Question #1

A rigid body is rotated first by an angle $\theta = \pi/3$ around the unit vector $\boldsymbol{r} = (1/\sqrt{3}) \cdot \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ and then by an angle $\phi = -\pi/3$ around the fixed $\boldsymbol{y}$-axis. What is the final orientation of the body?

Since the rotation of ϕ is performed on the First y axis, must be pre-multiplied to $R(\theta, r)$.

$$R(\theta,r) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} + \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \cdot \frac{1}{2} + \begin{bmatrix} 0 & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} \\ -1/\sqrt{3} & 1/\sqrt{3} & 0 \end{bmatrix} \cdot \frac{\sqrt{3}}{2} =$$

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} + \begin{bmatrix} 1/3 & -1/6 & -1/6 \\ -1/6 & 1/3 & -1/6 \\ -1/6 & -1/6 & 1/3 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}$$

$$R = R_y(\phi) R(\theta,r) = \begin{bmatrix} 1/2 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 0.622 & -0.744 & -0.244 \\ 0.666 & 0.666 & -0.333 \\ 0.4106 & 0.0446 & 0.9106 \end{bmatrix}$$

Question #2

An initial orientation $\boldsymbol{R}_i$ and a final orientation $\boldsymbol{R}_f$ are defined by

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 0.5 & -\sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0.5 \end{pmatrix}, \qquad \boldsymbol{R}_f = \boldsymbol{I}.$$

Find the two sequences of ZYZ Euler angles that represent the rotation from $\boldsymbol{R}_i$ to $\boldsymbol{R}_f$.

We know that $${}^{i}R_f = {}^{i}R_0 \, {}^{0}R_f = {}^{0}R_i^T \, {}^{0}R_f = R_i^T = \begin{bmatrix} 0 & -1 & 0 \\ 1/2 & 0 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix}$$

The ZYZ euler angles MAT is

$$R_z(\alpha) R_y(\beta) R_z(\gamma) = \begin{bmatrix} -s\alpha s\gamma + c\alpha c\beta c\gamma & -s\alpha c\gamma - s\gamma c\alpha c\beta & s\beta c\alpha \\ s\alpha c\beta c\gamma + s\gamma c\alpha & -s\alpha s\gamma c\beta + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

I solve ${}^{i}R_f = R_z\alpha \, R_y\beta \, R_z\gamma$ For α, β, γ :

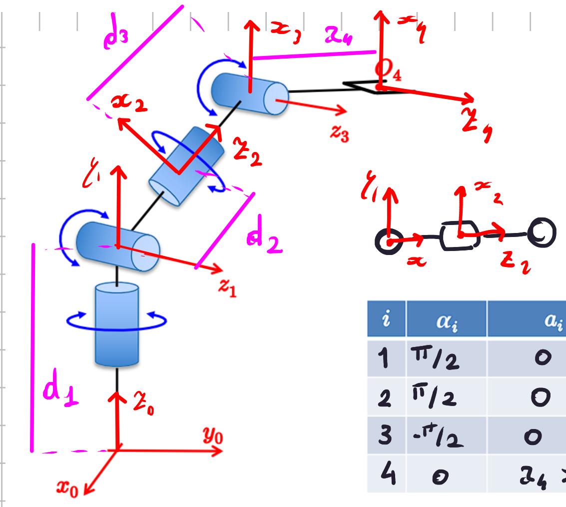
$$\cos\beta = 1/2 \Rightarrow \sin\beta = \pm \sqrt{3}/2 \Rightarrow \beta = \pm \frac{\pi}{3}$$

$$\begin{cases} -s\beta c\gamma = -\sqrt{3}/2 \\ s\beta s\gamma = 0 \end{cases} \Rightarrow \begin{cases} \mp \sqrt{3}/2 \, c\gamma = -\sqrt{3}/2 \\ \pm \sqrt{3}/2 \, s\gamma = 0 \end{cases} \Rightarrow \begin{cases} \cos\gamma = \pm 1 \\ \sin\gamma = 0 \end{cases} \Rightarrow \gamma = \begin{cases} 0 & \text{if } \beta = \pi/3 \\ \pi & \text{if } \beta = -\pi/3 \end{cases}$$

$$\begin{cases} s\beta c\alpha = 0 \\ s\beta s\alpha = \sqrt{3}/2 \end{cases} \Rightarrow \begin{cases} \pm \frac{\sqrt{3}}{2} c\alpha = 0 \\ \pm \sqrt{3}/2 \, s\alpha = \sqrt{3}/2 \end{cases} \Rightarrow \begin{cases} \cos\alpha = 0 \\ \sin\alpha = \pm 1 \end{cases} \Rightarrow \alpha = \begin{cases} \pi/2 & \text{if } \beta = \pi/3 \\ -\pi/2 & \text{if } \beta = -\pi/3 \end{cases} \Rightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} \\ \frac{\pi}{3} \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} -\frac{\pi}{2} \\ -\pi/3 \\ \pi \end{pmatrix}$$

Question #3

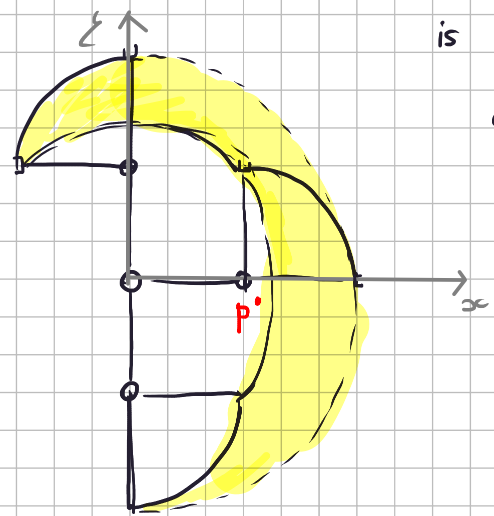
For the 4R robot with a spherical shoulder of Fig. 1, complete the assignment of Denavit-Hartenberg (D-H) frames and fill in the associated table of parameters *[for this, use the extra sheet distributed]*. Keep the quantities that are already defined in the figure unchanged. If needed, provide the transformation between the last D-H frame and the standard frame of an end-effector gripper.



i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1>0$	q_1
2	$\pi/2$	0	$d_2>0$	q_2
3	$-\pi/2$	0	$d_3>0$	q_3
4	0	$a_4>0$	0	q_4

Question #4

A 2R planar robot with links of equal length L has limited joint ranges as follows: $q_1 \in [-\pi/2, \pi/2]$, $q_2 \in [0, \pi/2]$. Draw the primary workspace $WS_1 \in \mathbb{R}^2$. For $L = 1.4$ [m], is the point $P = (1.6, -0.2)$ reachable by the robot end effector?



is P reachable? I solve the IK:

$\cos q_2 = -0.3367 \Rightarrow \sin q_2 = \pm 0.94161$

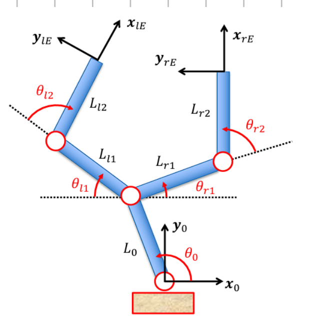
$$\Rightarrow q_2 = \begin{cases} \mathrm{atan2}\{0.94161, -0.3367\} = 1.9142 \notin [0, \pi/2] \\ \mathrm{atan2}\{-0.94161, -0.3367\} = -1.9142 \notin [0, \pi/2] \end{cases}$$

$\Rightarrow$ P is not reachable

Question #5

A branched two-arm planar robot having 5 dofs is sketched in Fig. 2, with generic labels for the link lengths and the actual definition of the joint angles. The sign convention for angles is the usual one (i.e., positive if counterclockwise). Determine the relative pose of the end-effector frame of the left arm with respect to that of the right arm, as expressed by the 4×4 homogeneous matrix ${}^{rE}\boldsymbol{T}_{lE}(\boldsymbol{q})$ with $\boldsymbol{q} = (\theta_0, \theta_{r1}, \theta_{r2}, \theta_{l1}, \theta_{l2})$. Check numerically the obtained symbolic expression when all the links have equal and unitary length and the two-arm robot is in the configuration $\boldsymbol{q}^* = (\pi/2, 0, 0, -\pi/2, 0)$ —the right arm is horizontal and the left one is vertical and upward.

I have to determine ${}^0T_{rE}$ and ${}^0T_{lE}$.



Let's denote ϕ_L and ϕ_R the angle between x_0 and x_{LE} and x_{RE}.

$\Rightarrow \phi_L = \pi - \theta_{l1} + \theta_{l2}$

$\phi_R = \theta_{r1} + \theta_{r2}$

$$\begin{cases} p_x^R = L_0\cos\theta_0 + L_{r1}\cos\theta_{r1} + L_{r2}\cos(\theta_{r1}+\theta_{r2}) \\ p_y^R = L_0\sin\theta_0 + L_{r1}\sin\theta_{r1} + L_{r2}\cos(\theta_{r1}+\theta_{r2}) \end{cases}$$

$$\begin{cases} p_x^L = L_0 \cos\theta_0 + L_{l1}\cos(\pi - \theta_{l1}) + L_{l2}\cos(\pi - \theta_{l1} + \theta_{l2}) \\ p_y^L = L_0 \sin\theta_0 + L_{l1}\sin(\pi - \theta_{l1}) + L_{l2}\sin(\pi - \theta_{l1} + \theta_{l2}) \end{cases}$$

$$\Rightarrow {}^0T_{LE} = \begin{bmatrix} \cos(\pi-\theta_{l1}+\theta_{l2}) & -\sin(\pi-\theta_{l1}+\theta_{l2}) & 0 & L_0\cos\theta_0 + L_{l1}\cos(\pi-\theta_{l1}) + L_{l2}\cos(\pi-\theta_{l1}+\theta_{l2}) \\ \sin(\pi-\theta_{l1}+\theta_{l2}) & \cos(\pi-\theta_{l1}+\theta_{l2}) & 0 & L_0\sin\theta_0 + L_{l1}\sin(\pi-\theta_{l1}) + L_{l2}\sin(\pi-\theta_{l1}+\theta_{l2}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_{RE} = \begin{bmatrix} \cos(\theta_{r1}+\theta_{r2}) & -\sin(\theta_{r1}+\theta_{r2}) & 0 & L_0\cos\theta_0 + L_{r1}\cos(\theta_{r1}) + L_{r2}\cos(\theta_{r1}+\theta_{r2}) \\ \sin(\theta_{r1}+\theta_{r2}) & \cos(\theta_{r1}+\theta_{r2}) & 0 & L_0\sin\theta_0 + L_{r1}\sin(\theta_{r1}) + L_{r2}\sin(\theta_{r1}+\theta_{r2}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow {}^{rE}T_{LE} = {}^0T_{rE}^{-1}\,{}^0T_{Le}$ For unitary link and $q^* = (\frac{\pi}{2}, 0, 0, -\frac{\pi}{2}, 0)$
$\theta_0 \quad \theta_{r1} \quad \theta_{r2} \quad \theta_{L1} \quad \theta_{L2}$

$$= {}^{rE}T_{LE}(q^*) = \begin{bmatrix} 0 & 1 & 0 & -2 \\ -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question #6

Figure 3 shows a planar RRP robot, with the definition of its joint variables. The task of interest is specified by the position $\boldsymbol{p} = (p_x, p_y)$ of the robot end effector and by the orientation α of the forearm w.r.t. the $\boldsymbol{x}$-axis. The associated direct kinematics is

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} l_1 c_1 + q_3 c_{12} \\ l_1 s_1 + q_3 s_{12} \\ q_1 + q_2 \end{pmatrix} = \boldsymbol{f}_r(\boldsymbol{q}).$$

Determine the analytic solutions to the inverse kinematics problem. Disregard any situation that is unfeasible or singular. Provide at least one solution for the following (feasible) input data: $l_1 = 1$ [m], $\boldsymbol{r}_d = (2, 1, \pi/6)$ [m,m,rad].

I denote $\cos\alpha = u$ and $\sin\alpha = v$

$$\begin{cases} p_x = L_1 \cos q_1 + q_3 u \\ p_y = L_1 \sin q_1 + q_3 v \end{cases} \Rightarrow (p_x - q_3 u)^2 + (p_y - q_3 v)^2 = L_1^2 \Rightarrow \text{solvable for } q_3$$

After knowing q_3 we solve

$$\begin{cases} p_x = L_1 \cos q_1 + q_3 u \\ p_y = L_1 \sin q_1 + q_3 v \end{cases}$$ For $\cos q_1$ and $\sin q_1$ and find q_1 by atan2.

$\Rightarrow$ in the end $q_2 = \alpha - q_1$.

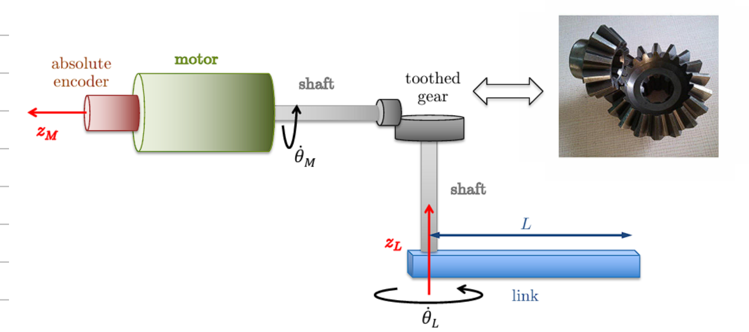
Let $L_1 = 1$, $p_x = 2$ $p_y = 1$ $\alpha = \frac{\pi}{6}$, $u = \frac{\sqrt{3}}{2}$ $v = \frac{1}{2}$

$(p_x - q_3 u)^2 + (p_y - q_3 v)^2 = L_1^2 \Rightarrow q_3 = 1.241$ (solved numerically)

$$\Rightarrow \begin{cases} p_x = L_1 \cos q_1 + q_3 u \\ p_y = L_1 \sin q_1 + q_3 v \end{cases} \Rightarrow \begin{cases} 2 = \cos q_1 + 1.241 \cdot \frac{\sqrt{3}}{2} \Rightarrow \cos q_1 = 0.925 \\ 1 = \sin q_1 + 1.241 \cdot \frac{1}{2} \Rightarrow \sin q_1 = 0.3795 \end{cases} \Rightarrow q_1 = 0.38 \quad q_2 = 0.14$$

Question #7

With reference to Fig. 4, a motor with inertia J_M drives a link through a gear with toothed wheels (a photo of this is also shown in the figure). The wheel on the motor shaft (aka, the *pinion*) has radius $r_M = 2$ [cm], while the radius of the wheel on the link rotation axis is $r_L = 10$ [cm]. The link has inertia $J_L = 0.3$ [kgm^2] around its rotation axis. Assuming that an optimal inertia matching is realized by the reduction ratio of this transmission, determine the torque τ_M that the motor needs to produce around its z_M axis in order to accelerate the link at $\ddot{\theta}_L = -5$ [rad/s^2]. Neglect dissipative effects as well as the inertia of the transmission components (and of the encoder).



The reduction ratio is $n_r = \frac{10}{2} = 5$ and $\theta_m = 5\theta_L$. If we assume optimal inertia matching, we have

$$5 = \sqrt{\frac{0.3}{J_m}} \Rightarrow J_m = 0.012 \ \mathrm{Kg\,m^2},$$

$$\Rightarrow \tau_m = \left(0.012 \cdot 5 + \frac{0.3}{5}\right) \cdot (-5) = -\frac{3}{5} \ \mathrm{Nm}$$

Question #8

An absolute encoder is mounted on the motor of the system shown in Fig. 4. If the link length is $L = 0.5$ [m], determine the minimum number of tracks n_t that the encoder needs to have in order to achieve at least a resolution of $\delta = 0.1$ [mm] at the link tip.

A res. of $\delta = 0.1$ mm $= 0.0001$ m is of 0.011 degrees by solving $\frac{2\pi L}{360} = \frac{\delta}{\Delta\theta_L} \Rightarrow \Delta\theta_L = 0.011°$ the encoder is on the motor. To detect $\Delta\theta_L$ on the link it should detect $\Delta\theta_m$ on the motor: $\Delta\theta_m = 5 \cdot \Delta\theta_L = 0.055$ degrees. Since the res. is $\Delta\theta_m = \frac{360}{2^{N_t}} \Rightarrow$ We have that $N_t = 13$ since $N_t = \left\lceil \log\left(\frac{360}{0.055}\right) \right\rceil$