

Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise #3

- i. Compute the 3×3 task Jacobian $\mathbf{J}_t(\mathbf{q})$ associated to the task vector function $\mathbf{r} = t(\mathbf{q})$ defined in Exercise #2.
- ii. Find all the singularities of the matrix $\mathbf{J}_t(\mathbf{q})$.
- iii. In a singular configuration \mathbf{q}_s , determine a basis for the null space $\mathcal{N}\{\mathbf{J}_t(\mathbf{q}_s)\}$ and a basis for the range space $\mathcal{R}\{\mathbf{J}_t(\mathbf{q}_s)\}$. Both bases should be *globally* defined, namely they should have a constant dimension for all possible \mathbf{q} such that $\mathbf{J}_t(\mathbf{q})$ is singular.
- iv. Set now $K = L = M = 1$ [m]. Find a task velocity $\dot{\mathbf{r}}_f \in \mathcal{R}\{\mathbf{J}_t(\mathbf{q}_s)\}$ and an associated joint velocity $\dot{\mathbf{q}}_f \in \mathbb{R}^3$ realizing it, i.e., such that $\mathbf{J}_t(\mathbf{q}_s) \dot{\mathbf{q}}_f = \dot{\mathbf{r}}_f$. Is this $\dot{\mathbf{q}}_f$ unique?

Exercise #5

Consider the elliptic path shown in Fig. 2, with major (horizontal) semi-axis of length $a > 0$ and minor (vertical) semi-axis of length $b < a$.

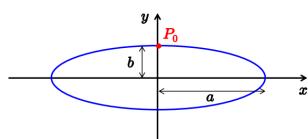


Figure 2: An elliptic path to be parametrized by $\mathbf{p}_d(s)$.

- Choose a smooth parametrization $\mathbf{p}_d(s) \in \mathbb{R}^2$, with $s \in [0, 1]$, of the full elliptic path starting at $P_0 = (0, b)$.
- Provide a timing law $s = s(t)$ that traces the path counterclockwise with a constant speed $v > 0$ on the path. What will be the motion time T for completing the full ellipse?

The elliptic path is $\mathbf{p}_d(s) = \begin{cases} a \cos(2\pi s + \frac{\pi}{2}) \\ b \sin(2\pi s + \frac{\pi}{2}) \end{cases} \quad s \in [0, 1]$

i choose $s(t) = vt \quad t \in [0, T]$. since $vT = 1 \Rightarrow T = \frac{1}{v}$.

- The following bounds on the norms of the velocity and of the acceleration should be satisfied along the resulting trajectory $\mathbf{p}_d(t) \in \mathbb{R}^2$, for all $t \in [0, T]$:

$$\|\dot{\mathbf{p}}_d(t)\| \leq V_{max}, \quad \|\ddot{\mathbf{p}}_d(t)\| \leq A_{max}, \quad \text{with } V_{max} > 0 \text{ and } A_{max} > 0.$$

Accordingly, what will be the maximum feasible speed v_f for this motion?

- Provide the numerical values of the maximum feasible speed v_f and of the resulting motion time T_f for the following data: $a = 1$, $b = 0.3$ [m]; $V_{max} = 3$ [m/s]; $A_{max} = 6$ [m/s²].

$$\dot{\mathbf{p}}_d = \mathbf{p}_d' \dot{s} = \mathbf{p}_d' v = \begin{pmatrix} -2\pi a v \cdot \sin(2\pi s + \frac{\pi}{2}) \\ 2\pi b \cdot v \cos(2\pi s + \frac{\pi}{2}) \end{pmatrix} \quad \text{the norm is}$$

$$\|\dot{\mathbf{p}}_d\| = \left[(2\pi v)^2 \left(a^2 \sin^2(2\pi s + \frac{\pi}{2}) + b^2 \cos^2(2\pi s + \frac{\pi}{2}) \right) \right]^{1/2} \leq \left[(2\pi v)^2 \cdot (a^2 + b^2) \right]^{1/2} \leq V_m \Rightarrow (2\pi)^2 v^2 \cdot (a^2 + b^2) \leq V_m^2 \Rightarrow$$

$$v \leq \sqrt{\frac{V_m^2}{(a^2 + b^2)(2\pi)^2}}.$$

$$\dot{s}: v \Rightarrow \ddot{s}: 0$$

$$\ddot{\mathbf{p}}_d = \mathbf{p}_d'' \dot{s}^2 + \mathbf{p}_d' \ddot{s} = \begin{pmatrix} -4\pi^2 a^2 v^2 \cos(2\pi s + \frac{\pi}{2}) \\ -4\pi^2 b \cdot v^2 \sin(2\pi s + \frac{\pi}{2}) \end{pmatrix} + 0 \leq A_m$$

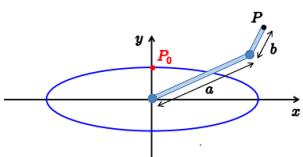
$$\|\ddot{\mathbf{p}}_d\| = \left[(4\pi^2 v^2)^2 (a^2 \cos^2(2\pi s + \frac{\pi}{2}) + b^2 \sin^2(2\pi s + \frac{\pi}{2})) \right]^{1/2} \leq (16\pi^4 v^4 (a^2 + b^2))^{1/2} \leq A_m \Rightarrow v \leq \sqrt[4]{\frac{A_m^2}{(a^2 + b^2) 16\pi^4}}$$

With the numerical data:

$$v \leq \sqrt{\frac{9}{1.09 \cdot (2\pi)^2}} \Rightarrow v \leq \sqrt[4]{\frac{36}{1.09 \cdot 16\pi^4}} \Rightarrow v_s = 0.381539 \Rightarrow T = 2.62 \text{ [sec]}$$

Exercise #6

A planar 2R robot has its base placed at the center of the ellipse of Fig. 2, as shown in Fig. 3. The robot has the first link of length a and the second link of length $b < a$, the same values of the semi-axes of the ellipse. The position $\mathbf{p} = \mathbf{f}(q)$ of its end effector (point P) should follow the trajectory $\mathbf{p}_d(t)$ defined in a parametric way in Exercise #5, with a path speed $v = 0.4 \text{ [s}^{-1}\text{]}$.



- What are the conditions on $a > b > 0$ in order for the robot to be able to reach all points of the desired trajectory $\mathbf{p}_d(t)$ while avoiding any robot singularity? Choose numerical values for a and for $b < a$ that satisfy these conditions and keep these values for the rest of this exercise.
- Choose an initial robot configuration $\mathbf{q}_n(0)$ so as to match the desired trajectory $\mathbf{p}_d(t)$ at time $t = 0$, i.e., with initial Cartesian error $\mathbf{e}(0) = \mathbf{p}_d(0) - \mathbf{f}(\mathbf{q}_n(0)) = 0$.

the ellipsoid should be contained in WS₁. each point of the ellipsoid should have norm greater than $2-b$. The closest point on $\mathbf{p}_d(t)$ have norm b so: $b > 2-b \Rightarrow 2b > 2 \Leftrightarrow b > \frac{2}{2}$. I choos $a=4$ and $b=3$.

in $t=0 \Rightarrow \mathbf{p}_d(0) = (0, 3)$. I solve the IK.

$$\begin{cases} 4C_1 + 3C_{12} = 0 \\ 4S_1 + 3S_{12} = 3 \end{cases} \Rightarrow \cos q_2 = -\frac{2}{3} \Rightarrow q_2 = 2.3 \Rightarrow \begin{cases} 4C_1 + 3(C_1C_2 - S_1S_2) = 0 \\ 4S_1 + 3(S_1C_2 + C_1S_2) = 3 \end{cases} \Rightarrow \begin{cases} 2C_1 - \sqrt{5}S_1 = 0 \\ 2S_1 + \sqrt{5}C_1 = 3 \end{cases} \Rightarrow$$

$$\begin{cases} C_1 = 0.745 \\ S_1 = \frac{2}{3} \end{cases} \Rightarrow q_1 = 0.73 \Rightarrow q(0) = q_0 = \begin{pmatrix} 0.73 \\ 2.3 \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} -4S_1 - 3S_{12} & -3S_{12} \\ 4C_1 + 3C_{12} & 3C_{12} \end{pmatrix} \Rightarrow \mathbf{J}(q_0) = \begin{pmatrix} -3 & -0.334 \\ 0 & -2.981 \end{pmatrix}$$

the nominal joint velocity should be

- iii. What nominal joint velocity command $\dot{\mathbf{q}} = \dot{\mathbf{q}}_n(t)$ should be given for $t \in [0, T]$ in order to execute perfectly the entire trajectory $\mathbf{p}_d(t)$ with matched initial conditions?

- iv. Choose another initial configuration $\mathbf{q}(0)$ such that $\mathbf{e}(0) \neq 0$, but with the y -component of the error $e_y(0) = 0$. Design a joint velocity control law $\dot{\mathbf{q}} = \dot{\mathbf{q}}_c(\mathbf{q}, t)$, with a feedback term depending on the current configuration \mathbf{q} , that will let $e_x(t)$ converge to zero with exponential decaying rate $r = 5$ and keep $e_y(t) = 0$ for all $t \geq 0$.

- v. With the available data, compute the numerical values of the initial nominal joint velocity command $\mathbf{q}_n(0) \in \mathbb{R}^2$ and of the initial joint velocity control law $\dot{\mathbf{q}}_c(\mathbf{q}(0), 0) \in \mathbb{R}^2$.

I choose \mathbf{q}^* s.t. $\mathbf{f}_r(\mathbf{q}^*) = (1, 3)$

$$\begin{cases} 4C_1 + 3C_{12} = 1 \\ 4S_1 + 3S_{12} = 3 \end{cases} \Rightarrow \begin{cases} C_2 = -\frac{5}{8} \\ S_2 = \frac{\sqrt{39}}{8} \end{cases} \Rightarrow q_2^* = 2.2459 \Rightarrow \begin{cases} 4C_1 + 3(C_1C_2 - S_1S_2) = 1 \\ 4S_1 + 3(S_1C_2 + C_1S_2) = 3 \end{cases} \Rightarrow \begin{cases} \frac{17}{8}C_1 - \frac{3\sqrt{39}}{8}S_1 = 1 \\ \frac{3\sqrt{39}}{8}C_1 + \frac{17}{8}S_1 = 3 \end{cases}$$

$$\Rightarrow \begin{cases} C_1 = 0.915 \\ S_1 = 0.403 \end{cases} \Rightarrow q_1^* = 0.4148 \Rightarrow q(0) = (0.4148, 2.2459)^T$$

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

T

$$\dot{\mathbf{e}} = \dot{\mathbf{p}}_d - \dot{\mathbf{p}} = \dot{\mathbf{p}}_d - \mathbf{J}\dot{\mathbf{q}} \quad \text{Since I want } \dot{\mathbf{e}} = -Ke \Rightarrow \dot{\mathbf{p}}_d - \mathbf{J}\dot{\mathbf{q}} = -Ke \Rightarrow$$

$$\mathbf{J}\dot{\mathbf{q}} = Ke + \dot{\mathbf{p}}_d \Rightarrow \dot{\mathbf{q}} = \mathbf{J}^{-1}Ke + \mathbf{J}^{-1}\dot{\mathbf{p}}_d \Rightarrow \begin{cases} \dot{e}_1 = -K_1 e_1 \\ \dot{e}_2 = -K_2 e_2 \end{cases} \quad \text{but}$$

control law

$$\dot{e}_i = -K_i e_i \Rightarrow \frac{de_i}{dt} = -K_i e_i \Rightarrow dt = -\frac{1}{K_i} \frac{1}{e_i} de_i \Rightarrow \int_0^t dt = -\frac{1}{K_i} \int_0^t \frac{1}{e_i} de_i \Rightarrow t = -\frac{1}{K_i} \ln(e_i) \Rightarrow \ln(e_i) = -K_i t$$

$$\Rightarrow e_i = e_i(0) \cdot \exp(-K_i t) \quad \text{so, since } e(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ we have:}$$

$$e(t) = \begin{pmatrix} -e^{-3t} \\ 0 \end{pmatrix} \Rightarrow \lim_{t \rightarrow \infty} e(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\dot{\vec{p}}_d(s) = \begin{pmatrix} -2\pi a v \cdot \sin(2\pi s + \frac{\pi}{2}) \\ 2\pi b \cdot v \cos(2\pi s + \frac{\pi}{2}) \end{pmatrix}, \quad \dot{\vec{p}}_d(0) = \begin{pmatrix} -9.589 \\ 0 \end{pmatrix} \quad J(q(0)) = \begin{pmatrix} -3 & -1.387 \\ 1 & -2.659 \end{pmatrix}$$

The initial command will be

$$\dot{q} = J^{-1}(K_e + \dot{\vec{p}}_d) = J^{-1}(q(0)) \cdot \left(\begin{pmatrix} -9 \exp(0) \\ 0 \end{pmatrix} + \begin{pmatrix} -9.589 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} -0.283 & 0.148 \\ -0.106 & -0.32 \end{pmatrix} \begin{pmatrix} -18 \\ 0 \end{pmatrix} = \begin{pmatrix} 5.094 \\ 1.908 \end{pmatrix}$$
