^w\boldsymbol{p}_e \in \mathbb{R}^3$.
- When the two prismatic joints are limited as $q_i \in [q_{i,m}, q_{i,M}]$, under the assumption that $q_{i,M} - q_{i,m} > 2L$, for $i = 1, 2$, and the revolute joint is in the range $q_3 \in [-3\pi/4, 0]$, sketch the primary workspace of this robot and locate the relevant points on its boundary.



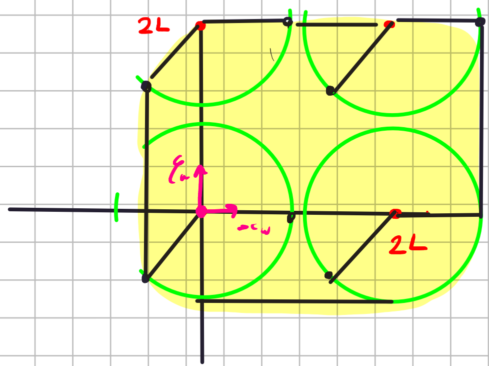
	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	q_1	$\pi/2$
2	$\pi/2$	0	q_2	$\pi/2$
3	0	L	0	q_3

$$
{}^wT_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
{}^3T_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0T_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \cos q_3 & -\sin q_3 & 0 & L\cos q_3 \\ \sin q_3 & \cos q_3 & 0 & L\sin q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \sin q_3 & \cos q_3 & 0 & -L\sin q_3 - q_2 \\ 0 & 0 & 1 & 0 \\ \cos q_3 & -\sin q_3 & 0 & L\cos q_3 + q_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\Rightarrow \text{I compute } {}^wT_e = {}^wT_0\, {}^0T_3\, {}^3T_e
$$

$$
{}^wT_e = \begin{bmatrix} -\sin q_3 & 0 & \cos q_3 & L\sin q_3 + q_2 \\ \cos q_3 & 0 & \sin q_3 & L\cos q_3 + q_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Exercise 4

With reference to the scheme in Fig. 3, assume that the three toothed gears of the transmission have radius, respectively, $r_m = 0.5$, $r_e = 40$, and $r_l = 10$ [cm]. The motor inertia is $J_m = 7.1 \cdot 10^{-4}$ kgm^2, while the inertia of the link around its rotation axis is denoted by J_l. An incremental encoder is mounted on the axis of the middle gear. Gravity is absent and inertia and friction of the gears are negligible.

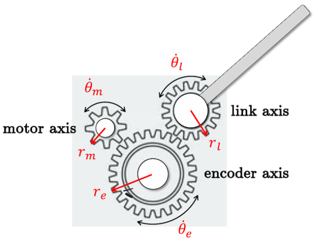


Figure 3: Transmission gears from motor to link, using an incremental encoder.

- What is the value of the link inertia J_l that optimizes torque transmission?
- With this J_l, what is the acceleration $\ddot{\theta}_l$ when the motor delivers on its axis a torque $\tau_m = 10$ [Nm]?
- For a link resolution of $0.01°$, how many pulses per turn (with quadrature) should the encoder have?
- With this resolution, what is the average speed $\dot{\theta}_m$ when the encoder increments 100 pulses per second?

$\theta_m = 80\,\theta_e$

the ratio is $n_r = \frac{40}{0.5} \cdot \frac{10}{40} = 20 \Rightarrow \theta_m = 20\,\theta_L \Rightarrow \theta_L = \frac{\theta_m}{20}$

Assuming n_r is optimal, the optimal J_L is such that $20 = \sqrt{J_L / 7.1 \cdot 10^{-4}} \Rightarrow J_L = 0.284$ kgm^2

IF $\tau_m = 10$ Nm $\Rightarrow 10 = \left(7.1 \cdot 10^{-4} \cdot 20 + \frac{0.284}{20}\right) \cdot \ddot{\theta}_L \Rightarrow \ddot{\theta}_L = 352 \frac{\text{rad}}{\text{second}}$

A link resolution of $0.01°$ is an encoder resolution of $\Delta\theta_e = \frac{10}{40} \cdot 0.01 = \frac{1}{400}°$

The res. $\Delta\theta_e$ is given by $\Rightarrow \frac{360°}{PPR \cdot 4} \Rightarrow PPR = 36\,000$

IF the encoder do 36 000 pulse/second he moves at 360 $\frac{\text{degrees}}{\text{second}}$

100 pulses $\Rightarrow \theta_e$ moves 1 degree $\Rightarrow \theta_m$ moves 80 degrees

$\Rightarrow 100 \frac{\text{pulses}}{\text{sec}} \Rightarrow \dot{\theta}_m = 80 \frac{\text{degrees}}{\text{second}}$

Exercise 5

The RPPR spatial robot shown in Fig. 4 has the DH parameters given in Tab. 1.

- Draw the corresponding DH frames (use the extra sheet) and give the values, or at least the signs, of the components of $\boldsymbol{q}$ in the shown configuration.
- Consider the task vector

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ \alpha \end{pmatrix} = \begin{pmatrix} \sin q_1 q_3 + L \cos q_1 \cos q_4 \\ -\cos q_1 q_3 + L \sin q_1 \cos q_4 \\ q_2 + L \sin q_4 \\ q_4 \end{pmatrix}. \qquad (1)$$

Solve the inverse kinematics problem in closed form for a given $\boldsymbol{r}_d \in \mathbb{R}^4$, determining also the possible singular situations. With $L = 1.5$ m, provide the numerical solutions for these data: $\boldsymbol{r}_{d1} = (2, 2, 4, -\pi/4)$, $\boldsymbol{r}_{d2} = (0, 0, 3, \pi/2)$, $\boldsymbol{r}_{d3} = (1, 1, 2, 0)$, and $\boldsymbol{r}_{d4} = (0, 1.5, 4, 0)$ [m,m,m,rad].

$q_4 = \alpha \Rightarrow q_2 = p_z - L \sin\alpha, \qquad \cos\alpha = u$

then $p_x^2 + p_y^2 = q_3^2 \sin^2 q_1 + L^2 \cos^2 q_1 u^2 + q_3^2 \cos^2 q_1 + L^2 u^2 \sin^2 q_1$

$p_x^2 + p_y^2 = q_3^2 + L^2 u^2 \Rightarrow q_3 = \pm\sqrt{p_x^2 + p_y^2 - L^2 u^2} = \pm v$

$x = \cos q_1$
$y = \sin q_1$

$$\Rightarrow \begin{cases} p_x = \pm v \sin q_1 + L u \cos q_1 \\ p_y = \mp v \cos q_1 + L u \sin q_1 \end{cases} \rightarrow \begin{cases} p_x = v \sin q_1 + L u \cos q_1 \\ p_y = -v \cos q_1 + L u \sin q_1 \end{cases} \quad \begin{cases} L u x + v y = p_x \\ -v x + L u y = p_y \end{cases}$$

$$\searrow \begin{cases} p_x = -v \sin q_1 + L u \cos q_1 \\ p_y = v \cos q_1 + L u \sin q_1 \end{cases} \quad \begin{cases} L u x - v y = p_x \\ v x + L u y = p_y \end{cases}$$

$$(*)\quad \begin{cases} L\omega x + \nu y = p_x \\ -\nu x + L\omega y = p_y \end{cases} \Rightarrow \det = L^2\omega^2 + \nu^2 \text{ should be not zero}$$

$$(**)\quad \begin{cases} L\omega x - \nu y = p_x \\ \nu x + L\omega y = p_y \end{cases}$$

$$(*) \Rightarrow \quad x = \overset{\cos q_1}{\overset{\shortparallel}{\frac{\nu p_y - L\omega p_x}{L^2\omega^2+\nu^2}}} \qquad y = \overset{\sin q_1}{\overset{\shortparallel}{\frac{L\omega p_y + \nu p_x}{L^2\omega^2+\nu^2}}}$$

Let $L = 1.5$, $r_d := 2, 2, 4, -\pi/4$

$$\Rightarrow q_4 = -\frac{\pi}{4} \qquad q_2 = p_z - L\sin\alpha = 5.06 \qquad \omega = \overset{\sqrt{2}/2}{\overset{\shortparallel}{\cos\left(-\frac{\pi}{4}\right)}} = 0.707$$

$$q_3 = \pm\sqrt{p_x^2 + p_y^2 - L^2\omega^2} = \pm\sqrt{4 + 4 - (1.5\cdot 0.707)^2} = \pm 2.622$$

$$\begin{cases} p_x = \pm\nu\sin q_1 + L\omega\cos q_1 \\ p_y = \mp\nu\cos q_1 + L\omega\sin q_1 \end{cases} \rightarrow \begin{cases} 2 = 2.622\cdot y + 1.06x \\ 2 = -2.622x + 1.06y \end{cases} \Rightarrow \begin{cases} x = \cos q_1 = -0.39 \\ y = \sin q_1 = 0.92 \end{cases} \Rightarrow q_1 = 1.971$$

$$\searrow \begin{cases} 2 = -2.622\cdot y + 1.06x \\ 2 = 2.622x + 1.06y \end{cases} \Rightarrow \begin{cases} x = \cos q_1 = 0.92 \\ y = \sin q_1 = -0.39 \end{cases} \Rightarrow q_1 = -0.4$$

$$\Rightarrow \begin{vmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{vmatrix} = \begin{bmatrix} 1.971 \\ 5.06 \\ 2.622 \\ -\pi/4 \end{bmatrix} \quad \begin{bmatrix} -0.4 \\ 5.06 \\ -2.622 \\ -\pi/4 \end{bmatrix}$$