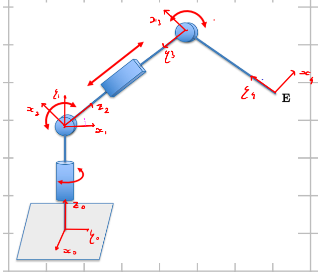
Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise 2



Consider again the robot in Exercise 1.

- Derive the expression of the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of this robot relating the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ to the linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ and angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of the end-effector frame. What is the generic rank of the lower 3×4 block $\boldsymbol{J}_A(\boldsymbol{q})$ of this matrix?
- Evaluate $\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0)$, again as a parametric function. At the same previously specified configuration $\boldsymbol{q} = \boldsymbol{q}_0$, provide answers/solutions to the following problems.
  - Find, if possible, a joint velocity $\dot{\boldsymbol{q}}_a \neq \boldsymbol{0}$ that produces no linear velocity ($\boldsymbol{v} = \boldsymbol{0}$) at the end-effector. Would then also $\boldsymbol{\omega} = \boldsymbol{0}$ follow?
  - Determine if the generalized Cartesian velocity $\boldsymbol{V} = \begin{pmatrix} \boldsymbol{v}^T & \boldsymbol{\omega}^T \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}^T$ is feasible. If so, provide a joint velocity $\dot{\boldsymbol{q}}_b \in \mathbb{R}^4$ that instantaneously realizes it.
  - Find, if possible, a non-zero generalized Cartesian force $\boldsymbol{F}_c = \begin{pmatrix} \boldsymbol{f}^T & \boldsymbol{m}^T \end{pmatrix}^T \in \mathbb{R}^6$ applied at the end-effector that can be statically balanced by zero joint forces/torques ($\boldsymbol{\tau} = \boldsymbol{0}$, with $\boldsymbol{\tau} \in \mathbb{R}^4$). If such a $\boldsymbol{F}_c \neq \boldsymbol{0}$ does not exist, explain why.

Joint	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	d_1	q_1
2	$\pi/2$	0	0	q_2
3	$-\pi/2$	0	q_3	0
4	0	a_4	0	q_4

$$ {}^0T_1 = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1T_2 = \begin{pmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^2T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3T_4 = \begin{pmatrix} c_4 & -s_4 & 0 & a_4c_4 \\ s_4 & c_4 & 0 & a_4s_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^0T_2 = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1c_2 & s_1 & c_1s_2 & 0 \\ s_1c_2 & -c_1 & s_1s_2 & 0 \\ s_2 & 0 & -c_2 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^0T_4 = \begin{pmatrix} c_1c_2 & -c_1s_2 & s_1 & q_3c_1s_2 \\ s_1c_2 & -s_1s_2 & -c_1 & q_3s_1s_2 \\ s_2 & c_2 & 0 & d_1-q_3c_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_4 & -s_4 & 0 & a_4c_4 \\ s_4 & c_4 & 0 & a_4s_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^0T_3 = \begin{pmatrix} c_1c_2 & s_1 & c_1s_2 & 0 \\ s_1c_2 & -c_1 & s_1s_2 & 0 \\ s_2 & 0 & -c_2 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1c_2 & -c_1s_2 & s_1 & q_3c_1s_2 \\ s_1c_2 & -s_1s_2 & -c_1 & q_3s_1s_2 \\ s_2 & c_2 & 0 & d_1-q_3c_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ \delta(q) = \begin{pmatrix} c_1c_2 & -c_1s_2 & s_1 & q_3c_1s_2 \\ s_1c_2 & -s_1s_2 & -c_1 & q_3s_1s_2 \\ s_2 & c_2 & 0 & d_1-q_3c_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_4c_4 \\ a_4s_4 \\ 0 \\ 1 \end{pmatrix} = \begin{cases} c_1(a_4c_{24}+q_3s_2) \\ s_1(a_4c_{24}+q_3s_2) \\ a_4s_{24}+d_1-q_3c_2 \end{cases} $$

$$ \Rightarrow J_L = \frac{d\delta}{dq} = \begin{pmatrix} -s_1(a_4c_{24}+q_3s_2) & c_1(-a_4s_{24}+q_3c_2) & c_1s_2 & -a_4c_1s_{24} \\ c_1(a_4c_{24}+q_3s_2) & s_1(-a_4s_{24}+q_3c_2) & s_1s_2 & -a_4s_1s_{24} \\ 0 & a_4c_{24}+q_3s_2 & -c_2 & a_4c_{24} \end{pmatrix} $$

$$ J_A = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 0 & s_1 & s_2 & c_1s_2 \\ 0 & -c_1 & -c_2 & s_1s_2 \\ 1 & 0 & 0 & -c_2 \end{pmatrix} \Rightarrow \text{the generic rank is } 3 $$

$$ v = \begin{pmatrix} -s_1(a_4c_{24}+q_3s_2) & c_1(-a_4s_{24}+q_3c_2) & c_1s_2 & -a_4c_1s_{24} \\ c_1(a_4c_{24}+q_3s_2) & s_1(-a_4s_{24}+q_3c_2) & s_1s_2 & -a_4s_1s_{24} \\ 0 & a_4c_{24}+q_3s_2 & -c_2 & a_4c_{24} \end{pmatrix} \Rightarrow \text{generic rank is } 3, $$

For the dimension theorem $\Rightarrow \dim R(J) + \dim N(J) = 4 \Rightarrow \dim N(J) = 1 \Rightarrow \exists\, \dot{q} \neq 0 : J\dot{q} = 0.$

$$ J_{0_L} = J_L(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 \\ 0 & a_4 & -1 & a_4 \end{pmatrix}. \quad \dot{q} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq 0 \text{ produces } v = 0 : \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 \\ 0 & a_4 & -1 & a_4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -a_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} $$

$$ J_A(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \Rightarrow J_A(0) \begin{pmatrix} 0 \\ 1 \\ -a_4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1+a_4 \\ 0 \end{pmatrix} \neq 0 \ (\text{iff } a_4 \neq 1). $$

V isn't feasible since the first row of J_0 is zero and $V_1 = 1$.

$$ J_0^T = \begin{pmatrix} 0 & a_4 & 0 & 0 & 0 & 1 \\ 0 & 0 & a_4 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & a_4 & 0 & 0 & -1 \end{pmatrix}. \quad J_0^T F = \begin{cases} a_4F_2 + F_6 = 0 \\ a_4F_3 - F_5 = 0 \\ -F_3 - F_5 = 0 \\ F_3 - F_6 = 0 \end{cases} \Rightarrow \begin{cases} F_2 = 0 \\ F_6 = 0 \\ F_6 = -F_5 \\ F_3 = F_6 \end{cases} \Rightarrow F^* = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ is balanced by } \tau = 0 $$

Exercise 3

For a planar RP robot with direct kinematics of the end-effector position given by

$$\boldsymbol{p} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \tag{1}$$

consider the planning of a rest-to-rest motion between an initial and a final Cartesian point, respectively, $\boldsymbol{p}_A = \begin{pmatrix} 4 & 3 \end{pmatrix}^T$ [m] at $t = 0$ and $\boldsymbol{p}_B = \begin{pmatrix} -3.5355 & 3.5355 \end{pmatrix}^T$ [m] at $t = T$. Optimization of the motion time T is being sought, in two different operative conditions as follows.

a. Define a joint trajectory $\boldsymbol{q}_a^*(t)$ that minimizes the motion time for this task under the bounds on the joint accelerations,

$$|\ddot{q}_1| \le A_1 = 200\,°/\mathrm{s}^2, \qquad |\ddot{q}_2| \le A_2 = 5\ \mathrm{m/s}^2. \tag{2}$$

Find the value of the minimum motion time T_a^* and draw the time profiles of the position, velocity and acceleration of the two robot joints.

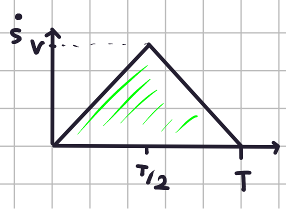
$$q_A = f^{-1}(p_A) \Rightarrow \begin{cases} q_2 c_1 = 4 \\ q_2 s_1 = 3 \end{cases} \Rightarrow \begin{cases} q_2 = 5 \\ q_1 = 0.643 \end{cases}. \qquad q_b = f^{-1}(p_b) = \begin{cases} q_1 = \frac{3}{4}\pi \\ q_2 = 5 \end{cases}$$

$$\Rightarrow \begin{cases} q_1(s) = 0.643 + s\underbrace{\left(\tfrac{3}{4}\pi - 0.643\right)}_{= L} \\ q_2(s) = 5 \end{cases} \qquad L = \|q_b - q_a\| = 1.713$$

For $s(t)$ i choose a b-c-b. I need to find $\max|\ddot{s}|$.

$$q'(s) = \begin{cases} L \\ 0 \end{cases} \qquad q''(s) = 0, \qquad \begin{matrix} \max q_1' = L \\ \max q_1'' = 0 \end{matrix} \Rightarrow \max|\ddot{s}| = \frac{A_1}{L} = 2.037$$

Since $|\dot{s}|$ is not bounded, i will use a bang-bang profile.



$$\text{Area} = \frac{VT}{2}, \quad V = T \cdot 1.01885 \Rightarrow \text{Area} = 1.038\,T^2 = \Delta s = 1 \Rightarrow T^2 = \frac{1}{1.038} \Rightarrow T = 0.981$$

b. Consider next the additional Cartesian bound on the norm of the end-effector acceleration,

$$\|\ddot{\boldsymbol{p}}\| \le A_c = 10\ \mathrm{m/s}^2. \tag{3}$$

Verify whether the previous solution $\boldsymbol{q}_a^*(t)$ satisfies the bound (3) or not. If not, propose a modified joint trajectory $\boldsymbol{q}_b^*(t)$ such that both bounds (2) and (3) will be satisfied, while trying to minimize the new motion time. Discuss the rationale of your choice and the supporting equations, provide the resulting motion T_b^*, and sketch the new time profiles of your solution.

$$p(s) = \begin{cases} 5\cos(0.643 + sL) \\ 5\sin(0.643 + sL) \end{cases} \qquad p'(s) = \begin{cases} -5L\sin(0.643 + sL) \\ 5L\cos(0.643 + sL) \end{cases} \qquad p''(s) = \begin{cases} -5L^2\cos(0.643 + sL) \\ -5L^2\sin(0.643 + sL) \end{cases} \Rightarrow \|p''(s)\| = \frac{83}{4}$$

$$\max\|\dot{p}\| = \max\|p'\| \cdot \max|\dot{s}| = 12.113 \cdot 1.102\,T$$

$$\ddot{p} = p''\dot{s}^2 + p'\ddot{s} \Rightarrow \max\|\ddot{p}\| \le \frac{83}{4} \cdot (1.102 \cdot T)^2 + |\ddot{s}| \cdot 12.113 \le 10 \Rightarrow |\ddot{s}| \le 1.7$$

$\Rightarrow$ new b-b profile

$$\Delta s = T \cdot \left(\frac{T}{2}|\ddot{s}|\right)\frac{1}{2} = \frac{T^2}{4} \cdot 1.7 = \frac{17}{40}T^2 = \Delta s = 1 \Rightarrow T^2 = \frac{40}{17} \Rightarrow T = 1.534$$