Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise #1

Given a smooth time-varying rotation matrix $\boldsymbol{R}(t) \in SO(3)$, provide a formula to determine the associated angular acceleration vector $\dot{\boldsymbol{\omega}}(t) \in \mathbb{R}^3$ as a function of $\boldsymbol{R}(t)$ and of the angular velocity $\boldsymbol{\omega}(t) \in \mathbb{R}^3$. Apply then this formula to compute $\boldsymbol{\omega}(t)$ and $\dot{\boldsymbol{\omega}}(t)$, given the following rotation matrix:

$$\boldsymbol{R}(t) = \begin{pmatrix} \cos t & 0 & \sin t \\ \sin^2 t & \cos t & -\sin t \cos t \\ -\sin t \cos t & \sin t & \cos^2 t \end{pmatrix}.$$

let $\omega = (\omega_x, \omega_y, \omega_z)$ to be the angular velocity. Since $S(\omega) = \dot{R}R^T$, ω can be easily found. In such case ω depends on $\dot{R}$ and R^T.

$\frac{d}{dt} S(\omega) = \ddot{R}R^T + \dot{R}\dot{R}^T$. For the given matrix:

$$\dot{R} = \begin{pmatrix} -\sin t & 0 & \cos t \\ 2\sin t\cos t & -\sin t & -\cos^2 t + \sin^2 t \\ -\cos^2 t + \sin^2 t & \cos t & -2\sin t\cos t \end{pmatrix} = \begin{pmatrix} -s & 0 & c \\ 2sc & -s & 1-2c^2 \\ 1-2c^2 & c & -2sc \end{pmatrix} \Rightarrow \ddot{R} = \begin{pmatrix} -c & 0 & -s \\ 2-4s^2 & -c & -2sc \\ 4sc & -s & 4s^2-2 \end{pmatrix}$$

$$\ddot{R}R^T = \begin{pmatrix} -c & 0 & -s \\ 2-4s^2 & -c & -2sc \\ 4sc & -s & 4s^2-2 \end{pmatrix} \cdot \begin{pmatrix} c & s^2 & -sc \\ 0 & c & s \\ s & -sc & c^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ c(2-6s^2) & s^2(2-4s^2)+c^2(2s^2-1) & 2-4s^2(-sc)-cs-2sc^3 \\ 4sc^2+s(4s^2-2) & 4s^2c-sc(4s^2-2) & -4s^2c^2-s^2+c^2(4s^2-2) \end{pmatrix}$$

$$\dot{R}\dot{R}^T = \begin{pmatrix} -s & 0 & c \\ 2sc & -s & 1-2c^2 \\ 1-2c^2 & c & -2sc \end{pmatrix} \begin{pmatrix} -s & 2sc & 1-2c^2 \\ 0 & -s & c \\ c & 1-2c^2 & -2sc \end{pmatrix} = \begin{pmatrix} 1 & -2s^2c+c-2c^3 & s(2c^2-1)-2sc^2 \\ -2s^2c+c-2c^3 & (2sc)^2+s^2+(1-2c)^2 & 2sc(1-2c^2)-sc-2sc(1-2c^2) \\ 4c^2s-s & 2sc(1-2c^2)-sc-2sc(1-2c^2) & (1-2c^2)^2+c^2+(2sc)^2 \end{pmatrix}$$

Exercise #4

Consider the 3×3 Jacobian of a 3R spatial robot, with generic link lengths $l_2 > 0$ and $l_3 > 0$:

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1(l_2c_2 + l_3c_3) & -l_2c_1s_2 & -l_3c_1s_3 \\ c_1(l_2c_2 + l_3c_3) & -l_2s_1s_2 & -l_3s_1s_3 \\ 0 & l_2c_2 & l_3c_3 \end{pmatrix}, \qquad \boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Find all (singular) configurations $\boldsymbol{q}^\circ$ where the rank of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ is equal to 2 and all configurations $\boldsymbol{q}^*$ where the rank is equal to 1. In a singularity with rank 1, determine a basis for each of the subspaces $\mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}^*)\}$, $\mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}^*)\}$, $\mathcal{R}\{\boldsymbol{J}^T(\boldsymbol{q}^*)\}$, and $\mathcal{N}\{\boldsymbol{J}^T(\boldsymbol{q}^*)\}$.

The determinant is $l_2 l_3 (l_2c_2 + l_3c_3) \cdot \sin(q_2 - q_3) \Rightarrow$ is zero if:

$q_2 = q_3$ or $c_2 = -\frac{l_3}{l_2} c_3$

IF $q_2 = q_3 = 0$:

$$J = \begin{pmatrix} -s_1(l_2+l_3) & 0 & 0 \\ c_1(l_2+l_3) & 0 & 0 \\ 0 & l_2 & l_3 \end{pmatrix} \Rightarrow \text{rank} = 2$$

then $q_3 = 0$, $q_2 = \arccos\left(-\frac{l_3}{l_2}\right) \Rightarrow c_3 = 1$ $c_2 = -\frac{l_3}{l_2} \Rightarrow$ i consider also $q_1 = \frac{\pi}{2}$

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -l_3 & l_3 \end{pmatrix} \Rightarrow \text{rank} = 1, \quad J^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -l_3 \\ 0 & 0 & l_3 \end{pmatrix}$$

$$J\dot{q} = \begin{pmatrix} 0 \\ 0 \\ l_3(\dot{q}_3 - \dot{q}_2) \end{pmatrix} \Rightarrow \mathcal{R}(J) = \text{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}. \quad J\dot{q} = 0 \Rightarrow l_3(\dot{q}_3 - \dot{q}_2) = 0 \Rightarrow \dot{q}_3 = \dot{q}_2 \Rightarrow \mathcal{N}(J) = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

$$J^T F = \begin{pmatrix} 0 \\ -l_3F_3 \\ l_3F_3 \end{pmatrix} \Rightarrow \mathcal{R}(J^T) = \text{Span}\left\{\begin{pmatrix} 0 \\ -l_3 \\ l_3 \end{pmatrix}\right\}. \quad J^T F = 0 \Rightarrow F_3 = 0 \Rightarrow \mathcal{N}(J^T) = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Exercise #5

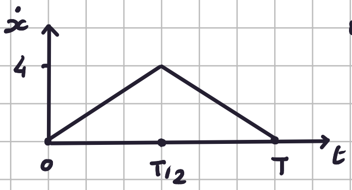
A mass $M = 2$ [kg] moves linearly under a bounded force u, with $|u| \leq U_{max} = 8$ [N], according to differential equation $M\ddot{x} = u$. The mass starts at $t = 0$ from $x_i = x(0) = 0$ with a negative velocity $\dot{x}_i = \dot{x}(0) = -2$ [m/s], and has to reach the final position $x_f = x(T) = 3$ [m] at rest (i.e., with $\dot{x}_f = \dot{x}(T) = 0$) in minimum time T. Determine the minimum time T and the associated optimal command $u^*(t)$. Sketch the time evolution of $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$.

$\ddot{x} = \frac{u}{M}$ $\qquad \max|\ddot{x}| = \frac{\max|u|}{M} = 4$

I can consider a bang-bang profile assuming $\dot{x}(0) = 0$ and then include the time needed to go from $\dot{x} = -2$ to $\dot{x} = 0$.

$$\Rightarrow \ddot{x}(t) = \begin{cases} 4 & t \in [0, T/2] \\ -4 & t \in [T/2, T] \end{cases}$$

$$\dot{x}(t) = \begin{cases} 4t & t \in [0, T/2] \\ 4\frac{T}{2} - 4t & t \in [T/2, T] \end{cases}$$



dist

$\Rightarrow L = T \cdot 2T \cdot \frac{1}{2} = T^2 = 3$

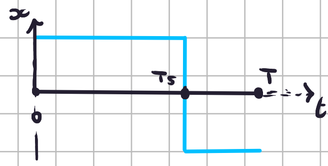
$\Rightarrow T = \sqrt{3}$

We have to add the time to go from $\dot{x} = -2$ to $\dot{x} = 0$

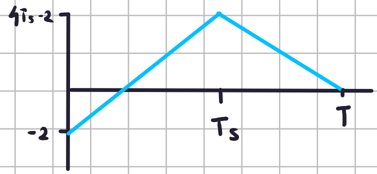
$-2 + 4t = 0 \Rightarrow t = 1/2 \Rightarrow$ at the end $T = \sqrt{3} + 1/2$ and the command is

$$u(t) = \begin{cases} 8 & \text{if } t \in [0, \frac{1}{2}(1+\sqrt{3})] \\ -8 & \text{if } t \in [\frac{1}{2}(1+\sqrt{3}), \sqrt{3}] \end{cases}$$

$$\ddot{x}(t) = \begin{cases} 4 & \text{For } t \in [0, T_s] \\ -4 & \text{For } t \in [T_s, T] \end{cases}$$



$$\Rightarrow \dot{x}(t) = \begin{cases} -2 + 4t & t \in [0, T_s] \\ -4t - 2 + 4T_s & t \in [T_s, T] \end{cases}$$



$\Rightarrow T - T_s =$

$$x(T) = -2T_s + 2T_s^2 - 2T^2 - 2T + 4T_sT + 2T_s^2 + 2T_s - 4T_s^2 = 3$$

$$\Rightarrow -2T^2 - 2T + 4T_sT = 3 \Rightarrow -2T^2$$