Since I have already taken the midterm, I will only be completing the exercises related to the second part of the course (from differential kinematics onwards).

Exercise 4

For a 4-dof robot, consider the task vector

$$r = f(q) = \begin{pmatrix} q_2 \cos q_1 + q_4 \cos(q_1 + q_3) \\ q_2 \sin q_1 + q_4 \sin(q_1 + q_3) \\ q_1 + q_3 \end{pmatrix}. \quad (2)$$

Determine all singular configurations for the corresponding analytic robot Jacobian $J(q)$. Moreover, find *if possible*:

- a joint velocity $\dot{q}_0 \neq 0$ such that $\dot{r} = 0$ when the robot is in a regular configuration;
- all joint velocities $\dot{q}$ such that $\dot{r} = 0$ when the robot is in a singular configuration;
- the direction(s) along which no task velocity can be realized when the robot is in the chosen singular configuration;
- a generalized task force $f_0 \neq 0$ that is statically balanced by the joint torque $\tau = 0$ when the robot is in a regular configuration;
- all generalized task forces f that can be statically balanced by zero joint torque when the robot is in the chosen singular configuration.

The analytic Jacobian is

$$J(q) = \begin{pmatrix} -q_2 s_1 - q_4 s_{13} & c_1 & -q_4 s_{13} & c_{13} \\ q_2 c_1 + q_4 c_{13} & s_1 & q_4 c_{13} & s_{13} \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The singular conf. happens if $q_3 = 0$ or $q_3 = \pi$, in such case, the columns 2 and 4 are linearly dependent. This happens also if $q_2 = 0$, in such case the col. 1 and 3 are lin. dep.

In a regular conf. such $q^* = (q_1 = 0,\ q_2 = 1,\ q_3 = \pi/2,\ q_4 = 1)$ we have:

$$J(q^*) = J^* = \begin{pmatrix} -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$ and $\operatorname{Ker} J^*$ is given by $x : J^* x = 0 \Rightarrow$

$$J^* \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{cases} -x_1 + x_2 - x_3 = 0 \\ x_1 + x_4 = 0 \\ x_1 + x_3 = 0 \end{cases} \quad \begin{cases} x_1 = -x_3 \\ x_2 = 0 \\ x_3 = x_4 \end{cases} \Rightarrow \dot{q} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$ realize $\dot{r} = 0$:

$$\dot{r} = \begin{pmatrix} -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now i consider a singular conf. $q_s = (0\ 0\ 0\ 0)^T \Rightarrow$

$$J_s = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \text{i study } \operatorname{Ker} J_s \Rightarrow J_s x = 0 \Rightarrow \begin{cases} x_2 + x_4 = 0 \\ x_1 + x_3 = 0 \end{cases} \Rightarrow \operatorname{Ker} J_s = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Regular conf. q^*: a force f_0 balances $\tau = 0$ if $J^{*T} f_0 = 0 \Rightarrow$

$$\operatorname{Ker} J^{*T} \Rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} x = 0 \Rightarrow \begin{cases} -x_1 + x_2 + x_3 = 0 \\ x_1 = 0 \\ -x_1 + x_3 = 0 \\ x_1 + x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \operatorname{Ker} J^{*T} = \{0\} \Rightarrow \begin{cases} f_0 \neq 0 \text{ s.t.} \\ J^{*T} f_0 = 0 \\ \text{doesn't exists.} \end{cases}$$

Now i study $\operatorname{Ker} J_s^T$:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} x = 0 \Rightarrow \begin{cases} x_3 = 0 \\ x_1 = 0 \end{cases} \Rightarrow \operatorname{Ker} J_s^T = \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

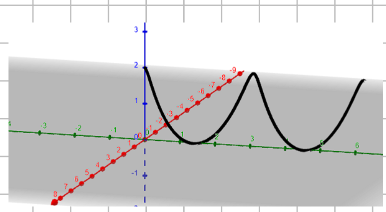
Exercise 5

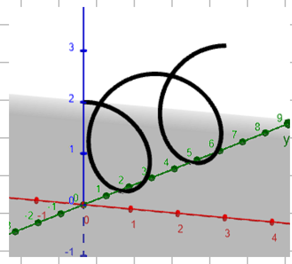
The end-effector of a robot manipulator should follow an helical path $\boldsymbol{p} = \boldsymbol{p}(s)$, parametrized by the scalar $s \geq 0$. The helix is right-handed, with radius $r = 0.4$ m and pitch $2\pi h$, with $h = 0.3$ m, starting from the position $\boldsymbol{p}_0 = (0, 0, 2r)$ at $s = 0$. Its axis passes through the point $C = (0, 0, r)$ and is parallel to the $\boldsymbol{y}$-axis. In the time interval $t \in [0, T]$, the robot end-effector should trace two complete turns of the helix, starting and ending its (rest-to-rest) motion with zero velocity, i.e., with $\dot{\boldsymbol{p}}(0) = \dot{\boldsymbol{p}}(T) = \boldsymbol{0}$.

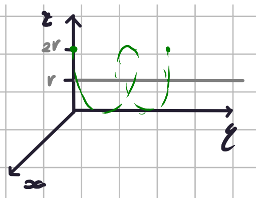
Plan a timing law $s = s(t)$ that minimizes the motion time T under the following bounds on the norm of the velocity and on the (absolute) tangential and normal accelerations,

$$\|\dot{\boldsymbol{p}}\| \leq V, \qquad |\ddot{\boldsymbol{p}}^T \boldsymbol{t}| \leq A, \qquad |\ddot{\boldsymbol{p}}^T \boldsymbol{n}| \leq A, \tag{3}$$

where $\boldsymbol{t} = \boldsymbol{t}(s)$ and $\boldsymbol{n} = \boldsymbol{n}(s)$ are the unit axes of the Frenet frame tangent and normal to the path. Determine the minimum time T^* when $V = 2$ m/s and $A = 4.5$ m/s^2. Sketch the profiles of $s(t)$, $\dot{s}(t)$ and $\ddot{s}(t)$ in the obtained time-optimal solution.







$$p(s) = \begin{cases} r\sin(4\pi s) \\ 2\pi h \cdot s \\ r + r\cos(4\pi s) \end{cases}$$

I want to plan a b-c-b profile. i need the length of the curve.

$$p'(s) = \frac{dp}{ds} = \begin{cases} 4\pi r \cdot \cos(4\pi s) \\ 2\pi h \\ -4\pi r \cdot \sin(4\pi s) \end{cases} \qquad \|p'(s)\| = \left(16\pi^2 r^2 \cdot \cos^2(4\pi s) + 16\pi^2 r^2 \cdot \sin^2(4\pi s) + 4\pi^2 h^2\right)^{1/2} =$$

$$= \left(16\pi^2 r^2 + 4\pi^2 h^2\right)^{1/2} = \sqrt{4\pi^2(4r^2 + h^2)} \approx 5.368$$

$$\Rightarrow L = \int_0^1 \|p'(s)\| ds = \int_0^1 5.368\, ds = 5.368$$

The tangent vector $t(s)$ is given by $\frac{p'(s)}{\|p'(s)\|} = \frac{1}{L}\begin{pmatrix} 4\pi r \cdot \cos(4\pi s) \\ 2\pi h \\ -4\pi r \cdot \sin(4\pi s) \end{pmatrix} = \begin{pmatrix} 0.936\cos(4\pi s) \\ 0.352 \\ -0.936\sin(4\pi s) \end{pmatrix}$

the normal vector $n(s)$ is $\frac{t'(s)}{\|t'(s)\|}$.

$$t'(s) = \begin{pmatrix} -4\pi \cdot 0.936\sin(4\pi s) \\ 0 \\ -4\pi \cdot 0.936\cos(4\pi s) \end{pmatrix} = \begin{pmatrix} -11.762 \cdot \sin(4\pi s) \\ 0 \\ -11.762 \cdot \cos(4\pi s) \end{pmatrix} \Rightarrow \|t'(s)\| = 11.762 \Rightarrow n(s) = \begin{pmatrix} -\sin(4\pi s) \\ 0 \\ -\cos(4\pi s) \end{pmatrix}$$

Now, i have to find the bounds on the timing law $s(t)$ that dont overflow the bounds on $\dot{p}$ and $\ddot{p}$.

$$\Rightarrow \|\dot{p}\| \leq V \Rightarrow \left\|\frac{dp}{ds}\frac{ds}{dt}\right\| \leq V \Rightarrow \|p'\| \cdot |\dot{s}| \leq V \Rightarrow |\dot{s}| \leq \frac{V}{5.368}$$

Now i compute $\ddot{p}(s)^T t(s)$. $\ddot{p}(s) = p''\dot{s}^2 + p'\ddot{s} = \dot{s}^2\begin{pmatrix} -63.165 \cdot \sin(4\pi s) \\ 0 \\ -63.165 \cdot \cos(4\pi s) \end{pmatrix} + \ddot{s}\begin{pmatrix} 5.026 \cdot \cos(4\pi s) \\ 2\pi h \\ -5.026 \cdot \sin(4\pi s) \end{pmatrix} =$

$$\left(\dot{s}^2\begin{pmatrix} -63.165 \cdot \sin(4\pi s) \\ 0 \\ -63.165 \cdot \cos(4\pi s) \end{pmatrix} + \ddot{s}\begin{pmatrix} 5.026 \cdot \cos(4\pi s) \\ 2\pi h \\ -5.026 \cdot \sin(4\pi s) \end{pmatrix}\right)^T t(s) = \ddot{s}\left(4.704\cos^2(4\pi s) + 0.663 + 4.704\sin^2(4\pi s)\right) = 5.367\ddot{s}$$

$$\Rightarrow |5.367\ddot{s}| \leq A \Rightarrow |\ddot{s}| \leq \frac{A}{5.367}$$

Now i have to consider $\ddot{p}^T n$.

$$\left(\dot{s}^2\begin{pmatrix}-63.165\cdot\sin(4\pi s)\\ 0\\ -63.165\cdot\cos(4\pi s)\end{pmatrix}+\ddot{s}\begin{pmatrix}5.026\cdot\cos(4\pi s)\\ 2\pi h\\ -5.026\cdot\sin(4\pi s)\end{pmatrix}\right)^T n(s)=$$

$$\left(\dot{s}^2\begin{pmatrix}-63.165\cdot\sin(4\pi s)\\ 0\\ -63.165\cdot\cos(4\pi s)\end{pmatrix}+\ddot{s}\begin{pmatrix}5.026\cdot\cos(4\pi s)\\ 2\pi h\\ -5.026\cdot\sin(4\pi s)\end{pmatrix}\right)^T\begin{pmatrix}-\sin(4\pi s)\\ 0\\ -\cos(4\pi s)\end{pmatrix}=\dot{s}^2\cdot 63.165$$

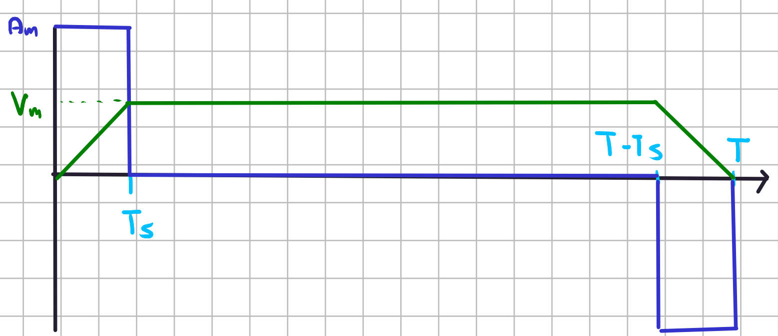
$$|\dot{s}^2\, 63.165|\leq A \Rightarrow \dot{s}^2\leq\frac{A}{63.165}\Rightarrow|\dot{s}|\leq 0.1258\sqrt{A}$$

the bounds for $\dot{s}$ is $V_{max}=\min\left\{0.1258\sqrt{A},\ \frac{V}{5.368}\right\}$ for $\ddot{s}$ is $\frac{A}{5.368}$

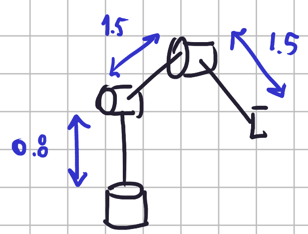
If $V=2$ and $A=4.5$ i get $V_{max}=0.2668$ and $A_{max}=0.838$

For a b-c-b profile, from the equations i get: $T_s=\frac{v_{max}}{a_{max}}$, $T=\frac{La_{max}+v_{max}^2}{a_{max}v_{max}}$

$T=20.438$ [sec]. $T_s=0.3183$



the WS_1 of the elbow type is a sphere of radius 3 [m] centered at 0.8 [m] from the base.



the analytical Jacobian of that manipulator is

$$J=\begin{pmatrix}-1.5s_1(c_2+c_{23}) & -1.5c_1(s_1+s_{23}) & -1.5c_1s_{23}\\ 1.5c_1(c_2+c_{23}) & 1.5s_1(s_2+s_{23}) & 1.5s_1s_{23}\\ 0 & 1.5(c_2+c_{23}) & 1.5c_{23}\end{pmatrix}$$

We have $\det J=-3.375\,s_3(c_2+c_{23})\Rightarrow J$ is singular if $q_3\in\{0,\pi\}$
to avoid the singularities, i choose a location for the base s.t. $p(s)$ is always inside WS_1 and never on the frontier.

Since the pitch is $2\pi h$, if i choose $(x_b, y_b)=(0, \pi h)$, the farest point on $p(s)$ from $(x_b, y_b, 0)$ is $p(0)$ and $p(1)$.

$p(0)=(0,0,2r)$. Is this inside WS_1?

$$WS_1=\left\{v:\ \|v-(0,\pi h,0.8)^T\|\leq 3\right\}\Rightarrow\|(0,-\pi h,2r-0.8)^T\|=\frac{3}{10}\pi<3\Rightarrow$$ the robot never occurs in singul.