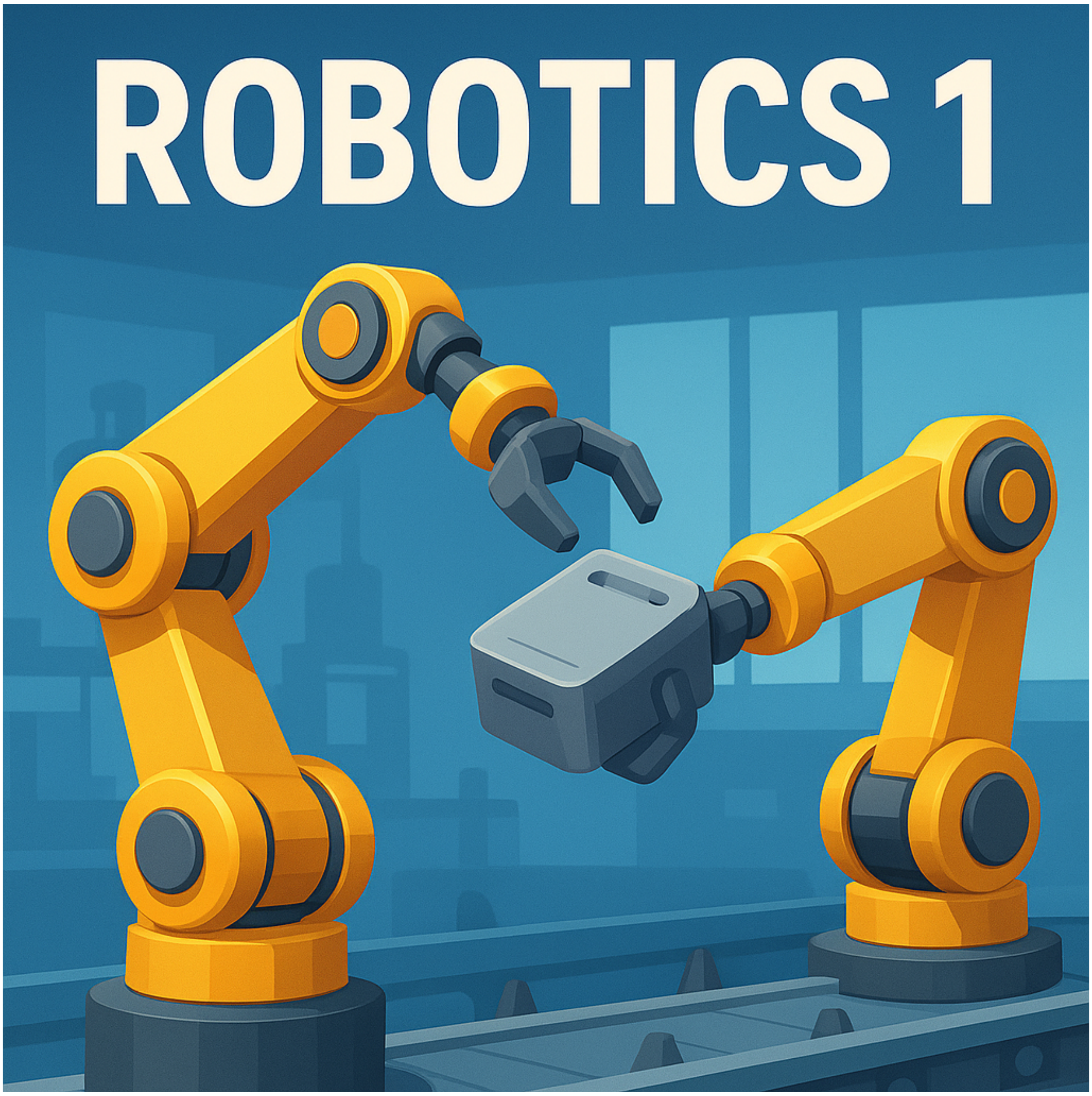
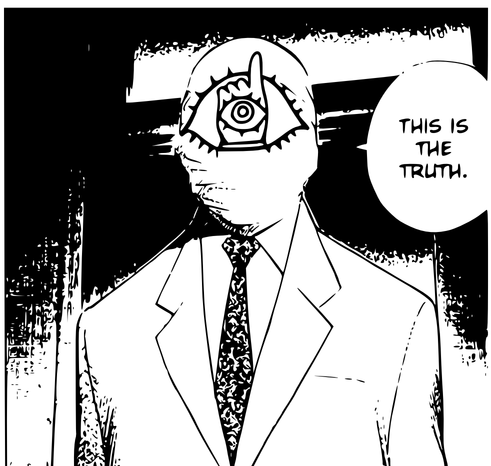
Marco Casu



Sapienza Università di Roma

Faculty of Information Engineering, Computer Science and Statistics
Department of Computer, Control and Management Engineering
Master's degree in Artificial Intelligence and Robotics

This document summarizes and presents the topics for the Robotics 1 course for the Master's degree in Artificial Intelligence and Robotics at Sapienza University of Rome. The document is free for any use. If the reader notices any typos, they are kindly requested to report them to the author.



CONTENTS

CHAPTER

1 INTRODUCTION

In this chapter we will see a brief introduction to the mathematical tools used in the main topics of the course. The topics presented in this section may seem somewhat unclear, as many concepts and definitions are only briefly introduced and deliberately not elaborated upon. They will be discussed in detail in their respective chapters.

1.1 About the End Effector Pose

A robot is made up of a series of arms connected to one another by joints, these joints can be **revolut** or **prismatic** (as shown in figure 1.1), a revolut joint rotate the link connected along 1 axis, the prismatic joint can make the link extend or contract, making them translate along 1 axis.

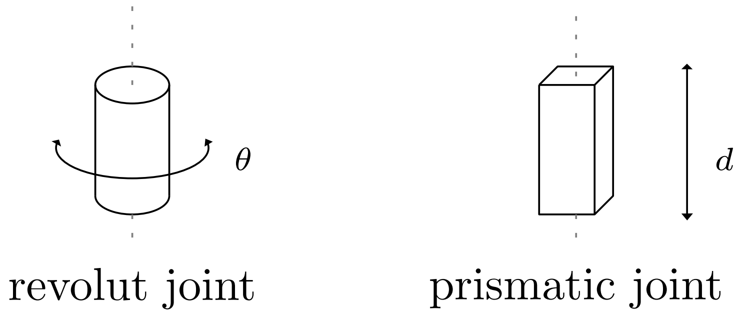


Figure 1.1: two types of joints (spatial representation)

It is important to know that that if the angle θ increase the joint is rotating counter clock wise. In a planar drawing, the joints are denoted as shown in image 1.2.

In the mathematical/geometrical model of a robotic arms, it is important the *kinematic skeleton*, the quantities involved are

- the current angle of the joints
- the length of the links

everything is defined respect to the base frame, usually denoted as Σ_0.

The robot shown in figure 1.3 is an *R4 robot* (4 revolut joints) with three links. With ${}^0\mathbf{p}_e$ and Σ_e we denote the position and the reference frame of the **end effector**, if there are a 0 supscript to a vector, we mean that is expressed in the base reference frame.

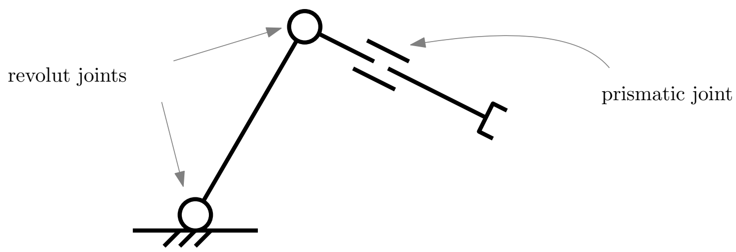


Figure 1.2: planar representation of the joints

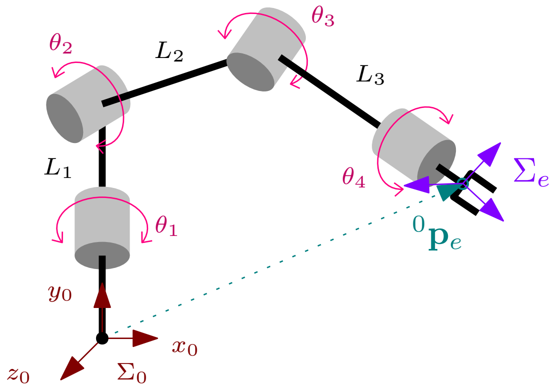


Figure 1.3: spatial R4 robot

With **Direct Kinematics**, we define the problem to find what are the **pose** (position and orientation) of the end effector, in function of the joint's angles.

$$Kin_p(\boldsymbol{\theta}) : \Sigma_0 \to \Sigma_e \tag{1.1}$$

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{pmatrix}^T \tag{1.2}$$

With Σ_e is denoted the reference frame of the end effector. How can we compute $Kin_p(\boldsymbol{\theta})$? This is given by an homogeneous 4×4 matrix defined as follows:

$$^0T_e = \begin{pmatrix} & {}^0R_e & & {}^0\mathbf{p}_e \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.3}$$

where

- ${}^0R_e \in SO(3)$ is the rotation matrix, and depends from $\boldsymbol{\theta}$
- ${}^0\mathbf{p}_e \in \mathbb{R}^3$ is the translation vector.

Recall: $SO(3)$ is the group of all the orthogonal 3×3 matrices with determinant equals to 1.

The matrix ${}^0T_e$ is obtained by multiplying n matrix (where n is the number of joints)

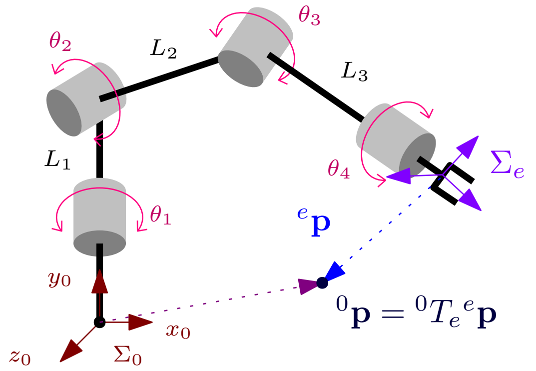
$${}^0T_e = {}^0A_1(\theta_1){}^1A_2(\theta_2)\dots{}^{n-1}A_n(\theta_n) = \tag{1.4}$$

$$\prod_{i=0}^{n-1} {}^iA_{i+1}(\theta_{i+1}). \tag{1.5}$$

Each *homogeneous matrix* ${}^{i-1}A_i$ describe the pose of the i-th joint's frame respect to the previous joint's frame, and depends from θ_i (the i-th joint's angle). A more detailed description of the matrix describing the direct kinematics will be given later.

If there are another frame Σ_w, the new matrix can be computed as follows

$${}^wT_e = {}^wT_0{}^0T_e. \tag{1.6}$$



the end effector position respect to the base frame is $^0T_e\mathbf{0}$

The **Inverse Kinematics** is the opposite problem, given a position $^0\mathbf{p}_e$ for the end effector, we want to find the values of $\boldsymbol{\theta}$ such that

$$^0\mathbf{p}_e = Kin_p(\boldsymbol{\theta}) \tag{1.7}$$

to find $\boldsymbol{\theta}$, we have to solve a non-linear system of equations, this is generally an undecidable problem, but for some specific cases, there exists a closed form, that can be found analytically, there are also numerical methods. Clearly, for the positions out of the work space, the system does not admit solutions (also this can be checked analytically).

1.2 About the End Effector Velocity

Let's now consider **Differential Kinematics**, that is the problem to find the end effector velocity in the workspace given the velocity of the joint's angles. Since the superposition principle is valid, the components resulting from the movement of each individual joint, which constitute the final velocity of the end effector, can be considered separately. It is important to know that the velocity component of the end effector given by a joint, is always orthogonal to the rotation axis of that joint.

The end effector have

- a linear velocity, usually denoted $\mathbf{v}$
- an angular velocity, usually denoted $\boldsymbol{\omega}$.

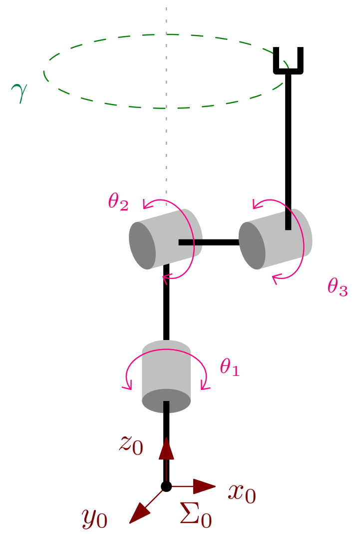


Figure 1.4: possible trajectory by moving θ_1

In the figure 1.4 the curve γ represent all possible positions where the end effector could lie if the angle θ_1 change, the linear velocity of the end effector is orthogonal to the z_0 axis. The velocity of the end effector doesn't depend only from the angular velocity, but also from the current configuration of the angles $\boldsymbol{\theta}$.

Even if the end effector is a rigid body, is sufficient to know the linear velocity of only one point and his angular velocity to compute the velocity of all the other points, since the following relation holds:

$$\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times \mathbf{r}_{12} \tag{1.8}$$

where

- $\mathbf{v}_1$ is the velocity of the first point
- $\mathbf{v}_2$ is the velocity of the second point
- $\boldsymbol{\omega}$ is the angular velocity of the rigid body
- $\mathbf{r}_{12}$ is the difference between the positions of the two points.

Let's analyze the velocity components of the end effector. If the i-th joint is changing is angle, the linear velocity of the end effector will have one component that is

$$\mathbf{v}_i = \mathbf{j}_i(\boldsymbol{\theta})\dot{\theta}_i \tag{1.9}$$

where $\mathbf{j}_i$ is a 3 components vector describing the direction of the velocity. Since the direction depends from the configuration, the vector $\mathbf{j}_i$ is in function of $\boldsymbol{\theta}$. This holds for all the angles θ_i, the resultant linear velocity of the end effector will be

$$\mathbf{v} = \sum_{i=1}^{n} \mathbf{j}_i(\boldsymbol{\theta})\dot{\theta}_i \tag{1.10}$$

it can be written in matrix form

$$\mathbf{v} = J_L(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \begin{pmatrix} \mathbf{j}_1(\boldsymbol{\theta}) & \dots & \mathbf{j}_n(\boldsymbol{\theta}) \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} \tag{1.11}$$

where $J_L(\boldsymbol{\theta})$ is a $3 \times n$ matrix called the **Jacobian Matrix**, where n is the number of joints. This description were given in terms of the linear velocity, but it holds also for the angular velocity of the end effector, indeed we have two Jacobian Matrix:

- we denote $J_L(\boldsymbol{\theta})$ the Jacobian matrix for the linear velocity
- we denote $J_A(\boldsymbol{\theta})$ the Jacobian matrix for the angular velocity

$$\boldsymbol{\omega} = J_A(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{1.12}$$

the matrix

$$J(\boldsymbol{\theta}) = \begin{pmatrix} J_L(\boldsymbol{\theta}) \\ J_A(\boldsymbol{\theta}) \end{pmatrix} \in Mat(6 \times n) \tag{1.13}$$

it's called **basic Jacobian**.

The Jacobian matrix is a mapping from the joint velocity space to the end effector velocity space. Let's ignore the angular velocity for now, suppose that we want to impose to the end effector a desired linear velocity (in a specific time instant)

$$\mathbf{v} = \mathbf{v}_d \in \mathbb{R}^3 \tag{1.14}$$

we need to find the values for the vector $\dot{\boldsymbol{\theta}}$ such that

$$\mathbf{v}_d = J_L(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{1.15}$$

if we have 3 joints, the matrix J is squared and can be inverted

$$\dot{\boldsymbol{\theta}} = J_L^{-1}(\boldsymbol{\theta})\mathbf{v}_d \tag{1.16}$$

but this is not the general case, if $n > 3$, the system of equations given in (1.15) could

- have zero solutions

- have infinite solutions

if the determinant of J_L^{-1} is zero, the system admit infinite solutions if and only if the desired velocity vector $\mathbf{v}_d$ is in the range space of J_L^{-1}

$$\det J_L^{-1} = 0 \implies \exists \text{ inf. sol.} \iff \mathbf{v}_d \in \text{Range}(J_L^{-1}) \tag{1.17}$$

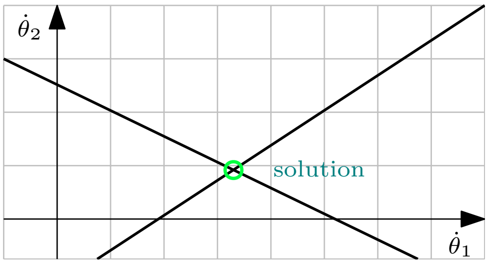
we remind that the range space of a matrix, is the set of all the linear combinations of the matrix's columns. If this isn't true, the system does not admit any solution, it means that no possible combination of velocity $\dot{\boldsymbol{\theta}}$ could realize the desired end effector velocity.

1.2.1 Singularity

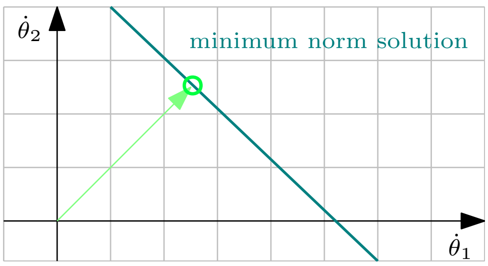
Let's talk about **singularities** in the joint velocity space, we will give a geometric example. Let's consider a 2R planar robot, with a fixed joints configuration $\boldsymbol{\theta}$, the Jacobian is a 2×2 matrix. Let $\mathbf{v}_d$ to be the desired velocity, the system is the following

$$\begin{pmatrix} v_d^x \\ v_d^y \end{pmatrix} = \begin{pmatrix} \mathbf{j}_1^T \\ \mathbf{j}_2^T \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \tag{1.18}$$

the two linear equation of the system is represented on the plane as two lines. If $\det J_L \neq 0$, there are only one solution, and is the intersection between the two lines.



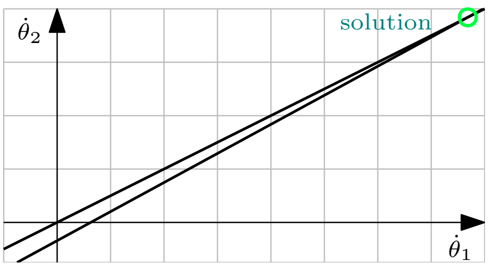
If $\det J_L = 0$, the two lines are parallel, so either they have no intersection, or they are the same line. If there are infinite solutions, we can choose the one with the smallest norm, since represents the "minimum energy" solution (the solution that requires the least joint rotation speed intensity).



We have a *singularity* when the determinant approaches zero

$$\det J_L \to 0 \tag{1.19}$$

The closer the determinant (in absolute value) gets to zero, the more "nearly" parallel the row vectors (and thus the lines they represent) become, which means the angle of intersection approaches zero. In this case the norm of the solution could be large.



This is true also because the following relations holds

$$J_L = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \implies \tag{1.20}$$

$$J_L^{-1} = \frac{1}{\det J_L} \begin{pmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{pmatrix} \tag{1.21}$$

$$\mathbf{v}_d = J_L \dot{\boldsymbol{\theta}} \tag{1.22}$$

$$\dot{\boldsymbol{\theta}} = J_L^{-1} \mathbf{v}_d \tag{1.23}$$

$$\dot{\boldsymbol{\theta}} = \frac{1}{\det J_L} \begin{pmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{pmatrix} \mathbf{v}_d \tag{1.24}$$

with $\det J_L \to 0$ the term $\frac{1}{\det J_L}$ (and with it, also $\dot{\boldsymbol{\theta}}$) became bigger and bigger. In this case, the required joint rotation velocity might not be achievable by the robotic arm's motors.

The previous example showed how certain algebraic relationships are connected to physical problems in robot joint control. Another similar example is the following, consider the robotic arm shown in figure 1.5.

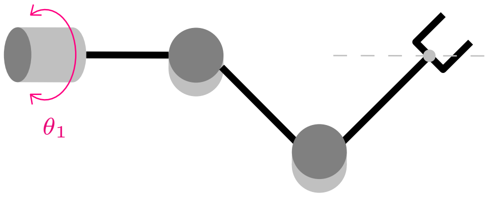


Figure 1.5: 3R spatial robot

Geometrically, it can be seen that by rotating only the first joint θ_1, the position of the end effector will not change, this condition holds when

$$J_L(\boldsymbol{\theta}) \begin{pmatrix} \dot{\theta}_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.25}$$

this is true if the vector $\begin{pmatrix} \dot{\theta}_1 & 0 & 0 \end{pmatrix}^T$ is in the kernel of the Jacobian matrix

$$\begin{pmatrix} \dot{\theta}_1 \\ 0 \\ 0 \end{pmatrix} \in \ker J_L(\boldsymbol{\theta}). \tag{1.26}$$

Therefore, the vectors contained in the kernel of the Jacobian matrix for the linear (or angular) velocity represent all possible combinations of individual joint velocities that would not change the position (or orientation) of the end effector.

1.3 Brief Overview of Planning and Control

When we want to control the end effector of a robotic arm, we want to know how to move the joints to get a specific position for the end effector, and also how to control the joints over the time to get a particular *trajectory* in the working space.

Consider a 2R planar robot, as shown in figure 1.6, where $\boldsymbol{\theta}$ is the angular position of the joints, and $\mathbf{p}_e = f(\boldsymbol{\theta})$ is the position of the end effector for some $f^2_{\mathbb{R}} \to \mathbb{R}^2$.

We would like to move the end effector from a certain starting point $\mathbf{p}_a \in \mathbb{R}^2$ to an another point $\mathbf{p}_b \in \mathbb{R}^2$. We could consider the segment line from $\mathbf{p}_a$ to $\mathbf{p}_b$ defined as follows:

$$\mathbf{p}(s) = s\mathbf{p}_b + (1-s)\mathbf{p}_a \qquad s \in [0,1]. \tag{1.27}$$

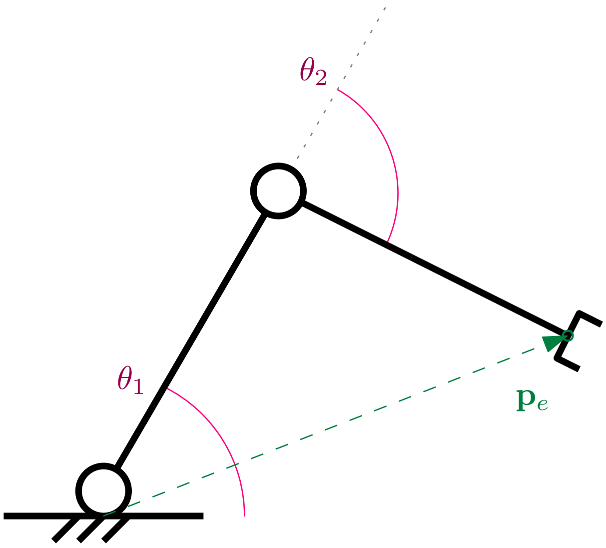


Figure 1.6: a 2R planar robot

Such a trajectory can be represented by a time-dependent function that starts from an initial time $t_0 = 0$ until a final time T, making s a monotonically increasing function of $t \in [0, T]$:

$$s : [0, 1] \longmapsto [0, T] \tag{1.28}$$
$$s(0) = 0 \tag{1.29}$$
$$s(1) = T \tag{1.30}$$
$$\mathbf{p}(s) = \mathbf{p}(s(t)) \tag{1.31}$$

We say that a trajectory is rest-to-rest if the velocity of the end effector at the start and at the end of that trajectory is zero:

$$\dot{\mathbf{p}}(s(0)) = \dot{\mathbf{p}}(s(T)) = \mathbf{0} \tag{1.32}$$

we need to include boundary conditions. Considering the chain rule, the derivative of $\mathbf{p}$ respect to the time t is

$$\dot{\mathbf{p}} = \frac{d\mathbf{p}}{dt} = \frac{d\mathbf{p}}{ds}\frac{ds}{dt} = \frac{d\mathbf{p}}{ds}\dot{s} \tag{1.33}$$

since

$$\frac{d\mathbf{p}}{ds} = \frac{d}{ds}\left(s\mathbf{p}_b + (1-s)\mathbf{p}_a\right) = \mathbf{p}_b - \mathbf{p}_a \tag{1.34}$$

we have

$$\dot{\mathbf{p}} = \frac{d\mathbf{p}}{ds}\dot{s} = \dot{s}(\mathbf{p}_b - \mathbf{p}_a) \tag{1.35}$$

the acceleration is

$$\ddot{\mathbf{p}} = \ddot{s}(\mathbf{p}_b - \mathbf{p}_a) + \dot{s} \cdot \mathbf{0} = \ddot{s}(\mathbf{p}_b - \mathbf{p}_a) \tag{1.36}$$

we have that

$$\dot{\mathbf{p}}(s(0)) = 0 \iff \dot{s}(0)(\mathbf{p}_b - \mathbf{p}_a) \iff \dot{s}(0) = 0 \tag{1.37}$$
$$\dot{\mathbf{p}}(s(T)) = 0 \iff \dot{s}(T)(\mathbf{p}_b - \mathbf{p}_a) \iff \dot{s}(T) = 0 \tag{1.38}$$

The starting velocity and the final velocity is zero, so the variation of the velocity is zero, this can be seen by the integral of the acceleration

$$\int_0^T \ddot{\mathbf{p}}dt = \int_0^T \ddot{s}(\mathbf{p}_b - \mathbf{p}_a)dt = (\mathbf{p}_b - \mathbf{p}_a)\int_0^T \ddot{s}dt = (\mathbf{p}_b - \mathbf{p}_a)(\dot{s}(T) - \dot{s}(0)) = 0. \tag{1.39}$$

Now we consider the *control aspects* of the problem, we denote $\mathbf{p}_e(t)$ the position of the end effector at the time t, and $\mathbf{p}_d(t)$ the **desired position** at time t.

$$\mathbf{p}_d(0) = \mathbf{p}_a. \tag{1.40}$$

We define the **error** such as the difference between the current position and the desired position:

$$\mathbf{e}(t) = \mathbf{p}_d(t) - \mathbf{p}_e(t) \tag{1.41}$$

The aim of the *control system* of the robot is to maintain $\mathbf{e}$ as close to zero as possible. This can be done by computing the initial error, and by giving to the system a new command $\dot{\boldsymbol{\theta}}(t)$ to correct it such that $\mathbf{e}(t) \to \mathbf{0}$. Let's denotate the error as follows

$$\mathbf{e}(t) = \begin{pmatrix} e_x(t) \\ e_y(t) \end{pmatrix} \tag{1.42}$$

For now, we will not discuss in detail how to control the error through the control of joint velocities $\dot{\boldsymbol{\theta}}$; it is sufficient to know that the following condition is required:

$$\dot{\mathbf{e}}(t) = -K\mathbf{e}(t) = \begin{pmatrix} -k_x & 0 \\ 0 & -k_y \end{pmatrix} \begin{pmatrix} e_x(t) \\ e_y(t) \end{pmatrix} \tag{1.43}$$

with $k_x, k_y > 0$. Why this conditions is required?

- if $e_x(t)$ is greater than zero, the condition $\dot{e}_x(t) = -k_x e_x(t)$ describes a decrease in error, making it approach zero
- if $e_x(t)$ is smaller than zero, the condition $\dot{e}_x(t) = -k_x e_x(t)$ describes an increase in error, making it approach zero
- same for e_y.

The system of equations

$$\dot{\mathbf{e}}(t) = \begin{pmatrix} -k_x & 0 \\ 0 & -k_y \end{pmatrix} \begin{pmatrix} e_x(t) \\ e_y(t) \end{pmatrix} \implies \begin{cases} \dot{e}_x(t) = -k_x e_x(t) \\ \dot{e}_y(t) = -k_y e_y(t) \end{cases} \tag{1.44}$$

admits exponential functions as a solution

$$e_x(t) = e_x(0)e^{-k_x t} \tag{1.45}$$

$$e_y(t) = e_y(0)e^{-k_y t} \tag{1.46}$$

if the initial error $\mathbf{e}(0)$ is not zero, then the error will approaches zero, without never reaching it. For practical applications it goes sufficiently fast to values very close to zero.

We introduce now an important concept in linear differential equations systems.

Definition 1 *Let $A \in M_{n,n}(\mathbb{R})$ to be a squared real-valued matrix. The **matrix exponential**, denoted e^A, is the $n \times n$ matrix defined as follows:*

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{1.47}$$

Given a linear system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1.48}$$

the solution is

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) \tag{1.49}$$

In some cases the exponential matrix can be computed easily, let's assume that A is diagonal

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \tag{1.50}$$

in this case we have that

$$A^k = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \times \cdots \times \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} = \begin{pmatrix} a_1^k & 0 & \cdots & 0 \\ 0 & a_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^k \end{pmatrix} \tag{1.51}$$

so

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} a_1^k & 0 & \cdots & 0 \\ 0 & a_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^k \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{a_1^k}{k!} & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{a_2^k}{k!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{a_n^k}{k!} \end{pmatrix} \tag{1.52}$$

$$= \begin{pmatrix} e^{a_1} & 0 & \cdots & 0 \\ 0 & e^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{a_n} \end{pmatrix}. \tag{1.53}$$

So, if a system is described by a diagonalizable matrix A there exists a diagonal matrix Λ and an invertible matrix T such that

$$A = T\Lambda T^{-1} \tag{1.54}$$

in this case we can easily calculate the exponential matrix

$$e^{At} = T^{-1}e^{\Lambda t}T. \tag{1.55}$$