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Fractional Calculus: A Survey of Useful Formulas

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Abstract. This paper presents a survey of useful, established formulas in Fractional Calculus, systematically collected for reference purposes.

1 Introduction

The correspondence between L'Hôpital and Leibniz, in 1695, about what might be a derivative of order $\frac{1}{2}$, led to the introduction of a generalisation of integral and derivative operators, known as *Fractional Calculus* (which in spite of its name covers irrational or even complex integration and differentiation orders).

Many expressions of Fractional Calculus have been published, but such results are scattered over the literature and use different notations. This paper intends to gather systematically some of the most useful formulas for reference purposes.

Section 2 presents the notation used and collects the definition and relevant properties of the main special functions that appear in Fractional Calculus. Section 3 collects some definitions of one-dimensional fractional integral and derivative operators and some of their properties. Section 4 is a table of fractional derivatives. Section 5 is a table of Laplace and Fourier transforms. Section 6 collects solutions of some systems of fractional equations. Section 7 collects some topics about fractional transfer functions. Section 8 is an introduction to fractional vector operators.

2 Notation and Special Functions

2.1 Notation

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floor of $x \in \mathbb{R}$ (largest integer not larger than x)	$\lfloor x \rfloor$	(1)
ceiling of $x \in \mathbb{R}$ (smallest integer not smaller than x)	$\lceil x \rceil$	(2)
variable of the $\mathscr{Z}$-transform (with some abuse of notation, it can be identified with the advance operator, and its inverse with the delay operator)	z	(3)
Heaviside function	$H(x) = \begin{cases} 1, \text{ if } x \geq x_0 \\ 0, \text{ if } x < x_0 \end{cases}$	(4)
Pringsheim notation of continued fraction (which need not have an infinite number of terms)	$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \dots}}}} = \left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \frac{b_4}{a_4}, \cdots\right] = \left[a_0, \frac{b_k}{a_k}\right]_{k=1}^{+\infty}$	(5)
Levi-Civita symbol	$\varepsilon_{lmn} = \begin{cases} +1, \text{ if } (l,m,n) = (1,2,3),(3,1,2),(2,3,1) \\ -1, \text{ if } (l,m,n) = (1,3,2),(3,2,1),(2,1,3) \\ 0, \text{ if } l = m \vee l = n \vee m = n \end{cases}$	(6)

2.2 Definitions of some Special Functions

Euler's gamma function $\Gamma(z) = \begin{cases} \int_0^{+\infty} e^{-y} y^{z-1} \mathrm{d}y, \text{ if } \Re(z) > 0 \\ \frac{\Gamma(z+n)}{(z)_n}, \text{ if } \Re(z) > -n, \ n \in \mathbb{N} \wedge z \notin \mathbb{Z}_0^- \end{cases}$	(7)
Pochhammer function $(\rho)_0 = 1$ and $(\rho)_k = \rho(\rho+1)...(\rho+k-1), \quad k \in \mathbb{N}$	(8)
Combinations of a things, b at a time $\binom{a}{b} = \begin{cases} \dfrac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}, \text{ if } a, b, a-b \notin \mathbb{Z}^- \\ \dfrac{(-1)^b \Gamma(b-a)}{\Gamma(b+1)\Gamma(-a)}, \text{ if } a \in \mathbb{Z}^- \wedge b \in \mathbb{Z}_0^+ \\ 0, \text{ if } [(b \in \mathbb{Z}^- \vee b-a \in \mathbb{N}) \wedge a \notin \mathbb{Z}^-] \vee (a, b \in \mathbb{Z}^- \wedge \lvert a \rvert > \lvert b \rvert) \end{cases}$	(9)
Beta function $B(x,y) = B(y,x) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$	(10)
Digamma function $\psi(x) = \frac{\mathrm{d}\log\Gamma(x)}{\mathrm{d}x} = \frac{1}{\Gamma(x)}\frac{\mathrm{d}\Gamma(x)}{\mathrm{d}x}$	(11)
Error function $\mathrm{erf}\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,\mathrm{d}t, \ z \in \mathbb{C}$	(12)
Complementary error function $\mathrm{erfc}\,(z) = 1 - \mathrm{erf}\,(z), \ z \in \mathbb{C}$	(13)
Mittag-Leffler function, or one-parameter Mittag-Leffler function $E_\alpha(z) = \displaystyle\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \ \Re(\alpha) > 0, \ z \in \mathbb{C}$	(14)

Generalized Mittag-Leffler function with two parameters $E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)},\ \Re(\alpha),\Re(\beta)>0,\ z\in\mathbb{C}$	(15)
Generalized Mittag-Leffler function with three parameters $E_{\alpha,\beta}^{\rho}(z)=\sum_{k=0}^{\infty}\frac{(\rho)_k\ z^k}{\Gamma(\beta+\alpha k)\ k!},\quad \alpha,\beta,\rho\in\mathbb{C},\ \Re(\alpha)>0,\ z\in\mathbb{C}$	(16)
Miller-Ross function $\mathscr{E}_z(\nu,a)=\sum_{k=0}^{+\infty}\frac{a^k z^{k+\nu}}{\Gamma(\nu+k+1)}=z^{\nu}E_{1,\nu+1}(az)\quad z\in\mathbb{C}$	(17)
Hypergeometric function ${}_pF_q(a_1,a_2,\dots a_p;b_1,b_2,\dots b_q;z)=$ $1+\sum_{k=1}^{\infty}\left[\frac{z^k}{k!}\prod_{n=0}^{k-1}\frac{(a_1+n)(a_2+n)\dots(a_p+n)}{(b_1+n)(b_2+n)\dots(b_q+n)}\right]$	(18)
Bessel functions of the first kind $J_{\alpha}(x)=\sum_{k=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{1}{2}x\right)^{2m+\alpha}$	(19)
Modified Bessel functions of the first kind $I_{\alpha}(x)=j^{-\alpha}J_{\alpha}(jx)$	(20)
Bessel functions of the second kind $Y_{\alpha}(x)=\frac{J_{\alpha}(x)\cos(\alpha\pi)-J_{-\alpha}(x)}{\sin(\alpha\pi)}$	(21)
Hermite polynomial $H_n(x)=e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x^2}\right)$	(22)

More information on this topic can be found e.g. in [8, 1, 9–11]

2.3 Properties of the Mittag-Leffler functions: special values

$E_{\alpha,1}(z)=E_{\alpha}(z),\quad E_1(z)=E_{1,1}(z)=\mathscr{E}_t(0,1)=e^z,\quad E_{\alpha,\beta}^{1}(z)=E_{\alpha,\beta}(z)$	(23)
$E_0(z)=\frac{1}{1-z},\quad \lvert z\rvert<1$ and $E_1(z)=E_{1,1}(z)=\mathscr{E}_t(0,1)=e^z$	(24)
$E_2(z^2)=E_{2,1}(z^2)=\cosh(\sqrt{z})$ and $E_{2,2}(z^2)=\frac{\sinh(z)}{z}$	(25)
$E_3(z)=\frac{1}{2}\left[e^{z^{\frac{1}{3}}}+2e^{-\frac{1}{2}z^{\frac{1}{3}}}\cos\left(\frac{\sqrt{3}}{2}z^{\frac{1}{3}}\right)\right]$	(26)
$E_4(z)=\frac{1}{2}\left[\cos\left(z^{\frac{1}{4}}\right)+\cosh\left(z^{\frac{1}{4}}\right)\right]$	(27)
$E_{1,2}(z)=\frac{e^z-1}{z}$ and $E_{2,2}(z)=\frac{\sinh(\sqrt{z})}{\sqrt{z}}$	(28)

2.4 Generalized exponential functions

Let $z, \lambda \in \mathbb{C}$, $\Re(\alpha) > 0$ and $n \in \mathbb{N}$. Then

$$E_\alpha\left(\lambda z^\alpha\right) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{k\alpha}}{\Gamma(\alpha k + 1)} \tag{29}$$

$$e_\alpha^{\lambda z} = z^{\alpha-1} E_{\alpha,\alpha}\left(\lambda z^\alpha\right) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{(k+1)\alpha-1}}{\Gamma(k\alpha+\alpha))} \tag{30}$$

$$e_{\alpha,n}^{\lambda z} = n! z^{\alpha-1} E_{\alpha,(n+1)\alpha}^{n+1}\left(\lambda z^\alpha\right) \tag{31}$$

The above functions satisfy the following properties:

${}^{\mathsf{C}}D_{a+}^{\alpha} E_\alpha(\lambda(z-a)^\alpha)(x) = \lambda E_\alpha(\lambda(x-a)^\alpha)$	(32)
$\lim_{z\to a+} E_\alpha\left(\lambda(z-a)^\alpha\right) = 1$	(33)
$\lim_{z\to a+} \left[(z-a)^{1-\alpha} e_\alpha^{\lambda(z-a)}\right] = \frac{1}{\Gamma(\alpha)}$	(34)
$\left(\frac{\partial}{\partial z}\right)^n \left[E_\alpha\left(\lambda z^\alpha\right)\right] = z^{-n} E_{\alpha,1-n}\left(\lambda z^\alpha\right)$	(35)
$\left(\frac{\partial}{\partial z}\right)^n \left[e_\alpha^{\lambda z}\right] = z^{\alpha-n-1} E_{\alpha,\alpha-n}\left(\lambda z^\alpha\right)$	(36)
$\left(\frac{\partial}{\partial \lambda}\right)^n \left[E_\alpha\left(\lambda z^\alpha\right)\right] = n! z^{\alpha n} E_{\alpha,\alpha n+1}^{n+1}\left(\lambda z^\alpha\right)$	(37)
$\left(\frac{\partial}{\partial \lambda}\right)^n \left[e_\alpha^{\lambda z}\right] = n! z^{\alpha n+\alpha-1} E_{\alpha,(n+1)\alpha}^{n+1}\left(\lambda z^\alpha\right)$	(38)
$e_{\alpha,n}^{\lambda z} = \frac{1}{n!}\left(\frac{\partial}{\partial \lambda}\right)^n \left[e_\alpha^{\lambda z}\right], \quad z \neq 0$	(39)

The generalized α-exponential functions do not have the index property, that is, in general

$$E_\alpha(\lambda z) E_\alpha(\mu z) \neq E_\alpha((\lambda+\mu) z); \quad e_\alpha^{\lambda z} e_\alpha^{\mu z} \neq e_\alpha^{(\lambda+\mu) z} \tag{40}$$

3 Fractional Derivatives and Integrals

Let $\alpha \in \mathbb{C} : \Re(\alpha) \in (n-1, n]$, $n \in \mathbb{N}$, and let $[a, b]$ be a finite interval in $\mathbb{R}$.

3.1 Definitions of some unidimensional fractional operators

Riemann-Liouville Left-sided Integral ${}^{\mathsf{RL}}I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(u)}{(x-u)^{1-\alpha}}\,\mathrm{d}u, \quad x \geq a$	(41)
Riemann-Liouville Right-sided Integral ${}^{\mathsf{RL}}I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(u)}{(u-x)^{1-\alpha}}\,\mathrm{d}u, \quad x \leq b$	(42)
Riemann-Liouville Left-sided Derivative ${}^{\mathsf{RL}}D_{a+}^{\alpha} f(x) = D^n\, {}^{\mathsf{RL}}I_{a+}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x \frac{f(u)}{(x-u)^{1-n+\alpha}}\,\mathrm{d}u, \quad x \geq a$	(43)

Riemann-Liouville Right-sided Derivative ${}^{\mathtt{RL}}D^{\alpha}_{b-}f(x) = D^n\,{}^{\mathtt{RL}}I^{n-\alpha}_{b-}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^b \frac{f(u)}{(u-x)^{1-n+\alpha}}\,\mathrm{d}u, \quad x \le b$	(44)
Caputo Left-sided Derivative ${}^{\mathtt{C}}D^{\alpha}_{a+}f(x) = {}^{\mathtt{RL}}I^{n-\alpha}_{a+}D^n f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{\frac{\mathrm{d}^n f(u)}{\mathrm{d}u^n}}{(x-u)^{1-n+\alpha}}\,\mathrm{d}u, \quad x \ge a$	(45)
Caputo Right-sided Derivative ${}^{\mathtt{C}}D^{\alpha}_{-}f(x) = (-D)^n\,{}^{\mathtt{L}}I^{n-\alpha}_{-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^\infty \frac{f(u)\mathrm{d}u}{(u-x)^{1-n+\alpha}}, \quad x < \infty$	(46)
Left-sided Finite-Difference $\Delta^{\alpha}_{h,a+}f(x) = \sum_{k=0}^{\lfloor\frac{x-a}{h}\rfloor}(-1)^k\binom{\alpha}{k}f(x-kh), \quad x \ge a$	(47)
Right-sided Finite-Difference $\Delta^{\alpha}_{h,b-}f(x) = \sum_{k=0}^{\lfloor\frac{b-x}{h}\rfloor}(-1)^k\binom{\alpha}{k}f(x+kh), \quad x \le b$	(48)
Grünwald-Letnikoff Left-sided Derivative ${}^{\mathtt{GL}}D^{\alpha}_{a+}f(x) = \lim_{h\to 0+}\frac{\Delta^{\alpha}_{h,a+}f(x)}{h^\alpha}, \quad x \ge a$	(49)
Grünwald-Letnikoff Right-sided Derivative ${}^{\mathtt{GL}}D^{\alpha}_{b-}f(x) = \lim_{h\to 0+}\frac{\Delta^{\alpha}_{h,b-}f(x)}{h^\alpha}, \quad x \le b$	(50)

Remark 1 Liouville integrals and derivatives are Riemann-Liouville integrals and derivatives for the particular case $a = -\infty$ or $b = +\infty$; that is to say, ${}^{\mathtt{L}}I^{\alpha}_{\pm}f(x) \overset{\text{def}}{=} {}^{\mathtt{RL}}I^{\alpha}_{\mp\infty\pm}f(x)$, and ${}^{\mathtt{L}}D^{\alpha}_{\pm}f(x) \overset{\text{def}}{=} {}^{\mathtt{RL}}D^{\alpha}_{\mp\infty\pm}f(x)$. Sometimes ${}^{\mathtt{L}}I^{\alpha}_{-}$ is named the fractional integral of Weyl, and ${}^{\mathtt{L}}D^{\alpha}_{-}$ the Weyl transform, W^α.

Remark 2 If D^α is any fractional derivative, the Miller-Ross sequential derivative of order $k\alpha, k \in \mathbb{Z}$ is given by

$$\mathcal{D}^\alpha = D^\alpha, \quad \mathcal{D}^{k\alpha} = D^\alpha\mathcal{D}^{(k-1)\alpha}. \tag{51}$$

Remark 3 If $f(t)$ has $\beta = \max\{0, \lfloor\alpha\rfloor\}$ continuous derivatives, and $D^\beta f(t)$ is integrable, then

$${}^{\mathtt{R}L}D^{\alpha}_{a\pm}f(x) = {}^{\mathtt{GL}}D^{\alpha}_{a\pm}f(x) \tag{52}$$

$${}^{\mathtt{C}}D^{\alpha}_{a+}f(x) = \left({}^{\mathtt{RL}}D^{\alpha}_{a+}\left[f(u) - \textstyle\sum_{k=0}^{\lceil\Re(\alpha)\rceil-1}\frac{f^{(k)}(a)}{k!}(u-a)^k\right]\right)(x) \tag{53}$$

$${}^{\mathtt{C}}D^{\alpha}_{b-}f(x) = \left({}^{\mathtt{RL}}D^{\alpha}_{b-}\left[f(u) - \textstyle\sum_{k=0}^{\lceil\Re(\alpha)\rceil-1}\frac{f^{(k)}(b)}{k!}(b-u)^k\right]\right)(x) \tag{54}$$

(53)–(54) are sometimes considered as the definitions of Caputo derivatives, since they can be applied to a larger set of functions than (45)–(46).

Remark 4 Whatever the definition employed, $I^0 f(x) = D^0 f(x) = f(x)$.

Remark 5 If $\alpha = m \in \mathbb{N}$, then

$${}^{\mathtt{RL}}D^{m}_{a+}f(x) = {}^{\mathtt{L}}D^{m}_{+}f(x) = {}^{\mathtt{C}}D^{m}_{a+}f(x) = {}^{\mathtt{GL}}D^{m}_{a+}f(x) = D^m f(x) = \frac{\mathrm{d}^m f(x)}{\mathrm{d}x^m} \tag{55}$$

$${}^{\mathtt{RL}}I^{m}_{b-}f(x) = {}^{\mathtt{L}}D^{m}_{-}f(x) = {}^{\mathtt{C}}D^{m}_{b-}f(x) = {}^{\mathtt{GL}}D^{m}_{b-}f(x) = (-1)^m D^m f(x) = (-1)^m\frac{\mathrm{d}^m f(x)}{\mathrm{d}x^m} \tag{56}$$

Remark 6 Some authors do not distinguish the definition employed by means of a superscript (GL, RL, C, L), but use different fonts for the operator instead (D, D, **D**, $\mathfrak{D}$, $\mathscr{D}$). The particular correspondence between fonts and definitions varies. Very often no indication at all is given, save perhaps in the accompanying text, and the reader is presumed to understand from the context which particular definition is intended.

Remark 7 In the literature, several alternative notations for operator D may be found:

$$D^{\alpha}_{a+}f(x) = (D^{\alpha}_{a+}f)(x) = {}_aD^{\alpha}_x f(x) = {}_aI^{-\alpha}_x f(x) = D^{\alpha}_{x-a}f(x) = \frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}(x-a)^{\alpha}} \quad (51)$$

$$D^{\alpha}_{b-}f(x) = (D^{\alpha}_{b-}f)(x) = {}_xD^{\alpha}_b f(x) = {}_xI^{-\alpha}_b f(x) = D^{\alpha}_{b-x}f(x) = \frac{\mathrm{d}^{\alpha}f(x)}{\mathrm{d}(b-x)^{\alpha}} \quad (52)$$

Only one of the two operators I and D needs to be used, since it is all a matter of changing the sign of α. In practice D is the one more often used.

3.2 Properties

3.2.1 Semigroup properties

For $\Re(\alpha) \in (n-1, n]$, $\Re(\beta) \in (m-1, m]$, $m, n \in \mathbb{N}$, and for suitable functions, we have

${}^{\mathtt{RL}}I^{\alpha}_{a+}\,{}^{\mathtt{RL}}I^{\beta}_{a+}f(x) = {}^{\mathtt{RL}}I^{\alpha+\beta}_{a+}f(x) \quad \text{and} \quad {}^{\mathtt{RL}}I^{\alpha}_{b-}\,{}^{\mathtt{RL}}I^{\beta}_{b-}f(x) = {}^{\mathtt{RL}}I^{\alpha+\beta}_{b-}f(x)$	(53)
${}^{\mathtt{RL}}D^{\alpha}_{a+}\,{}^{\mathtt{RL}}D^{\beta}_{a+}f(x) = {}^{\mathtt{RL}}D^{\alpha+\beta}_{a+}f(x) - \sum_{j=1}^{m} {}^{\mathtt{RL}}D^{\beta-j}_{a+}f(a)\frac{(x-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)}$,	(54)
${}^{\mathtt{RL}}D^{\beta}_{a+}\,{}^{\mathtt{RL}}I^{\alpha}_{a+}f(x) = I^{\alpha-\beta}_{a+}f(x) \quad \text{and} \quad {}^{\mathtt{RL}}D^{\beta}_{b-}\,{}^{\mathtt{RL}}I^{\alpha}_{b-}f(x) = {}^{\mathtt{RL}}I^{\alpha-\beta}_{b-}f(x)$	(55)
${}^{\mathtt{RL}}D^{\alpha}_{a+}\,{}^{\mathtt{RL}}I^{\alpha}_{a+}f(x) = {}^{\mathtt{RL}}D^{\alpha}_{b-}\,{}^{\mathtt{RL}}I^{\alpha}_{b-}f(x) = f(x)$	(56)
${}^{\mathtt{RL}}I^{\alpha}_{a+}\,{}^{\mathtt{RL}}D^{\alpha}_{a+}f(x) = f(x) - \sum_{j=1}^{n} \frac{({}^{\mathtt{RL}}I^{n-\alpha}_{a+}f)^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j}$	(57)
${}^{\mathtt{RL}}I^{\alpha}_{b-}\,{}^{\mathtt{RL}}D^{\alpha}_{b-}f(x) = f(x) - \sum_{j=1}^{n} \frac{(-1)^{n-j}({}^{\mathtt{RL}}I^{n-\alpha}_{b-}f)^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(b-x)^{\alpha-j}$	(58)
${}^{\mathtt{RL}}D^{m}\,{}^{\mathtt{RL}}D^{\alpha}_{a+}f(x) = {}^{\mathtt{RL}}D^{\alpha+m}_{a+}f(x), \quad m \in \mathbb{N}$	(59)
${}^{\mathtt{RL}}D^{m}\,{}^{\mathtt{RL}}D^{\alpha}_{b-}f(x) = (-1)^{m}\,{}^{\mathtt{RL}}D^{\alpha+m}_{b-}f(x), \quad m \in \mathbb{N}$	(60)
${}^{\mathtt{C}}D^{\alpha}_{a+}\,{}^{\mathtt{RL}}I^{\alpha}_{a+}f(x) = {}^{\mathtt{C}}D^{\alpha}_{b-}\,{}^{\mathtt{RL}}I^{\alpha}_{b-}f(x) = f(x), \quad \Re(\alpha) \notin \mathbb{N} \vee \alpha \in \mathbb{N}$	(61)
${}^{\mathtt{C}}D^{\alpha}_{a+}\,{}^{\mathtt{RL}}I^{\alpha}_{a+}f(x) = f(x) - \frac{{}^{\mathtt{RL}}I^{\alpha+1-n}_{a+}f(a)}{\Gamma(n-\alpha)}(x-a)^{n-\alpha}, \ \Re(\alpha) \in \mathbb{N} \wedge \Im(\alpha) \neq 0$	(62)
${}^{\mathtt{C}}D^{\alpha}_{b-}\,{}^{\mathtt{RL}}I^{\alpha}_{a+}f(x) = f(x) - \frac{{}^{\mathtt{RL}}I^{\alpha+1-n}_{b-}f(b)}{\Gamma(n-\alpha)}(b-x)^{n-\alpha}, \ \Re(\alpha) \in \mathbb{N} \wedge \Im(\alpha) \neq 0$	(63)
${}^{\mathtt{RL}}I^{\alpha}_{a+}\ {}^{\mathtt{C}}D^{\alpha}_{a+}f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^{k}$	(64)
${}^{\mathtt{RL}}I^{\alpha}_{b-}\ {}^{\mathtt{C}}D^{\alpha}_{b-}f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^{k}f^{(k)}(b)}{k!}(b-x)^{k}$	(65)

${}^{\mathtt{L}}I_{+}^{\alpha}\,{}^{\mathtt{L}}I_{+}^{\beta}f(x) = {}^{\mathtt{L}}I_{+}^{\alpha+\beta}f(x)$ and ${}^{\mathtt{L}}I_{-}^{\alpha}\,{}^{\mathtt{L}}I_{-}^{\beta}f(x) = {}^{\mathtt{L}}I_{-}^{\alpha+\beta}f(x)$	(66)
${}^{\mathtt{L}}D_{+}^{\alpha}\,{}^{\mathtt{L}}I_{+}^{\alpha}f(x) = f(x)$ and ${}^{\mathtt{L}}D_{-}^{\alpha}\,{}^{\mathtt{L}}I_{-}^{\alpha}f(x) = f(x)$	(67)
${}^{\mathtt{L}}D_{+}^{\beta}\,{}^{\mathtt{L}}I_{+}^{\alpha}f(x) = {}^{\mathtt{L}}I_{0+}^{\alpha-\beta}f(x)$ and ${}^{\mathtt{L}}D_{-}^{\beta}\,{}^{\mathtt{L}}I_{-}^{\alpha}f(x) = {}^{\mathtt{L}}I_{-}^{\alpha-\beta}f(x)$	(68)
$D^{m}\,{}^{\mathtt{L}}I_{+}^{\alpha}f(x) = {}^{\mathtt{L}}I_{+}^{\alpha-m}f(x)$ and $D^{m}\,{}^{\mathtt{L}}I_{-}^{\alpha}f(x) = (-1)^{k}\,{}^{\mathtt{L}}I_{-}^{\alpha-m}f(x),\ \Re(\alpha) > m$	(69)
$D^{m}\,{}^{\mathtt{L}}D_{+}^{\alpha}f(x) = {}^{\mathtt{L}}D_{+}^{\alpha+m}f(x)$ and $D^{m}\,{}^{\mathtt{L}}D_{-}^{\alpha}f(x) = (-1)^{k}\,{}^{\mathtt{L}}D_{-}^{\alpha+m}f(x)$	(70)

3.2.2 Integration by parts

If $\Re(\alpha) > 0$, for suitable functions, we have the following properties

$\int_a^b f(x)\left({}^{\mathtt{RL}}I_{a+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_a^b g(x)\left({}^{\mathtt{RL}}I_{b-}^{\alpha}f\right)(x)\,\mathrm{d}x$	(71)
$\int_a^b f(x)\left({}^{\mathtt{RL}}D_{a+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_a^b g(x)\left({}^{\mathtt{RL}}D_{b-}^{\alpha}f\right)(x)\,\mathrm{d}x$	(72)
$\int_{-\infty}^{\infty} f(x)\left({}^{\mathtt{L}}I_{+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} g(x)\left({}^{\mathtt{L}}I_{-}^{\alpha}f\right)(x)\,\mathrm{d}x$	(73)
$\int_{-\infty}^{\infty} f(x)\left({}^{\mathtt{L}}D_{+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} g(x)\left({}^{\mathtt{L}}D_{-}^{\alpha}f\right)(x)\,\mathrm{d}x$	(74)
$\int_{-\infty}^{0} f(x)\left({}^{\mathtt{L}}I_{0+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_{0}^{\infty} g(x)\left({}^{\mathtt{L}}I_{-}^{\alpha}f\right)(x)\,\mathrm{d}x$	(75)
$\int_{0}^{\infty} f(x)\left({}^{\mathtt{L}}D_{0+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_{0}^{\infty} g(x)\left({}^{\mathtt{L}}D_{-}^{\alpha}f\right)(x)\,\mathrm{d}x$	(76)
$\int_a^b f(x)\left({}^{\mathtt{C}}D_{a+}^{\alpha}g\right)(x)\,\mathrm{d}x = \int_a^b g(x)\left({}^{\mathtt{C}}D_{b-}^{\alpha}f\right)(x)\,\mathrm{d}x + \left[f(x)\left({}^{\mathtt{RL}}I_{a+}^{\alpha}g\right)(x)\right](b)$ $- \left[g(x)\left({}^{\mathtt{RL}}I_{b-}^{\alpha}f\right)(x)\right](b) \quad (0 < \Re e(\alpha) < 1)$	(77)

3.2.3 Leibniz formula and derivative of the composition of two functions

$\left[{}^{\mathtt{RL}}D_{a+}^{\alpha}(fg)\right](x) = \sum_{j=0}^{\infty}\binom{\alpha}{j}{}^{\mathtt{RL}}D_{a+}^{\alpha-j}f(x)(D^{j}g)(x)$	(78)
$\left[{}^{\mathtt{RL}}D_{a+}^{\alpha}(f(g))\right](x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}f(g(x))$ $+\sum_{j=1}^{\infty}\binom{\alpha}{j}\frac{j!(x-a)^{j-\alpha}}{\Gamma(j+1-\alpha)}\sum_{i=1}^{j}[D^{i}f(g)](x)\sum\prod_{r=1}^{j}\frac{1}{a_r!}\left(\frac{(D^{r}g)(x)}{r!}\right)^{a_r}$	(79)

3.3 Fractional Taylor Formulas

Among the different generalizations to the fractional case of Taylor series in the literature, we present the following:

Taylor's Formula/ Serie	The remainder	
$f(x) = \sum_{j=0}^{m-1} \frac{{}^{\mathrm{RL}}D_{a+}^{\alpha+j} f(x_0)}{\Gamma(\alpha+j+1)} (x-x_0)^{\alpha+j}$ $+R_m(x),\ \alpha>0$	$R_m(x) =$ ${}^{\mathrm{RL}}I_{a+}^{\alpha+m}\, {}^{\mathrm{RL}}D_{a+}^{\alpha+m} f(x)$	(80)
$f(x) = \sum_{j=0}^{m} \frac{\Gamma(\alpha)\ c_j(x_0)}{\Gamma((j+1)\alpha)} (x-x_0)^{(j+1)\alpha-1}$ $+R_m(x),$ where $\alpha \in [0,1]$ and $c_j(x) = (x-x_0)^{1-\alpha}\ \left[{}^{\mathrm{RL}}D_a^{\alpha} f\right]^j (x)$	$R_m(x) = \frac{\left({}^{\mathrm{RL}}D_{a+}^{\alpha}\right)^{(m+1)x} f(\xi)}{\Gamma((m+1)\alpha+1)}\ \cdot$ $(x-a)^{(m+1)\alpha}, \quad \xi \in [a,x]$	(81)
$f(x) = f(a) + \frac{{}^{\mathrm{C}}D_{a+}^{\alpha} f(a)}{\Gamma(\alpha+1)}(x-a)^{\alpha}$ $+\frac{{}^{\mathrm{C}}D_{a+}^{\alpha}\ {}^{\mathrm{C}}D_{a+}^{\alpha} f(a)}{\Gamma(2\alpha+1)}(x-a)^{2\alpha} + ...,$		(82)
$f(x) = \sum_{k=0}^{m-1} a_k\, x^{\alpha_k} + R_m(x),$ where $x>0$ and $a_k = \frac{D^{(\alpha_k)} f(0)}{\Gamma(\alpha_k+1)}$	$R_m(x) = \frac{1}{\Gamma(\alpha_m+1)} \cdot$ $\cdot\ \int_0^x (x-z)^{\alpha_m-1}\, D^{(\alpha_k)} f(z)\, \mathrm{d}z$	(83)

In (83), $\alpha_0 = 0$ and the α_k, $(k = 1, ..., m)$ are an increasing sequence of real numbers such that $0 < \alpha_k - \alpha_{k-1}$, and $D^{(\alpha_k)} = {}^{\mathrm{RL}}\ I_{0+}^{1-(\alpha_k-\alpha_{k-1})}\, {}^{\mathrm{RL}}D_{0+}^{1+\alpha_{k-1}}$. For more information see e.g. [1–7].

4 Analytical Expressions of Some Fractional Derivatives

$f(x), x>a$	${}^{\mathrm{RL}}D_{a+}^{\alpha} f(x)$	
k	$\frac{k(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$	(53)
$(x-a)^{\beta},\ \Re(\beta) > -1$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\ (x-a)^{\beta-\alpha}$	(54)
$e^{\lambda x},\ \lambda \neq 0$	$e^{\lambda a}(x-a)^{-\alpha} E_{1,1-\alpha}(\lambda(x-a)) =$ $e^{\lambda a} \mathscr{E}_{x-a}(-\alpha, \lambda)$	(55)
$(x \pm p)^{\lambda},\ a \pm p > 0$	$\frac{(a\pm p)^{\lambda}}{\Gamma(1-\alpha)}\ (x-a)^{-\alpha}\, {}_2F_1\left(1, -\lambda, 1-\alpha; \frac{a-x}{a\pm p}\right)$	(56)
$(x-a)^{\beta}(x\pm p)^{\lambda},$ $\Re(\beta) > -1 \wedge\ a \pm p > 0$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\ (a\pm p)^{\lambda}(x-a)^{\beta-\alpha}\cdot$ $\cdot{}_2F_1\left(\beta+1, -\lambda; \beta-\alpha; \frac{a-x}{a\pm p}\right)$	(57)
$(x-a)^{\beta}(p-x)^{\lambda},$ $p > x > a \wedge \Re(\beta) > -1$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\ (p-a)^{\lambda}(x-a)^{\beta-\alpha}\cdot$ $\cdot{}_2F_1\left(\beta+1, -\lambda; \beta-\alpha; \frac{x-a}{p-a}\right)$	(58)
$(x-a)^{\beta}\, e^{\lambda x},\ \Re(\beta) > -1$	$\frac{\Gamma(\beta+1)e^{\lambda a}}{\Gamma(\beta+1-\alpha)}\ (x-a)^{\beta-\alpha}\cdot$ $\cdot{}_1F_1(\beta+1, \beta+1-\alpha; \lambda(x-a))$	(59)
$\sin(\lambda(x-a))$	$\frac{(x-a)^{-\alpha}}{2i\Gamma(1-\alpha)}\cdot[{}_1F_1(1, 1-\alpha, i\lambda(x-a)) -$ ${}_1F_1(1, 1-\alpha, -i\lambda(x-a))]$	(60)

$f(x), x > a$	${}^{\mathrm{RL}}D^{\alpha}_{a+}f(x)$	
$\cos(\lambda(x-a))$	$\frac{(x-a)^{-\alpha}}{2\Gamma(1-\alpha)}\cdot[{}_1F_1(1,1-\alpha,i\lambda(x-a))+$ ${}_1F_1(1,1-\alpha,-i\lambda(x-a))]$	(61)
$(x-a)^{\beta}\sin(\lambda(x-a))$, $\Re(\beta) > -2$	$\frac{\Gamma(\beta+1)}{2i\Gamma(\beta+1-\alpha)}\cdot[{}_1F_1(\beta,\beta-\alpha,i\lambda(x-a))-$ ${}_1F_1(\beta,\beta-\alpha,-i\lambda(x-a))](x-a)^{\beta-\alpha}$	(62)
$(x-a)^{\beta}\cos(\lambda(x-a))$, $\Re(\beta) > -1$	$\frac{\Gamma(\beta+1)}{2\Gamma(\beta+1-\alpha)}\cdot[{}_1F_1(\beta,\beta-\alpha,i\lambda(x-a))+$ ${}_1F_1(\beta,\beta-\alpha,-i\lambda(x-a))](x-a)^{\beta-\alpha}$	(63)
$(x-a)^{\beta}\ \ln(x-a)$, $\Re(\beta) > -1$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\,(x-a)^{\beta-\alpha}\cdot$ $\cdot\,[\ln(x-a)+\psi(\beta+1)-\psi(\beta+1-\alpha)]$	(64)
$(x-a)^{\beta-1}\ E_{\mu,\beta}((x-a)^{\mu})$, $\Re(\beta) > 0 \wedge \Re(\mu) > 0$	$(x-a)^{\beta-1-\alpha}E_{\mu,\ \beta-\alpha}((x-a)^{\mu})$	(65)

$f(x),\ x > a$	${}^{\mathrm{RL}}D^{\alpha}_{a+}f(x),\ \Re(\alpha)\geq 0$	
$(p-x)^{-\frac{1}{2}},\ p > x$	$\sqrt{\frac{a}{\pi x}}\,\frac{1}{p-x},$ for $a=0 \wedge \alpha=\frac{1}{2}$	(66)
$x\,e^{\mu x}$	$x\,\mathscr{E}_x(-\alpha,\mu)+\alpha\,\mathscr{E}_x(1-\alpha,\mu),$ for $a=0$	(67)
$\mathscr{E}_x(\mu,\nu),\ \mu > -1$	$\mathscr{E}_x(\mu-\alpha,\nu),$ for $a=0$	(68)
$x^{\lambda}\mathscr{E}_x(\mu,\nu),\ \lambda+\mu > -1$	$\frac{\Gamma(\lambda+\mu+1)x^{\lambda+\mu-\alpha}}{\Gamma(\mu+1)\Gamma(\lambda+\mu+1-\alpha)}\cdot$ $\cdot{}_2F_2(\lambda+\mu+1,1,\mu+1,\lambda+\mu-\alpha+1;\nu x)$ for $a=0$	(69)
$x\mathscr{E}_x(\mu,\nu),\ \mu > -2$	$x\mathscr{E}_x(\mu-\alpha,\nu)+\alpha\mathscr{E}_x(\mu-\alpha+1,\nu),$ for $a=0$	(70)
$(x-a)^{-\alpha-1}\sin(2\lambda(x-a))$,	$\sqrt{\pi}\left(\frac{x-a}{2\lambda}\right)^{-(\alpha+\frac{1}{2})} sin(\lambda(x-a))\cdot$ $\cdot J_{-(\alpha+\frac{1}{2})}(\lambda(x-a))$	(71)
$(x-a)^{-\alpha-1}\cos(2\lambda(x-a))$,	$\sqrt{\pi}\left(\frac{x-a}{2\lambda}\right)^{-(\alpha+\frac{1}{2})} cos(\lambda(x-a))\cdot$ $\cdot J_{-(\alpha+\frac{1}{2})}(\lambda(x-a))$	(72)
$e_{\alpha}^{(\lambda(z-a))}(x)$	$\lambda e_{\alpha}^{(\lambda(x-a))}$	(73)

$f(x),\ \Re(\lambda) > 0 \wedge \mu > 0$	${}^{\mathrm{L}}D^{\alpha}_{+}f(x)$	
$(b-ax)^{\gamma-1}$, $\Re(\gamma-\alpha) < 1 \wedge a > 0 \wedge ax < b)$	$\frac{\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)a^{-\alpha}}(b-ax)^{\gamma-1-\alpha},$ for $\Re(\alpha)\geq 0$	(74)
$e^{\lambda x}$	$\lambda^{\alpha}\,e^{\lambda x}$	(75)
$\sin(\mu x)$	$\mu^{\alpha}\ \sin\left(\mu x+\frac{\pi\alpha}{2}\right),$ for $\Re(\alpha) > -1$	(76)
$\cos(\mu x)$	$\mu^{\alpha}\ \cos\left(\mu x+\frac{\pi\alpha}{2}\right),$ for $\Re(\alpha) > -1$	(77)
$e^{\lambda x}\ \sin(\mu x)$	$(\lambda^2+\mu^2)^{\alpha/2}e^{\lambda x}\ \sin\left(\mu x+\alpha\arctan\frac{\mu}{\lambda}\right)$	(78)
$e^{\lambda x}\ \cos(\mu x)$	$(\lambda^2+\mu^2)^{\alpha/2}\,e^{\lambda x}\ \cos\left(\mu x+\alpha\arctan\frac{\mu}{\lambda}\right)$	(79)

$f(x)\ \ (\Re(\lambda) > 0,\ \mu > 0)$	${}^{\mathrm{L}}D^{\alpha}_{-}f(x)\ \ (\Re(\alpha)\geq 0)$	
$(b+ax)^{\gamma-1}$, $(\Re(\gamma-\alpha) < 1, \lvert\arg(\frac{a}{b})\rvert < \pi)$	$\frac{\Gamma(1+\alpha-\gamma)}{\Gamma(1-\gamma)a^{-\alpha}}(b+ax)^{\gamma-1-\alpha}$	(80)
$e^{-\lambda x}$	$\lambda^{\alpha}\,e^{-\lambda x}$	(81)

$f(x)\quad (\Re(\lambda)>0,\ \mu>0)$	${}^{\mathtt{L}}D^{\alpha}_{-}f(x)\quad (\Re(\alpha)\geq 0)$	
$\sin(\mu x)$	$\mu^{\alpha}\,\sin\left(\mu x-\frac{\pi\alpha}{2}\right)$	(82)
$\cos(\mu x)$	$\mu^{\alpha}\,\cos\left(\mu x-\frac{\pi\alpha}{2}\right)$	(83)
$e^{-\lambda x}\,\sin(\mu x)$	$(\lambda^2+\mu^2)^{\alpha/2}e^{-\lambda x}\,\sin\left(\mu x-\alpha\arctan\frac{\mu}{\lambda}\right)$	(84)
$e^{-\lambda x}\,\cos(\mu x)$	$(\lambda^2+\mu^2)^{\alpha/2}\,e^{-\lambda x}\,\cos\left(\mu x-\alpha\arctan\frac{\mu}{\lambda}\right)$	(85)

The reader can find more information on this matter e.g. in [8,1,9,10,26].

Remark 8 Riemann-Liouville derivatives can be formulated resorting to generalised functions [9], in which case the following additional results can be established:

${}^{RL}D^{\alpha}_{a+}H(x-p)=\begin{cases}\dfrac{(x-\max\{a,p\})^{-\alpha}}{\Gamma(1-\alpha)}, & \text{if } x>p\\ 0, \text{ if } a\leq x\leq p\end{cases},\ a\in[-\infty,+\infty[$	(86)
${}^{RL}D^{\alpha}_{a+}\frac{\mathrm{d}^n\delta(x-p)}{\mathrm{d}x^n}=\begin{cases}\dfrac{(x-p)^{-\alpha-n-1}}{\Gamma(-\alpha-n)}, & \text{if } x>p\geq a\\ 0, \text{ if } a\leq x\leq p\vee p<a\end{cases},\ n\in\mathbb{N}, a\in[-\infty,+\infty[$	(87)

5 Laplace and Fourier Transforms

It is well known that the Laplace and Fourier transforms, for suitable functions, are given by

$\mathscr{L}\varphi(s)=(\mathscr{L}\varphi(t))\,(s)=\hat{\varphi}(s)=\int_0^{+\infty}e^{-st}\varphi(t)\,\mathrm{d}t$	(88)
$\mathcal{F}\varphi(\kappa)=(\mathcal{F}\varphi(x))\,(\kappa)=\hat{\varphi}(\kappa)=\int_{-\infty}^{\infty}e^{i\kappa x}\varphi(x),\mathrm{d}x$	(89)

5.1 Some properties

In connection with the fractional operators we have the following properties for $\Re(\alpha)>0$, $\Re(\alpha)\in(n-1,n]$, and suitable functions

$\mathscr{L}\,{}^{\mathtt{RL}}D^{\alpha}_{0+}f(s)=s^{\alpha}\mathscr{L}f(s)-\sum_{k=0}^{n-1}s^{n-k-1}D^k\,{}^{\mathtt{RL}}I^{n-\alpha}_{0+}f(0)$	(88)
$\mathscr{L}\,{}^{\mathtt{C}}D^{\alpha}_{0+}f(s)=s^{\alpha}\mathscr{L}f(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}D^kf(0)$	(89)
$\mathcal{F}\,{}^{\mathtt{L}}I^{\alpha}_{+}f(\kappa)=\dfrac{\mathcal{F}f(\kappa)}{(-i\kappa)^{\alpha}}\quad\text{and}\quad\mathcal{F}\,{}^{\mathtt{L}}I^{\alpha}_{-}f(\kappa)=\dfrac{\mathcal{F}f(\kappa)}{(i\kappa)^{\alpha}},\quad(0<\Re(\alpha)<1)$	(90)
$\mathcal{F}\,{}^{\mathtt{L}}D^{\alpha}_{+}f(\kappa)=(-i\kappa)^{\alpha}\mathcal{F}f(\kappa)\quad\text{and}\quad\mathcal{F}\,{}^{\mathtt{L}}D^{\alpha}_{-}f(\kappa)=(i\kappa)^{\alpha}\mathcal{F}f(\kappa)$	(91)

where $(\mp i\kappa)^\alpha = |\kappa|^\alpha e^{\mp\alpha\pi i\ \mathrm{sgn}(\kappa)/2}$.

5.2 Some Laplace transforms

$\mathscr{L}\{f(t)\}(s)$	$f(t)$	
$\frac{k!s^{\alpha-\beta}}{(s^\alpha\mp a)^{k+1}}$	$t^{\alpha k+\beta-1}\frac{\mathrm{d}^k E_{\alpha,\beta}(\pm at^\alpha)}{\mathrm{d}(\pm at^\alpha)^k}$	(92)
$\frac{1}{s^\alpha-\lambda}$	$e_\alpha^{\lambda t}$	(93)
$\frac{n!s^{\alpha-1}}{(s^\alpha-\lambda)^{n+1}}$	$t^{\alpha n}\left(\frac{\partial}{\partial\lambda}\right)^n E_\alpha(\lambda t^\alpha)$	(94)
$\frac{n!}{(s^\alpha-\lambda)^{n+1}}$	$\left(\frac{\partial}{\partial\lambda}\right)^n e_\alpha^{\lambda z}$	(95)
$\frac{s^{\alpha-\beta}}{s^\alpha\mp a}$	$t^{\beta-1}E_{\alpha,\beta}(\pm at^\alpha)$	(96)
$\frac{s^{\alpha-1}}{s^\alpha\mp a}$	$E_\alpha(\pm at^\alpha)$	(97)
$\frac{1}{s^\alpha\mp a}$	$t^{\alpha-1}E_{\alpha,\alpha}(\pm at^\alpha)$	(98)
$\frac{s^{1-\beta}}{s\mp a}$	$t^{\beta-1}E_{1,\beta}(\pm at)=\mathscr{E}_t(\beta-1,\pm a)$	(99)
$\frac{1}{s^\beta}$	$t^{\beta-1}E_{1,\beta}(0)=\mathscr{E}_t(\beta-1,0)=\frac{t^{\beta-1}}{\Gamma(\beta)}$	(100)
$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$	(101)
$\frac{1}{s\sqrt{s}}$	$2\sqrt{\frac{t}{\pi}}$	(102)
$\frac{1}{s^n\sqrt{s}}$, $(n=1,2,\cdots)$	$\frac{2^n t^{n-\frac{1}{2}}}{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}$	(103)
$\frac{s}{(s-a)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{\pi t}}e^{at}(1+2at)$	(104)
$\sqrt{s-a}-\sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}}\left(e^{bt}-e^{at}\right)$	(105)
$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\mathrm{erfc}\left(a\sqrt{t}\right)$	(106)
$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}\mathrm{erf}\left(a\sqrt{t}\right)$	(107)
$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\tau^2}\mathrm{d}\tau$	(108)
$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a}e^{a^2t}\mathrm{erf}\left(a\sqrt{t}\right)$	(109)
$\frac{1}{\sqrt{s}(s+a^2)}$	$\frac{2}{a\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\tau^2}\mathrm{d}\tau$	(110)
$\frac{b^2-a^2}{(s-a^2)\left(\sqrt{s}+b\right)}$	$e^{a^2t}\left[b-a\,\mathrm{erf}\left(a\sqrt{t}\right)\right]$ $-be^{b^2t}\mathrm{erfc}\left(b\sqrt{t}\right)$	(111)
$\frac{1}{\sqrt{s}\left(\sqrt{s}+a\right)}$	$e^{a^2t}\mathrm{erfc}\left(a\sqrt{t}\right)$	(112)
$\frac{1}{(s+a)\sqrt{s+b}}$	$\frac{1}{\sqrt{b-a}}e^{-at}\mathrm{erf}\left(\sqrt{b-a}\sqrt{t}\right)$	(113)
$\frac{b^2-a^2}{\sqrt{s}(s-a^2)\left(\sqrt{s}+b\right)}$	$e^{a^2t}\left[\frac{b}{a}\mathrm{erf}\left(a\sqrt{t}\right)-1\right]+e^{b^2t}\mathrm{erfc}\left(b\sqrt{t}\right)$	(114)
$\frac{(1-s)^n}{s^{n+\frac{1}{2}}}$	$\frac{n!}{(2n)!\sqrt{\pi t}}H_{2n}\left(\sqrt{t}\right)$	(115)
$\frac{(1-s)^n}{s^{n+\frac{3}{2}}}$	$\frac{n!}{(2n+1)!\sqrt{\pi}}H_{2n+1}\left(\sqrt{t}\right)$	(116)
$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s}}$	$ae^{-at}\left[I_1(at)+I_0(at)\right]$	(117)
$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-\frac{1}{2}(a+b)t}I_0\left(\frac{a-b}{2}t\right)$	(118)
$\frac{\Gamma(k)}{(s+a)^k(s+b)^k}$, $(k\geq0)$	$\sqrt{\pi}\left(\frac{t}{a-b}\right)^{k-\frac{1}{2}}e^{-\frac{1}{2}(a+b)t}I_{k-\frac{1}{2}}\left(\frac{a-b}{2}t\right)$	(119)
$\frac{1}{(s+a)^{\frac{1}{2}}(s+b)^{\frac{3}{2}}}$	$te^{-\frac{1}{2}(a+b)t}\left[I_0\left(\frac{a-b}{2}t\right)+I_1\left(\frac{a-b}{2}t\right)\right]$	(120)

$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$	$\frac{1}{t}e^{-at}I_1\left(at\right)$	(121)
$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}}$, $(k>0)$	$\frac{k}{t}e^{-\frac{1}{2}(a+b)t}I_k\left(\frac{a-b}{2}t\right)$	(122)
$\frac{\left(\sqrt{s+a}+\sqrt{s}\right)^{-2\nu}}{\sqrt{s}\sqrt{s+a}}$, $(\nu>-1)$	$\frac{1}{a^\nu}e^{-\frac{1}{2}at}I_\nu\left(\frac{a}{2}t\right)$	(123)
$\frac{1}{\sqrt{s^2+a^2}}$	$J_0\left(at\right)$	(124)
$\frac{1}{\sqrt{s^2-a^2}}$	$I_0\left(at\right)$	(125)
$\frac{\left(\sqrt{s^2+a^2}-s\right)^\nu}{\sqrt{s^2+a^2}}$, $(\nu>-1)$	$a^\nu J_\nu\left(at\right)$	(126)
$\frac{1}{(s^2+a^2)^k}$, $(k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}}J_{k-\frac{1}{2}}\left(at\right)$	(127)
$\left(\sqrt{s^2+a^2}-s\right)^k$, $(k>0)$	$\frac{ka^k}{t}J_k\left(at\right)$	(128)
$\frac{\left(\sqrt{s^2-a^2}+s\right)^\nu}{\sqrt{s^2-a^2}}$, $(\nu>-1)$	$a^\nu I_\nu(at)$	(129)
$\frac{1}{(s^2-a^2)^k}$, $(k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}}I_{k-\frac{1}{2}}(at)$	(130)
$\frac{1}{s\sqrt{s+1}}$	$\mathrm{erf}(\sqrt{t})$	(131)
$\frac{1}{s+\sqrt{s^2+a^2}}$	$\frac{J_1(at)}{at}$	(132)
$\frac{1}{\left(s+\sqrt{s^2+a^2}\right)^n}$, $n\in\mathbb{N}$	$\frac{nJ_n(at)}{a^nt}$	(133)
$\frac{1}{\sqrt{s^2+a^2}\left(s+\sqrt{s^2+a^2}\right)}$	$\frac{J_1(at)}{a}$	(134)
$\frac{1}{\sqrt{s^2+a^2}\left(s+\sqrt{s^2+a^2}\right)^n}$	$\frac{J_n(at)}{a^n}$	(135)
$\frac{k}{s^2+k^2}\coth\frac{\pi s}{2k}$	$\lvert\sin kt\rvert$	(136)
$\frac{1}{s}e^{-k/s}$	$J_0\left(2\sqrt{kt}\right)$	(137)
$\frac{1}{\sqrt{s}}e^{-k/s}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	(138)
$\frac{1}{\sqrt{s}}e^{k/s}$	$\frac{1}{\sqrt{\pi t}}\cosh 2\sqrt{kt}$	(139)
$\frac{1}{s\sqrt{s}}e^{-k/s}$	$\frac{1}{\sqrt{\pi k}}\sin 2\sqrt{kt}$	(140)
$\frac{1}{s\sqrt{s}}e^{k/s}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	(141)
$\frac{1}{s^\nu}e^{-k/s}$, $(\nu>0)$	$\left(\frac{t}{k}\right)^{\frac{\nu-1}{2}}J_{\nu-1}\left(2\sqrt{kt}\right)$	(142)
$\frac{1}{s^\nu}e^{k/s}$, $(\nu>0)$	$\left(\frac{t}{k}\right)^{\frac{\nu-1}{2}}I_{\nu-1}\left(2\sqrt{kt}\right)$	(143)
$e^{-k\sqrt{s}}$, $(k>0)$	$\frac{k}{2\sqrt{\pi t^3}}e^{-\frac{k^2}{4t}}$	(144)
$\frac{1}{s}e^{-k\sqrt{s}}$, $(k\geq 0)$	$\mathrm{erfc}\left(\frac{k}{2\sqrt{t}}\right)$	(145)
$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}$, $(k\geq 0)$	$\frac{1}{\sqrt{\pi t}}e^{-\frac{k^2}{4t}}$	(146)
$\frac{1}{s\sqrt{s}}e^{-k\sqrt{s}}$, $(k\geq 0)$	$2\sqrt{\frac{t}{\pi}}e^{-\frac{k^2}{4t}}-k\,\mathrm{erfc}\left(\frac{k}{2\sqrt{t}}\right)$	(147)
$\frac{a\,e^{-k\sqrt{s}}}{s\left(a+\sqrt{s}\right)}$, $(k\geq 0)$	$-e^{ak}e^{a^2t}\mathrm{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$ $+\mathrm{erfc}\left(\frac{k}{2\sqrt{t}}\right)$	(148)
$\frac{e^{-k\sqrt{s}}}{\sqrt{s}\left(a+\sqrt{s}\right)}$, $(k\geq 0)$	$e^{ak}e^{a^2t}\mathrm{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$	(149)
$\frac{s^{\alpha\rho-\beta}}{(s^\alpha-\lambda)^\rho}\quad(\Re(\beta)>0,\ \lvert\lambda s^{-\alpha}\rvert<1)$	$t^{\beta-1}E^{\rho}_{\alpha,\beta}\left(\lambda t^\alpha\right)$	(150)
$\frac{1}{(s^\alpha-\lambda)}\quad(\lvert\lambda s^{-\alpha}\rvert<1)$	$e^{\lambda t}_{\alpha}$	(151)
$\frac{n!}{(s^\alpha-\lambda)^{n+1}}\quad(\lvert\lambda s^{-\alpha}\rvert<1)$	$\left(\frac{\partial}{\partial\lambda}\right)^n e^{\lambda t}_{\alpha}$	(152)

$\frac{n!}{(s^\alpha-\lambda)^{n+1}} \quad (\lvert\lambda s^{-\alpha}\rvert < 1)$	$t^{\alpha n} e^{\lambda z}_{\alpha,n}$	(153)

6 Systems of Fractional Equations

Bellow we present the explicit solutions of the Riemann-Liouville and Caputo systems of linear fractional differential equations involving the following matrices

$$\mathbf{A} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & . \\ \ldots & \ldots & . \\ \ldots & \ldots & . \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} ; \quad \bar{\mathbf{B}}(x) = \begin{pmatrix} b_1(x) \\ \ldots \\ \ldots \\ \ldots \\ b_n(x) \end{pmatrix} \tag{154}$$

These kind of systems are a important tool in many applied areas, in particular in the application to the *state-variable* technique in control theory.

Problem	Solution	
${}^{\mathtt{RL}}D^\alpha_{a+}\bar{Y}(x) = \mathbf{A}\bar{Y}(x)$	$\bar{Y}_h(x) = e^{\mathbf{A}(x-a)}_\alpha \bar{\mathbf{C}}$, where $\bar{\mathbf{C}}$ is a constant matrix	(155)
${}^{\mathtt{RL}}D^\alpha_{a+}\bar{Y}(x) = \mathbf{A}\bar{Y}(x)$ $\bar{Y}(x_0) = \bar{Y}_0 \quad (x_0 > a)$	$\bar{Y}(x) = e^{\mathbf{A}(x-a)}_\alpha \left(e^{\mathbf{A}(x_0-a)}_\alpha\right)^{-1} \bar{Y}_0$	(156)
${}^{\mathtt{RL}}D^\alpha_{a+}\bar{Y}(x) = \mathbf{A}\bar{Y}(x)$ $\lim\limits_{x\to a+} \left[(x-a)^{1-\alpha}\bar{Y}(x)\right] = \bar{Y}_0$	$\bar{Y}(x) = e^{\mathbf{A}(x-a)}_\alpha \bar{Y}_0$	(157)
${}^{\mathtt{RL}}D^\alpha_{a+}\bar{Y}(x) = \mathbf{A}\bar{Y} + \bar{\mathbf{B}}(x)$	$\bar{Y}(x) = e^{\mathbf{A}(x-a)}_\alpha \bar{C} + \int_a^x e^{\mathbf{A}(x-\xi)}_\alpha \bar{\mathbf{B}}(\xi)d\xi$	(158)
${}^{\mathtt{C}}D^\alpha_{a+}\bar{Y}(x) = \mathbf{A}\bar{Y}(x)$ $\bar{Y}(a) = \bar{b}, \ (\bar{b} \in \mathbb{R}^n)$	$\bar{Y}(x) = E_\alpha(A(x-a)^\alpha)\bar{b}$	(159)
${}^{\mathtt{C}}D^\alpha_{a+}\bar{Y}(x) = \mathbf{A}\bar{Y}(x) + \bar{\mathbf{B}}(x)$	$\bar{Y}(x) = E_\alpha(A(x-a)^\alpha)\bar{C}$ $+ \int_a^x e^{\mathbf{A}(x-\xi)}_\alpha \bar{\mathbf{B}}(\xi)d\xi$	(160)

where

$$E_\alpha\left(\mathbf{A}z\right) = \textstyle\sum_{k=0}^{\infty} \mathbf{A}^k \frac{z^{\alpha k}}{\Gamma(k\alpha+1)} \quad \text{and} \quad E_\alpha\left(\mathbf{A}z\right) = \textstyle\sum_{k=0}^{\infty} \mathbf{A}^k \frac{z^{\alpha k}}{\Gamma(k\alpha+1)} \tag{161}$$

are, respectively, the natural matrix generalizations of the above mentioned Riemann-Liouville and Caputo α-exponential.

As to the issue of numerically solving fractional differential equations, see [25].

7 Transfer Functions

7.1 Discrete transfer function approximations

These approximations are discrete transfer functions, i.e. transfer functions that depend on z^{-1}, the inverse of the $\mathscr{Z}$-transform variable of time, which can be identified with the delay operator. The $\mathscr{Z}$-transform of a function sampled with sampling interval h is

$$\mathscr{Z}\left\{f(t)\right\}(z)=\sum_{k=0}^{+\infty}z^{-k}f(kh) \tag{162}$$

Unless otherwise noted, the approximations below correspond to ${}^{\mathtt{GL}}D^{\alpha}_{0+}f\left(x\right)$ and are truncated after an arbitrary number N of terms.

Euler or Grünwald-Letnikoff approximation, causal ${}^{\mathtt{GL}}D^{\alpha}_{a+}f\left(x\right)\approx\frac{\Delta^{\alpha}_{h,a+}f(x)}{h^{\alpha}},\quad(x>a)$. Therefore, $s^{\alpha}\approx\frac{1}{h^{\alpha}}\sum_{k=0}^{\left\lfloor\frac{x-a}{h}\right\rfloor}(-1)^{k}\begin{pmatrix}\alpha\\k\end{pmatrix}\mathrm{z}^{-k},\quad(x>a)$	(163)
Euler or Grünwald-Letnikoff approximation, anti-causal ${}^{\mathtt{GL}}D^{\alpha}_{b-}f\left(x\right)\approx\frac{\Delta^{\alpha}_{h,b-}f(x)}{h^{\alpha}},\quad(x<b)$. Therefore, $s^{\alpha}\approx\frac{1}{h^{\alpha}}\sum_{k=0}^{\left\lfloor\frac{b-x}{h}\right\rfloor}(-1)^{k}\begin{pmatrix}\alpha\\k\end{pmatrix}\mathrm{z}^{k},\quad(x<b)$	(164)
Tustin approximation (truncated MacLaurin series) $s^{\alpha}\approx\left(\frac{2}{h}\right)^{\alpha}\sum_{k=0}^{N}\sum_{n=0}^{k}\frac{\mathrm{z}^{-k}(-1)^{n}\Gamma(\alpha+1)\Gamma(-\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(n+1)\Gamma(k-n+1)\Gamma(-\alpha-k+n+1)}$	(165)
Tustin approximation (truncated continued fraction expansion) $s^{\alpha}\approx\left(\frac{2}{h}\right)^{\alpha}\left[1;\frac{2\alpha}{-\frac{1}{\mathrm{z}^{-1}}-\alpha},\frac{\alpha^{2}-k^{2}}{-\frac{2k+1}{\mathrm{z}^{-1}}}\right]_{k=1}^{N}$	(166)
First-order backwards finite difference approximation (truncated continued fraction expansion) $s^{\alpha}\approx\frac{1}{h^{\alpha}}\left[0;\frac{1}{1},\frac{\alpha\mathrm{z}^{-1}}{1},\frac{-\frac{k(k+\alpha)}{(2k-1)2k}\mathrm{z}^{-1}}{1},\frac{-\frac{k(k-\alpha)}{2k(2k+1)}\mathrm{z}^{-1}}{1}\right]_{k=1}^{N}$	(167)
Impulse response approximation $s^{\alpha}\approx\frac{h^{-\alpha}}{\Gamma(1-\alpha)}-\frac{h^{-\alpha-1}}{\Gamma(-\alpha)}+\sum_{k=1}^{N}\frac{(kh)^{-\alpha-1}}{\Gamma(-\alpha)}\mathrm{z}^{-k}$	(168)
Step response approximation $s^{\alpha}\approx\frac{h^{-\alpha}}{\Gamma(1-\alpha)}-\frac{h^{-\alpha-1}}{\Gamma(-\alpha)}+\sum_{k=1}^{N}a_{k}\mathrm{z}^{-k}$ $a_{k}=\frac{(kh)^{-\alpha}}{\Gamma(1-\alpha)}-\sum_{n=0}^{k-1}a_{n},\ k=1,2,\ldots N$	(169)

Remark 9 In every machine there will be a $k_{\max}\in\mathbb{N}$ which is the largest integer for which $\Gamma(k_{\max})$ does not yet return infinity. Because $\Gamma(x)$ grows very fast, $k_{\max}$ may be relatively small; if $\left\lfloor\frac{|x-a|}{h}\right\rfloor>k_{\max}$, the summations in (163)–(164) will thereby be truncated. To avoid this, the following approximation can be used instead of (163), assuming that $k_{\max}$ is even:

$$s^{\alpha}\approx\frac{1}{h^{\alpha}}\sum_{k=0}^{k_{\max}}(-1)^{k}\begin{pmatrix}\alpha\\k\end{pmatrix}f(t-kh)+\sum_{i=2}^{m}\frac{1}{(ih)^{\alpha}}\sum_{k=\left\lceil\frac{k_{\max}}{2}\right\rceil}^{k_{\max}}(-1)^{k}\begin{pmatrix}\alpha\\k\end{pmatrix}f(t-kih)$$

$$+\frac{1}{[(m+1)h]^{\alpha}} \sum_{k=\lceil \frac{k_{\max}}{2} \rceil}^{\lfloor \frac{t-c}{(m+1)h} \rfloor} (-1)^k \binom{\alpha}{k} f(t-k(m+1)h), \tag{170}$$

where $\lfloor \frac{t-c}{mh} \rfloor > k_{\max} \geq \left\lfloor \frac{t-c}{(m+1)h} \right\rfloor$ and $m \in \mathbb{N}$.

The expression (164) would be handled in a similar manner.

Remark 10 Each k adds two terms to the continued fraction in (167). A truncated MacLaurin series of a first-order backwards finite difference returns the Euler approximation (163).

Remark 11 A weighted average of approximations (163) and (165) is sometimes used [12]. The particular case of weights $\frac{3}{4}$ for (163) and $\frac{1}{4}$ for (165) is known as the Al-Alaoui operator [13].

Remark 12 Approximations (168)–(169) return the exact impulse and step responses at the interval h. From there on, either the impulse or the step response is followed; it is impossible to follow both. For $x = 0$, the output is always far from the exact value.

7.2 CRONE or Oustaloup approximation

$$s^{\alpha} \approx C \prod_{m=1}^{N} \frac{1+\frac{s}{\omega_{z,m}}}{1+\frac{s}{\omega_{p,m}}}, \qquad \text{where} \quad C = \frac{j\omega_C^{\alpha}}{\prod_{m=1}^{N} \frac{1+\frac{j\omega_C}{\omega_{z,m}}}{1+\frac{j\omega_C}{\omega_{p,m}}}}$$

$$\omega_C \in [\omega_l, \omega_h], \quad \omega_{z,m} = \omega_l \left(\frac{\omega_h}{\omega_l}\right)^{\frac{2m-1-\alpha}{2N}}, \quad \omega_{p,m} = \omega_l \left(\frac{\omega_h}{\omega_l}\right)^{\frac{2m-1+\alpha}{2N}} \tag{171}$$

Remark 13 The N stable real poles and the N stable real zeros of (7.2) are recursively placed in $[\omega_l, \omega_h]$, and verify

$$\frac{\omega_{z,m+1}}{\omega_{z,m}} = \frac{\omega_{p,m+1}}{\omega_{p,m}} = \left(\frac{\omega_h}{\omega_l}\right)^{\frac{1}{N}} \tag{172}$$

It is advisable to make $N \geq \left\lfloor \log_{10} \frac{\omega_h}{\omega_l} \right\rfloor$. Typically the approximation will be acceptable in $\left[10\,\omega_l, \frac{\omega_h}{10}\right]$. Frequency ω_C, at which the gain will be exact, is arbitrary, but it is reasonable to make $\omega_l \ll \omega_C \ll \omega_h$ (e.g. $\omega_C = \sqrt{\omega_l \omega_h}$). Or, if $1 \in [\omega_l, \omega_h]$, calculations can be simplified making $\omega_C = 1 \Rightarrow C = \frac{j^{\alpha}}{\prod_{m=1}^{N} \frac{1+\frac{j}{\omega_{z,m}}}{1+\frac{j}{\omega_{p,m}}}}$.

Remark 14 CRONE is an acronym of *Commande Robuste d'Ordre Non-Entier*, French for Non-Integer Order Robust Control.

7.3 Matsuda approximation

Given the frequency behaviour $G(j\omega)$ of transfer function $G(s)$ (which may be fractional), at frequencies $\omega_0, \omega_1, \ldots \omega_N$ (which do not need to be ordered),

$$G(s) \approx d_0(\omega_0) + \frac{s-\omega_0}{d_1(\omega_1)+} \frac{s-\omega_1}{d_2(\omega_2)+} \frac{s-\omega_2}{d_3(\omega_3)+} \ldots = \left[d_0(\omega_0); \frac{s-\omega_{k_1}}{d_k(\omega_k)}\right]_{k=1}^{N} \tag{173}$$

where $d_0(\omega) = |G(j\omega)|$ and $d_k(\omega) = \frac{\omega-\omega_{k-1}}{d_{k-1}(\omega)-d_{k-1}(\omega_{k-1})}$ $(k = 1, 2, \ldots N)$.

Approximation (173) only works if all orders involved are real.

7.4 General comments on approximations

Remark 15 The following applies to all approximations.

- s^α can be approximated as $\frac{1}{s^{-\alpha}}$, which may be useful one approximation is stable and causal and the other is not.
- $\frac{1}{s^{-\alpha}}$ can be approximated as $s^\alpha = s^{\lceil\alpha\rceil} s^{\alpha-\lceil\alpha\rceil}$ or as $s^\alpha = s^{\lfloor\alpha\rfloor} s^{\alpha-\lfloor\alpha\rfloor}$, to limit approximation orders to the $[-1, 1]$ range.
- Discrete approximations of s^α can be converted into continuous approximations and continuous ones into discrete ones using the Tustin method or any other such method. Notice that usually continuous-time approximations outperform discrete approximations.
- Transfer function $\frac{b_1 s^{\beta_1}+b_2 s^{\beta_2}+\ldots+b_m s^{\beta_m}}{a_1 s^{\alpha_1}+a_2 s^{\alpha_2}+\ldots+a_m s^{\alpha_m}}$ can be approximated finding approximations for s^{β_1}, s^{β_2}, …and s^{β_2}, and linearly combining them. But it can also be approximated as a whole, save if the CRONE approximation is used.
- To find more information on the topic consider in the section consult, for instance, [14–18] and the references included in it.

7.5 Stability of fractional transfer functions

7.6 Frequency responses of some fractional transfer functions

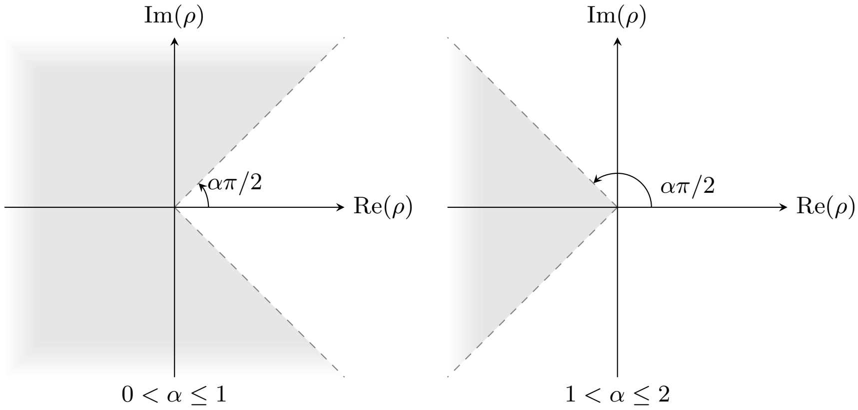

Fig. 1. In grey: regions of the complex plane where the roots ρ of polynomial $\sum_{k=0}^{n} a_k \rho^k$ must lie, for the commensurate transfer function $\dfrac{\sum_{k=0}^{m} b_k s^{k\alpha}}{\sum_{k=0}^{n} a_k s^{k\alpha}}$ to be stable

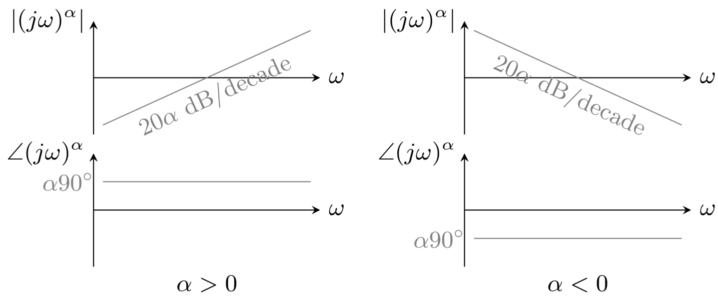

Fig. 2. Bode diagram of s^α

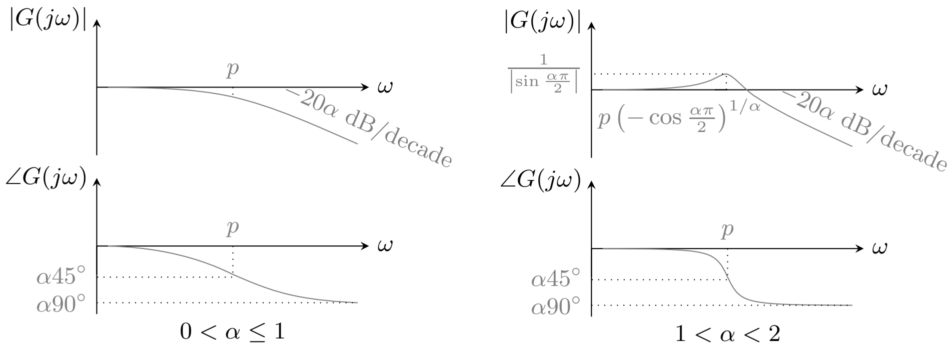

Fig. 3. Bode diagram of $G(s) = \frac{1}{\left(\frac{s}{p}\right)^\alpha + 1}$

8 An Introduction to Fractional Vector Operators

In this section we provide some topics about the fractional multidimensional operators. A first capital contribution was introduced in 1936 by Riesz. He generalized the

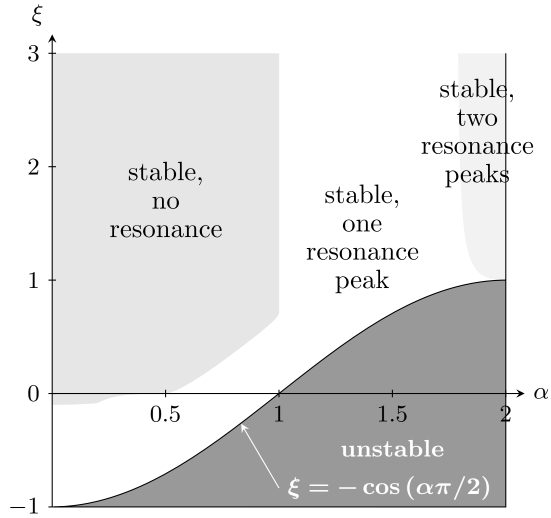

Fig. 4. Frequency behaviour of $\frac{1}{\left(\frac{s}{a}\right)^{2\alpha}+2\xi\left(\frac{s}{a}\right)^{\alpha}+1}$

Riemann-Liouville integral looking for a solution for some problem in potencial theory in connection with partial differential equations for parabolic and hyperbolic cases. He gave two n-dimensional integral operators which are known as Riesz potencial (see e.g. [19,20,1,10]). The inverse operator of I_α given by Riesz, corresponding to the parabolic case, is usually considered as a Fractional Laplacian. However commonly in the literature the fractional Laplacian is introduced by the following property

$$\left[\mathcal{F}\left(-\Delta\right)^{\frac{1}{2}} f(\mathbf{x})\right](\mathbf{k}) = |\mathbf{k}|^{-\alpha}\, \mathcal{F}f(\mathbf{k}), \tag{174}$$

where $\mathcal{F}$ is the Fourier transform, but it is well known that there are several operators verifying this condition.

On the other hand, the relevance of vectorial calculus in the scientific field is well known. We gather here some definitions of the extension of this operators to the fractional case.

Let us consider a general vector

$$\mathbf{F}(\mathbf{x}) = F_s(x)\mathbf{e_s} = F_x\mathbf{e_x} + F_y\mathbf{e_y} + F_z\mathbf{e_z},\ s \in \{1,2,3\} \tag{175}$$

where $\mathbf{e_s}$ are the orthogonal unit vectors. We will use D_s^α to denote any fractional differential operator with respect to the variable x_s.

Gradient of scalar function G		
Classical gradient $= \nabla G$	$\mathrm{grad}\, G = \frac{\partial G}{\partial x}\mathbf{e_1} + \frac{\partial G}{\partial y}\mathbf{e_2} + \frac{\partial G}{\partial z}\mathbf{e_3}$	(176)
Fractional gradient. Definition 1	$\mathrm{grad}^\alpha f(x) = \mathbf{e_s} D_s^\alpha f(x)$	(177)
Fractional gradient. Definition 2	$\mathrm{grad}^\alpha f(x) = \frac{\mathbf{e_s}}{\Gamma(1+\alpha)} D_s^\alpha f(x)$	(178)
Divergence of vector function F		
Classical divergence $\nabla.\mathbf{F}(\mathbf{x})$	$\mathrm{div}\, \mathbf{F}(\mathbf{x}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$	(179)

Fractional divergence. Definition 1	$\mathrm{div}^{\alpha}\mathbf{F}(\mathbf{x}) = \mathbf{e_s} D_s^{\alpha} F_s(x)$	(180)
Fractional divergence. Definition 2	$\mathrm{div}^{\alpha}\mathbf{F}(\mathbf{x}) = \frac{\mathbf{e_s}}{\Gamma(1+\alpha)} D_s^{\alpha} F_s(x)$	(181)
Curl of vector function F		
Classical Curl	$\mathrm{curl}\,\mathbf{F} = \mathbf{e_l}\varepsilon_{lmn} D_m F_n$	(182)
Fractional Curl. Definition 1	$\mathrm{curl}^{\alpha}\mathbf{F} = \mathbf{e_l}\varepsilon_{lmn} D_m^{\alpha} F_n$	(183)
Fractional Curl. Definition 2	$\mathrm{curl}^{\alpha}\mathbf{F} = \frac{\mathbf{e_l}}{\Gamma(\alpha+1)}\varepsilon_{lmn} D_m^{\alpha} F_n$	(184)
Fractional Curl. Definition 3	$\mathrm{curl}^{\alpha}\mathbf{F} = \frac{\mathbf{e_l}}{\Gamma(\alpha+1)}\varepsilon_{lmn} D_m I_n^{1-\alpha} F_n$	(185)

To get more information about this matter see e.g. [21–24] and the references therein.

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