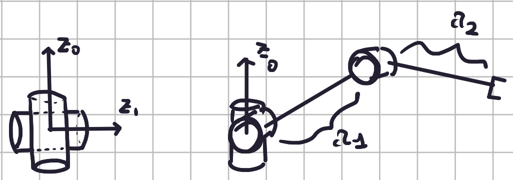
Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise 2

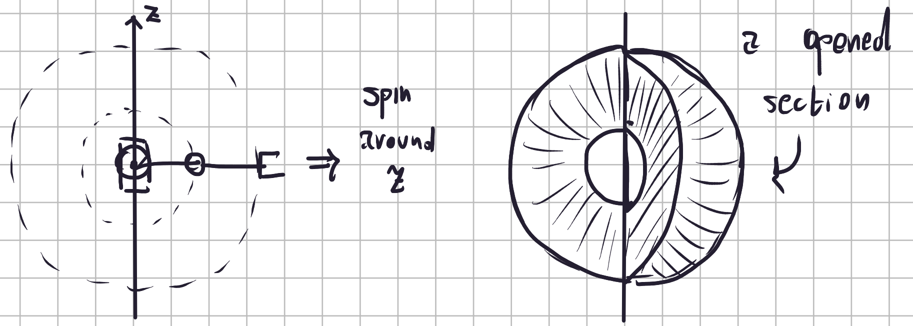
Table 1 contains the Denavit-Hartenberg (D-H) parameters of a robot with three revolute joints.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	θ_1
2	0	$a_2 > 0$	0	θ_2
3	0	$a_3 > 0$	0	θ_3

- Sketch a skeleton of this robot and of its reachable workspace.
- Assign the frames to the robot links according to the above table.
- Compute the 3×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ associated to the velocity of the origin of the last D-H frame and determine all its singularities.
- In a singular configuration $\boldsymbol{\theta}_s$, let $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{\theta}_s)$; find a basis for the subspaces $\mathcal{N}(\boldsymbol{J}_s)$ and $\mathcal{R}(\boldsymbol{J}_s)$ and provide an interpretation in terms of robot motion.



the reachable WS, if we fix q_1 is a ball in $\mathbb{R}^2$ with a hole inside of radius $|a_2 - a_1|$. by letting this shape spin by q_1, we gete a shell (ball with a spherical hole inside)



$$
{}^0T_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad
{}^1T_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad
{}^2T_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad
{}^0T_2 = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 & a_2c_1c_2 \\ s_1c_2 & -s_1s_2 & -c_1 & a_2s_1c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0T_3 = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 & a_2c_1c_2 \\ s_1c_2 & -s_1s_2 & -c_1 & a_2s_1c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow DK = \begin{cases} a_3c_1c_2c_3 - a_3c_1s_2s_3 + a_2c_1c_2 \\ a_3s_1c_2c_3 - a_3s_1s_2s_3 + a_2s_1c_2 \\ a_3s_2c_3 + a_3c_2s_3 + a_2s_2 \end{cases}
$$

to get the DK for the pos, i consider only the last row of ${}^0T_3$

I re-arrange the DK equations:

$$
f_r(q) = \begin{cases} \cos q_1 (a_3 \cos(q_2+q_3) + a_2 \cos q_2) \\ \sin q_1 (a_3 \cos(q_2+q_3) + a_2 \cos q_2) \\ a_3 \sin(q_2+q_3) + a_2 \sin q_2 \end{cases} \Rightarrow
$$

$$
J(q) = \begin{pmatrix} -s_1(a_3c_{23} + a_2c_2) & c_1(-a_3s_{23} - a_2s_2) & -a_3c_1s_{23} \\ c_1(a_3c_{23} + a_2c_2) & s_1(-a_3s_{23} - a_2s_2) & -a_3s_1s_{23} \\ 0 & a_3c_{23} + a_2c_2 & a_3c_{23} \end{pmatrix}
$$

J is singular if the 2 arm are Fully extended: $q_2 = q_3 = 0$:

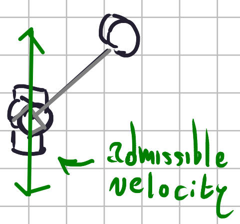
$$J(q_2 = q_3 = 0) = \begin{pmatrix} -s_1(a_3 + a_2) & 0 & 0 \\ c_1(a_3 + a_2) & 0 & 0 \\ 0 & a_3 c_{23} + a_2 c_2 & a_3 c_{23} \end{pmatrix} \Rightarrow \text{rank} = 2$$

Further more, if $a_2 = a_3 = a$ and $q_2 = 0$ and $q_3 = \pi$ we have

$$J(q_2 = 0, q_3 = \pi) = \begin{pmatrix} -s_1(-a_3 + a_2) & 0 & 0 \\ c_1(-a_3 + a_2) & 0 & 0 \\ 0 & -a_3 + a_2 & -a_3 \end{pmatrix} \xrightarrow{a_2 = a_3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a_3 \end{pmatrix}$$

Let's consider this conf. $q_2 = 0$, $q_3 = \pi$ with $a_2 = a_3 = a$

$\Rightarrow J_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, the range is $\mathcal{R}(J_s) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow$ we can achive only speed in the z direction by moving q_3.



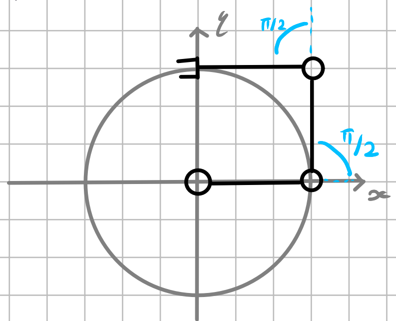
For $\mathcal{N}(J_s)$:
solve $J_s \dot{q} = 0 \Rightarrow \{ -\dot{q}_3 = 0 \Rightarrow \mathcal{N}(J_s) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\Rightarrow$ the robot can't move on the x, y plane.

Exercise 4

For a 3R planar robot with unit length links, can you define a circular path of diameter $d = 1$ m for its end-effector position $\boldsymbol{p} \in \mathbb{R}^2$ so that the robot will certainly trace such path without crossing a singular configuration? If so, provide an example. If not, explain why.

Yes, let's consider the Following initial configuration $q_i = (0, \pi/2, \pi/2)$



by letting Fixed q_2, q_3 and changing q_1 from 0 to 2π the ee Follows the unit circle in the plane.

The DK is $f_r(q) = \begin{cases} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{cases}$, with $q(s) = \begin{cases} q_1(s) = 0 \\ q_2(s) = \pi/2 \\ q_3(s) = 2\pi s \end{cases}$ $s \in [0, 1]$

$$\Rightarrow p(s) = f_r(q(s)) = \begin{cases} \cos(0) + \cos(\pi/2) + \cos(\pi/2 + q_3) \\ \sin(0) + \sin\frac{\pi}{2} + \sin(\frac{\pi}{2} + q_3) \end{cases} = \begin{cases} 1 + \sin(-q_3(s)) \\ 1 + \cos(-q_3(s)) \end{cases}$$

mi son reso conto ora che 1 doveva essere il diametro e non il raggio, quindi rifaccio l'esercizio usando lo stesso ragionamento

I need an initial conf. with $q_1 = 0$ and $f_r(q) = (0, 1/2)$

$$\begin{cases} 1 + C_2 + C_{23} = 0 \\ S_2 + S_{23} = 1/2 \end{cases} \Rightarrow \begin{cases} C_2 + C_{23} = 1 \\ S_2 + S_{23} = 1/2 \end{cases} \Rightarrow c_2 c_{23} + s_2 s_{23} = -\frac{3}{8} \Rightarrow \cos q_3 = -\frac{3}{8}$$

$$\Rightarrow \sin q_3 = \pm\sqrt{1 + (3/8)^2} = \pm\frac{\sqrt{73}}{8} \Rightarrow q_3^+ = 1.908471634$$

$$\Rightarrow 1 + \cos(q_2) + \cos(q_2 + 1.908471634) = 0 \Rightarrow q_2^+ = 1.6611$$

$$\Rightarrow q_i = \begin{pmatrix} 0 \\ 1.6611 \\ 1.9084 \end{pmatrix}, \quad q_f = \begin{pmatrix} 2\pi \\ 1.6611 \\ 1.9084 \end{pmatrix} \Rightarrow q(s) = \begin{pmatrix} 2\pi s \\ 1.6611 \\ 1.9084 \end{pmatrix} \quad s \in [0, 1]$$

$$f_r(q(s)) = \begin{cases} C_1 + C_{12} + C_{123} \\ S_1 + S_{12} + S_{123} \end{cases} = \begin{cases} \cos(2\pi s) + \cos(2\pi s + 1.6611) + \cos(2\pi s + 1.6611 + 1.9084) \\ \sin(2\pi s) + \sin(2\pi s + 1.6611) + \sin(2\pi s + 1.6611 + 1.9084) \end{cases}$$

this is a circle of radius R near 0.5, the error R-0.5 is due to numerical approximation

Exercise 5

Consider a point-to-point path planning problem for a 2R planar robot with unit length links. The robot should move its end-effector between the two Cartesian positions $\boldsymbol{p}_i = (0.6, -0.4)$ and $\boldsymbol{p}_f = (1, 1)$ [m]. Moreover, the Cartesian path should have tangent direction at the start and at the end specified respectively by the vectors $\boldsymbol{p}'_i = (-2, 0)$ and $\boldsymbol{p}'_f = (2, 2)$, where $\boldsymbol{p}' = d\boldsymbol{p}/ds$.

- Define a solution path $\boldsymbol{q}(s)$ directly in the joint space.
- Within the chosen class of interpolating functions, how many solution paths exists? Does any of these paths cross a kinematic singularity?
- On the chosen solution path, define a rest-to-rest timing law $s(t)$ that completes the motion in $T = 3$ s and has continuous acceleration $\ddot{s}(t)$ in the (open) time interval $(0, T)$.
- What is the value of the resulting joint velocity $\dot{\boldsymbol{q}}(t)$ at the midtime $t = T/2$?
- What is the value of the resulting end-effector velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{p}}(t)$ at the midtime $t = T/2$?

To plan directly in the joint space, i need to find q_i, q_f, $\dot{q}_i$ and $\dot{q}_f$ by using the IK and the inverse Jacobian.

$$\text{Since} \quad f_r(q) = \begin{cases} C_1 + C_{12} \\ S_1 + S_{12} \end{cases}$$

Case $p_i = (0.6, -0.4)$

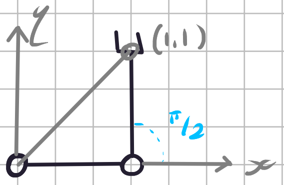
$$\begin{cases} C_1 + C_{12} = 0.6 \\ S_1 + S_{12} = -0.4 \end{cases} \Rightarrow C_1 C_{12} + S_1 S_{12} = -\frac{37}{50} \Rightarrow \cos q_2 = -\frac{37}{50} \Rightarrow \sin q_2 = \pm\sqrt{1 + (\frac{37}{50})^2} = \pm 1.244$$

$$\Rightarrow q_2^+ = \pi + \tan^{-1}\left(\frac{1.244}{-37/50}\right) = 2.1074$$

$$\begin{cases} C_1 + C_{12} = 0.6 \\ S_1 + S_{12} = -0.4 \end{cases} \Rightarrow q_1 = -0.13499 \Rightarrow q_i = \begin{pmatrix} -0.13 \\ 2.107 \end{pmatrix}$$

Case $p_f = (1,1)$)

In this case the IK can be found by geometric reasoning :



$$\Rightarrow q_f = (0, \pi/2)$$

Now i consider the Jacobian matrix :

$$J(q) = \begin{pmatrix} -s_1 - s_{12} & -s_{12} \\ c_1 + c_{12} & c_{12} \end{pmatrix} \Rightarrow J(q_i) = \begin{pmatrix} -0.788 & -0.918 \\ 0.596 & -0.395 \end{pmatrix}, \quad J(q_f) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

the initial and final directions can be scaled (the length dont matter)

$p_i' = (-1, 0) \qquad p_f' = (1, 1)$

$$\Rightarrow \quad q_i' = J^{-1} p_i' = \begin{pmatrix} -0.46 & 1.0694 \\ -0.694 & -0.917 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.46 \\ 0.694 \end{pmatrix}$$

$$q_f' = J^{-1} p_f' = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\Rightarrow$ i have 4 boundary conditions with q_i, q_i', q_f, q_f'. So i plan $q_1(s), q_2(s)$ using two polynomial of degree 3.

With these conditions and a 3-degree polynomial, only 1 solution exists.

q_1)

I have $q_1(0) = -0.13$, $q_1'(0) = 0.46$, $q_1(1) = 0$, $q_1'(1) = 1$

I consider $q_1(s) = as^3 + bs^2 + cs + d \Rightarrow q_1'(s) = 3as^2 + 2bs + c$

$$\begin{cases} d = -0.13 \\ c = 0.46 \\ a + b + c + d = 0 \\ 3a + 2b + c = 1 \end{cases} \Rightarrow \begin{cases} a = 6/5 \\ b = -\frac{153}{100} \end{cases} \Rightarrow q_1(s) = \frac{6}{5}s^3 - \frac{153}{100}s^2 + 0.46s - 0.13$$

q_2)

I have $q_2(0) = 2.107$, $q_2'(0) = 0.694$, $q_2(1) = \frac{\pi}{2}$, $q_2'(1) = -2$

$$\begin{cases} d = 2.107 \\ c = 0.694 \\ a + b + 0.694 + 2.107 = \pi/2 \\ 3a + 2b + 0.694 = -2 \end{cases} \Rightarrow \begin{cases} a = -0.233 \\ b \approx -1 \end{cases} \Rightarrow q_2(s) = -0.233s^3 - s^2 + 0.694s + 2.107 \qquad s \in [0, 1]$$

instead of s we consider $s = \frac{\sigma}{L}$ and $\sigma \in [0, L]$

To define a rest-to-rest that completes the path in 3 seconds i consider a trapezoidal profile for $\dot{s}(t)$. $s(0)=0$, $s(T)=1$. The length of the path is $L = \| (1,1)^T - (0.6,-0.4)^T \| = \sqrt{53}/5 \approx 1.456$

For the b-c-b profile we have that: $T = (LA + V^2)(AV)^{-1} \Rightarrow$

$3 = \left(\frac{\sqrt{53}}{5}A + V^2\right)(AV)^{-1}$ we have no limits so i choose $A = 5$, $V = 0.5021$

$\Rightarrow T_s = 0.10042$

$$\Rightarrow \sigma(t) = \begin{cases} 5/2\, t^2 & t \in [0, 0.10042] \\ 0.5021t - 0.0252 & t \in [0.10042, 2.89958] \\ -\frac{5}{2}(t-3)^2 + \sqrt{53}/5 & t \in [2.89958, 3] \end{cases}$$

At $t = T/2 \Rightarrow \sigma(T/2) = 0.72795 \Rightarrow$

$\dot{q}(t) = \frac{dq}{d\sigma} \cdot \frac{d\sigma}{dt}$ For each component.

$$\dot{q}_1(\sigma) = \left(\frac{18}{5} \cdot \frac{25}{53}\sigma(t)^2 - \frac{153}{50}\sigma(t) + 0.46\right) \cdot 0.5021 = 0.852\sigma^2 - 1.536\sigma + 0.23$$

$$\dot{q}_2(\sigma) = \left(-\frac{699}{100} \cdot \frac{25}{53}\sigma^2 - \frac{4}{\sqrt{53}}\sigma + 0.694\right) \cdot 0.5021 = -1.655\sigma^2 - 0.275\sigma + 0.348$$

$$\dot{q}_1(T/2) = \dot{q}_1(\sigma(T/2)) = -0.436$$
$$\dot{q}_2(T/2) = \dot{q}_2(\sigma(T/2)) = -0.726$$
$$\Rightarrow \dot{q}(T/2) = \begin{pmatrix} -0.436 \\ -0.726 \end{pmatrix}$$

Now i need $q(T/2)$.

$$q_1(\sigma(T/2)) = -0.02$$
$$q_2(\sigma(T/2)) = -0.233 \cdot \left(\frac{0.727}{1.456}\right)^3 - \left(\frac{0.727}{1.456}\right)^2 + 0.69 \cdot 0.727 + 2.107 = 2.33$$

$$\Rightarrow J(q(\sigma(T/2))) = \begin{pmatrix} -0.719 & -0.73 \\ 0.326 & -0.673 \end{pmatrix} \Rightarrow \dot{p}(T/2) = \begin{pmatrix} -0.719 & -0.73 \\ 0.326 & -0.673 \end{pmatrix} \begin{pmatrix} -0.436 \\ -0.726 \end{pmatrix} = \begin{pmatrix} 0.804 \\ 0.346 \end{pmatrix}$$