1 Introduction

The following is a derivation of the equations that describe the voltage across a capacitor while it is charging or discharging. The steps are intended to be very explicit and show each step.

A capacitor is a simple devices that uses two plates that are separated by a small distance. Current does not flow through a capacitor like it does a resistor. In the case of a resistor, as current flows due to a DC voltage, the electrons that enter the resistor on one side will be the same as the electrons that exit the other side. For a capacitor, electrons will enter via a terminal and then will get trapped on one of the plates. As the two plates are not electrically connected the electron cannot transfer across. Instead, electrons that are on the other plate will be pulled off and will travel through the second terminal and along to the rest of the circuit. This causes a build up of electrons on one plate and loss of electrons on the other. Each plate will have an equal quantity of charge, q, but one plate will have a positive charge and the other will have a negative charge. The capacitance of a capacitor can be calculated if the amount of charge, q, is known and the resulting voltage, V_c, is measured. Then, the capacitance of the capacitor, C is calculated as

$$C = \frac{q}{V_c}. \tag{1}$$

2 Charging Capacitor

The case of the charging capacitor is very simple. There is a power supply which is set to output a current, I, at V_s volts. The current flows from the power supply to the resistor, with resistance R. The voltage drop across the resistor is simply IR. The current then flows to the capacitor which has a voltage of V_c across the two pins. Finally, the capacitor is connected to the power supply.

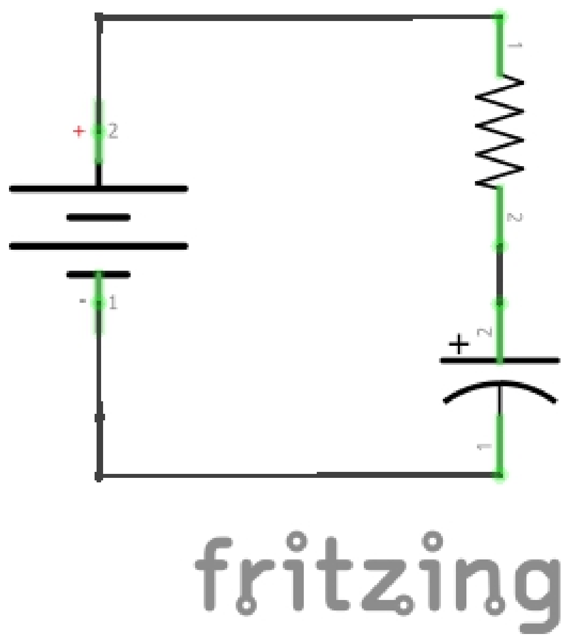

Figure 1: Circuit for a charging capacitor. Current flows from the power supply, through the resistor, to the capacitor and finally back to the power supply.

Kirchhoff's loop rule can be used to create an equation that describes our circuit. Kirchhoff's loop rule simply states that the sum of all the voltages around any closed loop will be zero, mathematically $\sum_n V_n = 0$. Some care is required as we add up the terms to consider if the voltage is higher or lower after the current flows through the component. The power supply is going to increase the voltage by V_s volts, the resistor will decrease the voltage by $V_r = IR$ volts and the capacitor will also decrease the voltage by V_c volts. Using the Kirchhoff's loop rule we end up with

$$0 = V_s - IR - V_c. \tag{2}$$

After reordering the terms, this becomes

$$V_s = IR + V_c. \tag{3}$$

Consider an imaginary plate that pass through a wire. If we count the total amount of charge, dq, that passes through the plate is some unit of time, dt, we can calculate the current, I, flowing as the amount of current to pass through the imaginary plate per unit time,

$$\begin{aligned} I &= \frac{dq}{dt} \\ &= C\frac{dV_c}{dt} \end{aligned} \tag{4}$$

The second line using the equation used to calculate capacitance, Eq. 1, to solve for the current flowing as a function of the voltage across the capacitor. This is done by solving Eq. 1 for q and substituting it into the first line. C is pulled outside because it is a constant with respect to dt.

Substituting the second line of Eq. 4 into Eq. 3 and be obtain

$$V_s = RC\frac{dV_c}{dt} + V_c. \tag{5}$$

This is starting to look a bit intimidating due to the Calculus terms involved but don't worry, they will be paired up with some even scarier integral signs soon. In the mean time, let's reorganize this equation to get all the voltage terms on one side and everything else on the other,

$$\frac{dt}{RC} = \frac{dV_c}{V_s - V_c}. \tag{6}$$

The equation is not in a form that we can perform an integral on it. When we perform integration we first need to evaluate the integral itself and then use two boundary limits to get the integral from the starting limit to the end limit. In this case we will say that the capacitor starts to charge at time $t = 0$ and we want to know what the voltage across the capacitor is at some arbitrary time $t = t'$. This gives us our integration limits and thus the integral looks like

$$\frac{1}{RC}\int_0^{t'} dt = \int_0^{t'} \frac{dV_c}{V_s - V_c}. \tag{7}$$

The terms RC can be pulled out of the integral on the left hand side, LHS, as they are constant and independent of time, dt. On the right hand side, RHS, V_s is independent of dV_c but it is in the denominator with V_c which is related to dV_c and thus that entire denominator term is stuck in the integral.

2.1 RHS

Because we are dashing and daring individuals, lets deal with the right hand side, RHS, of the integral first. This will be done using a simple substitution,

$$V_s - V_c = w \tag{8}$$

and

$$dV_c = -dw. \tag{9}$$

The second part of the integration used to determine how a small change of V_c, which is just dV_c, is related to a small change of w, which is dw.

Making the substitution into the integral,

$$\begin{aligned} \int_0^{t'} \frac{dV_c}{V_s - V_c} &= \int_0^{t'} \frac{-dw}{w} \\ &= -ln(w)\Big|_{t=0}^{t'}. \end{aligned} \tag{10}$$

Now that the integration has been performed we can undo our substitution to get the solution,

$$-ln(w)\Big|_{t=0}^{t'} = -ln(V_s - V_c)\Big|_{t=0}^{t'}. \tag{11}$$

Now we need to evaluate at the limits. The voltage across the capacitor at $t = 0$ is $V_c = 0$ since this is the moment in time when the capacitor starts to charge. As a result, $-ln(V_s - V_c) = -ln(V_s)$ at $t = 0$. At $t = t'$ the capacitor has been charging for some time and as a result it will have a voltage of V_c across the leads and thus we get $-ln(V_s - V_c)$. During integration, evaluation at the limits is always done as "final minus initial," or in this case, the result for $t = t'$ minus the result for $t = 0$,

$$
\begin{aligned}
-ln(V_s - V_c)\bigg|_{t=0}^{t'} &= -ln(V_s - V_c) + ln(V_s) \\
&= -ln\left(\frac{V_s - V_c}{V_S}\right)
\end{aligned}
\tag{12}
$$

The second line of this is the solution that we are after.

2.2 LHS

Now we need to deal with the LHS of Eq. 7. This integral is very easy since $\int dt = t$, thus

$$
\frac{1}{RC}\int_0^{t'} dt = \frac{1}{RC}t\bigg|_{t=0}^{t'}. \tag{13}
$$

Evaluating at the limits proves easy as well,

$$
\frac{1}{RC}\ t\bigg|_{t=0}^{t'} = \frac{t}{RC}. \tag{14}
$$

And with that, the LHS integration is complete.

2.3 Putting it all together

Now that both integrations have been performed, we can set the results of Eq. 12 equal to Eq. 14,

$$
-ln\left(\frac{V_s - V_c}{V_s}\right) = \frac{t}{RC}. \tag{15}
$$

We want to solve for V_c which is in the logarithm. First we will move the negative to the other side by multiplying through by -1 and then we will exponentiate each side,

$$
\frac{V_s - V_c}{V_s} = e^{-t/RC}. \tag{16}
$$

The last thing that needs to be done is to use algebra to solve for V_c,

$$\begin{aligned} V_c &= V_s - V_s e^{-t/RC} \\ &= V_s \left(1 - e^{-t/RC}\right) \\ &= V_s \left(1 - e^{-t/\tau}\right). \end{aligned} \tag{17}$$

The last line is written using the time constant of the circuit which is defined as $\tau = RC$.

2.4 Current flow

The capacitor creates a voltage that is in the opposite direction to the power supply. As a result the net voltage, V_n, that is drives current around the circuit is equal to the difference between the voltages of the power supply and the capacitor, or

$$V_n = V_s - V_c. \tag{18}$$

We can solve for this specifically by substituting the first line of Eq. 17 into the equation to obtain

$$V_n = V_s e^{-t/RC}. \tag{19}$$

Lets consider the capacitor and all the wires as ideal, that is they have no resistance. Then, the only source of resistance in our circuit is from the resistance. The equation that relates current to voltage is

$$V = IR. \tag{20}$$

R is the resistance for the entire circuit, which in our case is just the resistor. Substituting the net voltage into this equation, the expected current flow through the circuit for a charging capacitor is

$$I = \frac{V_s}{R} e^{-t/RC}. \tag{21}$$

The maximum current occurs when the capacitor is not charged at $t = 0$. As the capacitor charges it creates a voltage opposing the current flow causing it to decrease as time progresses.

The experiment uses an Arduino to power the experiment the circuit. The Arduino Uno is only designed to provide a maximum of 20 mA. Since the current is at a maximum when the capacitor is not charged, it acts like a direct short for a short instant of time, we need to ensure that the resistor has a large enough resistance to keep the current low enough that the Arduino can safely supply it. This is done by

$$\begin{aligned} R &= \frac{V}{I} \\ &= \frac{5}{0.02} \frac{\text{Volts}}{\text{Amps}} \\ &= 250 \text{ Ohms.} \end{aligned} \tag{22}$$

To be safe we should ensure that the current is well below the 20 mA limit and as a result I would recommend using a 500 Ohm resistor at the smallest.

3 Discharging Capacitor

The circuit and derivation of the equation for the discharging capacitor is simpler than that of the charging capacitor. For the circuit, the capacitor is the voltage source, V_c, that drives a current, I, which passes through a resistor, R, and back to the capacitor.

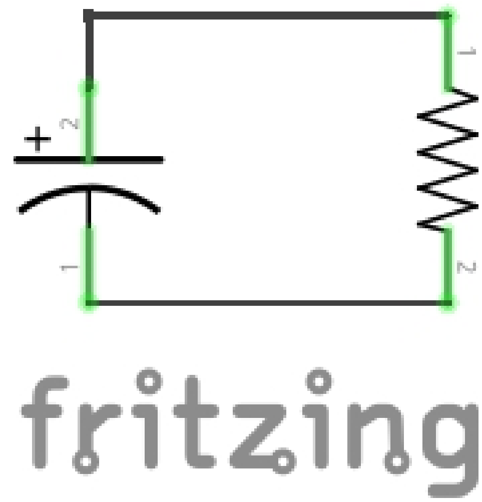

Figure 2: Circuit for a discharging capacitor. Current flows from the capacitor, through the resistor and back to the capacitor.

From Kirchhoff's loop rule we obtain the equation

$$0 = V_c - IR. \tag{23}$$

As with the charging capacitor, 4, case we have an equation to relate current flow to the charge on the capacitor. Since the capacitor is discharging in this case, current flow means that the charge on the capacitor is decreasing and as a result there is a negative sign that must be introduced,

$$I = \frac{-dq}{dt} = -C\frac{dV_c}{dt}. \tag{24}$$

Substituting this result into the Kirchhoff's loop rule result and we obtain

$$V_c = -RC\frac{dV_c}{dt}. \tag{25}$$

We again must use algebra to get all the voltages to the LHS and all the other terms to the RHS. This result can then be integrated,

$$\frac{-1}{RC}\int_0^{t'} dt = \int_0^{t'} \frac{dV_c}{V_c}. \tag{26}$$

The process of doing both of these integrations are the same as that of the charging case with the exception that integration of the LHS here is simple enough that we do not require any substitution. Performing the integrations we get

$$\frac{-1}{RC}t\Big|_{t=0}^{t'} = ln(V_c)\Big|_{t=0}^{t'}. \tag{27}$$

Now, evaluating at the limits

$$\frac{-t}{RC} = ln(V_c) - ln(V_0) \tag{28}$$

and simplifying the RHS

$$\frac{-t}{RC} = ln\left(\frac{V_c}{V_0}\right). \tag{29}$$

We are interested in a result for V_c so we will move that to the RHS. To deal with the logarithm we will again exponentiate each side,

$$\frac{V_c}{V_0} = e^{-t/RC}. \tag{30}$$

Finally, this result can be simplified using algebra and solved using both RC and τ,

$$\begin{aligned} V_c &= V_0 e^{-t/RC} \\ &= V_0 e^{-t/\tau}. \end{aligned} \tag{31}$$

3.1 Current Flow

I'm going to leave this as an exercise for the reader. The same process used for the charging capacitor applies here but now there is no power supply and the voltage for the circuit is provided by the capacitor. Also, as a hint, you should end up with a result very similiar to Eq. 21.