Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

**Exercise #2**

A planar RPR robot is shown in Fig. 2, together with the definition of the joint coordinates[1]. The third link has length $L = 0.6$ [m]. The robot has to execute two different tasks, with the end effector placed at the point $P_d = (2, 0.4)$ [m] and pointing downward.

a) In the first task, the robot end effector should start moving inside a tube with a vertical speed $v = -2.5$ [m/s]. Determine the initial joint velocity $\dot{q} \in \mathbb{R}^3$ that realizes this instantaneous motion.

b) In the second task, the robot should keep its initial configuration in the presence of an horizontal force $f = 15$ [N] and a torque $\mu = 6$ [Nm] applied to its end effector. Determine the joint commands $\tau \in \mathbb{R}^3$ (two torques and a force) needed for static balance.

Comments that justify intuitively some of the obtained results are welcome!

The DK of the manipulator is

$$f_r(q) = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{cases} q_2 c_1 + L c_{13} \\ q_2 s_1 + L s_{13} \\ q_1 + q_3 \end{cases}.$$ I solve the IK for $r_d = \begin{pmatrix} 2 \\ 0.4 \\ -\pi/2 \end{pmatrix}$

$$\begin{cases} 2 = q_2 c_1 + 0.6 c_{13} \\ 0.6 = q_2 s_1 + 0.6 s_{13} \\ -\pi/2 = q_1 + q_3 \end{cases} \Rightarrow \begin{cases} q_2 c_1 = 2 \\ q_2 s_1 = 1.2 \end{cases} \Rightarrow q_2^2 = 4 + 1.2^2 \Rightarrow q_2^* = \frac{2}{5}\sqrt{34} \text{ and } \begin{array}{l} q_1^* = \operatorname{atan2}\{1.2, 2\} = 0.54041 \Rightarrow \\ q_3^* = -2.1112 \end{array}$$

the Jacobian matrix is $J = \begin{pmatrix} -q_2 s_1 - L s_{13} & c_1 & -L s_{13} \\ q_2 c_1 + L c_{13} & s_1 & L c_{13} \\ 1 & 0 & 1 \end{pmatrix}$ In $q^*$: $J(q^*) = \begin{pmatrix} -0.4 & 0.9002 & 0.6 \\ 2 & 0.4353 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

For the first task, we want to realize $\dot{r} = (0, -2.5, 0)^T$. So:

$$\dot{q} = J^{-1}(q^*)\dot{r} = \begin{pmatrix} -1.006 \\ -1.1182 \\ 1.006 \end{pmatrix}$$

For the second task, the robot should resist a twist of $w = (15, 0, 6)$, So it should provide a torque that generate $w_d = (-15, 0, -6)$.

Since $J^T F = \tau$ we have:

$$J^T(q^*) w_d = \begin{pmatrix} -0.4 & 2 & 1 \\ 0.9002 & 0.4353 & 0 \\ 0.6 & 0 & 1 \end{pmatrix} \begin{pmatrix} -15 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ -13.5 \\ -15 \end{pmatrix}$$

**Exercise #3**

Plan a smooth rest-to-rest trajectory along a linear path from point $A = (1, 1, 1)$ [m] to point $B = (-1, 5, 0)$ [m], with simultaneous and coordinated change of orientation from

$$R_A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

to

$$R_B = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The total motion time is $T = 2.5$ [s]. The trajectory should be continuous up to the acceleration for all $t \in [0, T]$. Determine the velocity $\boldsymbol{v}_M \in \mathbb{R}^3$, acceleration $\boldsymbol{a}_M \in \mathbb{R}^3$, angular velocity $\boldsymbol{\omega}_M \in \mathbb{R}^3$, and angular acceleration $\dot{\boldsymbol{\omega}}_M \in \mathbb{R}^3$ attained at the time instant(s) when these four vectors assume, respectively, their maximum value in norm. Compute also the absolute orientation $R_{mid} \in SO(3)$ at the midpoint of the planned trajectory.

$\Delta p = B - A$

Linear Path) I consider $p(s) = A + \Delta p(-2\tau^3 + 3\tau^2)$ $\tau = \frac{t}{T} \in [0, 1]$, $t \in [0, T]$.

Explicitly:

$$p(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}\left(-\frac{16}{125}t^3 + \frac{12}{25}t^2\right) \qquad \ddot{p}(t) = \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}\left(-\frac{96}{125}t + \frac{24}{25}\right)$$

$$\dot{p}(t) = \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}\left(-\frac{48}{125}t^2 + \frac{24}{25}t\right) \qquad \dddot{p}(t) = \begin{pmatrix} 1.536 \\ -3.072 \\ 0.768 \end{pmatrix}$$

i consider $v^2(t) = \|\dot{p}(t)\|^2 = \left( \frac{48^2}{125^2} t^4 + \left(\frac{24}{25}\right)^2 t^2 - 2\left(\frac{48 \cdot 24}{125 \cdot 25}\right) t^3 \right) 21$

$\dot{v}^2(t) = 12.386t^3 + 38.707t - 46.448t^2$

$\dot{v}^2(t) = 0 \Rightarrow t = 1.25 \Rightarrow \max|\dot{q}| = v(1.25) = 2.7495$

Now i consider $a^2(t) = \|\ddot{p}(t)\|^2 = \left( \left(\frac{96}{125}\right)^2 t^2 + \left(\frac{24}{25}\right)^2 - \frac{2304}{3125} t \right) 21$ $\Rightarrow \max|\ddot{q}| = a(0.625) = 14.5152$

$\dot{a}^2(t) = \left( 2\left(\frac{96}{125}\right)^2 t - \frac{2304}{3125} \right) 21 = 0 \Rightarrow t = 0.625$

Angular path )

I consider ${}^A R_B = R_A^T R_B = \begin{bmatrix} 0 & -1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$. Now i look for $r, \theta$ s.t. $R(\theta, r) = {}^A R_B$

$\Rightarrow \operatorname{trace}({}^A R_B) = 1 + 2\cos\theta_{AB} \Rightarrow -\frac{1}{\sqrt{2}} = 1 + 2\cos\theta_{AB} \Rightarrow \cos\theta_{AB} = -\frac{2+\sqrt{2}}{4} \Rightarrow \sin\theta_{AB} = 0.521 \Rightarrow \theta_{AB} = 2.5935$

then : $r = \frac{1}{2\sin\theta_{AB}} \begin{pmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{pmatrix} = 0.9595 \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ -1/\sqrt{2} + 1 \end{pmatrix} = \begin{pmatrix} -0.6784 \\ 0.6784 \\ 0.28103 \end{pmatrix}$

I consider the trajectory $R(t) = R_A R(\theta(t), r)$ where

$\theta(t) = \theta_{AB}\left(-2\frac{t^3}{T^3} + 3\frac{t^2}{T^2}\right)$ so $R(0) = R_A$ $\quad R(T) = R_A {}^A R_B = R_B$.

it's a rotation along $r$ so the angular speed is $\omega(t) = r \cdot \dot{\theta}(t)$

$\omega(t) = r \cdot (-0.9959t^2 + 2.48976t)$, $\quad \dot{\omega}(t) = r(-1.9959t + 2.48976)$

the max speed is reached at $t = \frac{T}{2} = 1.25$

the max accel. is reached at $t = 0$ or $t = 2.5$

$\omega(1.25) = \begin{pmatrix} -1.0555 \\ 1.0555 \\ 0.4372 \end{pmatrix}$, $\max\|\omega\| = 1.5554$

$\dot{\omega}(T) = \begin{pmatrix} -1.696 \\ 1.696 \\ 0.7025 \end{pmatrix} \Rightarrow \max\|\dot{\omega}\| = 2.5$

$R_{mid} = R(1.25) = R_A R(\theta(1.25), r) = R_A R(1.2967, r) = R_A(rr^T + (I - rr^T) \cdot 0.2706 + S(r) \cdot 0.9626)$

$= R_A \cdot \begin{pmatrix} 0.6062 & -0.606 & 0.5139 \\ -0.065 & 0.6062 & 0.7924 \\ -0.792 & -0.513 & 0.3282 \end{pmatrix} = \begin{pmatrix} -0.065 & 0.6062 & 0.7924 \\ 0.6062 & -0.606 & 0.5139 \\ 0.792 & 0.5139 & -0.328 \end{pmatrix}$