Chapter 5

Fractional Variational Principles

The calculus of variations is dealing with the problem of extremising functionals. As it is known this problem represents a generalization of the problem of finding extremes of functions of several variables, or it is the problem of finding extremes of functions of an infinite number of variables. The roots of the calculus of variations can be seen in the works of the Greek thinkers.

The calculus of variations gives us analytical techniques to deal with important issues like constructing the geodesic between two given points on a given surface, obtaining the curve between two given points in the plane that yields a surface of revolution of minimum area when revolved around a given axis, dealing with the brachistochrone problem, and so on. The calculus of variations is used both to generate interesting differential equations, and to prove the existence of solutions, even in the case when these solutions cannot be obtained analytically. The Lagrangian and Hamiltonian formulation of dynamical systems represents one of the most important principles in physics. The classical Hamiltonian and Lagrangian mechanics are described in terms of derivatives of integer order. However, due to the presence of the friction the physical world is rather nonconservative. As it is known, the notion of dissipativity is a concept in system theory both from theoretical and practical points of view. Dissipativity is deeply connected to the notion of energy. When time evolves, a given dissipative system absorbs a fraction of its supplied energy and transforms it, for example, into heat. On the other hand the presence of frictional forces in physical models increases the complexity in the mathematical tools required to deal with them.

The fractional variational principles represent an emerging part of fractional calculus having an important impact in several areas of physics and engineering. In this context we refer to the works of Riewe [481, 482], Agrawal [15, 16, 18, 19], Klimek [317, 318], Atanackovič *et al.* [42], Baleanu *et al.* [54, 55, 61, 62, 70, 414], Jumarie [298, 299], Cresson *et al.* [153, 154], Tarasov *et al.* [542, 544], Golmankhaneh [239], Herrmann [273], and Rabei *et al.* [462–464, 468, 470].

Particularly, the fractional variational principles are deeply connected to the fractional quantization procedure and to control theory, see Baleanu *et al.* [56–60, 63, 66–68, 71, 74, 77–79, 82, 145, 274, 292, 417, 467], Torres *et al.* [34, 201, 411], Cresson [152], Agrawal *et al.* [20–25, 413, 444], Laskin [331], Biswas and Sen [97], Jelicic and Petrovacki [295], Herrmann [272], Goldfain [238], and Tricaud and Chen [559].

The chapter is basically divided into two main sections. The first section contains an introductory part and the survey of results. After that, the fractional Euler-Lagrange equations for discrete and continuous systems are presented. The fractional Lagrangian formulation for field systems is discussed, followed by the fractional Euler-Lagrange equations in the presence of a delay. The results for both Riemann-Liouville and Caputo derivatives are listed. The recently established fractional discrete Euler-Lagrange equations are presented and some results about the fractional Lagrange-Finsler geometry are briefly illustrated. At the end of the first section of this chapter, some illustrative applications are shown. We mention the fractional variational principles with Riesz derivatives, the multi-order and the multi-term fractional variational approach with Hilfer derivatives, a fractional Lagrangian approach of the Schrödinger equation, the fractional generalization of two equivalent Lagrangians, the Euler-Lagrange equations in fractional space, the multi-time fractional Lagrangian equations, the generalization of the well known Faddeev-Jackiw formalism and the fractional variational calculus with generalized boundary conditions.

The second part of the chapter is devoted to fractional Hamiltonian dynamics. After a short introduction and an overview of the main results, we concentrate on the fractional Hamiltonian analysis of discrete and continuous systems. Here we present direct methods to obtain the fractional canonical equations with Riemann-Liouville, Caputo and Riesz-Caputo derivatives. A special part is devoted to the fractional Hamiltonian formulation of the constrained systems. The fractional Hessian matrix is presented and the reduced phase space is discussed. After that, the fractional generalization of Ostrogradski's construction is presented. In the application part we start by illustrating some examples for the discrete fractional constrained systems. After that, the fractional Hamiltonian formulation in fractional time is presented together with a fractional generalization of the Nambu mechanics. The fractional counterpart of the supersymmetric classical model and the fractional optimal control formulation with and without delay are also illustrated. Multi-time Hamiltonian equations, the Hamilton-Jacobi formulation with Caputo derivative and fractional dynamics on the extended phase space are described.

5.1 Fractional Euler-Lagrange Equations

5.1.1 *Introduction and survey of results*

The Euler-Lagrange equation, developed by Leonhard Euler and Joseph-Louis Lagrange in the 1750s, represents the key formula of the calculus of variations. It gives a way to answer the question for functions which extremize a given cost functional. In the Lagrangian approach, the kinetic and the potential energies are expressed in terms of generalized coordinates and generalized velocities corresponding to each particle. The classical Lagrangian represents the difference between the kinetic and the potential energy. For example, the corresponding Lagrangian for dissipative systems depends explicitly on time, which implies that the Hamiltonian depends explicitly on time too. As a result, one of the good candidates to describe the behavior of the dissipative systems is the fractional calculus. The first application of fractional calculus reported by Niels Henrik Abel in 1823 was related to the formulation of the tautochrone problem [501]. It was proved that the fractional derivatives are the infinitesimal generators of a class of translation invariant convolution semigroups which appear universally as attractors [280]. However, the fractional Leibniz rule and the chain rule become more complicated than the forms encountered in the classical case and the integration by parts formula involves both the left and the right derivatives. As a result, during the last few years, all these properties of fractional calculus made the fractional variational principles an interesting area for many researchers from various fields. We notice that some physical interpretations of the fractional integrals and fractional derivatives have been proposed during the last years [277, 429, 534, 540, 541, 551, 552].

Non-local theories have been investigated and applied in several physical problems [88, 242–244, 346, 349]. It is well known that, due to the form of the fractional differential operators, the fractional Lagrangians and Hamiltonians are typical examples of non-local theories [16, 62, 468].

Several methods to obtain the fractional Euler-Lagrange equations and the corresponding Hamiltonians have been proposed. These equations are new from the mathematical point of view because both the left and the right derivatives are involved.

The first attempt to formulate the fractional generalization of Lagrangian and Hamiltonian equations is due to Riewe [481, 482] who claimed that, by making use of the fractional derivatives of various orders, it is possible to choose Lagrangians that result in a wide range of dissipative Euler-Lagrange equations [481].

The symmetric fractional derivatives were investigated and the Euler-Lagrange equations for models depending on sequential derivatives of this type were obtained in [317]. These investigations were performed in both the Lagrangian and the Hamiltonian formalism.

In [16] an extension of the classical calculus of variations for systems containing Riemann-Liouville fractional derivatives was discussed.

In [62] the Lagrangians linear in velocities were investigated by using the fractional calculus with Riemann-Liouville derivatives and the corresponding Euler-Lagrange equations were obtained.

The Euler-Lagrange equations for fields were investigated within the fractional Lagrangian formulation in [70].

The Euler-Lagrange equations and the transversality conditions for fractional variational problems were discussed in [19].

A generalization of Noether's theorem leading to conservation laws for fractional Euler-Lagrangian equation was obtained in [42].

A version of Noether's theorem for fractional variational problems with Riesz-Caputo derivatives was reported in [227].

In [298] an extension of the Lagrange analytical mechanics was proposed to deal with dynamics of fractal nature. This approach used a slight modification of the Riemann-Liouville derivative definition.

The fractional generalization of nonholonomic constraints defined by equations with fractional derivatives was examined in [544] and the corresponding equations of motion were derived by using a variational principle.

In [34] the Euler-Lagrange fractional equations and the sufficient optimality conditions for problems of the calculus of variations with functionals containing both fractional derivatives and fractional integrals of Riemann-Liouville type were investigated.

The fractional Euler-Lagrange equations with Riemann-Liouville fractional derivatives in the presence of delay were derived in [68]; the fractional variational principles with Caputo derivatives in the presence of delay derivatives were examined in [292].

The optimal control problems in the presence of delay in the state variables as well as the presence of the Riemann-Liouville fractional derivatives of the state variables were investigated in [291].

Fractional embedding of differential operators and Lagrangian systems were explained in [152].

Fractional Euler-Lagrange equations were reported in the presence of the elements of Berezin algebra in [71].

A class of fractional differential equations which were obtained by using the fractional variational principles was illustrated in [77].

The fractional multi-time Lagrangian equations were derived for dynamical systems within Riemann-Liouville derivatives [64].

The fractional Faddeev-Jackiw formalism was constructed in [240].

Based on the conventions established in [565, 566], a nonholonomic deformation of Fedosov type quantization of fractional Lagrange-Finsler geometries was done in [80]. The constructions are obtained for a fractional almost Kähler model encoding equivalently all data for fractional Euler-Lagrange equations with Caputo derivative [80].

5.1.2 *Fractional Euler-Lagrange equations for discrete and continuous systems*

5.1.2.1 *Fractional Euler-Lagrange equations for discrete systems*

Assume that α_j $(j = 1, \ldots, n_1)$ and β_k $(k = 1, \ldots, n_2)$ denote two sets of positive real numbers, $\alpha_{\max} = \max(\alpha_1, \ldots, \alpha_{n_1}, \beta_1, \ldots, \beta_{n_2})$, and M represents an integer fulfilling $M - 1 \leq \alpha < M$. In addition we assume that

$L_f\left(t, q^\rho, {}^{\mathrm{RL}}D_{a+}^{\alpha_1} q^\rho, \ldots, {}^{\mathrm{RL}}D_{a+}^{\alpha_{n1}} q^\rho, {}^{\mathrm{RL}}D_{b-}^{\beta_1} q^\rho, \ldots, {}^{\mathrm{RL}}D_{b-}^{\beta_{n2}} q^\rho\right)$ denotes a function having continuous first and second (partial) derivatives with respect to all its arguments.

Theorem 5.1 (see [16]**).** *Let $J[q^\rho]$ be a functional of the type*

$$J[q^\rho] = \int_a^b L_f\left(t, q^\rho, {}^{\mathrm{RL}}D_{a+}^{\alpha_1} q^\rho, \ldots, {}^{\mathrm{RL}}D_{a+}^{\alpha_{n1}} q^\rho, {}^{\mathrm{RL}}D_{b-}^{\beta_1} q^\rho, \ldots, {}^{\mathrm{RL}}D_{b-}^{\beta_{n2}} q^\rho\right) dt, \tag{5.1.1}$$

defined on the set of n functions q^ρ, $\rho = 1, \ldots, n$ which have continuous left Riemann-Liouville fractional derivatives of orders α_j, $j = 1, \ldots, n_1$, and right Riemann-Liouville fractional derivatives of orders β_j, $j = 1, \ldots, n_2$, in $[a, b]$ and obey the boundary conditions $(q^\rho)^{(j)}(a) = q^\rho_{aj}$ and $(q^\rho)^{(j)}(b) = q^\rho_{bj}$, $j = 1, \ldots, M-1$. A necessary condition for $J[q^\rho]$ to admit an extremum for given functions $q^\rho(t)$, $\rho = 1, \ldots, n$, is that $q^\rho(t)$ satisfy Euler-Lagrange equations

$$\frac{\partial L}{\partial q^\rho} + \sum_{j=1}^{n_1} {}^{\mathrm{RL}}D_{b-}^{\alpha_j} \frac{\partial L_f}{\partial\, {}^{\mathrm{RL}}D_{a+}^{\alpha_j} q^\rho} + \sum_{j=1}^{n_2} {}^{\mathrm{RL}}D_{a+}^{\beta_j} \frac{\partial L_f}{\partial\, {}^{\mathrm{RL}}D_{b-t}^{\beta_j} q^\rho} = 0. \tag{5.1.2}$$

We mention that when α_j becomes an integer, ${}^{\mathrm{RL}}D_{a+}^{\alpha_j}$ and ${}^{\mathrm{RL}}D_{b-}^{\alpha_j}$ are replaced by the ordinary derivatives $(d/dt)^{\alpha_j}$ and $(-d/dt)^{\alpha_j}$, respectively. We notice that the method initiated by Agrawal [16] was applied by Baleanu and co-workers to several problems of physical interest [53, 54, 62, 70].

As an example, we consider a mechanical system described by the classical Lagrangian

$$L(x, y, z) = \dot{x}\dot{z} + yz^3. \tag{5.1.3}$$

Making use of (5.1.3), the classical solutions of the Euler-Lagrange equations can be written as

$$x(t) = at + b, \quad z(t) = 0. \tag{5.1.4}$$

We notice that $y(t)$ has an undetermined evolution and a and b are constants to be determined by making use of the initial conditions.

The next step is to generalize (5.1.3) to the fractional case. Among several possibilities we choose

$$L_f = ({}^{\mathrm{RL}}D^{\alpha}_{a+}x)\,{}^{\mathrm{RL}}D^{\alpha}_{a+}z + yz^3. \tag{5.1.5}$$

Thus, the fractional Euler-Lagrange equations of (5.1.5) become

$${}^{\mathrm{RL}}D^{\alpha}_{b-}({}^{\mathrm{RL}}D^{\alpha}_{a+}z) = 0, \quad z^3 = 0, \quad {}^{\mathrm{RL}}D^{\alpha}_{b-}({}^{\mathrm{RL}}D^{\alpha}_{a+}x) + 3yz^2 = 0. \tag{5.1.6}$$

From (5.1.6) we observe that $z = 0$, and we conclude that y is undetermined. We notice that the equation for $x(t)$ reads as follows

$${}^{\mathrm{RL}}D^{\alpha}_{b-}({}^{\mathrm{RL}}D^{\alpha}_{a+}x) = 0. \tag{5.1.7}$$

The solution of equation (5.1.7), when $1 < \alpha < 2$, is given by

$$\begin{aligned} x(t) = {} & A(t-a)^{\alpha-1} + B(t-a)^{\alpha-2} \\ & + C(t-a)^{\alpha}\,{}_2F_1\left(1, 1-\alpha, 1+\alpha, \frac{t-a}{b-a}\right) \\ & + D(t-a)^{\alpha}\,{}_2F_1\left(1, 2-\alpha, 1+\alpha, \frac{t-a}{b-a}\right), \end{aligned} \tag{5.1.8}$$

where ${}_2F_1$ is the Gauss hypergeometric function and A, B, C, D are real constants. When $\alpha \to 1^+$ and $a = 0$, the classical linear solution of one-dimensional space is recovered, namely

$$x(t) = A + Ct. \tag{5.1.9}$$

In conclusion we notice that the fractional dynamics includes, as a particular case, the classical one.

5.1.3 *Fractional Lagrangian formulation of field systems*

It is well known that the Euler-Lagrange equations are used to describe central force motion, scattering, perturbation theory, Noether's theorem and so on. Extensions to continuous and relativistic systems and classical electrodynamics were also investigated. Since the Euler-Lagrange equations are fundamental for the field theory, we briefly present their fractional generalization in the following.

We notice that a covariant form of the action would involve a classical Lagrangian density $\mathcal{L}$ via

$$S = \int \mathcal{L} d^4 x = \int \mathcal{L} d^3 x dt, \tag{5.1.10}$$

where

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi). \tag{5.1.11}$$

The classical covariant Euler-Lagrange equation corresponding to (5.1.10) is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0, \tag{5.1.12}$$

where ϕ denotes the field variable.

The next step is to present the fractional generalization of (5.1.10) as it was reported in [70]. For these reasons, we consider the following action S, namely

$$S = \int \mathcal{L}\left[\phi(x), ({}^{\mathrm{RL}}D^{\alpha_k}_{a_k-}\phi(x)), ({}^{\mathrm{RL}}D^{\alpha_k}_{a_k+}\phi(x)), x\right] d^3 x dt. \tag{5.1.13}$$

Here $0 < \alpha_k \leq 1$, and the a_k correspond to x_1, x_2, x_3 and t respectively. In the following the limits of integration are $-\infty$ and $+\infty$, respectively.

By using the corresponding fractional integration by parts formula, the fractional Euler-Lagrange equation corresponding to (5.1.13) has the form

$$\frac{\partial \mathcal{L}}{\partial \phi} + \sum_{k=1}^{4} \left\{ ({}^{\mathrm{RL}}D^{\alpha_k}_{-\infty+}) \frac{\partial \mathcal{L}}{\partial({}^{\mathrm{RL}}D^{\alpha_k}_{\infty-})\phi} + ({}^{\mathrm{RL}}D^{\alpha_k}_{\infty-}) \frac{\partial \mathcal{L}}{\partial({}^{\mathrm{RL}}D^{\alpha_k}_{-\infty+})\phi} \right\} = 0, \tag{5.1.14}$$

see [70].

As it is expected for $\alpha_k \to 1$, Eq. (5.1.14) becomes the usual Euler-Lagrange equation for the classical fields [70].

5.1.4 *Fractional Euler-Lagrange equations with delay*

We believe that the inclusion of a delay (see, e.g., [26, 27] and the references therein) in a fractional Lagrangian can give better results in many problems involving dynamics of complex systems. For example the fractional generalization of the Bloch equation [366, 368] that includes both fractional derivatives and time delays was investigated in [92]. The existence of the fractional derivative on the left side of the Bloch equation encodes a degree of system memory in the dynamic model for magnetization. The appearance of a time delay on the right side of the equation balances the equation by adding a degree of system memory [92]. Some other results on combined use of fractional derivatives and delay can be found in [76, 91, 144, 497] as well as in the references therein. In the following we present some results of fractional variational principles with both fractional derivatives and delay.

5.1.4.1 *Riemann-Liouville fractional Euler-Lagrange equations with delay*

First we state a lemma used later in the proofs, namely

Lemma 5.1. *Let $\alpha > 0$, $p, q \geq 1$, $r \in T = (t_1, t_2)$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$).*

(a) *If $\varphi \in L_p(t_1, t_2)$ and $\psi \in L_q(t_1, t_2)$, then*

$$\int_{t_1}^{r} \varphi(t)({}^{\mathtt{RL}}I^{\alpha}_{t_1+}\psi)(t)dt = \int_{t_1}^{r} \psi(t)({}^{\mathtt{RL}}I^{\alpha}_{r-}\varphi)(t)dt \tag{5.1.15}$$

and hence, if $g \in {}^{\mathtt{RL}}I^{\alpha}_{t_2-}(L_p)$ and $f \in {}^{\mathtt{RL}}I^{\alpha}_{t_1+}(L_q)$, then

$$\int_{t_1}^{r} g(t)({}^{\mathtt{RL}}D^{\alpha}_{t_1+}f)(t)dt = \int_{t_1}^{r} f(t)({}^{\mathtt{RL}}D^{\alpha}_{r-}g)(t)dt \tag{5.1.16}$$

(b) *If $\varphi \in L_p(t_1, t_2)$ and $\psi \in L_q(t_1, t_2)$ then*

$$\int_{r}^{t_2} \varphi(t)({}^{\mathtt{RL}}I^{\alpha}_{t_1+}\psi)(t)dt = \int_{r}^{t_2} \psi(t)({}^{\mathtt{RL}}I^{\alpha}_{t_2-}\varphi)(t)dt + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{r} \psi(t)\left(\int_{r}^{t_2} \phi(s)(s-t)^{\alpha-1}ds\right)dt \tag{5.1.17}$$

and hence, if $g \in {}^{\mathtt{RL}}I^{\alpha}_{t_2-}(L_p)$ and $f \in {}^{\mathtt{RL}}I^{\alpha}_{t_1+}(L_q)$, then

$$\begin{aligned}\int_{r}^{t_2} g(t)({}^{\mathtt{RL}}D^{\alpha}_{t_1+}f)(t)dt \\ &= \int_{r}^{t_2} f(t)({}^{\mathtt{RL}}D^{\alpha}_{t_2-}g)(t)dt \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_{t_1}^{r} ({}^{\mathtt{RL}}D^{\alpha}_{t_1+}f)(t)\left(\int_{r}^{t_2} ({}^{\mathtt{RL}}D^{\alpha}_{t_2-}g)(s)(s-t)^{\alpha-1}ds\right)dt\end{aligned} \tag{5.1.18}$$

which implies

$$\int_r^{t_2} g(t)({}^{\mathrm{RL}}D^{\alpha}_{t_1+}f)(t)dt \tag{5.1.19}$$
$$= \int_r^{t_2} f(t)({}^{\mathrm{RL}}D^{\alpha}_{t_2-}g)(t)dt$$
$$- \frac{1}{\Gamma(\alpha)}\int_{t_1}^{r} f(t){}^{\mathrm{RL}}D^{\alpha}_{r-}\left(\int_r^{t_2} ({}^{\mathrm{RL}}D^{\alpha}_{t_2-}g)(s)(s-t)^{\alpha-1}ds\right)dt.$$

The proof can be seen in [68].

The next step is to analyze a modified problem when both the fractional derivatives and delay appear in the Lagrangian. The starting point is the one-dimensional problem [68]: Minimize

$$J(y) = \int_{t_1}^{t_2} F(t, y(t), {}^{\mathrm{RL}}D^{\alpha}_{t_1+}y(t), y(t-\tau), y'(t-\tau))dt, \tag{5.1.20}$$

in such a way that

$$y(t_2) = b, \qquad y(t) = \phi(t) \quad (t \in [t_1-\tau, t_1]), \tag{5.1.21}$$

where $t_1 < t_2$ and $0 < \tau < t_2 - t_1$.

By using the corresponding delay notations [191], namely

$$y_\tau = y(t-\tau), \quad y'_\tau = y'(t-\tau). \tag{5.1.22}$$

Eq. (5.1.20) is written as

$$J(y) = \int_{t_1}^{t_2} F(t, y(t), {}^{\mathrm{RL}}D^{\alpha}_{t_1+}y(t), y_\tau, y'_\tau)dt. \tag{5.1.23}$$

We are now in a position to formulate the following theorem.

Theorem 5.2. *Let $J(y)$ be a functional of the form* [68]

$$J(y) = \int_{t_1}^{t_2} F(t, y(t), {}^{\mathrm{RL}}D^{\alpha}_{t_1+}y(t), y(t-\tau), y'(t-\tau))dt, \tag{5.1.24}$$

defined on a set of continuous functions $y(t)$ which have continuous left Riemann-Liouville derivatives of order α in $[t_1, t_2]$ and satisfy the boundary conditions $y(t_1) = y(t_2) = b$, $y(t) = \phi(t)$, $t \in [t_1-\tau, t_1]$ and $y(t_2) = a_2$. The *necessary condition for J(y) to possess an extremum for a given function y(t) is that y(t) fulfills the Euler-Lagrange equations.*

$$F_y(t) + F_{y_\tau}(t+\tau) + {}^{\mathrm{RL}}D^{\alpha}_{t_2-\tau,-}\frac{\partial F(t)}{\partial({}^{\mathrm{RL}}D^{\alpha}_{t_1+}y(t))} - \frac{dF_{y'_\tau}(t+\tau)}{dt}$$
$$-\frac{1}{\Gamma(\alpha)}{}^{\mathrm{RL}}D^{\alpha}_{t_2-\tau,-}\int_{t_2-\tau}^{t_2}\left[{}^{\mathrm{RL}}D^{\alpha}_{t_2-}\frac{\partial F(t)}{\partial({}^{\mathrm{RL}}D^{\alpha}_{t_1+}y(t))})\right](z)(z-t)^{\alpha-1}dz = 0 \tag{5.1.25}$$

for $t_1 \le t \le t_2 - \tau$,

$$F_y(t) + {}^{\mathrm{RL}}D^{\alpha}_{t_2-}\left(\frac{\partial F(t)}{\partial({}^{\mathrm{RL}}D^{\alpha}_{t_1+}y(t))}\right) = 0, \tag{5.1.26}$$

for $t_2 - \tau \le t \le t_2$, *as well as the boundary condition*

$$F_{y'_\tau}(t+\tau)\eta(t)\mid_{t_1}^{(t_2-\tau)^-} = 0. \tag{5.1.27}$$

From (5.1.25), (5.1.26) and (5.1.27) we conclude that when the delay terms are absent and in the limit $\alpha \to 1$ the classical results are reobtained.

We briefly mention the generalization of Theorem 5.2 with fixed end points and several functions. To this end, let us assume that the functional $J(y_1, y_2, \ldots, y_n)$ has the form [68]

$$\begin{aligned} &J(y_1, y_2, \ldots, y_n) \\ &= \int_{t_1}^{t_2} F(t, y_1(t), \ldots y_n(t), {}^{\mathrm{RL}}D^{\alpha}_{t_1+}y_1(t), \ldots, {}^{\mathrm{RL}}D^{\alpha}_{t_1+}y_n(t), \\ &\qquad y_1(t-\tau), \ldots, y_n(t-\tau), y'_1(t-\tau), \ldots, y'_n(t-\tau))dt, \end{aligned} \tag{5.1.28}$$

and that it fulfills the boundary conditions

$$y_i(t_2) = y_{it_2}, \quad y_i(b) = y_{ib}, \quad y_i(t) = \phi_i(t)\ (i = 1, \ldots, n;\ t \in [t_1 - \tau, t_1]) \tag{5.1.29}$$

where $t_1 < t_2$ and $0 < \tau < t_2 - t_1$. Besides we assume that F possesses continuous partial derivatives with respect to all of its parameters and that the functions ϕ_i, $i = 1, 2, \ldots n$, are smooth.

Theorem 5.3. *A necessary condition for the curve $y_i = y_i(t), i = 1, \cdots, n$, fulfilling the boundary conditions (5.1.29) to be extremal for the functional (5.1.28) is that*

$$\begin{aligned} 0 &= F_{y_i}(t) + F_{(y_i)_\tau}(t+\tau) \\ &\quad + {}^{\mathrm{RL}}D^{\alpha}_{t_2-\tau,-} \frac{\partial F(t)}{\partial({}^{\mathrm{RL}}D^{\alpha}_{t_1+}y_i(t))} - \frac{dF_{(y_i)'_\tau}(t+\tau)}{dt} \\ &\quad - \frac{1}{\Gamma(\alpha)} {}^{\mathrm{RL}}D^{\alpha}_{t_2-\tau,-} \int_{t_2-\tau}^{t_2} \left[{}^{\mathrm{RL}}D^{\alpha}_{t_2-} \left(\frac{\partial F(t)}{\partial({}^{\mathrm{RL}}D^{\alpha}_{t_1+}y_i(t))} \right) \right](z)(z-t)^{\alpha-1} dz \end{aligned} \tag{5.1.30}$$

for $t_1 \le t \le t_2 - \tau$,

$$F_{y_i}(t) + {}^{\mathrm{RL}}D^{\alpha}_{t_2-}\left(\frac{\partial F(t)}{\partial({}^{\mathrm{RL}}D^{\alpha}_{t_1+}y_i(t))}\right) = 0 \tag{5.1.31}$$

for $t_2 - \tau \le t \le t_2$, and that the boundary conditions

$$\left(F_{(y_i)'_\tau}\right)(t+\tau)\eta(t) \,\Big|_{t_1}^{(t_2-\tau)^-} = 0 \tag{5.1.32}$$

are fulfilled for $i = 1, \ldots, n$.

For more details about this topic see [68].

5.1.4.2 *Caputo fractional Euler-Lagrange equations with delay*

The case when the Riemann-Liouville derivative is replaced by a Caputo derivative is of interest in its own right from both the theoretical and the applied point of view. Therefore, we now discuss the following problem [292]: Minimize

$$J(y) = \int_a^b F(t, y(t), {}^{\mathrm{C}}D_{a+}y(t), {}^{\mathrm{C}}D^{\beta}_{b-}y(t), y(t-\tau), y'(t-\tau))dt \tag{5.1.33}$$

such that

$$y(b) = c, \qquad y(t) = \phi(t) \quad (t \in [a-\tau, a]), \tag{5.1.34}$$

where $a < b$, $0 < \tau < b - a$, $0 < \alpha < 1$, $0 < \beta < 1$, and where c is a constant and F is a function with continuous first and second partial derivatives with respect to all of its arguments. The corresponding results are contained in the following theorem.

Theorem 5.4. *Let $J(y)$ be a functional of the form*

$$J(y) = \int_a^b F(t, y(t), {}^{\mathtt{C}}D_{a+}^{\alpha} y(t), {}^{\mathtt{C}}D_{b-}^{\beta} y(t), y(t-\tau), y'(t-\tau))dt, \quad (5.1.35)$$

with $0 < \alpha, \beta < 1$, defined on a set of continuous functions $y(t)$ which have continuous left Caputo derivatives of order α and right derivatives of order β in $[a, b]$ and satisfy the conditions $y(t) = \phi(t)$ $(t \in [a-\tau, a])$ and $y(b) = c$. Moreover, let $F : [a, b] \times \mathbb{R}^5 \to \mathbb{R}$ have continuous partial derivatives with respect to all its arguments. Then, the necessary condition for $J(y)$ to possess an extremum for a given function $y(t)$ is that $y(t)$ satisfies the Euler-Lagrange equations

$$\begin{aligned} 0 = & \frac{\partial F}{\partial y(t)}(t) + {}^{\mathtt{RL}}D_{b-\tau,-}^{\alpha} \left(\frac{\partial F}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha} y(t)} \right)(t) + {}^{\mathtt{RL}}D_{a+}^{\beta} \left(\frac{\partial F}{\partial {}^{\mathtt{C}}D_{b-}^{\beta} y(t)} \right)(t) \\ & + \frac{\partial F}{\partial y(t-\tau)}(t+\tau) - \frac{d}{dt} \frac{\partial F}{\partial y'(t-\tau)}(t+\tau) \\ & - \frac{1}{\Gamma(\alpha)} {}^{\mathtt{RL}}D_{b-\tau,-}^{\alpha} \left(\int_{b-\tau}^{b} \left({}^{\mathtt{RL}}D_{b-}^{\alpha} \frac{\partial F}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha} y(t)} \right)(s)(s-t)^{\alpha-1} ds \right) \end{aligned} \quad (5.1.36)$$

for $a \leq t \leq b - \tau$,

$$\begin{aligned} 0 = & \frac{\partial F}{\partial y(t)}(t) + {}^{\mathtt{RL}}D_{b-}^{\alpha} \left(\frac{\partial F}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha} y(t)} \right)(t) \\ & + {}^{\mathtt{RL}}D_{b-\tau,+}^{\beta} \left(\frac{\partial F}{\partial\, {}^{\mathtt{C}}D_{b-}^{\beta} y(t)} \right)(t) \\ & - \frac{1}{\Gamma(\beta)} {}^{\mathtt{RL}}D_{b-\tau,+}^{\beta} \left(\int_{a}^{b-\tau} \left({}^{\mathtt{RL}}D_{a+}^{\beta} \frac{\partial F}{\partial\, {}^{\mathtt{C}}D_{b-}^{\beta} y(t)} \right)(s)(s-t)^{\beta-1} ds \right) \end{aligned} \quad (5.1.37)$$

for $b - \tau \leq t \leq b$, and the transversality condition

$$\frac{\partial F}{\partial y'(t-\tau)}(t+\tau)\eta(t) \ \Big|_a^{b-\tau} = 0, \quad (5.1.38)$$

for any admissible function η satisfying $\eta(t) = 0$ $(t \in [a-\tau, a])$ and $\eta(b) = 0$.

The generalization of the above theorem reads as follows (see [292]).

Theorem 5.5. *Let $J(y_1, y_2, ..., y_n)$ be a functional of the form*

$$\begin{aligned} &J(y_1, y_2, ..., y_n) \\ &\quad = \int_a^b F(t, y_1(t), y_2(t), ..., y_n(t), {}^{\mathrm{C}}D^{\alpha}_{a+}y_1(t), {}^{\mathrm{C}}D^{\alpha}_{a+}y_2(t), \ldots, {}^{\mathrm{C}}D^{\alpha}_{a+}y_n(t), \\ &\qquad {}^{\mathrm{C}}D^{\beta}_{b-}y_1(t), {}^{\mathrm{C}}D^{\beta}_{b-}y_2(t), \ldots, {}^{\mathrm{C}}D^{\beta}_{b-}y_n(t), \\ &\qquad y_1(t-\tau), y_2(t-\tau), \ldots, y_n(t-\tau), \\ &\qquad y_1'(t-\tau), y_2'(t-\tau), \ldots, y_n'(t-\tau))dt, \end{aligned} \tag{5.1.39}$$

defined on a set of continuous functions $y_i(t)$, $i = 1, 2, \ldots, n$ that have left Caputo fractional derivatives of order α and right Caputo fractional derivatives of order β in the interval $[a, b]$ and satisfy the conditions

$$y_i(b) = c_i, \quad y_i(t) = \phi_i(t) \qquad (i = 1, 2 \ldots, n; t \in [a-\tau, a]), \tag{5.1.40}$$

where $a < b$, $0 < \tau < b - a$, the c_i are constant and the $\phi_i(t)$ are smooth. Assume also that $F : \mathbb{R}^{4n+1} \to \mathbb{R}$ has continuous second partial derivatives with respect to all of its arguments. Then, the necessary conditions for the curves $y_i(t)$, $i = 1, 2, \ldots, n$, satisfying the conditions (5.1.40) to be extremal for the functional (5.1.39) are

$$\begin{aligned} 0 = &\frac{\partial F}{\partial y_i(t)}(t) + {}^{\mathrm{RL}}D^{\alpha}_{b-\tau,-}\left(\frac{\partial F}{\partial {}^{\mathrm{C}}D^{\alpha}_{a+}y_i(t)}\right)(t) \\ &+ {}^{\mathrm{RL}}D^{\beta}_{a+}\left(\frac{\partial F}{\partial {}^{\mathrm{C}}D^{\beta}_{b-}y_i(t)}\right)(t) + \frac{\partial F}{\partial y_i(t-\tau)}(t+\tau) - \frac{d}{dt}\frac{\partial F}{\partial y_i'(t-\tau)}(t+\tau) \\ &- \frac{1}{\Gamma(\alpha)}{}^{\mathrm{RL}}D^{\alpha}_{b-\tau,-}\left(\int_{b-\tau}^{b}\left({}^{\mathrm{RL}}D^{\alpha}_{b-}\frac{\partial F}{\partial {}^{\mathrm{C}}D^{\alpha}_{a+}y_i(t)}\right)(s)(s-t)^{\alpha-1}ds\right) \end{aligned} \tag{5.1.41}$$

for $a \le t \le b - \tau$ and $i = 1, 2, \ldots, n$,

$$\begin{aligned} 0 = &\frac{\partial F}{\partial y_i(t)}(t) + {}^{\mathrm{RL}}D^{\alpha}_{b-}\left(\frac{\partial F}{\partial {}^{\mathrm{C}}D^{\alpha}_{a+}y_i(t)}\right)(t) \\ &+ {}^{\mathrm{RL}}D^{\beta}_{b-\tau,+}\left(\frac{\partial F}{\partial {}^{\mathrm{C}}D^{\beta}_{b-}y_i(t)}\right)(t) \\ &- \frac{1}{\Gamma(\beta)}{}^{\mathrm{RL}}D^{\beta}_{b-\tau,+}\left(\int_{a}^{b-\tau}\left({}^{\mathrm{RL}}D^{\beta}_{a+}\frac{\partial F}{\partial {}^{\mathrm{C}}D^{\beta}_{b-}y_i(t)}\right)(s)(t-s)^{\beta-1}ds)\right) \end{aligned} \tag{5.1.42}$$

for $b-\tau \le t \le b$ and $i = 1, 2, \ldots, n$, and the transversality conditions

$$\frac{\partial F}{\partial y_i'(t-\tau)}(t+\tau)\eta_i(t) \mid_a^{b-\tau} = 0, \qquad i = 1, 2, \ldots, n, \tag{5.1.43}$$

for any admissible functions η_i satisfying $\eta_i(t) = 0$ for $t \in [a-\tau, a]$ and $\eta_i(b) = 0$.

An example and more details about this topic are given in [292].

5.1.5 *Fractional discrete Euler-Lagrange equations*

The aim of this part is to present the formula of integration by parts as well as the corresponding discrete fractional Euler-Lagrange equations with the help of the right fractional sum and difference following the time scale calculus. This kind of calculus represents a unification of the theory of difference and differential equations, it unifies the integral and differential calculus with the calculus of finite differences [1, 2, 11, 36, 83, 101, 102, 283, 303]. As a result, we obtain a formalism to investigate a discrete-continuous dynamical system.

Let us assume that $\alpha > 0$ and $\sigma(s) = s + 1$. Then, the αth (forward) fractional sum of f was defined in [399] and it was used in [43] and [44] as

$$\Delta^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t - \sigma(s))^{(\alpha-1)} f(s). \tag{5.1.44}$$

If $\alpha > 0$ and $\rho(s) = s - 1$, then we define the αth (backward) fractional sum of f as

$$\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (\rho(s) - t)^{(\alpha-1)} f(s). \tag{5.1.45}$$

Using the notation

$$N_a = \{a, a+1, a+2, a+3, \ldots\} \quad \text{and} \quad {}_bN = \{b, b-1, b-2, b-3, \ldots\},$$

a by-part formula relating these two operators was obtained in [2].

Proposition 5.1. *Let $\alpha > 0$, and consider $a, b \in \mathbb{R}$ such that $a < b$ and $b \equiv a + \alpha(\mathrm{mod} 1)$. If f is defined on N_a and g is defined on ${}_bN$, then we obtain*

$$\sum_{s=a+\alpha}^{b} (\Delta^{-\alpha} f)(s) g(s) = \sum_{s=a}^{b-\alpha} f(s) \nabla^{-\alpha} g(s). \tag{5.1.46}$$

A by-part formula for fractional differences was also obtained in [2]:

Proposition 5.2. *Let $\alpha > 0$ be non-integer and let us assume that $b \equiv a + n - \alpha (\mathrm{mod} 1)$. If f is defined on ${}_bN$ and g is defined on N_a, then we have*

$$\sum_{s=a+(n-\alpha)+1}^{b-n+1} f(s)\Delta^\alpha g(s) = \sum_{s=a+n-1}^{b-(n-\alpha)+1} g(s)\nabla^\alpha f(s). \tag{5.1.47}$$

We mention that when $\alpha = 1$ we obtain the classical counterpart of the formula, namely

$$\begin{aligned} \sum_{s=a}^{b-1} f(s)\Delta g(s) &= f(s)g(s)|_a^b - \sum_{s=a+1}^{b} g(s)\nabla f(s) \\ &= f(s)g(s)|_a^b - \sum_{s=a}^{b-1} g(s+1)\Delta f(s). \end{aligned} \tag{5.1.48}$$

The key point in this subsection is to consider the functional [2] $J : S \to \mathbb{R}$ with

$$J(y) = \sum_{s=a-\alpha}^{b} L(s, y(s), \Delta^\alpha y(s)), \tag{5.1.49}$$

where

$$a, b \in \mathbb{R}, \quad 0 < \alpha < 1, \quad b \equiv a - \alpha \ (\mathrm{mod} 1),$$

$$L : (N_{a-\alpha} \cap \ {}_bN) \times (\mathbb{R}^n)^2 \to \mathbb{R},$$

and

$$S = \{y : N_{a-\alpha} \cap {}_bN \to \mathbb{R}^n : y(a) = y_0 \text{ and } y(b+\alpha) = y_1\}.$$

Moreover, we assume that the function y fits the discrete time scales N_a and $N_{a-\alpha}$. That is, $y(s) = y(s-\alpha)$ for all $s \in N_a$.

From (5.1.49) we obtain the corresponding Euler-Lagrange equations as (*cf.* [2])

$$\frac{\partial L(s)}{\partial y} + \nabla^\alpha \frac{\partial L(s)}{\partial \Delta^\alpha y} = 0. \tag{5.1.50}$$

5.1.6 *Fractional Lagrange-Finsler geometry*

Since we are dealing with fractional differential geometry, some adequate symbols should be used. As a result, the left and the right Caputo derivatives were denoted in [565, 566] as

$$ {}_{1x}\overset{\alpha}{\underline{\partial}}_{x} f(x) = \frac{1}{\Gamma(s-\alpha)} \int_{1x}^{x} (x - \ x')^{s-\alpha-1} \left(\frac{\partial}{\partial x'}\right)^{s} f(x')dx', \qquad (5.1.51a) $$

$$ {}_{x}\overset{\alpha}{\underline{\partial}}_{2x} f(x) = \frac{1}{\Gamma(s-\alpha)} \int_{x}^{2x} (x' - x)^{s-\alpha-1} \left(-\frac{\partial}{\partial x'}\right)^{s} f(x')dx', \quad (5.1.51b) $$

respectively, where, as usual, $s = \lceil \alpha \rceil$ denotes the smallest integer not less than α. For example, we emphasize that the integral is considered from ${}_{1}x$ to x in the symbol of partial derivative ${}_{1x}\overset{\alpha}{\underline{\partial}}_{x}$. We put α over a symbol in order to emphasize that the constructions are considered within fractional calculus with $\alpha \in (0, 1)$.

We recall (see, e.g., [304]) that a Lagrange space $L^n = (M, L)$ of integer dimension n is defined by a Lagrange fundamental function $L(x, y)$, namely a regular real function $L : TM \to \mathbb{R}$, where TM is the tangent bundle over the manifold M, such that the Hessian

$$ {}_{L}g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} \qquad (5.1.52) $$

is not degenerate.

We say that a Lagrange space L^n is a Finsler space F^n if and only if its fundamental function L is positive and two-homogeneous with respect to the variables y^i, i.e. $L = F^2$. In the following we work with Lagrange spaces and their fractional generalizations, considering the Finsler spaces to form a more particular, homogeneous subclass.

Definition 5.1. A (target) fractional Lagrange space $\underline{\overset{\alpha}{L^n}} = (\overset{\alpha}{\underline{M}}, \overset{\alpha}{L})$ of fractional dimension $\alpha \in (0, 1)$, is defined by a regular real function $\overset{\alpha}{L} : \overset{\alpha}{\underline{T}}M \to \mathbb{R}$, such that the fractional Hessian is [81]

$$ {}_{L}\overset{\alpha}{g}_{ij} = \frac{1}{4} \left(\overset{\alpha}{\underline{\partial}}_{i} \overset{\alpha}{\underline{\partial}}_{j} + \overset{\alpha}{\underline{\partial}}_{j} \overset{\alpha}{\underline{\partial}}_{i} \right) \overset{\alpha}{L} \neq 0. \qquad (5.1.53) $$

We consider the coefficients ${}_L\overset{\alpha}{g}{}^{ij}$ as being inverse to the ${}_L\overset{\alpha}{g}_{ij}$ given in Eq. (5.1.53). Any $\overset{\alpha}{\underline{L}}{}^n$ can be associated to a prime integer Lagrange space L^n [81].

The notion of nonlinear connection (N-connection) on $\overset{\alpha}{\underline{L}}{}^n$ can be defined in analogy to that on nonholonomic fractional manifolds [565, 566] by considering the fractional tangent bundle $\overset{\alpha}{\underline{T}}M$.

Definition 5.2. An N-connection $\overset{\alpha}{\mathbf{N}}$ on $\overset{\alpha}{\underline{T}}M$ is defined by a nonholonomic distribution (Whitney sum) with conventional h- and v-subspaces, $\underline{h\overset{\alpha}{T}}M$ and $\underline{v\overset{\alpha}{T}}M$, when

$$\overset{\alpha}{\underline{T}}\overset{\alpha}{\underline{T}}M = \underline{h\overset{\alpha}{T}}M \oplus \underline{v\overset{\alpha}{T}}M. \tag{5.1.54}$$

Locally, a fractional N-connection is defined by a set of coefficients, $\overset{\alpha}{\mathbf{N}} = \{{}^\alpha N_i^a\}$, where

$$\overset{\alpha}{\mathbf{N}} = {}^\alpha N_i^a(u)(dx^i)^\alpha \otimes \overset{\alpha}{\underline{\partial}}_a. \tag{5.1.55}$$

Proposition 5.3 (see [81]**).** *There is a class of N-adapted fractional (co-) frames linearly depending on* ${}^\alpha N_i^a$*,*

$${}^\alpha\mathbf{e}_\beta = \left[{}^\alpha\mathbf{e}_j = \overset{\alpha}{\underline{\partial}}_j - {}^\alpha N_j^a \overset{\alpha}{\underline{\partial}}_a,\ {}^\alpha e_b = \overset{\alpha}{\underline{\partial}}_b\right], \tag{5.1.56}$$

$${}^\alpha\mathbf{e}^\beta = [{}^\alpha e^j = (dx^j)^\alpha,\ {}^\alpha\mathbf{e}^b = (dy^b)^\alpha + {}^\alpha N_k^b (dx^k)^\alpha]. \tag{5.1.57}$$

The above mentioned bases are nonholonomical and they are characterized by

$$[{}^\alpha\mathbf{e}_\alpha,\ {}^\alpha\mathbf{e}_\beta] = {}^\alpha\mathbf{e}_\alpha {}^\alpha\mathbf{e}_\beta - {}^\alpha\mathbf{e}_\beta {}^\alpha\mathbf{e}_\alpha = {}^\alpha W_{\alpha\beta}^\gamma {}^\alpha\mathbf{e}_\gamma,$$

where the nontrivial nonholonomy coefficients ${}^\alpha W_{\alpha\beta}^\gamma$ are calculated as ${}^\alpha W_{ib}^a = \overset{\alpha}{\underline{\partial}}_b {}^\alpha N_i^a$ and ${}^\alpha W_{ij}^a = {}^\alpha\Omega_{ji}^a = {}^\alpha\mathbf{e}_i {}^\alpha N_j^a - {}^\alpha\mathbf{e}_j {}^\alpha N_i^a$, respectively. The values ${}^\alpha\Omega_{ji}^a$ define the coefficients of the N-connection curvature.

We consider the values $y^k(\tau) = dx^k(\tau)/d\tau$, for $x(\tau)$ parametrizing the smooth curves on a given manifold M such that $\tau \in [0, 1]$. The fractional counterparts of these configurations are determined by replacing $d/d\tau$ with the fractional Caputo derivative $\overset{\alpha}{\underline{\partial}}_\tau = {}_{1\tau}\overset{\alpha}{\underline{\partial}}_\tau$ when ${}^\alpha y^k(\tau) = \overset{\alpha}{\underline{\partial}}_\tau x^k(\tau)$.

Using the techniques developed in [402, 564, 567] but for the case of the fractional derivatives and integrals, we formulate the following theorem:

Theorem 5.6. *Any $\overset{\alpha}{L}$ defines the fundamental geometric objects determining canonically a nonholonomic fractional Riemann–Cartan geometry on $\underline{\overset{\alpha}{T}}M$ being satisfied the properties* [81]*:*

(1) *The fractional Euler-Lagrange equations*

$$\overset{\alpha}{\underline{\partial}}_\tau\left({}_{1y^i}\overset{\alpha}{\underline{\partial}}_i \overset{\alpha}{L}\right) - {}_{1x^i}\overset{\alpha}{\underline{\partial}}_i \overset{\alpha}{L} = 0$$

are equivalent to the fractional nonlinear geodesic (equivalently, semi–spray) equations

$$\left(\overset{\alpha}{\underline{\partial}}_\tau\right)^2 x^k + 2\overset{\alpha}{G}{}^k(x, {}^\alpha y) = 0,$$

where

$$\overset{\alpha}{G}{}^k = \frac{1}{4}\, {}_L\overset{\alpha}{g}{}^{kj}\left[y^j\, {}_{1y^j}\overset{\alpha}{\underline{\partial}}_j\left({}_{1x^i}\overset{\alpha}{\underline{\partial}}_i \overset{\alpha}{L}\right) - {}_{1x^i}\overset{\alpha}{\underline{\partial}}_i \overset{\alpha}{L}\right]$$

defines the canonical N-connection

$${}^\alpha_L N^a_j = {}_{1y^j}\overset{\alpha}{\underline{\partial}}_j \overset{\alpha}{G}{}^k(x, {}^\alpha y). \tag{5.1.58}$$

(2) *There is a canonical (Sasaki type) metric structure,*

$${}_L\overset{\alpha}{\mathbf{g}} = {}^\alpha_L g_{kj}(x,y)\ {}^\alpha e^k \otimes\ {}^\alpha e^j +\ {}^\alpha_L g_{cb}(x,y)\ {}^\alpha_L \mathbf{e}^c \otimes\ {}^\alpha_L \mathbf{e}^b, \tag{5.1.59}$$

where the preferred frame structure (defined linearly by ${}^\alpha_L N^a_j$) is ${}^\alpha_L \mathbf{e}_\nu = ({}^\alpha_L \mathbf{e}_i, e_a)$.

(3) *There is a canonical metrical distinguished connection*

$${}^\alpha_c \mathbf{D} = (h\ {}^\alpha_c D, v\ {}^\alpha_c D) = \{{}^\alpha_c \mathbf{\Gamma}^\gamma{}_{\alpha\beta} = ({}^\alpha \widehat{L}^i{}_{jk},\ {}^\alpha \widehat{C}^i_{jc})\}\ ,$$

which is a linear connection preserving under parallelism the splitting (5.1.54) and metric compatible, i.e. ${}^\alpha_c \mathbf{D}\left({}_L\overset{\alpha}{\mathbf{g}}\right) = 0$, when

$${}^\alpha_c \mathbf{\Gamma}^i{}_j =\ {}^\alpha_c \mathbf{\Gamma}^i{}_{j\gamma}\ {}^\alpha_L \mathbf{e}^\gamma = \widehat{L}^i{}_{jk} e^k + \widehat{C}^i_{jc}\ {}^\alpha_L \mathbf{e}^c,$$

for $\widehat{L}^i{}_{jk} = \widehat{L}^a{}_{bk}, \widehat{C}^i_{jc} = \widehat{C}^a_{bc}$ *in* ${}^{\alpha}_{c}\mathbf{\Gamma}^a{}_b = {}^{\alpha}_{c}\mathbf{\Gamma}^a{}_{b\gamma}\ {}^{\alpha}_{L}\mathbf{e}^{\gamma} = \widehat{L}^a{}_{bk} e^k + \widehat{C}^a_{bc}\ {}^{\alpha}_{L}\mathbf{e}^c$, *and*

$$
\begin{aligned}
{}^{\alpha}\widehat{L}^i_{jk} &= \frac{1}{2}\ {}^{\alpha}_{L}g^{ir}\left({}^{\alpha}_{L}\mathbf{e}_k\ {}^{\alpha}_{L}g_{jr} + {}^{\alpha}_{L}\mathbf{e}_j\ {}^{\alpha}_{L}g_{kr} - {}^{\alpha}_{L}\mathbf{e}_r\ {}^{\alpha}_{L}g_{jk}\right), \quad (5.1.60)\\
{}^{\alpha}\widehat{C}^a_{bc} &= \frac{1}{2}\ {}^{\alpha}_{L}g^{ad}\left({}^{\alpha}e_c\ {}^{\alpha}_{L}g_{bd} + {}^{\alpha}e_c\ {}^{\alpha}_{L}g_{cd} - {}^{\alpha}e_d\ {}^{\alpha}_{L}g_{bc}\right)
\end{aligned}
$$

are the generalized Christoffel indices.

5.1.7 *Applications*

5.1.7.1 *Fractional variational principles with Riesz derivatives*

For $0 < \alpha < 1$ we define the Riesz derivative

$$
{}^{\mathtt{R}}D^{\alpha}_{[a,b]}q(t) = \frac{1}{2}\left({}^{\mathtt{RL}}D^{\alpha}_{a+}q(t) - {}^{\mathtt{RL}}D^{\alpha}_{b-}q(t)\right)
$$

and the Riesz-Caputo derivative

$$
{}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q(t) = \frac{1}{2}\left({}^{\mathtt{C}}D^{\alpha}_{a+}q(t) - {}^{\mathtt{C}}D^{\alpha}_{b-}q(t)\right),
$$

respectively.

Let us assume that $L_f(t, q, {}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q(t))$ is a function with continuous first and second (partial) derivatives with respect to all its arguments. Then, the fractional Euler-Lagrange equations with Riesz's derivatives are described by the following theorem [20].

Theorem 5.7. *Let $J[q]$ be a functional of the form*

$$
J[q] = \int_a^b L_f(t, q, {}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q(t))dt
$$

defined on the set of functions which have continuous Riesz-Caputo fractional derivative of order α in $[a, b]$ and which satisfy the boundary conditions $q(a) = q_a$ and $q(b) = q_b$. Then a necessary condition for $J[q]$ to have a maximum for a given function $q(t)$ is that $q(t)$ satisfies the Euler-Lagrange equation

$$
\frac{\partial L_f}{\partial q} - {}^{\mathtt{R}}D^{\alpha}_{[a,b]}\left(\frac{\partial L_f}{\partial\, {}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q}\right) = 0. \qquad (5.1.61)
$$

5.1.7.2 *Multi-order and multi-term fractional variational formulations with Hilfer derivatives*

Hilfer [280, 557] has introduced the left two-parameter fractional derivative

$$ {}^{\mathtt{Hi}}D_{a+}^{\alpha,\beta}y(t) = {}^{\mathtt{RL}}I_{a+}^{(1-\beta)(n-\alpha)}\, D^n\, {}^{\mathtt{RL}}I_{a+}^{\beta(n-\alpha)}y(t) \tag{5.1.62} $$

and the right two-parameter fractional derivative

$$ {}^{\mathtt{Hi}}D_{b-}^{\alpha,\beta}y(t) = {}^{\mathtt{RL}}I_{b-}^{(1-\beta)(n-\alpha)}\, (-D)^n\, {}^{\mathtt{RL}}I_{b-}^{\beta(n-\alpha)}y(t), \tag{5.1.63} $$

where $\alpha \notin \mathbb{N}$ denotes the order of the fractional derivative, $n = \lceil\alpha\rceil$, and $\beta \in [0,1]$ represents a parameter. It is easily seen that these derivatives reduce to the Caputo operators for $\beta = 0$ and to the Riemann-Liouville operators for $\beta = 1$.

A functional in terms of multiple fractional Hilfer derivatives [280, 557] can be defined as [28]

$$ \begin{aligned} J[y] = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.1.64) \\ & \int_a^b F(t, y, {}^{\mathtt{Hi}}D_{a+}^{\alpha_{11},\beta_{11}}y, \ldots, {}^{\mathtt{Hi}}D_{a+}^{\alpha_{1n},\beta_{1n}}y, {}^{\mathtt{Hi}}D_{b-}^{\alpha_{21},\beta_{21}}y, \ldots, {}^{\mathtt{Hi}}D_{b-}^{\alpha_{2m},\beta_{2m}}y)dt. \end{aligned} $$

For this functional, the corresponding Euler-Lagrange equation and the terminal conditions have been derived in [28]; they take the forms

$$ \begin{aligned} 0 = \frac{\partial F}{\partial y} &+ \sum_{j=1}^{n} {}^{\mathtt{Hi}}D_{b-}^{\alpha_{1j},1-\beta_{1j}} \frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{a+}^{\alpha_{1j},\beta_{1j}}y} \\ &+ \sum_{k=1}^{m} {}^{\mathtt{Hi}}D_{a+}^{\alpha_{2k},1-\beta_{2k}} \frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{b-}^{\alpha_{2k},\beta_{2k}}y}, \end{aligned} \tag{5.1.65} $$

$$ \begin{aligned} 0 = &\left[\sum_{j=1}^{n} {}^{\mathtt{RL}}I_{b-}^{(1-\beta_{1j})(1-\alpha_{1j})} \frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{a+}^{\alpha_{1j},\beta_{1j}}y}\, {}^{\mathtt{RL}}I_{a+}^{\beta_{1j}(1-\alpha_{1j})}\eta(t)\right]_{t=a} \\ &- \left[\sum_{k=1}^{m} {}^{\mathtt{RL}}I_{a+}^{(1-\beta_{2k})(1-\alpha_{2k})} \frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{b-}^{\alpha_{2k},\beta_{2k}}y}\, {}^{\mathtt{RL}}I_{b-}^{\beta_{2k}(1-\alpha_{2k})}\eta(t)\right]_{t=a}, \end{aligned} \tag{5.1.66} $$

and

$$ \begin{aligned} 0 = &\left[\sum_{j=1}^{n} {}^{\mathtt{RL}}I_{b-}^{(1-\beta_{1j})(1-\alpha_{1j})} \frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{a+}^{\alpha_{1j},\beta_{1j}}y}\, {}^{\mathtt{RL}}I_{a+}^{\beta_{1j}(1-\alpha_{1j})}\eta(t)\right]_{t=b} \\ &- \left[\sum_{k=1}^{m} {}^{\mathtt{RL}}I_{a+}^{(1-\beta_{2k})(1-\alpha_{2k})} \frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{b-}^{\alpha_{2k},\beta_{2k}}y}\, {}^{\mathtt{RL}}I_{b-}^{\beta_{2k}(1-\alpha_{2k})}\eta(t)\right]_{t=b}. \end{aligned} \tag{5.1.67} $$

When the functional has multiple functions, then it can be written as

$$J[y] = \int_a^b F(t, \mathbf{y}, {}_aD_t^{\alpha_1,\beta_1}\mathbf{y}, {}_tD_b^{\alpha_2,\beta_2}\mathbf{y})dt, \tag{5.1.68}$$

where $\mathbf{y} = [\,y_1(t), \cdots, y_m(t)\,]^T$ represents an m-dimensional vector. For this functional the Euler-Lagrange equation and the terminal conditions are written as [28]

$$0 = \frac{\partial F}{\partial \mathbf{y}} + {}^{\mathtt{Hi}}D_{b-}^{\alpha_1,1-\beta_1}\frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{a+}^{\alpha_1,\beta_1}\mathbf{y}} + {}^{\mathtt{Hi}}D_{a+}^{\alpha_2,1-\beta_2}\frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{b-}^{\alpha_2,\beta_2}\mathbf{y}}, \tag{5.1.69}$$

$$\begin{aligned} 0 = &\left[{}^{\mathtt{RL}}I_{b-}^{(1-\beta_1)(1-\alpha_1)}\frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{a+}^{\alpha_1,\beta_1}y_j}\,{}^{\mathtt{RL}}I_{a+}^{\beta_1(1-\alpha_1)}\eta_j(t)\right]_{t=a} \\ &- \left[{}^{\mathtt{RL}}I_{a+}^{(1-\beta_2)(1-\alpha_2)}\frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{b-}^{\alpha_2,\beta_2}y_j}\,{}^{\mathtt{RL}}I_{b-}^{\beta_2(1-\alpha_2)}\eta_j(t)\right]_{t=a} \end{aligned} \tag{5.1.70}$$

for $j = 1, 2 \ldots, m$, and

$$\begin{aligned} 0 = &\left[{}^{\mathtt{RL}}I_{b-}^{(1-\beta_1)(1-\alpha_1)}\frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{a+}^{\alpha_1,\beta_1}y_j}\,{}^{\mathtt{RL}}I_{a+}^{\beta_1(1-\alpha_1)}\eta_j(t)\right]_{t=b} \\ &- \left[{}^{\mathtt{RL}}I_{a+}^{(1-\beta_2)(1-\alpha_2)}\frac{\partial F}{\partial\, {}^{\mathtt{Hi}}D_{b-}^{\alpha_2,\beta_2}y_j}\,{}^{\mathtt{RL}}I_{b-}^{\beta_2(1-\alpha_2)}\eta_j(t)\right]_{t=b} \end{aligned} \tag{5.1.71}$$

for $j = 1, \ldots, m$.

We notice that when a functional contains both multiple fractional derivatives and multiple functions, the necessary condition for this function can be obtained by making use of the same procedure used here.

5.1.7.3 *A fractional Lagrangian approach of Schrödinger equations*

The variant of the Schrödinger equation containing fractional derivatives was proposed by several researchers. In particular, in [190] the authors solved the fractional Schrödinger equation by making use of the quantum Riesz fractional operator suggested in [331]. A Caputo version of the time-fractional Schrödinger equation was suggested in [418], and a generalized fractional Schrödinger equation with space-time fractional derivatives was developed in [575]. Moreover, in [413] the fractional Schrödinger equation was obtained with a fractional variational principle and a fractional Klein-Gordon equation.

The classical Schrödinger equation in 1+1 dimensions is given by

$$i\hbar\frac{d\psi}{dt} + \frac{\hbar^2}{2m}\triangle\psi - V(x)\psi = 0. \tag{5.1.72}$$

The expression of the classical Lagrangian corresponding to (5.1.72) is given by

$$\mathcal{L} = \frac{i\hbar}{2}(\psi^\dagger\frac{d\psi}{dt} - \psi\frac{d\psi^\dagger}{dt}) - \frac{\hbar^2}{2m}(\nabla\psi\nabla\psi^\dagger) - V(x)\psi\psi^\dagger. \tag{5.1.73}$$

Based on (5.1.73), the Lagrangian density for the fractional Schrödinger equation can be written in the form

$$\begin{aligned}\mathcal{L}' = {} & \frac{i\hbar^\alpha}{2}[\psi^\dagger({}^{\mathtt{RL}}D^\alpha_{a+,t}\psi) + \psi({}^{\mathtt{RL}}D^\alpha_{b-,t}\psi^\dagger)] + \frac{\hbar^{2\alpha}}{2m^\alpha}[{}^{\mathtt{RL}}D^\alpha_{a+,x}\psi({}^{\mathtt{RL}}D^\alpha_{b-,x}\psi^\dagger)] \\ & - V(x)\psi\psi^\dagger.\end{aligned} \tag{5.1.74}$$

The corresponding equations for ψ and $\psi^\dagger$ are

$$i\hbar^\alpha({}^{\mathtt{RL}}D^\alpha_{a+,t}\psi) + \frac{\hbar^{2\alpha}}{2m^\alpha}{}^{\mathtt{RL}}D^\alpha_{a+,x}({}^{\mathtt{RL}}D^\alpha_{a+,x}\psi) - V(x)\psi = 0, \tag{5.1.75}$$

$$i\hbar^\alpha({}^{\mathtt{RL}}D^\alpha_{b-,t}\psi^\dagger) + \frac{\hbar^{2\alpha}}{2m^\alpha}{}^{\mathtt{RL}}D^\alpha_{b-,x}({}^{\mathtt{RL}}D^\alpha_{b-,x}\psi^\dagger) - V(x)\psi = 0, \tag{5.1.76}$$

respectively.

5.1.7.4 *Fractional Lagrangians which differ by a fractional Riesz derivative*

Following [58], let us consider the classical Lagrangian $L(q^\sigma(t), \dot{q}^\sigma(t))$, $\sigma \in \{1, \ldots, n\}$, together with its fractional counterpart

$$L_f(q^\sigma(t), {}^{\mathtt{RC}}D^\alpha_{[a,b]}q^\sigma(t)), \tag{5.1.77}$$

with the aim of presenting the generalization of the notion of the equivalent Lagrangians in the fractional case. We add ${}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q(t)$ to (5.1.77) and we get a new fractional Lagrangian,

$$L = L_f(q^\sigma(t), {}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q^\sigma(t)) + C({}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q^m(t)). \tag{5.1.78}$$

Here m represents the given coordinate while C denotes a real constant. The new fractional Euler-Lagrange equations arising from (5.1.78) have been shown in [58] to take the form

$$\frac{\partial L_f}{\partial q^\sigma(t)} + {}^{\mathtt{R}}D^{\alpha}_{[a,b]}\frac{\partial L_f}{\partial\, {}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q^\sigma(t)} = 0, \tag{5.1.79}$$

$$\frac{\partial L_f}{\partial q^m(t)} + {}^{\mathtt{R}}D^{\alpha}_{[a,b]}\frac{\partial L_f}{\partial\, {}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q^m(t)} + \frac{C}{2}\left(\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)}\right) = 0, \tag{5.1.80}$$

$$\frac{\partial L_f}{\partial q^\delta(t)} + {}^{\mathtt{R}}D^{\alpha}_{[a,b]}\frac{\partial L_f}{\partial\, {}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q^\delta(t)} = 0, \tag{5.1.81}$$

for $\sigma = 1, \ldots, m-1$ and $\delta = m+1, \ldots, n$. We notice that the last term of (5.1.80) characterizes the fractional dynamics [58]. However, when $\alpha = 1$ we recover the classical case.

5.1.7.5 *Euler-Lagrange equations in fractional space*

Let us now consider the action function of the form

$$S = \frac{1}{\Gamma(\alpha)}\int_a^b L_f(t, {}^{\mathtt{RL}}D^{\beta}_{a+}q, {}^{\mathtt{RL}}D^{\gamma}_{b-}q)(t-\tau)^{\alpha-1}d\tau, \tag{5.1.82}$$

where $0 \le \alpha \le 1$, $0 \le \gamma \le 1$, $0 \le \beta \le 1$. From [415] we conclude that the Euler-Lagrange equations corresponding to (5.1.82) are given by

$$\begin{aligned} 0 = \frac{\partial L_f}{\partial q} &+ \frac{1}{(t-\tau)^{\alpha-1}}\,{}^{\mathtt{RL}}D^{\beta}_{b-}\left(\frac{\partial L_f}{\partial\, {}^{\mathtt{RL}}D^{\beta}_{a+}}(t-\tau)^{\alpha-1}\right) \\ &+ \frac{1}{(t-\tau)^{\alpha-1}} + {}^{\mathtt{RL}}D^{\gamma}_{a+}\left(\frac{\partial L_f}{\partial\, {}^{\mathtt{RL}}D^{\gamma}_{b-}}(t-\tau)^{\alpha-1}\right). \end{aligned} \tag{5.1.83}$$

When $\beta = 1$ and $\gamma = 1$, the classical results are reobtained, namely

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\alpha-1}{t-\tau}\left(\frac{\partial L}{\partial \dot q}\right) = 0. \tag{5.1.84}$$

5.1.7.6 *Multi time fractional Lagrangian equations*

We consider in the following the multi time variable defined as

$$(t^\nu) = (t^1, t^2, \ldots, t^p) \in \mathbb{R}^p, \qquad \nu = 1, \ldots, p. \tag{5.1.85}$$

Let us denote the limits of the range of integration by $a = (a^\nu)$ and $b = (b^\nu)$, and define the functions as

$$q^i : \mathbb{R}^p \to \mathbb{R}, \quad (t^1, t^2, \ldots, t^p) \mapsto q^i(t^1, t^2, \ldots, t^p), \quad i = 1, \ldots, n. \tag{5.1.86}$$

Following [64], the multi time Lagrangian is written as

$$L_f : \mathbb{R}^{p+n+np} \to \mathbb{R}, \quad (t^\nu, q^i, ({}^{\mathrm{RL}}D^\alpha_{a+,t^\nu})q^i) \mapsto L_f(t^\nu, q^i,\ ({}^{\mathrm{RL}}D^\alpha_{a+,t^\nu})q^i) \tag{5.1.87}$$

and as a result, the corresponding fractional Euler-Lagrangian equations of (5.1.87) have the form

$$\frac{\partial L_f}{\partial q^i} + {}^{\mathrm{RL}}D^\alpha_{b-,t^\nu} \frac{\partial L_f}{\partial ({}^{\mathrm{RL}}D^\alpha_{a+,t^\nu} q^i)} = 0. \tag{5.1.88}$$

5.1.7.7 *Fractional Faddeev-Jackiw formalism*

The Faddeev-Jackiw formalism [212] is one of the most frequently used approaches to deal with the reduction of constrained and unconstrained systems. In the Faddeev-Jackiw procedure we start from a first order Lagrangian in time derivative and after that we write down the Lagrange density

$$L(q^i, \dot{q}^i, p_i, \dot{p}_i) = \frac{1}{2} \begin{pmatrix} p_i & q^i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{p}^i \\ \dot{q}_i \end{pmatrix} - H(p_i, q^i). \tag{5.1.89}$$

It is possible to construct a first order in time derivatives Lagrangian in such a way that the configuration space coincides with the Hamiltonian phase space. This property can be achieved by enlarging the given n-dimensional configuration space to a $2n$-dimensional configuration space. In the fractional mechanics approach let us consider, as in [240], the fractional Lagrangian

$$L_f = \frac{1}{2}(q^i \, {}^{\mathrm{RL}}D^\alpha_{b-} p_i + p_i \, {}^{\mathrm{RL}}D^\alpha_{a+} q^i) - H(q^i, p_i) \tag{5.1.90}$$

that can be written in matrix notation as

$$L_f = \frac{1}{2}\begin{pmatrix} \xi_n \ \xi_m \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} {}^{\mathtt{RL}}D^{\alpha}_{a+}\xi_n \\ {}^{\mathtt{RL}}D^{\alpha}_{b-}\xi_m \end{pmatrix} - H(q,p), \tag{5.1.91}$$

where $\xi_i = p_i$ for $i = 1,\ldots,n$ and $\xi_i = q^i$ for $i = n+1,\ldots,2n$. The corresponding Euler-Lagrange equations of (5.1.91) have the form [240]

$$\frac{\partial L_f}{\partial q^i} + {}^{\mathtt{RL}}D^{\alpha}_{b-}\frac{\partial L_f}{\partial\, {}^{\mathtt{RL}}D^{\alpha}_{a+}q^i} = 0, \tag{5.1.92}$$

$$\frac{\partial L_f}{\partial p_i} + {}^{\mathtt{RL}}D^{\alpha}_{a+}\frac{\partial L_f}{\partial\, {}^{\mathtt{RL}}D^{\alpha}_{b-}p_i} = 0. \tag{5.1.93}$$

The equations that we have obtained are equivalent with the fractional Hamilton equations, namely

$$\begin{pmatrix} {}^{\mathtt{RL}}D^{\alpha}_{a+}\xi^n \\ {}^{\mathtt{RL}}D^{\alpha}_{b-}\xi^m \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial H/\partial \xi^n \\ \partial H/\partial \xi^m \end{pmatrix}. \tag{5.1.94}$$

As a result, the basic Faddeev-Jackiw brackets become

$$\{q^i, p_j\}_{FJ} = \delta_{ij}, \qquad \{q^i, q^j\}_{FJ} = \{p_i, p_j\}_{FJ} = 0. \tag{5.1.95}$$

Finally, in bracket notation we obtain the following equations [240]

$$\begin{pmatrix} {}^{\mathtt{RL}}D^{\alpha}_{a+}\xi^n \\ {}^{\mathtt{RL}}D^{\alpha}_{b-}\xi^m \end{pmatrix} = \{\xi^m, \xi^n\}\begin{pmatrix} \partial H/\partial \xi^n \\ \partial H/\partial \xi^m \end{pmatrix}, \tag{5.1.96}$$

where

$$\{\xi^m, \xi^n\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.1.97}$$

We notice that this matrix is symmetric while in the classical case it is anti-symmetric [240].

5.1.7.8 *Fractional variational calculus with generalized boundary condition*

We consider the type of fractional variational calculus induced by functionals of the form

$$J(y) = {}^{\mathrm{RL}}I^{\gamma}_{a+} L_f(x, y(x), {}^{\mathrm{RL}}D^{\alpha}_{a+}y, y(a)). \tag{5.1.98}$$

According to [275], the corresponding Euler-Lagrange equation becomes

$$\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L_f}{\partial y} + {}^{\mathrm{C}}D^{\alpha}_{x-}\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L_f}{\partial {}^{\mathrm{RL}}D^{\alpha}_{a+}y}\right) = 0 \tag{5.1.99}$$

with the natural boundary condition (transversality conditions)

$$\left.\left(\frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L_f}{\partial\, {}^{\mathrm{C}}D^{\alpha}_{a+}y}\right)\right|_{t=x} = 0. \tag{5.1.100}$$

If $y(a)$ is given then we have $\eta(a) = 0$, but in the opposite case we obtain the boundary condition

$$\int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\frac{\partial L_f}{\partial y(a)}dt = 0. \tag{5.1.101}$$

We notice that these conditions are only necessary for an extremum [275].

As a special case, we consider the following problem. If y is a local extremizer to

$$J(y) = \int_a^b L_f(t, y(t), {}^{\mathrm{RL}}D^{\alpha}_{a+}y)dt, \tag{5.1.102}$$

and if we consider $\gamma = 1$ and $x = b$ in (5.1.99), (5.1.100) and (5.1.101), respectively, then, according to [275], we obtain the fractional Euler-Lagrange equation as

$$\frac{\partial L_f}{\partial y} + {}^{\mathrm{C}}D^{\alpha}_{b-}\left(\frac{\partial L_f}{\partial\, {}^{\mathrm{RL}}D^{\alpha}_{a+}y}\right) = 0 \tag{5.1.103}$$

for all $t \in [a, b]$, with the corresponding boundary condition

$$\left.\left(\frac{\partial L_f}{\partial {}^{\mathrm{RL}}D^{\alpha}_{a+}y}\right)\right|_{t=b} = 0. \tag{5.1.104}$$

The sufficient conditions for the existence of an extremum were discussed elsewhere [275]. More details about this topic can be found in [19, 21, 274, 375].

5.2 Fractional Hamiltonian Dynamics

5.2.1 *Introduction and overview of results*

The classical dynamics of Hamiltonian systems is characterized by conservation of phase-space volume under time evolution. This aspect is a cornerstone of conventional statistical mechanics of Hamiltonian systems. The main characteristic of the fractional Hamiltonian is its nonlocality. For these reasons, several techniques were applied to discuss this issue in the fractional case.

A Hamiltonian formalism was developed for non-local field theories in d space-time dimensions by considering auxiliary $(d+1)$-dimensional field theories which are local with respect to the evolution time [242]. The physical reduced space of non-local theories around the fixed points of these systems were analyzed in [243]. The space-time non-commutative field theories are acausal and the unitarity is lost as it was shown in [233, 519].

The higher-derivatives theories [236, 422] appear naturally as corrections to general relativity and cosmic strings.

The Hamiltonian treatment of non-local theories and Ostrogradski's formalism [236, 270] was discussed in [88] such that we recast the second class constraints into first class constraints by invoking the boundary Poisson bracket.

The passage from the Lagrangian containing fractional derivatives to the Hamiltonian was achieved in [468]. Also, the fractional Hamilton's equations of motion were obtained in a similar manner to the classical mechanics [468].

The definition of the fractional Hessian matrix within Riemann-Liouville fractional derivatives was introduced in [53].

Fractional Hamiltonian equations in terms of Riesz fractional derivatives were obtained in [20].

Using the fact that an extremum of variation of generalized action can lead to the fractional dynamics in the case of systems with long-range interaction and long-term memory function, the generalized Noether's theorem and Hamiltonian type equation were investigated in [545].

A generalized classical mechanics with fractional derivatives based on the time-clock randomization of momenta and coordinates taken from the conventional phase space was considered in [535]. The fractional equations of motion were derived using the Hamiltonian formalism.

It was proved in [240] that for the fractional constraint Hamiltonian formulation, by using Dirac brackets, we obtain the same equations as those obtained from fractional Euler-Lagrange equations.

The fractional Hamilton-Jacobi formulations for systems containing Riesz fractional derivatives was derived in [469]. The classical Nambu mechanics was generalized to involve fractional derivatives using two approaches, namely by using the fractional exterior derivative as well as by extending the standard velocities to the fractional ones [241].

5.2.2 *Fractional Hamiltonian analysis for discrete and continuous systems*

5.2.2.1 *A direct method within Riemann-Liouville fractional derivatives*

In the following, we introduce the meaning of the fractional Hamiltonian. As a starting point, for simplicity, below we consider the following form of the fractional Euler-Lagrange equations

$$\frac{\partial L_f}{\partial q^\rho(t)} + {}^{\mathrm{RL}}D^{\alpha}_{b-}\frac{\partial L_f}{\partial\, {}^{\mathrm{RL}}D^{\alpha}_{a+}q^\rho(t)} = 0, \quad 0<\alpha<1, \quad \rho = 1,\ldots,N, \tag{5.2.1}$$

where L_f denotes the corresponding fractional Lagrangian given as

$$L_f(q^\rho, {}^{\mathrm{R}L}D^{\alpha}_{a+}q^\rho, t), \quad 0<\alpha<1. \tag{5.2.2}$$

In the following we use (5.2.1) to define the generalized momenta as in [468], *viz.*

$$p_{\alpha_\rho} = \frac{\partial L_f}{\partial\, {}^{\mathrm{RL}}D^{\alpha}_{a+}q^\rho}, \quad \rho = 1,\ldots,N. \tag{5.2.3}$$

As a consequence of (5.2.1) and (5.2.3), a fractional Hamiltonian function is defined as

$$H_f = p_{\alpha_\rho}\, {}^{\mathrm{RL}}D^{\alpha}_{a+}q^\rho - L_f. \tag{5.2.4}$$

The canonical equations corresponding to (5.2.4) are

$$\frac{\partial H_f}{\partial t} = -\frac{\partial L_f}{\partial t}, \tag{5.2.5}$$

$$\frac{\partial H_f}{\partial p_{\alpha_\rho}} = {}^{\mathtt{RL}}D^{\alpha}_{a+} q^{\rho}, \tag{5.2.6}$$

$$\frac{\partial H_f}{\partial q^{\rho}} = {}^{\mathtt{RL}}D^{\alpha}_{b-} p_{\alpha_\rho}, \quad 0 < \alpha < 1, \quad \rho = 1, \ldots, N. \tag{5.2.7}$$

For the derivation of these equations and more results on this topic see [468].

5.2.2.2 *A direct method with Caputo fractional derivatives*

In the following, we present briefly the Hamiltonian formulation with Caputo's fractional derivatives as developed in [61]. Let us consider the fractional Lagrangian

$$L_f(q, {}^{\mathtt{C}}D^{\alpha}_{a+} q, t), \quad 0 < \alpha < 1. \tag{5.2.8}$$

By making use of (5.2.8), the canonical momenta p_α become

$$p_\alpha = \frac{\partial L_f}{\partial\, {}^{\mathtt{C}}D^{\alpha}_{a+} q}. \tag{5.2.9}$$

As a result we define the fractional canonical Hamiltonian as

$$H_f = p_\alpha \,({}^{\mathtt{C}}D^{\alpha}_{a+} q) - L_f. \tag{5.2.10}$$

The expressions of the fractional Hamilton equations are

$$\frac{\partial H_f}{\partial t} = -\frac{\partial L_f}{\partial t}, \tag{5.2.11a}$$

$$\frac{\partial H_f}{\partial p_\alpha} = {}^{\mathtt{C}}D^{\alpha}_{a+} q, \tag{5.2.11b}$$

$$\frac{\partial H_f}{\partial q} = {}^{\mathtt{RL}}D^{\alpha}_{b-} p_\alpha. \tag{5.2.11c}$$

In [61] a detailed description of this formulation is provided.

We mention that the fractional order Hamiltonian equations corresponding to the functional with fractional integral and fractional derivative in the Caputo sense, namely

$$J(y) = {}^{\mathtt{RL}}I^{\alpha}_{a+} L_f(t, q, {}^{\mathtt{C}}D^{\beta}_{a+} q), \qquad t \in [a, b], \quad q(a) = q_o, \tag{5.2.12}$$

and are given by [274]

$$\frac{\partial H_f}{\partial t} = -\frac{\partial L_f}{\partial t}, \tag{5.2.13a}$$

$$\frac{\partial H_f}{\partial p_\beta} = {}^{\mathtt{C}}D^{\beta}_{a+}q, \tag{5.2.13b}$$

$$\frac{\partial H_f}{\partial q} = (x-t)^{1-\alpha}\,{}^{\mathtt{RL}}D^{\beta}_{x-}\left((x-t)^{\alpha-1}\frac{\partial Lf}{\partial\,{}^{\mathtt{C}}D^{\beta}_{a+}q}\right). \tag{5.2.13c}$$

5.2.2.3 *A direct method within Riesz-Caputo fractional derivatives*

The fractional canonical momenta of the fractional Lagrangian

$$L = L_f(q, {}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q) \tag{5.2.14}$$

are defined as

$$p_\alpha = \frac{\partial L_f}{\partial\,{}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q}. \tag{5.2.15}$$

As a result, the corresponding fractional Hamiltonian becomes

$$H_f(q, p_\alpha) = p_\alpha({}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q) - L_f. \tag{5.2.16}$$

By using (5.2.14), (5.2.15) and (5.2.16) we obtain the fractional Hamilton's equations (see [20])

$${}^{\mathtt{R}}D^{\alpha}_{[a,b]}p_\alpha = -\frac{\partial H_f}{\partial q}, \qquad {}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q = \frac{\partial H_f}{\partial p_\alpha}. \tag{5.2.17}$$

5.2.3 *Fractional Hamiltonian formulation for constrained systems*

5.2.3.1 *Fractional Hessian matrix*

The fractional canonical equations described in the previous paragraphs are valid for the case when no primary constraints exist. As it is well known, several dynamical systems of physical interest have constraints [53, 54, 56].

In the classical case, Dirac's formalism provides a way to manage the constraints, namely they are divided into first and second class type, the

Dirac's bracket is introduced and the second class constraints are eliminated [236].

Let us consider the equations of motion which can be obtainable using the variational principle that states the extremum of the action S,

$$S = \int L dt \tag{5.2.18}$$

under certain given boundary conditions. Lagrangian theories are divided into two parts. If the determinant of the Hessian matrix

$$H_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \tag{5.2.19}$$

is nonzero the theory is called nonsingular. In the opposite case the theory is called singular.

The main aim is to define properly the fractional Hessian of a given fractional singular Lagrangian. We have to have in mind the restriction to recover the same rank as in the classical case while $\alpha \to 1$.

In the following we consider the case where the fractional Lagrangian to start with is

$$L_f\left(t, q^1, \cdots, q^n, {}^{\mathtt{RL}}D_{a+}^{\alpha} q^1, \cdots, {}^{\mathtt{RL}}D_{a+}^{\alpha} q^n\right). \tag{5.2.20}$$

Using (5.2.19) and (5.2.20) a natural definition of the fractional Hessian matrix is

$$H_{ij} = \frac{\partial^2 L_f}{\partial\, {}^{\mathtt{RL}}D_{a+}^{\alpha} q^i \partial\, {}^{\mathtt{RL}}D_{a+}^{\alpha} q^j}. \tag{5.2.21}$$

We notice that the main advantage of using (5.2.21) is that the classical result of the Hessian matrix is recovered in the limit process.

Nevertheless, if ${}^{\mathtt{RL}}D_{a+}^{\alpha}$ is replaced by ${}^{\mathtt{RL}}D_{b-}^{\alpha}$ in (5.2.20) the obtained fractional Lagrangian

$$L_f\left(t, q^1, \cdots, q^n, {}^{\mathtt{RL}}D_{b-}^{\alpha} q^1, \cdots, {}^{\mathtt{RL}}D_{b-}^{\alpha} q^n\right) \tag{5.2.22}$$

leads us to define the fractional Hessian matrix as

$$H_{ij} = \frac{\partial^2 L_f}{\partial\, {}^{\mathtt{RL}}D_{b-}^{\alpha} q^i \partial\, {}^{\mathtt{RL}}D_{b-}^{\alpha} q^j}. \tag{5.2.23}$$

From (5.2.21) and (5.2.23) we conclude that the representation of the fractional Hessian matrix depends on the fractional derivatives involved in the given Lagrangian.

5.2.3.2 *The reduced phase space*

Finding the reduced phase space in the fractional case is a fundamental step in identification of the true degrees of freedom for a given physical system. For a given fractional Lagrangian $L_f = L_f(q^i, {}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q^i)$, $i = 1, \ldots, n$, involving Riesz derivatives we suppose that $\Phi_m(p^i_\alpha, q^i) = 0$, $m = 1, \ldots, M$, with $M < n$, i.e. we have M primary constraints. Therefore, the fractional Hamiltonian equations are obtained from the fractional variational principle as (see [58])

$$\delta \int_{t1}^{t2} \left(p^i_\alpha ({}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q^i) - H_f - u^m \Phi_m \right) dt = 0. \tag{5.2.24}$$

The new variables u^m appear as Lagrange multipliers enforcing the primary constraints $\Phi_m = 0$. By using (5.2.24) we notice that the theory is invariant under $H_f \to H_f + c^m \Phi_m$, because this change merely results in changing $u^m \to u^m + c^m$. Finally, we obtain

$${}^{\mathtt{R}}D^{\alpha}_{[a,b]}p_{\alpha j} = -\frac{\partial H_f}{\partial q^j} - u^m \frac{\partial \Phi_m}{\partial q^j}, \quad {}^{\mathtt{RC}}D^{\alpha}_{[a,b]}q^j = -\frac{\partial H_f}{\partial p_{\alpha j}} + u^m \frac{\partial \Phi_m}{\partial p_{\alpha j}}. \tag{5.2.25}$$

5.2.3.3 *Fractional Ostrogradski's approach*

In the classical calculus the k-th order derivative of $\phi(t)$ is given by the Faà di Bruno formula [5]

$$\frac{d^k}{dt^k} F(h(t)) = k! \sum_{m=1}^{k} F^{(m)}(h(t)) \sum \prod_{r=1}^{k} \frac{1}{a_r!} \left(\frac{h^{(r)}(t)}{r!} \right)^{a_r}, \tag{5.2.26}$$

where the sum $\sum$ extends over all combinations of non-negative integer values of $a_1, \ldots, a_k$ such that the constraints

$$\sum_{r=1}^{k} r a_r = k \tag{5.2.27}$$

and

$$\sum_{r=1}^{k} a_r = m \tag{5.2.28}$$

are satisfied. As a result, the fractional derivative of a composition function is given by [59]

$$
\begin{aligned}
&{}^{\mathrm{C}}D^{\alpha}_{a+}F(h(t)) \qquad (5.2.29)\\
&\quad= \frac{(t-a)^{-p}}{\Gamma(1-p)}(F(h(t)-F(h(a)))\\
&\qquad+\sum_{k=1}^{\infty}\binom{\alpha}{k}k!\frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}\sum_{m=1}^{k}F^{(m)}(h(t))\sum\prod_{r=1}^{k}\frac{1}{a_r!}\left(\frac{h^{(r)}(t)}{r!}\right)^{a_r},
\end{aligned}
$$

where $\sum$ and the coefficients a_r have the same meaning as above. This form of the fractional generalization of the Faà di Bruno formula is appropriate for the so called fractional Ostrogradski Hamiltonian formulation [59].

In this line of thought we consider the dynamical variable $q(t)$ to be a 1+1 dimensional field $Q(x,t)$ such that the chirality condition [88]

$$\frac{dQ(x,t)}{dt} = \partial_x Q(x,t), \qquad (5.2.30)$$

is fulfilled. We notice that $Q(x,t) = q(x+t)$ assures the one-to-one correspondence between $q(t)$ and $Q(x,t)$ [88, 349].

Ostrogradski's coordinates are defined by

$$Q^{(n)}(t) = (\partial_x)^n Q(x,t)\mid_{x=x_0}, \qquad (5.2.31)$$

where the discontinuity curve $x_0(t) = x_0$ is a constant [88], which implies that

$$Q(x,t) = \sum_{n=0}^{\infty}\frac{(x-x_0)^n}{n!}Q^{(n)}(t). \qquad (5.2.32)$$

A corresponding boundary Poisson bracket was introduced in [88] as

$$\{F(t),G(t)\} = \sum_{k,l=0}^{\infty} c_{k,l}\int_{-\infty}^{\infty} dx(\partial_x)^{k+l}\left[\frac{\delta F(t)}{\delta Q^{(k)}}(x,t)\frac{\delta G(t)}{\delta P^{(l)}}(x,t)\right]-(F\leftrightarrow G), \qquad (5.2.33)$$

where the coefficients $c_{k,l}$ are constants and normalized in such a manner to satisfy the Jacobi identity.

The classical canonical relation becomes

$$\{Q(x,t),P(x',t)\} = \delta_R(x-x'), \qquad (5.2.34)$$

and Ostrogradski's momenta $P_{(n)}(t)$ are defined as

$$P_{(n)}(t) = \int_{-\infty}^{\infty} dx \frac{(x-x_0)^n}{n!} P(x,t), \tag{5.2.35}$$

therefore we conclude that [59, 88, 349]

$$P(x,t) = \sum_{n=0}^{\infty} P_{(n)}(t)(-\partial_x)^n \delta_R(x-x_0). \tag{5.2.36}$$

By using (5.2.36) the expression for $P_{(n)}(t)$ is

$$P_{(n)}(t) = \sum_{m=n}^{\infty} (-\partial_t)^{m-n} \frac{\partial L_f[Q](t)}{\partial Q^{(m+1)}(t)}, \tag{5.2.37}$$

where $L_f[Q](t)$ denotes the fractional counterpart of a given classical Lagrangian [59, 72].

As a result the Hamilton equations are written as

$$\dot{P}_{(n)}(t) + P_{(n-1)}(t) = \frac{\partial L'_f[Q](t)}{\partial Q^{(n)}(t)}, \qquad n \in \mathbb{N}, \tag{5.2.38}$$

$$\dot{P}_{(0)}(t) = \frac{\partial L'_f[Q](t)}{\partial Q^{(0)}(t)}. \tag{5.2.39}$$

By inspection we conclude that (5.2.39) is nothing but the Euler-Lagrange equation corresponding to the Lagrangian $L_f[Q](t)$. Finally, the expression of the corresponding Hamiltonian becomes

$$H = \sum_{n=0}^{\infty} P_{(n)}(t) Q^{(n+1)}(t) - L'_f[Q](t), \tag{5.2.40}$$

see [59, 72] where additional details on this topic may also be found.

5.2.4 *Applications*

5.2.4.1 *Discrete fractional constrained systems*

In the following we give some illustrative examples of fractional constrained dynamics as well as the exact solutions of their fractional Euler-Lagrange and fractional Hamiltonian equations.

Example 5.1. Let us consider the classical Lagrangian

$$L = \frac{1}{2}(\dot{q}_1(t) + \dot{q}_2(t))^2. \tag{5.2.41}$$

By making use of (5.2.41) we calculate canonical momenta as

$$p_{1\alpha} = p_{2\alpha} = \dot{q}_1(t) + \dot{q}_2(t). \tag{5.2.42}$$

The corresponding fractional Lagrangian is written as

$$L_f = \frac{1}{2}({}^{\mathtt{C}}D^{\alpha}_{a+}q_1(t) + {}^{\mathtt{C}}D^{\alpha}_{a+}q_2(t))^2, \tag{5.2.43}$$

where $0 < \alpha < 1$. Therefore, the fractional Hamiltonian corresponding to (5.2.43) becomes

$$H_f = \frac{(p_{1\alpha})^2}{2} + \lambda(p_{2\alpha} - p_{1\alpha}), \tag{5.2.44}$$

where $\lambda = {}^{\mathtt{C}}D^{\alpha}_{a+}q_2(t)$. As a result, the fractional canonical equations are

$${}^{\mathtt{C}}D^{\alpha}_{a+}q_2 = \lambda,\ {}^{\mathtt{C}}D^{\alpha}_{a+}q_1 = p_{1\alpha} - \lambda,\ {}^{\mathtt{RL}}D^{\alpha}_{b-}p_{1\alpha} = 0,\ {}^{\mathtt{RL}}D^{\alpha}_{b-}p_{2\alpha} = 0. \tag{5.2.45}$$

The Hamilton's equations in the extended phase-space are equivalent with the fractional Euler-Lagrange equations on the surface of constraint $p_{2\alpha} - p_{1\alpha} = 0$. The solutions of (5.2.45) are

$$p_{1\alpha} = p_{2\alpha} = (b-t)^{\alpha-1}, \qquad p_{3\alpha} = 0, \tag{5.2.46}$$

$$q_1(t) + q_2(t) = b_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(b-t)^{\alpha-1}}{(t-\tau)^{1-\alpha}}d\tau, \quad \lambda = {}^{\mathtt{C}}D^{\alpha}_{a+}q_2(t). \tag{5.2.47}$$

Example 5.2. The classical Lagrangian is given by

$$L = \dot{q}_1\dot{q}_2 - q_3(q_1 + q_2). \tag{5.2.48}$$

and its fractional counterpart becomes

$$L_f = ({}^{\mathtt{C}}D^{\alpha}_{a+}q_1)({}^{\mathtt{C}}D^{\alpha}_{a+}q_2) - q_3(q_1 + q_2). \tag{5.2.49}$$

The Euler-Lagrange equations are given by

$$ {}^{\mathtt{RL}}D^{\alpha}_{b-}({}^{\mathtt{C}}D^{\alpha}_{a+}q_2) - q_3 = 0, \quad {}^{\mathtt{RL}}D^{\alpha}_{b-}({}^{\mathtt{C}}D^{\alpha}_{a+}q_1) - q_3 = 0, \quad q_1 + q_2 = 0. \tag{5.2.50}$$

By using (5.2.50), we obtain

$$ {}^{\mathtt{RL}}D^{\alpha}_{b-}({}^{\mathtt{C}}D^{\alpha}_{a+}q_2) = 0, \quad q_1 = -q_2, \quad q_3 = 0. \tag{5.2.51}$$

The canonical momenta associated to q_1, q_2 and q_3 have the forms

$$p_{\alpha 1} = {}^{\mathtt{C}}D^{\alpha}_{a+}q_2, \qquad p_{\alpha 2} = {}^{\mathtt{C}}D^{\alpha}_{a+}q_1, \qquad p_{\alpha 3} = 0. \tag{5.2.52}$$

The canonical Hamiltonian is given by

$$H_f = p_{\alpha 1}p_{\alpha 2} + q_3(q_1 + q_2) + \lambda p_{\alpha 3}, \tag{5.2.53}$$

where $\lambda = {}^{\mathtt{C}}D^{\alpha}_{a+}q_3$. The canonical equations in the extended phase space are

$$p_{\alpha 1} = {}^{\mathtt{C}}D^{\alpha}_{a+}q_2, \qquad p_{\alpha 2} = {}^{\mathtt{C}}D^{\alpha}_{a+}q_1, \qquad \lambda = {}^{\mathtt{C}}D^{\alpha}_{a+}q_3, \quad {}^{\mathtt{RL}}D^{\alpha}_{b-}p_{\alpha 1} = q_3,$$

$$ {}^{\mathtt{RL}}D^{\alpha}_{b-}p_{\alpha 2} = q_3, \quad {}^{\mathtt{RL}}D^{\alpha}_{b-}p_{\alpha 3} = q_1 + q_2, \qquad p_{\alpha 3} = 0, \qquad p_{\lambda} = 0.$$

Finally, the solution of (5.2.54) is given as

$$\begin{array}{lll} q_3(t) = 0, & q_1(t) + q_2(t) = 0, & p_{\alpha 3} = 0, \\ p_{\lambda} = 0, & p_{\alpha 1} = (b-t)^{\alpha-1}, & p_{\alpha 2} = -(b-t)^{\alpha-1}, \\ \multicolumn{3}{c}{q_2(t) = q_2(a) + {}^{\mathrm{RL}}I^{\alpha}_{a^+}(b-t)^{\alpha-1}.} \end{array} \tag{5.2.54}$$

5.2.4.2 *Fractional Hamiltonian formulation in fractional time*

We define the Lagrangian L_f as

$$L = L_f(t, {}^{\mathtt{RL}}D^{\beta}_{a+}q, {}^{\mathtt{RL}}D^{\gamma}_{b-}q)(t-\tau)^{\alpha-1}. \tag{5.2.55}$$

By using (5.2.55), we define the canonical momenta as

$$p^{\alpha}_{\mu} = \frac{\partial L_f}{\partial\, {}^{\mathtt{RL}}D^{\mu}_{a+}q}, \tag{5.2.56}$$

$$p^{\alpha}_{\nu} = \frac{\partial L_f}{\partial\, {}^{\mathtt{RL}}D^{\nu}_{b-}q}. \tag{5.2.57}$$

The canonical Hamiltonian has the form

$$H_f = p_\mu^\alpha \, {}^{\mathtt{RL}}D_{a+}^\mu q + p_\nu^\alpha \, {}^{\mathtt{RL}}D_{a+}^\nu q - L_f \tag{5.2.58}$$

and the canonical equations become

$$\begin{aligned} &\frac{\partial H_f}{\partial p_\mu^\alpha} = {}^{\mathtt{RL}}D_{a+}^\mu q, && \frac{\partial H_f}{\partial p_\nu^\alpha} = {}^{\mathtt{RL}}D_{b-}^\nu q, \\ &\frac{\partial H_f}{\partial q} = {}^{\mathtt{RL}}D_{b-}^\mu p_\mu^\alpha + {}^{\mathtt{RL}}D_{a+}^\nu p_\nu^\alpha, && \frac{\partial H_f}{\partial \tau} = -\frac{\partial L_f}{\partial \tau}, \end{aligned} \tag{5.2.59}$$

see [416].

5.2.4.3 *Fractional Nambu mechanics*

In 1973, Nambu proposed a generalization of the classical Hamiltonian formalism and he introduced so called the classical Nambu bracket [420]. This bracket, involving a dynamical quantity and two or more Hamiltonians, describes the time-evolution of a quantity in a generalization of Hamilton's equations of motion for special physical systems. Nambu brackets in phase space describe the generic classical evolution of systems with many independent integrals of motion beyond those required to complete integrability of a given system. The superintegrable systems are described within Nambu's mechanics. The conditions when a Killing-Yano tensor becomes a Nambu tensor were presented in [52].

In the following we introduce the fractional generalization of the Nambu mechanics as presented in [65]. We introduce firstly the basic classical notions and after that we present the fractional generalization.

The Poincaré-Cartan 1-form $\Omega^{(1)}$ is given by

$$\Omega^{(1)} = p\ dq - H(p,q)dt.$$

For the three-dimensional case we define $\Omega^{(2)}$ as

$$\Omega^{(2)} = qdp \wedge dr - H_1 dH_2 \wedge dt.$$

The fractional generalization of the Poincaré-Cartan 1-form can be defined as

$$\Omega_\alpha^{(1)} = p(dq)^\alpha - H(p,q)(dt)^\alpha. \tag{5.2.60}$$

Note that $\Omega_\alpha^{(1)}$ is a fractional 1-form that can be called a fractional Poincaré-Cartan 1-form. We notice that we have

$$d\Omega_\alpha^{(1)} = \{(dp)^\alpha + {}^{\mathrm{RL}}D^\alpha_{a+,q}H(dt)^\alpha\} \wedge \{(dq)^\alpha - {}^{\mathrm{RL}}D^\alpha_{a+,p}H(dt)^\alpha\}.$$

Thus, by generalized Pfaffian equations, we obtain

$$\frac{(dq)^\alpha}{(dt)^\alpha} = {}^{\mathrm{RL}}D^\alpha_{a+,p}H, \tag{5.2.61a}$$

$$\frac{(dp)^\alpha}{(dt)^\alpha} = -{}^{\mathrm{RL}}D^\alpha_{a+,q}H, \tag{5.2.61b}$$

which are called the fractional Hamiltonian equations.

Besides, the fractional generalization of the 2-form is defined as

$$\Omega_\alpha^{(2)} = q(dp)^\alpha \wedge (dr)^\alpha - H_1 d^\alpha H_2 \wedge (dt)^\alpha, \tag{5.2.62}$$

and as a result the fractional exterior derivative of the fractional 2-form becomes

$$d^\alpha\Omega_\alpha^{(2)} = d^\alpha(q(dp)^\alpha \wedge (dr)^\alpha) - d^\alpha(H_1 d^\alpha H_2 \wedge (dt)^\alpha). \tag{5.2.63}$$

After some direct calculations we obtain

$$\begin{aligned}
&d^\alpha\Omega_\alpha^{(2)} \\
&= ({}^{\mathrm{RL}}D^\alpha_{a+,q}H_1\,{}^{\mathrm{RL}}D^\alpha_{a+,p}H_2 - {}^{\mathrm{RL}}D^\alpha_{a+,p}H_1\,{}^{\mathrm{RL}}D^\alpha_{a+,q}H_2)(dq)^\alpha \wedge (dp)^\alpha \wedge (dt)^\alpha \\
&\quad + ({}^{\mathrm{RL}}\mathrm{D}^\alpha_{\mathrm{a+,q}}\mathrm{H}_1\,{}^{\mathrm{RL}}\mathrm{D}^\alpha_{\mathrm{a+,r}}\mathrm{H}_2 - {}^{\mathrm{RL}}\mathrm{D}^\alpha_{\mathrm{a+,r}}\mathrm{H}_1\,{}^{\mathrm{RL}}\mathrm{D}^\alpha_{\mathrm{a+,q}}\mathrm{H}_2)(\mathrm{dq})^\alpha \wedge (\mathrm{dr})^\alpha \wedge (\mathrm{dt})^\alpha \\
&\quad + ({}^{\mathrm{RL}}D^\alpha_{a+,p}H_1\,{}^{\mathrm{RL}}D^\alpha_{a+,r}H_2 - {}^{\mathrm{RL}}D^\alpha_{a+,r}H_1\,{}^{\mathrm{RL}}D^\alpha_{a+,p}H_2)(dp)^\alpha \wedge (dr)^\alpha \wedge (dt)^\alpha.
\end{aligned}$$

By using the definitions

$$\begin{aligned}
\frac{\partial^\alpha(H_1,H_2)}{\partial^\alpha(p,r)} &= ({}^{\mathrm{RL}}D^\alpha_{a+,p}H_1\,{}^{\mathrm{RL}}D^\alpha_{a+,r}H_2 - {}^{\mathrm{RL}}D^\alpha_{a+,r}H_1\,{}^{\mathrm{RL}}D^\alpha_{a+,p}H_2), \\
\frac{\partial^\alpha(H_1,H_2)}{\partial^\alpha(q,r)} &= ({}^{\mathrm{RL}}D^\alpha_{a+,q}H_1\,{}^{\mathrm{RL}}D^\alpha_{a+,r}H_2 - {}^{\mathrm{RL}}D^\alpha_{a+,r}H_1\,{}^{\mathrm{RL}}D^\alpha_{a+,q}H_2), \\
\frac{\partial^\alpha(H_1,H_2)}{\partial^\alpha(q,p)} &= ({}^{\mathrm{RL}}D^\alpha_{a+,q}H_1\,{}^{\mathrm{RL}}D^\alpha_{a+,p}H_2 - {}^{\mathrm{RL}}D^\alpha_{a+,p}H_1\,{}^{\mathrm{RL}}D^\alpha_{a+,q}H_2),
\end{aligned} \tag{5.2.64}$$

we obtain the generalized Pfaffian equations

$$(dq)^\alpha - \frac{\partial^\alpha(H_1,H_2)}{\partial^\alpha(p,r)}(dt)^\alpha = 0, \tag{5.2.65}$$

$$(dp)^\alpha - \frac{\partial^\alpha(H_1, H_2)}{\partial^\alpha(q, r)}(dt)^\alpha = 0, \tag{5.2.66}$$

$$(dr)^\alpha - \frac{\partial^\alpha(H_1, H_2)}{\partial^\alpha(q, p)}(dt)^\alpha = 0. \tag{5.2.67}$$

By using the analogy to the integer order situation, these equations are called fractional Nambu equations [65].

5.2.4.4 *A fractional supersymmetric model*

Now let us recall the development of [71] and consider the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\Phi^2(x) + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \Phi'(x)\bar{\psi}\psi. \tag{5.2.68}$$

As it is known, this Lagrangian characterizes pseudoclassical systems being supersymmetric [69, 71].

By inspection we conclude that the fractional counterpart of (5.2.68) is

$$L_f = \frac{1}{2}({}^{\mathrm{RL}}D^\alpha_{a+}x)^2 - \frac{1}{2}\Phi(x)^2 + \frac{i}{2}(\bar{\psi}{}^{\mathrm{RL}}D^\alpha_{a+}\psi - ({}^{\mathrm{RL}}D^\alpha_{a+}\bar{\psi})\psi) - \Phi'(x)\bar{\psi}\psi. \tag{5.2.69}$$

The fractional Hessian matrix corresponding to (5.2.69) has rank one, therefore we have two primary constraints. The expressions of the canonical momenta are written as

$$p = {}^{\mathrm{RL}}D^\alpha_{a+}x, \quad \pi = \frac{\partial_r L}{\partial\, {}^{\mathrm{RL}}D^\alpha_{a+}\psi} = -\frac{i}{2}\bar{\psi}, \quad \bar{\pi} = \frac{\partial_r L}{\partial\, {}^{\mathrm{RL}}D^\alpha_{a+}\bar{\psi}} = -\frac{i}{2}\psi. \tag{5.2.70}$$

By making use of (5.2.70) we obtain two second-class constraints as

$$\chi_1 = \pi + \frac{i}{2}\bar{\psi} = 0, \qquad \chi_2 = \bar{\pi} + \frac{i}{2}\psi = 0. \tag{5.2.71}$$

Finally, the total fractional Hamiltonian is given by

$$H_T = \frac{p^2}{2} + \frac{\Phi^2(x)}{2} + \Phi'(x)\bar{\psi}\psi + (\pi + \frac{i}{2}\bar{\psi})\lambda_1 + (\bar{\pi} + \frac{i}{2}\psi)\lambda_2. \tag{5.2.72}$$

The Hamiltonian equations are then written as

$${}^{\mathrm{RL}}D^\alpha_{a+}x = \frac{\partial H_f}{\partial p}, \quad {}^{\mathrm{RL}}D^\alpha_{a+}\psi = \frac{\partial H_f}{\partial \pi}, \quad {}^{\mathrm{RL}}D^\alpha_{a+}\bar{\psi} = \frac{\partial H_f}{\partial \bar{\pi}}, \tag{5.2.73}$$

$$ {}^{\text{RL}}D^{\alpha}_{b-}p = \frac{\partial H_f}{\partial x}, \quad {}^{\text{RL}}D^{\alpha}_{b-}\pi = \frac{\partial H_f}{\partial \psi}, \quad {}^{\text{RL}}D^{\alpha}_{b-}\bar{\pi} = \frac{\partial H_f}{\partial \bar{\psi}}. \tag{5.2.74} $$

Finally, by using (5.2.73) we conclude that

$$ {}^{\text{RL}}D^{\alpha}_{a+}x = {}^{\text{RL}}D^{\alpha}_{a+}x, \quad {}^{\text{RL}}D^{\alpha}_{a+}\psi = \lambda_1, \quad {}^{\text{RL}}D^{\alpha}_{a+}\psi = \lambda_2. \tag{5.2.75} $$

5.2.4.5 *Fractional optimal control formulation*

The main problem investigated here is to minimize the performance index

$$ J(u) = \int_a^b F(x,u,t)dt \tag{5.2.76} $$

such that

$$ {}^{\text{RL}}D^{\alpha}_{a+}x = G(x,u,t), \tag{5.2.77} $$

where the terminal conditions $x(a) = c$ and $x(b) = d$ are given.

The corresponding fractional order counterpart formulation of this problem for the case of scalar variables and functions was proposed in [18]. More details on this topic can be found in [18, 24, 25].

A modified performance index can be defined as

$$ \bar{J}(u) = \int_a^b [H(x,u,t) - \lambda^{\text{T}}\, {}^{\text{RL}}D^{\alpha}_{a+}x]dt, \tag{5.2.78} $$

where $H(x,u,\lambda,t)$ represents the Hamiltonian

$$ H(x,u,\lambda,t) = F(x,u,t) + \lambda^{\text{T}} G(x,u,t), \tag{5.2.79} $$

and λ denotes the vector of Lagrange multipliers. By making use of (5.2.78), (5.2.79) and the fractional integration by parts, the necessary conditions for the fractional control problem become [18]

$$ {}^{\text{RL}}D^{\alpha}_{b-}\lambda = \frac{\partial H}{\partial x}, \tag{5.2.80} $$

$$ \frac{\partial H}{\partial u} = 0, \tag{5.2.81} $$

$$ {}^{\text{RL}}D^{\alpha}_{a+}x = \frac{\partial H}{\partial \lambda}. \tag{5.2.82} $$

As an example, we present the following problem [25]:

Minimize

$$J = \frac{1}{2}\int_0^2 [{}^{\mathrm{RL}}D_{0+}^{\alpha}\, {}^{\mathrm{RL}}D_{0+}^{\alpha}\theta]^2 dt \tag{5.2.83}$$

with respect to the dynamic constraint ${}^{\mathrm{RL}}D_{0+}^{\alpha}\, {}^{\mathrm{RL}}D_{0+}^{\alpha}\theta(t) = u(t)$. Introducing the change of variables $\theta(t) = x_1(t)$ and ${}^{\mathrm{RL}}D_{0+}^{\alpha}\theta(t) = x_2(t)$, the modified performance index in (5.2.78) becomes

$$J = \int_0^2 [H(\mathbf{x}, u, \lambda) - \lambda^{\mathrm{T}}\, {}^{\mathrm{RL}}D_{0+}^{\alpha}\mathbf{x}(t)]dt \tag{5.2.84}$$

with the Hamiltonian

$$H(\mathbf{x}, u, \lambda) = \frac{1}{2}u^2(t) + \lambda^T(A\mathbf{x}(t) + \mathbf{b}u(t)). \tag{5.2.85}$$

In this system,

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad \lambda(t) = \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix}, \\ \mathbf{b} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{5.2.86}$$

By making use of (5.2.80)–(5.2.82), we finally obtain the system of equations

$${}^{\mathrm{RL}}D_{2-}^{\alpha}\lambda_1 = 0, \qquad {}^{\mathrm{RL}}D_{2-}^{\alpha}\lambda_2 - \lambda_1 = 0, \qquad u + \lambda_2 = 0, \tag{5.2.87a}$$

$${}^{\mathrm{RL}}D_{0+}^{\alpha}x_1 - x_2 = 0, \qquad {}^{\mathrm{RL}}D_{0+}^{\alpha}x_2 - u = 0. \tag{5.2.87b}$$

The terminal conditions $x_1(0) = x_2(0) = 0$ and $x_1(2) = x_2(2) = 1$ are considered in [25] where, in particular, a method based on a Grünwald-Letnikov approximation scheme is presented to solve (5.2.87) numerically, and where the exact classical solution of (5.2.87) is shown to have the form

$$x_1(t) = \frac{t^2}{4}, \qquad x_2(t) = \frac{t}{2}, \tag{5.2.88}$$

$$\lambda_1(t) = 0, \qquad \lambda_2(t) = -u(t) = -\frac{1}{2}. \tag{5.2.89}$$

For the numerical solution of this problem, the numerical results of [25] indicated that the variables $x_1(t)$ and $x_2(t)$ and the control variable $u(t)$ converges for several values of α as the number of grid points N is increased. On the other hand, it was concluded that near the end point $t = 2$, the value of $u(t)$ grows rapidly. The results show that as α approaches to 1, the analytical solution is recovered, as expected. Finally, it was reported that this result was found to be consistent with the analytical solution and with other results presented in [24] and [25].

5.2.4.6 *The fractional optimal control approach with delay*

Next, as in [293], we find the optimal control variable $u(t)$ which minimizes the performance index

$$\begin{aligned} J(y,u) = \int_a^b & F[t, u(t), {}^{\mathsf{C}}D_{a+}^{\alpha_1} y(t), {}^{\mathsf{C}}D_{a+}^{\alpha_2} y(t), \dots, {}^{\mathsf{C}}D_{a+}^{\alpha_n} y(t), \\ & {}^{\mathsf{C}}D_{a+}^{\beta_1} y(t), {}^{\mathsf{C}}D_{a+}^{\beta_2} y(t), \dots, {}^{\mathsf{C}}D_{a+}^{\beta_m} y(t), y(t), y'(t), \dots, y^{(k)}(t), \\ & y(t-\tau), y'(t-\tau), \dots, y^{(k)}(t-\tau)]dt, \end{aligned} \tag{5.2.90}$$

subject to the constraint

$$\begin{aligned} 0 = G[& t, u(t), {}^{\mathsf{C}}D_{a+}^{\alpha_1} y(t), {}^{\mathsf{C}}D_{a+}^{\alpha_2} y(t), \dots, {}^{\mathsf{C}}D_{a+}^{\alpha_n} y(t), \\ & {}^{\mathsf{C}}D_{b-}^{\beta_1} y(t), {}^{\mathsf{C}}D_{b-}^{\beta_2} y(t), \dots, {}^{\mathsf{C}}D_{b-}^{\beta_m} y(t), y(t), y'(t), \dots, y^{(k)}(t), \\ & y(t-\tau), y'(t-\tau), \dots, y^{(k)}(t-\tau)], \end{aligned} \tag{5.2.91}$$

in such a way that

$$y^{(l)}(b) = c_l \quad (l = 0, 1, \dots, k-1) \text{ and } y(t) = \phi(t) \quad (t \in [a-\tau, a]), \tag{5.2.92}$$

where $a < b$, $0 < \tau < b-a$, $\alpha_i \in \mathbb{R}$ $(i = 1, 2, ..., n)$, $\beta_j \in \mathbb{R}$ $(j = 1, 2, ..., m)$. Here the c_l are constants and F and G are functions from $[a-\tau, b] \times \mathbb{R}^{n+m+2k+3}$ to $\mathbb{R}$ possessing continuous first and second partial derivatives with respect to all of their arguments.

We define a modified performance index as

$$\hat{J}(y,u) = \int_a^b (F + \lambda(t)G)\ dt, \tag{5.2.93}$$

where λ represents a Lagrange multiplier or an adjoint variable.

According to [291], the Euler-Lagrange equations are

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} {}^{\mathtt{RL}}D_{b-\tau,-}^{\alpha_i}\left(\frac{\partial F}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha_i}y(t)}\right)(t) + \sum_{i=1}^{n} {}^{\mathtt{RL}}D_{b-\tau,-}^{\alpha_i}\left(\lambda\frac{\partial G}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha_i}y(t)}\right)(t)\\
&\quad + \sum_{j=1}^{m} {}^{\mathtt{RL}}D_{a+}^{\beta_j}\left(\frac{\partial F}{\partial\, {}^{\mathtt{C}}D_{b-}^{\beta_j}y(t)}\right)(t) + \sum_{j=1}^{m} {}^{\mathtt{RL}}D_{a+}^{\beta_j}\left(\lambda\frac{\partial G}{\partial\, {}^{\mathtt{C}}D_{b-}^{\beta_j}y(t)}\right)(t)\\
&\quad + \sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}\left(\frac{\partial F}{\partial y^{(p)}(t)}\right)(t) + \sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}\left(\lambda\frac{\partial G}{\partial y^{(p)}(t)}\right)(t)\\
&\quad + \sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}\left(\frac{\partial F}{\partial y^{(p)}(t-\tau)}\right)(t+\tau)\\
&\quad + \sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}\left(\lambda\frac{\partial G}{\partial y^{(p)}(t-\tau)}\right)(t+\tau)\\
&\quad - \sum_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}{}^{\mathtt{RL}}D_{b-\tau,-}^{\alpha_i}\left(\int_{b-\tau}^{b}({}^{\mathtt{RL}}D_{b-}^{\alpha_i}\left(\frac{\partial F}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha_i}y(t)}\right)(s)(s-t)^{\alpha_i-1}ds\right)\\
&\quad - \sum_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}{}^{\mathtt{RL}}D_{b-\tau,-}^{\alpha_i}\left(\int_{b-\tau}^{b}({}^{\mathtt{RL}}D_{b-}^{\alpha_i}\left(\lambda\frac{\partial G}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha_i}y(t)}\right)(s)(s-t)^{\alpha_i-1}ds\right)\\
&\quad + \frac{\partial F}{\partial u(t)}(t) + \lambda(t)\frac{\partial G}{\partial u(t)}(t) \qquad (5.2.94)
\end{aligned}
$$

for $a \le t \le b-\tau$ and

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} {}^{\mathtt{RL}}D_{b-}^{\alpha_i}\left(\frac{\partial F}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha_i}y(t)}\right)(t) + \sum_{i=1}^{n} {}^{\mathtt{RL}}D_{b-}^{\alpha_i}\left(\lambda\frac{\partial G}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha_i}y(t)}\right)(t)\\
&\quad + \sum_{j=1}^{m} {}^{\mathtt{RL}}D_{b-\tau,-}^{\beta_j}\left(\frac{\partial F}{\partial\, {}^{\mathtt{C}}D_{b-}^{\beta_j}y(t)}\right)(t) + \sum_{j=1}^{m} {}^{\mathtt{RL}}D_{b-\tau,-}^{\beta_j}\left(\lambda\frac{\partial G}{\partial\, {}^{\mathtt{C}}D_{b-}^{\beta_j}y(t)}\right)(t)\\
&\quad + \sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}\left(\frac{\partial F}{\partial y^{(p)}(t)}\right)(t) + \sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}\left(\lambda\frac{\partial G}{\partial y^{(p)}(t)}\right)(t)\\
&\quad + \sum_{j=1}^{m}\frac{1}{\Gamma(\beta_j)}{}^{\mathtt{RL}}D_{b-\tau,+}^{\beta_j}\left(\int_{a}^{b-\tau}{}^{\mathtt{RL}}D_{a+}^{\beta_j}\left(\frac{\partial F}{\partial\, {}^{\mathtt{C}}D_{b-}^{\beta_j}y(t)}\right)(s)(t-s)^{\beta_j-1}ds\right)
\end{aligned}
$$

$$+ \sum_{j=1}^{m} \frac{1}{\Gamma(\beta_j)} {}^{\mathtt{RL}}D_{b-\tau,+}^{\beta_j} \left(\int_a^{b-\tau} {}^{\mathtt{RL}}D_{a+}^{\beta_j} \left(\lambda \frac{\partial L}{\partial\, {}^{\mathtt{C}}D_{b-}^{\beta_j} y(t)} \right)(s)(t-s)^{\beta_j - 1} ds \right)$$

$$+ \frac{\partial F}{\partial u(t)}(t) + \lambda(t) \frac{\partial G}{\partial u(t)}(t) \tag{5.2.95}$$

for $b - \tau \le t \le b$, whereas the transversality conditions are

$$\begin{aligned}
0 &= \sum_{p=1}^{k} \sum_{q=0}^{p-1} (-1)^q \frac{d^q}{dt^q} \left(\frac{\partial F}{\partial y^{(p)}(t)} \right)(t) \eta^{p-q-1}(t) \Big|_a^{b-\tau} \\
&\quad + \sum_{p=1}^{k} \sum_{q=0}^{p-1} (-1)^q \frac{d^q}{dt^q} \left(\lambda \frac{\partial G}{\partial y^{(p)}(t)} \right)(t) \eta^{p-q-1}(t) \Big|_a^{b-\tau} \\
&\quad + \sum_{p=1}^{k} \sum_{q=0}^{p-1} (-1)^q \frac{d^q}{dt^q} \left(\frac{\partial F}{\partial y^{(p)}(t-\tau)} \right)(t+\tau) \eta^{p-q-1}(t+\tau) \Big|_a^{b-\tau} \\
&\quad + \sum_{p=1}^{k} \sum_{q=0}^{p-1} (-1)^q \frac{d^q}{dt^q} \left(\lambda \frac{\partial G}{\partial y^{(p)}(t-\tau)} \right)(t+\tau) \eta^{p-q-1}(t+\tau) \Big|_a^{b-\tau} \\
&\quad + \sum_{p=1}^{k} \sum_{q=0}^{p-1} (-1)^q \frac{d^q}{dt^q} \left(\frac{\partial F}{\partial y^{(p)}(t)} \right)(t) \eta^{p-q-1}(t) \Big|_{b-\tau}^{b} \\
&\quad + \sum_{p=1}^{k} \sum_{q=0}^{p-1} (-1)^q \frac{d^q}{dt^q} \left(\lambda \frac{\partial G}{\partial y^{(p)}(t)} \right)(t) \eta^{p-q-1}(t) \Big|_{b-\tau}^{b},
\end{aligned} \tag{5.2.96}$$

where η denotes any admissible function fulfilling $\eta(t) \equiv 0$ for $t \in [a-\tau, a]$ and $\eta^{(l)}(b) = 0$ for $l = 0, 1, \ldots, k-1$.

5.2.4.7 *Fractional multi time Hamiltonian equations*

In 1935, the multi time Hamilton equations were introduced into classical mechanics by de Donder and Weyl [164, 578]. A quantization of field theory based on the de Donder-Weyl covariant Hamiltonian formalism was investigated in [301]. Following [64], we now present its fractional version by using the generalized canonical momenta

$$p_i^{\nu} = \frac{\partial L_f}{\partial\, {}^{\mathtt{RL}}D_{a+,t^{\nu}}^{\alpha} q^i} \tag{5.2.97}$$

and the Hamiltonian defined as

$$H = {}^{\mathtt{RL}}D^{\alpha}_{a+t^{\nu}} q^i \frac{\partial L}{\partial\, {}^{\mathtt{RL}}D^{\alpha}_{a+,t^{\nu}} q^i} - L_f(t^{\nu}, q^i, \; {}^{\mathtt{RL}}D^{\alpha}_{a+,t^{\nu}} q^i). \tag{5.2.98}$$

The transformations $D^{\alpha}_{a+,t^{\nu}} q^i \rightarrow p^{\nu}_i$ defines the corresponding fractional Legendre transform. Therefore, by using the fractional Legendre transformation we get the fractional multi time Hamiltonian equations as

$$\begin{aligned} \frac{\partial H}{\partial q^i} &= {}^{\mathtt{RL}}D^{\alpha}_{b-,t^{\nu}} p^{\nu}_i, \qquad i = 1, \ldots, n, \quad \nu = 1, \ldots, p, \\ \frac{\partial H}{\partial p^{\nu}_i} &= {}^{\mathtt{RL}}D^{\alpha}_{a+,t^{\nu}} q^i, \qquad i = 1, \ldots, n, \quad \nu = 1, \ldots, p, \\ \frac{\partial H}{\partial t^{\nu}} &= -\frac{\partial L_f}{\partial t^{\nu}}, \qquad \nu = 1, \ldots, p. \end{aligned} \tag{5.2.99}$$

By inspection we conclude that Eqs. (5.2.99) are $np + n$ equations and they are equivalent to the fractional Euler-Lagrange equation on $\mathbb{R}^n$.

5.2.4.8 *Hamilton-Jacobi formulation with Caputo fractional derivative*

In order to determine the Hamilton-Jacobi partial differential equation with fractional Caputo derivative, we start with the fractional Hamiltonian written as

$$H = p_{\beta} \; {}^{\mathtt{C}}D^{\beta}_{a+} q - L_f(t, q, \; {}^{\mathtt{C}}D^{\beta}_{a+} q). \tag{5.2.100}$$

The next step is to consider the canonical transform with a generating function $S(I^{1-\beta}_{a+} q, P_{\beta}, t)$, therefore the new Hamiltonian becomes

$$K = P_{\beta} \; {}^{\mathtt{C}}D^{\beta}_{a+} Q - L'_f(t, Q, \; {}^{\mathtt{C}}D^{\beta}_{a+} Q), \tag{5.2.101}$$

where Q, P_{β} denote the new canonical coordinates and L' is the new Lagrangian. As shown in [276], these coordinates fulfill the fractional Hamilton's principle

$$\delta(I^{\alpha}(p_{\beta} \, {}^{\mathtt{C}}D^{\beta}_{a+} q - H)) = 0 \quad \text{and} \quad \delta(I^{\alpha}(P_{\beta} \, {}^{\mathtt{C}}D^{\beta}_{a+} Q - K)) = 0. \tag{5.2.102}$$

We can say that this is satisfied if we have

$$\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}(p_{\beta} \; {}^{\mathtt{C}}D^{\beta}_{a+} q - H) = \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}(P_{\beta} \; {}^{\mathtt{C}}D^{\beta}_{a+} Q - K) + \frac{dF}{dt}, \tag{5.2.103}$$

where the function F has the form

$$F = S({}^{\mathtt{RL}}I_{a+}^{1-\beta}q, P_\beta, t) - P_\beta \, {}^{\mathtt{RL}}I_{a+}^{1-\beta}Q. \tag{5.2.104}$$

By direct calculations, see [276], we obtain the Hamilton-Jacobi equations of motion as

$$p_\beta = (x-t)^{1-\alpha}\Gamma(\alpha)\frac{\partial S}{\partial \, {}^{\mathtt{RL}}I_{a+}^{1-\beta}q}, \tag{5.2.105}$$

$${}^{\mathtt{RL}}I_{a+}^{1-\beta}Q = \frac{\partial S}{\partial P_\beta}, \tag{5.2.106}$$

and

$$\begin{aligned} K + (\Gamma(\alpha)(x-t)^{1-\alpha} - 1)P_\beta \, {}^{\mathtt{C}}D_{a+}^{\beta}Q + \frac{(t-a)^{-\beta}}{\Gamma(1-\beta)}Q(a)P_\beta \\ = H + \Gamma(\alpha)(x-t)^{1-\alpha}\frac{\partial S}{\partial t} + \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-a)^{-\beta}}{\Gamma(1-\beta)}q(a)p_\beta. \end{aligned} \tag{5.2.107}$$

When the transformed Hamiltonian K is identically zero we obtain from the Hamiltonian equations that $Q = E_1$ and $P_\beta = E_2$ where E_1, E_2 are two constants. As a result, the generating function satisfies the partial differential equation

$$H + \Gamma(\alpha)(x-t)^{1-\alpha}\frac{\partial S}{\partial t} = \frac{(t-a)^{-\beta}}{\Gamma(1-\beta)}E_1E_2 - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\frac{(t-a)^{-\beta}}{\Gamma(1-\beta)}q(a)p_\beta. \tag{5.2.108}$$

Therefore, this is the fractional Hamilton-Jacobi equation for the fractional variational problem given by the Caputo derivative. Finally, we notice, see also [276], that the generating function is

$$\begin{aligned} S = \int_{t_1}^{t_2} \Bigg(\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}L(t, q, \, {}^{\mathtt{C}}D_{a+}^{\beta}q) - \frac{(t-a)^{1-\beta}(x-t)^{\alpha-1}}{\Gamma(\alpha)\Gamma(1-\beta)}E_1E_2 \\ - \frac{(t-a)^{-\beta}(x-a)^{\alpha-1}(x-t)^{\alpha-1}}{(\Gamma(\alpha))^2\Gamma(1-\beta)}q(a)p_\beta \Bigg) dt. \end{aligned} \tag{5.2.109}$$

For more details about fractional Hamilton–Jacobi equation see Refs. [461, 465, 466] and the references therein.

5.2.4.9 *Fractional dynamics on extended phase space*

The main idea is to start with a given Hamiltonian $H(q,p)$ and to find $L^q(q,\dot{q})$ as a solution of the differential equation [421]

$$H(q, \frac{\partial L^q}{\partial \dot{q}}) - \dot{q}\frac{\partial L^q}{\partial \dot{q}} + L^q = 0. \tag{5.2.110}$$

It is known [421] that the extended Hamiltonian has the form

$$H(q,\pi_q,p,\pi_p) = \dot{q}\pi_q + \dot{p}\pi_p - L(q,\dot{q},p,\dot{p}) \tag{5.2.111}$$

and that the final Hamiltonian reads

$$H(q,\pi_q,p,\pi_p) = \sum_{n=0}^{N} \frac{1}{n!}\left[\frac{\partial^n H}{\partial p^n}\pi_q^n - \frac{\partial^n H}{\partial q^n}\pi_p^n\right]. \tag{5.2.112}$$

From here on we follow [73] and first denote by $H_f(q,p) = p\,{}^{\mathtt{C}}D_{a+}^{\alpha} q - L_f^q$ the fractional Hamiltonian where L_f^q represents the fractional Lagrangian. By using a similar technique as in the classical case we then obtain

$$H_f(q, \frac{\partial L^q}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha} q}) - {}^{\mathtt{C}}D_{a+}^{\alpha} q \frac{\partial L_f^q}{\partial\, {}^{C}D_{a+}^{\alpha} q} + L_f^q = 0, \tag{5.2.113}$$

and

$$H_f(p, \frac{\partial L_f^p}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha} p}) - {}^{\mathtt{C}}D_{a+}^{\alpha} p \frac{\partial L_f^p}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha} p} + L_f^p = 0. \tag{5.2.114}$$

Therefore, the fractional extended Lagrangian becomes

$$\begin{aligned} &L(q, {}^{\mathtt{C}}D_{a+}^{\alpha} q, p, {}^{\mathtt{C}}D_{a+}^{\alpha} p) \\ &\quad = -p\,{}^{\mathtt{C}}D_{a+}^{\alpha} q - q\,{}^{\mathtt{C}}D_{a+}^{\alpha} p + L_f^q(q, {}^{\mathtt{C}}D_{a+}^{\alpha} q) + L_f^p(p, {}^{\mathtt{C}}D_{a+}^{\alpha} p), \end{aligned} \tag{5.2.115}$$

and this implies the fractional Euler-Lagrange equations as

$${}^{\mathtt{RL}}D_{b-}^{\alpha} \frac{\partial L_f^q}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha} q} - \frac{\partial L_f^q}{\partial q} - {}^{\mathtt{RL}}D_{b-}^{\alpha} p - {}^{\mathtt{C}}D_{a+}^{\alpha} p = 0 \tag{5.2.116}$$

and

$${}^{\mathtt{RL}}D_{b-}^{\alpha} \frac{\partial L_f^p}{\partial\, {}^{\mathtt{C}}D_{a+}^{\alpha} p} - \frac{\partial L_f^p}{\partial p} - {}^{\mathtt{RL}}D_{b-}^{\alpha} q - {}^{\mathtt{C}}D_{a+}^{\alpha} q = 0. \tag{5.2.117}$$

Having in mind that the fractional extended momenta are

$$\pi_q^f = \frac{\partial L_f^q}{\partial\, {}^{\mathrm{C}}D_{a+}^{\alpha} q} - p, \qquad \pi_p^f = \frac{\partial L_f^q}{\partial\, {}^{\mathrm{C}}D_{a+}^{\alpha} p} - q, \tag{5.2.118}$$

the fractional extended Hamiltonian was reported in [73] as

$$H_f(q, \pi_q^f, p, \pi_p^f) = \sum_{n=0}^{N} \frac{1}{n!} \left[\frac{\partial^n H_f}{\partial p^n} (\pi_q^f)^n - \frac{\partial^n H_f}{\partial q^n} (\pi_p^f)^n \right]. \tag{5.2.119}$$

We notice [73] that the corresponding fractional generalization of the Poisson brackets can be written as

$$\{F, G\}_f = \frac{\partial F}{\partial q} \frac{\partial G}{\partial \pi_q^f} - \frac{\partial F}{\partial \pi_q^f} \frac{\partial G}{\partial q} + \{F, G\} + \frac{\partial F}{\partial p} \frac{\partial G}{\partial \pi_p^f} - \frac{\partial F}{\partial \pi_p^f} \frac{\partial G}{\partial p}. \tag{5.2.120}$$