Exercise 1

Consider the rotation matrix

$$R_d = \frac{1}{3}\begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ -1 & -2 & -2 \end{pmatrix}.$$

Find, if possible, all angle-axis pairs (θ, r) that provide the desired orientation R_d. At the end, check your results by verifying that $R(\theta, r) = R_d$.

$$2rr^T - I = \begin{bmatrix} 2r_x^2 - 1 & 2r_xr_y & 2r_xr_z \\ 2r_xr_y & 2r_y^2 - 1 & 2r_yr_z \\ 2r_xr_z & 2r_yr_z & 2r_z^2 - 1 \end{bmatrix}$$

I have to solve the system in r, θ.

The trace is $\frac{1}{3}(-2+1-2) = -1 = 1 + 2\cos\theta \Rightarrow 2\cos\theta = -2 \Rightarrow \cos\theta = -1$

$\Rightarrow \sin\theta = \pm\sqrt{1-1} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = \pm\pi \Rightarrow$ We solve $R = 2rr^T - I$ for r:

$$\begin{cases} r_xr_z = -\frac{1}{6} \\ r_xr_y = \frac{1}{3} \\ r_yr_z = -\frac{1}{3} \end{cases}$$

$r_z = -\frac{1}{6r_x}$

$r_x(2r_x) = 1/3 \Rightarrow 2r_x^2 = 1/3 \Rightarrow r_x^2 = \frac{1}{6} \Rightarrow r_x = \pm\sqrt{1/6}$

$r_y = 2r_x$ $\quad r_y = \pm 2\sqrt{1/6}$ $\quad r_z = -\frac{1}{3}\left(\pm 2\sqrt{1/6}\right)^{-1}$

Exercise 2

The end-effector of a robot undergoes a change of orientation between an initial R_i and a final R_f, as specified by

$$R_i = \begin{pmatrix} 0 & 0.5 & -\frac{\sqrt{3}}{2} \\ -1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0.5 \end{pmatrix}, \qquad R_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Provide a minimal representation of the relative rotation between the initial and the final orientation using YXY Euler angles $(\alpha_1, \alpha_2, \alpha_3)$. At the end, check your solutions by performing the direct computation.

I compute ${}^iR_f$ as ${}^0R_i^T\,{}^0R_f = \begin{bmatrix} 0 & 0 & -1 \\ 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \end{bmatrix}$

The YXY Euler angles is expressed by:

$$R_y(\alpha_1)R_x(\alpha_2)R_y(\alpha_3) = \begin{bmatrix} \cos\alpha_1 & 0 & \sin\alpha_1 \\ 0 & 1 & 0 \\ -\sin\alpha_1 & 0 & \cos\alpha_1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_2 & -\sin\alpha_2 \\ 0 & \sin\alpha_2 & \cos\alpha_2 \end{bmatrix}\begin{bmatrix} \cos\alpha_3 & 0 & \sin\alpha_3 \\ 0 & 1 & 0 \\ -\sin\alpha_3 & 0 & \cos\alpha_3 \end{bmatrix}$$

$$\begin{bmatrix} \cos\alpha_1 & \sin\alpha_1\sin\alpha_2 & \sin\alpha_1\cos\alpha_2 \\ 0 & \cos\alpha_2 & -\sin\alpha_2 \\ -\sin\alpha_1 & \cos\alpha_1\sin\alpha_2 & \cos\alpha_1\cos\alpha_2 \end{bmatrix}\begin{bmatrix} \cos\alpha_3 & 0 & \sin\alpha_3 \\ 0 & 1 & 0 \\ -\sin\alpha_3 & 0 & \cos\alpha_3 \end{bmatrix}$$

$$\begin{bmatrix} C1C2 - S1S2S3 & S1S2 & C1S3 + S1C2C3 \\ S2S3 & C2 & -S2C3 \\ -S1C3 - C1C2S3 & C1S2 & -S1S3 + C1C2C3 \end{bmatrix}$$

y solve :

$$
\begin{bmatrix} c1c2 - s1s2s3 & s1s2 & c1s3 + s1c2c3 \\ s2s3 & c2 & -s2c3 \\ -s1c3 - c1c2s3 & c1s2 & -s1s3 + c1c2c3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \end{bmatrix}
$$

$$
\Rightarrow \cos\alpha_2 = -\sqrt{3}/2 \Rightarrow \sin\alpha_2 = \pm 1/2 \Rightarrow \alpha_2 = \operatorname{atan2}\left\{\pm 1/2, -\sqrt{3}/2\right\} = \begin{cases} 5/6\,\pi \\ -5/6\,\pi \end{cases} = \pm\frac{5}{6}\pi
$$

$$
\begin{cases} \sin\alpha_1 \sin\alpha_2 = 0 \\ \cos\alpha_1 \sin\alpha_2 = -1/2 \end{cases} \Rightarrow \begin{cases} \pm\frac{1}{2}\sin\alpha_1 = 0 \\ \pm\frac{1}{2}\cos\alpha_1 = -1/2 \end{cases} \Rightarrow \begin{cases} \sin\alpha_1 = 0 \\ \cos\alpha_1 = \mp 1 \end{cases} \Rightarrow \begin{cases} \alpha_2 = 5/6\,\pi \Rightarrow \alpha_1 = \pi \\ \alpha_2 = -5/6\,\pi \Rightarrow \alpha_1 = 0 \end{cases}
$$

$$
\Rightarrow \begin{cases} \sin\alpha_2 \sin\alpha_3 = 1/2 \\ -\sin\alpha_2 \cos\alpha_3 = 0 \end{cases} \quad \begin{cases} \pm\frac{1}{2}\sin\alpha_3 = 1/2 \\ \mp 1/2 \cos\alpha_3 = 0 \end{cases} \Rightarrow \begin{cases} \sin\alpha_3 = \pm 1 \\ \cos\alpha_3 = 0 \end{cases} \to \begin{cases} \alpha_2 = 5/6\,\pi \Rightarrow \sin\alpha_3 = 1 \\ \alpha_2 = -5/6\,\pi \Rightarrow \sin\alpha_3 = -1 \end{cases}
$$

$$
\to \begin{cases} \alpha_3 = \pi/2 \\ \alpha_3 = -\pi/2 \end{cases}
$$

$$
\Rightarrow \alpha = \begin{bmatrix} \pi \\ 5/6\,\pi \\ \pi/2 \end{bmatrix} \vee \alpha = \begin{bmatrix} 0 \\ -5/6\,\pi \\ -\pi/2 \end{bmatrix}
$$

Exercise 4

A large 6R robot manipulator is mounted on the ceiling of an industrial cell and holds firmly a cylindric object in its jaw gripper. The world frame RF_w of the cell is placed on the floor, at about the cell center. The robot base frame RF_0 is defined by ${}^w\boldsymbol{T}_0$, while its end-effector frame RF_e has the origin O_e at the center of the grasped object. The robot direct kinematics is expressed in symbolic form by ${}^0\boldsymbol{T}_e(\boldsymbol{q})$, in terms of the joint variables $\boldsymbol{q}$. A camera is placed in the cell and its frame RF_c, having the origin O_c at the center of the image plane and the $\boldsymbol{z}_c$ unit vector along the focal axis of the camera, is defined by ${}^w\boldsymbol{T}_c$.

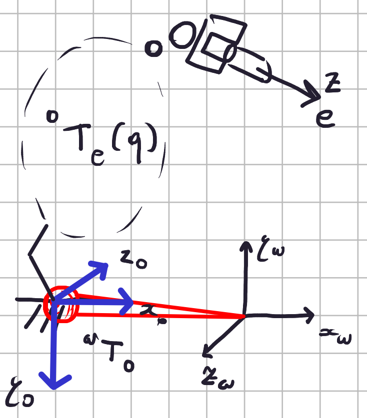
Figure 2 details the placement of the end-effector frame RF_e and of the camera frame RF_c. The robot should hold the object in front of the camera, with the major axis of the cylinder aligned to the camera focal axis and its center at a distance $d > 0$ from O_c. Define the task kinematics equation, to be solved for the joint variables $\boldsymbol{q}$, when the transformation matrices and the object-camera offset are given by

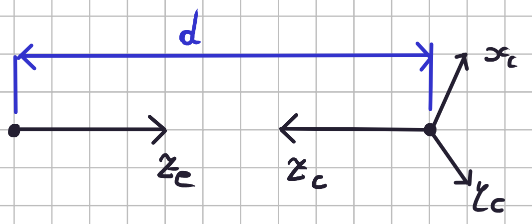
$$
{}^w\boldsymbol{T}_0 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 3.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad {}^w\boldsymbol{T}_c = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 2 \\ 0 & -1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad d = 1 \text{ [m]}.
$$

Discuss also whether the robot is kinematically redundant for the task or not.

We have to impose conditions on ${}^cT_e$,

the desired condition is





$\Rightarrow {}^cT_w$ should be :

$$
{}^cT_c = \begin{bmatrix} \boxed{A} & \boxed{B} & \begin{matrix} 0 \\ 0 \\ -1 \end{matrix} & \begin{matrix} 0 \\ 0 \\ d \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

z axis aligned

d distance

$\Rightarrow$ A and B are two arbitrary vector such that $\begin{bmatrix} A & B & \begin{matrix} 0 \\ 0 \\ -1 \end{matrix} \end{bmatrix} \in SO(3)$.

The following conditions hold

$$
{}^{c}T_e = \begin{bmatrix} \boxed{A} & \boxed{B} & \begin{matrix} 0 \\ 0 \\ -1 \end{matrix} & \begin{matrix} 0 \\ 0 \\ d \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{c}T_w \, {}^{w}T_0 \, {}^{0}T_e(q) = {}^{w}T_c^{-1} \, {}^{w}T_0 \, {}^{0}T_e(q)
$$

$$
\Rightarrow \begin{bmatrix} \boxed{A} & \boxed{B} & \begin{matrix} 0 \\ 0 \\ -1 \end{matrix} & \begin{matrix} 0 \\ 0 \\ d \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.7 & -4.59 \\ 0 & 1 & 0 & -1 \\ -0.7 & 0 & 0.7 & -0.353 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot {}^{0}T_e(q)
$$

$\Rightarrow$ in this problem we have to impose a position for the end effector, and an orientation, but the orientation is not totally defined, since if we set the z axis aligned we have infinite solutions (any rotation for the ee along his z axis), so the set of solutions is a 1-mainfold.

Exercise 3

A DC motor is used to move a link of length $L = 0.7$ [m], as shown in Fig. 1. The motor mounts on its axis an absolute encoder and uses as transmission elements an Harmonic Drive having a flexspline with $N_{FS} = 160$ teeth and a gear with two toothed wheels of radius $r_1 = 2$ and $r_2 = 4$ [cm], respectively.

- Compute the reduction ratio $n_r > 1$ of the transmission system. Which is the direction of rotation of the link when the motor angular position θ_m is turning counterclockwise?
- Determine the resolution of the absolute encoder that allows distinguishing two link tip positions that are $\Delta r = 0.1$ [mm] away. What should be the minimum number of tracks N_t of the encoder?
- If the link has an angular range $\Delta\theta_{max} = 180°$, how many turns of the motor are needed to cover the entire range? With a multi-turn absolute encoder, what is the minimum number of bits for counting all these turns?
- If the motor inertia is $J_m = 1.2 \cdot 10^{-4}$ [kgm^2], determine the optimal value of the link inertia J_l around the axis at its base which minimizes the motor torque τ_m needed for a desired link acceleration $\ddot{\theta}$. What is then the value of τ_m (in [Nm]) for $\ddot{\theta} = 7$ [rad/s^2]?

The reduction ratio of the harmonic drive is $n_{hd} = N_{FS}/2 = 80$

Every 80 revolution of the DC motor, the HD performs one revolution.

Every 2 rotation of r_1, r_2 performs a rotation

$\Rightarrow$ the final reduction ratio is $n_r = 160$

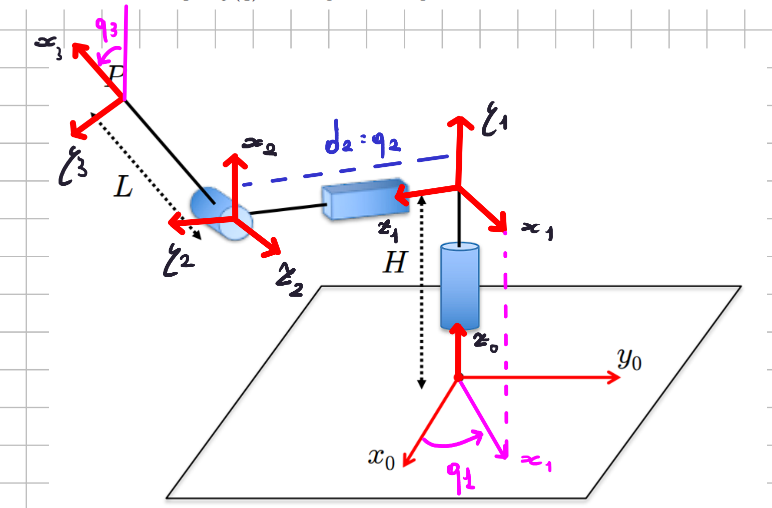
$\theta = 160\,\theta_m$. The rotation is clockwise.

$$
\tau_m = J_m n_r \ddot{\theta} + \frac{1}{n_r} J_l \ddot{\theta} \Rightarrow n_r = \sqrt{J_l / J_m} \Rightarrow n_r^2 = J_l / J_m \Rightarrow J_l = J_m n_r^2
$$

$$
= 1.2 \cdot 10^{-4} \cdot 160^2 = 3.072
$$

Exercise 5

For the spatial RPR robot of Fig. 3, complete the assignment of Denavit-Hartenberg (DH) frames and fill in the associated table of parameters. The origin of the last frame should be placed at the point P. Moreover, the frame assignment should be such that all constant DH parameters are *non-negative* and the value of the joint variables q_i, $i = 1, 2, 3$, are *strictly positive* in the shown configuration. Compute then the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ for the position of point P.



	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	H	q_1
2	$\pi/2$	0	q_2	$\pi/2$
3	0	L	0	q_3

$$\begin{bmatrix} cq_1 & 0 & sq_1 & 0 \\ sq_1 & 0 & -cq_1 & 0 \\ 0 & 1 & 0 & H \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_3 & -sq_3 & 0 & Lcq_3 \\ sq_3 & cq_3 & 0 & Lsq_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & sq_1 & cq_1 & q_2 sq_1 \\ 0 & -cq_1 & sq_1 & -q_2 cq_1 \\ 1 & 0 & 0 & H \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_3 & -sq_3 & 0 & Lcq_3 \\ sq_3 & cq_3 & 0 & Lsq_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} S_1S_3 & S_1C_3 & C_1 & LS_1S_3 + q_2S_1 \\ -C_1S_3 & -C_1C_3 & S_1 & -LC_1S_3 - q_2C_1 \\ C_3 & -S_3 & 0 & LC_3 + H \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \underset{\text{position}}{f(q)} = \begin{array}{l} p_x = L\sin q_1 \sin q_3 + q_2 \sin q_1 \\ p_y = -L\cos q_1 \sin q_3 - q_2 \cos q_1 \\ p_z = L\cos q_3 + H \end{array}$$

Exercise 6

For the spatial RPR robot of Fig. 3, provide the closed-form expression of the inverse kinematics for the position $\boldsymbol{p}$ of point P. Assuming for simplicity that the joints have unlimited ranges, how many inverse kinematics solutions are there in the regular case? Compute the numerical values of all inverse solutions $\boldsymbol{q}$ when $\boldsymbol{p} = (3, 4, 1.5)$ [m] and the geometric parameters of the robot are $H = L = 1$ [m]. Check the solutions!

$$\begin{cases} p_x = L\sin q_1 \sin q_3 + q_2 \sin q_1 \\ p_y = -L\cos q_1 \sin q_3 - q_2 \cos q_1 \\ p_z = L\cos q_3 + H \end{cases} \Rightarrow$$

$$\cos q_3 = (p_z - H)\frac{1}{L} \Rightarrow \cos^2 q_3 = (p_z - H)^2 \frac{1}{L^2} \Rightarrow$$

$$\sin^2 q_3 = 1 - 1/L^2 (p_z - H)^2 \Rightarrow \sin q_3 = \pm\sqrt{1 - 1/L^2 (p_z - H)^2}$$

$$\Rightarrow q_3 = \operatorname{arctan}\left\{\pm\sqrt{1 - 1/L^2 (p_z - H)^2},\ (p_z - H)1/L\right\}$$ 2 solutions for q_3

I denote $\sqrt{1 - 1/L^2 (p_z - H)^2} = \alpha \Rightarrow \sin q_3 = \pm\alpha$. I square and sum the first 2 equations.

$$p_x^2 + p_y^2 = L^2\alpha^2 \sin^2 q_1 + q_2^2 \sin^2 q_1 \pm L\alpha q_2 \sin^2 q_1 + L^2\alpha^2 \cos^2 q_1 + q_2^2 \cos^2 q_1 \pm L\alpha q_2 \cos^2 q_1 = L^2\alpha^2 + q_2^2 \pm L\alpha q_2 - p_x^2 - p_y^2 = 0$$

$$q_2^2 \pm L\alpha q_2 + (L^2\alpha^2 - p_x^2 - p_y^2) = 0$$

$$\Delta = L^2\alpha^2 - 4(L^2\alpha^2 - p_x^2 - p_y^2) \Rightarrow q_2 = \frac{\mp L\alpha \pm \sqrt{\Delta}}{2} \rightarrow \begin{cases} -\frac{1}{2}(L\alpha + \sqrt{\Delta}) & \text{se } \sin q_3 = \alpha \\ -\frac{1}{2}(L\alpha - \sqrt{\Delta}) & \text{se } \sin q_3 = \alpha \\ +\frac{1}{2}(L\alpha + \sqrt{\Delta}) & \text{se } \sin q_3 = -\alpha \\ +\frac{1}{2}(L\alpha - \sqrt{\Delta}) & \text{se } \sin q_3 = -\alpha \end{cases} \quad \text{4 soluzioni per ora}$$

$$\Rightarrow \begin{cases} p_x = \pm L \sin q_1 \alpha + q_2 \sin q_1 \\ p_y = \mp L \cos q_1 \alpha - q_2 \cos q_1 \end{cases} \Rightarrow \begin{cases} \sin q_1 = (q_2 \pm L\alpha)^{-1} p_x \\ \cos q_1 = (-q_2 \mp L\alpha)^{-1} p_y \end{cases} \Rightarrow$$

if $\sin q_3 = \alpha$:

$$\begin{cases} q_2 = -\frac{1}{2}(L\alpha + \sqrt{\Delta}) \\ q_2 = -\frac{1}{2}(L\alpha - \sqrt{\Delta}) \end{cases}$$

$$\begin{cases} \sin q_1 = (q_2 + L\alpha)^{-1} p_x \\ \cos q_1 = (-q_2 - L\alpha)^{-1} p_y \end{cases}$$

if $\sin q_3 = -\alpha$:

$$\begin{cases} q_2 = +\frac{1}{2}(L\alpha + \sqrt{\Delta}) \\ q_2 = +\frac{1}{2}(L\alpha - \sqrt{\Delta}) \end{cases}$$

$$\begin{cases} \sin q_1 = (q_2 - L\alpha)^{-1} p_x \\ \cos q_1 = (-q_2 + L\alpha)^{-1} p_y \end{cases}$$

$$\Rightarrow \sin q_3 \begin{cases} \alpha \begin{cases} \to q_2 = \\ \to q_2 = \end{cases} \\ -\alpha \begin{cases} \to q_2 = \\ \to q_2 = \end{cases} \end{cases}$$

In the regular case there are 4 solutions.

Let $p = (3, 4, 1.5)$.

$$\sin q_3 = \pm\sqrt{1 - \tfrac{1}{L^2}(p_z - H)^2} = \pm\sqrt{1 - \tfrac{1}{4}} = \pm \sqrt{3}/2$$

$$\cos q_3 = (p_z - H)\frac{1}{L} = (1.5 - 1)\frac{1}{1} = 1/2$$

$$\Rightarrow q_3 = \text{atan2}\left\{\pm\sqrt{3}/2, 1/2\right\} = \pm\frac{\pi}{3}$$

Case $q_3 = \pi/3$:

$$\Delta = L^2\alpha^2 - 4(L^2\alpha^2 - p_x^2 - p_y^2) = (\pi/3)^2 - 4([\pi/3]^2 - 9 - 16) \approx 96.7101 \Rightarrow \pm\sqrt{\Delta} = \pm 9.8341$$

$$\begin{cases} q_2 = -\frac{1}{2}(L\alpha + \sqrt{\Delta}) \\ q_2 = -\frac{1}{2}(L\alpha - \sqrt{\Delta}) \end{cases} \begin{cases} q_2 = -\frac{1}{2}(\frac{\pi}{3} + 9.8341) \\ q_2 = -\frac{1}{2}(\frac{\pi}{3} - 9.8341) \end{cases} = \begin{cases} q_2 = -5.44 \\ q_2 = 4.39 \end{cases}$$

$$\begin{cases} \sin q_1 = (q_2 + L\alpha)^{-1} p_x \\ \cos q_1 = (-q_2 - L\alpha)^{-1} p_y \end{cases}$$

$$\nearrow \begin{cases} \sin q_1 = (-5.44 + \frac{\pi}{3})^{-1} \cdot 3 \\ \cos q_1 = (5.44 - \pi/3)^{-1} \cdot 4 \end{cases} = \begin{cases} \sin q_1 = -0.6829 \\ \cos q_1 = 0.9105 \end{cases} \Rightarrow q_1 = \tan^{-1}\left(-\frac{0.6829}{0.9105}\right)$$

$$q_1 = -0.6435$$

$$\searrow \begin{cases} \sin q_1 = (4.39 + \frac{\pi}{3})^{-1} \cdot 3 \\ \cos q_1 = (-4.39 - \pi/3)^{-1} \cdot 4 \end{cases} = \begin{cases} \sin q_1 = 0.5517 \\ \cos q_1 = -0.7356 \end{cases} \Rightarrow q_1 = 2.498$$

the first two solutions are:

$$\begin{bmatrix} 0.6431 \\ -5.44 \\ \pi/3 \end{bmatrix} \qquad \begin{bmatrix} 2.498 \\ 4.39 \\ \pi/3 \end{bmatrix}$$