Exercise 2

For the 3-dof planar robot in Fig. 2, one can define a task vector $\boldsymbol{r} = (p_x, p_y, \alpha)$ containing the position of the robot tip and the angle $\alpha \in (-\pi, \pi]$ of the last link with respect to the axis x_w of the world frame. Using the joint variables defined in the figure, find all inverse kinematics solutions of this robot for a given task vector $\boldsymbol{r}_d$. Determine also the singular cases and explain what happens then. Evaluate your solution with the numerical data $L = 1$ m and $\boldsymbol{r}_d = (1, 0, -\pi/4)$ [m,m,rad]. Finally, compute the task Jacobian matrix $\boldsymbol{J}(\boldsymbol{q}) = \partial \boldsymbol{r}/\partial \boldsymbol{q}$ and find its singularities.

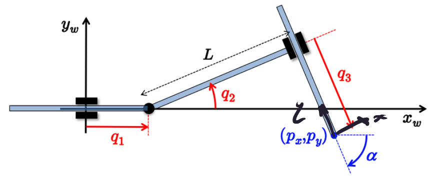

Figure 2: A 3-dof planar robot with the definition of joint and task variables.

I define some intermediate frames to compute $f_r(q)$.

$$
{}^wT_2 = \begin{bmatrix} 1 & 0 & 0 & q_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
{}^2T_3 = \begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
=
\begin{bmatrix} C_2 & -S_2 & 0 & LC_2 \\ S_2 & C_2 & 0 & LS_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^3T_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
=
\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^wT_e = \begin{bmatrix} 1 & 0 & 0 & q_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C_2 & -S_2 & 0 & LC_2 \\ S_2 & C_2 & 0 & LS_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
=
\begin{bmatrix} C_2 & -S_2 & 0 & LC_2+q_1 \\ S_2 & C_2 & 0 & LS_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
=
$$

$$
\begin{bmatrix} S_2 & C_2 & 0 & S_2q_3+LC_2+q_1 \\ -C_2 & S_2 & 0 & -C_2q_3+LS_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\Rightarrow f_r(q) = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} q_3\sin q_2 + L\cos q_2 + q_1 \\ -q_3\cos q_2 + L\sin q_2 \\ q_2 - \pi/2 \end{pmatrix}
$$

$$
\Rightarrow J = \begin{bmatrix} 1 & q_3C_2 - LS_2 & \sin q_2 \\ 0 & q_3\sin q_2 + L\cos q_2 & -\cos q_2 \\ 0 & 1 & 0 \end{bmatrix}
\Rightarrow
$$

Singular if $q_2 = \pm\frac{\pi}{2}$ since $\det J = \cos q_2$

$$
\Rightarrow J(q_2 = \pi/2) = \begin{bmatrix} 1 & -L & 1 \\ 0 & q_3 & 0 \\ 0 & 1 & 0 \end{bmatrix}
$$

$\Rightarrow \text{rank } J = 2 \neq 3$.

Exercise 3

A robot joint should move from q_i at $t = 0$, with a generic initial velocity v_i, to q_f at $t = T$, using a trajectory $q(t)$ that has continuous acceleration in the open interval $(0, T)$. Choose the motion profile and determine analytically the value of the final velocity v_f to be attained at $t = T$ so that the resulting initial acceleration is $\ddot{q}(0) = 0$. Provide then the expression of the corresponding maximum values of $|\dot{q}(t)|$ and $|\ddot{q}(t)|$ in the closed interval $[0, T]$. Using the numerical data $q_i = -0.5$, $q_f = 1$ [rad] and $T = 3$ s, apply your results to the two cases *i)* $v_i = 0$ and *ii)* $v_i = 1$ [rad/s].

the boundary conditions are:

$q(0) = q_i$, $\quad \dot{q}(0) = v_i$, $\quad q(T) = q_f$, $\quad \ddot{q}(0) = 0.$

I choose a 3-degree pol. $\quad * \; \Delta q = q_f - q_i$

$q(t) = at^3 + bt^2 + ct + d \qquad \dot{q}(t) = 3at^2 + 2bt + c \qquad \ddot{q}(t) = 6at + 2b$

$$\begin{cases} q(0) = d = q_i \\ \dot{q}(0) = c = v_i \\ \ddot{q}(0) = 2b = 0 \\ aT^3 + bT^2 + cT + d = q_f \end{cases} \Rightarrow \begin{cases} b = 0 \\ c = v_i \\ d = q_i \\ a = \dfrac{\Delta q - v_i T}{T^3} \end{cases} \Rightarrow q(t) = \frac{\Delta q - v_i T}{T^3} \cdot t^3 + v_i t + q_i \Rightarrow \dot{q}(t) = 3\frac{\Delta q - v_i T}{T^3} t^2 + v_i$$

$$\Rightarrow v_f = \dot{q}(T) = 3\frac{\Delta q - v_i T}{T^3} T^2 + v_i = \frac{3}{T}(\Delta q - v_i T) + v_i$$

The maximum velocity is reached when $\ddot{q}(t) = 0$

$$\ddot{q}(t) = \frac{\Delta q - v_i T}{T^3} \cdot 6t = 0 \Rightarrow t = 0 \Rightarrow \max|\dot{q}| = v_i$$

and $\max \ddot{q}$ is obtained at $t = T \Rightarrow \max|\ddot{q}| = \dfrac{\Delta q - v_i T}{T^2} \cdot 6$

Now: consider $q_i = -0.5$, $q_f = 1$, $T = 3 \Rightarrow \Delta q = 0.5$

Case $v_i = 0$)

$q(t) = \frac{1}{54} t^3 - 0.5$, $\quad \dot{q}(t) = \frac{3}{54} t^2$, $\quad v_f = \dot{q}(3) = \frac{1}{2} = \max|\dot{q}|$

$\ddot{q}(t) = \frac{6}{54} t$, $\quad \max|\ddot{q}| = \frac{1}{3}$

Case $v_i = 1$)

$q(t) = -\frac{5}{54} t^3 + t - 0.5$, $\quad \dot{q}(t) = -\frac{5}{18} t^2 + 1 \qquad \ddot{q}(t) = -\frac{5}{9} t$

$\max|\ddot{q}| = \frac{5}{3}$, $\quad \max|\dot{q}| = \frac{3}{2}$, $\quad v_f = \frac{3}{2}$