Exercise 1 [6 points]

The initial orientation of a rigid body with respect to a basis reference frame is given by the matrix

$$\boldsymbol{R}_i = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

The final desired orientation $\boldsymbol{R}_f$ is expressed in terms of roll-pitch-yaw angles $(\alpha, \beta, \gamma) = (\pi/3, \pi/3, -\pi/2)$ in the sequence ZYX around the fixed axes associated to the initial orientation. Find a pair $(\boldsymbol{r}, \theta)$ such that the relative change of orientation of the body is represented by the axis-angle method associated to the unit vector $\boldsymbol{r}$ and angle θ. Comment on how the same result can be obtained when the unit vector $\boldsymbol{r}$ is expressed in terms of the basis reference frame, rather than in the frame associated to $\boldsymbol{R}_i$.

I compute ${}^iR_S = {}^0R_i^T\,{}^0R_S$. The RPY angles matrix ${}^0R_S$ is

$${}^0R_S = R_x(-\tfrac{\pi}{2})R_y(\tfrac{\pi}{3})R_z(\tfrac{\pi}{3}) = \begin{bmatrix} 0.25 & -0.433 & 0.866 \\ -0.433 & 0.75 & 0.5 \\ -0.866 & -0.5 & 0 \end{bmatrix}$$

$$\Rightarrow {}^iR_S = {}^0R_i^T R_S = \begin{bmatrix} -0.435 & -0.659 & 0.612 \\ 0.433 & -0.75 & -0.5 \\ 0.789 & 0.047 & 0.6123 \end{bmatrix}$$

I have to find r and θ s.t. $R(\theta, r) = {}^iR_S$.

Since $\operatorname{trace} R(\theta, r) = 1 + 2\cos\theta$ i consider:

$$1 + 2\cos\theta = -0.5727 \Rightarrow \cos\theta = -0.78 \Rightarrow \sin\theta = \pm\sqrt{1-0.78^2} = \pm 0.625$$

$$\theta = \operatorname{atan2}\{-0.78, \pm 0.625\}$$

$$\Rightarrow \theta \neq 0 \Rightarrow r = \frac{1}{2\sin\theta}\begin{bmatrix} R_{32}-R_{23} \\ R_{13}-R_{31} \\ R_{21}-R_{12} \end{bmatrix} \Rightarrow r = \frac{1}{\pm 1.25}\begin{bmatrix} 0.57 \\ -0.177 \\ -0.091 \end{bmatrix}$$

$$r = \left(\frac{57}{125}, -\frac{177}{1250}, -\frac{91}{1250}\right), \quad \sin\theta = 0.625$$

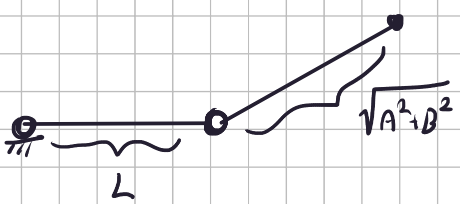
$$r = \left(-\frac{57}{125}, \frac{177}{1250}, \frac{91}{1250}\right), \quad \sin\theta = -0.625$$

Exercise 3 [6 points]

Consider the planar 2R robot in Fig. 2, with the numerical data $L = 0.4$, $A = 0.4$, and $B = 0.3$ [m]. An end-effector frame RF_e is attached at point P to the gripper, with the $\boldsymbol{z}_e$ axis along the approach direction.

The robot is equivalent to the following:

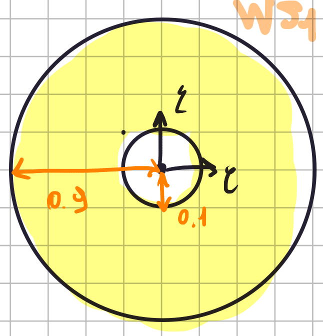
without considering the ee orientation.

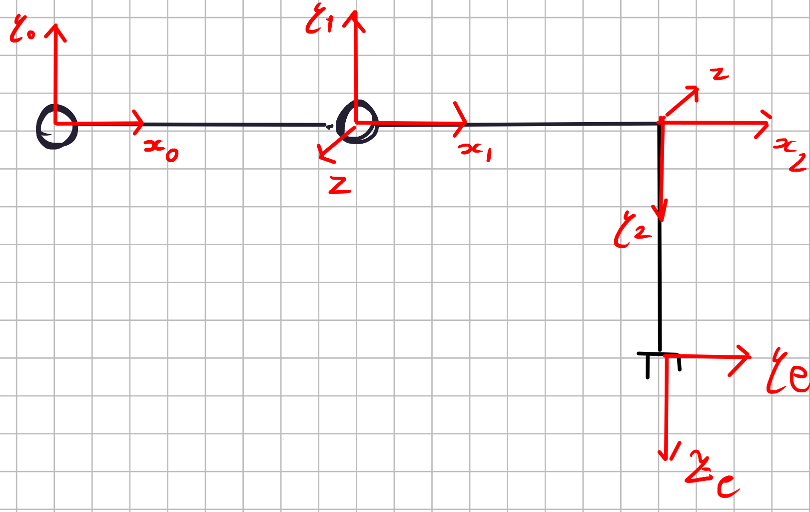


$$WS_1 = \{p \in \mathbb{R}^2 : \underbrace{|L - \sqrt{A^2+B^2}|}_{1/10} \leq \|p\| \leq \underbrace{L + \sqrt{A^2+B^2}}_{9/10}\}$$

$$WS_2 = \emptyset$$

$\Rightarrow$





	α_i	a_i	d_i	θ_i
1	0	L	0	q_1
2	$-\pi$	A	0	q_2

$$^2T_e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & B \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_2 = \begin{bmatrix} C_1 & -S_1 & 0 & LC_1 \\ S_1 & C_1 & 0 & LS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & S_2 & 0 & AC_2 \\ S_2 & -C_2 & 0 & AS_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_1C_2 - S_1S_2 & C_1S_2 + S_1C_2 & 0 & AC_1C_2 - AS_1S_2 + LC_1 \\ S_1C_2 + C_1S_2 & S_1S_2 - C_1C_2 & 0 & AS_1C_2 + AC_1S_2 + LS_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} 0 \\ B \\ 0 \\ 1 \end{matrix}$$

The position of the ee is given by the last column of $^0T_2\,^2T_e$:

$$\begin{cases} p_x = L\cos q_1 + \sqrt{A^2+B^2}\cos(q_1+q_2) \\ p_y = L\sin q_1 + \sqrt{A^2+B^2}\sin(q_1+q_2) \end{cases} \quad \text{i dende } \sqrt{A^2+B^2} = u = 1/2$$

$$\Rightarrow \cos q_2 = \frac{p_x^2+p_y^2-(L^2+u^2)}{2Lu} = \frac{p_x^2+p_y^2-0.41}{0.4} \Rightarrow \sin q_2 = \pm\sqrt{1-\cos q_2}$$

For $p_1 = (0, -0.9)$ I have $\cos q_2 = 1 \Rightarrow \sin q_2 = 0 \Rightarrow q_2 = 0$

We find q_1 by solving:

$$\begin{cases} 0.4\cos q_1 + \frac{1}{2}\cos q_1 = 0 \\ 0.4\sin q_1 + \frac{1}{2}\sin q_1 = -0.9 \end{cases} \Rightarrow \begin{matrix} \cos q_1 = 0 \\ \sin q_1 = -1 \end{matrix} \Rightarrow q_1 = -\pi/2$$

$$q = (-\pi/2, 0)$$

For $p_3 = (0,0)$ there aren't solutions, is out of the WS_1 : $\|p_3\| < 0.1$.

Since $\|p_2\| = 0.806$, there will be 2 regular solutions.

$$\cos q_2 = \frac{p_x^2+p_y^2-0.41}{0.4} = \frac{3}{5} \Rightarrow \sin q_2 = \pm\sqrt{10}/5 \Rightarrow \begin{cases} q_2 = 0.811 \\ q_2 = -0.811 \end{cases}$$

$$\Rightarrow \quad q_1 = \text{atan2}(p_y, p_x) - \text{atan2}(l \sin q_2, L + l \cos q_2)$$

$$q_1 = \pi + \text{atan}\frac{0.7}{-0.4} - \text{atan}\frac{\frac{1}{2}\cdot\frac{\sqrt{10}}{5}}{0.4 + \frac{1}{2}\cdot\frac{3}{5}} = 1.665$$

$$q_1 = \pi + \text{atan}\frac{0.7}{-0.4} - \text{atan}\frac{-\frac{1}{2}\cdot\frac{\sqrt{10}}{5}}{0.4 + \frac{1}{2}\cdot\frac{3}{5}} = 2.514$$

$$\Rightarrow \quad q = \begin{cases} (0.81,\ 1.665) \\ (-0.81,\ 2.514) \end{cases}$$

These solutions are given by considering a robot with 2 straigth arms. Since this have a 90° turn on the link, we have to add this bias on the solutions.

For the equivalent $q_2 = -\text{atan}\left(\frac{0.3}{0.4}\right) = -\text{atan}\frac{3}{4}$

$$\Rightarrow \text{ the solution for } p_1 \text{ is } \quad q = \left(-\frac{\pi}{2},\ \text{arctan}\frac{3}{4}\right) = \left(-\frac{\pi}{2},\ 0.643\right)$$

For p_2 we have

$$q = \begin{cases} (0.81,\ 1.665 + \text{atan}(3/4)) \\ (-0.81,\ 2.514 + \text{atan}(3/4)) \end{cases} = \begin{cases} (0.81,\ 2.3) \\ (-0.81,\ -3.12) \end{cases}$$

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