Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise 4

The kinematics of a 4-dof robot manipulator is characterized by the DH parameters in Tab. 1. Build the geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ that relates the joint velocities $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ to the six-dimensional *twist* vector composed by a velocity $\boldsymbol{v} = \boldsymbol{v}_4 \in \mathbb{R}^3$ of the origin of the last (end-effector) DH frame and by an angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}_4 \in \mathbb{R}^3$ of the same frame:

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	$\pi/2$	0	0	q_2
3	$-\pi/2$	0	q_3	0
4	0	a_4	0	q_4

I have to compute the homogeneus transformations.

$$^0T_1 = \begin{pmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad ^1T_2 = \begin{pmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad ^2T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad ^3T_4 = \begin{pmatrix} C_4 & -S_4 & 0 & a_4C_4 \\ S_4 & C_4 & 0 & a_4S_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^0T_2 = \begin{pmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C_1C_2 & S_1 & C_1S_2 & 0 \\ S_1C_2 & -C_1 & S_1S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^0T_3 = {^0T_2}\,{^2T_3} = \begin{pmatrix} C_1C_2 & S_1 & C_1S_2 & 0 \\ S_1C_2 & -C_1 & S_1S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C_1C_2 & -C_1S_2 & S_1 & q_3C_1S_2 \\ S_1C_2 & -S_1S_2 & -C_1 & q_3S_1S_2 \\ S_2 & C_2 & 0 & -q_3C_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For the complete position DK i consider only the last column of $^0T_4$

$$f_p(q) = \begin{pmatrix} C_1C_2 & -C_1S_2 & S_1 & q_3C_1S_2 \\ S_1C_2 & -S_1S_2 & -C_1 & q_3S_1S_2 \\ S_2 & C_2 & 0 & -q_3C_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_4C_4 \\ a_4S_4 \\ 0 \\ 1 \end{pmatrix} = \begin{cases} a_4C_1C_2C_4 - a_4C_1S_2S_4 + q_3C_1S_2 \\ a_4S_1C_2C_4 - a_4S_1S_2C_4 + q_3S_1S_2 \\ a_4(S_2C_4 + C_2S_4) - q_3C_2 \end{cases} = \begin{cases} C_1(a_4C_{24} + q_3S_2) \\ S_1(a_4C_{24} + q_3S_2) \\ a_4S_{24} - q_3C_2 \end{cases} = P_{0,E}$$

I need to compute $P_{1,E}, P_{2,E}, P_{3,E}, z_1, z_2, z_3$

We recall that $P_{0,i} = {^0T_i} \cdot (0\ 0\ 0\ 1)^T$ so:

$$P_{1,E} = P_{0,E} - P_{0,1} = P_{1,E} \qquad P_{3,E} = P_{0,E} - P_{0,3} = \begin{cases} C_1 a_4 C_{24} \\ S_1 a_4 C_{24} \\ a_4 S_{24} \end{cases} \qquad S_3 = \begin{pmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ C_1 & S_1 & 0 \end{pmatrix}$$

$$P_{2,E} = P_{0,E} - P_{0,2} = P_{1,E}$$

skew symmetric matrices

$$\text{then: } z_1 = \begin{pmatrix} S_1 \\ -C_1 \\ 0 \end{pmatrix} \quad z_2 = \begin{pmatrix} C_1S_2 \\ S_1S_2 \\ -C_2 \end{pmatrix} \quad z_3 = \begin{pmatrix} S_1 \\ -C_1 \\ 0 \end{pmatrix} \Rightarrow S_1 = \begin{pmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ C_1 & S_1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & C_2 & S_1S_2 \\ -C_2 & 0 & -C_1S_2 \\ -S_1S_2 & C_1S_2 & 0 \end{pmatrix},$$

Contributions

the joint 1 is revolut, the linear contr. J_{L1} is given by

$$J_{L1} = z_0 \times P_{0,E} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} C_1(a_4C_{24} + q_3S_2) \\ S_1(a_4C_{24} + q_3S_2) \\ a_4S_{24} - q_3C_2 \end{pmatrix} = \begin{pmatrix} -S_1(a_4C_{24} + q_3S_2) \\ C_1(a_4C_{24} + q_3S_2) \\ 0 \end{pmatrix}$$

$$J_{A1} = z_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$J_{L2} = S_1 \, p_{1,E} = \begin{pmatrix} 0 & 0 & -c_1 \\ 0 & 0 & -s_1 \\ c_1 & s_1 & 0 \end{pmatrix} \begin{pmatrix} c_1(a_4 c_{24} + q_3 s_2) \\ s_1(a_4 c_{24} + q_3 s_2) \\ a_4 s_{24} - q_3 c_2 \end{pmatrix} = \begin{pmatrix} c_1(q_3 c_2 - a_4 s_{24}) \\ s_1(q_3 c_2 - a_4 s_{24}) \\ a_4 c_{24} + q_3 s_2 \end{pmatrix}$$

$$J_{A2} = \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}$$

$$J_{L3} = \begin{pmatrix} c_1 s_2 \\ s_1 s_2 \\ -c_2 \end{pmatrix}$$

$$J_{A3} = 0$$

$$J_{L4} = \begin{pmatrix} 0 & 0 & -c_1 \\ 0 & 0 & -s_1 \\ c_1 & s_1 & 0 \end{pmatrix} \begin{pmatrix} c_1 a_4 c_{24} \\ s_1 a_4 c_{24} \\ a_4 s_{24} \end{pmatrix} = \begin{pmatrix} -a_4 c_1 s_{24} \\ -a_4 s_1 s_{24} \\ a_4 c_{24} \end{pmatrix}$$

$$J_{A4} = \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix}$$

$$\Rightarrow J_L = \begin{bmatrix} -s_1(a_4 c_{24} + q_3 s_2) & c_1(q_3 c_2 - a_4 s_{24}) & c_1 s_2 & -a_4 c_1 s_{24} \\ c_1(a_4 c_{24} + q_3 s_2) & s_1(q_3 c_2 - a_4 s_{24}) & s_1 s_2 & -a_4 s_1 s_{24} \\ 0 & a_4 c_{24} + q_3 s_2 & -c_2 & a_4 c_{24} \end{bmatrix}$$

$$J_A = \begin{bmatrix} 0 & s_1 & 0 & s_1 \\ 0 & -c_1 & 0 & -c_1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

We have a singular conf. if $q_1 = 0$ and $q_2 = \pi/2$, in such case, the second and fourth cols become lin. dependent:

$$J(q_1 = 0, q_2 = \tfrac{\pi}{2}, q_3 = 0) = \begin{bmatrix} 0 & -a_4 s_{24} & 1 & -a_4 s_{24} \\ a_4 c_{24} & 0 & 0 & 0 \\ 0 & a_4 c_{24} & -1 & a_4 c_{24} \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \text{let } q_s = \begin{pmatrix} 0 \\ \pi/2 \\ 0 \\ -\pi/2 \end{pmatrix}$$

$$\Rightarrow J(q_s) = J_s = \begin{bmatrix} 0 & 0 & 1 & 0 \\ a_4 & 0 & 0 & 0 \\ 0 & a_4 & -1 & a_4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

i studi $\mathrm{Ker}\, J_s$ by solving $J_s \dot{q} = 0$

the rank of J_s is 3, the max rank is 4.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ a_4 & 0 & 0 & 0 \\ 0 & a_4 & -1 & a_4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \dot{q} \Rightarrow \begin{cases} \dot{q}_3 = 0 \\ a_4 \dot{q}_1 = 0 \\ a_4(\dot{q}_2 + \dot{q}_4) - \dot{q}_3 = 0 \\ -\dot{q}_2 - \dot{q}_4 = 0 \\ \dot{q}_1 = 0 \end{cases} \Rightarrow \begin{cases} \dot{q}_1 = 0 \\ \dot{q}_2 = -\dot{q}_4 \\ \dot{q}_3 = 0 \end{cases} \Rightarrow \text{a basis is } \operatorname{Ker} J_s = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

the non realizable vectors are given by $J_s^T x = 0$:

$$\begin{pmatrix} 0 & a_4 & 0 & 0 & 0 & 1 \\ 0 & 0 & a_4 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & a_4 & 0 & -1 & 0 \end{pmatrix} x = 0 \Rightarrow \begin{cases} a_4 x_2 + x_6 = 0 \\ a_4 x_3 - x_5 = 0 \\ -x_3 = 0 \\ x_1 + a_4 x_3 - x_5 = 0 \end{cases} \begin{cases} x_6 = -a_4 x_2 \\ x_5 = 0 \\ x_3 = 0 \\ x_1 = 0 \end{cases} \Rightarrow \text{basis: } \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -a_4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$