Fractional calculus: basic theory and applications

(Part I)

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Outline

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4. Numerical methods
5. Applications:
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   b) Transport in fusion plasmas
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1. Introduction

Here we introduce the notion of fractional integral as a straightforward generalization of the standard, integer-order integral, and define the fractional derivative as the inverse operation. To motivate the concept we discuss two examples: Abel's equation and heat diffusion. The concepts discussed here are further elaborated in the next section.

What is a fractional derivative?

$\frac{d^n f}{dx^n}$ L'Hopital (1695):
"What if n=1/2?"

Leibniz (1695):
"This is an apparent paradox from which, one day, useful consequences will be drawn"

It is a usual practice to extend mathematical operations, originally defined for a set of objects, to a wider set of objects

$$\sqrt{r} \quad r \in R^+ \quad \Rightarrow \quad \sqrt{s} \quad s \in R$$

$$n! \quad n \in N^+ \quad \Rightarrow \quad \Gamma(s) \quad s \in R$$

$$\partial_x^n \phi \quad n \in N^+ \quad \Rightarrow \quad \partial_x^\alpha \phi \quad \alpha \in C$$

This mathematical "games" have, eventually, important physical applications

Fractional integrals

Integer order integration

$$ {}_aD_x^{-n}\,\phi = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} dx_n\; \phi(x_n) $$

Interchanging the order of integration this can be rewritten as

$$ {}_aD_x^{-n}\,\phi = \frac{1}{(n-1)!}\int_a^x (x-y)^{n-1}\,\phi(y)\,dy $$

Extending this expression for non-integer n=ν we get

Riemann-Liouville fractional integral

$$ {}_aD_x^{-\nu}\,\phi = \frac{1}{\Gamma(\nu)}\int_a^x (x-y)^{\nu-1}\,\phi(y)\,dy $$

Example: $ {}_0D_x^{-\nu}\, x^{\mu} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}\, x^{\mu+\nu} $

Fractional derivatives

Having defined the fractional integral, define the fractional derivative as the inverse operation

$$ {}_aD_x^{\mu}\;{}_aD_x^{-\mu}\,\phi = \phi $$

$$ {}_aD_x^{\mu}\,\phi = \frac{d^N}{dx^N}\left[{}_aD_x^{-\nu}\,\phi\right] \qquad \nu = N-\mu \qquad N = \text{smallest integer} > \mu $$

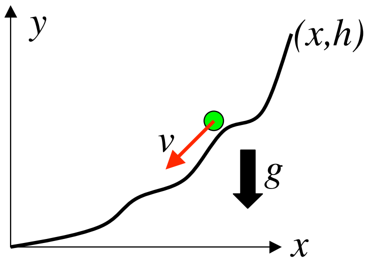
$$ {}_aD_x^{m}\,\phi = \frac{d^N}{dx^N}\left[{}_aD_x^{-(N-m)}\,\phi\right] = \frac{d^m}{dx^m}\phi $$

Riemann-Liouville fractional derivative

$$ {}_aD_x^{\alpha}\,\phi = \frac{1}{\Gamma(m-\alpha)}\,\partial_x^m \int_a^x \frac{\phi(y)}{(x-y)^{\alpha+1-m}}\,dy \qquad m-1 < \alpha \le m $$

Examples: $ {}_0D_x^{\mu}\, x^{\lambda} = \dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\, x^{\lambda-\mu} $ $\qquad {}_{-\infty}D_x^{\mu}\, e^{ikx} = (ik)^{\mu}\, e^{ikx}$

An example: Abel's equation



Let *T(h)* be the time it takes a particle to go down sliding on the curve $x = \phi(y)$ What is the relation between the function *T(h)* and $\phi(y)$?

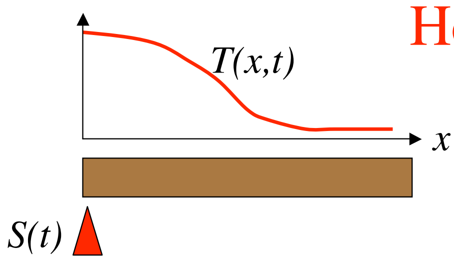
Energy conservation $$\frac{1}{2}v^2 = g(h-y)$$

$$v = \frac{ds}{dt} = \frac{\sqrt{1+\phi'^2}\,dy}{dt} \qquad \int_0^h \frac{\sqrt{1+\phi'^2}}{\sqrt{2g(h-y)}}dy = \int_0^T dt = T$$

$$f(y) = \sqrt{\frac{1}{2g}}\sqrt{1+\phi'^2} \qquad T(h) = \int_0^h \frac{f(y)}{\sqrt{h-y}}dy$$

$$T = \sqrt{\pi}\ {}_0D_h^{-1/2} f \qquad f = \frac{1}{\sqrt{\pi}}\ {}_0D_y^{1/2} T$$

Heat diffusion



Consider a bar that is heated on one end by a time dependent source *S(t)*. What is the relation between *S(t)* and the temperature *T(x,t)* of the bar?

Heat diffusion equation $\quad$ Boundary conditions $\quad$ Initial conditions

$$\partial_t T = \kappa\, \partial_x^2\, T \qquad T(x=0,t) = S(t) \qquad T(x,t=0) = 0$$

Laplace transform $$\hat{T}(x,s) = \int_0^\infty e^{-st}\, T(x,t)\, dt \qquad s\hat{T} = \kappa\, \partial_x^2\, \hat{T}$$

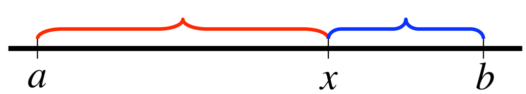
$$T = \frac{x}{\sqrt{2\pi\kappa}} \int_0^t e^{-\frac{x^2}{4\kappa(t-\tau)}} \frac{S(\tau)}{(t-\tau)^{3/2}}\, d\tau \qquad \Gamma(t) = \int_0^\infty T(x,t)\, dx$$

$$\Gamma(t) = \sqrt{2\kappa}\ {}_0D_t^{-1/2}\, S(t)$$

2. Fractional calculus

Here we present the precise definitions of the left and right Riemann-Liouville (RL) fractional integrals and derivatives, and briefly discuss some of the main properties of these operators. Of particular interest is the singular behavior of the RL operators at the boundaries, and the definition of the fractional derivatives in the Caputo sense. Further details on the material discussed here can be found in [Samko-etal-1993], [Podlubny-1999] and references therein.

Riemann-Liouville integrals



$\alpha > 0$

Left integral $$ {}_aD_x^{-\alpha}\,\phi = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1}\,\phi(y)\,dy \qquad x > a $$

Right integral $$ {}_xD_b^{-\alpha}\,\phi = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1}\,\phi(y)\,dy \qquad x < b $$

Given the reflection operator $Q\,\phi\,(x) = \phi\,(a+b-x)$

$$ Q\left[{}_aD_x^{-\alpha}\,\phi\right] = {}_xD_b^{-\alpha}\left[Q\,\phi\right] \qquad Q\left[{}_xD_b^{-\alpha}\,\phi\right] = {}_aD_x^{-\alpha}\left[Q\,\phi\right] $$

$$Q\left[{}_aD_x^{-\alpha}\,\phi\right] = \frac{1}{\Gamma(\alpha)}\int_a^{a+b-x}\frac{\phi(u)}{(a+b-x-u)^{1-\alpha}}\,du \qquad z=-u+a+b$$

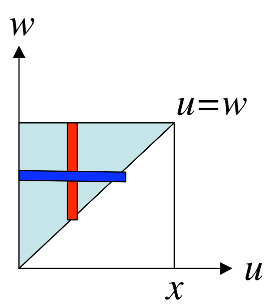
$$=\frac{1}{\Gamma(\alpha)}\int_x^{b}\frac{\phi(a+b-z)}{(z-x)^{1-\alpha}}\,dz={}_xD_b^{-\alpha}\left[Q\,\phi\right]$$

Semi-group property

The fractional integrals satisfy the following important semigroup property

$$\boxed{\begin{aligned} {}_aD_x^{-\alpha}\;{}_aD_x^{-\beta}\;\phi &= {}_aD_x^{-\alpha-\beta}\;\phi \\ {}_xD_b^{-\alpha}\;{}_xD_b^{-\beta}\;\phi &= {}_xD_b^{-\alpha-\beta}\;\phi \end{aligned}} \qquad \begin{aligned} \alpha &> 0 \\ \beta &> 0 \end{aligned}$$

$${}_aD_x^{-\alpha}\;{}_aD_x^{-\beta}\;\phi = \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^x\frac{dw}{(x-w)^{1-\alpha}}\int_a^w\frac{\phi(u)}{(w-u)^{1-\beta}}du$$



Fubini's theorem

$$\int_a^x dw\int_a^w du\cdots=\int_a^x du\int_u^x dw\cdots$$

$$=\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^x du\;\;\phi(u)\underbrace{\int_u^x\frac{dw}{(x-w)^{1-\alpha}(w-u)^{1-\beta}}}_{\frac{B(\alpha,\beta)}{(x-u)^{1-\alpha-\beta}}}$$

Beta function $\quad B(\alpha,\beta)=\dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$=\frac{1}{\Gamma(\alpha+\beta)}\int_a^x\frac{\phi(u)}{(x-u)^{1-\alpha-\beta}}\,du={}_aD_x^{-(\alpha+\beta)}\phi$$

Riemann-Liouville derivatives

$n=[\alpha]+1$

[α]=integral part of α

Left derivative $\quad {}_aD_x^{\alpha}\,\phi=\dfrac{1}{\Gamma(n-\alpha)}\,\dfrac{\partial^n}{\partial x^n}\displaystyle\int_a^x \frac{\phi(u)}{(x-u)^{\alpha-n+1}}\,du$

Right derivative $\quad {}_xD_b^{\alpha}\,\phi=\dfrac{(-1)^n}{\Gamma(n-\alpha)}\,\dfrac{\partial^n}{\partial x^n}\displaystyle\int_x^b \frac{\phi(u)}{(u-x)^{\alpha-n+1}}\,du$

Reciprocity

$${}_aD_x^{\alpha}\;{}_aD_x^{-\alpha}\phi=\phi$$

$${}_aD_x^{\alpha}\;{}_aD_x^{-\alpha}\phi=\frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\,\frac{\partial^n}{\partial x^n}\int_a^x dz\int_a^z du\,\frac{\phi(u)}{(x-z)^{\alpha-n+1}(z-u)^{1-\alpha}}$$

Exchanging the order of the integrals and using the definition of the Beta function we get:

$$=\frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\,\frac{\partial^n}{\partial x^n}\int_a^x du\,\phi(u)\left[\frac{B(\alpha,n-\alpha)}{(x-u)^{1-n}}\right]\quad=\frac{1}{\Gamma(n)}\,\frac{\partial^n}{\partial x^n}\int_a^x du\,\frac{\phi(u)}{(x-u)^{1-n}}$$

using $\quad \dfrac{1}{(n-1)!}\displaystyle\int_a^x du\,\frac{\phi(u)}{(x-u)^{1-n}}={}_aD_x^{-n}$

$$=\frac{\partial^n}{\partial x^n}\,{}_aD_x^{-n}\phi=\phi$$

However

$$ {}_aD_x^{-\alpha}\;{}_aD_x^{\alpha}\phi = \phi(x) - \sum_{j=1}^{k}\left[{}_aD_x^{\alpha-j}\phi\right]_{x=a}\frac{(x-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} \qquad j-1\le\alpha<j $$

The previous formulae can be generalized as

$$ {}_aD_x^{\alpha}\;{}_aD_x^{-\beta}\phi = {}_aD_x^{\alpha-\beta}\phi \qquad \alpha\ge\beta\ge 0 $$

$$ {}_aD_x^{-\alpha}\;{}_aD_x^{\beta}\phi = {}_aD_x^{\beta-\alpha}\phi(x) - \sum_{j=1}^{k}\left[{}_aD_x^{\beta-j}\phi\right]_{x=a}\frac{(x-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} $$

Composition of fractional derivatives

$$ \frac{\partial^n}{\partial x^n}\left[{}_aD_x^{\alpha}\,\phi\right] = {}_aD_x^{n+\alpha}\,\phi \qquad \alpha\ge 0 $$

$$ {}_aD_x^{\alpha}\left[\frac{\partial^n\phi}{\partial x^n}\right] = {}_aD_x^{\alpha+n}\phi(x) - \sum_{j=0}^{n-1}\frac{\phi^{(j)}(a)(x-a)^{j-\alpha-n}}{\Gamma(1+j-\alpha-n)} \qquad n-1\le\beta<n $$

$$ {}_aD_x^{\alpha}\left[{}_aD_x^{\beta}\phi\right] = {}_aD_x^{\alpha+\beta}\phi(x) - \sum_{j=1}^{m}\left[{}_aD_x^{\beta-j}\phi\right]_{x=a}\frac{(x-a)^{-j-\alpha}}{\Gamma(1-\alpha-j)} \qquad m-1\le\alpha<m $$

In particular, fractional derivatives commute

$$ {}_aD_x^{\alpha}\left[{}_aD_x^{\beta}\phi\right] = {}_aD_x^{\beta}\left[{}_aD_x^{\alpha}\phi\right] $$

if and only if

$$ \begin{cases} \left[{}_aD_x^{\beta-j}\phi\right]_{x=a} = 0 & j=1,\cdots n \\ \left[{}_aD_x^{\alpha-j}\phi\right]_{x=a} = 0 & j=1,\cdots m \end{cases} $$

Behavior near the lower terminal

Let $\phi(x) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(a)}{k!}(x-a)^k$ $\qquad {}_aD_x^\alpha\, \phi = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(a)}{k!}\, {}_aD_x^\alpha (x-a)^{k-\alpha}$

using

$${}_aD_x^\alpha (x-a)^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}(x-a)^{k-\alpha} \qquad {}_aD_x^\alpha\, \phi = \sum_{k=0}^{\infty} \phi^{(k)}(a)\frac{(x-a)^{k-\alpha}}{\Gamma(k+1-\alpha)}$$

$$\lim_{x\to a}\, {}_aD_x^\alpha\, \phi = \lim_{x\to a} \sum_{k=0}^{m-1} \frac{\phi^{(k)}(a)}{\Gamma(k+1-\alpha)} \frac{1}{(x-a)^{\alpha-k}} \qquad m-1 \le \alpha < m$$

$\lim_{x\to a}\, {}_aD_x^\alpha\, \phi = \infty$ unless $\phi^{(k)}(a) = 0 \quad k = 1,\cdots m-1$

Fourier-Laplace transforms

Fourier $\quad \hat{\phi}(k) = \int_{-\infty}^{\infty} e^{ikx}\, \phi(x)\, dx \qquad \phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\hat{\phi}(k)\, dx$

$$F\left[{}_{-\infty}D_x^\alpha\, \phi\right] = (-i\,k)^\alpha\, \hat{\phi}(k) \qquad F\left[{}_xD_{+\infty}^\alpha\, \phi\right] = (i\,k)^\alpha\, \hat{\phi}(k)$$

Laplace $\quad \hat{\phi}(s) = \int_0^{\infty} e^{-st}\, \phi(t)\, dt \qquad \phi(t) = \int_{c-i\infty}^{c+i\infty} e^{st}\hat{\phi}(s)\, ds$

$$L\left[{}_0D_t^\alpha\, \phi\right] = s^\alpha\, \hat{\phi}(s) - \sum_{k=0}^{n-1} s^k \left[{}_0D_t^{\alpha-k-1}\, \phi\right]_{t=0} \qquad n-1 \le \alpha < n$$

These are natural generalizations of the Fourier-Laplace transform of regular derivatives

Note that we have taken $\quad a = -\infty \qquad b = \infty$

Some limitations of the RL definition

Although a well-defined mathematical object, the Riemman-Liouville definition of the fractional derivative has some problems when it comes to apply it in physical problems. In particular:

The derivative of a constant is not zero

$$ {}_aD_x^{\alpha}\, A = \frac{A}{\Gamma(1-\alpha)}\frac{1}{(x-a)^{\alpha}} $$

This might be an issue when using RL operators for writing evolution equations

The RL derivative is in general singular at the lower limit

$$ \lim_{x\to a}\ {}_aD_x^{\alpha}\,\phi = \infty $$

unless $\phi^{(k)}(a) = 0$

This might be an issue when applying boundary conditions

The Laplace transform of the RL derivative depends on the fractional derivative at zero

$$ L\left[{}_0D_t^{\alpha}\,\phi\right] = s^{\alpha}\,\hat{\phi}(s) - \sum_{k=0}^{n-1} s^k \left[{}_0D_t^{\alpha-k-1}\,\phi\right]_{t=0} $$

This might be an Issue when solving Initial value problems

Caputo fractional derivative

These problems can be resolved by defining the fractional operators in the Caputo sense. Consider the case $1 < \alpha < 2$

$$ {}_aD_x^{\alpha}\,\phi - \underbrace{\frac{\phi(a)}{\Gamma(1-\alpha)}\frac{1}{(x-\alpha)^{\alpha}} - \frac{\phi'(a)}{\Gamma(2-\alpha)}\frac{1}{(x-\alpha)^{\alpha-1}}}_{\text{singular terms}} = \underbrace{\sum_{k=0}^{\infty}\frac{\phi^{(k+2)}(a)(x-a)^{k+2-\alpha}}{\Gamma(k+3-\alpha)}}_{\text{regular terms}} $$

$$ {}_aD_x^{\alpha}\left[\phi(x) - \phi(a) - \phi^{(1)}(a)\right] = \sum_{k=0}^{\infty}\frac{\phi^{(k+2)}(a)(x-a)^{k+2-\alpha}}{\Gamma(k+3-\alpha)} $$

Define the Caputo derivative by subtracting the singular terms

$$ {}_0^CD_x^{\alpha}\phi = {}_aD_x^{\alpha}\left[\phi(x) - \phi(a) - \phi^{(1)}(a)\right] $$

using

$$ \int_a^x \frac{\phi''(u)}{(x-u)^{\alpha-1}}du = \sum_{k=0}^{\infty}\frac{\phi^{(k+2)}(a)}{k!}\int_a^x\frac{(x-a)^k}{(x-u)^{\alpha-1}}du $$

$$ {}_a^CD_x^{\alpha}\,\phi = \frac{1}{\Gamma(2-\alpha)}\int_a^x \frac{\phi''(u)}{(x-u)^{\alpha-1}}\,du $$

In general

$$ {}_a^C D_x^\alpha \phi = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{\partial_u^n \phi}{(x-u)^{\alpha-n+1}} \, du $$

$$ {}_x^C D_b^\alpha \phi = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_a^x \frac{\partial_u^n \phi}{(x-u)^{\alpha-n+1}} \, du $$

and as expected: $ {}_a^C D_x^\alpha A = 0$

$$ \lim_{x \to a} \; {}_a D_x^\alpha \phi = 0 $$

$$ L\left[{}_0 D_t^\alpha \phi \right] = s^\alpha \hat{\phi}(s) - \sum_{k=0}^{n-1} s^k \left[{}_0 \phi^{(\alpha-k-1)} \right]_{t=0} $$

Note that in an infinite domain

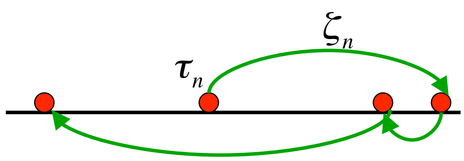
$$ {}_{-\infty}^{C} D_x^\alpha \phi = {}_{-\infty} D_x^\alpha \phi \qquad {}_x^C D_\infty^\alpha \phi = {}_x D_\infty^\alpha \phi $$

3. Fractional diffusion and random walks

Here we discuss the close connection between fractional calculus (in particular fractional diffusion equations) and the theory of continuous time random walks (CTRW). We discuss the space-time scaling properties of the solution of the fractional diffusion equation and the relations with the Levy stable distributions. We present two examples of Levy distributions one in the context of chaotic advection [del-Castillo-Negrete-1998, 2000] and another in the context of financial mathematics [Mantegna-Stanley-1994]. CTRWs were originally introduced in [Montroll-Weiss-1965]. Further details can be found in the review papers [Montroll-Shlesinger-1984], [Metzler-Klafter-2000], and references therein. For a discussion in the context of turbulent transport in plasmas see [del-Castillo-Negrete-etal-2004]. For a readable introduction to stochastic processes and Levy distributions see [Paul-Baschnagel-1999], [Bouchard- Georges-1990] and references therein. For a discussion on the Green's function of the fractional diffusion equation see [Mainardi et al. Fractional Calculus and Applied Analysis, 4, 153-192 (2001)].

The continuous time random walk

[Montroll-Weiss-1965]



τ_n = waiting time $\quad \psi(\tau)$ = waiting time pdf

ζ_n = jump $\quad \lambda(\zeta)$ = jump size pdf

Master equation

$$P(x,t) = \underbrace{\delta(x)\int_t^{\infty}\psi(t')\,dt'}_{\substack{\text{Contribution from particles} \\ \text{that have not moved during } (0,t)}} + \underbrace{\int_0^t \psi(t-t')\left[\int_{-\infty}^{\infty}\lambda(x-x')\,P(x',t')\,dx'\right]dt'}_{\substack{\text{Contribution from particles located at } x' \\ \text{and jumping to } x \text{ during } (0,t)}}$$

The Montroll-Weiss master equation can be solved using Laplace-Fourier transforms

$$\hat{\tilde{P}}(k,s) = \frac{1-\tilde{\psi}(s)}{s}\,\frac{1}{1-\tilde{\psi}(s)\,\hat{\lambda}(k)}$$

We want to take the long time, and large scale limit. To do this, introduce the small parameters epsilon and delta and consider

$$\psi^*(t) = \frac{1}{\varepsilon}\psi\left(\frac{t}{\varepsilon}\right) \qquad \lambda^*(x) = \frac{1}{\delta}\lambda\left(\frac{x}{\delta}\right)$$

Then , the limit of interest is $\quad \varepsilon \to 0 \quad \delta \to 0$

In terms of Laplace transforms then

$$L[\psi^*(t)](s) = \hat{\psi}(\varepsilon s) \qquad F[\lambda^*(x)](k) = \hat{\lambda}(\delta k)$$

rescaled MW equation $\quad \hat{\tilde{P}}^*(k,s) = \dfrac{1-\tilde{\psi}(\varepsilon s)}{s}\,\dfrac{1}{1-\tilde{\psi}(\varepsilon s)\,\hat{\lambda}(\delta k)}$

Gaussian Markovian case

Trapping pdf $\psi(t) = \mu\, e^{-\mu\tau}$ Jumps pdf $\lambda(x) = \frac{1}{\sqrt{2\pi\sigma}}\, e^{-x^2/2\sigma}$

$\tilde{\psi}(\varepsilon s) = \frac{1}{1+\varepsilon s < t >} \approx 1 - \varepsilon s < t > + \cdots$ 1/μ and σ transport scales

$\hat{\lambda}(\delta k) = e^{-\langle x^2\rangle k^2\delta^2/2} \approx 1 - \delta^2\langle x^2\rangle k^2/2 + \cdots$

Continuum distinguished limit $\varepsilon \to 0 \quad \delta \to 0 \qquad \chi = \frac{\langle x^2\rangle}{2\langle t\rangle}\frac{\delta^2}{\varepsilon}$ finite

$$s\hat{\tilde{P}} - 1 = -\chi\, k^2\, \hat{\tilde{P}}$$

Laplace transform $L\left[\partial_t \tilde{P}\right] = s\,\tilde{P} - \delta(x)$

Fourier transform $F\left[\partial^2{}_x \hat{P}\right] = -k^2\,\hat{P}$

$$\boxed{\partial_t P = \chi\, \partial_x^2 P}$$

Non-Gaussian, non-Markovian case

Trapping pdf with "memory" $\psi(t) \sim t^{-(\beta+1)}$ $\tilde{\psi}(\varepsilon s) \approx 1 - c_1\left(\varepsilon s\right)^\beta + \cdots$

Jumps pdf $\lambda(x) \sim |x|^{-(1+\alpha)}$ $\hat{\lambda}(\delta k) \approx 1 - c_2\left(\delta|k|\right)^\alpha + \cdots$

Continuum distinguished limit $\varepsilon \to 0 \quad \delta \to 0 \qquad \chi = \frac{\langle x^2\rangle}{2\langle t\rangle}\frac{\delta^\alpha}{\varepsilon^\beta}$

$$s^\beta\, \hat{\tilde{P}} - s^{\beta-1} = -\chi\,|k|^\alpha\, \hat{\tilde{P}}$$

Laplace transform $L\left[\partial_t^\beta \tilde{P}\right] = s^\beta\,\tilde{P} - s^{\beta-1}\,\delta(x)$

Fourier transform $F\left[\partial_{|x|}^\alpha \hat{P}\right] = -|k|^\alpha\,\hat{P}$

$$\boxed{\partial_t^\beta P = \chi\, \partial_{|x|}^\alpha P}$$

Fractional diffusion equation

Solution of fractional diffusion equation

Consider the initial value problem of the asymmetric fractional diffusion equation in an infinite domain

$$\partial_t \phi = \left[l \; {}_{-\infty}D_x^{\alpha} + r \; {}_{x}D_{\infty}^{\alpha} \right] \phi \qquad \phi(x, t=0) = \phi_0(x)$$

$$l = -\frac{(1-\theta)}{2\cos(\alpha\pi/2)} \qquad r = -\frac{(1+\theta)}{2\cos(\alpha\pi/2)} \qquad -1 \le \theta \le 1$$

Using $F\left[{}_{-\infty}D_x^{\alpha}\, \phi\right](k) = (-i\,k)^{\alpha}\, \phi(k)$ $\quad F\left[{}_{x}D_{\infty}^{\alpha}\, \phi\right](k) = (i\,k)^{\alpha}\, \phi(k)$

$$\hat{\phi}(k) = \hat{\phi}_0(k)\, \exp\left\{ t\left[l(-ik)^{\alpha} + r(ik)^{\alpha} \right] \right\}$$

Then, using $\quad (\mp i\,k)^{\alpha} = |k|^{\alpha} \exp\left(-i \frac{\alpha\pi}{2} \frac{k}{|k|} \right)$

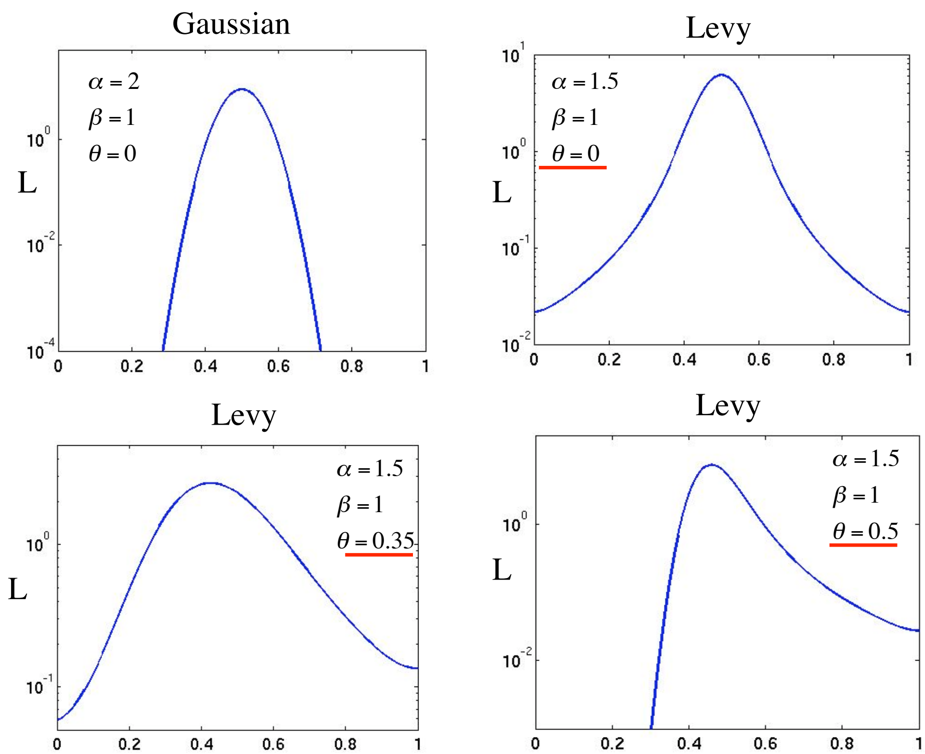
We can write the solution as $\quad \hat{\phi}(k) = \hat{\phi}_0(k)\, \hat{L}_{\alpha\theta}(k)$

where $\quad \hat{L}_{\alpha\theta}(k) = \exp\left\{ -\,t|k|^{\alpha} \left[1 + i\,\theta \frac{k}{|k|} \tan\left(\frac{\alpha\pi}{2} \right) \right] \right\}$

And using the convolution theorem

$$\phi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} L_{\alpha\theta}(x - x')\, \phi_0(x')\, dx'$$

Some examples



Probabilistic interpretation

$\phi(x,t) =$ Probability of being at x at time t

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{L_{\alpha\theta}(x-x')}_{\text{Transition probability}}\,\phi_0(x')\,dx'$$

In the case alpha=2, theta=0 the solution reduces to

$$\hat{L}_{\alpha\theta}(k) = \exp\left\{-t\,k^2\right\} \qquad L_{20}(x) = \frac{1}{2\sqrt{\pi t}}\exp\left\{-\frac{x^2}{4t}\right\}$$

Which as expected corresponds to a normalized Gaussian with variance $\sigma^2 = 2t$

Consistent with the Brownian random walk for $\alpha = 2$ the transition probability is Gaussian.

What is probabilistic interpretation for $\alpha \neq 2$?

Levy distributions

Consider a set of independent, identically distributed random variables

$$\{l_n\}_{n=1,2\cdots N} \qquad \mathrm{Prob}\left(l < l_n < l + dl\right) = p(l)$$

Given this, we would like to know what is the probability distribution of the normalized sum

$$S_N = \frac{1}{B_N}\sum_{n=1}^{N} l_n - A_N \qquad \mathrm{Prob}\left(x < S_N < x + dx\right) = P(x) = ??$$

In the context of a random walk the answer to this question gives the probability of finding a particle at a given point at a given time provided we know the probability distribution of the individual steps

An answer to this question is provided by the Central Limit Theorem according to which:

$$P(x) \underset{\mathrm{N}\to\infty}{\Rightarrow} \text{Gaussian} \qquad \text{provided} \qquad \left\langle l^2 \right\rangle = \int_{-\infty}^{\infty} l^2\, p(l)\, dl < \infty$$

However, what happens if $\left\langle l^2 \right\rangle = \infty$?????

A probability density is stable if there are constants a and b such that

$$p(a_1\, l + b_1) * p(a_1\, l + b_1) = \int_{-\infty}^{\infty} dl\, p\left[a_1 (z - l) + b_1\right] p(a_2 l + b_2) = p(az + b)$$

The importance of the stable distributions is given by the following Theorem (Levy-Khintchine)

A probability density L(x) can be a limiting distribution

$$S_N = \frac{1}{B_N}\sum_{n=1}^{N} l_n - A_N \qquad \mathrm{Prob}\left(x < S_N < x + dx\right) \underset{N\to\infty}{\Rightarrow} L(x)$$

only if it is stable

The Gaussian is stable, and according to the CLT we know that is the attractor of of all processes by finite variance.

However, the family of stable distributions is much bigger.

Levy-Khintchine canonical representation:

A probability density $$L_{\alpha\beta}(k) = \langle e^{ikx} \rangle = \int_{-\infty}^{\infty} dx\, L_{a\beta}(x)\, e^{ikx}$$

Is stable if and only if $$L_{\alpha\theta}(k) = \exp\left[i\gamma k + -c\,|k|^{\alpha}\left(1 + i\theta \frac{k}{|k|}\tan\left(\frac{\alpha\pi}{2}\right)\right)\right]$$

$0 < \alpha \le 2$ $\quad -1 < \theta < 1$

But according to our previous calculation this is the Green's fuunction of the space fractional diffusion equation.

So we have gone full circle: from stochastic process to fractional diffusion equations and back to stochastic processes. We first derived the fractional diffusion equation as a continuum limit for non-Gaussian random walks. Then, we solved the fractional equation, and showed that the solution is given in terms of Levy stable distributions.

Space-time fractional diffusion equation

$$ {}_0^C D_t^{\beta}\, \phi = \left[l\; {}_{-\infty}D_x^{\alpha} + r\; {}_x D_{\infty}^{\alpha} \right] \phi \qquad \phi(x, t=0) = \delta(x)$$

Symmetric case $$l = r = -\frac{1}{2\cos(\alpha\pi/2)}$$

Solution in terms of Fourier-Laplace transforms $$\phi(k,s) = \frac{s^{\beta-1}}{s^{\beta} + \chi|k|^{\alpha}}\, \phi(k, t=0)$$

$$\left[E_{\beta}(-c\, t^{\beta})\right](s) = \frac{s^{\beta-1}}{s^{\beta} + c} \qquad E_{\beta}(z) = \sum_n \frac{z^n}{\Gamma(\beta n + 1)}$$

Mittag-Leffler function

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ikx}\, E_{\beta}\left(-\chi\, |k|^{\alpha}\, t^{\beta}\right) dk$$

Self-similarity and non-diffusive scaling

Similarity variable $\eta = t^{-\beta/\alpha} x$ Fundamental solution $\phi(x,t) = t^{-\beta/\alpha} K(\eta)$

$$K(\eta) = \frac{1}{\pi}\int_0^\infty \cos(\eta z)\, E_\beta(-\chi z^\alpha)\, dz$$

$$\phi(x,\lambda t) = \frac{1}{\lambda^{\beta/\alpha}}\phi\left(\frac{x}{\lambda^{\beta/\alpha}}, t\right)$$ Self-similar scaling

Moments $\langle x^n \rangle = \int x^n \phi(x,t)\, dx = t^{n\beta/\alpha} \int \eta^n K(\eta)\, d\eta$

$$\langle x^n \rangle \sim t^{n\beta/\alpha} \qquad \frac{2\beta}{\alpha} \begin{cases} > 1 & \text{Super-diffusion} \\ < 1 & \text{Sub-diffusion} \end{cases}$$

α determines the asymptotic scaling in space

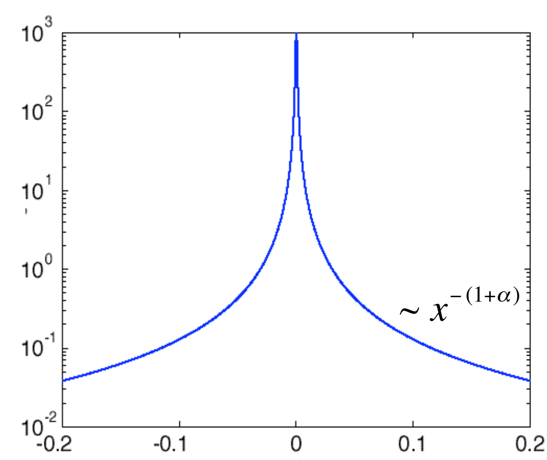
$$\phi(x,t) = t^{-\beta/\alpha} K(\eta)$$ $$\eta = t^{-\beta/\alpha} x$$ Space similarity variable

$$K(\eta) = \frac{1}{\pi}\int_0^\infty \cos(\eta z)\, E_\beta(-\chi z^\alpha)\, dz$$

$\eta \to \infty$ $\qquad K(\eta) \sim \eta^{-(1+\alpha)}$

For fixed t and large x

$$\phi(x, t_0) \sim x^{-(1+\alpha)}$$



β determines the asymptotic scaling in time

$$\phi(x,t) = |x|^{-1} \zeta^{-\beta/\alpha} K\left(\zeta^{-\beta/\alpha}\right)$$

$\zeta = t|x|^{-\alpha/\beta}$ Time similarity variable

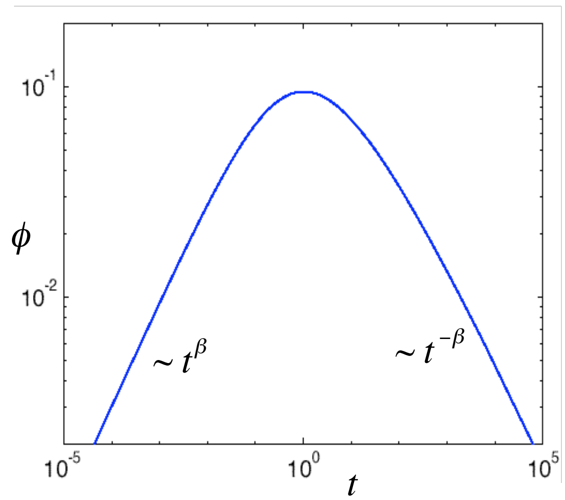
$$K(\eta) = \frac{1}{\pi}\int_0^\infty \cos(\eta z) E_\beta\left(-\chi z^\alpha\right) dz$$

$$\eta \to \infty \qquad K(\eta) \sim \eta^{-(1+\alpha)}$$

$$\eta \to 0 \qquad K(\eta) \sim 1 + \eta^{-(1-\alpha)}$$

For fixed x:

$$\phi(x_0,t) \sim \begin{cases} t^\beta \text{ for } t \sim 0 \\ t^{-\beta} \text{ for } t \to \infty \end{cases}$$

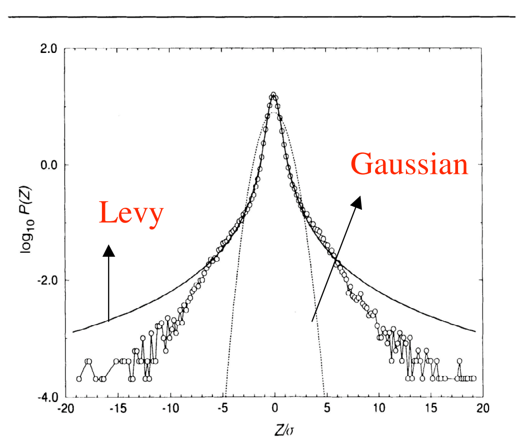


An example from finance

The simplest models of the fluctuations of assets prices assume an underlying geometric Brownian motion, and lead to Gaussian distribution of price variations.

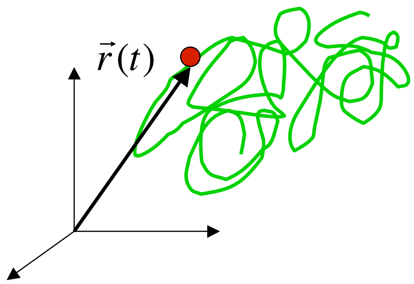
In 1963 by Mandelbrot pointed out the existence of "fat tails" due to large events. Based on this he proposed to use Levy stable distributions to model prices.

Mategna and Stanley showed that the probability distribution of price variations of the S&P500 strongly departs from a Gaussian and is well-fitted near the center by a Levy stable distribution of index α=1.4.



[Mategna-Stanley, *Nature* 1995]

Passive scalar diffusion



$\delta\vec{r}(t) = \vec{r}(t) - \vec{r}(0)$

$\langle\ \rangle$ = ensemble average

$M(t) = \langle \delta\vec{r} \rangle$ = mean

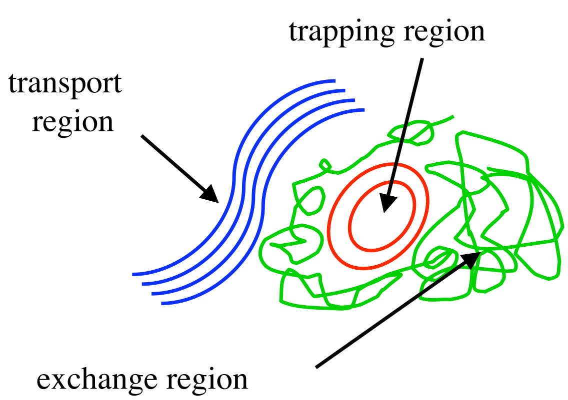
$\sigma^2(t) = \left\langle \left[\delta\vec{r} - \langle \delta\vec{r} \rangle \right]^2 \right\rangle$ = variance

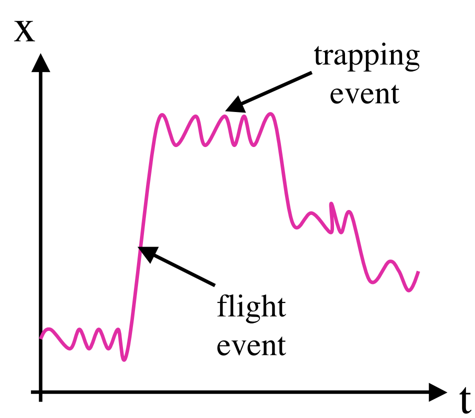
$P(\delta\vec{r}, t)$ = probability distribution

homogenous, isotropic turbulence → Brownian random walk → $\lim\limits_{t \to \infty} \begin{cases} M(t) = V\,t \\ \sigma^2(t) = D\,t \\ P(\delta\vec{r}, t) = \text{Gaussian} \end{cases}$

V= transport velocity D=diffusion coefficient

Coherent structures can give rise to anomalous diffusion

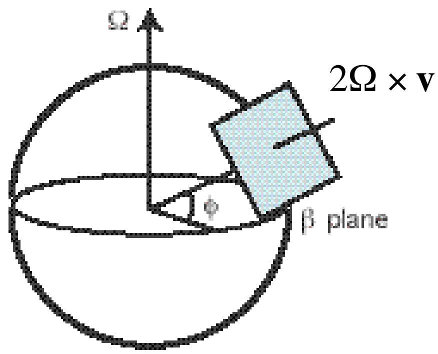


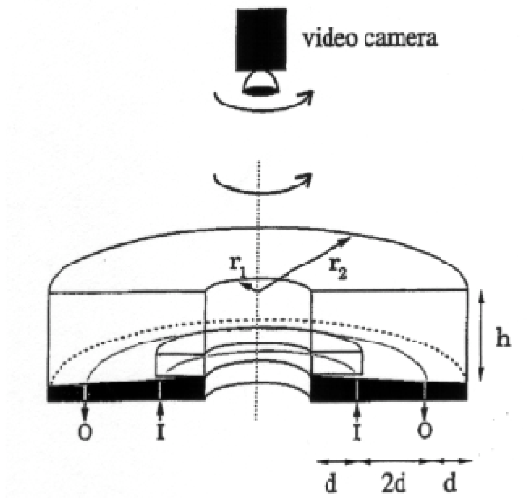


Coherent structures → correlations → $\lim\limits_{t \to \infty} \begin{cases} \sigma^2(t) \sim t^{\gamma} \\ P(x, t) = \text{non - Gaussian} \end{cases}$

An example from chaotic advection

Quasigeostrophic flow



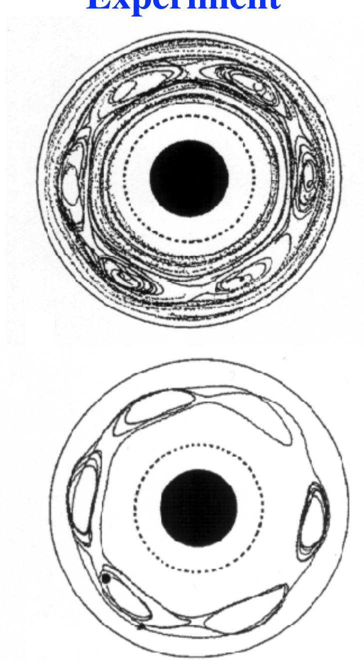


The rotating annulus experiment

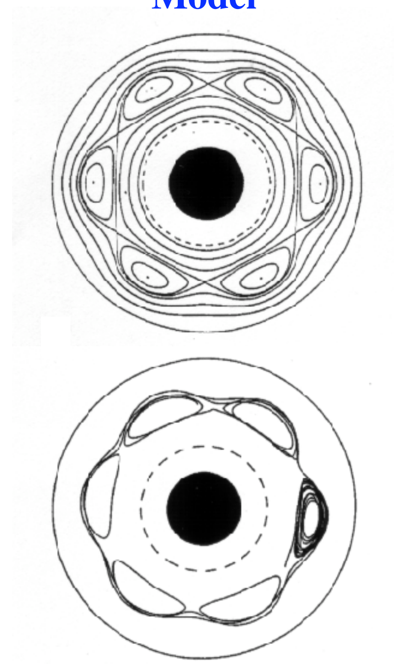
Solomon, Weeks, Swinney,
Phys. Rev. Lett. **71**, 3975 (1993).

Test particle transport

Experiment



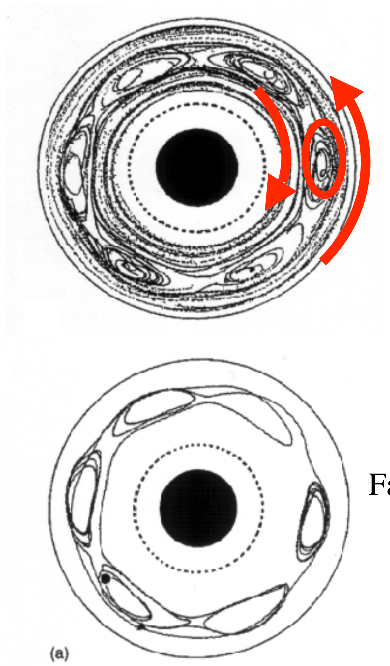
Model



Solomon, Weeks, Swinney, PRL, **71**, 3975 (1993)

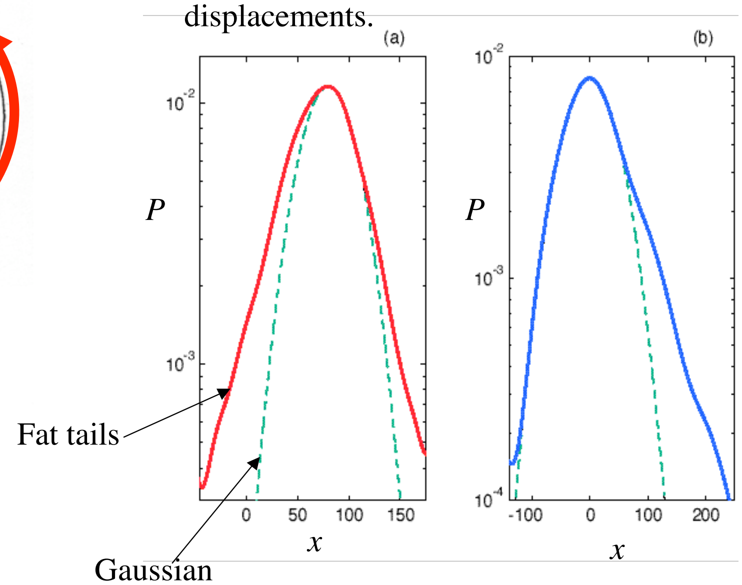
[del-Castillo-Negrete-1998]

Non-Gaussian distribution functions



Solomon, Weeks, Swinney, PRL, **71**, 3975 (1993)

Probability distribution functions of particle displacements.

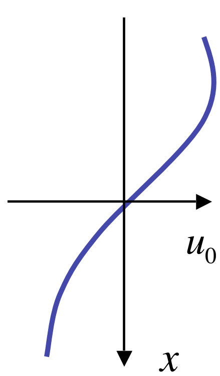


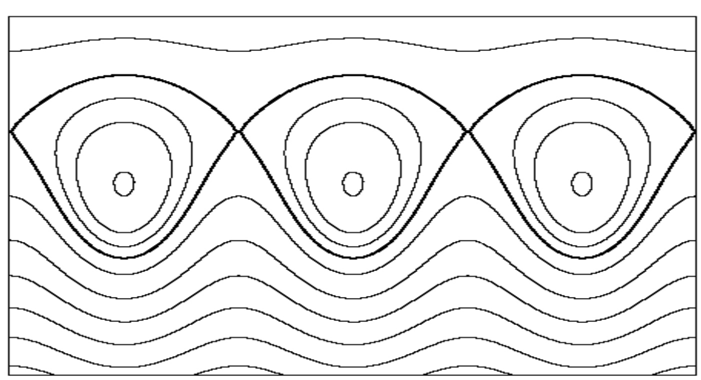
[del-Castillo-Negrete-1998]

Hamiltonian chaotic advection model

$$\frac{dx}{dt} = -\frac{\partial \phi}{\partial y} \qquad \frac{dy}{dt} = \frac{\partial \phi}{\partial x}$$

$$\phi = \underbrace{-\ln\left[\cosh x\right]}_{\text{tanh(x) zonal flow}} + \underbrace{\sum_{j=1}^{2} \varepsilon_j \; \varphi_j(x) \cos k_j (y - c_j t)}_{\text{regular neutral modes}}$$

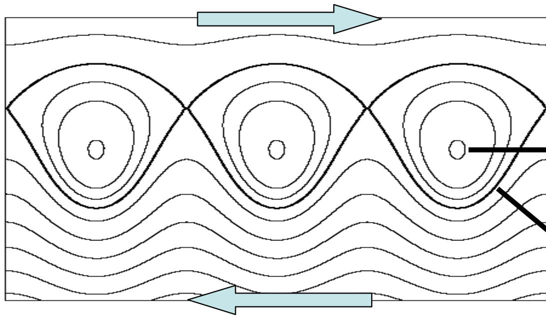




[del-Castillo-Negrete-1998]

integrable flow

$$\varphi_1 = -\ln\left[\cosh x\right] + \varepsilon_1\, \varphi_1(x)\cos k_1 y + c_1 x$$



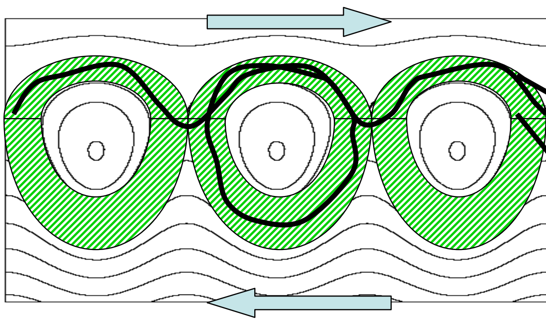
shear flow transport

trapping region

separatrix

chaotic transport

$$\varphi = \varphi_1 + \underbrace{\varepsilon_2\, \varphi_2(x)\cos\left(k_2 y - \omega t\right)}_{\text{nonintegrable perturbation}}$$

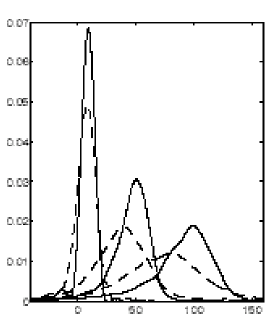


chaotic orbit

exchange region
stochastic layer

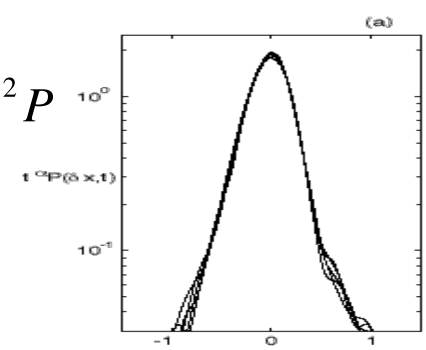
Self-similar probability distributions

P



δx

$t^{\gamma/2} P$



$$\eta = \left(\delta x - \left\langle \delta x \right\rangle\right) t^{-\gamma/2}$$

$$X = \delta x - \left\langle \delta x \right\rangle$$

collapse of curves $\Rightarrow$ $P^*(X,t) = t^{-\gamma/2} F\left(X\, t^{-\gamma/2}\right)$

$$\left\langle X^n \right\rangle \sim t^{n\gamma/2} \qquad P^*(X,t) = \lambda^{\gamma/2} P^*(\lambda^{\gamma/2} X, \lambda t)$$