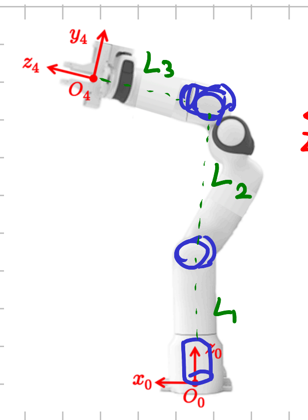
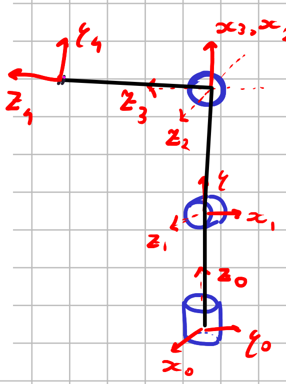
Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise 2

Make reference to the robot in Exercise 1.





i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	L_1	q_1
2	0	L_2	0	q_2
3	$-\pi/2$	0	0	q_3
4	0	0	L_3	q_4

$$ {}^0T_1 = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1T_2 = \begin{pmatrix} c_2 & -s_2 & 0 & L_2c_2 \\ s_2 & c_2 & 0 & L_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^2T_3 = \begin{pmatrix} c_3 & 0 & -s_3 & 0 \\ s_3 & 0 & c_3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3T_4 = \begin{pmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & L_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^0T_2 = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & -s_2 & 0 & L_2c_2 \\ s_2 & c_2 & 0 & L_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1c_2 & -c_1s_2 & s_1 & L_2c_1c_2 \\ s_1c_2 & -s_1s_2 & -c_1 & L_2s_1c_2 \\ s_2 & c_2 & 0 & L_2s_2+L_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^0T_3 = \begin{pmatrix} c_1c_2 & -c_1s_2 & s_1 & L_2c_1c_2 \\ s_1c_2 & -s_1s_2 & -c_1 & L_2s_1c_2 \\ s_2 & c_2 & 0 & L_2s_2+L_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_3 & 0 & -s_3 & 0 \\ s_3 & 0 & c_3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1c_{23} & -s_1 & -c_1s_{23} & L_2c_1c_2 \\ s_1c_{23} & c_1 & -s_1s_{23} & L_2s_1c_2 \\ s_{23} & 0 & c_{23} & L_2s_2+L_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ f_r(q) = \begin{pmatrix} c_1c_{23} & -s_1 & -c_1s_{23} & L_2c_1c_2 \\ s_1c_{23} & c_1 & -s_1s_{23} & L_2s_1c_2 \\ s_{23} & 0 & c_{23} & L_2s_2+L_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ L_3 \\ 1 \end{pmatrix} = \begin{cases} c_1(L_2c_2 - L_3s_{23}) \\ s_1(L_2c_2 - L_3s_{23}) \\ L_3c_{23} + L_2s_2 + L_1 \end{cases} $$

a. Derive the expression of the 6×4 geometric Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ of this robot, relating the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ to the linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ and angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of the end-effector frame.

b. Find all configurations at which the upper 3×4 block $\boldsymbol{J}_L(\boldsymbol{q})$ of the geometric Jacobian loses rank.

c. Find all configurations at which the lower 3×4 block $\boldsymbol{J}_A(\boldsymbol{q})$ of the geometric Jacobian loses rank.

d. In the configuration $\boldsymbol{q}_0 = \boldsymbol{0}$, check if the linear Cartesian velocity $\boldsymbol{v}_b = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$ is feasible. Provide a joint velocity $\dot{\boldsymbol{q}}_b \in \mathbb{R}^4$ that instantaneously realizes $\boldsymbol{v}_b$ or, at least, that minimizes the norm of the error w.r.t. the Cartesian velocity $\boldsymbol{v}_b$. If such a joint velocity exists, is it unique?

$$ J_L = \frac{\partial f_r}{\partial q} = \begin{pmatrix} -s_1(L_2c_2 - L_3s_{23}) & -c_1(L_2s_2 + L_3c_{23}) & -c_1L_3c_{23} & 0 \\ c_1(L_2c_2 - L_3s_{23}) & -s_1(L_2s_2 + L_3c_{23}) & -s_1L_3c_{23} & 0 \\ 0 & L_2c_2 - L_3s_{23} & -L_3s_{23} & 0 \end{pmatrix} $$

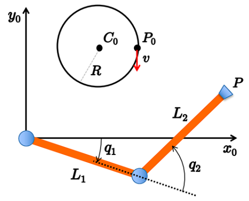
$$ J_A = \begin{pmatrix} 0 & s_1 & s_1 & -c_1s_{23} \\ 0 & -c_1 & -c_1 & -s_1s_{23} \\ 1 & 0 & 0 & c_{23} \end{pmatrix} $$

I consider $\det(J_A J_A^T) = 2s_{23}^2 \Rightarrow J_A$ is singular if $q_2+q_3 \in \{0, \pi\}$

For $\det(J_L J_L^T) = L_2^2L_3^2(L_2c_2 - L_3s_{23})^2 c_3^2 = 0 \iff \begin{cases} q_3 \in \{-\frac{\pi}{2}, \frac{\pi}{2}\} \\ c_2 = \frac{L_3}{L_2}s_{23} \Rightarrow q_2 = \arccos\left\{\frac{L_3}{L_2}\sin(q_2+q_3)\right\} \end{cases}$

in $q_0 = 0 \Rightarrow J_L(q_0) = \begin{pmatrix} 2L_3^2 & 0 & -L_2L_3 \\ 0 & L_2^2 & 0 \\ -L_2L_3 & 0 & L_2^2 \end{pmatrix}$, $J_L^{-1}v_b = \begin{pmatrix} -2L_3^2 & 0 & 1/(L_2L_3) \\ 0 & -2L_2^2 & 0 \\ 1/L_2L_3 & 0 & 2/L_3^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2L_3^2 + (L_2L_3)^{-1} \\ 0 \\ (L_2L_3)^{-1} + 2L_2^{-2} \end{pmatrix} = \dot{q}_b$

Exercise 3



Consider the planar 2R robot in Fig. 2, shown together with the geometric data of its desired task. The robot end-effector should follow a desired trajectory made by a circular path of radius R centered at $\boldsymbol{C}_0 = \begin{pmatrix} C_{0,x} & C_{0,y} \end{pmatrix}^T$, to be executed clockwise with a continuous, possibly time-varying desired scalar speed $v(t) > 0$, starting at time $t = 0$ from the path point $\boldsymbol{P}_0 = \begin{pmatrix} P_{0,x} & P_{0,y} \end{pmatrix}^T = \begin{pmatrix} C_{0,x}+R & C_{0,y} \end{pmatrix}^T$.

Assuming that the robot is commanded by the joint velocity $\dot{\boldsymbol{q}}$, define a single control law that guarantees the following properties:

- when the initial robot configuration $\boldsymbol{q}_0 = \boldsymbol{q}(0)$ at $t = 0$ is matched with the Cartesian point $\boldsymbol{P}_0$, there is a perfect reproduction of the desired trajectory for all $t \geq 0$;
- if there is no such initial matching, the Cartesian trajectory tracking error will converge to zero exponentially and in a decoupled way with respect to its components expressed in a reference frame $RF_r(t) = (\boldsymbol{x}_r(t), \boldsymbol{y}_r(t))$ that is moving with the desired position and has the axis $\boldsymbol{x}_r(t)$ always tangent to the path.

let $p_d(t) = \begin{cases} C_{ox} + R\cos(-\omega t) \\ C_{oy} + R\sin(-\omega t) \end{cases} \Rightarrow \dot{p}_d(t) = \begin{cases} \omega R \sin(-\omega t) \\ -\omega R\cos(-\omega t) \end{cases} \Rightarrow \|\dot{p}_d\| = \sqrt{2(\omega R)^2} = v \Rightarrow \omega = \frac{v}{R\sqrt{2}}$

$$J = \begin{pmatrix} -(L_1 s_1 + L_2 s_{12}) & -L_2 s_{12} \\ L_1 c_1 + L_2 c_{12} & L_2 c_{12} \end{pmatrix} \Rightarrow \dot{q}_d = J^{-1}\dot{p}_d = \frac{1}{L_1 L_2 s_2}\begin{pmatrix} L_2 c_{12} & L_2 s_{12} \\ -L_1 c_1 - L_2 c_{12} & -(L_1 s_1 + L_2 s_{12}) \end{pmatrix}\begin{pmatrix} \omega R \sin(-\omega t) \\ -\omega R\cos(-\omega t) \end{pmatrix}$$

$e = p_d - p \Rightarrow \dot{e} = \dot{p}_d - \dot{p} = J\dot{q}_d - J\dot{q}$ I consider as a command $\dot{q} = \dot{q}_d + J^{-1}Ke$

where $K = \text{diag}\{k_1, k_2\}$ so.

$$\dot{e} = \dot{p}_d - \dot{p} = J\dot{q}_d - J\dot{q} = J\dot{q}_d - J(\dot{q}_d + J^{-1}Ke) = Ke \Rightarrow \dot{e} = -Ke \Rightarrow \begin{cases} \dot{e}_x = -k_1 e_x \\ \dot{e}_y = -k_2 e_y \end{cases} \Rightarrow \begin{cases} e_x(t) = e_{ox} \cdot \exp(-k_1 t) \\ e_y(t) = e_{oy} \cdot \exp(-k_2 t) \end{cases}$$

Where $\begin{pmatrix} e_{ox} \\ e_{oy} \end{pmatrix} = e(0)$, so if $e(0) = 0 \Rightarrow e(t) = 0 \ \forall t \Rightarrow 0 = p_d - p \Rightarrow p = p_d$.

a. Using next the following numerical data

$$L_1 = L_2 = 0.5, \qquad \boldsymbol{C}_0 = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}, \qquad R = 0.15 \text{ [m]}, \qquad v = 3 \text{ [m/s]},$$

determine the value $\boldsymbol{q}_0 = \boldsymbol{q}(0)$ of an initial configuration and the value of the commanded velocity $\dot{\boldsymbol{q}}(0)$ at $t = 0$ that are needed for perfect reproduction of the desired trajectory.

b. In addition, with the robot in the initial configuration

$$\boldsymbol{q}_{\text{off}} = \begin{pmatrix} 0 \\ \pi/6 \end{pmatrix} \text{ [rad]} \neq \boldsymbol{q}_0,$$

using the two time constants $\tau_{r,x} = 0.1$ and $\tau_{r,y} = 0.05$ [s] for the desired exponential transients of the trajectory tracking error components in the frame $RF_r(t)$, determine the initial value $\dot{\boldsymbol{q}}(0)$ of the control law that satisfies the above mentioned properties.

In $q_0 \Rightarrow p(0) = \begin{cases} \frac{1}{2}(c_1 + c_{12}) \\ \frac{1}{2}(s_1 + s_{12}) \end{cases} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad e(0) = p_d(0) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -\omega R \end{pmatrix} = \begin{pmatrix} -1 \\ -v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 \\ -3/\sqrt{2} \end{pmatrix}$

$J = \begin{pmatrix} 0 & 0 \\ 1 & 1/2 \end{pmatrix} \Rightarrow$ isn't invertible. I consider $\dot{q} = \dot{q}_d + J^{\#}Ke$ with $K = 2I$

$$J^{\#}(q_0) = \begin{pmatrix} 0 & 0 \\ 1 & 1/2 \end{pmatrix}^{\#} = \begin{pmatrix} 0 & 0.8 \\ 0 & 0.4 \end{pmatrix}. \qquad \dot{q}_d = \begin{pmatrix} 0 & 0.8 \\ 0 & 0.4 \end{pmatrix}\begin{pmatrix} 0 \\ -3/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1.697 \\ -0.848 \end{pmatrix}$$

$$\Rightarrow \dot{q}(0) = \begin{pmatrix} -1.697 \\ -0.848 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1/2 \end{pmatrix}^{\#} \cdot 2 \begin{pmatrix} -1 \\ -3/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -5.091 \\ -2.545 \end{pmatrix}$$