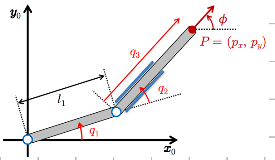
Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise #2

For the planar RRP robot in Fig. 2, define the direct kinematics $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$ from the joint variables $\boldsymbol{q} = (q_1, q_2, q_3)$ to the task variables $\boldsymbol{r} = (p_x, p_y, \phi)$, derive the associated Jacobian $\boldsymbol{J}(\boldsymbol{q})$, and find all its kinematic singularities. With $l_1 = 0.5$ [m], compute in static conditions the joint torque/force vector $\boldsymbol{\tau}$ (with units [Nm,Nm,N]) that balances a force/moment vector $\boldsymbol{F} = \begin{pmatrix} 0 & 1.5 & -4.5 \end{pmatrix}^T$ [N,N,Nm] applied to the robot end-effector, first in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & 0 & 3 \end{pmatrix}^T$ [rad,rad,m] and then in a singular configuration $\boldsymbol{q}_s$ among those found.



the DK is: $f_r(q) = \begin{cases} l_1 c_1 + q_3 c_{12} \\ l_1 s_1 + q_3 s_{12} \\ q_1 + q_2 \end{cases} \Rightarrow J = \begin{bmatrix} -l_1 s_1 - q_3 s_{12} & -q_3 s_{12} & c_{12} \\ l_1 c_1 + q_3 c_{12} & q_3 c_{12} & s_{12} \\ 1 & 1 & 0 \end{bmatrix}$ now i compute the determinant

$$\begin{matrix} -l_1 s_1 - q_3 s_{12} & -q_3 s_{12} & c_{12} & -l_1 s_1 - q_3 s_{12} & -q_3 s_{12} \\ l_1 c_1 + q_3 c_{12} & q_3 c_{12} & s_{12} & l_1 c_1 + q_3 c_{12} & q_3 c_{12} \\ 1 & 1 & 0 & 1 & 1 \end{matrix} \Rightarrow \det J = -q_3 s_{12}^2 + l_1 c_1 c_{12} + q_3 c_{12}^2 - q_3 c_{12}^2 + l_1 s_{12} s_1 + q_3 s_{12}^2 = l_1 c_2$$

$\Rightarrow J$ is singular $\Leftrightarrow c_2 = 0 \Leftarrow q_2 = \pm \frac{\pi}{2}$

Since $J^T F = \tau$ to balance $\vec{F} = (0 \;\; 1.5 \;\; -4.5)^T$ i solve $-J^T F = \tau$. With $l_1 = \frac{1}{2}$ and $q = (\frac{\pi}{2} \;\; 0 \;\; 3)^T$

$$-J^T \begin{pmatrix} 0 \\ 1.5 \\ -4.5 \end{pmatrix} = -\begin{pmatrix} -3.5 & 0 & 1 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1.5 \\ -4.5 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 4.5 \\ -1.5 \end{pmatrix}$$

then i consider a singular conf. $q_s = (0, \frac{\pi}{2}, 0)$:

$$-J^T \begin{pmatrix} 0 \\ 1.5 \\ -4.5 \end{pmatrix} = -\begin{pmatrix} 0 & 1/2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1.5 \\ -4.5 \end{pmatrix} = \begin{pmatrix} 15/4 \\ 9/2 \\ -3/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 7.5 \\ 9 \\ -3 \end{pmatrix}$$

Exercise #3

The Jacobian of a 3R spatial robot relating $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to the velocity $\boldsymbol{v} \in \mathbb{R}^3$ of its end-effector is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1(c_2 + c_{23}) & -c_1(s_2 + s_{23}) & -c_1 s_{23} \\ c_1(c_2 + c_{23}) & -s_1(s_2 + s_{23}) & -s_1 s_{23} \\ 0 & c_2 + c_{23} & c_{23} \end{pmatrix}, \qquad q_2 = \frac{\pi}{2} \;\; q_3 = -\frac{\pi}{2} \qquad q_2 = q_3 = 0$$

where the shorthand notation has been used (e.g., $c_{23} = \cos(q_2 + q_3)$). This matrix may have rank 1, 2, or 3, depending on the configuration $\boldsymbol{q}$. In each of these cases, define a basis for the null space $\mathcal{N}\{\boldsymbol{J}\}$ and for the range space $\mathcal{R}\{\boldsymbol{J}\}$ of the Jacobian. Find a configuration $\boldsymbol{q}_s$ with rank $\boldsymbol{J}(\boldsymbol{q}_s) = 2$ such that the end-effector velocity $\boldsymbol{v}_s = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T$ is feasible. Determine then a joint velocity $\dot{\boldsymbol{q}}_s$ such that $\boldsymbol{J}(\boldsymbol{q}_s)\dot{\boldsymbol{q}}_s = \boldsymbol{v}_s$. Sketch graphically the situation.

Case rank = 3) Since it is a 3×3 matrix, if rank = 3 then $\dim N(J) = 0 \Rightarrow N(J) = \{0\}$

$\det J = (c_2 + c_{23})(s_2 + s_{23}) c_{23} - (c_2 + c_{23})^2 s_{23} (c_1^2 + s_1^2 s_{23})$ is zero if $c_2 + c_{23} = 0$

Case rank = 1) $\quad s_{23} = 0 \quad s_2 = -s_{23} \quad c_2 = -c_{23} \quad 1, -1 \quad q_2 = 0 \quad q_3 = \pi$

$q_2 = \frac{\pi}{2} \quad q_3 = 0$

$$J = \begin{pmatrix} 0 & -2c_1 & -c_1 \\ 0 & -2s_1 & -s_1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank} = 1, \quad J\dot{q} = \begin{cases} -2c_1 \dot{q}_2 - c_1 \dot{q}_3 = 0 \\ -2s_1 \dot{q}_2 - s_1 \dot{q}_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} -2c_1 \dot{q}_2 + c_1 2 \dot{q}_2 = 0 \\ \frac{-2s_1 \dot{q}_2}{s_1} = \dot{q}_3 \end{cases} \Rightarrow N(J) = \left\{ \begin{pmatrix} a \\ b \\ -2b \end{pmatrix} a, b \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$$

$\Rightarrow$ a basis for $R(J)$ is $B = \left\{ \begin{pmatrix} -c_1 \\ -s_1 \\ 0 \end{pmatrix} \right\}$

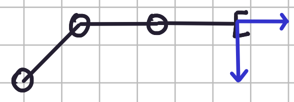
Case rank = 2) if $s_{23} = s_2 = 0$ $(q_2 = q_3 = 0)$ we have:

$$J = \begin{pmatrix} -2s_1 & 0 & 0 \\ 2c_1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \Rightarrow \text{rank} = 2. \quad J\dot{q} = 0 \Rightarrow \begin{cases} -2s_1 \dot{q}_1 = 0 \\ 2c_1 \dot{q}_1 = 0 \\ 2\dot{q}_2 + \dot{q}_3 = 0 \end{cases} \quad \begin{cases} \dot{q}_1 = 0 \\ \dot{q}_2 = -\dot{q}_3 \frac{1}{2} \end{cases} \Rightarrow \text{basis for } N(J): B = \left\{ \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \right\}$$

$\Rightarrow$ a base for $R(J)$ is $B = \left\{ \begin{pmatrix} -2s_1 \\ 2c_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

let $q = (\frac{\pi}{4}, 0, 0)$:

$$J = \begin{pmatrix} -\sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \Rightarrow J\dot{q} = \begin{pmatrix} -\sqrt{2}\dot{q}_1 \\ \sqrt{2}\dot{q}_1 \\ \dot{q}_2 + \dot{q}_3 \end{pmatrix} = v \quad \text{this is } v_s = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ if } \begin{cases} \dot{q}_1 = 1/\sqrt{2} \\ \dot{q}_2 = 1 \\ \dot{q}_3 = -1 \end{cases}$$



Exercise #4

A 2R planar robot has to perform in a coordinated way a rest-to-rest motion from $\boldsymbol{q}_s = \begin{pmatrix} 0 & -\pi/2 \end{pmatrix}^T$ to $\boldsymbol{q}_g = \begin{pmatrix} -\pi/2 & \pi/2 \end{pmatrix}^T$, while guaranteeing continuity of acceleration at all times. Plan a joint trajectory in the presence of bounds $|\dot{q}_i| \le V_i$ on joint velocities and $|\ddot{q}_i| \le A_i$ on joint accelerations (for $i = 1, 2$), so as to complete the motion task in minimum time T^* within the chosen class of trajectories. Provide the value of T^* for $V_1 = 1$, $V_2 = 2$ [rad/s] and $A_1 = 1.5$, $A_2 = 2$ [rad/s^2].

I consider:

$$q_1(t) = -\frac{\pi}{2T^2}\left(-\frac{2}{T}t^3 + 3t^2\right) \quad \text{and} \quad q_2(t) = -\frac{\pi}{2} + \frac{\pi}{T^2}\left(-\frac{2}{T}t^3 + 3t^2\right)$$

$$\Rightarrow \dot{q}_1(t) = -\frac{\pi}{2T^2}\left(-\frac{6}{T}t^2 + 6t\right) \quad \text{and} \quad \dot{q}_2(t) = \frac{\pi}{T^2}\left(-\frac{6}{T}t^2 + 6t\right)$$

the max (or min) for a 2-degree polynomial that is zero on a and b is always in $\frac{a+b}{2}$ so:

$$\max|\dot{q}_1| = \left|\dot{q}_1\left(\tfrac{T}{2}\right)\right| = \left|-\frac{\pi}{2T}\left(-\frac{3}{2}+3\right)\right| = \frac{3\pi}{4T} \le V_1 \qquad \max|\dot{q}_2| = \left|\dot{q}_2\left(\tfrac{T}{2}\right)\right| = \frac{3\pi}{2T} \le V_2 \Rightarrow \begin{cases} T \ge \frac{3\pi V_1}{4} \\ T \ge \frac{3\pi V_2}{2} \end{cases}$$

$$\text{Since } \ddot{q}_1 = -\frac{\pi}{2T^2}\left(-\frac{12}{T}t + 6\right) \Rightarrow \max|\ddot{q}_1| = |\ddot{q}_1(0)| = \frac{3\pi}{T^2} \le A_1$$

$$\ddot{q}_2 = \frac{\pi}{T^2}\left(-\frac{12}{T}t + 6\right) \Rightarrow \max|\ddot{q}_2| = |\ddot{q}_2(0)| = \frac{6\pi}{T^2} \le A_2$$

$$\Rightarrow \begin{cases} \frac{T^2}{3\pi} \ge A_1 \\ \frac{T^2}{6\pi} \ge A_2 \end{cases} \Rightarrow \begin{cases} T \ge \sqrt{3\pi A_1} \\ T \ge \sqrt{6\pi A_2} \end{cases}$$

$$\Rightarrow T^* = \max\left\{ \frac{3\pi V_1}{4}, \frac{3\pi V_2}{2}, \sqrt{3\pi A_1}, \sqrt{6\pi A_2} \right\}$$

IF $V_1 = 1$, $V_2 = 2$, $A_1 = 3/2$, $A_2 = 2 \Rightarrow T^* = \max\left\{ \frac{3}{4}\pi, 3\pi, \sqrt{\frac{9}{2}\pi}, \sqrt{9\pi} \right\} = 3\pi$

Exercise #5

This is in the form of a Questionnaire. Please answer with formulas and/or clear and short texts.

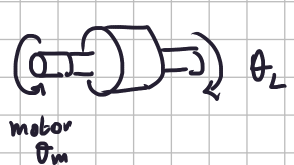
A) Which of the following matrices represents a rotation and which not? Motivate your answers.

$$R_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad R_2 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}, \quad R_3 = \begin{pmatrix} -\sqrt{0.5} & \frac{1}{\sqrt{2}} & 0 \\ \sqrt{0.5} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

B) An Harmonic Drive with a circular spline having 150 inner teeth is used as reduction element in a robot joint. An absolute encoder is mounted on the motor side of the joint. How many bits should have this encoder in order to provide an angular resolution better than or equal to 0.0002 rad on the link side of the transmission?

A) R1 is not a rotation matrix since it's determinant is -1. R3 have determinant 1 and the columns are linearly independent and of norm 1 so it's a rotation matrix. For the same reason, also R2 is a rotation matrix.

B) the red. ratio is $n = \frac{N_{FS}}{2}$ and $N_{CS} = N_{FS} + 2 \Rightarrow n = 74$



$\theta_m = n\theta_L \Rightarrow \theta_m = 74\theta_L$

0.0002 rad. on the link side is 0.0148 on the motor side.

$$\Rightarrow 0.0148 = \frac{2\pi}{2^{N_b}} \Rightarrow 2^{N_b} = \frac{2\pi}{0.0148} \Rightarrow \text{bits} = \left\lceil \log_2\left(\frac{2\pi}{0.0148}\right) \right\rceil = 9$$