**Question #6** [all students]

For the 4-dof planar RRPR robot in Fig. 2, with the joint variables $q = (q_1, q_2, q_3, q_4)$ defined therein, derive the Jacobian $J(q)$ associated to the 3-dimensional task vector $r = (p_x, p_y, \alpha)$, where $p = (p_x, p_y) \in \mathbb{R}^2$ gives the position of the final flange center P and $\alpha \in \mathbb{R}$ is the orientation of the last robot link w.r.t. the axis x_0. Find all singular configurations q_s of this task Jacobian matrix. For one such q_s, let $J_s = J(q_s)$ and determine a basis for $\mathcal{R}\{J_s\}$ and one for $\mathcal{N}\{J_s\}$.
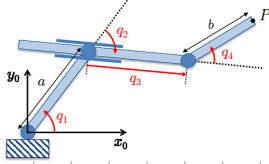
$$f_r(q) = \begin{cases} a c_1 + q_3 c_{12} + b c_{124} \\ a s_1 + q_3 s_{12} + b s_{124} \\ q_1 + q_2 + q_4 \end{cases} \Rightarrow J(q) = \begin{pmatrix} -a s_1 - q_3 s_{12} - b s_{124} & -q_3 s_{12} - b s_{124} & c_{12} & -b s_{124} \\ a c_1 + q_3 c_{12} + b c_{124} & q_3 c_{12} + b c_{124} & s_{12} & b c_{124} \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

IF $q_3 = b$ and $q_4 = \pi$ we have the following configuration:

We use $\begin{cases} \sin(x + \pi) = -\sin x \\ \cos(x + \pi) = -\cos x \end{cases}$

moving q_2 dont modify the ee - lin. vel
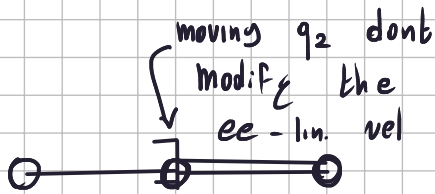
$$\Rightarrow J(q^*) = \begin{pmatrix} -a s_1 & 0 & c_{12} & b s_{12} \\ a c_1 & 0 & s_{12} & -b c_{12} \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow J(q^*) \cdot \begin{pmatrix} 0 \\ \dot{q}_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix} \leftarrow \text{Zero linear velocity.}$$

We consider $q_s = (0, 0, b, \pi) \Rightarrow J(q_s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ a & 0 & 0 & -b \\ 1 & 1 & 0 & 1 \end{bmatrix} = J_s$

$$J(q_s) \dot{q} = \begin{cases} \dot{q}_3 \\ a\dot{q}_1 - b\dot{q}_4 \\ \dot{q}_1 + \dot{q}_2 + \dot{q}_4 \end{cases} \Rightarrow \mathcal{R}(J) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Now i compute the kernel

$$J(q_s)\dot{q} = 0 \Rightarrow \begin{cases} \dot{q}_3 = 0 \\ a\dot{q}_1 - b\dot{q}_4 = 0 \\ \dot{q}_1 + \dot{q}_2 + \dot{q}_4 = 0 \end{cases} \begin{cases} \dot{q}_3 = 0 \\ \dot{q}_1 = \frac{b}{a}\dot{q}_4 \\ \dot{q}_4(1 + \frac{b}{a}) = -\dot{q}_2 \end{cases} \Rightarrow \mathcal{N}(J_s) = \left\{ \begin{pmatrix} \frac{b}{a}x \\ -(1+\frac{b}{a})x \\ 0 \\ x \end{pmatrix} \ x \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} b/a \\ -1-b/a \\ 0 \\ 1 \end{pmatrix} \right\}$$

Two planar 2R robots, named A and B and having both unitary link lengths, are in the static equilibrium shown in Fig. 3. The two D-H configurations w.r.t. their base frames are, respectively, $q_A = (3\pi/4, -\pi/2)$ [rad] and $q_B = (\pi/2, -\pi/2)$ [rad]. Robot A pushes against robot B as in the figure, with a force $F \in \mathbb{R}^2$ having norm $\|F\| = 10$ [N]. Compute the joint torques $\tau_A \in \mathbb{R}^2$ and $\tau_B \in \mathbb{R}^2$ (both in [Nm]) that keep the two robots in equilibrium.
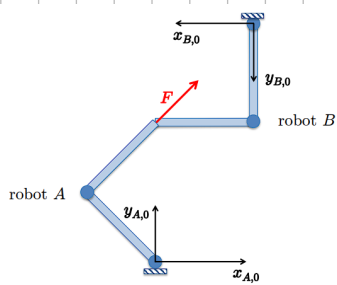
First, we notice that F have an angle of $45°$ respect to $x_{A,0}$, so in the RF_A we have that

$$F_A = \left(10\cos\frac{\pi}{4}, 10\sin\frac{\pi}{4}\right)^T = \left(5\sqrt{2}, 5\sqrt{2}\right)^T$$

if a robot is in a static equilibrium, then $\tau = J^T(q)F$, so i have to compute J and J^T. For a 2R we have :

$$J(q) = \begin{pmatrix} -S_1 - S_{12} & -S_{12} \\ C_1 + C_{12} & C_{12} \end{pmatrix} \Rightarrow J^T(q) = \begin{pmatrix} -S_1 - S_{12} & C_1 + C_{12} \\ -S_{12} & C_{12} \end{pmatrix} \text{ with } q_A = \begin{pmatrix} \frac{3}{4}\pi \\ -\pi/2 \end{pmatrix} \Rightarrow J^T(q_A) = \begin{pmatrix} -\sqrt{2} & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

$$\Rightarrow \tau_A = J^T(q_A)F_A = \begin{pmatrix} -\sqrt{2} & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 5\sqrt{2} \\ 5\sqrt{2} \end{pmatrix} = \begin{pmatrix} -10 \\ 0 \end{pmatrix} \text{ [Nm]}$$

For the robot B, is applying a Force F_B. I need to compute F_A in RF_A. By geometric inspecting is easy to see that $F_A = -\left(5\sqrt{2}, 5\sqrt{2}\right)$ in RF_B. So F_B in RF_B is $\left(5\sqrt{2}, 5\sqrt{2}\right)^T$. For $q_B = \left(\pi/2, -\pi/2\right)^T$ we have

$$J^T(q_B) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \tau_B = J^T(q_B)F_B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5\sqrt{2} \\ 5\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 5\sqrt{2} \end{pmatrix} \text{ [Nm]}$$

With reference to Fig. 4, a planar 2R robot with link lengths $l_1 = 0.5$ and $l_2 = 0.4$ [m] should intercept and follow a target that moves at constant speed $v = 0.3$ [m/sec] along a line passing through the point $P_0 = (-0.8, 1.1)$ [m] and making an angle $\beta = -20°$ with the axis x_0. The robot starts at rest from the configuration $q_s = (\pi, 0)$ [rad] (in DH terms) as soon as the target enters the workspace. The rendez-vous occurs after $T = 2$ s, with the robot end effector and the target having the same final velocity. Plan a coordinated joint space trajectory for this task.

the direction of the target is $\bar{v} = \left(\cos(-\frac{1}{9}\pi), -\sin(\frac{1}{9}\pi)\right)^T = \left(0.939, -0.342\right)^T$

$\Rightarrow v = 0.3 \cdot \bar{v} = \left(0.2817, -0.1026\right)^T$

the trajectory of the target is $p(t) = P_0 + vt$ with $P_0 = (-0.8, 1.1)$.

the workspace is the circle of radius $l_1 \cdot l_2 = 0.9$. When $p(t)$ enters in the workspace? I solve $\|p(t)\| = 0.9 \Rightarrow t^* = 2.15$

At $t = t^*$ we have $p(t^*) = \begin{pmatrix} -0.1943 \\ 0.8734 \end{pmatrix}$. We now consider $t = 0$ the time

where p enters the workspace so $p(t) = \begin{pmatrix} -0.1943 \\ 0.8734 \end{pmatrix} + t\left(0.2817, -0.1026\right)^T$

After 2 seconds i want the robot to get to p.

$$P(2) = \begin{pmatrix} -0.1943 \\ 0.8734 \end{pmatrix} + 2 \left(0.2817, -0.1026 \right)^T = \begin{pmatrix} 0.3691 \\ 0.6742 \end{pmatrix} = P_f$$

the Final conf. For the robot should be $q_f = \vec{\delta}_r^{-1}(P_f)$.
the Final velocity should be $v = \left(0.2817, -0.1026 \right)^T$ so $\dot{q}_f = \bar{J}^{-1}(q_f) v$.

For q_f, i consider the IK

$$\begin{cases} 0.5 C_1 + 0.4 C_{12} = 0.3691 \\ 0.5 S_1 + 0.4 S_{12} = 0.6742 \end{cases} \Rightarrow \cos q_2 = 0.451 \Rightarrow \sin q_2 = 1.096 \Rightarrow q_2 = 1.18 \Rightarrow q_1 = 0.497$$

$$\Rightarrow q_f = \begin{pmatrix} 0.497 \\ 1.18 \end{pmatrix} \Rightarrow J(q_f) = \begin{pmatrix} -0.636 & -0.397 \\ 0.397 & -0.042 \end{pmatrix} \Rightarrow \dot{q}_f = J^{-1} v = \begin{pmatrix} -0.285 \\ -0.252 \end{pmatrix}$$

So i have 4 boundary conditions.

$q(0) = q_i \qquad q(2) = q_f$

$\dot{q}(0) = 0 \qquad \dot{q}(2) = \dot{q}_f$

$\Rightarrow$ i use a 3 degree polynomial for q_1 and q_2

$\dot{q}_1(t) = \frac{3\pi}{2T}\left(\frac{t}{T} - \frac{t^2}{T^2} \right) \qquad\qquad \dot{q}_2(t) = -\frac{\pi^2}{2T}\sin\left(\frac{\pi}{T}t\right) \qquad\qquad \frac{3\pi}{2T}\left(\frac{t}{T} - \frac{t^2}{T^2} \right)$

$\ddot{q}_1(t) = \frac{3\pi}{2T^2} - \frac{6\pi}{2T^3}t \qquad\qquad \ddot{q}_2(t) = -\frac{\pi^3}{2T^2}\cos\left(\frac{\pi}{T}t\right)$

$q_1(0) = \frac{\pi}{4} \qquad \dot{q}_1(0) = 0 \qquad \ddot{q}_1(0) = \frac{3\pi}{2T^2} \qquad\qquad q_1(T) = \frac{\pi}{2} \qquad \dot{q}_1(T) = 0 \qquad \ddot{q}_1(T) = \frac{6\pi^{-2}}{T^2}$

$q_2(0) = 0 \qquad \dot{q}_2(0) = 0 \qquad \ddot{q}_2(0) = -\frac{\pi^3}{2T^2} \qquad\qquad q_2(T) = -\frac{\pi}{2} \qquad \dot{q}_2(T) = 0 \qquad \ddot{q}_2(T) = \frac{\pi^3}{2T^2}$

For the maximum absolute velocity i consider $\ddot{q}_1$ and $\ddot{q}_2$ and put the derivatives to 0.

$\ddot{q}_1(t) = \frac{3\pi}{2T^2} - \frac{6\pi}{2T^3}t = 0 \Rightarrow t = \frac{T}{2} \Rightarrow \dot{q}_1\left(\frac{T}{2}\right) = \frac{3\pi}{8T}$

$$\ddot{q}_2(t) = -\frac{\pi^3}{2T^2}\cos\left(\frac{\pi}{T}t\right) = 0 \Rightarrow t\frac{\pi}{T} = \frac{\pi}{2} \Rightarrow t = \frac{T}{\pi}\frac{\pi}{2} = \frac{T}{2} \Rightarrow \dot{q}_2\left(\frac{T}{2}\right) = -\frac{\pi^2}{2T}\sin\frac{\pi}{2} = -\frac{\pi^2}{2T} \Rightarrow V_2 = \frac{\pi^2}{2T}$$

to find A_1, A_2 i have to put to zero the jerk:

$$\dddot{q}_1(t) = -\frac{2}{T^3} = 0 \Rightarrow \text{i consider } \ddot{q}_1 \text{ at the boundaries} \Rightarrow$$

$$\ddot{q}_1(0) = \frac{3\pi}{2T^2} - 2\frac{t}{T^3} = \frac{3\pi}{2T^2}$$

$$\ddot{q}_1(T) = \frac{3\pi}{2T^2} - 2\frac{t}{T^3} = \frac{3\pi}{2T^2} - 2\frac{1}{T^2}$$

$\left.\right\}$ Since $T > 0 \Rightarrow A_1 = \frac{3\pi}{2T^2}$

$$\dddot{q}_2(t) = \frac{\pi^4}{2T^3}\sin\left(\frac{\pi}{T}t\right) = 0 \Rightarrow t = 0 \Rightarrow \ddot{q}_2(0) = -\frac{\pi^3}{2T^2} \Rightarrow A_2 = \frac{\pi^3}{2T^2}$$

if $V_1 = 4 \Rightarrow \qquad \frac{3}{8}\frac{\pi}{T} = 4 \Rightarrow T = 0.294$

$\qquad V_2 = 8 \Rightarrow \frac{\pi^2}{2T} = 8 \Rightarrow T = 0.6168$

$\qquad A_1 = \frac{3\pi}{2T^2} = 20 \Rightarrow T = 0.485$

$\qquad A_2 = \frac{\pi^3}{2T^2} = 40 \Rightarrow T = 0.6225$

$\left.\right\}$ i consider the largest $\Rightarrow T = 0.6225$

Question #10 [all students]

Consider again the task in Question #8. The robot is commanded by the joint velocity $\dot{q} \in \mathbb{R}^2$. Once the rendez-vous has been accomplished, design a feedback control law that will let the robot follow the moving target and react to position errors e_t and e_n that may occur along the tangent and normal directions to the linear path, respectively with the prescribed decoupled dynamics $\dot{e}_t = -3\,e_t$ and $\dot{e}_n = -10\,e_n$. Provide the explicit expression of all terms in the control law.

In RF_0 we have $p(t) = p_0 + vt = \begin{pmatrix}0.2817\\-0.1026\end{pmatrix}t + \begin{pmatrix}-0.8\\1.1\end{pmatrix}$. the cartesian error is

$$^0e = \begin{pmatrix}e_n\\e_t\end{pmatrix} = p_d - f(q).$$ The frame of the target is given by $R_z\left(-\frac{\pi}{5}\right)$ so

I use ^{0} as apice for RF_0 and ^{P} for Frenet frame of the target.

if ^Pw is a vector $\quad ^0w = {}^0R_P\,{}^Pw$.

$$^0R_P = R_z\left(-\frac{\pi}{5}\right) = \begin{bmatrix}0.939 & 0.342\\-0.342 & 0.939\end{bmatrix}$$

I denote $\dot{p}_d = v$ the vel. of the target.

$$\dot{q} = J^{-1}(q)\left(v + {}^0R_P\,{}^PK\,{}^Pe\right)$$

Where $\quad {}^0e = p_d - f(q)$

$$^Pe = {}^0R_P^T\,{}^0e = \begin{bmatrix}0.939 & -0.342\\0.342 & 0.939\end{bmatrix}(p_d - f(q))$$

where ^PK is the gain matrix in the target frame. Since $^0e = p_d - f(q)$ then $^0\dot{e} = v - \dot{p}$

but $\dot{p} = J(q)\dot{q} = $ ← velocity of ee

$$J(q)J^{-1}(q)\left(v + {}^0R_P\,{}^PK\,{}^Pe\right) = v + {}^0R_P\,{}^PK\,{}^Pe \Rightarrow$$

$K_1 = 3$
$K_2 = 10$
⇑
$\begin{pmatrix}-K_1 e_t\\-K_2 e_n\end{pmatrix}$

$$^0\dot{e} = -{}^0R_P\,{}^PK\,{}^Pe = -{}^0R_P\,{}^PK\,{}^0R_P^T\,{}^0e = -K\,{}^0e \Rightarrow$$

$^0e = {}^0R_P\,{}^Pe$

$$^P\dot{e} = {}^PR_0\,{}^0\dot{e} = {}^PR_0\left(-{}^0R_P\right){}^PK\,{}^0R_P^T\,{}^0e = {}^PR_0\left(-{}^0R_P\right){}^PK\,{}^0R_P^T\,{}^0R_P\,{}^Pe = -{}^PK\,{}^Pe = \begin{pmatrix}-K_1 e_t\\-K_2 e_n\end{pmatrix}$$

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