Exercise 1

The end-effector of a robot manipulator has an initial orientation specified by the ZXY Euler angles $(\alpha, \beta, \gamma) = (\pi/2, \pi/4, -\pi/4)$ [rad] and should reach a final orientation specified by an axis-angle pair $(\boldsymbol{r}, \theta)$, with $\boldsymbol{r} = (0, -\sqrt{2}/2, \sqrt{2}/2)$ and $\theta = \pi/6$ rad. What is the required rotation matrix $\boldsymbol{R}_{if}$ between these two orientations? Represent $\boldsymbol{R}_{if}$ by the RPY-type angles (ϕ, χ, ψ) around the fixed-axes sequence YXY.

We have to compute ${}^{w}R_i$ and ${}^{w}R_f$ and consider ${}^{i}R_f = {}^{w}R_i^T \, {}^{w}R_f$.

$$ {}^{w}R_i = R_z(\alpha) R_x(\beta) R_y(\gamma) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} $$

$$ \Rightarrow {}^{w}R_i = \begin{bmatrix} 1/2 & -0.707 & 1/2 \\ 0.707 & 0 & -0.707 \\ 1/2 & 0.707 & 0.5 \end{bmatrix} $$

We define now ${}^{w}R_f$ as $rr^T + (I - rr^T)\cos\theta + S(r)\sin\theta$

$$ rr^T = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}, \quad S(r) = \begin{bmatrix} 0 & -\sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 & 0 \\ \sqrt{2}/2 & 0 & 0 \end{bmatrix} $$

$$ \Rightarrow {}^{w}R_f = \begin{bmatrix} 0.866 & -0.353 & -0.353 \\ 0.353 & 0.933 & -0.066 \\ 0.353 & -0.066 & 0.933 \end{bmatrix} \Rightarrow {}^{i}R_f = {}^{w}R_i^T \, {}^{w}R_f \approx \begin{bmatrix} 0.85 & 0.44 & 0.24 \\ -0.36 & 0.2 & 0.9 \\ 0.35 & -0.87 & 0.337 \end{bmatrix} $$

The matrix that represents the RPY angles (ϕ, χ, ψ) around YXY is

$$ \begin{bmatrix} c\psi & 0 & s\psi \\ 0 & 1 & 0 \\ -s\psi & 0 & c\psi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\chi & -s\chi \\ 0 & s\chi & c\chi \end{bmatrix} \begin{bmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix} = $$

$$ \begin{bmatrix} c\psi & s\psi s\chi & s\psi c\chi \\ 0 & c\chi & -s\chi \\ -s\psi & c\psi s\chi & c\psi c\chi \end{bmatrix} \begin{bmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix} \Rightarrow \text{i have to solve} $$

$$ \begin{bmatrix} c\psi c\phi - s\psi c\chi s\phi & s\psi s\chi & c\psi s\phi + c\phi s\psi c\chi \\ s\chi s\phi & c\chi & -s\chi c\phi \\ -s\psi c\phi - s\phi c\psi c\chi & c\psi s\chi & -s\phi s\psi + c\phi c\psi c\chi \end{bmatrix} = \begin{bmatrix} 0.85 & 0.44 & 0.24 \\ -0.36 & 0.2 & 0.9 \\ 0.35 & -0.87 & 0.337 \end{bmatrix} $$

$\Rightarrow \cos\chi \approx 0.202632$ and $\sin^2\chi \sin^2\phi + \sin^2\chi \cos^2\phi = -0.362^2 + 0.9097^2 = 0.6965$

$\Rightarrow \sin^2\chi \underbrace{(\sin^2\phi + \cos^2\phi)}_{1} = 0.6965 \Rightarrow \sin\chi = \pm\sqrt{0.6965} = 0.8343$

$$\vartheta^{+,-} = \operatorname{atan2}\left\{\pm 0.8345,\ 0.2026\right\} = \begin{cases} \vartheta^{+} = 1.3325 \\ \vartheta^{-} = -1.3325 \end{cases}$$

then, i consider $\begin{cases} s\vartheta s\phi = -0.36 \\ -s\vartheta c\phi = 0.9 \end{cases} \Rightarrow \begin{cases} s\phi = -\dfrac{0.36}{s\vartheta} \\ c\phi = -\dfrac{0.9}{s\vartheta} \end{cases} \Rightarrow \begin{cases} s\phi = -\dfrac{0.36}{\pm 0.97} \\ c\phi = -\dfrac{0.9}{\pm 0.97} \end{cases}$

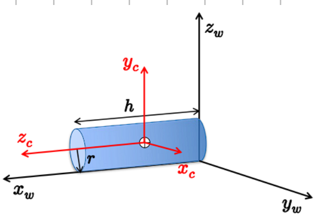
$\Rightarrow \sin\phi^{+} = -\dfrac{36}{97}$ $\quad \sin\phi^{-} = \dfrac{36}{97}$ $\quad \phi^{+} = -2.76$

$\cos\phi^{+} = -\dfrac{90}{97}$ $\quad \cos\phi^{-} = \dfrac{90}{97}$ $\quad \Rightarrow \quad \phi^{-} = 0.38$

Analogously we find ψ^{+} and ψ^{-}.

Exercise 2

A cylinder of height h and radius r lies on the plane (x_w, y_w) in the initial pose shown in Fig. 1, with a frame $RF_c = (x_c, y_c, z_c)$ attached to the geometric center of its body. The cylinder rolls without slipping by a ground distance $d > 0$ in the y_w-direction, and rotates then by an angle ϑ around the original z_w-axis. Finally, a rotation φ is performed around the current direction of the z_c-axis. Determine the expression of the elements of the homogeneous transformation matrix ${}^w\boldsymbol{T}_c(h, r, d, \vartheta, \varphi)$ that characterizes the final pose of the cylinder. Evaluate then ${}^w\boldsymbol{T}_c$ for $h = 0.5$, $r = 0.1$, $d = 1.5$ [m] and $\vartheta = \pi/3$, $\varphi = -\pi/2$ [rad]. *Hint: Check your intermediate results with simpler data.*



The center of the cylinder in our world frame is $[\frac{h}{2}, 0, r]$. When it rolls, it moves along x_c by d, and rotate around z_w by d/r radiants. The transformation from the world frame to the cylinder frame is:

$${}^wT_i = \begin{bmatrix} 0 & 0 & 1 & h/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first roll is described by:

$${}^cT_d = \begin{bmatrix} \cos(-\frac{d}{r}) & \sin\frac{d}{r} & 0 & d \\ -\sin\frac{d}{r} & \cos(-\frac{d}{r}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then a rotation by θ around z_w is performed. $T_\theta = \begin{vmatrix} R_z(\theta) & 0 \\ 0 & 1 \end{vmatrix}$. Since is performed on the z_w axis, will be the first in the matrix product.

The final transformation is:

$$T_\varphi = \begin{vmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Thend end pos of the cylinder is ${}^wT_c = T_\theta\, {}^wT_i\, T_d\, T_\varphi$

For the given numerical values we obtain

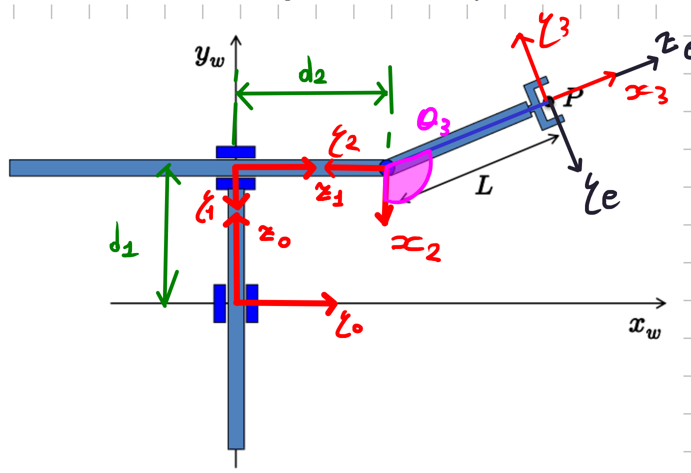
$${}^wT_c \approx \begin{bmatrix} 0.56 & 0.65 & 0.5 & -1.17 \\ -0.32 & -0.37 & 0.86 & 0.96 \\ 0.75 & -0.65 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 3

Consider the PPR planar robot with a 2-jaw gripper in Fig. 2, shown together with the world frame RF_w.

- Assign the link frames and fill in the associated table of parameters according to the Denavit–Hartenberg (DH) convention (use the extra sheet). The origin of the last DH frame should be placed at the gripper's center (point P). Choose the frames so that there is **no** axis pointing inside the sheet.
- Determine the homogeneous transformation matrices ${}^wT_0$ and ${}^3T_e$, respectively between the world frame RF_w and the zero-th DH frame RF_0 and between the last DH frame RF_3 and the end-effector frame RF_e placed at the gripper, with the usual convention (z_e in the approach direction and y_e in the open/close slide direction of the jaws).
- Provide the direct kinematics for the end-effector position ${}^w p_e \in \mathbb{R}^3$.
- When the two prismatic joints are limited as $q_i \in [q_{i,m}, q_{i,M}]$, under the assumption that $q_{i,M} - q_{i,m} > 2L$, for $i = 1, 2$, and the revolute joint is in the range $q_3 \in [-3\pi/4, 0]$, sketch the primary workspace of this robot and locate the relevant points on its boundary.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	q_1	0
2	$-\pi/2$	0	q_2	$-\pi/2$
3	0	L	0	q_3



$${}^wT_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^3T_e = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ The DK is given by

$${}^wT_e = {}^wT_0\,{}^0A_1\,{}^1A_2\,{}^2A_3\,{}^3T_w$$

We have to compute ${}^0A_1$, ${}^1A_2$, ${}^2A_3$

$${}^0A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2A_3 = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 & L\cos q_3 \\ \sin q_3 & \cos q_3 & 0 & L\sin q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_2 \\ 1 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^wT_2 = {}^wT_0\,{}^0A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_2 \\ 1 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & q_2 \\ 1 & 0 & 0 & q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^wT_3 = {}^wT_2\,{}^2A_3 = \begin{bmatrix} 0 & -1 & 0 & q_2 \\ 1 & 0 & 0 & q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 & L\cos q_3 \\ \sin q_3 & \cos q_3 & 0 & L\sin q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -sq_3 & -cq_3 & 0 & -L\sin q_3 + q_2 \\ cq_3 & -sq_3 & 0 & L\cos q_3 + q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^wT_e = \begin{bmatrix} -sq_3 & -cq_3 & 0 & -L\sin q_3 + q_2 \\ cq_3 & -sq_3 & 0 & L\cos q_3 + q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & C_3 & S_3 & -LS_3 + q_2 \\ 0 & S_3 & C_3 & LC_3 + q_1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The position of the EE is $p = \begin{bmatrix} -L\sin q_3 + q_2 \\ L\cos q_3 + q_1 \\ 0 \end{bmatrix}$