

Exercise 1 [10 points]

Consider the orientation obtained by a (partial) Euler sequence with a rotation of an angle α around z , followed by a rotation of an angle β around the current y . Find three angles ϕ , χ , and ψ such that the product $R_\alpha(\phi)R_\chi(\chi)R_z(\psi)$ returns the same final orientation. Give the procedure for solving this problem in general, determine the singular cases, and provide then a numerical value of the sought triple of angles when $\alpha = \pi/4$, $\beta = -\pi/3$ [rad]. Check the result.

use it for R_{xyz}

$$R_z(\alpha)R_y(\beta) = \begin{bmatrix} c\alpha c\beta & -s\alpha & s\beta c\alpha \\ s\alpha c\beta & c\alpha & s\alpha s\beta \\ -s\beta & 0 & c\beta \end{bmatrix} \quad \text{i have to solve for } \phi, \chi, \psi \text{ the equation of the two matrices}$$

$$R_x(\phi)R_y(\chi)R_z(\psi) = \begin{bmatrix} c\chi & s\chi s\psi & s\chi c\psi \\ s\phi s\chi & -s\phi s\psi c\chi + c\phi c\psi & -s\phi c\chi c\psi - s\psi c\phi \\ -s\chi c\phi & s\phi c\psi + s\psi c\phi c\chi & -s\phi s\psi + c\phi c\chi c\psi \end{bmatrix}$$

$$c\chi = c\alpha c\beta \Rightarrow \pm s\chi = \pm \sqrt{1 - c^2\alpha c^2\beta} \Rightarrow \chi = \arctan2\left\{\pm \sqrt{1 - c^2\alpha c^2\beta}, c\alpha, c\beta\right\}$$

$$\begin{cases} s\chi s\psi = -s\alpha \\ s\chi c\psi = s\beta c\alpha \end{cases} \Rightarrow \begin{cases} s\psi = \mp \frac{s\alpha}{s\chi} \\ c\psi = \pm \frac{s\beta c\alpha}{s\chi} \end{cases} \Rightarrow \psi = \arctan2\left\{\mp \frac{s\alpha}{s\chi}, \pm \frac{s\beta c\alpha}{s\chi}\right\} \begin{cases} \nearrow \arctan2\left\{-\frac{s\alpha}{s\chi}, \frac{s\beta c\alpha}{s\chi}\right\} \text{ if } +s\chi \\ \searrow \arctan2\left\{\frac{s\alpha}{s\chi}, -\frac{s\beta c\alpha}{s\chi}\right\} \text{ if } -s\chi \end{cases}$$

$$\begin{cases} s\phi s\chi = s\alpha c\beta \\ -s\chi c\phi = -s\beta \end{cases} \Rightarrow \begin{cases} s\phi = \pm \frac{s\alpha s\beta}{s\chi} \\ c\phi = \pm \frac{s\beta}{s\chi} \end{cases} \Rightarrow \begin{cases} \text{if } +s\chi \Rightarrow \phi = \arctan2\left\{\frac{s\alpha s\beta}{s\chi}, \frac{s\beta}{s\chi}\right\} \\ \text{if } -s\chi \Rightarrow \phi = \arctan2\left\{-\frac{s\alpha s\beta}{s\chi}, -\frac{s\beta}{s\chi}\right\} \end{cases}$$

$$\text{let } \alpha = \frac{\pi}{4}, \beta = -\frac{\pi}{3} : c\chi = \frac{\sqrt{2}}{4} \Rightarrow \pm s\chi = \pm \frac{\sqrt{15}}{4} \Rightarrow \chi = \pm 1.2094$$

$$\psi = \arctan2\left\{\mp \frac{s\alpha}{s\chi}, \pm \frac{s\beta c\alpha}{s\chi}\right\} \begin{cases} \nearrow \arctan2\left\{-\frac{2\sqrt{7}}{7}, -\frac{\sqrt{21}}{7}\right\} = -2.28 \text{ (if } \chi = 1.2094) \\ \searrow \arctan2\left\{\frac{2\sqrt{7}}{7}, \frac{\sqrt{21}}{7}\right\} = 0.85 \text{ (if } \chi = -1.2094) \end{cases}$$

$$\text{if } +s\chi \Rightarrow \phi = \arctan2\left\{\frac{s\alpha s\beta}{s\chi}, \frac{s\beta}{s\chi}\right\} = \arctan2\left\{-\frac{\sqrt{21}}{7}, -\frac{\sqrt{42}}{7}\right\} = -2.52$$

$$\text{if } -s\chi \Rightarrow \phi = \arctan2\left\{-\frac{s\alpha s\beta}{s\chi}, -\frac{s\beta}{s\chi}\right\} = \arctan2\left\{\frac{\sqrt{21}}{7}, \frac{\sqrt{42}}{7}\right\} = 0.615$$

$$\Rightarrow \begin{bmatrix} \phi \\ \chi \\ \psi \end{bmatrix} = \begin{bmatrix} -2.52 \\ 1.2094 \\ -2.28 \end{bmatrix} \vee \begin{bmatrix} 0.615 \\ -1.2094 \\ 0.85 \end{bmatrix}$$

Exercise 2 [10 points]

Let a first rotation be defined by an angle γ around \mathbf{x} , followed by a rotation of an angle δ around the unit vector $\mathbf{v} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$ expressed in the original frame. Determine the resulting rotation matrix $\mathbf{R}(\gamma, \delta)$ in symbolic form. For a numerical case with $\gamma = -\pi/2$, $\delta = \pi/3$ [rad], extract the invariant axis \mathbf{r} of the total rotation and the corresponding angle θ . Check the result.

$$\mathbf{R}(\delta, \mathbf{v}) = \mathbf{v}\mathbf{v}^T + (\mathbf{I} - \mathbf{v}\mathbf{v}^T) \cos \delta + \mathbf{S}(\mathbf{v}) \sin \delta$$

$$\mathbf{R}_\gamma = \mathbf{R}_\infty(\gamma) \Rightarrow \mathbf{R}(\gamma, \delta) = \mathbf{R}(\delta, \mathbf{v}) \mathbf{R}(\gamma, \mathbf{x})$$

$$\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\mathbf{R}(\delta, \mathbf{v}) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cos \delta + \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \sin \delta = \begin{bmatrix} \frac{1}{2}(1+\cos \delta) & \frac{1}{2}(1-\cos \delta) & -\frac{1}{2}\sin \delta \\ \frac{1}{2}(1-\cos \delta) & \frac{1}{2}(1+\cos \delta) & -\frac{1}{2}\sin \delta \\ \frac{1}{\sqrt{2}}\sin \delta & \frac{1}{\sqrt{2}}\sin \delta & \cos \delta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{2}(1+\cos \delta) & \frac{1}{2}(\cos \delta - 1) & -\frac{1}{\sqrt{2}}\sin \delta \\ \frac{1}{2}(\cos \delta - 1) & \frac{1}{2}(1+\cos \delta) & -\frac{1}{\sqrt{2}}\sin \delta \\ \frac{1}{\sqrt{2}}\sin \delta & \frac{1}{\sqrt{2}}\sin \delta & \cos \delta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & -\sin \gamma \\ 0 & \sin \gamma & \cos \gamma \end{bmatrix} =$$

$$\mathbf{R} = \begin{bmatrix} \frac{1}{2}(1+\cos \delta) & \frac{1}{2}(\cos \delta - 1)\cos \gamma - \frac{1}{\sqrt{2}}\sin \delta \sin \gamma & -\frac{1}{2}(\cos \delta - 1)\sin \gamma - \frac{1}{\sqrt{2}}\sin \delta \cos \gamma \\ \frac{1}{2}(\cos \delta - 1) & \frac{1}{2}(\cos \delta + 1)\cos \gamma - \frac{1}{\sqrt{2}}\sin \delta \sin \gamma & -\frac{1}{2}(1+\cos \delta)\sin \gamma - \frac{1}{\sqrt{2}}\sin \delta \cos \gamma \\ \frac{1}{\sqrt{2}}\sin \delta & \frac{1}{\sqrt{2}}\sin \delta \cos \gamma + \cos \delta \sin \gamma & \cos \delta \cos \gamma - \frac{1}{\sqrt{2}}\sin \delta \sin \gamma \end{bmatrix}$$

For $\delta = \frac{\pi}{3}$
 $\gamma = -\frac{\pi}{2}$
 We have:

$$\begin{bmatrix} 0.75 & 0.61 & -0.25 \\ -0.25 & 0.61 & 0.75 \\ 0.61 & -0.5 & 0.61 \end{bmatrix}$$

To find \mathbf{r}, θ such that $\mathbf{R} = \mathbf{R}(\theta, \mathbf{r})$ i know that

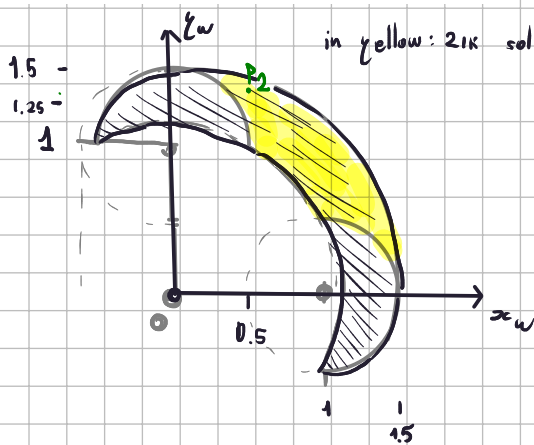
$$1 + 2\cos \theta = \text{trace } \mathbf{R} \Rightarrow \cos \theta = 0.485 \Rightarrow \sin \theta = \pm 0.874$$

$$\Rightarrow \mathbf{r} = \frac{1}{2\sin \theta} \begin{bmatrix} \mathbf{R}_{32} - \mathbf{R}_{23} \\ \mathbf{R}_{13} - \mathbf{R}_{31} \\ \mathbf{R}_{21} - \mathbf{R}_{12} \end{bmatrix} \Rightarrow \pm 0.572 \begin{bmatrix} -1.25 \\ -0.86 \\ -0.86 \end{bmatrix} \Rightarrow \theta = 1.064 \quad \mathbf{r} = \begin{bmatrix} -0.715 \\ -0.431 \\ -0.431 \end{bmatrix} \text{ or } \theta = -1.064 \quad \mathbf{r} = \begin{bmatrix} 0.715 \\ 0.431 \\ 0.431 \end{bmatrix}$$

Exercise 3 [10 points]

Consider the 2R planar robot in Fig. 1, with $L_1 = 1$, $L_2 = 0.5$ [m]. The joint variables have a limited range: $\theta_1 \in [0, \pi/2]$, $\theta_2 \in [-\pi/2, \pi/2]$ [rad].

- Sketch the primary workspace of this robot, localizing the relevant points on its boundary.
- Indicate the region of the workspace where two inverse kinematics solutions exist.
- For each of the following five points, specify whether there are 0, 1, 2, or ∞ inverse kinematics solutions: $P_1 = (0.1, 1.5)$, $P_2 = (0.5, 1.3)$, $P_3 = (-0.4, 1.1)$, $P_4 = (1.0, 1.0)$, $P_5 = (1.0, -0.3)$ [m].



P_1) Since $\|P_1\| > L_1 + L_2 \Rightarrow P_1 \notin WS_1$

P_2) $\|P_2\| = 1.35$ and $P_2 \in WS_1$

P_4) is for sure inside. $P_4 \in WS_1$

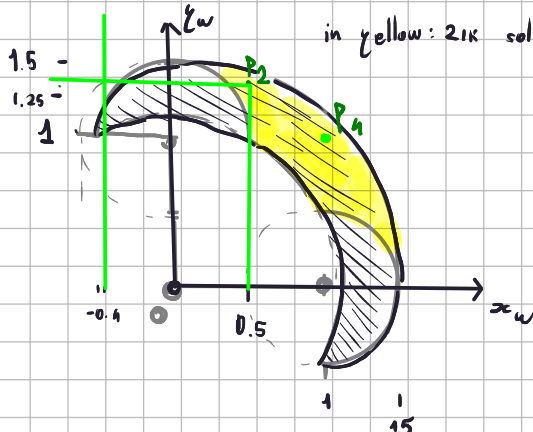
P_5) is not, since the only one point on the line $x = 1$ in the WS have $z = -0.5$.

For P_3 i can't see, i have to solve the IK

$$\Rightarrow \cos q_2 = 0.12 \Rightarrow \sin q_2 = \pm 0.992$$

$$\Rightarrow q_2 = \pm \arcsin\left(\frac{\pm 0.992}{0.12}\right) = \pm 1.4504 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

I look for q_1 .



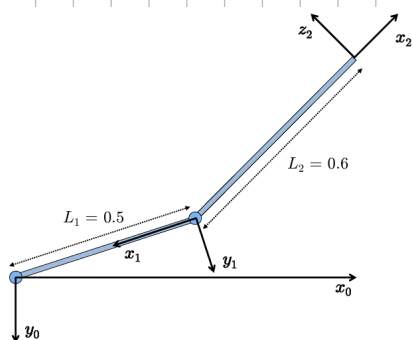
$$\pm q_1 = \operatorname{atan2}\{1.1, -0.4\} - \operatorname{atan2}\left\{\pm \frac{1}{2} \cdot 0.992, 1 + \frac{1}{2} \cdot 0.12\right\} =$$

$$= 1.91 - 0.43 = 1.48 \in [0, \pi/2] \Rightarrow P_3 \in WS_1 \text{ with only one sol: } q = \begin{pmatrix} 1.48 \\ 1.45 \end{pmatrix}$$

$$1.91 + 0.43 = 2.34 \notin [0, \pi/2]$$

Exercise 4 [10 points]

Figure 2 shows an unusual but feasible choice of Denavit-Hartenberg (D-H) frames for a 2R planar robot. Provide the corresponding D-H table of parameters and the direct kinematics of this robot as an homogeneous transformation matrix ${}^0T_2(q)$. Evaluate then this matrix in numerical form at $q^* = (\pi/2, -\pi/2)$ [rad] and draw the robot in this configuration.



	α_i	a_i	d_i	θ_i
1	$-\pi$	$-L_1$	0	q_1
2	$-\pi$	L_2	0	q_2

$${}^0T_1 = \begin{bmatrix} c_1 & s_1 & 0 & -L_1 c_1 \\ s_1 & -c_1 & 0 & -L_1 s_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1T_2 = \begin{bmatrix} c_2 & s_2 & 0 & L_2 c_1 \\ s_2 & -c_2 & 0 & L_2 s_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_2(q) = \begin{bmatrix} c_1 c_2 + s_1 s_2 & c_1 s_2 - s_1 c_2 & 0 & L_2 c_1 c_2 + L_2 s_1 s_2 - L_1 c_1 \\ s_1 c_2 - c_1 s_2 & s_1 s_2 + c_1 c_2 & 0 & L_2 s_1 c_2 - L_2 c_1 s_2 - L_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^0T_2(q^*) = \begin{bmatrix} -1 & 0 & 0 & -L_2 \\ 0 & -1 & 0 & -L_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 5 [10 points]

The differential equations of a DC motor are given in slide #14 of the block 03.CompsActuators.pdf. With the motor unloaded and starting from rest, if we apply a constant armature voltage \bar{v}_a , the motor will start rotating and then reach a steady-state condition, with a constant angular velocity $\bar{\omega}$ and a constant produced torque $\bar{\tau}$. What are the expressions of $\bar{\omega}$ and $\bar{\tau}$ in terms of the system parameters and \bar{v}_a ? If we attach a load with inertia $I_L > 0$ to the motor shaft through a transmission with reduction ratio $n_r > 1$ and assume no dissipative terms on the load side, will the steady-state velocity of the motor change? And what will be the velocity ω_L of the load at steady state?

$\dot{\theta} = \omega$
 $\omega \text{ const} \Rightarrow \ddot{\theta} = 0$

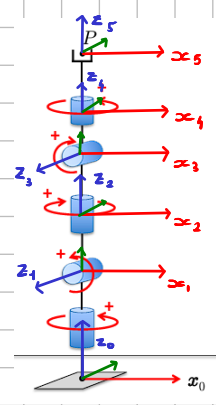
$$\begin{cases} v = Ri + L \frac{di}{dt} + K_v \omega \\ \tau = I_m \frac{d\omega}{dt} + F_m \omega + \tau_{load} \end{cases} \Rightarrow \begin{cases} v = Ri + L \frac{di}{dt} + K_v \omega \\ \tau = I_m \frac{d\omega}{dt} + F_m \omega + \tau_{load} \end{cases} \begin{cases} v = Ri + K_v \omega \\ \tau = F_m \omega = K_t i \end{cases}$$

$$\begin{cases} v = Ri + K_v \omega \\ \tau = F_m \omega = K_t i \end{cases} \Rightarrow \begin{cases} \omega = \frac{v - Ri}{K_v} \\ \tau = \left(\frac{v - Ri}{K_v} \right) F_m \end{cases}$$

$$\hookrightarrow \frac{di}{dt} = 0$$

Exercise 6 [20 points]

The 5R robot in Fig. 3 is shown in its zero configuration (i.e., for $q = 0$), with indication of the positive joint rotations. Assign the D-H frames consistently with these specifications and fill the corresponding table of parameters (specifying also the signs of the non-zero constant parameters). The origin of the last D-H frame should be at point P . Evaluate then numerically the position and the orientation of the last frame at $q = 0$, when all the non-zero kinematic lengths of the links are unitary.

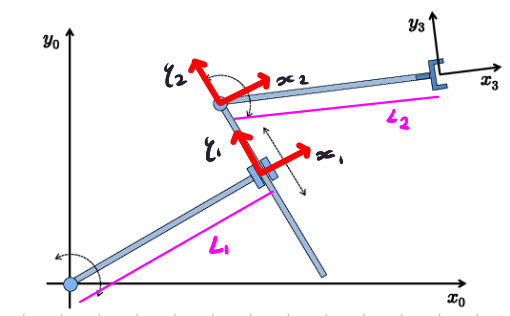


	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$-\pi/2$	0	$d_2 > 0$	q_2
3	$\pi/2$	0	$d_3 > 0$	q_3
4	$-\pi/2$	0	$d_4 > 0$	q_4
5	0	0	$d_5 > 0$	q_5

if $d_i = 1$, then
 $S_r(0) = \begin{pmatrix} 0 & 0 & 5 \end{pmatrix}^T$

Exercise 7 [30 points]

Consider the planar RPR robot in Fig. 4, with the first and third joint revolute and the second prismatic. a. Determine the task kinematics $r = f_r(q)$ for $r = (p, \phi)$, being $p = (p_x, p_y) \in \mathbb{R}^2$ the position of the end-effector and $\phi \in (-\pi, \pi]$ its orientation angle with respect to x_0 . [Hint: Use D-H joint variables.] b. Solve analytically the inverse kinematics problem for $r_d = (p_{dx}, p_{dy}, \phi_d)$ in the regular case only.



$${}^0T_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2T_3 = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_3 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_3 = \begin{bmatrix} c_1 & -s_1 & 0 & L_1 c_1 - q_2 s_1 \\ s_1 & c_1 & 0 & L_1 s_1 + q_2 c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & L_2 c_3 \\ s_3 & c_3 & 0 & L_2 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} c_1 c_3 - s_1 s_3 & -(c_1 s_3 + s_1 c_3) & 0 & L_2 c_1 c_3 - L_2 s_1 s_3 + L_1 c_1 - q_2 s_1 \\ s_1 c_3 + c_1 s_3 & c_1 c_3 - s_1 s_3 & 0 & L_2 s_1 c_3 + L_2 c_1 s_3 + L_1 s_1 + q_2 c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} c_{13} & -s_{13} & 0 & L_2 c_{13} + L_1 c_1 - q_2 s_1 \\ s_{13} & c_{13} & 0 & L_2 s_{13} + L_1 s_1 + q_2 c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow S_r(q) = \begin{cases} p_x = L_2 c_{13} + L_1 c_1 - q_2 s_1 \\ p_y = L_2 s_{13} + L_1 s_1 + q_2 c_1 \\ \phi = q_1 + q_3 \end{cases} \quad \begin{array}{l} \text{I solve} \\ \text{it for} \\ q_1, q_2, q_3 \end{array}$$

$$\phi = q_1 + q_3 \Rightarrow \begin{cases} p_x = L_2 \cos \phi + L_1 \cos q_1 - q_2 \sin q_1 \\ p_y = L_2 \sin \phi + L_1 \sin q_1 + q_2 \cos q_1 \end{cases} \Rightarrow \quad \text{I square and sum}$$

$$p_x^2 + p_y^2 = L_2^2 \cos^2 \phi + L_2^2 \sin^2 \phi + L_1^2 \cos^2 q_1 + L_1^2 \sin^2 q_1 + q_2^2 \cos^2 q_1 + q_2^2 \sin^2 q_1 + 2L_1 L_2 \cos q_1 \cos \phi - 2L_1 L_2 \sin q_1 \sin \phi - 2q_2 L_1 \cos q_1 \sin \phi + 2q_2 L_1 \sin q_1 \cos \phi \Rightarrow$$

$$\Rightarrow p_x^2 + p_y^2 = L_2^2 + L_1^2 + q_2^2 + L_1 L_2 \cos q_3 + q_2 L_2 \sin q_3$$