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Finite difference scheme for the solution of two-dimensional convection-diffusion equation with time fractional derivative

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Abstract
In this work, we propose a finite difference scheme for solving the two-dimensional time fractional convection-diffusion equation. In this approach the time fractional derivative of mentioned equation which is described in the Caputo sense, is approximated by a scheme of order $O(\tau^{2-\alpha})$, $0 < \alpha < 1$, and spatial derivatives are replaced with a second order finite difference scheme, so the proposed scheme is of order $O(\tau^{2-\alpha} + h_x^2 + h_y^2)$. We analyze the unconditional stability of proposed scheme using the technique of Fourier analysis. A numerical example demonstrates the theoretical result.
Keywords: Time fractional convection-diffusion, Caputo fractional derivative, unconditional stability, two-dimensional.

Mathematics Subject Classification: 65N06, 65N12.

1 Introduction

In recent years there has been a growing interest in the field of fractional calculus [9, 16, 21, 22]. Fractional differential equations have attracted increasing attention because they have applications in various fields of science and engineering [5]. Many phenomena in fluid mechanics, viscoelasticity, chemistry, physics, finance and other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e., the theory of derivatives and integrals of fractional order. Some of the most applications are given in the book of Oldham and Spanier [20], the book of Podlubny [22] and the papers of Metzler and Klafter [15], Bagley and Trovik [1]. Many considerable works on the theoretical analysis [8, 25] have been carried on, but analytic solutions of most fractional differential equations can not be obtained explicitly. So many authors have resorted to numerical solution strategies based on convergence and stability analysis[2, 3, 5, 6, 11, 23, 24, 26, 27]. Liu has carried on so many work on the finite difference method of fractional differential equations [13, 14, 12]. There are several definitions of a fractional derivative of order $\alpha > 0$ [20, 21]. The two most commonly used are the Riemann-Liouville and Caputo. The difference between the two definitions is in the order of evaluation[19].
We start with recalling the essentials of the fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. We give some basic definitions and properties of the fractional calculus theory.

Definition 1.1. *For $\mu \in \mathbb{R}$ and $x > 0$, a real function $f(x)$ is said to be in the space C_μ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C(0, \infty)$ and for $m \in \mathbb{N}$ it is said to be in the space C_μ^m if $f^m \in C_\mu$.*

Definition 1.2. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ for a function $f(x) \in C_\mu$, $\mu \geq -1$, is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad \alpha > 0, \quad x > 0, \quad J^0 f(x) = f(x).$$

Also we have the following properties

- $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
- $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
- $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

Definition 1.3. *If m be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$ is defined as*

$$ {}_0^C D_t^\alpha f(x) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x (x-t)^{m-\alpha-1} f^{(n)}(t) d\tau, & m-1 < \alpha < m, \quad m \in \mathbb{N}, \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & m = \alpha. \end{cases} \tag{1}$$

In this paper we consider the two-dimensional fractional convection-diffusion equation

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} + \frac{\partial u(x,y,t)}{\partial x} + \frac{\partial u(x,y,t)}{\partial y} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + f(x,y,t), \tag{2}$$
$$0 \leq t \leq T \quad , \quad 0 < x, y < L,$$

with the boundary conditions

$$\begin{aligned} u(0,y,t) &= \varphi_1(y,t), \qquad & u(L,y,t) &= \varphi_2(y,t), \\ u(x,0,t) &= \psi_1(x,t), \qquad & u(x,L,t) &= \psi_2(x,t), \end{aligned} \tag{3}$$

and the initial condition

$$u(x,y,0) = w(x,y), \qquad 0 \leq x, \ \ y \leq L, \tag{4}$$

where $0 < \alpha < 1$ and $\frac{\partial^\alpha u}{\partial t^\alpha}$ is in the Caputo sense (1). We propose a finite difference approximation of order two in space variable and first order in time component for Eq. (2). We prove the unconditionally stability property of proposed scheme. The outline of paper is as following: In Section 2, we introduce the derivation of new method for the solution of Eq. (2). This scheme is based on approximating the time derivative of mentioned equation by a scheme of order $\mathcal{O}(\tau^{2-\alpha})$ and spatial derivative with a second-order finite difference scheme. In this section we obtain the matrix form of the proposed method and prove the unconditional stability property of method using the Fourier method. In Section 3 we report the numerical experiments of solving Eq. (2) with the method developed in this paper for a test problem. Finally a conclusion is given in Section 4.

2 Main result

Firstly, we introduce the following notations

$$\delta_x^2 u_{n,m}^k = \frac{u_{n+1,m}^k - 2u_{n,m}^k + u_{n-1,m}^k}{h_x^2}, \qquad \delta_y^2 u_{n,m}^k = \frac{u_{n,m+1}^k - 2u_{n,m}^k + u_{n,m-1}^k}{h_y^2},$$

$$\delta_x u_{n,m}^k = \frac{u_{n+1,m}^k - u_{n-1,m}^k}{2h_x}, \qquad \delta_y u_{n,m}^k = \frac{u_{n,m+1}^k - u_{n,m-1}^k}{2h_y},$$

and

$$x_n = nh_x, \quad n = 0,1,\ldots,M_x, \quad y_m = mh_y, \quad m = 0,1,\ldots,M_y, \quad t_k = k\tau, \quad k = 0,1,\ldots,N,$$

where $h_x = L/M_1$ and $h_y = L/M_2$ are space steps and $\tau = T/N$ is time step. Also we define

$$D_\tau^\alpha \omega^n = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[\omega^n - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\omega^k - a_{n-1}\omega^0\right],$$

$$a_k = (k+1)^{1-\alpha} - k^{1-\alpha}.$$

Now, for the discretization, we need the following Lemmas,

Lemma 2.1. *[23]. Suppose* $0 < \alpha < 1$ *and* $y \in C^2[0,t_n]$*, we have*

$$\left|\frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}\frac{y'(s)}{(t_n-s)^\alpha}ds - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0 y(t_n) - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})y(t_k) - a_{n-1}y(0)\right]\right|$$

$$\leq \frac{1}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha})\right]\max_{0\leq t_n\leq t}|y''(t)|\,\tau^{2-\alpha},$$

$$a_k = (k+1)^{1-\alpha} - k^{1-\alpha}.$$

Lemma 2.2. *[3]. Let* $0 < \alpha < 1$ *and* $a_m = (m+1)^{1-\alpha} - m^{1-\alpha}$, $m = 0,1,\ldots$ *then,*

$$1 = a_0 > a_1 > a_2 > \ldots > a_m \to 0, \quad as \;\; m \to \infty.$$

Considering (2) at the point $(x_n.y_m,t_k)$, we obtain

$$\begin{aligned} &{}_0^C D_t^\alpha u(x_n,y_m,t_k) + \frac{\partial u(x_n,y_m,t_k)}{\partial x} + \frac{\partial u(x_n,y_m,t_k)}{\partial y} \\ &\qquad = \frac{\partial^2 u(x_n,y_m,t_k)}{\partial x^2} + \frac{\partial^2 u(x_n,y_m,t_k)}{\partial y^2} + f(x_n,y_m,t_k), \qquad (5) \\ &0 \leq n \leq M_1, \quad 0 \leq m \leq M_2, \quad 0 \leq k \leq N. \end{aligned}$$

Define the grid functions

$$U_{n,m}^k = u(x_n,y_m,t_k), \quad f_{n,m}^k = f(x_n,y_m,t_k), \quad 0 \leq n \leq M_1, \quad 0 \leq m \leq M_2\,, \quad 0 \leq k \leq N.$$

By using the Taylor series expansion and Lemma 2.1, we have

$$
\begin{aligned}
&{}_0^C D_t^\alpha u(x_n, y_m, t_k) = D_\tau^\alpha U_{n,m}^k + (E_t)_{n,m}^k,\\
&\frac{\partial^2 u(x_n, y_m, t_k)}{\partial x^2} = \delta_x^2 U_{n,m}^k - (E_x^1)_{n,m}^k, \qquad \frac{\partial u(x_n, y_m, t_k)}{\partial x} = \delta_x U_{n,m}^k - (E_x^2)_{n,m}^k,\\
&\frac{\partial^2 u(x_n, y_m, t_k)}{\partial y^2} = \delta_y^2 U_{n,m}^k - (E_y^1)_{n,m}^k, \qquad \frac{\partial u(x_n, y_m, t_k)}{\partial y} = \delta_y U_{n,m}^k - (E_y^2)_{n,m}^k,\\
&0 \le n \le M_1, \quad 0 \le m \le M_2, \quad 1 \le k \le N,
\end{aligned} \tag{6}
$$

where there are constants C_x^1, C_x^2, C_y^1 and C_y^2 such that

$$
\left|(E_t)_{n,m}^k\right| \le C_t \tau^{2-\alpha}, \quad \left|(E_x^1)_{n,m}^k\right| \le C_x^1 h_x^2, \quad \left|(E_x^1)_{n,m}^k\right| \le C_x^2 h_x^2,
$$

$$
\left|(E_y^1)_{n,m}^k\right| \le C_y^1 h_y^2, \quad \left|(E_y^2)_{n,m}^k\right| \le C_y^2 h_y^2.
$$

By substituting (6) into (5), we have

$$
D_\tau^\alpha U_{n,m}^k = \delta_x^2 U_{n,m}^k - \delta_x U_{n,m}^k + \delta_y^2 U_{n,m}^k - \delta_y U_{n,m}^k + f_{n,m}^k + (E)_{n,m}^k, \tag{7}
$$

and

$$
(E)_{n,m}^k = -(E_t)_{n,m}^k - (E_x^1)_{n,m}^k + (E_x^2)_{n,m}^k - (E_y^1)_{n,m}^k + (E_y^2)_{n,m}^k.
$$

Thus we get

$$
\left|(E)_{n,m}^k\right| \le C(\tau^{2-\alpha} + h_x^2 + h_y^2),
$$

where C is a positive constant. Omitting the small term $(E)_{n,m}^k$ and by considering the boundary and initial conditions, we get the following finite difference scheme

$$
\begin{cases}
\left(-\dfrac{1}{2h_x} - \dfrac{1}{h_x^2}\right) U_{n-1,m}^k + \left(\dfrac{2}{h_x^2} + \dfrac{2}{h_y^2} + \mu\right) U_{n,m}^k + \left(\dfrac{1}{2h_x} - \dfrac{1}{h_x^2}\right) U_{n+1,m}^k\\
+ \left(-\dfrac{1}{2h_y} - \dfrac{1}{h_y^2}\right) U_{n,m-1}^k + \left(\dfrac{1}{2h_y} - \dfrac{1}{h_y^2}\right) U_{n,m+1}^k\\
= \mu \left[\displaystyle\sum_{r=1}^{k-1} (a_{k-r-1} - a_{k-r}) U_{n,m}^r + a_{k-1} U_{n,m}^0\right] + f_{n,m}^k\\
U_{0,m}^k = \varphi_1(y_m, t_k), \ U_{L,m}^k = \varphi_2(y_m, t_k), \ 0 \le m \le M_2, \ 0 \le k \le N,\\
U_{n,0}^k = \psi_1(x_n, t_k), \ U_{n,L}^k = \psi_2(x_n, t_k), \ 0 \le n \le M_1, \ 0 \le k \le N,\\
U_{n,m}^0 = w(x_n, y_m), \quad 0 \le n \le M_1, \ 0 \le m \le M_2,
\end{cases} \tag{8}
$$

where $\mu = \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}$. If we put

$$
b_1 = -\frac{1}{2h_x} - \frac{1}{h_x^2}, \qquad b_2 = \frac{1}{2h_x} - \frac{1}{h_x^2}, \qquad a = \frac{2}{h_x^2} + \frac{2}{h_y^2} + \mu,
$$

$$
c_1 = -\frac{1}{2h_y} - \frac{1}{h_y^2}, \qquad c_2 = \frac{1}{2h_y} - \frac{1}{h_y^2},
$$

and define

$$U^k=\left[U^k_{1,1},U^k_{2,1},\dots,U^k_{M_1-1,1},U^k_{1,2},U^k_{2,2},\dots,U^k_{M_1-1,2}\ ,\dots,U^k_{1,M_2-1},U^k_{2,M_2-1},\dots,U^k_{M_1-1,M_2-1}\right],$$

then we can write (8) in the matrix-vector form as follows

$$\mathbf{A}\mathbf{U}^k=\mathbf{G}+\mathbf{F}$$

in which

$$\mathbf{G}=\begin{pmatrix}\mu\left[\sum_{r=1}^{k-1}(a_{k-r-1}-a_{k-r})U^r_{1,1}+a_{k-1}U^0_{1,1}\right]\\ \vdots\\ \mu\left[\sum_{r=1}^{k-1}(a_{k-r-1}-a_{k-r})U^r_{M_1-1,1}+a_{k-1}U^0_{M_1-1,1}\right]\\ \mu\left[\sum_{r=1}^{k-1}(a_{k-r-1}-a_{k-r})U^r_{1,2}+a_{k-1}U^0_{1,2}\right]\\ \vdots\\ \mu\left[\sum_{r=1}^{k-1}(a_{k-r-1}-a_{k-r})U^r_{M_1-1,2}+a_{k-1}U^0_{M_1-1,2}\right]\\ \vdots\\ \mu\left[\sum_{r=1}^{k-1}(a_{k-r-1}-a_{k-r})U^r_{1,M_2-1}+a_{k-1}U^0_{1,M_2-1}\right]\\ \vdots\\ \mu\left[\sum_{r=1}^{k-1}(a_{k-r-1}-a_{k-r})U^r_{M_1-1,M_2-1}+a_{k-1}U^0_{M_1-1,M_2-1}\right]\end{pmatrix},$$

$$\mathbf{F}=\big(f^k_{1,1}-c_1U^k_{1,0}-b_1U^k_{0,1},f^k_{2,1}-c_1U^k_{2,0},\dots,f^k_{M_1-2,1}-c_1U^k_{M_1-2,0},f^k_{M_1-1,1}-c_1U^k_{M_1-1,0}-b_2U^k_{M_1,1}$$

$$,f^k_{1,2}-b_1U^k_{0,2},f^k_{2,2},\dots,f^k_{M_1-2,2},f^k_{M_1-1,2}-b_2U^k_{M_1,2},\dots$$

$$,\dots,f^k_{1,M_2-2}-b_1U^k_{0,M_2-2},f^k_{2,M_2-2},\dots,f^k_{M_1-2,M_2-2},f^k_{M_1-1,M_2-2}-b_2U^k_{M_1,M_2-2},$$

$$f^k_{1,M_2-1}-c_2U^k_{1,M_2}-b_1U^k_{0,M_2-1},f^k_{2,M_2-1}-c_2U^k_{2,M_2},\dots,$$

$$\dots,f^k_{M_1-2,M_2-1}-c_2U^k_{M_1-2,M_2},f^k_{M_1-1,M_2-1}-c_2U^k_{M_1-1,M_2}-b_2U^k_{M_1,M_2-1}\big)^T,$$

$$\mathbf{A}=\begin{pmatrix}M&H&&\\N&M&H&\\&N&\ddots&H\\&&N&M\end{pmatrix}_{(M_1-1)\times(M_2-1)},\quad M=\begin{pmatrix}a&b_2&0&\cdots&0\\b_1&a&b_2&\ddots&\vdots\\0&b_1&\ddots&\ddots&0\\\vdots&\ddots&\ddots&a&b_2\\0&\cdots&0&b_1&a\end{pmatrix}_{(M_1-1)},$$

$$N=\begin{pmatrix}c_1&0&\cdots&0\\0&c_1&\ddots&\vdots\\\vdots&\ddots&\ddots&0\\0&\cdots&0&c_1\end{pmatrix}_{(M_1-1)},\qquad H=\begin{pmatrix}c_2&0&\cdots&0\\0&c_2&\ddots&\vdots\\\vdots&\ddots&\ddots&0\\0&\cdots&0&c_2\end{pmatrix}_{(M_1-1)}.$$

Now, we will use Fourier analysis for the stability of difference scheme (8). Let $u_{n,m}^k$ and $U_{n,m}^k$ be the approximation and exact solutions for the finite difference scheme (8), respectively. We define

$$\rho_{n,m}^k = U_{n,m}^k - u_{n,m}^k,$$

$$k = 0, 1, \ldots, N, \quad n = 1, 2, \ldots, M_1, \quad m = 1, 2, \ldots, M_2,$$

and

$$\rho^k = [\rho_{1,1}^k, \rho_{2,1}^k, \ldots, \rho_{M_1-1,1}^k, \rho_{1,2}^k, \rho_{2,2}^k, \ldots, \rho_{M_1-1,2}^k, \ldots\ldots, \rho_{1,M_2-1}^k, \rho_{2,M_2-1}^k, \ldots, \rho_{M_1-1,M_2-1}^k]^T,$$

Obviously, from (8) we can obtain

$$\left(-\frac{1}{2h_x} - \frac{1}{h_x^2}\right)\rho_{n-1,m}^k + \left(\frac{2}{h_x^2} + \frac{2}{h_y^2} + \mu\right)\rho_{n,m}^k + \left(\frac{1}{2h_x} - \frac{1}{h_x^2}\right)\rho_{n+1,m}^k$$

$$+\left(-\frac{1}{2h_y} - \frac{1}{h_y^2}\right)\rho_{n,m-1}^k + \left(\frac{1}{2h_y} - \frac{1}{h_y^2}\right)\rho_{n,m+1}^k = \mu\left[\sum_{r=1}^{k-1}(a_{k-r-1} - a_{k-r})\rho_{n,m}^r + a_{k-1}\rho_{n,m}^0\right]. \tag{9}$$

Now, we define the grid function

$$\rho^k(x,y) = \begin{cases} \rho_{n,m}^k & (x,y) \in \Omega_1, \\ 0 & (x,y) \in \Omega_2, \end{cases}$$

where

$$\Omega_1 = \left\{(x,y)\,\middle|\, x_{n-\frac{1}{2}} < x \le x_{n+\frac{1}{2}}\ ,\ y_{m-\frac{1}{2}} < y \le y_{m+\frac{1}{2}},\ \ n = 1, 2, \ldots, M_1 - 1,\ m = 1, 2, \ldots, M_2 - 1\right\},$$

$$\Omega_2 = \left\{(x,y)\,\middle|\, 0 \le x \le \tfrac{h_x}{2}\ ,\ or\ \ L - \tfrac{h_x}{2} < x \le L,\ or\ \ 0 \le y \le \tfrac{h_y}{2},\ \ L - \tfrac{h_y}{2} < y \le L\right\}.$$

Then $\rho^k(x,y)$ can be expanded in Fourier series as follows

$$\rho^k(x,y) = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} d_k(l_1,l_2)e^{2\pi i(l_1x/L + l_2y/L)}, \quad k = 1, 2, \ldots, N,$$

where

$$d_k(l_1,l_2) = \frac{1}{L}\int_0^L\int_0^L \rho^k(x,y)e^{-2\pi i(l_1x/L + l_2y/L)}dxdy.$$

Applying Parseval equality, we can obtain

$$\|\rho^k\|_2^2 = \sum_{j=1}^{M_1-1}\sum_{l=1}^{M_2-1} h_xh_y\left|\rho_{j,l}^k\right|^2 = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|d_k(l_1,l_2)|^2, \quad k = 0, 1, \ldots, N, \tag{10}$$

where

$$\|\rho^k\|_2 = \left(\int_0^L\int_0^L \left|\rho^k(x,y)\right| dxdy\right)^{\frac{1}{2}}.$$

Based on the above analysis, we can assume that the solution of difference equation (9) has the following form

$$\rho_{j,l}^{k} = d_k e^{i(\sigma_1 j h_x + \sigma_2 l h_y)},$$
$$\sigma_1 = \frac{2\pi l_1}{L}, \qquad \sigma_2 = \frac{2\pi l_2}{L}. \tag{11}$$

Substituting (11) into (9) and by simplification, we can obtain

$$d_k = \frac{\mu\left[\sum_{r=1}^{k-1}\left(a_{k-r-1}-a_{k-r}\right)d_r + a_{k-1}d_0\right]}{\mu + \frac{4}{h_x^2}\sin^2\frac{\theta}{2} + \frac{4}{h_y^2}\sin^2\frac{\beta}{2} + \frac{i}{h_x}\sin\theta + \frac{i}{h_y}\sin\beta}, \tag{12}$$
$$k = 0, 1, \ldots, N, \qquad \theta = \sigma_1 h_x, \qquad \beta = \sigma_2 h_y.$$

Proposition 2.3. *Assume d_k for $k = 1, 2, \ldots, N$ are the solutions of the Eq. (12), then*

$$|d_k| \le |d_0|, \qquad k = 1, 2, \ldots, N.$$

Proof. We will use mathematical induction for the proof. For $k = 1$, from (12), we obtain that

$$|d_1|^2 \le \frac{\mu^2 |d_0|^2}{\left(\mu + \frac{4}{h_x^2}\sin^2\frac{\theta}{2} + \frac{4}{h_y^2}\sin^2\frac{\beta}{2}\right)^2 + \left(\frac{1}{h_x}\sin\theta + \frac{1}{h_y}\sin\beta\right)^2} \le |d_0|^2,$$

so

$$|d_1| \le |d_0|.$$

Assume that

$$|d_j| \le |d_0|, \qquad j = 1, 2, \ldots, k-1,$$

then

$$|d_k| \le \frac{\mu\,|d_0|\left[\sum_{r=1}^{k-1}\left(a_{k-r-1}-a_{k-r}\right) + a_{k-1}\right]}{\left|\mu + \frac{4}{h_x^2}\sin^2\frac{\theta}{2} + \frac{4}{h_y^2}\sin^2\frac{\beta}{2} + \frac{i}{h_x}\sin\theta + \frac{i}{h_y}\sin\beta\right|}$$

$$= \frac{\mu\,|d_0|\left[(1-a_{k-1}) + a_{k-1}\right]}{\left|\mu + \frac{4}{h_x^2}\sin^2\frac{\theta}{2} + \frac{4}{h_y^2}\sin^2\frac{\beta}{2} + \frac{i}{h_x}\sin\theta + \frac{i}{h_y}\sin\beta\right|}$$

$$= \frac{\mu\,|d_0|}{\left|\mu + \frac{4}{h_x^2}\sin^2\frac{\theta}{2} + \frac{4}{h_y^2}\sin^2\frac{\beta}{2} + \frac{i}{h_x}\sin\theta + \frac{i}{h_y}\sin\beta\right|} \le |d_0|.$$

□

Theorem 2.4. *The finite difference method (8) is unconditionally stable.*

Proof. By considering (10) and using Proposition 3.1, we can obtain,

$$\|\rho^k\|_2^2 = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|d_k(l_1,l_2)|^2 \le \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|d_0(l_1,l_2)|^2 = \|\rho^0\|_2^2, \quad k=1,2,\dots,N.$$

Thus we obtain that

$$\|\rho^k\|_2 \le \|\rho^0\|_2, \quad k=1,2,\dots,N,$$

this show that the finite difference method (8) is unconditionally stable. □

3 Numerical results

In this section we present the numerical results of finite difference method (8) on a test problem. We test the accuracy and stability of the method described in this paper by performing the mentioned method for different values of h_x, h_y and τ. We performed our computations using **Matlab** 7 software on a Pentium IV, 2800 MHz CPU machine with 2 Gbyte of memory. We calculated the computational orders of the method presented in this article in time variable with [17]

$$\mathrm{C}_1 - \text{order} = \log_2\left(\frac{\|L_\infty(2\tau,h)\|}{\|L_\infty(\tau,h)\|}\right),$$

and in space variables with [5]

$$\mathrm{C}_2 - \text{order} = \log_2\left(\frac{\|L_\infty(16\tau,2h)\|}{\|L_\infty(\tau,h)\|}\right).$$

Example 3.1. *We consider the initial-boundary value problem*

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} + \frac{\partial u(x,y,t)}{\partial x} + \frac{\partial u(x,y,t)}{\partial y} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + \frac{t^{1-\alpha}e^{x+y}}{(1-\alpha)\Gamma(1-\alpha)}$$

$$\varphi_1(y,t) = te^y, \quad \varphi_2(y,t) = te^{1+y}, \quad 0 \le y \le 1, \quad 0 < t < 1,$$

$$\psi_1(x,t) = te^x, \quad \psi_2(x,t) = te^{1+x}, \quad 0 \le x \le 1, \quad 0 < t < 1,$$

$$w(x,y) = 0, \quad 0 \le y \le 1, \quad 0 \le x \le 1,$$

with the exact solution

$$u(x,y,t) = te^{x+y}, \qquad 0 < t < 1, \qquad 0 \le x,y \le 1.$$

Table 1
Errors and computational orders obtained with $\tau = \frac{1}{10}$

	$\alpha = 0.25$		$\alpha = 0.45$		
$h_x = h_y$	L_∞	C_1-order	L_∞	C_1-order	CPU time(s)
1/4	1.8570×10^{-3}	–	1.8524×10^{-3}	–	0.1250
1/8	5.2125×10^{-4}	1.8329	5.2013×10^{-4}	1.8324	0.1880
1/16	1.3142×10^{-4}	1.9877	1.3114×10^{-4}	1.9877	0.5000
1/32	3.2927×10^{-5}	1.9968	3.2857×10^{-5}	1.9968	13.9370

We solve this problem with the method presented in this article with several values of h_x, h_y and α for $L = 1$ at final time $T = 1$ with $\tau = 1/10$. The L_∞ error, C_1-Order and CPU time (s) are shown in Table 1. Table 1 shows that the proposed method has good results in both accuracy and CPU time. Computational orders in Table 1 confirm the second order of accuracy in space components. Figure 1 presents the graphs of exact and approximate solutions and absolute error obtained with $\alpha = 0.25$, $h_x = h_y = 1/32$ and $\tau = 1/10$.

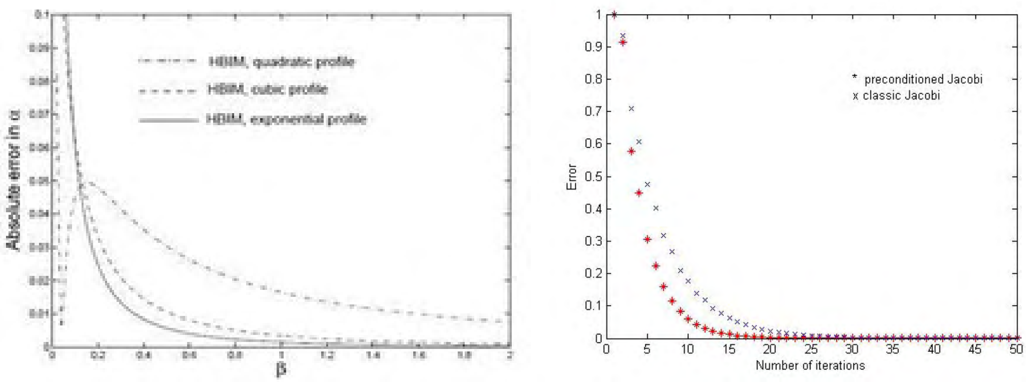

Figure 1: Error (left panel) and approximate solution (right panel) obtained with $h_x = h_y = 1/32$, $\tau = 1/20$ and $\alpha = 0.25$ at $T = 1$.

Table 2

Errors obtained with $\tau = h_x^2 = h_y^2$ and different values of α

h	$\alpha = 0.25$	$\alpha = 0.55$	$\alpha = 0.75$	$\alpha = 0.95$
1/10	3.3508×10^{-4}	3.3419×10^{-3}	3.3435×10^{-3}	3.3532×10^{-3}
1/15	1.4952×10^{-4}	1.4912×10^{-4}	1.4919×10^{-4}	1.4962×10^{-4}
1/20	8.4224×10^{-5}	8.4000×10^{-5}	8.4041×10^{-5}	8.4284×10^{-4}

Table 3

Numerical results obtained with $\alpha = 0.2$ and $\alpha = 0.45$

	$\alpha = 0.2$		$\alpha = 0.45$	
	L_∞	C_2-Order	L_∞	C_2-Order
$h_x = h_y = \tau = \frac{1}{4}$	1.8586×10^{-3}	–	1.8526×10^{-3}	–
$h_x = h_y = \frac{1}{8}, \tau = \frac{1}{64}$	5.2164×10^{-4}	1.2706	5.2013×10^{-4}	1.2702
$h_x = h_y = \frac{1}{16}, \tau = \frac{1}{1024}$	1.3152×10^{-4}	1.3778	1.3114×10^{-4}	1.3778
$h_x = h_y = \tau = \frac{1}{8}$	5.2164×10^{-4}	–	5.2014×10^{-4}	–
$h_x = h_y = \frac{1}{16}, \tau = \frac{1}{128}$	1.3152×10^{-4}	1.3778	1.3114×10^{-4}	1.3778

Tables 1, 3 show that the computational orders are close to theoretical orders, i.e the order of method is $\mathcal{O}(\tau^{2-\alpha})$ in time variable and $\mathcal{O}(h_x^2 + h_y^2)$ in space variables.

4 Conclusion

In this article, we constructed a finite difference scheme for the solution of two-dimensional convection-diffusion equation with time fractional derivative equation and proved the unconditional stability property by Fourier analysis. Numerical results confirmed the theoretical results of proposed scheme, i.e the method has second-order of accuracy in space component and is of order $\mathcal{O}(\tau^{2-\alpha})$ in time variable.

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Order stars and order arrows

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Abstract

Order stars have become a fundamental tool for the understanding of order and stability properties of numerical methods for ordinary differential equations. Order arrows were originally proposed to complement the use of order stars. This survey retraces their discovery and principal achievements.

Keywords: Stiff differential equations, Order stars, Order arrows, A-stability

Mathematics Subject Classification: 65L05, 65L06, 65L20

1 Introduction

Numerical methods for differential equations, when applied to the linear test problem, generate a sequence of approximations which satisfy a difference equation of the form

$$P_0(z)y_n + P_1(z)y_{n-1} + \cdots + P_k(z)y_{n-k} = 0, \tag{1}$$

where $P_0, P_1, \cdots, P_k$ are polynomials. The stability of the difference equation (1) is determined by the values of w satisfying

$$\Phi(w, z) = 0, \tag{2}$$

where $\Phi(w, z) = P_0(z)w^k + P_1(z)w^{k-1} + \cdots + P_k(z)$. A stability function $\Phi(w, z)$ has order p if $\Phi(\exp(z), z) = O(z^{p+1})$ and it is the generalized Padé approximation to $\exp(z)$ if $p = \sum_{i=0}^{k} \deg(P_i) + k - 1$.

In the case of one-step methods, such as Runge–Kutta, Collocation, Rosenbrock type or multiderivative formulas, (1) reduces to

$$P_0(z)y_n + P_1(z)y_{n-1} = 0, \tag{3}$$

and the stability of the difference equation (3) is determined by the value of the rational function

$$R(z) = -\frac{P_1(z)}{P_0(z)}.$$

We focus on the stability polynomial $\Phi(w, z)$ as the main characteristic of a method. The order star is the set of points z in the complex plane such that

$$\Phi(w \exp(z), z) = 0, \tag{4}$$

and such that $|w| > 1$ ("the order star") or such that $|w| < 1$ ("the dual order star"). The "order arrows" are the lines made up from points z in the complex plane satisfying (4) for which w is real and positive. The rays emanating from origin with increasing value of w are "up-arrows" and those emanating from origin with decreasing w are "down-arrows".

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2 Properties of order stars and order arrows

Order stars were introduced by Hairer, Nørsett and Wanner in [6] to analyze barriers on order and possible orders compatible with A-stability. Order arrows were introduced by Butcher as an alternative tool for studying stability and order barriers [3]. References are also made to [1, 2] which include applications and a discussion of order arrows. The specific motivation to introduce order arrows was a result referred to as the Butcher–Chipman conjecture [4] which was eventually proved using order arrows in [3].

Close to the origin, the order star of a rational approximation R of order p consists of $p+1$ sectors of width $\pi/(p+1)$, separated by $p+1$ sectors of the dual order star, each of the same width. Fingers are called as the connected components of each of sectors. If r sectors join together to one finger, it is called a finger of multiplicity r. There are n poles inside of a bounded finger with multiplicity of n and also every bounded dual finger with multiplicity of m should be contained m zeros. Furthermore, rational approximation R is A-acceptable if and only if it has no poles in $\mathbb{C}^-$ and order star of it has no intersection with the imaginary axis.

For order arrows of rational approximation R, an up-arrow emanating from zero follows a path in the complex plane which, unless it meets other up-arrows at a stagnation point, either terminates at a pole or diverges to $-\infty$. Similarly a down-arrow, unless it meets other down-arrows, terminates at a zero or diverges to $+\infty$. In the event that an up-arrow or a down-arrow, respectively, meets other arrows at a stagnation point, further down-arrows, or up-arrows respectively follow their own paths to a pole or a zero or $\pm\infty$. Also, there are $p+1$ up-arrows and $p+1$ down-arrows emanating, alternately, from the origin. The angle between each up-arrow and the next down-arrow emanating from origin is $\pi/(p+1)$. A necessary condition for $R(z)$ to be A-acceptable is that up-arrows emanating from zero cannot be tangential to the imaginary axis, and cannot cross from the right half-plane to the left half-plane. Add to these restrictions, the fact that an up-arrow from zero that terminates at a pole, cannot leave zero in a negative direction.

For arrow diagrams on a Riemann surface the behaviour of arrows emanating from origin is similar, with the complication that some arrows can rise or descend in the sheet system around branch points.

An example of order star and order arrows for a seventh order rational approximation to exp function in the form

$$R(z) = \frac{N(z)}{(1-\frac{1}{6}\,z+\frac{1}{30}\,z^2)^3(1-\frac{1}{5}\,z)},$$

is presented in Figure 1. Note that the poles are marked with filled circles $\bullet$ and the zeros are marked with circles $\circ$. In the left diagram, the dual order star, which can also be described as the "relative stability region", is the interior of the unshaded region. The order star is the interior of the shaded region. In the right diagram, arrows are attached to the paths to show the direction of increasing $\exp(-z)R(z)$. We note that order star of R has intersection with the imaginary axis and also, an up-arrows is tangential to the imaginary axis. So, this rational approximation is not A-stable.

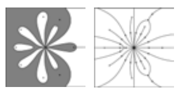

Figure 1: Order star (left) and order arrows (right) of a seventh order rational approximation to exp function with denominator $(1-\frac{1}{6}\,z+\frac{1}{30}\,z^2)^3(1-\frac{1}{5}\,z)$.

3 Proof of Dahlquist second barrier by order arrows

Many of the results counted among the successes of order star theory, can also be analyzed using order arrows. These include the Ehle barrier, the Dahlquist second barrier and the Daniel–Moore barrier. Also, some order barriers for rational approximations to the exponential function are obtained by order arrows in [2]. Here, we will discuss Dahlquist second barrier by order arrows.

The general form of a linear k-step method for an ODE $y' = f(y)$ is

$$y_n = \sum_{i=1}^{k} \alpha_i y_{n-i} + h \sum_{j=0}^{k} \beta_j f(y_{n-j}), \tag{5}$$

where y_n is the numerical approximation to the exact solution at the point x_n and α_i, $1 \le i \le k$, and β_j, $0 \le j \le k$ are fixed numbers. The values of α_i and β_j are chosen to obtain the highest possible order and characterize a method.

It was shown how order of the method (5) is related to A-stability, this is now known as the "Dahlquist second barrier", which limits the order of an A-stable linear k-step method to 2. Although this result has been proved in [5], we use order arrows to obtain simpler and more illuminating proof.

Theorem 3.1. *An A-stable linear multistep method cannot has order greater than 2.*

Proof. The stability polynomial of the method (5) is $\rho(w) - z\sigma(w)$ where

$$\rho(w) = w^k - \sum_{i=1}^{k} \alpha_i w^{k-i}, \qquad \sigma(w) = \sum_{i=0}^{k} \beta_i w^{k-i}.$$

Rewriting $\rho(w) - z\sigma(w)$ as a polynomial in terms of w, it takes the form (2) where $\deg(P_i) \le 1$ for $1 \le i \le k$ and $\deg(P_0) \le \delta$ which δ is 0 for explicit methods and 1 for implicit ones. Assume t, $t \le \delta$, up-arrows emanating from origin terminate at poles (zeros of P_0). Because, for an A-stable method, there is no crossing and tangential up-arrows to the imaginary axis, we must have

$$\frac{2(t+1)\pi}{p+1} > \pi,$$

or

$$p \le 2t.$$

Hence, an A-stable multistep method must be implicit and of order $p \le 2$.

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Construction of efficient second derivative methods for stiff differential systems

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Abstract
Second derivative multistep methods and their modified form have been proposed for integrating stiff initial-value problems for first-order ordinary differential equations. In this context, we investigate these methods from second derivative general linear methods point of view and construct some perturbations of these methods. After satisfying order conditions, the remaining free parameters are used to maximize the angle α of $A(\alpha)$-stability of the methods. Numerical results demonstrate the efficiency of the perturbed methods.
Keywords: Multistep methods, Second derivative methods, General linear methods, $A(\alpha)$-stability, Stiff systems.

Mathematics Subject Classification: 65L05.

1 Introduction

Many codes have been introduced for solving initial value problems

$$\begin{cases} y'(x) = f(y(x)), & x \in [x_0, \overline{x}], \\ y(x_0) = y_0, \end{cases} \tag{1}$$

in class of linear multistep such as [2, 3]. To improve stability properties of these methods, Hojjati et. al [5] proposed the extension of these methods which utilizes second derivative of the solution and a future point at x_{n+k+1}, the so called extended second derivative multistep methods (ESDMMs) and their modified form (MESDMMs).

In the other hand to have more accurate methods Butcher and Hojjati in [4] extended general linear methods (GLMs) to second derivative general linear methods (SGLMs). These methods studied more by Abdi and Hojjati in [1].

In this paper, to find efficient methods with extensive absolute stability region for stiff systems, considering MESDMMs in SGLMs structure, we introduce perturbations of MESDMMs which preserves their orders and improves the stability properties of MESDMMs.

2 Reforming MESDMMs as multistage-multivalue methods

We recall the ESDMMs and MESDMMs, that have been introduced by Hojjati et. al [5]. The algorithm based on ESDMMs is defined by the following three steps:

(i) Compute $\overline{y}_{n+k}$ as the solution of the conventional second derivative BDF method

$$\overline{y}_{n+k}+\sum_{j=0}^{k-1}\alpha_j y_{n+j}=h\beta_k\overline{f}_{n+k}+h^2\gamma_k\overline{g}_{n+k},$$

where $\overline{f}_{n+k}=f(\overline{y}_{n+k})$ and $\overline{g}_{n+k}=g(\overline{y}_{n+k})$.
(ii) Compute $\overline{y}_{n+k+1}$ as the solution of the same second derivative BDF advanced one step, that is,

$$\overline{y}_{n+k+1}+\alpha_{k-1}\overline{y}_{n+k}+\sum_{j=0}^{k-2}\alpha_j y_{n+j+1}=h\beta_k\overline{f}_{n+k+1}+h^2\gamma_k\overline{g}_{n+k+1},$$

where $\overline{f}_{n+k+1}=f(\overline{y}_{n+k+1})$ and $\overline{g}_{n+k+1}=g(\overline{y}_{n+k+1})$.
(iii) Evaluating $\overline{g}_{n+k+1}=g(\overline{y}_{n+k+1})$, compute y_{n+k} as the solution of

$$y_{n+k}+\sum_{j=0}^{k-1}\widehat{\alpha}_j y_{n+j}=h\widehat{\beta}_k f_{n+k}+h^2(\widehat{\gamma}_k g_{n+k}-\widehat{\gamma}_{k+1}\overline{g}_{n+k+1}).$$

Considering stage (iii) as follows

$$y_{n+k}+\sum_{j=0}^{k-1}\widehat{\alpha}_j y_{n+j}=h(\widehat{\beta}_k-\beta_k)\overline{f}_{n+k}+h\beta_k f_{n+k}+h^2(\widehat{\gamma}_k-\gamma_k)\overline{g}_{n+k}-h^2\widehat{\gamma}_{k+1}\overline{g}_{n+k+1}+h^2\gamma_k g_{n+k},$$

the Jacobian matrix in each of the 3 stages is the same. This method is well known as MESDMMs. The overall methods are k-step method of order $k+2$. It is A-stable up to order 6 and $A(\alpha)$–stable up to order 14. MESDMMs can be reformed as SGLMs

$$\begin{aligned}Y^{[n]}&=h(\mathbf{A}\otimes\mathbf{I}_m)f(Y^{[n]})+h^2(\overline{\mathbf{A}}\otimes\mathbf{I}_m)g(Y^{[n]})+(\mathbf{U}\otimes\mathbf{I}_m)y^{[n-1]},\\ y^{[n]}&=h(\mathbf{B}\otimes\mathbf{I}_m)f(Y^{[n]})+h^2(\overline{\mathbf{B}}\otimes\mathbf{I}_m)g(Y^{[n]})+(\mathbf{V}\otimes\mathbf{I}_m)y^{[n-1]},\end{aligned}\tag{2}$$

with $\mathbf{A},\overline{\mathbf{A}},\mathbf{U},\mathbf{B},\overline{\mathbf{B}},\mathbf{V}$ are given by

$$\mathbf{A}=\begin{bmatrix}\beta_k & 0 & 0\\ -\alpha_{k-1}\beta_k & \beta_k & 0\\ \widehat{\beta}_k-\beta_k & 0 & \beta_k\end{bmatrix},\quad \overline{\mathbf{A}}=\begin{bmatrix}\gamma_k & 0 & 0\\ -\alpha_{k-1}\gamma_k & \gamma_k & 0\\ \widehat{\gamma}_k-\gamma_k & -\widehat{\gamma}_{k+1} & \gamma_k\end{bmatrix},$$

$$\mathbf{U}=\begin{bmatrix}-\alpha_{k-1} & -\alpha_{k-2} & \cdots & -\alpha_1 & -\alpha_0\\ \alpha_{k-1}\alpha_{k-1}-\alpha_{k-2} & \alpha_{k-1}\alpha_{k-2}-\alpha_{k-3} & \cdots & \alpha_{k-1}\alpha_1-\alpha_0 & \alpha_{k-1}\alpha_0\\ -\widehat{\alpha}_{k-1} & -\widehat{\alpha}_{k-2} & \cdots & -\widehat{\alpha}_1 & -\widehat{\alpha}_0\end{bmatrix},$$

$$\mathbf{B}=\begin{bmatrix}\widehat{\beta}_k-\beta_k & 0 & \beta_k\\ 0 & 0 & 0\\ \vdots & \vdots & \vdots\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \overline{\mathbf{B}}=\begin{bmatrix}\widehat{\gamma}_k-\gamma_k & -\widehat{\gamma}_{k+1} & \gamma_k\\ 0 & 0 & 0\\ \vdots & \vdots & \vdots\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\mathbf{V}=\begin{bmatrix}-\widehat{\alpha}_{k-1} & -\widehat{\alpha}_{k-2} & \cdots & -\widehat{\alpha}_1 & -\widehat{\alpha}_0\\ 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix},$$

where $\mathbf{A},\overline{\mathbf{A}}\in\mathbb{R}^{3\times3}$, $\mathbf{U}\in\mathbb{R}^{3\times k}$, $\mathbf{B},\overline{\mathbf{B}}\in\mathbb{R}^{k\times3}$, $\mathbf{V}\in\mathbb{R}^{k\times k}$.

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3 PMESDMMs and FPMESDMMs

In the perturbed MESDMMs, the coefficient matrices $\mathbf{B}$ and $\overline{\mathbf{B}}$, which will be denoted by B and $\overline{B}$ respectively, take the forms

$$B = \mathbf{B} + \mathsf{B}, \quad \overline{B} = \overline{\mathbf{B}} + \overline{\mathsf{B}}.$$

where $\mathsf{B} = [\mathsf{b}_{ij}]$ and $\overline{\mathsf{B}} = [\overline{\mathsf{b}}_{ij}]$. We consider two different cases for which $\mathsf{b}_{1j} = \overline{\mathsf{b}}_{1j} = 0$, which we call perturbed MESDMM (PMESDMM) and $\mathsf{b}_{1j} \neq 0 \neq \overline{\mathsf{b}}_{1j}$, which we call fully perturbed MESDMM (FPMESDMM), respectively. After satisfying order conditions, the remaining free parameters for B and $\overline{\mathsf{B}}$ will be used to maximize the angle α of $A(\alpha)$-stability of the methods.

MESDMMs are L-stable for $k = 1, 2, 3, 4$. Hence, we search for perturbed methods for $k = 5, 6, \ldots, 12$. In Table 1 we can recognize the improvement in the stability region in PMESDMMs and FPMESDMMs compared with MESDMMs.

Table 1: Angles of $A(\alpha)$-stability of ESDMM, MESDMM, PMESDMM and FPMESDMM for $k = 5, 6, \ldots, 12$.

k	5	6	7	8	9	10	11	12
α for ESDMM	89.81°	88.35°	85.28°	80.47°	73.58°	63.98°	50.36°	29.90°
α for MESDMM	89.86°	88.49°	85.83°	81.81°	76.34°	69.19°	59.37°	44.24°
α for PMESDMM	89.99°	89.81°	88.70°	86.77°	86.02°	84.31°	77.43°	76.13°
α for FPMESDMM	90°	89.85°	89.12°	88.48°	87.40°	86.16°	82.05°	77.59°

As an example we will take the case where $k = 9$, the corresponding stability regions are plotted in Fig. 1.

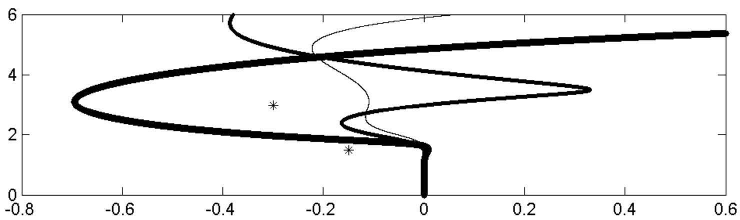

Figure 1: Stability regions near the origin of FPMESDMM (thin line), PMESDMM (medium line) and MESDMM (thick line) for $k = 9$ and the points $h_1(-3 + 30i)$, $h_2(-3 + 30i)$ with $h_1 = 0.1$ and $h_2 = 0.05$.

To illustrate the effect of this improvement in the stability properties, consider the following stiff initial value problem

$$\begin{aligned} y_1' &= -3y_1 - 30y_2 + 32e^{-x}, \quad y_1(0) = 1, \\ y_2' &= \ \ 30y_1 - 3y_2 - 28e^{-x}, \quad y_2(0) = 1, \end{aligned} \tag{3}$$

which has the exact solution $y_1(x) = y_2(x) = e^{-x}$. Eigenvalues of the Jacobian of the system are $-3 \pm 30i$. We apply MESDMM and its perturbed and fully perturbed versions on this problem with fixed stepsizes $h_1 = 0.1$ and $h_2 = 0.05$ for $k = 9$. Considering the absolute stability regions, as it is shown in Figure 1, $h_1(-3 \pm 30i)$ lies outside of the stability region for MESDMM while it lies inside of this region for PMESDMM and FPMESDMM. However, $h_2(-3 \pm 30i)$ lies inside of the stability region for the all mentioned methods. We report the global error at the end of the interval of integration $[0, 60]$ in Table 2. As we expect, for $h = h_1$, the results show that the absolute instability of MESDMM manifests itself in the form of a violently growing error, whereas the other two methods give good approximations.

Table 2: The global error at the end of the interval of integration $[0, 60]$ for MESDMM, PMESDMM and FPMESDMM applied to problem (3).

k	h	MESDMM	PMESDMM	FPMESDMM
9	0.1	6.6985e+14	5.8835e−38	5.9250e−38
	0.05	9.7594e−38	9.7601e−38	9.7640e−38

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Superconvergence of a finite element approximation to the solution of double discrete barrier option

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Abstract
The paper presents a numerical study on the behavior of the finite element method in solving the double discrete barrier option pricing. Such an analysis highlights that if the finite element method is carried out following directions given by [1] and the computational mesh is aligned with the options payoff, then the solutions obtained are superconvergent at the boundaries of the finite elements. To the best of our knowledge, superconvergent approximations of the proposed model have never been observed so far, and are somehow unexpected as the initial solutions of the problems as well in discrete times, considered have various kinds of irregularities.
Keywords: Finite Element Method, Vanilla Option, Double Barrier Option, Superconvergence

1 Introduction

A very popular approach to derivative pricing is the use of mathematical models based on partial differential equations. In particular, among the most commonly employed models, there is the famous Black-Scholes (BS) model, which has been originally proposed in [2].
Contrary to the FDM, which performs a direct discretization of the differential operators, the FEM is based on the weak integral formulation of the equation being solved.

The standard theory of the finite element method predicts convergence in the L^2 norm as $O(h^{r+1})$, where r is the degree of r polynomial used in each element. In particular, for parabolic partial differential equations in one space variable, Bakker [1] has shown that, if the FEM is performed with a uniform mesh, if the variational integrals are computed using Gauss-Lobatto quadrature and if the initial solution is collocated at Gauss-Lobatto nodes, then at the boundaries of the finite elements the convergence rate is $O(h^{2r})$.

2 Discrete Double Barrier Option

The mathematical model of the proposed option pricing at n discrete times satisfies the Black-Scholes partial differential equation

$$\frac{\partial V}{\partial \tau} - rS\,\frac{\partial V}{\partial S} - \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + rV = 0, \qquad S\varepsilon(0,\infty) \tag{1}$$

with the following condition:

$$V(S,T) = \max(S-E,0) \tag{2}$$

$$\lim_{S\to+\infty} V(S,t) = S - Ke^{-rt}$$

$$V(S,t_1) = V(S,t_2) = \ldots = V(S,t_n) = 0, \qquad L \le S \le U \tag{3}$$

$$V(0,t) = 0 \tag{4}$$

As we see in relation (3) we have N unregular point at discrete times t_1, t_2,. . . , t_n which the option is zeros when $S > U$ and $S < L$. Then from the financial point, with the high probability the stock price S_{max} never fail to the barrier U and thus between discrete barrier times the option value is zero and just for the $S\epsilon(T-\delta t, T)$ the option value behaves like vanilla option.
Then we have:

$$\lim_{S\to+\infty} V(S,t) = \begin{cases} S - Ke^{-rt}, & S\epsilon(T-\delta t, T) \\ 0 & O.W. \end{cases} \tag{5}$$

3 Finite Element Approximation

In this section we will present the FEM directly by considering the BS equation. The proposed problem's domain changed from the domain $S\epsilon[S_{min}, S_{max}]$ to the interval $y\epsilon[0,1]$ by the following change of variable:

$$\phi(y) = (S_{max} - S_{min})y + S_{min} \tag{6}$$

and equation (1) is changed to:

$$\frac{\partial u(y,\tau)}{\partial \tau} = z_1(y)\frac{\partial^2 u(y,\tau)}{\partial y^2} + z_2(y)\frac{\partial u(y,\tau)}{\partial y} - ru(y,\tau), \tag{7}$$

which $z_1(y)$ and $z_2(y)$ are defiened as below:

$$z_1(y) = \frac{1}{2}\sigma^2\left(\frac{\varphi(y)}{\varphi'(y)}\right)^2 \tag{8}$$

$$z_2(y) = r\frac{\varphi(y)}{\varphi'(y)} - \frac{1}{2}\sigma^2\frac{\varphi^2(y)\varphi''(y)}{(\varphi'(y))^3}. \tag{9}$$

The weak formulation of the BS equation is then obtained, and finally the solution to the proposed option pricing problems is found by solving an algebraic system. First of all we define the FEM intervals of equal length h, whose boundaries are $y_0, y_1, ..., y_N$, and define I_j is the finite element $[y_{j-1}, y_j], j = 1, 2, .., N$, as well note that the srike price E should be one of the boundary nodes of the interval I_j .
We want to approximate the solution as a linear combination of piecewise polynomial functions of degree smaller or equal to r. Then we choose $N_1 = rN + 1$ nodes of $\eta_0, \eta_1, ..., \eta_{N_1}$ obtained by considering the Gauss-Lobatto nodes of each finite element $I_j, j = 1, 2, .., N$, and r denotes the degree of polynomials used in this paper. we multiply the relation (1) in $w_j(y)$ and integrate in the whole domain, to reach the weak form solution as follows:

$$\int_I \left(\frac{\partial u(y,\tau)}{\partial \tau} - z_1(y)\frac{\partial^2 u(y,\tau)}{\partial y^2} - z_2(y)\frac{\partial u(y,\tau)}{\partial y} + ru(y,\tau)\right)w_j(y)dy = 0, \tag{10}$$

which j goes from 1 to N_1 and I denotes the domain $[0,1]$.
Using integration by parts equation (5) is rewritten as follows:

$$\int_I \left[\left(\frac{\partial u(y,\tau)}{\partial \tau} + \left(\frac{dz_1(y)}{dy} - z_2(y)\right)\frac{\partial u(y,\tau)}{\partial y} + ru(y,\tau)\right)w_j(y) - z_1(y)\frac{\partial u(y,\tau)}{\partial y}\frac{\partial w_j(y)}{\partial y}\right]dy = 0, \tag{11}$$

which comes from integration by parts. The weak formulation for each shape function from node 1 to N_1 is made.
We set in relation (6) the following approach:

$$u(y,\tau)=\sum_{j=1}^{N_1} w_j(y)u_j(\tau). \tag{12}$$

By tranforming each inetrval $I_j, j=1,2,..,N$ to interval $[-1,1]$, we choose lobatto points in this interval, and then converting the mentioned transforming, we construct unequall distance in the whole domain $[0,1]$ and evaluating each integral by Gausse Lobatto polynomials, we obtain a system of N_1-2 linear ordinary equations whose unknowns are the functions $u_j(\tau), \quad j=1,2,...,N_1$. The coefficient matrix of integral form of relation (6) for the shape functions $w_j(y)$ are written as follows:

$$M_{i,j} = \int_I w_i(y)w_j(y)dy \tag{13}$$

$$H_{i,j} = \int_I (\frac{dz_1(y)}{dy}-z_2(y))w_i(y)\frac{dw_j(y)}{dy}dy \tag{14}$$

$$K_{i,j} = \int_I z_1(y)\frac{dw_i(y)}{dy}\frac{dw_j(y)}{dy}dy \qquad i=2,3,...,N_1-1, j=1,2,...,N_1. \tag{15}$$

By defining the transpose vector of $U(t)=[u_1(t),u_2(t),...,u_{N_1}(t)]$, the system of time dependent of the equation (6) is formulated as:

$$M\frac{dU(\tau)}{dt}+(rM+H-K)U(\tau)=0, \tag{16}$$

we have obtained a system of N_1-2 ordinary differential equations in the N_1 unknown functions u_1, u_2, ..., u_{N_1} and we add boundary conditions at $S=S_0$ and $S=S_{\max}$ have not been imposed yet.
By adding the boundary conditions to the first and last row of above matrices (8)-(10) as follows:

$$u_1(\tau)=0, \qquad u_{N_1}(\tau)=S_{max}-Ee^{-r\tau}. \tag{17}$$

Note that the initial condition is imposed by collocation at the Gauss-Lobatto nodes.

4 Results

In this section we directly focus onto the case of piecewise quadratic FEM but also other approximation spaces could be considered. In below, we tabulated input parameters in table 1 and by using quadratic FEM in tables 1 and report the errors and their ratios for discrete double barrier option. In order to reaching superconvergence it is very important to insert the barriers L and U in FEM boundary nodes. Because of this we know the stock prices do not exceed from $S=S_{max}$ and then from the financial point we conclude the the stock price also do not fail from $S=S_{min}$, then by choosing proper values of S_{min} and S_{max} we can adjust the nodes which the barriers L and U are in FEM nodes.

Volatility σ	Interest Rate r	Maturity T	Strike Price E	Down Barrier L	Up Barrier U
0.2	0.05	0.5	80	60	120

Table 1: Parameters and data for the discrete double barrier option

4.1 The quadratic FEM

By testing quadratic FEM, we see the rate of convergence is clearly 4, as ratio is very closed to 16, and the results agree with the theoretical one by Bakker [1].

Number of Finite Elements	Error	Ratio
256	5.75e-008	
128	9.82e-007	17.06
64	1.60e-005	16.34
32	2.79e-004	17.40
16	6.48e-003	23.19

Table 2: Superconvergence of quadratic FEM on discrete double barrier option by choosing $S_{max} = 200$ and $S_{min} = 50$

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The modified variational iteration method and Elzaki transform for solving heat-like equations

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Abstract

In this paper, we present a reliable combination of modified variational iteration method and elzaki transform to investigate some Heat-Like equations. The proposed variational iteration method is applied to the reformulated first and second order initial value problem which leads the solution in terms of transformed variables, and the series solution is obtained by making use of the inverse transformation. The results show the efficiency of this method.

Keywords: Modified variational iteration elzaki transform method, Heat-like equations.

Mathematics Subject Classification: 35J60, 35A05

1 Introduction

Linear and nonlinear partial differential equations are of fundamental importance in science and engineering. In recent years, many research workers have paid attention to find the solutions of linear and nonlinear differential equations by using various methods. Among these are the Adomian decomposition method, Variational iteration method, the extended tanh method, the homotopy perturbation method and so on [1, 2, 3, 4].
In this work, we will use the modified form of Elzaki variational iteration method is called Modified Variational Iteration Elzaki Transform Method will be employed in a straight forward manner. This method provides an effective and efficient way of solving a wide range of linear and nonlinear operator equations. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for linear and nonlinear partial differential equations. This article considers the effectiveness of the Modified Variational Iteration Elzaki transform method in solving partial differential equations .

2 The Modified Variational Iteration Elzaki Transform Method

Consider a general nonlinear non-homogenous partial differential equation with initial conditions of the form:

$$L[u(x,t)] + R[u(x,t)] + N[u(x,t)] = g(x,t) \tag{1}$$

With the initial condition

$$u(x,0) = h(x) \quad , \quad u_t(x,0) = f(x) \tag{2}$$

where L is Linear operator, R is linear operator less than L, N is nonlinear operator, $g(x,t)$ is inhomogeous term. we assume in this work L is operator $\frac{\partial^2}{\partial t^2}$.
Taking elzaki transform on both sides of equation (1), to get:

$$E[Lu(x,t)] + E[Ru(x,t)] + E[Nu(x,t)] = E[g(x,t)]$$

Using the differentiation property of elzaki transforms and initial conditions (2), we have:

$$E[u(x,t)] = v^2 E[f(x,t)] + v^2 h(x) + v^3 f(x) - v^2 E[Ru(x,t) + Nu(x,t)] \tag{3}$$

Applying the inverse elzaki transform on both sides of equation (3), to find:

$$u(x,t) = G(x,t) - E^{-1}\{v^2 E[Ru(x,t) + Nu(x,t)]\} \tag{4}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions.
Derivative by $\frac{\partial}{\partial t}$ both side (4)

$$u_t(x,t) - \frac{\partial}{\partial t}G(x,t) + \frac{\partial}{\partial t}E^{-1}\{v^2 E[Ru(x,t) - \frac{\partial}{\partial t}Nu(x,t)]\} = 0$$

By the correction function of the irrational method

$$u_{n+1} = u_n - \int_0^t (u_n)_t(x,\xi) - \frac{\partial}{\partial \xi}G(x,\xi) + \frac{\partial}{\partial \xi}E^{-1}\{p^2 E[Ru_n(x,\xi) - \frac{\partial}{\partial \xi}Nu_n(x,\xi)]\} \tag{5}$$

Solution u is given by

$$u = \lim_{n\to\infty} u_n \tag{6}$$

3 Iliustrative Examples

In this section, we solve the Heat-Like equations with the modified variational iteration elzaki transform method, therefore we have:

Example 3.1. *We consider the following one dimensional heat-like equation*

$$u_t = \frac{1}{2}x^2 u_{xx} \quad , \quad 0 < x < 1, t > 0$$

With initial condition

$$u(x,0) = x^2$$

We apply the modified variational iteration elzaki transform method therefore have:

$$u_{n+1} = u_n - \int_0^t (u_n)_\xi - \frac{\partial}{\partial \xi}E^{-1}\{vE[\frac{1}{2}x^2(u_n)_{xx}]\}d\xi$$

Therefore solution is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = x^2(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + \cdots) = x^2 e^t$$

Example 3.2. *We consider the following two dimensional heat-like equation*

$$u_t = \frac{1}{2}(y^2 u_{xx} + x^2 u_{yy}) \quad , \quad 0 < x, y < 1, t > 0$$

With initial condition

$$u(x,y,0) = y^2$$

We apply the modified variational iteration elzaki transform method therefore have:

$$u_{n+1} = u_n - \int_0^t (u_n)_\xi - \frac{\partial}{\partial \xi} E^{-1}\{vE[\frac{1}{2}(y^2(u_n)_{xx} + x^2(u_n)_{yy})]\}d\xi$$

Therefore the solution is given by

$$\begin{aligned} u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t) &= y^2(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots + \frac{t^{2n}}{(2n)!} + \cdots) \\ &+ x^2(t + \frac{t^2}{3!} + \frac{t^5}{5!} + \cdots + \frac{t^{2n+1}}{(2n+1)!} + \cdots) \\ &= y^2 \cosh t + x^2 \sinh t \end{aligned}$$

Example 3.3. *We consider the following three dimensional heat-like equation*

$$u_t = x^4y^4z^4 + \frac{1}{36}(x^2u_{xx} + y^2u_{yy} + z^2u_{zz}) \quad , \quad 0 < x,y,z < 1, t > 0$$

With initial condition

$$u(x,y,z,0) = 0$$

We apply the modified variational iteration elzaki transform method therefore have:

$$u_{n+1} = u_n - \int_0^t (u_n)_\xi - x^4y^4z^4 - \frac{\partial}{\partial \xi} E^{-1}\{vE[\frac{1}{36}(x^2(u_n)_{xx} + y^2(u_n)_{yy} + +z^2(u_n)_{zz})]\}d\xi$$

Hence the solution is given by

$$\begin{aligned} u(x,y,z,t) = \lim_{n\to\infty} u_n(x,y,z,t) &= x^4y^4z^4(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + \cdots) \\ &= x^4y^4z^4(e^t - 1) \end{aligned}$$

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Solution of boundary value problems by Spline radial basis functions

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Abstract

In this paper, we convert the linear and nonlinear differential equations with separated boundary conditions into optimal control problems. Then we construct a convergent approximate solution which satisfies the exact boundary conditions. some examples are solved and the optimal error are given.

Keywords: boundary value problems, least square approximation, thin plate splines, radial basis functions

Mathematics Subject Classification: 34A45

1 Introduction and Preliminaries

Boundary value problems manifest themselves in very branches of science. For example engineering, technology, control and optimization theory. In the present paper a spline radial basis function is used to solve the boundary value problems.
Consider a boundary value problems in the general form

$$\begin{cases} g(x,y,y',...,y^{(m)}) = 0 \ \ , \ a \leq x \leq b \\ \sum_{j=0}^{m-1} \alpha_{i,j} y^{(j)}(a_i) = A_i \ \ , \ 0 \leq i \leq m-1 \end{cases} \tag{1}$$

where $a \leq a_i \leq b$ and A_i are real values. Our presentation finds a sequence of functions $\{v_k\}$ of the form

$$v_k(t) = \sum_{i=0}^{n} c_i \phi_i(t), \tag{2}$$

which satisfy the exact separated boundary conditions, where the $\phi_i(t)$'s are basis functions.

2 Thin Plate Spline Radial Basis Functions

d-th order thin plate spline Radial Basis Function is defined as

$$\phi_k^{(d)}(t) = \begin{cases} r^{2k-d} log(r) & k \geq d \ \ , \ \ d \in 2\mathbb{N} \\ r^{2k-d} & k \geq d \ , \ d \in 2\mathbb{N}+1 \end{cases} \tag{3}$$

where $r = \|x\|$ is the Euclidean norm. these functions are basic solutions of the equation $\Delta^k \phi = 0$ in d-dimensions [4].

3 Explanation of the Method

Consider the differential equation

$$g(t, y(t), y'(t), ..., y^{(m)}(t)) = 0, \quad a < t < b \tag{4}$$

with the general separated boundary conditions

$$\begin{cases} \alpha_{00}y(a_0) + \alpha_{01}y'(a_0) + ... + \alpha_{0,m-1}y^{(m-1)}(a_0) = A_0 \\ \alpha_{10}y(a_1) + \alpha_{11}y'(a_1) + ... + \alpha_{1,m-1}y^{(m-1)}(a_1) = A_1 \\ \vdots \\ \alpha_{m-1,0}y(a_{m-1}) + ... + \alpha_{m-1,m-1}y^{m-1}(a_{m-1}) = A_{m-1} \end{cases} \tag{5}$$

and let $g : \mathbb{R}^{m+2} \to \mathbb{R}$ be a continuous function . We convert the problem to an optimal control problem

$$\min_y \int_a^b [g(t, y(t), y'(t), ..., y^{(m)}(t))]^2 dt$$

and separated boundary conditions

$$\sum_{j=0}^{m-1} \alpha_{i,j} y^{(j)}(a_i) = A_i \quad , \; 0 \le i \le m-1$$

The actual solution of (3)-(4) is a function v such that

$$\begin{cases} \|g(t, v(t), v'(t), ..., v^{(m)}(t))\|^2_{L^2([a,b])} = 0 \\ \sum_{j=0}^{m-1} \alpha_{i,j} v^{(j)}(a_i) = A_i \quad , \; 0 \le i \le m-1 \; . \end{cases}$$

The sketch of the method is delineated as follows:
For a fixed $n \in \mathbb{N}$, consider a uniform partition $a < a + h < a + 2h < \ldots < a + nh = b$ on $[a, b]$ where $h = \dfrac{b-a}{n}$. Define

$$\phi_i(t) = \phi_k^{(d)}(\frac{t-a}{h} - i), \quad (i = 0, 1, \ldots, n-1),$$

where $\phi_k^{(d)}$ is a scaling function and $\phi_i, (\; i = 0, 1, \ldots, n-1)$ are translations and dilations of $\phi_k^{(d)}$. Let

$$v_k(t) = \sum_{i=0}^{n} c_i \phi_i(t), \tag{6}$$

where the coefficients $\{c_i\}$ are determined from the conditions

$$\sum_{j=0}^{m-1} \alpha_{i,j} v_k^{(j)}(a_i) = A_i \quad , \; 0 \le i \le m-1 \tag{7}$$

and the following least square problem:

$$\min_{c_i} \|g(t, v_k(t), v_k'(t), \ldots, v_k^{(m)}(t))\|^2_{L^2([a,b])}.$$

The minimization problem is equivalent to the following system:

$$\begin{cases} \frac{\partial}{\partial c_i} \|g(t, v_k(t), v_k'(t), \ldots, v_k^{(m)}(t))\|^2_{L^2([a,b])} = 0, \quad (i = 0, 1, \ldots, n-1) \\ \sum_{j=0}^{m-1} \alpha_{i,j} v_k^{(j)}(a_i) = A_i \quad , \; 0 \le i \le m-1 \; . \end{cases}$$

4 An Example

To illustrate the efficiency of the presented method, the solution a numerical example is presented. In this example the resulting minimization problem was solved by maple 15.

Example 4.1 ([5], Example 1). *Consider the linear boundary value problem*

$$y^{(4)}(x) + xy(x) = -(8 + 7x + x^3)\exp(x), \qquad 0 \le x \le 1 \tag{8}$$
$$y(0) = y(1) = 0$$
$$y''(0) = 0, \; y''(1) = -4\exp(1)$$

with exact solution

$$y(x) = x(1 - x)\exp(x) \tag{9}$$

we use the method and obtain this results:

Table 1: least square error(LSE) with $\phi_i(r) = r^9$.

$y_k^{(j)}$	$LSE(n = 4)$	$LSE(n = 5)$
$j = 0$	$3.9966e - 9$	$3.0212e - 18$
$j = 1$	$4.4446e - 7$	$3.6281e - 16$
$j = 2$	$5.4086e - 5$	$5.7914e - 14$
$j = 3$	$7.1875e - 3$	$1.1819e - 11$
$j = 4$	1.1610	$3.9071e - 9$

Table 2: least square error(LSE) with $\phi_i(r) = r^{11}$.

$y_k^{(j)}$	$LSE(n = 7)$
$j = 0$	$6.4785e - 25$
$j = 1$	$1.6659e - 22$
$j = 2$	$5.0581e - 20$
$j = 3$	$5.0581e - 17$
$j = 4$	$1.0507e - 14$

5 Conclusion

The existence and uniqueness of the solution is guaranteed by Agarwal s book [1]. Comparing the obtained results with other works [2, 3, 5], this method was clearly reliable in comparing with grid points techniques where solution is defined at grid points only. Moreover the method yields a good result even for small n.

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The general solution to system of linear quaternion matrix equations with applications

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Abstract

We in this paper establish necessary and sufficient conditions for the existence of and an explicit expression for a common solution to six linear quaternion matrix equations $XA_1 = C_1$, $XA_2 = C_2$, $XA_3 = C_3$, $XA_4 = C_4$, $A_5XB_5 = C_5$, $A_6XB_6 = C_6$. As an application, centrosymmetric solution to certain system of quaternion matrix equations is presented.

Keywords: The general solution, Quaternion matrix equations, Centrosymmetric solution.

Mathematics Subject Classification: 15A24, 15A33, 15A09

1 Introduction

We denote the set of all $m \times n$ matrices over the quaternion field

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k | i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$; the symbols I and A^+ stand for the identity matrix with the appropriate size and a reflexive inverse of matrix A over $\mathbb{H}$ by which satisfies simultaneously $AA^+A = A$ and $A^+AA^+ = A^+$, respectively. Moreover, R_A and L_A denote the two projectors $L_A = I - A^+A$, $R_A = I - AA^+$ induced by A.

Researches on solvability conditions and general solutions to systems of linear matrix equations have been actively ongoing for more than 30 years.

In this paper we investigate system

$$XA_1 = C_1, \quad XA_2 = C_2, \quad XA_3 = C_3, \quad XA_4 = C_4, \quad A_5XB_5 = C_5, \quad A_6XB_6 = C_6, \tag{1}$$

and also have a good observation that investigating the general solution to system (4) can give the necessary and sufficient condition for the existence and expression centrosymmetric solution to system

$$XA_1 = C_1, \qquad XA_3 = C_3, \qquad A_5XB_5 = C_5. \tag{2}$$

over $\mathbb{H}$. The quaternion matrices have many applications in quantum mechanics [1] and signal processing [2].

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2 The general solution to system (4)

In this section, we consider the general solution of system (4) over $\mathbb{H}$. We begin with the following.

Lemma 2.1. *[3] Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{r\times s}$, $C \in \mathbb{H}^{m\times s}$ be known and $x \in \mathbb{H}^{n\times r}$ unknown. Then the following statements are equivalent:*
(i)The system

$$AXB = C, \tag{3}$$

is consistent.
(ii)$R_AC = 0,\ CL_B = 0$.
In that case, the general solution of (3) is

$$X = A^+CB^+ + L_AY_1 + Y_2R_B,$$

where Y_1 and Y_2 are any matrices over $\mathbb{H}$ with appropriate dimensions.

Lemma 2.2. *[4] Let $A_1 \in \mathbb{H}^{m\times r}$, $A_2 \in \mathbb{H}^{m\times s}$, $C_1 \in \mathbb{H}^{n\times r}$, $C_2 \in \mathbb{H}^{n\times s}$, be known and $X \in \mathbb{H}^{n\times m}$ unknown, $T = R_{A_1}A_2$, $Q = A_2L_T$, then*

$$XA_1 = C_1, \qquad XA_2 = C_2, \tag{4}$$

is consistent if and only if

$$C_iA_i^+A_i = C_i, \quad i = 1, 2; \qquad (C_2A_2^+ - C_1A_1^+)Q = 0.$$

In that case, the general solution of (4) can be expressed as the following

$$X = C_1A_1^+ + (C_2A_2^+ - C_1A_1^+)A_2T^+R_{A_1} + YR_TR_{A_1},$$

where Y is an arbitrary matrix over $\mathbb{H}$ with appropriate size.

Theorem 2.3. *Let $A_1 \in \mathbb{H}^{m\times r}$, $A_2 \in \mathbb{H}^{m\times s}$, $A_3 \in \mathbb{H}^{m\times k}$, $A_4 \in \mathbb{H}^{m\times l}$, $A_5 \in \mathbb{H}^{t\times n}$, $A_6 \in \mathbb{H}^{p\times n}$, $C_1 \in \mathbb{H}^{n\times r}$, $C_2 \in \mathbb{H}^{n\times s}$, $C_3 \in \mathbb{H}^{n\times k}$, $C_4 \in \mathbb{H}^{n\times l}$, $C_5 \in \mathbb{H}^{t\times d}$, $C_6 \in \mathbb{H}^{p\times q}$, $B_5 \in \mathbb{H}^{m\times d}$, $B_6 \in \mathbb{H}^{m\times q}$ be known and $X \in \mathbb{H}^{n\times m}$ unknown;*

$$\begin{array}{lllll} A = R_{A_1}A_3, & T = R_{A_1}A_2, & S = R_TR_{A_1}, & B = R_TA, & C = R_{A_1}A_4, \\ D = R_BSA_4, & Q = A_2L_T, & G = R_DR_BSB_5, & R = R_DR_BSB_6, & \\ H = R_GR, & J = A_6L_{A_5}, & & & \end{array}$$

$$\begin{aligned} \Psi &= C_3 - C_1A_1^+A_3 - (C_2A_2^+ - C_1A_1^+)A_2T^+A, \\ \Phi &= C_4 - C_1A_1^+A_4 - (C_2A_2^+ - C_1A_1^+)A_2T^+C - \Psi B^+SA_4, \\ E &= C_5 - A_5C_1A_1^+B_5 - A_5(C_2A_2^+ - C_1A_1^+)A_2T^+R_{A_1}B_5 - A_5\Psi B^+SB_5 \\ &\quad - A_5\Phi D^+R_BSB_5, \\ F &= C_6 - A_6C_1A_1^+B_6 - A_6(C_2A_2^+ - C_1A_1^+)A_2T^+R_{A_1}B_6 - A_6\Psi B^+SB_6 \\ &\quad - A_6\Phi D^+R_BSB_6 - A_6A_5^+EG^+R, \end{aligned}$$

Then the following statements are equivalent:
(i) System (4) is solvable.
(ii) The following equalities are all satisfied

$$\begin{array}{lllll} C_iL_{A_i} = 0, & i = 1, \cdots, 4, & R_{A_5}C_5 = 0, & C_5L_{B_5} = 0, & C_6L_{B_6} = 0, \\ R_{A_6}C_6 = 0, & (C_2A_2^+ - C_1A_1^+)Q = 0, & \Phi L_D = 0, & \Psi L_B = 0, & \\ R_{A_6}F = 0, & R_{A_5}E = 0, & EL_G = 0, & FL_R = 0, & R_JFL_H = 0. \end{array}$$

In that case, the general solution of the system (4) can be expressed as

$$\begin{aligned}X = {} & C_1A_1^+ + (C_2A_2^+ - C_1A_1^+)A_2T^+R_{A_1} + \Psi B^+S + \Phi D^+R_BS + A_5^+EG^+R_DR_BS \\ & +L_{A_5}J^+FL_H(RL_H)^+R_DR_BS - A_6^+JJ^+FL_H(RL_H)^+RH^+R_GR_DR_BS \\ & +A_6^+FH^+R_GR_DR_BS + ZR_GR_DR_BS - A_6^+A_6ZHH^+R_GR_DR_BS \\ & -A_6^+JWRH^+R_GR_DR_BS + A_6^+JWRL_H(RL_H)^+RH^+R_GR_DR_BS \\ & +L_{A_5}WR_DR_BS - L_{A_5}J^+JWRL_H(RL_H)^+R_DR_BS.\end{aligned}$$

where W, Z are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

3 Centrosymmetric solution to the system (4)

In this section, we use the results of theorem 2.3 to establish the solvability conditions and expressions of centrosymmetric solution to (4).

Definition 3.1. *The matrix $A = (a_{ij}) \in \mathbb{H}^{m\times r}$ is called centrosymmetric if $a_{ij} = a_{m-i+1,n-j+1}$ i.e., $A = A^{\#}$, where $A^{\#} = (a_{m-i+1,n-j+1})$.*

Now we consider the centrosymmetric solution to the system (4).

Theorem 3.2. *Let $A_1 \in \mathbb{H}^{m\times r}$, $A_3 \in \mathbb{H}^{m\times k}$, $A_5 \in \mathbb{H}^{t\times n}$, $C_1 \in \mathbb{H}^{n\times r}$, $C_3 \in \mathbb{H}^{n\times k}$, $C_5 \in \mathbb{H}^{t\times d}$, $B_5 \in \mathbb{H}^{m\times d}$, be known and $X \in \mathbb{H}^{n\times m}$ unknown;*

$$\begin{aligned}& A = R_{A_1}A_3, \quad T = R_{A_1}A_1^{\#}, \quad S = R_TR_{A_1}, \quad B = R_TA, \quad C = R_{A_1}A_3^{\#}, \\ & D = R_BSA_3^{\#}, \quad Q = A_1^{\#}L_T, \quad G = R_DR_BSB_5, \quad R = R_DR_BSB_5^{\#}, \\ & H = R_GR, \quad J = A_5^{\#}L_{A_5},\end{aligned}$$

$$\begin{aligned}\Psi = {} & C_3 - C_1A_1^+A_3 - (C_1^{\#}A_2^+ - C_1A_1^+)A_1^{\#}T^+A, \\ \Phi = {} & C_3^{\#} - C_1A_1^+A_3^{\#} - (C_1^{\#}A_2^+ - C_1A_1^+)A_1^{\#}T^+C - \Psi B^+SA_3^{\#}, \\ E = {} & C_5 - A_5C_1A_1^+B_5 - A_5(C_1^{\#}A_2^+ - C_1A_1^+)A_1^{\#}T^+R_{A_1}B_5 - A_5\Psi B^+SB_5 \\ & -A_5\Phi D^+R_BSB_5, \\ F = {} & C_5^{\#} - A_5^{\#}C_1A_1^+B_5^{\#} - A_5^{\#}(C_1^{\#}A_2^+ - C_1A_1^+)A_1^{\#}T^+R_{A_1}B_5^{\#} - A_5^{\#}\Psi B^+SB_5^{\#} \\ & -A_5^{\#}\Phi D^+R_BSB_5^{\#} - A_5^{\#}A_5^+EG^+R,\end{aligned}$$

Then the following statements are equivalent: (i) System (4) is solvable. (ii) The following equalities are all satisfied

$$\begin{aligned}& C_iL_{A_i} = 0, \quad i = 1, \cdots, 4, \quad R_{A_5}C_5 = 0, \quad C_5L_{B_5} = 0, \quad C_5^{\#}L_{B_5^{\#}} = 0, \\ & R_{A_5^{\#}}C_5^{\#} = 0, \quad (C_1^{\#}A_2^+ - C_1A_1^+)Q = 0, \quad \Phi L_D = 0, \quad \Psi L_B = 0, \\ & R_{A_5^{\#}}F = 0, \quad R_{A_5}E = 0, \quad EL_G = 0, \quad FL_R = 0, \quad R_JFL_H = 0.\end{aligned}$$

In that case, the general solution of the system (4) can be expressed as

$$X = \frac{1}{2}(X_1 + X_1^{\#})$$

where

$$\begin{aligned}X_1 = {} & C_1A_1^+ + (C_1^{\#}A_2^+ - C_1A_1^+)A_1^{\#}T^+R_{A_1} + \Psi B^+S + \Phi D^+R_BS \\ & + A_5^+EG^+R_DR_BS + L_{A_5}J^+FL_H(RL_H)^+R_DR_BS \\ & - A_6^+JJ^+FL_H(RL_H)^+RH^+R_GR_DR_BS + A_6^+FH^+R_GR_DR_BS \\ & + ZR_GR_DR_BS - A_6^+A_5^{\#}ZHH^+R_GR_DR_BS - A_6^+JWRH^+R_GR_DR_BS \\ & + A_6^+JWRL_H(RL_H)^+RH^+R_GR_DR_BS + L_{A_5}WR_DR_BS \\ & - L_{A_5}J^+JWRL_H(RL_H)^+R_DR_BS.\end{aligned}$$

and W, Z are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

4 Main Result

We have derived necessary and sufficient conditions for the existence of and the expression for the general solution to system (4) over $\mathbb{H}$. As an application, we have presented necessary and sufficient condition for the existence and the general expressions of centrosymmetric solution to system (4).

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Numerical solution of nonlinear ordinary differential equations using Bernstein polynomials and their orthonormal duals

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Abstract

This paper is concerned with the construction of a biorthogonal basis for the Bernstein polynomials to form a biorthogonal Bernstein system in the unit interval. The structure of the biorthogonal basis raise to simple computations. To this end, we construct operational matrix of derivatives which ease solution an ODE. Finally, we consider some test problems with known solutions to show efficiency and accuracy of our method.

Keywords: Bernstein polynomials; biorthogonal system; operational matrix of derivative; Block Pulse functions

Mathematics Subject Classification: 53A15

1 Introduction

Differential equations appear in several branches of applied mathematics. Analytical and numerical treatments of these equations have drawn much attention for many researchers. In general many classical methods used to produce good approximations for these equations. For example in many papers, the orthogonal polynomials [7], semi-orthogonal wavelets[4], and oblique multiwavelets [5] used for solving differential equations. The Bernstein polynomials have been used recently to solve differential equations and in general many initial value problems [3] and system of high order linear integro-differential equations [2]. Other works in which these polynomials have been used can be found in [1]. In many works as [6, 7] Bernstein polynomials have been used to numerical approximation in different problems. In recent years various operational matrices for the polynomials have been developed to cover the numerical methods used to solving initial value problems. In this article we use Bernstein operational matrices for numerical solution of the second order ordinary differential equation. We introduce a biorthogonal system which includes two sets of linear independent functions. One set consists of $n+1$ Bernstein functions $b_i(x), \quad i = 0, 1, \ldots, n$. Second set consists of $n+1$ block pulse functions. Both sets defined on unit interval.

Definition 1.1. *The Bernstein basis polynomials of degree n are defined by*

$$\beta_i(x) = \binom{n}{i} x^i (1-x)^{n-i},\ i = 0, 1, \ldots n.$$

Definition 1.2. *The ith member of a set of n Block pulse functions defined as*

$$\psi_i(x) = \begin{cases} 1 & \frac{i}{n+1} \leq x \leq \frac{i+1}{n+1} \\ 0 & o.w. \end{cases} \qquad i = 0, 1, \ldots n. \tag{1}$$

Let $\Psi(x)$ and $\beta(x)$ are two vectors defined by

$$\Psi(x) = [\psi_0(x),\ \psi_1(x),\ \cdots,\ \psi_n(x)]^T \tag{2}$$

$$\beta(x) = [b_0(x),\ b_2(x),\ \ldots,\ b_n(x)]^T, \tag{3}$$

then we have $\beta(x) = B\Delta_n(x)$, [2], where

$$B = \begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & (-1)^1\binom{n}{0}\binom{n-0}{2} & \cdots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0\binom{n}{1} & (-1)^1\binom{n}{1}\binom{n-1}{1} & \cdots & (-1)^{n-1}\binom{n}{1}\binom{n-1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & (-1)^0\binom{n}{i} & \cdots & (-1)^{n-i}\binom{n}{i}\binom{n-i}{n-i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^0\binom{n}{n} \end{bmatrix},$$

$$\Delta_n(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}.$$

Now we introduce a set of $n+1$ functions orthogonal to $n+1$ Bernstein polynomials of the vector $\beta(x)$. Let

$$\theta_i(x) = a_{i0}\psi_0(x) + a_{i1}\psi_1(x) + \cdots + a_{in}\psi_n(x),\ i = 0,\ 1, \cdots,\ n,$$

where $a_{ij},\ i,j = 0,\ 1, \cdots,\ n$, are unknown. Then we define the vector function $\alpha(x)$ by:

$$\alpha(x) = [\theta_0(x),\ \theta_1(x), \cdots,\ \theta_n(x)]^T. \tag{4}$$

So we can write $\alpha(x) = A\Psi(x)$, where A is the matrix of unknown coefficients in (3). To determine the unknown matrix A we use the biorthogonality relation

$$\langle \theta_i, b_i \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases},\quad i,j = 0,\ 1\ \cdots,\ n. \tag{5}$$

Rewriting (5) in the matrix form yields:

$$\langle \beta, \alpha^T \rangle = I,$$

where I is identity matrix of the rank $n+1$. The differentiation of the vector $\alpha(x)$ can be expressed as:

$$\alpha'(x) = \mathbf{D}\alpha(x),$$

where $\mathbf{D}$ is the $(n+1)\times(n+1)$ operational matrix of derivatives for Bernstein polynomials. For computing the entries of $\mathbf{D}$ we use (5) .
Now we consider the following general ordinary differential equation:

$$y''(x) = f\big(x, y(x), y'(x)\big),\ x \in [0,1], \tag{6}$$

$$y'(0) = y_0,\ y'(1) = y_1. \tag{7}$$

Here f, is a known function, y_0, y_1 are given real numbers and y is the unknown function to be found. The existence of solution of (5) with Neumann boundary conditions is studied in [9] using the quasi-linearization method. Now we apply our method which is based on biorthogonal basis to solve this system.

2 Main Result

We successfully applied our proposed method which is based on construction of biorthogonal basis for Bernstein polynomial. It turns out that our method is easy to use and very accurate comparing to the well known methods like semi-orthogonal B-spline wavelets [4].

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Boundary value problems for real order differential equations

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Abstract
In this paper irrational differential equations, their invariant functions and boundary or initial value problems including these equations are investigated and soltion of these problems are introduced.
Keywords: Fractional order derivative, rational order derivative, irrational order derivative

1 Fractional order differential equations

We consider the α-fractional order derivation of function $f(x)$ in the following form:

$$D^\alpha f(x) = D^n \left(\int_0^x \frac{(x-\xi)^{n-\alpha-1}}{(n-\alpha-1)!} f(\xi) d\xi \right) \qquad n \in \mathbb{Z}, \quad n-1 \leq \alpha < n \tag{1}$$

We introduce the following function for this kind of derivation

$$h_p(x) = \sum_{k=0}^{\infty} \frac{x^{kp-1}}{(kp-1)!} = \frac{x^{p-1}}{(p-1)!} + \frac{x^{2p-1}}{(2p-1)!} + \frac{x^{3p-1}}{3(p-1)!} + \dots \tag{2}$$

where p is a the step of derivation and it is a rational number. It is easy to see that

$$D^{np} h_p(x) = h_p(x) \tag{3}$$

It means this function is invariant to this kind of derivation. For example, we consider the following fractional order differential equation:

$$y^{(\frac{2}{3})} - 3y^{(\frac{1}{3})} + 2y = 0 \tag{4}$$

where $p = \frac{1}{3}$, $np = \frac{2}{3}$. By considering (2) we can write the solution of this equation as the follows:

$$y(x) = h_p(x, r) = \sum_{k=1}^{\infty} \frac{r^k x^{kp-1}}{(kp-1)!} \tag{5}$$

By considering the relation $D^{np} h_p(x, r) = r^n h_p(x, r)$, we have:

$$r^2 h_{\frac{1}{3}}(x, r) - 3r h_{\frac{1}{3}}(x, r) + 2h_{\frac{1}{3}}(x, r) = 0 \tag{6}$$

consequently, we obtain the charactersitic equation $r^2 - 3r + 2 = 0$.
Hence, the general solution of equation (4) will be in the following form

$$y(x) = C_1 h_{\frac{1}{3}}(x,1) + C_2 h_{\frac{1}{3}}(x,2) = C_1 \sum_{k=1}^{\infty} \frac{1^k x^{\frac{k}{3}-1}}{(\frac{k}{3}-1)!} + C_2 \sum_{k=1}^{\infty} \frac{2^k x^{\frac{k}{3}-1}}{(\frac{k}{3}-1)!}.$$

We can consider the following boundary value problem:

$$\begin{array}{lll} y'(x) = 0 & , & x \in (1,4) \subset \mathbb{R} \\ y(1) = y_1 & , & y(4) = y_2 \end{array}$$

The solution in fractional functional space$C_{\frac{1}{3}}^{(2)}(1,4)$ is, $y(x) = 2y_2 - y_1 + 2\frac{y_1 - y_2}{\sqrt{x}}$.

Corresponding to fractional order derivation of differential equation, we consider the fractional functional space. For example, if the order of O.D.E be np ($n \in \mathbb{N}, p \in \mathbb{Q}^+$), we use the functioal space $C_p^{(n)}(I)$. This linear functional space contains all of real functions which np fractional order derivation of them are continous and bounded on real interval $I \subset \mathbb{R}$. This space is a linear normed space with the following norm:

$$\forall f \in C_p^{(n)}(I): \quad \|f\| = \sum_{j=0}^{n} \max |f^{(jp)}(x)| \tag{7}$$

2 Irrational order O.D.E

Suppose α be a positive real number and m be natural number. similarly, we cosider $C_\alpha^{(m\alpha)}(J)$ as a linear functional space with the following norm, more ever This functional space contains all of the real functions which α-real order derivation of them are continuos and bouned on interval $J \subset \mathbb{R}$.

$$\forall f \in C_\alpha^{(m\alpha)}(J): \quad \|f\| = \sum_{s=0}^{m} \max_{x \in J} |f^{(s\alpha)r}(x)| \tag{8}$$

This functional space is a linear normed space. Similarly we introduce the following function which is invariant to α-real derivation:

$$a_\alpha(x,\lambda) = \sum_{k=1}^{\infty} \frac{\lambda^k x^{k\alpha-1}}{(k\alpha-1)!} \tag{9}$$

If the differential equation is given in the form of

$$\sum_{j \leq m} a_j D^{j\alpha} y(x) = 0, \tag{10}$$

whre $\alpha \in \mathbb{R}^+$, $m \in \mathbb{N}$, $a_j, j = 0,1,\dots,m$ are real constat. Then the general solution of this equation will be in the followng form:

$$y(x) = \sum_{k=1}^{m} C_k a_\alpha(x,\lambda_k) \tag{11}$$

where c_k; $k = 1,2,\dots,m$ are arbitrary constant.
Differential equation (10) can be given with the following boundary conditions, respectively:

$$\sum_{j=0}^{m} \left\{ \alpha_{ij}^{(1)} D^{j\alpha} y(x)|_{x=x_1} + \alpha_{ij}^{(2)} D^{j\alpha} y(x)|_{x=x_2} \right\} = \alpha_j \qquad i = 1,2,\dots,m \tag{12}$$

$$D^{j\alpha}y(x)|_{x=x_1} = \alpha_j, \qquad j = 0, 1, \dots m-1 \tag{13}$$

For example, we consider the following differential equation

$$D^{\sqrt{2}}y(x) - 3y(x) = 0, \qquad x \in (2,4) \tag{14}$$

The solution of this equation is written in $C^{(\sqrt{2})}_{\sqrt{2}}(2,4)$. Therefore, we will have:

$$y(x) = a_{\sqrt{2}}(x,\lambda) = \sum_{k=1}^{\infty} \frac{\lambda^k x^{k\sqrt{2}-1}}{(k\sqrt{2}-1)!} \tag{15}$$

The charactristic equation is given by

$$\lambda a_{\sqrt{2}}(x,\lambda) - 3a_{\sqrt{2}}(x,\lambda) = 0 \tag{16}$$

consequently $\lambda = 3$ and the general solution is

$$y(x) = c_0 a_{\sqrt{2}}(x,3) \tag{17}$$

If this equation is given with the following boundary condition

$$y(2) + y(4) = 1, \qquad x \in (2,4) \tag{18}$$

Then the arbitrary constant c_0 is found by

$$c_0 = \frac{1}{a_{\sqrt{2}}(2,3) + a_{\sqrt{2}}(4,3)}$$

And the solution of boundary value problem (14) , (2.7) is

$$y(x) = \frac{a_{\sqrt{2}}(x,3)}{a_{\sqrt{2}}(2,3) + a_{\sqrt{2}}(4,3)} = \sum_{k=1}^{\infty} \frac{3^k x^{k\sqrt{2}-1}}{(k\sqrt{2}-1)![a_{\sqrt{2}}(2,3) + a_{\sqrt{2}}(4,3)]} \tag{19}$$

Finally, we consider the following $\sqrt{2}$-real order differential equation:

$$D^{3\sqrt{2}}y(x) = 0; \qquad x \in (2,4) \tag{20}$$

where $\alpha = \sqrt{2}$ is the step of irrational order derivation, we seek the solution of this equation in $C^{(3\sqrt{2})}_{\sqrt{2}}(2,4)$. Hence we have

$$y(x) = c_1 \frac{x^{\sqrt{2}-1}}{(\sqrt{2}-1)!} + c_2 \frac{x^{2\sqrt{2}-1}}{(2\sqrt{2}-1)!} + c_3 \frac{x^{3\sqrt{2}-1}}{(3\sqrt{2}-1)!} \tag{21}$$

we consider the following results:

$$\begin{aligned} D^{\sqrt{2}}y(x) &= c_1 \frac{x^{-1}}{(-1)!} + c_2 \frac{x^{\sqrt{2}-1}}{(\sqrt{2}-1)!} + c_3 \frac{x^{2\sqrt{2}-1}}{(2\sqrt{2}-1)!} \\ &= c_2 \frac{x^{\sqrt{2}-1}}{(\sqrt{2}-1)!} + c_3 \frac{x^{2\sqrt{2}-1}}{(2\sqrt{2}-1)!} \end{aligned}$$

Notice that the first term is vanished, becaus $(-1)! = \infty$

$$D^{2\sqrt{2}}y(x) = c_2 \tfrac{x^{-1}}{(-1)!} + c_3 \tfrac{x^{\sqrt{2}-1}}{(\sqrt{2}-1)!} = c_3 \tfrac{x^{\sqrt{2}-1}}{(\sqrt{2}-1)!}$$

$$D^{3\sqrt{2}}y(x) = c_3 \tfrac{x^{-1}}{(-1)!} = 0$$

we conclude that the general solution satisfies in equation (20).
The equation can be considered with the following boundary conditions:

$$
\begin{aligned}
&y(2) - y(4) = 3 \\
&D^{\sqrt{2}} y(x)|_{x=2} + D^{\sqrt{2}} y(x)|_{x=4} = 6 \\
&D^{2\sqrt{2}} y(x)|_{x=2} - D^{2\sqrt{2}} y(x)|_{x=4} = 0
\end{aligned}
$$

Similarly, this equation can be considered with the initial conditions. Then we will have an initial value problem as follows:

$$
\begin{aligned}
&D^{3\sqrt{2}} y(x) = 0 \\
&y(2) = 5, \qquad D^{\sqrt{2}} y(x)|_{x=2} = 0, \qquad D^{2\sqrt{2}} y(x)|_{x=2} = 3
\end{aligned}
$$

General solution of this equation will be in the form of (21) and the three arbitrary constants c_1, c_2, c_3 is determined by three initial conditions of this problem.

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A matrix approach to solving a system of fractional differential equations

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Abstract
In this paper, we present approximate solution for a system of fractional differential equations by extending the application of the shifted Chebyshev operational matrix.
Keywords: Fractional differential equation, Chebyshev polynomials, operational matrix, Caputo derivative.

Mathematics Subject Classification: 34A08

1 Introduction

Fractional calculus has been the focus of many studies due to their application in various field such as physics, chemistry and engineering [1]. Unlike the linear FDE, nonlinear FDE does not have an exact analytic solution, and so, some techniques such the Adomian's decomposition method, analytical homotopy method, and Predictor-corrector method be used for obtaining an analytical approximation solution to linear and nonlinear FDE [2]. Recently, solving FDE using orthogonal polynomials have also received considerable attention. Using this method reduces the differential equation to a system of algebraic equations. The operational matrix of fractional derivative has been determined for some type of orthogonal polynomials such as Chebyshev polynomials and Legendre polynomials . Paraskevopoulos has suggested the operational matrix of integration by using these polynomials as a basis in ODEs [3]. In this article we intend to extend the application of the shifted chebychev polynomials to solve a system of FDE. The method is based on the solving the system by using shifted Chebyshev operational matrix method.

2 Preliminaries and notations

We recall some basic definition and properties of the fractional calculus theory, which are used in this article.

Definition 2.1. *The Caputo fractional integral operator of order $\alpha > 0$ of a function $f(x)$ is defined as*

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \tag{1}$$

Properties of defined operator can be found in [1].

Definition 2.2. *let $T_n(t) = \cos(n \arccos(t))$ be the well-known Chebyshev polynomials of the first kind of degree n which defined on $[-1,1]$. We defined $T_i(\frac{2x}{L}-1)$ as the shifted Chebyshev polynomials on $[0, L]$ which denoted by $\tau_i(x)$ and satisfying the orthogonally relation*

$$\int_0^L \tau_j(x)\tau_k(x)\omega_L(x)\,dx = \delta_{kj}h_k$$

Where δ_{kj} is Kronecker delta function, $\omega_L(x) = \frac{1}{\sqrt{Lx-X^2}}$ and $h_0 = \pi$, $h_k = \frac{\pi}{2}$ $(k \geq 1)$. The shifted Chebyshev polynomials can be determined with the following recursive formula

$$\tau_{i+1}(x) = 2(\frac{2x}{L} - 1)\tau_i(x) - \tau_{i-1}(x), \; i = 1, 2, \cdots . \tag{2}$$

where $\tau_0(x) = 1$ and $\tau_1(x) = \frac{2x}{L} - 1$. A function $y(x)$, square integrable in $[0, L]$, may be approximated in terms of shifted Chebyshev polynomials as $y_N(x) = \sum_{i=0}^{N} c_i\tau_i(x) = \mathbf{C}\phi(x)$, at which, we have

$$c_i = \frac{1}{h_i}\int_0^L y(x)\tau_i(x)\omega_L(x)\,dx \qquad i = 0, 1, 2, \cdots . \tag{3}$$

Where $\mathbf{C} = [c_0, c_1, \cdots, c_N]$ and $\phi(x) = [\tau_0, \tau_1, \cdots, \tau_N]^T$. The derivative of the vector $\phi(x)$ can be expressed by $\frac{d^\nu \phi(x)}{dx^\nu} = \mathbf{D}^{(\nu)}\phi(x)$ where $\mathbf{D}^{(\nu)}$ is the $(N+1)\times(N+1)$ operational matrix defined as follow. The first rows of are all zero and for other elements $\mathbf{D_{ij}}^{(\nu)} = [s_\nu(i,j)]$ where

$$s_\nu(i,j) = \sum_{k=\lceil\nu\rceil}^{i} \frac{(-1)^{i-k}2i(i+k-1)!\Gamma(k-\nu+\frac{1}{2})}{\eta_j L^\nu \Gamma(k+\frac{1}{2})(i-k)!\Gamma(k-\nu-j+1)\Gamma(k+j-\nu+1)}. \tag{4}$$

3 Main Result

In this section, we apply Chebyshev operational matrix (COM) method to solve a linear system of FDE. Consider the linear system of FDE

$$D_*^{\alpha_i} y_i + \sum_{j=1}^{k_i} a_{ij}y_j = g_i(x), \quad m_i - 1 \leq \alpha_i < m_i, \quad i = 1, 2, \cdots, n \tag{5}$$

with initial conditions

$$y_i^{(j)}(0) = d_{ij}, \quad i = 1, 2, \cdots, n, \quad j = 0, 1, \cdots, m_i - 1, \tag{6}$$

where a_{ij}, k_i and d_{ij} are real constants. To solve initial value problem (5)-(6) we approximate $y_i(x)$ and $g_i(x)$ by the shifted Chebyshev polynomials as

$$y_i(x) \simeq \sum_{k=0}^{N} c_{ik}\,\tau_k(x) = \mathbf{C}_i\phi(x), \quad g_i(x) \simeq \sum_{k=0}^{N} g_{ik}\,\tau_k(x) = \mathbf{G}_i\phi(x), \tag{7}$$

where vector $\mathbf{G}_i = [g_{i0}, g_{i1}, \cdots, g_{iN}]$ is known, but $\mathbf{C}_i = [c_{i0}, c_{i1}, \cdots, c_{iN}]$ is unknown vector, and so we have

$$D_*^{\alpha_i} y_i(x) \simeq \mathbf{C}_i\mathbf{D}^{(\alpha_i)}\phi(x), \quad i = 1, 2, \cdots, n. \tag{8}$$

Now, we employ Eqs. (7)-(8) and calculate the residual $R_N^i(x)$ for equations (5) as follow

$$R_N^i(x) = (\mathbf{C}_i\mathbf{D}^{(\alpha_i)} + \sum_{j=1}^{k_i} a_{ij}\mathbf{C}_j - \mathbf{G}_i)\phi(x), \quad i = 1, 2, \cdots, n. \tag{9}$$

Using tau method [4] we generate $(N - m_i + 1)$ linear equations by applying

$$\langle R_N^i(x) \, . \, \tau_j(x) \rangle \,=\, \int_0^L R_N^i(x)\, \tau_j(x)\, \mathrm{d}x \,=\, 0, \quad j \,=\, 0,\, 1,\, \cdots,\, N - m_i. \tag{10}$$

Considering also initial conditions (6) and using Eq. (7)), we get

$$y_i^{(j)}(0) \,=\, \mathbf{C}_i \mathbf{D}^{(j)} \phi(0) \,=\, d_{ij}, \quad j \,=\, 0,\, 1,\, \cdots,\, m_i - 1. \tag{11}$$

Finally, we solve n(N+1) algebraic equations which obtain from (10) and (11) and find unknown coefficients of the vector$\mathbf{C}_i$ and consequently obtain $y_i(x)$, $(i \,=\, 1,\, 2,\, \cdots,\, n)$.

Now, let us consider the following nonlinear system of FDE

$$D_*^{\alpha_i} y_i(x) \,=\, \mathbf{F}_i(x, y_1, \cdots, y_n), \quad m_i - 1 \leq \alpha_i < m_i, \quad i \,=\, 1,\, 2,\, \cdots,\, n \tag{12}$$

With initial conditions (5), where $\mathbf{F}_i$ is a nonlinear function. We use COM method and get

$$\mathbf{C}_i \mathbf{D}^{(\alpha_i)} \phi(x) \,=\, \mathbf{F}_i(x,\, \mathbf{C}_1\phi(x),\, \cdots,\, \mathbf{C}_n\phi(x)), \quad (i \,=\, 1,\, 2,\, \cdots,\, n). \tag{13}$$

This equation is satisfied exactly at $N - m_i + 1$ first root of τ_{N+1} and so we approximate (13) at these points. These equations together with Eq. (11) generate n(N+1) nonlinear equations. We can solve these resultant equations by using Newton's iteration method. Consequently, the approximate solution $y_i(x)$, $\quad (i \,=\, 1,\, 2,\, \cdots,\, n)$ can be obtained.

Example 3.1. *As the first example, we consider the following linear system of FDEs*

$$\begin{cases} D_*^{1.5} y_1(x) \,=\, y_2(x), & y_1(0) \,=\, 1,\, y_1'(0) \,=\, 1, \\ D_*^{0.5} y_2(x) \,=\, -y_2(x) - y_1(x) + 1 + x, & y_2(0) \,=\, 0. \end{cases}$$

Now, we apply the describe technique with $N \,=\, 2$ and approximate $y_1(x)$ and $y_2(x)$ as

$$y_i(x) \simeq \sum_{k=0}^{2} c_{ik}\, \tau_k(x) \,=\, \mathbf{C}_i \phi(x), \quad i \,=\, 1,\, 2,$$

where $\mathbf{C}_1 \,=\, [c_{10}, c_{11}, c_{12}]$ and $\mathbf{C}_2 \,=\, [c_{20}, c_{21}, c_{22}]$, using (10) and (11)we have

$$\begin{array}{ll} \frac{64c_{21}}{\pi\sqrt{\pi}} - c_{20} \,=\, 0, & \frac{8c_{21}}{\pi\sqrt{\pi}} - \frac{32c_{22}}{9\pi\sqrt{\pi}} + c_{20} + c_{10} - g_{20} \,=\, 0, \\ \frac{16c_{21}}{\pi\sqrt{\pi}} + \frac{64c_{22}}{5\pi\sqrt{\pi}} + c_{21} + c_{11} - g_{21} \,=\, 0, & c_{10} - c_{11} + c_{12} \,=\, 1, \\ 2c_{11} - 8c_{12} \,=\, 1, & c_{20} - c_{21} + c_{22} \,=\, 0. \end{array} \tag{14}$$

Finally, by solving algebraic system we obtain $c_{10} = \frac{3}{2}$, $c_{11} = \frac{1}{2}$, $c_{12} = 0$, $c_{20} = 0$, $c_{21} = 0$, $c_{22} = 0$. hence, we can write $y_1(x) \,=1 + x$ and $y_2(x) \,= 0$, which is coinciding by the exact solution.

Example 3.2. *Consider the following nonlinear system of FDE*

$$\begin{cases} y_1'(x) \,=\, y_2(x), & y_1(0) = 0, \\ D_*^{0.75} y_2(x \,=\, 6 + x^6 - y_1^3 - \frac{2}{x} y_2(x), & y_2(0) = 0. \end{cases}$$

We apply our technique and approximate $y_i(x)$, $(i \,=\, 1,\, 2)$ as the previous example. Using Eqs. (10)and (13) yields

$$(\mathbf{C}_1 \mathbf{D}^{(1)} - \mathbf{C}_2)\phi(x) \,=\, 0, \tag{15}$$

$$\mathbf{C}_2 \mathbf{D}^{(0.75)} \phi(x) + \frac{2}{x} \mathbf{C}_2 \phi(x) + (\mathbf{C}_1 \phi(x))^3 - 6 - x^6 \,=\, 0. \tag{16}$$

and using (11) we get

$$2c_{11} - c_{20} \,=\, 0,\, 8c_{12} - c_{21} \,=\, 0,\, c_{10} - c_{11} + c_{12} \,=\, 0,\, c_{20} - c_{21} + c_{22} \,=\, 0. \tag{17}$$

we calculate Eq. (16) at $x_0 = \frac{1}{2} + \frac{1}{2}\cos(\frac{\pi}{6})$, $x_1 = \frac{1}{2}$ *as the first and the second root of* $\tau_3(x)$ *and find*

$$\begin{array}{l} 4.01028c_{21} + 5.414385c_{22} + 2.014359c_{22} + (c_{10} + 0.86602c_{11} + 0.5c_{12})^3 \ = \ 6.65967, \\ 1.90745c_{21} - 5.71929c_{22} + 4c_{20} - 0.5 + (c_{10} - c_{11})^3 \ = \ 0.5. \end{array} \tag{18}$$

solving Eqs. (17) and (36) yields

$$c_{10} = 0.37855,\ c_{11} = 0.50153,\ c_{12} = 0.12297, c_{20} = 1.00306,\ c_{21} = 0.983798,\ c_{22} = -0.01926.$$

Thus, we get $y_1(x) = 0.01926x + 0.98380x^2$, $y_2(x) = 2.12168x - 0.15408x^2$, *which agreement with those found in [5].*

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Numerical solution of linear Fredholm integro-differential equations by using biorthogonal multiscaling functions

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Abstract

An effective method base upon biorthogonal multiscaling funcions is proposed for the solution of Fredholm integro-differential equations. The properties of multiscaling functions are first presented. These properties together with collocation method are then utilized to reduse the integro-differential equation to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the new technique.

Keywords: Biorthogonal multiscaling functions, Collocation methods, Fredholm integro-differential equations,

Mathematics Subject Classification: 65R20

1 Introduction

In the present paper, we introduce a new numerical method to solve the following linear Fredholm integro-differntial equation

$$y'(x) = f(x) + y(x) + \int_0^1 k(x,t)y(t)dt, \quad x \in [0,1], \tag{1}$$

$$y(0) = y_0 \tag{2}$$

where th function $f(x) \in L^2([0,1])$, the kernel $k(x,t) \in L^2([0,1]\times[0,1])$ are known and $y(x)$ is the unknown function to be determined.
For linear first-order Fredholm integro-differential equation of the form (1), Linz [1] cosidered numerical methods by transforming it into a second kind integral equation and in [2], Volk applied projection methods.
In recent years, wavelets have found their way into many different fields of science and engineerring. wavelet methods, by using the Daubechies orthonormal scaling functions were presented in [3]. Also solving differential and integro-differential equations by using wavelet method has been discussed in many papers [4, 5]. For this purpose different approaches such as finite element method(FEM), boundary element method(BEM), Galerkin and collocation methods are used.
In this paper, we use biorthogonal multiscaling functions with multiplicity 4 for solving Fredholm integro-differential equations. These multiscaling functions have been constructed in [6]. Our method consists of reducing the given Fredholm integro-differential equation to a set of algebraic by expanding the equation as multiscaling functions with unknown coefficients. The properties of these multiscaling functions are utilized to evaluate the unknown coefficients.

2 Properties of biorthogonal multiscaling functions

2.1 The biorthogonal multiscaling functions

The use of biorthogonal scaling functin vector $\phi = (\phi^1(t), \phi^2(t), \phi^3(t), \phi^4(t))$ appears to be attractive since it's component functions have small support on $[-1, 1]$, accuracy of order 8 and have symmetry properties. Moreover ϕ is a cardinal Hermite interpolant, that is

$$\begin{aligned} &\phi(k) = \delta_k[1,0,0,0]^T, && \phi'(k) = \delta_k[0,1,0,0]^T, \\ &\phi''(k) = \delta_k[0,0,1,0]^T, && \phi'''(k) = \delta_k[0,0,0,1]^T, \qquad \forall k \in \mathbb{Z}. \end{aligned}$$

2.2 Function approximation

A function $f(x)$ defined over $[0, 1]$ may be represented by multiscaling functions as

$$f(x) = \sum_{k=0}^{2^J} \sum_{m=0}^{4} c_{J,k} \phi_{J,k}^m(x) = C^T \phi_J(x), \tag{3}$$

where ϕ_J and C are $n \times 1$ vectors with $n = 4(2^J + 1)$ and given by

$$\phi_J(x) = \left[\phi_{J,0}^1(x), \phi_{J,0}^2(x), \phi_{J,0}^3(x), \phi_{J,0}^4(x)|...|\phi_{J,2^J}^1(x), \phi_{J,2^J}^2(x), \phi_{J,2^J}^3(x), \phi_{J,2^J}^4(x)\right]^T,$$

$$C = \left[c_{J,0}^1, c_{J,0}^2, c_{J,0}^3, c_{J,0}^4|...|c_{J,2^J}^1, c_{J,2^J}^2, c_{J,2^J}^3, c_{J,2^J}^4\right]^T,$$

with

$$\begin{aligned} c_{Jk}^1 &= f(k/2^J), \\ c_{Jk}^2 &= f'(k/2^J), \\ c_{Jk}^3 &= f''(k/2^J), \\ c_{Jk}^4 &= f'''(k/2^J), \end{aligned} \tag{4}$$

for $k = 0, 1..., 2^J$.

3 Description of Numerical Method

In this section, we solve the first order integro-differential equation(1) by utilizing biorthogonal multiscaling functions. For this purpose, by using equation(2) we approximate the functions involved by

$$y(x) = Y^T \phi_J(x), \tag{5}$$
$$y'(x) = Y^T \phi'_J(x) = Y^T D \phi_J(x), \tag{6}$$
$$f(x) = F^T \phi_J(x), \tag{7}$$
$$K(x,t) = \phi_J(t)^T K \phi_J(x), \tag{8}$$

where Y, Y_0 and F are the coefficients which are defined similar to (2). Also D is $n \times n$ operational matrix of derivative for multiscaling functions and K is a known $n \times n$ matrix.
Using equations (5)-(2) in equation (1) we get

$$Y^T D \phi_J(x) = F^T \phi_J(x) + Y^T \phi_J(x) + \int_0^1 Y^T \phi_J(t) \phi_J^T(t) K \phi_J(x) dt. \tag{9}$$

Then by considering

$$P = \int_0^1 \phi_J(t)\phi_J^T(t)dt,$$

one can write (5) as

$$Y^T D\phi_J(x) = F^T\phi_J(x) + Y^T\phi_J(x) + Y^T PK\phi_J(x). \tag{10}$$

Collocating both sides of (4) at the $N-1$ points τ_i, $i = 1, 2, ..., N-1$, gives

$$R(\tau_i) = Y^T D\phi_J(\tau_i) - F^T\phi_J(\tau_i) - Y^T\phi_J(\tau_i) - Y^T PK\phi_J(\tau_i) = 0, \quad i = 1, 2, ..., N-1, \tag{11}$$

where τ_i, $i = 1, 2, ..., N-1$ are some points on the interval $[0, 1]$.
Using equation (2) in equation (2) leads to

$$Y^T\phi_J(0) = 0. \tag{12}$$

Equation (8) together with equation (9) give a system of linear equations of dimension $4(2^J+1)$ which can be solved for y_k, $k = 1, 2, ..., 4(2^J+1)$. So the unknown function $y(x)$ can be found. Also we used the well-known software Maple to solve the system of linear equations.

3.1 Numerical results

The method is applied to a linear Fredholm integro-differential eqation with exact solution. figure 1 demonstrates the validity and applicability of the technique.

Example 3.1. *Consider the integro-differential equation*

$$y'(x) = xe^x + e^x - x + \int_0^1 xy(t)dt,$$
$$y(0) = 0.$$

the exact solution for this problem is $y(x) = xe^x$.

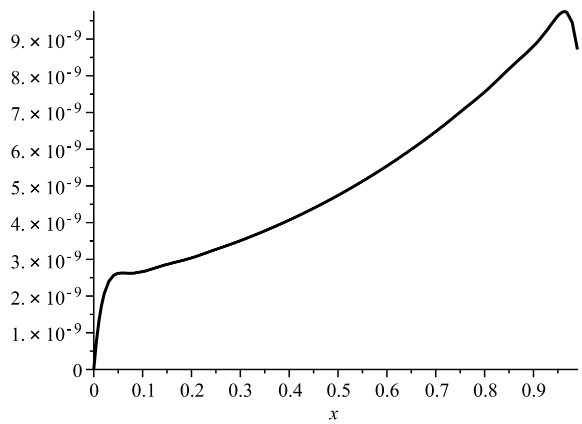

Figure 1: The error function for Example 1, for J = 2.

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New modified homotopy perturbation method and its convergence

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Abstract

There are some methods to obtain approximate solutions of functional equations. One of them is Homotopy perturbation method. However, computing coefficients of p^i is time-consuming and challengeable. In order to deal with the problem, a new modified Homotopy perturbation method is introduced, which doesn't need computations of the coefficients of p^i. Here, the method has been applied to solve some examples and the results were compared with Homotopy perturbation method. Moreover, convergence of the method has been discussed comprehensively.

Keywords: Homotopy Perturbation Method, Functional Equation, New Iterative Method, Convergence.

Mathematics Subject Classification: 65Q20

1 Introduction

Homotopy perturbation method introduced by He in 1998 ,well addressed in [1], has been known as a powerful device for solving different kind of equation, this is because of further developments and improvement applied by himself and other researches [2-12]. In this method the solution is considered as the summation of an infinite series, which usually converge rapidly to the solution. Recently a new technique has been proposed for solving linear or nonlinear functional equations [13-14]. In this paper, this method is used to present a new modification of HPM. To illustrate the

basic concept of modified Homotopy perturbation method, consider the following nonlinear differential equation

$$A(u) - f(r) = 0, r \in \Omega, \tag{1}$$

with boundary conditions

$$B(u, \partial u/\partial(n) = 0, r \in \Gamma, \tag{2}$$

where A is a general functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and is the boundary of the domain . Generally speaking, the operator A can be divided into two parts L and N, where L is a linear, while N is a nonlinear operator. Eq. (1), therefore, can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \tag{3}$$

We construct a homotopy $v(r,p) : \Omega \times [0,1] \longrightarrow \mathbf{R}$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0. \tag{4}$$

or

$$L(v) - L(u_0) = p[f(r) - L(u_0) - N(v)], \tag{5}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial approximation for the solution of Eq. (1), which satisfies the boundary conditions. According to HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of Eq. (5) can be written as a power series in p :

$$v = \sum_{i=0}^{\infty} v_i p^i. \tag{6}$$

By substituting (6), into (5), the following equation will be derived.

$$\sum_{i=0}^{\infty} L(v_i)p^i - L(u_0) = p[f(r) - L(u_0) - N(\sum_{i=0}^{\infty} v_i p^i)]. \tag{7}$$

If $p = 1$ the nonlinear operator$N$ can be decomposed as

$$N(\sum_{i=0}^{\infty} v_i) = N(v_0) + \sum_{i=1}^{\infty}[N(\sum_{j=0}^{i} v_i) - N(\sum_{i=0}^{i-1} v_j)].$$

So

$$\sum_{i=0}^{\infty} L(v_i)p^i - L(u_0) = f(r) - u_0 N(v_0) + \sum_{i=1}^{\infty}[N(\sum_{j=0}^{i} v_i) - N(\sum_{i=0}^{i-1} v_j)].$$

By following iterative scheme, v_i's will be obtained.

$$L(v_0) = L(u_0), \tag{8}$$

$$L(v_1) = f(r) - u_0 - N(v_0),$$

$$L(v_{i+1}) = N(\sum_{j=0}^{i} v_i) - N(\sum_{i=0}^{i-1} v_j), i = 1, 2, ...$$

2 The convergence of New modified HPM

Lemma 2.1. *New modified HPM for finding the solution of Eq. (1) is equivalent to determining the fallowing sequence*

$$S_n = v_1 + ... + v_n, \tag{9}$$

$$s_0 = 0,$$

by using the iterative scheme:

$$S_{n+1} = -L^{-1}N(s_n + v_0) - u_0 + L^{-1}(f(r)). \tag{10}$$

Theorem 2.2. *Let $\mathbb{B}$ be a Banach space.*
a) $\sum_{i=0}^{\infty} v_i$ obtained by (10), convergence to $s \in \mathbb{B}$, if

$$\exists(0 < \lambda < 1), s.t(\forall n \in \mathbf{N} \Rightarrow \|v_n\| \leqslant \lambda \|v_{n-1}\|). \tag{11}$$

b)If N is continuous, then $s = \sum_{i=0}^{\infty} v_i$ satisfies in

$$s = -L^{-1}N(s + v_0) - u_0 + L^{-1}(f(r)). \tag{12}$$

Proof.

$$\|s_{n+1}-s_n\| = \|v_{n+1}\| \leqslant \lambda\|v_n\| \leqslant \lambda^2\|v_{n-1}\| \leqslant \dots \leqslant \lambda^{n+1}\|v_0\|. \tag{13}$$

For any $n, m \in \mathbf{N}, n \geq m$, we derive

$$\begin{aligned} \|s_n - s_m\| &= \|(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \dots + (s_{m+1} - s_m)\| \\ &\leq \|s_n - s_{n-1}\| + \|s_{n-1} - s_{n-2}\| + \dots + \|s_{m+1} - s_m)\| \\ &\leq \lambda^n\|v_0\| + \lambda^{n-1}\|v_0\| + \dots + \lambda^{m+1}\|v_0\| \\ &\leq (\lambda^n + \lambda^{n-1} + \dots + \lambda^{m+1})\|v_0\| \\ &\leq \lambda^{m+1}(1 + +\lambda + \lambda^2 + \dots + \lambda^n + \dots)\|v_0\| \\ &\leq \frac{\lambda^{m+1}}{1-\lambda}\|v_0\|. \end{aligned} \tag{14}$$

So

$$u = \lim_{n,m\to\infty} \|s_n - s_m\| = 0.$$

Then $\{s_n\}$, is Cauchy sequence in Banach space, and it is convergent, i.e.,

$$\exists s \in \mathbb{B}, s.t \lim_{n\to\infty} s_n = \sum_{i=0}^{\infty} v_i = s.$$

b)From Eq. (10), we have

$$\lim_{n\to\infty} S_{n+1} = -L^{-1}N(\lim_{n\to\infty} s_n + v_0) - u_0 + L^{-1}(f(r)),$$

$$\Rightarrow s = -L^{-1}N(\sum_{i=0}^{\infty} v_i) - u_0 + L^{-1}(f(r)),$$

So

$$s = -L^{-1}N(s + v_0) - u_0 + L^{-1}(f(r)).$$

□

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Numerical solution of the nonlinear Fredholm integral equations of the second kind by radial basis functions

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Abstract

The numerical solution of nonlinear Fredholm integral equations of the second kind is developed by radial basis functions, based on Legendre-Gauss-Lobatto nodes and weights.The application of RBF-collocation method yields the nonlinear system of equations which can be solved by the Newton's iteration method. Methods applied on a test problem, computed solution are compared with the analytic solution and the comparison is made among some RBF methods to demonstrate the accuracy and easy implementation of presented methods.

Keywords: Radial basis function (RBF), Fredholm integral equations, Multiquadric (MQ), Inverse multiquadric (IMQ), Gaussian (GA)

Mathematics Subject Classification: 65R20

1 Introduction

We consider the nonlinear Fredholm integral equation of the second kind,

$$y(x) = f(x) + \lambda \int_a^b k(x,t)F(x,y(t))dt, \tag{1}$$

where λ is constant, and $f(x)$ and $k(x,t)$ are assumed to be defined on the interval $a \leq x, t \leq$ b .We assume that (1) has a unique solution to be determined .

1.1 Introduction to RBFs

Mathematical modeling of many problems in science and engineering leads to nonlinear equations such as differential equations and integral equations [1-3]. A radial basis function is positive-value function that its value depends on distance from a few points that these points are called centers.

Let $R^+ = \{x \in R, x \geq 0\}$, and $\phi : R^+ \to R$ be a continuous function with $\phi(0) \geq 0$. A radial basis function on R^d is a function Φ of the form

$$\Phi(x) = \phi(\|x - x_i\|), \tag{2}$$

where $x, x_i \in R^d$, $\|.\|$ denotes the Euclidean distance between x, x_i.

If $\phi(x)$ be a RBF on R and $f \in L^2[a,b]$, then for every set of distinct points $x_1, ..., x_N$ in [a,b] with interpolation by a single function $\phi(x)$ we have

$$f(x) = \sum_{i=1}^{N} c_i \phi(\|x - x_i\|), \tag{3}$$

c_i can be determined from the solution of following systems of equations:

$$f(x_j) = \sum_{i=1}^{N} c_i \phi(\|x_j - x_i\|) \quad j = 1, ..., N. \tag{4}$$

In order to explain RBF methods, suppose that the center set $\{x_i\}_{i=1}^N$ in the given domain $\Lambda \subseteq R$ is given. The center points can have an arbitrary distribution. The arbitrary grid structure is one of the major differences between the RBF method and other methods. Such a mesh free methods yield high flexibility particularly when the domain is irregular.

2 The approximate solution of nonlinear Fredholm integral equations

We approximate the solution (1) with interpolation by radial basis functions ($\phi(x)$)

$$y(x) = \sum_{i=1}^{N} c_i \phi(\|x - x_i\|) \tag{5}$$

By substituting (5) in (1) we obtain

$$\sum_{i=1}^{N} c_i \phi(\|x - x_i\|) = f(x) + \lambda \int_a^b k(x,t) F(x, \sum_{i=1}^{N} c_i \phi(\|x - x_i\|)) dt. \tag{6}$$

For obtaining $c_i, i = 1, ..., N$ in (6) , by collocating at the points $x_j, j = 1, ..., N$ we have

$$\sum_{i=1}^{N} c_i \phi(\|x_j - x_i\|) = f(x_j) + \lambda \int_a^b k(x_j,t) F(x_j, \sum_{i=1}^{N} c_i \phi(\|x_j - x_i\|)) dt. \tag{7}$$

For computation of the above integral, we apply Legendre-Gauss-Lobatto quadrature. Let $L_N(x)$ be the shifted Legendre polynomial of order N on [0,1]. Then $x_1 = 0 < x_2 < ... < x_{N-1} < x_N = 1$, and $x_i, 2 \leq i \leq N-1$ which are the zeros of $L'(x)$, where $L'(x)$ is the derivative of $L_N(x)$ with respect to $x \in [0,1]$. We approximate the integral of f(x) on [0,1] as

$$\int_0^1 f(x) dx = \sum_{i=1}^{N} w_i f(x_i), \tag{8}$$

where x_i are Legendre-Gauss-Lobatto nodes and the weights w_i are given in [4]

$$x_i = \frac{2}{N(N+1)[L_N(x_i)]^2}, \quad i = 1, ..., N \tag{9}$$

It is well known from [5] that the integration in (7) is exact whenever f(x) is a polynomial of degree $\leq 2N + 1$. Therefore (7) can be written as

$$\sum_{i=1}^{N} c_i \phi(\|x_j - x_i\|) = f(x_j) + \lambda \sum_{r=1}^{N} w_r k(x_j, x_r) F(x_j, \sum_{i=1}^{N} c_i \phi(\|x_j - x_i\|)) \quad j = 1, ..., N$$

This is a nonlinear system of equations that can be solved by Newton's iteration method to obtain the $c_i (i = 1(1)N)$.

3 Main Result

In order to illustrate the performance of the RBF method in solving nonlinear Fredholm integral equations and justify the accuracy and efficiency of our method, we consider an example. This has been solved by presented method with different values of N and $\mu > 0$ where μ is shape parameter that plays an important role for the application of RBF methods. There is experimental evidence that shows the accuracy of RBF methods significantly depends on the amount of this shape parameter, but the optimal shape parameter choice for any equations is open problem yet. In order to analyze the error of the method the following notations are introduced:

$$e_{max} = max\{|y(x_i) - y_N(x_i)| : x_i = \frac{i}{1000}, i = 1, ..., 1000\}, \tag{10}$$

where y_N is numerical solution and e_{max} approximates $\|y - y_N\|$.

Example 3.1. *The following nonlinear Hammerstein integral equation is considered [6,7],*

$$y(x) = \int_0^1 xt[y(t)]^2 dt - \frac{5}{12}x + 1, \quad 0 \leq x \leq 1$$

This equation has the exact solution $y(x) = 1 + \frac{1}{3}x$.
Errors of the numerical results are given in Table 1.

N	$MQ = \sqrt{r^2+\mu^2}$ $(\mu = 2)$	$IMQ = \frac{1}{\sqrt{r^2+\mu^2}}$ $(\mu = 2)$	$GA = exp(\frac{-r^2}{\mu^2})$ $(\mu = 1)$
3	2.06×10^{-3}	5.72×10^{-3}	2.15×10^{-2}
4	2.84×10^{-4}	5.98×10^{-4}	6.53×10^{-3}
5	3.50×10^{-5}	1.50×10^{-4}	5.57×10^{-4}
6	5.34×10^{-6}	1.68×10^{-5}	1.35×10^{-4}
7	7.88×10^{-7}	4.41×10^{-6}	1.70×10^{-5}
8	1.27×10^{-7}	4.88×10^{-7}	2.10×10^{-6}
9	2.03×10^{-8}	1.37×10^{-7}	1.62×10^{-7}
10	3.49×10^{-9}	1.49×10^{-8}	1.37×10^{-7}

Table 1: Numerical results for different RBFs of Example 1

The shape parameter μ in the Table 1 for all the calculations has been chosen experimentally. The numerical results demonstrate the good accuracy of this scheme.

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A numerical solution for fractional partial differential equations via a semi-discrete scheme and collocation method

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Abstract

In this paper, a numerical method for solving time-fractional diffusion equations and space-fractional diffusion equations is introduced. The finite difference scheme and Chebyshev collocation method is applied to solve this problems. Also, to simplify application of the method, the matrix form of suggested method is obtained. Illustrative example show that the proposed method is very efficient and accurate.

Keywords: Chebyshev polynomials, Gauss-Lobatto points, Fractional diffusion equation, Finite difference.

Mathematics Subject Classification: 35R11

1 Introduction and Preliminaries

The use of fractional partial differential equations (FPDEs) in mathematics, physics, engineering and chemistry has become increasingly popular in recent years [1, 3].

In this paper we consider The space fractional diffusion equations (SFDEs) and the time fractional diffusion equations (TFDEs).
For instance, Saadatmandi and Dehghan [4] used tau approach to solve SFDEs and Lin and Xu [2] have solved TFDEs by using finite difference and spectral approximation. This work presents a numerical method to solve this kind of problems using finite difference scheme and collocation method using Chebyshev polynomials.

Definition 1.1 The Caputo fractional derivatives with respect to x and t can be defined by [1]:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} \frac{\partial^m u(s,t)}{\partial s^m}\, ds, \quad m-1<\alpha\le m.$$

and

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = \frac{1}{\Gamma(m-\beta)} \int_0^t (t-\tau)^{m-\beta-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m}\, d\tau, \quad m-1<\beta\le m.$$

2 Description of the method

In this section, we consider SFDEs and TFDEs of the forms

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + p(x,t),\ 0<x<1,\ 0\le t\le T,\ 1<\alpha\le 2, \tag{1}$$

with initial condition

$$u(x,0) = f(x), \quad 0 < x < 1, \tag{2}$$

and boundary conditions

$$u(0,t) = g_0(t),\ 0 < t \le T, \quad u(1,t) = g_1(t),\ 0 < t \le T, \tag{3}$$

and

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} - \frac{\partial^2 u(x,t)}{\partial x^2} = v(x,t), \quad 0 < x < 1, \quad 0 < t \le T, \quad 0 < \beta \le 1, \tag{4}$$

$$u(x,0) = q(x), 0 < x < 1, \tag{5}$$

$$u(0,t) = u(1,t) = 0, 0 \le t \le T. \tag{6}$$

Note that for $\alpha = 2$ and $\beta = 1$ equations (1) and (4) are the classical diffusion equation.
Here we describe the process of solving the space fractional diffusion equations as in 1-4.

Let $t_n = n\ \Delta t$, $n = 0, 1, ..., M$ where $\Delta t = \frac{T}{M}$, $t_0 = 0$, $t_M = T$. First, we use a finite difference technique and $\theta-$weighted scheme to discritize the time derivative.

$$u_{n+1}(x) - u_n(x) = \frac{d(x)\Delta t}{\Gamma(2-\alpha)} \int_0^x (x-s)^{1-\alpha}\ [\theta \frac{d^2 u_{n+1}(s)}{ds^2} + (1-\theta)\frac{d^2 u_n(s)}{ds^2}]ds + \Delta t p_n(x), \tag{7}$$

where $\theta \in [0,1]$, $u_n(x) = u(x, t_n)$ and $p_n(x) = p(x, t_n)$. Now we expand $u_n(x)$ by shifted Chebyshev polynomials:

$$u_n(x) = \sum_{i=0}^{N} r_i^n\ T_i^*(x), \quad n = 1, ..., M, \tag{8}$$

where $T_i^*(x) = T_i(2x-1)$ and $r_0^n, r_1^n, ..., r_N^n$ are unknown coefficients.
From equation (7), (8) and collocation points $x_k = \frac{1}{2}\cos(\frac{k\pi}{N}) + \frac{1}{2}$, $k = 0, 1, ..., N$ we obtain

$$\sum_{i=0}^{N} r_i^{n+1}\ T_i^*(x_k) - \sum_{i=0}^{N} r_i^n\ T_i^*(x_k) = \frac{d(x_k)\Delta t}{\Gamma(2-\alpha)} \int_0^{x_k} (x_k - s)^{1-\alpha} [\theta \sum_{i=0}^{N} r_i^{n+1} T_i^{*''}(s)$$

$$+ (1-\theta)\sum_{i=0}^{N} r_i^n\ T_i^{*''}(s)]ds + \Delta t\ p_n(x_k), n = 0, ..., M-1, k = 1, ..., N-1. \tag{9}$$

Also boundary conditions (3) for $n = 0, 1, ..., M-1$ are used to obtain

$$u_{n+1}(x_N) = \sum_{i=0}^{N} r_i^{n+1}\ T_i^*(x_N) = g_0(t_{n+1}), \quad u_{n+1}(x_0) = \sum_{i=0}^{N} r_i^{n+1}\ T_i^*(x_0) = g_1(t_{n+1}). \tag{10}$$

Therefor eqautions (9) and (10) generate a set of (N+1) algebraic equations, which can be solved for unknown coefficients.
Clearly $u_0(x)$ can be obtained from the initial condition (2) as follows:

$$u_0(x) = u(x, t_0) = f(x).$$

Note that to discritize the time derivative in (4) we obtained

$$\frac{\partial^\beta u(x, t_{n+1})}{\partial t^\beta} = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} [(j+1)^{1-\alpha} - j^{1-\alpha}](u(x, t_{k-j+1}) - u(x, t_{k-j}))$$

3 The matrix form of the proposed method

In order to find the matrix form of suggested method, first by using (9) for $k = 1, 2, ..., N-1$, we obtain the preliminary matrices $\mathbf{T}^*$, $\mathbf{Q}$ and finally by using (10) the matrix form for this method is achieved as follows:

$$\mathbf{A}[r]^{n+1} = \mathbf{B}[r]^n + \mathbf{v_n}, \tag{11}$$

where the matrix elements of $\mathbf{A}$ are

$$a_{ij} = \begin{cases} T^*_{j-1}(x_0), & i = 1,\ 1 \le j \le N+1, \\ T^*_{j-1}(x_N), & i = N+1,\ 1 \le j \le N+1, \\ T^*_{j-1}(x_{i-1}), & 2 \le i \le N,\ j = 1, 2, \\ T^*_{j-1}(x_{i-1}) - \theta\ \Delta t\ d(x_i) \sum_{k=0}^{j-[\alpha]-1} (-1)^k\ 2^{2(j-k-1)} & \\ \frac{(j-1)(2j-k-3)!\ (j-k-1)!}{k!\ (2j-2k-2)!\ \Gamma(j-k-\alpha)}\ x_{i-1}^{j-k-\alpha-1}, & 2 \le i \le N,\ 3 \le j \le N+1. \end{cases}$$

Matrix $\mathbf{B}$ is obtained by adding two zero rows to the first and last row of matrix $((1-\theta)\Delta t\ \mathbf{Q} + \mathbf{T}^*)$, where the matrices elements of $\mathbf{T}^*$ and $\mathbf{Q}$ are $t_{ij} = T^*_{j-1}(x_i)$ and

$$q_{ij} = \begin{cases} 0, & 1 \le i \le N-1, j = 1, 2, \\ & \\ d(x_i) \sum_{k=0}^{j-[\alpha]-1} (-1)^k\ 2^{2(j-k-1)} \frac{(j-1)(2j-k-3)!\ (j-k-1)!}{k!\ (2j-2k-2)!\ \Gamma(j-k-\alpha)}\ x_i^{j-k-\alpha-1}, & \\ & 1 \le i \le N-1, 3 \le j \le N+1. \end{cases}$$

Also $[r]^n = [r_0^n, r_1^n, ..., r_N^n]^T$ and $\mathbf{v_n} = [g_1(t_{n+1}), \Delta t\ p_n(x_1), ..., \Delta t\ p_n(x_{N-1}), g_0(t_{n+1})]^T$.

4 Numerical example

Example 4.1 [4] Consider the problem (1) for $\alpha = 1.8$ with initial condition $u(x, 0) = x^2 - x^3$ and boundary conditions $u(0, t) = 0$, $u(1, t) = 0$.
Let $d(x) = \Gamma(1.2)x^{1.8}$ and $p(x, t) = 3x^2(2x-1)e^{-t}$. The exact solution for this problem is $u(x, t) = x^2(1-x)e^{-t}$.
We solved this equation by using proposed method and in Table 1 we compared our results with results of obtained in [4].

Table 1: Comparison of absolute errors for $u(x, 2)$ for example 4.2.

x	Method [4] with m=5	present method(N=3)
0.0	0.0	0.0
0.1	4.47×10^{-6}	3.15×10^{-7}
0.2	2.78×10^{-7}	4.23×10^{-7}
0.3	5.81×10^{-6}	2.94×10^{-7}
0.4	1.02×10^{-5}	5.70×10^{-6}
0.5	1.17×10^{-5}	6.28×10^{-5}
0.6	1.08×10^{-5}	1.05×10^{-5}
0.7	8.54×10^{-6}	2.43×10^{-6}
0.8	6.06×10^{-6}	8.50×10^{-6}
0.9	3.67×10^{-6}	7.68×10^{-7}
1.0	0.0	0.0

5 Conclusion

In this paper, finite difference scheme and Chebyshev collocation method have been successfully applied to find the solution of the space fractional diffusion equations and time fractional diffusion equations. Sine using matrix form of the method is more convenient for application of collocation method, thus the matrix form of the proposed method was obtained. The results and comparison of the our proposed method and other methods indicate that this scheme is accurate and efficient approach for the solution of this problems.

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Solving fuzzy linear equations using weighted fuzzy arithmetic

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Abstract

There are different methods to solve the fuzzy linear equation $\overline{AX} + \overline{B} = \overline{C}$, where $\overline{A}, \overline{B}$ and $\overline{C}$ are triangular fuzzy numbers. In this paper, first we define a new weighted fuzzy arithmetic operations, then using this new definition we will introduce a modified method for solving fuzzy linear equations which gives a better solution.
Keywords: Fuzzy linear equations, New Fuzzy Arithmetic.

Mathematics Subject Classification: 03E72

1 Introduction

We define a new type of arithmetic operations on traingular fuzzy numbers. Suppose $\overline{X} = (x_1, x_2, x_3)$ and $\overline{Y} = (y_1, y_2, y_3)$ are traingular fuzzy numbers and $\circledast \in \{\oplus, \otimes, \ominus, \oslash\}$.
We define
$\overline{Z}_\lambda = \overline{X} * \overline{Y} = (B - k - W_\lambda(A, B-k), B, B+k+W_\lambda(B+k, C))$
where $\overline{Z} = (A, B, C)$ is the result of using existing arithmetic operation, called classic method, $k = Min\{B - A, C - B\}$ and weight function W_λ is $W_\lambda(r, s) = \lambda|s - r|$ where $0 < \lambda < 1$.
It can be easily shown that $\overline{Z}_\lambda$ is a traingular fuzzy number and $\overline{Z}_\lambda[\alpha] \subseteq \overline{Z}[\alpha]$.

Lemma 1.1. *If $\lambda_1, \lambda_2 \in (0, 1)$ and $\lambda_1 < \lambda_2$ then $\overline{Z}_{\lambda_1}[\alpha] \subseteq \overline{Z}_{\lambda_2}[\alpha]$*

Lemma 1.2. *For any $r, s \in R, r \leqslant s$ we have*
$(s - r) - W_\lambda(r, s) = W_{1-\lambda}(r, s)$

Example 1.3. *Let $\overline{X} = (-3, -2, 2)$ and $\overline{Y} = (4, 5, 8)$ we compute some classic and weighted operations where $\lambda = 1/2$*

1. *Summation:* $\overline{X} + \overline{Y} = (1, 3, 10)$ *and* $\overline{X} \oplus \overline{Y} = (1, 3, 7.5)$
2. *Division:*$\overline{X}/\overline{Y} = (-3/4, -2/5, 1/2)$ *and* $\overline{X} \oslash \overline{Y} = (-3/4, -2/5, 9/40)$

2 Main Result

Using new weighted fuzzy arithmetic, we try to refine the solution of the fuzzy linear equation,$\overline{A}$ $\overline{X} + \overline{B} = \overline{C}$, where $\overline{A}, \overline{B}$ and $\overline{C}$ are triangular fuzzy numbers. Let $\overline{A} = (a_1, a_2, a_3), \overline{B} = (b_1, b_2, b_3)$ and $\overline{C} = (c_1, c_2, c_3)$. We assume zero does not belong to the support of $\overline{A}$.

The classic solution is

$\overline{X} = \frac{\overline{C}-\overline{B}}{\overline{A}} = (A, \frac{c_2-b_2}{a_2}, B)$ where

$A = Min\{\frac{c_1-b_3}{a_1}, \frac{c_1-b_3}{a_3}, \frac{c_3-b_1}{a_1}, \frac{c_3-b_1}{a_3}, \frac{c_2-b_2}{a_2}\}$
and
$B = Max\{\frac{c_1-b_3}{a_1}, \frac{c_1-b_3}{a_3}, \frac{c_3-b_1}{a_1}, \frac{c_3-b_1}{a_3}, \frac{c_2-b_2}{a_2}\}$

we can write $\overline{X}[\alpha] = [A + \alpha(\frac{c_2-b_2}{a_2} - A), B - \alpha(B - \frac{c_2-b_2}{a_2}]$
Now using weighted fuzzy arithmetic we have:

$\overline{X}_\lambda = (\frac{c_2-b_2}{a_2} - k - W_\lambda(\frac{c_2-b_2}{a_2} - k, A), \frac{c_2-b_2}{a_2}, \frac{c_2-b_2}{a_2} + k + W_\lambda(\frac{c_2-b_2}{a_2} + k, B))$ where $k = min\{\frac{c_2-b_2}{a_2} - A, B - \frac{c_2-b_2}{a_2}\}$
So

$\overline{X}_\lambda[\alpha] = [\frac{c_2-b_2}{a_2} - k - W_\lambda(\frac{c_2-b_2}{a_2} - k, A) + \alpha(k + W_\lambda(\frac{c_2-b_2}{a_2} - k, A)), \frac{c_2-b_2}{a_2} + k + W_\lambda(\frac{c_2-b_2}{a_2} + k, B) - \alpha(k + W_\lambda(\frac{c_2-b_2}{a_2} + k, B))]$.

Theorem 2.1. *$\overline{X}_\lambda$ is a triangular fuzzy number and $\overline{X}_\lambda[\alpha] \subseteq \overline{X}[\alpha]$*

Thus we have a better solution.

Example 2.2. *Let $\overline{A} = (1,3,7), \overline{B} = (-3,-1,2)$ and $\overline{C} = (3,5,10)$ using classic method we get $\overline{X} = (1/7, 2, 13)$ and by modified weighted method for $\lambda = 1/2$ we have $\overline{X}_\lambda = (1/7, 2, 59/7)$*

Our general strategy for solving fuzzy linear equations will be: 1. the solution is $\overline{X}_c$ when it exists and $\overline{X}_c \leq \overline{X}_e \leq \overline{X}_I$.

2.If $\overline{X}_c$ fails to exist , the solution is $\overline{X}_e$ and 3. If $\overline{X}_c$ fails to exist and $\overline{X}_e$ is difficult to construct, use $\overline{X}_I$ as the approximation solution. [1,3], and now we have a solution $\overline{X}_\lambda$ that is more accurate than $\overline{X}_I$.

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Numerical solution of an inhomogeneous heat equation by the product integration method

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Abstract

We reduce an inhomogeneous heat equation to an integral equation and apply the product integration method to solve it. Convergence analysis of the product integration method is studied in [2, 3, 4]. Numerical implementation of the method is illustrated by benchmark problem originated from heat conduction.
Keywords: Product integration technique, Weakly singular Volterra integral equation, Heat equation, Numerical solution

Mathematics Subject Classification: 45G05; 30E20; 65R20.

1 Introduction

In this paper we consider the inhomogeneous heat equation in one spatial dimension: find $u(x,t)$ such that

$$u_t = u_{xx} + f(x,t) \qquad 0 < x < \infty, \quad 0 < t, \tag{1}$$

$$u(x,0) = \phi(x) \quad 0 < x < \infty, \tag{2}$$

$$u_x(0,t) + \alpha(t)u(0,t) = g(t), \qquad 0 < t, \tag{3}$$

and

$$|u(x,t)| \leq C_1 \exp\left\{C_2|x|^{1+\gamma}\right\}, \qquad \gamma < 1. \tag{4}$$

Where $u(x,t)$ is the temperature, $C_i, i = 1,2$, are positive constants and the data f, ϕ, α and g satisfy the following Assumption **A.** and are known. In [1] it is shown that this problem has a unique solution.

Assumption A. we shall assume that the source function $f(x,t)$ is bounded over each domain considered and that $f(x,t)$ is uniformly Holder continuous on each compact subset of the domain under consideration. We shall also assume that the initial-boundary data is piecewise-continuous.

2 Reduction to an integral equation

We give some definitions, lemmas and theorems associated with this section

Definition 2.1. *The fundamental solution of heat equation denote by $K(x,t)$, and the Neumann's function denote by $N(x,\xi,t)$*

$$K(x,t) := \frac{\exp\left\{-\frac{x^2}{4t}\right\}}{\sqrt{4\pi t}}, \; N(x,\xi,t) := K(x-\xi,t) + K(x+\xi,t), \quad 0 < t, x \in \mathbf{R}. \tag{5}$$

Lemma 2.2. *For any integrable function f that satisfies $|f(x)| \leq C_1 \exp\{C_2 x^2\}$, where C_1 and C_2 are positive constants, then $\lim_{t\downarrow 0} \int_{-\infty}^{\infty} K(x-\xi,t)f(\xi)d\xi = f(x),\ 0 < t$, at the point x of continuity of f.*

Proof. See Lemma 3.4.3 of [1]. □

Lemma 2.3. *At a point of continuity of g, $\lim_{x\downarrow 0} -2\int_0^t \frac{\partial K}{\partial x}(x,t-\tau)g(\tau)d\tau = g(t)$.*

Proof. See Lemma 4.2.1 of [1]. □

Lemma 2.4. *For bounded continuous f in $-\infty < x < \infty,\ 0 \leq t$, which is uniformly Holder continuous with exponent $\alpha,\ 0 < \alpha < 1$, with respect to x, the potential $z(x,t) := \int_0^t \int_{-\infty}^{\infty} K(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau$, possesses the following properties*
1.z, z_x, z_t and z_{xx} are continuous; 2.$z_t = z_{xx} + f(x,t) \quad -\infty < x < \infty,\ 0 \leq t$.

Proof. See Lemma 19.2.1 of [1]. □

Theorem 2.5. *The problem of determining the unique solution u of problem (13)- (1.3), is equivalent to the problem of determining a unique, piecewise-continuous solution ψ to the integral equation,$\psi(t) = d(t)+2\alpha(t)\int_0^t K(0,t-\tau)\psi(\tau)d\tau, \quad 0 < t$, where $d(t) = g(t)-\alpha(t)\int_0^{\infty} N(0,\xi,t)\phi(\xi)d\xi - \alpha(t)\int_0^t\int_0^{\infty} N(0,\xi,t-\tau)f(\xi,\tau)d\xi d\tau$. And the solution u has the representation*

$$u(x,t) = -2\int_0^t K(x,t-\tau)\psi(\tau)d\tau + \int_0^{\infty} N(x,\xi,t)\phi(\xi)d\xi + \int_0^t\int_0^{\infty} N(x,\xi,t-\tau)f(\xi,\tau)d\xi d\tau. \quad (6)$$

Proof. Consider the problem (13)-(1.3). According Lemmas 2.1-2.3 we find $u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t)$, such that u_1, u_2 satisfy heat equation $u_t = u_{xx}$, and u_3 is true of inhomogeneous equation (13). For this purpose we let$u_1(x,t) = -2\int_0^t K(x,t-\tau)\psi(\tau)d\tau, u_2(x,t) = \int_0^{\infty} N(x,\xi,t)\phi(\xi)d\xi, u_3(x,t) = \int_0^t\int_0^{\infty} N(x,\xi,t-\tau)f(\xi,\tau)d\xi d\tau$, where $\psi(t)$ is a piecewise-continuous function. From Lemma 2.2 we have

$$u_x(0,t) = u_{1x}(0,t) + u_{2x}(0,t) + u_{3x}(0,t) = u_{1x}(0,t) = \lim_{x\downarrow 0} -2\int_0^t \frac{\partial K}{\partial x}(x,t-\tau)\psi(\tau)d\tau = \psi(t). \quad (7)$$

For $0 < x < \infty,\ 0 < t$, Lemma 2.1 leads

$$u(x,0) = u_2(x,0) = \lim_{t\downarrow 0}\int_0^{\infty} N(x,\xi,t)\phi(\xi)d\xi = \lim_{t\downarrow 0}\int_{-\infty}^{\infty} K(x-\xi,t)\phi_e(\xi)d\xi = \phi(x), \quad (8)$$

where ϕ_e is the even extension of ϕ to $-\infty < x < \infty$. We show that u possesses equation (13). From Lemma 2.3

$$u_t - u_{xx} = u_{3t} - u_{3xx} = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\int_0^t\int_{-\infty}^{\infty} K(x-\xi,t-\tau)f_e(\xi,\tau)d\xi d\tau = f(x,t),$$

here f_e is the even extension of f to $-\infty < x < \infty$.
By substitution u in (1.3) and use of (7) we obtain

$$\begin{aligned}\psi(t) &= g(t) - \alpha(t)u_1(0,t) - \alpha(t)u_2(0,t) - \alpha(t)u_3(0,t)\\ &= d(t) + 2\alpha(t)\int_0^t K(0,t-\tau)\psi(\tau)d\tau, \quad 0 < t, \qquad (9)\end{aligned}$$

From consideration of chapter3 of [1] the solution u in the class (1.3) is unique, and hence the proof is completed. □

3 Product integration technique

We explain the product integration method for the following equation

$$u(t) = g(t) + \int_0^t p(t,\tau)k(t,\tau,u(\tau))d\tau, \quad t \in [0,b], \tag{10}$$

where p is weakly singular and k is smooth. We introduce $N+1$ grid points $0 \le t_0 < t_1 < ... < t_N \le b$ in t. Our goal is to compute $u(t)$ at the grid points and the numerical approximation to $u(t_n)$ is written as u_N^n. The basic point in product integration technique are

1. Sample the system of Volterra integral equations at points t_n in the grid, $u(t_n) = g(t_n) + \int_0^{t_n} p(t_n,\tau)k(t_n,\tau,u(\tau))d\tau$.

2. Use the Lagrange interpolation polynomial, $L_N(k,t_n;\tau) = \sum_{j=0}^N l_{N,j}(\tau)K(t_n,t_j,u(t_j))$, to approximate $k(t_n,\tau,U(\tau))$ and obtain the algorithm $u_N^{(n)} = g(t_n)+\sum_{j=0}^N \omega_j(t_n)k(t_n,t_j,U_N^{(j)})$, where $\omega_j(t) = \int_0^t p(t,\tau)l_{N,j}(\tau)d\tau$.

3. Evaluate $u_N^{(j)}$ from Step 2. and obtain $u_N(t) = g(t) + \sum_{j=0}^N \omega_j(t)k(t,t_j,u_N^{(j)})$, as a Nystrom approximation for $u(t)$.

4 Numerical results

For $u(x,t) = \exp\{-x\} + \exp\{-t\}$ in the problem (13)-(1.3) we have $f(x,t) = -(\exp\{-x\} + \exp\{-t\})$, $g(t) = \exp\{-t\}$, $\phi(t) = \exp\{-x\}+1$. The integral equation associated with this problem is, $\psi(t) = 2\sqrt{\frac{t}{\pi}} - 1 + \frac{1}{\sqrt{\pi}}\int_0^t \frac{\psi(\tau)}{\sqrt{t-\tau}}d\tau$, which has the exact solution $\psi \equiv 1$. Table 1. shows relative errors of $\widetilde{\psi}$ at $t = 0.01i, i = 1,...,10$ with $b = 0.1$, ψ is exact solution and $\widetilde{\psi}$ is evaluated by product integration technique.
Suppose $\widetilde{u}$ is the approximated solution evaluated by product integration method in the first integral of (1.5). Let $R = (r_{ij})$ is an 5×5 matrix with r_{ij} = the relative error of $\widetilde{u}$ at $(0.02i, 0.02j)$, then

$$R = \begin{bmatrix} neg & \text{2.3D-16} & \text{8.1D-16} & \text{4.4D-15} & \text{5.5D-15} \\ neg & \text{1.2D-16} & neg & \text{2.7D-15} & \text{4.3D-15} \\ \text{1.2D-16} & \text{3.5D-16} & \text{3.5D-16} & \text{9.5D-16} & \text{9.5D-15} \\ \text{1.2D-16} & \text{2.4D-16} & \text{1.2D-16} & \text{3.6D-16} & \text{1.9D-15} \\ \text{1.2D-16} & neg & neg & \text{7.3D-16} & \text{3.4D-15} \end{bmatrix}$$

Where *neg* means negligible, and for example $2.3D-16$ means 2.3×10^{-16}. All of programs written by Mathematica programming.

i	$\left\lvert\frac{\psi-\widetilde{\psi}}{\psi}\right\rvert_{t=0.01i}$
1	*negligible*
2	4.44089×10^{-16}
3	6.66134×10^{-16}
4	8.88178×10^{-16}
5	8.88178×10^{-16}
6	3.10862×10^{-15}
7	1.33227×10^{-15}
8	3.33067×10^{-15}
9	1.9984×10^{-15}
10	2.68674×10^{-14}

Table 1: relative errors of $\widetilde{\psi}$

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Existence and uniqueness of solutions for integral equations with singular kernel

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Abstract

In this paper, we prove the existence and uniqueness of solutions for some nonlinear functional-integral equations by using generalized Lipschitz condition. As application, we study integral equations with singular kernel.

Keywords: Nonlinear functional-integral equation, iterative method, Fixed point theorem.

Mathematics Subject Classification: 47H30; 54H25.

1 Introduction

In this work, we try to prove the existence and uniqueness of the solutions of the following nonhomogeneous nonlinear Volterra integral equation

$$u(x) = f(x) + \varphi\Big(\int_a^x F(x,t,u(t))dt\Big) \equiv Tu, \tag{1}$$

where $x, t \in I = [a,b]$, $-\infty < a < b < \infty$.

2 Basic Concepts

We assume that $f : [a,b] \to \mathbb{R}$ is continuous and F is a mapping on the domain $D = \{(x,t,u) : x,t \in [a,b]\ , u \in X\}$. Through this article, we consider the complete metric space (X,d), which $d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|$, for all $f, g \in X$ and assume that φ is a bounded linear transformation on X.

Definition 2.1. *Let δ denote the class of those functions $\beta : [0,\infty) \to [0,1)$ which satisfies the condition*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0.$$

Definition 2.2. *Let $\mathfrak{B}$ denote the class of those functions $\phi : [0,\infty) \to [0,\infty)$ which satisfies the following condition:*
(i) ϕ is increasing,
(ii) for each $x > 0$, $\phi(x) < x$,
(iii) $\beta(x) = \frac{\phi(x)}{x} \in \delta$, $x \neq 0$.

For example, $\phi(t) = \mu t$, where $0 \le \mu < 1$, $\phi(t) = \frac{t}{t+1}$ and $\phi(t) = \ln(1+t)$ are in $\mathfrak{B}$.

3 Existence and uniqueness of the Solution of Nonlinear Integral Equations

Theorem 3.1. *Consider the integral equation (13) such that,*
(i) $F : D \to R$ *and* $f : [a,b] \to \mathbb{R}$ *are continuous.*
(ii) $\varphi : X \to X$ *is a bounded linear transformation.*
(iii) There exists a integrable function $p : [a,b] \times [a,b] \to R$ *such that*

$$|F(x,t,u) - F(x,t,v)| \leq p(x,t)\phi(|u-v|), \tag{2}$$

for each $x, t \in [a,b]$ *and* $u, v \in R$.
(iv) $\sup_{x\in[a,b]} \int_a^b p^2(x,t)dt \leq \frac{1}{\|\varphi\|^2(b-a)}$.
Then the integral equation (13) has a unique fixed point u *in* X.

Note. We define (2) as a generalized Lipschitz condition.

Proof. Consider the iterative scheme

$$u_{n+1}(x) = f(x) + \varphi\Big(\int_a^x F(x,t,u_n(t))dt\Big) \equiv Tu_n, \qquad n = 0, 1, \cdots . \tag{3}$$

where $u_0 \in X$ is an arbitrary initial guess. So,

$$\begin{aligned}
|Tu_n - Tu_{n-1}| &= |\varphi\Big(\int_a^x F(x,t,u_n(t))dt\Big) - \varphi\Big(\int_a^x F(x,t,u_{n-1}(t))dt\Big)| \\
&\leq |\varphi\Big(\int_a^x F(x,t,u_n(t)) - F(x,t,u_{n-1}(t))dt\Big)| \\
&\leq \|\varphi\| \, |\int_a^x F(x,t,u_n(t)) - F(x,t,u_{n-1}(t))dt| \\
&\leq \|\varphi\| \int_a^x |F(x,t,u_n(t)) - F(x,t,u_{n-1}(t)|dt \\
&\leq \|\varphi\| \int_a^b p(x,t)\phi(|u_n(t) - u_{n-1}(t)|)dt \\
&\leq \|\varphi\| \Big(\int_a^b p^2(x,t)dt\Big)^{\frac{1}{2}} \Big(\int_a^b \phi^2(|u_n(t) - u_{n-1}(t)|)dt\Big)^{\frac{1}{2}}.
\end{aligned} \tag{4}$$

As the function ϕ is increasing then

$$\phi(|u_n(t) - u_{n-1}(t)|) \leq \phi(d(u_n, u_{n-1})),$$

so, we obtain

$$\begin{aligned}
d^2(u_{n+1}, u_n) &\leq \|\varphi\|^2 \Big(\sup_{x\in[a,b]} \int_a^b p^2(x,t)dt\Big)\Big(\int_a^b \phi^2(d(u_n,u_{n-1}))dt\Big) \\
&\leq \phi^2(d(u_n, u_{n-1})).
\end{aligned}$$

Therefore

$$\begin{aligned}
d(u_{n+1}, u_n) \leq \phi(d(u_n, u_{n-1})) &= \frac{\phi(d(u_n,u_{n-1}))}{d(u_n,u_{n-1})} d(u_n, u_{n-1}) \\
&= \beta(d(u_n, u_{n-1}))d(u_n, u_{n-1}),
\end{aligned} \tag{5}$$

and so the sequence $\{d(u_{n+1}, u_n)\}$ is decreasing and bounded. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(u_{n+1}, u_n) = r$. Assume $r > 0$. Then from (1.3) we have

$$\frac{d(u_{n+1}, u_n)}{d(u_n, u_{n-1})} \leq \beta(d(u_n, u_{n-1})), \quad n = 1, 2, \cdots .$$

The above inequality yields $\lim_{n\to\infty} \beta(d(u_{n+1}, u_n)) = 1$. Then $\beta \notin \delta$ and this is contradiction. So $r = 0$ and then $\lim_{n\to\infty} d(u_{n+1}, u_n) = 0$. Now we show that $\{u_n\}$ is a Cauchy sequence. On the contrary, assume that

$$\limsup_{n,m\to\infty} d(u_n, u_m) > 0. \tag{6}$$

By the triangle inequality and relation (1.3) we have

$$\begin{aligned} d(u_n, u_m) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{m+1}) + d(u_{m+1}, u_m) \\ &\leq d(u_n, u_{n+1}) + (\beta(d(u_n, u_m))d(u_n, u_m)) + d(u_{m+1}, u_m), \end{aligned}$$

hence

$$d(u_n, u_m) - (\beta(d(u_n, u_m))d(u_n, u_m)) \leq d(u_n, u_{n+1}) + d(u_{m+1}, u_m).$$

Therefore we obtain

$$d(u_n, u_m) \leq (1 - \beta(d(u_n, u_m)))^{-1}[d(u_n, u_{n+1}) + d(u_{m+1}, u_m)].$$

Since $\limsup_{n,m\to\infty} d(u_n, u_m) > 0$ and $\lim_{n\to\infty} d(u_{n+1}, u_n) = 0$ then

$$\limsup_{n,m\to\infty}((1 - \beta(d(u_n, u_m)))^{-1}) = +\infty,$$

from which we obtain $\limsup_{n,m\to\infty} \beta(d(u_n, u_m)) = 1$. But since $\beta \in \delta$, we get $\limsup_{n,m\to\infty} d(u_n, u_m) = 0$. This contradicts (1.3) and shows $\{u_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, then there exists $u \in X$ such that $\lim_{n\to\infty} u_n = u$. Now by taking the limit of both sides of (2), we have

$$\begin{aligned} u = \lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty}(f(x) + \varphi(\int_a^x F(x, t, u_n(t))dt)) \\ &= f(x) + \varphi(\int_a^x F(x, t, \lim_{n\to\infty} u_n(t))dt) \\ &= f(x) + \varphi(\int_a^x F(x, t, u(t))dt). \end{aligned}$$

So, there exists a unique solution $u \in X$ such that $Tu = u$. □

4 Illustrative Examples

Example 4.1. *Consider the following singular Volterra integral equation*

$$u(x) = f(x) + \lambda \int_0^x (x - t)^{-\alpha} u(t)dt, \qquad x, t \in [0, T], \tag{7}$$

where $|\lambda| < 1$ and $0 < \alpha < \frac{1}{2}$. Then

$$|F(x, t, u) - F(x, t, v)| = |\lambda(u - v)(x - t)^{-\alpha}| \leq |\lambda||u - v||(x - t)|^{-\alpha}.$$

Put $p(x, t) = (x - t)^{-\alpha}$ and $\phi(t) = \lambda t$. On the other hand

$$\sup_{x\in[0,T]} \int_0^T p^2(x, t)dt = \sup_{x\in[0,T]} \int_0^T |(x - t)|^{-2\alpha}dt = \frac{T^{1-2\alpha}}{1 - 2\alpha}.$$

Therefore if $T^{\frac{1}{2}-\alpha} \leq (1 - 2\alpha)^{\frac{1}{2}}$, then the equation (3) has a stable unique solution in complete metric space $C[0, T]$. Note that we can't obtain these results by using the usual Lipschitz condition.

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Sinc-Galerkin method for numerical solution of time-dependent linear convection-diffusion equation

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Abstract
We apply a standard implicit finite difference scheme to discretize in temporal direction and Sinc-Galerkin in spatial direction. The exponential convergence analysis of our approach can be obtained by existent theorems in Sinc-Galerkin method. The presented method applied on the test problems, our numerical results have been compared with the other methods, these result verified efficiency of the method.

Keywords: Convection-diffusion equation; Sinc-Galerkin method; Discretization in time; Numerical results.

Mathematics Subject Classification: 65M22; 35k10; 35k57.

1 Introduction

We consider one-dimensional parabolic convection-diffusion equation:

$$u_t - \epsilon u_{xx} + a(x)u_x + b(x)u = f(x,t), \qquad 0 < x < 1, \qquad t > 0 \tag{1}$$

with boundary and initial conditions:

$$u(0,t) = 0, \qquad u(1,t) = 0 \qquad u(x,0) = g(x), \tag{2}$$

where ϵ is a diffusion coefficient or singular perturbation parameter satisfying $0 < \epsilon \ll 1, a(x)$ is the velocity, $f(x,t) - b(x)u$ is the reaction term.
Recently, there has been a lot of effort in developing numerical methods for solution of this equation [1], [3], [4], [6], [7]. These method are based on finite difference scheme exponentially fitted method, splines collocation method, exponential splines and B-spline collocation method.
In this paper we discretized the one-dimensional parabolic convection-diffusion equation in the time direction and then the Sinc-Galerkin method is applied. The Sinc method, which was developed by F.Stenger [5], is based on the Whittaker-Shannon-Kotel' nikov sampling theorem for entire functions. This method has many advantages over classical methods that use polynomials as bases. For example, in the presence of singularities, it gives a much better rate of convergence and greater accuracy than polynomial methods.

2 Description of the method

First, we recall notations and definitions of the Sinc function.
- $sinc(z) = sin(\pi z)/(\pi z), \qquad z \in C$ Not that $|sinc(z)| \leq 1$ for any z.

- $S(k,h)(z) = sinc[(z-kh)/h], \quad z \in C, h > 0.$
- $C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(hk)S(k,h)(x), h > 0,$
  here, $C(f,h)(x)$ is called the Whittaker cardinal expansion of $f(x)$ whenever this series converges.

The properties of Whittaker cardinal expansions have been studied and are thoroughly surveyed in [2, 5]. These properties are derived in the infinite strip D of the complex plane where for $D_d = \{\zeta = \xi + i\eta : |\eta| < d \leq \frac{\pi}{2}, d > 0\}$.
We consider the conformal map

$$\phi(x) = \ln(\frac{x-a}{x-b}). \tag{3}$$

The map ϕ carries the eye-shaped region

$$D_E = \{z = x + iy; |\arg(\frac{z-a}{b-z})| < d \leq \frac{\pi}{2}\}, \tag{4}$$

onto D_d. The basis function on (a,b) are then given by $S(j,h)\circ\phi(x)$. Notice that these functions exhibit Kronecker delta behavior on the grid points $x \in (a,b)$ defined by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \qquad h = \sqrt{\frac{\pi d}{\alpha M}}, \qquad 0 < \alpha \leq 1. \tag{5}$$

The Sinc-Galerkin method requires that the derivatives of composite sinc functions be evaluated at the nodes. We need the following lemma.
Lemma 1.([5]). Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d, given by (22). Then

$$\delta_{jk}^{(0)} = [S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{6}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{(k-j)}}{k-j}, & j \neq k, \end{cases} \tag{7}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{(k-j)}}{(k-j)^2}, & j \neq k. \end{cases} \tag{8}$$

Secondly, we discretize equation (1) in time direction by means of the implicit Euler method with uniform step size $\Delta t = T/p$ where T is total time and p is chosen as a positive integer, to get the following system of linear ordinary differential equations:

$$-\epsilon z_j^{''}(x) + a(x)z_j^{'}(x) + (b(x) + \frac{1}{\Delta t})z_j(x) = f(x,t_j) + \frac{1}{\Delta t}z_{j-1}(x), \tag{9}$$

$$z_j(0) = 0, \qquad z_j(1) = 0, \qquad j = 1,2,\ldots,p \tag{10}$$

where $z_j(x) = u(x,t_j)$ is solution of Eq. (9) at jth time level.

Now in each time level we can apply the Sinc-Galerkin method to approximate the solution of linear boundary value problem (9) and (10). The approximate solution for $z_j(x)(j = 1,2,\ldots,p)$ is represented by formula

$$z_m^j(x) = \sum_{r=-M}^{N} c_r^j S_r(x), \qquad m = M + N + 1, \tag{11}$$

where $S_r(x)$ is function $S(r,h)\circ\phi(x)$ for some fixed step size h. The unknown coefficient c_r^j in relation (11) are determined by orthogonalizing the residual with respect to the basis function, i.e,

$$\prec -\epsilon z_j^{''}(x), S_k \succ + \prec a(x)z_j^{'}(x), S_k \succ + \prec (b(x)+\frac{1}{\Delta t})z_j(x), S_k \succ = \tag{12}$$

$$\prec (\frac{1}{\Delta t}z_{j-1}(x)+f(x,t_j)), S_k \succ .$$

The inner product used for the sinc-Galerkin method is defined by

$$\prec f,\eta \succ = \int_a^b f(x).\eta(x)\omega(x)dx.$$

For the case of second-order problem, it is convenient to take $\omega(x)=\frac{1}{\phi'(x)}$.
By applying the approximations defined in theorem 4.4 [2] for each terms of relation (12), we obtain the discrete sinc-Galerkin system for the determination of the unknown coefficients $\{c_r^j\}_{r=-M}^N$ for $j=1,2,\dots,p$ as:

$$\sum_{r=-M}^{N}\left\{\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kr}^{(i)}\frac{g_{2,i}(x_r)}{\phi'(x_r)}c_r^j-\sum_{i=0}^{1}\frac{1}{h^i}\delta_{kr}^{(i)}\frac{g_{1,i}(x_r)}{\phi'(x_r)}c_r^j\right\}$$

$$+(\frac{1}{\Delta t}+b(x_k))\frac{\omega(x_k)}{\phi'(x_k)}c_k^j=\frac{\omega(x_k)}{\phi'(x_k)}(f(x_k,t_j)+\frac{1}{\Delta t}z_{j-1}(x_k)) \quad -M\le k\le N \quad j=1,2,\dots,p, \tag{13}$$

where
$g_{2,2}=-\epsilon\omega(\phi')^2,\quad g_{2,1}=-\epsilon\omega\phi''-2\epsilon\omega'\phi',\quad g_{2,0}=-\epsilon\omega'',\quad g_{1,1}=a(x)\omega\phi',\quad g_{1,0}=(a(x)\omega)'$,
for each value of j in (13), we have one system of linear equations that can be solved easily.

3 Numerical results

In this section, we discuss the numerical results obtained in the integration of some problems of type (1).
Example 1. We take $a(x)=1, b(x)=0, T=1$ and we determine $g(x)$ and $f(x,t)$ to get exact solution, given by:

$$u_e(x,t)=e^{-t}(C_1+C_2x-e^{-(1-x)/\epsilon}), \qquad C_1=e^{-1/\epsilon}, C_2=1-e^{-1/\epsilon}.$$

Table 1: *Maximum pointwise error for example $M=N=16$ and $\Delta t=.1$.*

$n\rightarrow$	16		32	
$\epsilon\downarrow$	present	Kakalbajoo [3]	present	Kakalbajoo [3]
2^0	1.5733×10^{-4}	1.5507×10^{-2}	1.5947×10^{-5}	6.5619×10^{-5}
2^{-2}	3.0378×10^{-3}	1.507×10^{-2}	1.4994×10^{-3}	1.3429×10^{-3}
2^{-4}	8.4205×10^{-3}	1.9314×10^{-2}	4.2473×10^{-3}	1.1076×10^{-2}
2^{-6}	1.1055×10^{-2}	3.8743×10^{-2}	5.5752×10^{-3}	1.8444×10^{-2}
2^{-8}	1.1188×10^{-2}	2.1546×10^{-1}	6.0213×10^{-3}	1.1948×10^{-1}
2^{-10}	1.1090×10^{-2}	4.0680×10^{-1}	5.9960×10^{-3}	2.7456×10^{-1}

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Variational iteration method for solving systems of linear delay differential equations

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Abstract
In this paper, using a model transformation approach a system of linear delay differential equations (DDEs) with multiple delays is converted to a non-delayed initial value problem. The variational iteration method (VIM) is then applied to obtain the approximate analytical solutions. Comparisons with the classical fourth-order Runge-Kutta method (RK4) verify that this method is very effective and convenient.
Keywords: Variational iteration method; delay differential equations; multiple delays; Runge-Kutta method.

Mathematics Subject Classification: 34K06, 74H10.

1 Introduction

The system of linear DDEs with multiple delays may be expressed as

$$\begin{cases} \dot{x}(t) = A_0 x(t) + \sum_{d=1}^{m} A_d x(t-\tau_d) + p(t) & t \in [0, t_f], \\ x(t) = \phi(t) & t \in [-\tau, 0], \end{cases} \tag{1}$$

where the matrices $A_0, A_1, \ldots, A_m \in \mathbb{R}^{n\times n}$, $x(t) = [x_1(t)\ \ x_2(t) \ldots \ \ x_n(t)]^T \in \mathbb{R}^n$, $\phi(t) = [\phi_1(t)\ \ \phi_2(t) \ldots \ \ \phi_n(t)]^T \in \mathbb{R}^n$ and $p(t) = [p_1(t)\ \ p_2(t) \ldots \ \ p_n(t)]^T \in \mathbb{R}^n$ is a known continuous function representing the external excitation. The delays $\tau_1, \tau_2, \ldots, \tau_m$ are known, positive and constant numbers. The set $[0, t_f]$ is a time interval and $\tau = \max\{\tau_1, \tau_2, \ldots, \tau_m\}$.

As we know, the variational iteration method (VIM) presented by He [1] is a powerful mathematical tool for finding solutions of linear and nonlinear problems and it can be implemented easily in practice.

2 Model transformation and solution method

Now we study the model (1) with multiple delays $\tau_1, \tau_2, \ldots, \tau_m$. We shall use a transformation technique that requires the technical assumption that the interval length $\Delta t = t_f$ is a rational multiple of all the delays $\tau_1, \tau_2, \ldots, \tau_m$:

$$\Delta t = q_1\tau_1, \qquad \Delta t = q_2\tau_2, \qquad \ldots \qquad , \Delta t = q_m\tau_m,$$

where $q_1, q_2, \ldots, q_m \in \mathbb{Q}$. With this assumption, we divide the time interval $[0, t_f]$ into l equidistant subintervals such that l is the minimum number of divisions of $[0, t_f]$ which satisfies all of the following conditions:

$$\frac{l\tau_1}{\Delta t} = z_1 \in \mathbb{N}, \quad \frac{l\tau_2}{\Delta t} = z_2 \in \mathbb{N}, \quad \ldots \quad , \frac{l\tau_m}{\Delta t} = z_m \in \mathbb{N}.$$

Let $T = \frac{\Delta t}{l}$ and $N = nl$. For each $t \in [0, T]$, we define

$$y_k(t) = x_j(t + (k - 1 - (j-1)l)T),$$
$$k = (j-1)l + 1, \dots, jl, \qquad j = 1, 2, \dots, n. \tag{2}$$

Then the problem (1) is converted to the following N-dimensional non-delayed system:

$$\begin{cases} \dot{y}(t) = Cy(t) + f(t) & t \in [0, T], \\ y(0) = y_0 \end{cases} \tag{3}$$

where $C \in \mathbb{R}^{N\times N}$, $y(t) = [y_1(t) \;\; y_2(t) \dots \;\; y_N(t)]^T \in \mathbb{R}^N$, $f(t) = [f_1(t) \;\; f_2(t) \dots \;\; f_N(t)]^T \in \mathbb{R}^N$ and the initial value $y_0 = [y_{10} \;\; y_{20} \dots \;\; y_{N0}]^T \in \mathbb{R}^N$ is determined as follows:

$$y_{k0} = \phi_j(0), \qquad k = (j-1)l + 1, \qquad j = 1, 2, \dots, n,$$
$$y_{k0} = y_{k-1}(T), \quad k = (j-1)l + 2, \dots, jl, \quad j = 1, 2, \dots, n.$$

Let $y(T) = [r_1 \;\; r_2 \dots \;\; r_N]^T \in \mathbb{R}^N$ denote the final value. The transition matrix method assumes the existence of two transition matrices $Y(t) \in \mathbb{R}^{N\times N}$ and $F(t) \in \mathbb{R}^{N\times N}$ such that

$$y(t) = Y(t)y(T), \tag{4}$$

$$f(t) = F(t)y(T). \tag{5}$$

According to (4) and (5), it is easy to see that the transition matrix $Y(t)$ must satisfy the following non-delayed initial value problem:

$$\begin{cases} \dot{Y}(t) = CY(t) + F(t) & t \in [0, T], \\ Y(T) = I \end{cases} \tag{6}$$

Now we construct the correction functional [1] for problem (6) as follows:

$$Y_{n+1}(t) = Y_n(t) + \int_T^t \lambda(s)(\dot{Y}_n(s) - C\widetilde{Y}_n(s) - F(s))ds.$$

By taking variation with respect to independent variable Y_n and noticing that $\delta Y_n(T) = 0$ and $\delta\widetilde{Y}_n = 0$, we get [1]

$$\delta Y_{n+1}(t) = \delta Y_n(t) + \lambda(s)\delta Y_n(s)|_{s=t} - \int_T^t \dot{\lambda}(s)\delta Y_n(s)ds = 0,$$

which imply the following stationary conditions

$$1 + \lambda(s)|_{s=t} = 0, \qquad \dot{\lambda}(s) = 0.$$

The general Lagrange multiplier [1], therefore, can be readily identified as $\lambda = -1$. As a result, we obtain the following iteration formula:

$$Y_{n+1}(t) = Y_n(t) - \int_T^t (\dot{Y}_n(s) - CY_n(s) - F(s))ds, \tag{7}$$

with initial approximation $Y_0(t) = I$. So we achieve the approximate value for $y(t)$ as

$$y(t) \simeq Y_n(t)y(T). \tag{8}$$

Incorporating the initial condition of the system (3) into (8), we can determine the unknown constants $r_1, r_2, \dots, r_N$. Finally, from (2), the solution of (1) can be easily obtained.

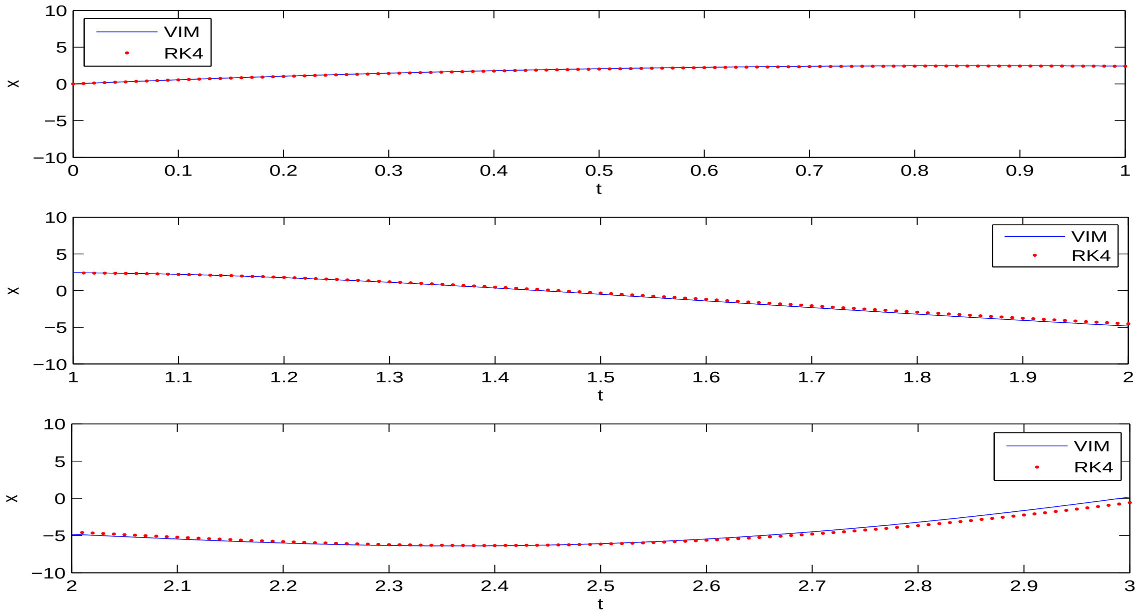

Figure 1: Comparison of the VIM with the numerical RK4 for Example 3.1.

3 Numerical results

In this section, we present two examples to show the efficiency of the method described in the previous section. The package MATLAB 9 is used for computation.

Example 3.1. *Consider the following multiple-delays scalar system [2]:*

$$\begin{cases} \dot{x}(t) = -x(t) - 5x(t-1) - 2x(t-2), & t \in [0,3] \\ \quad x(t) = \sin(t), & t \in [-2,0] \end{cases}$$

Figure 1 depicts the approximate analytical solution of Example 3.1 after using 5 iterations by the the iteration formula (7). Comparison with the numerical RK4 ($h = 0.01$) shows good agreement between two methods.

Example 3.2. *Consider a single-delay second order system as follows [3]:*

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 1.66 & -0.697 \\ 0.93 & -0.330 \end{bmatrix} x(t-1) + \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, & t \in [0,2] \\ x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & t \in [-1,0] \end{cases}$$

where $x(t) = [x_1(t) \;\; x_2(t)]^T$.

Figure 2 depicts the approximate analytical solution of Example 3.2 after using 8 iterations by the iteration formula (7). Comparison with the numerical RK4 ($h = 0.01$) shows good agreement between two methods.

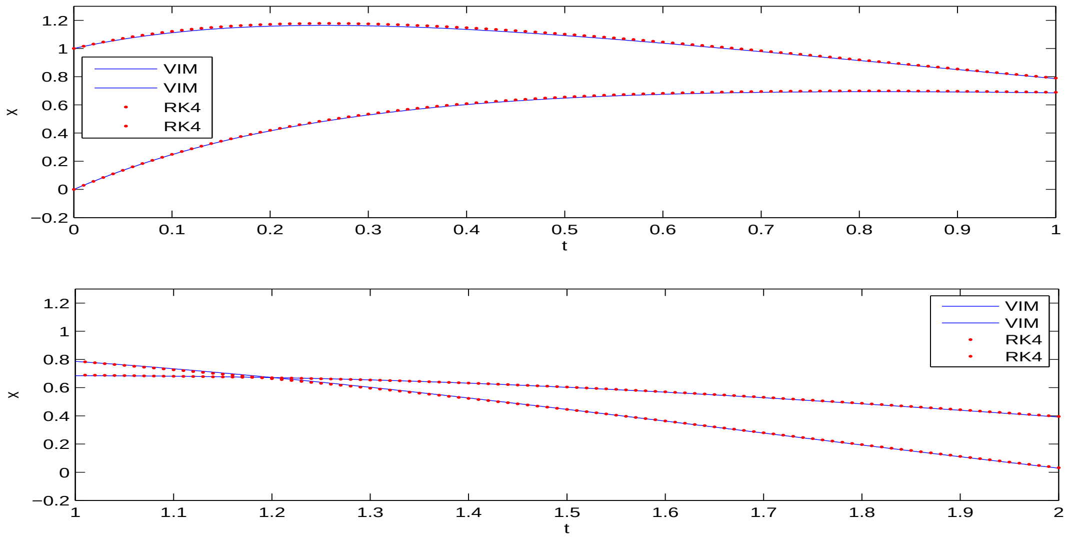

Figure 2: Comparison of the VIM with the numerical RK4 for Example 3.2.

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Multi-symplectic wavelet collocation method for the Zakharov system

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Abstract

In this paper, we develop a novel multi-symplectic wavelet collocation method for solving multisymplectic Hamiltonian system with periodic boundary conditions. Based on the autocorrelation function of Daubechies scaling functions, collocation method is conducted for the spatial discretization. The obtained semi-discrete system is proved to have semi-discrete multi-symplectic conservation laws and semi-discrete energy conservation laws. Then, appropriate symplectic scheme is applied for time integration, which leads to full-discrete multi-symplectic conservation laws. Numerical experiments for the Zakharov system.

Keywords: Zakharov system, Multi-symplectic, Wavelet collocation method

Mathematics Subject Classification: 53A15

1 Introduction

The evolution equations describing the interaction between Langmuir and ion acoustic waves in a plasma ([1],[2]) are

$$i\partial_t \phi + \partial_{xx}\phi + 2\phi\psi = 0$$

$$\partial_{tt}\psi - \partial_{xx}\psi + \partial_{xx}(|\phi|^2) = 0, \tag{1}$$

where the complex unknown function $\phi(x,t)$ represents the slowly varying envelope of the highly oscillatory electric field, and the unknown real function $\psi(x,t)$ denotes the fluctuation of the ion density about its equilibrium value.
We will concentrate on the Zakharov system subject to initial-boundary conditions

$$\phi|_{t=0} = \phi_0(x),$$

$$\psi|_{t=0} = \psi_0(x),$$

$$\partial_t\psi|_{t=0} = \psi_1(x)$$

$$\phi(x_L,t) = \phi(x_R,t),$$

$$\psi(x_L,t) = \psi(x_R,t)$$

where, $\phi_0(x)$, $\psi_0(x)$ and $\psi_1(x)$ are given initial values.

To reformulate Eq. (1) as a Hamiltonian system on multisymplectic structure, set $\phi = u + iv$, and introduce the new variables $\partial_x u = p$,$\partial_t v = q$, $\partial_t\psi = \partial_{xx} f$, and $\partial_x f = g$. Separating into real and imaginary parts, Eq. (1) can be rewritten as the following firstorder system

$$
\begin{aligned}
-\partial_t v + \partial_x p &= -2u\psi, \\
\partial_t u + \partial_x q &= -2v\psi, \\
-\partial_x u &= -p, \\
-\partial_x v &= -q, \\
\partial_t f &= \psi - (u^2 + v^2), \\
-\partial_t \psi + \partial_x g &= 0, \\
-\partial_x f &= -g
\end{aligned}
\tag{2}
$$

or the standard Hamiltonian PDEs:

$$M\partial_t z + K\partial_x z = \nabla z S(z), \tag{3}$$

where, $z = (u, v, p, q, \psi, f, g)^T$ is the state variable, M and K are two skew-symmetric matrices and $S : R^d \to R$ is a scalar-valued smooth function.

Implement for the Zakharov system (2) yields the local energy conservation law

$$
\begin{aligned}
\partial_t E + \partial_x F &= 0, \\
E &= -\psi(u^2 + v^2) + \frac{1}{2}(p^2 + q^2) + \frac{1}{2}(\psi^2 + g^2), \\
F &= -(p\partial_t u + q\partial_t v + g\partial_t f)
\end{aligned}
\tag{4}
$$

and the local momentum conservation law

$$
\begin{aligned}
\partial_t I + \partial_x G &= 0, \\
I &= v\partial_x u - f\partial_x \psi, \\
G &= -\psi(u^2 + v^2) - \frac{1}{2}(p^2 + q^2) + \frac{1}{2}(\psi^2 - g^2) - v\partial_t u + f\partial_t \psi
\end{aligned}
\tag{5}
$$

Applying the wavelet collocation method for spatial discretization of multi-symplectic system (3), we obtain

$$M(z_l)_t + K(I_J z)_x|_{x=x_l} = \nabla z S(z_l), \tag{6}$$

To obtain the equations for z_l , the crucial step is to express the one order spatial partial derivatives at collocation points x_l in terms of the values z_l . Making one-time differential and evaluating the resulting expressions at collocation points, we obtain

$$\frac{\partial I_J u_i(x,t)}{\partial x}|_{x_l} = \Sigma_{m=0}^{N-1} \frac{u_{il}\partial\theta(2^J x - m)}{\partial x}|_{x_l} = (B_1 U_i)_l, \tag{7}$$

where B_1 is a $N \times N$ matrix with elements

$$(B_1)_{lm} = \frac{d\theta(2Jx - m)}{dx}|_{x_l} = 2^J \theta^{'}(l - m).$$

Combining (6) with the differentiation matrix B_1, we obtain the wavelet collocation discretization for the multi-symplectic PDEs (3)

$$M\frac{d}{dt} z_l + K\Sigma_{m=l-(M-1)}^{l+(M-1)} (B_1)_{lm} z_m = \nabla_z S(z_l) \tag{8}$$

We discrete (8) with the implicit midpoint scheme and obtain the multi-symplectic wavelet collocation method for the multisymplectic PDEs (3)

$$M\frac{z_l^{n+1}-z_l^n}{\tau}+K\Sigma_{m=l-(M-1)}^{l+(M-1)}(B_1)_{lm}z_m^{n+1/2}=\nabla S_z(z_l^{n+1/2}) \tag{9}$$

where $z_l^{n+1/2}=\dfrac{z_l^{n+1}+z_l^n}{2}$.

Applying multi-symplectic wavelet collocation method for the multi-symplectic formulation of the Zakharov system, we obtain

$$\begin{aligned}
V^{n+1} &= V^n+\tau(B_1^2U^{1/2}+2U^{1/2}\psi^{1/2})\\
U^{n+1} &= U^n+\tau(B_1^2V^{1/2}-2V^{1/2}\psi^{1/2})\\
F^{n+1} &= F^n+\tau(\psi^{1/2}-((U^{1/2})^2+(V^{1/2})^2)\\
\psi^{n+1} &= \psi^n-B_1\tau F^{1/2}
\end{aligned}$$

The corresponding discrete ECL (4) for the MSWCM (2) and the discrete MCL(5) are similar to formuls in [2].

2 Main Result

We get the MSWCM for multi-symplectic PDEs by using wavelet collocation method in space and symplectic scheme in time. The collocation method is based on the autocorrelation functions of Daubechies scaling functions, and the obtained spatial differentiation matrix is a sparse circulant matrix with limited-bandwidth, which leads to less computation. The MSWCM has full-discrete multi -symplectic conservation laws that we emplement wavelet collocation method on the Zakharov system.

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An application of a compact finite difference method in image denoising

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Abstract

In this study, we have briefly presented Rudin-Osher-Fatemi (ROF) model for a classical denoising problem in image recovery. First, a six-order compact finite difference scheme has been given to approximate first and second-order derivative. Then, a third-order total variation diminishing Runge-Kutta (TVD-RK3) method has been given to implement our schemes to proposed model. A numerical example is given to illustrate our method. The obtained results show that our method is effective.

Keywords: Compact finite difference, image denoising, total variation diminishing Runge-Kutta

Mathematics Subject Classification: 65L12, 68U10

1 Introduction

The reconstruction of an original image u describing a true scene from an observed image f is an important problem in image analysis. In general, the degradation connecting u to f is the resulting of two phenomena: The first phenomenon is deterministic (for example, a blurred image is precisely what would be recorded in the camera if the photographer forgot to adjust lens). The second phenomenon is random, indeed the noise may be introduced by the image recoding, image transmission, etc. In this study, we consider the second case. An image denoising problem can be described as $f = u + r$, where the noise denoted by r is the additive Gaussian noise with standard deviation σ.

The Rudin-Osher-Fatemi (ROF) model is a classical denoising model which has been firstly introduced by Rudin et al. in [5]. The ROF model subject to initial data $u(x, y, 0)$ is

$$u_t = -\nabla \left(\frac{\nabla u}{|\nabla u|} \right) + \lambda(u - f), \tag{1}$$

where $\lambda > 0$ is a scale parameter. The presence of $\frac{1}{|\nabla u|}$ implies that Eq. (1) is not well defined at points where $\nabla u = 0$, therefore we slightly perturb $\frac{1}{|\nabla u|}$ as $\frac{1}{\sqrt{|\nabla u|^2+\beta}}$, where β is a small positive regularization parameter. Let u be the real image, then the Eq. (1) can be written as

$$u^t = \frac{u^{xx}\left((u^y)^2+\beta\right) - 2u^{xy}u^xu^y + u^{yy}\left((u^x)^2+\beta\right)}{\left((u^x)^2+(u^y)^2+\beta\right)^{3/2}} - \lambda(u-f) \tag{2}$$

with $u(x,y,0)$ given as initial value. It is shown that as $t\to\infty$, the solution of Eq. (1) tends to true u (see [5]).

A sixth-order compact finite difference scheme (CFD6) is given to find solution of the Burgers' equation in [6]. Recently, Bastani and Salkuyeh have applied the CFD6 to approximate the Fisher's equation [1]. We use the CFD6 to approximate the spatial derivative in Eq. (1), then third order total variation diminishing Runge-Kutta (TVD-RK3) method (see [2]) is given to solve the obtained equation.

2 Illustration of the method

In this section, the CFD6 has been given to approximate the spatial first and second-order derivative. Then, TVD-RK3 is briefly presented to approximate the obtained differential system. To do this, we define grid points by selecting h and k as step sizes along the x-axis (y-axis) and t-axis respectively, where $h = x_{i+1} - x_i$, $i = 0, 1, \ldots, N-1$ and $n = t/k$. The boundary condition in this study is considered as the Dirichlet, the implications of the CFD6 with Neumann boundary condition will be the subject of a follow-up paper. Finally, a numerical experiment has been given to illustrate the effectiveness of the method.

2.1 Sixth order compact finite schemes

A fourth order compact finite difference has been given by Lele [4] to approximate the first order derivative du/dx at interior point x_i as

$$\alpha u'_{i-1} + u'_i + \alpha u'_{i+1} = b\frac{u_{i+2}-u_{i-2}}{4h} + a\frac{u_{i+1}-u_{i-1}}{2h}, \tag{3}$$

where

$$a = \frac{2}{3}(\alpha+2), \quad b = \frac{1}{3}(4\alpha-1). \tag{4}$$

To obtain the coefficients a and b, Taylor series coefficients of various orders have been matched. Lele showed that by selecting $\alpha = 1/3$, the six-order compact finite difference scheme obtains as follows:

$$\frac{1}{3}u'_{i-1} + u'_i + \frac{1}{3}u'_{i+1} = \frac{1}{9}\frac{u_{i+2}-u_{i-2}}{4h} + \frac{14}{9}\frac{u_{i+1}-u_{i-1}}{2h}, \tag{5}$$

similarly, to obtain suitable scheme for boundary condition, we use Taylor series and get the following schemes in boundary points x_1, x_2, x_{N-1} and x_N,

$$\begin{cases} u'_1 + 5u'_2 = \frac{1}{h}(-\frac{197}{60}u_1 - \frac{5}{12}u_2 + 5u_3 - \frac{5}{3}u_4 + \frac{5}{12}u_5 - \frac{1}{20}u_6) \\ \frac{2}{11}u'_1 + u'_2 + \frac{2}{11}u'_3 = \frac{1}{h}(-\frac{20}{33}u_1 - \frac{35}{132}u_2 + \frac{34}{33}u_3 - \frac{7}{33}u_4 + \frac{2}{33}u_5 - \frac{1}{132}u_6) \\ \frac{2}{11}u'_{N-2} + u'_{N-1} + \frac{2}{11}u'_N = \frac{1}{h}(\frac{20}{33}u_N + \frac{35}{132}u_{N-1} - \frac{34}{33}u_{N-2} + \frac{7}{33}u_{N-3} \\ \qquad\qquad - \frac{2}{33}u_{N-4} + \frac{1}{132}u_{N-5}) \\ 5u'_{N-1} + u'_N = \frac{1}{h}(\frac{197}{60}u_N + \frac{5}{12}u_{N-1} - 5u_{N-2} + \frac{5}{3}u_{N-3} - \frac{5}{12}u_{N-4} + \frac{1}{20}u_{N-5}) \end{cases} \tag{6}$$

The Eqs. (6) and (5) can be written in matrix form

$$M_1U^{'} = N_1U. \tag{7}$$

Similary, for the second-order derivative at $x_1,\ldots,x_N$, the following schemes are given in [1]

$$\begin{cases} 10u_1^{''} + u_2^{''} = \frac{12}{h^2}(\frac{115}{36}u_1 - \frac{1555}{144}u_2 + \frac{89}{6}u_3 - \frac{773}{72}u_4 + \frac{151}{36}u_5 - \frac{11}{16}u_6), \\ u_{i-1}^{''} + 10u_i^{''} + u_{i+1}^{''} = \frac{12}{h^2}(u_{i-1} - 2u_i + u_{i+1}), \quad i = 2,\ldots,N-1 \\ u_{N-1}^{''} + 10u_N^{''} = \frac{12}{h^2}(-\frac{11}{16}u_{N-5} + \frac{151}{36}u_{N-4} - \frac{773}{72}u_{N-3} + \frac{89}{6}u_{N-2} \\ \qquad\qquad - \frac{1555}{144}u_{N-1} + \frac{115}{36}u_N) \end{cases} \tag{8}$$

which can be written as

$$M_2U^{''} = N_2U. \tag{9}$$

By applying the Eqs. (7) and (9) to (1), the following system can be obtain

$$U_t = \mathcal{L}(U), \tag{10}$$

where $\mathcal{L}$ is a nonlinear operator. Now, we can apply the TVD-RK3 schemes to Eq. (9) as follows:

$$\begin{aligned} u^{(1)} &= u^n + k\mathcal{L}(u^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}k\mathcal{L}(u^{(1)}) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}k\mathcal{L}(u^{(2)}) \end{aligned} \tag{11}$$

where n and k are the step of the method and step size of the time, respectively.

2.2 Numerical tests

In this section, we consider a gray image $u_{50\times 50}$ to investigate our method. All computations are carried out using some codes produced in Matlab 7. To investigate our method, the signal to noise ratio (SNR) is applied to measure the level of noise. The SNR is given as [3]

$$\text{SNR} = 10\log_{10}\left(\sum_{i,j} U(i,j)^2 \Big/ \sum_{i,j} r(i,j)^2\right), \tag{12}$$

where U is the real image and r is the noise. The larger value of the SNR is better of the denoised image. The SNR values are given for $k = 0.001$ and $\beta = 10$ in Table 1.

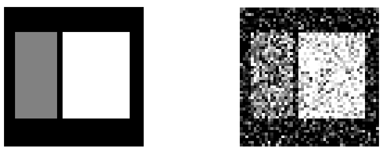

Figure 1: Original image, noisy image with Gaussian noise $\sigma = 0.1$.

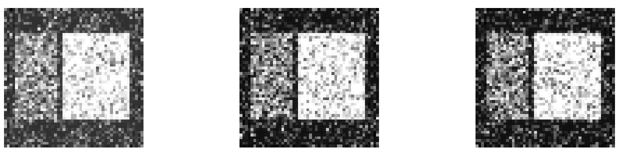

Figure 2: Denoised image for $\lambda = 0.01$, $\lambda = 0.1$ and $\lambda = 1$ at $t = 0.1$.

λ	0.01	0.1	1
SNR	7.61	7.58	7.44

Table 1: SNR for the various value of λ

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The existence of periodic solutions for the nonlinear fifth order autonomous ordinary differential equations

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Abstract

In this paper, we study the existence of periodic solutions for the autonomous nonlinear ordinary differential equations of order fifth. Our method is based on the evaluation of Brouwer's degree theory and making use of the homotopy invariance property of the topological degree [1,2]. For this, we prove a lemma about the third order ODE systems and then present a theorem about the sufficient conditions for the existence of periodic solutions for the fifth order ODE's.

1 Introduction

The existence of nontrivial periodic solution of autonomous nonlinear third order differential equations of the form

$$x''' + f(x, x', x'') = 0, \qquad f(x, -x', x'') = -f(x, x', x''),$$

has been investigated in [3]. Here, we generalize the results in [3-5] for the fifth order autonomous equation.

It is interesting to note that the existence of periodic solutions of nonlinear autonomous differential equations has not been extensively investigated. The Poincare-Bendixon theorem, which is a powerful tool for the investigation of periodic solutions of second order differential equations, is not applicable for third and higher order systems. In what follows, we use the idea of Brouwer's degree theory to prove the existence of periodic solutions of higher order systems. The numerical results obtained demonstrates the validity of analytical method used.

For $C \in \mathbb{R}^n$, define $|C|_\infty = \max\{|c_i| : i = 1, ..., n\}$ and for a closed domain D, define

$$|f|_D = \max_{v \in D} |f(v)|,$$
$$\|f\|_D = \max_{v \in D} |f(v)|_\infty,$$

for real valued and vector functions, respectively. For $c > 0$, define open subset $\Omega_c = \{C : |C|_\infty < c\}$. The Brouwer's degree of the mapping f respect to Ω_c and 0 is denoted be $\deg(f, \Omega_c, 0)$.

Lemma 1.1. *Here we consider the third order system:*

$$\begin{cases} x''' = -a_1^2 x' + f_1(x, y, x', y', x'', y''), \\ y''' = -a_2^2 y' + f_2(x, y, x', y', x'', y''). \end{cases} \tag{1}$$

where f_1 and f_2 are smooth enough to ensure the uniqueness and existence of solutions. Without loss of generality we can assume $a_1 > a_2 > 0$. Suppose there exists $k \in (1,2)$ and $c > 0$ such that

$$k < \frac{a_1}{a_2} < \frac{2}{k-1},$$

$$\delta a_2^2 c > (k+1)M,$$

where

$$\omega_0 = \frac{\pi}{a_1 + a_2}, \qquad \omega_1 = \frac{(k+1)\pi}{a_1 + a_2}$$

$$\delta = \min\left\{ \left| \begin{array}{cc} \sin(a_1\omega_0) & 0 \\ 0 & \sin(a_2\omega_0) \end{array} \right|, \left| \begin{array}{cc} \sin(a_1\omega_1) & 0 \\ 0 & \sin(a_2\omega_1) \end{array} \right| \right\},$$

and

$$M = \max\{|f_1|_D, |f_2|_D\}$$

$$D = \Big\{(x,y,x',y',x'',y'') : |x| \le \frac{2c}{a_2}, |y| \le \frac{2c}{a_2}, |x'| \le 2c, |y'| \le 2c, |x''| \le 2ca_1, |y''| \le 2ca_1\Big\},$$

then there exists ω, $\omega_0 < \omega < \omega_1$ and $c_1 \neq 0, c_2 \neq 0$, such that if

$$\begin{array}{ll} x(0) = -\frac{c_1}{a_1}, & y(0) = \frac{c_2}{a_2} \\ x''(0) = a_1 c_1, & y''(0) = a_2 c_2, \end{array}$$

then $x'(0) = y'(0) = x'(\omega) = y'(\omega) = 0$.

Proof. For brevity, we introduce the following notation

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad F(X, X', X'') = \begin{pmatrix} f_1(x,y,x',y',x'',y'') \\ f_2(x,y,x',y',x'',y'') \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Next, we consider the following system

$$\begin{cases} X''' + A^2 X' = F(X, X', X''), \\ X(0) = -A^{-1}C, \\ X(0) = 0, \\ X'(0) = AC, \end{cases} \tag{2}$$

where C is the vector $(c_1, c_2)^T$.
The solution of (1) is given by

$$X'(t, C) = \sin(At)C + G'(t, X, X', X''),$$

$$G'(t, X, X', X'') = A^{-1}\int_0^t \sin\big(A(t-s)\big)F\big(X(s), X'(s), X''(s)\big)ds.$$

Again we can write

$$\deg(\sin(A\omega_j)C, \Omega_c, 0) = (-1)^j,$$

$$|\sin(A\omega_j)C|_\infty \ge \delta c > 0, \qquad (j = 0, 1),$$

$$\|G(\omega_j, X, X', X'')\|_D < \delta c,$$

which implies:

$$\deg(X'(\omega_j, C), \Omega_c, 0) = (-1)^j, \quad (j = 0, 1).$$

Again by the homotopy invariance property of topological degree, we conclude, that there exists $\omega, \omega_0 < \omega < \omega_1$ and C, $|C|_\infty = c$, such that $X'(\omega, C) = 0$. ☐

Now, using Lemma 1.1, the following result on the existence of periodic solutions of the fifth order equation in the following theorem can be shown.

Theorem 1.2. *We consider the fifth order equation*

$$x^{(5)} + f(x, x', x'', x''', x^{(4)}) = 0, \tag{3}$$

where f is continuous and

$$f(x, -x', x'', -x''', x^{(4)}) = -f(x, x', x'', x''', x^{(4)}).$$

Assuming there exists $a, b > 0$ and $k \in (1, 2)$ such that

$$k + \frac{1}{k} < \frac{a}{\sqrt{b}} < \frac{k-1}{2} + \frac{2}{k-1}, \tag{4}$$

and for a domain $D \subset \mathbb{R}^4$, containing the origin such that if

$$M = \left| ax''' + bx' - f(x, x', x'', x''', x^{(4)}) \right|_D,$$

is sufficiently small, then Eq. (٣) has a periodic solution.

Proof. Assume,

$$a_1^2 = \frac{a+\Delta}{2}, \qquad a_2^2 = \frac{a-\Delta}{2},$$

where $\Delta = \sqrt{a^2 - 2b}$. Now, we write Eq. (3) in the equivalent form

$$\begin{cases} x''' = -a_1^2 x' + \varepsilon y', \\ \\ y''' = -a_2^2 y' + \frac{1}{\varepsilon} g(x, y, x', y', x'', y''), \end{cases} \tag{5}$$

where $g = ax''' + bx' - f(x, x', x'', x''', x^{(4)})$.
Choosing M and ε small compared to a_2, by Lemma 1.1, a non-zero solution is obtained with the condition

$$x'(0) = y'(0) = x'(\omega) = y'(\omega) = 0.$$

Next, we extend the solution $(x(t), y(t))$ periodically to $[0, 2\omega]$, by

$$\begin{cases} x = x(t) \\ \\ y = y(t) \end{cases} t \in [0, \omega] \qquad \begin{cases} x = x(2\omega - t) \\ \\ y = y(2\omega - t) \end{cases} t \in [\omega, 2\omega].$$

□

Example 1. Let the following fifth order equation

$$x^{(5)} + 5x''' + 4x' + x'x^2 + \frac{3}{2}(x''')^3 x^4 = 0,$$

or the equivalent system with given initial conditions,

$$\begin{cases} x' = y \\ y' = z \\ z' = u \\ u' = v \\ v' = -5u - 4y - x^2 y - \frac{3}{2} u^3 v \end{cases}$$

$$x(0) = -0.2, \quad y(0) = 0, \quad z(0) = 0.50307, \quad u(0) = 0, \quad v(0) = -1.1.$$

This system has a periodic solution with period $\omega = 6.30277$, where

$$|x(\omega)| = -0.2, \quad |y(\omega)| < 10^{-7}, \quad |z(\omega)| = 0.05307, \quad u(\omega) < 10^{-7}, \quad v(\omega) = -1.1.$$

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A direct method for the numerical solution of nonlinear two-dimensional Fredholm integral equations

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Abstract

By a direct method and using two-dimensional rationalized Haar (RH) functions, the numerical solution of nonlinear second kind two-dimensional Fredholm integral equations (2DFIE) is reduced to solving a nonlinear system of algebraic equations. Also, a numerical example is presented to show the efficiency of the method.

Keywords: Nonlinear two-dimensional integral equations; Fredholm integral equations; Rationalized Haar functions.

Mathematics Subject Classification: 65R20.

1 Introduction

The numerical solution of nonlinear one-dimensional Fredholm equations using a basis of Haar functions was considered by Razzaghi and Ordokhani in [1]. Here, we consider the nonlinear 2DFIE of the second kind

$$u(x,t) = f(x,t) + \int_0^1 \int_0^1 k(x,t,y,z)G(u(y,z))dydz, \quad (x,t) \in \Omega = [0,1] \times [0,1], \tag{1}$$

where $u(x,t)$ is an unknown real valued function and $f(x,t)$ and $k(x,t,y,z)$ are given continuous functions defined, respectively on Ω and $\Omega \times \Omega$, and $G(u(x,t))$ is a polynomial of $u(x,t)$ with constant coefficients. The aim of this paper is to reduce (13) to a nonlinear system of algebraic equations by applying RH functions. We assume k is such that (13) possesses a unique solution $u(x,t) \in C(\Omega)$. Also, for convenience, we assume that

$$G(u(y,z)) = [u(y,z)]^p, \tag{2}$$

where p is a positive integer, but the method can be easily extended and applied to any nonlinear two-dimensional FIE of the form (13).

2 RH functions

Definition 2.1. *The Haar wavelet is the function defined on the real line* $\mathbb{R}$ *as*

$$H(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \\ 0, & \textit{otherwise}. \end{cases}$$

Definition 2.2. *The RH functions $h_l(x)$, for $l = 2^i + j$ with $i \in \mathbb{Z}^{\geq 0}$ and $j = 0, 1, \ldots, 2^i - 1$, are the functions defined on the interval $[0, 1)$ as*

$$h_l(x) = H(2^i x - j)|_{[0,1)}. \tag{3}$$

Also, we define $h_0(x) = 1$ for all $t \in [0, 1)$.

By simple calculations, we obtain

$$\langle h_l, h_q \rangle = \int_0^1 h_l(x).h_q(x)dx = \begin{cases} 2^{-i}, & l = q = 2^i + j \text{ with } i \in \mathbb{Z}^{\geq 0} \text{ and } j \in J_i, \\ 1, & l = q = 0, \\ 0, & l \neq q, \end{cases} \tag{4}$$

where $J_i = \{0, 1, \ldots, 2^i - 1\}$ for any $i \in \mathbb{Z}^{\geq 0}$ and $\langle , \rangle$ denotes the scalar product.

A function $f(x)$ in $\mathfrak{L}^2([0, 1))$ can be approximated in terms of RH functions as

$$f(x) \simeq \sum_{l=0}^{m-1} f_l h_l(x) = \mathbf{h}^t(x)\mathbf{f},$$

where the RH coefficients f_l are given by $f_l = \langle f, h_l\rangle/\langle h_l, h_l\rangle$, $m = 2^{\alpha+1}$, and α is a non-negative integer. The RH coefficients vector $\mathbf{f}$ and RH functions vector $\mathbf{h}(x)$ are defined as $\mathbf{f} = [f_0, f_1, \ldots, f_{m-1}]^t$ and $\mathbf{h}(x) = [h_0(x), h_1(x), \ldots, h_{m-1}(x)]^t$.

We define the sequence $\{h_{lq}\}_{l,q=0}^{\infty}$ of two-dimensional RH functions on $[0, 1)\times[0, 1)$ as $h_{lq}(x, t) = h_l(x).h_q(t)$, where the functions $h_l(x)$, $l = 0, 1, \ldots,$ are one-dimensional RH functions defined by (17).

Similarly, any function $u(x, t)$ in $\mathfrak{L}^2([0, 1) \times [0, 1))$ can be approximated as:

$$u(x, t) \simeq \sum_{l,q=0}^{m-1} u_{lq} h_{lq}(x, t) = \mathbf{h}^t(x, t)\mathbf{u}, \tag{5}$$

where

$$\mathbf{h}(x, t) = \mathbf{h}(x) \otimes \mathbf{h}(t), \tag{6}$$

and $\mathbf{u} = [u_{00}, u_{01}, \ldots, u_{0,m-1}, \ldots, u_{m-1,0}, u_{m-1,1}, \ldots, u_{m-1,m-1}]^t$. In (2.1), $\otimes$ denotes the Kronecker product. The RH coefficients u_{lq} in (8) are given by

$$u_{lq} = \langle h_l(x), \langle u(x, t), h_q(t)\rangle\rangle/\langle h_l, h_l\rangle.\langle h_q, h_q\rangle.$$

3 Nonlinear 2DEIE of the second kind

Consider the second kind two-dimensional nonlinear FIE (13) with assumption (2). Approximating functions $u(x, t)$, $f(x, t)$ and $[u(x, t)]^p$ by RH functions, as we described in Eq. (8), we obtain

$$u(x, t) \simeq \mathbf{h}^t(x, t)\mathbf{u}, \quad f(x, t) \simeq \mathbf{h}^t(x, t)\mathbf{f} \quad [u(x, t)]^p \simeq \mathbf{h}^t(x, t)\mathbf{u}^{(p)}, \tag{7}$$

where $\mathbf{u}$, $\mathbf{f}$ and $\mathbf{u}^{(p)}$ are RH coefficients vectors of functions $u(x, t)$, $f(x, t)$ and $[u(x, t)]^p$, respectively. By defining the Newton-Cotes nodes $(x_i, t_j) = ((2i - 1)/2m, (2j - 1)/2m)$ for $i, j = 1, 2, \ldots, m$, the following lemma is satisfied.

Lemma 3.1. *Let p be a positive integer and $\mathbf{u}$ and $\mathbf{u}^{(p)}$ are RH coefficients vectors of functions $u(x, t)$ and $[u(x, t)]^p$, that are defined on Ω, respectively. Then,*

$$\mathbf{u}^{(p)} = \frac{1}{m^2}\mathbf{Q}^{-1}[\sum_{i,j=1}^{m} (\sigma_{ij})^p h_{00}(x_i, t_j), \ldots, \sum_{i,j=1}^{m} (\sigma_{ij})^p h_{m-1,m-1}(x_i, t_j)],$$

where

$$\sigma_{ij} = \sum_{l,q=0}^{m-1} u_{lq} h_{lq}(x_i, t_j), \quad i, j = 1, 2, \ldots, m.$$

Also, the kernel $k(x, t, y, z)$ may be approximated as

$$k(x, t, y, z) \simeq \sum_{i,l,j,q=0}^{m-1} h_{il}(x, t) k_{iljq} h_{jq}(y, z) = \mathbf{h}^t(x, t)\mathbf{K}\mathbf{h}(y, z), \tag{8}$$

where $\mathbf{K}$ is the RH coefficients matrix of k. We can write $\mathbf{K}$ in the blocked form $\mathbf{K} = [\mathbf{k}^{(i,j)}]_{i,j=1}^{m}$, where $\mathbf{k}^{(i,j)} = [k_{i-1,l-1,j-1,q-1}]_{l,q=1}^{m}$. The coefficients k_{iljq} in (20) are given by

$$k_{iljq} = \langle h_i(x), \langle h_l(t), \langle h_j(y), \langle k(x, t, y, z), h_q(z)\rangle\rangle\rangle\rangle / \langle h_i, h_i\rangle.\langle h_j, h_j\rangle.\langle h_l, h_l\rangle.\langle h_q, h_q\rangle.$$

Substituting (14) and (20) into (13) and then replacing $\simeq$ with $=$, give

$$\mathbf{h}^t(x, t)\mathbf{u} \simeq \mathbf{h}^t(x, t)\mathbf{K}\mathbf{Q}\mathbf{u}^{(p)} + \mathbf{h}^t(x, t)\mathbf{f}, \tag{9}$$

where

$$\mathbf{Q} = \int_0^1 \int_0^1 \mathbf{h}(y, z)\mathbf{h}^t(y, z)dydz. \tag{10}$$

Eq. (1.3) yields that $\mathbf{Q}$ is a diagonal matrix of the form $\mathbf{Q} = \mathrm{diag}\,(\mathbf{x}\otimes\mathbf{x})$ in which $\mathbf{x}$ is the m-vector defined by $\mathbf{x} = [1, 1, 2^{-1}, 2^{-1}, \underbrace{2^{-2}, \ldots, 2^{-2}}_{2^2}, \ldots, \underbrace{2^{-\alpha}, \ldots, 2^{-\alpha}}_{2^\alpha}]^t$.

The inner product of both the sides of (1.6) by $\mathbf{h}(x, t)$ and then using (30) gives

$$\mathbf{u} = \mathbf{W}\mathbf{u}^{(p)} + \mathbf{f}, \tag{11}$$

where $\mathbf{W} = \mathbf{K}\mathbf{Q}$. Note that to obtain (35) we replaced $\simeq$ by $=$ in (1.6). Since $\mathbf{u}^{(p)}$ is a vector whose elements are nonlinear functions of the elements of the vector $\mathbf{u}$, Eq. (35) is a nonlinear system of m^2 algebraic equations with m^2 unknowns u_{ij}.

4 Numerical examples

The method is applied to an example and the resulted nonlinear system is solved by a Newton method. The L_2 error and L_2 rate of convergence are defined to be, respectively:

$$\|e_m\|_2 := \left(\int_0^1 \int_0^1 |u(x, t) - \mathbf{h}^t(x, t)\mathbf{u}|^2 dxdt\right)^{1/2}, \qquad \rho_m := \log_2(\|e_m\|_\infty / \|e_{2m}\|_\infty).$$

In tables, NI denotes the number of iterations in Newton method. The initial guess in Newton's method is considered to be $\mathbf{u}^{(0)} = \mathbf{f}$.

Example 4.1. *[2] Consider the following nonlinear 2DFIE*

$$u(x, t) = \frac{1}{(1 + x + t)^2} - \frac{x}{6(1 + t)} + \int_0^1 \int_0^1 \frac{x}{1 + t}(1 + z + y)u^2(y, z)dydz, \quad 0 \le x, t \le 1,$$

The exact solution is $u(x, t) = \frac{1}{(1+x+t)^2}$. *Table 1 illustrates the results for this example.*

Table 1: Numerical results for Example 4.1.

$(x,t)=(\frac{1}{2^l},\frac{1}{2^l})$	$m=2$	$m=4$	$m=8$	$m=16$	$m=32$
l=1	8.7×10^{-2}	5.1×10^{-2}	2.8×10^{-2}	1.5×10^{-2}	7.6×10^{-3}
l=2	2.7×10^{-2}	1.1×10^{-1}	6.5×10^{-2}	3.4×10^{-2}	1.8×10^{-2}
l=3	1.7×10^{-1}	1.3×10^{-2}	1.1×10^{-1}	5.9×10^{-2}	3.1×10^{-2}
l=4	3.2×10^{-1}	1.4×10^{-1}	4.9×10^{-3}	8.0×10^{-2}	4.2×10^{-2}
l=5	4.1×10^{-1}	2.3×10^{-1}	9.1×10^{-2}	1.5×10^{-3}	5.0×10^{-2}
l=6	4.7×10^{-1}	2.9×10^{-1}	1.4×10^{-1}	5.3×10^{-2}	4.3×10^{-4}
NI	3	3	3	3	3
$\|e_m\|_2$	7.98×10^{-2}	4.21×10^{-2}	2.14×10^{-2}	1.07×10^{-2}	5.4×10^{-3}
ρ_m	-	0.92	0.98	0.99	1.00

Table 2: Properties of matrix H and computing times (seconds).

m	Number of entries of H	Number of zeros	Percentage of zeros	Computing time
2	16	0	0	0.282000
4	256	112	44	0.000000
8	4096	3072	75	0.016000
16	65536	59136	90	0.234000
32	1048576	1011712	96	10.843000

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Solving the one-phase Stefan problem using the homotopy perturbation method

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Abstract
The main aim of this paper is to solve the one-phase Stefan problem using an analytical method, the homotopy perturbation method. After introducing the Stefan problem and stating the existence and uniqueness of it, we present the homotopy perturbation method and then apply it to the problem.

Keywords: Stefan Problem, Homotopy Perturbation Method.

Mathematics Subject Classification: 35R35

1 Introduction

Phase-change, or the Stefan problems in which material melts or solidifies occur in a wide variety of natural and industrial processes. Mathematically, these are special cases of moving-boundary problems, in which the location of the front between the solid and liquid is not known beforehand, but must be determined as a part of the solution [2]. The basic partial differential equation is heat transfer equation, nevertheless, solving the problem is not straightforward due to the moving boundary. In general, when solving the problem, the technique should be able to track the moving boundary. Stefan problems model [3, 14], many real world and engineering situations. Examples include solidification of metals, freezing of water and food, crystal growth, casting, welding, melting, ablation, etc. Many numerical methods have been used for solving the Stefan problems. Crank [2] as well as Lynch and O Neill [13] provide a comprehensive summary of the numerical methods used for this type of problems. Phase change problems have always remained as an active area of research. Analytical progress in the solution of Stefan problems has remained very limited and usually not available. In one-dimensional Stefan problem we wish to determine the free boundary (sufficiently smooth) which we denote by Γ_s and is given by $x = s(t)$ and the temperature solution $u(x,t)$ in the heat conduction domain $D = (0, s(t)) \times (0,T]$, where $T > 0$ is a given arbitrary final time of interest. We have a fixed boundary at $x = 0$. We denote the closure of the domain D by $\overline{D} = [0, s(t)] \times [0,T]$. In this paper, we consider the one-phase Stefan problem and seek a solution $(u(x,t), s(t))$, which satisfies the one-dimensional heat equation,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad (x,t) \in D, \tag{1}$$

subject to the initial condition,

$$u(x,0) = \varphi(x) \geq 0, \quad x \in [0,b], \quad b = s(0) > 0, \tag{2}$$

the Dirichlet boundary conditions on the fixed and moving boundary $x = s(t)$,

$$u(0,t) = f(t) \geq 0, \quad t \in (0,T], \tag{3}$$

$$u(s(t),t) = 0, \quad t \in (0,T], \tag{4}$$

and the Stefan condition,

$$\frac{\partial u}{\partial x}(s(t),t) = -\dot{s}(t), \quad t \in (0,T]. \tag{5}$$

The following theorem from [1] shows that the above problem has a unique solution:

Theorem 1.1. *If f and φ are nonnegative piecewise-continuous functions such that, when $b > 0$, either $\varphi \neq 0$ or $f \neq 0$ in $0 \leq t < \epsilon$ for each positive ϵ or, when $b = 0$ and φ is irrelevant, $f \neq 0$ in $0 \leq t < \epsilon$ for each positive ϵ, then there exists a unique solution (s,u) to the above Stefan problem.*

Proof. Refer to [1]. □

The paper is organized as follows: Section 2 introduces the homotopy perturbation method completely. Section 3 is devoted to present the application of this method to the Stefan problem. At last in section 4, numerical example is given to demonstrate the accuracy of the method.

2 Main Result

2.1 Homotopy Perturbation Method (HPM)

Homotopy perturbation method was first proposed by He [4]. The method is a powerful and efficient tool for finding solutions of linear and non-linear equations. It has been used to obtain the solutions of a large class of linear and non-linear equations [5, 6, 7, 8, 9, 10, 11]. To illustrate the basic ideas of this method, we consider the following non-linear functional equation:

$$A(u) - f(r) = 0, \; r \in \Omega, \tag{6}$$

With the following boundary condition:

$$B(u, \frac{\partial u}{\partial n}) = 0, \; r \in \Gamma = \partial\Omega, \tag{7}$$

where A is a general functional operator, B a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A can be decomposed into two operators L and N, where L is linear, and N is nonlinear operator. Eq. (6) can be, therefore, written as follows:

$$L(u) + N(u) - f(r) = 0. \tag{8}$$

Using the homotopy technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \to \mathbf{R}$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \; p \in [0,1], \; r \in \Omega, \tag{9}$$

or equivalently, we get:

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0. \tag{10}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation for the solution of equation (2.1), which satisfies the boundary conditions. Obviously, from Eqs. (8) and (9) we will have:

$$H(v,0) = L(v) - L(u_0) = 0, \tag{11}$$

$$H(v,1) = A(u) - f(r) = 0, \tag{12}$$

The changing values of p from zero to unity are just that of $u(r,p)$ from $u_0(r)$ to $u(r)$.In topology, this is called homotopy. According to HPM, we can first use the embedding parameter pas a small parameter, and assume that the solution of Eqs. (8) and (9) as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots, \tag{13}$$

Setting $p = 1$, results in the approximation to the solution of Eq. (6)

$$u = lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{14}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method, which has eliminated limitations of the traditional perturbation techniques. The series (14) is convergent for more cases. Some criteria are suggested for convergence of the series (14), in [4].

2.2 Application of HPM to the free boundary Problem

In this section, we will apply the homotopy perturbation method to the free boundary Problem (1)-(5) According to the HPM, we construct the following simple homotopy for equation (1)

$$\begin{aligned} H(v,p) &= (1-p)v_t + p(v_t - v_{xx}) \\ &= v_t - pv_{xx} = 0, \end{aligned} \tag{15}$$

where $p \in [0,1]$ is an embedding parameter. When $p = 0$, (15) is an ordinary differential equation, $v_t = 0$ which is easy to solve; and if , $p = 1$ it turns out to be the equation (1). The basic assumption is that the solution can be written as a power series in p ,

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + ... = \sum_{i=0}^{\infty} p^i v_i. \tag{16}$$

In this paper, we assumed that the free boundary $s(t)$ is a known function. Now, by substituting (16) in (1), we get,

$$(v_0)_t + p(v_1)_t + p^2(v_2)_t + ... - p((v_0)_{xx} + p(v_1)_{xx} + p^2(v_2)_{xx} + ...) = 0. \tag{17}$$

In order to obtain the coefficients of various powers of p , we need to compare the different powers of p . Since resulting equations are ordinary differential equation of first order, we need an initial condition for any equation. Comparing the terms with identical powers of p in (17) together with initial condition (2), the following series of equations can be obtained

$$\begin{aligned} p^0: \quad & (v_0)_t = 0 \\ & v_0(x,0) = \varphi(x), \end{aligned} \tag{18}$$

$$\begin{aligned} p^i: \quad & (v_i)_t = (v_{i-1})_{xx}, \ for\ i = 1,2,..., \\ & v_i(x,0) = 0. \end{aligned} \tag{19}$$

By taking the limit of $v(x,t)$, we can obtain the solution of the free boundary problem as follows,

$$u(x,t) = Lim_{p\to 1}[v_0 + pv_1 + p^2 v_2 + ...]. \tag{20}$$

In this moment, we must show that the resulting solution $u(x,t)$ satisfies the conditions (3)-(5).

2.3 Numerical example

Now, we are ready to apply the HPM to the calculation of the Stefan problem through the test by numerical example. Consider the problem (1)-(5) with the following parameters:

$$\begin{array}{c} \varphi(x) = -1 + exp(1 - \frac{1}{\sqrt{2}} - \frac{x}{\sqrt{2}}), \quad s(0) = b = \sqrt{2} - 1, \\ f(t) = -1 + exp(1 - \frac{1}{\sqrt{2}} + \frac{t}{2}), \end{array} \tag{21}$$

which correspond to the exact solution [12]

$$\begin{array}{c} u(x,t) = -1 + exp(1 - \frac{1}{\sqrt{2}} + \frac{t}{2} - \frac{x}{\sqrt{2}}), \\ s(t) = \sqrt{2} - 1 + \frac{t}{\sqrt{2}}. \end{array} \tag{22}$$

Precisely, $\varphi(x)$ and $f(t)$ are piecewise continuous and nonnegative. Then, by theorem 1, this problem has a unique solution $(s(t), u(x,t))$. As mentioned in the previous section, we suppose that $s(t)$ is a known function. Now, from relations (18) and (21), we get,

$$\begin{array}{ll} p^0: & (v_0)_t = 0 \\ & v_0(x,0) = -1 + exp(1 - \frac{1}{\sqrt{2}} - \frac{x}{\sqrt{2}}). \end{array} \tag{23}$$

Solving this boundary value problem yields,

$$v_0 = -1 + exp(1 - \frac{1}{\sqrt{2}} - \frac{x}{\sqrt{2}}) = -1 + A(x), \tag{24}$$

where,

$$A(x) = exp(1 - \frac{1}{\sqrt{2}} - \frac{x}{\sqrt{2}}), \tag{25}$$

For the first exponent of p by (19), we can write,

$$\begin{array}{ll} p^1: & (v_1)_t = (v_0)_{xx}, \\ & v_1(x,0) = 0, \end{array} \tag{26}$$

The solution of this problem is

$$v_1 = A(x)\frac{t}{2}. \tag{27}$$

By a similar argument, we obtain,

$$\begin{array}{ll} v_2 = \frac{1}{2!}A(x)\frac{t^2}{2^2}, & v_3 = \frac{1}{3!}A(x)\frac{t^3}{2^3}, \\ v_4 = \frac{1}{4!}A(x)\frac{t^4}{2^4}, & v_5 = \frac{1}{5!}A(x)\frac{t^5}{2^5}, \end{array} \tag{28}$$

Now, we can calculate the solution of the Stefan problem,mentioned above,

$$\begin{aligned} u(x,t) &= Lim_{p\to 1}[v_0 + pv_1 + p^2 v_2 + ...] \\ &= -1 + Lim_{p\to 1}\textstyle\sum_{k=0}^{\infty} A(x)\frac{p^k}{k!}\frac{t^k}{2^k} \\ &= -1 + \textstyle\sum_{k=0}^{\infty} A(x)\frac{1}{k!}\frac{t^k}{2^k} \\ &= -1 + A(x)exp(\tfrac{t}{2}). \end{aligned} \tag{29}$$

By substituting $A(x)$ from (25)in the above relation, we have,

$$u(x,t) = -1 + exp(1 - \frac{1}{\sqrt{2}} + \frac{t}{2} - \frac{x}{\sqrt{2}}). \tag{30}$$

This is the exact solution of the stefan problem corresponding to the data (21), as mentioned in [12]. Now, we can simply show that this solution also satisfies the conditions (3)-(5), as following,

$$\begin{aligned} u(0,t) &= -1 + exp(1 - \tfrac{1}{\sqrt{2}} + \tfrac{t}{2}) = f(t), \\ u(s(t),t) &= -1 + e^0 = 0, \\ \tfrac{\partial u}{\partial x}(s(t),t) &= -\tfrac{1}{\sqrt{2}}e^0 = -\dot{s}(t). \end{aligned} \tag{31}$$

The homotopy perturbation method is an efficient technique for solving various scientific and engineering problems. It is seen that HPM is a powerful and accurate method for finding the solution of one-phase Stefan problem. Moreover, it is straight forward and avoids the hectic work of calculations. The authors believe that the procedure as described in the present paper will be applicable to linear and nonlinear Stefan problems and it will considerably benefit to engineers and scientists working in this field.

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General solution of Fisher equations by matrix differential transformation method

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Abstract
In this paper, we introduce a new method based on Differential Transformation Method as a Matrix Differential Transformation Method(MDTM) to apply for matrix partial differential equations(MPDEs) and employed to solving matrix Fisher equations. We begin by showing how the MDTM applies to a linear part and non-linear part of any MPDEs and give various examples of MPDEs to illustrate the sufficiency of the method for this kind of nonlinear MPDEs. The results obtained are in good agreement with the exact solution. These results show that the technique introduced here is powerful, accurate and easy to apply.
Keywords: Nonlinear differential equations; Matrix Fisher equations; Matrix differential transformation method.

Mathematics Subject Classification: 93C10; 35N10; 65F05; 93C15.

1 Introduction

Many problems in the fields of physics, engineering and biology are modeled by matrix differential equations(MDEs)[1] and matrix partial differential equations(MPDEs).

The differential transform method is a semi-numerical-analytic-technique that formalizes the Taylor series in a totally different manner. It was first introduced by Zhou in a study about electrical circuits [2].

In this paper, we extended differential transformation method to apply for MPDEs, and derive analytic approximations for some important matrix nonlinear equations. The equations under consideration are matrix Fisher equation. These equations are formulated as follows:

$$\text{Matrix Fisher Equation:} \qquad u_t = Au_{xx} + u(I-u) + B(x,t), \tag{1}$$

where $u(x,t) \in R^{n\times n}$, and $A \in R^{n\times n}$ is constant matrix, $I \in R^{n\times n}$ is Identity matrix and $B(x,t) \in R^{n\times n}$ is known function matrix. Similarities, like the appearance of u_t, u_{xx} and u_{xxx}, in these equations motivated us to study them as a class of nonlinear MPDEs in one single work.

2 Basic definitions

The basic definitions of matrix differential transformation are introduced as follows:

2.1 One-dimensional matrix differential transform

With reference to the articles [3, 4], We introduce in this section the basic definition of the One-dimensional matrix differential transformation:

Definition 2.1. *If $u(t) \in \mathbb{R}^{n\times n}$ is matrix analytical function in the domain T, then it will be differentiated continuously with respect to time t,*

$$\frac{d^k u(t)}{dt^k} = \phi(t,k), \quad \forall t \in T. \tag{2}$$

for $t = t_i$, where $\phi(t,k) = \phi(t_i,k)$, where k belongs to the set of non-negative integer, denoted as the K domain. Therefor, Eq.(2) can be written as

$$U_i(k) = \phi(t_i,k) = \left[\frac{d^k u(t)}{dt^k}\right]_{t=t_i}, \quad \forall k \in K, \tag{3}$$

where $U_i(k) \in \mathbb{R}^{n\times n}$ is called the spectrum of $u(t)$ at $t = t_i$, in the K domain.

Definition 2.2. *If $u(t) \in \mathbb{R}^{n\times n}$ can be expressed by Taylor's series about fixed point t_i, then $u(t)$ can be represented as*

$$u(t) = \sum_{k=0}^{\infty} \frac{u^{(k)}(t_i)}{k!}(t-t_i)^k. \tag{4}$$

If $u_n(t)$ is be the n-partial sums of a Taylor's series (4), then

$$u_n(t) = \sum_{k=0}^{n} \frac{u^{(k)}(t_i)}{k!}(t-t_i)^k + R_n(t). \tag{5}$$

where $u_n(t)$ is called the n-th Taylor polynomial for $u(t)$ about t_i and $R_n(t)$ is remainder term.

If $U(k)$ is defined as

$$U(k) = \frac{u^{(k)}(t_i)}{k!}, \qquad \text{where} \quad k = 0, 1, .., \infty \tag{6}$$

then Eq (4) reduce to

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_i)^k. \tag{7}$$

and the n-partial sums of a Taylor's series (7) reduce to

$$u_n(t) = \sum_{k=0}^{n} U(k)(t-t_i)^k + R_n(t). \tag{8}$$

The $U(k)$ defined in Eq (6), is called the matrix differential transform of matrix function $u(t)$.

For simplicity assume that $t_0 = 0$, then solution (7) reduce to

$$u(t) = \sum_{k=0}^{n} t^k U(k) + R_{n+1}(t). \tag{9}$$

From the above definitions, it can be found that the concept of the one-dimensional matrix differential transform is derived from the Taylor series expansion. With Eq.(6) and Eq.(7), the fundamental mathematical operations performed by one-dimensional matrix differential transform can readily be obtained.

3 Applications and numerical results

This section is devoted to computational results. We applied the method presented in this paper and solved some examples. These examples are chosen such that there exist exact solutions for them.

3.1 Matrix Fisher Equation

For Example consider the Eq. (10) that is spesial type of the matrix Fisher equation , as follow:

$$u_t = Au_{xx} + u(I-u) + B(x,t),$$

subject to initial condition

$$u(x,0) = \begin{bmatrix} \cos(x) & -1 \\ 0 & e^x \end{bmatrix}, \tag{10}$$

where

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},\ B(x,t) = \begin{bmatrix} \cos^2(x)e^{(2t)} & 1-\cos(x)e^t \\ -\cos(x)e^t & e^{2(x+t)} \end{bmatrix},$$

the exact solution of above equation obtained from proposed method

$$u(x,t) = \begin{bmatrix} \cos(x)e^t & -1 \\ 0 & e^{(x+t)} \end{bmatrix}.$$

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5 Conclusions

In this paper, we have shown that the matrix differential transformation method(MDTM) can be used successfully for solving the linear and nonlinear system of partial differential equations. The results of the test examples show that the matrix differential transformation method results are equal to Taylor's series solution of exact solution that is not affected by computation round of errors. The accuracy of the obtained solution can be improved by taking more terms in the solution. In many cases, the series solutions obtained with MDTM can be written in exact closed form.

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A new approach for determining the solution of the Volterra integral equations with convolution kernel

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Abstract

In this paper, we present a new and an efficient method for determining the solution of the Volterra integral equations of the second kind with convolution kernel. This method is based on converting the integral equation to a linear ordinary differential equation which needs specified boundary conditions. For determining boundary conditions, we use the integration technique. This method is very stable and can be applied to approximate a solution of the Volterra integral equation with smooth and weakly singular kernels.

Keywords: Taylor series expansion; Volterra integral equation; weakly singular kernel

Mathematics Subject Classification: 65R20

1 Introduction

In this paper, we consider the second kind Volterra integral equations of the form

$$y(x) + \lambda \int_0^x k(x,t)y(t)dt = f(x), \qquad 0 \le x \le 1. \tag{1}$$

In Eq. (3), the functions $k(x,t)$, $f(x)$ and λ are given, and $y(t)$ is the solution to be determined. We assume that Eq. (3) has a unique solution. However, the necessary and sufficient conditions for existence and uniqueness of the solution of the Eq. (3) could be found in [1]. Furthermore, assume that the kernel $k(x,t) = k(\mid x-t \mid)$ with k continuous in $I = [0,1]$ and decreases as $\mid x-t \mid$ increases from zero or that $k(x,t) = a(x,t)\kappa(x-t)$ with $a(x,t)$ which is continuous for x, $t \in I$ and κ is weakly singular, meaning that $\kappa(x-t) = O(\mid x-t \mid^{-\alpha})$, $0 < \alpha < 1$. In this paper, a novel, simple, and an efficient approach is proposed to determine approximate solutions of the second kind Volterra integral equations. We use the Taylor series expansion of the unknown function for obtaining the solution. Then for determining specified boundary conditions, for transformed linear ordinary differential equation, we employ the integration method. This method is more stable than derivative method and can be applied to approximate the solution of the Volterra integral equation with smooth and weakly singular kernels. Also, it is shown that when the present method is applied to the Volterra integral equation with a rapidly decaying kernel, it provides more accurate results than those obtained by the method in [2].

2 Determination of approximate solution

Consider the second kind Volterra integral equations (3) and without loss of generality suppose that $\lambda = 1$. In order to determine the approximate solution, we employ Taylor's expansion of the unknown function $y(t)$ at x

$$y(t) = y(x) + y'(x)(t-x) + \cdots + \frac{1}{n!} y^{(n)}(x)(t-x)^n + R_n(t,x), \tag{2}$$

where $R_n(t,x)$ denotes Lagrange remainder

$$R_n(t,x) = \frac{y^{(n+1)}(\zeta)}{(n+1)!}(t-x)^{n+1}, \tag{3}$$

for some point ζ between x and t. In genral, the Lagrange remainder $R_n(t,x)$ becomes sufficiently small when n is large enough. In particular, if the desired solution $y(t)$ is a polynomial of the degree equal to or less than n, then $R_n(t,x) = 0$. Substituting Eq. (2) for $y(t)$ in the integrand into Eq. (3) leads to

$$y(x) + \sum_{j=0}^{n} \frac{(-1)^j}{j!} y^{(j)}(x) \int_0^x k(x,t)(x-t)^j dt = f(x), \tag{4}$$

where in the above derivation, the Lagrange remainder has been dropped due to sufficiently small truncated error. Moreover, a notation $y^{(0)}(x) = y(x)$ is adopted. In Eq. (4), $y^{(j)}(x)$ for $j = 0, \ldots, n$ are unknown functions. In order to obtain these unknown functions, we consider the above equation as a linear equation for $y(x)$ and its derivatives up to n. Consequently, other n independent linear equations for $y(x)$ and its derivatives up to n are needed. These equations can be obtained by the integration of both sides of Eq. (3) n times as follows

$$\int_0^x (x-t)^{i-1} y(t) dt + \int_0^x \int_t^x (x-s)^{i-1} k(s,t) y(t) ds dt = f_{(i)}(x), \quad i = 1, \cdots, n, \tag{5}$$

where

$$f_{(i)}(x) = \int_0^x (x-t)^{i-1} f(t) dt. \tag{6}$$

Now, inserting Eq. (2) for $y(t)$ into Eq. (5), we can get

$$\int_0^x (x-t)^{i-1} \sum_{j=0}^{n} \frac{(-1)^j}{j!} y^{(j)}(x)(x-t)^j dt + \int_0^x k_i(x,t) \sum_{j=0}^{n} \frac{(-1)^j}{j!} y^{(j)}(x)(x-t)^j dt = f_{(i)}(x), \tag{7}$$

where

$$k_i(x,t) = \int_t^x (x-s)^{i-1} k(s,t) ds, \qquad i = 1, \cdots, n. \tag{8}$$

Hence, Eqs. (4) and (7) form a system of linear equations for the unknowns $y(x)$ and its derivatives up to n. Introducing

$$C(x) = \begin{bmatrix} c_{00}(x) & c_{01}(x) & \ldots & c_{0n}(x) \\ c_{10}(x) & c_{11}(x) & \ldots & c_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ c_{n0}(x) & c_{n1}(x) & \ldots & c_{nn}(x) \end{bmatrix} \tag{9}$$

$$F(x) = \begin{bmatrix} f(x) \\ f_{(1)}(x) \\ \vdots \\ f_{(n-1)}(x) \end{bmatrix}, \; Y(x) = \begin{bmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n)}(x) \end{bmatrix} \tag{10}$$

the above system composed of Eqs. (4) and (7) can be rewritten as

$$C(x)Y(x) = F(x), \tag{11}$$

where in (9), the first row refers to coefficients of $y^{(j)}(x)$ in Eq. (4) for $j = 0, \dots, n$ and the other rows refer to coefficients of $y^{(j)}(x)$ in Eq. (7) for $j = 0, \dots, n$. Application of Cramer s rule to the resulting system yields an approximate solution of Eq. (4). It is also noted that not only $y(x)$ but also $y^{(j)}(x)$ for $j = 1, ..., n$ are determined via solving the resulting system.

3 Numerical examples

In this section, we present numerical results for some examples to show the efficiency and the accuracy of the presented method in Section 2. All computations are carried out using Mathematica 5.

Example 1. Consider the second kind Volterra integral equation (3) with $\lambda = -\frac{2}{\pi}$ and $k(x,t) = \frac{1}{4+(x-t)^2}$. To evaluate the accuracy of the approximation produced by the presented method, $f(x)$ is chosen such that the exact solution is $y(x) = 1 + x^2 + x^5$. Numerical results (error between the exact and approximate value of $y(x)$) with $n = 2, 4, 6$ are given in Table 1.

x	$n=2$		$n=4$		$n=6$	
	method in [2]	Our approx.	method in [2]	Our approx.	method in [2]	Our approx.
0.1	2.7031×10^{-7}	2.2779×10^{-12}	2.5536×10^{-8}	1.1102×10^{-15}	1.2401×10^{-9}	6.6613×10^{-16}
0.2	0.1749×10^{-4}	5.8522×10^{-10}	9.3845×10^{-7}	6.8833×10^{-15}	7.9976×10^{-8}	8.8817×10^{-16}
0.3	0.2007×10^{-3}	1.4894×10^{-8}	0.1625×10^{-4}	3.4239×10^{-13}	9.2362×10^{-7}	1.1102×10^{-15}
0.4	0.1133×10^{-2}	1.4630×10^{-7}	0.1024×10^{-3}	5.7105×10^{-12}	5.3223×10^{-6}	1.1102×10^{-15}
0.5	0.4330×10^{-2}	8.4963×10^{-7}	0.4104×10^{-3}	4.8663×10^{-11}	2.1147×10^{-5}	2.2204×10^{-16}
0.6	0.0129	3.5277×10^{-6}	0.1254×10^{-2}	2.6865×10^{-10}	6.6936×10^{-5}	1.1102×10^{-15}
0.7	0.0324	1.1593×10^{-5}	0.3197×10^{-2}	1.0880×10^{-9}	1.8206×10^{-4}	6.6613×10^{-16}
0.8	0.0717	3.2044×10^{-5}	0.7150×10^{-2}	3.4732×10^{-9}	4.4439×10^{-4}	3.1086×10^{-15}
0.9	0.1441	7.7487×10^{-5}	0.0144	9.1148×10^{-9}	9.9894×10^{-4}	3.1086×10^{-15}
1.0	0.2681	1.6840×10^{-4}	0.2712	2.0098×10^{-8}	2.1016×10^{-3}	1.7763×10^{-15}

Table 1. Comparison of the absolute errors for approximate solutions in Example 1.

Example 2. In this example, we consider (3) with a weakly singular kernel $k(x,t) = (x-t)^{\frac{-1}{3}}$ and $\lambda = -\frac{1}{10}$. The function $f(x)$ is chosen so that (3) has the known exact solution $y(x) = (x(1-x))^2$. Numerical results (error between the exact and approximate value of $y(x)$) with $n = 0, 2, 4$ are given in Table 2.

x	$n=0$	$n=2$	$n=4$	$n=6$
0.1	1.4376×10^{-4}	1.0174×10^{-7}	1.7347×10^{-18}	3.4694×10^{-18}
0.2	7.0219×10^{-4}	1.1037×10^{-6}	6.9388×10^{-18}	2.0816×10^{-17}
0.3	1.5080×10^{-3}	4.0671×10^{-6}	6.9388×10^{-18}	4.1633×10^{-17}
0.4	2.1851×10^{-3}	9.3796×10^{-6}	6.9388×10^{-18}	6.9388×10^{-18}
0.5	2.3293×10^{-3}	1.6119×10^{-5}	1.3877×10^{-17}	6.9388×10^{-18}
0.6	1.6430×10^{-3}	2.1545×10^{-5}	3.4694×10^{-17}	5.5511×10^{-17}
0.7	5.3768×10^{-5}	2.0667×10^{-5}	2.7755×10^{-17}	6.2450×10^{-17}
0.8	2.1741×10^{-3}	5.8556×10^{-6}	3.8163×10^{-17}	5.8980×10^{-17}
0.9	4.3314×10^{-3}	3.3506×10^{-5}	1.1102×10^{-16}	2.9316×10^{-16}
1.0	5.1566×10^{-3}	1.1134×10^{-4}	1.4922×10^{-16}	1.0052×10^{-15}

Table 2. Comparison of the absolute errors for approximate solutions in Example 2.

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Numerical solutions of two-point linear boundary value problems under uncertainty

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Abstract

In this paper, fuzzy differential transform method (FDTM) is used to solve fuzzy two point linear boundary value problem with fuzzy boundary conditions under lateral type of H-differentiability. Using proposed method leads to convert the original problem to linear system of crisp equations. Also, some numerical example is given to illustrate the utility of FDTM.

Keywords: Fuzzy two point linear boundary value problem, Fuzzy differential transform method (FDTM), Lateral type of H-differentiability.

Mathematics Subject Classification: 34A07; 34B05

1 Introduction

The topic of fuzzy differential equations (FDEs) has been rapidly growing in recent years. The concept of fuzzy derivative was first introduced by Chang and Zadeh [6]; it was followed up by Dubois and Prade [7], who used the extension principle in their approach. Other methods have been discussed by Goetschel and Voxman [8]. Kandel and Byatt [11, 12] applied the concept of fuzzy differential equation (FDE) to the analysis of fuzzy dynamical problems. The FDE and the initial value problem (Cauchy problem) were rigorously treated by O. Kaleva [9, 10], S. Seikkala [13], O. He and W. Yi [14], Kloeden [15] and by other researchers. The numerical methods for solving fuzzy differential equations are applied in references [1, 2, 3, 4, 5]. Differential transform method is different from the traditional high order Taylor series method, which requires symbolic computation of necessary derivatives of the data function and is computationally expensive for higher order.

2 Fuzzy two point linear boundary value problem

In this section we consider the fuzzy two point linear boundary value problem with boundary conditions as following:

$$\begin{cases} x''(t) = a_1(t)x'(t) + a_0(t)x(t) + g(t), \\ \widetilde{x}(0) = \widetilde{b}_0, \quad \widetilde{x}(l) = \widetilde{b}_1 \end{cases} \tag{1}$$

where $a_0(t), a_1(t)$, are continuous on some interval $T = [t_0, \infty)$ and $g(t)$ is a fuzzy-valued function. In order to solve Eq.(1) we describe the fuzzy differential transform method. Let us consider $x(t)$ is differentiable of order k over time domain T, then

$$\underline{X}_i(k,r) = \frac{d^k \underline{x}(t;r)}{dt^k}|_{t=t_i}, \quad \overline{X}_i(k,r) = \frac{d^k \overline{x}(t;r)}{dt^k}|_{t=t_i}, \quad \forall k \in K = \{0,1,2,\dots\} \tag{2}$$

when $x(t)$ is (i)-differentiable and

$$\begin{cases} \underline{X}_i(k,r) = \frac{d^k \overline{x}(t;r)}{dt^k}|_{t=t_i}, & \overline{X}_i(k,r) = \frac{d^k \underline{x}(t;r)}{dt^k}|_{t=t_i}, \quad k \text{ is odd} \\ \underline{X}_i(k,r) = \frac{d^k \underline{x}(t;r)}{dt^k}|_{t=t_i}, & \overline{X}_i(k,r) = \frac{d^k \overline{x}(t;r)}{dt^k}|_{t=t_i}, \quad k \text{ is even} \end{cases} \tag{3}$$

when $x(t)$ is (ii)-differentiable. Notice that $\underline{X}_i(t;r)$ and $\overline{X}_i(t;r)$ are called the lower and upper specturm of $x(t)$ at $t = t_i$, respectively. The objective of this section is to find the solution of Eq.(1) at the equally spaced grid points $[t_0, t_1, \ldots, t_N]$ where $t_i = a + ih$ for each $i = 0, 1, \ldots, N$ and $h = \frac{b-a}{N}$. That is, the domain of interest is divided to N sub-domain and the fuzzy approximation functions in each sub-domain are $x_i(t;r)$ for $i = 0, 1, \ldots, N-1$, respectively.
From the initial conditions the following can be obtained:

$$\underline{X}(0;r) = \underline{x}(0;r), \quad \overline{X}(0;r) = \overline{x}(0;r), \quad 0 \le r \le 1$$

In the first sub-domain, $\underline{x}(t;r)$ and $\overline{x}(t;r)$ can be described by $\underline{x}(0;r) = \underline{x}_0(r)$ and $\overline{x}(0;r) = \overline{x}_0(r)$, respectively. They can be represented in terms of their n-th order Taylor series with respect to t_0, that is

$$\underline{x}(t_0;r) = \underline{X}_0(0;r) + \underline{X}_0(1;r)(t-t_0) + \underline{X}_0(2;r)(t-t_0)^2 + \ldots + \underline{X}_0(N;r)(t-t_0)^N$$

$$\overline{x}(t_0;r) = \overline{X}_0(0;r) + \overline{X}_0(1;r)(t-t_0) + \overline{X}_0(2;r)(t-t_0)^2 + \ldots + \overline{X}_0(N;r)(t-t_0)^N$$

Additionally, using Taylor series for $x(t_1;r)$ leads to obtain:

$$\underline{x}(t_1;r) = \sum_{j=0}^{N} \underline{X}_0(j;r)h^j \qquad \overline{x}(t_1;r) = \sum_{j=0}^{N} \overline{X}_0(j;r)h^j$$

The final value $x_0(t_1)$ of th first sub-domain is the initial value of the second sun-domain, i.e., $x_1(t_1;r) = X_1(0) = x_0(t_1;r)$. In a similar manner $x(t_2;r)$ can be represented as

$$\underline{x}(t_2;r) \approx \underline{x}_1(t_2;r) = \sum_{j=0}^{N} \underline{X}_1(j;r)h^j \qquad \overline{x}(t_1;r) \approx \overline{x}_1(t_2;r) = \sum_{j=0}^{N} \overline{X}_1(j;r)h^j$$

Hence, the solution on the grid points t_{i+1} can be obtained as follows:

$$\underline{x}(t_{i+1};r) \approx \underline{x}_i(t_{i+1};r) = \sum_{j=0}^{N} \underline{X}_i(j;r)h^j \qquad \overline{x}(t_{i+1};r) \approx \overline{x}_i(t_{i+1};r) = \sum_{j=0}^{N} \overline{X}_i(j;r)h^j$$

Now, based on type of differentiability, we get four corresponding cases as following:

CaseI. Let us consider $x(t)$ and $x'(t)$ are (i)-differentiable and without loss of generality $a_0(t)$ and $a_1(t)$ are positive real functions for all $t \in T$, then Eq.(1) will be as follows:

$$\begin{cases} (k+1)(k+2)\underline{X}(k+2;r) = \sum_{l=0}^{N} A_1(l)(k-l+1)\underline{X}(k-l+1;r) + \\ \sum_{l=0}^{N} A_0(l)\underline{X}(k-l;r) + \underline{G}(k;r), \\ (k+1)(k+2)\overline{X}(k+2;r) = \sum_{l=0}^{N} A_1(l)(k-l+1)\overline{X}(k-l+1;r) + \\ \sum_{l=0}^{N} A_0(l)\overline{X}(k-l;r) + \overline{G}(k;r), \\ \underline{X}(0;r) = \underline{b}_0(r), \ \overline{X}(0;r) = \overline{b}_0(r), \quad 0 \le r \le 1 \\ \sum_{k=0}^{N} l^k \underline{X}(k;r) = \underline{b}_1(r), \ \sum_{k=0}^{N} l^k \overline{X}(k;r) = \overline{b}_1(r), \quad 0 \le r \le 1. \end{cases} \tag{4}$$

CaseII. Let us consider $x(t)$ is (i)-differentiable and $x'(t)$ is (ii)-differentiable, then Eq.(1) will be as follows:

$$\begin{cases} (k+1)(k+2)\overline{X}(k+2;r) = \sum_{l=0}^{N} A_1(l)(k-l+1)\underline{X}(k-l+1;r)+ \\ \sum_{l=0}^{N} A_0(l)\underline{X}(k-l;r) + \underline{G}(k;r), \\ (k+1)(k+2)\underline{X}(k+2;r) = \sum_{l=0}^{N} A_1(l)(k-l+1)\overline{X}(k-l+1;r)+ \\ \sum_{l=0}^{N} A_0(l)\overline{X}(k-l;r) + \overline{G}(k;r), \\ \underline{X}(0;r) = \underline{b}_0(r),\ \overline{X}(0;r) = \overline{b}_0(r), \quad 0 \le r \le 1 \\ \sum_{k=0}^{N} l^k \underline{X}(k;r) = \underline{b}_1(r),\ \sum_{k=0}^{N} l^k \overline{X}(k;r) = \overline{b}_1(r), \quad 0 \le r \le 1. \end{cases} \tag{5}$$

Case III. Let us consider $x(t)$ is (ii)-differentiable and $x'(t)$ is (i)-differentiable, then Eq.(1) will be as follows:

$$\begin{cases} (k+1)(k+2)\overline{X}(k+2;r) = \sum_{l=0}^{N} A_1(l)(k-l+1)\overline{X}(k-l+1;r)+ \\ \sum_{l=0}^{N} A_0(l)\underline{X}(k-l;r) + \underline{G}(k;r), \\ (k+1)(k+2)\underline{X}(k+2;r) = \sum_{l=0}^{N} A_1(l)(k-l+1)\underline{X}(k-l+1;r)+ \\ \sum_{l=0}^{N} A_0(l)\overline{X}(k-l;r) + \overline{G}(k;r), \\ \underline{X}(0;r) = \underline{b}_0(r),\ \overline{X}(0;r) = \overline{b}_0(r), \quad 0 \le r \le 1 \\ \sum_{k=0}^{N} l^k \underline{X}(k;r) = \underline{b}_1(r),\ \sum_{k=0}^{N} l^k \overline{X}(k;r) = \overline{b}_1(r), \quad 0 \le r \le 1. \end{cases} \tag{6}$$

Case IV. Let us consider $x(t)$ and $x'(t)$ are (ii)-differentiable, then Eq.(1) will be as follows:

$$\begin{cases} (k+1)(k+2)\underline{X}(k+2;r) = \sum_{l=0}^{N} A_1(l)(k-l+1)\overline{X}(k-l+1;r)+ \\ \sum_{l=0}^{N} A_0(l)\underline{X}(k-l;r) + \underline{G}(k;r), \\ (k+1)(k+2)\overline{X}(k+2;r) = \sum_{l=0}^{N} A_1(l)(k-l+1)\underline{X}(k-l+1;r)+ \\ \sum_{l=0}^{N} A_0(l)\overline{X}(k-l;r) + \overline{G}(k;r), \\ \underline{X}(0;r) = \underline{b}_0(r),\ \overline{X}(0;r) = \overline{b}_0(r), \quad 0 \le r \le 1 \\ \sum_{k=0}^{N} l^k \underline{X}(k;r) = \underline{b}_1(r),\ \sum_{k=0}^{N} l^k \overline{X}(k;r) = \overline{b}_1(r), \quad 0 \le r \le 1. \end{cases} \tag{7}$$

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Higher-order asymptotic formula for the eigenvalues of Sturm-Liouville problem with indefinite weight function in the case of $y(a) = y'(b) = 0, y'(a) = y(b) = 0$

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Abstract
In this paper, we investigate the asymptotic behavior of the differential equation

$$y'' + (\lambda r(x) - q(x))y = 0, \qquad 0 \le x \le 1,$$

where $[0, 1]$ contains a finite number of zeros of $r(x)$, λ is a real parameter and the function $q(x)$ is bounded and integrable in $[0, 1]$. Using a technique used previously in [7], we derive the higher-order asymptotic distribution of the positive eigenvalues associated with this equation for the SLP (in the boundary condition of, $y(0) = y'(1) = 0$ or $y'(0) = y(1) = 0$).
Keywords: Sturm-Liouville problem, eigenvalues, asymptotic

Mathematics Subject Classification: 34L20

1 Introduction

We study the indefinite Sturm-Liouville spectral problem

$$y'' + (\lambda\, r(x) - q(x))y = 0, \qquad a \le x \le b \tag{1}$$

$$y'(a) = y(b) = 0 \vee y(a) = y'(b) = 0,$$

defined on the interval $[a, b]$ where λ is a real parameter, $r(x), q(x)$ are real and integrable on $[a, b]$; moreover,

$$\int_a^b \sqrt{r_+(t)}dt > 0, \quad where \quad r_+(x) = max\{r(x), 0\}. \tag{2}$$

It follows from [2] that the spectrum of this problem is discrete and has no finite accumulation points; moreover, only finitely many eigenvalues lie the outside the real and imaginary axes. In what follows, we shall assume that λ is a positive parameter. In [2] it was shown that the asymptotics of the eigenvalues is of the form

$$\lambda_n \sim \frac{n\pi}{\int_a^b \sqrt{r_+(t)}dt}.$$

Our goal is to refine the asymptotics under the additional assumptions of smoothness of the functions $r(x)$ and $q(x)$. In addition, we assume that $r(x)$ has a finite number of zeros, which are called

turning points.
The outline of our paper is as follows. First, we find the asymptotics of eigenvalues for one turning point. Next, using a technique previously in [7], we derive the higher order asymptotic distribution of the positive eigenvalues in the case of two turning points. Finally, in the case of an arbitrary finite number of turning points it can be reduced to the two cases discussed above.

2.The case of one turning point

First, consider the case

$$r(x) = (x - x_\nu)^{l_\nu} h(x), \qquad h(x) > 0.$$

To simplify the formulas, we assume that x varies on the closed interval with endpoints a and b, where $r(a) < 0$ and $r(b) > 0$. The turning point x_ν lies between a and b.
We distinguish four different types of turning points:

$$T_\nu = \begin{cases} I & \text{if } l_\nu \text{ is even and } r(x) < 0 \text{ in } [a,b] \\ II & \text{if } l_\nu \text{ is even and } r(x) > 0 \text{ in } [a,b] \\ III & \text{if } l_\nu \text{ is odd and } r(x) < 0 \text{ in } [x_\nu, b] \\ IV & \text{if } l_\nu \text{ is odd and } r(x) > 0 \text{ in } [x_\nu, b] \end{cases}$$

is called of type x_ν. By Langer's transformation we can make zero of $r(x)$ the origin. To be specific, let us define the Langer's transformation $\xi(x)$ for different type of TP.
For a turning point of Type I :

$$\xi_I(x) = \begin{cases} -\{\int_x^{x_\nu} (-r)^{1/2}(t)dt\}^{\frac{2}{\ell+2}} & x \le x_\nu \\ -\{\int_{x_\nu}^{x} (-r(t))^{1/2}dt\}^{\frac{2}{\ell+2}} & x_\nu \le x. \end{cases}$$

For a turning point of Type II :

$$\xi_{II}(x) = \begin{cases} \{\int_x^{x_\nu} r^{1/2}(t)dt\}^{\frac{2}{\ell+2}} & x \le x_\nu \\ \{\int_{x_\nu}^{x} r(t)^{1/2}dt\}^{\frac{2}{\ell+2}} & x_\nu \le x. \end{cases}$$

For a turning point of Type III :

$$\xi_{III}(x) = \begin{cases} \{\int_x^{x_\nu} r^{1/2}(t)dt\}^{\frac{2}{\ell+2}} & x \le x_\nu \\ -\{\int_{x_\nu}^{x} (-r(t))^{1/2}dt\}^{\frac{2}{\ell+2}} & x_\nu \le x. \end{cases}$$

For a turning point of Type IV :

$$\xi_{IV}(x) = \begin{cases} -\{\int_x^{x_\nu} (-r(t))^{1/2}dt\}^{\frac{2}{\ell+2}} & x \le x_\nu \\ \{\int_{x_\nu}^{x} (r(t))^{1/2}dt\}^{\frac{2}{\ell+2}} & x_\nu \le x. \end{cases}$$

From [9] we rewrite showing the connection between the argument of complex valued solution of (14) in the interval containing one of the turning point say, x_ν, and the argument of complex valued solution of Sturm-Liouville equation with one turning point in $x = 0$ in the same interval. In fact the following result illustrates a crucial relationship between a general problem (14) with a turning point at x_ν to a transformed problem in which is mapped to $x = 0$.We show that such a transformation preserves the argument of any fixed complex valued solution.

Theorem 1.1. *Let z be a strictly complex -valued solution of the differential equations*

$$y'' + (\rho^2\ r(x) - q(x))y = 0, \qquad x \in [0,1] \tag{3}$$

and W be a solution of

$$W'' + (u^2(-1)^{M_\nu}\xi^{\ell_\nu} - R_\nu(\xi))W = 0, \qquad \xi \in [c,d] \tag{4}$$

then on the interval $[x_\nu - \epsilon, x_\nu + \epsilon]$

$$argW(\xi(x)) = \arg z,$$

where $r(x) = \prod_{j=1}^{n}(x - x_j)^{l_j}\phi_0(x)$ *and*

$$R_\nu(\xi) = (\frac{dx}{d\xi})^{1/2}\frac{d^2}{d\xi^2}\{\frac{1}{(\frac{dx}{d\xi})^{1/2}}\} + (\frac{dx}{d\xi})^2 q(x(\xi)).$$

$M_k =$ *the number of turning of type (III) or (IV) in* $(x_k, 1)$ *,or one can see that* $(-1)^{M_k} = (-1)^{l_n+\cdots+l_{k-1}}$, $c < 0 < d$, $u^2 = \frac{(\ell_\nu+2)^2}{4}\rho^2$, *the transformation* $\xi(x)$ *is Langer's transformation.*

Proof: For proof see [9].

3.The main result

We begin by consolidating some results from [5, 9] for completeness. For a complex-valued solution $\Omega(x, \lambda)$, of

$$y'' + \lambda x^\alpha y = 0, \qquad (E_0)$$

we form the logarithmic derivative $r_0(x, \lambda) = \Omega'(x, \lambda)/\Omega(x, \lambda)$, a quantity that exists for each $x \in [a, b]$ since the real and imaginary parts of Ω are linearly independent solution of (E_0). The quantity $r_1(x, \lambda)$ is defined by setting

$$r_1(x, \lambda) = -\int_x^b q(t)e^{2\int_x^t r_0(s,\lambda)ds}dt,$$

while the $r_n(x, \lambda)$ are defined recursively $(for \quad n \geq 1)$ by

$$r_{n+1}(x, \lambda) = \int_x^b r_n^2(x, \lambda)\exp(2\sum_{l=0}^{n}\int_x^t r_l(s, \lambda)ds)dt$$

It follows $(cf.[4])$ that the function

$$r(x, \lambda) = \sum_{n=0}^{\infty} r_n(x, \lambda) := S(x, \lambda) + iT(x, \lambda)$$

is a series solution (in x) of the Riccati equation

$$v' = q - \lambda x^\alpha - v^2$$

from which one can reconstruct solutions of (14) with Neumann condition $y'(a) = y'(b) = 0$ via the following result:

Theorem 1.2. *(see Harris-Talarico[4]) There exists* λ_0 *such that any real valued solution of*

$$y'' + (\lambda x^\alpha - q(x))y = 0 \qquad (5)$$

can be expressed as :

$$Z(x, \lambda) = c_1 e^{\int_a^x S(t,\lambda)dt}\cos(c_2 + \int_a^x T(t, \lambda)dt)$$

for $x \in [a, b]$ $(a < 0 < b)$ *and* $|\lambda| \geq \lambda_0$ *where* $c_1, c_2 \in \Re$. *If* $Z(., \lambda)$ *satisfies*

$$y(a)\cos\gamma + y'(a)\sin\gamma = 0 \qquad (6)$$

then

$$\begin{aligned} c_2 =: c_2^a &= \frac{\pi}{2} && if \quad \gamma = 0 \\ &= \arctan(\frac{1}{T(a, \lambda)}S(a, \lambda) + \cot\gamma) && if \gamma \neq 0 \qquad (7) \end{aligned}$$

Similarly, if Z satisfies

$$y(b)\cos\beta + y'(b)\sin\beta = 0 \tag{8}$$

then

$$\begin{aligned} c_2 =: c_2^b &= n\pi + \frac{\pi}{2} && if \quad \beta = 0 \\ &= n\pi + \arctan\Big(\frac{1}{T(b,\lambda)}S(b,\lambda) + \cot\beta\Big) && if \quad \beta \neq 0 \end{aligned} \tag{9}$$

for all integer n.

It follows from (7) and (9) that the eigenvalues of (7),(6) and (8),i.e., our problem (14), are the values of λ for which

$$c_2^a + \int_a^b T(t,\lambda)dt = c_2^b \tag{10}$$

We see from [3], that the asymptotic distribution of the eigenvalues of (7),(6) and (8) is therefore determined by the following transcendental equation:

$$\begin{aligned} n\pi + \arctan\Big(\frac{1}{T(b,\lambda)}S(b,\lambda) + \cot\beta\Big) &= \int_a^b T(t,\lambda)dt + \arctan\Big(\frac{1}{T(a,\lambda)}S(a,\lambda) + \cot\gamma\Big) \\ &= \Im\int_a^b r(t,\lambda)dt + \arctan\Big(\frac{1}{T(b,\lambda)}S(b,\lambda) + \cot\gamma\Big) \\ &= \Im\Big(\int_a^b r_0(t,\lambda)dt + \int_a^b r_1(t,\lambda)dt + \cdots\Big) \\ &+ \arctan\Big(\frac{1}{T(a,\lambda)}S(a,\lambda) + \cot\gamma\Big) \qquad (11) \\ &= \arg\Omega(b,\lambda) - \arg\Omega(a,\lambda) - \frac{\pi}{2k}\int_0^b xq(x)J_\nu^2(k^{-1}\lambda^{1/2}x^k)dx + \cdots \\ &+ \arctan\Big(\frac{1}{T(a,\lambda)}S(a,\lambda) + \cot\gamma\Big). \end{aligned}$$

Note that we use the following result from [8],

$$\Im\int_a^b r(t,\lambda)dt = \arg\Omega(b,\lambda) - \arg\Omega(a,\lambda) - \frac{\pi}{2k}\int_0^b xq(x)J_\nu^2(k^{-1}\lambda^{1/2}x^k)dx.$$

By applying the above relation to approximate eigenvalues in the case of $\gamma = 0, \beta = \frac{\pi}{2}$ or $\gamma = \frac{\pi}{2}, \beta = 0$.

Theorem 1.3. *Consider the differential equation (1) on $[a,b]$ under condition (2). Then the positive eigenvalues admit the following asymptotic representation:*
Case1: $\gamma = 0, \beta = \frac{\pi}{2}$
(a) Let x_ν be of type IV. Then

$$\sqrt{\lambda_n} = \frac{n\pi - \frac{\pi}{4}}{\int_a^b\sqrt{r_+(t)}dt} - \frac{1}{n\pi}\Big(\frac{4(\nu-1)^2-1}{8\int_a^b\sqrt{r_+(t)}dt} - \frac{1}{2}H(b)\Big) + O\Big(\frac{1}{n^2}\Big)$$

where

$$H(b) = \int_{x_\nu}^b \Big(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}(\tilde{r}^{-1/4})\Big)\frac{\tilde{r}}{r^{\frac{1}{2}}}dx,$$

and

$$\tilde{r} = (\frac{d\xi}{dx})^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$

(b) Let x_ν be of type III. Then

$$\sqrt{\lambda_n} = \frac{n\pi + \frac{\pi}{4}}{\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{n\pi}(\frac{4(\nu-1)^2-1}{8\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{2}H(a)) + O(\frac{1}{n^2})$$

where

$$H(a) = \int_a^{x_\nu} (\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}(\tilde{r}^{-1/4}))\frac{\tilde{r}}{r^{\frac{1}{2}}}dx,$$

and

$$\tilde{r} = (\frac{d\xi}{dx})^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$

(c) Let x_ν be of type II. Then

$$\sqrt{\lambda_n} = \frac{n\pi}{\int_a^b \sqrt{r(t)}dt} - \frac{1}{n\pi}\{\frac{4(\nu-1)^2-1}{8\int_a^{x_\nu} \sqrt{r(t)}dt} + \{\frac{4(\nu-1)^2-1}{8\int_{x_\nu}^b \sqrt{r(t)}dt} - \frac{1}{2}H(a) - \frac{1}{2}H(b)\} + O(\frac{1}{n^2})$$

where $H(a)$ and $H(b)$ are defined above. Case1: $\gamma = 0, \beta = \frac{\pi}{2}$
(a) Let x_ν be of type IV. Then

$$\sqrt{\lambda_n} = \frac{n\pi - \frac{\pi}{4}}{\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{n\pi}(\frac{4(\nu-1)^2-1}{8\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{2}H(b)) + O(\frac{1}{n^2})$$

where

$$H(b) = \int_{x_\nu}^b (\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}(\tilde{r}^{-1/4}))\frac{\tilde{r}}{r^{\frac{1}{2}}}dx,$$

and

$$\tilde{r} = (\frac{d\xi}{dx})^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$

(b) Let x_ν be of type III. Then

$$\sqrt{\lambda_n} = \frac{n\pi + \frac{\pi}{4}}{\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{n\pi}(\frac{4(\nu-1)^2-1}{8\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{2}H(a)) + O(\frac{1}{n^2})$$

where

$$H(a) = \int_a^{x_\nu} (\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}(\tilde{r}^{-1/4}))\frac{\tilde{r}}{r^{\frac{1}{2}}}dx,$$

and

$$\tilde{r} = (\frac{d\xi}{dx})^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$

(c) Let x_ν be of type II. Then

$$\sqrt{\lambda_n}=\frac{n\pi}{\int_a^b\sqrt{r(t)}dt}-\frac{1}{n\pi}\{\frac{4(\nu-1)^2-1}{8\int_a^{x_\nu}\sqrt{r(t)}dt}+\{\frac{4(\nu-1)^2-1}{8\int_{x_\nu}^b\sqrt{r(t)}dt}-\frac{1}{2}H(a)-\frac{1}{2}H(b)\}+O(\frac{1}{n^2})$$

where $H(a)$ and $H(b)$ are defined above.

Proof: Let us proof only case $(1a)$, from [14] for $\alpha=4n\pm1$ and $x<0$ we know that $\Omega(x,\lambda)=e^{i\pi/2}|x|^{1/2}H_\nu^{(1)}(e^{ik\pi}\rho|x|^k)$, and from $[1,(9.2.7)]$, Hankel's expansion the form

$$H_\nu^{(1)}(z)=\sqrt{\frac{2}{\pi z}}e^{i(z-\frac{1}{2}\nu\pi-\frac{\pi}{4})}M(z),\quad(-\pi<\arg z<\pi),$$

where $M(z)$ is bounded away from the origin and $M(z)\to1$ as $|z|\to\infty$ in the sector.

Now since $\Omega(x,\lambda)=i\rho^{-\nu}e^{-ik\pi\nu}(e^{ik\pi}\rho|x|^k)^\nu H_\nu^{(1)}(e^{ik\pi}\rho|x|^k)$, where $k=\frac{\alpha+2}{2}$, $\nu=\frac{1}{\alpha+2}$ and $k\nu=\frac{1}{2}$, by using ([1],(9.1.30)) we find $\Omega'(x,\lambda)=ike^{ik\pi}\rho|x|^{k-3/2}xH_{\nu-1}^{(1)}(e^{ik\pi}\rho|x|^k)$. Therefore

$$\frac{\Omega'(x,\lambda)}{\Omega(x,\lambda)}=k\rho e^{ik\pi}x\mid x\mid^{k-2}\ \frac{H_{\nu-1}^{(1)}(z)}{H_\nu^{(1)}(z)}$$

.

$$=k\rho e^{ik\pi}x|x|^{k-2}(i+\frac{8\nu-4}{8z}+O(\frac{1}{z^2})),\qquad z=e^{ik\pi}\rho|x|^k.$$

Then

$$\Re\frac{\Omega'(x,\lambda)}{\Omega(x,\lambda)}=k\rho x|x|^{k-2}-k\frac{8\nu-4}{8|x|}+O(\frac{1}{z^2}).$$

and

$$\Im\frac{\Omega'(x,\lambda)}{\Omega(x,\lambda)}=0.$$

On the other hand,

$$\frac{S(a,\lambda)}{T(a,\lambda)}=\frac{\Re r_0(a,\lambda)+\Re r_1(a,\lambda)+\cdots}{\Im r_0(a,\lambda)+\Im r_1(a,\lambda)+\cdots}=\frac{\Re\frac{\Omega'(a,\lambda)}{\Omega(a,\lambda)}+\Re\{-\int_a^b q(t)e^{2\int_a^t r_0(s,\lambda)ds}dt\}+\cdots}{\Im\frac{\Omega'(a,\lambda)}{\Omega(a,\lambda)}+\Im\{-\int_a^b q(t)e^{2\int_a^t r_0(s,\lambda)ds}dt\}+\cdots}.$$

Now we have:

$$\int_a^b q(t)e^{2\int_a^t r_0(s,\lambda)ds}dt=\int_a^0 q(t)e^{2\int_a^t r_0(s,\lambda)ds}dt+\int_0^b q(t)e^{2\int_a^0 r_0(s,\lambda)ds+2\int_0^t r_0(s,\lambda)ds}dt$$

$$=\frac{1}{|\Omega(a,\lambda)|^2}\int_a^0 q(t)|\Omega(t,\lambda)|^2e^{2i\{\arg\Omega(t,\lambda)-\arg\Omega(a,\lambda)\}}dt$$

$$+\int_0^b q(t)|\frac{\Omega(0,\lambda)}{\Omega(a,\lambda)}|^2e^{2i\{\arg\Omega(0,\lambda)-\arg\Omega(a,\lambda)\}}\times|\frac{\Omega(t,\lambda)}{\Omega(0,\lambda)}|^2e^{2i\{\arg\Omega(t,\lambda)-\arg\Omega(0,\lambda)\}}dt$$

$$=\frac{e^{-2i\arg\Omega(a,\lambda)}}{|\Omega(a,\lambda)|^2}\{\int_a^0 q(t)|\Omega(t,\lambda)|^2e^{2i\arg\Omega(t,\lambda)}dt+\int_0^b q(t)|\Omega(t,\lambda)|^2e^{2i\arg\Omega(t,\lambda)}dt\}=O(1).$$

Similarly for $\alpha=4n\mp1$ and $x>0$ we know that $\Omega(x,\lambda)=x^{1/2}H_\nu^{(1)}(\rho x^k)$. Now since $\Omega(x,\lambda)=\rho^{-\nu}(\rho x^k)^\nu H_\nu^{(1)}(\rho x^k)$, by using ([1],(9.1.30)) we find $\Omega'(x,\lambda)=kx^{k-1/2}\rho H_{\nu-1}^{(1)}(\rho x^k)$. Therefore

$$\frac{\Omega'(x,\lambda)}{\Omega(x,\lambda)}=k\rho x^{k-1}\frac{H_{\nu-1}^{(1)}(z)}{H_\nu^{(1)}(z)}=k\rho x^{k-1}(i+\frac{8\nu-4}{8z}+O(\frac{1}{z^2})).$$

Then

$$\Re\frac{\Omega'(x,\lambda)}{\Omega(x,\lambda)} = k\rho x^{k-1}\frac{8\nu-4}{8\rho x^k} = k\frac{8\nu-4}{8x} + O(\frac{1}{z^2}),$$

and

$$\Im\frac{\Omega'(x,\lambda)}{\Omega(x,\lambda)} = k\rho x^{k-1} + O(\frac{1}{z^2}),$$

whence

$$\Re\frac{\Omega'(b,\lambda)}{\Omega(b,\lambda)} = k\frac{8\nu-4}{8b} + O(\frac{1}{z^2}),$$

and

$$\Im\frac{\Omega'(b,\lambda)}{\Omega(b,\lambda)} = k\rho b^{k-1} = \sqrt{\lambda}b^{k-1} + O(\frac{1}{z^2}),$$

also

$$\Re r_1(b,\lambda) = \Im r_1(b,\lambda) = 0.$$

By substitution in (11) and using :

$$\arctan x = x - \frac{x^3}{3} + \cdots ,$$

we have (the particulars of the calculations are omitted):

$$\sqrt{\lambda_n} = \frac{n\pi - \frac{\pi}{4}}{\int_a^b \sqrt{r_+(t)}dt} - \frac{1}{n\pi}(\frac{4(\nu-1)^2-1}{8\int_a^b \sqrt{r(t)}dt} - \frac{1}{2}H(b)) + O(\frac{1}{n^2}).$$

2 Main Result

The cases of two and n turning points

From now then, without losing generalization, we suppose that the coefficients $q(x)$ and $r(x)$ satisfy:
(i) $r(x)$ is real and has in [0, 1] n zeros x_ν of order $l_\nu \in$ N ,$1 \le \nu \le n$ where $0 < x_1 < x_2 < ... < x_n < 1$.
(ii) The function $\phi_0 : I \to R - \{0\}, x \to\ r(x)\prod_{\nu=1}^{n}(x-x_\nu)^{-l_\nu}$ is twice continuously differentiable.
(iii) $q(x)$ is bounded and integrable in I.
We shall use the symbol $\Omega_{IV}(\xi, u)$ to signify the complex-valued solution of

$$W^{''} + u^2\xi^{l_\nu}W = 0,$$

where ξ is corresponding Langer's transformation of turning point of type IV. We will use the symbols $\Omega_I(\xi, u)$,$\Omega_{II}(\xi, u)$ and $\Omega_{III}(\xi, u)$in similar case.
Now we can derive the following results on the distribution of the eigenvalues of (14) with Neumann boundary condition .

We consider only the following case:

$r(0) < 0$**,**$r(1) < 0$**.**

1.a $T_1 = IV$**,**$T_2 = III$**.**

We suppose that the weight function $r(x)$ has in $[0, 1]$ two zeros x_1 and x_2 where x_1 of type IV and x_3 of type III. By (11) the distribution of positive eigenvalue satisfies :

$$\begin{aligned} n\pi &= \Im\int_0^1 \frac{y'}{y}dx) = \Im(\int_0^{\alpha_{12}} \frac{y'}{y}dx + \int_{\alpha_{12}}^1 \frac{y'}{y}dx) + \arctan(\frac{S(0,\lambda)}{T(0,\lambda)}) - \arctan(\frac{S(1,\lambda)}{T(1,\lambda)}) \\ &= \rho\int_0^1 \sqrt{r_+(t)}dt + \frac{\pi}{2} - \frac{1}{4\rho}(\frac{8\nu_1 - 4}{\int_{x_1}^{x_2}\sqrt{r(t)}dt} + \frac{8\nu_2 - 4}{\int_{x_1}^{x_2}\sqrt{r(t)}dt}) + \frac{1}{4\rho}(\frac{4\nu_1^2 - 1}{\int_{x_1}^{x_2}\sqrt{r(t)}dt} + \frac{4\nu_2^2 - 1}{\int_{x_1}^{x_2}\sqrt{r(t)}dt}) \\ &- \frac{1}{2u}P(x_1, x_2) + O(\frac{1}{u^2}), \end{aligned} \tag{12}$$

where $\alpha_{12} \in (x_1, x_2)$ is such that $\int_{x_1}^{\alpha_{12}} \sqrt{r(t)}dt = \int_{\alpha_{12}}^{x_2} \sqrt{r(t)}dt$ (the existence of α_{12} follows by Intermediate Value Theorem), and

$$P(x_1, x_2) = P_{IV}(x_1, \alpha_{12}) + P_{III}(x_2, \alpha_{12}) = \int_{x_1}^{\alpha_{12}} E_{IV}(x)dx + \int_{\alpha_{12}}^{x_2} E_{III}(x)dx,$$

$$P_{IV}(x_1, \alpha_{12}) = \int_0^{d_{12}} \frac{R_{IV}(\xi)}{\xi^{\frac{l}{2}}}d\xi = \int_{x_1}^{\alpha_{12}} (\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}(\tilde{r}^{-1/4}))\frac{\tilde{r}}{r^{\frac{1}{2}}}dx := \int_{x_1}^{\alpha_{12}} E_{IV}(x)dx,$$

and

$$\tilde{r} = (\frac{d\xi}{dx})^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}, \xi(\alpha_{12}) = d_{12}, \xi(x_1) = 0, \xi(0) = c_{12}$$

$$P_{III}(x_2, \alpha_{12}) = \int_{\alpha_{12}}^{x_2} (\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}}\frac{d^2}{dx^2}(\tilde{r}^{-1/4}))\frac{\tilde{r}}{r^{\frac{1}{2}}}dx := \int_{\alpha_{12}}^{x_2} E_{III}(x)dx.$$

By inversion, we get

$$\rho_n = \frac{n\pi - \frac{\pi}{2}}{\int_0^1 \sqrt{r_+(t)}dt} - \frac{1}{n\pi}\{\frac{[4(\nu_1-1)^2 - 1] + [4(\nu_2-1)^2 - 1]}{4\int_{x_1}^{x_2}\sqrt{r(t)}dt} - \frac{1}{2}P(x_1, x_2)\} + O(\frac{1}{n^2}).$$

2.a $T_1 = IV$**,**$T_2 = T_3 = ... = T_{n-1} = II$**,**$T_n = III$**.**
By applying the same method and using theorem (1),(2) we get:

$$\begin{aligned} n\pi &= \rho\int_0^1 \sqrt{r_+(t)}dt - \frac{(n-1)\pi}{2} + \frac{n\pi}{2} + \frac{1}{4\rho}(\frac{4(\nu_1-1)^2 - 1}{\int_{x_1}^{x_2}\sqrt{r(t)}dt} + \frac{4(\nu_2-1)^2 - 1}{\int_{x_1}^{x_2}\sqrt{r(t)}dt} + \frac{4(\nu_2-1)^2 - 1}{\int_{x_2}^{x_3}\sqrt{r(t)}dt} \\ &+ \frac{4(\nu_3-1)^2 - 1}{\int_{x_2}^{x_3}\sqrt{r(t)}dt} + \frac{4(\nu_3-1)^2 - 1}{\int_{x_3}^{x_4}\sqrt{r(t)}dt} + \cdots + \frac{4(\nu_{n-1}-1)^2 - 1}{\int_{x_{n-2}}^{x_{n-1}}\sqrt{r(t)}dt} + \frac{4(\nu_{n-1}-1)^2 - 1}{\int_{x_{n-1}}^{x_n}\sqrt{r(t)}dt} \\ &+ \frac{4\nu_n^2 - 1}{\int_{x_{n-1}}^{x_n}\sqrt{r(t)}dt}) - \frac{1}{2u}P(x_1, x_2, \cdots, x_n) + O(\frac{1}{u^2}), \end{aligned}$$

where

$$P(x_1, x_2, \cdots, x_n) = \int_{x_1}^{\alpha_{12}} E_{IV}(x)dx + \sum_{i=1}^{n-2}\int_{\alpha_{i(i+1)}}^{x_{i+1}} E_{II}^-(x) + \sum_{i=1}^{n-2}\int_{x_{i+1}}^{\alpha_{i+1(i+2)}} E_{II}^+(x) + \int_{\alpha_{(n-1)n}}^1 E_{III}(x)dx$$

and by inversion:

$$\rho_n = \frac{n\pi - \frac{\pi}{2}}{\int_0^1 \sqrt{r_+(x)}dx} - \frac{1}{n\pi}[\frac{[4(\nu_1-1)^2-1]+[4(\nu_2-1)^2-1]}{4\int_{x_1}^{x_2}\sqrt{r(t)}dt} + \frac{[4(\nu_2-1)^2-1]+[4(\nu_3-1)^2-1]}{4\int_{x_2}^{x_3}\sqrt{r(t)}dt}$$
$$+ \cdots + \frac{[4(\nu_{n-1}-1)^2-1]+(4(\nu_n-1)^2-1)}{4\int_{x_{n-1}}^{x_n}\sqrt{r(t)}dt} - \frac{1}{2}P(x_1,x_2,x_3,...,x_n)] + O(\frac{1}{n^2}). \quad (12)$$

REMARK. Note that the reader can obtain asymptotic distribution of eigenvalues in different types of (TP) by consideration of combination of the above cases.

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Jacobi pseudospectral method for a class of singular boundary value problems arising in physiology

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Abstract
In this paper, we presents a computational technique for solution to a class of singular boundary value problems arising in physiology. The method is based upon Jacobi pseudospectral method. Numerical examples are presented to demonstrate the validity and applicability of the method, and a comparison is made with existing methods in the literature.
Keywords: Boundary value problems; Singular points; Jacobi pseudospectral method; Physiology.

Mathematics Subject Classification: 65M70

1 Introduction

We consider a class of nonlinear singular boundary value problems of the following form [6, 11, 10]

$$y''(x) + \frac{m}{x}y'(x) = f(x,y), \quad 0 \le x \le 1, \tag{1}$$

$$y'(0) = 0, \tag{2}$$

$$ay(1) + by'(1) = \gamma, \tag{3}$$

Where $a,\ b$ are real numbers and we assume that $f(x,y) \in \{[0,1] \times \mathbb{R}\}$ is continuous, $\frac{\partial f}{\partial y}$ exists and continuous, and $\frac{\partial f}{\partial y} \ge 0, \quad \forall x \in [0,1]$. For the case $m = 2$, $a = \gamma$ and $b = 1$ the existence and uniqueness of the solution 1-4 has been given in [1].
This work is based on the Michaelis-Menten kinetics [2] for the steady state oxygen diffusion in spherical cells, in which

$$f(x,y) = f(y) = \frac{ny(x)}{y(x)+k}, \quad k > 0, \quad n > 0. \tag{4}$$

A similar problem arise in the study of the distribution of heat sources in the human head [3, 4] in which

$$f(x,y) = f(y) = -ne^{-nky(x)}, \quad k > 0, \quad n > 0. \tag{5}$$

Point wise bounds and uniqueness results are given in [2] for this problem with $f(x,y)$ of the form given by (4) and (5). Pandy and Singh [5] have used finite difference method (FD) based on uniform mesh, Kanth and Bhattacharya [6] used cubic spline of order $O(h^4)$ for solving 1-4

approximately.
The objective of this paper is to use Jacobi pseudospectral method for approximation singular boundary value problem 1-4. Theoretical studies and numerical experiences have confirmed that for problems with smooth solution Jacobi pseudospectral methods converge faster than other methods [7].

2 Jacobi Orthogonal Polynomials

The Jacobi polynomials play important roles in mathematical analysis and its applications. In particular, the Legendre and Chebyshev approximations have been used successfully for spectral and pseudospectral methods. Consider the Jacobi orthogonal polynomials $p_n^{(\alpha,\beta)}(t)$ with respect to weight function $w(t) = (1-t)^\alpha(1+t)^\beta, \quad \alpha, \beta > -1$ on the interval $[-1, 1]$, that is

$$\int_{-1}^{1} w(t)p_n^{(\alpha,\beta)}(t)p_m^{(\alpha,\beta)}(t)dt = \frac{2^{n+\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\delta_{nm}.$$

The polynomials satisfy a three-term recurrence relation [8]

$$\begin{aligned} p_n^{(\alpha,\beta)}(t) = &\frac{(2n+\alpha+\beta-1)(\alpha^2-\beta^2+(2n+\alpha+\beta-2)(2n+\alpha+\beta)t)}{2n(n+\alpha+\beta)(2n+\alpha+\beta-2)}p_{n-1}^{(\alpha,\beta)}(t)- \\ &\frac{(n+\alpha-1)(n+\beta^2-1)(2n+\alpha+\beta)}{n(n+\alpha+\beta)(2n+\alpha+\beta-2)}p_{n-2}^{(\alpha,\beta)}(t), \quad n = 2, 3, \cdots \end{aligned} \tag{6}$$

$$p_0^{(\alpha,\beta)}(t) = 1, \quad p_1^{(\alpha,\beta)}(t) = \frac{\alpha+\beta}{2} + \frac{\alpha+\beta+2}{2}t.$$

The collocation points t_j and weights w_j may be determined by the method outlined by Golub and Welsch [9]. The approach is based on determining the eigenvalues and normalized eigenvectors of a modified tridiagonal Jacobi matrix.
The Jacobi pseudospectral method [7] expand the function $y \in L^2[-1, 1]$ by using interpolations of degree N of the form

$$y(t) \simeq P_N(t) = \sum_{j=0}^{N} L_j(t)f(t_j), \quad t \in [-1, 1], \tag{7}$$

where $t_j, j = 0, 1, \cdots, N$, are a set of distinct collocation points in the interval [-1,1], and $L_j(t)$ are the Lagrange polynomial that satisfy the property $L_j(t_k) = \delta_{jk}$, i.e

$$L_j(t) = \prod_{k=0_{k\neq j}}^{N}\left(\frac{t-t_k}{t_j-t_k}\right).$$

3 Discretization of Singular Boundary Value Problem

In this section, we solve the singular boundary value problem (1)-(4) by using Jacobi pseudospectral method. By change the variable $x = \frac{t+1}{2}, \quad t \in [-1, 1]$ we transfer the interval $[-1, 1]$ to $[0, 1]$. From Eq. (1) we have that boundary value problem is singular at point $x = 0$. Now we collocate Eq. (1) in the points $x_r, \quad r = 1, 2, \cdots, N-1$, which at this points Eq. (1) is not singular. For this purpose, we first substitute Eq. (3) for $y(x)$ in Eqs. (1)-(4), i.e

$$\sum_{j=0}^{N} y_j L_j^{(2)}(x) + \frac{m}{x}\sum_{j=0}^{N} y_j L_j^{(1)}(x) = f\left(x, \sum_{j=0}^{N} y_j L_j(x)\right), \tag{8}$$

$$y'(0) = 0, \tag{9}$$

$$ay(1) + by'(1) = \gamma. \tag{10}$$

Now collocate the equation (10) at $x = x_r, \quad r = 1, 2, \cdots, N-1$ and by using the differential matrices obtained in section 2, the above equations can be written as follows

$$\sum_{j=0}^{N} D_{rj}^{(2)} y_j + \frac{m}{x_r} \sum_{j=0}^{N} D_{rj}^{(1)} y_j = f\left(x_r, y_r\right), \quad r = 1, 2, \cdots, N-1, \tag{11}$$

$$\sum_{j=0}^{N} D_{0j}^{(1)} y_j = 0, \tag{12}$$

$$a y_N + b \sum_{j=0}^{N} D_{Nj}^{(1)} y_j = \gamma. \tag{13}$$

where $D_{rj}^{(1)}$ and $D_{rj}^{(2)}$ are the same derivatives $L_j^{(1)}(x)$ and $L_j^{(2)}(x)$ in points x_r, respectively. Now equations (11)-(13) are a system of nonlinear equations that can be solved by Newton iterative method.

4 Main Result

In this section, we implement Jacobi pseudospectral method on two physical model examples:
(i) oxygen diffusion
(ii) nonlinear heat conduction model of human head.
These problems already been studied by Asaithambi and Goodman [10], Pandy and singh [5], Kanth and Bhattacharya [6]. The numerical results show that present method approximates the solution very well and computational time is less than other methods.

4.1 Example 1

The first example is a nonlinear heat conduction model of the human head, which correspond to Eqs. (1)-(4) and (5) with $n = 1, k = 1$. Numerical results by our method are given respectively in tables 1 and 2 for the following two cases:
(i)$a = b = 1$ and $\gamma = 0$,
(ii)$a = 0.1, b = 1$ and $\gamma = 0$.
In tables 3, 4, shows numerical results for case (ii) in $x = 0.5$.
Tables 5 and 6, shows convergence order for cases (i) and (ii) at $x = 0.5$.

Table 1: Numerical results for $a = b = 1, \gamma = 0$ and for $N = 10$

x_i	$\alpha = 0, \beta = 0$	$\alpha = -0.5, \beta = -0.5$	$\alpha = 1, \beta = 1$	$\alpha = 0, \beta = 1$
0.0	0.367516815121393	0.367516815130241	0.367516815092856	0.367516815128108
0.1	0.366362329232813	0.366362329241781	0.366362329203807	0.366362329238599
0.2	0.362894066111464	0.362894066120302	0.362894066082534	0.362894066117687
0.3	0.357097545713269	0.357097545722227	0.357097545684154	0.357097545719327
0.4	0.348948420614143	0.348948420623277	0.348948420584652	0.348948420620162
0.5	0.338412148742389	0.338412148751497	0.338412148712711	0.338412148748713
0.6	0.325443522426073	0.325443522435232	0.325443522396077	0.325443522432369
0.7	0.309986040224137	0.309986040233593	0.309986040193502	0.309986040230406
0.8	0.291971103047934	0.291971103057541	0.291971103016847	0.291971103054517
0.9	0.271317010149861	0.271317010159514	0.271317010118219	0.271317010156426
1.0	0.247927723308331	0.247927723318267	0.247927723276353	0.247927723315102

Table 2: Numerical results for $a = 0.1, b = 1, \gamma = 0$ and for $N = 10$

x_i	$\alpha = 0, \beta = 0$	$\alpha = -0.5, \beta = -0.5$	$\alpha = 1, \beta = 1$	$\alpha = 0, \beta = 1$
0.0	1.14703901932921	1.14703901932961	1.14703901932794	1.14703901932949
0.1	1.14650964241007	1.14650964241047	1.14650964240880	1.14650964241035
0.2	1.14492050209157	1.14492050209197	1.14492050209029	1.14492050209185
0.3	1.14226856357048	1.14226856357089	1.14226856356921	1.14226856357076
0.4	1.13854874836453	1.13854874836493	1.13854874836325	1.13854874836480
0.5	1.13375390332520	1.13375390332561	1.13375390332391	1.13375390332548
0.6	1.12787475670641	1.12787475670682	1.12787475670511	1.12787475670669
0.7	1.12089986072506	1.12089986072547	1.12089986072376	1.12089986072534
0.8	1.11281551986781	1.11281551986822	1.11281551986649	1.11281551986810
0.9	1.10360570409944	1.10360570409986	1.10360570409812	1.10360570399974
1.0	1.09325194510806	1.09325194510848	1.09325194510672	1.09325194510835

Table 3: Numerical results for case (i) at $x = 0.5$

N	$\alpha = 0, \beta = 0$	$\alpha = 0, \beta = 1$
7	1.133753896033203661	1.133753899327222492
8	1.133753903082830832	1.133753903189032278
9	1.133753903309080138	1.1337539033167161802
10	1.133753903325202031	1.133753903325479864
11	1.133753903325783677	1.133753903325801217
12	1.133753903325820331	1.133753903325821023
13	1.133753903325821775	1.133753903325821816
14	1.133753903325821860	1.133753903325821862
15	1.133753903325821864	1.133753903325821864
16	1.133753903325821853	1.133753903325821837

Table 4: Numerical results for case (ii) at $x = 0.5$

N	$\alpha = 1, \beta = 1$	$\alpha = \frac{-1}{2}, \beta = \frac{-1}{2}$
7	1.133753886300674245	1.133753900228479211
8	1.133753902699228666	1.133753903228810427
9	1.133753903278466673	1.133753903319468685
10	1.133753903323910654	1.133753903325600367
11	1.133753903325694116	1.133753903325808979
12	1.133753903325816360	1.133753903325821366
13	1.133753903325821525	1.133753903325821836
14	1.133753903325821848	1.133753903325821863
15	1.133753903325821863	1.133753903325821864
16	1.133753903325821875	1.133753903325821852

Table 5: Convergence order for case (ii) at $x = 0.5$

$\log(\frac{E_N}{E_{N+1}})$	$\alpha = 0, \beta = 0$	$\alpha = 0, \beta = 1$	$\alpha = 1, \beta = 1$	$\alpha = \frac{-1}{2}, \beta = \frac{-1}{2}$
$\log(\frac{E_8}{E_9})$	3.44	3.41	3.34	3.50
$\log(\frac{E_9}{E_{10}})$	2.64	2.68	2.54	2.69
$\log(\frac{E_{10}}{E_{11}})$	3.32	3.30	2.24	3.38
$\log(\frac{E_{11}}{E_{12}})$	2.76	2.79	2.68	2.82
$\log(\frac{E_{12}}{E_{13}})$	2.33	3.22	3.16	3.27
$\log(\frac{E_{13}}{E_{14}})$	2.84	2.86	2.77	2.87
$\log(\frac{E_{14}}{E_{15}})$	3.07	2.82	3.13	—

Table 6: Convergence order for case (i) at $x = 0.5$

$\log(\frac{E_N}{E_{N+1}})$	$\alpha = 0, \beta = 0$	$\alpha = 0, \beta = 1$	$\alpha = 1, \beta = 1$	$\alpha = \frac{-1}{2}, \beta = \frac{-1}{2}$
$\log(\frac{E_8}{E_9})$	2.76	2.72	2.67	2.78
$\log(\frac{E_9}{E_{10}})$	3.94	2.43	2.32	2.38
$\log(\frac{E_{10}}{E_{11}})$	2.71	2.72	2.62	2.80
$\log(\frac{E_{11}}{E_{12}})$	2.50	2.51	2.41	2.61
$\log(\frac{E_{12}}{E_{13}})$	2.64	2.63	2.58	2.65
$\log(\frac{E_{13}}{E_{14}})$	2.52	2.53	2.46	2.51
$\log(\frac{E_{14}}{E_{15}})$	2.64	2.64	2.57	2.70

4.2 Example 2

The second example is a nonlinear diffusion problem, which correspond to Eqs. (1)-(4) with $n = 0.76129, k = 0.03119, a = \gamma = 5, b = 1$. Numerical results by our method are given in table 7 for $N = 4, 8, 12$, and numerical results with other methods be given in table 8.

Table7: Numerical results by other methods

x_i	$N = 4$	$N = 8$	$N = 12$
0.0	0.828482909076037	0.828483290342087	0.828483290359688
0.1	0.829705667327250	0.829706092416702	0.829706092433806
0.2	0.833374270777634	0.833374733578988	0.833374733591007
0.3	0.839489494171558	0.839489913935258	0.839489913953708
0.4	0.848052487852403	0.848052784973619	0.848052784996068
0.5	0.859064777762555	0.859064927153894	0.859064927169235
0.6	0.872528265443409	0.872528319947140	0.872528319958288
0.7	0.888445228035366	0.888445305604946	0.888445305623196
0.8	0.906818318277838	0.906818548045666	0.906818548066807
0.9	0.927650564509241	0.927650988835169	0.927650988365581
1.0	0.950945370667002	0.950945798478812	0.950945798496480

Table 8: Numerical results by other methods

x_i	$Cubic\ spline[6]\ method\ for\ h = \frac{1}{60}$	$Panndy\ and\ singh[5]$
0.0	0.8284832730	0.8284831497
0.1	0.8297060752	0.8297060742
0.2	0.8333747169	0.8333747157
0.3	0.8394898186	0.8394898966
0.4	0.8480527704	0.8480527684
0.5	0.8590649140	0.8590649116
0.6	0.8725283084	0.8725283056
0.7	0.8884452959	0.8884452928
0.8	0.9068185402	0.9068185369
0.9	0.9276509825	0.9276509791
1.0	0.9509457946	0.9509457914

5 Conclusion

A method for approximating the studied singular boundary value problem based on pseudospectral methods is presented. The cubic spline [9] and the method given in [4] have the disadvantage that the number of system be large and also the approximation is not good. Only a limited number of α, β have been used in this paper and further work is needed to fully explore the possibilities that different pseudospectral discretizations be used offer. The numerical results show that the proposed method is very accurate and needs less computational efforts. All computations in this paper have been performed by software Maple 7 with 30-digit arithmetic.

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Load balanced parallel block QR decomposition

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Abstract

This paper introduces a new parallel QR decomposition using Householder transformation and Givens rotation. We try to parallelize the first stage of Boleng and Raghavan's algorithm [1] using parallel block Householder transformation. Load balancing technique be used here considers total computational works. The new hybrid algorithm has two stages, IT (The Internal Transformations Stage) and BR (The Balanced Rotations Stage). In the first stage, each processor uses block Householder transformation. The second stage annihilates remaining elements using Givens rotations. Since we are desinging with shared memory, parallelism does not defray extra communication costs.

Keywords: QR decomposition, Load balanced, Householder transformation, Givens rotation

Mathematics Subject Classification: 53A15

1 Introduction

Definition 1.1. *Any real $m \times n$ matrix A can be decomposed as*

$$A = QR$$

where Q is an $m \times m$ orthogonal matrix and R is an $m \times n$ upper triangular matrix. A is assumed to be full rank: rank(A) = rank (R)=n. Householder transformation and Givens rotation can be used to compute the decomposition.

Definition 1.2. *A Householder transformation is a transformation that takes a vector and reflects it about some plane or hyperplane. We can use this operation to calculate the QR decomposition of an $m \times n$ matrix A. A Householder matrix is presented by*

$$H = I - \beta v v^T,$$

where H is an $m \times m$ matrix and $\beta = 2/v^T v$ (v is a column vector). H is orthogonal and symmetric. The Houlseholder transformation introduces large number of zeroes in one matrix operation but can not be carried out in parallel in a straight forward way.

Definition 1.3. *The QR decomposition can also be computed with a series of Givens rotations. Each rotation zeroes an element in the subdiagonal of the matrix, forming the R matrix. An $m \times m$*

matrix J(i,j,c,s) of the form

$$J(i,j,c,s) = \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & c & \dots & s & & \\ & & \vdots & \ddots & \vdots & & \\ & & -s & \dots & c & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{bmatrix}$$

is called Givens matrix where $c^2 + s^2 = 1$ *and*

$$c = a_{jk}/\sqrt{a_{jk}^2 + a_{ik}^2}, \qquad s = a_{ik}/\sqrt{a_{jk}^2 + a_{ik}^2}$$

The number of needed Givens matrices is

$$r = n(n-1)/2$$

and the number of householder matrices is n when $m > n$ *, or n-1 when* $m = n$.

1.1 The Internal Transformations Stage (IT)

During this stage, the rows of the matrix are divided amoung the processors with each processor getting a block of size $(m/p \times n)$ and performing block Householder transformations that we will introduce in below.

1.1.1 Extention Of the Householder Transformation

Let us consider a full rank matrix V and introduce the matrix extension of Householder transformation

$$H(V) = I - 2V(V^TV)^{-1}V^T.$$

Theorem 1.4. *For any full column rank* $(m/p \times r)$ *matrix A,*

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

where A_1 *is an* $r/p \times r$ *nonsingular marix, if we choose*

$$V_A = \begin{bmatrix} A_1 + X \\ A_2 \end{bmatrix}$$

where X is given by

$$X = P^T\sqrt{D}PA_1$$

with $\sqrt{D} = diag_{i=1}{}^r\sqrt{d_i}$ *where the nonnegative scalar* d_i *and the orthogonal matrix P are defined by*

$$I + (A_2A_1^{-1})^T(A_2A_1^{-1}) = P^T diag_{i=1}{}^r d_i P,$$

then

$$H(V_A)A = \begin{bmatrix} -X \\ 0_{(m-r)\times r} \end{bmatrix}$$

where I is an $r \times r$ *identity matrix and* $0_{(m\times r)\times r}$ *is an* $(m \times r) \times r$ *zero matrix.*

Proof. See [2]. ☐

$$\begin{bmatrix} X & X & X & X & X & X & X & X & X & X & X & X \\ 0 & X & X & X & X & X & X & X & X & X & X & X \\ 0 & 0 & X & X & X & X & X & X & X & X & X & X \\ 0 & 0 & 0 & X & X & X & X & X & X & X & X & X \\ 0 & 0 & 0 & 0 & X & X & X & X & X & X & X & X \\ X & X & X & X & X & X & X & X & X & X & X & X \\ 0 & X & X & X & X & X & X & X & X & X & X & X \\ 0 & 0 & X & X & X & X & X & X & X & X & X & X \\ 0 & 0 & 0 & X & X & X & X & X & X & X & X & X \\ 0 & 0 & 0 & 0 & X & X & X & X & X & X & X & X \\ X & X & X & X & X & X & X & X & X & X & X & X \\ 0 & X & X & X & X & X & X & X & X & X & X & X \\ 0 & 0 & X & X & X & X & X & X & X & X & X & X \\ 0 & 0 & 0 & X & X & X & X & X & X & X & X & X \\ 0 & 0 & 0 & 0 & X & X & X & X & X & X & X & X \end{bmatrix}$$

The matrix A after completion of the IT stage. Example using m=15, n=12 and p=3.

1.1.2 Application To The Block QR Decomposition

Let us consider a block matrix

$$M = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}$$

where A_1 is an $r \times r$ nonsingular matrix. Then by choosing $H_1 = H(V_A)$, we obtain

$$H_1 M = \begin{bmatrix} -X & B_1^* \\ 0_{(m-r)\times r} & B_2^* \end{bmatrix}$$

which is an upper block triangular matrix. The Householder transformation matrix can be

$$H_2 = \begin{bmatrix} H_2^1 & 0_{m\times(n-r)} \\ 0_{(m-r)\times r} & H_2^1 \end{bmatrix},$$

where H_2^1 and H_2^2 are the Householder transformation matrices which act, respectly, on X and B_2^*. The application of this result leads to parallelize the QR decomposition of the matrix M. It authorizes to calculate H_2^1 and H_2^2 in two completely independent processors.

1.2 The Balanced Rotations Stage (BR)

In this stage, remaining elements are annihilated by using Givens rotations. This consist of $m/p-1$ steps. During each step, we should consider the following sequences:

- The rows of block 1 are assigned as pivot rows to the processors.
- Processors vanish all the elements in the remaining blocks that are of the same length as their assigned pivot rows. Each step annihilates the first diagonal in block 2 to p.

The next step is to perform both the IT and BR stages on the submatrix as shown in figure 2 by Y entires. The above two stages are repeated to annihilate the elements in blocks 2 to p until the matrix is fully decomposed.

$$
\begin{bmatrix}
X & X & X & X & X & X & X & X & X & X & X & X \\
0 & X & X & X & X & X & X & X & X & X & X & X \\
0 & 0 & X & X & X & X & X & X & X & X & X & X \\
0 & 0 & 0 & X & X & X & X & X & X & X & X & X \\
0 & 0 & 0 & 0 & Y & Y & Y & Y & Y & Y & Y & Y \\
0 & 0 & 0 & 0 & Y & Y & Y & Y & Y & Y & Y & Y \\
0 & 0 & 0 & 0 & Y & Y & Y & Y & Y & Y & Y & Y \\
0 & 0 & 0 & 0 & Y & Y & Y & Y & Y & Y & Y & Y \\
0 & 0 & 0 & 0 & Y & Y & Y & Y & Y & Y & Y & Y \\
0 & 0 & 0 & 0 & Y & Y & Y & Y & Y & Y & Y & Y \\
0 & 0 & 0 & 0 & Y & Y & Y & Y & Y & Y & Y & Y \\
0 & 0 & 0 & 0 & Y & Y & Y & Y & Y & Y & Y & Y \\
0 & 0 & 0 & 0 & Y & Y & Y & Y & Y & Y & Y & Y \\
0 & 0 & 0 & 0 & Y & Y & Y & Y & Y & Y & Y & Y \\
0 & 0 & 0 & 0 & Y & Y & Y & Y & Y & Y & Y & Y
\end{bmatrix}
$$

The matrix A after completion of the BR stage. Example using m=15, n=12 and p=3.

1.3 Details of Load Balancing Technique

Load balancing refers to the practice of distributing works among tasks so that all tasks are kept busy all of the time. It can be considered a minimization of task idle time. One of the ways for load balancing the BR stage is using the Karl Friedrich Gauss idea for obtaining sum of the integers 1 to 100. Applying this idea, the processors assignment is done in a cyclic way. Cyclic length is 2p.

2 Main Result

This paper introduced a new parallel QR decomposition using modest number of processors. Most of the parallel algorithms need at least $m/2$ processors. For the problems with large sizes, this number of processors are not available, but this algorithm could approach to optimal results by using a few processers. We know that Householder transformation can not be parallelized, however we would parallelize each block of matrix utilizing block Householder transformation. Load balancing technique helped to balance the work of each processor. Therefore, these methods for decomposing can reduce computational time and numerical calculation.

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Numerical solution of Schrodinger equation based on the homotopy analysis method

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Abstract

In this paper, we consider a numerical approach to find the approximate solution of a linear or nonlinear Schrodinger equation by using the homotopy analysis method (HAM). For this purpose, we prove two theorems to show the convergence of the HAM for solving this equation in linear or nonlinear case. Then, we solve two examples numerically to illustrate the convergence of the method by the proposed algorithm.

Keywords: Homotopy analysis method, Schrodinger equation, convergence.

Mathematics Subject Classification: 65M99, 35J10

1 Introduction

The Schrodinger equation is the important partial differential equation with many applications in hydrodynamics, optics, chemistry and physics [1-3]. We consider the linear Schrodinger equation of the form,

$$u_t + iu_{xx} = 0, \quad u(x,0) = f(x), \quad i^2 = -1, \tag{1}$$

and nonlinear Schrodinger equation of the form,

$$iu_t + \frac{1}{2}u_{xx} + \gamma|u|^2 u = 0, \quad t \geq 0, \quad u(x,0) = f(x), \quad i^2 = -1. \tag{2}$$

In the HAM, we consider the following differential equation:

$$N[u(x,t)] = 0, \tag{3}$$

where N is a nonlinear operator, x, t denote the independent variables and u is an unknown function. By means of the HAM [4], we construct the zeroth-order deformation equation

$$(1-q)L[\Phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\Phi(x,t;q)], \tag{4}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, L is a linear operator and $H(x,t)$ is an auxiliary function. $\Phi(x,t;q)$ is an unknown function and $u_0(x,t)$ is an initial guess of $u(x,t)$. It is obvious that when $q = 0$ and $q = 1$, we have:

$$\Phi(x,y;0) = u_0(x,t), \qquad \Phi(x,t;1) = u(x,t). \tag{5}$$

By Taylor's theorem, we expand $\Phi(x,t;q)$ in a power series of the embedding parameter q as follows:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(x,t)R_m(\overrightarrow{u}_{m-1}), \tag{6}$$

$$R_m(\overrightarrow{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N\Phi(x,t;q)}{\partial q^{m-1}}\Big|_{q=0} \tag{7}$$

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{8}$$

It should be emphasized that $u_m(x,t)$ for $m \ge 1$ is governed by the linear equation (6) with linear boundary conditions that come from the original problem.

2 Main Result

2.1 Linear form

In order to solve Eq. (1), we consider the initial approximation

$$u_0(x,t) = f(x), \tag{9}$$

nonlinear operator

$$N[\Phi(x,t;q)] = \frac{\partial \Phi(x,t;q)}{\partial t} + i\frac{\partial^2 \Phi(x,t;q)}{\partial x^2}, \tag{10}$$

and the linear operator

$$L[\Phi(x,t;q)] = \frac{\partial \Phi(x,t;q)}{\partial t}. \tag{11}$$

Under the initial condition $u_m(x,t) = 0$ and

$$R_m(\overrightarrow{u}_{m-1}) = \frac{\partial u_{m-1}(x,t)}{\partial t} + i\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2}, \tag{12}$$

the solution of the mth-order deformation equation (6) for $m \ge 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + hH(x,t)L^{-1}[R_m(\overrightarrow{u}_{m-1})]. \tag{13}$$

Theorem 2.1. *If the series solution $u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$ obtained from the HAM of Eq. (1) is convergent then it converges to the exact solution Eq. (1).*

2.2 Nonlinear form

In order to solve Eq. (2) by means of HAM, we consider [1]

$$L(\Phi(x,t;q)] = i\frac{\partial\phi(x,t;q)}{\partial t}, \tag{14}$$

$$N[\Phi(x,t,q)] = i\Phi_t(x,t;q) + \frac{1}{2}\Phi_{xx}(x,t;q) + \gamma|\Phi(x,t;q)|^2\Phi(x,t;q). \tag{15}$$

$$u_m(x,t) = \chi_m u_{m-1}(x,t) - ihH(x,t)\int R_m(\overrightarrow{u}_{m-1})dt + c_1(x), \tag{16}$$

$$R_m(\overrightarrow{u}_{m-1}) = i\frac{\partial u_{m-1}}{\partial t} + \frac{1}{2}\frac{\partial^2 u_{m-1}}{\partial x^2} + \gamma\sum_{i=0}^{m-1}\sum_{k=0}^{m-i-1}|u_i||u_k|u_{m-k-i-1}, \tag{17}$$

Theorem 2.2. *If the series solution $u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$ obtained from the HAM of Eq. (2) is convergent then it converges to the exact solution of Eq. (2).*

3 Numerical examples

In this section, we solve a linear and a nonlinear Schrodinger equation via the HAM numerically by applying the following algorithm where ϵ is a given positive value and *sum* is the approximate value of the solution for equations (1) or (2) at the given point (x,t). The programs have been provided by Maple.

Algorithm 1.
1) Read n, $x \in \mathbb{R}$, $t \in [0,T]$ and $f(x)$,
2) Put $sum = 0$ and $u(x,0) = f(x)$,
3) For $m = 1(1)n$ do
 3.a) Evaluate $u_m(x,t)$ via (13) for equation (1) or via (16) for equation (2),
 3.b) Set $sum = sum + u_m(x,t)$,
4) write n and *sum*.

Example 3.1. *Consider (1) with initial condition [1]*

$f(x) = 1 + cosh(2x).$

We assume $h = -1$ and $H(x,t) \equiv 1$. Table 1 shows the results of this method for the points $(x_1,t_1) = (\frac{\pi}{2}, 0.25)$, $(x_2,t_2) = (\frac{3\pi}{2}, 0.75)$ for $n = 2,4,8,15$ and 20.

Table 1

	$(\frac{\pi}{2}, 0.25)$	$(\frac{3\pi}{2}, 0.75)$
$n=2$	$6.795976640 - 11.59195328i$	$-21684.38383 - 18587.47186i$
$n=4$	$7.278974697 - 9.659961066i$	$-773.4779940 - 9293.735934i$
$n=8$	$7.263162264 - 9.754260686i$	$-6038.545359 - 564.2625349i$
$n=15$	$7.263159088 - 9.754292341i$	$-6132.831604 - 874.3525262i$
$n=20$	$7.263159088 - 9.754292336i$	$-6132.819225 - 874.3547170i$

Example 3.2. *Consider the nonlinear Schrodinger equation with zero trapping potential [1],*

$$iu_t = -\frac{1}{2}u_{xx} - |u|^2 u, \quad t \geq 0, \tag{18}$$

under the initial condition

$$u(x,0) = e^{ix}. \tag{19}$$

We assume $h = -1$ *and* $H(x,t) \equiv 1$. *Table 2 shows the results of this method for the points* $(x_1, t_1) = (\frac{\pi}{2}, 0.25)$, $(x_2, t_2) = (\frac{3\pi}{2}, 0.75)$ *for* $n = 2, 4, 6$ *and* 8.

Table 2

	$(\frac{\pi}{2}, 0.25)$	$(\frac{3\pi}{2}, 0.75)$
$n = 2$	$-0.120000000 + 0.9921875000i$	$0.3750000000 - 0.9296875000i$
$n = 4$	$-0.1246744792 + 0.9921875000i$	$0.3662109375 - 0.9305114750i$
$n = 6$	$-0.1246747335 + 0.9921976670i$	$0.3662727356 - 0.9305076120i$
$n = 8$	$-0.1246747334 + 0.9921976670i$	$0.3662725287 - 0.9305076220i$

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A numerical method for solving nonstiff Volterra integro-differential equations

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Abstract

In this paper, we introduce a family of extended multistep collocation methods by using super-future points for Volterra integro-differential equations, with the aim of increasing the order of classical one-step collocation methods and multistep collocation methods without increasing the computational cost. We discuss the order of the constructed methods and present the stability analysis.

Keywords: Collocation methods, Volterra integro-differential equations, Stability.

Mathematics Subject Classification: 65R20

1 Introduction

we consider Volterra Integro-Differential Equations (VlDEs) of the form

$$y'(t) = g(t, y(t)) + \int_0^t K(t, \tau, y(\tau))d\tau, \qquad t \in I := [0, T], \qquad y(0) = y_0, \tag{1}$$

where the known function g and the kernel K are assumed sufficiently smooth.

One of the most popular numerical methods for solving this kind of equations are collocation methods which the approximate solution in every subinterval depends on the fix number m of the collocation points. These methods are of convergence order m for any choice of collocation parameters.

Recently, a family of multistep collocation methods for (1) are proposed, which are constructed by adding interpolation conditions in the previous r step points [5].

We are interested in a new class of multistep collocation methods, which depends on a fixed number r of previous time steps and $2m$ collocation points in the previous and next subintervals, in order to construct higher order methods with extensive stability regions.

A similar method has been successfully applied for solving ODEs and nonlinear VIEs [7].

2 Construction of method

Let $I_h = \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\}$ be a partition of the interval $[0, T]$ with constant stepsize $h := t_{n+1} - t_n$, $n = 0, 1, \cdots, N-1$. In this section, we describe construction of the new methods. These methods compute the approximated solution of (1) in the subinterval $[t_n, t_{n+1}]$ by using the approximated values of the solution in the r previous steps and its first derivative in the m collocation points in the subinterval $[t_n, t_{n+1}]$ and the same number of collocation points in the next subinterval $[t_{n+1}, t_{n+2}]$ are used.

To construct the methods, we seek a collocation polynomial of the form

$$u_n(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + h \sum_{j=1}^{m} \psi_j(s) U_{n,j} + h \sum_{j=1}^{m} \chi_j(s) U_{n+1,j}, \quad n = r-1, \cdots, N-1, \quad (2)$$

where

$$\begin{aligned} U_{n,j} &= u_n'(t_{n,j}), \\ U_{n+1,j} &= u_n'(t_{n+1,j}). \end{aligned} \quad (3)$$

3 Linear stability analysis

In this section, we analyze the stability properties of the new method with respect to the basic test equation []

$$y'(t) = g(t) + \xi y(t) + \eta \int_0^t y(\tau) d\tau, \quad t > 0, \qquad y(0) = y_0, \quad (4)$$

where $\xi, \eta \in \mathbb{C}$. The solution of (4) is stable if $Re(\lambda_1), Re(\lambda_2) < 1$ where $\lambda_{1,2} = (\xi \pm \sqrt{\xi^2 + 4\eta})/2$. We observe that, in particular, if ξ and η are real, this condition reduces to $\xi, \eta < 0$. As usual, we look for sufficient conditions for the stability of the the numerical solution of (4).

To state the main results of stability properties of the method, Let us define

$$\hat{\varphi}_k = \int_0^1 \varphi_k(s) ds, \quad \hat{\psi}_j = \int_0^1 \psi_j(s) ds, \quad \hat{\chi}_j = \int_0^1 \chi_j(s) ds,$$

$$\bar{\varphi}_{ik} = \int_0^{c_i} \varphi_k(s) ds, \quad \bar{\psi}_{ij} = \int_0^{c_i} \psi_j(s) ds, \quad \bar{\chi}_{ij} = \int_0^{c_i} \chi_j(s) ds,$$

$$\tilde{\varphi}_{ik} = \varphi_k(1 + c_i), \quad \tilde{\psi}_{ij} = \psi_j(1 + c_i), \quad \tilde{\chi}_{ij} = \chi_j(1 + c_i),$$

$$\varphi(1) = [\varphi_0(1), \cdots, \varphi_{r-1}(1)]^T, \quad \psi(1) = [\psi_1(1), \cdots, \psi_m(1)]^T,$$

$$\chi(1) = [\chi_1(1), \cdots, \chi_m(1)]^T, \quad u = [1, 1, \cdots, 1]^T \in \mathbb{R}^m,$$

and consider the matrices

$$E = \begin{bmatrix} \multicolumn{2}{c}{\varphi(1)^T} \\ \mathbf{I}_{r-1} & \mathbf{0}_{r-1,1} \end{bmatrix}, \quad F = \begin{bmatrix} \psi(1)^T \\ \mathbf{0}_{r-1,m} \end{bmatrix}, \quad G = \begin{bmatrix} \chi(1)^T \\ \mathbf{0}_{r-1,m} \end{bmatrix},$$

$$\mathbf{U}_n = [U_{n,1}, \cdots, U_{n,m}]^T, \quad \mathbf{y}_n^{(r)} = [y_n, \cdots, y_{n-r+1}]^T.$$

Theorem 3.1. *Applying the test equation (4) to the new method, leads to the following recurrence relation*

$$\begin{bmatrix} h\mathbf{U}_{n+1} \\ h\mathbf{U}_n \\ \mathbf{y}_{n+1}^{(r)} \end{bmatrix} = R(z,w) \begin{bmatrix} h\mathbf{U}_n \\ h\mathbf{U}_{n-1} \\ \mathbf{y}_n^{(r)} \end{bmatrix} + \mathbf{G}_n,$$

where $z := \xi h$, $w = \eta h^2$,

$$R(z,w) = [Q(z,w)]^{-1} M(z,w)$$

and

$$Q(z,w) = \left[\begin{array}{c|c|c} \mathbf{I}_m - z\tilde{\chi} - w\hat{\chi} - w\bar{\chi} & -z\tilde{\psi} - w\hat{\psi} - w\bar{\psi} & -z\tilde{\varphi} - w\hat{\varphi} - w\bar{\varphi} \\ \hline \mathbf{0}_{m,m} & \mathbf{I}_m & \mathbf{0}_{m,r} \\ \hline \mathbf{0}_{r,m} & \mathbf{0}_{r,m} & \mathbf{I}_r \end{array} \right],$$

$$M(z,w) = \left[\begin{array}{c|c|c} \mathbf{I}_m - z\tilde{\chi} - w * \bar{\chi} & -z\tilde{\psi} - w\bar{\psi} & -z\tilde{\varphi} - w\bar{\varphi} \\ \hline \mathbf{I}_m & \mathbf{0}_{m,m} & \mathbf{0}_{m,r} \\ \hline G & F & E \end{array} \right].$$

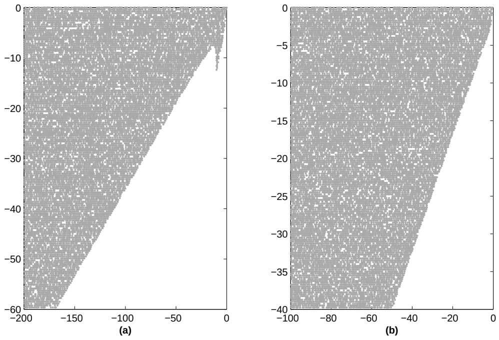

Figure 1: Stability regions in the (z,w)-plane with (a)$r = 2$, $m = 2$, $c = [\frac{7}{10}, 1]$, (b)$r = 3$, $m = 2$, $c = [\frac{7}{10}, 1]$

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Solving stiff system of fractional differential equations by fractional complex transform

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Abstract

In this paper, we apply fractional complex transform to convert the stiff systems of fractional differential equations to the stiff systems of differential equations. Then, we find solutions for them by variational iteration method (VIM).

Keywords: Variational iteration, Fractional complex transform, Stiff system of differential equation, Jumarie's derivative

Mathematics Subject Classification: 34A05, 34A15

1 Introduction

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering. The fractional complex transform was first proposed by He and Li [1]. We extend the fractional complex transform method [2] to solve the stiff systems of fractional ordinary differential equations (ODEs).

2 Main Result

Jumarie's derivative [3] is a modified Riemann -Liouville derivative defined as

$$D_z^{\gamma} f(z) = \begin{cases} \frac{1}{\Gamma(-\gamma)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z (z-\tau)^{-\gamma-1} (f(\tau) - f(0)) \mathrm{d}\tau, & \gamma < 0, \\ \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z (z-\tau)^{-\gamma} (f(\tau) - f(0)) \mathrm{d}\tau, & 0 < \gamma < 1, \\ \left(f^{(\gamma-n)}(z) \right)^{(n)}, & n \leq \gamma < n+1, \quad n \geq 1, \end{cases} \tag{1}$$

where $f(z)$ is a real continuous (but not necessarily differentiable) function. The fundamental mathematical operations and results of Jumarie's derivative are given in [3]. In this section, we review some of them.

$$D_z^{\gamma} c = 0, \quad \gamma > 0, \quad c = \text{constant}.$$

$$D_z^{\gamma}\Big(c\, f(z)\Big) = c\, D_z^{\gamma} f(z), \quad \gamma > 0, \quad c = \text{constant}.$$

$$D_z^{\gamma} z^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\gamma)} z^{\beta-\gamma}, \quad \beta > \gamma > 0.$$

$$D_z^{\gamma}\Big(f(z)\, g(z)\Big) = \Big(D_z^{\gamma} f(z)\Big)\, g(z) + f(z)\, \Big(D_z^{\gamma} g(z)\Big).$$

$$D_z^{\gamma}\Big(f(z(t))\Big) = f'_z(z).z^{(\gamma)}(t) = f_z^{(\gamma)}(z)(z'_t)^{\gamma}.$$

3 Examples

The fractional complex transform [1] can convert a fractional differential equation (FDEs) into its differential partner.
We consider linear and nonlinear stiff systems of FDEs. In this section, α is a parameter describing the order of the fractional Jumarie's derivative [3].

Example 3.1. *Consider the linear system of FDEs*

$$D_t^{\alpha}\mathbf{x}(t) = \begin{pmatrix} D_t^{\alpha}x_1(t) \\ D_t^{\alpha}x_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 95 \\ -1 & -97 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad 0 < \alpha \leq 1 \tag{2}$$

with initial conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3}$$

By the fractional complex transform

$$T = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{4}$$

and using Jumarie's chain rule [3], we have

$$D_t^{\alpha}\mathbf{x}(t) = D_T\mathbf{x}(t)\frac{d^{\alpha}T}{dt^{\alpha}}, \tag{5}$$

$$D_T\mathbf{x}(t) = \begin{pmatrix} D_T x_1(T) \\ D_T x_2(T) \end{pmatrix} = \begin{pmatrix} -1 & 95 \\ -1 & -97 \end{pmatrix}\begin{pmatrix} x_1(T) \\ x_2(T) \end{pmatrix}, \tag{6}$$

with initial conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{7}$$

The exact solution is

$$\begin{cases} x_1(T) = \frac{1}{47}(95e^{-2T} - 48e^{-96T}), \\ x_2(T) = \frac{1}{47}(48e^{-96T} - 48e^{-2T}). \end{cases} \tag{8}$$

Hence,

$$\begin{cases} x_1(t) = \frac{1}{47}(95e^{\frac{-2t^{\alpha}}{\Gamma(1+\alpha)}} - 48e^{\frac{-96t^{\alpha}}{\Gamma(1+\alpha)}}), \\ x_2(t) = \frac{1}{47}(48e^{\frac{-96t^{\alpha}}{\Gamma(1+\alpha)}} - 48e^{\frac{-2t^{\alpha}}{\Gamma(1+\alpha)}}). \end{cases}$$

The variational iteration formulae can be easily constructed as follows [4]:

$$\begin{aligned} x_1^{(n+1)}(T) &= x_1^{(n)}(T) - \int_0^T e^{S-T}\Big(x_1'^{(n)}(S) + x_1^{(n)}(S) - 95x_2^{(n)}(S)\Big)dS, \\ x_2^{(n+1)}(T) &= x_2^{(n)}(T) - \int_0^T e^{97(S-T)}\Big(x_2'^{(n)}(S) + x_1^{(n)}(S) + 97x_2^{(n)}(S)\Big)dS. \end{aligned} \tag{9}$$

From [4], we have

$$\begin{aligned} x_1^{(2)}(T) &= \frac{-41587200 - 442415\,e^{-97T} + 97e^{-T}(437903 + 428640\,T)}{446976}, \\ x_2^{(2)}(T) &= \frac{-1996185600 - 9409\,e^{-T}(216719 + 214320\,T) + \gamma}{2081120256}, \end{aligned}$$

where $\gamma = e^{-97T}(2124043727 + 2059884240\,T)$. *Hence,*

$$\begin{aligned} x_1^{(2)}(t) &= \frac{-41587200 - 442415\,e^{-97t/\Gamma(1+\alpha)} + \beta}{446976}, \\ x_2^{(2)}(t) &= \frac{-1996185600 - 9409\,\omega + \gamma}{2081120256}, \end{aligned}$$

where
$\beta = 97e^{-t/\Gamma(1+\alpha)}(437903 + 428640\,t/\Gamma(1+\alpha))$,
$\omega = e^{-t/\Gamma(1+\alpha)}(216719 + 214320\,t/\Gamma(1+\alpha))$ *and*
$\gamma = e^{-97t/\Gamma(1+\alpha)}(2124043727 + 2059884240\,t/\Gamma(1+\alpha))$.

4 Conclusion

The fractional complex transform is very simple and the use of this method does not need the knowledge of fractional calculus.

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J-Normal matrices

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Abstract

In this work we give some properties of J-normal matrices. In particular, a list of nearly 20 conditions is given, each is equivalent to matrix A being J-normal.

Keywords: J-normal matrix, J-unitary matrix, J-Hermitian matrix, indefinite inner product

Mathematics Subject Classification: 15A24, 15B99

1 Introduction

For $J = I_r \oplus (-I_{n-r}) = diag(1,\dots,1,-1,\dots,-1)$, $0 < r < n$, concider $\mathbb{C}^n$ endowed with an indefinite inner product $[\cdot,\cdot]$ defined by $[x, y] =< Jx, y >$, where $< x, y >= y^*x$. Let $M_n(\mathbb{C})$ denote the algebra of $n \times n$ matrices over the field $\mathbb{C}$ of complex numbers. A matrix $A \in M_n(\mathbb{C})$ is said to be J-normal if $A^{[*]}A = AA^{[*]}$, where $A^{[*]} = JA^*J$ denotes the J-adjoint of A. A matrix $A \in M_n(\mathbb{C})$ which equals to its J-adjoint is called J-Hermitian. A J-normal matrix $U \in M_n(\mathbb{C})$ is said to be J-unitary if $UU^{[*]} = U^{[*]}U = I_n$.(see [1] and [3])

In this paper we present a list of conditions on $A \in M_n(\mathbb{C})$, each of which is equivalent to A being J-normal, except that some involve indicated restrictions on A (e.g. nonsingularity). Section 2 of this paper contains nearly 20 such conditions. The outline of proofs and/or comments to most of the conditions are given in Section 3.

We conclude this section by introducing the notation. We use polar decomposition of A into a unitary times a positive semidefinite matrix and the J-Toeplitz decomposition

$$A = H + iK, \quad H = H^{[*]}, \quad K = K^{[*]}$$

with

$$H = \tfrac{1}{2}(A + A^{[*]}), \quad K = \tfrac{1}{2i}(A - A^{[*]}).$$

2 Conditions

1. $p(A)$ is J-normal for any polynomial.
2. A^T is J-normal.
3. $\bar{A}$ is J-normal.
4. $A^{[*]}$ is J-normal.
5. A^{-1} is J-normal (for invertible A).
6. $A^{-1}A^{[*]}$ is J-unitary (for invertible A).
7. $A = A^{[*]}AA^{[*]^{-1}}$ (for invertible A).
8. A commutes with $A^{-1}A^{[*]}$ (for invertible A).
9. $U^{[*]}AU$ is J-normal for any J-unitary U.

(Henceforth, let $H = \frac{1}{2}(A + A^{[*]})$, $K = \frac{1}{2}(A - A^{[*]})$.)

10. $HK = KH$.
11. $AH = HA$.
12. $AH + HA^{[*]} = 2H^2 (= HA + A^{[*]}H)$.
13. $AK = KA$.
14. $AK - KA^{[*]} = 2K^2 (= KA - A^{[*]}K)$.
15. $H^{-1}A + A^{[*]}H^{-1} = 2I (= AH^{-1} + H^{-1}A^{[*]})$(as long as H is nonsingular).
16. $K^{-1}A - A^{[*]}K^{-1} = 2I (= AK^{-1} - K^{-1}A^{[*]})$(as long as K is nonsingular).
17. $H^2 - K^2 = A^{[*]}A (or AA^{[*]})$.

(Henceeforth, assume $A = VP$ is a polar decomposition of A such that V and P commute with J.)

18. $VP = PV$.
19. $AV = VA$.
20. $AP = PA$.
21. In the Toeplitz decompositon of A,

$$H^2 + K^2 = A^{[*]}A (or AA^{[*]}).$$

3 Proofs and commments

Conditions 4, 7, 8 can easily be seen to be equivalent to A being J-normal by resorting to the defining equation for J-normality and elementary properties of $A^{[*]}$.

Proof of condition 1. Let $p(A) = a_nA^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I$. We have

$$(AB)^{[*]} = J(AB)^*J = JB^*A^*J = JB^*JJA^*J = B^{[*]}A^{[*]},$$
$$(A^K)^{[*]} = (A\dots A)^{[*]} = J(A\dots A)^*J = JA^*\dots A^*J = (JA^*J)\dots(JA^*J) = (A^{[*]})^K,$$
$$(A+B)^{[*]} = J(A+B)^*J = J(A^*+B^*)J = JA^*J + JB^*J = A^{[*]} + B^{[*]},$$

for $B \in M_n(\mathbb{C})$ and $K \geq 1$. Let A is J-normal, then

$$A^K(A^K)^{[*]} = A^K(A^{[*]})^K = A\dots AA^{[*]}\dots A^{[*]} = \underbrace{(AA^{[*]})\dots(AA^{[*]})}_{K\ times} = (AA^{[*]})^K$$
$$= (A^{[*]}A)^K = A^{[*]}A\dots A^{[*]}A = A^{[*]}\dots A^{[*]}A\dots A = (A^{[*]})^KA^K = (A^K)^{[*]}A^K$$

for $K \geq 1$, and if $K_1 > K_2$ then

$$A^{K_1}(A^{K_2})^{[*]} = A^{K_1-K_2}(A^{K_2}(A^{K_2})^{[*]}) = A^{K_1-K_2}((A^{K_2})^{[*]}A^{K_2}) = A^{K_1-K_2-1}A(A^{[*]})^{K_2}A^{K_2}$$
$$= A^{K_1-K_2-1}(A^{[*]})^{K_2}AA^{K_2} = \cdots = (A^{[*]})^{K_2}A^{K_1} = (A^{K_2})^{[*]}A^{K_1}.$$

Hence $p(A)$ is J-normal.
Proof of condition 2. We have

$$(A^T)^{[*]} = J(A^T)^*J = J(A^*)^TJ = (JA^*J)^T = (A^{[*]})^T.$$

Thus,
$$(A^T)^{[*]}A^T = A^T(A^T)^{[*]} \Leftrightarrow (A^{[*]})^TA^T = A^T(A^{[*]})^T \Leftrightarrow AA^{[*]} = A^{[*]}A.$$

Proof of condition 3. We have

$(\bar{A})^{[*]} = J(\bar{A})^* J = (\overline{JA^*J}) = (\overline{A^{[*]}})$.
Thus,
$(\bar{A})^{[*]}\bar{A} = \bar{A}(\bar{A})^{[*]} \Leftrightarrow (\overline{A^{[*]}})\bar{A} = \bar{A}(\overline{A^{[*]}}) \Leftrightarrow AA^{[*]} = A^{[*]}A$.
Proof of condition 5. We have

$AA^{-1} = I \Rightarrow (AA^{-1})^{[*]} = I$,
and
$(AA^{-1})^{[*]} = J(AA^{-1})^* J = J(A^{-1})^* A^* J = J(A^{-1})^* JJA^* J = (A^{-1})^{[*]}A^{[*]}$.
So
$(A^{[*]})^{-1} = (A^{-1})^{[*]}$.
Thus,
$A^{-1}(A^{-1})^{[*]} = (A^{-1})^{[*]}A^{-1} \Leftrightarrow A^{[*]}A^{-1}(A^{-1})^{[*]} = A^{-1} \Leftrightarrow AA^{[*]}A^{-1}(A^{-1})^{[*]} = I \Leftrightarrow$
$AA^{[*]}A^{-1} = A^{[*]} \Leftrightarrow AA^{[*]} = A^{[*]}A$.
Proof of condition 6.

$(A^{-1}A^{[*]})(A^{-1}A^{[*]})^{[*]} = I \Leftrightarrow (A^{-1}A^{[*]})(A(A^{-1})^{[*]}) = I \Leftrightarrow A^{[*]}A(A^{[*]})^{-1} = A \Leftrightarrow$
$A^{[*]}A = AA^{[*]}$.
Proof of condition 9.

$(U^{[*]}AU)(U^{[*]}AU)^{[*]} = (U^{[*]}AU)^{[*]}(U^{[*]}AU) \Leftrightarrow U^{[*]}AA^{[*]}U = U^{[*]}A^{[*]}AU \Leftrightarrow$
$AA^{[*]} = A^{[*]}A$.
Proof of condition 10.

$HK = KH \Leftrightarrow \frac{1}{4}(A + A^{[*]})(A - A^{[*]}) = \frac{1}{4}(A - A^{[*]})(A + A^{[*]}) \Leftrightarrow A^{[*]}A - AA^{[*]}$
$= AA^{[*]} - A^{[*]}A \Leftrightarrow AA^{[*]} = A^{[*]}A$.
Proof of condition 11.

$AH = HA \Leftrightarrow \frac{1}{2}A(A + A^{[*]}) = \frac{1}{2}(A + A^{[*]})A \Leftrightarrow A^2 + AA^{[*]} = A^2 + A^{[*]}A \Leftrightarrow$
$AA^{[*]} = A^{[*]}A$.
Proof of condition 12.

$AH + HA^{[*]} = 2H^2 \Leftrightarrow \frac{1}{2}A(A + A^{[*]}) + \frac{1}{2}(A + A^{[*]})A^{[*]} = \frac{1}{2}(A + A^{[*]})^2 \Leftrightarrow AA^{[*]} = A^{[*]}A$.
Proof of condition 13.

$AK = KA \Leftrightarrow \frac{1}{2}A(A - A^{[*]}) = \frac{1}{2}(A - A^{[*]})A \Leftrightarrow A^2 - AA^{[*]} = A^2 - A^{[*]}A \Leftrightarrow$
$AA^{[*]} = A^{[*]}A$.
Proof of condition 14.

$AK - KA^{[*]} = 2K^2 \Leftrightarrow \frac{1}{2}A(A - A^{[*]}) - \frac{1}{2}(A - A^{[*]})A^{[*]} = \frac{1}{2}(A - A^{[*]})^2 \Leftrightarrow AA^{[*]} = A^{[*]}A$.

Conditions 15 and 16 are respectively equivalent to 12 and 14 for the cases H nonsingular, K nonsingular.

Proof of condition 17.

$H^2 - K^2 = AA^{[*]} \Leftrightarrow \frac{1}{4}(A + A^{[*]})(A + A^{[*]}) - \frac{1}{4}(A - A^{[*]})(A - A^{[*]}) = AA^{[*]} \Leftrightarrow$
$\frac{1}{2}(A^{[*]}A + AA^{[*]}) = AA^{[*]} \Leftrightarrow A^{[*]}A = AA^{[*]}$.
Proof of condition 18. $PJ = JP$ implies that $P^*J = JP^*$, thus $P^{[*]}P = JP^*JP = P^*JJP = P^*P = P^2$. also, $VJ = JV$ implies that $V^{[*]} = V^*$ and $VV^{[*]} = VV^* = I$. If V commutes with P, then $AA^{[*]} = VPP^{[*]}V^{[*]} = VP^2V^{[*]} = P^2VV^{[*]} = P^2$ and $A^{[*]}A = P^{[*]}V^{[*]}VP = P^{[*]}P = P^2$, so A is J-normal. If A is J-normal, then $AA^{[*]} = A^{[*]}A$ implies that $P^2 = VP^2V^{[*]}$. Observe that,

P^2 and $VP^2V^{[*]}$ are both positive semidefinite square matrices with obvious respective positive semidefinite square roots P and $VPV^{[*]}$. But theorem (7.2.6) of [2] says that such a square root is unique, so $P = VPV^{[*]}$, or $VP = PV$.

Conditions 19 and 20 are clearly equivalent to condition 18.

Proof of condition 21. Condition 21 is an immediate consequence of the identity,

$H^2 + K^2 = \frac{1}{2}(AA^{[*]} + A^{[*]}A)$
which holds for any square A.

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On the order of weighted approximation of unbounded functions

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Abstract

The generalization of Gadjiev -Ibragimov operators by means of Kantorovich and Kirov-Popova is considered and the order of weighted approximation of unbounded functions by them are obtained.

Keywords: weighted modulus of continuity, Gadjiev-Ibargimov operators, weighted space, linear positive operators.

Mathematics Subject Classification: 41A36, 41A58.

1 Introduction

Korovkin and some other authors have developed results for the approximation of bounded and unbounded function by sequence of positive linear operators. In [7], a sequence of linear positive operators was established which named Gadjiev-Ibragimov operators and it contains of well-known operators as Bernstein, Szasz, Bernstein-Cholodwsky and Baskakov. These operators were investigated by many authors, for example [8],[5],[9],[2] and [4]. In fact, Gadjiev-Ibragimov operator is based on Taylor expansion of function $K_n(x,t,u)$ which $K_n(x,t,u)$ be a sequence of entire functions with respect to $u \in \mathbb{R}$, for fixed $x,t \in [0,A],\ A > o$. In [5], Gadjiev-Ibragimov operators generalized to unbounded function defined on semi axis $\mathbb{R}_0 = [0,\infty)$.

We consider Kirov-Popova [10] and Kantorovich type [8] generalization of Gadjiev-Ibragimov operators as following form and then obtain the degree of weighted approximation of unbounded function by these operators.

- Kirov-Popova generalization :

$$G_n^{[r]}(f;x) = \sum_{\upsilon=0}^{\infty}\sum_{i=0}^{r} f^{(i)}\left(\frac{\upsilon}{n^2\psi_n(0)}\right)\frac{(x-\frac{\upsilon}{n^2\psi_n(0)})^i}{i!}\,p_{\upsilon,n}(x), \tag{1}$$

  where $p_{\upsilon,n}(x) = K_n^{(\upsilon)}(x,t,u)\frac{(-\alpha_n\psi_n(0))^\upsilon}{\upsilon!}$ and $f \in C^{(r)}(\mathbb{R}_0)$ and $x \in \mathbb{R}_0$. These operators is called the r-th order of the family (1)where $K_n(x,t,u)$ be sequence of entire function of variables $x,t,u \in \mathbb{R}_0$ has following conditions:

  - $K_n(x,0,0) = 1$, for every $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$,
  - $\left\{(-1)^\upsilon \frac{\partial^\upsilon K_n(x,t,u)}{\partial u^\upsilon}|_{u=u_1;t=0}\right\} \geq 0,\ \ (\upsilon \in \mathbb{N}\cup\{0\},\ x \in \mathbb{R}_0)$
  - $\frac{\partial^\upsilon}{\partial u^\upsilon}K_n(x,t,u)|_{u=u_1;t=0} = -nx\left[\frac{\partial^{\upsilon-1}}{\partial u^{\upsilon-1}}K_{n+m}(x,t,u)|_{u=u_1;t=0}\right]$ $\upsilon \in \mathbb{N}$ and for any fixed $u = u_1$

where $n+m$ is natural number and m is a constant independent of υ. Also $\varphi_n(x)$ and $\psi_n(x)$ are two sequences of the class $C(\mathbb{R}_0)$ such as $\varphi_n(0)=0$, for any $t\in\mathbb{R}_0$, $\psi_n(t)>0$ and

$$\lim_{n\to\infty}\frac{1}{n^2\psi_n(0)}=0,$$

and $\{\alpha_n\}$ is a sequence of positive numbers satisfying the condition

$$\frac{\alpha_n}{n}=1+O(\frac{1}{n^2\psi_n(0)}).$$

- Kantorovich type generalization:

$$G_n^*(f;x)=n^2\psi_n(0)\sum_{\upsilon=0}^{\infty}P_\upsilon(\alpha_n,\psi_n;K_n)\int_{I_{n,\upsilon}}f(t)dt \tag{2}$$

where $I_{n,\upsilon}:=[\frac{\upsilon}{n^2\psi_n(0)},\frac{\upsilon+1}{n^2\psi_n(0)}]$, $n\in\mathbb{N},\upsilon\in\mathbb{N}\cup\{0\}$,$P_\upsilon(\alpha_n,\psi_n;K_n)=$ $(\frac{\partial^\upsilon}{\partial u^\upsilon}K_n(x,t,u)|_{\substack{u=\alpha_n\psi_n(0)\\ t=0}})\frac{(-\alpha_n\psi_n(t))}{\upsilon!}$ and $f\in\mathcal{M}_{loc}(\mathbb{R}_0)$, the class of all measurable functions on $\mathbb{R}_0$ and bounded on every compact subinterval of $\mathbb{R}_0$ (see[8]).

2 Main Result

Theorem 2.1. *Let $G_n^{[r]}(f;x)$ be family of operators defined as (3) and $f\in C_\rho^0(\mathbb{R}_0)$. Then for a sufficiently large n the inequality*

$$\sup_{x\in\mathbb{R}_0}\frac{|L_n^{[r]}(f;x)-f(x)|}{(1+x^2)(1+x^{r+3})}\leq K\eta(f^{(r)};\frac{1}{n^2\psi_n(0)})$$

where K is positive constant and $\eta(f;\delta)$ be weighted modulus of continuity.

Theorem 2.2. *If $f\in C_\rho^0(\mathbb{R}_0)$, then the inequality*

$$\sup_{x\in[0,\infty)}\frac{|G^*(f;x)-f(x)|}{(1+x^2)(1+x^{\frac{7}{2}})}\leq K\eta(f;\frac{1}{\sqrt{n^2\psi_n(0)}})$$

where K is a positive constant.

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Price dynamics and dividend structure

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Abstract
In this article we will introduce concept of market and briefly explain the Black Scholes model for option pricing and give price dynamics for dividend structure
Keywords: Black-Scoles model, Option pricing, Stochastic differential equations

1 Introduction and Preliminaries

Let $W_1(t), W_2(t), ..., W_m(t)$ be m independent Wiener prosses defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\{\mathcal{F}_t\}_{t\in I}$ be the filtration given by $\mathcal{F}_t = \sigma(W_j(s) : s \leq t, 1 \leq j \leq m)$

A market is an $R^{n+1}-$valued Ito process $X_t = (X_t^\circ, X_t^1, ..., X_t^n)$ for $\circ \leq t \leq T$ and the components are specified by

$$dX_t^\circ = \gamma(t)X_t^\circ dt, \quad X_\circ^\circ = 1; \tag{1.1}$$

$$dX_t^{(i)} = \sum_{j=1}^{m} \sigma_{ij}(t)dW_j(t) + f_i(t)dt, \quad 1 \leq i \leq n, \tag{1.2}$$

with adapted stochastic processes $\gamma(t), \sigma_{ij}(t)$, and $f_i(t)$ satisfying conditions to be specified later.

Here $X_t^{(\circ)}$ refers to the unit price of the safe investment and $X_t^{(i)}, 1 \leq i \leq n$, refers to the unit price of the ith risky investment. In practial problems, the stochastic processes $X_t^{(i)}, 1 \leq i \leq n$, are solutions of stochastic differential equations. Since the solutions are Ito processes, they can be written in the form of Equation (1.2). The solution of Equation (1.1) is given by

$$X_t^{(\circ)} = e^{\int_\circ^t \gamma(s)ds}, \tag{1.3}$$

Define

$$\widehat{X}_t = (X_t^1, X_t^2, ..., X_t^n), \tag{1.4}$$

which is the risky part of the investment. A portfolio p(t) is said to be self-financing if its value $V_p(t)$ satisfies the following equality

$$V_p(t) = V_p(0) + \int_0^t p(s).dX_s,$$

This equation means that no money is brought in or taken out from the system at any time, a fact suggesting the term self-financing for the portfolio.
Let us now list some popular claims and see which of them will fall into the framework above.

$$\mathcal{X} = max\{S(T) - K, 0\} \qquad (European\ call\ option)$$

$$\mathcal{X} = max\frac{1}{T}\int_0^T S(t)dt - K, 0 \quad (Asian\ option).$$

2 Main results

Theorem 2.1. *(Black-Scoles Equation) Assume that the market is specified* $dB(t) = rB(t)dt$

$$dS(t) = \alpha(t, S(t))S(t)dt + \sigma(t, S(t))S(t)dW(t)$$

and that we want to price a contingent claim of the form $\mathcal{X} = \Phi(S(T))$. *Then the only pricing function of the form* $\Pi(t) = F(t, S(t))$ *which is consistent with the absence of arbitrage is when F is the solution cf the following boundary value problem in the domain* $[0, T] \times R^+$,

$$F_t(t, s) + rsF_s(t, s) + \frac{1}{2}s^2\sigma^2(t, s)F_{ss}(t, s) - rF(t, s) = 0,$$

$$\textit{such that} \quad F(T, s) = \Phi(s).$$

We consider an underlying asset ("the stock") with price process S, over a fixed time interval [0,T] and dividend function $\delta[s]$. We take as given a number of deterministic points in time, $T_1, ..., T_n$ where

$$0 < T_n < T_{n-1} < ... < T_1 < T.$$

Theorem 2.2. *The pricing function F(t, s) is determined by the following recursive procedure:*
i) On the interval $[T_1, T]$ *F solves the boundary value problem*

$$F_t(t, s) + rsF_s(t, s) + \frac{1}{2}s^2\sigma^2(t, s)F_{ss}(t, s) - rF(t, s) = 0,$$

$$\textit{such that} \quad F(T, s) = \Phi(s).$$

ii) On every half-open interval $[T_{i+1}, T_i)$ *we have* $F(t, s) = F^i(t, s)$ *for i=1,2,... ,n - 1, over the closed interval* $[T_{i+1}, T_i]$, *solves the boundery value problem*

$$F^i_t(t, s) + rsF^i_s(t, s) + \frac{1}{2}s^2\sigma^2(t, s)F^i_{ss}(t, s) - rF^i(t, s) = 0,$$

$$F^i(T_i, s) = F^{i-1}(T_i, s - \delta[s])$$

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Analytical solution for the generalized Kuramoto-Sivashinsky equation by differential transform method

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Abstract
In this paper, the numerical solution of the generalized Kuramoto-Sivashinsky equation is presented by differential transform method (DTM). The DTM is a powerful and efficient technique for finding solutions of nonlinear equations without the need of a linearization process. In this approach the solution is found in the form of a rapidly convergent series with easily computed components.
Keywords: Kuramoto-Sivashinsky equation; Differential transform method.

Mathematics Subject Classification: 35A22, 83C15

1 Introduction

In this paper, we consider the generalized Kuramoto-Sivashinsky (GKS) equation [1]-[2] in the form

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \tag{1}$$

where α, β and γ are real constants. This equation is also called KdV-Burgers-Kuramoto (KBK) equation. The GKS equation, one of the most important nonlinear partial differential equations (NPDEs), occupies a prominent position in describing physical processes in motion of turbulence and other unstable process systems. It can be used to describe long waves on a viscous fluid flowing down along an inclined plane [3], unstable drift waves in plasma [4] and stress waves in fragmented porous media.

2 Differential transform method

The basic definition and the fundamental theorems of the DTM and its applicability for various kinds of differential equations are given in [5]-[8]. For convenience of the reader, we present a review of the DTM.

The differential transform function of the function $w(x, y)$ is the following form:

$$W(k, h) = \frac{1}{k!h!}[\frac{\partial^{(k+h)} w(x, y)}{\partial x^k \partial y^h}]_{(x=x_0,\, y=y_0)}, \tag{2}$$

where $w(x,y)$ is the original function and $W(k,h)$ is the transformed function.

The inverse differential transform of $W(k,h)$ is defined as

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)(x-x_0)^k(y-y_0)^h. \tag{3}$$

Combining Eq. (2) and Eq. (3), and when (x_0,y_0) are taken as $(0,0)$, the function $w(x,y)$ in Eq. (3) is expressed as the following

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}[\frac{\partial^{(k+h)}w(x,y)}{\partial x^k\partial y^h}]_{(x=x_0,\,y=y_0)}x^ky^h, \tag{4}$$

and Eq. (3) is shown as

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^ky^h. \tag{5}$$

In real applications, the function $w(x,y)$ by a finite series of Eq. (5) can be written as

$$w(x,y)=\sum_{k=0}^{n}\sum_{h=0}^{m}W(x,y)x^ky^h. \tag{6}$$

The fundamental mathematical operations performed by two dimensional differential transform method can readily be obtained and are listed in Table 1.

Table 1: The operations for the two-dimensional differential transform method.

Original function	Transformed function
$w(x,y)=u(x,y)\mp v(x,y),$	$W(k,h)=U(k,h)\mp V(k,h)$
$w(x,y)=\alpha u(x,y)$	$W(k,h)=\alpha U(k,h)$
$w(x,y)=\frac{\partial u(x,y)}{\partial x}$	$W(k,h)=(k+1)U(k+1,h)$
$w(x,y)=\frac{\partial u(x,y)}{\partial y}$	$W(k,h)=(h+1)U(k,h+1)$
$w(x,y)=\frac{\partial^{(r+s)}u(x,y)}{\partial x^r\partial y^s}$	$W(k,h)=(k+1)(k+2)...(k+r)(h+1)(h+2)...(h+s)U(k+r,h+s)$
$w(x,y)=u(x,y)v(x,y)$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s)$
$w(x,y)=x^my^n$	$W(k,h)=\delta(k-m,h-n)=\delta(k-m)\delta(h-n)=\{\begin{array}{ll}1, & k=m,h=n\\ 0, & \text{otherwise}\end{array}$
$w(x,y)=\frac{\partial u(x,y)}{\partial x}\frac{\partial v(x,y)}{\partial y}$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(h-s+1)U(k-r+1,s)V(r,h-s+1)$
$w(x,y)=u(x,y)v(x,y)z(x,y)$	$W(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}U(r,h-s-p)V(t,s)Z(k-r-t,p)$
$w(x,y)=u(x,y)\frac{\partial v^2(x,y)}{\partial x^2}$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+2)(k-r+1)U(r,h-s)V(k-r+2,s)$
$w(t)=t$	$W(k)=\delta(k-1)$

3 Numerical results

In this part, DTM will be applied for solving two special equations, namely GKS and KS with specific initial conditions. The results reveal that the method is very effective and simple.

Example 3.1: We show an accuracy test for the generalized Kuramoto-Sivashinsky equation

$$u_t + uu_x + u_{xx} + 4u_{xxx} + u_{xxxx} = 0,$$

subject to the initial condition:

$$u(x,0) = 15 - 15\left(\mathrm{Tanh}(\frac{1}{2}(x+10)) + \mathrm{Tanh}^2(\frac{1}{2}(x+10)) - \mathrm{Tanh}^3(\frac{1}{2}(x+10))\right), \tag{7}$$

Using the DTM, we have

$$(h+1)U(k,h+1) + \sum_{r=0}^{k}\sum_{s=0}^{h}(1+k-r)U(1+k-r,s)U(r,h-s) + (1+k)(2+k)U(2+k,h) +$$
$$4(1+k)(2+k)(3+k)\ U(3+k,h) + (1+k)(2+k)(3+k)(4+k)U(4+k,h) = 0. \tag{8}$$

Substituting Eq. (7) into Eq. (8) and by a recursive method, the results are listed as follows:

$$\begin{aligned}
&U(0,0) = 2.47305\times 10^{-7},\ U(1,0) = -4.94576\times 10^{-7},\ U(2,0) = 4.94525\times 10^{-7}, \ldots\\
&U(0,1) = 2.96745\times 10^{-6},\ U(1,1) = -5.9343\times 10^{-6},\ \ U(2,1) = 5.93339\times 10^{-6}, \ldots\\
&U(0,2) = 0.0000178029,\quad U(1,2) = -0.0000356004,\quad U(2,2) = 0.0000355922, \ldots
\end{aligned}$$

Consequently substituting all $U(k,h)$ into Eq. (6), we achieve the closed form series solution as

$$u(x,t) = 15 - 15\left(\mathrm{Tanh}(\frac{1}{2}(x-6t+10)) + \mathrm{Tanh}^2(\frac{1}{2}(x-6t+10)) - \mathrm{Tanh}^3(\frac{1}{2}(x-6t+10))\right),$$

which is the exact solution of the problem [9].

Example 3.2: Now we consider the Eq. (1) for $\alpha = -1$, $\beta = 0$, $\gamma = 1$:

$$u_t + uu_x - u_{xx} + u_{xxxx} = 0,$$

subject to the initial condition:

$$u(x,0) = 5 + \frac{15}{19\sqrt{19}}\left(-3\ \mathrm{Tanh}(\frac{1}{2\sqrt{19}}\ (x+25)) + \mathrm{Tanh}^3(\frac{1}{2\sqrt{19}}\ (x+25))\right), \tag{9}$$

Similarly the previous example and employing the DTM

$$u(x,t) = 5 + \frac{15}{19\sqrt{19}}\left(-3\ \mathrm{Tanh}(\frac{1}{2\sqrt{19}}\ (x-5t+25)) + \mathrm{Tanh}^3(\frac{1}{2\sqrt{19}}\ (x-5t+25))\right),$$

which is the exact solution of the problem [9].

4 Conclusion

In this work, we have successfully developed DTM to obtain an approximation to the solution of the Kuramoto-Sivashinsky equation. It is apparent that this method is a very influential and efficient technique. There is no need for linearization or perturbations; large computational work and round-off errors are avoided. The results obtained demonstrate the reliability of the algorithm and its applicability to some partial differential equations. It provides more realistic series solutions that converge very rapidly in real physical problems.

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General linear methods for chemical stiff ODEs

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Abstract

In this paper, we solve numerically some stiff chemical problems using general linear methods (GLMs). As GLMs possess the order and stability properties of multistep and multivalue methods simultaneous, we expect they are efficient to apply on stiff chemical problems.

Keywords: General linear methods, stiff ODEs, Stability, Consistency, Convergence.

Mathematics Subject Classification: 65L05

1 Introduction

The most widely used numerical methods for solving ODEs are the linear multistep and Rung–Kutta methods. However these traditional methods have well-known disadvantages. In 1966, Butcher [3] introduced general linear methods (GLMs) as a unifying framework for the traditional methods to study the properties of consistency, stability and convergence, and to formulate new methods with clear advantages. Burrage and Butcher [1] represented the method by four matrices which generally denoted by A, U, B and V. GLMs for numerical soution of a system of ordinary differential equation

$$y' = f(y(x)), \quad y : \mathbb{R} \to \mathbb{R}^m, \quad f : \mathbb{R}^m \to \mathbb{R}^m \tag{1}$$

are characterized by (p, q, r, s) where p and q are respectively order and stage order of the method, r is the number of input and output approximations and s is the number of internal stages. Let $Y^{[n]} = [Y_i^{[n]}]_{i=1}^s$ be an approximation of stage order q to the vector $y(x_{n-1}+ch) = [y(x_{n-1}+c_ih)]_{i=1}^s$ and $f(Y^{[n]}) = [f(Y_i^{[n]})]_{i=1}^s$. Let also denote by $y^{[n-1]} = [y_i^{[n-1]}]_{i=1}^r$ and $y^{[n]} = [y_i^{[n]}]_{i=1}^r$ the input and output vectors at the step number n, respectively. A GLM used for the numerical solution of (1.2) is given by

$$\begin{aligned} Y^{[n]} &= h(A \otimes I_m)f(Y^{[n]}) + (U \otimes I_m)y^{[n-1]}, \\ y^{[n]} &= h(B \otimes I_m)f(Y^{[n]}) + (V \otimes I_m)y^{[n-1]}, \end{aligned} \tag{2}$$

where $n = 1, 2, \cdots, N$, h is the stepsize and m is dimension of the system and notation $\otimes$ is the Kronecker product. Here $A \in \mathbb{R}^{s\times s}$, $U \in \mathbb{R}^{s\times r}$, $B \in \mathbb{R}^{r\times s}$ and $V \in \mathbb{R}^{r\times r}$. It is convenient to write coefficients of the method, that is elements of A, U, B and V as a partitioned $(s + r) \times (s + r)$ matrix

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right].$$

2 Order conditions and stability for GLMs

To formulate order and stage order conditions for SGLMs, we assume that components of the input vector $y_i^{[n-1]}$ for the next step satisfy

$$y_i^{[n-1]} = \sum_{k=0}^{p} h^k \alpha_{ik} y^{(k)}(x_{n-1}) + O(h^{p+1}), \quad i = 1, 2, ..., r, \tag{3}$$

for some real α_{ik}, $i = 1, 2, ..., r$, $k = 0, 1, 2, ..., p$. The values of α_{ik} must be chosen so that the stage values within the current step with stepsize h, are given by

$$Y_i^{[n]} = y(x_{n-1} + ch) + O(h^{q+1}), \quad i = 1, 2, ..., s, \tag{4}$$

and the output values computed at the end of current step are given by

$$y_i^{[n]} = \sum_{k=0}^{p} h^k \alpha_{ik} y^{(k)}(x_{n-1}) + O(h^{p+1}), \quad i = 1, 2, ..., r, \tag{5}$$

for the same numbers α_{ik}. Let us introduce the notation

$$e^{cz} = \left[e^{c_1 z}, e^{c_2 z} \cdots e^{c_s z} \right],$$

and define the vector $w = w(z)$ with elements given by

$$w_i = \sum_{k=0}^{p} \alpha_{ik} z^k, \quad i = 1, 2, ..., r.$$

Theorem 2.1. *[4] Assume that $y^{[n-1]}$ satisfies (16). Then the GLM (22) of order and stage order $q = p$ satisfies (17) and (10) iff*

$$\exp(cz) = zA\exp(cz) + Uw(z) + O(z^{p+1}), \tag{6}$$
$$\exp(z)w(z) = zB\exp(cz) + Vw(z) + O(z^{p+1}). \tag{7}$$

3 Application of GLMs

In [2] Butcher by considering GLMs in diagonally implicit multi-stage form, which the matrix A has the lower triangular form $A = \begin{bmatrix} \lambda & & & \\ a_{21} & \lambda & & \\ \vdots & \vdots & \ddots & \\ a_{s1} & a_{s2} & \cdots & \lambda \end{bmatrix}$, has divided GLMs in to four types, depending on the nature of differential system to be solved and the computer architecture that is used to implement these methods. Types 1 and 2 are those with arbitrary a_{ij}, where $\lambda = 0$ and $\lambda > 0$, respectively. Such methods are appropriate for nonstiff or stiff differential systems in a sequential computing environment. For type 3 or 4 methods, $A = \lambda I$, where $\lambda = 0$ or $\lambda > 0$, respectively. Such methods are appropriate for nonstiff or stiff differential systems in a parallel computing environment.

4 Construction of GLMs with $p = q = r = s = 3$

we consider methods with $r = s = p = q = 3$ and $U = I$ and $V{=}uv^T$, where $v = [v_1, v_2, \cdots, v_r]^T$ and $u = [1, 1, \cdots, 1]^T \in \mathbb{R}^r$, such that $V = vu$. Here, we present a single example characterized by $\lambda = \frac{1}{2}$ and $c = [-1 \;\; 0 \;\; 1]^T$. The coefficients of the method are given by the partitioned matrix

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{ccc|ccc} \frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ \frac{5}{4} & \frac{1}{2} & 0 & 0 & 1 & 0 \\ \frac{7}{5} & \frac{4}{5} & \frac{1}{2} & 0 & 0 & 1 \\ \hline \frac{5}{4} & \frac{1}{5} & -\frac{1}{12} & \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{17}{20} & \frac{7}{60} & -\frac{1}{6} & \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{23}{30} & \frac{2}{15} & -\frac{1}{20} & \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \end{array}\right].$$

5 Numerical experiments

In this section, we apply the constructed method in the previous section on a chemical stiff IVPs. The Belousov-Zhabotinskii reaction [5] may be represented by the following scheme of homogeneous chemical reactions

$$\begin{cases} A + Y \longrightarrow X, & k_1 = 4.72 \\ X + Y \longrightarrow P, & k_2 = 3 \times 10^9 \\ B + X \longrightarrow 2X + Z, & k_3 = 1.5 \times 10^4 \\ 2X \longrightarrow Q, & k_4 = 4 \times 10^7 \\ Z \longrightarrow Y, & k_5 = 1 \end{cases}$$

Letters A, . . . , Z denote species taking part in the reactions and constants ki denote the reaction rates. Obviously, the rate constants differ by several orders of magnitude which indicates the likeliness of the corresponding ODE system being stiff. The initial conditions are given by initial concentrations of species at t = 0 $A = B = 0.066, Y = X = P = Q = 0, Z = 0.002$. The ODEs system modeling of the reaction scheme is

$$\begin{cases} \acute{y_1} = -k_1y_1y_2, & y_1(0) = 0.066, \\ \acute{y_2} = -k_1y_1y_2 - k_2y_3y_2 + k_5y_6, & y_2(0) = 0, \\ \acute{y_3} = -k_2y_3y_2 + k_3y_3y_5 - 2k_4y_3^2 + k_1y_1y_2, & y_3(0) = 0, \\ \acute{y_4} = k_2y_3y_2, & y_4(0) = 0, \\ \acute{y_5} = -k_3y_5y_3, & y_5(0) = 0.066, \\ \acute{y_6} = k_3y_5y_3 - k_5y_6, & y_6(0) = 0.002, \\ \acute{y_7} = k_4y_3^2, & y_7(0) = 0. \end{cases}$$

The obtained solution of this problem at t = 40 is reported in Table 1.

Table 1: The results of problem Belousov-Zhabotiskii reaction at t = 40.

y_i	Solution at $t = 40$
y_1	0.061274900027305
y_2	0.000000721219470
y_3	0.000000000177657
y_4	0.004723908486329
y_5	0.066001356012250
y_6	0.000000042276675
y_7	0.000000082351743

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Second derivative general linear methods in Nordsieck form for IVPs

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Abstract
In this paper, we review the structure of second derivative general linear methods (SGLMs) in the Nordsieck form with variable stepsize and also discuss about the error constants and the local discretization error of the SGLMs.
Keywords: General linear methods, Second derivative general linear methods, Nordsieck methods, Variable stepsize, Local discretization error.

Mathematics Subject Classification: 65L05

1 Introduction

In 1966 Butcher [1] introduced general linear methods (GLMs) for the numerical solution of the initial value problem for ordinary differential equations (ODEs)

$$\begin{aligned} y'(x) &= f(y(x)), \qquad x \in [x_0, X], \\ y(x_0) &= y_0, \end{aligned} \tag{1}$$

where $y : \mathbb{R} \to \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}^m$, by the equations

$$\begin{aligned} Y^{[n]} &= h(\mathbf{A} \otimes I_m) f(Y^{[n]}) + (\mathbf{U} \otimes I_m) y^{[n-1]}, \\ y^{[n]} &= h(\mathbf{B} \otimes I_m) f(Y^{[n]}) + (\mathbf{V} \otimes I_m) y^{[n-1]}, \end{aligned} \tag{2}$$

where notation $\otimes$ is the Kronecker product and $\mathbf{A} \in \mathbb{R}^{s\times s}$, $\mathbf{U} \in \mathbb{R}^{s\times r}$, $\mathbf{B} \in \mathbb{R}^{r\times s}$ and $\mathbf{V} \in \mathbb{R}^{r\times r}$, and also $Y^{[n]} = [Y_i^{[n]}]_{i=1}^s$ is an approximation of the stage order q to the vector $y(x_{n-1} + ch) = [y(x_{n-1} + c_i h)]_{i=1}^s$, $f(Y^{[n]})) = [f(Y_i^{[n]})]_{i=1}^s$, $y^{[n-1]} = [y_i^{[n-1]}]_{i=1}^r$ and $y^{[n]} = [y_i^{[n]}]_{i=1}^r$. The vectors $y^{[n-1]}$ and $y^{[n]}$ are called input and output vectors at the step number n, respectively, and $c = [c_1 \ \ c_2 \ \ \cdots \ \ c_s]^T$ is called the abscissa vector.

Second derivative general linear methods (SGLMs) as an extension of GLMs have been introduced by Butcher and Hojjati [2]. An SGLM is characterized by six matrices denoted by $\mathbf{A}, \overline{\mathbf{A}} \in \mathbb{R}^{s\times s}$, $\mathbf{U} \in \mathbb{R}^{s\times r}$, $\mathbf{B}, \overline{\mathbf{B}} \in \mathbb{R}^{r\times s}$ and $\mathbf{V} \in \mathbb{R}^{r\times r}$. In an SGLM, the vectors of stage values, input and output values are related by

$$\begin{aligned} Y^{[n]} &= h(\mathbf{A} \otimes I_m) f(Y^{[n]}) + h^2(\overline{\mathbf{A}} \otimes I_m) g(Y^{[n]}) + (\mathbf{U} \otimes I_m) y^{[n-1]}, \\ y^{[n]} &= h(\mathbf{B} \otimes I_m) f(Y^{[n]}) + h^2(\overline{\mathbf{B}} \otimes I_m) g(Y^{[n]}) + (\mathbf{V} \otimes I_m) y^{[n-1]}, \end{aligned} \tag{3}$$

where $g(\cdot) = f'(\cdot)f(\cdot)$ and $g(Y^{[n]}) = [g(Y_i^{[n]})]_{i=1}^s$. In the Nordsieck representation of SGLMs, the vectors $y^{[n-1]}$ and $y^{[n]}$ are in the following form

$$y^{[n-1]} = \begin{bmatrix} y(x_{n-1}) \\ hy^{'}(x_{n-1}) \\ \vdots \\ h^p y^{(p)}(x_{n-1}) \end{bmatrix}, \qquad y^{[n]} = \begin{bmatrix} y(x_n) \\ hy^{'}(x_n) \\ \vdots \\ h^p y^{(p)}(x_n) \end{bmatrix}.$$

We will assume that the method (3) has order p and stage order $q = p$ and is in the Nordsieck form. This means that if

$$y_{n-1} = y(x_{n-1}) + O(h^{p+1}), \qquad y_i^{[n-1]} = h^i y^{(i)}(x_{n-1}) + O(h^{p+1}),$$

$i = 1, 2, \cdots, p$, then

$$Y_i = y(x_{n-1} + c_i h) + O(h^{q+1}),$$

$i = 1, 2, \cdots, s$, and

$$y_n = y(x_n) + O(h^{p+1}), \qquad y_i^{[n]} = h^i y^i(x_n) + O(h^{p+1}),$$

$i = 1, 2, \cdots, p$, $x_n = x_0 + nh, n = 1, 2, \cdots, N.$
Denote Z, e, e_j as the $(p+1)$ dimensional vectors with component number i equal to $z^{i-1}, 1, \delta_{ij}$, respectively.

Theorem 1.1. *[2] An SGLM in the Nordsieck form has order p and stage order $q = p$ if and only if*

$$\begin{aligned} \exp(cz) &= z\mathbf{A}\exp(cz) + z^2\overline{\mathbf{A}}\exp(cz) + \mathbf{U}Z + O(z^{p+1}), \\ \exp(z)Z &= z\mathbf{B}\exp(cz) + z^2\overline{\mathbf{B}}\exp(cz) + \mathbf{V}Z + O(z^{p+1}). \end{aligned}$$

Corollary 1.2. *[2] An equivalent conditions for order and stage order p is that $\mathbf{U}$ and $\mathbf{V}$ are related to $\mathbf{A}, \overline{\mathbf{A}}, \mathbf{B}, \overline{\mathbf{B}}$ by*

$$\mathbf{U} = C - \mathbf{A}CK - \overline{\mathbf{A}}CK^2, \qquad \mathbf{V} = E - \mathbf{B}CK - \overline{\mathbf{B}}CK^2,$$

where $C = \begin{bmatrix} \mathrm{e} & c & \frac{c^2}{2!} & \cdots & \frac{c^p}{p!} \end{bmatrix}, K = [0 \;\; e_1 \;\; e_2 \;\; \cdots \;\; e_{p-1} \;\; e_p]_{(p+1)\times(p+1)}$ *and* $E = \exp(K)$.

2 SGLMs with variable stepsize

On the nonuniform grid

$$x_0 < x_1 < x_2 < \cdots < x_N$$

the Nordsieck representation of method (3) takes the form

$$\begin{aligned} Y^{[n]} &= h_n(\mathbf{A}\otimes I_m)f(Y^{[n]}) + h_n^2(\overline{\mathbf{A}}\otimes I_m)g(Y^{[n]}) + (\mathbf{U}\otimes I_m)\widetilde{y}^{[n-1]}, \\ y^{[n]} &= h_n(\mathbf{B}\otimes I_m)f(Y^{[n]}) + h_n^2(\overline{\mathbf{B}}\otimes I_m)g(Y^{[n]}) + (\mathbf{V}\otimes I_m)\widetilde{y}^{[n-1]}, \end{aligned} \tag{4}$$

where $h_n = x_n - x_{n-1}$ and $\widetilde{y}^{[n-1]}$ is an approximation of order p to the vector

$$\widetilde{y}(x_{n-1}, h_n) = \begin{bmatrix} y(x_{n-1}) \\ h_n y^{'}(x_{n-1}) \\ \vdots \\ h_n^p y^{(p)}(x_{n-1}) \end{bmatrix}.$$

Since

$$\begin{bmatrix} y(x_{n-1}) \\ h_n y'(x_{n-1}) \\ \vdots \\ h_n^p y^{(p)}(x_{n-1}) \end{bmatrix} = \begin{bmatrix} I_m & 0 & \cdots & 0 \\ 0 & \delta_n I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \delta_n^p I_m \end{bmatrix} \begin{bmatrix} y(x_{n-1}) \\ h_{n-1} y'(x_{n-1}) \\ \vdots \\ h_{n-1}^p y^{(p)}(x_{n-1}) \end{bmatrix},$$

where $\delta_n = h_n/h_{n-1}$, the vector $\widetilde{y}^{[n-1]}$ appearing in (4) is defind by the formula

$$\widetilde{y}^{[n-1]} = (D(\delta_n) \otimes I_m) y^{[n-1]},$$

and $D(\delta_n)$ is the rescaling matrix given by

$$D(\delta_n) = \operatorname{diag}(1, \delta_n, \cdots, \delta_n^p).$$

Thus we obtain the following representation of SGLMs in the Nordsieck form

$$\begin{aligned} Y^{[n]} &= h_n(\mathbf{A} \otimes I_m) f(Y^{[n]}) + h_n^2(\overline{\mathbf{A}} \otimes I_m) g(Y^{[n]}) + (\mathbf{U} D(\delta_n) \otimes I_m) y^{[n-1]}, \\ y^{[n]} &= h_n(\mathbf{B} \otimes I_m) f(Y^{[n]}) + h_n^2(\overline{\mathbf{B}} \otimes I_m) g(Y^{[n]}) + (\mathbf{V} D(\delta_n) \otimes I_m) y^{[n-1]}. \end{aligned} \tag{5}$$

3 Finding error constant for SGLMs

Assume that the method (4) has the order and stage order p, and denote $F = f(Y^{[n]})$ and $G = g(Y^{[n]})$. Replacing $y_n, y^{[n]}, Y^{[n]}, F, G$ in (4) by $\widehat{y}_n = y(x_n), \widehat{y}^{[n]}, \widehat{Y}^{[n]}, \widehat{F}$ and $\widehat{G}$, where

$$\widehat{y}^{[n]} = \begin{bmatrix} y(x_n) \\ h_n y'(x_n) \\ \vdots \\ h_n^p y^{(p)}(x_n) \end{bmatrix}, \widehat{Y}^{[n]} = \begin{bmatrix} y(x_{n-1} + c_1 h_n) \\ y(x_{n-1} + c_2 h_n) \\ \vdots \\ y(x_{n-1} + c_s h_n) \end{bmatrix},$$

$$\widehat{F} = \begin{bmatrix} y'(x_{n-1} + c_1 h_n), \\ y'(x_{n-1} + c_2 h_n) \\ \vdots \\ y'(x_{n-1} + c_s h_n) \end{bmatrix}, \widehat{G} = \begin{bmatrix} y''(x_{n-1} + c_1 h_n) \\ y''(x_{n-1} + c_2 h_n) \\ \vdots \\ y''(x_{n-1} + c_s h_n) \end{bmatrix},$$

we obtain

$$\widehat{y}_n = \widehat{y}_{n-1} + h_n(b^T \otimes I)\widehat{F} + h_n^2(\overline{b}^T \otimes I)\widehat{G} + (v^T D(\delta_n) \otimes I)\widehat{y}^{[n-1]} + \widetilde{\beta} h_n^{p+1} y^{p+1}(x_n) + O(h_n^{p+2}),$$

$$\widehat{y}^{[n]} = h_n(B \otimes I)\widehat{F} + h_n^2(\overline{B} \otimes I)\widehat{G} + (V D(\delta_n) \otimes I)\widehat{y}^{[n-1]} + (\beta \otimes I) h_n^{p+1} y^{p+1}(x_n) + O(h_n^{p+2}).$$

where $b^T = \mathbf{B}(1,:),\ \overline{b}^T = \overline{\mathbf{B}}(1,:),\ B = \overline{\mathbf{B}}(2:r,:),\ v^T = \mathbf{V}(1, 2:r),\ V = \mathbf{V}(2:r, 2:r)$.

The error constants $\widetilde{\beta}$ and β can be computed similarly as in [3, 4]. This leads to

$$\widetilde{\beta} = \tfrac{1}{(p+1)!} - (\tfrac{1}{p!} b^T c^p + \tfrac{1}{(p-1)!} \overline{b}^T c^{p-1}),$$

$$\beta = \begin{bmatrix} \frac{1}{p!} & \frac{1}{(p-1)!} & \cdots & \frac{1}{1!} \end{bmatrix}^T - (\tfrac{1}{p!} B c^p + \tfrac{1}{(p-1)!} \overline{B} c^{p-1}).$$

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Solution of nonlinear multi-order fractional differential equations by Legendre wavelet method

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Abstract

In this paper we present a computational method for solving a class of nonlinear multi-order fractional differential equations which is based on Legendre wavelets. In the proposed method, we present a new technique for computation of nonlinear terms in such equations.

Keywords: Legendre wavelets, multi-order fractional differential equations.

Mathematics Subject Classification: 34A08

1 Introduction and Preliminaries

In this paper we will apply the Legendre wavelet method for solving a generalized form of nonlinear multi-order fractional differential equations as:

$$\hat{\beta}_0(x)D_*^{\alpha}y(x)+\sum_{i=1}^{s}\hat{\beta}_i(x)D_*^{\alpha_i}y(x)+\hat{\gamma}(x)F(y(x))=f(x), \tag{1}$$

where $\hat{\beta}_0(x)\neq 0,\ n-1<\alpha\leq n,\ 0<\alpha_1<\alpha_2<\ldots<\alpha_s<\alpha;$ n and s are fixed positive integers, F is a analytical function, D_*^{α} denotes Caputo fractional derivative of order α, f is a known function of x, and y is an unknown function to be determined later. The Riemann-Liouville fractional integration operator of order $\alpha>0$ is defined as [1]:

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau, \qquad I^0f(t)=f(t), \tag{2}$$

and the fractional derivative of order $\alpha>0$ is defined as [1]:

$$D^{\alpha}f(t)=\left(\frac{d}{dt}\right)^n I^{n-\alpha}f(t), \qquad (n-1<\alpha\leq n), \tag{3}$$

where n is an integer. The Caputo derivative of order $\alpha>0$ is defined as [1]:

$$D_*^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, \qquad (n-1<\alpha\leq n), \tag{4}$$

where n is an integer.

2 Main results

2.1 Legendre wavelets

Legendre wavelets $\psi_{n',m'}(t) = \psi(k, \hat{n}, m', t)$ have four arguments; $k \in \mathbb{N}, n' = 1, 2, \cdots, 2^{k-1}$, and $\hat{n} = 2n' - 1$, moreover m' is the degree of the Legendre polynomial of the first kind and t is the normalized time i.e. $t \in [0, 1]$. They are defined on the interval $[0, 1]$ as [2]:

$$\psi_{n',m'}(t) = \begin{cases} 2^{k/2}\sqrt{m' + \frac{1}{2}}p_{m'}(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}}{2^k} \\ 0, & otherwise \end{cases}, \tag{5}$$

where $m' = 0, 1, \cdots, M - 1$, and M is a fixed positive integer. The coefficient $\sqrt{m' + 1/2}$ in (5) is for orthonormality, Here, $P_{m'}(t)$ are the well-known Legendre polynomials of of degree m' which are orthogonal with respect to the weight function $w(t) = 1$ on the interval [-1, 1].

2.2 Function approximation

An arbitrary function $f(t) \in L^2(R)$ defined over [0,1] may be expanded into Legendre wavelet basis as:

$$f(t) = \sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty} c_{n',m'}\psi_{n',m'}(t), \tag{6}$$

where $c_{n',m'} = (f(t), \psi_{n',m'}(t))$ in which (,) denotes the inner product.
If the infinite series in (6) is truncated, then (6) can be written as

$$f(t) \approx \sum_{n'=1}^{2^{k-1}}\sum_{m'=0}^{M-1} c_{n',m'}\psi_{n',m'}(t) = C^T\Psi(t), \tag{7}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}]^T$$

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M-1}(t), \psi_{20}(t), \ldots, \psi_{2M-1}(t), \ldots, \psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}M-1}(t)]^T \tag{8}$$

Taking the collocation points $t_i = \frac{(2i-1)}{2^k M}$ $(i = 1, 2, \ldots, 2^{k-1}M)$, we define the wavelet matrix $\Phi_{m\times m}$ as , $\Phi_{m\times m} = [\Psi(\frac{1}{2m}), \Psi(\frac{3}{2m}), \ldots, \Psi(\frac{2m-1}{2m})]$.

2.3 Operational matrix of fractional integration

The fractional integration of order α of the vector $\Psi(t)$ can be expressed as:

$$(I^\alpha\Psi)(t) = P^\alpha\Psi(t), \tag{9}$$

where P^α is the $m \times m$ operational matrix of fractional integration of order α. It is shown that the matrix P^α for Legendre wavelet can be approximated as [2]:

$$P^\alpha \approx P^\alpha_{m\times m} = \Phi P^\alpha_B \Phi^{-1}, \tag{10}$$

where P^α_B is the operational matrix of fractional integration of order α of the Block Pulse functions (BPF) [3]. Also, we define an m-set of Block Pulse Functions (BPF) as, $b_i(t) = \begin{cases} 1, & \frac{i}{m} \leq t < \frac{(i+1)}{m}, \\ 0, & otherwise, \end{cases}$ $, (i = 0, 1, 2, \ldots, m-1)$.
Legendre wavelets may be expanded into an m-term BPF as $\Psi_m(t) = \Phi_{m\times m}B_m(t)$, such that $B_m(t) = [b_0(t), b_1(t), \ldots, b_i(t), \ldots, b_{m-1}(t)]^T$.

3 Description of the numerical method

In this section, the Legendre wavelets expansions together with their operational matrix of fractional order integration are used for numerical solution of (1). For solving this problem we assume:

$$D_*^{\alpha} y(x) = K_m^T \Psi_m(x), \tag{11}$$

where K_m^T is an unknown vector. By using initial conditions and (6), we have:

$$D_*^{\alpha_i} y(x) = K_m^T P_{m\times m}^{\alpha-\alpha_i} \Psi_m(x) + \sum_{j=0}^{n-\hat{i}-1} I^{j+\hat{i}-\alpha_i} c_{\hat{i}+j}, \qquad i = 1, 2, \ldots, s_1, \tag{12}$$

where $\hat{i} = \lceil \alpha_i \rceil$ is the smallest integer not less than α_i. Also we have:

$$y(x) = K_m^T P_{m\times m}^{\alpha} \Psi_m(x) + \sum_{j=0}^{n-1} I^j c_j. \tag{13}$$

Now, by using (7) and (10) we have:

$$I^j c_j = C_j^T \Phi_{m\times m} P_B^j B_m(x), \qquad j = 0, 1, \ldots n-1, \qquad F^0 = I_{m\times m}, \tag{14}$$

and since $\Psi_m = \Phi_{m\times m} B_m(x)$, from (13) and (14) we obtain:

$$y(x) = \left[K_m^T P_{m\times m}^{\alpha} \Phi_{m\times m} + \sum_{j=0}^{n-1} C_j^T \Phi_{m\times m} P_B^j \right] B_m(x) = [a_1, a_2, \ldots, a_m] B_m(x). \tag{15}$$

and it can be shown that:

$$F(y(x)) = [F(a_1), F(a_2), \ldots, F(a_m)] B_m(x). \tag{16}$$

Now, by substituting (11), (12), (16) into (1) we obtain a nonlinear algebraic equation. Now, by taking collocation points $t_i = \frac{(2i-1)}{2^k M}$ $(i = 1, 2, \ldots, m)$, this equation is transformed into a nonlinear system of algebraic equations with m unknowns a_i $(i = 1, 2, \ldots, m)$. Solving this system, and determining a_i, we get the numerical solution of problem (1).

Example 3.1. *Consider the following nonlinear multi-order fractional differential equation:*

$$D_*^2 y(x) + D_*^{\alpha} y(x) + \arcsin(y(x)) + y(x) = f(x), \qquad 0 < \alpha \le 1,$$

where, $f(x) = \cos(x) + x$, and the initial conditions $y(0) = 0$ and $y'(0) = 0$. The exact solution of this problem for $\alpha = 1$ is $y(x) = \sin(x)$. Fig.1 shows the behavior of the numerical solution for some different values of α, for $m = 128$. The absolute errors of $y(x)$ at some sample points $x \in [0, 1]$ for different values of m are shown in Table 1.

Figure 1: Numerical solution (Nu.S) and exact solution (Ex.S).

Table 1: Absolute errors for different values of m

m	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
12	8.06×10^{-3}	2.32×10^{-2}	2.03×10^{-2}	2.80×10^{-3}	2.29×10^{-2}
32	2.04×10^{-4}	3.67×10^{-3}	4.69×10^{-3}	2.17×10^{-3}	8.50×10^{-3}
128	7.66×10^{-4}	2.15×10^{-4}	1.93×10^{-3}	2.54×10^{-4}	2.12×10^{-3}

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A computational method for solving two dimensional nonlinear Fredholm integral equations

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Abstract
In this paper, an extension of the operational Tau method is considered for solving 2DNFIEs (Two Dimensional Nonlinear Fredholm Integral Equations). We developed this method with arbitrary polynomial bases to obtain approximate solution of these equations. Some theoretical results are given to simplify and reduce the computation.
Keywords: Two dimensional nonlinear Fredholm integral equation, Operational Tau method

Mathematics Subject Classification: 65R20, 45B05, 45G10

1 Introduction

A 2DNFIE may be arisen in applied sciences and engineering such as modeling of piezoelectric materials and utilizing of these materials in nano-tubes [1], optimal control [2], calculation of plasma physics [3] and theory of electrostatic problem of axial translation of a grid elliptical disc-inclusion [4].
In this work, we consider a 2DNFIE of the form

$$u(x,t) - \lambda \int_0^{T_2}\int_0^{T_1} k(x,t,y,z)u^k(y,z)dydz = f(x,t), \quad (x,t) \in \Omega = [0,T_1] \times [0,T_2], \tag{1}$$

where $f(x,t)$ and $k(x,t,y,z)$ are given smooth real valued functions defined on Ω and $X = \Omega \times \Omega$ respectively and λ, T_1 and T_2 are given real constants.

2 Preliminary results of the Tau method

The operational approach to the Tau method proposed by Ortiz and Samara [5] based on the use of three simple matrices

$$\mu = [\mu_{ij}]_{i,j=0}^{\infty},\ \mu_{ij} = \delta_{i+1,j},\ \iota = [\iota_{ij}]_{i,j=0}^{\infty}, \iota_{ij} = \frac{1}{i+1}\delta_{i+1,j},\ \eta = [\eta_{ij}]_{i,j=0}^{\infty}, \eta_{ij} = (j+1)\delta_{i,j+1}$$

having the following properties:

Lemma 2.1. *[6] If $y_N(x) = \underline{a}_N X_x$ where $\underline{a}_N = (a_0, \cdots, a_N, 0, \cdots)$, $X_x = (1, \cdots, x^N, \cdots)^T$, then*
(a) $xy_N(x) = \underline{a}_N \mu X_x$;
(b) $\int_0^x y_N(t)dt = \underline{a}_N \iota X_x$.

Corollary 2.2. *[5] Generally, under assumptions of Lemma 2.1, we have*
(a) $x^i X_x = \mu^i X_x$;
(b) $\int_0^x X_t dt = \iota X_x$ *where* $X_t = (1, t, \cdots, t^N, \cdots)^T$.

3 Implementation of the Tau method

To solve Eq. (1) by the operational Tau method, we assume that f and k are polynomials. Let $\underline{\phi}_x = \{\phi_i(x) : i \in \mathcal{N} = \{0, 1, \ldots, N\}\}$ be a polynomial basis given by $\underline{\phi}_x = \hat{\Phi}\underline{X}_x$ where $\hat{\Phi}$ is a nonsingular lower triangular coefficients matrix and $\underline{X}_x = (1, x, \cdots, x^N)^T$ is the standard basis. In this work, we assume the approximate solution has the truncated series form

$$u_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} u_{i(N+1)+j}\phi_i(x)\phi_j(t) = \underline{u}\underline{\phi}_{x,t} = \underline{u}\Phi\underline{X}_{x,t}, \tag{2}$$

where $\underline{u} = (u_0, u_1, \cdots, u_{(N+1)^2-1})^T$ is the vector of unknown coefficients, $\underline{\phi}_{x,t} = \underline{\phi}_x \otimes \underline{\phi}_t$,
$\Phi = \hat{\Phi} \otimes \hat{\Phi}$ is a nonsingular lower triangular coefficients matrix with columns $\Phi_j = [\phi_{ij}]_{i=0}^{(N+1)^2-1}$ and $\underline{X}_{x,t} = \underline{X}_x \otimes \underline{X}_t$ where $\otimes$ denotes the kronecker product.
In the remaining part of this paper, we assume that μ and ι are $(N+1) \times (N+1)$ matrices.

Lemma 3.1. *If $u_N(x,t) = \underline{u}\Phi\underline{X}_{x,t}$, then*
(a) *The effect of $x^p t^q$ on the coefficients of $u_N(x,t)$ is equivalent to post multiplication of $\underline{u}\Phi$ by $\tilde{\mu}_x^p\tilde{\mu}_t^q$, i.e.*

$$x^p t^q u_N(x,t) = \underline{u}\Phi\tilde{\mu}_x^p\tilde{\mu}_t^q\underline{X}_{x,t} \quad or \quad x^p t^q \underline{X}_{x,t} = \tilde{\mu}_x^p\tilde{\mu}_t^q\underline{X}_{x,t},$$

where $\tilde{\mu}_x = \mu \otimes I$ with the elements $(\tilde{\mu}_x)_{ij} = \delta_{i+N+1,j},\ i = 0, 1, \cdots, N(N+1)-1$, $\tilde{\mu}_t = I \otimes \mu$ with the elements $(\tilde{\mu}_t)_{ii} = \mu,\ i = 0, 1, \cdots, N$ and I is an $(N+1) \times (N+1)$ identity matrix;
(b) $\int_0^t\int_0^x \underline{X}_{y,z}dydz = \tilde{\iota}\underline{X}_{x,t}$ *where $\tilde{\iota} = \iota \otimes \iota$ with the elements*

$$(\iota)_{ij} = \begin{cases} \frac{1}{i+1}\iota & , j = i+1,\ i = 0, 1, \cdots, N-1, \\ 0 & , otherwise. \end{cases}$$

Remark 3.2. *The elements of the matrices $\tilde{\mu}_x^p$ and $\tilde{\mu}_t^q$ are determined by*

$$(\tilde{\mu}_x^p)_{ij} = \begin{cases} 1 & , j = i + p(N+1),\ i = 0, 1, \cdots, (N-p+1)(N+1)-1, \\ 0 & , otherwise. \end{cases}$$

and

$$(\tilde{\mu}_t^q)_{ij} = \begin{cases} \mu^q & , i = j,\ i = 0, 1, \cdots, N, \\ 0 & , otherwise. \end{cases} \quad with \quad (\mu^q)_{ij} = \delta_{i+q,j},\ i = 0, 1, \cdots, N-q$$

Remark 3.3. *The matrix $\tilde{\mu}_x^p\tilde{\mu}_t^q$ in Lemma 3.10 has the following simple form*

$$\tilde{\mu}_x^p\tilde{\mu}_t^q = \tilde{\mu}_t^q\tilde{\mu}_x^p = \begin{pmatrix} \bar{0} & B \\ \bar{0} & \bar{0} \end{pmatrix},$$

where $B = diag(\mu^q)$ is an $(N-p+1) \times (N-p+1)$ matrix.

Now, we proceed to convert Eq. (1) to the corresponding nonlinear system of algebraic equations. To write a matrix representation for the integral part of Eq. (1), we state the following lemma and theorem.

Lemma 3.4. *Let $u_N(x,t) = \underline{u}\underline{\phi}_{x,t} = \underline{u}\Phi\underline{X}_{x,t}$ and $v_N(x,t) = \underline{v}\underline{\phi}_{x,t} = \underline{v}\Phi\underline{X}_{x,t}$ where $\underline{v}$ is a vector similar to $\underline{u}$ with elements v_i, then $u_N(x,t)v_N(x,t) = \underline{u}\Phi V\underline{X}_{x,t} = \underline{u}\Phi V\Phi^{-1}\underline{\phi}_{x,t}$ with*

$$V = \begin{pmatrix} \underline{v}\Phi_0 & \underline{v}\Phi_1 & \cdots & \underline{v}\Phi_{(N+1)^2-1} \\ 0 & \underline{v}\Phi_0 & \cdots & \underline{v}\Phi_{(N+1)^2-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \underline{v}\Phi_0 \end{pmatrix},$$

where $\Phi_j = [\phi_{ij}]_{i=0}^{(N+1)^2-1}$.

Corollary 3.5. *If $u_N(x,t) = \underline{u}\underline{\phi}_{x,t} = \underline{u}\Phi\underline{X}_{x,t}$, then $u_N^k(x,t) = \underline{u}\Phi U^{k-1}\underline{X}_{x,t} = \underline{u}\Phi U^{k-1}\Phi^{-1}\underline{\phi}_{x,t}$ where U is an upper triangular matrix with elements*

$$U_{ij} = \begin{cases} \sum_{r=0}^{(N+1)^2-1} u_r\phi_{rj} & , \ i \le j, \\ 0 & , \ i > j. \end{cases}$$

Theorem 3.6. *Let*

$$\begin{aligned} k_N(x,t,y,z) &= \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} k_{ijmn}\phi_i(x)\phi_j(t)\phi_m(y)\phi_n(z) \\ &= \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} \tilde{k}_{ijmn}x^i t^j y^m z^n. \end{aligned}$$

Then, we have

$$\int_0^{T_2}\int_0^{T_1} k(x,t,y,z)u^k(y,z)dydz \simeq \int_0^{T_2}\int_0^{T_1} k_N(x,t,y,z)u_N^k(y,z)dydz = \underline{u}\Phi U^{k-1}K\underline{X}_{x,t},$$

with

$$K = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} \tilde{k}_{ijmn}\underline{\xi}^{(m,n)}(T_1,T_2)\underline{e}^T_{i(N+1)+j+1},$$

where $\underline{\xi}^{(m,n)}(x,t) = \tilde{\mu}_z^n\tilde{\mu}_y^m\tilde{\iota}\underline{X}_{x,t}$ and $\underline{e}_{i(N+1)+j+1}$ is $\big(i(N+1)+j+1\big)^{th}$ coordinate unit vector.

To simplify the results of Theorem 3.6, we state the following Lemmas.

Lemma 3.7. *For the $(N+1)\times(N+1)$ matrix ι, we have*

$$(\mu^n\iota)_{r,r+n+1} = \begin{cases} \frac{1}{r+n} & , \ r,n = 1,2,\cdots,N+1, \ r+n+1 \le N+1, \\ 0 & , \ otherwise. \end{cases}$$

Lemma 3.8. *If $\tilde{\mu}_y = \mu\otimes I$, $\tilde{\mu}_z = I\otimes\mu$ and $\tilde{\iota} = \iota\otimes\iota$, then for $r = 0,1,\cdots,N-m-1$, we have*

$$[\tilde{\mu}_z^n\tilde{\mu}_y^m\tilde{\iota}]_{rs} = \begin{cases} \frac{1}{r+m+1}\mu^n\iota & , \ s = r+m+1, \ r = 0,1,\cdots,N-m-1, \\ 0 & , \ otherwise. \end{cases}$$

To convert Eq. (1) to a matrix form, we assume that the right-hand side of (1) has the form

$$f(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} f_{i(N+1)+j}\phi_i(x)\phi_j(t) = \underline{f}\ \underline{\phi}_{x,t} = \underline{f}\Phi\underline{X}_{x,t} \tag{3}$$

where $\underline{f} = (f_0, f_1, \cdots, f_{(N+1)^2-1})^T$.
By the above lemmas and theorems we have provided all requirements to convert Eq. (1) to the corresponding matrix representation. Hence, substituting from (2) and (3) in (1) and using Theorem 3.6, yield the nonlinear system

$$\underline{u}\Phi - \lambda\underline{u}\Phi U^{k-1}K = \underline{f}\Phi. \tag{4}$$

Eq. (5) can be expressed in arbitrary polynomial bases $\underline{\phi}_{x,t}$ of the form

$$\underline{u} - \lambda\underline{u}\Phi U^{k-1}K\Phi^{-1} = \underline{f}.$$

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A new spectral-collocation method for solving multi order fractional differential equations

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Abstract

In this paper a new operational matrix of fractional order derivative for Chebyshev polynomial is presented. Shifted Chebyshev polynomials and their properties are employed for deriving a general procedure for forming this matrix. The application of the proposed operational matrix for solving multi-order fractional differential equation is explained. The obtained results demonstrate efficiency and capability of the proposed method.

Keywords: Spectral method, Operational matrix, Chebyshev polynomial, collocation, multi-order fractional differential equation

Mathematics Subject Classification: 34A08, 65M70

1 Introduction

Orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem.

In the last several decades, many researchers found that derivatives of non-integer order are very suitable for the description of various physical phenomena such as damping laws, diffusion process, etc. These findings invoked the growing interest of studies of the fractional calculus in various fields such as physics, chemistry and engineering. For these reasons we need reliable and efficient techniques for the solution of fractional differential equations.

Definition 1.1. *The fractional derivative of order $\alpha > 0$ for function $f(x)$ in the Caputo sense is defined as [2]*

$$D^{\alpha}y(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1} f^m(t)dt$$

for $m-1 < \alpha \leq m, \quad m \in \mathbb{N}, \quad x > 0, \quad f \in C_{-1}^{m}$.

2 Shifted Chebyshev Polynomials

In order to use well known Chebyshev polynomials $T_m(t)$ on the interval $[0,1]$ we define the so-called shifted Chebyshev polynomials by introducing a change of variable $t = 2x-1$. So the shifted

Chebyshev polynomials of order m is denote by $\tilde{T}_m(x)$ and it can be obtained as follows [1]

$$\tilde{T}_{m+1}(x) = 2(2x-1)\tilde{T}_m(x) - \tilde{T}_{m-1}(x), \quad m = 1,\ 2,\ 3,\ ... \tag{1}$$

where $\tilde{T}_0(x) = 1$ and $\tilde{T}_1(x) = 2x - 1$.
A function $y(x)$ defined in $[0,1]$ can be expressed in terms of shifted Chebyshev polynomials as

$$y(x) = \sum_{i=0}^{m} c_i \tilde{T}_i(x) = C^T \Psi(x) \tag{2}$$

where the shifted Chebyshev coefficient vector C and the shifted Legendre vector $\Psi(x)$ are given by

$$\begin{aligned} C^T &= [c_0,\ c_1,\ ...,\ c_m] \\ \Psi(x) &= [\tilde{T}_0(x),\ \tilde{T}_1(x),\ ...,\tilde{T}_m(x)] \end{aligned} \tag{3}$$

In (3) the coefficients c_i are given by

$$c_i = \frac{4\gamma_i}{\pi} \int_0^1 \frac{\tilde{T}_i(x) y(x)}{\sqrt{1-(2x-1)^2}} dx \tag{4}$$

and $\gamma_m = \begin{cases} 2 & m = 0\,, \\ 1 & m \geq 1\,. \end{cases}$

Theorem 2.1. *Let $\Psi(x)$ be the shifted Chebyshev vector defined in (2) and $\alpha > 0$, Then fractional derivative of this vector can be expressed by*

$$D^\alpha \Psi(t) = D^{(\alpha)} \Psi(t) \tag{5}$$

Where $D^{(\alpha)}$ is the $(m+1) \times (m+1)$ operational matrix of fractional derivative and it can be derived by using the properties of shifted Chebyshev polynomials and fractional order derivatives of polynomials.

3 Applications

In this section, in order to show the high importance of operational matrix of derivative and fractional derivatives, we apply them to solve fractional differential equation. Consider the nonlinear fractional multi order differential equation

$$D^\alpha y(x) = F\left(x,\ y(x), D^{\beta_1} y(x), ..., D^{\beta_k} y(x)\right), \tag{6}$$

with the initial conditions

$$y^{(i)}(0) = d_i\,,\ i = 0, ..., n \tag{7}$$

Where $n < \alpha \leq n+1\,,\ 0 < \beta <_1 \beta_2 < ... < \beta_k < \alpha$ and D^α denote denotes the Caputo fractional derivative of order α. For solving solve this multi order differential equation we approximate $y(x)$, $D^\alpha y(x)$ and $D^{\beta_i} y(x), i = 1, 2, ..., k$ by using the shifted Chebyshev polynomials and their operational matrix of fractional derivatives as follow

$$\begin{aligned} y(x) &= C^T \Psi(x), \\ D^\alpha y(x) &\simeq C^T D^{(\alpha)} \Psi(x), \\ D^{\beta_i} y(x) &\simeq C^T D^{(\beta_i)} \Psi(x), \quad j = 1, 2, ..., k \end{aligned} \tag{8}$$

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By substituting Eq. (8) in Eq. (6) we have

$$C^T D^{(\alpha)}\Psi(x) = F\left(x,\, C^T\Psi(x), C^T D^{(\beta_1)}\Psi(x), ..., C^T D^{(\beta_k)}\Psi(x)\right), \tag{9}$$

Moreover by substituting Eq.(8) in Eq. (7) we get

$$C^T\Psi(0) = d_0, \quad C^T D^{(\beta_i)}\Psi(0) = d_i\,, \qquad i = 0, 1, ..., n \tag{10}$$

To find the solution $y(x)$, we first collocate Eq. (9) at $m-n$ points. For suitable collocation points we use the first $m-n$ roots of shifted Chebyshev polynomial $\tilde{T}_m(x)$. These equations together with Eq. (7) generate $m+1$ nonlinear equations which can be solved using Newton's iterative method for unknown C. Consequently solution $y(x)$ can be calculated by substituting the derived vector C given in Eq. (8).

4 Illustrative Examples

In this section, we demonstrate the effectiveness of the proposed method with two illustrative examples.

Example 4.1. *Consider the following nonlinear initial value problem [3]*

$$\begin{cases} D^3 y(x) + D^{\frac{5}{2}} y(x) + y^2(x) = x^4, \\ y(0) = y^{'}(0) = 1, \; y^{''}(0) = 2. \end{cases} \tag{11}$$

By applying the technique described in section 3 with m=2, we approximate solution as

$$y(x) \simeq c_0\tilde{T}_0(x) + c_1\tilde{T}_1(x) + c_2\tilde{T}_2(x) + c_3\tilde{T}_3(x) = C^T\Psi(x) \tag{12}$$

Moreover by applying initial conditions we have a system of nonlinear equation. By solving this nonlinear system we obtain

$$\mathrm{C} = [0.375, 0.5, 0.125, -2.91 \times 10^{-19}] \tag{13}$$

For this vector C we get

$$\mathrm{y(x)} = \mathrm{C^T\Psi(x)} = x^2 - 9.31 \times 10^{-18}\, x^3 - 5.238 \times 10^{-18}\, x + 2.91 \times 10^{-19}, \tag{14}$$

which is the exact solution of this problem.

Example 4.2. *Consider the following nonlinear initial value problem [4]*

$$\begin{cases} D^{\alpha} y(x) + y(x) = 0, \qquad 0 < \alpha < 1 \\ y(0) = y^{'}(0) = 1. \end{cases} \tag{15}$$

The exact solution of this problem $y(x) = \sum_{k=0}^{\infty} \frac{(-x^k)^{\alpha}}{\Gamma(k\alpha+1)}$. We applied the method presented in Section 3 and solved this problem. Fig. 1 shows the numerical results for and $m = 20$ and $\alpha = 0.5, 0.75, 0.95$ and 1. As Fig. 1 shows, when α approach to integer 1 the numerical solution converge to exact solution $y(x) = e^{-x}$.

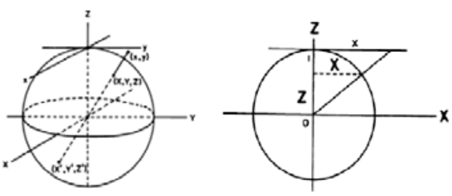

Figure 1: The numerical results for $m = 12$ and $\alpha = 0.25$ (Dots), $\alpha = 0.5$ (Dash Dots), $\alpha = 0.75$ (Dash), $\alpha = 1$ (Line).

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Application of block pulse functions to numerical solution of a stochastic SIR model

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Abstract
In this paper, we present a new technique for solving numerically stochastic SIR model based on block pulse functions. Numerical simulations are presented to illustrate our mathematical findings.
Keywords: Block pulse functions; Stochastic operational matrix; Stochastic SIR model; Itô integral; Brownian motion.

Mathematics Subject Classification: Primary: 65C30, 60H35, 65C20; Secondary: 60H20, 68U20.

1 Introduction

In modeling the spread process of infectious diseases, many classical epidemic models have been proposed and studied, such as SIR, SEIR and SIRS models. The SIR infections disease model is an important biologic model and has been studied by many authors. Most models for the transmission of infectious diseases descend from the classical SIR model of Kermack and McKendrick established in 1927, see [3]. One of the most basic SIR models is as follows

$$\begin{cases} \dot{S(t)} = \Lambda - \beta S(t)I(t) - \mu S(t), \\ \dot{I(t)} = \beta S(t)I(t) + (\mu + \epsilon + \gamma)I(t), \\ \dot{R(t)} = \gamma I(t) - \mu R(t), \end{cases} \tag{1}$$

where the parameter Λ, β, ϵ, μ, γ are positive constants, and $S(t)$, $I(t)$, $R(t)$ denote the number of the individuals susceptible to the disease, of infected members and of members who have been removed from the possibility of infection through full immunity, respectively. In this paper, our approach to include stochastic perturbation is analogous to that of Imhof and Walcher [1]. Here we assume that stochastic perturbations are of a white noise type which are directly proportional to S(t), I(t), R(t), influenced on the $\dot{S(t)}$, $\dot{I(t)}$, $\dot{R(t)}$ in the model (1). By this way, the model (1) will be deduced to the form:

$$\begin{cases} dS(t) = (\Lambda - \beta S(t)I(t) - \mu S(t))dt + \sigma_1 S(t)dB_1(t), \\ dI(t) = (\beta S(t)I(t) + (\mu + \epsilon + \gamma)I(t))dt + \sigma_2 I(t)dB_2(t), \\ dR(t) = (\gamma I(t) - \mu R(t))dt + \sigma_3 R(t)dB_3(t), \end{cases} \tag{2}$$

where $B_i(t)$ are independent standard Brownian motions and $\sigma_i^2 \geq 0$ represent the intensities of $B_i(t), i = 1, 2, 3.$[2]

In the present work, we use block pulse functions and stochastic integration operational matrix to solve this system of differential equations. The paper [4] solves stochastic Volterra integral equations by Block Pulse Functions (BPFs) and [5] apply this method for solving m-dimensional stochastic It'o Volterra integral equations.

2 Integral and stochastic integral operational matrix for Block pulse functions

The block pulse functions are defined on the time interval $[0, T)$ by

$$\phi_i(t) = \begin{cases} 1 & (i-1)\frac{T}{m} \leq t < i\frac{T}{m}, \\ 0 & elsewhere. \end{cases} \tag{3}$$

where, $i = 1, ..., m$ and for convenient we put $h = \frac{T}{m}$.
The set of block-pulse functions may be written as a vector $\Phi(t)$ of dimension m

$$\Phi(t) = [\phi_1(t), ..., \phi_m(t)]^T \quad t \in [0, T). \tag{4}$$

The integral of BPFs is defined by

$$\int_0^t \Phi(s)ds \simeq P\Phi(t), \tag{5}$$

where, P is operational matrix of integration that is given by

$$P = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{6}$$

So,

$$\int_0^t f(s)ds \simeq \int_0^t F^T\Phi(s)ds \simeq F^T P\Phi(t). \tag{7}$$

The Itô integral of BPFS is defined by

$$\int_0^t \Phi(s)dB(s) \simeq P_S\Phi(t), \tag{8}$$

where stochastic operational matrix of integration is given by

$$P_S = \begin{pmatrix} B(\frac{h}{2}) & B(h) & B(h) & \dots & B(h) \\ 0 & B(\frac{3h}{2}) - B(h) & B(2h) - B(h) & \dots & B(2h) - B(h) \\ 0 & 0 & B(\frac{5h}{2}) - B(2h) & \dots & B(3h) - B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B(\frac{(2m-1)h}{2}) - B((m-1)h) \end{pmatrix}_{m\times m} \tag{9}$$

So, the Itô integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s)dB(s) \simeq \int_0^t F^T\Phi(s)dB(s) \simeq F^T P_S\Phi(t). \tag{10}$$

3 Application of BPFs to solve SIR model

Here we apply this method for Eq.(1)

$$\begin{cases} S(t) = S_0(t) + \int_0^t \Lambda - \beta S(t)I(t)dt - \int_0^t \mu S(t)dt + \int_0^t \sigma_1 S(t)dB_1(t), \\ I(t) = I_0(t) + \int_0^t \beta S(t)I(t)dt + \int_0^t (\mu+\epsilon+\gamma)I(t)dt + \int_0^t \sigma_2 I(t)dB_2(t), \\ R(t) = R_0(t) + \int_0^t \gamma I(t)dt - \int_0^t \mu R(t)dt + \int_0^t \sigma_3 R(t)dB_3(t), \end{cases} \tag{11}$$

we approximate function $S(t), S_0(t), I(t), I_0(t), R(t), R_0(t)$ by BPFs,

$$S(t) \simeq \bar{S}(t) = S^T\Phi(t) = \Phi^T(t)S, \tag{12}$$

$$S_0(t) \simeq S_0^T\Phi(t) = \Phi^T(t)S_0, \tag{13}$$

$$I(t) \simeq \bar{I}(t) = I^T\Phi(t) = \Phi^T(t)I, \tag{14}$$

$$I_0(t) \simeq I_0^T\Phi(t) = \Phi^T(t)I_0, \tag{15}$$

$$R(t) \simeq \bar{R}(t) = R^T\Phi(t) = \Phi^T(t)R, \tag{16}$$

$$R_0(t) \simeq R_0^T\Phi(t) = \Phi^T(t)R_0, \tag{17}$$

Substituting (12-17) into (11), we get

$$\begin{cases} S^T\Phi(t) = S_0^T\Phi(t) + \int_0^t \Lambda - \beta\int_0^T S^T\Phi(t)\Phi^T(t)Idt - \mu\int_0^t S^T\Phi(t)dt + \sigma_1\int_0^t S^T\Phi(t)dB_1(t), \\ I^T\Phi(t) = I_0^T\Phi(t) + \int_0^t \beta S^T\Phi(t)\Phi^T(t)Idt + (\mu+\epsilon+\gamma)\int_0^t I^T\Phi(t)dt + \sigma_2\int_0^t I^T\Phi(t)dB_2(t), \\ R^T\Phi(t) = R_0^T\Phi(t) + \gamma\int_0^t I^T\Phi(t)dt - \mu\int_0^t R^T\Phi(t)dt + \sigma_3\int_0^t R^T\Phi(t)dB_3(t), \end{cases} \tag{18}$$

We replace $S^T\Phi(t)\Phi^T(t)I$ by $S^T\tilde{I}\Phi(t)$. So,

$$\begin{cases} S^T\Phi(t) \simeq S_0^T\Phi(t) + \int_0^t \Lambda - \beta S^T\tilde{I}P\Phi(t) - \mu S^TP\Phi(t) + \sigma_1 S^TP_S\Phi(t), \\ I^T\Phi(t) \simeq I_0^T\Phi(t) + \beta S^T\tilde{I}P\Phi(t) + (\mu+\epsilon+\gamma)I^TP\Phi(t) + \sigma_2 I^TP_S\Phi(t), \\ R^T\Phi(t) \simeq R_0^T\Phi(t) + \gamma I^TP\Phi(t) - \mu R^TP\Phi(t) + \sigma_3 R^TP_S\Phi(t), \end{cases} \tag{19}$$

where, $\tilde{X} = diag(X)$, $S^T\tilde{I}P$ are $2m-$ vectors. By replacing $\simeq$ with $=$, we have

$$\begin{cases} S^T = S_0^T + \int_0^t \Lambda - \beta S^T\tilde{I}P - \mu S^TP + \sigma_1 S^TP_S, \\ I^T = I_0^T + \beta S^T\tilde{I}P + (\mu+\epsilon+\gamma)I^TP + \sigma_2 I^TP_S, \\ R^T = R_0^T + \gamma I^TP - \mu R^TP + \sigma_3 R^TP_S. \end{cases} \tag{20}$$

After solving nonlinear system (20) we find S; I; R and finally we approximate S(t); I(t); R(t) of (11).

4 Main results

Because we can not solve some SVIEs analytically, in this article we present a new technique for solving systems of SVIEs numerically. Here, we consider block pulse functions and their operational matrix of integration. The benefits of this method are lower cost of setting up the system of equations without any integration, also, the computational cost of operations is low. These advantages make the method very simple. Efficiency of this method and good degree of accuracy is confirmed by a numerical example in SIR model.

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A new approach for determining the convergence control parameter in HAM

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Abstract
In this paper, the Homotopy Analysis Method (HAM) is used for solving a initial boundary value problem (I-BVP) of nonlinear ordinary differential equation (ODE). The aim of this paper is to introduce a different approach instead of $\hbar$-curve process can be determined the values of the convergence control parameter $\hbar$ at the any order of approximations of HAM series solutions, by minimizing the norm of residual function, automatically. The numerical results demonstrate the efficiency and performance of the proposed approach.
Keywords: Initial boundary value problem; Homotopy Analysis Method; convergence control parameter $\hbar$.

Mathematics Subject Classification: 65H20; 65L05

1 Introduction

Nowadays the HAM [5, 6, 7], is a well known method to provides the analytic solution of various types of nonlinear ODE's, PDE's and IE's [1, 2, 3, 4]. The HAM provides a convenient way to control the convergence of the series solutions by means of the convergence-control parameter $\hbar$ that is determined by plotting the $\hbar$-curve, as suggested by Liao [7]. But the $\hbar$-curve approach can not give the"'optimal" value of $\hbar$ to accelerate the convergence of series solution. In this paper, a kind of residual function is defined, which can be used to find the values convergence control parameters. An alternative approach is presented, in which the approximate values of $\hbar$ is determined by a minimizing the residual function for M order of approximation of homotopy series solution. This paper is organized as follows. In section 2, the description of new approach for finding values of $\hbar$ is presented. In Section 3, one examples of the I-BVP by using the proposed approach is presented, and the results are compared with conventional $\hbar$-curve approach. We conclude in Section 4.

2 Description of method to I-BVPs

In this, we consider the following type of nonlinear initial value problems in a finite domain as follows:

$$F(x, y, y', ..., y^{(n)}) = 0, \quad a \leq x \leq b \tag{1}$$

$$y(a) = \alpha_0, ..., y^{(n)}(a) = \alpha_n \tag{2}$$

in case $\alpha_i s$ are given constants. In order to obtain a convergent series solution to the nonlinear problem (2.1) and (2.2), HAM first construct the zeroth order deformation equation

$$((1-p)L[\phi(x,p) - y_0(x)] = p\hbar N[x, \phi(x,p)] \tag{3}$$

where nonlinear operator N is given by

$$N[y(x)] = F(x, y, y', ..., y^{(n)}) \tag{4}$$

and $y(x)$ is an unknown function, $p \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is a convergence-control parameter, and $\phi(x,p)$ is an unknown function and $y_0(x)$ is an initial guess of the solution $y(x)$ which is satisfied the conditions (2.2), and L is an auxiliary linear operator which must be properly chosen. From (2.3), when $p = 0$ and $p = 1$, we have $\phi(x,0) = y_0(x)$ and $\phi(x,1) = y(x)$, respectively. By expanding the function $\phi(x,p)$ in a Taylor series with respect to the embedding parameter p gives

$$\phi(x,p) = y_0(x) + \sum_{m=1}^{\infty} y_m(x)p^m \tag{5}$$

where $y_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x,p)}{\partial p^m}|_{p=0}$. If the above series converges at $p = 1$, we have the so-called HAM-series solution

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) \tag{6}$$

which must satisfy the original equation $N[y(x)] = 0$, as proved by Liao in [7]. Define the vector $\overrightarrow{y}_m = (y_0, y_1, ..., y_m)$, all terms in the series solution (2.6), as pointed out by Liao, are obtained by the so-called higher-order deformation equation

$$L[y_m(x) - \chi_m y_{m-1}(x)] = p\hbar R_m[\overrightarrow{y}_{m-1}(x), x] \tag{7}$$

and its initial condition $y_m(a) = 0, y'_m(a) = 0, ..., y_m^{(n)}(a) = 0$, $m \geqslant 1$, where

$$\chi_m = \begin{cases} 0, m \leq 1, \\ 1, m > 1 \end{cases}$$

and

$$R_m[\overrightarrow{y}_{m-1}(x), x] = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,p)]}{\partial p^{m-1}}|_{p=0}. \tag{8}$$

In this way, the convergence of the series (2.6) depends on the parameter $\hbar$. To find a proper convergence control parameter $\hbar$, to get a convergent series solution, there is a classic way of plotting the $\hbar$-curves". However, the $\hbar$-curves tell us only a graphically region and can not gives the suitable value of $\hbar$, but proposed approach can be performed by minimizing process, automatically.
It should be noted that based on the zeroth order deformation Equation (2.3), Mth-order HAM approximation of the solution $y(x)$, for a $\hbar$, as follows:

$$\widehat{y}_M(x,\hbar) = y_0(x) + \sum_{m=1}^{M} y_m(x,\hbar). \tag{9}$$

In order to determine a reasonable $\widehat{y}_M(x,\hbar)$, the following function of x and $\hbar$, is definied:

$$\psi_M(x) = N[x, \widehat{y}_M(x,\hbar)]. \tag{10}$$

Consider $\| \, . \, \|$ denots the L_2-norm, and $a = x_1 < x_2 < ... < x_n = b$ is a partition of $[a, b]$, then define so-called residual function:

$$\Lambda(\hbar) = \| \, (\psi_M(x_1, \hbar), ..., \psi_M(x_n, \hbar)) \|. \tag{11}$$

The value of $\hbar$ at the Mth-order of approximation is given by searches for a local minimum in the residual function $\Lambda(\hbar)$ starting from an automatically selected point.
One can prove the following proposition:

Proposition 2.1. *If the homotopy solution series converges for a value of convergence-control paramete $\hbar$, then $\forall x \in [a, b]$,*
$lim_{M\to\infty} \psi_M(x, \hbar) = 0.$

3 Numerical Examples

In this section, one differential equation is employed to illustrate the validity of new approach described in Section 2. The convergence and efficiency of this approach are investigated by comparing it with the traditional HAM approach.

Example 3.1. *Consider the following I-BVP*

$$\begin{cases} y'(x) + y(x)^2 = 1, & 0 \leq x \leq 1, \\ y(0) = 0 \end{cases} \tag{12}$$

.
which has the exact solution in the form of $y(x) = \tanh x$. For the zeroth order deformation Equation (2.3), the auxiliary linear operator L is given by $L = \frac{d}{dx}$ with the property $L[c_1] = 0$ where c_1 is arbitrary integration constant, and the nonlinear operator N is given by $N[\phi(x, p)] = \frac{d\phi(x,p)}{dx} + \phi(x, p)^2 - 1$. In view of the condition, the initial guess is determined as $y_0(x) = x$. In this example, By using new approach with $M = 5$, the proper value for $\hbar$ obtained as -0.76357. Table 1 show the comparison of the exact solution and the approximate solution obtained by HAM at $\hbar = -0.76357$ and $\hbar = -1$ for any $x \in [0, 1]$. As it can be seen in Table 1, absolute errors of the new approach for any $x \in [0, 1]$ are fixed. For new approach, the residual function $\Lambda(\hbar)$ decreases quickly as the order of approximation increases, as shown in Table 2.

Table 1. The results for 5-term approximate of HAM at $\hbar = -0.76357$ and $\hbar = -1$ in E. 3.1.

	$\hbar = -0.76357$	$\hbar = -1$
x	$\lvert y - \widehat{y}_5 \rvert$	$\lvert y - \widehat{y}_5 \rvert$
0	0	0
0.1	2.29828e-07	3.88578e-16
0.2	1.48776e-06	2.89574e-12
0.3	3.46845e-06	5.52547e-10
0.4	4.70855e-06	2.26384e-08
0.5	4.17091e-06	3.98151e-07
0.6	2.49066e-06	4.09421e-06
0.7	1.28058e-06	2.90373e-05
0.8	1.17677e-06	1.56807e-04
0.9	1.46961e-06	6.87408e-04
1.0	1.3726e-06	2.55616e-03

Table 2. The values of $\hbar$ given by the new approach at M^{th}-order approximate of HAM in E. 3.1.

M	$\hbar$	$\Lambda(\hbar)$
1	-0.662548	0.21284
3	-0.742592	2.60928e-3
5	-0.763573	5.76669e-5
7	-0.775779	1.45392e-6
9	-0.783703	3.84063e-08
11	-0.805316	4.09567e-09
13	-0.935184	3.86508e-06
15	-0.999587	1.64762e-05
16	-1.	7.13056e-06
18	-1.	1.27079e-06
20	-1.	2.16449e-07

4 Conclusion

In this paper, we applied HAM for approximate solving of the I-BVP. A value the convergence-control parameter has also been given. The results show that this method is useful for finding an accurate approximation of the exact solution. Also, this method can be used for solving a class of nonlinear differential equations.

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An adaptive solution for operator equations by using frames of translates

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Abstract

In this paper we seek an approximated solution for the operator equation $Lu = f$, where $L : H \to H$ is a bounded, invertible and symmetric operator on a separable Hilbert space H. First we find a solution based on the knowledge of a frame of translates for H. Then we design an adaptive algorithm to approximate this solution and investigate its computational complexity.

Keywords: Hilbert space, frame, adaptive algorithm,

Mathematics Subject Classification: 39B42, 65K15

1 Introduction

For a Hilbert space H and a countable set Λ, we assume that $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ is a frame for H. This means that there exist constants $0 < A_\Psi \leq B_\Psi < \infty$ such that

$$A_\Psi \|f\|_H^2 \leq \sum_\lambda |\langle f, \psi_\lambda\rangle|^2 \leq B_\Psi \|f\|_H^2, \quad \forall f \in H. \tag{1}$$

For the frame Ψ, let $F : \ell_2(\Lambda) \to H$ be the synthesis operator $F((c_\lambda)_\lambda) = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda$ and $F^* : H^* \to \ell_2(\Lambda)$ be the analysis operator $F^*(f) = (\langle f, \psi_\lambda\rangle)_\lambda$. Also let $S := FF^* : H^* \to H$ be the frame operator

$$S(f) = \sum_\lambda \langle f, \psi_\lambda\rangle \psi_\lambda.$$

Note that F^* is the adjoint of F and because of (1), F is bounded. In fact we have

$$\|F\| = \|F^*\| \leq B_\Psi^{\frac{1}{2}}.$$

It was shown in [4], for the frame Ψ, S is a self adjoint positive invertible operator satisfying $A_\Psi I_H \leq S \leq B_\Psi I_H$ and $B_\Psi^{-1} I_H \leq S^{-1} \leq A_\Psi^{-1} I_H$. Also, the sequence

$$\widetilde{\Psi} = (\widetilde{\psi}_\lambda)_{\lambda\in\Lambda} = (S^{-1}\psi_\lambda)_{\lambda\in\Lambda}$$

is a frame, that is called canonical dual frame, for H with bounds B_Ψ^{-1}, A_Ψ^{-1}. Every $u \in H$ has the expansion

$$f = \sum_\lambda \langle u, \widetilde{\psi}_\lambda\rangle \psi_\lambda = \sum_\lambda \langle u, \psi_\lambda\rangle \widetilde{\psi}_\lambda. \tag{2}$$

Since $Ker(F) = (Ran(F^*))^\perp$ we have, $\ell_2(\Lambda) = Ran(F^*) \oplus Ker(F)$. Thus the orthogonal projection Q of a sequence $(c_\lambda)_{\lambda\in\Lambda} \in \ell_2(\Lambda)$ onto the $Ran(F^*)$ is given by $Q(c_\lambda)_{\lambda\in\Lambda} \in \ell_2(\Lambda) = (\langle \sum_\lambda c_\lambda S^{-1}\psi_\lambda, \psi_j\rangle)_{j\in\Lambda}$, that is $Q = F^*S^{-1}F : \ell_2(\Lambda) \to \ell_2(\Lambda)$.

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If $\phi \in H$ and the sequence $(\phi_\lambda)_\lambda = (T_\lambda \phi)_\lambda = (\phi(. - \lambda))_\lambda$ is a frame for H, then we have a frame of translates for H. For more details see [4].
Our goal is to construct an adaptive algorithm in order to give an approximated solution for the operator equation

$$Lu = f, \tag{3}$$

where $L : H \to H$ is a bounded, invertible and symmetric operator on a separable Hilbert space. In general it is impossible to find the exact solution of the problem (3), because the separable Hilbert space H is infinite dimensional. As typical example we think of linear differential or integral equations in variational form. In [1, 5, 6], some iterative adaptive methods for solving this system has been developed.

2 Motivation and some basic facts

Lemma 2.1. *Let $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ be a frame for H and L be as in (3). Then the sequence $\Phi = (L\Psi_\lambda)_{\lambda\in\Lambda}$ is a frame for H.*

Theorem 2.2. *Let $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ be a frame for H and $\Phi = (L\Psi_\lambda)_{\lambda\in\Lambda}$. The solution u of the problem (3) has the expansion*

$$u = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle (S')^{-1}\phi_\lambda, \tag{4}$$

where s' is the frame operator of the frame Φ.

Lemma 2.3. *Let $\phi \in H$ and $\Phi = (\phi_\lambda)_\lambda = (T_\lambda\phi)_\lambda$ be a frame of translates for H. Then $S'T_\lambda = T_\lambda S'$ and $S'^{-1}T_\lambda = T_\lambda S'^{-1}$.*

By the general definition, the canonical dual frame associated with $(T_\lambda\phi)_\lambda$ is given by $(S'^{-1}T_\lambda\phi)_\lambda$. Using the previous lemma we have $(S'^{-1}T_\lambda\phi)_\lambda = (T_\lambda S'^{-1}\phi)_\lambda$. We use this result in order to find $(S'^{-1}T_\lambda\phi)_\lambda$. In fact calculation of $(S'^{-1}T_\lambda\phi)_\lambda$ only requires that we find $S'^{-1}\phi$; the rest of the functions in the family obtained by translations.
Because of (4), it is enough to solve the following problem in order to give a solution to (3).

$$S'x = \phi_l, \tag{5}$$

since the solution of (5) is the canonical dual element $x = S'^{-1}\phi_l = \widetilde{\phi}_l$, then our goal is to give an adaptive solution to (5). You can see some techniques in [2, 3] for computing the canonical dual frames.
Because of (2), the equation (5) is equivalent to $\sum_\lambda \langle x, \widetilde{\phi}_\lambda\rangle S'\phi_\lambda = \phi_l$, that is $\sum_\lambda \langle x, \widetilde{\phi}_\lambda\rangle \langle S'\phi_\lambda, \phi_{\lambda'}\rangle = \langle \phi_l, \phi_{\lambda'}\rangle, \quad \forall \lambda' \in \Lambda$ or

$$AX = \Phi_l, \tag{6}$$

where A is a matrix defined by $A_{\lambda',\lambda} = \langle S'\phi_\lambda, \phi_{\lambda'}\rangle$ and X is the column vector $(\langle x, \widetilde{\phi}_\lambda\rangle)_\lambda$ and Φ_l is the column vector $(\langle \phi_l, \phi_\lambda\rangle)_\lambda$. It is not difficult to show that $A = F^*SF$, hence the matrix A defines a bounded linear operator from $\ell_2(\Lambda)$ to $\ell_2(\Lambda)$ and also A is symmetric and positive definite.

For a tolerance $\epsilon > 0$ and a vector $V \in \ell_2(\Lambda)$ let V_N be the vector obtained by replacing all but the N largest coefficients in modulus of V by zeros, for the smallest $N \in \mathbf{N}$ such that $\|V - V_N\|_{\ell_2(\Lambda)} \le \epsilon$,. V_N is called the best N-term approximation for V. Now let $V \in \ell^2(\Lambda)$, for each $n \ge 1$ let v_n^* be the n-th largest of the number $|v_\lambda|$ and let $V^* := (v_n^*)_{n=1}^\infty$. For each $0 < \tau < 2$ we let $\ell_\tau^\omega(\Lambda)$ denote the collection of all vectors $V \in \ell^2(\Lambda)$ for which $|V|_{\ell_\tau^\omega(\Lambda)} := sup_{n\ge1}\{n^{\frac{1}{\tau}} v_n^*\}$ is finite. This expression defines a quasi norm for $\ell^{\omega}_{\tau}(\Lambda)$ and defining $\|V\|_{\ell^{\omega}_{\tau}(\Lambda)} := |V|_{\ell^{\omega}_{\tau}(\Lambda)} + \|V\|_{\ell^2(\Lambda)}$, we have a norm for $\ell^{\omega}_{\tau}(\Lambda)$. Also

$$|V+W|_{\ell^{\omega}_{\tau}(\Lambda)} \preceq (|V|_{\ell^{\omega}_{\tau}(\Lambda)} + |W|_{\ell^{\omega}_{\tau}(\Lambda)}), \tag{7}$$

where $a \preceq b$ means that $a \leq cb$, for a constant c.

3 Main Result

Definition 3.1. *We say that the matrix A is compressible of order $s > 0$, if there are two positive sequences $(\alpha_j)_{j\geq 0}$ and $(\beta_j)_{j\geq 0}$, that are both summable ,and for every $j \geq 0$ there exists a matrix A_j with at most $2^j\alpha_j$ nonzero entries per row and per column such that*

$$\|A - A_j\| \leq 2^{-js}\beta_j. \tag{8}$$

Assumption: we assume that $\Lambda \subset \mathbf{R}^n$ is separated that means $inf_{\lambda\neq\lambda'}\|\lambda-\lambda'\|_{\mathbf{R}^n}) > \delta > 0$ and the matrix A is compressible of order s for $s < s^* = r - n$, for some $r > n$.
It was shown in [5], such a compressible matrix maps $\ell^{\omega}_{\tau}(\Lambda)$ boundedly into itself for $\tau = (\frac{1}{2}+s)^{-1}$. Following [8], for an accuracy $\epsilon > 0$ and a finitely supported vector W with $N = \#(spport(W))$ we introduce two basic numerical ingredient that we will used in our algorithm.

COARSE $[W,\epsilon] \to (\Lambda, \bar{W})$. The output $\bar{W}$ of **COARSE**, by construction, satisfies $\|W - \bar{W}\|_{\ell_2(\Lambda} \leq \epsilon$. Also if $V \in \ell^{\omega}_{\tau}(\Lambda), \tau = (s+\frac{1}{2})^{-1}$, for some $s > 0$ then the outputs $\bar{W}$, Λ of **COARSE** $[W,(1-d)\epsilon]$ requires at most $2N$ arithmetic operations and $NlogN$ sorts,where $N = \#supp(W)$. Moreover,

$$|\bar{W}|_{\ell^{\omega}_{\tau}(\Lambda)} \preceq |V|_{\ell^{\omega}_{\tau}(\Lambda)}. \tag{9}$$

APPLY $A[W,\epsilon] \to (\Lambda, \bar{W})$. The outputs $\bar{W}$, Λ of **APPLY** $A[W,\epsilon]$ have the following properties:
(i) $\|AW - \bar{W}\|_{\ell_2(\Lambda)} \leq \epsilon$. Moreover if $W \in \ell^{\omega}_{\tau}(\Lambda)$ with $\tau = (s+\frac{1}{2})^{-1}$ then **(ii)** $\#(\Lambda) \preceq |W|^{\frac{1}{s}}_{\ell^{\omega}_{\tau}(\Lambda)}\epsilon^{-\frac{1}{s}}$ and $|\bar{W}|_{\ell^{\omega}_{\tau}(\Lambda)} \preceq |W|_{\ell^{\omega}_{\tau}(\Lambda)}$. **(iii)** The number of arithmetic operations needed to compute $\bar{W}$ is at most a fixed multiple of $(\epsilon^{-\frac{1}{s}}|W|^{\frac{1}{s}}_{\ell^{\omega}_{\tau}(\Lambda)} + N)$.
Now we prepare to construct our adaptive algorithm. Assuming Q is bounded on $\ell^{\omega}_{\tau'}(\Lambda)$ for $\tau' = (\frac{1}{2}+s')^{-1}$, $0 < s < s'$ (hence Q is bounded on $\ell^{\omega}_{\tau'}$, [7])we construct our algorithm for the target accuracy $\epsilon > 0$. At first, for some $0 < d < \frac{1}{3}$ and $\rho := \|I - \alpha A\| < 1$ (since A is a positive definite matrix this real number α exists) set

$$K := min\{k \in \mathbf{N} : \quad 3\rho^k < d \quad min\{1, [C_1C_2|I-Q|_{\ell^{\omega}_{\tau'}\to\ell^{\omega}_{\tau'}}]^{\frac{s}{s'-s}}\}\},$$

where C_1, C_2 are two constants induced from (7) and (9) for τ'.
SOLVE $[\epsilon, A, \Phi_l] \to (X_\epsilon, \Lambda_\epsilon)$
(i) Set $i = 0, X^0 = 0, \Lambda_0 = \emptyset, \epsilon_0 := \|A|^{-1}_{Ran(F^*_\Phi)}\| \quad \|\Phi_l\|_{\ell_2(\Lambda)}$.
(ii) If $\epsilon_i \leq \epsilon$ stop and set $X_\epsilon := X^i$, otherwise
(ii.1) $i := i+1, \quad \epsilon_i := 3\rho^K\frac{\epsilon_{i-1}}{d}$
(ii.2) $\Phi^i_l :=$ **COARSE** $[\Phi_l, \frac{d\epsilon_i}{6\alpha K}]$
(ii.3) $V^{(i,0)} := X^{i-1}$
(ii.4) For $j = 1, ..., K$ compute
(1) $W^{j-1} :=$ **APPLY** $A[V^{(i,j-1)}, \frac{d\epsilon_i}{6\alpha K}])$
(2) $V^{(i,j)} := V^{(i,j-1)} + \alpha(\Phi^i_l - W^{j-1})$
(iii) $X^i :=$ **COARSE** $[V^{(i,K)}, (1-d)\epsilon_i]$ and go to **(ii)**.

Theorem 3.2. *If X be a solution for (6) then the following inequalities hold for the algorithm* **SOLVE***:*

$$\|Q(X - X_\epsilon)\| \leq \epsilon \ , \quad \|QX + (I - Q)X^{i-1} - V^{(i,K)}\| \leq \frac{2}{3} d\epsilon_i, \qquad (i \geq 1).$$

Theorem 3.3. *Assume that the solution X of (6) belongs to ℓ^ω_τ. Then $\#(supp(X_\epsilon)) \preceq \epsilon^{-\frac{1}{s}} |X|^{\frac{1}{s}}_{\ell^\omega_\tau}$. Also the number of arithmetic operations needed to compute X_ϵ is bounded by a multiple of $\epsilon^{-\frac{1}{s}} |X|^{\frac{1}{s}}_{\ell^\omega_\tau}$.*

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Numerical solution of a class of fractional differential equations

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Abstract
In this paper, a class of fractional differential equations is converted to Volterra integral equations with weakly singular kernels, i.e. Generalized Abel integral equations and then the resulted integral equations are solved by using the techniques of product integration. Numerical results are presented to show the efficiency and simplicity of the proposed method.
Keywords: Fractional differential equations, Volterra integral equations with weakly singular kernel, Generalized Abel integral equations

Mathematics Subject Classification: 34A08, 45E10, 45D05

1 Introduction

Fractional differential equations (FDEs) have been the focus of many researches, due to their applicability to model a large number of physical and engineering problems, especially in fluid mechanics, viscoelasticity and biology [3]. Consequently a considerable attention has been made to solve FDEs. Since there is not exact analytical solution in many cases, several numerical techniques have been proposed for solving FDEs. Some of these methods are homotopy analysis method [2], finite element method [6], Legendre polynomial approximation along with variational iteration method [7] and Pseudo-spectral methods [4].

Several definitions of fractional derivatives exist in literature, among them the Riemann-Liouville and Caputo are the most common. Each definition utilize Riemann-Liouville fractional integral operator (See Def. 1.1) and integer order derivatives. The difference of these definitions is in the order of evaluation.

Definition 1.1. *The Riemann-Liouville fractional integral operator of order $q > 0$ is defined by [5]*

$$I^{(q)}y(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t \frac{y(x)}{(t-x)^{1-q}} dx, & q > 0, \\ y(t), & q = 0, \end{cases} \tag{1}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 1.2. *The Riemann-Liouville fractional differential operator of order $q > 0$ is defined by [5]*

$$D^q y(t) = y^{(q)}(t) = \begin{cases} \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{y(x)}{(t-x)^{q-n+1}} dx, & 0 \leq n-1 < q < n, \\ \frac{d^n y(t)}{dt^n}, & q = n. \end{cases} \tag{2}$$

It can be easily seen that for $\beta > -1$ and $q > 0$ we have

$$D^q t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta+q-1)} t^{\beta-q} & q-\beta \notin \mathbb{N} \\ 0 & q-\beta \in \mathbb{N} \end{cases} \tag{3}$$

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In this paper, we aim to effectively employ product integration rules to the following class of fractional differential equations,

$$a_0\tilde{y}^{(n)}(t) + \sum_{k=1}^{m} a_k \tilde{y}^{(q_k)}(t) = f(t), \tag{4}$$

$$\tilde{y}^{(i)}(0) = \beta_i, \quad i = 0, \ldots, n-1, \tag{5}$$

where $\tilde{y}^{(q_k)}$ is the fractional derivative in the Riemann-Liouville sense, $m \in \mathbb{N}$, $a_k \in \mathbb{R}$, $\beta_k \in \mathbb{R}$, $n-1 < q_k < n$, $n \in \mathbb{N}$ and f is a given function. To do this, we first rewrite Eq. (4) along with (5) as an equivalent integral equation with singular kernel i.e. generalized Abel equation and then we apply the product integration techniques to solve it numerically. We also discuss the solvability of the problem and the error analysis for the proposed method. The resulted linear systems for numerical solutions are triangular and so the method is very fast and simple.

2 Applications

At first, we show that without loss of generality, the initial conditions in (5) can be assumed homogeneous.
Let $y(t) = \tilde{y}(t) - \sum_0^{n-1} \beta_i t^i$, then by the linearity of the fractional derivative, the following FDE with homogeneous initial conditions is obtained.

$$a_0 y^{(n)}(t) + \sum_{k=1}^{m} a_k y^{(q_k)}(t) = g(t), \tag{6}$$

$$y^{(i)}(0) = 0 \quad i = 0 \ldots n-1, \tag{7}$$

where $g(t) = f(t) + \sum_{k=0}^{n-1} \beta_k D^{q_k} t^k$ and for each k, $D^{q_k} t^k$ is determined by (3).

Based on Def. 1.2, the fractional differential equation (6) can be written as

$$a_0 y^{(n)}(t) + \sum_{k=1}^{m} a_k \frac{d^n}{dt^n} \int_0^t \frac{y(x)}{(t-x)^{q_k-n+1}} dx = g(t), \tag{8}$$

By integration n times we have the following weakly singular integral equation which is a special case of generalized Abel equation:

$$a_0 y(t) + \sum_{k=1}^{m} a_k \int_0^t \frac{y(x)}{(t-x)^{q_k-n+1}} dx = F(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1}, \tag{9}$$

where $c_i, \quad i = 0 \ldots n-1$, are determined by initial conditions (7).
Also $F(t) = \int_0^t \int_0^{x_n} \int_0^{x_{n-1}} \cdots \int_0^{x_2} f(x_1) dx_1 dx_2 \cdots dx_n$, $(ntimes)$. Let $G_k(t) = \int_0^t \frac{y(x)}{(t-x)^{q_k-n+1}} dx$. By (7), we have $G_k^{(i)} = 0, i = 0, \ldots, (n-1)$ and so all c_i's are zero.

Remark 2.1. *Except for the cases where $F(t)$ is computed explicitly by integration n times, we can consider*

$$F(t) = \frac{1}{\Gamma(n)} \int_0^t (t-x)^{n-1} g(x) dx \tag{10}$$

which can be view as an integral operator defined by Def. (1.1) or proved by induction easily.

Remark 2.2. *When $a_0 = 0$ in the FDE (6), the integral equation (9) is a Volterra integral equation of the first kind, otherwise it is of second kind. In each case, the linear system obtained has special structure and can be solved easily.*

Now we present a method of solving (9). Suppose we seek the solution in the interval $[0, T]$. Consider an evenly distributed partition of the interval to N subinterval, $t_i := ih, i := 0, 1, \dots, N$, $h := \frac{T}{N}$, and let $y_i := y(t_i)$, $F_i = F(t_i)$, then write the Eq. (9) in t_i

$$a_0 y_i + \sum_{k=1}^{m} a_k \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{P_j y(x)}{(t_i - x)^{q_k - n+1}} dx = F_i, \quad i = 1, \dots, N, \tag{11}$$

We approximate $y(x)$ in each subinterval of integration $[t_j, t_{j+1}]$ by a suitable interpolation polynomial, say a straight line, and then any integral in the above system is exactly determined. If we assume that $P_j y(x) = \frac{1}{h}((x - t_j)y_{j+1} - (x - t_{j+1})y_j)$, then

$$a_0 y_i + \sum_{k=1}^{m} a_k \sum_{j=0}^{i-1} \frac{A_{i,j,k} y[j+1] - B_{i,j,k} y[j]}{-Q_k(-Q_k + 1)} = F_i, \quad i = 1, \dots, N, \tag{12}$$

where $A_{i,j,k} = (Q_k - w_i^j)(h(w_i^j - 1))^{-Q_k} + (hw_i^j)^{-Q_k} w_i^j$, $B_{i,j,k} = (w_i^j - 1)(h(w_i^j - 1))^{-Q_k} - (hw_i^j)^{-Q_k}(w_i^j - 1 + Q_k)$, $w_i^j = i - j, Q_k = q_k - n$.

It is seen from (12), that this linear system is lower triangular and in the sense of computational cost, the proposed method is efficient.

3 Numerical examples

In order to illustrate the effectiveness of the proposed method, we consider the following examples.

Example 3.1. *Following [5], we first consider a simple fractional differential equation,*

$$y^{(1.5)} = 12t - 6t^{0.5}, \tag{13}$$

subject to

$$y(0) = 0, \quad y'(0) = 0. \tag{14}$$

The exact solution of this equation is $y(t) = 12\Gamma(2)t^{2.25}/\Gamma(3.5) - 6\Gamma(1.5)t^2/\Gamma(3)$. *We solve the linear system (12) for* $N = 20$ *and the results are shown in Table 1.*

Example 3.2.

$$y''(t) + y^{(3/2)}(t) = 4\sqrt{t}/\sqrt{\pi} + 2, \quad y(0) = 0, \quad y'(0) = 0. \tag{15}$$

4 Conclusions

In this paper, we propose a numerical method for solving a class of fractional differential equations by using product integration techniques. Numerical examples show that the proposed method is quiet efficient and the linear systems are easy to solve. Computations are made by using Maple.

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Bifurcation analysis of a simplified BAM neural network model with time delays

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Abstract

In this paper, a five-neuron bidirectional associative memory (BAM) neural network with two time delays is studied. Since the study of Hopf bifurcation is very important for the design and application of BAM neural networks, we investigate that Hopf bifurcation occurs and a family of periodic solutions appear when the sum of two delays passes through a critical value.

Keywords: Neural Network, Hopf bifurcation, Periodic solutions, Time delay.

Mathematics Subject Classification: 68T05, 37G15, 34D20, 37L10.

1 Introduction

The bidirectional associative memory (BAM) networks were first introduced by Kasko [3, 5]. The properties of periodic solutions are significant in many applications. It is well known that BAM NNs are able to store multiple patterns, but most of NNs have only one storage pattern or memory pattern. BAM NNs have practical applications in storing paired patterns or memories and have the ability of searching the desired patterns through both forward and backward directions.

The delayed BAM neural network is described by the following system:

$$\begin{cases} \dot{x_i}(t) = -\mu_i x_i(t) + \sum_{j=1}^{m} c_{ji} f_i(y_j(t-\tau_{ji})) + I_i & (i = 1, 2, \ldots, n) \\ \dot{y_j}(t) = -\upsilon_j y_j(t) + \sum_{i=1}^{n} d_{ij} g_j(x_i(t-\sigma_{ij})) + J_j & (j = 1, 2, \ldots, m) \end{cases} \tag{1}$$

where c_{ji} and d_{ij} are the connection weights through the neurons in two layers: the X-layer and the Y-layer. The stability of internal neuron processes on the X-layer and Y-layer are described by μ_i and υ_j, respectively. On the X-layer, the neurons whose states are denoted by $x_i(t)$ receive the input I_i and the inputs outputted by those neurons in the Y-layer via activation function f_i, while the similar process happens on the Y-layer. Also, τ_{ji} and σ_{ij} correspond to the finite time delays of neural processing and delivery of signals. For further details, see [3, 5].

Since a great number of periodic solutions indicate multiple memory patterns, the study of Hopf bifurcation is very important for the design and application of BAM NNs. In fact, various local periodic solutions can arise from the different equilibrium points of BAM NNs by applying Hopf bifurcation technique. But the exhaustive analysis of the dynamics of such a large system is complicated, so some authors have studied the dynamical behaviours of simplified systems [1, 2, 4, 6, 7, 8, 9, 10].

Motivated by the above, in this paper, we consider the following five-neuron BAM neural network:

$$\begin{cases} \dot{x_1}(t) = -\mu_1 x_1(t) + c_{11} f_1(y_1(t-\tau_2)) + c_{31} f_1(y_3(t-\tau_2)) \\ \dot{x_2}(t) = -\mu_2 x_2(t) + c_{22} f_2(y_2(t-\tau_2)) + c_{32} f_2(y_3(t-\tau_2)) \\ \dot{y_1}(t) = -\upsilon_1 y_1(t) + d_{11} g_1(x_1(t-\tau_1)) + d_{21} g_1(x_2(t-\tau_1)) \\ \dot{y_2}(t) = -\upsilon_2 y_2(t) + d_{12} g_2(x_1(t-\tau_1)) + d_{22} g_2(x_2(t-\tau_1)) \\ \dot{y_3}(t) = -\upsilon_3 y_3(t) + d_{13} g_3(x_1(t-\tau_1)) + d_{23} g_3(x_2(t-\tau_1)) \end{cases} \tag{2}$$

where $\mu_i > 0 (i = 1, 2)$ and $\upsilon_j > 0 (j = 1, 2, 3)$. The time delay from the X-layer to another Y-layer is τ_1, while the time delay from the Y-layer back to the X-layer is τ_2. In the next section, we state our main results on the Hopf bifurcation analysis of the system (2). We should mention that it is the first time to deal with (2).

2 Main Result

To establish the main results for system (2.1), it is necessary to make the following assumption:

$$\text{(H1)} \quad f_i,\, g_j \in C^n, \quad f_i(0) = g_i(0) = 0, \quad (i = 1, 2;\, j = 1, 2, 3)$$

It is easy to see that the origin is an equilibrium point of (2). Under the hypothesis (H1) and letting $u_1(t) = x_1(t-\tau_1)$, $u_2(t) = x_2(t-\tau_1)$, $u_3(t) = y_1(t)$, $u_4(t) = y_2(t)$, $u_5(t) = y_3(t)$ and $\tau = \tau_1 + \tau_2$, we can get the associated characteristic equation of (2):

$$\lambda^5 + a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e + (a_1\lambda^3 + b_1\lambda^2 + c_1\lambda + d_1)e^{-\lambda\tau} + (a_2\lambda + b_2)e^{-2\lambda\tau} = 0 \tag{3}$$

Now, by assuming

$$\text{(H2)} \quad a_1 = b_1 = c_1 = d_1 = 0,$$

it can be proved that $\lambda = i\omega (\omega > 0)$ is a root of (3) if and only if $z = \omega^2$ satisfies

$$z^5 + pz^4 + qz^3 + rz^2 + sz + v = 0. \tag{4}$$

Then by assuming $h(z) = z^5 + pz^4 + qz^3 + rz^2 + sz + v$ and z_k^*, $k = 1, 2, 3, 4, 5$ as the positive roots of (4), we have

$$\tau_0 = min_{k \in \{1,\ldots,5\}} \frac{1}{2\omega_k} [cos^{-1}(\frac{a_2\omega_k^6 + (ab_2 - a_2 b)\omega_k^4 + (da_2 - cb_2)\omega_k^2 + eb_2}{-b_2^2 - a_2^2\omega_k^2})],$$

where $\omega_k = \sqrt{z_k^*}$.

Letting $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ and $\alpha(\tau_0) = 0$, $\omega(\tau_0) = \omega_0$, we can state the following theorem:

Theorem 2.1. *Suppose $z_0 = \omega_0^2, h'(z_0) \neq 0$. Then, at $\tau = \tau_0$, $\pm i\omega_0$ is a pair of simple purely imaginary roots of (3) and $\frac{dRe(\lambda(\tau_0))}{d\tau} \neq 0$.*

Proof. By differentiating equation (3) with respect to τ, we can easily prove this theorem. □

Theorem 2.2. *Assume that (H1) and (H2) hold. (a) if $v < 0$, then the zero solution of system (2) is asymptotically stable for all $\tau \in [0, \tau_0)$. (b) if $v < 0$ and $h'(z_0) \neq 0$, then system (2) undergoes a Hopf bifurcation at the zero solution when τ passes through τ_0.*

Proof. It should be noted that when $v < 0$, the equation (4) has at least one positive root. By using this fact and bifurcation theory, this theorem follows from Theorem 2.1. □

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Exact solutions of the nonlinear equations using $(\frac{G'}{G})$-expansion method

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Abstract

In this paper, we obtain the exact solutions of Lienard equation using $(\frac{G'}{G})$-expansion method. The solutions obtained here are expressed in hyperbolic functions.

Keywords: Lienard equation, $(\frac{G'}{G})$-expansion method, Hyperbolic function solution.

1 Introduction

The research area of nonlinear equations has been very active for the past few decades. There are many kinds of nonlinear equations that appear in various areas of physical and mathematical sciences [1, 2, 3, 4, 12, 14, 18]. Much effort has been made on the construction of exact solutions of such nonlinear equations. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optimal fiber, biology, solid state physics, chemical physics and geometry.

In recent years, many powerful and efficient methods to find analytic solutions of nonlinear equations have been presented by a diverse group of scientists. These methods include the tanh-function method, homogeneous balance method, the extended tanh-function method [5, 9] and the sine-cosine method [11, 12].

Recently, the $(\frac{G'}{G})$-expansion method, first introduced by Wang et al. [13], has been widely used to obtain exact solutions of nonlinear equations[1, 2, 3, 4, 13, 14, 15, 18]. This method is based on the explicit linearization of nonlinear evolution equations for travelling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a set of algebric equations by simple algebraic computation. In this paper, we utilize the $(\frac{G'}{G})$-expansion method and obtain exact solutions of Lienard equation [16]:

$$\begin{aligned}&u''(x) - u(x) + 4u^3(x) - 3u^5(x) = 0\\&u(0) = \tfrac{\sqrt{2}}{2}\\&u'(0) = \tfrac{\sqrt{2}}{4}\end{aligned}$$

2 Main Result

3 The basic idea of $(\frac{G'}{G})$-expansion method

In this section we recall the basic idea of the $(\frac{G'}{G})$-expansion method.

The $(\frac{G'}{G})$-expansion method is based on the assumption that the travelling wave solution of ODE's as follow:

$$N(u, u', u'', u^{(3)}, u^{(4)}, ...) = 0. \tag{1}$$

can be expressed by a polynomial in $(\frac{G'}{G})$ as

$$u(x) = \sum_{i=0}^{n} A_i \left(\frac{G'}{G}\right)^i, \quad A_n \neq 0, \tag{2}$$

where $G = G(x)$ satisfies the second-order linear ODE

$$G'' + \lambda G' + \mu G = 0, \tag{3}$$

and A_i $(i = 0, 1, 2, ..., n)$, λ, μ are constants to be determined later. The general solution of (3) can easily be written as [4]

$$\frac{G'}{G} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}x + c_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}x}{c_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}x + c_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}x}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{c_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2}x - c_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2}x}{c_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2}x + c_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2}x}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0, \end{cases} \tag{4}$$

where c_1 and c_2 are arbitrary constants.
Now $u(x)$ can be determined explicitly by using the following three steps:

- Step (1). By considering the homogeneous balance between the highest nonlinear terms and the highest order derivatives of $u(x)$ in Eq.(5), the positive integer n in (2) is determined.
- Step (2). By substituting (2) with Eq.(4) into (5) and collecting all terms with the same powers of $(\frac{G'}{G})$ together, the left hand side of Eq.(5) is converted into a polynomial. After setting each coefficient of this polynomial to zero, we obtain a set of algebraic equations in terms of A_i $(i = 0, 1, 2, ..., n)$, c, λ, μ.
- Step (3). Solving the system of algebraic equations and then substituting the results and the general solutions of (4) into (2) gives travelling wave solutions of (5).

4 Exact solutions of the Lienard equation

In this section, we will utilize the $(\frac{G'}{G})$-expansion method to obtain exact solutions of Lienard equation. Let us consider the Lienard equation:

$$u''(x) - u(x) + 4u^3(x) - 3u^5(x) = 0. \tag{5}$$

By the balancing procedure we get $n = \frac{1}{2}$, By the transformation $u(x) = v^{\frac{1}{2}}(x)$, Eq.(5) becomes:

$$\frac{1}{2}v''v - \frac{1}{4}v'^2 - v^2 + 4v^3 - 3v^4 = 0 \tag{6}$$

With the balancing procedure we get $n = 1$, therefore the solution of (6) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$v(x) = A_0 + A_1 \left(\frac{G'}{G}\right), \quad A_1 \neq 0, \tag{7}$$

where G is the solution of (3). Substituting (7) into (6) and making use of (4) and equating each coefficient of this polynomial to zero, we obtain a set of nonlinear algebraic equations for A_0,A_1,c,λ and μ. Solving this system using *Mathematica*, we obtain

$$\begin{array}{llll} Case1: & A_0 = \frac{2-\lambda}{4}, & A_1 = -\frac{1}{2}, & \lambda = \pm 2\sqrt{1+\mu} \\ Case2: & A_0 = \frac{2+\lambda}{4}, & A_1 = \frac{1}{2}, & \lambda = \pm 2\sqrt{1+\mu} \end{array}. \tag{8}$$

Using these values (Case1) of A_0 and A_1, in (7) we obtain

$$v(x) = \frac{2-\lambda}{4} - \frac{1}{2}(\frac{G'}{G}). \tag{9}$$

Substituting the general solutions of (3) into (9) we obtain travelling wave solution, namely,

$$v(x) = \frac{1}{2} - \frac{\sqrt{\lambda^2-4\mu}}{4}\left(\frac{c_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}x + c_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}x}{c_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}x + c_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}x}\right); \tag{10}$$

where c_1 and c_2 are arbitrary constants. Using the transformation $u(x)=v^{\frac{1}{2}}(x)$, we have

$$u(x) = \left\{\frac{1}{2} - \frac{\sqrt{\lambda^2-4\mu}}{4}\left(\frac{c_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}x + c_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}x}{c_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}x + c_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}x}\right)\right\}^{\frac{1}{2}}; \tag{11}$$

In particular, if we choose $c_2 \neq 0, \quad c_1^2 < c_2^2$, then the solution (11) gives the solitary wave solution

$$u(x) = \left\{\frac{1}{2} - \frac{\sqrt{\lambda^2-4\mu}}{4}\left[\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}x + x_0)\right]\right\}^{\frac{1}{2}}, \tag{12}$$

Where $x_0 = \tanh^{-1}\frac{c_1}{c_2}$, With consider initial conditions and using (8) , we obtain:

$$u(x) = \sqrt{\frac{1-\tanh(x)}{2}}, \tag{13}$$

Using values (8) (Case2) of A_0 and A_1, in (7) we obtain

$$v(x) = \frac{2+\lambda}{4} + \frac{1}{2}(\frac{G'}{G}). \tag{14}$$

Substituting the general solutions of (3) into (14) we obtain travelling wave solution, namely,

$$v(x) = \frac{1}{2} + \frac{\sqrt{\lambda^2-4\mu}}{4}\left(\frac{c_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}x + c_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}x}{c_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}x + c_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}x}\right); \tag{15}$$

where c_1 and c_2 are arbitrary constants. Using the transformation $u(x)=v^{\frac{1}{2}}(x)$, we have

$$u(x) = \left\{\frac{1}{2} + \frac{\sqrt{\lambda^2-4\mu}}{4}\left(\frac{c_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}x + c_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}x}{c_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}x + c_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}x}\right)\right\}^{\frac{1}{2}}; \tag{16}$$

In particular, if we choose $c_2 \neq 0, \quad c_1^2 < c_2^2$, then the solution (16) gives the solitary wave solution

$$u(x) = \left\{\frac{1}{2} + \frac{\sqrt{\lambda^2-4\mu}}{4}\left[\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}x + x_0)\right]\right\}^{\frac{1}{2}}, \tag{17}$$

Where $x_0 = \tanh^{-1}\frac{c_1}{c_2}$, With consider initial conditions and using (8) , we obtain:

$$u(x) = \sqrt{\frac{1+\tanh(x)}{2}}, \tag{18}$$

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Numerical solution of fourth order ordinary differential equation by quintic Spline in the Neumann problem

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Abstract

In this paper, we use quintic spline method to investigate a numerical technique for solving linear fourth -order boundary -value problem in Neumann condition.we are considered numerical illustrations on the interval $[0, 1]$ when $\alpha = 0$.Also consider the maximum absolute errors in the solution at grid points and tabulated in tables for different choices of step size.

Keywords: fourth-order ordinary differentiation equation, Neumann problem, quintic spline

Mathematics Subject Classification: 56L10

1 Introduction

We consider the fourth-order boundary value problem of the form:

$$LU \equiv (t^{\alpha}u'')'' + au = g \tag{1}$$

where

$$\begin{gathered} 0 \leq \alpha \leq 4,\ t \in [0, b],\ g \in L_2(0, b),\ a = const \\ u''(0) = u'''(0) = u''(b) = u'''(b) = 0 \end{gathered} \tag{2}$$

In [1] and [2] proved to generalized solution for Neumman problem in W_{α}^{2} and shows Existence and Uniqueness of the generalized solution (1) by various $\alpha \in [0, b]$. Now, we obtain the numerical solution (1) by Quintic spline in Neumman condition and we calculate the maximum absolute error in the solution.

2 Quintic Spline Function

. we consider a uniform mesh, Δ, with nodal points, $x_{i-\frac{1}{2}}$, on $[a, b]$, such that

$$\Delta : a = x_0 < x_{\frac{1}{2}} < ... < x_N = b$$

$$x_{i-\frac{1}{2}} = a + (i - \frac{1}{2})h, \quad i = 1, ..., N$$

and step length is: $h = \frac{b-a}{N}$.

we denote a function value,$u(x_i)$ by u_i.

A Quintic spline function $S_i(x)$ interpolating to a function $u(x)$ on $[a,b]$ is defined as following condition :

1.In each subinterval,$[x_i, x_{i+1}]$,$S_i(x)$ is a polynomial of, at most, degree five;

2. The first-fourth derivatives of,$S_i(x)$ are continuous on,$[a,b]$.

3. for every $i = 0, 1, ..., N$ we consider $S_i(x) = u(x_i), i = 0, ..., N$.

The spline function,$S_i(x)$, for $x \in [x_i, x_{i+1}]$is defined by:

$$S_i(x) = \sum_{k=0}^{5} a(i,k)(x - x_i)^k \tag{3}$$

where $k = 0, 1, 2, 3, 4, 5$ are constants to be determined. We further require that the values of the first-second -third- and fourth-order derivatives are the same for the pair of segments that join at each point(x_i, u_i)

To derive an expression for the coefficients of Equation (5) in terms of $u_{i-\frac{1}{2}}, u_{i+\frac{1}{2}}, M_{i-\frac{1}{2}}$, $M_{i+\frac{1}{2}}, F_{i-\frac{1}{2}}, F_{i+\frac{1}{2}}$, we first denote:[3]

$$\begin{array}{llll} (i)S_i(x_{i-\frac{1}{2}}) = u_{i-\frac{1}{2}} & (ii)S_i(x_{i+\frac{1}{2}}) = u_{i+\frac{1}{2}} & (iii)S_i''(x_{i-\frac{1}{2}}) = M_{i-\frac{1}{2}} & (iv)S_i^{''}(x_{i+\frac{1}{2}}) = M_{i+\frac{1}{2}} \\ (v)S_i^4(x_{i-\frac{1}{2}}) = F_{i-\frac{1}{2}} & (vi)S_i^4(x_{i+\frac{1}{2}}) = F_{i+\frac{1}{2}} & & \end{array} \tag{4}$$

From algebraic manipulation, we get the $a(i,k)$[3]

the continuity of the first derivative implies:

$$\begin{aligned} M_{i-\frac{3}{2}} + 22M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}} &= \frac{h^2}{240}(7F_{i-\frac{3}{2}} - 254F_{i-\frac{1}{2}} + 7F_{i+\frac{1}{2}}) \\ &+ \frac{24}{h^2}(u_{i-\frac{3}{2}} - 2u_{i-\frac{1}{2}}) + u_{i+\frac{1}{2}}, \qquad i = 2, ..., N-1 \end{aligned} \tag{5}$$

and the continuity of the third derivative yield:

$$M_{i-\frac{3}{2}} - 2M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}} = \frac{h^2}{24}(F_{i-\frac{3}{2}} + 22F_{i-\frac{1}{2}} + F_{i+\frac{1}{2}}), \;\; i = 2, ..., N-1 \tag{6}$$

subtracting equation (6) from equation (5) we obtain:

$$M_{i-\frac{1}{2}} = \frac{1}{h^2}(u_{i-\frac{3}{2}} - 2u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}}) - \frac{h^2}{1920}(F_{i-\frac{3}{2}} + 158F_{i-\frac{1}{2}} + F_{i+\frac{1}{2}}) \tag{7}$$

Elimination of M_i's between Equations (6) and(7) leads to the following useful relation

$$\begin{aligned} &u_{i-\frac{5}{2}} - 4u_{i-\frac{3}{2}} + 6u_{i-\frac{1}{2}} - 4u_{i+\frac{1}{2}} + u_{i+\frac{3}{2}} \\ &\qquad = \frac{h^4}{1920}(F_{i-\frac{5}{2}} + 236F_{i-\frac{3}{2}} + 1446F_{i-\frac{1}{2}} + 236F_{i+\frac{1}{2}} + F_{i+\frac{3}{2}}) \quad i = 3, ..., N-2 \end{aligned} \tag{8}$$

It will be the main relation for our method. we consider Equation (1) subject to boundary conditions We discretize the given system in Equation (1) at the grid points,$x_{i-\frac{1}{2}}$,$i = 3, 4, ..., N-2$,and use the spline relation . We obtain the$(N-2)$linear algebraic equation in the(N)unknowns,$u_{i-\frac{1}{2}}$,$i =$

$1, ..., N$ as: '

$$\begin{aligned}(1+\frac{a}{1920}h^4)u_{i-\frac{5}{2}}+(-4+\frac{236a}{1920}h^4)u_{i-\frac{3}{2}}+(6+\frac{1446a}{360}h^4)u_{i-\frac{1}{2}}\\ +(-4+\frac{236a}{1920}h^4)u_{i+\frac{1}{2}}+(1+\frac{a}{1920}h^4)u_{i+\frac{3}{2}}\\ =\frac{h^4}{1920}(g_{i-\frac{5}{2}}+236g_{i-\frac{3}{2}}+1446g_{i-\frac{1}{2}}+236g_{i+\frac{1}{2}}+g_{i+\frac{3}{2}})\ \ i=3,...,N-2\end{aligned} \tag{9}$$

To obtain the unique solution of the above system, we need four more equations.

when $i = 1, 2, N-1, N$ the expressions $u_{-\frac{1}{2}}$, $u_{-\frac{3}{2}}$,$u_{N+\frac{1}{2}}$, $u_{N+\frac{3}{2}}$, $g_{-\frac{1}{2}}$, $g_{-\frac{3}{2}}$, $g_{N+\frac{1}{2}}$, $g_{N+\frac{3}{2}}$are out of $[0, 1]$, by using parameters $0 < r, s < 1$ and Lagrange multipliers ,they interpolate by $x_{i-\frac{1}{2}}$ when $i = 3, ..., N-2$ and the boundary formulas associated with boundary conditions .

$$t=\frac{u_{N+1/2}-u_{N-1/2}}{u_{N+1/2}-u_{N-3/2}-h^2u''(1)-h^3u'''(1)}$$
$$r=\frac{u_{N+3/2}-u_{N-1/2}}{u_{N+3/2}-u_{N-5/2}-h^2u''(1)-h^3u'''(1)}$$

for $i = 1, 2, N-1, N$ we have:

$$\begin{aligned}(a_{i,i}+b_{i,i}h^4)u_{i-\frac{5}{2}}+(a_{i,i+1}+b_{i,i+1}h^4)u_{i-\frac{3}{2}}+(a_{i,i+2}+b_{i,i+2}h^4)u_{i-\frac{1}{2}}\\ +(a_{i,i+3}+b_{i,i+3}h^4)u_{i+\frac{1}{2}}+(a_{i,i+4}+b_{i,i+4}h^4)u_{i+\frac{3}{2}}\\ =\frac{h^4}{1920}(g_{i-\frac{5}{2}}+236g_{i-\frac{3}{2}}+1446g_{i-\frac{1}{2}}+236g_{i+\frac{1}{2}}+g_{i+\frac{3}{2}})\end{aligned} \tag{10}$$

The scheme of Equation (9,10) along with boundary formulae yields the diagonal linear system of order $N \times N$ and may be written in matrix form as:

$AU = C + T$, that $A\overline{U} = C$, $AE = T$, $U = (u_{i-\frac{1}{2}})$, $T = (t_{i-\frac{1}{2}})$, $F = (a)$.

$A\overline{U} = (\overline{u}_{i-\frac{1}{2}})$, matrix A cab be denoted by $A = (A0 + h^4BF)$[3].

$c = [c_{\frac{1}{2}}, ..., c_{N-\frac{1}{2}}]^T$ for $i = 1, ..., N$, we have: $c_{i-\frac{1}{2}} = \frac{1}{1920}h^4(g_{i-5/2} + 236g_{i-3/2} + 1446g_{i-1/2} + 236g_{i+1/2} + g_{i+3/2})$

3 Numerical illustration

In order to test the utility of the proposed method we have solved the following example. The exact solutions are known to us. The maximum absolute errors are tabulated.

Example 3.1. *Consider the linear boundary value problem:*

$u^4(x) - 16u(x) = 96e^{2x}$ *and* $0 < x < 1$ $u''(0) = 20$,$u''(1) = 32e^2$, $u'''(0) = 52$,$u'''(1) = 76e^2$

with the theoretical solution $u(x) = (3x+2)e^{2x}$. *This problem has been solved using the Quintic spline with different values of N=10, 15, 20, 100. The maximum absolute errors in the solution are tabulated.*

Table 1: The maximum absolute errors in the solution of example 1:

step lengths	*our Method*
1/10	*1.2(-2)*
1/15	*1.1(-3)*
1/20	*3.0(-4)*
1/100	*2.1(-5)*

4 Main Result

We investigated numerical solutions of Neumann problem (1).

We have developed quintic spline for solving of fourth-order boundary-value problems. This approach has some advantages over finite difference methods that it provides continuous approximations. We applied our method to solve some problems and we compared theoretical solution with numerical solution.

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A new iterative solution method for solving multiple linear systems

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Abstract

In this paper, a new iterative solution method is proposed for solving multiple linear systems $A^{(i)}x^{(i)} = b^{(i)}$, for $1 \leq i \leq s$, where the coefficient matrices $A^{(i)}$ and the right-hand sides $b^{(i)}$ are different and arbitrary in general. This method is based on the global least squares(GL-LSQR) method. A linear operator $\mathcal{L} : \mathbb{R}^{n\times s} \to \mathbb{R}^{n\times s}$ is defined to connect all the linear systems together. To approximate all numerical solutions of the multiple linear systems simultaneously, we apply the GL-LSQR method for the operator $\mathcal{L}$ and obtain the approximate solutions recursively. We compare the presented method with the well-known LSQR method when it is applied for s systems independently. Finally, some numerical experiments on test matrices are presented to show the efficiency of the new method.

1. Introduction

We want to solve, using global least squares(GL-LSQR) method, the following linear systems:

$$A^{(i)}x^{(i)} = b^{(i)}, \qquad 1 \leq i \leq s \tag{1}$$

where $A^{(i)}$ are arbitrary matrices of order n, and in general $A^{(i)} \neq A^{(j)}$ and $b^{(i)} \neq b^{(j)}$ for $i \neq j$.

In this paper, we propose a new method to solve the linear systems (1) simultaneously, where the coefficient matrices and right-hand sides are arbitrary. We define a linear operator $\mathcal{L} : \mathbb{R}^{n\times s} \to \mathbb{R}^{n\times s}$ to connect all the linear systems (1) together. Then we apply the GL-LSQR method [2] for the linear operator $\mathcal{L}$ and obtain recursively the approximate solutions simultaneously. In the new method , the linear operator $\mathcal{L}$ will be reduced to a lower global bidiagonal, namely $\mathcal{L}$-Bidiag, matrix form. We obtain a recurrence formula for generating the sequence of approximate solutions. Our new method has certain advantages over the existent methods such as [1]. In the new method, the coefficient matrices $A^{(i)}$ and right-hand sides $b^{(i)}$ are arbitrary. Also we do not need to store the basis vectors, we do not need to predetermine a subspace dimension and the approximate solutions and residuals are cheaply computed at every stage of the algorithm because they are updated with short-term recurrence.

We use the following notations. For X and Y two matrices in $\mathbb{R}^{n\times s}$, we consider the following inner product $< X, Y >_F = tr(X^TY)$, where $tr(.)$ denotes the trace of a matrix. The associated norm is the Frobenious norm denoted by $||.||_F$. The notation $X \perp_F Y$ means that $< X, Y >_F = 0$ and $X(:, i)$ means that the ith column of X. Finally, we use the notation $*$ for the following product:

$$\mathcal{V} * y = \sum_{j=1}^{m} y_j V_j, \tag{2}$$

where $\mathcal{V} = [V_1, V_2, ..., V_m], V_j \in \mathbb{R}^{n\times s}$ for $1 \leq j \leq m$, and $y \in \mathbb{R}^m$.
By the same way, we define

$$\mathcal{V} * T = [\mathcal{V} * T(:, 1), \mathcal{V} * T(:, 2), ..., \mathcal{V} * T(:, m)], \tag{3}$$

where T is the $m \times m$ matrix. It is easy to show that the following relations are satisfied:

$$\mathcal{V} * (y + z) = \mathcal{V} * y + \mathcal{V} * z, \quad (\mathcal{V} * T) * y = \mathcal{V} * (Ty), \tag{4}$$

where y and z are two vectors of $\mathbb{R}^m$.

The GL-LSQR algorithm can be referred to [2].

2. The GL-LSQR-like operator method

For solving the linear systems (1), we define the following linear operator

$$\mathcal{L} \quad : \quad \mathbb{R}^{n\times s} \longrightarrow \mathbb{R}^{n\times s} \tag{5}$$

$$\mathcal{L}(X) \;=\; [A^{(1)}X(:,1),...,A^{(s)}X(:,s)], \qquad \mathcal{L}^T(X) = [{A^{(1)}}^T X(:,1),...,{A^{(s)}}^T X(:,s)], \tag{6}$$

where $A^{(j)}$, $j = 1,...,s$ are the coefficient matrices of the multiple linear systems (1). Therefore, the linear systems (1) is written as:

$$\mathcal{L}(X) = B, \tag{7}$$

where B is an $n\times s$ rectangular matrix whose columns are $b^{(1)}, b^{(2)},...,b^{(s)}$ the right hand sides of the linear systems (1).

Definition 1: Let $\mathcal{L}$ be linear operator (5) and $\mathcal{V}_m = [V_1, V_2,...,V_m] \in \mathbb{R}^{n\times ms}$. Then

$$\mathcal{L}(\mathcal{V}_m) = [\mathcal{L}(V_1), \mathcal{L}(V_2),...,\mathcal{L}(V_m)] \in \mathbb{R}^{n\times ms}.$$

where $V_j \in \mathbb{R}^{n\times s}$, $j = 1, 2,...,s$.

To approximate the solution of the block operator equation (7), we present a new algorithm, will be referred to $\mathcal{L}$-GL-LSQR algorithm, which is based on the Global-Bidiag-like procedure, will be referred to $\mathcal{L}$-Bidiag. The $\mathcal{L}$-Bidig procedure reduces the linear operator $\mathcal{L}$ to the lower bidiagonal matrix form. This procedure can be described as follows.

$\mathcal{L}$-Bidiag (starting matrix B; reduction to lower bidiagonal matrix form):

$$\left.\begin{array}{c} \beta_1 U_1 = B, \;\; \alpha_1 V_1 = \mathcal{L}^T(U_1) \\ \beta_{i+1}U_{i+1} = \mathcal{L}(V_i) - \alpha_i U_i \\ \alpha_{i+1}V_{i+1} = \mathcal{L}^T(U_{i+1}) - \beta_{i+1}V_i \end{array}\right\}, \qquad i = 1, 2, ..., \tag{8}$$

where $U_i, V_i \in \mathbb{R}^{n\times s}$. The scalars $\alpha_i \geq 0$ and $\beta_i \geq 0$ are chosen so that $||U_i||_F = ||V_i||_F = 1$.

We define

$$\begin{array}{c} \mathcal{U}_m \equiv [U_1, U_2,...,U_m], \\ \mathcal{V}_m \equiv [V_1, V_2,...,V_m], \end{array} \qquad T_m \equiv \begin{bmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_m & \alpha_m \\ & & & \beta_{m+1} \end{bmatrix}.$$

According to notation $*$ and by using the definition 1, the recurrence relations (8) may be rewritten as:

$$\mathcal{U}_{m+1} * (\beta_1 e_1) \;=\; B, \tag{9}$$

$$\mathcal{L}(\mathcal{V}_m) \;=\; \mathcal{U}_{m+1} * T_m. \tag{10}$$

The quantities generated from the linear operator $\mathcal{L}$ and B by the $\mathcal{L}$-Bidiag process will now be used to solve the block least squares problem,

$$\min_X ||B - \mathcal{L}(X)||_F.$$

Let the quantities

$$X_m = \mathcal{V}_m * y_m, \tag{11}$$

$$R_m = B - \mathcal{L}(X_m), \tag{12}$$

be defined, where $y_m \in \mathbb{R}^m$. According to linearity of the operator , it is easy to show that

$$\mathcal{L}(X_m) = \mathcal{L}(\mathcal{V}_m) * y_m.$$

Also it readily follows from (15), (16) and properties of product $*$ the equation

$$R_m = B - \mathcal{L}(\mathcal{V}_m) * y_m = \mathcal{U}_{m+1} * (\beta_1 e_1) - (\mathcal{U}_{m+1} * T_m) * y_m = \mathcal{U}_{m+1} * (\beta_1 e_1 - T_m y_m),$$

holds to working accuracy.

To minimize the mth residual $||R_m||_F$, since $\mathcal{U}_{m+1}$ is F-orthonormal and by using the proposition 3, we choose y_m so that

$$||R_m||_F = ||\beta_1 e_1 - T_m y_m||_2, \tag{13}$$

is minimum. This minimization problem is carried out by applying the QR decomposition [3].

The main steps of the $\mathcal{L}$-GL-LSQR algorithm can be summarized as follows.

Algorithm: $\mathcal{L}$-GL-LSQR algorithm

1. Set $X_0 = 0$

$\beta_1 = ||B||_F$, $U_1 = B/\beta_1$, $\alpha_1 = ||\mathcal{L}^T(U_1)||_F$, $V_1 = \mathcal{L}^T(U_1)/\alpha_1$.
Set $W_1 = V_1$, $\bar{\phi}_1 = \beta_1$, $\bar{\rho}_1 = \alpha_1$
For $i = 1, 2, ...$ until convergence, Do:
    $\hat{W}_i = \mathcal{L}(V_i) - \alpha_i U_i$
    $\beta_{i+1} = ||\hat{W}_i||_F$
    $U_{i+1} = \hat{W}_i/\beta_{i+1}$
    $S_i = \mathcal{L}^T(U_{i+1}) - \beta_{i+1}V_i$
    $\alpha_{i+1} = ||S_i||_F$
    $V_{i+1} = S_i/\alpha_{i+1}$
    $\rho_i = (\bar{\rho}_i^2 + \beta_{i+1}^2)^{1/2}$
    $c_i = \bar{\rho}_i/\rho$
    $s_i = \beta_{i+1}/\rho_i$
    $\theta_{i+1} = s_i\alpha_{i+1}$
    $\bar{\rho}_{i+1} = c_i\alpha_{i+1}$
    $\phi_i = a_i\bar{\phi}_i$
    $\bar{\phi}_{i+1} = c_i\bar{\phi}_i$
    $\phi_i = c_i\bar{\phi}_i$
    $\bar{\phi}_{i+1} = -s_i\bar{\phi}_i$
    $X_i = X_{i-1} + (\phi_i/\rho_i)W_i$
    $W_{i+1} = V_{i+1} - (\theta_{i+1}/\rho_i)W_i$
    If $|\bar{\phi}_{i+1}|$ is small enough then stop
EndDo.

As we observe, the $\mathcal{L}$-GL-LSQR algorithm has certain advantages, we obtain simultaneously the approximate solution of the multiple linear systems (1). Also the residual norm is cheaply computed at every stage of the algorithm.

3. Numerical experiments

In this section, all the numerical experiments were computed in double precision with some MATLAB codes. For all the examples the initial guess X_0 was taken to be zero matrix. The numerical experiments contains tridiagonal matrices of the form

$$A^{(i)} = tridiag(-1 - ih, 2, -1 + ih), \qquad i = 1, 2, ..., s,$$

where $h = \frac{1}{n+1}$, $n = dim(A^{(i)})$.
We consider the right-hand side $b^{(i)}$=$rand(n, 1)$, where function *rand* creates an $n \times 1$ random matrix. All the tests were stopped as soon as, $|\bar{\phi}| \leq 10^{-7}$.

To validate our claim we display the convergence history in Fig. 1.

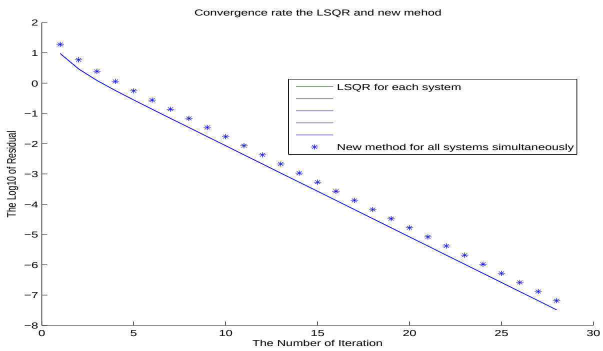

Figure 1: Convergence history of the new algorithm for the first matrices set with s=4 and n=3000.

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Approximate solutions of an inverse nonlinear diffusion problem with a nonlocal constraint

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Abstract

In this paper, we use potential system form of the nonlinear diffusion equation, for obtaining approximate solution of the one dimensional inverse nonlinear diffusion equation with a nonlocal constraint in place of initial boundary condition. By using the invariance surface conditions related to the equivalence transformations, we reduce the system under consideration to a new system of partial differential equations, where the new independent variables are similarity variables of the equivalence transformations. Under suitable hypothesis, by expanding the dependent variables of reduced system in series, we will find an approximate solution.
Keywords: Diffusion equation, Equivalence transformations, Non-classical symmetries, Infinitesimal transformations, Invariance surface conditions, unperturbed system.

Mathematics Subject Classification: 35R30

1 Introduction

The formulation of the physical problem considered in this paper is given in dimensionless form as

$$u_t = \frac{1}{Cu^2}(u^2 u_x)_x, \quad 0 < x < \infty,\ 0 < t < T, \tag{1}$$

subjected to the given boundary condition

$$u(0,t) = g(t), \qquad 0 \le t \le T, \tag{2}$$

and the nonlocal boundary condition

$$h(t) = \int_0^\infty u(x,t)dx, \tag{3}$$

where $g(t)$ and $h(t)$ are known functions and $f, g \in C[0,T], g(t) \neq 0$ and $u(x,t)$ must be determined in $0 \le t \le T$ $(T \ll 1)$. Here, $u(x,t)$ is the concentration of substance.[1] We reformulate the nonlocal boundary condition (3) to the separated Neumann type condition

$$u_x(0,t) = \frac{(h(t)'}{(g(t))^2}. \tag{4}$$

Here, we assume that

$$\lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} u_x(x,t) = 0. \tag{5}$$

The following equation used to describe the heat conduction in homogeneous isotropic rigid body

$$\tau\partial_t \mathbf{q} + (1 - \tau L^{'} L^{-1} T_t)\mathbf{q} + L\nabla T = 0, \tag{6}$$

When (6) is combined with the energy equation we obtain the *hyperbolic heat equation*,

$$\partial_t T + \tau\partial_t^2 T = \alpha\nabla^2 T. \tag{7}$$

Equation (7) is known as a hyperbolic heat equation because of the additional term that modifies parabolic Fourier heat equation, $\partial_t T = \alpha\nabla^2 T$.
In this paper, we restrict ourselves to the unidimensional case and assume $L = L_0 T^2$. In nondimensional form, our potential system can be written as

$$\begin{aligned} \tau v_t - \frac{2\tau v}{u} u_t + u^2 u_x + v = 0, \\ u_t + \frac{1}{\hat{C}u^2} v_x = 0, \end{aligned} \tag{8}$$

where

$$u = \frac{T}{T_0}, \quad v = \frac{q\ x}{L_0 T_0^3}.$$

For $\tau = 0$, (8) becomes

$$\begin{aligned} u^2 u_x + v = 0, \\ u_t + \frac{1}{\hat{C}u^2} v_x = 0, \end{aligned} \tag{9}$$

where, we call a unperturbed system, which is reducible to the following nonlinear diffusion equation

$$u_t \quad = \quad \frac{1}{Cu^2}(u^2 u_x)_x. \tag{10}$$

We will determine the infinitesimal transformations of the form which are admitted by equations (9)

$$\begin{aligned} x^* &= x + \varepsilon\xi(t, x), \\ t^* &= t + \varepsilon\tau(t, x), \\ u^* &= u + \varepsilon\varphi(t, x, u), \\ v^* &= v + \varepsilon\psi(t, x, u, v). \end{aligned}$$

Suppose that Γ is a vector field on the space $\mathbb{R}^2 \times \mathbb{R}^2$ of independent variables x, y, and dependent variables u and v

$$\Gamma = \tau(t, x)\frac{\partial}{\partial t} + \xi(t, x)\frac{\partial}{\partial x} + \varphi(t, x, u, v)\frac{\partial}{\partial u} + \psi(t, x, u, v)\frac{\partial}{\partial v}. \tag{11}$$

The solution of system (9) defines an submanifold $\mathbb{V} \subset \mathbb{R}^2 \times \mathbb{R}^2$ and will be invariant under the one parameter subgroup generated by Γ if and only if $\mathbb{V}$ is an invariant submanifold of this group. For this purpose the solution (9) must be satisfies in the invariant surface conditions

$$\tau u_t + \xi u_x = \varphi, \qquad \tau v_t + \xi v_x = \psi\ . \tag{12}$$

For the system (9), (12) to be compatible, the first order prolongation $\Gamma^{(1)}$ of the vector field Γ

$$\begin{aligned} \Gamma^{(1)} = \Gamma &+ [D_x\varphi - u_x D_x\xi - u_t D_x\tau]\frac{\partial}{\partial u_x} + [D_t\varphi - u_x D_t\xi - u_t D_t\tau]\frac{\partial}{\partial u_t} \\ &+ [D_x\varphi - v_x D_x\xi - v_t D_x\tau]\frac{\partial}{\partial v_x} + [D_t\psi - v_x D_t\xi - v_t D_t\tau]\frac{\partial}{\partial v_t} \end{aligned}$$

must be tangent to the first order system (12),

$$\Gamma^{(1)}\{\tau u_t + \xi u_x - \varphi = 0\}, \quad \Gamma^{(1)}\{\tau v_t + \xi v_x - \psi = 0\}. \tag{13}$$

The variables u_t, u_x, v_t and v_x found from (9), (12) must be substituted into (13), then we get two polynomials in u and v. These two identities enable the infinitesimal transformations to be derived in the form

$$\xi = (-3a_1 + a_2)x + a_3,\ \tau = (2a_2 - 7a_1)x - a_4,\ \varphi = -a_1 u,\ \psi = -a_2 v, \tag{14}$$

where a_1, a_2, a_3, and a_4 arbitrary constants.

In order to solve the inverse diffusion problem (1)-(4), we solve the system (9) with equivalence transformations. From [2], and (14) we get the following infinitesimal components for the system (9)

$$\xi = 1, \quad \tau = -t, \quad \phi_1 = -u, \quad \psi = -3v. \tag{15}$$

Therefore, the vector field Γ is in the form

$$\Gamma = \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 3v\frac{\partial}{\partial v}. \tag{16}$$

The invariance surface conditions associated with the vector field Γ can be written as

$$\frac{\partial u}{\partial x} - t\frac{\partial u}{\partial t} = -u, \tag{17}$$

$$\frac{\partial v}{\partial x} - t\frac{\partial v}{\partial t} = -3v. \tag{18}$$

The general solution of the (17)-(18), may be obtained by solving the characteristic equation[4]

$$\frac{dx}{1} = \frac{dt}{-t} = \frac{du}{-u} = \frac{dv}{-3v}, \tag{19}$$

where

$$u = \exp(-x)\ \psi(t,\sigma), \tag{20}$$

$$v = \exp(-3x)\ \chi(t,\sigma), \tag{21}$$

$$\sigma = t\ \exp(x). \tag{22}$$

By substituting (20)-(22) in (6)-(7), we get a reduced system (RS) of partial differential equations in the form

$$\begin{gathered} -\psi^3 + \sigma\psi^2\frac{\partial\psi}{\partial\sigma} + \chi = 0, \\ t\psi^2\frac{\partial\Psi}{\partial t} + \sigma\psi^2\frac{\partial\psi}{\partial\sigma} - \frac{3t}{\hat{C}}\chi + \frac{t\sigma}{\hat{C}}\frac{\partial\chi}{\partial\sigma} = 0. \end{gathered} \tag{23}$$

Assuming that ψ and χ are analytic functions of its arguments, it is possible to expand them in the form

$$\psi = \psi_0(\sigma) + t\psi_1(\sigma) + t^2\psi_2(\sigma) + ..., \tag{24}$$

$$\chi = \chi_0(\sigma) + t\chi_1(\sigma) + t^2\chi_2(\sigma) + ..., \tag{25}$$

by truncating (25)-(26) at order n, substituting in (RS), neglecting the higher powers of t (in this paper $n = 3$) and by substituting (25)-(26) in (22)-(24) and equating the coefficient of t^i, $i = 0, 1, 2$

to zero and neglecting the higher powers of t, we get a system of six ordinary differential equations (ODEs) in the unknowns ψ_0, ψ_1, ψ_2, χ_0, χ_1, χ_2, with the variable σ.
By solve the above system and substituting ψ_0, χ_0, ψ_1, χ_1, ψ_2, χ_2 in (25), from (20)we obtain the approximate solution of (1)-(4), for $t \ll 1$ in the form

$$u \approx \exp(-x)[1 + t(\frac{3}{C} + \frac{I_1}{\sigma}) + t^2(\frac{I_2}{\sigma^2} - \frac{2I_1}{\sigma^{C^2}} + \frac{1}{2}(\frac{3}{C} - \frac{18}{C^2}))],$$

where I_1, I_2 are integration constants. Now, We can determine I_1, I_2 from (2) and (4).

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Comparison between Sinc collocation method based on double exponential transformation and radial basis function for integral equations

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Abstract

In this paper comparison between two numerical methods based on collocation for numerical solution of Fredholm and Volterra integral equations are considered. The first, sinc collocation method based on double exponential transformations**(DE)** and the other one, approximation by means of radial Basis Function(RBF) are applied. some remarks with respect to their computational cost and stability and implementation are discussed. Examples are presented to illustrated effectiveness of two methods.

Keywords: Volterra and Fredholm integral equations, Sinc collocation method, Double exponential transformation , Radial Basis Function.

Mathematics Subject Classification: 53A15

1 Introduction

In this paper, we consider the second kind Fredholm integral equation of the form :

$$u(x) - \lambda \int_a^b k(x,t)u(t)dt = f(x), \qquad a \leq x \leq b \tag{1}$$

where a, b, are real constants $f(x), k(x,t)$ are given functions and $u(x)$ is to be determined. in the above equation if the upper bound is set $b = x$ we get Volterra integral equations. the use of equation (1) has increased and occurs in various areas of engineering, mechanics, physics, chemistry, astronomy, biology, economies, potential theory, electrostatic, mathematics problems of radiative equilibrium, etc. Many methods are usually use to handle(1)such as the successive approximation, Radial basis function, sinc collocation method, Adomian's decomposition ,Homotopy perturbation, Chebyshev and taylor collocation , Haar Wavelet, Tau series methods ,etc.[4,6,17,18]

The main purpose of the present research is considering the two numerical solution methods and comparison between this methods based on computational cost and stability and implementation:

Firstly, Double Exponential transformation in the Sinc collocation method for Fredholm and Voltera integral equations. The Double Exponential transformation, abbreviated as **DE** was first propose by Takahasi and Mori [16] in 1974 for one dimensional numerical integration and it has come to be widely used in applications. It is known that the double exponential transformation gives an optimal result for numerical evaluation of definite integral of an analytic function [14,15]. In 1997, Sugihara [13,8] established the "Meta-Optimality " of the DE formulas in a mathematically rigorous manner, and since then it has turned out that DE transformation is also useful for other various kind of numerical methods. Indeed it has been demonstrated that the use of the Sinc method in cooperate with the DE transformation gives highly efficient numerical methods for approximation of function, indefinite numerical integration and solution of differential equations. Recently, Muhammad et al. in [9,10] established a method of indefinite numerical integration based on DE transformation incorporated into Sinc expansion of the integrand which gives results with high efficiency [11,12].

In the standard setup of the Sinc numerical methods, the error are known to be $O(\exp(-k\sqrt{N}))$ with some $k > 0$, where N is the number of nodes or bases which is used in the methods. However, Sugihara has recently in [8,13] found that the errors in the Sinc numerical methods are $O(\exp(-cN/logN))$ with some $c > 0$, which is also meaningful practically.

Secondly, Radial Basis Function(RBF) as a meshless methods are part of an emerging field of mathematics. the history of (RBF) approximation goes back to 1986 when multiquadric RBFs first studied by Roland and Hardy[7].Todays the literature on different aspects of RBF approximation is extensive. RBFs are used not only for interpolation or approximation of data sets[1], but also as a powerful tools for solving mathematics problems e.g ODEs and PDEs equations.the readers can find more details about RBF and related methods in[1,17] and references cited therein.

The layout of the paper is as follows: in section 2,3 we give basic definitions, assumptions and preliminaries of the Sinc and RBFs approximations and related topics, respectively. section 4 says main theorem about error convergence. section 5 contains the details of our numerical implementation and some experimental results. finally section 6 gives main results.

2 Sinc Collocation method based on DE Transformation

Let f be a function defined on$\mathbb{R}$ and $h > 0$ is step size then the Whittaker cardinal defined by the series

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x) \tag{2}$$

whenever this series convergence, and

$$S(j,h)(x) = \frac{\sin[\pi(x-jh)/h]}{\pi(x-jh)/h}, k = 0, \pm1, \pm2, .. \tag{3}$$

where $S(j,h)(t)$ is known asjth Sinc function evaluated at$t$.

Throughout of this paper, let $d > 0$, and D_d denote the region $\{z = x + iy \quad |y| < d\}$ in the complex plan $\mathcal{C}$ and ϕ the conformal map of a simply connected domain D in the complex domain onto D_d, such that $\phi(a) = -\infty, \phi(b) = \infty$, where a, b are boundary points of D with $a, b \in \partial D$. Let ψ denote the inverse map of ϕ, and let the arc Γ, with end points $a, b \quad (a, b \in \Gamma)$, given by $\Gamma = \psi(-\infty, \infty)$. For $h > 0$, let the points x_k on Γ given by $x_k = \psi(kh), \quad k \in Z$.

Moreover, let us consider $H^1(D_d)$ be the family of all functions g analytic in D_d, such that

$$N_1(g, D_d) = \lim_{\epsilon\to0} \int_{\partial D_{d(\epsilon)}} |g(t)||dt| < \infty,$$
$$D_{d(\epsilon)} = \{t \in C, \quad |Ret| < \tfrac{1}{\epsilon}, \quad |Imt| < d(1-\epsilon)\}.$$

We recall the following definitions from [1,6,11], that will become instrumental in establishing our useful formulas:

Definition 2.1. *A function g is said to be decay double exponentially, if there exist constants α and C, such that:*

$$|g(t)| \le C\exp(-\alpha\exp|t|), \quad t \in (-\infty, \infty)$$

equivalently, a function g is said to be decay double exponentially with respect to conformal map ϕ, if there exist positive constants α and C such that:

$$|g(\phi(t))\acute{\phi}(t)| \le C\exp(-\alpha\exp|t|), \quad t \in (-\infty, \infty).$$

Here, we suppose that $K_{\phi}^{\alpha}(D_d)$ denote the family of functions g where $g(\phi(t))\acute{\phi}(t)$ belongs to $H^1(D_d)$ and decays double exponentially with respect to ϕ. If f belongs to $K_{\phi}^{\alpha}(D_d)$ with respect to ϕ, then we have the following formulas for definite and indefinite integrals based on DE transformation which is given and fully discussed in [5,8]:

$$\int_a^b f(x)dx = h\sum_{j=-N}^{j=N} f(\phi(jh))\acute{\phi}(jh) + O(\exp(\frac{-2\pi dN}{log(2\pi dN/\alpha)}),$$

and

$$\begin{aligned}\int_a^s f(x)dx &= h\textstyle\sum_{j=-N}^{j=N} f(\phi(jh))\acute{\phi}(jh)(\frac{1}{2}+\frac{1}{\pi}si(\frac{\pi\phi^{-1}(s)}{h}-j\pi)) \\ &+O(\tfrac{logN}{N}\exp(-\tfrac{\pi dN}{log(\pi dN/\alpha)})),\end{aligned}$$

where $Si(t)$ is the Sine integral defined by:

$$Si(t) = \int_0^t \frac{\sin w}{w}dw,$$

and the mesh size h satisfies $h = \frac{1}{N}\log(\pi dN/\alpha)$.

To apply this approximation for equation(1), first use definite integration for second term, so :

$$\int_a^x k(x,t)u(t)dt \simeq h\sum_{i=-N}^{i=N} k(x,\phi(ih))\phi'(ih)(\frac{1}{2}+\frac{1}{\pi}si(\frac{\pi\phi^{-1}(x)}{h}-j\pi))u_j \tag{4}$$

Where $u_j, j = -N...N$ is an approximation of exact value $u(x_j)$ and

$$\phi(t) = \frac{b-a}{2}\tanh(\frac{\pi}{2}\sinh t) + \frac{a+b}{2}, \tag{5}$$

$$\phi'(t) = \frac{b-a}{2}\frac{\pi/2\cosh(t)}{\cosh^2(\pi/2\sinh(t))} \tag{6}$$

if we replace the second term of (1) by right-hand side of (4) we obtain:

$$u(x) - h\sum_{i=-N}^{i=N} k(x,\phi(ih))\phi'(ih)(\frac{1}{2}+\frac{1}{\pi}si(\frac{\pi\phi^{-1}(x)}{h}-j\pi))u_j \simeq f(x) \tag{7}$$

To find unknown $u_j = u(x_j), j = -N...N$, we can apply the Sinc collocation points x_k as $x_k = \phi(kh), k = -N...N$, so we have following linear system of $(2N+1)(2N+1)$ unknown u_j.

$$u(x_k) - h\sum_{j=-N}^{j=N} k(x_k,\phi(jh))\phi'(jh)(\frac{1}{2}+\frac{1}{\pi}si(\pi(k-j)))u_j \simeq f(x_k), \quad k,j=-N..N \tag{8}$$

By solving system of linear equations, we obtain approximate solution u_j which corresponds to the exact solution $u(x_j)$ at the Sinc points $x_k = \phi(kh)$.

To obtain an approximation in arbitrary x we use a method [4] similar to the Nyström for the Volterra integro differential equation as:

$$u_N(x) = f(x) + \lambda h\sum_{i=-N}^{i=N} k(x,\phi(ih))\phi'(ih)(\frac{1}{2}+\frac{1}{\pi}si(\frac{\pi\phi^{-1}(x)}{h}-j\pi)))u_j \tag{9}$$

By using the notations:

$$\begin{aligned}\mathbf{A} &= [k(x_k,\phi(jh))\phi'(jh)(\tfrac{1}{2}+\tfrac{1}{\pi}si(\pi(k-j)))] \\ \tilde{\mathbf{u}} &= (u_{-N},...,u_N)^t \quad ,\mathbf{g} = (g(x_{-N}),...,g(x_N))^t\end{aligned}$$

the system (8) can be show in matrix form

$$(I - \lambda h\mathbf{A})\tilde{\mathbf{u}} = \mathbf{g} \tag{10}$$

Similarly we can apply approximation to Fredholm integral equation and take similar system of linear equations.

3 Radial Basis Function method

Definition 3.1. *Let $\mathbb{R}^+ = \{x \in R, x \geq 0\}$ be the non-negative half-line and let $\phi : \mathbb{R}^+ \longrightarrow R$ be a continuous function with $\phi(0) \geq 0$, A radial basis function on R^d is a function of the form $\phi(\|X - X_i\|)$ where $X, X_i \in R^d$and $\|.\|$ denotes the Eucliden norm between X, X_i.*

Given data points $(x_i, f_i)_{i=1}^M$, a radial basis function is defined by:

$$S(X) = \sum_{i=1}^{N} \lambda_i \phi(\|X - X_i\|), \qquad \lambda \in R \tag{11}$$

There are two main groups of RBFs, piecwise smooth and infinity smooth. Some example of both are given in table 1. Piecwise smooth RBFs lead to an algebraic rate of convergence, whereas the infinity smooth RBF yield a spectral or even faster rate of convergence.

$NameofRBF$	$Definition$
Multiquadrics(MQ)	$\sqrt{r+c^2}$
Inverse Multiquadrics(IMQ)	$1/\sqrt{r+c^2}$
Gaussian(GA)	$\exp(-cr)$

Table 1. Some examples of RBFs

To approximate function $u(x)$, we can apply two cases[2,3], the first approximation is obtained by simply applying thr RBFs, i.e

$$u_M(x) = \sum_{i=1}^{M} w_i \phi_i(x), \qquad \phi_i(x) = \sqrt{(x - c_i)^2 + a_i^2} \tag{12}$$

The second approximation is based on the first order

$$u_M(x) = \sum_{i=1}^{M} w_i \phi_i'(x), \qquad \phi_i'(x) = \frac{(x - c_i)}{\sqrt{(x - c_i)^2 + a_i^2}} \tag{13}$$

Where m is the number of RBFc(or centers). $\phi_i s$ selection are based on observations from the numerical experiments with $\alpha = 1$ in MQ radial basis functions.

By selecting N interpolating points $\{(x_i, f_i)\}_{i=1}^M$ we obtain :

$$u_M(x_j) = u_j = \sum_{i=1}^{M} w_i \phi_i(x_j), \qquad j = 1, ..., M \tag{14}$$

above linear system of equations are showed by following notation:

$$\mathbf{A}\tilde{\mathbf{W}} = \mathbf{U} \tag{15}$$

Where

$$A_{i,j} = (\phi_i(x_j)), \quad i,j = 1,...,M$$

$$W = (w_1,...,w_M)^t, \qquad U = (u_1,...,u_M)^t$$

In order to use RBFs method for equation (1) we apply (12), then we have:

$$\sum_{i=1}^{M} \{\phi_i(x) - \lambda \int_a^b k(x,t)\phi_i(t)dt\} w_i \simeq f(x) \tag{16}$$

to ontain $w_i s$ we use a set of collocation points $\{x_j\}_{j=1}^M$, so we take a system of M equations and M variables. then approximation at an arbitrary point by (12) or (13) attained.

4 Error Analysis

In this section we give two theorem about convergence analysis and error bounds.

Theorem 4.1. [9]*Let $u(x)$ be the exact solution of* (1) *and $k(x,.) \in K_\phi^\alpha(D_d)$. if $u(\phi(x))$ is analytic and bounded onD_d,and $u_N(x)$ be approximate solution, then there exists a constant C independent of$N$ such that:*

$$\sup_{x\in(a,b)} |u(x) - u_N(x)| \le (C\sqrt{N}\mu_N + C')\frac{\log(N)}{\sqrt{N}} \exp(-\frac{\pi d N}{\log(\pi d N/\alpha)}) \tag{17}$$

where :

$$\mu_N = \|(I - \lambda A)^{-1}\|, \qquad h = \frac{1}{N}\log(\pi d N/\alpha) \tag{18}$$

Theorem 4.2. [19]*Let$\{x_i\}_{i=1}^M$ be M points in D which is convex, let*

$$h = max_{x\in D} min_{1\le i\le M} \|x - x_i\|_2 \tag{19}$$

when $\phi(\eta) < c(1 - |\eta|)^{-2+\alpha}$ for any $u(x)$ satisfies $\int (u(\eta)^2/\phi(\eta)d\eta < \infty$we have:

$$\|u_M^{(\alpha)} - u^{(\alpha)}\|_\infty \le ch^{1-\alpha} \tag{20}$$

where $\phi(x)$ is RBFs and the constant c depends on RBFs, d is space dimension , 1 and α are nonnegative integer, it can be seen that not only RBFs itself but also its any order derivative a good convergence.

5 Numerical Experiments

In this section, Examples are presented to illustrated effectiveness of two methods. all examples are selected from[2,3]in order to do manigniful comparision, to do computation in RBF method base on[2,3] we take :

$$c_i = \frac{1}{2}(1 + \cos(\frac{(i-1)\pi}{M-1}), \quad i = 1,...,M \tag{21}$$

and $a_i = \beta d_i, i = 1,...,M$ where d_i is the distance from ith center to the nearest center and $\beta > 0$ is a constant.note that value of β make better approximations.

Also for the error comparison, the accuracy of approximation are calculated by two means:

$$RMS = \sqrt{\frac{1}{M_t}\sum_{1\le i\le M_t}[u(x_i) - u_M(x_i)]}, \quad \|En\| = max_{1\le i\le M}|u(x_i) - u_M(x_i)| \tag{22}$$

where RMS is root mean squared error and M_t is number of test points ($M_t = M$) We Consider the following two test problems:

Example 5.1. *Consider the Volterra integral equation[]:*

$$u(x) + \int_0^x xtu(t)dt = \frac{(2-x)\exp(-x^2) + x}{2}, \quad 0 \le x \le 1 \tag{23}$$

with the exact solution: $u(x) = \exp(-x^2)$.

To compare results, we take equal numbers of basical function in each methods, which showed by $M_t = 2N + 1$, also to have a same comparison. We concentrate on 4 parameters Run-Time (column T in second), RMS error (column RMS), Norm infinity (column $\|.\|_\infty$), Condition number (column Cond based on infinity norm) in two methods. Results in Sinc collocation method and RBF method are shown in Table 2, Table 3, respectively.

N	M_t	$T(s)$	$\|.\|_\infty$	RMS	$Cond$
4	9	1.36	6.25-E005	5.23E-005	2.10E+00
8	17	2.27	4.83E-006	2.77E-006	2.12E+00
16	34	6.76	1.59E-008	8.71E-009	2.12E+00

Table 2. Results for Example 1 by Sinc collocation method.

N	M_t	$T(s)$	$\|.\|_\infty$	RMS	$Cond$
4	9	11.76	2.945-E04	1.41E-04	2.14E+07
8	17	37.67	4.93E-05	2.89E-05	2.80E+09
16	34	147.46	3.71E-06	6.82E-06	1.03E+10

Table 3. Results for Example 1 by RBF method.

Numerical results which are presented for two methods, show simplicity and very good accuracy of the methods. the results supported the confidence in applying methods to problems. The schemes work in a similar fashion, by decreasing number of basical functions the errors have been improved. but at the Run time error and condition number increasing the parameter N a difference have been occurred. In the Sinc collocation method Run time error and condition number which is showed by μ_N demonstrates the good manner of this method.

Example 5.2. *Consider the integral equation [2, 3]:*

$$u(x) + \int_0^x (x-t)u(t)dt = 1, \quad 0 \le x \le 1 \tag{24}$$

with the exact solution: $u(x) = \cos(x)$.

N	M_t	$T(s)$	$\|.\|_\infty$	RMS	$Cond$
4	9	1.44	6.10-E04	3.92 E-04	2.18E+00
8	17	2.29	4.67E-06	2.99 E-06	2.19E+00
16	34	8.60	5.04E-09	2.98 E-09	2.19E+00

Table 4. Results for Example 2 by Sinc collocation method.

N	M_t	$T(s)$	$\|.\|_\infty$	RMS	$Cond$
4	9	12.26	9.10-E05	5.52E-05	1.58E+07
8	17	41.39	1.43E-05	8.40E-06	2.05E+08
16	34	166.69	3.59E-03	1.88E-03	2.72E+12

Table 5. Results for Example 2 by RBF method.

In this example, results show good approximation based on Sinc collocation method. Clearly these remarkable differences are mostly due to structure of coefficient matrix and special property (Toplizet matrix) which is an important factor in comparison.

Example 5.3. *Consider the integral equation[2,3]:*

$$u(x) - \int_0^1 x^2 \exp(t(x-1))u(t)dt = (1-x)\exp(x) + x, \quad 0 \le x \le 1 \tag{25}$$

with the exact solution: $u(x) = \exp(x)$.

N	M_t	$T(s)$	$\|.\|_\infty$	RMS	$Cond$
4	9	1.20	2.82-E05	1.63E-05	4.82E+00
8	17	2.02	2.98-09	1.14E-09	4.82E+00
16	34	5.50	1.32E-09	5.28E-10	4.82E+00

Table 6. Results for Example 3 by Sinc collocation method.

N	M_t	$T(s)$	$\|.\|_\infty$	RMS	$Cond$
4	9	11.68	2.78-E06	1.30E-06	1.86E+11
8	17	53.01	1.31-06	6.22E-07	1.57E+12
16	34	919.16	2.48E-08	1.34E-08	1.38E+17

Table 7. Results for Example 3 by RBF method.

however, results show that methods are practically well, but Sinc collocation method gives better accuracy than the RBF method at the expense of more computational effort.

6 Main Result

Compare Sinc collocation method to RBF meshless method have the following advantages:

1-The numerical methods demonstrate the good accuracy of these schemes.

2-By decreasing M_t, RMS and $\|.\|$ columns are closely similar but Run time and condition number columns are different, also have a remarkable difference in value which is very considerable factor.

3-determining tha parameter α in Sinc collocation method and c_i, β in RBF method is still computationally intensive.

4-Based on results, Sinc collocation method gives better accuracy at the computational cost, also the implemention and coding are very easy.

7 Conclusion

We apply the Sinc collocation method based on double exponential transformation and RBF method to integral equations, we observe here that significant differences have been obtained compared with numerical results reported by tables. Sinc collocation method in Run time and condition number is very better than RBF method. Also we can improve the accuracy of the solution by selecting the appropriate shape parameters and selecting the large values of N. Results show the high accuracy of method by taking this view that storing in time and memory is another useful property in the Sinc method. In addition this method is portable to other area of problems and easy to programming.

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A meshless approximate solution of Mixed Volterra-Fredholm integral equations

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Abstract

This paper presents a meshless method using RBF collocation scheme for numerical solution of mixed Volterra-Fredholm integral equations where the region of integration is a non-rectangular domain. This method requires only a scattered data of nodes in the domain. The proposed scheme is simple and computationally attractive. Applications are demonstrated through illustrative examples.
Keywords: Mixed Volterra-Fredholm integral equation, Radial Basis functions, Numerical treatment.

Mathematics Subject Classification: 45A99

1 Introduction and Preliminaries

The solution of the mixed Volterra-Fredholm integral equations has been a subject of considerable interest. Consider the following mixed Volterra-Fredholm integral equation

$$u(x,t) - \int_0^t \int_\Omega K(x,t,\xi,s)u(\xi,s)d\xi ds = f(x,t), \quad (x,t) \in \Omega \times [0,T] \tag{1}$$

where $u(x,t)$ is an unknown function. The functions $f(x,t)$ and $K(x,t,\xi,s)$ are continuous on $\Omega \times [0,T]$, where Ω is a compact subset of $\mathbb{R}^n, n = 1,2,3$, with convenient norm $\|.\|$. Equation (1) can be written in the form

$$u - \mathcal{T}u = f$$

where the integral operator $\mathcal{T} : C(\Omega \times [0,T]) \to C(\Omega \times [0,T])$ is defined as

$$(\mathcal{T}u)(x,t) = \int_0^t \int_\Omega K(x,t,\xi,s)u(\xi,s)d\xi ds.$$

Throughout of the paper, we suppose that $f \in C(\Omega \times [0,T])$ and $K \in C(\Omega^2 \times D)$, where $D = \{(t,s) : 0 \le s \le t \le T\}$. Then equation (1) possesses a unique solution[3].

2 Radial basis function approximation

Radial basis functions were introduced in [1] and they form a primary tool for multivariate interpolation. They are also receiving increased attention for solving PDE's in irregular domains. An RBF depends only on the distance to a center point x_j and is of the form $\phi(\|x - x_j\|)$. The RBF may also have a shape parameter c. This parameter is a free parameter for controlling the shape

of functions. In this paper we have selected the parameter c experimentally.
Now we consider some definitions for the convergence on native space.
Definition 1: We define the fill distance of a given set $\mathcal{X} = \{x_1, ..., x_n\}$ consisting of pairwise distinct points in Ω

$$h_{\mathcal{X},\Omega} = \sup_{x\in\Omega} \min_{x_j\in\mathcal{X}} \|x - x_j\|.$$

Definition 2: The definition of the native space is

$$\mathcal{N}_\phi = \{f \in L_2(\mathbb{R}^s) \cap C(\mathbb{R}^s) : \frac{\hat{f}}{\sqrt{\hat{\phi}}} \in L_2(\mathbb{R}^s)\}$$

where $\hat{\phi}$ is a Fourier transform of ϕ.

Theorem(See[2]) Let $\Omega \subseteq \mathbb{R}^2$ and suppose that the points $\mathcal{X} = \{x_1, ..., x_n\}$ are distinct. Denote the interpolate to $u \in \mathcal{N}_\phi$ on $\mathcal{X}$ by u_n. Then there is a positive constant C such that for every $x \in \Omega$ and for the Gaussian radial basis function we have:

$$\|u - u_n\|_{L_\infty(\Omega)} \leq \exp(\frac{-C|\log h_{\mathcal{X},\Omega}|}{h_{\mathcal{X},\Omega}})\|u\|_{\mathcal{N}_\phi}.$$

3 The proposed method

Suppose $0 = t_0 < t_1 < ... < t_M = T$ be a scattered set of data in $[0, T]$ and $x_0, x_1, ..., x_N$ be a scattered set of nodes in Ω. We assume that Ω has a non-rectangular shape. For approximating the solution of (1) consider:

$$u^{M,N}(x,t) = \sum_{j=0}^{M}\sum_{k=0}^{N} c_{k,j}\phi_k(x)\phi_j(t), \qquad (x,t) \in \Omega \times [0,T] \tag{2}$$

as an approximation for the exact solution $u(x, t)$, where:

$$\phi_k(x) = \phi(\|x - x_k\|), \quad \phi_j(t) = \phi(|t - t_j|),$$

$$k = 0, ..., N, \; j = 0, ..., M.$$

Then by replacing $u^{M,N}(x, t)$ from (2) in (1) we have

$$\sum_{j=0}^{M}\sum_{k=0}^{N} c_{k,j}[\phi_k(x)\phi_j(t) - \int_0^t\int_\Omega K(x,t,\xi,s)\phi_k(\xi)\phi_j(s)d\xi ds] = f(x,t), \tag{3}$$

$$(x,t) \in \Omega \times [0,T].$$

Now the interval $[0, t]$ must be converted to a fixed interval $[-1, 1]$ by a simple linear transformation of the form

$$s(t,\theta) = \frac{t}{2}\theta + \frac{t}{2},$$

then equation (1) takes the following form :

$$\sum_{j=0}^{M}\sum_{k=0}^{N} c_{k,j}[\phi_k(x)\phi_j(t) - \int_{-1}^1\int_\Omega \frac{t}{2}K(x,t,\xi,s(t,\theta))\phi_k(\xi)\phi_j(s(t,\theta))d\xi d\theta] = f(x,t). \tag{4}$$

Without lose of generality we assume that:

$$\Omega = \{\xi = (\sigma,\tau) \in \mathbb{R}^2 : -1 \leq \tau \leq 1, v_1(\tau) \leq \sigma \leq v_2(\tau)\}.$$

Now the interval $[v_1(\tau), v_2(\tau)]$ is converted to the fixed interval [-1,1] by the following linear transformation:

$$\sigma(\tau, z) = \frac{v_2(\tau) - v_1(\tau)}{2} z + \frac{v_2(\tau) + v_1(\tau)}{2}.$$

So (4) becomes:

$$\sum_{j=0}^{M} \sum_{k=0}^{N} c_{k,j} [\phi_k(x)\phi_j(t) - \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K_1(x, t, \sigma(\tau, z), \tau, s(t, \theta))\phi_k(\sigma(\tau, z))\phi_j(s(t, \theta)) dz d\tau d\theta] = f(x, t) \tag{5}$$

where

$$K_1(x, t, \sigma(\tau, z), \tau, s(t, \theta)) = \frac{t}{2} \frac{v_2(\tau) - v_1(\tau)}{2} K(x, t, \sigma(\tau, z), \tau, s(t, \theta)).$$

Now assume that (5) holds at $(x_i, t_r), i = 0, ..., N, r = 0, ..., M$. So we have

$$\sum_{j=0}^{M} \sum_{k=0}^{N} c_{k,j} [\phi_k(x_i)\phi_j(t_r) - \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} K_1(x_i, t_r, \sigma(\tau, z), \tau, s(t_r, \theta))\phi_k(\sigma(\tau, z))\phi_j(s(t_r, \theta)) d\xi d\theta] = f(x_i, t_r) \tag{6}$$

Using a m points quadrature formula with the points $\{\theta_l\}, \{\tau_p\}, \{z_q\}$ in the interval $[-1, 1]$ and wights $\{w_l\}, \{w_p\}, \{w_q\}$ for numerical integration in (6) yields

$$\sum_{j=0}^{M} \sum_{k=0}^{N} c_{k,j} [\phi_k(x_i)\phi_j(t_r) - \sum_{l=1}^{m}\sum_{p=1}^{m}\sum_{q=1}^{m} K_1(x_i, t_r, \sigma(\tau_p, z_q), s(t_r, \theta_l))\phi_k(\sigma(\tau_p, z_q))\phi_j(s(t_r, \theta_l)) w_p w_q w_l] = f(x_i, t_r) \tag{7}$$

Solving (7) leads to the quantities $c_{k,j}$ and then the values of $u(x, t)$ at any point of $\Omega \times [0, T]$ can be approximated by

$$u^{M,N}(x, t) = \sum_{j=0}^{M} \sum_{k=0}^{N} c_{k,j} \phi_k(x)\phi_j(t), \quad (x, t) \in \Omega \times [0, T].$$

If $\Omega = \Omega_1 \cup \Omega_2 \cup ...\Omega_d$, where $\Omega_i,\ 1 \leq i \leq d$ are disjoint domains of the first or second kinds, we have:

$$\sum_{j=0}^{M} \sum_{k=0}^{N} c_{k,j} [\phi_k(x_i)\phi_j(t_r) - \sum_{e=1}^{d}\sum_{l=1}^{m}\sum_{p=1}^{m}\sum_{q=1}^{m} K_1(x_i, t_r, \sigma_e(\tau_p, z_q), s(t_r, \theta_l))\phi_k(\sigma_e(\tau_p, z_q))\phi_j(s(t_r, \theta_l)) w_p w_q w_l] = f(x_i, t_r) \tag{8}$$

where

$$\sigma_e(\tau, z) = \frac{v_{2,e}(\tau) - v_{1,e}(\tau)}{2} z + \frac{v_{2,e}(\tau) + v_{1,e}(\tau)}{2}. \tag{9}$$

As we mentioned $v_{1,e}$ and $v_{2,e}$ are continuous functions in to the sub-domain Ω_e.

Remark: The aim of this transformation is only for integration and there is no transformation for $u(x, t)$ and $f(x, t)$ and (x, t) is still in $\Omega \times [0, T]$. Moreover the points in $u(x, t)$ and the points for integration are different. So we can't use the pseudospectral methods.

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Highly oscillatory integrals of a general class: A review of most recent numerical methods

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Abstract

Steepest descent is a most efficient numerical method for highly oscillatory integrals. We review some very recent progresses and extension of this method and present some open questions and projects for further researches.

Keywords: highly oscillatory integrals, steepest descent method, asymptotic expansion

Mathematics Subject Classification: 65D30, 41A60

1 Introduction

A glance at dozens of papers, that published in very hight quality scientific journals on applied mathematics, reveals that highly oscillatory integral of the general form

$$I[f] := \int_a^b f(x)e^{i\omega g(x)}\,dx, \tag{1}$$

has been the subject of very active researches in the last few years (see, e.g., [1, 2, 3, 4, 5] and references therein). Among the new results and progresses, the author of this paper has also a considerable contribution that presented in a recent international conference ([8]). A corresponding manuscript has also been submitted to the Journal of Computational and Applied Mathematics that is still under review.

Traditional numerical quadrature methods such as Newton-Cotes and Gaussian quadratures have very poor accuracy when applied to the highly oscillatory integral (1). This is because that the corresponding integrand oscillates violently in the integration interval for higher values of the *frequency* ω.

Treatments of this phenomenon have essence in asymptotic expansion formulae of $I[f]$, which studied in the field of harmonic analysis. Steepest descent is a very efficient and rapid method for numerical approximation of $I[f]$. The method dates back to rather far past, but just recently some progresses and extensions have been obtained (see [6, 2, 1]). The aim of this paper (talk) is to review some most recent progresses on the steepest descent method and present some topics for future researches.

2 Main Result

The idea of steepest descent method, being rather old, is to replace the path of integration with a so-called *steepest path* in the complex plane such that the integrand never oscillates along the new path. The idea deploys the Cauchy Theorem, so the integrand should be analytic, i.e., both functions f and g should be analytic in a large enough region D in the complex plane containing the interval $[a, b]$. The combination of this method with the n-point Gauss-Laguerre quadrature rule introduces the so-called *numerical steepest descent method* that converges with the highest rate with comparison to other numerical methods.

In order to find the steepest path, initially we assume that g has not any stationary points in D, that is $g'(x) \neq 0$ for $x \in D$. Then g is invertible in D. If further the equation $g(x) = z$ can be solved for x by analytical means, then the steepest path, passing $x \in [a, b]$, can be find as

$$h_x(p) := g^{-1}(g(x) - ip). \tag{2}$$

Considerable extensions of this method are due to Huybrechs's paper [6], where he discussed the cases when f and/or g is nonanalytic, g has some stationary points in D, and $g^{-1}(z)$ can not be found easily by analytical means. In the following, we bring the summaries.

2.1 In the presence of stationary points

Theorem 2.1. *[6] Assume that f and g are analytic in a sufficiently large region $D \subset \mathbb{C}$ and that the equation $g'(x) = 0$ has l solutions $\xi_i \in (a, b)$. Define $r_i := (\min_{k>1} g^{(k)}(\xi_i) \neq 0) - 1$ and $r := \max_i r_i$. Then the numerical steepest descent method converges with the order $\mathcal{O}(\omega^{-2n-1/(r+1)})$.*

2.2 Nonanalytic integrand

Theorem 2.2. *[6] Assume that g is analytic and that $g^{(k)}(\xi) = 0$, $k = 1, \ldots, r$, and $g^{(r+1)}(\xi) \neq 0$. Let f be sufficiently smooth, and let $f_\xi(x)$ be the Hermite interpolating polynomial of degree $s(r + 1) - 1$ that satisfies*

$$f_\xi^{(k)}(\xi) = f^{(k)}(\xi), \quad j = 0, \ldots, s(r + 1) - 1.$$

Then the numerical steepest descent method converges with the order $\mathcal{O}(\omega^{-s-1/(r+1)})$.

Theorem 2.3. *[6] Assume that f and $\widetilde{g}$ are analytic and that $\widetilde{g}^{(k)}(\xi) = 0$, $k = 1, \ldots, r$, and $\widetilde{g}^{(r+1)}(\xi) \neq 0$. Let $g_\xi(x)$ be the Hermite interpolating polynomial of degree $(s + 1)(r + 1) - 1$ that satisfies*

$$g_\xi^{(k)}(\xi) = \widetilde{g}^{(k)}(\xi), \quad j = 0, \ldots, (s + 1)(r + 1) - 1.$$

Then the numerical steepest descent method converges with the order $\mathcal{O}(\omega^{-s-1/(r+1)})$.

2.3 The case when the integration interval is $[0, \infty)$

Theorem 2.4. *[7] Let f and g be analytic in a closed simply connected and sufficiently large complex region D which contains the half-line $[0, +\infty)$, and that g is uniquely invertible on D. Further assume that the following conditions hold:*

(a) $f(z) = f_1(z)f_2(z)$, where $f_1(z)$ is an analytic bounded function in D with $f_1(x) \in L^1([0, \infty))$, and $f_2(z)$ is an analytic function in D vanishing as $|z| \to \infty$.

(b) $|g^{-1}(z)|$ tends to infinity, with the restriction $|g^{-1}(z)| = \mathcal{O}(e^{\omega_0|z|})$ for some $\omega_0 > 0$, as $|z| \to +\infty$.

Then

$$I[f] = e^{i\omega g(0)} \int_0^\infty f(h(p))e^{-\omega p}h'(p)\, dp. \tag{3}$$

Further, the numerical steepest descent method converges with the order $\mathcal{O}(\omega^{-2n-1})$.

However there are some difficulties when applying the numerical steepest descent method for (1). The main obstacle in that the stationary points, should be found analytically, so the method is not fully automatic.

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Application of the Exp-function method for solving the combined KdV–mKdV and Gardner–KP equations

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Abstract
In this article, we establish the exact solutions for the combined KdV–mKdV and Gardner–KP equations. The exp-function method (EFM) is used to construct solitary and soliton solutions of nonlinear evolution equations. This method is developed for searching exact travelling wave solutions of nonlinear partial differential equations. It is shown that the EFM, with the help of symbolic computation, provides a straightforward and powerful mathematical tool for solving nonlinear evolution equations in mathematical physics.

Keywords: Nonlinear partial differential equation; Combined KdV–mKdV and Gardner–KP equations; Travelling wave solution;

Mathematics Subject Classification: 53A15

1 Introduction

In the recent decade, the study of nonlinear partial differential equations in modelling physical phenomena has became an important tool. Nonlinear phenomena play a fundamental role in applied mathematics and physics. Also, the investigation of the travelling wave solutions plays an important role in nonlinear sciences. Here, we use an effective method for constructing a range of exact solutions for the following nonlinear partial differential equations which was first presented by He ([1]). A new method called the Exp-function method (EFM) is presented to look for traveling wave solutions of nonlinear evolution equations (NLEEs). The EFM has successfully been applied to many situations. For example, He and Wu [2] have solved the nonlinear wave equations by the EFM. Authors of [3] have applied the EFM for the modified KdV and the generalized KdV equations. The positive and negative models of the Gardner equation [4, 5], or the combined KdV-mKdV equations are given by

$$\mathrm{u_t + 6uu_x \pm 6u^2u_x + u_{xxx} = 0,} \tag{1}$$

which describe internal solitary waves in shallow seas. Those two models will be classified as positive Gardner equation and negative Gardner equation depending on the sign of the cubic nonlinear term. The Gardner equation (1), like the KdV and the mKdV equation, is completely integrable with a Lax pair and inverse scattering transform [6]. Kadomtsov and Petviashivilli extended the KdV equation to obtain the Kadomtsov-Petviashivilli (KP) equation

$$\mathrm{(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0.} \tag{2}$$

Kadomtsov and Petviashivilli relaxed the restriction that the waves be strictly one dimensional [6], namely the x-direction of the KdV equation to derive the completely integrable KP equation (2). In this article, we used the EFM to investigate the Gardner–KP (GKP) equations [6] given by

$$\mathrm{(u_t + 6uu_x \pm 6u^2u_x + u_{xxx})_x + u_{yy} = 0,} \tag{3}$$

that will be shown to be completely integrable. We want to obtain analytical solutions of nonlinear combined KdV–mKdV and Gardner–KP (GKP) equations and to determine the accuracy of the EFM in solving these kinds of problems. We first consider the nonlinear equation of the form:

$$\mathcal{N}(u, u_t, u_x, u_{xx}, , u_{yy}, u_{tt}, u_{tx}, u_{ty}, ...) = 0, \tag{4}$$

and introduce the transformation $u(x, y, t) = u(\eta)$, $\quad \eta = x + y - ct$, where c is constant to be determined later. Therefore Eq. (4) is reduced to an ODE as follows

$$\mathcal{M}(u, -cu', u', u'', u'', ...) = 0. \tag{5}$$

The EFM is based on the assumption that travelling wave solutions as in [2] can be expressed in the form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{6}$$

where c, d, p and q are positive integers which could be freely chosen, a_n's and b_m's are unknown constants to be determined. To determine the values of c and p, we balance the linear term of highest order in Eq. (5) with the highest order nonlinear term. Also, to determine the values of d and q, we balance the linear term of lowest order in Eq. (5) with the lowest order nonlinear term.

2 Applications

The combined KdV–mKdV equations:

Case 1: Positive Gardner equation

$$u_t + 6uu_x + 6u^2u_x + u_{xxx} = 0, \tag{7}$$

and we use transformation $u = v - \frac{1}{2}$, to convert the models (7) into the modified KdV equation

$$v_t - \frac{3}{2}v_x + 6v^2v_x + v_{xxx} = 0. \tag{8}$$

(I) The first set is:

$$a_1 = 0, \quad a_{-1} = 0, \quad a_0 = a_0, \quad b_0 = 0, \quad b_{-1} = b_{-1}, \quad b_1 = \frac{a_0^2}{4b_{-1}}, \quad c = -\frac{1}{2}, \tag{9}$$

$$v_1(x, t) = \frac{a_0}{b_{-1}\exp(-x - \frac{1}{2}t) + \frac{a_0^2}{4b_{-1}}\exp(x + \frac{1}{2}t)}. \tag{10}$$

If we choose $a_0 = 2b_{-1}$, then we will get

$$v_{1,1}(x, t) = \operatorname{sech}\left(x + \frac{1}{2}t\right), \qquad u_{1,1}(x, t) = -\frac{1}{2} + \operatorname{sech}\left(x + \frac{1}{2}t\right). \tag{11}$$

Case 2: Negative Gardner equation

Let

$$u_t + 6uu_x - 6u^2u_x + u_{xxx} = 0, \tag{12}$$

and use transformation $u = v + \frac{1}{2}$, which converts the model (12) into the modified KdV equation

$$v_t + \frac{3}{2}v_x - 6v^2v_x + v_{xxx} = 0, \tag{13}$$

(I) The first set is:

$$a_1 = 0, \quad a_{-1} = a_{-1}, \quad a_0 = a_0, \quad b_0 = 2a_0, \quad b_{-1} = -2a_{-1}, \quad b_1 = 0, \quad c = 1, \tag{14}$$

$$v_1(x,t) = \frac{a_{-1}\exp(-x+t) + a_0}{-2a_{-1}\exp(-x+t) + 2a_0}. \tag{15}$$

If we choose $a_0 = a_{-1}$, then we will get

$$v_{1,1}(x,t) = \frac{1}{2}\coth\left(\frac{x-t}{2}\right). \qquad u_{1,1}(x,t) = \frac{1}{2} + \frac{1}{2}\coth\left(\frac{x-t}{2}\right). \tag{16}$$

The Gardner–KP equations:

Case 1: Positive Gardner–KP (GKP) equation
Consider

$$(u_t + 6uu_x + 6u^2u_x + u_{xxx})_x + u_{yy} = 0, \tag{17}$$

and we use transformation $u = v - \frac{1}{2}$, to reduced to the model (17) into the Gardner–KP equation

$$(v_t - \frac{3}{2}v_x + 6v^2v_x + v_{xxx})_x + v_{yy} = 0, \tag{18}$$

(I) The first set is:

$$a_1 = 0, \quad a_{-1} = 0, \quad a_0 = a_0, \quad b_0 = 0, \quad b_{-1} = b_{-1}, \quad b_1 = \frac{a_0^2}{4b_{-1}}, \quad c = \frac{1}{2}, \tag{19}$$

$$v_1(x,y,t) = \frac{a_0}{b_{-1}\exp(-x-y+\frac{1}{2}t) + \frac{a_0^2}{4b_{-1}}\exp(x+y-\frac{1}{2}t)}. \tag{20}$$

If we choose $a_0 = 2b_{-1}$, then we will get

$$v_{1,1}(x,y,t) = \operatorname{sech}\left(x+y-\frac{1}{2}t\right), \qquad u_{1,1}(x,y,t) = -\frac{1}{2} + \operatorname{sech}\left(x+y-\frac{1}{2}t\right). \tag{21}$$

Case 2: Negative Gardner–KP equation
Consider

$$(u_t + 6uu_x - 6u^2u_x + u_{xxx})_x + u_{yy} = 0, \tag{22}$$

and we use transformation $u = v + \frac{1}{2}$, to convert the model (22) into the Gardner–KP equation

$$(v_t + \frac{3}{2}v_x - 6v^2v_x + v_{xxx})_x + v_{yy} = 0, \tag{23}$$

(I) The first set is:

$$a_1 = 0, \quad a_{-1} = a_{-1}, \quad a_0 = a_0, \quad b_0 = 2a_0, \quad b_{-1} = -2a_{-1}, \quad b_1 = 0, \quad c = 2, \tag{24}$$

$$v_1(x,y,t) = \frac{a_{-1}\exp(-x-y+ct) + a_0}{-2a_{-1}\exp(-x-y+ct) + 2a_0}. \tag{25}$$

If we choose $a_0 = a_{-1}$, then we will get

$$v_{1,1}(x,y,t) = \frac{1}{2}\coth\left(\frac{x+y-2t}{2}\right), \quad u_{1,1}(x,y,t) = \frac{1}{2} + \frac{1}{2}\coth\left(\frac{x+y-2t}{2}\right). \tag{26}$$

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A new two-step method with nine-order convergence for solving nonlinear equations

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Abstract
In this paper, based on Frontini s method with an ninth-order convergence for solving the simple roots of nonlinear equations by Hermite interpolation methods. Per iteration this method requires two evaluations of the function and three evaluation of its first derivative, which implies that the efficiency index of the developed method is 1.552, Numerical comparisons are made to show the performance of the derived method, as shown in the illustrative examples.
Keywords: Nonlinear equation; Two-step; Three-step; Convergence order; Efficiency index.

Mathematics Subject Classification: 65B99, 65N99

1 Introduction

In this paper, we consider iterative methods to find a simple root of a nonlinear equation $f(\alpha) = 0$, where $f : D \subset \Re \longrightarrow \Re$ for an open interval D is a scalar function.
To improve the local order of convergence and efficiency index, many modified methods have been proposed in the open literature, see $[2-9]$ and references therein.
In this paper, based on Frontini s method and Hermite interpolation methods, we derive a new ninth-order method. For the computational cost, it requires the evaluations of only two functions and three first-order derivative per iteration. This gives 1.552 as an efficiency index of the derived method. The new method is comparable with Newton's method and other known methods. The efficacy of the method is tested on a number of numerical examples.

2 Description of the methods

We use of Frontini s method with third-order convergence for the first. Therefore, a new scheme is derived as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'\left(x_n - \frac{f(x_n)}{2f'(x_n)}\right)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'\left(y_n - \frac{f(y_n)}{2P_f(x_n,k_n,y_n)}\right)}. \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{1}$$

where

$$\begin{aligned} P_f(x_n,k_n,y_n) &= \frac{6(k_n-y_n)}{(x_n-y_n)(x_n-3k_n+2y_n)}f(y_n) \\ &- \frac{(x_n-y_n)^2}{(x_n-y_n)(x_n-3k_n+2y_n)}f^{'}(k_n) \\ &- \frac{(k_n-y_n)(2x_n-3k_n+y_n)}{(x_n-y_n)(x_n-3k_n+2y_n)}f^{'}(x_n) \\ &+ \frac{6(k_n-y_n)}{(x_n-y_n)(x_n-3k_n+2y_n)}f^{'}(x_n). \end{aligned} \tag{2}$$

3 The analysis of convergence

We prove the following convergence theorem for the method Eq.(1.3)

Theorem: Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to R$ for an open interval I. If x_0 is sufficiently close to α, then the two-step iterative method Eq.(1.3) has ninth-order convergence and satisfies the following error equation :

$$e_{n+1} = ((1/256)c_3^4 + c_2^8 - (1/16)c_2^2c_3^3 + (3/8)c_3^2c_2^4 - c_3c_2^6)e_n^9 + O(e_n^{10}), \tag{3}$$

where $e_n = x_n - \alpha$ for $n = 0, 1, 2, \ldots$, and $c_n = \frac{f^{(n)}(\alpha)}{n!f^{'}(\alpha)}$.

Remark. The order of convergence of the iterative method Eq.(1.3) is 9. This method requires two evaluations of the function, namely, $f(x_n)$ and $f(y_n)$ and three evaluations of first derivatives $f^{'}(x_n)$ and $f^{'}(k_n)$ and $f^{'}\left(y_n - \frac{f(y_n)}{2P_f(x_n,k_n,y_n)}\right)$. We take into account the definition of efficiency index [?] as $p^{1/d}$, where p is the order of the method and d is the number of function evaluations per iteration required by the method. If we suppose that all the evaluations have the same cost, we have that the efficiency index of the method Eq.(1.3) is $\sqrt[5]{9} \approx 1.552$.

4 Numerical examples

In this section, the obtained theoretical results are confirmed by numerical experiments and compared with Algorithms1 and 2 presented recently by Noor et al.[10] and Algorithm3:[11] whose order of convergence of these methods is nine,
The computational results presented in Table 1 shows that in almost all of cases, the presented methods converge more rapidly than Algorithm1, Algorithm2 and Algorithm3. This means that the new methods have better efficiency in computing process than Algorithm1, Algorithm2 and Algorithm3 as the compared other methods, and furthermore, the formula Eq.(1.3) produces the ninth-order methods.

Table 1: Comparison of iterative methods.

Function	Method	$\lvert x_k - x_{k-1}\rvert$	$\lvert f(x_k)\rvert$
$f_1(x), x_0 = 1.6$	Matinfar	4.263989161756854e-011	2.220446049250313e-016
	Algorithm1	3.084354993632132e-011	2.220446049250313e-016
	Algorithm2	4.958472521465751e-009	2.220446049250313e-016
	Algorithm3	7.963061042844899e-002	2.220446049250313e-015
$f_2(x), x_0 = 1.4$	Matinfar	2.287947609147523e-012	1.776356839400251e-015
	Algorithm1	4.556728327997917e-010	1.776356839400251e-015
	Algorithm2	1.597025920396789e-007	1.776356839400251e-015
	Algorithm3	5.257190105866805e-002	9.992504601541441e-010
$f_3(x), x_0 = 8.2$	Matinfar	3.552713678800501e-015	0
	Algorithm1	3.712585794346524e-012	0
	Algorithm2	6.519035977703425e-009	0
	Algorithm3	1.094326942315398e-001	9.769962616701378e-015
$f_4(x), x_0 = 3.2$	Matinfar	3.317186525464422e-010	0
	Algorithm1	1.112975045458597e-009	0
	Algorithm2	9.976341126449739e-005	0
	Algorithm3	1.017950785620059e-005	0
$f_5(x), x_0 = -3$	Matinfar	2.220446049250313e-015	0
	Algorithm1	8.007465829074079e-006	0
	Algorithm2	6.046056337090100e-004	0
	Algorithm3	7.060924525070789e-007	0
$f_6(x), x_0 = 8.9$	Matinfar	7.105427357601002e-015	0
	Algorithm1	7.923598256454056e-004	0
	Algorithm2	2.131628207280301e-014	0
	Algorithm3	7.488168664337991e-005	0
$f_7(x), x_0 = 3.1$	Matinfar	4.440892098500626e-016	8.881784197001252e-016
	Algorithm1	0	8.881784197001252e-016
	Algorithm2	0	1.776356839400251e-015
	Algorithm3	3.706772666589586e-010	8.881784197001252e-016

5 Conclusions

In this work we presented an approach which can be used to constructing of ninth-order iterative methods that do not require the computation of second or higher derivatives. Also, we proposed a new two-step iterative method for solving nonlinear equations. We showed that the new two-step iterative method has ninth-order convergence. Numerical examples also show that the numerical results of our new two-step method, in less iterations, improve the results of other existing three-step methods with ninth-order convergence.

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On the positivity step size coefficient of the classical explicit fourth-order Runge-Kutta method

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Abstract

This paper deals with the numerical solution of initial value problems, for systems of ordinary differential equations, by classical explicit fourth-order Runge-Kutta method(we will refer to it as the classical fourth-order method) which is positivity preserving. Our main objective is the preservation of positivity in the numerical solution of linear and nonlinear positive problems while maintaining a sufficient degree of accuracy and computational efficiency. We also pay particular attention to monotonicity property. We obtain new results for positivity which are important in practical applications. We provide some numerical examples to illustrate our results.

Keywords: Ordinary differential equations, Initial value problems, Advection equation, Runge-Kutta methods, Heun's explicit third-order method, Positivity, Monotonicity.

1 Introduction

Consider an initial value problem for a positive system of ordinary differential equations (ODEs) of type

$$U'(t) = F\big(t, U(t)\big), \ \ (t \geq 0), \ \ U(0) = U_0. \tag{1}$$

With positivity, we mean, the component-wise non-negativity of the initial vector, is preserved in time for the exact solution ($U(t) \geq 0,\ t > 0$ if $U_0 \geq 0$). There are many problems of practical interest that can be modelled by positive ODEs. For example positive ODEs arise from modelling chemical reactions or semi-discrete form of advection-diffusion equations (see e.g. [2]). In both cases, the components of the unknown can denote concentrations or densities which are physical quantities and they need to remain positive.

Solving a positive ODE numerically with a non-negative initial vector, it is a natural demand that the resulting numerical approximations $U_n \approx U(t_n), t_n = n\Delta t, \Delta t$ being the time step, should be non-negative. Therefore, we need to analyze numerical methods from the point of view of positivity.

As our numerical method, we consider the Heun's explicit third-order scheme

$$\begin{aligned}
&U_{n_1} = U_n, \quad U_{n_2} = U_n + \tfrac{1}{2}\Delta t F(t_n +, U_{n_1}),\\
&U_{n_3} = U_n + \tfrac{1}{2}\Delta t F(t_n + \tfrac{1}{2}\Delta t, U_{n_2}), \quad U_{n_4} = U_n + \Delta t F(t_n + \tfrac{1}{2}\Delta t, U_{n_3})\\
&U_{n+1} = U_n + \tfrac{1}{6}\Delta t(F(t_n, U_{n_1}) + 2F(t_n + \tfrac{1}{2}\Delta t, U_{n_2}) + 2F(t_n + \tfrac{1}{2}\Delta t, U_{n_3}) + F(t_n + \Delta t, U_{n_4})).
\end{aligned} \tag{2}$$

In the literature, we can find several papers devoted to discussing positivity property (e.g., [1, 4, 5]). In [2], positivity results have been presented for some Runge-Kutta methods. In [3], a step size condition has been obtained for monotonicity ($\|U_n\| \leq \|U_0\|$ for all $n \geq 1, U_0 \in \mathbb{R}^m$) with arbitrary convex function $\|.\|$, for general linear methods. $\|.\|$ is a convex function on $\mathbb{V}$ (the vector space on which the differential equation is defined) if $\|\lambda v + (1-\lambda)w\| \leq \lambda\|v\| + (1-\lambda)\|w\|$ for $0 \leq \lambda \leq 1$ and $v, w \in \mathbb{V}$. Usually, *step size coefficients* γ are determined such that monotonicity, in the sense of mentioned above, is present for all Δt with $0 < \Delta t \leq \gamma\tau_0$ ($\tau_0 > 0$ is a maximal step size such that $\|v + \Delta t F(t, v)\| \leq \|v\|$ for all t, $0 < \Delta t \leq \tau_0$ and $v \in \mathbb{R}^m$), see e.g.[3, 5]. General monotonicity of Runge-Kutta methods presented in [3] shows that the maximal step size coefficient γ for Heun's rule, is equal to 0. Monotonicity-preserving methods, can prevent the occurrence of negative values where even very small negative values are unacceptable, as for example, in the advective transport of chemical species. On the other hand, positivity preservation may be obtained from monotonicity-preserving methods see e.g.[1].

Applying the Heun's rule to special nonlinear ODEs (positive semi-discrete systems arising 1D with limited third-order upwind-biased spatial discretization), shows that the necessity of the step size restriction on positivity in general theory for Heun's method is somewhat too strict. From this practical point of view, the question arises whether it is theoretically possible to have positivity preservation for Heun's method. To answer this question, the Heun's method is applied to mentioned above nonlinear ODEs and some results are achieved theoretically that coincide with numerical experiments. Here, we focus on positivity for this method.

In the second section, general positivity results are presented for Heun's method. In the third section, the special positivity results are obtained for Heun's method. The numerical results have been shown in fourth section with respect to positivity.

2 General results on positivity for Heun's explicit third-order method

In this section, we study the general positivity for Heun's method. In many papers, one starts from an assumption about F which, $\tau_0 \geq 0$, to be the maximal step size such that positivity holds for the forward Euler method i.e.

$$U + \Delta t F(t, U) \geq 0 \ \ (\text{for all } t \text{ and } U \geq 0), \tag{3}$$

whenever $0 < \Delta t \leq \tau_0$ and $U \in \mathbb{R}^m$. As we can see in [2], Diagonally implicit Runge-Kutta methods can be written as follows:

$$v_i = \sum_{j=0}^{i-1}\Big(p_{ij} + q_{ij}\Delta t F(t_n + c_{j+1}\Delta t, v_j)\Big) + q_i \Delta t F(t_n + c_{i+1}\Delta t, v_i), \quad v_0 = U_n \tag{4}$$

for $i = 1, \ldots, s$, and finally set $U_{n+1} = v_s$. If $\sum_{j=0}^{i-1} p_{ij} = 1$ and $q_s = 0$, this is just another way of writing the s-stage diagonally implicit form of general Runge-Kutta method. If $q_i = 0$ for all indices i, the method is explicit. This form is theoretically convenient because the whole process in Runge-Kutta methods is written in terms of linear combinations of scaled forward and backward Euler steps.

We shall determine step size coefficients γ, such that the positivity is valid for (2) under the step size restriction $\Delta t \leq \gamma(\kappa)\tau_0$. Following an idea of Shu-Osher [4, 5] for explicit Runge-Kutta met

General positivity theorem : If all parameters p_{ij}, q_{ij} and q_i with $0 \leq j < i \leq s$ are non-negative, then method (3) will be positive for any F satisfying

i. there is an α such that $v + \Delta t F(t, v) \geq 0$ for all $t \geq 0$, $v \geq 0$ and $\alpha\Delta t \leq 1$,
ii. for any $v \geq 0$, $t \geq 0$ and $\Delta t > 0$, the equation $u = v + \Delta t F(t, u)$ has a unique solution that depends continuously on Δt and v, under the step size restriction $\alpha\Delta t \leq \min_{0\leq j<i\leq s}(p_{ij}/q_{ij})$, with

convention $p_{ij}/0 = +\infty$ for $p_{ij} \geq 0$.

With this theorem we have an empty positivity interval ($\gamma = 0$) for inhomogeneous linear systems. because this method can not be written as convex combinations of Euler steps, under the assumption (3) of the ODE, with non-negative coefficients p_{ij}, q_{ij}. A proof for the non-existence of coefficients $p_{ij}, q_{ij} \geq 0$ is given in [5].

General monotonicity results have been obtained in [3]. In that paper it has been shown that the obtained step size coefficient ($\gamma = 0$) is necessary for monotonicity in the maximum norm. It follows that the Shu-Osher form (4) is optimal.

Lemma 1. Monotonicity with step size coefficient γ implies positivity with the same step size coefficient. ([1])

3 Special results on positivity for Heun's explicit third-order method

In this section, we obtain the largest step size for Heun's method, whenever the underlying ODE possesses the related positivity preserving property. Let us consider

$$U_i' = \frac{q_i\big(U(t)\big)}{\Delta x}\big(U_{i-1}(t) - U_i(t)\big)\ \ , \ \ i = 1, 2 \dots, m, \tag{5}$$

with the nonlinear function $q_i(U)$ satisfying

$$q_i(U) \geq 0 \quad \text{for any vector U}, \tag{6}$$

and $\Delta x = \frac{1}{m},\ U = [U_1, U_2, \dots, U_m]^T,\ U_0 = U_m$. This special semi-discrete system arises from a linear advection problem after discretization using a flux limiter.

Lemma 2. Assuming (7) and Lipschitz continuity for q_i in (6) with respect to U, this non-linear system is positive.([1])

In the following we assume that there is a maximal step size $\tau_0 > 0$ under which positivity holds for the forward Euler method,

$$U + \Delta t \frac{q_i(U)}{\Delta x}\big(U_{i-1} - U_i\big) \geq 0 \qquad \text{for all } 0 < \Delta t \leq \tau_0,\ U \geq 0, \tag{7}$$

and we shall determine γ such that the positivity is valid for (2) under the step size restriction $\Delta t \leq \gamma\tau_0$. Application of (2) to (6) with $\nu_i^l = \Delta t \frac{q_i(U_l)}{\Delta x}$ and $l = n_1, n_2, n_3, n_4$, gives

$$\begin{aligned}
&U_i^{n+1} = \tfrac{1}{24}\nu_i^{n_4}\nu_{i-1}^{n_3}\nu_{i-2}^{n_2}\nu_{i-3}^{n_1}U_{i-4}^n + \tfrac{1}{12}\big(\nu_i^{n_3}\nu_{i-2}^{n_1}\nu_{i-1}^{n_2} + \nu_i^{n_4}\nu_{i-1}^{n_3}\nu_{i-2}^{n_2}\\
&-\tfrac{1}{2}\nu_i^{n_4}\nu_{i-1}^{n_3}\nu_{i-2}^{n_2}\nu_{i-3}^{n_1} - \tfrac{1}{2}\nu_i^{n_4}\nu_{i-1}^{n_3}\nu_{i-2}^{n_2}\nu_{i-2}^{n_1} - \tfrac{1}{2}\nu_i^{n_4}\nu_{i-1}^{n_3}\nu_{i-2}^{n_2}\nu_{i-1}^{n_1} - \tfrac{1}{2}\nu_i^{n_4}\nu_i^{n_3}\\
&\nu_{i-2}^{n_1}\nu_{i-1}^{n_2}\big)U_{i-3}^n + \tfrac{1}{6}\big(\nu_i^{n_2}\nu_{i-1}^{n_1} + \nu_i^{n_3}\nu_{i-1}^{n_2} - \tfrac{1}{2}\nu_i^{n_3}\nu_{i-1}^{n_2}\nu_{i-2}^{n_1} - \tfrac{1}{2}\nu_i^{n_3}\nu_{i-1}^{n_2}\nu_{i-1}^{n_1}\\
&-\tfrac{1}{2}\nu_i^{n_3}\nu_{i-1}^{n_1}\nu_i^{n_2} + \nu_i^{n_4}\nu_{i-1}^{n_3} - \tfrac{1}{2}\nu_i^{n_4}\nu_{i-1}^{n_3}\nu_{i-2}^{n_2} + \tfrac{1}{4}\nu_i^{n_4}\nu_{i-1}^{n_3}\nu_{i-2}^{n_2}\\
&\nu_{i-2}^{n_1} - \tfrac{1}{2}\nu_i^{n_4}\nu_{i-1}^{n_3}\nu_{i-1}^{n_2} + \tfrac{1}{4}\nu_i^{n_4}\nu_{i-1}^{n_3}\nu_{i-1}^{n_2}\nu_{i-2}^{n_1} + \tfrac{1}{4}\nu_i^{n_4}\nu_{i-1}^{n_3}\nu_{i-1}^{n_2}\nu_{i-1}^{n_1}\\
&-\tfrac{1}{2}\nu_i^{n_4}\nu_i^{n_3}\nu_{i-1}^{n_2} + \tfrac{1}{4}\nu_i^{n_4}\nu_i^{n_3}\nu_{i-1}^{n_2}\nu_{i-2}^{n_1} + \tfrac{1}{4}\nu_i^{n_4}\nu_i^{n_3}\nu_{i-1}^{n_2}\nu_{i-1}^{n_1} + \tfrac{1}{4}\nu_i^{n_4}\nu_i^{n_3}\nu_i^{n_2}\nu_{i-1}^{n_1}\big)U_{i-2}^n\\
&\qquad +\tfrac{1}{6}\big(\nu_i^{n_1} + 2\nu_i^{n_2} - \nu_i^{n_2}\nu_{i-1}^{n_1} + \nu_i^{n_2}\nu_i^{n_1} + 2\nu^{n_3} - \nu_i^{n_3}\nu_{i-1}^{n_2} + \tfrac{1}{2}\nu_i^{n_3}\nu_{i-1}^{n_2}\nu_{i-1}^{n_1} - \nu_i^{n_3}\nu_i^{n_2}\\
&\tfrac{1}{2}\nu_i^{n_3}\nu_i^{n_2}\nu_{i-1}^{n_1} + \tfrac{1}{2}\nu_i^{n_3}\nu_i^{n_2}\nu_i^{n_1} + \nu_i^{n_4} - \nu_i^{n_4}\nu_{i-1}^{n_3} + \tfrac{1}{2}\nu_i^{n_4}\nu_{i-1}^{n_3}\nu_{i-1}^{n_2} - \tfrac{1}{4}\nu_i^{n_4}\\
&\nu_{i-1}^{n_3}\nu_{i-1}^{n_2}\nu_{i-1}^{n_1} - \nu_i^{n_4}\nu_i^{n_3} + \tfrac{1}{2}\nu_i^{n_4}\nu_i^{n_3}\nu_{i-1}^{n_2} - \tfrac{1}{4}\nu_i^{n_4}\nu_i^{n_3}\nu_{i-1}^{n_2}\nu_{i-1}^{n_1} + \tfrac{1}{2}\\
&\nu_i^{n_4}\nu_i^{n_3}\nu_i^{n_2} - \tfrac{1}{4}\nu_i^{n_4}\nu_i^{n_3}\nu_i^{n_2}\nu_{i-1}^{n_1} - \tfrac{1}{4}\nu_i^{n_4}\nu_i^{n_3}\nu_i^{n_2}\nu_i^{n_1}\big)U_{i-1}^n\\
&+\big(1 - \tfrac{1}{6}\nu_i^{n_1} - \tfrac{1}{3}\nu_i^{n_2} + \tfrac{1}{6}\nu_i^{n_2}\nu_i^{n_1} - \tfrac{1}{3}\nu_i^{n_3} + \tfrac{1}{6}\nu_i^{n_3}\nu_i^{n_2} - \tfrac{1}{12}\nu_i^{n_3}\nu_i^{n_2}\nu_i^{n_1}\\
&-\tfrac{1}{6}\nu_i^{n_4} + \tfrac{1}{6}\nu_i^{n_4}\nu_i^{n_3} - \tfrac{1}{12}\nu_i^{n_4}\nu_i^{n_3}\nu_i^{n_2} + \tfrac{1}{24}\nu_i^{n_4}\nu_i^{n_3}\nu_i^{n_2}\nu_i^{n_1}\big)U_i^n
\end{aligned} \tag{8}$$

Theorem 1. Sufficient for scheme (2) applied to (6), to be positive is $0 \leq \Delta t \frac{q_i(U)}{\Delta x} \leq \gamma, \gamma = 1$, for all $U \in \mathbb{R}^m$ and $i = 1, 2, \ldots, m$.

It is possible to derive nonlinear positivity results for a class of diagonally implicit Runge-Kutta methods.

4 Numerical experiments

In this section, we have considered the scalar linear advection equation in one dimension $U_t + U_x = 0$ with $t > 0$, $0 < x < 1$ and a periodic boundary condition. We have discretized in space on uniformly distributed grid points $x_i = i\Delta x$, and $\Delta x = \frac{1}{500}$ by means of the flux form. The general discretization written out in full gives

$$U_i' == \frac{1}{\Delta x}\Big(1 - \psi(\theta_{i-1}) + \frac{1}{\theta_i}\psi(\theta_i)\Big)\big(U_{i-1} - U_i\big) \qquad i = 1, 2, \ldots, 500,$$

where θ_i is the ratio $\theta_i = \frac{U_i - U_{i-1}}{U_{i+1} - U_i}$, for $i = 1, 2, \ldots, 500$.
Here ψ is the *limiter function* with $\psi(\theta) = \max\Big(0, \min\big(1, \frac{1}{3} + \frac{1}{6}\theta, \theta\big)\Big)$. For details see $[2, p.\ 216]$.

Table 1 gives some numerical solutions with fixed time step sizes Δt and two initial profiles, viz. the peaked function $U_0(x,t) = sin^{100}(\pi x)$ and the block function $U_0(x,t) = 1$ for $0.3 \leq x \leq 0.7$ and 0 otherwise. Our final time is $t_f = 1$. Considering, an approximate solution positive if the smallest component is greater than -10^{-25}, the Heun's rule performs well up to CFL numbers = 1 but it's results quickly deteriorates when applied with larger and larger CFL numbers.

Table 1: Results for the scalar linear advection. N denotes the number of time steps.

	1D advection with smooth profile		1D advection with non-smooth profile	
N	$\min_{i,n}(U_i^n)$	$\max_{i,n}(U_i^n)$	$\min_{i,n}(U_i^n)$	$\max_{i,n}(U_i^n)$
400	-6.80e+080	6.80e+080	-2.47e+082	2.47e+082
450	-1.28e+040	1.28e+040	-1.33e+041	1.33e+041
500	1.33e-203	1.00	-9.46e-037	1+3e-014
550	1.10e-203	1.00	0.00	1+5e-014
600	9.26e-204	9.991e-001	0.00	1.00

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An improved expansion-iterative method for numerical solving of Fredholm-Voltrra integral equation

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Abstract

This article proposes a simple efficient method for solving Fredholm-Voltrra Integral Equation of the second kind. By using Block-Pulse functions and their improved operational matrix of integration, second kind Fredholm-Voltrra integral Equation can be reduced to a linear lower triangular system which can be directly solved by forward substitution. Numerical examples show that the approximate solutions have a good degree of accuracy.

Keywords: Block-pulse functions, Operational matrix, Voltrra-Fredholm integral equations system, Expansion method.

1 Introduction

Consider the following Fredholm-Volterra integral equation **(F-VIE)** of the second kind

$$\mu\Phi^{(n)}(x,t) - \lambda\int_{-1}^{1} k(x,y)\Phi^{(n-1)}(y,t)dy - \lambda\int_{0}^{t} d(t,z)\Phi^{(n-1)}(x,z)dz = f(x,t), \tag{1}$$

for $0 \le t \le 1$ and $|x| \le 1$ where $k,\ d$ and f are given functions, μ, λ are constant parameters and $\phi(x,t)$ is the unknown function to be determined.

2 Review of block-pulse functions

Definition 2.1. *An m-set of* **BPFs** *is defined as follows: [1, 2]* $\phi_i(t) = \begin{cases} 1, & \frac{i}{m} \le t < \frac{i+1}{m}, \\ 0, & o.w \end{cases}$

Assume that $k(s,t)$ is a function of two variables in $\ell^2([0,1)\times[0,1))$. We can simply expand the k with respect to **BPFs** as $k(s,t) \simeq \phi^T(s)K\psi(t)$, where $\phi(s)$ and $\psi(t)$ are m_1 and m_2 dimensional **BPF** vectors respectively, and K is the $m_1 \times m_2$ block-pulse coefficient matrix with $k_{ij}, i = 0,1,...,m_1-1,\ j = 0,1,...,m_2$ as follows: $k_{ij} = m_1 m_2 \int_0^1\int_0^1 k(s,t)\phi_i(s)\psi_j(t)dsdt$, For convenience, we put $m_1 = m_2 = m$

Definition 2.2. *Computing* $\int_0^1 \phi_i(\tau)d\tau$ *follows*

$$\int_0^t \phi_i(\tau)d\tau = \begin{cases} 0, & t < ih, \\ t - ih, & ih \le t < (i+1)h, \\ h, & (i+1)h \le t < 1. \end{cases} \simeq \left[0,...,0,\frac{h}{2},h,...,h\right]\phi(t) = P\phi,$$

By using this matrix we can express the integral of a function $f(t)$ into its Block-Pulse series [1]:

$$g(t) = \int_0^t f(\tau)d\tau \cong \int_0^t F^T\Phi(t)dt \cong F^T P\Phi(t) = [\overline{g}_1 \ \overline{g}_2 \ldots \overline{g}_m]\Phi(t), \tag{2}$$

on the other hand $\overline{g}_i$ can be written as follows:

$$\overline{g}_i = \int_0^{(i-1)h} f(t)dt + \frac{1}{2}\int_{(i-1)h}^{ih} f(t)dt = \frac{1}{2}[g((i-1)h) + g(ih)], \tag{3}$$

In order to improve the accuracy of Block-Pulse coefficients obtained from the integration rule, we can first use the Lagrange's interpolation formula, with three points $t_0 = (i-2)h, t_1 = (i-1)h$ and $t_2 = ih$, and then approximate $g(t)$ in the subinterval $t \in [(i-1)h, ih)$:

$$\begin{aligned}\overline{g}(t) = g((i-2)h)\frac{(t-(i-1)h)(t-ih)}{2h^2} - g((i-1)h)\frac{(t-(i-2)h)(t-ih)}{h^2}\\ + g(ih)\frac{(t-(i-2)h)(t-(i-1)h)}{2h^2}\end{aligned} \tag{4}$$

and by employing these approximations, the i'th Block-Pulse coefficient of $g(t)$ is evaluated more precisely as follows:

$$\overline{g}_i = \frac{1}{h}\int_{(i-1)h}^{ih} \overline{g}(t)dt = -\frac{1}{12}g((i-2)h) + \frac{8}{12}g((i-1)h) + \frac{5}{12}g(ih)$$

Finally, we can obtain the improved operational matrix $\overline{p}$ in the form of

$$\overline{P} = \frac{h}{2}\begin{pmatrix} 1 & \frac{13}{6} & 2 & 2 & \cdots & 2 \\ 0 & \frac{5}{6} & \frac{13}{6} & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \frac{13}{6} \\ 0 & 0 & 0 & \cdots & 0 & \frac{5}{6} \end{pmatrix} \tag{5}$$

3 Numerical solution of integral equation

Consider the Fredholm-Volterra integral equation Eq. (1) Approximating functions

$$\Phi^{(n-1)}(y,t), \Phi^{(n-1)}(x,z), \Phi^{(n)}(x,t), f(x,t)$$

and functions $k(x,y), d(t,z)$ with respect to **BPFs** gives where the matrix K and D are **BPFs** coefficients of $k(x,y)$ and $d(t,z)$ respectively. $f(x,t) \cong \Psi^T(x)f\Psi(t)$, $k(x,y) \cong \Psi^T(x)K\Psi(y)$. By substituting approximating functions $\Phi^{(n-1)}(y,t)$ and $k(x,y)$, in (a) gives

$$\begin{aligned}\int_{-1}^1 k(x,y)\Phi^{(n-1)}(y,t)dy &= \int_{-1}^1 \Psi^T(x)K\Psi(y)\Psi^T(y)\Phi^{(n-1)}\Psi(t)dy\\ &= \Psi^T(x)K\left\{\int_{-1}^1 \Psi(y)\Psi^T(y)dy\right\}\Phi^{(n-1)}\Psi(t)\\ &= \Psi^T(x)K(h)\Phi^{(n-1)}\Psi(t)\end{aligned}$$

By substituting approximating functions $\Phi^{(n-1)}(x,z)$ and $d(t,z)$, in (b) gives

$$\begin{aligned}\int_0^t d(t,z)\Phi^{(n-1)}(x,z)dz &= \int_0^t \Phi^{(n-1)}(x,z)d(t,z)dz\\ &= \int_0^t \Psi^T(x)\Phi^{(n-1)}\Psi(z)\Psi^T(z)d^T\Psi(t)dz\\ &= \Psi^T(x)\Phi^{(n-1)}\left\{\int_0^t \Psi(z)\Psi^T(z)dz.d^T\Psi(t)\right\}\end{aligned}$$

Suppose K_i be the ith row of the constant matrix K^T and R_j be the jth row of the improved integration operational matrix $\overline{P}$, by the previous relations and assuming that $m_1 = m_2 = m$ we will have,

$$\int_0^t \Psi(z)\Psi^T(z)dz.d^T\Psi(t) = \int_0^t \Phi(z)\Phi^T(z)dz.d^T\Psi(t) \tag{6}$$

because

$$\begin{pmatrix} R_1\Psi(t) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & R_m\Psi(t) \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix} \Psi(t) = \begin{pmatrix} \frac{h}{2}k_{1,1} & \frac{13h}{12}k_{2,1} & hk_{3,1} & \cdots & hk_{m,1} \\ 0 & \frac{5h}{12}k_{2,2} & \frac{13h}{12}k_{3,2} & \cdots & hk_{m,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{5}{12}hk_{m,m} \end{pmatrix} \Psi(t)$$

$$\int_0^t d(t,z)\Phi^{(n-1)}(x,z)dz \cong \Psi^T(x)\Phi^{(n-1)}B\Psi(t) \tag{7}$$

Assuming $\lambda = -1$ and $\mu = 1$ in Eq. (1), and Using (7) in Eq. (1) gives

$$\Psi^T(x)\Phi^{(n)}\Psi(t) + \Psi^T(x)K(I)\Phi^{(n-1)}\Psi(t) + \Psi^T(x)\Phi^{(n-1)}.B.\Psi(t) \cong \Psi^T(x)f\Psi(t) \tag{8}$$

Note that $\Phi^{(n)}$ is a diagonal matrix. Also, $K(I)\Phi^{(n-1)}$ and $\Phi^{(n-1)}.B$ are $m \times m$ matrices. $\Phi^{(n)} \cong \widehat{U}^T$, $K(I)\Phi^{(n-1)} \cong \widehat{V}^T$, $\Phi^{(n-1)}.B \cong \widehat{W}^T$ where $\widehat{U}^T, \widehat{V}^T$ and $\widehat{W}^T$ are m-vectors with components equal to the diagonal entries of matrices $\Phi^{(n)}, K(I)\Phi^{(n-1)}$ and $\Phi^{(n-1)}.B$ respectively.Then gives $\widehat{U}^T + \widehat{V}^T + \widehat{W}^T \cong f$ Replacing $\cong$ with $=$ and computing $\widehat{U}^T, \widehat{V}^T$ and $\widehat{W}^T$ follows

$$\begin{pmatrix} \Phi_1^{(n)} \\ \Phi_2^{(n)} \\ \Phi_3^{(n)} \\ \vdots \\ \Phi_m^{(n)} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_m \end{pmatrix} - \begin{pmatrix} \frac{h}{2}k_{1,1} & \frac{13}{12}hk_{2,1} & hk_{3,1} & \cdots & hk_{m,1} \\ 0 & \frac{5}{12}hk_{2,2} & \frac{13}{12}hk_{3,2} & \cdots & hk_{m,2} \\ 0 & 0 & \frac{5}{12}hk_{3,3} & \cdots & hk_{m,3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{5}{12}hk_{m,m} \end{pmatrix} \begin{pmatrix} \Phi_1^{(n-1)} \\ \Phi_2^{(n-1)} \\ \Phi_3^{(n-1)} \\ \vdots \\ \Phi_m^{(n-1)} \end{pmatrix}$$

can be written as $\Phi^{(n)} = Q - A\Phi^{(n-1)}$, $Q = (f_1, ..., f_m)^T$ for $m = 1, 2, 3, ...$

Now, considering the initial guess $\Phi^{(0)} = 0$, where 0 is the zero m-vector, or $\Phi^{(0)} = Q$, and using the recurrence relation $\Phi^{(n)} = Q - A\Phi^{(n-1)}$, one may steadily increase the degree of approximation until convergence is reached to a sufficient accuracy. To do this, $||\Phi^{(n)} - \Phi^{(n-1)}|| < \varepsilon$ or $\frac{||\Phi^{(n)} - \Phi^{(n-1)}||}{||\Phi^{(n)}||} < \varepsilon$ for arbitrary small ε, may be considered as stopping condition, where $||.||$ is an arbitrary vector norm. In the next section, it will be studied how one knows what is sufficient accuracy.

Example 1. Consider the following **(F-VIE)** of the second kind. (see [3])

$$\phi^{(n)}(x,t) + \int_0^1 (\frac{y}{2+x})\phi^{(n)}(y,t)dy + \int_0^t (e^{t+z})\phi^{(n-1)}(x,z)dz = e^{x-t} + \frac{e^{-t}}{2+x} + te^{x+t}$$

with the exact solution $\phi(x,t) = e^{x-t}$ for $0 \le t \le 1$ and $T = 1, m = 4$.

$$||\Phi_{11}^{(2)} - \Phi_{11}^{(1)}|| = 0.052064, \ ||\Phi_{11}^{(3)} - \Phi_{11}^{(2)}|| = 0.004124, \ ||\Phi^{(4)} - \Phi^{(3)}|| = 0.000227$$

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Comparison between homotopy perturbation, homotopy analysis and Adomian decomposition methods for solving the Benney-Lin equation

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Abstract

This article applies the Homotopy Perturbation (HPM), Homotopy Analysis (HAM) and Adomian Decomposition Methods (ADM) to obtain approximate analytical solutions of the Benney-Lin equation. The comparison between the solutions of this problem will be demonstrated by these three methods. Graphical representation shows that these three methods are most effective, accurate and convenient to solve this problem.

Keywords: Benney-Lin equation, Homotopy Perturbation Method, Homotopy Analysis Method, Adomian Decomposition Method.

Mathematics Subject Classification: Primary: 35R99; Secondary: 35B20, 35B30

1 Introduction

In this work, it will be implemented the HPM, HAM and ADM to obtain approximate solutions of the following partial differential equation:

$$\frac{\partial}{\partial t}u + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \beta(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4}) + \alpha\frac{\partial^5 u}{\partial x^5} + \frac{\partial u}{\partial x} = 0, \tag{1}$$

subject to the following initial condition:

$$u(x,0) = k(x),$$

which is called the Benney-Lin equation[1] where $\beta > 0$, $\alpha \in \mathbb{R}$. The Benney-Lin equation was first introduced By Benney[2] and later by Lin[3]. It is an important general equation that describes the evolution of long waves in various problem in fluid dynamics. In this paper, two different values for $k(x)$ will be studied:

Case 1: $k(x) = 1 - 2\mu^2 tan(\mu x)$, **Case 2:** $k(x) = 1 - 2\mu^2 tanh^2(\mu x)$.

2 Implementation the HPM, HAM and ADM

2.1 Using HPM

According to the HPM, we construct the following homotopy for equation (1):

$$L(\varphi) - L(u_0) + pL(u_0) + pN(\varphi) = 0, \tag{2}$$

were $L(\varphi)=\frac{\partial}{\partial t}\varphi$ and $N(\varphi)=\varphi\frac{\partial\varphi}{\partial x}+\frac{\partial^3\varphi}{\partial x^3}+\beta(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^4\varphi}{\partial x^4})+\alpha\frac{\partial^5\varphi}{\partial x^5}+\frac{\partial\varphi}{\partial x}$ with the initial approximation $u_0=u(x,0)=k(x)$. Suppose that the solution of Eq. (1) has the form:

$$\varphi(x,t;p)=u_0(x,t)+\sum_{i=1}^{\infty}u_i(x,t)p^i. \tag{3}$$

Substituting (3) in (2) and comparing the coefficient of identical degrees of p, the approximate solution of (1) can be obtained by setting $p=1$ in (3). For example in **Case 1** results in:

$u_0(x,t)=1-2\mu^2tan(\mu x)$,

$u_1(x,t)=240\alpha t\mu^7tan^6(\mu x)+48\beta t\mu^6tan^5(\mu x)+(12t\mu^5+480\alpha t\mu^7)tan^4(\mu x)+(4\beta t\mu^4$ $+80\beta t\mu^6-4t\mu^5)tan^3(\mu x)+(16t\mu^5+272\alpha t\mu^7+4t\mu^3)tan^2(\mu x)+(-4t\mu^5+32\beta t\mu^6$ $+4\beta t\mu^4)tan(\mu x)+32\alpha t\mu^7+4t\mu^5+4t\mu^3$,

and the approximate solution of Eq. (1) is $u(x,t)\simeq u_0(x,t)+u_1(x,t)+u_2(x,t)$.

2.2 Using HAM

According to the HAM, we construct the following homotopy for equation (1):

$$(1-p)(L(\varphi)-L(u_0))=p\hbar N(\varphi), \tag{4}$$

where $L(\varphi)=\frac{\partial}{\partial t}\varphi$ and $N(\varphi)=\frac{\partial}{\partial t}\varphi+\varphi\frac{\partial\varphi}{\partial x}+\frac{\partial^3\varphi}{\partial x^3}+\beta(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^4\varphi}{\partial x^4})+\alpha\frac{\partial^5\varphi}{\partial x^5}+\frac{\partial\varphi}{\partial x}$.
Choosing the initial approximation $u_0(x,0)=k(x)$ and setting:

$$\varphi(x,t;p)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)p^m, \tag{5}$$

where $u_m(x,t)=\frac{1}{m!}\frac{\partial^m\varphi(x,t;p)}{\partial p^m}$. Putting (5) in (4), differentiating m-times with respect to p and setting $p=0$ one obtain:

$u_0(x,t)=1-2\mu^2tan(\mu x)$,

$u_1(x,t)=-240\alpha t\hbar\mu^7tan^6(\mu x)-48\beta t\hbar\mu^6tan^5(\mu x)+(-12t\hbar\mu^5-480\hbar\alpha t\mu^7)tan^4(\mu x)$ $+(-4\beta t\hbar\mu^4-80\beta t\hbar\mu^6+4t\hbar\mu^5)tan^3(\mu x)+(-16t\hbar\mu^5-272\alpha t\hbar\mu^7-4t\hbar\mu^3)tan^2(\mu x)$ $+(4t\hbar\mu^5-32\beta t\hbar\mu^6-4\beta t\hbar\mu^4)tan(\mu x)-32\alpha t\hbar\mu^7-4t\hbar\mu^5-4t\hbar\mu^3$.

Therefore the approximate solution of Eq. (1) is $u(x,t)\simeq u_0(x,t)+u_1(x,t)+u_2(x,t)$.

2.3 Using ADM

Consider the following differential equation:

$$L(u(x,t))+R(u(x,t))+N(u(x,t))=f(x,t).$$

According to the ADM we assume:

$$L(u)=\frac{\partial}{\partial t}u,\quad N(u)=0,\quad R(u)=u\frac{\partial u}{\partial x}+\frac{\partial^3u}{\partial x^3}+\beta(\frac{\partial^2u}{\partial x^2}+\frac{\partial^4u}{\partial x^4})+\alpha\frac{\partial^5u}{\partial x^5}+\frac{\partial u}{\partial x},$$

then

$u_0(x,t)=1-2\mu^2tan(\mu x)$,

$u_1(x,t)=-\int_0^t(u_0\frac{\partial u_0}{\partial x}+\frac{\partial^3u_0}{\partial x^3}+\beta(\frac{\partial^2u_0}{\partial x^2}+\frac{\partial^4u_0}{\partial x^4})+\alpha\frac{\partial^5u_0}{\partial x^5}+\frac{\partial u_0}{\partial x})\,dt$,

$u_2(x,t)=-\int_0^t(u_1\frac{\partial u_1}{\partial x}+\frac{\partial^3u_1}{\partial x^3}+\beta(\frac{\partial^2u_1}{\partial x^2}+\frac{\partial^4u_1}{\partial x^4})+\alpha\frac{\partial^5u_1}{\partial x^5}+\frac{\partial u_1}{\partial x})\,dt$.

Therefore the approximate solution of Eq. (1) is $u(x,t)\simeq u_0(x,t)+u_1(x,t)+u_2(x,t)$. See Fig. 1 and Table. 1.

3 Numerical Results

In this section, numerical results are shown for two cases, when $\alpha = 1$, $\beta = 1$, $\mu = 1$, $\hbar = -1$, $t = 0.001$. The approximate solutions will be obtained by 3-terms HPM, HAM and ADM. According to the data in Tables 1, 2 and Figs. 1, 2, we can see that at $t = 0.001$, the values of the approximate solutions by HPM, HAM and ADM are quite close and almost the same.

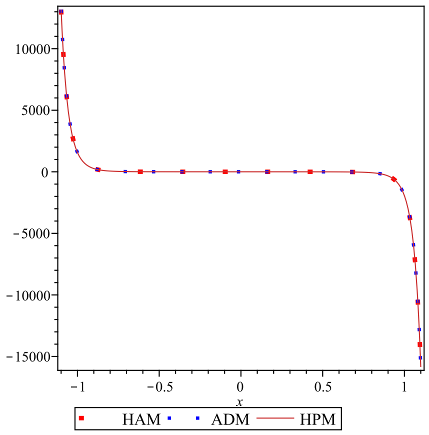

Figure 1: Comparison between the results obtained by HAM, ADM and HPM for **Case 1.**

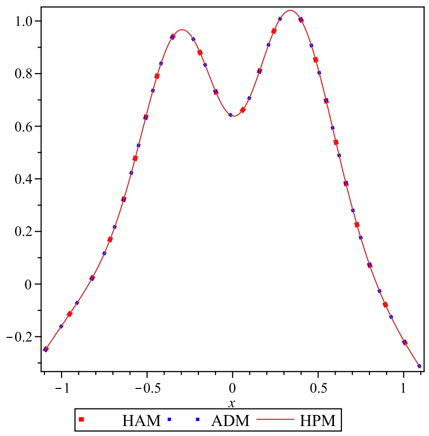

Figure 2: Comparison between the results obtained by HAM, ADM and HPM for **Case 2.**

4 Conclusion

The basic goal of this work has been to solve the Benney-Lin equation by powerful analytic methods, such as HPM, HAM and ADM. Obtaining an approximate analytic solution for this problem and comparing the results by these methods are the other our aims. The numerical solutions are shown graphically, and the results show that the HPM and HAM are in excellent agreement with the results obtained by the ADM. It may be concluded that the homotopy methodology is very powerful and simple techniques for wide classes of problems and can be also easy to be extended to the other non-linear evaluation equations with the aid of Mathematica, Matlab or Maple software.

Table 1: Numerical approximate solutions of **Case 1** when $t = 0.001$, $Error1 = HPM - ADM$, $Error2 = HAM - ADM$

$t = 0.001$					
x	HPM	HAM	ADM	$Error1$	$Error2$
-0.2	1.52932614	1.52932614	1.52921202	0.00011412	0.00011412
-0.1	1.26178962	1.26178962	1.26173401	0.00005561	0.00005561
0	1.02326800	1.02326800	1.02324357	0.00002443	0.00002443
0.1	0.78302407	0.783002407	0.78301421	0.00000986	0.00000986
0.2	0.50681935	0.50681935	0.50680812	0.00001123	0.00001123

Table 2: Numerical approximate solutions of **Case 2** when $t = 0.001$, $Error1 = HPM - ADM$, $Error2 = HAM - ADM$

$t = 0.001$					
x	HPM	HAM	ADM	$Error1$	$Error2$
-0.2	0.89409486	0.89409456	0.89413977	0.00004492	0.0004492
-0.1	0.73295749	0.73295749	0.73314713	0.00018965	0.00018965
0	0.63924800	0.63924800	0.63951285	0.00026485	0.00026485
0.1	0.71011424	0.71011424	0.71034782	0.00023357	0.00023357
0.2	0.89092668	0.89092668	0.89104961	0.00012293	0.00012293

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Monotonicity of the ground state energy in a circular quantum dot

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Abstract
In this paper we investigate a minimization problem related to the ground state energy of the Schrödinger operator and we prove a monotonicity result corresponding to the principal eigenvalue in a ball in $\mathbb{R}^n$.
Keywords: Quantum Dots, Minimization Problems, Eigenvalue Problems

Mathematics Subject Classification: 35Q93, 35Q40, 35P15, 35J10

1 Introduction

In this paper we have modeled a quantum dot structure with a lateral size which is determined by the size of the complement of a set D embedded in a large area Ω . We have mathematically solved Schrödinger equation for the lowest energy level of structures with different optional shapes where all of them have the same depth times the size of the well (area in 2-dimensions) aiming for an optimized energy level.

Now we state the mathematical equations modeling the above physical optimization problem. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and $D \subset \Omega$ be a set of positive measure. In this paper we denote the Lebesgue measure by $|.|$. Consider the governing Schrödinger equation

$$-\triangle u + \frac{\kappa}{|D|}\chi_D u = \lambda u \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

where $\kappa > 0$, $\lambda = \lambda(D)$ is the first eigenvalue (ground state energy), $u = u(x)$ is a corresponding wave function and $\chi_D(x)$ is the characteristic function of D.

It is well known that the first eigenvalue λ is obtained by the following variational formulation [1]

$$\lambda = \min_{\substack{u\in H_0^1(\Omega)\\ \|u\|_{L^2(\Omega)}=1}} \|u\|^2_{H_0^1(\Omega)} + \frac{\kappa}{|D|}\int_D u^2 dx. \tag{2}$$

Mathematically, we are interested in the optimization problem

$$\inf_{D\subset\Omega;\ \beta|\Omega|\le|D|\le|\Omega|} \lambda(D) = \inf_{\substack{D\subset\Omega;\ \beta|\Omega|\le|D|\le|\Omega|\\ u\in H_0^1(\Omega);\ \|u\|_{L^2(\Omega)}=1}} \|u\|^2_{H_0^1(\Omega)} + \frac{\kappa}{|D|}\int_D u^2 dx, \tag{3}$$

where $\beta \in (0, 1)$. It should be noted that the last equality in (3) can be verified easily. Let us remind that in (3), as a physical interpretation, $\kappa\backslash|D|$ represents the depth of the quantum dot (well) and $|D|$ stands for the size of the well.

Such optimization problems have been quite attractive to mathematician in the past decades [2, 3, 4]. It should be noted that in [5] the optimization problem was considered when $|D|$ is constant whereas in this paper we have $|D| \in [\beta|\Omega|, |\Omega|]$.

2 Preliminaries

In this section we review some well known results with an eye on the minimization problem (3). By the regularity theory for elliptic partial differential equations [1], we have next lemma.

Lemma 2.1. *Let u be an eigenfunction corresponding to the first eigenvalue of (1). Then $u \in H^2(\Omega) \cap C^{1,\gamma}(\Omega) \cap C(\overline{\Omega})$ for some $\gamma \in (0,1)$ and $u > 0$ in Ω.*

Assume Ω is a ball in $\mathbb{R}^N$ centered at the origin and $f : \Omega \to \mathbb{R}$ be a Lebesgue measurable function. Then, $f^* : \Omega \to \mathbb{R}$ and $f_* : \Omega \to \mathbb{R}$ represent the Schwartz decreasing and increasing rearrangements of f. That is, functions f^* and f are equimeasurable in the sense that

$$|\{x \in \Omega : f(x) \geq c\}| = |\{x \in \Omega : f^*(x) \geq c\}| \qquad \forall c \in \mathbb{R} \tag{4}$$

and we have similar argument for f_*. Additionally, f^* is a radial function which is decreasing as a function of $r = \|x\|$, whereas f_* is a radial function which is increasing as a function of r. We will apply two classic rearrangement inequalities derived in [6] and [7].

Lemma 2.2. *Suppose Ω is a ball in $\mathbb{R}^n$. Then*

$$\int_\Omega f^* g_* dx \leq \int_\Omega f g dx \leq \int_\Omega f^* g^* dx,$$

where f and g are nonnegative functions.

Lemma 2.3. *Assume Ω is a ball in $\mathbb{R}^n$, $p \geq 1$ and $0 \leq u \in W_0^{1,p}(\Omega)$. Then we have $u^* \in W_0^{1,p}(\Omega)$ and*

$$\int_\Omega |\nabla u|^p dx \geq \int_\Omega |\nabla u^*|^p dx. \tag{5}$$

Throughout this paper, a ball centered at the origin with radius r will be denoted by $\mathcal{B}(0,r)$.

3 Study of The Mathematical Modeling Equations

This section is devoted to optimization problem (3). Let us state an existence result.

Theorem 3.1. *There exists $D \subset \Omega, \quad \beta|\Omega| \leq |D| \leq |\Omega|$ which is a solution of problem (3).*

From a physical point of view, it is important for us to know the shape of the optimal set D. Such questions have been addressed in [5, 8]. To this end, hereafter we assume Ω is a ball centered at the origin. More precisely, without loss of generality we take $\Omega = \mathcal{B}(0,1)$.

Lemma 3.2. *Let $\Omega = \mathcal{B}(0,1)$, then (3) has a solution $\widetilde{D}$ with eigenvalue $\lambda(\widetilde{D})$ associated to a decreasing and radially symmetric function $\widetilde{u}$ such that $\widetilde{D}$ is a rotationally symmetric set as a neighborhood of $\partial\Omega$.*

Now we can state the main theorem of this paper that identifies configuration of the optimal set D exactly which is worth due to the physical applications.

Theorem 3.3. *Let Ω be as in lemma 3.2 and D be a rotationally symmetric minimizer of the optimization problem (3) corresponding to radially decreasing eigenfunction u. Then, we have $|D| = \beta|\Omega|$.*

Corollary 3.4. *If $\Omega = \mathcal{B}(0,1)$, then the optimal solution D of (3) is unique modulo sets of measure zero.*

Corollary 3.5. *Let μ_Ω be the principle eigenvalue of the Laplace's operator with the Dirichlet's boundary condition. If in (3) we have $\beta \to 0$ then $\lambda(D) \to \mu_\Omega$ where D is the minimizer such that $|D| = \beta|\Omega|$.*

4 Physical Interpretations

let us state the physical interpretations of the results in case $\mathbb{R}^2$. As explained in the previous section we have modeled a class of quantum dots all being of different circular shapes but all of them have equal depth times the area. While the exact functional behavior of the first energy level on the characteristics of the structure is not explicitly known, our results (theorem 3.3 and corollary 3.4) manifest increasing energy with simultaneously decreasing the depth and increasing the area of the dot.

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Positive solutions of an initial value problem for nonlinear fractional differential equations

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Abstract

In this paper, we investigate the existence and multiplicity of positive solutions for the nonlinear fractional differential equation boundary value problem

$$D^{\alpha}_{0+}u(t) + D^{\beta}_{0+}u(t) = f(t, u(t)) \qquad (1.1)$$

$$u(0) = 0, 0 < t < 1,$$

where $0 < \beta < \alpha < 1$, D^{α}_{0+} is the standard Riemann-Liouville differentiation and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. By using some fixed-point results on cones, some existence and multiplicity results of positive solutions are obtained.

Keywords: Boundary value problem, Fixed point, Fractional differential equation, Green function, Positive solution.

Mathematics Subject Classification: 26A33, 34A08, 34K37.

1 Introduction

Fractional differential equations have been of great interest recently. You know, fractional calculus appears in various field of science covering many known classical fields, such as Abel integral equation and viscoelasticity, analysis of feedback amplifiers, capacitor theory, fractances, generalized voltage divider, etc. Basic approach to fractional calculus and applications in mechanics, probability and statics is given in [1] and references there in. The comprehensive treatment of the classic fractional equations techniques such as Laplace and Fourier transform method, method of Green function, Mellin transform and some numerical techniques are given in [2] and [3]. In some papers, for nonlinear problems, techniques of functional analysis such as fixed point theory, the Banach contraction principle, Leray-Schauder theory and so on are applied for solving such kind of the problems (see for example, [4, 5, 6]).

Definition 1.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by*

$$I^{\alpha}f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)\,d\tau.$$

Definition 1.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by*

$$D^{\alpha}f(t) := \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t (t-\tau)^{n-\alpha-1}f(\tau)\,d\tau,$$

where $n = [\alpha] + 1$ and the right side is pointwise defined on $(0, \infty)$.

Definition 1.3. *Two-parametric Mittag-Leffler function is defined by*

$$E_{(\alpha,\beta)}(z) = \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}$$

for all $\alpha > 0$ *and* $\beta > 0$.

Let P be a cone in a Banach space E. The map $\theta : P \to [0,\infty]$ is said to be a nonnegative continuous concave functional whenever θ is continuous and

$$\theta(tx + (1-t)y) \geq t\theta(x) + (1-t)\theta(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$ ([9]). We need the following fixed point theorems for obtaining our results.

Lemma 1.4. *([7]) Let* E *be a Banach space,* P *a cone in* E *and* Ω_1, Ω_2 *two bounded open balls of* E *centered at the origin with* $\overline{\Omega_1} \subset \Omega 2$. *Suppose that* $A : P \cap (\overline{\Omega_2} \backslash \Omega 1) \to P$ *is a completely continuous operator such that either*
(i) $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$, *or*
(ii) $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$
holds. Then A *has a fixed point in* $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Lemma 1.5. *([8]) Let* P *be a cone in a real Banach space* E, c, b *and* d *positive real numbers,* $P_c = \{x \in P : \|x\| \leq c\}$, θ *a nonnegative concave functional on* P *such that* $\theta(x) \leq \|x\|$ *for all* $x \in \overline{P_c}$ *and* $P(\theta, b, d) = \{x \in P : b \leq \theta(x),\ \|x\| \leq d\}$. *Suppose that* $A : \overline{P_c} \to \overline{P_c}$ *is completely continuous and there exist constants* $0 < a < b < d \leq c$ *such that*
(c_1) $\{x \in P(\theta, b, d) : \theta(x) > b\} \neq \emptyset$ *and for some* $x \in P(\theta, b, d)$ *we have* $\theta(Ax) > b$,
(c_2) $\|Ax\| < a$ *for all* x *with* $\|x\| \leq a$,
(c_3) $\theta(Ax) > b$ *for all* $x \in P(\theta, b, c)$ *with* $\|Ax\| > d$.
Then A *has at least three fixed points* x_1, x_2 *and* x_3 *such that* $\|x_1\| < a$, $b < \theta(x_2)$, $a < \|x_3\|$ *with* $\theta(x_3) < b$.

2 Main Result

As you know, there is an integral form of the solution for the equation (1.1). Suppose that the functions u and f are continuous on $[0,1]$. Then $u(t) = \int_0^t G(t-\tau) f(\tau, u(\tau))\, d\tau$ is a solution for the equation (1.1), where $G(t) = t^{\alpha-1} E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})$ and $E_{\alpha,\beta}$ is the two-parameter function of the Mittag-Leffler type (see [3]). Now, we give an equivalent solution for the equation (1.1). In fact, if we apply the Laplace transform to equation (1.1), then by using a calculation and finding the inverse Laplace transform we get that $u(t) = t^{\alpha-1} E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta}) * f(t, u(t))$ is an equivalent solution for the equation (1.1). In this way, note that

$$D^\alpha u(t) + D^\beta u(t) = (D^\alpha G(t) + D^\beta G(t)) * f(t, u(t)),$$

where $G(t) = t^{\alpha-1} E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})$. But, we have

$$D^\alpha G(t) + D^\beta G(t) = t^{-1} E_{\alpha-\beta,0}(-t^{\alpha-\beta}) + t^{\alpha-\beta-1} E_{\alpha-\beta,\alpha-\beta}(-t^{\alpha-\beta})$$

$$= E_{\alpha-\beta,0}(-t^{\alpha-\beta}) - E_{\alpha-\beta,0}(-t^{\alpha-\beta}) - \frac{1}{t}\frac{1}{\Gamma(\alpha-\beta)}.$$

Since $\lim_{t\to 0} \frac{1}{t}\frac{1}{\Gamma(\alpha-\beta)} = \delta(t)$, we get $D^\alpha G(t) + D^\beta G(t) = \delta(t)$ and so

$$D^\alpha u(t) + D^\beta u(t) = \delta(t) * f(t, u(t)) = f(t, u(t)).$$

Now, we establish some results on existence and multiplicity of positive solutions for the problem (1.1). Let $E = (C[0,1], \|.\|_\infty)$ be endowed via the order $u \leq v$ if and only if $u(t) \leq v(t)$ for all $t \in [0,1]$. Consider the cone $P = \{u \in E \mid u(t) \geq 0\}$ and the nonnegative continuous concave functional $\theta(u) = \inf_{\frac{1}{2}<t<1} \mid u(t) \mid$. Now, we give our first result.

Lemma 2.1. *Define $T : P \to P$ by $Tu(t) := \int_0^t G(t-\tau) f(\tau, u(\tau))\, d\tau$, where $G(t) = t^{\alpha-1} E_{\alpha-\beta,\alpha}(-t^{\alpha-\beta})$ and $E_{\alpha,\beta}(z)$ is the two-parameter function of the Mittag-Leffler type. Then T is completely continuous.*

Theorem 2.2. *Suppose that in the problem (1.1) there exists a positive real number $r > 0$ such that*
(A_1) $f(t,u) \leq \alpha r$ for all $(t,u) \in [0,1] \times [0,r]$,
(A_2) $f(t,u) \geq 0$ for all $t \in [0,1]$ with $u(t) = 0$.
Then the problem (1.1) has a positive solution u such that $0 \leq \|u\| \leq r$.

Example 2.3. *Consider the nonlinear fractional differential equation boundary value problem*

$$D^{3/2}u(t) + D^{1/2}u(t) + u(t) + \sin t = 0, \;\; u(0) = 0 \;\; (0 < t < 1).$$

Put $r = 2$ and $\alpha = \frac{3}{2}$. Since $f(t,u) = u(t) + \sin t \leq u + 1 \leq 3 = \alpha r$ for all $(t,u) \in [0,1] \times [0,2]$ and $f(t,u) = u + \sin t \geq 0$ for all $(t,u) \in [0,1] \times \{0\}$, by using Theorem 1.2 we get that this problem has a positive solution u with $0 \leq \|u\| \leq 2$.

Theorem 2.4. *Suppose that in the problem (1.1) there exist positive real numbers $0 < a < b < c$ such that*
(A_1) $f(t,u) < \alpha a$ for all $(t,u) \in [0,1] \times [0,a]$,
(A_2) $f(t,u) > Nb$ for all $(t,u) \in [\frac{1}{2}, 1] \times [b,c]$, where

$$N^{-1} = \inf_{\frac{1}{2}<t<1} \mid \int_0^t G(t-s)\, ds \mid,$$

(A_3) $f(t,u) \leq \alpha c$ for all $(t,u) \in [0,1] \times [0,c]$.
Then the problem (1.1) has at least there positive solutions u_1, u_2 and u_3 such that $\sup_{0\leq t\leq 1} |u_1(t)| < a$, $b < \inf_{\frac{1}{2}\leq t\leq 1} |u_2(t)| < \sup_{\frac{1}{2}\leq t\leq 1} |u_2(t)| \leq c$, $a < \sup_{0\leq t\leq 1} |u_3(t)| \leq c$ and $\inf_{\frac{1}{2}\leq t\leq 1} |u_3(t)| < b$.

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Determination of a control function in one-dimensional parabolic equations by using Lagrange functions

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Abstract
The problem of finding the solution of partial differential equations with source control parameter has appeared increasingly in physical phenomena, for example, in the study of heat conduction process, thermo-elasticity, chemical diffusion and control theory. In this paper, we consider two inverse heat conduction problem. In this work, except for unknown function in heat equation, other function in problem is unknown also. For solving of this inverse problem, an additional condition is used at given area of problem. Some numerical examples will be given in the last problem.
Keywords: Parabolic Inverse problem, Heat conduction equation, Lagrange functions, Control function.

1 Introduction

In this paper, we will consider two inverse problems for finding a source parameter p(t) in the following semilinear parabolic equation:

$$u_t = u_{xx} + p(t)u + q(x,t) \quad 0 \le x \le L, 0 \le t \le T \tag{1}$$

with initial condition:

$$u(x,0) = f(x) \quad 0 \le x \le L, \tag{2}$$

and boundary conditions:

$$u(0,t) = g_0(t) \quad 0 \le t \le T, \tag{3}$$

$$u(1,t) = g_1(t) \quad 0 \le t \le T, \tag{4}$$

with over extra additional condition:

$$u(x_1,t) = k_1(t) \quad 0 \le x \le L, 0 \le t \le T, \tag{5}$$

is assumed that $f, g_0, g_1, q \quad and \quad k_1$ are known functions.
x_1, T and L are identified positive numbers and $u(x,t)$ and p(t) are unknown functions. Or with additional condition which describes oversimplification over a portion of the spatial domain:

$$\int_0^1 k(x)u(x,t)dx = E(t) \qquad 0 < t \le T \tag{6}$$

where k and E are known functions, and the functions $u \quad and \quad p$ are unknown.
It is assumed that, for some constant $\rho > 0$, the kernel k(x) satisfies [3].

$$\int_0^1 |k(x)|dx \leq \rho$$

this type of equation has many applications,for example cannon et.al.[1] formulated a backward Euler finite difference via a transformation and proved the convergence order of $O(\triangle t, h^2)$ .Dehghan and saadatmandi in [2] proposed a tau method for the solution of problem (1-5). The method of lines based on a applying the standard finite difference schemes and the runge-kutta formula presented in [3] for this problem.
before defining solutions of problem, we present the following lemma:

Lemma 1.1. *If $g_0(t) \neq 0 \quad and \quad g_1(t) \neq 0$ and $u(x,t)$ be continuous ,of the solutions the inverse problem (1)-(5) then:*

$$\exists \phi(x) \qquad \frac{g_0(t)}{\phi(0)} = \frac{g_1(t)}{\phi(1)} = \frac{k_1(t)}{\phi(x_1)}$$

for the inverse problem (1)-(5) we consider u(x,t) as the following

$$u(x,t) = k(x,t)f(x), \tag{7}$$

we impose the conditions to find k(x,t) whit the conditions to (2)-(4) for solution (5) we have:

$$\begin{cases} k(x,0) = 1, \\ k(0,t) = \frac{g_0(t)}{f(0)} = G_0(t), & 0 \leq t \leq T \\ k(1,t) = \frac{g_1(t)}{f(1)} = G_1(t), & 0 \leq t \leq T \\ k(x_1,t) = \frac{k_1(t)}{f(x_1)} = k_1^*(t), & 0 \leq t \leq T \end{cases} \tag{8}$$

we define $k(x,t)$ as follows:

$$k(x,t) = L_0(x)G_0(t) + L_1(x)G_1(t) + L_*(x)k_1^*(x) \qquad t > 0 \tag{9}$$

where L_i is a one dimensional language function and we have:

$$L_0(x) = \frac{(x-1)(x-x_1)}{x_1} \quad L_1(x) = \frac{(x)(x-x_1)}{1-x_1} \quad L_*(x) = \frac{(x)(x-1)}{x_1(x_1-1)} \tag{10}$$

it is clear k(x,t) is true in conditions (8).
using of u(x,t) in the previous section, we have of (1):

$$p(t) = \frac{\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - q(x,t)}{u(x,t)} \tag{11}$$

such that functions u and p satisfy (1)-(6).
Eq (6) can be interpreted as a weighted thermal energy contained in apportion of the spatial domain at time t. The applications of this inverse problem and some similar parameter identification problems are discussed in [4,5,6,7]. Dehgan in [8,9] presented four finite differencre schemes for the solution of this problem and similar higher dimensional problems are investigated. Some numerical methods are presented in [13,14,15] for solving this problem.

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Solving a general nonlinear Fredholm integro-differential equation under the mixed conditions

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Abstract

In this paper, the integral mean value method is employed to handle a general nonlinear Fredholm integro-differential equation under the mixed conditions. The application of the method is based on the integral mean value theorem for integrals. By using the integral mean value method, an integro-differential equation is transformed to an ordinary differential equation, then by solving it, the obtained solution is transformed to a system of nonlinear algebraic equations to calculate the unknown values.

Keywords: Nonlinear Fredholm integro-differential equations; Integral mean value method; Mixed conditions.

Mathematics Subject Classification: 45Jxx

1 Introduction

In this paper, a review of the integral mean value method is given and then its application is extended to the general nonlinear Fredholm integro-differential. Also, a comparison with two analytic methods, namely Adomian decomposition method (ADM) and homotopy perturbation method (HPM), is given.

We consider the mth-order nonlinear Fredholm integro-differential equation with variable coefficients

$$\sum_{r=0}^{m} f_r(x)u^{(r)}(x) = g(x) + \lambda \int_a^b K(x,t)F(u(t))dt, \quad x \in [a,b], \tag{1}$$

where $F(u(x))$ is a function of $u(x)$, $K(x,t)$ is the kernel of the integro-differential equation, λ is a parameter, $g(x)$ is the data function, $f_r(x)$ is a function with respect to x, and $u(x)$ is the unknown function that will be determined. We consider the Eq. (1) under the mixed conditions

$$\sum_{r=0}^{m-1} \left(a_{rj}u^{(r)}(a) + b_{rj}u^{(r)}(b) + c_{rj}u^{(r)}(c)\right) = \mu_j, \quad j = 0,1,2,...,m-1, \tag{2}$$

where a_{rj}, b_{rj}, c_{rj} and μ_j are functions with respect to x, and c, $a < c < b$, is a constant. For the linear case, it is assumed that $F(u(x)) = u(x)$. Several numerical and analytical methods were used such as the successive approximation method [1, 2].

2 A review of the integral mean value method

The integral mean value method proposed first by Loghmani et al [3, 4], in 2011, for solving the integral and integro-differential equations. This method is based on the mean value theorem for integrals and is very suitable for the equations with a separable kernel. The integral mean value method transforms an integro-differential equation to an ordinary differential equation. Then the solution of the obtained ODE is transformed to a system of algebraic equations. By calculating the solutions of the algebraic equations and substituting them into the solution of the ODE, the solution of the governing equation is obtained. To handle the Eq. (1), under the mixed condition (2), by using the integral mean value method, we can express the procedure as follows

1. Applying the mean value theorem for the Eq. (1) yields

$$\sum_{r=0}^{m} f_r(x)u^{(r)}(x) = g(x) + \lambda(b-a)K(x,\xi)F(u(\xi)), \ \ x \in [a,b], \tag{3}$$

   where $\xi = \theta a + (1-\theta)b$, $0 < \theta < 1$.

2. Solving the Eq. (3) under the mixed conditions (2) leads

$$u(x) = U(x;\xi,u(\xi)), \ \ x \in [a,b]. \tag{4}$$

   For this step, we use an ODE solver tool of Mathematica, namely DSolve/NDSolve, to find the exact/numerical solution of the Eq. (3) under the mixed conditions (2).

3. Substituting ξ into (4) gives

$$u(\xi) = U(\xi;\xi,u(\xi)). \tag{5}$$

4. Cosindering the Eqs. (1), (3) and (4), we find

$$g(x) + \lambda(b-a)K(x,\xi)F(u(\xi)) = g(x) + \lambda\int_a^b K(x,t)F(U(t;\xi;u(\xi)))dt. \tag{6}$$

5. Substituting ξ into (6) gives

$$(b-a)K(\xi,\xi)F(u(\xi)) = \int_a^b K(\xi,t)F(U(t;\xi,u(\xi)))dt. \tag{7}$$

6. Solving the algebraic system including the Eqs. (5) and (7) provides the values of $u(\xi)$ and ξ. Substituting them into the Eq. (4) gives the solution of the Eq. (1) under the mixed conditions (2). To achieve this purpose, for the numerical solutions, we use the Newton's method. However, for the exact solutions of the algebraic system, we prefer to use a solver tool of Mathematica, namely *Solve*.

Remark 2.1. *The first main step of the integral mean value method, in handling an integro-differential problem, is to solve an ordinary differential equation under the given conditions. In this study, the corresponding ODE is a linear ODE, denoted by Eq. (3). The second main step is to solve/handel an algebraic system including the Eqs. (5) and (7). Therefore, the method breaks down when one of the mentioned steps comes to a deadlock.*

3 Test examples

To show the efficiency of the present procedure described in the previous part, we present some examples.

Example 3.1. *Consider the following nonlinear Fredholm integro-differential problem*

$$\begin{cases} u''(x) + xu'(x) + u(x) = \frac{7}{4}x + \int_0^1 xtu^2(t)dt, \\ \\ 2u(\frac{1}{2}) + u'(1) = 2, \;\; u'(\frac{1}{2}) + u(0) = 1. \end{cases} \tag{8}$$

The exact solution is $u(x) = x$. Applying the integral mean value method, we find

$$U(x;\xi,u(\xi)) = \frac{\alpha(x;\xi,u(\xi))}{\beta}, \tag{9}$$

where

$$\begin{aligned} \alpha(x;\xi,u(\xi)) &= e^{-\frac{x^2}{2}}\left(-1+4\xi u^2(\xi)\right)\left(4e^{\frac{5}{8}} - 6\sqrt{2e\pi}\mathit{Erfi}\left(\tfrac{1}{2\sqrt{2}}\right)\right) + 4e^{\frac{5}{8}}x\left(7+4\xi u^2(\xi)\right) \\ &+x\left(7+4\xi u^2(\xi)\right)\left(2e^{\frac{1}{2}}\left(-5+2\sqrt{2\pi}\mathit{Erfi}\left(\tfrac{1}{2\sqrt{2}}\right)\right) - \sqrt{2\pi}\left(\mathit{Erfi}\left(\tfrac{1}{2\sqrt{2}}\right) - \mathit{Erfi}\left(\tfrac{1}{\sqrt{2}}\right)\right)\right) \\ &+2e^{\frac{1-4x^2}{8}}\sqrt{2\pi}\left(-1+4\xi u^2(\xi)\right)\mathit{Erfi}\left(\tfrac{1}{\sqrt{2}}\right) - 2e^{\frac{1}{8}}x\left(7+4\xi u^2(\xi)\right)\left(-2+\sqrt{2\pi}\mathit{Erfi}\left(\tfrac{1}{\sqrt{2}}\right)\right) \\ &+2\sqrt{2}e^{\frac{-x^2}{2}}\left(-e^{\frac{1}{8}}\sqrt{\pi} - 2e^{\frac{5}{8}}\sqrt{\pi} + 3\sqrt{e\pi}\right)\left(-1+4\xi u^2(\xi)\right)\mathit{Erfi}\left(\tfrac{x}{\sqrt{2}}\right), \end{aligned}$$

and

$$\begin{aligned} \beta = 8\Big(&-10\sqrt{e} + 4e^{\frac{5}{8}} + 4\sqrt{2e\pi}\mathit{Erfi}\left(\tfrac{1}{2\sqrt{2}}\right) + \sqrt{2\pi}\left(-\mathit{Erfi}\left(\tfrac{1}{2\sqrt{2}}\right) + \mathit{Erfi}\left(\tfrac{1}{\sqrt{2}}\right)\right) \\ &+e^{\frac{1}{8}}\left(4 - 2\sqrt{2\pi}\mathit{Erfi}\left(\tfrac{1}{\sqrt{2}}\right)\right)\Big), \end{aligned}$$

and also

$$\begin{cases} \xi = 0.6299605249474366, \\ u(\xi) = 0.6299605249474358. \end{cases} \tag{10}$$

Thus, from the (4) and (10), the approximate solution of the Eq. (8) is obtained as follows

$$u_{approx}(x) = 0.9999999999999997x + \gamma(x), \tag{11}$$

where $\gamma(x) = e^{-\frac{x^2}{2}}\left(4.6936117498711335 + 0.700955417031777\mathit{Erfi}\left(\frac{x}{\sqrt{2}}\right)\right) \times 10^{-16}$.
Now, we try to construct series pattern solution of the Eq. (8) by using the ADM/HPM. Unfortunately, in testing the different kind of the linear operators, we find that the ADM and HPM are not suitable tools to handle the Eq. (8). For instance, choosing the linear operator

$$L(u(x)) = \frac{d^2}{dx^2}u(x), \tag{12}$$

it is impossible to define the integral inverse of (12) according to the given conditions. Other selections to the linear operator, need significant more computation time and computer hardware requirements. Thus, in this case, it is not prefer to use the ADM/HPM to handle the Eq. (8).

Remark 3.2. *It is easy to see that the values of ξ and $u(\xi)$ in (10) are convergent to $1/\sqrt[3]{4}$ and for this value of them the (9) gives the exact solution of the Eq. (8).*

Example 3.3. *Consider the following nonlinear Fredholm integro-differential problem*

$$\begin{cases} u''(x) = -\frac{1}{6}x + \int_0^1 xtu^4(t)dt, \\ \\ 2xu(\frac{1}{2}) + u'(1) = x + 1, \;\; u'(\frac{1}{2}) + u(0) = 1. \end{cases} \tag{13}$$

The exact solution is $u(x) = x$. Employing the integral mean value method, we get

$$U(x;\xi,u(\xi)) = \frac{1}{144}\left(-9 + 54\xi u^4(\xi) + x\left(149 - 30\xi u^4(\xi)\right) + 4x^3\left(-1+6\xi u^4(\xi)\right)\right), \tag{14}$$

and

$$\begin{cases} \xi = \frac{1}{\sqrt[5]{6}}, \\ u(\xi) = \frac{1}{\sqrt[5]{6}}. \end{cases} \tag{15}$$

Thus, from the (4) and (15), the exact solution of the Eq. (13) is obtained as follows

$$u(x) = x. \tag{16}$$

To construct series pattern solution of the Eq. (13), by using the ADM/HPM, we choose the linear operator as follows

$$L(u(x)) = \frac{d^2}{dx^2}u(x), \tag{17}$$

therefore, returning to (17), we find $u_0(x) = x$. *Starting by this initial guess, the ADM/HPM gives* $u_n(x) = 0,\ n \geq 1$, *therefore, using the ADM/HPM, the exact solution of the Eq. (13) is obtained as follows*

$$u(x) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x) = x. \tag{18}$$

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A practical review of the Adomian decomposition method

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Abstract

In this paper, a practical review of the Adomian decomposition method, to extend the procedure to handle the strongly nonlinear problems, is given. To achieve this purpose, a new and simple way for generation of the Adomian polynomials, for a general nonlinear function, is proposed. The efficiency of the approach will be shown by applying the procedure on several interesting and important problems. The Mathematica program generating the Adomian polynomials based on the procedure in this paper is designed.

Keywords: Adomian decomposition method; Series solution; Strongly nonlinear problems.

Mathematics Subject Classification: 35C10

1 Introduction

The so-called Adomian polynomials are used to deduce the recursive relation during the implementation of the Adomian decomposition method (ADM) while solving nonlinear problems. The main aim of the present paper is to propose an explicit formulate to calculate the Adomian polynomials and the Adomian series of a general nonlinear function. In this respect, we first outline the modifications of some definitions as already given in [1].

Definition 1.1. *Let u be a function of the parameter λ, whose Maclaurin series is given by*

$$u(\lambda) = \sum_{n=0}^{+\infty} u_n \lambda^n. \tag{1}$$

This series is called the parametric series of u.

Definition 1.2. *Let ϕ be a function of the parameter λ, then the mth-order parametric derivative of ϕ is given by*

$$D_m[\phi] = \frac{1}{m!} \frac{\partial^m \phi}{\partial \lambda^m}\bigg|_{\lambda=0}, \tag{2}$$

where $m \geq 0$ is an integer.

Definition 1.3. *The mth-order Adomian polynomial of ϕ is given by*

$$A_m(\phi(u)) = D_m\left[\phi(u(\lambda))\right], \tag{3}$$

where $m \geq 0$ is an integer and $A_m(\phi(u)) = A_m(\phi(u); u_0, u_1, ..., u_m)$.

Remark 1.4. *For the case $0 \leq \lambda \leq 1$, the parametric series (1) and parametric derivative (2) reduce to homotopy series and homotopy derivative respectively [1].*

2 Adomian polynomials for a general function

In this section, we first outline two theorems as already given in [2, 3]. Then by using them, we propose a new theorem which provides a new and simple way to calculate the Adomian polynomials for a general function. Here, we mention them with a minor modification. For simplicity, we use the following notation

$$\widehat{u}_{m,n} = \sum_{i=n}^{m} u_i \lambda^i,$$

Theorem 2.1. *For the power-law function $f(u) = u^k$, the corresponding mth-order Adomian polynomial is given by*

$$A_m(u^k) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \dots \sum_{r_{k-2}=0}^{r_{k-3}} u_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} u_{r_{k-2}-r_{k-1}} u_{r_{k-1}}, \quad (4)$$

where $m \geq 0$ and $k \geq 0$ are positive integers.

Proof. Refer to [2]. □

Theorem 2.2. *For parametric series (1), it holds*

$$D_m\left[f(u(\lambda))\right] = D_m\left[f(\widehat{u}_{m,0})\right],$$

where f is a nonlinear function.

Proof. Refer to [3]. □

Corollary 2.3. *From the Theorem 1, we find*

$$u^k(\lambda) = \left(\sum_{n=0}^{+\infty} u_n \lambda^n\right)^k = u_0^k + \sum_{m=1}^{+\infty} A_m(u^k)\lambda^m, \quad (5)$$

where the Adomian polynomials $A_m(u^k)$ are given by (4).

Remark 2.4. *It is clear that $A_m(u^k)$ in Theorem 1 can easily be calculated by an simple code by using a symbolic software such as Mathematica. For this respect, the following sample code can be used in Mathematica.*

<u>**Code. 1**</u>

$$\begin{gathered}\textbf{ADPforPowerLaw}[m\_, k\_] := \text{Module}[\{\}, D_{1,j\_} := u_j; D_{i\_,0} := u_0^i; D_{2,j\_} := \sum_{r=0}^{j} u_{j-r}u_r; \\ D_{i\_,order\_} := \sum_{r=0}^{order} u_{order-r} D_{i-1,r}; \text{Print}[''A''_m,''\,('', u^k, '')\,='', \text{Expand}[D_{k,m}]]]; \end{gathered} \quad (6)$$

For instance, using ADPforPowerLaw[4,6], the $A_4(u^6)$ is calculated as follows

$$A_4(u^6) = 15u_0^2u_1^4 + 60u_0^3u_1^2u_2 + 15u_0^4u_2^2 + 30u_0^4u_1u_3 + 6u_0^5u_4.$$

Corollary 2.5. *From the Theorem 2, for $m \geq n$, we find*

$$D_m\left[(f(\widehat{u}_{\infty,n})\right] = D_m\left[f(\widehat{u}_{m,n})\right],$$

where f is a nonlinear operator.

Corollary 2.6. *From the Corollary 2 and Theorem 1, for $m \geq k$, we find*

$$D_m\left[(\hat{u}_{m,1})^k\right] = \sum_{r_1=0}^{m-1} \sum_{\substack{r_2=0\\ r_2\neq r_1}}^{r_1} \sum_{\substack{r_2=0\\ r_2\neq r_1}}^{r_1} \cdots \sum_{\substack{r_{k-2}=0\\ r_{k-2}\neq r_{k-3}}}^{r_{k-3}} \sum_{\substack{r_{k-1}=0\\ r_{k-1}\neq r_{k-2}}}^{r_{k-2}} u_{r_{k-1}} \prod_{j=0}^{k-2} u_{r_j - r_{j+1}}, \tag{7}$$

where $r_0 = m$.

Corollary 2.7. *It is easy to see that*

$$D_m\left[(\hat{u}_{m,1})^k\right] = \begin{cases} 0, & m < k, \\ u_1^k, & m = k. \end{cases} \tag{8}$$

Corollary 2.8. *Let $kn \geq m+1$ and $n \geq 1$, we find*

$$D_m\left[(\widehat{u}_{\infty,n})^k\right] = 0.$$

Remark 2.9. *An implementation in Mathematica for $D_m\left[(\hat{u}_{m,1})^k\right]$ in (7) is given by the following sample code.*

Code. 2

$$\mathbf{DnonU0}[m\_, k\_] := \text{Module}[\{\}, D_{1,j\_} := \text{If}[j > 0, u_j, 0]; D_{i\_,0} := 0; D_{2,j\_} := \sum_{r=1}^{j-1} u_{j-r}u_r;$$
$$D_{i\_,order\_} := \sum_{r=2}^{order-1} u_{order-r} D_{i-1,r}; \text{Print}[''A''_m,''(''{,}u^k_{m,''{,}1''},'')='', \text{Expand}[D_{k,m}]]];$$

An alternative way is to use the Theorem 1 by taking $u_0 = 0$. To achieve this purpose, In Code 1, in the last command, the Expand$[D_{k,m}]$ is replaced by Expand$[D_{k,m}]/.u_0 \to 0$.

3 Main results

Theorem 3.1. *Let $f(u)$ has an expanding in Taylor series with respect to u_0, it holds*

$$A_m\left(f\left(u\right)\right) = \sum_{k=1}^{m} \frac{f^{(k)}(u_0)}{k!} D_m\left[(\widehat{u}_{m,1})^k\right]. \tag{9}$$

Proof. Expanding $f(u)$ in Taylor series with respect to u_0, one has

$$f(u) = f(u_0) + \sum_{k=1}^{+\infty} \frac{f^{(k)}(u_0)}{k!}\left(u - u_0\right)^k. \tag{10}$$

From the (10), we have

$$A_m\left[f\left(u\right)\right] = D_m\left[\sum_{k=1}^{\infty} \frac{f^{(k)}(u_0)}{k!}\left(u(\lambda) - u_0\right)^k\right],$$

recalling the Corollaries 5 and 2, we find

$$A_m\left(f\left(u\right)\right) = \sum_{k=1}^{m} \frac{f^{(k)}(u_0)}{k!} D_m\left(\left(u(\lambda) - u_0\right)^k\right) = \sum_{k=1}^{m} \frac{f^{(k)}(u_0)}{k!} D_m\left((\widehat{u}_{m,1})^k\right).$$

This ends the proof. □

Remark 3.2. *The expression $D_m\left((\widehat{u}_{m,1})^k\right)$ in (9) can easily be calculated by the (7) or by Theorem 1 with taking $u_0 = 0$.*

Corollary 3.3. *From the Theorem 3, we find*

$$f(u(\lambda)) = f(u_0) + \sum_{m=1}^{+\infty} w_m \lambda^m, \tag{11}$$

where $w_m = A_m(f(u))$ and $A_m(f(u))$ can be calculated by (9).

Remark 3.4. *To calculate $A_m(f(u))$, the following sample code can be used in Mathematica.*

Code. 3

$$\begin{aligned}
&\textbf{AdomianPolynomial}[function_, m_] := \text{Module}[\{\}, D_{1,j_,x_} := \text{If}[j > 0, x_j, 0];\\
&D_{i_,0,x_} := 0; D_{2,j_,x_} := \sum_{r=1}^{j-1} x_{j-r}x_r;\ D_{i_,order_,x_} := \sum_{r=2}^{order-1} x_{order-r}D_{i-1,r,x};\\
&HD_{order_} := \text{If}[m == 0, function/.u \to u_0,\\
&\sum_{n=1}^{order} \left((\text{D}[function, \{u, n\}]/.u \to u_0)\,/n! * D_{n,order,u}\right]);\\
&\text{Print}[''A''_m,''\,('', function,''\,)='', \text{Expand}[HD_m]]];
\end{aligned}$$

Using AdomianPolynomial$[f[u], 4]$, the corresponding Adomian polynomial is given as follows

$$A_4(f(u)) = u_4 f'(u_0) + \tfrac{1}{48}\left(24u_2^2 + 48u_1u_3\right) f''(u_0) + \tfrac{1}{2}u_1^2u_2 f'''(u_0) + \tfrac{1}{24}u_1^4 f^{(4)}(u_0).$$

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Strong order of stochastic Runge–Kutta methods for both commuting and non–commuting stochastic differential equations

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Abstract
In this paper we will show that in the multi–Wiener process case there can be an order reduction of a stochastic Runge–Kutta (SRK) method down to 0.5 if the functions drift and diffusion do not all commute, even if higher Stratonovich integrals are used in the method. Numerical results for two test problem with the Burrage and Platen methods will be given to illustrate the theoretical results.
Keywords: Runge–Kutta methods; Stochastic differential equations.

Mathematics Subject Classification: 60H10, 65L06, 60H35.

1 Introduction

Consider the autonomous stochastic differential equation (SDE) is given by

$$dy(t) = g_0(y(t))\ dt + \sum_{j=1}^{d} g_j(y(t))\ dW_j(t)\ ,\ y(t_0) = y_0\ , t \in [t_0, T]\ ,\ y \in \mathbb{R}^m, \tag{1}$$

where $W_j(t)$ $(j = 1, \ldots, d)$ are independent Wiener processes. Equation (1) can be written as stochastic integral equation

$$y(t) = y_0 + \int_{t_0}^{t} g_0(y(s))\ ds + \sum_{j=1}^{d} \int_{t_0}^{t} g_j(y(s))\ dW_j(s), \tag{2}$$

where $d-$integrals in (2) cannot be considered as a Riemann–Stieltjes integral (see [1]). For problem (1) SRK methods with s-stage given by

$$y_{n+1} = y_n + \sum_{k=0}^{d}\sum_{j=1}^{s} z_j^{(k)}\ g_k(Y_j), \qquad Y_i = y_n + \sum_{k=0}^{d}\sum_{j=1}^{s} Z_{ij}^{(k)}\ g_k(Y_j), \qquad i = 1, 2, \ldots, s, \tag{3}$$

where $Z^{(k)} = (Z_{ij}^{(k)})$ for $i, j = 1, 2, \ldots, s$ and ${z^{(k)}}^T = (z_1^{(k)}, \ldots, z_s^{(k)})$ for $k = 0, 1, \ldots, d$. In order to derive SRK methods with strong order p, the Stratonovich Taylor series expansion of the exact solution and the Stratonovich Taylor series expansion of SRK method (3) is necessary. Hence the local truncation error can be written as (see [2])

$$L(t) = y(t) - Y(t) = \sum_{t \in T^\star} e(t)\ F(t) y_0,$$

where $F(t)(y_0)$ is the elementary differential associated with the rooted $d+1$-colored trees and $e(t)$ is the coefficient of local truncation error for tree t.

2 Strong order bounds for SRK methods

Consider the case of (3) in which only for $k = 0, \dots, d$ the first Stratonovich integral J_i $(i = 0, \dots, d)$ appear. In this case the method is written as

$$z^{(k)} = J_k \gamma^{(k)}, \qquad Z^{(k)} = J_k B^{(k)}, \qquad J_0 = h. \tag{4}$$

Hence by the convergence theorem in [1] the following theorem can be stated.

Theorem 2.1. *For problems with more than one Wiener process, the maximum strong order of a SRK with only random variable J_k is 0.5. Only in the case $d = 1$ is strong order is 1.*

In spite of this result, if there is a commute property between the g_j in the sense that

$$[g_i, g_j] \equiv g_i^{'}(y)\ g_j(y) - g_j^{'}(y)\ g_i(y) = 0, \qquad \forall\ i \neq j, \quad i, j = 1, \dots, d,$$

then the following result is hold.

Theorem 2.2. *For problems with d Wiener processes in which*

$$[g_i, g_j] \equiv g_i^{'}(y)\ g_j(y) - g_j^{'}(y)\ g_i(y) = 0, \qquad \forall\ i \neq j, \quad i, j = 1, \dots, d,$$

then a strong order 1 is attainable by methods of the form (4)when

$$\gamma^{(i)} e = 1, \quad i = 0, \dots, d, \quad \gamma^{(i)} B^{(j)} e + \gamma^{(j)} B^{(i)} e = 1, \quad \forall i, j = 1, \dots, d.$$

In order to increase strong order bound (of 0.5) Burrage showed that when $d = 1$ a strong order 1.5 was attainable if an additional random variable representing a second order Stratonovich integral $\frac{J_{10}}{h}$ was also present in Runge–Kutta method. The natural extension of this is to have

$$z^{(k)} = J_k \gamma^{(k)} + \frac{J_{k0}}{h} \delta^{(k)}, \quad Z^{(k)} = J_k B^{(k)} + \frac{J_{k0}}{h} D^{(k)}, \qquad k = 0, 1, \dots, d,$$

with $D^{(0)} = 0,\ \delta^{(0)} = 0$. Hence the following theorem is hold.

Theorem 2.3. *For general problems with more than one Wiener process the maximum strong order of an SRK with random variables representing J_k and $\frac{J_{k0}}{h}$ $(k = 1, \dots, d)$ is 0.5 irrespective of the number of stages. Only in the case $d = 1$ is maximum strong order 1.5 attainable.*

3 Numerical results

In this section, numerical results from the implementation of Platen and Burrage and CL methods are implemented (see [1] for further details). These methods should have strong order 1, 1 and 1.5 respectively, for problems $d = 1$. In fact the method *Platen* is given by

$$B^{(0)} = B^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^{(0)^{(T)}} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad \gamma^{(1)^{(T)}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

while method *Burrage* has parameters

$$B^{(0)} = B^{(1)} = \begin{pmatrix} 0 & 0 \\ \frac{2}{3} & 0 \end{pmatrix}, \qquad \gamma^{(0)^{(T)}} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix}, \qquad \gamma^{(1)^{(T)}} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

The parameters for the 4–stage method CL are

$$B^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.7242916356 & 0 & 0 & 0 \\ 0.4237353406 & -0.1994437050 & 0 & 0 \\ -1.578475506 & 0.840100343 & 1.738375163 & 0 \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2.702000410 & 0 & 0 & 0 \\ 1.757261649 & 0 & 0 & 0 \\ -2.918524118 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^{(0)^{(T)}} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{pmatrix},$$

$$\gamma^{(1)^{(T)}} = \begin{pmatrix} -0.7800788474 & 0.07363768240 & 1.486520013 & 0.2199211524 \end{pmatrix},$$

$$\delta^{(1)^{(T)}} = \begin{pmatrix} 1.693950844 & 1.636107882 & -3.024009558 & -0.3060491602 \end{pmatrix}.$$

The implementation determines the mean square error for each stepsize at the end of the interval of integration is defined by

$$MS = \frac{1}{K}\sum_{i=1}^{K} \left(y_N^{(i)} - y^{(i)}(t_N)\right)^2,$$

where $y_N^{(i)}$ is the numerical approximation and $y^{(i)}(t_N)$ is the exact solution of SDE at t_N in the i–th simulation over all K simulations. In each case, the order of convergence is calculated as mean square error with stepsize h divided by mean square error $\frac{h}{2}$. Thus the errors should be reduce by a factor 2^p, where p is the order of method.

Test Problem 1. Consider

$$dy(t) = (1 - y^2(t))odW(t), \quad y(0) = 0, \quad t \in [0, 1],$$

with the exact solution

$$y(t) = \tan h(W(t) + \arctan h(0)).$$

h	*Platen*	*Burrage*	*CL*
$\frac{1}{25}$	0.03792116	0.02353555	0.01566250
$\frac{1}{50}$	0.01571780	0.00929187	0.00748598
$\frac{1}{100}$	0.00746319	0.00444777	0.00044492
$\frac{1}{200}$	0.00395955	0.00233910	0.00040415
$\frac{1}{400}$	0.00195239	0.00119672	0.00046121

Table 1: Mean square errors for Test Problem 1, with $K = 500$.

Test Problem 2. Consider

$$dy(t) = -\frac{3}{2}y(t)\ dt + y(t)\ odW_1(t) + y(t)\ odW_2(t), \quad y(0) = 1, \quad t \in [0, 1],$$

with the exact solution

$$y(t) = -\frac{3}{2}t + W_1(t) + W_2(t).$$

h	*Burrage*	*Platen*	*CL*
$\frac{1}{4}$	0.87866249	0.62577442	0.60249574
$\frac{1}{8}$	0.37108759	0.26138955	0.19648668
$\frac{1}{16}$	0.29225039	0.15753197	0.09962375
$\frac{1}{32}$	0.07895903	0.05984274	0.03743084
$\frac{1}{64}$	0.03124931	0.02389529	0.00967278
$\frac{1}{128}$	0.01140892	0.00778277	0.00710453

Table 2: Mean square errors for Test Problem 2, with $K = 500$.

h	*Platen*	*Burrage*	*CL*
$\frac{1}{25}, \frac{1}{50}$	2.49842930	2.53291856	2.09224444
$\frac{1}{50}, \frac{1}{100}$	2.10604312	2.08910758	9.825451767
$\frac{1}{100}, \frac{1}{200}$	1.88485812	1.90148776	1.10087839
$\frac{1}{200}, \frac{1}{400}$	2.02805279	1.95459255	0.87628195
Average	2.12934583	2.11952661	3.47371413

Table 3: Convergence ratios for Test problem 1, with $K = 500$.

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Preconditioned basic iterative methods for M and H-matrices

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Abstract

Linear systems with M or H-matrices often appear in a wide variety of areas. In this paper we give preconditioners for solving the systems with nonsingular H-matrix or M-matrix .We show that these preconditioner increase the convergence rate of the basics iterative methods such as Gauss-Seidel and Jacobi methods. Numerical example are also given to illustrate our results.

Keywords: Jacobi and Gauss-Seidel matrices; SOR method; AOR method; Spectral radius; M-matrix; $H-$matrix; L-matrix; Positive definite matrix; Convergence.

Mathematics Subject Classification: 65F10, 65F15

1 Introduction

Let us consider the iterative methods for the linear system

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a known nonsingular matrix, $b \in \mathbb{R}^n$ is known and $x \in \mathbb{R}^n$ is unknown. If A split into

$$A = M - N, \tag{2}$$

where M is nonsingular, then the basic iterative method for solving (13) can be expressed in the form,

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \qquad k = 0, 1, 2, ... \tag{3}$$

As it is well known, the above iterative process is convergence to the unique solution $x = A^{-1}b$ for each initial value $x^{(0)}$, if and only if the spectral radius of the iteration matrix $M^{-1}N$ satisfies $\rho(M^{-1}N) < 1$. We may write

$$A = D - L - U, \tag{4}$$

where D is the diagonal matrix, $-L$ and $-U$ are strictly lower and upper triangular parts of A, respectively. $M = D$, $N = L + U$ and $M = D - L$, $N = U$ when D is nonsingular leads to the classical Jacobi and classical Gauss-Seidel methods. Then the corresponding iteration matrices of the classical Jacobi and Gauss-Seidel methods are given by $J = D^{-1}(L + U)$ and $G = (D - L)^{-1}U$. The preconditioned iterative methods have been proposed in [8]. The main idea of these preconditioned iterative methods is to transform the original system into the preconditioned form

$$PAx = Pb, \tag{5}$$

where $P \in \mathbb{R}^{n\times n}$ is nonsingular, and has unit diagonal entries. We call the basic iterative methods corresponding to the preconditioned system (1.3) the preconditioned iterative methods, such as the preconditioned Jacobi method, the preconditioned Gauss-Seidel method, etc. Here we consider the following five types of preconditioners $P = P_k$ $(1 \leq k \leq 5)$ and studied the convergence analysis of them and state the comparison theorems for them. The preconditioner P_1 is of the form $P_1 = I + S_1$, where

$$S_1 = \begin{pmatrix} 0 & -\alpha_1 a_{12} & 0 & \dots & 0 \\ 0 & 0 & -\alpha_2 a_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\alpha_{n-1} a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{6}$$

The preconditioned P_2 is of the form $P_2 = I + S_2$, where

$$S_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\alpha_n a_{n1} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{7}$$

The preconditioned P_3 is of the form $P_3 = I + S_3$, where

$$S_3 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ -\alpha_2 a_{21} & 0 & \dots & 0 \\ -\alpha_3 a_{31} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\alpha_n a_{n1} & 0 & \dots & 0 \end{pmatrix}. \tag{8}$$

The preconditioned P_4 is of the form $P_4 = I + S_4$, where

$$S_4 = \begin{pmatrix} 0 & 0 & \dots & -\alpha_1 a_{1n} \\ 0 & 0 & \dots & -\alpha_2 a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -\alpha_{n-1} a_{n-1,n} \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{9}$$

The preconditioned P_5 is of the form $P_5 = I + S_5$, where

$$S_5 = \begin{pmatrix} 0 & & & -c_1 k_1 & & \\ & 0 & & & -c_2 k_2 & \\ & & \ddots & & & \\ & -c_r k_r & & \ddots & & \\ & & & & \ddots & \\ & & & -c_n k_n & & 0 \end{pmatrix}. \tag{10}$$

The preconditioner P_1 was first introduced by Gunawardena et al. [3] when α_i $(1 \leq i \leq n-1)$, and it has been studied by Kohno et al. [6] and Wu et al. [10] for $0 \leq \alpha_i \leq 1$ $(1 \leq i \leq n-1)$. The preconditioner P_2 was first introduced by Evans et al. [2] when $\alpha_n = 1$, and it has been studied by Yun [12] and Li et al. [7] when $0 < \alpha_n \leq 1$. The preconditioner P_3 was first introduced by Milaszewics [8] when $\alpha_i = 1$ $(2 \leq i \leq n)$, and it has been studied by Yun [11, 13] for $\alpha_i = 1$ $(2 \leq i \leq n)$ and Huang et al. [5] for $0 \leq \alpha_i \leq 1$ $(2 \leq i \leq n)$. The preconditioner P_4 was introduced by Dehghan and Hajarian [1]. The preconditioner P_5 was introduced by Ting-Zhu. Huang et al, [4], which is a generalization of methods that introduced by Zhang et al. [14] and Noutsos and Tzoumas [9].

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Some notes on multi-order fractional integro-differential equations

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Abstract
In this paper our aim is to study multi-order fractional integro-differential equations. Existence and uniqueness of solution of this kinds of equations have been discussed. At the end os paper we will present a method for solving them.
Keywords: Fractional integro-differential equation; Multi-order fractional differential equation; Fractional differential transform method.

Mathematics Subject Classification: 26A33; 65R20; 34A08

1 Introduction

Many authors have studied multi-order fractional differential equations in their works and many papers have been published, for example we can refer to the most recent paper which the authors applied spline collocation methods for linear multi-term fractional differential equations [14]. We refer also to the K. Diethelm which have extensively study in solution of multi-order fractional differential equations (see, e.g., [3, 4, 5]). It is somewhat surprising that multi-order fractional integro-differential equations have received rather less attention, so we decided to study this equations which is a general problem. We will consider the fractional multi-order integro-differential equation of the form

$$D^{\mu_1}y(x) + D^{\mu_2}y(x) + ... + D^{\mu_n}y(x) = f(x,y) + \int_a^x k_1(x,t)g_1(y(t))dt + \int_a^b k_2(x,t)g_2(y(t))dt, \quad (1)$$

$m_i - 1 < \mu_i \leq m_i,\ a < x < b,\ and\ m_i \in \mathbb{N}$, with nonlocal boundary conditions

$$\sum_{j=1}^{m} \left(\gamma_{ij}y^{(j-1)}(a) + \eta_{ij}y^{(j-1)}(b)\right) = d_i, \qquad i = 1,2,...,m. \quad (2)$$

Where D^{μ_i} denotes a differential operator with fractional order μ_i, there are various types of definitions for the fractional derivative of order $\mu > 0$, the most commonly used definitions among various definitions of fractional derivative of order $\mu > 0$ are the Riemann-Liouville and Caputo formulas, ones which use fractional integration and derivative of whole order. The difference between two definitions is in the order of evaluation. Riemann-Liouville fractional integration of order μ is defined as

$$J^{\mu}_{x_0}f(x) = \frac{1}{\Gamma(\mu)}\int_{x_0}^{x}(x-t)^{\mu-1}f(t)dt, \qquad \mu > 0,\ x > 0. \quad (3)$$

The following equations define Riemann-Liouville and Caputo fractional derivatives of order μ, respectively

$$D^{\mu}_{x_0}f(x) = \frac{d^m}{dx^m}[J^{m-\mu}_{x_0}f(x)] \quad (4)$$

$$D^{\mu}_{*x_0}f(x) = J^{m-\mu}_{x_0}[\frac{d^m}{dx^m}f(x)] \tag{5}$$

where $m-1<\mu\leq m$ and $m\in\mathbb{N}$. From (7) and (4), we have

$$D^{\mu}_{x_0}f(x) = \frac{1}{\Gamma(m-\mu)}\frac{d^m}{dx^m}\int_{x_0}^{x}(x-t)^{m-\mu-1}f(t)dt, \quad x>x_0 \tag{6}$$

also we have

$$\mu_1>\mu_2>...>\mu_n>0, \quad \mu_i=\frac{\beta_i}{\alpha_i}$$

Here α_i is order of fraction. $f(x,y)$ and $k_i(x,t)$ $(i=1,2)$ are holomorphic functions, γ_{ij}, η_{ij}, and d_i $(i=1,2,...,m)$ are constants, (note: here $y(x)$ is of class $C^m[a,\ b]$). About existence and uniqueness of solution of fractional differential equations and integro differential equations have received very little attention, in the resent paper the author have studied fractional order integro-differential equation with non-local and global boundary conditions by converting it to the corresponding well known Fredholm integral equation of second kind [6]. Also fractional differential equations have been discussed in [1, 12, 15, 17]. In [8, 10, 11] Banach fixed point theorem is used for fractional integro-differential equation.

2 Existence and Uniqueness of Solution

Existence and uniqueness of solution of Eq. (1) in special case

$$D^{\mu_1}y(x) = f(x,y)+\int_a^x k_1(x,t)g_1(y(t))dt+\int_a^b k_2(x,t)g_2(y(t))dt, \tag{7}$$

has been proved in [6]. By considering Eq. (1) in the following form

$$D^{\mu_1}y(x) = \underbrace{-D^{\mu_2}y(x)-...-D^{\mu_n}y(x)+f(x,y)}_{g(x,y)}+\int_a^x k_1(x,t)g_1(y(t))dt+\int_a^b k_2(x,t)g_2(y(t))dt, \tag{8}$$

we will have an equation like as Eq. (7) so existence and uniqueness can be concluded. In the abow form with initial condition we know $g(x,y)\in C[a,\ b]$ (see [9] Theorem 2.2), so existence and uniqueness can be refereed to [11], also for considering the generally well-posed necessity of fractional order differential equations with local and nonlocal conditions see[7]. Now section we present numerical results.

3 Numerical approach

In the among exited numerical method fractional differential transform method(to study this method see [2, 13]) have shown to be efficient to solve fractional integro-differential equations (see, e.g., [13]). Here we will applied this method to considered equation (1). We have $\mu_i=\frac{\beta_i}{\alpha_i}$, by setting $\alpha=gcd\{\alpha_1,...,\alpha_n\}$ implies $\mu_i^{'}=\frac{\beta_i^{'}}{\alpha}$. We can obtain fractional differential transformation of Eq. (1) in the following form

$$\begin{aligned}\frac{\Gamma(1+\frac{k}{\alpha}+\mu_1^{'})}{\Gamma(1+\frac{k}{\alpha})}Y(k+\alpha\mu_1^{'}) \quad &+ \quad \frac{\Gamma(1+\frac{k}{\alpha}+\mu_2^{'})}{\Gamma(1+\frac{k}{\alpha})}Y(k+\alpha\mu_2^{'})+...\\ &+ \quad \frac{\Gamma(1+\frac{k}{\alpha}+\mu_n^{'})}{\Gamma(1+\frac{k}{\alpha})}Y(k+\alpha\mu_n^{'})=I(k)+F(k)\end{aligned} \tag{9}$$

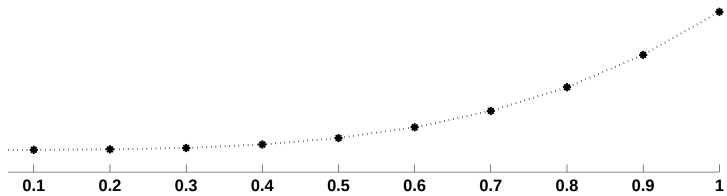
where $F(K)$ and $I(k)$ are transformation of $f(x, y)$. For showing application of method we present an example:

Example 1. Consider the following linear fractional integro-differential equation with the given nonlocal condition

$$D^{\frac{1}{3}}y(x) + D^{\frac{1}{4}}y(x) = \frac{3}{2}\frac{x^{\frac{2}{3}}}{\Gamma(\frac{2}{3})} + \frac{4}{3}\frac{x^{\frac{3}{4}}}{\Gamma(\frac{3}{4})} - 1 + e^{x^2} - x^2e^{x^2} + \int_0^x x^2e^{xt}y(t)dt \quad (10)$$

$$y(0) + y(1) = 1, \quad (11)$$

with exact solution $y(x) = x$. From following figure we can see the numerical solutions are in a good agreement with the exact solution.



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Efficient solution of a free boundary value problem in finance

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Abstract

In the financial derivative markets, the great importance should be given to the evaluation of the pricing of the options. Due to this, the problem of pricing American options under the jump-diffusion model is considered.
Therefore, first, the pricing American options was shown as a free boundary value problem for a partial integro-differential equation , then an efficient numerical method for solving the free boundary value problem is discussed.
Keywords: Partial integro-differential equations, Derivative market, American Options, Initial and Free Boundary Value Problems, Method of Lines, Penalty Method.

Mathematics Subject Classification: 91G60, 91G80, 91G20, 93C20

1 Introduction

To price financial derivatives under actual market conditions, more complex models are required. Black and Scholes [2] developed a formula to compute the price of an option which is used in many practical implementations since 1975 in the finance industry. Extensions of this model include among others stochastic volatility models, Lévy models and jump-diffusion models [4]. Under certain assumptions the jump-diffusion models for the pricing of American options leads to a Partial Integro-Differential Equation (PIDE) involving a non-local integral term. Then the valuation of the Pricing American Ooptions (PAO) can often be reduced to the study of a free boundary value problem for a PIDE. For this important, The paper is organized as follows: First, we introduce the PIDE in a general form and then specifically for Jump diffusion model. In the next section, we will describe the penalty method to solve the Free Boundary Value Problem (FBVP) derived from the jump process. Finally, we will discuss how the method of Lines can be used to study the numerical valuation of American Options and to determine the early exercise frontier.

2 The model: The FBVP for the PIDE

In this section, we try to develop a pricing model which its underlying asset model is a combination of diffusion and jump terms:

$$dS = (\mu - \gamma k)Sdt + \sigma SdW + (J-1)Sdp \qquad 0 < t < T \tag{1}$$

Where S denotes the underlying asset price, μ is the drift rate, σ is the volatility, dW is the standard Winner process, J is the jump size. dp is the independent Poisson process with density of $\gamma > 0$

Now consider a portfolio that consists of an option with price of $V = V(S,t)$, and $-\Delta$ of the underlying asset with abovementioned model.
it is easy to show that V satisfies the following partial integro-differential equation :

$$\begin{array}{lll} \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} + (r-\gamma k)S\frac{\partial V}{\partial S} - (r-\gamma)V + \gamma\int_0^\infty V(JS,t)p(J)dJ = 0, & if, & S > B(t) \\ V(S,t) = K - S, & if, & 0 \leq S \leq B(t) \end{array} \tag{2}$$

also, the initial and boundary conditions rule to this option:

$$\begin{array}{c} V(S,T) = max\{K-S,0\}, \\ V(0,t) = 0, \\ \lim_{S->\infty} V(S,t) = 0, \\ V(G(t),t) = K - B(t), \\ \frac{\partial V(B(t),t)}{\partial S} = -1, \\ B(T) = K, \end{array} \tag{3}$$

where $B(t)$ is the (free moving) exercise boundarie.

3 Penalty Method to Solve the FBVP

To solve the FBVP, we first fix boundary of the function by adding a penalty term to the PIDE, and then rewrite the problem. Nielsen et al. [1], and Zvan et al. [3] proposed the following penalty term for the American option:

$$\frac{\epsilon C}{V(S,t)+\epsilon-K+S}, \tag{4}$$

where $C > rK$ is a constant, and $0 < \epsilon << 1$. Now the problem 2-3, by adding this term to the PIDE, is transformed as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} + (r-\gamma k)S\frac{\partial V}{\partial S} - (r-\gamma)V + \gamma\int_0^\infty V(JS,t)p(J)dJ + \frac{\epsilon C}{V(S,t)+\epsilon-K+S} = 0, \qquad 0 < S < \infty \tag{5}$$

also, the initial and boundary conditions rule to this option:

$$\begin{array}{c} V(S,T) = max\{K-S,0\}, \\ V(0,t) = 0, \\ \lim_{S->\infty} V(S,t) = 0 \end{array} \tag{6}$$

The boundary of the problem 5-6 is fixed and it's solution will be described in the next section.

4 The Method of Lines

In order to apply the method of lines to the problem 5-6, we first need to obtain an acceptable approximation of the integral term. The integral term in the PIDE is on a semi-infinite interval and can be solved by using the Gauss-Laguerre method.
Now, to design an acceptable approximation of the derivatives in the PIDE, we can transform $0 < S < \infty$ to $0 < S < S_\infty$. Then the interval $[0, S_\infty]$ is partitioned into N subintervals with length of $\Delta S = S_\infty/N$ for each, and apply approximation of the centered-difference formula for second order derivative, and the Euler method for first order derivative of S.
Consequently the problem 5-6 is transformed to an ordinary non-linear system of differential equations.
Finally, we complete solution method with solving this non-linear system by using the fourth-order Runge-Kutta method.

5 Conclusions

First, the American option pricing models can be derived from underlying asset models with known volatility but unknown and stochastic drift. Second, amongst other useful applications not mentioned here that can be subject for future researches, is studying co-integration and consistency with methods discussed in this article.

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Galerkin and collocation methods for the solution of Kelin-Gordon equation using interpolating scaling functions

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Abstract

A numerical technique is presented for the solution of Klein-Gordon equation. This method uses interpolating scaling functions. The method consists of expanding the required approximate solution as the elements of interpolating scaling functions. Using the operational matrix of derivatives, we reduce the problem to a set of algebraic equations. Some numerical examples are included to demonstrate the validity and applicability of the technique. The method is easy to implement and produces accurate results.

Keywords: Galerkin method; Klein-Gordon equation; Interpolating Scaling function; Operational matrix of derivative.

Mathematics Subject Classification: 65L60; 34B15

1 Introduction

In this article, we present numerical scheme for solving initial-value problem of the one-dimensional nonlinear Klein-Gordon equation as

$$u_{tt} + \alpha u_{xx} + g(u) = f(x,t), \qquad x \in \Omega = [0,1] \subset R, 0 < t \leq 1, \tag{1}$$

with

$$\begin{aligned} u(x,0) = g_1(x), \qquad u_t(x,0) = g_2(x), \qquad x \in \Omega, \\ u(0,t) = h_1(t), \qquad u(1,t) = h_2(t), \qquad x \in \delta\Omega,\ 0 < t \leq 1. \end{aligned} \tag{2}$$

where $u = u(x,t)$ represents the wave displacement at (x,t) and α is a known constant and $g(u)$ is the nonlinear force. The nonlinear Klein-Gordon equation appears in many types of nonlinearities. The Klein-Gordon equation plays an important role in mathematical physics [1, 2]. The equation has attracted much attention in studying solitons and condensed matter physics [4], in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations [3].

In this paper interpolating scaling functions (ISF) are constructed. These bases are used to construct Alpert multiwavelets [5]. Operational matrix of derivative is derived in [6]. In the present work we use interpolation property and solve the Klein-Gordon equation. Interpolating scaling functions are used to reduce Klein-Gordon equation to a system of nonlinear equation and Newton method are applied to solve the system at nonlinear equation. Recently Lakestani and Nemati Saray used these bases to solve Telegraph equation[6] and nonlinear generalized Burgers-Huxley equation [6].

2 Interpolating scaling function

Suppose P_r is the Legendre polynomial of order r and r is any fixed nonnegative integer number. Let τ_k denotes the roots of P_r for $k = 0, ..., r-1$. The interpolating scaling functions (ISF) are given by [6]

$$\phi^k(t) := \begin{cases} \sqrt{\frac{2}{\omega_k}} L_k(2t-1), & t \in [0,1], \\ 0, & otherwise, \end{cases}$$

where ω_k is the Gauss-Legendre quadrature weight and $L_k(t)$ is the lagrange interpolating polynomial defined by [6] for $k = 0, ..., r-1$. We can expand any polynomial g of degree less than r with the functions with an orthonormal basis on $[0,1]$ as following,

$$g(t) = \sum_{k=0}^{r-1} d_k \phi^k(t),$$

where the coefficients are given by

$$d_k = \sqrt{\frac{\omega_k}{2}} g(\hat{\tau_k}), \quad \hat{\tau_k} = \frac{\tau_k + 1}{2}, \qquad k = 0, ..., r-1.$$

Let $\phi^k_{nl}(t)$ defined by

$$\phi^k_{nl}(t) = 2^{(n/2)} \phi^k(2^n t - l) \tag{3}$$

where $k = 0, ..., r-1$, $l = 0, ..., 2^n - 1$, and n is any fixed nonnegative integer number.

For any two fixed nonnegative integer numbes r and n, a function f(t) defined over $[0,1)$ may be represented by ISF expansion as,

$$f(t) = \sum_{k=0}^{r-1} \sum_{l=0}^{2^n-1} s^k_{nl} \phi^k_{nl}(t) = S^T \Phi(t) \tag{4}$$

Also a function $g(x,t)$ of two independent variables for $0 \le x \le 1$, and $0 \le t \le 1$, may be expanded in terms of interpolating scaling functions as

$$g(x,t) = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \Phi_i(x) \Phi_j(t) = \Phi^T(x) G \Phi(t), \tag{5}$$

such that G is an $N \times N$ matrix with $N = r2^n$. Let the derivative of $f(t)$ in (17) be given by

$$\frac{d}{dt} f(t) = \sum_{k=0}^{r-1} \sum_{l=0}^{2^n-1} \tilde{s}^k_{nl} \phi^k_{nl}(t) = \tilde{S}^T \Phi(t). \tag{6}$$

We express relation between S and $\tilde{S}$ by

$$\tilde{S} = DS, \tag{7}$$

where D is the operational matrix for derivatives of the scaling function. The matrix D can be expressed as a block tridiagonal matrix which is obtained from

$$D = 2^n \begin{bmatrix} \underline{R} & H & & & & \\ -H^T & R & H & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -H^T & R & H \\ & & & & -H^T & \overline{R} \end{bmatrix},$$

here, each block is an $r \times r$ matrix and for $k, i = 1, ..., r$, we have

$$[\underline{R}]_{ki} = \frac{1}{2}\phi^i(1)\phi^k(1) - \phi^i(0)\phi^k(0) - q_{ki}, \quad [R]_{ki} = \frac{1}{2}\phi^i(1)\phi^k(1) - \frac{1}{2}\phi^i(0)\phi^k(0) - q_{ki},$$

$$\left[\overline{R}\right]_{ki} = \phi^i(1)\phi^k(1) - \frac{1}{2}\phi^i(0)\phi^k(0) - q_{ki}, \quad [H]_{ki} = \frac{1}{2}\phi^i(0)\phi^k(1).$$

3 Galerkin method based on interpolating scaling function (GCM)

The solution $u(x,t)$ of (2.1) can be approximated as

$$u(x,t) \simeq \Phi^T(t)U\Phi(x). \tag{8}$$

By using (6), we obtain

$$u_t(x,t) \simeq \Phi^T(t)DU\Phi(x), \quad u_{tt}(x,t) \simeq \Phi^T(t)D^2U\Phi(x), \quad u_{xx}(x,t) = \Phi^T(t)UD^2\Phi(x). \tag{9}$$

We suppose that

$$\hat{g}(x,t) = g(u(x,t)), \tag{10}$$

Using (6), we can approximate $\hat{g}(x,t)$ as the following form

$$\hat{g}(x,t) \simeq \Phi^T(t)G\Phi(x), \tag{11}$$

where G is a $(N \times N)$ matrix. The entries of this matrix are obtained by

$$G_{i,j} = 2^{-n}\sqrt{\frac{\omega_k}{2}}\sqrt{\frac{\omega_{k'}}{2}}\hat{g}\left(2^{-n}(\hat{\tau}_k + l), 2^{-n}(\hat{\tau}_{k'} + l')\right), \quad i = rl + (k+1), j = rl' + (k'+1).$$

Using (9), (11) and independence property of the entries of vectors $\Phi(t)$ and $\Phi(x)$, we get

$$\Upsilon = D^2U + \alpha UD^2 + G - F. \tag{12}$$

One can obtain the equation (12) has $(N-1) \times (N-2)$ independent equations, because the rank of D^2 is $N-2$.

By using initial and boundary conditions (2.2), we have N^2 nonlinear equations, which can be solved for $U_{i,j}$, $i, j = 1, \cdots, N$. This nonlinear system of equations solve by Newton method and we will get the approximated solution of the Klein-Gordon equation.

3.1 Collocation method interpolating scaling functions (CCM)

In this method, Υ in (12) is given by another manner. For applying collocation method, we need to have some collocation points. These points are given by $x_i = \frac{i}{N-1}$ for $i = 0, \cdots, N-1$. By putting (9)-(10) in equation (2.1), we have

$$\Phi^T(t)D^2U\Phi(x) + \alpha\Phi^T(t)UD^2\Phi(x) + g(\Phi^T(t)U\Phi(x)) - \Phi^T(t)F\Phi(x) = 0, \tag{13}$$

Using collocation points to obtain N^2 equations but we have $(N-2) \times (N-2)$ independent equations because the rank of D^2 is $N-2$. By using initial and boundary conditions (2.2), we give N^2 equations. These system of equations are nonlinear and Newton method is used to solve them.

4 Numerical Result

Consider the following Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = 6xt(x^2 - t^2) + x^6t^6. \tag{14}$$

$$\begin{cases} u(x,0) = 0, & u_t(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) = 0, & u(1,t) = t^3, \quad 0 \le t \le 1. \end{cases}$$

The analytical solution is given in [7] as

$$u(x,t) = x^3t^3.$$

Table 1 consist of norm infinity and L_2 norm of example 1 for $n = 1, 2$. Also we show that the methods represented in this paper is the better than the collocation method used in [7].

Table 1. norm infinity and L_2 norm of errors.

t	GCM L_2 $n=1$	L_∞ $n=1$	CCM L_2 $n=1$	L_∞ $n=1$	[7] L_2 $n=3$	L_∞ $n=3$
0.2	1.66×10^{-20}	3.79×10^{-20}	3.07×10^{-18}	1.10×10^{-17}	6.3×10^{-8}	9.2×10^{-8}
0.4	5.74×10^{-20}	2.90×10^{-20}	3.87×10^{-18}	1.71×10^{-17}	9.1×10^{-7}	1.4×10^{-6}
0.6	8.56×10^{-20}	1.10×10^{-19}	2.13×10^{-18}	2.42×10^{-17}	4.2×10^{-6}	5.8×10^{-6}
0.8	7.25×10^{-20}	2.53×10^{-19}	2.93×10^{-18}	1.40×10^{-17}	6.1×10^{-6}	7.3×10^{-6}
1.0	1.27×10^{-19}	4.00×10^{-20}	4.85×10^{-18}	4.18×10^{-18}	5.5×10^{-6}	7.5×10^{-6}

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A numerical solution of non-linear Fredholm integro-differential equations

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Abstract
In this paper, we will develop a new method to find a numerical solution for the general form of the Non Linear Fredholm Integro-Differential Equations (NFIDEs). To this end, we will present our method based on the matrix form of the (NFIDEs). The corresponding unknown coefficients of our method have been determined by using the computational aspects of matrices. Finally the accuracy of the method has been verified by presenting some numerical computation.
Keywords: Fredholm Integro-Differential Equations, Matrix Forms, Numerical Solutions.

Mathematics Subject Classification: 65R20

1 Introduction

In 1981, Ortiz and Samara [1] proposed an operational technique for finding a numerical solution of a non-linear ordinary differential equations with some supplementary conditions based on the Tau method [2]. Various techniques have been described in a series of papers [3] for the case of linear ordinary differential eigenvalue problems.The object of this paper is to present a similar operational approach by using the Adomian decomposition method for the integral parts of non-linear Fredholm integro-differential equations of the second kind with initial conditions.

1.1 Non-linear Fredholm integro-differential equations

Consider the following non-linear Fredholm integro-differential equation with the given initial conditions

$$Dy(x)-\int_0^a (k_1(x,t)F_1(y(t))+k_2(x,t)F_2(y(t))+\cdots+k_m(x,t)F_m(y(t)))dt=f(x), \quad x\in[0,a] \quad (1)$$

$$y^{(j)}(0)=d_j, \quad j=0,1,\cdots,n_d-1 \quad (2)$$

where D is a linear differential operator of order n_d with polynomial coefficients $p_i(x)$, that is

$$D=\sum_{i=0}^{n_d} p_i(x)\frac{d^i}{dx^i}, \quad p_i(x)=\sum_{j=0}^{\alpha_i} p_{ij}x^j. \quad (3)$$

And assume that $f(x)$ and $k_r(x,t)$ for $r=1,2,\cdots,m$ in (1) are polynomials, otherwise they can be approximated by polynomials to any degree of accuracy (by Lagrange interpolation, Taylor series or any other suitable method).

2 Matrix representation for $Dy(x)$

The effect of differentiation or shifting on the coefficients $\underline{a}_n = [a_0, a_1, \cdots, a_n, 0, 0, 0, \cdots]$ of a polynomial $y_n(x) = \underline{a}_n \underline{\underline{X}}$ is the same as that of post-multiplication of $\underline{a}_n$ by either the matrix η or the matrix μ defined by

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & \\ & 0 & 1 & 0 & \\ & & 0 & 1 & \vdots \\ & & & 0 & \\ & & \cdots & & \ddots \end{bmatrix} \quad \text{and} \quad \eta = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 2 & 0 & & \vdots \\ 0 & 0 & 3 & 0 & \\ & & \cdots & & \ddots \end{bmatrix}.$$

Proposition 2.1. *The structure of the matrix* Π *is as follows*

$$\Pi = \begin{bmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \cdots & \pi_{1,m_1} & 0 & & & \\ \vdots & & & \vdots & & & \ddots & & \\ \pi_{n_d+1,1} & \pi_{n_d+1,2} & \pi_{n_d+1,3} & \cdots & & \pi_{n_d+1,m_{n_d+1}} & & 0 & \\ 0 & \pi_{n_d+2,2} & \pi_{n_d+2,3} & \cdots & & & & \pi_{n_d+2,m_{n_d+2}} & 0 \\ & \ddots & & & & & & & \ddots \end{bmatrix}$$

where

$$m_i = \begin{cases} \max\{\alpha_0 + i, \alpha_1 + i - 1, \cdots, \alpha_{i-1} + 1\} & if \quad i = 1, 2, \cdots, n_d + 1 \\ m_{n_d+1} + i - n_d + 1 & if \quad i = n_d + 2, n_d + 3, \cdots \end{cases}$$

and

$$\pi_{i,j} = \sum_{k=0}^{i-1} \frac{(i-1)!}{(i-1-k)!} \hat{p}_{k,j-i+k} \quad i = 1, 2, \cdots \quad j = 1, 2, \cdots, m_i$$

$$\hat{p}_{i,j} = \begin{cases} p_{i,j} & if \quad j = 0, 1, \cdots, \alpha_i \\ 0 & if \quad j < 0 \ or \ j > \alpha_i. \end{cases}$$

Lemma 2.2. *Let $y_n(x)$ be a polynomial of the following form:*

$$y_n(x) = \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{\infty} a_i x^i.$$

Then we have

$$i) \ \frac{d^r}{dx^r} y_n(x) = \underline{a}_n \eta^r \underline{\underline{X}}, \quad r = 1, 2, 3, \cdots$$

$$ii) \ x^r y_n(x) = \underline{a}_n \mu^r \underline{\underline{X}}, \quad r = 1, 2, 3, \cdots$$

where $\underline{a}_n = [a_0, a_1, \ldots, a_n, 0, 0, \ldots]$.

Theorem 2.3. *If the operator D and the polynomial $y_n(x)$ are of the forms (3),(??) then $Dy_n(x) = \underline{a}_n \Pi \underline{\underline{X}}$, where*

$$\Pi = \sum_{i=0}^{n_d} \eta^i p_i(\mu).$$

Proof. See [1] □

3 Error estimation

In this section an error estimator for the approximate solution of (1) and (2) is obtained. Let us call $e_n(x) = y(x) - y_n(x)$ as the error function of the approximate solution $y_n(x)$ to $y(x)$, where $y(x)$ is the the exact solution of (1) and (2). Hence $y_n(x)$ satisfies the following problem:

$$Dy_n(x) - \int_0^a \Big(\sum_{i=1}^m k_i(x,t)F_i(y_n(t))\Big)dt = f(x) + H_n(x), \quad x \in [0,a] \tag{4}$$

with

$$y_n^{(j)}(0) = d_j, \quad j = 0,1,\cdots,n_d - 1. \tag{5}$$

The perturbation term $H_n(x)$ can be obtained by substituting the computed solution $y_n(x)$ into the equation

$$H_n(x) = Dy_n(x) - \int_0^a \Big(\sum_{i=1}^m k_i(x,t)F_i(y_n(t))\Big)dt - f(x). \tag{6}$$

We proceed to find an approximation $e_{n,N}(x)$ to the error function $e_n(x)$ in the same way as we did before for the solution of problem (1) and (2). Subtracting (4) and (5) from (1) and (2) respectively and taking two terms of expansion $F(y(t))$ around $y_n(t)$, the error function $e_n(x)$ satisfies the problem

$$De_n(x) - \int_0^a \Big(\sum_{i=1}^m k_i(x,t)\{e_n(t)F_i'(y_n(t)) + \frac{1}{2}e_n^2(t)F_i''(y_n(t))\}\Big)dt = -H_n(x), \quad x \in [0,a] \tag{7}$$

with

$$e_n^{(j)}(0) = 0, \quad j = 0,1,\cdots,n_d - 1. \tag{8}$$

Remark 3.1. *Note that in the following tables, the notations Exact, App., Abs.Err. and Esti.Err. have been used for the exact solution, approximate solution obtained by our method, absolute error and estimate error of the approximate solution respectively.*

Example 3.2.

$$(1+x)y^{'}(x) - 2y(x) + \int_0^1 4xty(t)^2dt = e^x + x^2e^x - \frac{1}{2}xe^2 - \frac{3}{2}x, \ \ y(0) = 0, \ \ 0 \le x < 1.$$

The exact solution is given by $y(x) = xe^x$. For the numerical results with $n = 5$ see the following Table.

n	x	Exact	App.A.T.	A.T.Err.	Est.Err.
5	0.00	0.000000	-.000233	2.325670e-04	2.325670e-04
5	.20	.244281	.244253	2.715360e-05	1.085065e-05
5	.40	.596730	.596731	6.963397e-07	1.488429e-08
5	.60	1.093271	1.093306	3.468065e-05	1.488429e-08
5	.80	1.780433	1.780506	7.372597e-05	1.085065e-05
5	1.00	2.718282	2.718181	1.011473e-04	2.325670e-04

Table 1: Table of example

4 Main results

we have solved an special class of Fredholm which is important in practical problems for example in biological systems. For solving this type of problems, we have designed remarkably simple method which has high accuracy in comparison with other existing methods and clarified the accuracy by solving numerical examples(see table).

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On the convergence of the Gl-GMRES method for solving the general coupled linear matrix equations

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Abstract
In this paper, we first propose the global generalized minimum residual (Gl-GMRES) method for solving a general class of coupled matrix equations. Then, some new theoretical results are presented by employing Schur complement. These results could be used to obtain new convergence properties of the Gl-GMRES method for solving the mentioned coupled linear matrix equations.
Keywords: Matrix equation, Gl-GMRES method, Schur complement.

Mathematics Subject Classification: 15A24, 65F10.

1 Introduction

Consider the general coupled matrix equations

$$\sum_{j=1}^{p} A_{ij}X_jB_{ij} = C_i, \qquad i = 1,\dots,p, \tag{1}$$

where $A_{ij} \in \mathbb{R}^{m\times m}, B_{ij} \in \mathbb{R}^{n\times n}$, and $C_i \in \mathbb{R}^{m\times n}$, $i,j = 1,2,\dots,p$, are large and sparse matrices, and $X_i \in \mathbb{R}^{m\times n}$, $i = 1,2,\dots,p$, are the unknown matrices. Such problems arise in linear control and filtering theory for continues or discrete-time large-scale dynamical systems. They also play an important role in image restoration and other problems; for more details see [1, 3, 5] and the references therein.

We present the linear operator $\mathcal{M}$ as follows:

$$\mathcal{M} : \mathbb{R}^{m\times n} \times \cdots \times \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times pn},$$

$$X = (X_1, X_2, \dots, X_p) \to \mathcal{M}(X) = (\mathcal{A}_1(X), \mathcal{A}_2(X), \dots, \mathcal{A}_p(X)),$$

where

$$\mathcal{A}_i(X) = \sum_{j=1}^{p} A_{ij}X_jB_{ij}, \qquad i = 1,2,\dots,p.$$

Using the linear operator $\mathcal{M}$, we rewrite Eq. (13) as

$$\mathcal{M}(X) = C, \tag{2}$$

where $C = (C_1, C_2, \dots, C_p)$. In the next section, we utilize the linear matrix operator $\mathcal{M}$ to present Gl-GMRES algorithm [3] for solving (13). More precisely, we focus on the solution of Eq. (1.3) instead of Eq. (13).

For two matrices Y and Z in $\mathbb{R}^{m\times n}$, the inner product $< Y, Z >_F$ is defined as $< Y, Z >_F = tr(Y^TZ)$, and the associated norm is the Frobenius norm denoted by $\|.\|_F$.

Definition 1.1. (*Bouyouli et al.* [2](*Let $A = [A_1, A_2, \ldots, A_p]$ and $B = [B_1, B_2, \ldots, B_\ell]$ be matrices of dimensions $n \times ps$ and $n \times \ell s$, respectively, where A_i and B_j are $n \times s$ matrices. Then the matrix $A^T \diamond B = [(A^T \diamond B)_{ij}]_{p\times\ell}$ is defined by $(A^T \diamond B)_{ij} =< A_i, B_j >_F$.*

Definition 1.2. (*Schur* [4](*Let M be a matrix partitioned into four blocks as:*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where the submatrix D is assumed to be square and nonsingular. The Schur complement of D in M, denoted by (M/D), is defined by $(M/D) = A - BD^{-1}C$.

2 Main Result

In the following, we introduce inner product $\circledcirc$ and its corresponding matrix norm. Then, the Gl-GMRES method for solving Eq. (1.3) is proposed. Finally, some new theorems are presented.

Definition 2.1. *Assume that $\overline{X} = (\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_p)$ and $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_p)$ are in $\mathbb{R}^{m\times pn}$. We define the inner product $< ., . >$ as follows:*

$$< \overline{X}, \widetilde{X} >= tr(\overline{X}^T \diamond \widetilde{X}). \tag{3}$$

Remark 2.2. *For $X = (\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_p)$ in $\mathbb{R}^{m\times pn}$, the norm of X is defined by $\|X\|^2 = tr(X^T \diamond X)$. Throughout this paper, a set of matrices in $\mathbb{R}^{m\times pn}$ is said to be orthonormal if it is orthonormal with respect to the scalar product (1.3).*

Recently, Beik and Salkuyeh [1] have introduced a new product denoted by $\circledcirc$ and defined as follows.

Definition 2.3. *Let $A = [A^{(1)}, A^{(2)}, \ldots, A^{(k)}]$, $B = [B^{(1)}, B^{(2)}, \ldots, B^{(\ell)}]$ be $m \times kpn$ and $m \times \ell pn$ matrices, respectively, where $A^{(i)} = [A_1^{(i)}, A_2^{(i)}, \ldots, A_p^{(i)}]$, $B^{(s)} = [B_1^{(s)}, B_2^{(s)}, \ldots, B_p^{(s)}]$ and $A_j^{(i)}, B_j^{(s)} \in \mathbb{R}^{m\times n}$ for $i = 1, 2, \ldots, k$, $s = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots, p$. The matrix $A^T \circledcirc B = [(A^T \circledcirc B)_{ij}]_{k\times\ell}$ is defined by $(A^T \circledcirc B)_{ij} = tr((A^{(i)})^T \diamond B^{(j)})$.*

Suppose that $X^{(0)} = (X_1^{(0)}, X_2^{(0)}, \ldots, X_p^{(0)})$ in $\mathbb{R}^{m\times pn}$ is a given initial approximate solution and consider the Eq. (1.3). As a natural way, we define the matrix Krylov subspace as follows

$$\mathcal{K}_k(\mathcal{M}, R^{(0)}) = span\left\{R^{(0)}, \mathcal{M}(R^{(0)}), \ldots, \mathcal{M}^{k-1}(R^{(0)})\right\}, \tag{4}$$

where $R^{(0)} = C - \mathcal{M}(X^{(0)})$.

1. *Global Arnoldi process.*

۱. Set $V_1 = R^{(0)}/\left\|R^{(0)}\right\|$.
۲. For $j = 1, 2, \ldots, k$ Do
۳. $W := \mathcal{M}(V_j)$
۴. For $i = 1, 2, \ldots, j$ Do
۵. $h_{ij} :=< W, V_i >$
۶. $W := W - h_{ij}V_i$
۷. End for
۸. $h_{j+1,j} := \|W\|$. If $h_{j+1,j} := 0$، then stop.
۹. $V_{j+1} := W/h_{j+1,j}$
۱۰. End for

Suppose that $\mathcal{V}_k = [V_1, V_2, \ldots, V_k]$ denotes the $m \times kpn$ where $V_i = [V_1^{(i)}, V_2^{(i)}, \ldots, V_p^{(i)}]$ for $i = 1, 2, \ldots, k$. Let $\overline{H}_k$ the $(k+1) \times k$ be an upper Hessenberg matrix where its nonzero entries h_{ij} are computed by Algorithm 2 and H_k is the $k \times k$ matrix obtained from $\overline{H}_k$ by deleting its last row. It is not difficult to see that the matrix $\mathcal{V}_k$, produced by Algorithm 2, is an orthonormal basis for the $\mathcal{K}_k(\mathcal{M}, R^{(0)})$, i.e., $\mathcal{V}_k^T \circledcirc \mathcal{V}_k = I_k$.

Starting from an initial guess $X^{(0)} \in \mathbb{R}^{m\times pn}$ and the corresponding residual $R^{(0)} = C - \mathcal{M}(X^{(0)})$, the Gl-GMRES algorithm computes the approximate solution $X^{(k)}$ such that $X^{(k)} \in X^{(0)} + \mathcal{K}_k(\mathcal{M}, R^{(0)})$.

Considering the orthonormal basis $\mathcal{V}_k = [V_1, V_2, \ldots, V_k]$ for $\mathcal{K}_k(\mathcal{M}, R^{(0)})$, we get

$$X^{(k)} = X^{(0)} + \sum_{i=1}^{k} V_i y_i^{(k)} = X^{(0)} + \mathcal{V}_k((y^{(k)} \otimes I_p) \otimes I_n), \tag{5}$$

where the vector $y^{(k)}$ in Eq. (5) is obtained by imposing the following orthogonality condition

$$R^{(k)} = C - \mathcal{M}(X_k) \quad \perp \quad \mathcal{K}_k(\mathcal{M}, \mathcal{M}(R_0)). \tag{6}$$

The orthogonality condition (9) shows that $X^{(k)}$ can be obtained as the solution of the minimization problem

$$\min_{X - X^{(0)} \in \mathcal{K}_k(\mathcal{M}, R^{(0)})} \|C - \mathcal{M}(X)\|. \tag{7}$$

Theorem 2.4. *The approximate solution $X^{(k)}$ computed by the Gl-GMRES algorithm is presented by $X^{(k)} = X^{(0)} + \mathcal{V}_k((y^{(k)} \otimes I_p) \otimes I_n)$ where $y^{(k)}$ is the solution of the least squares problem $\min_{y\in\mathbb{R}^k} \left\|\beta e_1 - \overline{H}_k y\right\|_2$, where $\beta = \left\|R^{(0)}\right\|$.*

Proof. See [1]. ☐

2. *Gl-GMRES(k) algorithm for (13).*

۱. *Choose $X^{(0)}$، and a tolerance ε. Compute $R^{(0)} = C - \mathcal{M}(X^{(0)})$، and $V_1 = R^{(0)}$.*

۲. *Construct the orthonormal basis $V_1, V_2, \ldots, V_k$ by Algorithm 2*

۳. *Determine $y^{(k)}$ as the solution of the least square problem:*

$$\min_{y\in\mathbb{R}^k} \left\|\beta e_1 - \overline{H}_k y\right\|_2.$$

Compute $X^{(k)} = X^{(0)} + \mathcal{V}_k((y^{(k)} \otimes I_p) \otimes I_n)$.

۴. *Compute the residual $R^{(k)}$ and $\left\|R^{(k)}\right\|$.*

۵. *If $\frac{\left\|R^{(k)}\right\|}{\left\|R^{(0)}\right\|} < \varepsilon$ Stop؛ else $R^{(0)} := R^{(k)}$، $V_1 := R^{(0)}$، and Go to ۲.*

Now, using the definition of the Schur complement and the properties of $\circledcirc$ product, we may establish the following theorems.

Theorem 2.5. *Assume that the Gl-GMRES method has been applied for solving Eq. (1.3). Suppose that $\mathcal{W}_k^T \circledcirc \mathcal{W}_k$ is a nonsingular matrix. Then the residual matrix $R^{(k)}$ can be expressed by the following Schur complement*

$$R^{(k)} = \left(\begin{array}{cc} R^{(0)} & \mathcal{W}_k \\ ((\mathcal{W}_k^T \circledcirc R^{(0)}) \otimes I_p) \otimes I_n & ((\mathcal{W}_k^T \circledcirc \mathcal{W}_k) \otimes I_p) \otimes I_n \end{array} \right) \Big/ (((\mathcal{W}_k^T \circledcirc \mathcal{W}_k) \otimes I_p) \otimes I_n), \tag{8}$$

where $\mathcal{W}_k = [\mathcal{M}(V_1), \mathcal{M}(V_2), \ldots, \mathcal{M}(V_k)]$.

Theorem 2.6. *Suppose that $R^{(k)}$ is the residual matrix obtained by the Gl-GMRES method to Eq. (1.3) . Let $\mathcal{W}_k^T \circledcirc \mathcal{W}_k$ be a nonsingular matrix. Then*

$$\left\| R^{(k)} \right\|_F^2 = \left((\widetilde{\mathcal{V}}_{k+1}^T \circledcirc \widetilde{\mathcal{V}}_{k+1}) \Big/ (\mathcal{W}_k^T \circledcirc \mathcal{W}_k) \right),$$

where $\mathcal{W}_k = [\mathcal{M}(V_1), \mathcal{M}(V_2), \ldots, \mathcal{M}(V_k)]$ and $\widetilde{\mathcal{V}}_{k+1} = [R^{(0)}, \mathcal{W}_k]$.

Theorem 2.7. *Let $\widetilde{\mathcal{V}}_{k+1}$ be defined as before. The residual $R^{(k)}$ satisfies the following relation*

$$\frac{4\chi(\widetilde{\mathcal{V}}_{k+1}^T \circledcirc \widetilde{\mathcal{V}}_{k+1})}{(1+\chi(\widetilde{\mathcal{V}}_{k+1}^T \circledcirc \widetilde{\mathcal{V}}_{k+1}))^2} \leq \frac{\left\| R^{(k)} \right\|_F^2}{\left\| R^{(0)} \right\|_F^2} \leq 1,$$

where $\chi(\widetilde{\mathcal{V}}_{k+1}^T \circledcirc \widetilde{\mathcal{V}}_{k+1})$ is the condition number of the matrix $\widetilde{\mathcal{V}}_{k+1}^T \circledcirc \widetilde{\mathcal{V}}_{k+1}$.

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An iterative algorithm for the generalized (P,Q)-reflexive solution of the coupled Sylvester-transpose matrix equations

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Abstract

In this paper, we present an iterative algorithm for solving the following coupled Sylvester-transpose matrix equations

$$\sum_{j=1}^{q}\left(A_{ij}X_jB_{ij}+C_{ij}X_j^TD_{ij}\right)=F_i, \qquad i=1,2,\ldots,p,$$

over the generalized (P_j,Q_j)-reflexive matrix X_j $(j=1,2,\ldots,q)$. When the problem is consistent, then the reflexive solution group of considered coupled Sylvester-transpose matrix equations can be obtained within finite iterative steps for any initial generalized (P_j,Q_j)-reflexive matrix $X_j^{(0)}$ $(j=1,2,\ldots,q)$, in the exact arithmetic.

Keywords: Linear matrix equation, (P,Q)-reflexive matrix, Iterative algorithm.

Mathematics Subject Classification: 15A24, 65F10.

1 Introduction

Throughout this paper, we use $tr(A), A^T$ to denote the trace and the transpose of the matrix A, respectively. Moreover, $\mathbb{R}^{n\times m}$ represents the set of all $n\times m$ real matrices and the set of all symmetric orthogonal matrices in $\mathbb{R}^{n\times n}$ is denoted by $\mathbb{SOR}^{n\times n}$. For two matrices Y and Z in $\mathbb{R}^{n\times s}$, the inner product $<Y,Z>_F$ is defined such that $<Y,Z>_F=tr(Y^TZ)$.

Definition 1.1. *A matrix $A\in\mathbb{R}^{n\times m}$ is said to be a generalized reflexive matrix with respect to $P\in\mathbb{SOR}^{n\times n}$ and $Q\in\mathbb{SOR}^{m\times m}$, if $A=PAQ$. We denote the set of all generalized (P,Q)-reflexive matrices $\mathbb{R}_r^{n\times m}(P,Q)$.*

Recently, the idea of conjugate gradient (CG) method has been developed for constructing iterative algorithms to compute the solution of different kinds of linear matrix equations over reflexive and anti-reflexive, generalized bisymmetric, generalized centro-symmetric, mirror-symmetric, skew-symmetric and (P,Q)-reflexive matrices, for more details see [1, 2, 3, 4] and the references therein. For instance, Wang and Wu [4] have presented an iterative algorithm for the following linear system of matrix equations

$$\begin{cases} \sum_{i=1}^{N} A_i^{(1)}X_iB_i^{(1)}=C^{(1)} \\ \vdots \\ \sum_{i=1}^{N} A_i^{(M)}X_iB_i^{(M)}=C^{(M)} \end{cases}$$

over the (P, Q)-reflexive matrix $X_\ell \in \mathbb{R}^{n\times m}$ $(A_\ell^{(i)} \in \mathbb{R}^{p\times n}, B_\ell^{(i)} \in \mathbb{R}^{m\times q}, C_\ell^{(i)} \in \mathbb{R}^{p\times q}, \ell = 1, 2, \ldots, N, i = 1, 2, \ldots, M)$.

Definition 1.2. *Assume that $A = [A_1, A_2, \ldots, A_k]$ and $B = [B_1, B_2, \ldots, B_k]$ where $A_i, B_i \in \mathbb{R}^{r_i\times s_i}$ for $i = 1, 2, \ldots, k$. Then the $k \times k$ matrix $A^T \circledast B$ is defined by*

$$A^T \circledast B = diag\left(< A_1, B_1 >_F, \ldots, < A_k, B_k >_F\right).$$

Using the $\circledast$ product, we may define the following inner product and its corresponding matrix norm which are utilized for obtaining the main results of this work.

Definition 1.3. *Suppose that $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_k)$ and $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_k)$ where $\Phi_i, \Psi_i \in \mathbb{R}^{r_i\times s_i}$ for $i = 1, 2, \ldots, k$. We define the inner product $< ., . >$ as follows:*

$$< \Phi, \Psi >= tr(\Phi^T \circledast \Psi). \tag{1}$$

Remark 1.4. *For $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_k)$, where $\Gamma_i \in \mathbb{R}^{r_i\times s_i}$ for $i = 1, 2, \ldots, k$, the norm of Γ is defined by $\|\Gamma\|^2 = tr(\Gamma^T \circledast \Gamma)$.*

In this work, we consider the following problem.

Problem I. For given matrices $A_{ij} \in \mathbb{R}^{r_i\times n_j}, B_{ij} \in \mathbb{R}^{m_j\times s_i}$, $C_{ij} \in \mathbb{R}^{r_i\times m_j}, D_{ij} \in \mathbb{R}^{n_j\times s_i}$, $F_i \in \mathbb{R}^{r_i\times s_i}$, $P_j \in \mathbb{SOR}^{n_j\times n_j}$ and $Q_j \in \mathbb{SOR}^{m_j\times m_j}$. Find the generalized the matrices $X_j \in \mathbb{R}_r^{n_j\times m_j}(P_j, Q_j)$, $j = 1, 2, \ldots, q$, such that

$$\sum_{j=1}^{q} \left(A_{ij}X_jB_{ij} + C_{ij}X_j^T D_{ij}\right) = F_i, \qquad i = 1, 2, \ldots, p. \tag{2}$$

2 Main Result

For simplicity, we utilize the following three linear operators. The first linear operator is denoted by $\mathcal{M}$ and defined as follows:

$$\begin{array}{l}\mathcal{M} : \mathbb{R}^{n_1\times m_1} \times \cdots \times \mathbb{R}^{n_q\times m_q} \to \mathbb{R}^{r_1\times s_1} \times \cdots \times \mathbb{R}^{r_p\times s_p}\\ X = (X_1, \ldots, X_q) \to \mathcal{M}(X) = (M_1(X), \ldots, M_p(X)),\end{array}$$

where

$$M_i(X) = \sum_{j=1}^{q} \left(A_{ij}X_jB_{ij} + C_{ij}X_j^T D_{ij}\right), \qquad i = 1, 2, \ldots, p.$$

Using the linear operator $\mathcal{M}(X)$, the matrix equations (13) can be written in the following form:

$$\mathcal{M}(X) = F,$$

where $F = (F_1, F_2, \ldots, F_p)$ and $F_i \in \mathbb{R}^{r_i\times s_i}$, $i = 1, 2, \ldots, p$.
The second linear operator is represented by $\mathcal{A}$ and defined such that

$$\begin{array}{l}\mathcal{A} : \mathbb{R}^{r_1\times s_1} \times \cdots \times \mathbb{R}^{r_p\times s_p} \to \mathbb{R}^{n_1\times m_1} \times \cdots \times \mathbb{R}^{n_q\times m_q}\\ Y = (Y_1, \ldots, Y_p) \to \mathcal{A}(Y) = (A_1(Y), \ldots, A_q(Y)),\end{array}$$

where

$$A_j(Y) = \sum_{i=1}^{p} \left(A_{ij}^T Y_i B_{ij}^T + D_{ij}Y_i^T C_{ij}\right), \qquad j = 1, 2, \ldots, q.$$

Moreover, we define the linear operator $\mathcal{D}$ as follows:

$$\begin{aligned}&\mathcal{D}:\mathbb{R}^{r_1\times s_1}\times\cdots\times\mathbb{R}^{r_p\times s_p}\to\mathbb{R}^{n_1\times m_1}\times\cdots\times\mathbb{R}^{n_q\times m_q}\\&Y=(Y_1,\ldots,Y_p)\to\mathcal{D}(Y)=(D_1(Y),\ldots,D_q(Y)),\end{aligned}$$

where $D_j(Y)=\frac{1}{2}\sum_{i=1}^{p}\left(A_{ij}^TY_iB_{ij}^T+D_{ij}Y_i^TC_{ij}+P_jA_{ij}^TY_iB_{ij}^TQ_j+P_jD_{ij}Y_i^TC_{ij}Q_j\right)$, and the matrices $P_j\in\mathbb{SOR}^{n_j\times n_j}$, $Q_j\in\mathbb{SOR}^{m_j\times m_j}$, for $j=1,2,\ldots,q$, are given.

Remark 2.1. *Assume that $X=(X_1,X_2,\ldots,X_q)$, $Y=(Y_1,Y_2,\ldots,Y_p)$ such that $X_j\in\mathbb{R}_r^{n_j\times m_j}(P_j,Q_j)$ and $Y_i\in\mathbb{R}^{r_i\times s_i}$ for $i=1,2,\ldots,p$ and $j=1,2,\ldots,q$. Moreover, suppose that the matrices $P_j\in\mathbb{SOR}^{n_j\times n_j}$, $Q_j\in\mathbb{SOR}^{m_j\times m_j}$, $j=1,2,\ldots,q$, are given. Then,*

$$<\mathcal{A}(Y),X>=<\mathcal{D}(Y),X>.$$

Proposition 2.2. *Suppose that $X=(X_1,X_2,\ldots,X_q)$, $Y=(Y_1,Y_2,\ldots,Y_p)$ and $\mathcal{M}(X),\mathcal{A}(Y)$ are defined as before. Then*

$$<\mathcal{M}(X),Y>=<X,\mathcal{A}(Y)>.$$

In the following, we propose an algorithm for solving Problem I by extending the idea of the conjugate gradient method [5].

3. *Algorithm for solving Problem I.*

۱. *Input matrices $A_{ij}\in\mathbb{R}^{r_i\times n_j}, B_{ij}\in\mathbb{R}^{m_j\times s_i}$، $C_{ij}\in\mathbb{R}^{r_i\times m_j}, D_{ij}\in\mathbb{R}^{n_j\times s_i}$, $F_i\in\mathbb{R}^{r_i\times s_i}$ $P_j\in\mathbb{SOR}^{n_j\times n_j}$ and $Q_j\in\mathbb{SOR}^{m_j\times m_j}$ for $i=1,2,\ldots,p$ and $j=1,2,\ldots,q$.*

۲. *Choose arbitrary matrix $X^{(1)}=(X_1^{(1)},\ldots,X_q^{(1)})$ such that $X_j^{(1)}\in\mathbb{R}_r^{n_j\times m_j}(P_j,Q_j)$ for $j=1,2,\ldots,q$.*

۳. *Calculate $R^{(1)}=F-\mathcal{M}(X^{(1)})$؛ Set $P^{(1)}=\mathcal{D}(R^{(1)})$ and $k=1$.*

۴. *If $R^{(k)}=0$ or $R^{(k)}\neq0$ and $P^{(k)}=0$ then Stop؛ Otherwise go to ۵.*

۵. *Compute*

$$X^{(k+1)}=X^{(k)}+\frac{\|R^{(k)}\|^2}{\|P^{(k)}\|^2}P^{(k)};$$

$$R^{(k+1)}=R^{(k)}-\frac{\|R^{(k)}\|^2}{\|P^{(k)}\|^2}\mathcal{M}(P^{(k)});$$

$$P^{(k+1)}=\mathcal{D}(R^{(k+1)})+\frac{\|R^{(k+1)}\|^2}{\|R^{(k)}\|^2}P^{(k)}.$$

۶. *Go to Step ۴*

Now, we may present some properties of Algorithm 2 in the following.

Lemma 2.3. *Assume that the sequences $\{R^{(k)}\}$ and $\{P^{(k)}\}$, $k=1,2,\ldots,s$ ($R^{(k)}\neq0$, $k=1,2,\ldots,s$) are produced by Algorithm 2. Then,*

$$<R^{(i)},R^{(j)}>=0,\qquad<P^{(i)},P^{(j)}>=0,\qquad i,j=1,2,\ldots,s,\ i\neq j.\tag{3}$$

Lemma 2.4. *Suppose that Problem I is consistent. Moreover, assume that the matrix group $X^*=(X_1^*,X_2^*,\ldots,X_q^*)$ is an arbitrary solution of Problem I. Then for any initial matrix group $X^{(1)}=(X_1^{(1)},X_2^{(1)},\ldots,X_q^{(1)})$ such that $X_j^{(1)}\in\mathbb{R}_r^{n_j\times m_j}(P_j,Q_j)$ ($j=1,2,\ldots,q$), the sequences $X^{(i)},R^{(i)}$ and $P^{(i)}$ (produced by Algorithm 2) satisfy the following equality*

$$<P^{(i)},X^*-X^{(i)}>=\|R^{(i)}\|^2,\qquad i=1,2,3,\ldots\ .\tag{4}$$

Remark 2.5. *From Lemma 2.1, it is deduced that if there exists a positive integer number k such that $P^{(k)} = 0$ and $R^{(k)} \neq 0$ then Problem I is not consistent.*

Theorem 2.6. *Let Problem I be consistent. Then, for any initial matrix group $X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \ldots, X_q^{(1)})$ such that $X_j^{(1)} \in \mathbb{R}_r^{n_j \times m_j}(P_j, Q_j)$ $(j = 1, 2, \ldots, q)$, and in the absence of roundoff errors, a solution of Problem I can be computed by Algorithm 2 within at most $m + 1$ iteration steps, where $m = \sum_{i=1}^{p} r_i s_i$.*

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Meijer's G-functions as the solution of Schrödinger equation

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Abstract

This paper directly obtains Meijer's G-functions as the solution of partial differential equation ($PDEs$) related to Schrödinger equation. It is illustrated that wave function Ψ, is the symmetric product form of Meijer's G-functions. We show that one of the three basic univalent Meijer's G-functions, $G^{1,0}_{0,2}$, exists in the solution.
Keywords: Meijer's G-functions; Partial differential equations; Univalent functions; Generalized fractional calculus

Mathematics Subject Classification: 30C45, 33C60, 33E20, 34B24

1 Introduction

The Meijer's G-function brings a number of great utilities in mathematical physics because of its analytical properties, and particularly because it can be expressed as a finite sum of generalized hypergeometric functions that have well-known series expansions. The prominent astrophysical thermonuclear functions $I_1(z,\nu)$ and $I_2(z,d,\nu)$ are expressed in terms of these G-functions representations [2]. Additionally, the Meijer's G-function is also used as the weight function to obtain the Gazeau-Klauder (photon-added) coherent states [3]. Due to the elegant and general properties of G-function, it has become possible to represent the solutions of many problems in these fields in their terms. Stated in this way, the problems gain a much more general character due to the great freedom of choice of the orders $m;n;p;q$ and the parameters of G-functions in comparison to other special functions. Simultaneously, the calculations become simpler and more unified. Evidence showing the importance of G-functions is given by the fact that the basic elementary functions and most of the special functions of mathematical physics, including the generalized hypergeometric functions, follow as its particular cases. Therefore, each result concerning the G-function has become a key leading to numerous particular results for the Bessel functions, confluent hypergeometric functions, classical orthogonal polynomials and others, see [1,5].

In the previous paper we have classified the univalent Meijer's G-functions into three types. Three basic univalent Meijer's G-functions are introduced, namely, $G^{1,0}_{0,2};G^{1,1}_{1,2};G^{1,1}_{1,1}$ and by successive applications of special transformations, a number of univalent Meijer's G-functions can be obtained [4].

Definition 1.1. A definition of Meijer's G-function is given by the following path integral in the complex plane, called Mellin-Barnes type integral [1]:

$$G^{m,n}_{p,q}\left({}^{a_1,\dots,a_p}_{b_1,\dots,b_q}\middle|z\right)=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\ \prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\ \prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s ds. \tag{1}$$

Here, an empty product means unity and the integers $m; n; p; q$ are called orders of the G-function, or the components of the order $(m; n; p; q)$, while a_p and b_q are called "parameters", and in general, they are complex numbers. The definition holds under the following assumptions: $0 \leq m \leq q$ and $0 \leq n \leq p$, where m, n, p, and q are the integer numbers. $a_j - b_k \neq 1, 2, 3, \dots$ for $k = 1, \dots, n$ and $j = 1, 2, \dots, m$ imply that no pole of any $\Gamma(b_j - s), j = 1, \dots, m$ coincides with any pole of any $\Gamma(1 - a_k + s), k = 1, \dots, n$.

The Meijer's G-function $y(z) = G^{m,n}_{p,q}(z|^{a_j}_{b_k})$ satisfies the linear ordinary differential equation of the generalized hypergeometric type whose order is equal to $\max(p, q)$.

$$[(-1)^{p-m-n} z \prod_{j=1}^{p}(z\frac{d}{dz} - a_j + 1) - \prod_{k=1}^{q}(z\frac{d}{dz} - b_k)]y(z) = 0. \tag{2}$$

The **Schrödinger equation**, describing non-relativistic quantum phenomena, is as follows:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V(\mathbf{r})\Psi = -i\hbar\frac{\partial\Psi}{\partial t},$$

where m is the mass of a subatomic particle, $\hbar$ is Plank's constant (divided by 2π), V is the potential energy of the particle, and $|\Psi(\mathbf{r}, t)|^2$ is the probability density of finding the particle at $\mathbf{r}$ at time t. Now introducing new method, we obtain the solution of this equation in terms of Meijer's G-functions.

2 Main Result

The method of modified separation of variables

The Separation of Variables is employed to obtain solutions to the many partial differential equations. The method consists of assuming the solution as a product of functions, each depending on one coordinate variable only [6]. The method proposed in this study,i.e. the modified separation of variables, leads to obtain the solution as a product of the Meijer's G-functions, with each one depending on one variable by making links between ordinary linear differential equation of the Meijer's G-functions and each one of the $ODEs$. Four steps are followed in this method, namely:

1) Choosing a convenient coordinates system by considering the boundary conditions and boundary surfaces.

2) Obtaining $ODEs$ from PDE by writing the solution as a product form of different variables and putting it into PDE.

3) Starting with Eq.(2), the orders m, n, p and q are chosen and the variable z is changed to $\bar{z}$, such that Eq.(2) converts into each of $ODEs$, and this is three times for time independent problem (e.g., x, y, z coordinates) and four times for time dependent problem (e.g., x, y, z, t). If this particular step is done, solving $ODEs$ is not required because Eq.(2) is an equation that does not need to be solved. It is solved, and the solutions are $MGFs$.

4) using boundary conditions and obtaining the exact solution.
A separation of variable $\psi(x, y, z, t) = X(x)Y(y)Z(z)T(t)$ yields the $ODEs$

$$\frac{d^2X}{dx^2} + \lambda X = 0, \qquad \frac{d^2Y}{dy^2} + \sigma Y = 0, \qquad \frac{d^2Z}{dz^2} + \nu Z = 0,$$

$$\tfrac{dT}{dt} + i\omega T = 0, \qquad \text{where} \qquad \omega = \tfrac{\hbar}{2\mu}(\lambda + \sigma + \nu).$$

Vanishing of ψ, BCs at $x=0$ and $x=a$, for all y,z; at $y=0$ and $y=b$ for all x,z; at $z=0$ and $z=c$ for all x,y, gives

$\psi(0,y,z,t)=\psi(a,y,z,t)=0,$ which means $X(0)=X(a)=0,$

$\psi(x,0,z,t)=\psi(x,b,z,t)=0,$ which means $Y(0)=Y(b)=0,$

$\psi(x,y,0,t)=\psi(x,y,c,t)=0,$ which means $Z(0)=Z(C)=0,$

Using the method of Modified separation of variables, we obtain leads to the three following solutions:

$X_n(x)=G^{1,0}_{0,2}(\frac{n^2\pi^2}{4a^2}x^2|^{-}_{\frac{1}{2},0}),$ $\lambda_n=(\frac{n\pi}{a})^2$ for $n=1,2,...$

$Y_m(y)=G^{1,0}_{0,2}(\frac{m^2\pi^2}{4b^2}y^2|^{-}_{\frac{1}{2},0}),$ $\sigma_m=(\frac{m\pi}{b})^2$ for $m=1,2,...$

$Z_l(z)=G^{1,0}_{0,2}(\frac{l^2\pi^2}{4c^2}z^2|^{-}_{\frac{1}{2},0}),$ $\nu_m=(\frac{l\pi}{c})^2$ for $l=1,2,...$

$T(t)=C_{lmn}G^{1,0}_{0,1}(i\omega_{lmn}t|^{-}_{0})$ where $\omega_{lmn}=\frac{\hbar}{2\mu}[(\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2+(\frac{l\pi}{c})^2].$

Therefore, the solution of Schrödinger equation consistent with the BCs is

$$\psi(x,y,z,t)=\sum_{l,m,n=1}^{\infty}A_{lmn}G^{1,0}_{0,1}(i\omega_{lmn}t|^{-}_{0})G^{1,0}_{0,2}(\frac{n^2\pi^2}{4a^2}x^2|^{-}_{\frac{1}{2},0})G^{1,0}_{0,2}(\frac{m^2\pi^2}{4b^2}y^2|^{-}_{\frac{1}{2},0})G^{1,0}_{0,2}(\frac{l^2\pi^2}{4c^2}z^2|^{-}_{\frac{1}{2},0}).$$

The constants $A_{l,m,n}$ are determined by the initial shape, $\psi(x,y,z,0)$ of the wave function.

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Sinc-collocation method for solving singular initial value problems

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Abstract

In some applicable problems we can observe singular initial value problems. In solving these problems most of numerical methods have difficulties and often could not pass the singular point successfully. In this paper we apply the sinc-collocation method for solving singular initial value problems. Ability of the sinc-collocation method in overcoming on the singular points difficulties makes it to be an efficient method in dealing with these equations.

Keywords: Sinc-collocation method; Singular; Initial value problem; nonlinear problems.

Mathematics Subject Classification: 91A12

1 Introduction

Sinc methods for the numerical solution of ODEs and PDEs have been extensively studied and found to be a very effective technique, particularly for problems with singular solutions and those on unbounded domains.

In this article we apply the sinc-collocation method to solve singular initial value problems:

$$\begin{aligned} \mathcal{L}(y) = p(x)y'' + q(x)y' + u(x)y = f(x, y), \\ y(0) = 0, \quad y'(0) = \alpha_0, \end{aligned} \tag{1}$$

where $p(x), q(x), u(x)$ and $f(x, y)$, are analytic.

2 Preliminaries

If we use sinc function, from [1] we can write

$$f(z) = \mathcal{C}(f, h) + E_{sinc}, \qquad E_{sinc}(h) = 0\left(\exp\left(-\frac{\pi d}{h}\right)\right), \tag{2}$$

where $f(z)$ is analytic on a strip domain $|Imz| < d$. If $f(x)$ be a real function, sinc expansion is defined on $-\infty < x < \infty$, while the equation that we want to solve is defined $a < x < \infty$, thus

$$\phi(z) = \ln(\sinh(z)), \tag{3}$$

has been used. The map ϕ carries the eye-shaped region

$$D_E = \left\{z = x + iy : |\arg(\sinh(z))| < d < \frac{\pi}{2}\right\}, \tag{4}$$

on to $D_d = \{\zeta = \xi + i\eta : |\eta| < d < \pi/2\}$.

$$h = \sqrt{\frac{\pi d}{\alpha N}}, \qquad 0 < \alpha \leq 1. \tag{5}$$

The h is the mesh size in D_d for the uniform grids kh, $-\infty < k < \infty$. The base functions on (a, ∞) are given by

$$S(j,h)p\phi(x) = sinc\left(\frac{\phi(x) - jh}{h}\right). \tag{6}$$

The sinc grid points $z \in (a, \infty)$ in D_E will be denoted by x because they are real. We have

$$x = \phi^{-1}(t) = \psi(t) = \ln(e^{kh} + \sqrt{e^{2kh} + 1}). \tag{7}$$

If $y(x)$ be a solution of (13) and we use finite terms of expansion from [2] we have,

$$|y(x) - \sum_{j=-N}^{N} y_j S(j,h)(\psi^{-1}(x))| \leq c\sqrt{N}\exp(-\sqrt{\pi d \alpha N}). \tag{8}$$

Lemma 2.1. *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E to D_d given by (5). Then*

$$\delta_{jk}^{(0)} = [S(j,h)o\phi(x)]_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{9}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)o\phi(x)]_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases} \tag{10}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)o\phi(x)]_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases} \tag{11}$$

Proof. Ref [3]. □

3 sinc-collocation method

For boundary conditions in (13) the sinc basis functions in (9) don't have a derivative when x tends to zero. Thus, we modify the sinc basis functions as

$$xS_j(x). \tag{12}$$

The derivative of the modified sinc basis functions are defined ax x approaches and are equal to zero [4]. Now we construct a polynomial $p(x)$ that satisfies (13). This polynomial is given by

$$p(x) = \lambda x^2 + \alpha_0 x. \tag{13}$$

The approximate solution for $y(x)$, in (13) with initial conditions in (13) is represented by

$$y_N(x) = u_N(x) + p(x), \tag{14}$$

$$u_N(x) = \sum_{j=-N}^{N} c_j x S_j(x). \tag{15}$$

In (9), λ is constant to be determined. It is noted that the approximate solution $y_N(x)$, satisfy initial conditions in (13). The $2N + 1$ coefficients $\{c_j\}_{(j=-N)}^{N}$ and the unknown λ determined by substituting $y_N(x)$ into (13) and evaluating the result at the Sinc points

$$x_j = \ln\left(e^{jh} + \sqrt{e^{2jh} + 1}\right), \qquad j = -N-1, \ldots, N. \tag{16}$$

Obviously by using (1.3)-(2.1) and (11) we have

$$u_N(x_{-N-1}) = 0, \quad u_N(x_j) = c_j x_j, \quad j = -N, \ldots, N, \tag{17}$$

$$u'_N(x_{-N-1}) = \sum_{k=-N}^{N} c_K x_{-N-1}\phi'(x_{-N-1})\delta^{(1)}_{k(-N-1)'} \tag{18}$$

$$u'_N(x_j) = \sum_{k=-N}^{N} c_k \left\{ x_j\phi'(x_j)\delta^{(1)}_{kj} + \delta^{(0)}_{kj} \right\}, \qquad j = -N, \ldots, N, \tag{19}$$

$$u''_N(x_j) = \sum_{k=-N}^{N} c_k \left\{ 2\phi'(x_j)\delta^{(1)}_{kj} + x_j\phi''(x_j)\delta^{(1)}_{kj} + x_j\phi'^2(x_j)\delta^{(2)}_{kj} \right\}, \quad j = -N, \ldots, N, \tag{20}$$

Substituting (11), (20)-(24) in (13) we obtain

$$p(x_j)y''(x_j) + q(x_j)y'(x_j) + u(x_j)y(x_j) = f(x_j, y(x_j)), \qquad j = -N-1, \ldots, N. \tag{21}$$

If (13) be linear then (1.4) gives $2N + 2$ linear algebraic equation and if (13) be nonlinear then (1.4) gives a nonlinear algebraic equation which can be solved for the unknown coefficients c_k and λ by using the Newton's method. Consequently, $y(x)$ given in (13) can be calculated.

4 Numerical examples

In this section, we present some examples to show efficiency and capability of the sinc-collocation method. The problems are solved with Matlab on a personal computer.

Example 4.1. *Consider*

$$\begin{aligned}&(x-1)(x^2+1)^3y'' + (x^2+1)^3y' - 2(x^2+1)^3y \\ &= (x+1)(-2x^4+3x^3-9x^2+3x+1), \quad y(0) = 0, \quad y'(0) = 1,\end{aligned}$$

with exact solution $y = \frac{x}{x^2+1}$.

Table 1. Using sinc-collocation method for example 4.1

x_i	*Exact solution*	*Present method*
0.000055756499727	5.575649955373497e-005	5.575646006258089e-005
0.113189301328983	0.111757484781056	0.111755169351787
2.513526255765731	0.343480467510085	0.343250764029825
3.595979060658014	0.258126636850558	0.258107831538438
4.320922485469001	0.219666546091880	0.219965153341444
6.860066119031846	0.142738113365142	0.143682866289368
7.585585022427243	0.129577083540655	0.129356943962129

In table 1, we report the exact and approximate value of solutions for $N = 100$.

Example 4.2. *Consider the equation*

$$\begin{aligned}&x(x+2)^3y'' - 3(x+2)^3y' + 2(x+2)^3y - (x+2)^3y^3 \\ &\qquad = -(x^6 - 5x^4 - 5x^3 + 15x^2 + 29x + 13), \qquad y(0) = 0, y'(0) = 1,\end{aligned}$$

with exact solution $y = \frac{x}{x^2+1}$.

Table 2. Using sinc-collocation method for example 4.1

x_i	Exact solution	Present method
0.000341994048834	3.419940088341139e-004	3.419949634674329e-004
0.113189301328983	0.111757484781056	0.111757707466690
0.466921020663844	0.383345795280065	0.383345610995369
0.881373587019543	0.496040051718977	0.496008042067412
1.472682704370908	0.464745433989774	0.463823446840111
2.872907663764246	0.310463835338246	0.306835892714598
4.683591269426189	0.204202377023045	0.194219033394465

In table 2, we report the exact and approximate value of solutions for $N = 75$.

5 Conclusion

In our article we cannot use sinc bases functions for our initial value problems thus; we modified the sinc basis functions in our research. Then we apply sinc-collocation method for solving singular initial value problems. Numerical examples highlight efficiency of sinc-collocation method in problems with singularity in equations.

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Alternating direction implicit scheme for two-dimensional parabolic equation

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Abstract
Inhomogeneous parabolic initial-boundary value problems arise in many practical problems. This paper presents finite difference method, based on alternating direction implicit scheme for solving an inhomogeneous two-dimensional time dependent diffusion equation with boundary condition. The systems of linear equations in this scheme are strictly diagonally dominant, so they are always solvable. The scheme described in this work is unconditionally stable. Numerical results are compared with theoretical solution and errors in the maximum norm are shown in the tables. The elapsed times needed to run the programs are also reported.
Keywords: Alternating direction implicit scheme, Finite difference schemes, Parabolic partial differential equations.

Mathematics Subject Classification: 65M06

1 Introduction

We want to find the approximation solution of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + sin(x)sin(y)e^{-t} - 4, \quad 0 \le t \le T, \quad 0 \le x, y \le 1 \tag{1}$$

with initial condition given by

$$u(x, y, 0) = sin(x)sin(y) + x^2 + y^2, \quad 0 \le x, y \le 1 \tag{2}$$

boundary conditions on the first points of intervals

$$u(x, 0, t) = x^2, \qquad 0 \le t \le T, \quad 0 \le x \le 1, \tag{3}$$

$$u(0, y, t) = y^2, \qquad 0 \le t \le T, \quad 0 \le y \le 1, \tag{4}$$

and boundary conditions on the last points of intervals

$$u(x, 1, t) = sin(1)sin(x)e^{-t} + x^2 + 1, \qquad 0 \le t \le T, \quad 0 \le x \le 1, \tag{5}$$

$$u(1, y, t) = sin(1)sin(y)e^{-t} + y^2 + 1, \qquad 0 \le t \le T, \quad 0 \le y \le 1. \tag{6}$$

Of the numerical approximation methods available for solving partial differential equations employing finite difference methods are more frequently used and more universally applicable than any other. Finite difference methods generally give solutions that are either as accurate as the

data warrant or as accurate as is necessary for the technical purposes for which the solutions are required. In both cases a finite difference solution is as satisfactory as one calculated from an analytical formula[6].
The remaining of this paper is organized as follows. In Section 2, the alternating direction implicit scheme is described. Comparison of the numerical works with the analytical solution is presented in Sections 3. Finally, the conclusions are summarized in Section 4.

2 The alternating direction implicit scheme

We divide the domain $[0,1]^2 \times [0,T]$ into an $M^2 \times N$ mesh with spatial step size $h = \frac{1}{M}$ in both x and y directions and the time step size $k = \frac{T}{N}$. so we have

$$x_i = ih, \quad i = 0,1,\dots,M, \tag{7}$$

$$y_i = jh, \quad j = 0,1,\dots,M, \tag{8}$$

$$t_n = nk, \quad n = 0,1,\dots,N, \tag{9}$$

for grid points (x,y,t), which M is even. We use $u^n_{i,j}$ to denote the finite difference approximations of $u(ih,jh,nk)$ of the Eq. (13).
The process of stepping from time t_n to t_{n+1} is carried out in two stages [2], [3], [5] and [6].
In the first half-time interval of this procedure, we use

$$\frac{\partial u}{\partial t}\,|^n_{i,j} = \frac{u^{n+\frac{1}{2}}_{i,j} - u^n_{i,j}}{\frac{k}{2}}, \tag{10}$$

$$\frac{\partial^2 u}{\partial x^2}\,|^n_{i,j} = \frac{u^{n+\frac{1}{2}}_{i-1,j} - 2u^{n+\frac{1}{2}}_{i,j} + u^{n+\frac{1}{2}}_{i+1,j}}{h^2}, \tag{11}$$

$$\frac{\partial^2 u}{\partial y^2}\,|^n_{i,j} = \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{h^2}. \tag{12}$$

In this half-time, we have

$$-\frac{s}{2}u^{n+\frac{1}{2}}_{i-1,j} + (1+s)u^{n+\frac{1}{2}}_{i,j} - \frac{s}{2}u^{n+\frac{1}{2}}_{i+1,j} = \frac{s}{2}u^n_{i,j-1} + (1-s)u^n_{i,j} + \frac{s}{2}u^n_{i,j+1} \tag{13}$$

for $i = 1,2,\dots M-1$, and for $j = 1,2,\dots M-1$ which $s = \frac{k}{h^2}$.
In the second half-time interval we have

$$\frac{\partial u}{\partial t}\,|^{n+\frac{1}{2}}_{i,j} = \frac{u^{n+1}_{i,j} - u^{n+\frac{1}{2}}_{i,j}}{\frac{k}{2}}, \tag{14}$$

$$\frac{\partial^2 u}{\partial x^2}\,|^{n+\frac{1}{2}}_{i,j} = \frac{u^{n+\frac{1}{2}}_{i-1,j} - 2u^{n+\frac{1}{2}}_{i,j} + u^{n+\frac{1}{2}}_{i+1,j}}{h^2}, \tag{15}$$

$$\frac{\partial^2 u}{\partial y^2}\,|^{n+\frac{1}{2}}_{i,j} = \frac{u^{n+1}_{i,j-1} - 2u^{n+1}_{i,j} + u^{n+1}_{i,j+1}}{h^2}. \tag{16}$$

Therefore we obtain

$$-\frac{s}{2}u^{n+1}_{i,j-1} + (1+s)u^{n+1}_{i,j} - \frac{s}{2}u^{n+1}_{i,j+1} = \frac{s}{2}u^{n+\frac{1}{2}}_{i-1,j} + (1-s)u^{n+\frac{1}{2}}_{i,j} + \frac{s}{2}u^{n+\frac{1}{2}}_{i+1,j} \tag{17}$$

for $i = 1,2,\dots M-1$, and for $j = 1,2,\dots M-1$. Values of $u^{n+1}_{i,j}$ on the boundaries $x = 0,1$ and $y = 0,1$ are given by $(3)-(6)$. At the same time, we divide the inhomogeneous part of the equation $(sin(x)sin(y)e^{-t} - 4)$ in the same mesh points and add them to the right side vector of our linear systems. These systems of linear algebraic equations are strictly diagonally dominant, therefore they are solvable. We solve them by using Thomas algorithm which is very fast. This scheme is unconditionally stable [1], [4].

3 Numerical Tests

The analytical solution of the partial differential equation satisfying (1) − (6) is

$$u(x, y, t) = sin(x)sin(y)e^{-t} + x^2 + y^2. \tag{18}$$

We use the alternating direction implicit scheme for $h = 0.1, T = 1$ and compare it with the analytical solution. The results are obtained and shown in Table 1. Also we report the maximum discretisation error and the elapsed time for running the programs. Inspection of this Table shows that the alternating direction implicit scheme is reasonably accurate and elapsed times are not so much.

s	Error in max norm	Elapsed time (sec)
1	1.8×10^{-3}	0.59
$\frac{1}{2}$	9.1×10^{-4}	1.01
$\frac{1}{4}$	4.6×10^{-4}	2.11
$\frac{1}{8}$	2.4×10^{-4}	4.99
$\frac{1}{16}$	1.2×10^{-4}	11.00
$\frac{1}{32}$	6.9×10^{-5}	32.43

Table 1. $h = 0.1$ and $T = 1$.

We test our programs with other data. In this test $h = 0.05, T = 1$ have been used and the results are shown in Table 2.

s	Error in max norm	Elapsed time (sec)
1	5.8×10^{-4}	12.02
$\frac{1}{2}$	29×10^{-4}	26.34
$\frac{1}{4}$	1.45×10^{-4}	62.72
$\frac{1}{8}$	7.34×10^{-5}	169.67
$\frac{1}{16}$	3.75×10^{-5}	533.34
$\frac{1}{32}$	1.96×10^{-5}	1797.27

Table 2. $h = 0.05$ and $T = 1$.

4 Conclusions

This paper has outlined alternating direction implicit scheme for study of an inhomogeneous two-dimensional parabolic partial differential equation. The systems of linear equations in this scheme are strictly diagonally dominant, so they are solvable. The scheme described in this work is unconditionally stable. As elapsed times of running of the programs have been shown, the use of this scheme is efficient and very fast.

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Application of parametric Spline for solution of boundary value problems

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Abstract
In this article, using parametric spline function approximation methods to obtain numerical solution of two-point singular boundary-value problems. We compare our results with the results produced by spline method. However, it is observed that our approach produce better numerical solutions in the sense that max$|x_i|$is a minimum.
Keywords: Differential equations, Singular boundary value problems, Numerical results.

Mathematics Subject Classification: 65L10

1 Introduction

The spline functions have been used by a number of authors to solve both initial and boundary value problems of ordinary and partial differential equations. The use of cubic spline for the solution of linear two point boundary value problems was first suggested by Bickley [1]. His main idea was to use the condition of continuity as a discretization equation for the linear two point boundary value problems. Leter,Fyfe [2]discussed the application of deferred corrections to the method suggested by Bickled by considering again the cas of (regular)linear boundary value problems. Bogalaev [3] used finite-element framework and achieved uniform first order accuracy at the nodes.

Natesan and Ramanujam [4] incorporated asymptotic approximation into a finite-difference scheme, whereas Mohanty and Navnit Jha [5] have developed compression spline based numerical method for solution of the problem. Aziz and Khan [6] solved this problem by quintic spline method.

In this manuscript the following two-order boundary value problem is consider:

$$\varepsilon \frac{d^2u}{dr^2} + a(r)\frac{du}{dr} + b(r)u = f(r) \tag{1}$$

With boundary conditions

$$u(0) = A, u(1) = B \tag{2}$$

Where $0 < \varepsilon < 1, 0 < r < 1, 0 < k < a(r), k, b(r) > 0$,where A,B are finite real constants.

Definition 1.1. *A function $S_\Delta(x)$ of class $C^2[a,b]$ which interpolates $u(x)$ at the mesh points x_i, depends on a parameter P, and reduces to cubic spline $S_\Delta(x)$, in $[a,b]$ as $P \to 0$, is termed a parametric cubic spline function.Since the parametric P can occur in $S_\Delta(x)$ in many ways, such a spline in not unique. The three parametric cubic spline derived from cubic spline by introducing the parametric in three different ways are termed as cubic spline in compression, cubic spline in tension and adaptive cubic spline.*

2 Numerical methods

If $S_\Delta(x)$ is a parametric cubic spline satisfying the differential equation

$$S_\Delta^{''}(x) + PS_\Delta(x) = [S_\Delta^{''}(x_{i-1}) + PS_\Delta(x_{i-1})]\frac{x_i - x}{h_i} + [S_\Delta^{''}(x_i) + PS_\Delta(x_i)]\frac{x - x_{i-1}}{h_i} \tag{3}$$

Where $x \in [x_{i-1}, x_i], S_\Delta(x_i) = u_i, h_i = x_i - x_{i-1}$and $P > 0$.

Solving the differential equation(3) and using interpolatory conditions at x_i and x_{i-1} to determine the constants of integration , we get after writing $w = h_i\sqrt{P}$,

$$S_\Delta(x) = zu_i + \bar{z}u_{i-1} + h_i^2[q_1(z)M_i + q_1(\bar{z})M_{i-1}]/w^2 \tag{4}$$

Where $z = (x - x_{i-1})/h_i, \bar{z} = 1 - z, q_1(z) = z - \frac{sinwz}{sinw}, q_1(0) = q_1(\mp 1) = 0, q_1(z)$is an odd function of z. For uniform mesh i.e.$h = h_i = h_{i+1}$,we obtain the following spline relations:

$$\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}),$$

$$\alpha m_{i+1} + 2\beta m_i + \alpha m_{i-1} = \frac{(\alpha + \beta)}{h}(u_{i+1} - u_{i-1}),$$

Where

$$m_i = S_\Delta^{'}(x_i), M_i = S_\Delta^{''}(x_i), \alpha = \frac{1}{w^2}(wcscw - 1), \beta = \frac{1}{w^2}(1 - wcotw).$$

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Estimate on solutions of Stokes-Boussinesq system in a tube

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Abstract
Estimates on the solutions for Stokes-Boussinesq system for a reactive flow are obtained, when the domain is an arbitrary tube in $\mathbb{R}^3$, and the data is front-like. We show that the front-like datum admits a solution which will stay front-like in time.
Keywords: Stokes-Boussinesq system, Initial value problem, Reactive-convective flow

Mathematics Subject Classification: 35L67

1 Introduction

The Boussinesq equation for a reactive flow, which describes the flame propagation in a gravitationally stratified medium is given by

$$\theta_t + u \cdot \nabla\theta - \Delta\theta = f(\theta). \tag{1}$$

The fluid velocity u can be prescribed (imposed from outside), hence not affected by the reaction-diffusion process; the more physically relevant approach, however, is to obtain the fluid velocity from some fluid equations with a force term depending on the reactant. In our case we assume the Stokes system

$$-\nu\Delta u + \nabla p = \theta\vec{\rho} \tag{2}$$
$$\operatorname{div} u = 0, \tag{3}$$

where $\nu > 0$ is the Prandtl number (inversely proportional to the Reynold number), $u = (u_1, u_2, u_3)$ is the velocity of an incompressible flow, p its pressure, and θ is the reaction progress (usually interpreted as temperature), normalized so that $0 \le \theta \le 1$.

The nonlinear function $f(\theta)$ is a Lipschitz function of ignition type. That is, there is a threshold temperature $0 < \vartheta_0 < 1$, such that

$$f(\theta) = 0 \quad \text{for} \quad \theta \le \vartheta_0 \quad \text{and } \theta \ge 1, \qquad f(\theta) > 0 \quad \text{on } (\vartheta_0, 1). \tag{4}$$

The vector $\vec{\rho} = \rho\vec{g}$ corresponds to the non-dimensional gravity $\vec{g}$ scaled by the Rayleigh number ρ.

The domain is the tube $D = (-\infty, +\infty) \times \Omega$, where Ω is a connected and bounded domain in $\mathbb{R}^2$. Points in D are presented by $(x, \tilde{x})$, where $x \in \mathbb{R}$ and $\tilde{x} \in \Omega$.

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We assume *Neumann* boundary condition $\frac{\partial\theta}{\partial n} = 0$ for θ, and the *no-slip* boundary condition $u = 0$ for u on the boundary.

Following [BCR06], we define the *bulk burning rate* $\bar{B}(t)$ and the *Nusselt number* $\bar{N}(t)$ by

$$\begin{aligned} B(t) &:= \frac{1}{|\Omega|}\int_D f(\theta)dx d\tilde{x}, & \bar{B}(t) &:= \frac{1}{t}\int_0^t B(s)ds \\ N(t) &:= \frac{1}{|\Omega|}\int_D |\nabla\theta|^2 dx d\tilde{x}, & \bar{N}(t) &:= \frac{1}{t}\int_0^t N(s)ds. \end{aligned} \tag{5}$$

Define also the *average flow* $\bar{U}(t)$ by

$$\bar{U}(t) := \frac{1}{t}\int_0^t ||u(s)||_\infty ds. \tag{6}$$

The *laminar front speed* c_0 is the (unique) number such that the equation

$$-c_0\Phi' = \Phi'' + f(\Phi), \quad \Phi(-\infty) = 1, \quad \Phi(+\infty) = 0 \tag{7}$$

has a solution with the range $0 < \Phi < 1$.

1.1 Results

Theorem 1.1. *Assume the above, and assume that the initial temperature $\theta_0(x, \tilde{x})$ approaches 1 and 0 sufficiently fast, as x approaches $-\infty$ and $+\infty$, respectively. Then there exists a constant C, depending on the domain D and the initial data θ_0, u_0, such that*

$$c_0 - C[\rho + \rho^2] + o(1) \le \bar{B}(t) \le c_0 + C[\rho + \rho^2] + o(1) \tag{8}$$

$$\bar{N}(t) \le \left[C\rho + \sqrt{\frac{c_0}{2} + C^2\rho^2}\right]^2 + o(1) \tag{9}$$

$$\bar{U}(t) \le C[\rho + \rho^2] + o(1) \tag{10}$$

as $t \to +\infty$. c_0 here is the laminar front speed as in (7).

Theorem 1.2. *With the above assumption, the solution would satisfy*

$$\Phi(x - c_0 t + x_1 + \bar{U}(t)t + C\sqrt{t}) - \frac{C}{\sqrt{t}} \le \theta(x, \tilde{x}, t) \le \Phi(x - c_0 t - x_2 + \bar{U}(t)t + C\sqrt{t}) + \frac{C}{\sqrt{t}} \tag{11}$$

with appropriate x_1, x_2.

2 Technical Lemmas

Lemma 2.1. *[BCR06] There exists a constant C_0 depending only on the initial data θ_0 such that*

$$\bar{N}(t) \le \frac{1}{2}\bar{B}(t) + \bar{U}(t) + C_0\left(\frac{1}{t} + \frac{1}{\sqrt{t}}\right) \tag{12}$$

$$\bar{B}(t) \le c_o + \bar{U}(t) + C_0\left(\frac{1}{t} + \frac{1}{\sqrt{t}}\right) \tag{13}$$

Lemma 2.2. *[LM09] There exists a constant C, depending only on the domain Ω, such that, for any solenoidal flow u which vanishes at the boundary of D, and satisfies*

$$-\Delta u + \nabla p = F \tag{14}$$

for some p, we have

$$||u||_\infty \le \frac{\sqrt{2}}{\sqrt{\nu\pi}}||\nabla u||_{L^2}^{\frac{1}{2}}||F||_{L^2}^{\frac{1}{2}} + C||\nabla u||_{L^2} \tag{15}$$

Lemma 2.3. *[LM09] There exists a function h, such that*

$$||\theta\vec{\rho} - \nabla h||_{L^2(D)} \leq \rho C_1 ||\nabla\theta||_{L^2(D)}, \tag{16}$$

with C_1 depending only on Ω.

Theorem 2.4. *There exist $C > 0$ and C_Ω, such that for all $t \geq 1$ we have*

$$\bar{U}(t) \leq C\left(\rho\sqrt{\bar{N}(t)} + \frac{1}{\sqrt{t}}||\nabla u_0||_{L^2}\right). \tag{17}$$

Proof. We sketch the proof here. For details see [MRaoofi]. Multiplying the fluid equation (2) by u and integrating over D, one obtains

$$\frac{1}{2}\frac{d}{dt}||u||^2_{L^2} + \nu||\nabla u||^2_{L^2} = \int_D (\theta\vec{\rho} - \nabla h)\cdot u \tag{18}$$

where h is as of lemma 7. Now using Hölder inequality, followed by Poincaré's inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u||^2_{L^2} + \nu||\nabla u||^2_{L^2} &\leq ||(\theta\vec{\rho} - \nabla h)||_{L^2}||u||_{L^2} \\ &\leq C_P||(\theta\vec{\rho} - \nabla h)||_{L^2}||\nabla u||_{L^2}\end{aligned} \tag{19}$$

Integrating over time and, using the so called Cauchy's inequality with ϵ, and using lemma 7, (19) gives us

$$||u(t)||^2_{L^2} - ||u(0)||^2_{L^2} + \nu\int_0^t ||\nabla u||^2_{L^2} \leq \frac{C_P^2C_1^2\rho^2}{4\epsilon}\int_0^t ||\nabla\theta||^2_{L^2} + \epsilon\int_0^t ||u||^2_{L^2}. \tag{20}$$

Using again Poincaré's inequality, and using $\epsilon = \frac{\nu}{2}$, and ignoring the first term in above inequality, we arrive at

$$\int_0^t ||\nabla u||^2_{L^2}dt \leq \left(\frac{C_P^2C_1^2\rho^2}{\nu^2}\int_0^t ||\nabla\theta||^2_{L^2}dt + \frac{2C_P^2}{\nu}||\nabla u(0)||^2_{L^2}\right) \tag{21}$$

2. Again Multiplying equation (2) by u_t and integrating over D, we obtain

$$||u_t||^2_{L^2} + \frac{\nu}{2}\frac{d}{dt}||\nabla u||^2_{L^2} = \int_D (\theta\vec{\rho} - \nabla h)\cdot u_t. \tag{22}$$

Integrating over time, we obtain in a similar fashion

$$\begin{aligned}&\int_0^t ||u_t||^2_{L^2} + \frac{\nu}{2}\left(||\nabla u(t)||^2_{L^2} - ||\nabla u(0)||^2_{L^2}\right) \\ &\qquad\leq \frac{C\rho^2}{4\epsilon}\int_0^t ||\nabla\theta||^2_{L^2} + \epsilon\int_0^t ||u_t||^2_{L^2},\end{aligned} \tag{23}$$

which gives us

$$\int_0^t ||u_t||^2_{L^2} \leq C\left(\rho^2\int_0^t ||\nabla\theta||^2_{L^2}dt + ||\nabla u(0)||^2_{L^2}\right) \tag{24}$$

3. Now we invoke lemma 2.2 to obtain

$$\begin{aligned}||u||_\infty \leq& C\left[(||u_t||^{\frac{1}{2}}_{L^2} + ||\theta\vec{\rho} - \nabla h||^{\frac{1}{2}}_{L^2})||\nabla u||^{\frac{1}{2}}_{L^2} + ||\nabla u||_{L^2}\right] \\ \leq& C\left(||u_t||_{L^2} + ||\theta\vec{\rho} - \nabla h||_{L^2} + ||\nabla u||_{L^2}\right).\end{aligned} \tag{25}$$

This gives us (17).

□

Proof of theorem 1. It would be a result of Lemma 2.1 and Theorem 2.4 after some calculations ([BCR06]).

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A representation of the general common solution to a system of real quaternion matrix equations with applications

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Abstract

This paper studies the system of linear real quaternion matrix equations $A_1X = C_1,\ A_2X = C_2,\ A_3X = C_3,\ A_4X = C_4,\ A_5XB_5 = C_5,\ A_6XB_6 = C_6$. We present some necessary and sufficient conditions for the existence of a solution to this system and give an expression of the general solution to the system when the solvability conditions are satisfied. As an application, centrosymmetric solution to a system of quaternion matrix equations are presented. Moreover, we give a numerical example to illustrate our results.

Keywords: Matrix equation, Quaternion field, Centrosymmetric matrix

Mathematics Subject Classification: 15A24, 15B33, 15A09

1 Introduction

It is known that linear matrix equations have been one of the main topics in matrix theory and its applications. The primary work in the investigation of a matrix equation (system) is to give solvability conditions and general solutions to the equation(s).

Throughout this paper, let $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}^{m\times n}$ stand, respectively, for the real number field, the complex number field, and the set of all $m \times n$ matrices over the real quaternion algebra

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

we denote the reflexive inverse of a matrix A by A^+, which satisfies both the conditions $AA^+A = A$ and $A^+AA^+ = A^+$. The matrices $L_A = I - A^+A$ and $R_A = I - AA^+$ are the orthogonal projectors induced by matrix A, where A^+ is any but fixed reflexive inverse of A, and I is the identity matrix of appropriate size.

Quaternions made further appearance ever since inassociative algebras, analysis, topology, and physics. Nowadays quaternion matrices play an important role in computer science, quantum physics, signal and color image processing, and so on [1, 2].

Many authors have studied the system of matrix equations,

$$\begin{cases} A_3XB_3 = C_3, \\ A_4XB_4 = C_4, \end{cases}$$

over the complex field [3],[4].

Motivated by the work mentioned above, we in this paper study the system

$$A_1X = C_1, \quad A_2X = C_2, \quad A_3X = C_3, \quad A_4X = C_4, \quad A_5XB_5 = C_5, \quad A_6XB_6 = C_6, \tag{1}$$

This paper is organized as follows. In Section 2, we derive necessary and sufficient conditions for the existence and an expression of the general solution to system (1). As applications, we in Section 3 give necessary and sufficient conditions for the existence of and expressions for centrosymmetric solutions to the system

$$A_1X = C_1, \quad A_2X = C_2, \quad A_5XB_5 = C_5. \tag{2}$$

2 The general solution to system (1)

Lemma 2.1. *let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{r\times s}$, $C \in \mathbb{H}^{m\times s}$ be known and $X \in \mathbb{H}^{n\times r}$ unknown. Then the matrix equation*

$$AXB = C, \tag{3}$$

is consistent if and only if

$$AA^+CB^+B = C.$$

In that case, the general solution of matrix equation (3) is

$$X = A^+CB^+ + L_AV + UR_B, \tag{4}$$

where U and V are any matrices with compatible dimensions over $\mathbb{H}$.

Lemma 2.2. *let $A_1 \in \mathbb{H}^{m\times n}$, $C_1 \in \mathbb{H}^{m\times r}$, $A_2 \in \mathbb{H}^{s\times n}$, $C_2 \in \mathbb{H}^{s\times r}$ be known and $X \in \mathbb{H}^{n\times r}$ unknown, $S = A_2L_{A_1}$, $G = R_SA_2$. Then the system*

$$\begin{cases} A_1X = C_1, \\ A_2X = C_2, \end{cases}$$

is consistent if and only if

$$A_iA_i^+C_i = C_i, \quad i = 1, 2; \qquad G(A_2^+C_2 - A_1^+C_1) = 0.$$

In that case, the general solution of the system is

$$X = A_1^+C_1 + L_{A_1}S^+A_2(A_2^+C_2 - A_1^+C_1) + L_{A_1}L_SY, \tag{5}$$

where Y is an arbitary matrix with compatible dimension over $\mathbb{H}$.

Now we give the fundamental theorem of this paper.

Theorem 2.3. *let $A_1 \in \mathbb{H}^{m\times n}$, $A_2 \in \mathbb{H}^{s\times n}$, $A_3 \in \mathbb{H}^{t\times n}$, $A_4 \in \mathbb{H}^{k\times n}$, $A_5 \in \mathbb{H}^{q\times n}$, $A_6 \in \mathbb{H}^{l\times n}$, $B_5 \in \mathbb{H}^{r\times p}$, $B_6 \in \mathbb{H}^{r\times u}$, $C_1 \in \mathbb{H}^{m\times r}$, $C_2 \in \mathbb{H}^{s\times r}$, $C_3 \in \mathbb{H}^{t\times r}$, $C_4 \in \mathbb{H}^{k\times r}$, $C_5 \in \mathbb{H}^{q\times p}$, $C_6 \in \mathbb{H}^{l\times u}$ be known and $X \in \mathbb{H}^{n\times r}$ unknown and $S = A_2L_{A_1}$, $A = A_3L_{A_1}$, $B = AL_S$, $C = A_4L_{A_1}$, $R = CL_SL_B$, $P = A_5L_{A_1}$, $N = PL_sL_BL_R$, $M = A_6L_{A_1}$, $Q = ML_SL_BL_R$, $G = QL_N$, $H = R_{B_5}B_6$,*

$$\begin{aligned}
\Psi &= C_3 - A_3A_1^+C_1 - A_3L_{A_1}S^+A_2(A_2^+C_2 - A_1^+C_1), && (6)\\
\Phi &= C_4 - A_4A_1^+C_1 - CS^+A_2(A_2^+C_2 - A_1^+C_1) - CL_SB^+\Psi, && (7)\\
E &= C_5 - A_5A_1^+C_1B_5 - PS^+A_2(A_2^+C_2 - A_1^+C_1)B_5 - PL_SB^+\Psi B_5 && (8)\\
&\quad -PL_SL_BR^+\Phi B_5, \\
F &= C_6 - A_6A_1^+C_1B_6 - MS^+A_2(A_2^+C_2 - A_1^+C_1)B_6 - ML_SB^+\Psi B_6 && (9)\\
&\quad -ML_SL_BR^+\Phi B_6 - QN^+EB_5^+B_6,
\end{aligned}$$

Then the system (1) is consistent if and only if

$$\begin{aligned}
&SS^+A_2(A_2^+C_2 - A_1^+C_1) = A_2^+C_2 - A_1^+C_1, \\
&R_{A_1}C_1 = 0, \quad R_{A_2}C_2 = 0, \quad R_{A_3}C_3 = 0, \quad R_{A_4}C_4 = 0, \\
&R_{A_5}C_5 = 0, \quad C_5L_{B_5} = 0, \quad R_{A_6}C_6 = 0, \quad C_6L_{B_6} = 0, \\
&R_B\Psi = 0, \quad R_R\Phi = 0, \quad R_NE = 0, \quad EL_B = 0, \\
&FL_{B_6} = 0, \quad R_QF = 0, \quad R_GFL_H = 0.
\end{aligned} \tag{10}$$

In that case, the general solution of the system (3) can be expressed as follows:

$$\begin{aligned}
X \;=\;& A_1^+C_1 + L_{A_1}S^+A_2(A_2^+C_2 - A_1^+C_1) + L_{A_1}L_sB^+\Psi + L_{A_1}L_sL_BR^+\Phi \\
&+L_{A_1}L_SL_BL_RN^+EB_5^+ + L_{A_1}L_SL_BL_RL_NG^+FB_6^+ \\
&-L_{A_1}L_SL_BL_RL_NG^+Q(R_GQ)^+R_GFH^+HB_6^+ + L_{A_1}L_SL_BL_R(R_GQ)^+R_GFH^+R_{B_5} \\
&+L_{A_1}L_SL_BL_RL_NZ - L_{A_1}L_SL_BL_RL_NG^+GZB_6B_6^+ + L_{A_1}L_SL_BL_RWR_{B_5} \\
&-L_{A_1}L_SL_BL_RL_NG^+QL_{(R_GQ)}WHB_6^+ - L_{A_1}L_SL_BL_R(R_GQ)^+R_GQWHH^+R_{B_5},
\end{aligned}$$

where Z, W are arbitrary matrices with compatible dimension over $\mathbb{H}$.

3 Various symmetric solutions to system (2)

Definition 3.1. *Let $A = (a_{ij}) \in \mathbb{H}^{m\times n}$, $A^{\#} = (a_{m-i+1,n-j+1}) \in \mathbb{H}^{m\times n}$. Then $A^{\#} = V_mAV_n$ where V_n is the $n\times n$ backward identity matrix.*
The matrix $A = (a_{ij}) \in \mathbb{H}^{m\times n}$ is called centrosymmetric if $a_{ij} = a_{m-i+1,n-j+1}$ i.e., $A = A^{\#}$.

Theorem 3.2. *let $A_1 \in \mathbb{H}^{m\times n}$, $A_2 \in \mathbb{H}^{t\times n}$, $A_5 \in \mathbb{H}^{q\times n}$, $B_5 \in \mathbb{H}^{r\times p}$, $C_1 \in \mathbb{H}^{m\times r}$, $C_2 \in \mathbb{H}^{t\times r}$, $C_5 \in \mathbb{H}^{q\times p}$, be known and $X \in \mathbb{H}^{n\times r}$ unknown and $S = A_1^{\#}L_{A_1}$, $A = A_2L_{A_1}$, $B = AL_S$, $C = A_2^{\#}L_{A_1}$, $R = CL_SL_B$, $P = A_5L_{A_1}$, $N = PL_sL_BL_R$, $M = A_5^{\#}L_{A_1}$, $Q = ML_SL_BL_R$, $G = QL_N$, $H = R_{B_5}B_5^{\#}$,*

$$\begin{aligned}
\Psi &= C_2 - A_2A_1^+C_1 - A_2L_{A_1}S^+A_1^{\#}((A_1^{\#})^+C_1^{\#} - A_1^+C_1), \\
\Phi &= C_2^{\#} - A_2^{\#}A_1^+C_1 - CS^+A_1^{\#}((A_1^{\#})^+C_1^{\#} - A_1^+C_1) - CL_SB^+\Psi, \\
E &= C_5 - A_5A_1^+C_1B_5 - PS^+A_1^{\#}((A_1^{\#})^+C_1^{\#} - A_1^+C_1)B_5 - PL_SB^+\Psi B_5 \\
&\quad -PL_SL_BR^+\Phi B_5, \\
F &= C_5^{\#} - A_5^{\#}A_1^+C_1B_5^{\#} - MS^+A_1^{\#}((A_1^{\#})^+C_1^{\#} - A_1^+C_1)B_5^{\#} - ML_SB^+\Psi B_5^{\#} \\
&\quad -ML_SL_BR^+\Phi B_5^{\#} - QN^+EB_5^+B_5^{\#},
\end{aligned}$$

Then the system (2) is consistent if and only if

$$\begin{aligned}
&SS^+A_1^{\#}((A_1^{\#})^+C_1^{\#} - A_1^+C_1) = (A_1^{\#})^+C_1^{\#} - A_1^+C_1, \\
&R_{A_1}C_1 = 0, \quad R_{A_1^{\#}}C_1^{\#} = 0, \quad R_{A_2}C_2 = 0, \quad R_{A_2^{\#}}C_2^{\#} = 0, \\
&R_{A_5}C_5 = 0, \quad C_5L_{B_5} = 0, \quad R_{A_5^{\#}}C_5^{\#} = 0, \quad C_5^{\#}L_{B_5^{\#}} = 0, \\
&R_B\Psi = 0, \quad R_R\Phi = 0, \quad R_NE = 0, \quad EL_B = 0, \\
&FL_{B_5^{\#}} = 0, \quad R_QF = 0, \quad R_GFL_H = 0.
\end{aligned}$$

In that case, the centrosymmetric solution of system (2) can be expressed as

$X = \frac{1}{2}(X_2 + X_2^{\#})$, *where*

$$\begin{aligned} X_2 &= A_1^+C_1 + L_{A_1}S^+A_1^{\#}((A_1^{\#})^+C_1^{\#} - A_1^+C_1) + L_{A_1}L_sB^+\Psi + L_{A_1}L_sL_BR^+\Phi \\ &+ L_{A_1}L_SL_BL_RN^+EB_5^+ + L_{A_1}L_SL_BL_RL_NG^+F(B_5^{\#})^+ + L_{A_1}L_SL_BL_RL_NZ \\ &- L_{A_1}L_SL_BL_RL_NG^+Q(R_GQ)^+R_GFH^+H(B_5^{\#})^+ + L_{A_1}L_SL_BL_RWR_{B_5} \\ &+ L_{A_1}L_SL_BL_R(R_GQ)^+R_GFH^+R_{B_5} - L_{A_1}L_SL_BL_RL_NG^+QL_{(R_GQ)}WH(B_5^{\#})^+ \\ &- L_{A_1}L_SL_BL_R(R_GQ)^+R_GQWHH^+R_{B_5} - L_{A_1}L_SL_BL_RL_NG^+GZB_5^{\#}(B_5^{\#})^+ \end{aligned}$$

where Z, W are arbitary matrices with compatible dimension over $\mathbb{H}$.

4 Main Result

We have derived some new necessary and sufficient conditions for the existence and the expression of the general solution to system (1) over $\mathbb{H}$. Using the results on system (1), we have established necessary and sufficient conditions for the existence of a centrosymmetric solution to system (2) over $\mathbb{H}$.

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Finite element solution of well-known Hirota-Satsuma coupled MKdV equation

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Abstract

In this paper the finite element method based on quintic B-splines have been investigated for solving numerically the Hirota-Satsuma coupled MKdV equation. Accuracy of the proposed method is shown numerically by calculating conservation laws, L_2 and L_∞ norms on studying of a soliton solution. It is shown that the collocation scheme for solutions of the MKdV equation gives rise to smaller errors and is quite easy to implement. Numerical experiments support these theoretical results.

Mathematics Subject Classification: 65M06, 65M12

1 Introduction

The effort in finding exact and numerical solutions to a nonlinear equation is important for the understanding of most nonlinear physical phenomena. In this paper, we consider a generalized Hirota-Satsuma coupled MKdV equations which were introduced by Wu et al. In [1], the authors introduced a 4×4 matrix spectral problem with three potentials and proposed a corresponding hierarchy of nonlinear equations. The coupled MKdV system is very complicated and not easy to solve by direct integration method and the homogeneous balance method. The generalized Hirota-Satsuma coupled KdV and coupled MKdV equations have been studied by many authors via different approaches [1-5]. Recently, Fan ([2]) has provided a suggestion to construct soliton solutions for these equation by using an extended tanh-function method and symbolic computation.Solitary solutions for various nonlinear wave equations have been investigated using different methods which can only solve special kind of nonlinear problems due to the limitations or shortcomings in the methods.

2 Problem Definition

In this paper we consider the well known Hirota-Satsuma coupled MKdV equation as

$$\begin{aligned}
&u_t - \frac{1}{2}u_{xxx} + 3u^2u_x - \frac{3}{2}v_{xx} - 3(uv)_x + 3\lambda u_x = 0,\\
&v_t + v_{xxx} + 3vv_x + 3u_xv_x - 3u^2v_x - 3\lambda v_x = 0,
\end{aligned} \tag{1}$$

with the following boundary and initial conditions

$$\begin{aligned}
&u(a,t) = \beta_1, u(b,t) = \beta_2, v(a,t) = 0, v(b,t) = 0,\\
&u_x(a,t) = 0, u_x(b,t) = 0, v_x(a,t) = 0, v_x(b,t) = 0, \qquad t \in (0,T],\\
&u(x,0) = f(x), \quad v(x,0) = -g(x), \quad a \le x \le b,
\end{aligned} \tag{2}$$

where λ is positive parameter and subscripts x and t denote differentiation and $f(x)$ and $g(x)$ are localized disturbance inside the considered interval and will be chosen later.

In sequence a numerical scheme based on finite element method is proposed to solve the problem (3)-(4).

3 Finite Element Solution

In this section a numerical solution of (3) will be derived by using the collocation method based on quintic B-splines. Collocation approximation can be expressed for $u(x,t)$ and $v(x,t)$ in terms of element parameters δ_m and σ_m, respectively, and quadratic B-splines $Q_m(x)$, $m = -2, \ldots, N+2$, as

$$u_N(x,t) = \sum_{m=-2}^{N+2} \delta_m(t) Q_m(x), \qquad v_N(x,t) = \sum_{m=-2}^{N+2} \sigma_m(t) Q_m(x), \tag{3}$$

where Q_m is the quintic B-splines and δ_m and σ_m are time-dependent parameters to be determined from the quintic B-spline collocation form of the Hirota-Satsuma equation. First we partition the interval $[a,b]$ as $a = x_0 < x_1 < \ldots < x_N = b$, $\quad h = x_m - x_{m-1}$, $\quad m = 1, 2, \ldots, N$. At the nodes x_m, the quintic B-splines Q_m, $m = -2, \ldots, N+2$, are defined by equations (5). For quintic B-splines near end boundaries, it is necessary to introduce 10-additional knots outside the solution domain to provide the support for the quintic B-spline functions, positioned at

$$x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} \quad \textit{and} \quad x_{N+1} < x_{N+2} < x_{N+3} < x_{N+4} < x_{N+5}.$$

The set of quintic B-splines $Q_{-2}, Q_{-1}, \ldots, Q_{N+2}$ forms a basis over the problem domain $[a,b]$, [6].

$$Q_m(x) = \frac{1}{h^5} \begin{cases} (x - x_{m-3})^5, & [x_{m-3}, x_{m-2}] \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 & [x_{m-2}, x_{m-1}] \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5 & [x_{m-1}, x_m] \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 & \\ +15(x - x_{m-1})^5 - 20(x - x_m)^5 & [x_m, x_{m+1}] \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5 & \\ -20(x - x_m)^5 + 15(x - x_{m+1})^5 & [x_{m+1}, x_{m+2}] \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5 & \\ -20(x - x_m)^5 + 15(x - x_{m+1})^5 + 6(x - x_{m+2})^5 & [x_{m+2}, x_{m+3}] \\ 0 & x < x_{m-3}, x_{m+3} < x \end{cases} \tag{4}$$

By using the approximation (3) and quintic B-splines (5), the nodal value u and v and its first and second derivatives $u^{'}$, $u^{''}$, $v^{'}$ and $v^{''}$ at the nodes x_m, $\quad m = -2, \ldots, N+2$, are obtained in terms of the element parameters. Finally substituting the collocation approximations in the system (3) and its evaluation at the knots give a nonlinear system of differential equations as

$$
\begin{aligned}
&(\delta^0_{m-2}+26\delta^0_{m-1}+66\delta^0_m+26\delta^0_{m+1}+\delta^0_{m+2})-\frac{30}{h^3}(\delta_{m+2}-2\delta_{m+1}+2\delta_{m-1}-\delta_{m-2})\\
&+\frac{15}{h^2}d^2_m(\delta_{m+2}+10\delta_{m+1}-10\delta_{m-1}-\delta_{m-2})-\frac{30}{h^2}(\sigma_{m-2}+2\sigma_{m-1}-6\sigma_m+2\sigma_{m+1}\\
&+\sigma_{m+2})-\frac{15}{h}v_m(\delta_{m+2}+10\delta_{m+1}-10\delta_{m-1}-\delta_{m-2})-\frac{15}{h}u_m(\sigma_{m+2}+10\sigma_{m+1}-10\\
&\sigma_{m-1}-\sigma_{m-2})+\frac{15}{h}\lambda(\delta_{m+2}+10\delta_{m+1}-10\delta_{m-1}-\delta_{m-2})=0,\\
&(\sigma^0_{m-2}+26\sigma^0_{m-1}+66\sigma^0_m+26\sigma^0_{m+1}+\sigma^0_{m+2})+\frac{60}{h^3}(\sigma_{m+2}-2\sigma_{m+1}+2\sigma_{m-1}-\\
&\sigma_{m-2})+\frac{15}{h}v'_m(\sigma_{m-2}+26\sigma_{m-1}+66\sigma_m+26\sigma_{m+1}+\sigma_{m+2})+\frac{75}{h^2}v'_m(\delta_{m+2}+10\delta_{m+1}\\
&-10\delta_{m-1}-\delta_{m-2})-\frac{15}{h}u^2_m(\sigma_{m+2}+10\sigma_{m+1}-10\sigma_{m-1}-\sigma_{m-2})-\frac{15}{h}\lambda(\sigma_{m+2}+10\\
&\sigma_{m+1}-10\sigma_{m-1}-\sigma_{m-2})=0.
\end{aligned}
\tag{5}
$$

where ^{0} denotes differentiation with respect to time. Replacing the time derivative of the parameter δ^0 and σ^0 by usual finite difference approximation $\delta^0=\dfrac{\delta^{n+1}-\delta^n}{\Delta t}$ and $\sigma^0=\dfrac{\sigma^{n+1}-\sigma^n}{\Delta t}$ and parameter δ and σ by the Crank-Nicolson formulation $\delta=\dfrac{\delta^{n+1}+\delta^n}{2}$ and $\sigma=\dfrac{\sigma^{n+1}+\sigma^n}{2}$, gives nonlinear recurrence relationship for time parameters. The numerical results of the Hirota-Satsuma equation for two test problems are derived. Accuracy of the proposed numerical method will be measured with discrete L_{2v} and $L_{\infty v}$, L_{2u} and $L_{\infty u}$ error norms. The resulting system consists of the $(2N+2)$ equations with the $(2N+2)$ unknown parameters. The elimination of parameters δ^{n+1}_{N+2} , δ^{n+1}_{N+1}, δ^{n+1}_{-2}, δ^{n+1}_{-1}, σ^{n+1}_{N+2}, σ^{n+1}_{N+1}, σ^{n+1}_{-2} and σ^{n+1}_{-1} from this system, using the boundary conditions $u(a,t)=\beta_1$, $u(b,t)=\beta_2$,$v(a,t)=v(b,t)=0$, $u_x(a,t)=u_x(b,t)=0$ and $v_x(a,t)=v_x(b,t)=0$ enables one to get a solvable the proposed $(2N+2)(2N+2)$ matrix system. The resulting pentadiagonal matrix system is easily and efficiently solved with a variant of the Thomas algorithms [1-3].

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Homotopy perturbation method to obtain solution of the fractional Sharma-Tasso-Olver equation

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Abstract

In this work, solution of the fractional Sharma-Tasso-Olver (FSTO) differential equation was investigated. The present study introduced a novel- and simple analytical method to obtain the solution of (FSTO) differential equation. In this approach, this solution was considered as a Taylor series expansion that converges rapidly to the nonlinear problem. The fractional derivatives were taken in the Caputo sense. Numerical results show that the (HPM) was relatively easy, to obtain the solution of (FSTO) differential equation.

Keywords: Sharma-Tasso-Olver equation, Homotopy perturbation method, Analytical approximate solution.

Mathematics Subject Classification: 35R11, 34A08, 35A20

1 Introduction

The HPM is a new approach for finding approximate analytical solutions for linear and non-linear problems. The method was first proposed by He, and was successfully applied to solve non-linear wave equation by He, fractional Zakharov-Kuznetsov equations by Yildirim and Gulkanat[1], fractional Fornberg-Whitham equation by Parveen Kumar Gupta and Mithilesh Singh[2], Riccati equation with fractional order by Alam Khan, Ara and Jamil[3], etc. In this paper, solution of the fractional Sharma-Tasso-Olver Equation was investigated by the Homotopy Peryurbation Method. This equation can be written as:

$$u_t^{\mu} + a(u^3)_x + \frac{3}{2}a(u^2)_{xx} + au_{xxx} = 0, \tag{1}$$

subject to the initial condition:

$$u(x,0) = 2k\frac{e^{kx+w}}{e^{kx+w}+r} - k, \tag{2}$$

where a is a real parameter, $u(x,t)$ is a unknown function, μ is a constant and lies in the interval $(0,1]$, t and x are time and spatial coordinates, respectively, k, w and r are free parameters.
The exact solution of (1) is[4]:

$$u(x,t) = 2k\frac{e^{kx-ak^3t+w}}{e^{kx-ak^3t+w}+r} - k. \tag{3}$$

2 Basic Definitions and Notations of the Fractional Calculus

In this section, some definitions and properties of the fractional calculus that will be used in this work are presented.

Definition 2.1. *The Gamma function is intrinsically tied in fractional calculus. The simplest interpretation of the gamma function is simply the generalization of the fractional for all real numbers. The definition of the gamma function is given by:*

$$\Gamma(\mu) = \int_0^\infty e^{-\xi}\xi^{\mu-1}\,d\xi. \tag{4}$$

Definition 2.2. *The Riemann-Liouville fractional integral operator J^μ of order μ on the usual Lebesgue space $L_1[a,b]$ is given by:*

$$J^\mu f(x) = \frac{1}{\Gamma(\mu)}\int_0^x (x-\xi)^{\mu-1} f(\xi)\,d\xi. \tag{5}$$

It has the following properties:

- $J^0 f(x) = f(x)$,
- $J^\mu (x-\sigma)^\gamma = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\gamma+1)}(x-\sigma)^{\mu+\gamma}, \quad \mu \geq 0, \quad \gamma > -1, \quad \sigma \in \mathbb{C}.$

Definition 2.3. *Let $f \in L_1[a,b]$, $m \in \mathbb{N}\cup\{0\}$, then the Caputo fractional derivative D_*^μ of $f(x)$ is defined as:*

$$\mathrm{D}_*^\mu f(x) = \begin{cases} J^{m-\mu} D_*^\mu f(x) = \frac{1}{\Gamma(m-\mu)}\int_0^x (x-\xi)^{m-\mu-1} f(\xi)\,d\xi & m-1<\mu<m, \\ \frac{d^m}{dx^m} f(x) & \mu = m. \end{cases} \tag{6}$$

It has the following basic properties:

- $D_*^{\frac{1}{2}} f(x) = J^{\frac{1}{2}} f'(x)$,
- $D_*^\mu (x-\sigma)^\gamma = \frac{\Gamma(\mu+1)}{\Gamma(\gamma-\mu+1)}(x-\sigma)^{\gamma-\mu}, \quad \mu \geq 0, \quad \gamma > -1, \quad \sigma \in \mathbb{C}.$

3 Solution of the Problem

To illustrate a basic concept of homotopy perturbation method, the Sharma-Tasso-Olver equation is considered as:

$$D_t^\mu u + aD_x u^3 + \frac{3}{2}aD_{xx}u^2 + aD_{xxx}u = 0, \tag{7}$$

with the initial condition (2). According to the HPM, the following HPM is presented:

$$(1-p)D_t^\mu u + p(D_t^\mu u + aD_x u^3 + \frac{3}{2}aD_{xx}u^2 + aD_{xxx}u) = 0. \tag{8}$$

Assume the solution of Eq. (8) as a power series in the form of:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + p^4u_4 + \ldots. \tag{9}$$

substituting (9) in (8) and equating the coefficients of like power of p, consequently, after solving these equations:

$u_0(x,t) = 2k\frac{e^{kx+w}}{e^{kx+w}+r} - k,$

$u_1(x,t) = -2ak^4 \frac{e^{kx+w} rt^{\mu}}{(e^{kx+w}+r)^2\Gamma(\mu+1)}$,
$u_2(x,t) = -2a^2k^7 \frac{e^{kx+w} rt^{2\mu}(e^{kx+w}-r)}{(e^{kx+w}+r)^3\Gamma(2\mu+1)}$.

The solution in a series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots . \tag{10}$$

4 Numerical Results

In this section, the space-and time fractional Sharma-Tasso-Olver equation are considered for numerical comparisons. In order to verify numerically whether the proposed methodology leads to higher accuracy, the approximate solution was evaluated using n-term approximation. In Fig. 1 $u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + u_5$ and the exact solution were drawn for $\mu = 1$, $k = 1$, $t = 0.1$, $a = 1$, $r = 1$, $w = \frac{1}{2}$. Fig. 2 shows the approximate solutions which were obtained for different values of μ and $k = 1$, $t = 0.1$, $a = 1$, $r = 1$, $w = \frac{1}{2}$ by using the HPM.

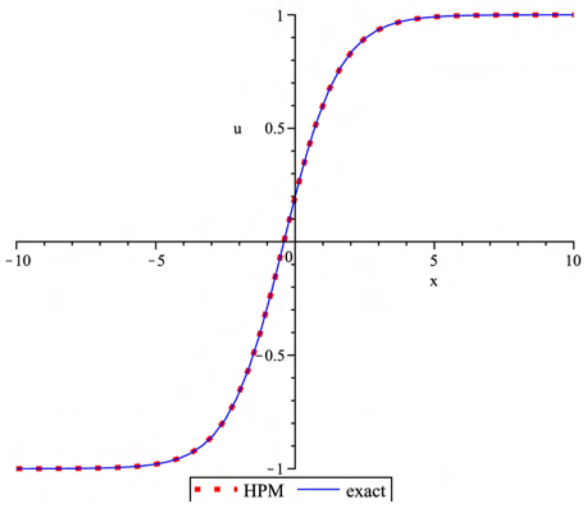

Figure 1: Comparing between HPM solution and the exact solution for $\mu = 1$.

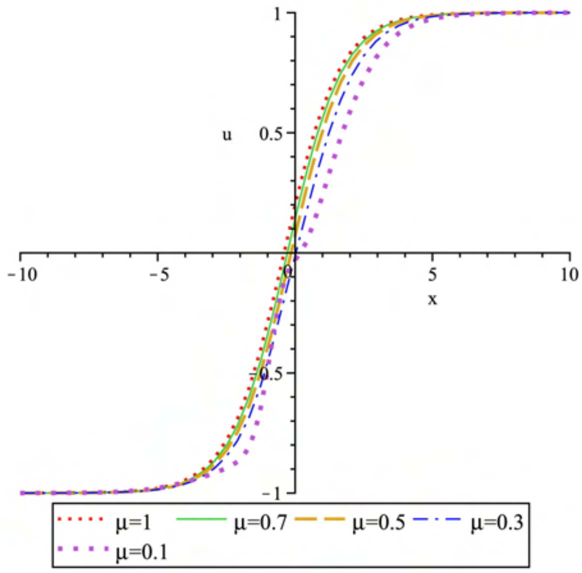

Figure 2: Plot solutions of equation (1) when $\mu = 1$, $\mu = 0.7$, $\mu = 0.5$, $\mu = 0.3$, $\mu = 0.1$, for $t = 0.1$

5 Discussion and Conclusions

In this paper, the homotopy perturbation method (HPM) was successfully employed to obtain the approximate solution of the fractional Sharma-Tasso-Olver equation. The numerical results showed that this method had very good accuracy and reduced the size of calculations compared with other perturbation techniques. It may be concluded that homotopy methodology was very powerful and efficient in finding analytical and numerical solutions for wide classes of linear and non-linear fractional differential equations.

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The Sinc-collocation method for solving a problem arising in chemical reactor theory

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Abstract
The Sinc-collocation method is presented for solving nonlinear two-point boundary value problems for second order differential equations with applications to chemical reactor theory. Some properties of the Sinc-collocation method required for our subsequent development are given and are utilized to reduce the computation of solution of these problems to some algebraic equations. The method is computationally attractive and applications are demonstrated through an illustrative example.

Keywords: Sinc function, Collocation method, Nonlinear boundary value problems, Chemical reactor.

Mathematics Subject Classification: 65L10, 65L60, 34B15

1 Introduction

Consider the mathematical model for an adiabatic tubular chemical reactor which processes an irreversible exothermic chemical reaction. For steady state solutions, this model can be reduced to

$$y'' - \lambda y' + \lambda\mu(\beta - \alpha)\exp(y) = 0, \tag{1}$$

with boundary conditions

$$\lambda y(0) - y'(0) = 0, \qquad y'(1) = 0. \tag{2}$$

The unknown y represents the steady state temperature of the reaction, and the parameters λ, μ and β represent the Peclet number, the Damkohler number and the dimensionless adiabatic temperature rise, respectively. This problem has been studied by several authors [1, 2, 3]. In [1], by using Green's function technique, the problem is converted into a Hammerstein integral equation and then the solution is obtained by using Adomian's method. Also in [3] a Chebyshev finite difference method is used for solving this problem. In this paper, a new approach of the solution based on sinc-collacation method, is used to approximate solution of the problem (13)-(2).

2 Sinc function properties

Sinc function properties are discussed thoroughly in [4]. The sinc function is defined on the whole real line, by $\mathrm{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. For any $h > 0$, the translated sinc functions with evenly spaced nodes are given by $S(j,h)(x) = \mathrm{Sinc}\left(\frac{x-jh}{h}\right)$. To construct an approximation on the interval (a,b), we consider the conformal map $\phi(z) = \ln\left(\frac{z-a}{b-z}\right)$. The map ϕ carries the eye-shaped region $D_E = \left\{z \in \mathbb{C} : \left|arg\left(\frac{z-a}{b-z}\right)\right| < d \leq \frac{\pi}{2}\right\}$ onto the infinite strip in the complex plane $D_S = \left\{w = u + iv : |v| < d \leq \frac{\pi}{2}\right\}$. For the sinc method, the basis functions on the interval (a,b) for $z \in D_E$ are derived from the composite translated sinc functions, $S_j(x) = S(j,h) \circ \phi(z) = \text{Sinc}\left(\frac{\phi(z)-jh}{h}\right)$. The function $z = \phi^{-1}(w) = (a+b\,e^w)/(1+e^w)$, is an inverse mapping of $w = \phi(z)$. We define the range of ϕ^{-1} on the real line as $\Gamma = \left\{\phi^{-1}(u) \in D_E : -\infty < u < \infty\right\}$. The sinc grid points $z_j \in (a,b)$ in D_E will be denoted by x_j because they are real. For the evenly spaced nodes $\{jh\}_{j=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_j = \phi^{-1}(jh) = \frac{a + b\,e^{jh}}{1 + e^{jh}}, \quad j = 0, \pm1, \pm2, ... \tag{3}$$

Definition 1. Let $L_\alpha(D_E)$ be the set of all analytic functions, for which there exists a constant, C, such that

$$| F(z) | \leq C \frac{| \rho(z) |^\alpha}{[1+ | \rho(z) |]^{2\alpha}}, \quad z \in D_E, \quad 0 < \alpha \leqslant 1, \tag{4}$$

where $\rho(z) = e^{\phi(z)}$.
Theorem 1. Let $F \in L_\alpha(D_E)$, let N be a positive integer, and let $h = \left(\frac{\pi d}{\alpha N}\right)$ then there exists positive constant c_1, independent of N, such that

$$\sup_{z\in\Gamma}\left|F(z) - \sum_{j=-N}^{N} F(z_j)S(j,h) \circ \phi(z)\right| \leq c_1 e^{-(\pi d \alpha N)^{1/2}}. \tag{5}$$

We also require derivatives of composite sinc functions evaluated at the nodes.

$$\delta_{ij}^{(n)} = \frac{d^n}{d\phi^n}[S(i,h) \circ \phi(x)]|_{x=x_j}. \tag{6}$$

In particular [4]

$$\delta_{ij}^{(0)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}, \quad \delta_{ij}^{(1)} = \frac{1}{h}\begin{cases} 0, & i = j, \\ \frac{(-1)^{j-i}}{j-i}, & i \neq j. \end{cases}, \quad \delta_{ij}^{(2)} = \frac{1}{h^2}\begin{cases} \frac{-\pi^2}{3}, & i = j, \\ \frac{-2(-1)^{j-i}}{(j-i)^2}. & i \neq j. \end{cases} \tag{7}$$

3 The sinc-collocation method

Let us consider the nonlinear Eq. (13), with boundary conditions (2). Assume $y(x)$ to be the exact solution of the boundary value problem (13) and let $y(x) \in L_\alpha(D_E)$. We approximate solution for $y(x)$, in Eq. (13) as

$$y(x) \simeq y_{-N-1}u(x) + \sum_{i=-N}^{N} y_i w(x)S(i,h) \circ \phi(x) + y_{N+1}v(x), \tag{8}$$

where $y_i = y(x_i)$ and $u(x), w(x), v(x)$ are polynomials given by

$$u(x) = (\lambda + 2)(1 - x)^3 - (3 + \lambda)(1 - x)^2, \qquad w(x) = x(1 - x), \qquad v(x) = 3x^2 - 2x^3.$$

Setting

$$\frac{d^r}{d\phi^r}[S_i(x)] = S_i^{(r)}(x), \qquad r = 1, 2,$$

we have

$$(w(x)S_i(x))' = w'(x)S_i(x) + w(x)\phi'(x)S_i^{(1)}(x), \tag{9}$$

$$(w(x)S_i(x))'' = w''(x)S_i(x) + (2w'(x)\phi'(x) + w(x)\phi''(x))S_i^{(1)}(x) + w(x)\phi'^2(x)S_i^{(2)}(x). \tag{10}$$

Also, we set

$$\phi^{(k)}(x_j) = \phi_j^{(k)}, \qquad w^{(k)}(x_j) = w_j^{(k)}, \quad u^{(k)}(x_j) = u_j^{(k)}, \quad v^{(k)}(x_j) = v_j^{(k)}, \qquad k = 0, 1, 2.$$

By using Eqs. (9)-(20) and having substituted $x = x_j$ for $j = -N-1, ..., N+1$, where x_j are sinc grid points given in (7), we have

$$y(x_j) \simeq y_{-N-1}u_j + y_j w_j + y_{N+1}v_j, \tag{11}$$

$$y'(x_j) \simeq y_{-N-1}u'_j + \sum_{i=-N}^{N} y_i(w'_j\delta_{i,j}^{(0)} + w_j\phi'_j\delta_{ij}^{(1)}) + y_{N+1}v'_j, \tag{12}$$

$$y''(x_j) \simeq y_{-N-1}u''_j + \sum_{i=-N}^{N} y_i \left\{ w''_j\delta_{ij}^{(0)} + \left(2w'_j\phi'_j + w_j\phi''_j\right)\delta_{ij}^{(1)} + w_j(\phi'_j)^2\delta_{ij}^{(2)} \right\} + y_{N+1}v''_j, \tag{13}$$

By using relations (21)-(14), and substituting $x = x_j$, $j = -N-1, ..., N+1$, we can rewrite Eq. (13) as

$$y''(x_j) - \lambda y'(x_j) + \lambda\mu(\beta - \alpha)\exp(y(x_j)) = 0, \tag{14}$$

The above nonlinear system consists of $2N+3$ equations with $2N+3$ unknown coefficients $\{y_i\}_{i=-N-1}^{N+1}$. Solving this nonlinear system by the well known Newton's method. Consequently $y(x)$ given in (11) can be calculated.

4 Illustrative Example

To validate the application of sinc-collocation method to (13), (2) we use particular values of the parameters, $\lambda = 10$, $\beta = 3$ and $\mu = 0.02$. The solutions of the given example is obtained for $\alpha = \frac{1}{2}$, $d = \frac{\pi}{2}$ and for different values of $N = 10, N = 20$. Table 1 gives a comparison of the results from the contraction mapping principle [1], the shooting method, the Adomian's method [1] and present method with $N = 10$ and $N = 20$.

Table 1: Results for Example

x	Contraction Principle	Shooting Method	Method of [1]	Present method $N = 10$	Present method $N = 20$
0.0	0.006079	0.006048	0.006048	0.0060485	0.0060484
0.2	0.018224	0.018192	0.018192	0.018191	0.018193
0.4	0.030456	0.030424	0.030424	0.030425	0.030425
0.6	0.042701	0.042669	0.042669	0.042670	0.042669
0.8	0.054401	0.054371	0.054371	0.054371	0.054372
1.0	0.061459	0.061458	0.061458	0.061459	0.061459

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Well-posedness of an evolution Volterra equation with completely monotonic kernel

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Abstract

An evolution Volterra equation which is a hyperbolic type integro-differential equation with a completely monotonic kernel is considered. Galerkin approximation method is used to prove existence and uniqueness of the solution together with regularity estimates. The method can be used also for parabolic type integro-differential equations.

Keywords: Volterra equation, Galerkin approximation, completely monotonic kernel, regularity, a priori estimate.

Mathematics Subject Classification: 45K05

1 Introduction

We study a model problem,

$$\ddot{u}(x,t) + Au(x,t) - \int_0^t K(t-s)Au(x,s)\,ds = f(x,t), \quad \text{in} \;\; \Omega \times (0,T), \tag{1}$$

together with initial-boundary conditions,

$$\begin{aligned} &u(x,t) = 0, && \text{on } \partial\Omega \times (0,T), \\ &u(x,0) = u^0(x), \;\; \dot{u}(x,0) = v^0(x), && \text{in } \Omega. \end{aligned}$$

We consider $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded polygonal domain. The approach presented here, also can be applied for mixed homogeneous Dirichlet and nonhomogeneous Neumann bounadry conditions, which are of especial importance, for example, in applied mechanics. In this case we set $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} = \partial\Omega$, $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \varnothing$ and $\mathrm{meas}(\Gamma_{\mathrm{D}}) \neq 0$.

Here, the operator A is an elliptic selfadjoint, positive definite unbounded linear operator. The kernel K is a completely monotone function, that is, $K \in L_1(0,\infty) \cap \mathcal{C}^2(0,\infty)$, such that

$$(-1)^k D_t^k K(t) \geq 0, \quad t \in (0,\infty), \;\; k = 0,1,2.$$

We denote $\|K\|_{L_1(0,\infty)} = \gamma$.

We note that, e.g., Mittag-Leffler type kernels, that are weakly singular and arise in fractional order viscoelasticity, are completely monotone, see [3]. We also notice that a completely monotone function b is of positive type, i.e., $b \in L_{1,loc}[0,\infty)$ such that for all $T > 0$ and $\varphi \in \mathcal{C}([0,T])$,

$$\int_0^T \int_0^t b(t-s)\varphi(t)\varphi(s)\,ds\,dt \geq 0.$$

Well-posedness and numerical treatment for integro-differential equations have been studied extensively, see, e.g., [1], [2], [3], and references therein. An abstract Volterra equation, as an abstract model for equations of linear viscoelasticity, has been studied in [4], where weakly singular kernels can not be treated. In [3], well-posedness and regularity of a model problem similar to (1) was studied in the framework of the semigroup of linear operators. The drawback of the framework is that this does not admit nonhomogeneous Neumann boundary condition. While in practice such boundary conditions are of special interest. Here we investigate existence, uniqueness and regularity of the solution of the model problem by means of the Galerkin approximation method, in a similar way as for hyperbolic PDE's in [5]. The approach presented here is applied for mixed homogeneous Dirichlet and nonhomogeneous Neumann bounadry conditions, for both hyperbolic and parabolic type integro-differential equations. Here, for short, we present the results for hyperbolic type problems.

Now we consider the problem (1) with mixed homogeneous Dirichlet and nonhomogeneous Neumann bounadry conditions, and we define a weak solution $u = u(x,t)$ that satisfies

$$u \in L_2((0,T);V), \quad \dot{u} \in L_2((0,T);H), \quad \ddot{u} \in L_2((0,T);V^*), \tag{2}$$

$$\begin{aligned} &\langle \ddot{u}(t), v\rangle + a(u(t),v) - \int_0^t K(t-s)a(u(s),v)\,ds \\ &\qquad = (f(t),v) + (g(t),v)_{\Gamma_{\mathrm{N}}}, \quad \forall v \in V, \text{ a.e. } t \in (0,T), \end{aligned} \tag{3}$$

$$u(0) = u^0, \quad \dot{u}(0) = v^0. \tag{4}$$

Here $(g(t),v)_{\Gamma_{\mathrm{N}}} = \int_{\Gamma_{\mathrm{N}}} g(t)\cdot v\,dS$, $H = L_2(\Omega)^d$, $H^{\Gamma_{\mathrm{N}}} = L_2(\Gamma_{\mathrm{N}})^d$, and $\langle\cdot,\cdot\rangle$ denotes the pairing of V^* and V, with $V = \{v \in H^1(\Omega)^d : v|_{\Gamma_{\mathrm{D}}} = 0\}$. We note that g is the surface load from the Neumann boundary condition.

2 Main Result

Let $\{(\lambda_j,\varphi_j)\}_{j=1}^\infty$ be the eigenpairs of the weak eigenvalue problem

$$a(\varphi,v) = \lambda(\varphi,v), \quad \forall v \in V.$$

It is known that $\{\varphi_j\}_{j=1}^\infty$ can be chosen to be an ON-basis in H and an orthogonal basis for V.

Now, for a fixed positive integer $m \in \mathbb{N}$, we seek a function of the form

$$u_m(t) = \sum_{j=1}^m d_j(t)\varphi_j$$

to satisfy the weak form (2)-(4), in the form,

$$\begin{aligned} &(\ddot{u}_m(t),\varphi_k) + a(u_m(t),\varphi_k) - \int_0^t \beta(t-s)a(u_m(s),\varphi_k)\,ds \\ &\qquad = (f(t),\varphi_k) + (g(t),\varphi_k)_{\Gamma_{\mathrm{N}}}, \quad k = 1,\ldots,m,\ t \in (0,T), \end{aligned}$$

with initial conditions

$$u_m(0) = \sum_{j=1}^m (u^0,\varphi_j)\varphi_j, \quad \dot{u}_m(0) = \sum_{j=1}^m (v^0,\varphi_j)\varphi_j. \tag{5}$$

Based on the definitions above, in a classical way, we need to prove existence, uniqueness and a priori estimates for the Galerkin approximation u_m.

Theorem 2.1. *For each $m \in \mathbb{N}$, there exists a unique Galerkin approximation u_m. Moreover, if $u^0 \in V, v^0 \in H$, $f \in L_2((0,T);H)$, $g \in W_1^1((0,T);H^{\Gamma_N})$, there is a constant $C = C(\Omega,\gamma,T)$ such that,*

$$\begin{aligned}\|u_m\|_{L_\infty((0,T);V)} + \|\dot{u}_m\|_{L_\infty((0,T);H)} + \|\ddot{u}_m\|_{L_2((0,T);V^*)} \\ \leq C\big\{\|u^0\|_V + \|v^0\| + \|g\|_{W_1^1((0,T);H^{\Gamma_N})} + \|f\|_{L_2((0,T);H)}\big\}.\end{aligned} \tag{6}$$

From the a priori estimate (6) we conclude boundedness (in the weak sense) of $\{u_m\}_{m=1}^\infty$, $\{\dot{u}_m\}_{m=1}^\infty$, and $\{\ddot{u}_m\}_{m=1}^\infty$ in $L_2((0,T);V), L_2((0,T);H)$, and $L_2((0,T);V^*)$, respectively. This implies the weak convergence, which is the main tool to prove the next theorem.

Theorem 2.2. *If $u^0 \in V$, $v^0 \in H$, $g \in W_1^1((0,T);H^{\Gamma_N})$, $f \in L_2((0,T);H)$, there exists a unique weak solution of (1). Moreover*

$$\begin{aligned}\|u\|_{L_\infty((0,T);V)} + \|\dot{u}\|_{L_\infty((0,T);H)} + \|\ddot{u}\|_{L_2((0,T);V^*)} \\ \leq C\big\{\|u^0\|_V + \|v^0\| + \|g\|_{W_1^1((0,T);H^{\Gamma_N})} + \|f\|_{L_2((0,T);H)}\big\}.\end{aligned}$$

Further, to get more regularity we need to specialize to the homogeneous Dirichlet boundary condition ($\Gamma_\mathrm{N} = \varnothing$), such that the elliptic regularity holds, that is,

$$\|u\|_{H^2} \leq C\|Au\|, \quad u \in H^2(\Omega)^d \cap V.$$

Here $H^2 = H^2(\Omega)$ is the standard Sobolev space.

Theorem 2.3. *We assume that $\Gamma_N = \varnothing$. If $u^0 \in H^2$, $v^0 \in V$, and $\dot{f} \in L_2((0,T);H)$, then for the unique weak solution of (1) we have*

$$\begin{aligned}\|u\|_{L_\infty((0,T);H^2)} + \|\dot{u}\|_{L_\infty((0,T);V)} + \|\ddot{u}\|_{L_\infty((0,T);H)} + \|\dddot{u}\|_{L_2((0,T);V^*)} \\ \leq C\big\{\|u^0\|_{H^2} + \|v^0\|_V + \|f\|_{H^1((0,T);H)}\big\}.\end{aligned}$$

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A homotopy based method for solving systems of linear equations

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Abstract
A new iterative method is proposed to solve systems of linear algebraic equations, $Ax = b$. The method is based on the concept of homotopy, where the system is transformed to an infinite number of easily solvable systems of equations. A comparative study of the method from computational cost viewpoint and speed of convergence shows that the new presented method competes well with classic iterative techniques. Also, using a convergence control parameter, an stationary modification of the method is presented which can extend the convergence cases of the presented homotopy method.

Keywords: System of linear equations; Homotopy analysis method; Iterative methods; Convergence control parameter.

Mathematics Subject Classification: 56F10, 93C05, 65H20

1 The Basic Idea

Consider the system of linear equations, $Ax = b$, where A is an invertible matrix of size n and $x, b \in \mathbb{R}^n$. Invertibility of A implies that the system has a unique solution. Our aim is to obtain this solution by using the homotopy idea. Suppose $v^{(\circ)}$ is an initial guess of the solution, using an *auxiliary matrixM* (probably related to A), we construct the homotopy equation

$$(1-q)M(x - v^{(\circ)}) + q(Ax - b) = 0. \tag{1}$$

For every $q \in [0, 1]$, equation (1) is a system of linear equations whose solution is dependent upon q. When $q = 0$ the system is equivalent to $M(x - v^{(\circ)}) = 0$, if M is an invertible matrix, this system leads to the obvious solution $v^{(\circ)}$. In the case where $q = 1$ the system will be equivalent to $Ax = b$, i.e. the original system under study.

Now, if we accept that the solution to the homotopy equation (1) could be represented as an infinite series in the form

$$x = x^{(\circ)} + x^{(1)}q + x^{(2)}q^2 + x^{(3)}q^3 + \cdots, \tag{2}$$

then substituting series (2) in (1) we would have

$$M(x^{(\circ)} + x^{(1)}q + x^{(2)}q^2 + \cdots - v^{(\circ)}) + q\{(A - M)(x^{(\circ)} + x^{(1)}q + \cdots) + Mv^{(0)} - b\} = 0. \tag{3}$$

The above equation holds for every $q \in [0, 1]$, so the left hand side expressions must be independent of q, therefore we have

$$\begin{aligned} &M(x^{(\circ)} - v^{(\circ)}) = 0, \\ &Mx^{(1)} = b - Ax^{(\circ)}, \\ &Mx^{(n)} = (M - A)x^{(n-1)}, \quad n \geq 2. \end{aligned} \tag{4}$$

One would have all $x^{(n)}$s by sequentially solving the above equations, then employing (2) a solution series can be constructed. Here it is important for M to be simple enough that one can easily solve the above system. From a theoretical point of view, in the case of invertible M, one has

$$\begin{aligned} x^{(\circ)} &= \quad v^{(\circ)}, \\ x^{(n)} &= \quad (I - M^{-1}A)^{(n-1)} M^{-1}(b - Av^{(\circ)}), \qquad n \geq 1. \end{aligned} \tag{5}$$

The series solution is then can be expressed as follows:

$$\begin{aligned} x \quad &= x^{(\circ)} + x^{(1)} + x^{(2)} + x^{(3)} + \cdots, \\ &= v^{(\circ)} + \textstyle\sum_{n=1}^{\infty} (I - M^{-1}A)^{(n-1)} M^{-1}(b - Av^{(\circ)}). \end{aligned} \tag{6}$$

2 Discussion on convergence

According to equation (6), the convergence condition for the homotopy method can be easily stated by using Neumann's series Theorem, as follows:

Theorem 3.2: *If the auxiliary matrix, M, is chosen such that the spectral radius of $I - M^{-1}A$ is less than one, i.e. $\rho(I - M^{-1}A) < 1$, then the solution series obtained by homotopy method, for solving system $Ax = b$, will converge to the exact solution.*

So we can summarize the main requirements for the auxiliary matrix as follows:

1. Systems (4) must be easy-to-solve,
2. $\rho(I - M^{-1}A) < 1$, i.e. insures convergence of the method,
3. $\rho(I - M^{-1}A) \ll 1$, i.e. insures rapid convergence of the solution series.

If one chooses M to be a diagonal, triangular or even tridiagonal matrix, then the first condition is fulfilled, but in order to make the other requirements satisfied we need to calculate spectral radius of $I - M^{-1}A$, which is not an economical advice. In the sequel we will study some special cases of matrices for which the second condition is automatically fulfilled

2.1 Some special cases of convergence

If the coefficient matrix A has some special structure then the convergence condition for the homotopy method is satisfied. Here are the families for which homotopy method is convergent:

1. Diagonally dominant matrices,
2. Hermitian positive definite matrices,
3. M-matrices,
4. Nonnegative matrices with regular splitting

From the above list the M-matrices fulfill our requirements by definition and the 4th case has been discussed in [1], we focus on two other cases.

2.1.1 Diagonally dominant matrices

There are two classes of diagonally dominant matrices where the homotopy method results in convergent series.
First class: Strictly row diagonally dominant matrices (SRDD)
Second class: Irreducibly row diagonally dominant matrices (IRDD)

Theorem 2.1: *If A is SRDD and M is the lower (or upper) triangular (with diagonal) part of A, then the homotopy method for solving the system $Ax = b$ is convergent.*

Lemma 2.2: [2] *IRDD matrices are nonsingular.*

Theorem 2.3: *If the coefficient matrix of the system $Ax = b$ is IRDD, then the homotopy method is convergent .*

2.1.2 Hermitian positive definite matrices

In the special case of Hermitian positive definite (HPD) matrices, the following theorem states that it is possible to ensure convergence of the homotopy method in some cases.
Theorem 2.4: [3] *Let A be a HPD matrix and suppose that there exist a splitting $A = M - N$ where $M + N^*$ is also positive definite, then $\rho(I - M^{-1}A) < 1$.* So if we can split a HPD matrix A, in a way which satisfies the conditions stated in the above theorem, then the homotopy method would converge. Of course we should keep in mind that this matrix splitting must result in easily solvable systems according to (4).

Now we come to the last comment on convergence problem, actually the cases we studied here are not the only cases where the homotopy method converges, there are examples of matrices which doesn't fit in any of the considered families but still the homotopy method is convergent on them. So the question: "*For which classes of matrices, the homotopy method is convergent?*" is still an *open* problem.

3 Comparison with classic iterative methods

The computational cost of the homotopy method, applied to a linear system of equations of size n, is of order ($\mathcal{O}^2$), so the amount of algebraic operations needed have the same order as classic iteration methods, thus comparable to them. Also it is verified that in cases where classic methods like Jacobi or Gauss-Seidel are convergent, so is the homotopy method. Moreover there cases where iterative techniques may diverge while the homotopy method successfully solves the system.

Theorem 3.1: *If the Jacobi (or Gauss-Seidel) method, for solving the system $Ax = b$, converges then so is the homotopy method.*

Similar theorems and corollaries could be stated for other iterative methods, like Richardson's method (which would be the same as Dr. Keramat's case [4]) and SOR method. In general if an iteration method uses the matrix splitting $A = B - C$ (thus the iteration matrix: $G = B^{-1}C$), then in applying the homotopy method, one can use M to be the same as B. With this choice of the auxiliary matrix the convergence criteria of the homotopy method would be the same as the criteria in the iterative method.

4 Adding a convergence control parameter

We can add a convergence control parameter, $\alpha \in \mathbb{R}$, to the homotopy equation (1), just like what is proposed by Prof. Shi Jun Liao in HAM [5], and we would gain new results which ensures convergence in some cases.
New systems, similar to (4), would be

$$\begin{aligned} &M(x^{(\circ)} - v^{(\circ)}) = 0, \\ &Mx^{(1)} = \alpha(b - Ax^{(\circ)}), \\ &Mx^{(n)} = (M - \alpha A)x^{(n-1)}, \qquad n \geq 2. \end{aligned} \tag{7}$$

The convergence condition in this *modified homotopy method* would change to $\rho(I - \alpha M^{-1}A) < 1$. So the main question we are concerned with, can be stated as follows:
In solving a system of equations $Ax = b$, using modified homotopy method, can we ensure the convergence by suitably choosing the auxiliary matrix M and the auxiliary parameter α.
If the answer to this question is "*Yes*", then for an arbitrary matrix B we must be able to find a scalar α such that $\rho(I - \alpha B) < 1$. Here we show, through an example, that the general answer to the above mentioned question is "*No*".

Example 4.1: If $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, we have $I - \alpha B = \begin{pmatrix} 1-\alpha & 0 & 0 \\ 0 & 1+\alpha & 0 \\ 0 & 0 & 1-2\alpha \end{pmatrix}$. So $\rho(I - \alpha B) = max\{|1-\alpha|, |1+\alpha|, |1-2\alpha|\} > 1$.

But there are cases where αs could be found to fulfill our requirements. When $M = I$ is chosen, then there do exist such αs in some special cases (e.g. positive definite matrices). Our reasoning is based on a theorem in [6], which we will also recall here.

Definition 4.2: A square matrix B is said to be definite if the real part of all eigenvalues of B are positive (negative).

Theorem 4.3: [6] *If A is a definite matrix then there exist a scalar α such that $\rho(I-\alpha A) < 1$.*

Although the above theorem ensures the ability of the modified homotopy method for solving the system of linear equations $Ax = b$, in the case where A is a definite matrix, but we still have problem on finding α. There are cases where we can choose, in advance, suitable αs according to the coefficient matrix at hand. Here we show that there are simple choices for α when the coefficient matrix is symmetric positive definite or SRDD and IRDD.

Theorem 4.4: *In the system of linear equations $Ax = b$, if A is a symmetric positive definite matrix, then by choosing $M = I$ and $0 < \alpha < \dfrac{2}{\|A\|_\infty}$, the modified homotopy method converges.*

Remark: Analytically speaking, every α which satisfies $0 < \alpha < \frac{2}{\|A\|_\infty}$, results in a convergent series solution, but numerically we can't let α to take small values. As α approaches zero the matrix $I - \alpha A$ tends to the identity matrix, thus the spectral radius gets closer to 1. So this would greatly reduce the speed of convergence.

Theorem 4.5: *If A is an IRDD or SRDD matrix with positive diagonal elements, then by choosing $M = I$ and $0 < \alpha < \min\limits_j \dfrac{2}{a_{jj}}$, the modified homotopy method converges.*

Note that if the matrix is SRDD or IRDD then the system would be equivalent with the one with positive diagonal elements.

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Presentation of analytic solutions for first kind Fredholm integral equations

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Abstract

As is known first kind of Fredholm integral equations can not be solved by successive approximation methods. Because in these equations, the unknown function appears only under the integral sign. Therefore this kind of integral equations are known as ill-posed.According to this fact,these equations are solved only by numerical methods. In this paper, we try to present analytic solutions as well for the first as to the second kind of Fredholm integral equations. Our method is based on Fredholm's general theory for second kind integral equations and the Hilbert-Schmidt theory about constructing eigenfunctions.
Keywords: Fredholm Integral Equations, Algebraic System, Eigenvalues, Eigenfunctions, Projection Operators, Matrix Expansion.

Mathematics Subject Classification: 45B05

1 Introduction

It is obvious that the second kind of Fredholm integral equations are solved by the general theory which was presented by Fredholm [2]. In fact by using this theory one can determine the solution in terms of different values of the parameter λ [2], [3] and [5]. But for the first kind of Fredholm integral equations there is no general theory about solving methods. In some special cases, this kind of integral equations are solved by making use of integral transforms, such as Laplace, Fourier and Mellin transforms [3] and [7]. Also in case where the kernel of the first kind integral equation has weak singularities, by using fractional derivative, at first it is changed to an equation with strong singularity kernels. Then by the Poiencare-Bertrand formula it is changed to second kind of Fredholm integral equations [8],[9]. In this paper we consider first kind Fredholm integral equations with continuous and differentiable kernel $K(t,x)$ as

$$\int_a^b K(t,x)u(x)\,dx = f(t),$$

where $u(x)$ and $f(t)$ are unknown function and nonhomogeneous term respectively.
In this case,we can not apply the above mentioned process, because its kernel is a differentiable function and no singularities in its kernel. Consequently we can not change to the second kind Fredholm integral equations.
The method which will be applied in this paper, is based on the general theory of the second kind

Frehholm integral equations and the general theory of Hilbert-Schmidt for constructing eigenvectors and eigenfunctions.

2 Investigation of Related Algebraic System

At first, we consider the related algebraic system of the first kind Fredholm integral equation as follows:

$$AX = B, \tag{1}$$

where $A = (a_{ij})_{i,j=1}^{n}$ is a $n \times n$ matrix, $X = (x_1, \cdots, x_n)^T$ and $b = (b_1, b_2, \cdots, b_n)^T$ are column vectors, $a_{ij} = a_{ji}$, $a_{ij}, b_i \in R$ and $i, j = 1, 2, \cdots, n$.

Now we are going to determine other eigenvalues and eigenvectors of matrix A. For this, the characteristic equation of the matrix A is

$$|A - \lambda I| = 0. \tag{2}$$

Since A is a symmetric matrix, its eigenvalues are real. We suppose $\lambda_k,\ k = 1, 2, \cdots, n$, are distinct roots of this characteristic equation ,that is

$$\lambda_p \neq \lambda_q \,;\, p \neq q \,,\, \lambda_k \neq 0 \tag{3}$$

If the corresponding eigenvectors are denoted by $X_k,\ k = 1, 2, \cdots, n$, (related to the respective eigenvalues λ_k) then for these vectors we have

$$(X^{(p)})^T X^{(q)} = \delta_{pq} = \begin{cases} 0, & p \neq q, \\ 1, & p = q. \end{cases} \tag{4}$$

Without loss of generality, we suppose these vectors to be orthogonal.

The projection operators of the matrix A is denoted by P_k and they are defined in the following form

$$P_k = X^{(k)} \left(X^{(k)}\right)^T, \qquad k = 1, \cdots, n.$$

From linear algebra [1], we have

$$A = \sum_{k=1}^{n} \lambda_k P_k = \sum_{k=1}^{n} \lambda_k X^{(k)} \left(X^{(k)}\right)^T, \tag{5}$$

and the column vectors X, B are written as follows

$$B = \sum_{k=1}^{n} \beta_k X^{(k)}, \tag{6}$$

and

$$X = \sum_{k=1}^{n} C_k X^{(k)}. \tag{7}$$

Considering relations (5), (6) and (7) and using (1) implies

$$\sum_{k=1}^{n} \lambda_k X^{(k)} \left(X^{(k)}\right)^T \sum_{m=1}^{n} C_m X^{(m)} = \sum_{k=1}^{n} \beta_k X^{(k)}.$$

According to the linear independency of the vectors $X^{(k)}$ we have

$$C_k = \frac{\beta_k}{\lambda_k}\ , \qquad k = 1, 2, \cdots, n. \tag{8}$$

hence, we get

$$X = \sum_{k=1}^{n} \frac{\beta_k}{\lambda_k} X^{(k)}. \tag{9}$$

3 Investigation of First Kind Fredholm Integral Equations

We consider the first kind Fredholm integral equation which related to the algebraic system (1). For this we consider this equation as

$$\int_a^b K(x,t)u(t)dt = f(x), \qquad x \in [a,b]. \tag{10}$$

Dividing the interval $[a,b]$ in n intervals of length h

$$h[K(x,t_1)u(t_1) + K(x,t_2)u(t_2) + \cdots + K(x,t_n)u(t_n)] = f(x).$$

where

$$t_0 = a,\ t_1 = a+h,\ t_2 = a+2h, \cdots, t_n = a+nh,\ h = \frac{b-a}{n}.$$

If this interval is parted also with respect to x and the points t_i are the partial points, we have

$$h\sum_{j=1}^{n} K(t_i,t_j)u(t_j) = f(t_i)\ , \qquad i = 1, \cdots, n. \tag{11}$$

Finally, if we put

$$K(t_i,t_j) = k_{ij}]\ , \qquad u(t_j) = u_j\ , \qquad f(t_i) = f_i$$

then the algebraic system is

$$h\sum_{j=1}^{n} k_{ij}u_j = f_i\ , \qquad i = 1, \cdots, n. \tag{12}$$

Therefore, we obtain an algebraic system which can be written in the matrix from

$$hKu = f, \tag{13}$$

where

$$f = (f_1, f_2, \cdots, f_n)^T,\ u = (u_1, \cdots, u_n)^T,\ K = (k_{ij})_{i,j=1}^{n}$$

Now, we consider the spectral expansion of the matrix K. For this, we have to find the unknown vector $u \neq 0$ such that

$$hKu = \rho u. \tag{14}$$

If we put

$$\rho = \frac{1}{\lambda}, \tag{15}$$

then we have the following algebraic system

$$(I - \lambda hK)u = 0. \tag{14.1}$$

Its determinant is

$$\begin{vmatrix} 1-\lambda hk_{11} & -\lambda hk_{12} & \dots & -\lambda hk_{1n} \\ -\lambda hk_{21} & 1-\lambda hk_{22} & \dots & -\lambda hk_{2n} \\ \vdots & \vdots & & \vdots \\ -\lambda hk_{n1} & -\lambda hk_{n2} & \dots & 1-\lambda hk_{nn} \end{vmatrix}. \tag{16}$$

If λ_k , $k = 1, 2, \cdots, n$ are the roots of this equation and $u^{(k)}$ are the solutions of (14), then for these solutions we will have

$$\left(u^{(p)}\right)^T u^{(q)} = \delta pq. \tag{17}$$

If these vectors are orthogonal, then the solution of (13) can be written as (9).

Remark 3.1. *If we denote the limit of (16) by $D(\lambda)$, as $n \to \infty$, then for $D(\lambda)$ and $D(x, y; \lambda)$ we have the following relation [2]*

$$D(x, y; \lambda) - \lambda K(x, y)D(\lambda) - \lambda \int_a^b K(x, t)D(t, y; \lambda)dt = 0 \ , \tag{18}$$

Note that

$$\lim_{n\to\infty} D(\lambda) = \triangle(\lambda) = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{\lambda^n}{n!} \int_a^b \int_a^b \cdots \int_a^b K \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \\ \xi_1 & \xi_2 & \cdots & \xi_n \end{pmatrix} d\xi_1 d\xi_2 \cdots \xi_n,$$

where

$$K \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_n \\ \xi_1 & \xi_2 & \dots & \xi_n \end{pmatrix} = \begin{vmatrix} k(\xi_1, \xi_1) & k(\xi_1, \xi_2) & \dots & k(\xi_1, \xi_n) \\ k(\xi_2, \xi_1) & k(\xi_2, \xi_2) & \dots & k(\xi_2, \xi_n) \\ \vdots & \vdots & & \vdots \\ k(\xi_n, \xi_1) & k(\xi_n, \xi_2) & \dots & k(\xi_n, \xi_n) \end{vmatrix}.$$

Convergence of these series has been proved by Hadamard [1].

4 Solving the First Kind Fredholm Integral Equation

Now for determining the eigenvalues and eigenvectors as well of the integral equation (10) as the algebraic system (13),(14) we insert $\rho u(x)$ instead of the right-hand side of (10), that is

$$\int_a^b K(x, t)u(t)dt = \rho u(x). \tag{19}$$

By using the following change of parameter

$$\rho = \frac{1}{\lambda}, \tag{20}$$

we will have

$$u(x) = \lambda \int_a^b K(x, t)u(t)dt. \tag{21}$$

The equation (21) is the same as equation (24) which has appeared on the page 48 of [2]. Its roots are the eigenfunctions of equation (21).

In the following part these eigenfunctions are calculated. For this, suppose λ_k , $k = 1, 2, \cdots, n$, are the solutions of $D(\lambda)$, that is $D(\lambda)$ has just n roots.

$$D(\lambda_k) = 0, \tag{22}$$

then from (18) we have

$$D(x, y; \lambda_k) = \lambda_k \int_a^b K(x, t) D(t, y; \lambda_k) dt.$$

If the variable y is fixed by y_0 we have

$$D(x, y_0; \lambda_k) = \lambda_k \int_a^b K(x, t) D(t, y_0; \lambda_k) dt, \tag{23}$$

where

$$u_k(x) = D(x, y_0; \lambda_k). \tag{24}$$

In fact, $u_k(x)$ are the eigenfunctions of (21). By orthonormality of these functions, that is

$$< u_p, u_q >= \delta_{pq}, \tag{25}$$

then we obtain for the kernel of integral equation

$$K(x, t) = \sum_\nu \frac{u_\nu(x) u_\nu(t)}{\lambda_\nu}. \tag{26}$$

This expression is the same expansion as given in (54) on page 134 of [2].
On the other hand, for the arbitrary function $f(x)$ we can write

$$f(x) = \sum_\nu f_\nu u_\nu(x). \tag{27}$$

The coefficients of this expansion can be determined as

$$f_\nu =< f, u_\nu > . \tag{28}$$

Suppose the following linear combination for unknown function $u(t)$

$$u(t) = \sum_\nu c_\nu u_\nu(t), \tag{29}$$

then from (10) we obtain

$$\sum_\nu \frac{u_\nu(x)}{\lambda_\nu} \int_a^b u_\nu(t) \sum_\mu c_\mu u_\mu(t) dt = \sum_\nu f_\nu u_\nu(x).$$

Regarding (25) implies

$$\sum_\nu \frac{u_\nu(x)}{\lambda_\nu} c_\nu = \sum_\nu f_\nu u_\nu(x),$$

or

$$\frac{c_\nu}{\lambda_\nu} = f_\nu,$$

hence these coefficients are

$$c_\nu = \lambda_\nu f_\nu. \tag{30}$$

Finally, we can determine the analytic solution of (10) from (29) as follows

$$u(t) = \sum_\nu \lambda_\nu f_\nu u_\nu(t). \tag{31}$$

Therefore by making use of the general Fredholm theory for the second kind integral equations, we can present the analytic solution of the first kind of Fredholm integral equations. In sum up we can conclude the following theorem.

Theorem 4.1. *If in the first kind Fredholm integral equation*

$$\int_a^b K(x,t)u(t)dt = f(x) \qquad ; \qquad x \in [a,b]$$

$K(x,t)$ is a self-adjoint kernel and $f(x)$ is a real-valued given integrable function on $[a,b] \times [a,b]$ and $[a,b]$ respectively, then its analytic solution is given by

$$u(t) = \sum_\nu \lambda_\nu f_\nu u_\nu(t),$$

where λ_ν are eigenvalues and $u_\nu(t)$ are eigenfunctions and f_ν are the coefficients of the expansion of $f(x)$.

Remark 4.2. *If in the equation (10), the kernel is a real valued function and degenerated, then the solution of (10) can be determined in suitable functional space. This solution can include arbitrary constants.*

Remark 4.3. *The assertion in Remark 22 will hold, if the kernel of (10) is not a self-adjoint kernel.*

Example 4.4. *We consider the integral equation*

$$\int_0^1 \frac{u(x\xi)}{1-\alpha\xi} d\xi = f(x) \qquad , \qquad \alpha > 1 \tag{32}$$

By changing the variable by $x\xi = t$ we have

$$\int_0^x \frac{u(t)}{x-\alpha t} dt = f(x) \qquad , \qquad \alpha > 1 \tag{33}$$

This integral equation is of Volterra kind, the singularity of kernel is inside the interval $[0,x]$, is changed to a second kind integral equation with weak singularity [4]. An integral can be taken in the Cauchy sense but not the singularity of a kernel [6,8,9].

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Multivariate quasi-interpolation scheme for solving the two-dimensional Burgers' equations

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Abstract
In this paper, we introduce the multivariate quasi-interpolation scheme for the numerical solution of the two-dimensional Burgers' equations. In this method, the unknown functions and their spatial derivatives are approximated by using multivariate quasi-interpolation scheme. In the time discretization of the equations, the Taylors series expansion is used. This method is applied on one experiment and the numerical results show the accuracy of the method.
Keywords: Nonlinear two-dimensional Burgers' equation; Taylors series expansion; Quasi-interpolation scheme; Radial basis functions.

Mathematics Subject Classification: 35L70; 33E99; 41A30; 30K05.

1 Introduction

Burgers' equation is a fundamental partial differential equation from fluid mechanics. It is used in various areas of applied mathematics and physics, such as modeling of gas dynamics, heat conduction, and acoustic waves [1, 2, 3].

There are different numerical methods for the solution of this equation such as finite differences, finite elements and Adomian method [4, 5, 6].

In this paper, we apply the multivariate quasi-interpolation scheme for the solution of the two-dimensional Burgers' equations.

The organization of this paper is as follows. In Section 2, we describe discretization of the time derivative and the multivariate quasi-interpolation scheme for the nonlinear two-dimensional Burgers' equations. In Section 3, we present our numerical experiment, and the last Section contains conclusions.

2 Multivariate quasi-interpolation scheme

Consider the two-dimensional Burgers' equations:

$$u_t + uu_x + vu_y = \mu(u_{xx} + u_{yy}), \tag{1}$$

$$v_t + uv_x + vv_y = \mu(v_{xx} + v_{yy}), \tag{2}$$

with the initial conditions:

$$u(x, y, 0) = f_1(x, y), \quad v(x, y, 0) = f_2(x, y), \quad (x, y) \in \Omega = [a, b] \times [c, d],$$

and the boundary conditions:

$$u(x,y,t) = g_1(x,y,t), \quad v(x,y,t) = g_2(x,y,t), \qquad (x,y) \in \partial\Omega,$$

where $u(x,y,t)$ and $v(x,y,t)$ are the two unknown variables. f_1, f_2, g_1 and g_2 are all known functions and $\mu > 0$ is a viscosity constant.

At first, we discretize the temporal derivative u_t of Eq. (1) by using Taylors' series similar to what we did in [7]. By using this approach, the time discretized form of Burgers' equation (1) is given as follows:

$$2u^{n+1} + \Delta t(u^n u_x^{n+1} + u_x^n u^{n+1} + v^n u_y^{n+1} + v^{n+1} u_y^n) - \mu\Delta t(u_{xx}^{n+1} + u_{yy}^{n+1})$$

$$= 2u^n + \mu\Delta t(u_{xx}^n + u_{yy}^n).$$

Also, the time discretized form of Eq. (2) is given as follows:

$$2v^{n+1} + \Delta t(u^n v_x^{n+1} + v_x^n u^{n+1} + v^n v_y^{n+1} + v^{n+1} v_y^n) - \mu\Delta t(v_{xx}^{n+1} + v_{yy}^{n+1})$$

$$= 2v^n + \mu\Delta t(v_{xx}^n + v_{yy}^n).$$

Now, we use the multivariate quasi-interpolation scheme [8, 9] for approximation of u and v. In this scheme, u^n and v^n are approximated as follows:

$$u^n(x,y) = \sum_{i=0}^{n_x}\sum_{j=0}^{n_y} u_{ij}^n \varphi_i(x)\psi_j(y), \qquad v^n(x,y) = \sum_{i=0}^{n_x}\sum_{j=0}^{n_y} v_{ij}^n \varphi_i(x)\psi_j(y), \tag{3}$$

where $N = n_x \times n_y$ is the number of the collocation points and u_{ij} and v_{ij} are the values of u and v at the intersection of the ith horizontal grid line and the jth vertical grid line, respectively; and the functions $\varphi_i(x)$ are defined as follows:

$$\varphi_0(x) = \frac{1}{2} + \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)}, \quad \varphi_1(x) = \frac{\phi_2(x) - \phi_1(x)}{2(x_2 - x_1)} - \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)},$$

$$\varphi_i(x) = \frac{\phi_{i+1}(x) - \phi_i(x)}{2(x_{i+1} - x_i)} - \frac{\phi_i(x) - \phi_{i-1}(x)}{2(x_i - x_{i-1})}, \quad 2 \leqslant i \leqslant n_x - 2,$$

$$\varphi_{n_x-1}(x) = \frac{(x_{n_x} - x) - \phi_{n_x-1}(x)}{2(x_{n_x} - x_{n_x-1})} - \frac{\phi_{n_x-1}(x) - \phi_{n_x-2}(x)}{2(x_{n_x-1} - x_{n_x-2})},$$

$$\varphi_{n_x}(x) = \frac{1}{2} + \frac{\phi_{n_x-1}(x) - (x_{n_x} - x)}{2(x_{n_x} - x_{n_x-1})},$$

that the function $\phi_i(x)$ is multiquadric radial basis function and is defined by $\phi_i(x) = \sqrt{(x - x_i)^2 + r^2}$.The parameter r is the shape parameter. The functions $\psi_j(y)$ are defined similar to the above equations but only the function $\phi_j(y) = \sqrt{(y - y_j)^2 + r^2}$ is used instead of $\phi_i(x)$. Also, the approximation of u_x^n, u_y^n, u_{xx}^n, u_{yy}^n, v_x^n, v_y^n, v_{xx}^n and v_{yy}^n are obtained by differential of Eqs. 3.
At the end, the $2N$ unknowns u_{ij} and v_{ij} are found by using the collocation method.

3 The numerical experiment

An exact solution of Burgers' equations (1) and (2) can be generated by using the Hopf-Cole transformation [10] which is

$$u(x,y,t) = \frac{3}{4} - \frac{1}{4[1+\exp(-4x+4y-t)/(32\mu)]},$$

$$v(x,y,t) = \frac{3}{4} + \frac{1}{4[1+\exp(-4x+4y-t)/(32\mu)]}.$$

The numerical computations were performed using $n_x = n_y = 4$, $\Delta t = 0.0001$, $r = 1.5$ and $\Omega = [0,1] \times [0,1]$. Table 1 shows root-mean-square error (RMSE) of u and v at $t = 0.05$ and $t = 0.1$ with different values of μ. E1 is RMSE of u and E2 is RMSE of v. RMSE of u is defined as follows:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(u_{num}(X_i) - u_{exa}(X_i))^2}{N}}$$

wherein X_i is the collocation point. RMSE of v is defined similar to the above equation.

RMSE	t	$\mu = 0.1$	$\mu = 0.02$	$\mu = 0.01$
E1	0.05	4.84434×10^{-7}	4.58031×10^{-4}	1.64173×10^{-3}
	0.1	8.57325×10^{-7}	8.94851×10^{-4}	3.64173×10^{-3}
E2	0.05	4.84434×10^{-7}	4.58031×10^{-4}	1.64173×10^{-3}
	0.1	8.57325×10^{-7}	8.94851×10^{-4}	3.64173×10^{-3}

Table 1: RMSE of u and v at $t = 0.05$ and $t = 0.1$ with different values of μ.

Table 2 gives the numerical and exact values of u and v at some typical points at time level $t = 0.1$ with $\mu = 0.04$. The 2 and 3 columns are related to u and the 4 and 5 columns are related to v.

Points	Numerical	Exact	Numerical	Exact
(0.1,0.1)	0.62020	0.62011	0.87980	0.87989
(0.5,0.1)	0.55245	0.55236	0.94755	0.94764
(0.9,0.1)	0.51597	0.51764	0.98403	0.98236
(0.3,0.3)	0.62019	0.62011	0.87981	0.87989
(0.7,0.3)	0.55152	0.55236	0.94848	0.94764
(0.1,0.5)	0.69048	0.69087	0.80952	0.80913
(0.5,0.5)	0.62024	0.62011	0.87976	0.87989
(0.9,0.5)	0.55246	0.55236	0.94754	0.94764
(0.3,0.7)	0.69134	0.69087	0.80866	0.80913
(0.7,0.7)	0.62021	0.62011	0.87979	0.87989
(0.1,0.9)	0.73129	0.72962	0.76871	0.77038
(0.5,0.9)	0.69049	0.69087	0.80951	0.80913
(0.9,0.9)	0.62020	0.62011	0.87980	0.87989

Table 2: Comparison of numerical solutions with the exact solutions for u and v at $t = 0.1$.

4 Conclusions

In this paper, we use quasi-interpolation scheme for the numerical solution of the two-dimensional Burgers' equations. This method is a meshless one and it does not require a mesh to discretize the domain of the problem under condition. Our experiment indicates that it is an accurate and efficient numerical method. Numerical results and RMSEs show that the proposed scheme is a appropriated tool to solve two-dimensional Burgers' equations.

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Fuzzy stochastic differential system

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Abstract
In this paper, we study fuzzy stochastic differential equation initial value problems (IVPs). We obtain the existence and uniqueness theorem for a solution of the fuzzy stochastic differential equation (FSDE) under the Lipschitz condition. We present characterization theorems for the solution of a FSDE under the m.s. derivative-based interpretation, by the solution of a system of ODEs. Numerical examples are provided which connect the new results with previous findings.
Keywords: Fuzzy number, Fuzzy stochastic differential system m.s. differentiability, Lipschitz condition

1 Introduction

Definition 1 [1]. Define

$$\rho(X,Y) = (ED^2(X,Y))^{1/2}, X, Y \in L^2.$$

The norm $\| X \|^2$ of an element $X \in L^2$ is defined by

$$\| X \|_2 = \rho(X,\hat{0}) = (E \| X \|^2)^{1/2}.$$

(L_2, ρ) is a complete metric space [2], Corollary 2.2 and ρ satisfies that

$$\rho(X+Z, Y+Z) = \rho(X,Y), \rho(\lambda X, \lambda Y) = | \lambda | \rho(X,Y), \tag{1}$$

$$\rho(\lambda X, kX) \leq | \lambda - k | \| X \|_2, \tag{2}$$

for any $X, Y, Z \in L_2$ and $\lambda, k \in R$.

2 Fuzzy stochastic differential equations (FSDE)

Let us first establish some definitions and notations on the fuzzy random vector. Let $X_1, \ldots, X_m$ be f.r.v.'s. $X = (X_1, \ldots X_m)$T is called an m-dimensional fuzzy random vector, where T denotes the transpose of the vector. It is a Borel measurable function $X : \Omega \to (E^n)^m = E^n \times \ldots \times E^n$. Let $L_2^m = \{X \mid X = (X_1, \ldots, X_m)^T X_i \in L_2, i = 1, 2, \ldots, m\}$. Define

$$\rho(X,Y) = \max_{1 \leq i \leq m} \rho(X_i, Y_i), \qquad X, Y \in L_2^m$$

The norm $\| X \|_2$ of an element $X \in L_2^m$ is defined by

$$\| X \|_2 = \rho(X, \hat{0}) = \max_{1 \leq i \leq m} \| X_i \|^2 .$$

By the completeness of (L_2, ρ) and Eq.(1), Eq.(2) a standard proof applies that (L_2^m, ρ) is a complete metric space and ρ satisfies that

$$\rho(X+Z, Y+Z) = \rho(X,Y), \quad \rho(\lambda X, \lambda Y) = \mid \lambda \mid \rho(X,Y), \tag{3}$$

$$\rho(\lambda X, kY) \leq \mid \lambda - k \mid \parallel X \parallel_2, \tag{4}$$

for any $X, Y, Z \in L_2^m$ and $\lambda, k \in R$.
A second-order m-dimensional vector f.s.p. is characterized by a mapping of the interval T into L_2^m. For the sake of convenience, we shall adopt the notation $X(t) : T \to L_2^m$ in what follows. The m.s. continuity, m.s. differentiation, and m.s. integration associated with a second-order m-dimensional f.s.p. are defined with respect to the metric ρ in L_2^m. Hence, an m-dimensional f.r.p. $X(t), t \in T$, is m.s. continuous at t, for example, if $\rho(X(t+h), X(t)) \to 0$, as $h \to 0$. In view of this definition, it is clear that the m-dimensional f.s.p. $X(t)$ is m.s. continuous at $t \in T$ if and only if each of its component processes is m.s. continuous at t. Similar definitions and observations can be made with regard to m.s. differentiation and m.s. integration of the second-order m-dimensional f.s.p. $X(t)$.
We cosider fuzzy stochastic differential equations by

$$\begin{cases} X'(t) = F(t, X(t)) & t \in T = [t_0, b] \\ X(t_0) = X_0 \end{cases} \tag{5}$$

where F is a mapping: $T \times L_2^m \to L_2^m$ and $X_0 \in L_2^m$. We now consider the solution of Eq.(5) in the mean square sense. From Theorem 2 in [2]we know that $X(t)$ is a solution of Eq.(5) if and only if it is m.s. continuous and satisfies the integral equation

$$X(t) = X_0 + \int_{t_0}^{t} F(s, X(s))ds. \tag{6}$$

Theorem 2 [1]. Let F be m.s. continuous with respect to t and there exists a $k > 0$ such that

$$\rho(F(t,X), F(t;Y)) \leq k\rho(X,Y) \tag{7}$$

for all $t \in T$ and $X, Y \in L_2^m$. Then Eq.(5) has a unique solution.

3 Characterization Theorem for the solutions of FSDEs by using ODEs

Theorem 3. Let $F : T \to L_2^1$ be m.s. differentiable. Denote
$[F(t)]^\alpha = [\underline{F}^\alpha(t), \overline{F}^\alpha(t)] \quad \alpha \in [0,1]$.
Then the boundary functions $\underline{F}^\alpha(t), \overline{F}^\alpha(t)$ are m.s. differentiable

$$[F'(t)]^\alpha = [(\underline{F}^\alpha(t))', (\overline{F}^\alpha(t))'], \quad \alpha \in [0,1]. \qquad \blacksquare$$

Proof: According to definition 12, we can defined a mapping $F : [t_0, T] \to L_2^1$ is m.s. differentiable at $t \in [t_0, T]$ if there exits a $F'(t) \in L_2^1$ such that the limits

$$\lim_{h\to 0}(\tfrac{F^{'}(t+h) - F^{'}(t)}{h}) = \lim_{h\to 0}(\tfrac{F^{'}(t) - F^{'}(t-h)}{h})$$

exist and equal to $F'(t)$. Now

$$[F^{'}(t+h) - F^{'}(t)]^\alpha = [(\underline{F}^\alpha)^{'}(t+h) - (\underline{F}^\alpha)^{'}(t), (\overline{F}^\alpha)^{'}(t+h) - (\overline{F}^\alpha)^{'}(t)]$$

and similarly for $[F^{'}(t) - F^{'}(t-h)]^{\alpha}$. Dividing by h and passing to the limit gives the theorem ■

Let us consider the fuzzy stochastic differential equations initial value problem (FSDE)(IVP)

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(t_0) = x_0, \end{cases} \tag{8}$$

where, $f : T \times L_2^m \to L_2^m$ and $x_0 \in L_2^m$. Then the above theorem shows us a way to translate the FSDE Eq.(8) into a system of ODE.
Let $[x(t)]^{\alpha} = [\underline{x}^{\alpha}(t), \overline{x}^{\alpha}(t)]$. Now, suppose $x(t)$ is m.s. differentiable according to Theorem 4, $[x'(t)]^{\alpha} = [(\underline{x}^{\alpha}(t))', (\overline{x}^{\alpha}(t))']$. Clearly, Eq.(8) translates into the following system of ODEs

$$\begin{cases} (\underline{x}^{\alpha}(t))' = \underline{f}^{\alpha}(t, \underline{x}^{\alpha}(t), \overline{x}^{\alpha}(t)), \\ (\overline{x}^{\alpha}(t))' = \overline{f}^{\alpha}(t, \underline{x}^{\alpha}(t), \overline{x}^{\alpha}(t)), \\ \qquad \underline{x}^{\alpha}(0) = \underline{x}_0^{\alpha}, \\ \qquad \overline{x}^{\alpha}(0) = \overline{x}_0^{\alpha}, \end{cases} \tag{9}$$

where

$$[f(t,x)]^{\alpha} = [\underline{f}^{\alpha}(t, \underline{x}^{\alpha}(t), \overline{x}^{\alpha}(t)), \overline{f}^{\alpha}(t, \underline{x}^{\alpha}(t), \overline{x}^{\alpha}(t))].$$

In the following theorem we show that the FSDE Eq.(8) will be equivalent to system Eq.(9). The numerical solutions of the ODEs are extremely well studied in the literature, so any numerical method we can consider for the system of ODEs, since the solution will be as well as solution of the FSDE under study. We can use the numerical methods directly on the ODEs obtained by the following theorem.

Theorem 4. Let us consider the FSDE Eq.(8) where $f : T \times L_2^m \to L_2^m$ is such that

1. $[f(t,x)]^{\alpha} = [\underline{f}^{\alpha}(t, \underline{x}^{\alpha}(t), \overline{x}^{\alpha}(t)), \overline{f}^{\alpha}(t, \underline{x}^{\alpha}(t), \overline{x}^{\alpha}(t))]$,
2. there exist $L > 0$ such that
   $|\underline{f}^{\alpha}(t, \underline{x}(t), \underline{y}(t)), \underline{f}^{\alpha}(t, \overline{x}(t), \overline{y}(t))| \leq L \max\{|\overline{x} - \underline{x}|, |\overline{y} - \underline{y}|\}$
   and
   $|\overline{f}^{\alpha}(t, \underline{x}(t), \underline{y}(t), \overline{f}^{\alpha}(t, \overline{x}(t), \overline{y}(t))| \leq L \ \max\{|\overline{x} - \underline{x}|, |\overline{y} - \underline{y}|\}$ for all $\alpha \in [0,1]$.

Then the FSDE Eq.(8) and the system of ODE Eq.(9) are equivalent ■

Proof: According to Theorems 1, 2 $\underline{f}^{\alpha}$ and $\overline{f}^{\alpha}$ are the continuity of the function f. Further, the Lipschitz property in condition (2), we can show property as follows:

$\max\{|\underline{f}^{\alpha}(t, \underline{x}(t), \underline{y}(t)), \underline{f}^{\alpha}(t, \overline{x}(t), \overline{y}(t))|, |\overline{f}^{\alpha}(t, \underline{x}(t), \underline{y}(t)), \overline{f}^{\alpha}(t, \overline{x}(t), \overline{y}(t))|\} \leq L\, max\{|\overline{x} - \underline{x}|, |\overline{y} - \underline{y}|\}$,

by the Hausdorff distance d property

$$\rho(x,y) = \max\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\},$$

finally

$$\rho(f(t, x(t)), (f(t, y(t)) \leq \rho(x,y). \tag{10}$$

According to Theorem 2, it shows FSDE Eq.(8) has a unique solution. By Theorem 3, we can show that the solution of FSDE are m.s. differentiable and so, implies the functions $(\overline{x}^{\alpha})$ and $(\underline{x}^{\alpha})$ are m.s. differentiable, and as a conclusion $((\overline{x}^{\alpha}), (\underline{x}^{\alpha}))$ is a solution of Eq.(9). Conversely. Let us suppose that we have a solution $((\overline{x}^{\alpha}), (\underline{x}^{\alpha}))$, with $\alpha \in [0,1]$ fixed, of the system Eq.(9). Also, the Eq.(10) implies the existence and uniqueness of the fuzzy stochastic differential solution x'. Now, since x is m.s. differentiable, $(\overline{x}^{\alpha})$,$(\underline{x}^{\alpha})$ the endpoints of $(x)^{\alpha}$ (which are obviously valid level sets of a fuzzy-valued function) is a solution of Eq.(9). Since the solution of Eq.(9) is unique, we have $(x)^{\alpha}$, that is the problems Eq.(8) and Eq.(9) are equivalent.

4 Numerical Example

Example 1.5. Consider the following second-order linear fuzzy stochastic differential equation

$$Y''(t) = 5Y'(t) - 6Y(t) + \gamma(t), \qquad t \in [0, b] \tag{11}$$

If $\gamma(t)$, $t \in [0, \infty)$, is a Gaussian f.s.p., then $\gamma(t) = E(\gamma(t)) + \xi(t)$ (see [2] Theorem 5.3), where $\xi(t)$ is a real-valued Gaussian s.p. with $E(\xi(t)) = 0$ for all $t \geq 0$. Suppose that $E(\xi(t)) = u$, for all $t \geq 0$, u is a LR-fuzzy number, i.e.

$$u(y) = \begin{cases} L(\frac{b-y}{b-a}) & if \quad a \leq y < b \\ 1 & if \quad b \leq y \leq c \\ R(\frac{y-c}{d-c}) & if \quad c \leq y \leq d \\ 0 & otherwise \end{cases}$$

where L, R :$[0,1] \to [0,1]$ are two fixed left-continuous and nonincreasing functions with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. Thus

$$[\gamma(t)]^\alpha = [f(\alpha) + \xi(t), g(r) + \xi(t)]$$

$$[\underline{\gamma(s)}]^\alpha + [\overline{\xi(s)}]^\alpha = f(\alpha) + g(\alpha) + 2 * \xi(t)$$

$$[\underline{\gamma(s)}]^\alpha - [\overline{\xi(s)}]^\alpha = f(\alpha) - g(\alpha)$$

where

$$f(\alpha) = b - (b-a)L^{-1}(\alpha), g(\alpha) = c + (d-c)R^{-1}(\alpha).$$

Hence the unique solution of Eq.(11) is

$$\underline{Y(t)}^\alpha = (1/2) * (e^{2*t}(3,-1) + e^{3*t}(-2,1)(\underline{Y}_0^\alpha + \overline{Y}_0^\alpha) - (1/14) * (e^{6*t}(1,1) + e^{6*t}(1,1) + e^{-t}(6,-1)(\underline{Y}_0^\alpha - \overline{Y}_0^\alpha) + \int_0^t (-e^{2*(t-s)} + e^{(t-s)})\xi(s)ds + (f(\alpha) + g(\alpha)((1/6)(e^{3*t} - 1) - (1/4)(e^{2*t} - 1)) - (1/84)(f(\alpha) - g(\alpha))((1/84)(e^{6*t} - 1) - (1/14)(1 - e^{-t}))$$

$$\overline{Y(t)}^\alpha = (1/2) * (e^{2*t}(3,-1) + e^{3*t}(-2,1)(\underline{Y}_0^\alpha + \overline{Y}_0^\alpha) + (1/14) * (e^{6*t}(1,1) + e^{6*t}(1,1) + e^{-t}(6,-1)(\underline{Y}_0^\alpha - \overline{Y}_0^\alpha) + \int_0^t (-e^{2*(t-s)} + e^{(t-s)})\xi(s)ds + (f(\alpha) + g(\alpha)((1/6)(e^{3*t} - 1) - (1/4)(e^{2*t} - 1)) - (1/84)(f(\alpha) + g(\alpha))((1/84)(e^{6*t} - 1) - (1/14)(1 - e^{-t}))$$

By using Runge-Kutta, we present the numerical solution of this example at $t = 2$ in Fig.1.

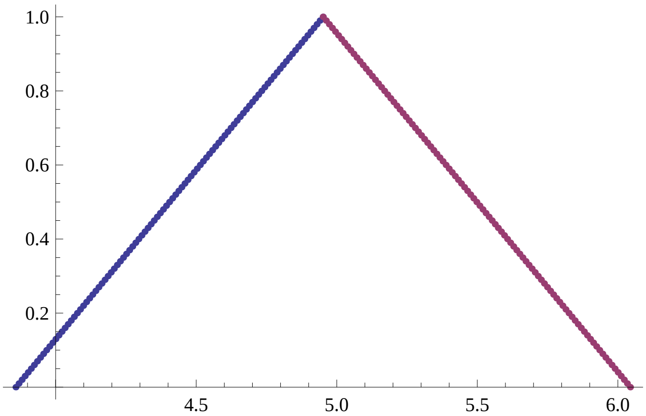

Fig1. Result of Example 1 for $t = 2$ by using Ruge-Kutta method

5 Conclusion

The main of our purpose in the paper is solve fuzzy stochastic differential equation by using method ODEs. That, we prove theorem for the expression of concept, this theorem shows i) if f is be fuzzy stochastic function ii) it satisfy the Lipschitz condition . Then the FSDE Eq.(8) and the system of ODE Eq.(9) are equivalent. We can solve this system by Euler or Rung-Kutta method.

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Some bounds for the generalized singular values

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Abstract

In this paper we give a short proof of a lower bound and a upper bound for the smallest and the largest generalized singular value by Diaz-Metcalf inequality. We can extend this process for finding a bound for eigenvalues of symmetric definite generalized eigenvalue problem ($Ax = \lambda Bx$).

Keywords: Generalized Eigenvalue Problem, Generalized Singular Value, Matrix Norm

Mathematics Subject Classification: 15A18, 15A60, 15A15.

1 Main Results

Theorem 1.1. *Let A and B be $n \times n$ matrices with complex(real) elements with singular values $\sigma_i(A)$ and $\sigma_i(B)$, $\quad i = 1, 2, \ldots, n$ and B be nonsingular.*
Let $K(A, B) = Cond(A)Cond(B)$ and $m \leq \sigma_n(B^{-1}A)$, $\quad M \geq \sigma_1(B^{-1}A)$. Then

$$\frac{1}{K(A,B)} \frac{\|A\|_E}{\|B\|_E} \leq m \leq \frac{\|A\|_E}{\|B\|_E}$$

and

$$\frac{\|A\|_E}{\|B\|_E} \leq M \leq K(A,B) \frac{\|A\|_E}{\|B\|_E}$$

Proof. We will apply the following result of Diaz and Metcalf inequality [3] which is a stronger form Pólya-Szegö and Kantrovich's inequality. For real numbers $a_k \neq 0$ and $b_k \neq 0$, $\quad (k = 1, \ldots, n)$ satisfy $m \leq \frac{a_k}{b_k} \leq M$, then $\sum_{k=1}^{n} a_k^2 + mM \sum_{k=1}^{n} b_k^2 \leq (M + m) \sum_{k=1}^{n} a_k b_k$.
Let $a_k = \sigma_k(A)$, $\quad b_k = \sigma_k(B)$, $\quad m = \frac{\sigma_n(A)}{\sigma_1(B)}$, $\quad M = \frac{\sigma_1(A)}{\sigma_n(B)}$. Then the Diaz-Metcalf's inequality follows, that

$$\sum_{k=1}^{n} \sigma_k^2(A) + mM \sum_{k=1}^{n} \sigma_k^2(B) \leq (M + m) \sum_{k=1}^{n} \sigma_k(A)\sigma_k(B) \tag{1}$$

We can proof of [5] :

$$\sigma_n(B^{-1}A) \geq \frac{\sigma_n(A)}{\sigma_1(B)} = m \tag{2}$$

$$\sigma_1(B^{-1}A) \leq \frac{\sigma_1(A)}{\sigma_n(B)} = M \tag{3}$$

$$\frac{M}{m} = \frac{\sigma_1(A)}{\sigma_n(A)} \frac{\sigma_1(B)}{\sigma_n(B)} = Cond(A)Cond(B) = K(A,B) = K \tag{4}$$

It holds that $\|A\|_E = \sum_{i=1}^{n} \sigma_i^2(A)$ and $\|B\|_E = \sum_{i=1}^{n} \sigma_i^2(B)$. By Cauchy-Schwarz inequality:

$$\sum_{i=1}^{n} \sigma_i(A)\sigma_i(B) \leq \|A\|_E\|B\|_E \tag{5}$$

By Diaz-Metcalf's inequality in (1) and inequality (5) we write:

$$\|A\|_E^2 + mM\|B\|_E^2 \leq (M+m)\|A\|_E\|B\|_E \tag{6}$$

Use of $M = Km$ in (4) and inequality (6). We shall analysis

$$0 \leq -Km^2\|B\|_E^2 + (K+1)m\|A\|_E\|B\|_E - \|A\|_E^2 \tag{7}$$

Thus

$$m_{\pm} = \frac{-(K+1)\|A\|_E\|B\|_E \pm (K-1)\|A\|_E\|B\|_E}{-2K\|B\|_E^2}$$

$$\frac{1}{K}\frac{\|A\|_E}{\|B\|_E} \leq m \leq \frac{\|A\|_E}{\|B\|_E}$$

and

$$\frac{\|A\|_E}{\|B\|_E} \leq M \leq K\frac{\|A\|_E}{\|B\|_E}$$

□

Example 1.2. *Let A and B be 2000×2000, random matrix is returned by Matlab software. 'gsvd' function in Matlab after $7.06189e+002s$ finds*

$$The \;\; smallest \;\; generalized \;\; singular \;\; value \;\; of \;\; (A,B) = 3.03609e-004$$

$$The \;\; largest \;\; generalized \;\; singular \;\; value \;\; of \;\; (A,B) = 1.75877e+003$$

If we have been condition numbers of A and B ($Cond(A) = 2.53057e+005$ and $Cond(B) = 1.37532e+005$)
by the above theorem, time is needed $0.02846s$ otherwise is needed $80.60833s$ for the following bounds:

$$2.874076e-011 \leq m \leq 1.00028$$

and

$$1.00028 \leq M \leq 3.48130e+010$$

Note. If matrices A and B can be artificially ill-conditioned, their entries should be properly scaled, see[2].

Corollary 1.3. *Let A and B be symmetric positive definite with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$ and $\lambda_1(B) \geq \lambda_2(B) \geq \ldots \geq \lambda_n(B)$ and*
$m \leq \lambda_n(B^{-1}A)$, $M \geq \lambda_1(B^{-1}A)$ then:

$$\frac{1}{K(A,B)}\frac{\|A\|_E}{\|B\|_E} \leq m \leq \frac{\|A\|_E}{\|B\|_E}$$

and

$$\frac{\|A\|_E}{\|B\|_E} \leq M \leq K(A,B)\frac{\|A\|_E}{\|B\|_E}$$

Proof. Since A and B are symmetric positive definite thus $|\lambda_i(A)| = \sigma_i(A)$ and $|\lambda_i(B)| = \sigma_i(B)$. Set $m = \frac{\lambda_n(A)}{\lambda_1(B)}$, $M = \frac{\lambda_1(A)}{\lambda_n(B)}$. Use of Diaz-Metcalf's inequality, Cauchy-Schwarz inequality and

$$\frac{\lambda_n(A)}{\lambda_1(B)} \leq \lambda_n(B^{-1}A), \quad \frac{\lambda_1(A)}{\lambda_n(B)} \geq \lambda_1(B^{-1}A)$$

□

2 Summary

Some applied problems need a bound for the smallest or the largest generalized singular value without gain exact values that spend so much time. In this paper we obtained some bounds for the generalized singular values of matrices A and B. After pointing out that bound of singular values how vary when A and B are square and B is nonsingular, we applied this results for symmetric definite generalized eigenvalue problem ($Ax = \lambda Bx$).

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Numerical method of second order fuzzy differential equation (SOFDE) by characterization theorem

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Abstract

In this paper, we study the problem regarding the initial value of Second-Order fuzzy differential equation. We present a Theorem, called Characterization Theorem, to be equivalent to system ODEs with SOFDE. By using this Theorem, we can solve SOFDE directly with numerical method. We prove this Theorem with two applied examples.

Keywords: Fuzzy differential equations, Fuzzy derivative, Second-Order Fuzzy Differential equation

1 Introduction

Definition 1 [2]. Let $\widetilde{f} : [t_0, T] \to E^n$ and $y_0 \in [t_0, T]$. We say that $\widetilde{f}$ is Hukuhara differentiable at y_0 if there exists an element $\widetilde{f}' \in E^n$ such that for all $h > 0$ sufficiently small, there are $\widetilde{f}(y_0 + h) \ominus \widetilde{f}(y_0)$, $\widetilde{f}(y_0) \ominus \widetilde{f}(y_0 - h)$ and the limits

$$\lim_{h\to 0}\left(\frac{\widetilde{f}(y_0+h) \ominus \widetilde{f}(y_0)}{h}\right) = \lim_{h\to 0}\left(\frac{\widetilde{f}(y_0) \ominus \widetilde{f}(y_0-h)}{h}\right) = \widetilde{f}'(y_0). \tag{1.1}$$

The fuzzy set $\widetilde{f}'(y_0)$ is called the Hukuhara derivative of $\widetilde{f}$ at y_0.

2 Second-Order fuzzy differential equation (SOFDE)

The following Second-Order problem has been studied in [1],

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & t \in [t_0, T], \\ x(t_0) = k_1, \quad x'(t_0) = k_2, \end{cases} \tag{2.2}$$

with $f : [t_0, T] \times E^n \times E^n \to E^n$ continuous and k_1, k_2 real constants. This problem is written in its integral form:

$$x(t) = k_1 + k_2(t - t_0) + \int_{t_0}^{t}\int_{t_0}^{z} f(s, x(s), x'(t))ds\, dz \quad t \in [t_0, T].$$

We know that the space $C(I, E^n)$ of continuous functions $x : I \to E^n$ is a complete metric space with the distance

$$H(x, y) = \sup\{d(x(t), y(t))e^{-\rho t}\},$$

where $\rho \in R$ is fixed [1]. By $C^1(I, E^n)$, we denote the set of continuous functions $x : I \to E^n$ such that $x' : I \to E^n$ exists as a continuous function. For $x, y \in C^1(I, E^n)$, we define the distance

$$H_1(x, y) = H(x, y) + H(x', y').$$

In the recent paper [1], the authors present a Theorem 2.1 for the existence of a unique solution $Eq.(2.2)$.

Theorem 1[1]. Let $f : [t_0, T] \times E^n \times E^n \to E^n$ be continuous, and suppose that there exist $M_1, M_2 > 0$ such that

$$d(f(t, x_1, x_2), f(t, y_1, y_2)) \leq M_1 d(x_1, y_1) + M_2 d(x_2, y_2) \tag{2.3}$$

for all $t \in [t_0, T], x_1, x_2, y_1, y_2 \in E^n$. Then the initial value problem $Eq.(2.3)$ has a unique solution on $[t_0, T]$.
The definition 1 is a straightforward generalization of the Hukuhara differentiability of a set-valued function. So if F is differentiable at $t_0 \in [t_0, T]$, then all its α-levels $F_\alpha(t) = [F(t)]^\alpha$ are Hukuhara differentiable at t_0 and $[F'(t_0)]^\alpha = DF_\alpha(t_0)$, where DF_α denotes the Hukuhara derivative of $F\alpha$.
Let us consider the fuzzy initial value problem SOFDE

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), \quad t \in [t_0, T], \\ x(t_0) = k_1, \quad x'(t_0) = k_2, \end{cases} \tag{2.4}$$

with $f : [t_0, T] \times E^n \times E^n \to E^n$ continuous, and k_1, k_2 real constants. Let $[x(t)]^\alpha = [\underline{x}^\alpha(t), \overline{x}^\alpha(t)]$. If $x(t), x'(t)$ is Hukuhara differentiable from Theorem 2 in [4], then $[x'(t)]^\alpha = [(\underline{x}^\alpha(t))', (\overline{x}^\alpha(t))']$ and $[x''(t)]^\alpha = [(\underline{x}^\alpha(t))'', (\overline{x}^\alpha(t))'']$. Clearly, $Eq.(2.4)$ translates into the following system of ODEs

$$\begin{cases} (\underline{x}^\alpha(t))'' = \underline{f}^\alpha(t, \underline{x}^\alpha(t), \overline{x}^\alpha(t), (\underline{x}^\alpha(t))', (\overline{x}^\alpha(t))'), \\ (\overline{x}^\alpha(t))'' = \overline{f}^\alpha(t, \underline{x}^\alpha(t), \overline{x}^\alpha(t), (\underline{x}^\alpha(t))', (\overline{x}^\alpha(t))'), \\ \underline{x}^\alpha(t_0) = \underline{k}_1, \\ \overline{x}^\alpha(t_0) = \overline{k}_1, \\ (\underline{x}^\alpha(t_0))' = \underline{k}_2, \\ (\overline{x}^\alpha(t_0))' = \overline{k}_2, \end{cases} \tag{2.5}$$

where

$$[f(t, x, x')]^\alpha = [\underline{f}^\alpha(t, \underline{x}^\alpha(t), \overline{x}^\alpha(t), (\underline{x}^\alpha(t))', (\overline{x}^\alpha(t))'), \overline{f}^\alpha(t, \underline{x}^\alpha(t), \overline{x}^\alpha(t), (\underline{x}^\alpha(t))', (\overline{x}^\alpha(t))')].$$

In the following theorem we show that the SOFDE $Eq.(2.4)$ will be equivalent to system $Eq.(2.5)$. The numerical solutions of the ODEs are extremely well studied in the literature, so we can consider any numerical method for the system of ODEs, since the solution will be the same as the solution of the SOFDE under study. We can use the numerical methods directly on the ODEs obtained by the following theorem.

Theorem 2 . Let us consider the SOFDE $Eq.(2.4)$ where $f : [t_0, t_0 + a] \times E^n \times E^n \to E^n$ is such that

1. $[f(t,x,x')]^\alpha = [\underline{f}^\alpha(t,\underline{x}^\alpha(t),\overline{x}^\alpha(t),(\underline{x}^\alpha(t))',(\overline{x}^\alpha(t))'),$
$\overline{f}^\alpha(t,\underline{x}^\alpha(t),\overline{x}^\alpha(t),(\underline{x}^\alpha(t))',(\overline{x}^\alpha(t))')],$

2. there exist $L > 0$ such that
$|\underline{f}^\alpha(t,\underline{x}(t),\underline{y}(t),(\underline{x}(t))',(\underline{y}(t))'),\underline{f}^\alpha(t,\overline{x}(t),\overline{y}(t),(\overline{x}(t))',(\overline{y}(t))')|$
$\leq L\max\{|\overline{x}-\underline{x}|,|\overline{y}-\underline{y}|,|\overline{x}'-\underline{x}'|,|\overline{y}'-\underline{y}'|\}$
and
$|\overline{f}^\alpha(t,\underline{x}(t),\underline{y}(t),(\underline{x}(t))',(\underline{y}(t))'),\overline{f}^\alpha(t,\overline{x}(t),\overline{y}(t),(\overline{x}(t))',(\overline{y}(t))')|$
$\leq L\ max\{|\overline{x}-\underline{x}|,|\overline{y}-\underline{y}|,|\overline{x}'-\underline{x}'|,|\overline{y}'-\underline{y}'|\}$ for all $\alpha\in[0,1]$,

3. $\underline{f}^\alpha \text{and} \overline{f}^\alpha$ are equi continuous.

Then the SOFDE $Eq.(2.4)$ and the system of ODE $Eq.(2.5)$ are equivalent.

Proof: The equicontinuous $\underline{f}^\alpha$ and $\overline{f}^\alpha$ implies the continuity of the function f. Further, the Lipschitz property in condition (2), we can show property as follows:

$$\sup\max\{|\underline{f}^\alpha(t,\underline{x}(t),\underline{y}(t),(\underline{x}(t))',(\underline{y}(t))'),\underline{f}^\alpha(t,\overline{x}(t),\overline{y}(t),(\overline{x}(t))',(\overline{y}(t))')|,$$

$$|\overline{f}^\alpha(t,\underline{x}(t),\underline{y}(t),(\underline{x}(t))',(\underline{y}(t))'),\overline{f}^\alpha(t,\overline{x}(t),\overline{y}(t),(\overline{x}(t))',(\overline{y}(t))')|\}$$

$$\leq L\sup\max\{|\overline{x}-\underline{x}|,|\overline{y}-\underline{y}|,|\overline{x}'-\underline{x}'|,|\overline{y}'-\underline{y}'|\},$$

by the Hausdorff distance d_H property

$$d_H(x,y)=\sup\max\{|\underline{x}-\underline{y}|,|\overline{x}-\overline{y}|\},$$

$$d_H(x',y')=\sup\max\{|\underline{x}'-\underline{y}'|,|\overline{x}'-\overline{y}'|\},$$

and by distance d property

$$d(u+w,v+z)\leq d(u,v)+d(w,z),$$

consequence

$$\sup\max\{|\underline{f}^\alpha(t,\underline{x}(t),\underline{y}(t),(\underline{x}(t))',(\underline{y}(t))'),\underline{f}^\alpha(t,\overline{x}(t),\overline{y}(t),(\overline{x}(t))',(\overline{y}(t))')|,$$

$$|\overline{f}^\alpha(t,\underline{x}(t),\underline{y}(t),(\underline{x}(t))',(\underline{y}(t))'),\overline{f}^\alpha(t,\overline{x}(t),\overline{y}(t),(\overline{x}(t))',(\overline{y}(t))')|\}$$

$$\leq L\sup\max\{|\underline{x}'-\underline{y}'|\}+L\sup\max\{|\overline{x}'-\overline{y}'|\},$$

finally

$$\sup\max\{|\underline{f}^\alpha(t,\underline{x}(t),\underline{y}(t),(\underline{x}(t))',(\underline{y}(t))'),\underline{f}^\alpha(t,\overline{x}(t),\overline{y}(t),(\overline{x}(t))',(\overline{y}(t))')|,$$

$$|\overline{f}^\alpha(t,\underline{x}(t),\underline{y}(t),(\underline{x}(t))',(\underline{y}(t))'),\overline{f}^\alpha(t,\overline{x}(t),\overline{y}(t),(\overline{x}(t))',(\overline{y}(t))')|\}$$

$$\leq Md(x,y)+M_1d(x',y'). \tag{2.6}$$

According to Theorem 1, it shows SOFDE $Eq.(2.4)$ to have a unique solution. By proposition 1, we can show that the solution of SOFDE is Hukuhara differentiable and so, implies the functions $(\overline{x}^\alpha)$ and $(\underline{x}^\alpha)$ are differentiable, and as a conclusion $((\overline{x}^\alpha),(\underline{x}^\alpha))$ is a solution of $Eq.(2.5)$. Conversely. In [3], Kaleva states that if we ensure that the solution $(\overline{x}^\alpha, \underline{x}^\alpha)$ of the system first order ODEs are valid level sets of a fuzzy number valued function and if the derivatives $((\overline{x}^\alpha)', (\underline{x}^\alpha)')$ are valid level sets of a fuzzy-valued function, then by using the stacking Theorem we can construct the solution of the FIVP . Now, we can expand this process for system $Eq.(2.5)$ and SOFDE $Eq.(2.4)$, this solution of system $Eq.(2.5)$ exists by Lipschitz condition (2), and the solution is a unique. Let us suppose that we have a solution $((\overline{x}^\alpha)', (\underline{x}^\alpha)')$, with $\alpha \in [0,1]$ fixed, of the system $Eq.(2.5)$. Also, the $Eq.(2.6)$ implies the existence and uniqueness of the fuzzy solution $\widetilde{x}$. Now, since $\widetilde{x}$ is Hukuhara differentiable, $(\overline{\widetilde{x}}^\alpha)$,$(\underline{\widetilde{x}}^\alpha)$ the endpoints of $(\widetilde{x})^\alpha$ (which are obviously valid level sets of a fuzzy-valued function) is a solution of $Eq.(2.5)$. Since the solution of $Eq.(2.5)$ is unique, we have $(\widetilde{x})^\alpha$, that is the problems $Eq.(2.4)$ and $Eq.(2.5)$ are equivalent.

3 Numerical Example

Consider the vibrating mass the spring constant is $k = 4\frac{lb}{ft}$, there is no damping force and the forcing function is $100\cos\varepsilon t$ for $\varepsilon > 0$ in [4]. The differential equation of motion is

$$\begin{cases} y''^\alpha(t) + 4y(t)^\alpha = 100\cos(\varepsilon t), \\ [y(0)]^\alpha = [-1+\alpha, 1-\alpha] \qquad 0 \le \alpha \le 1, \\ [y'(0)]^\alpha = [-1+\alpha, 1-\alpha] \end{cases} \tag{3.7}$$

The unique solution is

$$y = [(-1+\alpha)\cos(2t) + \tfrac{-1+\alpha}{2}\sin(2t) + \Psi(t), (1-\alpha)\cos(2t) + \tfrac{1-\alpha}{2}\sin(2t) + \Psi(t)]$$

for

$$\Psi(t) = \tfrac{100}{4-\varepsilon^2}(\cos(\varepsilon t) - \cos(2t)).$$

By using Runge-Kutta, we present the numerical solution of this example at $t = 2$ in Table 1.

r	$\underline{y}$	$\overline{y}$	$Exact_p$	$Exact_b$	$Error_p$	$Error_b$
$r=0$	-0.587346	1.57145	-0.587561	1.57171	0.000215275	0.000251586
$r=0.1$	-0.479406	1.46351	-0.482071	1.46622	0.00266468	0.00270099
$r=0.2$	-0.371466	1.35557	-0.37658	1.36072	0.00511409	0.0051504
$r=0.3$	-0.263526	1.24763	-0.271089	1.25523	0.0075635	0.00759981
$r=0.4$	-0.155586	1.13969	-0.165599	1.14974	0.0100129	0.0100492
$r=0.5$	-0.047646	1.03175	-0.0601083	1.04425	0.0124623	0.0124986
$r=0.6$	0.060294	0.923814	0.0453823	0.938762	0.0149117	0.014948
$r=0.7$	0.168234	0.815874	0.150873	0.833271	0.0173611	0.0173974
$r=0.8$	0.276174	0.707934	0.256364	0.727781	0.0198105	0.0198468
$r=0.9$	0.384114	0.599994	0.361854	0.62229	0.0222599	0.0222962
$r=1$	0.492054	0.492054	0.467345	0.5168	0.0247093	0.0247457

Table.1 Ruge-Kutta method with $t = 2$ for example 2

4 Conclusion

In this paper, we present a Theorem which shows that the Second-Order Fuzzy Differential SOFDE and the system ODEs are equivalent. We solve the example [4]using Runge-Kutta method.

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A well-posedness of the Sine-Gordon equation using homotopy perturbation method

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Abstract

In this paper, we investigate a well-posedness of the nonlinear sine-Gordon equation based on the homotopy perturbation method (HPM). At first, we prove the existence and uniqueness of the solution, and then the stability is analyzed, such that we create a small change in the initial conditions. In this case, the HPM is applied to solve the nonlinear sine-Gordon equation.

Keywords: Sine-Gordon equation, Homotopy perturbation method, Well-posedness, Stability analysis.

Mathematics Subject Classification: 35L70, 65N12.

1 Introduction

The sine-Gordon equation is a nonlinear hyperbolic partial differential equation (PDE). An interested property of the sine-Gordon equation is the existence of soliton and multisoliton solutions [1], such that it is a well-known soliton equation.

Consider the one-dimensional nonlinear sine-Gordon equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} - \sin(u(x,t)), \quad (x,t) \in [a,b] \times [t_0, T], \tag{1}$$

with the initial conditions

$$\begin{cases} u(x,t_0) = f_0(x), & x \in [a,b], \\ \dfrac{\partial u}{\partial t}(x,t_0) = f_1(x), & x \in [a,b], \end{cases} \tag{2}$$

and the boundary conditions

$$\begin{cases} u(a,t) = g_0(t), & t \geq t_0, \\ u(b,t) = g_1(t), & t \geq t_0. \end{cases} \tag{3}$$

The main property of Eq. (1) is the conservation of energy. The energy (E(t)), of Eq. (1) is given by the following expression [2]:

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} [(u_x)^2 + (u_t)^2 + 2(1 - \cos(u))]\, dx. \tag{4}$$

1.1 Homotopy perturbation method

We introduce a variable parameter $p \in [0,1]$ in Eq. (1) such that

$$u_{tt} = u_{xx} - \sin(pu) = 0. \tag{5}$$

Based on the perturbation technique used in [3], we assume the solution of Eq. (5) can be expressed in terms of p as follows:

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \dots. \tag{6}$$

Setting $p = 1$, results in the approximate solution of Eq. (1) as follows:

$$u^* = \lim_{p\to 1} u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + \dots. \tag{7}$$

To obtain the approximate solution of Eq. (1), we consider the Taylor series expansion of $\sin(u)$ as follows:

$$\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} + \dots + (-1)^{n-1}\frac{u^{2n-1}}{(2n-1)!} + \dots. \tag{8}$$

Using the HPM [3], we can construct the homotopy $u(r,p) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$u_{tt} - y_{0tt} + py_{0tt} + p(-u_{xx} + \sin(pu)) = 0, \tag{9}$$

where y_0 is the initial approximation. Substituting Eqs. (6) and (8) into Eq. (9), and equating the coefficients of the terms with the identical power of p, is implied that (in this paper, we have obtained the fourth-order approximation):

$$\begin{aligned}
&p^0 : u_{0tt} - y_{0tt} = 0,\ y_0 = f_0(x) + tf_1(x),\\
&p^1 : u_{1tt} - u_{0xx} + y_{0tt} = 0,\ u_1(x,t_0) = u_{1t}(x,t_0) = 0,\\
&p^2 : u_{2tt} - u_{1xx} + u_0 = 0,\ u_2(x,t_0) = u_{2t}(x,t_0) = 0,\\
&p^3 : u_{3tt} - u_{2xx} + u_1 = 0,\ u_3(x,t_0) = u_{3t}(x,t_0) = 0,\\
&p^4 : u_{4tt} - u_{3xx} + u_2 - \frac{u_0^3}{3!} = 0,\ u_4(x,t_0) = u_{4t}(x,t_0) = 0.
\end{aligned} \tag{10}$$

Then, $\{u_i\}_{i=0}^4$ is obtained by simple integral from Eqs. (10). Thus, we have

$$u_{approx} = u_0 + u_1 + u_2 + u_3 + u_4.$$

2 A well-posedness of the problem

In this section, we show that the sine-Gordon equation is a well-posed problem. According to Hadamard, a problem is well-posed if, it has a solution, the solution is unique and depends continuously on initial data and parameters.

2.1 Existence and uniqueness of the solution

By integrating two times from Eq. (1) with respect to t and using the initial conditions, we obtain

$$u(x,t) = G(x,t) + \int_0^t (t-\tau)\frac{\partial^2 u(x,\tau)}{\partial x^2}\,d\tau - \int_0^t (t-\tau)\sin(u(x,\tau))\,d\tau, \tag{11}$$

where $G(x,t) = f_0(x) + tf_1(x)$. Thus $u(x,t)$ is a solution of Eq. (1).
Let $u_1(x,t)$ and $u_2(x,t)$, be two solutions of Eq. (1). We show that $u_1(x,t) = u_2(x,t)$.
Using the idea in [4], we assume $v(x,t) = u_1(x,t) - u_2(x,t)$, then v satisfies the wave equation with initial conditions: $v(x,0) = v_t(x,0) = 0$, and boundary conditions $v(a,t) = v(b,t) = 0$, our goal is to prove: $v(x,t) = 0$. From Eq. (4), we have

$$E(t) = \frac{1}{2}\int_{\mathbb{R}}[(v_x)^2 + (v_t)^2 + 2(1-\cos(v))]\,dx.$$

We prove that $E(t) = 0$. Differentiating with respect to t we obtain

$$\begin{aligned}\dot{E}(t) &= \frac{1}{2}\int_a^b [2v_t(v_{xx} - \sin v) + 2v_x v_{xt} + 2v_t \sin v]\,dx \\ &= \frac{1}{2}\int_a^b [2(v_t v_{xx} + v_x v_{xt})]\,dx \\ &= \int_a^b \frac{\partial}{\partial x}(v_t v_x)\,dx = [v_x(x,t)v_t(x,t)]_a^b \\ &= [v_x(b,t)v_t(b,t) - v_x(a,t)v_t(a,t)] = 0.\end{aligned}$$

Since $\dot{E}(t) = 0$, E(t) is constant, and as $E(0) = 0$, we conclude that $E(t) = 0$. Then $v_t(x,t) = 0$ and $v(x,t) = v(x,t) - v(x,0) = \int_a^b v_t(x,t)dt = 0$. Thus $u_1(x,t) = u_2(x,t)$.

2.2 Stability analysis

Consider Eq. (1) with $t_0 = 0$, and the initial conditions [2, 5]:

$$f_0(x) = 0, \quad f_1(x) = 4\mathrm{sech}(x),$$

with the exact solution $u(x,t) = 4\arctan(\mathrm{sech}(x)t)$, we create a small change in the initial conditions as follows:

$$f_0(x) = 0, \quad f_1(x) = 4\mathrm{sech}(x) + \varepsilon\sin(\varepsilon x),$$

where $0 < \varepsilon \ll 1$. Using the HPM, we obtain the analytic solution as follows:

$$\begin{aligned}v(x,t) &= \frac{-1}{362880}\frac{1}{\cosh^9(x)}\Big(t(60480t^2\cosh^9(x)\varepsilon^3\sin(\varepsilon x) - 3024t^4\cosh^9(x)\varepsilon^5\sin(\varepsilon x) \\ &\quad +60480t^2\cosh^9(x)\varepsilon\sin(\varepsilon x) + 72t^6\cosh^9(x)\varepsilon^7\sin(\varepsilon x) - 9072t^4\cosh^9(x)\varepsilon^3\sin(\varepsilon x) \\ &\quad +36288t^4\cosh^8(x)\varepsilon^2\cos(\varepsilon x)^2 - t^8\cosh^9(x)\varepsilon^9\sin(\varepsilon x) + 216t^6\cosh^9(x)\varepsilon^5\sin(\varepsilon x) \\ &\quad -3024t^4\cosh^9(x)\varepsilon\sin(\varepsilon x) - 1451520t^4\cosh^7(x)\varepsilon\sin(\varepsilon x) - 161280t^8 \\ &\quad -362880\cosh^9(x)\varepsilon\sin(\varepsilon x) - 1451520\cosh^8(x) - 290304t^4\cosh^4(x) \\ &\quad -36288t^4\cosh^8(x)\varepsilon^2 + 576t^6\cosh^8(x) + 35136t^6\cosh^6(x) - 221184t^6\cosh^4(x) \\ &\quad +241920t^8\cosh^2(x) + 6560t^8\cosh^6(x) - 4t^8\cosh^8(x) - 92736t^8\cosh^4(x) \\ &\quad +3024t^4\cosh^9(x)\varepsilon^3\sin(\varepsilon x)\cos^2(\varepsilon x) + 483840t^2\cosh^6(x) + 207360t^6\cosh^2(x))\Big).\end{aligned}$$

In this case, the following is implied:

$$\|v(x,t) - u(x,t)\| < \varepsilon, \quad x \in \mathbb{R},\ t \in [0,1).$$

Thus a small change in the initial data leads to a small change in the solution. Since the HPM is applied at $t \in [0,1]$, using change of variable $t = (T - t_0)y + t_0$, we can apply HPM for each interval $[t_0, T]$.

3 Conclusions

We have investigated a well-posedness of the sine-Gordon equation. For stability of the problem, we have created a small change in the initial conditions and then have applied the HPM for obtaining the analytic solution of the nonlinear sine-Gordon equation. According to Hadamard, the conditions of existence, uniqueness and stability hold for Eq. (1). Thus we conclude that the sine-Gordon equation is a well-posed problem.

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Solution of the nonlinear Lane-Emden type equations arising in astrophysics

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Abstract
We developed the sinc-Galerkin method based on double exponential transformation (DE-transformation) for solution of Lane-Emden type equations arising in astrophysics.The convergence rate $O(\exp(-cN/\log N))$ where N is a parameter representing the number of terms in the sinc approximation is attained. To illustrate the reliability of this method, some specific equations are considered as test examples.
Keywords: Lane-Emden type equations, Sinc function, Galerkin method, Nonlinear singular boundary-value problems.

Mathematics Subject Classification: 65L10; 65L20; 65L70

1 Introduction

We consider the following Lane-Emden equations which are nonlinear singular boundary value problems with mixed boundary conditions

$$L(y(x)) \equiv y''(x) + \frac{2}{x}y'(x) + y^n = 0, \quad x \geq 0 \tag{1}$$

$$\begin{aligned} &y(0) = \alpha, \\ &\beta y(1) + \gamma y'(1) = \delta, \end{aligned} \tag{2}$$

where the parameter n has physical significance in the range $0 \leq n \leq 5$. Lane-Emden equation (1) with boundary conditions (2) has analytical solutions for $n = 0, 1, 5$[1] and for other values of n, we have to use numerical or approximate methods, for example [2-11].
In this paper, following [12] we developed the sinc-Galerkin method for equation(1) which is based on DE transformation [13].
Let $t = \phi(z)$ denote a conformal map which maps the simply connected domain D with boundary ∂D onto a strip region $D_d \equiv \{t = \nu + i\omega : |Imt < d|\}$.
To construct approximation for Lane-Emden equation(1) on the interval (0,1), we employ a trans-

formation $t = \phi$, whose inverse $\phi^{-1} = \psi$, defined by

$$t = \phi(z) = \log(\frac{1}{\pi}\log(\frac{z}{1-z}) + \sqrt{\frac{1}{\pi^2}\log(\frac{z}{1-z})^2 + 1}), \tag{3}$$

$$z = \psi(t) = \phi^{-1}(t) = \frac{1}{2}\tanh(\frac{\pi}{2}\sinh t) + \frac{1}{2}. \tag{4}$$

Corresponding to the uniform grid points defined by $t_k = kh, k = 0, \pm1, \pm2, \cdots$, in D_d we specifically have the sinc grid points $z_k = \psi(kh) = \phi^{-1}(kh), k = 0, \pm1, \pm2, \cdots$, for the DE transformation.

Sinc methods are based on the sinc approximation expressed as

$$f(z) = \sum_{k=-N}^{N} f(z_j)S(j,h)\circ\phi(z) + O(\exp(-\frac{\pi d N}{\log(\pi d N \alpha)})), \quad z_j = \phi^{-1}(jh) \tag{5}$$

where

$$S(j,h)(t) = \frac{\sin[(\frac{\pi}{h})(t-jh)]}{(\frac{\pi}{h})(t-jh)} \tag{6}$$

The following DE formula proposed by Takahasi-Mori, is well known[14]:

$$\int_a^b f(x)dx = h\sum_{k=-N}^{N}\frac{f(x_j)}{\phi'(x_j)} + O(\exp(-\frac{2\pi d N}{\log(2\pi d N\alpha)})), \quad x_j = \phi^{-1}(jh), \tag{7}$$

with mesh size $h = \frac{\log(2\pi dN/\lambda)}{N}$.

2 Sinc-Galerkin method based on the DE transformation

We approximate the solution $y(x)$ of Lane-Emden equation (1) by

$$y_N(x) \equiv \alpha(1-x) + \frac{\gamma\alpha+\delta}{\beta+2\gamma}x^2 + \sum_{k=-N}^{N} c_k(x-1)S(k,h)\circ\phi(x) + A\omega(x), \tag{8}$$

$$\omega(x) = x(1 - \frac{(\beta+\gamma)x}{\beta+2\gamma}) \tag{9}$$

The unknown coefficients in (8) determined in the sinc-Galerkin method by orthogonalizing the following residual

$$(Ly_N, S(k,h)\circ\phi) = 0, k = -N, -N+1, \ldots, N, \tag{10}$$

with respect to the 2N+2 basis functions where the inner product is defined as

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\frac{1}{\phi'(x)}dx. \tag{11}$$

If we use the DE formula (3) for evaluation of the inner products in (1) and $y_N(x_k) = (x_k - 1)c_k$ then we have a discrete sinc-Galerkin system as follows:

$$\begin{aligned}
&\sum_{k=-N}^{N}(\frac{1}{h^2}\delta_{jk}^{(2)} + \frac{1}{h}\delta_{jk}^{(1)}((\frac{1}{\phi'})'(x_k) - \frac{2}{x_k\phi'(x_k)}))(x_k-1)c_k \\
&+ (\frac{1}{\phi'(x_j)}(\frac{1}{\phi'})''(x_j) - \frac{1}{\phi'(x_j)}(\frac{2}{x\phi'})'(x_j))(x_j-1)c_j \\
&+ \frac{(\alpha(1-x_j) + \frac{\gamma\alpha+\delta}{\beta+2\gamma}x_j^2 + (x_j-1)c_j + A\omega(x_j))^n}{(\phi'(x_j))^2} \\
&= 0, j = -N, -N+1, \ldots, N,
\end{aligned} \tag{12}$$

where

$$\delta_{jk}^{(m)} = h^m \frac{d^m}{dt^m} S(j,h)(t)|_{t=kh}, m = 0, 1, 2. \tag{13}$$

3 Main Results

The numerical results verified that the present method is an applicable method and converges to the exact solution rapidly and with $O(\exp(-cN/\log N))$ accuracy.

Example 3.1. *For $n = 0$, Eq.(1) has exact solution $y(x) = 1 - \frac{x^2}{6}$. The obtained results with the sinc-Galerkin method for $N = 10$ are given in Table1.*

Example 3.2. *For $n = 1$, Eq.(1) has exact solution $y(x) = \frac{\sin(x)}{x}$. The obtained results with the sinc-Galerkin method for $N = 5$ are given in Table1.*

Example 3.3. *For $n = 5$, Eq.(1) has exact solution $y(x) = (1 + \frac{x^2}{3})^{\frac{-1}{2}}$. The obtained results with the sinc-Galerkin method for $N = 5$ are given in Table 1.*

Table 1: *Absolute errors in the solutions at grid points for examples 1, 2, and 3.*

x	$h = \frac{1}{N}\log(\frac{\pi N}{2})$	Errors Ex.1	Errors Ex.2	Errors Ex.3
0.0	0.275	$1.55E-15$	0.00	0.00
0.5	0.412	$1.66E-14$	$4.01E-5$	$1.43E-3$
1.0	0.412	$1.31E-14$	$9.56E-6$	$7.09E-3$

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A predictor-corrector scheme for solving Riccati differential equations of fractional order

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Abstract

In this paper, we extend the Adams-Bashforth-Moulton algorithm to solve the Riccati differential equations of fractional order. Numerical illustrations to demonstrate utility of the method are given. This method are based on numerical integration techniques applied to an equivalent nonlinear and Volterra integral equation. The classical approach leads to an algorithm with very high arithmetic complexity. Therefore we derive an alternative that leads to lower complexity without sacrificing too much precision.

Keywords: Riccati equation; Fractional derivative; predictor-corrector algorithm.

Mathematics Subject Classification: 26A33, 34A08

1 Introduction

Fractional differential equations have been found to be effective to describe some physical phenomena such as damping laws, rheology, diffusion processes, and so on. Oldham and Spanier [2] have played a key role in the development of the subject. The algorithm is a predictor-corrector (more precisely, PECE) method investigated in a more detailed way in [1]. It can be interpreted in the spirit of the classical Adams-Bashforth-Moulton schemes for first-order equations Specifically we analyse the discretization error of this approach under various assumptions on the given data. It must be emphasized that other algorithms for certain fractional differential equations are available, but these typically have a restricted applicability in the sense that they normally encounter difficulties when handling non-linear equations. The numerical solutions, therefore turn out to be important. As a pursuance to this we extend the fractional predictor-corrector scheme to solve Riccati equation of fractional order.

2 Preliminaries

We enlist below some definitions [3] and basic results.

Definition 2.1. *A real function $f(t)$, $t > 0$ is said to be in the space $C_\alpha, \alpha \in \Re$ if there exists a real number $p\,(> \alpha)$, such that $f(t) = x^p f_1(t)$ where $f_1(t) \in C[0, \infty)$. Clearly $C_\alpha \subset C_\beta$ if $\beta \leq \alpha$.*

Definition 2.2. *A function $f(t)$, $t > 0$ is said to be in the space $C_\alpha^m, m \in N \bigcup \{0\}$, if $f^{(m)} \in C_\alpha$.*

Definition 2.3. *The (left sided) Riemann-Liouville fractional integral of order $\mu > 0$, of a function $f \in C_\alpha, \alpha \geq -1$ is defined as:*

$$\begin{aligned} I^\mu f(t) &= \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\mu}} d\tau, \ \mu > 0, \ t > 0, \\ I^0 f(t) &= f(t). \end{aligned} \tag{1}$$

Definition 2.4. *The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^m$, $m \in I\!N \bigcup \{0\}$, is defined as:*

$$D_*^\mu f(t) = \begin{cases} \left[I^{m-\mu} f^{(m)}(t) \right] & m-1 < \mu < m, \quad m \in I\!N, \\ \frac{d^m}{dt^m} f(t) & \mu = m. \end{cases} \tag{2}$$

Note that

$$(i) \qquad I^\mu t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)} t^{\gamma+\mu}, \qquad \mu > 0, \ \gamma > -1, \ t > 0.$$

$$(ii) \qquad I^\mu D_*^\mu f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, \qquad m-1 < \mu \leq m, \quad m \in I\!N.$$

3 Predictor-corrector scheme for fractional differential equations

In this section we recall the fundamental algorithm that we shall later use to solve our differential equations. It has been investigated further in [1]. The algorithm will be a generalization of the well known second-order Adams- Bashforth-Moulton method for first-order initial value problems. Thus we momentarily concentrate our attention on the differential equation

$$D^\alpha y(t) = f(t, y(t)), \qquad 0 \leq t \leq T, \tag{3}$$

equipped with initial conditions

$$y^{(k)}(0) = y_0^{(k)}, \qquad k = 0, 1, \cdots, m-1, \quad \alpha \in (m-1, m], \tag{4}$$

where f is in general a nonlinear function of its arguments. Our approach is based on the analytical property that the initial value problem (3), (1.1) is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} f(\tau, y(\tau)) d\tau. \tag{5}$$

we assume that we are working on a uniform grid $\{t_n = nh : n = 0, 1, \cdots, N\}$ with some integer N and h := T/N. The basic idea is, assuming that we have already calculated approximations $y_h(t_j) \approx y(t_j)(j = 1, 2, \cdots, n)$, that we try to obtain the approximation $y_h(t_{n+1})$ by means of the equation

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, y_h^p(t_{n+1}) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_n(t_j)). \tag{6}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & \text{if } 1 \le j \le n. \\ 1, & \text{if } j = n+1, \end{cases} \tag{7}$$

The remaining problem is the determination of the predictor formula that we require to calculate the value $y_h^P(t_{n+1})$. This approach gives us the predictor $y_h^P(t_{n+1})$ as

$$y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_n(t_j)). \tag{8}$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}((n+1-j)^\alpha - (n-j)^\alpha). \tag{9}$$

Our basic algorithm, the fractional Adams-Bashforth-Moulton method, is completely described now by eqs. (6) and (8) with the weights $a_{j,n+1}$ and $b_{j,n+1}$ being defined according to (7) and (9), respectively.
Error in this method is

$$max_{j=0,1,\cdots,N} |y(t_j) - y_h(t_j)| = O(h^p), \tag{10}$$

where $p = min(2, 1+\alpha)$.

4 Illustrative examples

We want to illustrate the properties of our scheme by using the example

Example 4.1. *Consider the following fractional Riccati equation :*

$$D_*^\alpha y(t) = -y^2 + 1, \qquad t > 0, \quad n-1 < \alpha < n, \tag{11}$$

with the initial condition, $y(0) = 0$. *The exact solution, when* $\alpha = 1$, *is* $u(t) = \frac{e^{2t}-1}{e^{2t}+1}$.

Consider the equation (2.1) according algorithm (6)

$$y_h(t_{n+1}) = \frac{h^\alpha}{\Gamma(\alpha+2)}(-y_h^p(t_{n+1})^2 + 1) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1}(-y_h(t_j)^2 + 1) \tag{12}$$

where

$$y_h^p(t_{n+1}) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1}(-y_h(t_j)^2 + 1) \tag{13}$$

and weights $a_{j,n+1}$ and $b_{j,n+1}$ being defined according to (7) and (9), respectively. we have taken the step size $h = 0.01$ and $h = 0.1$ in this example. The results of our computation agreement with the analytical solution.
Fig. 1 shows exact solution and the solution obtained using of Eq. (2.1) for $\alpha = 0.85$ and $h = 0.1$ Fig. 2 shows exact solution and the solution y(t) of Eq. (2.1) for $\alpha = 0.95$ and $h = 0.1$ Fig. 3 shows exact solution and the solution y(t) of Eq. (2.1) for $\alpha = 0.95$ and $h = 0.01$

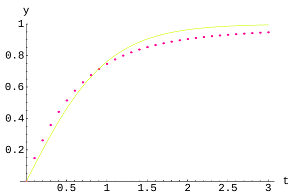

Fig.1.$\alpha = 0.85$ $h = 0.1$

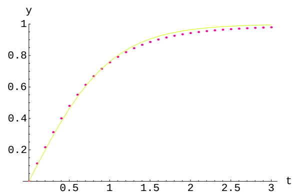

Fig.2.$\alpha = 0.95$ $h = 0.1$

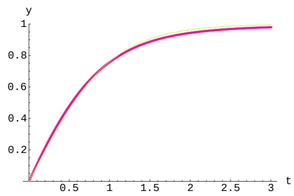

Fig.3.$\alpha = 0.95$ $h = 0.01$

5 Conclusion

We have shown that the Adams-Bashforth-Moulton method can be a very useful and efficient tool for the approximate solution of initial value problems involving Riccati fractional differential equations. In order to fully exploit the capabilities of the approach it seems to be useful to tune the parameters according to the specific requirements of the problem under consideration. Mathematica has been used for computations in this paper.

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A numerical solution for two-dimensional inverse parabolic problem

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Abstract
In this paper we consider a two-dimensional inverse heat conduction problem (IHCP) which is severely ill-posed, i.e., the solution does not depend continuously on the data. We propose a numerical approach based on the finite-difference method and the least-squares scheme to solve this problem in the presence of noisy data, then to regularize the resultant ill-conditioned linear system of equations, we apply the Tikhonov regularization.
Keywords: Inverse heat conduction problem, Finite difference method, Consistency, Stability, Convergency, Least-square method, Tikhonov regularization Method.

Mathematics Subject Classification: 35R30

1 Introduction

Inverse heat conduction problems have numerous important applications in various branches of engineering and science, including among others, estimation of unknown boundary heat fluxes , and thermophysical properties of materials.

Fortunately, many methods have been reported to solve IHCPs, and among the most versatile methods the following can be mentioned: Tikhonov regularization, iterative regularization mollification, BFM (Base Function Method), SFDM (Semi Finite Difference Method), and FSM (Function Specification Method).

2 mathematical formulation

In this section, let us consider the following two-dimensional IHCP

$$U_t = U_{xx} + U_{yy}, \qquad 0 < x < 1, 0 < y < 1, \quad 0 < t < T, \tag{1a}$$
$$U(0, y, t) = g_0(y, t), \qquad 0 \le y \le 1, 0 < t < T, \tag{1b}$$
$$U(1, y, t) = g_1(y, t), \qquad 0 \le y \le 1, 0 < t < T, \tag{1c}$$
$$U(x, 0, t) = h_0(x, t), \qquad 0 \le x \le 1, 0 < t < T, \tag{1d}$$
$$U(x, 1, t) = h_1(x, t), \qquad 0 \le x \le 1, 0 < t < T, \tag{1e}$$
$$U(x, y, 0) = f(x, y), \qquad 0 \le x \le 1, 0 \le y \le 1, \tag{1f}$$

and the over-specified condition

$$U(a, y, t) = q(y, t), \qquad 0 \le a \le 1, 0 \le y \le 1, 0 < t < T, \tag{1g}$$

where T is a given constant, $f(x, y)$, $g_1(y, t)$, $h_0(x, t)$, and $h_1(x, t)$ are known functions, while the function $g_0(y, t)$ is unknown which remain to be determined. Note that, for an unknown function $g_0(y, t)$ we must therefore provide additional information (1g) to provide a unique solution to the inverse problem (1).

3 Overview of the Method

The application of the present numerical method to find the solution of the inverse problem (1) can be divided into the following steps.

3.1 Finite difference method for discretizing

We start by dividing the domain $[0, 1]^2 \times [0, T]$ into an $M^2 \times N$ mesh with spatial step size $\Delta h = 1/M$ in both x-and y-directions and the time-step size $\Delta t = T/N$, respectively. The grid points (x, y, t) are given by:

$$x_i = i\Delta h, \qquad i = 0, 1, 2, ..., M, \tag{2}$$

$$y_j = j\Delta h, \qquad j = 0, 1, 2, ..., M, \tag{3}$$

$$t_n = n\Delta t, \qquad n = 0, 1, 2, ..., N, \tag{4}$$

where M and N are integers. Note that $u_{i,j,n}$ used to denote the finite difference approximation of $u(i\Delta h, j\Delta h, n\Delta t)$. We assume that:

$$\Delta x = x_{i+1} - x_i, \tag{5}$$

$$\Delta y = y_{j+1} - y_j. \tag{6}$$

In the case $\Delta x = \Delta y = \Delta h$, we have $r = \frac{\Delta t}{\Delta h^2}$.

Therefore, equation (1a) can be discretized as the follows,

$$u_{i,j,n+1} - u_{i,j,n} = \frac{\Delta t}{\Delta x^2}(u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n}) + \frac{\Delta t}{\Delta y^2}(u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n}),$$

and

$$u_{i,j,n+1} = [ru_{i-1,j,n} + (1 - 4r)u_{i,j,n} + ru_{i+1,j,n}] + [ru_{i,j-1,n} + ru_{i,j+1,n}]. \tag{7}$$

Remark 3.1. *In this work the polynomial form proposed for the unknown function $g_0(y, t)$ before performing the calculation. Therefore $g_0(y, t)$ approximated as*

$$g_0(y, t) = \sum_{i=0}^{\gamma} \sum_{j=0}^{\iota} a_{i,j} y^i t^j, \tag{8}$$

where $a_{i,j}$ are constants which remain to be determined simultaneously for each interval.

Substitution the (8) into (7), therefore the solution of this equation is

$$U_{i,j,n}; i, j = 1, 2, \ldots, (M - 1),\ n = 1, \ldots, N.$$

3.2 Least-squares minimization technique and the Tikhonov regularization method

The estimated coefficients $a_{i,j};\ i = 0, \ldots, \gamma, j = 0, \ldots, \iota$ can be determined by using least squares method when the sum of the squares of the deviation between the calculated $U_{\lambda,j,n+1}$ and the measured $q(j\Delta y, (n+1)\Delta t))$ at $x = a = \lambda\Delta x$ is less than a small number. The error in the estimates $E(a_{0,0}, a_{0,1}, ..., a_{\gamma,\iota})$ can be expressed as

$$E(a_{0,0}, a_{0,1}, ..., a_{\gamma,\iota}) = \sum_{v=0}^{\mu-1}\sum_{j=0}^{M}(U_{\lambda,j,v+1} - q(j\Delta y, (v+1)\Delta t))^2, \tag{9}$$

which is to be minimized.To obtain the minimum value of $E(a_{0,0}, a_{0,1}, ..., a_{\gamma,\iota})$, with respect to $a_{0,0}, a_{0,1}, ..., a_{\gamma,\iota}$, differentiation of $E(a_{0,0}, a_{0,1}, ..., a_{\gamma,\iota})$, with respect to $a_{0,0}, a_{0,1}, ..., a_{\gamma,\iota}$, will be performed. Thus the linear system corresponding to the values of $a_{i,j}$ can be expressed as

$$\Lambda\Theta = B. \tag{10}$$

The Matrix Λ is ill-conditioned. On the other hand, as $g_0(y,t)$ is affected by measurement errors, the estimate of Θ by (10) will be unstable so that the Tikhonov regularization method must be used to control this measurement errors. In an IHCP there are two sources of error in the estimation; the first source is the unavoidable bias deviation, and the second source of error is the variance due to the amplification of measurement errors, [1]. Therefore, we compare Tikhonov method and SVD method by considering total error S defined by

$$S = \left[\frac{1}{N-1}\sum_{i=1}^{N}(\widehat{\Phi_i} - \Phi_i)^2\right]^{\frac{1}{2}}, \tag{11}$$

where N ,Φ and $\widehat{\Phi}$ are the number of estimated values , the estimated values and the exact values, respectively.

4 Main Result

In this example, let us consider the following two-dimensional inverse problem, for estimating unknown boundary condition $g_0(y,t)$

$$U_t = U_{xx} + U_{yy}, \qquad 0 < x < 1, 0 < y < 1, \quad 0 < t < T, \tag{12a}$$
$$U(0,y,t) = g_0(y,t), \qquad 0 \le y \le 1, 0 < t < T, \tag{12b}$$
$$U(1,y,t) = e^{-t}(\sin 1 + \cos y), \qquad 0 \le y \le 1, 0 < t < T, \tag{12c}$$
$$U(x,0,t) = e^{-t}(\sin x + 1), \qquad 0 \le x \le 1, 0 < t < T, \tag{12d}$$
$$U(x,1,t) = e^{-t}(\sin x + \cos 1), \qquad 0 \le x \le 1, 0 < t < T, \tag{12e}$$
$$U(x,y,0) = \sin x + \cos y, \qquad 0 \le x \le 1, 0 \le y \le 1, \tag{12f}$$

and the overspecified condition

$$U(0.1,y,t) = e^{-t}(\sin 0.1 + \cos y), \qquad 0 \le y \le 1, 0 < t < T. \tag{12g}$$

The exact solution of this problem is

$$U(x,y,t) = e^{-t}(\sin x + \cos y),\ \ g_0(y,t) = e^{-t}(\cos y), \qquad 0 \le x \le 1, 0 < t < T.$$

Table 1 shows the comparison between the exact solution and approximate solution result from our method by Tikhonov regularization 0th, 1st , 2nd and SVD with noisy data. Furthermore, we compare two methods with computation total error.

y	Exact	Tikhonov 0	Tikhonov 1	Tikhonov 2	SVD
0.1	0.955989	0.989554	0.959390	0.958599	0.989560
0.2	0.941638	0.971284	0.940893	0.940179	0.971291
0.3	0.917877	0.945030	0.914534	0.913948	0.945037
0.4	0.884946	0.910791	0.880313	0.879905	0.910798
0.5	0.843172	0.868567	0.838231	0.838050	0.868575
0.6	0.792974	0.818358	0.788287	0.788384	0.818366
0.7	0.734852	0.760165	0.730482	0.730907	0.760172
0.8	0.669388	0.693986	0.664816	0.665618	0.693994
0.9	0.597236	0.619823	0.591288	0.592518	0.619830
S		0.0050837596	0.0035470110	0.0035381104	0.0050845387

Table 1. The comparison between exact , Tikhonov and SVD solution for $(g_0(y, 0.04))$ *with noisy data (0.01*rand(1)).*

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A methodology of solution for solving Saint-Venant equations by finite element method

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Abstract

Solving the Saint-Venant equations by finite element method (FEM) takes a long CPU time. Hence, we can apply the fast numerical methods such as finite difference method. But, these methods are not stable most of the time and moreover verification of the stability for these methods is very hard. In this paper, we present a methodology to solution of these equations. In addition, we split the discretized system into two simple systems so that each of them can be solved through direct methods.To demonstrate the performance of the method, some examples are presented.

Keywords: Hyperbolic partial differential equation, Saint-Venant equations, Finite element method.

Mathematics Subject Classification: 35L04, 65L60.

1 Introduction

Unsteady flow is of great interest to hydraulic engineers. Such flows can be described by the Saint-Venant equations which consist of the conservation of mass and momentum equations. These equations also are nonlinear hyperbolic partial differential equations. However, a general closed-form solution of these equations is not available, except for certain special simplified conditions and they must be solved using an appropriate numerical technique [1]. In order to cut down on the problem of CPU time for solving the Saint-Venant equations by FEM, one of the solution is the adaptive methods. In [2], some adaptive methods have been applied for solving the Saint-Venant equations by FEM. However, these methods would have significant errors, if we want to predict the value of surface elevation for a long time. In this paper, we present a methodology to solve the problem even for fairly large channel in suitable time with a few error. This error is related to the discretization of the equations and using the finite dimensional space of FEM. We consider the initial-boundary value Saint-Venant problem for unsteady flow in an open channel having no lateral inflow or outflow for one dimensional as:

$$\begin{cases} \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(\frac{Q^2}{A}) + gA\frac{\partial h}{\partial x} + \frac{gn^2|Q|Q}{R^{4/3}A} & = & 0 & \text{momentum equation,} \\ \frac{\partial h}{\partial t} + \frac{1}{B}\frac{\partial Q}{\partial x} & = & 0 & \text{continuity equation,} \\ Q(x,0) & = & Q^0 & 0 \le x < L, \\ h(x,0) & = & h^0 & 0 \le x \le L, \\ Q(L,t) & = & 0 & t \ge 0, \\ h(0,t) & = & h_0 & t > 0, \end{cases} \tag{1}$$

in which x = distance along the channel length, t = time, A = flow area, B = top water surface width, g = acceleration due to gravity, Q = discharge, h = water surface elevation, R = hydraulic radius, n = Manning coefficient, and L = length of channel, also h^0, h_0 and Q^0 are positive constant scalers. In general A and R are the functions of h (i.e. $A = A(h)$, $R = R(h)$).

2 Main Result

The discrete form of (1) is as follows:

$$\begin{cases} \frac{1}{\Delta t}Q(x,t_{n+1}) + \frac{\partial}{\partial x}(\frac{2Q(x,t_n)Q(x,t_{n+1})}{A(x,t_n)}) & + \quad gA(x,t_n)\frac{\partial h(x,t_{n+1})}{\partial x} + \\ \quad + \frac{gn^2|Q(x,t_n)||Q(x,t_{n+1})}{R^{4/3}(x,t_n)A(x,t_n)} & = \quad \frac{1}{\Delta t}Q(x,t_n) + \frac{\partial}{\partial x}(\frac{Q^2(x,t_n)}{A(x,t_n)}), \\ \frac{1}{\Delta t}h(x,t_{n+1}) + \frac{1}{B}\frac{\partial Q(x,t_{n+1})}{\partial x} & = \quad \frac{1}{\Delta t}h(x,t_n), \\ Q(x,0) & = \quad Q^0, \qquad\qquad 0 \leq x < L, \\ h(x,0) & = \quad h^0, \qquad\qquad 0 \leq x \leq L, \\ Q(L,t_n) & = \quad 0, \qquad\qquad n = 0,\dots,N, \\ h(0,t_n) & = \quad h_0, \qquad\qquad n = 1,\dots,N. \end{cases} \tag{2}$$

The variational form of problem (2) is that find $Q(x,t_{n+1}) \in V = \{Q(x,t_k) \in H^1(\Omega) : Q(L,t_k) = 0, \quad k = 0,...,N\}$, and $h(x,t_{n+1}) \in H = \{h(x,t_k) \in H^1(\Omega) : h(0,t_k) = h_0, \quad k = 1,...,N\}$ such that

$$\begin{array}{rcll} d(h,v) + m(Q,v) + b(Q,v) & = & (\alpha,v)_0 & \forall v \in H, \\ s(h,e) + w(Q,e) & = & (\beta,e)_0 & \forall e \in V, \end{array} \tag{3}$$

which $\Omega = [0,L]$, $(.,.)_0$ is an inner product in the $L_2(\Omega)$ space, and the bilinear forms on $V \times H$ are given respectively by

$$\begin{array}{rcl} m(Q,v) & = & \int_\Omega(\frac{1}{\Delta t} + \frac{gn^2|Q(x,t_n)|}{R^{4/3}(x,t_n)A(x,t_n)})Q(x,t_{n+1})vdx, \\ b(Q,v) & = & -2\int_\Omega \frac{Q(x,t_n)}{A(x,t_n)}Q(x,t_{n+1})v'dx + \frac{2Q(x,t_n)Q(x,t_{n+1})}{A(x,t_n)}v|_{\partial\Omega}, \\ d(h,v) & = & -g\int_\Omega h(x,t_{n+1})(A(x,t_n)v)'dx + gA(x,t_n)h(x,t_{n+1})v|_{\partial\Omega}, \\ s(h,e) & = & \frac{1}{\Delta t}\int_\Omega h(x,t_{n+1})edx, \\ w(Q,e) & = & \frac{-1}{B}\int_\Omega Q(x,t_{n+1})e'dx + \frac{1}{B}Q(x,t_{n+1})e|_{\partial\Omega}, \\ (\alpha,v) & = & \int_\Omega \alpha v dx, \end{array} \tag{4}$$

where $\partial\Omega$ is the boundary of Ω and $v|_{\partial\Omega}$ the restriction of v on $\partial\Omega$. In order to be stable the solution of Saint-Venant equations obtained by finite element method, we need to consider a very fine mesh, that is not commodious. We present a methodology to first solve the problem of CPU time so that we can find the solution in a short time and second solve the discretized system directly for very big systems. In continue for simplicity we abbreviate $U(x_i,t_j)$ by U_i^j. The general matrix form of the variational form of problem in time $t = t_j$ reads:

$$\begin{bmatrix} D & E \\ S & W \end{bmatrix} \begin{bmatrix} \mathbf{h}^j \\ \mathbf{Q}^j \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}, \tag{5}$$

in which, $\mathbf{h}^j = [h_1^j, h_2^j, \dots, h_M^j]^T$, $\mathbf{Q}^j = [Q_0^j, Q_1^j, \dots, Q_{M-1}^j]^T$, and $F = [F_1^j, F_2^j, \dots, F_M^j]^T$, $G = [G_0^j, G_1^j, \dots, G_{M-1}^j]^T$ where $F_k^j = (\alpha,\psi_k), k = 1,\dots,M$, and $G_k^j = (\beta,\varphi_k)$, $k = 0,\dots,M-1$, and also the entries of the matrices D, E, S and W are respectively defined as follows:

$$\begin{array}{rcll} D_{ij} & = & d(\psi_i,\psi_j), & i,j = 1,\dots,M, \\ E_{ij} & = & m(\varphi_i,\psi_j) + b(\varphi_i,\psi_j), & i = 0,\dots,M-1,\ j = 1,\dots,M, \\ S_{ij} & = & s(\psi_i,\varphi_j), & i = 1,\dots,M,\ j = 0,\dots,M-1, \\ W_{ij} & = & w(\varphi_i,\varphi_j), & i,j = 0,\dots,M-1. \end{array}$$

For example, it is readily seen that the matrix S can be written as:

$$S_{ij} = \frac{\Delta x}{6\Delta t} \begin{cases} 1 & i = j, \\ 4 & i = j+1, \\ 1 & i = j+2. \end{cases}$$

We observe that S does not depend on the time. For solving the system (9), we cancel the first row of D, E, S and W. Also, the last column of D, E, S and W are transformed to the right hand. In continue, we show these reduced matrices with superscript r. Hence, this system is converted to the following system:

$$\begin{bmatrix} D^r & E^r \\ S^r & W^r \end{bmatrix} \begin{bmatrix} \mathbf{h}^{j,r} \\ \mathbf{Q}^{j,r} \end{bmatrix} = \begin{bmatrix} F^r \\ G^r \end{bmatrix}, \tag{6}$$

where

$$\mathbf{h}^{j,r} = \begin{bmatrix} h_1^j & h_2^j & \ldots & h_{M-1}^j \end{bmatrix}^T, \quad \mathbf{Q}^{j,r} = \begin{bmatrix} Q_0^j & Q_1^j & \ldots & Q_{M-2}^j \end{bmatrix}^T.$$

Two removed equations are as follows:

$$\begin{array}{r} \frac{gA^0}{2} h_2^j + aQ_0^j + bQ_1^j + cQ_2^j = F_1, \\ \frac{\Delta x}{6\Delta t} h_1^j - \frac{1}{2B} Q_0^j + \frac{1}{2B} Q_1^j = G_0. \end{array} \tag{7}$$

The simplified system (1) is as follows:

$$\begin{array}{ll} (a) & D^r \mathbf{h}^{j,r} = F^r - E^r \mathbf{Q}^{j,r}, \\ (b) & W^r \mathbf{Q}^{j,r} = G^r - S^r \mathbf{h}^{j,r}. \end{array} \tag{8}$$

Since the matrix $E^r(W^r)^{-1}S^r - D^r$ is a nonsingular matrix [3], we can obtain

$$\mathbf{h}^{j,r} = (E^r(W^r)^{-1}S^r - D^r)^{-1}(E^r(W^r)^{-1}G^r - F^r).$$

We note that $\mathbf{h}^{j,r}$ is in terms of h_M^j and Q_{M-1}^j. By substituting the values $\mathbf{h}^{j,r}$ in (3b), $\mathbf{Q}^{j,r}$ is computed with respect to h_M^j and Q_{M-1}^j. Now, by (2), h_M^j and Q_{M-1}^j are determined and finally the computed values of $\mathbf{h}^j$ and $\mathbf{Q}^j$ are obtained.

Example 2.1. *We consider an open channel with rectangular cross section that its bottom width is 6.1 m. The bottom slope is 0.00008, Manning coefficient* $\mathrm{n} = 0.013$ *and the length of channel is 20 m. The initial conditions in the channel are 5.79 m-depth and a steady discharge of 126* m^3/s*. The water surface level in reservoir is constant at the up-stream end and also the sluice gate at the downstream end of the channel is suddenly closed at time* $t = 0$*. Figure 1 shows the flow depth in the channel at time* $t = 1\,sec$ *with custom methodology (meth1) and our methodology (meth2) of solution. In addition, we present the CPU time and the number of partitions (N) with meth1 and meth2 for* $t = 1\,sec$ *(Table 1). As we observe, using meth2 takes short CPU time.*

Table 1: CPU times and the number of partitions at t=1 sec.

N	$meth1$	$meth2$
50	1173.46	0.46
100	1694.58	0.50
150	2238.34	0.62

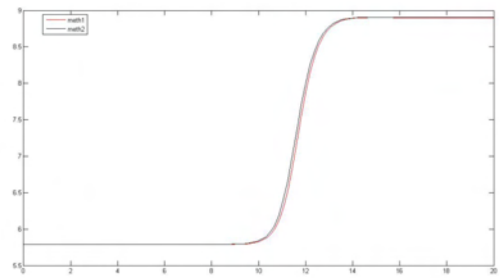

Figure 1: Flow depth in the rectangular channel at time $t = 1$ sec.

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