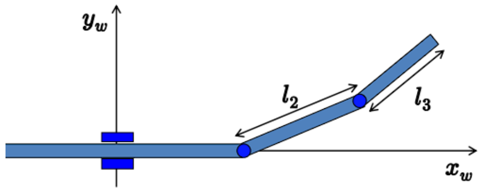
Exercise 1

For the PRR planar robot in Fig. 1, consider first the task vector $\boldsymbol{r}$ made by the position $\boldsymbol{p} \in \mathbb{R}^2$ of the end-effector and by its angle $\phi \in \mathbb{R}$ with respect to the $\boldsymbol{x}_w$ axis. Compute the corresponding Jacobian $\boldsymbol{J}_{\boldsymbol{r}}(\boldsymbol{q})$ and find all its singularities. With the robot in a generic singular configuration $\boldsymbol{q}_s$:

a) provide the expression of all joint velocities $\dot{\boldsymbol{q}}$ that produce no task velocity $\dot{\boldsymbol{r}}$;

b) determine all task velocities $\dot{\boldsymbol{r}}$ that cannot be instantaneously realized.



The DK of the manipulator is : $f_r(q) = r = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \begin{cases} q_1 + l_2 c_2 + l_3 c_{23} \\ l_2 s_2 + l_3 s_{23} \\ q_2 + q_3 \end{cases}$

$$J(q) = \begin{pmatrix} 1 & -l_2 s_2 - l_3 s_{23} & -l_3 s_{23} \\ 0 & l_2 c_2 + l_3 c_{23} & l_3 c_{23} \\ 0 & 1 & 1 \end{pmatrix}$$

we have $\det J = \begin{matrix} 1 & -l_2 s_2 - l_3 s_{23} & -l_3 s_{23} & 1 & -l_2 s_2 - l_3 s_{23} \\ 0 & l_2 c_2 + l_3 c_{23} & l_3 c_{23} & 0 & l_2 c_2 + l_3 c_{23} \\ 0 & 1 & 1 & 0 & 1 \end{matrix} - \begin{matrix} 1 & -l_2 s_2 - l_3 s_{23} & -l_3 s_{23} & 1 & -l_2 s_2 - l_3 s_{23} \\ 0 & l_2 c_2 + l_3 c_{23} & l_3 c_{23} & 0 & l_2 c_2 + l_3 c_{23} \\ 0 & 1 & 1 & 0 & 1 \end{matrix} = l_2 c_2 \Rightarrow$

J is singular $\iff \cos(q_2) = 0 \Rightarrow q_2 = \pm \pi/2$. I consider $q_s = (0, \frac{\pi}{2}, 0) \Rightarrow$

$\dot{q}_1 - q_2(l_2 + l_3) - l_3 q_3$

$- q_2(l_2 + l_3)$

$$J(q_s) = J_s = \begin{pmatrix} 1 & -l_2 - l_3 & -l_3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } J_s \dot{q} = \begin{pmatrix} \dot{q}_1 - l_2 \dot{q}_2 - l_3(\dot{q}_2 + \dot{q}_3) \\ 0 \\ \dot{q}_2 + \dot{q}_3 \end{pmatrix} \Rightarrow R(J_s) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The velocities $\dot{r}$ that cannot be realized are $\mathbb{R}^3 \setminus R(J_s) = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Next, consider only the two-dimensional task vector $\boldsymbol{r} = \boldsymbol{p}$ for the same robot and find all singularities of the corresponding Jacobian $\boldsymbol{J}_p(\boldsymbol{q})$. When the robot is in a configuration $\boldsymbol{q}_s$ with all strictly positive joint values and such that the matrix $\boldsymbol{J}_p(\boldsymbol{q}_s)$ loses rank:

c) provide the expression of all forces $\boldsymbol{f} \in \mathbb{R}^2$ applied to the end-effector that need no joint force/torque $\boldsymbol{\tau} \in \mathbb{R}^3$ to be balanced;

d) determine the $\boldsymbol{\tau}$ that statically balances a force $\boldsymbol{f} = (3 \;\; 1)^T$ [N] applied to the end-effector.

if $f_r(q) = p$ then $J = \begin{pmatrix} 1 & -l_2 s_2 - l_3 s_{23} & -l_3 s_{23} \\ 0 & l_2 c_2 + l_3 c_{23} & l_3 c_{23} \end{pmatrix}$. Singular iF $q_2 = \pm \frac{\pi}{2}$ Since the matrix become $J_s = \begin{pmatrix} 1 & -l_2 - l_3 c_3 & -l_3 c_3 \\ 0 & -l_3 s_3 & -l_3 s_3 \end{pmatrix}$ and $J_s^2 = J_s^3 - l_2 J_s^1$. Is the same singularity of the complete Jacobian since that was related to the linear components. $q_s = (1, \frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow$

$J_s = \begin{pmatrix} 1 & -l_2 & 0 \\ 0 & -l_3 & -l_3 \end{pmatrix} \Rightarrow J^T = \begin{pmatrix} 1 & 0 \\ -l_2 & -l_3 \\ 0 & -l_3 \end{pmatrix}$. I need the F s.t. $J^T F = 0 \Rightarrow \text{Ker}(J^T)$

$$J^T F = \begin{cases} F_1 = 0 \\ -l_2 F_1 - l_3 F_2 = 0 \\ -l_3 F_2 = 0 \end{cases} \Rightarrow F = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{Ker } J^T = \{0\}.$$

Then i want to find τ that balances $F = (3 \;\; 1)^T$:

$$-J^T F = \tau \Rightarrow \begin{cases} -3 \\ l_2 3 + l_3 \\ l_3 \end{cases} \Rightarrow \tau = (-3 \quad 3 l_2 + l_3 \quad l_3)^T$$

Exercise 2

A cylindrical robot has the direct kinematics of its end-effector position expressed by

$$\boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} q_3 \cos q_2 \\ q_3 \sin q_2 \\ q_1 \end{pmatrix}.$$

When the desired position is $\boldsymbol{p}_d = \begin{pmatrix} 1 & -1 & 3 \end{pmatrix}^T$ [m], provide the first few iterations of a Newton algorithm for the numerical solution of the inverse kinematics problem in the following two cases:

a) starting from the initial guess $\boldsymbol{q}_a^{[0]} = \begin{pmatrix} -2 & 0.7\pi & \sqrt{2} \end{pmatrix}^T$ [m,rad,m];

b) starting from the initial guess $\boldsymbol{q}_b^{[0]} = \begin{pmatrix} 2 & \pi/4 & \sqrt{2} \end{pmatrix}^T$ [m,rad,m].

In case of convergence, the algorithm should stop as soon as $\|\boldsymbol{e}^{[k]}\| = \|\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q}^{[k]})\| \le \epsilon = 0.1$ mm.

the Jacobian is $J(q) = \begin{pmatrix} 0 & -q_3 s_2 & c_2 \\ 0 & q_3 c_2 & s_2 \\ 1 & 0 & 0 \end{pmatrix}$

Case $q^0 = \begin{pmatrix} -2 & 0.7\pi & \sqrt{2} \end{pmatrix}^T$

$$q^1 = q^0 + J^{-1}(q^0)(p_d - f(q^0)) = \begin{pmatrix} -2 \\ 0.7\pi \\ \sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ -0.572 & -0.415 & 0 \\ -0.587 & 0.809 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} -0.831 \\ 1.1441 \\ -2 \end{pmatrix} \right) = \begin{pmatrix} 3 \\ 2.0415 \\ -1.385 \end{pmatrix}$$

$$e^1 = p_d - f(q^1) = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0.516 \\ -1.015 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.484 \\ 0.015 \\ 0 \end{pmatrix} \Rightarrow \|e^1\| \approx 0.484$$

$$q^2 = q^1 + J^{-1}(q^1)(p_d - f(q^1)) = \begin{pmatrix} 3 \\ 2.0415 \\ -1.385 \end{pmatrix} + J(q^1)^{-1} \begin{pmatrix} 0.484 \\ 0.015 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2.3555 \\ -1.601 \end{pmatrix} \Rightarrow e^2 = p_d - f(q^2) = \begin{pmatrix} 1 - 1.13 \\ -1 + 1.13 \\ 3 - 3 \end{pmatrix} = \begin{pmatrix} 0.13 \\ 0.13 \\ 0 \end{pmatrix}$$

$$\Rightarrow \|e^2\| = 0.18$$

Exercise 3

A 2R planar robot with link lengths $l_1 = 1.2$, $l_2 = 0.8$ [m] is at rest at $t = 0$ in the configuration $\boldsymbol{q}_0 = \boldsymbol{0}$ (stretched along the $\boldsymbol{x}_0$ axis). A pointwise target moves at constant speed $v = 1.5$ m/s on a straight line with an angle $\delta = 15°$ from the $\boldsymbol{x}_0$ axis, being in $\boldsymbol{p}_0 = \begin{pmatrix} -2 & 1 \end{pmatrix}^T$ [m] at $t = 0$ and entering after in the robot workspace. Solve the following rendez-vous problem:

a) define a trajectory that will bring the robot end-effector on the target when the latter crosses the $\boldsymbol{y}_0$ axis; the end-effector should have then the same velocity $\boldsymbol{v}_t \in \mathbb{R}^2$ of the target;

b) provide the rendez-vous time $t_{rv} > 0$ and the expression of the command $\dot{\boldsymbol{q}}(t) \in \mathbb{R}^2$, $t \in [0, t_{rv}]$.

How would you modify the velocity command $\dot{\boldsymbol{q}}(t)$ as a function of $\boldsymbol{q}(t)$ so as to reach the target at the rendez-vous position if the robot starts from a configuration close but different from $\boldsymbol{q}_0$?

let $P(t)$ to be the position of the target over time.

$$\dot{P} = \begin{cases} v\cos\delta \\ v\sin\delta \end{cases} \approx \begin{cases} 1.448 \\ 0.388 \end{cases} \Rightarrow P(t) = \begin{pmatrix} -2+1.448t \\ 1+0.388t \end{pmatrix}.$$ $P(t)$ crosses the y axis when $P_x(t) = 0$

$$\Rightarrow \quad -2+1.448t = 0 \Rightarrow t^* = 1.381.$$

let $r(t)$ be the position of the robot. $r(0) = (2, 0)$

I want the trajectory to:

$$p_d(0) = (2, 0) \qquad p_d(t^*) = P(t^*) \qquad \dot{p}_d(0) = 0 \qquad \dot{p}_d(t^*) = \dot{P}$$

I use a 3 degree polynomial Func.

X coordinate) $\quad p_x = at^3 + bt^2 + ct + d$

$$\begin{cases} p_x(0) = d = 2 \\ p_x(t^*) = 2.633a + 1.907b + 2 = -4 \\ \dot{p}_x(0) = c = 0 \\ \dot{p}(t^*) = 3 \cdot 1.907a + 2 \cdot 1.381b = 1.448 \end{cases} \Rightarrow \begin{cases} a = 5.315 \\ b = -10.484 \end{cases}$$

Y coordinate) $\quad p_y = at^3 + bt^2 + ct + d$

$$\begin{cases} p_y(0) = d = 0 \\ p_y(t^*) = 2.633a + 1.907b = 1.535 \\ \dot{p}_y(0) = c = 0 \\ \dot{p}_y(t^*) = 3 \cdot 1.907a + 2 \cdot 1.381b = 0.388 \end{cases} \Rightarrow \begin{cases} a = -0.362 \\ b = 2.133 \end{cases}$$

the expression for $\dot{q}$ should be $\dot{q} = J^{-1}(q)\dot{p}_d$

IF the robot starts from $q^0 \neq 0$ i add a feedback term for the error: $e = p_d - r$ where $r = S_r(q)$ actual pose.

$$\dot{e} = \dot{p}_d - \dot{r} = \dot{p}_d - J(q)\dot{q} \qquad \text{i choose} \quad \dot{q} = J^{-1}(q)(\dot{p}_d + Ke) \quad \text{where} \quad K = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\text{so}: \quad \dot{e} = \dot{p}_d - J(q)\dot{q} = \dot{p}_d - J(q)J^{-1}(q)(\dot{p}_d + Ke) = \dot{p}_d - \dot{p}_d - Ke = -Ke$$

$$\Rightarrow \dot{e} = -Ke \Rightarrow \begin{cases} \dot{e}_x = -2e_x \\ \dot{e}_y = -2e_y \end{cases} \Rightarrow \begin{cases} e_x = e_x^0 \cdot \exp(-2t) \\ e_y = e_y^0 \cdot \exp(-2t) \end{cases} \Rightarrow \lim_{t\to\infty} e = 0$$