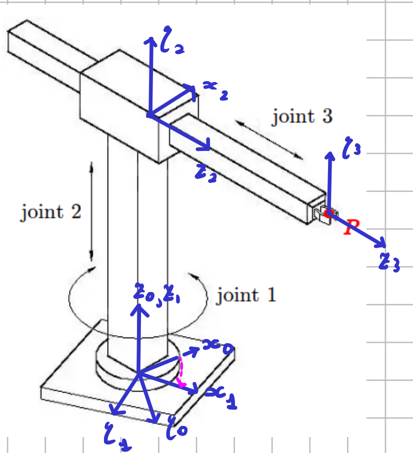
Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise 2

A 3-dof cylindrical robot is shown in Fig. 2.

a) Assign the frames according to the Denavit-Hartenberg (D-H) convention, so that all constant D-H parameters are non-negative. Provide the table with the corresponding parameters. The first frame should be placed on the floor at the robot base and the origin of the last frame should be at point P. Compute position and orientation of the end-effector frame as given by the homogeneous matrix ${}^0\boldsymbol{T}_3(\boldsymbol{q})$.



JOINT	α_i	a_i	d_i	θ_i
1	0	0	0	q_1
2	$\pi/2$	0	q_2	$\pi/2$
3	0	0	q_3	0

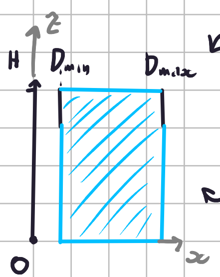
$$\Rightarrow {}^0T_1 = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1T_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^2T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow {}^0T_2 = \begin{pmatrix} -s_1 & 0 & c_1 & 0 \\ c_1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^0T_3 = \begin{pmatrix} -s_1 & 0 & c_1 & q_3 c_1 \\ c_1 & 0 & s_1 & q_3 s_1 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{cases} q_3 c_1 \\ q_3 s_1 \\ q_2 \end{cases} \quad \text{For the IK: } \begin{cases} x = q_3 c_1 \\ y = q_3 s_1 \\ z = q_2 \end{cases} \Rightarrow q_3^2 = x^2 + y^2 \Rightarrow q_3 = \pm\sqrt{x^2+y^2} \Rightarrow \begin{matrix} c_1 = \pm x/\sqrt{x^2+y^2} \\ s_1 = \pm y/\sqrt{x^2+y^2} \end{matrix}$$

$$\Rightarrow q_1 = \pm \operatorname{atan2}\{y, x\}.$$

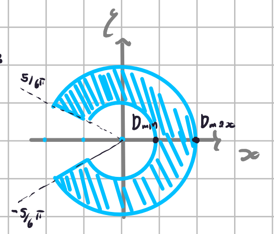
Is a cylindric with an hole inside and without a slice of $2/6\,\pi$



section of Workspace

this rotated from $-5/6\,\pi$ to $5/6\,\pi$

Top view:



In these bounds we have 1 solution since q_3 can never be negative.

The Jacobian is
$$\frac{dp}{dq} = J(q) = \begin{pmatrix} -q_3 s_1 & 0 & c_1 \\ q_3 c_1 & 0 & s_1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \det J = q_3 \Rightarrow J \text{ is singular} \iff q_3 = 0.$$

In a generic $q_s = (q_1, q_2, 0) \Rightarrow J_s = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & s_1 \\ 0 & 1 & 0 \end{pmatrix}$. I study $\operatorname{Ker}(J_s)$

$$J_s \dot{q} = 0 \Rightarrow \begin{cases} \dot{q}_3 c_1 = 0 \\ \dot{q}_3 s_1 = 0 \\ \dot{q}_2 = 0 \end{cases} \Rightarrow \begin{matrix} \dot{q}_2 = 0 \\ \dot{q}_3 = 0 \end{matrix} \Rightarrow \operatorname{Ker} J_s = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Now i study $\operatorname{Ker} J_s^T$

$$J_s^T F = 0 \Rightarrow \begin{cases} F_3 = 0 \\ F_1 c_1 + F_2 s_1 = 0 \end{cases} \Rightarrow \begin{cases} F_3 = 0 \\ F_1 = -\tan(q_1) F_2 \end{cases} \Rightarrow \operatorname{Ker} J_s^T = \operatorname{Span}\left\{ \begin{pmatrix} -\tan q_1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

The null space of the Jacobian matrix represents physically the velocity of the joints that produces no end effector velocity. In a singular configuration, moving the first joint will not change the position of the end effector. The null space of the Jacobian transpose represent all the forces that statically balances a zero torque on the joints.

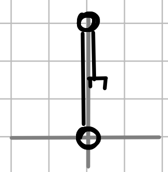
Exercise 3

Consider the following trajectory planning problem for a 2R planar robot with links of length $L_1 = 2$ and $L_2 = 1$ [m]. The robot should perform a rest-to-rest motion in a given time $T = 2$ s, moving its end-effector from point $A = (0, 1)$ to point $B = (3, 0)$ [m]. The initial direction of the end-effector motion from point A is specified by the tangent vector in Cartesian space $\boldsymbol{dp}/ds|_A = (5, 0)$, where s is a suitable scalar that parametrizes the path. Similarly, the final approach direction to point B is specified by $\boldsymbol{dp}/ds|_B = (0, -1)$.

a) Design a joint trajectory that solves this problem, providing the analytic expression of the various terms in the complete solution and some plots that help illustrating it.

b) Suppose now that the joint velocities are limited: $|\dot q_1| \le V_1 = 2$, $|\dot q_2| \le V_2 = 3$ [rad/s]. Verify whether the trajectory that has been planned in item a) is feasible. If this is not the case, determine a convenient, possibly minimum, motion time T^* that solves the same problem and complies also with these bounds.

Since we are interested only in the final directions, we want that $t(0) = \dfrac{p'(0)}{\|p'(0)\|} = \begin{pmatrix}1\\0\end{pmatrix}$ and $t(1) = \dfrac{p'(1)}{\|p'(1)\|} = \begin{pmatrix}0\\-1\end{pmatrix}$. Let f to be the DK. For B:

$q_B = f^{-1}(B) = \begin{pmatrix}0\\0\end{pmatrix}$. For A:  $\Rightarrow q_A = \begin{pmatrix}\pi/2\\ \pi\end{pmatrix}$

$J(q_A) = \begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix} \Rightarrow J(q_A)\dot q = \begin{pmatrix}-\dot q_1 + \dot q_2\\ 0\end{pmatrix} \Rightarrow$ to realize $p' = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \dfrac{dq}{ds}(0) = \begin{pmatrix}-a\\0\end{pmatrix}$ $a \in \mathbb{R}$

$J(q_B) = \begin{pmatrix}0 & 0\\ 3 & 1\end{pmatrix} \Rightarrow J(q_B)\dot q = \begin{pmatrix}0\\ 3\dot q_1 + \dot q_2\end{pmatrix} \Rightarrow \dfrac{dq}{ds}(1) = \begin{pmatrix}-a\\0\end{pmatrix}$ $a \in \mathbb{R}$

Conditions: $q_1(0) = \pi/2$ $\quad q_1(1) = 0$ $\quad q_2(0) = \pi$ $\quad q_2(1) = 0$

$q_1'(0) = 0$ $\quad q_1'(1) = -1$ $\quad q_2'(0) = 1$ $\quad q_2'(1) = 0$

q_1) I use a 3-degree pol.

$$\begin{cases} d = \frac{\pi}{2} \\ c = 0 \\ 3a + 2b = -1 \\ a + b + \frac{\pi}{2} = 0 \end{cases} \Rightarrow q_1(s) = 2.14s^3 - 3.71s^2 + \frac{\pi}{2}$$

q_2)

$$\begin{cases} d = \pi \\ c = 1 \\ 3a + 2b + 1 = 0 \\ a + b = -1 - \pi \end{cases} \Rightarrow q_2(s) = 7.28s^3 - 11.424s^2 + s + \pi$$

Now i decide a timing law for $s(t)$. I consider a b-c-b profile.

$s(t) = \frac{1}{4}(-t^3 + 3t^2)$ $\quad t \in [0, 2]$. Now i consider

$q_1' = 6.42s^2 - 7.42s$ $\qquad q_1'' = 12.84s - 7.42 = 0 \Rightarrow s = 0.577$

$\max|q_1'| = |q_1'(0.577)| = 2.1439$

$|\dot q_1| \le V_1 \Rightarrow |q_1' \dot s| \le V_1 \Rightarrow |\dot s| \le \dfrac{V_1}{|q_1'|} \Rightarrow \max|\dot s| \le \dfrac{V_1}{\max|q_1'|} \Rightarrow |\dot s| \le 0.9328$

$q_2' = 21.84s^2 - 22.848s + 1$ $\qquad q_2'' = 43.68s - 22.848 = 0 \Rightarrow s = \frac{34}{65}$

$\max|q_2'| = 4.3756 \Rightarrow \max|\dot s| \le \dfrac{V}{\max|q_2'|} = 0.6 \Rightarrow |\dot s| \le 0.6$

$$\Rightarrow \quad \dot{s} = \frac{1}{4}\left(-3t^2+6t\right) \Rightarrow \max|\dot{s}| = |\dot{s}(t^*)| = \frac{3}{4} > 0.6$$

$$\ddot{s} = \frac{1}{4}\left(-6t+6\right) = 0 \Rightarrow t^* = 1$$

$\Rightarrow$ i have to scale the time. $\Rightarrow K = \dfrac{0.75}{0.6} = \dfrac{5}{4}$

$$\Rightarrow s(t) = \frac{1}{4}\left(-(kt)^3 + 3(kt)^2\right) \quad t \in \left[0, \frac{5}{4}T\right] = [0, 2.5]$$

Exercise 4

The kinematics of a 3R spatial robot is specified through the D-H parameters given in Tab. 1. The robot has its base mounted on the floor, defined by the plane (x_0, y_0).

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0.7	q_1
2	0	0.5	0	q_2
3	0	0.5	0	q_3

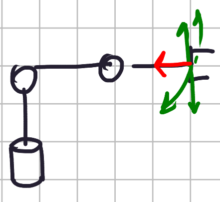
$${}^0T_1 = \begin{pmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0.7 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1T_2 = \begin{pmatrix} C_2 & -S_2 & 0 & 0.5C_2 \\ S_2 & C_2 & 0 & 0.5S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^2T_3 = \begin{pmatrix} C_3 & -S_3 & 0 & 0.5C_3 \\ S_3 & C_3 & 0 & 0.5S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^1T_3 = \begin{pmatrix} C_{23} & -S_{23} & 0 & 0.5(C_2+C_{23}) \\ S_{23} & C_{23} & 0 & 0.5(S_2+S_{23}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{the DK is} \quad p = f_r(q) = \begin{cases} 0.5(C_2+C_{23})\cdot C_1 \\ 0.5(C_2+C_{23})\cdot S_1 \\ 0.5(S_2+S_{23})+0.7 \end{cases}$$

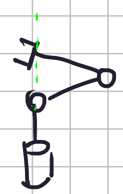
the robot Jacobian is
$$J = \begin{pmatrix} -0.5S_1(C_2+C_{23}) & -0.5C_1(S_2+S_{23}) & -0.5C_1S_{23} \\ 0.5C_1(C_2+C_{23}) & -0.5S_1(S_2+S_{23}) & -0.5S_1S_{23} \\ 0 & 0.5(C_2+C_{23}) & 0.5C_{23} \end{pmatrix}$$

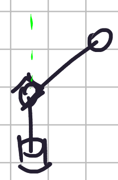
the robot is in a singular conf. if

- the arm is fully stretched $\Rightarrow q_3 = 0$



- the end effector is on the z_0 axis





$$\Rightarrow p_x^2 + p_y^2 = 0 \Rightarrow 0.5^2(C_2+C_{23})^2 = 0 \Rightarrow C_2 + C_{23} = 0 \Rightarrow \begin{cases} q_3 = \pi \Rightarrow C_{23} = -C_2 \\ C_2 = -C_{23} \Rightarrow \end{cases}$$

the WS is a sphere of radius 1 centered in $P_0 = (0, 0, 0.7)$, the trajectory is an elicoid with the axis coincident to z_0 $\Rightarrow$ Since $p(s)$ doesn't touch z_0, the second type singul. $\begin{cases} q_3 = \pi \\ C_2 = -C_{23} \end{cases}$ never occurs. We have to show that $p(s)$ is not on the boundaries of WS_1.

$\partial WS_1 = \left\{ v : \|v - P_0\| = 1 \right\}$. $\left\| p(s) - \begin{pmatrix} 0 \\ 0 \\ 0.7 \end{pmatrix} \right\| = \sqrt{0.5 + 0.04s^2 - 0.1s}$ $\quad s \in [0,3]$

$\Rightarrow \sqrt{0.5 + 0.04s^2 - 0.1s} = 1 \Rightarrow -1.5 + 0.04s^2 - 0.1s = 0 \Rightarrow s = \frac{15}{2}$ but $s \in [0,3] \Rightarrow p(s) \in WS_1$ and never singular.

Since $\dot{s} = 1$, $s(t) = t$. if $q(0) = (0, \frac{\pi}{6}, -\frac{\pi}{2})$ we have:

$p_0 = f_r(q(0)) = \left(\frac{1+\sqrt{3}}{4}, 0, \frac{19 - 5\sqrt{3}}{20} \right)^T$ is not on $p(s)$ since

$\sqrt{p_x^2(s) + p_y^2(s)} = r \;\; \forall s$ and $\sqrt{p_{0_x}^2 + p_{0_y}^2} = \frac{1+\sqrt{3}}{4} \neq r = \frac{1}{2}$.

Let ${}^0T_F(s)$ to be the matrix of changing coordinates: $T = \begin{pmatrix} t & n & b & p(s) \\ 0 & 0 & 0 & 1 \end{pmatrix}$, so

if ${}^0e$ is the error in the RF_0 and ${}^Fe$ in the Frenet Frame:

${}^0e = {}^0T_F \, {}^Fe$ and ${}^Fe = {}^0T_F^T \, {}^0e$.

$p'(s) = \begin{pmatrix} -r2\pi \sin 2\pi s \\ r2\pi \cos 2\pi s \\ h \end{pmatrix} \Rightarrow \|p'\| = \sqrt{4\pi^2 r^2 + h^2} = 3.167$

$t(s) = \frac{p'(s)}{\|p'(s)\|} = \frac{1}{3.167} \begin{pmatrix} -r2\pi \sin 2\pi s \\ r2\pi \cos 2\pi s \\ h \end{pmatrix} \Rightarrow t'(s) = \frac{1}{3.167} \begin{pmatrix} -r4\pi^2 \cos 2\pi s \\ -r4\pi^2 \sin 2\pi s \\ 0 \end{pmatrix} \Rightarrow \|t'\| = 6.232$

$n(s) = \frac{1}{19.736} \begin{pmatrix} -r4\pi^2 \cos 2\pi s \\ -r4\pi^2 \sin 2\pi s \\ 0 \end{pmatrix}$. The skew-symmetric mat of t is $S(t) = \begin{pmatrix} 0 & -0.126 & 0.991 \cos 2\pi s \\ 0.126 & 0 & 0.991 \sin 2\pi s \\ -0.991 \cos 2\pi s & -0.991 \sin 2\pi s & 0 \end{pmatrix}$

$b = t \times n = \begin{pmatrix} 0 & -0.126 & 0.991 \cos 2\pi s \\ 0.126 & 0 & 0.991 \sin 2\pi s \\ -0.991 \cos 2\pi s & -0.991 \sin 2\pi s & 0 \end{pmatrix} \frac{1}{19.736} \begin{pmatrix} -r4\pi^2 \cos 2\pi s \\ -r4\pi^2 \sin 2\pi s \\ 0 \end{pmatrix} =$ $\;*\; \frac{1}{19.736} \cdot r4\pi^2 \approx 1$

$\begin{pmatrix} 0 & -0.126 & 0.991 \cos 2\pi s \\ 0.126 & 0 & 0.991 \sin 2\pi s \\ -0.991 \cos 2\pi s & -0.991 \sin 2\pi s & 0 \end{pmatrix} \begin{pmatrix} -\cos 2\pi s \\ -\sin 2\pi s \\ 0 \end{pmatrix} = \begin{pmatrix} 0.126 \cdot \sin(2\pi s) \\ -0.126 \cdot \cos(2\pi s) \\ 0.991 \end{pmatrix}$

I consider a control law of the following type:

I denote $p_d = p(t)$ the desired pos and p the actual pos.

$\Rightarrow {}^0e = p_d - p \qquad {}^Fe = {}^0R_F^T \, {}^0e$

$\Rightarrow {}^F\dot{e} = {}^0R_F^T \, {}^0\dot{e} + \left(S(\omega) {}^0R_F \right)^T {}^0e =$

$${}^{0}R_F^T\,{}^{0}\dot{e} + \left(S(\omega)\,{}^{0}R_F\right)^T {}^{0}e = {}^{0}R_F^T\left({}^{0}\dot{e} + S^T(\omega)\,{}^{0}e\right) = {}^{0}R_F^T\left(\dot{p}_d - J\dot{q} + S^T(\omega)\,{}^{0}e\right)$$

${}^{0}R_F^T\left(\dot{p}_d - J\dot{q} + S^T(\omega)\,{}^{0}e\right)$ i want that this is $-K\,{}^{F}e$ so

$${}^{0}R_F^T\left(\dot{p}_d - J\dot{q} + S^T(\omega)\,{}^{0}e\right) = -K\,{}^{F}e \Rightarrow$$

$${}^{0}R_F^T\,\dot{p}_d - {}^{0}R_F^T J\dot{q} + {}^{0}R_F^T S^T\,{}^{F}R_0\,{}^{0}e = -K\,{}^{F}e \Rightarrow$$

$${}^{0}R_F^T\,\dot{p}_d + K\,{}^{F}e + {}^{0}R_F^T S^T\,{}^{F}R_0\,{}^{0}e = {}^{0}R_F^T J\dot{q} \Rightarrow \dot{q} = \left({}^{0}R_F^T J\right)^{-1}\left({}^{0}R_F^T\,\dot{p}_d + K\,{}^{F}e + {}^{0}R_F^T S^T\,{}^{F}R_0\,{}^{0}e\right)$$

$$\dot{q} = \left({}^{0}R_F^T J\right)^{-1}\left({}^{0}R_F^T\,\dot{p}_d + K\,{}^{F}e + {}^{0}R_F^T S^T\,{}^{F}R_0\,{}^{0}e\right) =$$

$$J^{-1}\,{}^{0}R_F\left({}^{0}R_F^T\,\dot{p}_d + K\,{}^{F}e + {}^{0}R_F^T S^T\,{}^{F}R_0\,{}^{0}e\right) =$$

$$J^{-1}\left(\dot{p}_d + {}^{0}R_F K\,{}^{F}e + S^T(\omega)\,{}^{F}R_0\,{}^{0}e\right)$$

$$\begin{matrix} -0.5S_1(C_2+C_{23}) & -0.5C_1(S_2+S_{23}) & -0.5C_1S_{23} & -0.5S_1(C_2+C_{23}) & -0.5C_1(S_2+S_{23}) \\ 0.5C_1(C_2+C_{23}) & -0.5S_1(S_2+S_{23}) & -0.5S_1S_{23} & 0.5C_1(C_2+C_{23}) & -0.5S_1(S_2+S_{23}) \\ 0 & 0.5(C_2+C_{23}) & 0.5C_{23} & 0 & 0.5(C_2+C_{23}) \end{matrix}$$

$$\frac{1}{8}s_1^2(c_2+c_{23})(s_2+s_{23})c_{23} - \frac{1}{8}c_1^2 s_{23}(c_2+c_{23})^2$$

$$-\frac{1}{8}(c_2+c_{23})^2 s_1 s_{23} + \frac{1}{8}c_{23}c_1^2(c_2+c_{23})(s_2+s_{23})$$

$$\frac{1}{8}(c_2+c_{23})\left[(s_2+s_{23})c_{23}s_1^2 - c_1^2 s_{23}(c_2+c_{23}) - (c_2+c_{23})s_1 s_{23} + c_{23}c_1^2(s_2+s_{23})\right]$$

$q_2 = \pm\pi/2 \qquad q_3 = 0, \pi$