Chapter 1

Preliminaries

This chapter is preliminary in character and contents. We give here the definitions and some properties of several fractional integrals and fractional derivatives of different types. Also we give the definition and properties of some special functions which will be used in this book.

More detailed information about the content of this chapter may be found, for example, in the works of Erdélyi *et al.* [209], Copson [149], Riesz [480], Doetsch [189], Sneddon [526], Zemanian [595], McBride [385], Samko *et al.* [501], Kiryakova [314], Podlubny [453], Butzer *et al.* [116–118], Kilbas *et al.* [309], Caponetto *et al.* [130], Diethelm [172], Mainardi [370], Monje *et al.* [407], Duarte Ortigueira [192] and Tarasov [543].

In general, the results we present in this chapter will be considered for "suitable functions". Precise details can be found, e.g., in the above mentioned references.

First of all, let $\Omega = [a,b]$ $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real axis $\mathbb{R}$. We denote by $L_p(a,b)$ $(1 \leq p \leq \infty)$ the set of those Lebesgue complex-valued measurable functions f on Ω for which $\|f\|_p < \infty$, where

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p} \quad (1 \leq p < \infty) \tag{1.0.1}$$

and

$$\|f\|_\infty = \operatorname{esssup}_{a \leq x \leq b} |f(x)|. \tag{1.0.2}$$

Here $\operatorname{esssup}|f(x)|$ is the essential maximum of the function $|f(x)|$.

1.1 Fourier and Laplace Transforms

In this section we present definitions and some properties of one- and multi-dimensional Fourier and Laplace transforms.

We begin with the one-dimensional case. The Fourier transform of a function $\varphi(x)$, of a real variable, is defined by

$$(\mathcal{F}\varphi)(\kappa) = \mathcal{F}[\varphi(x)](\kappa) = \hat{\varphi}(\kappa) = \int_{-\infty}^{\infty} e^{i\kappa x}\varphi(x)dx, \tag{1.1.1}$$

with $x, \kappa \in \mathbb{R}$. The inverse Fourier transform is given by the formula

$$(\mathcal{F}^{-1}g)(x) = \mathcal{F}^{-1}[g(\kappa)](x) = \frac{1}{2\pi}\hat{g}(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\kappa x}g(\kappa)d\kappa. \tag{1.1.2}$$

Each of the transforms (1.1.1) and (1.1.2) is inverse to the other one,

$$\mathcal{F}^{-1}\mathcal{F}\varphi = \varphi, \quad \mathcal{F}\mathcal{F}^{-1}g = g, \tag{1.1.3}$$

also the following simple relation is valid

$$(\mathcal{F}\mathcal{F}\varphi)(x) = \varphi(-x). \tag{1.1.4}$$

The rate of decrease of $(\mathcal{F}\varphi)(x)$ at infinity is connected with the smoothness of the function $\varphi(x)$.

Other well known properties of the Fourier transform are

$$\mathcal{F}[D^n\varphi(x)](\kappa) = (-i\kappa)^n(\mathcal{F}\varphi)(\kappa) \quad (n \in \mathbb{N}) \tag{1.1.5}$$

and

$$D^n(\mathcal{F}\varphi)(\kappa) = (i\kappa)^n\mathcal{F}[\varphi(x)](\kappa) \quad (n \in \mathbb{N}) \tag{1.1.6}$$

where D^n denotes the classical differential operator of order n.

The Fourier convolution operator of two functions h and φ is defined by the integral

$$h * \varphi = (h * \varphi)(x) = \int_{-\infty}^{\infty} h(x-t)\varphi(t)dt \quad (x \in \mathbb{R}). \tag{1.1.7}$$

It has the commutative property

$$h * \varphi = \varphi * h. \tag{1.1.8}$$

and is connected to the Fourier transform operator by

$$\left(\mathcal{F}(h * \varphi)\right)(\kappa) = (\mathcal{F}h)(\kappa) \cdot (\mathcal{F}\varphi)(\kappa). \tag{1.1.9}$$

The n-dimensional Fourier transform of a function $\varphi(\mathbf{x})$ of $\mathbf{x} \in \mathbb{R}^n$ is defined by

$$(\mathcal{F}\varphi)(\kappa) = \mathcal{F}[\varphi(\mathbf{x})](\kappa) = \hat{\varphi}(\kappa) = \int_{\mathbb{R}^n} e^{i\kappa\cdot\mathbf{x}}\varphi(\mathbf{x})d\mathbf{x}, \tag{1.1.10}$$

with $\mathbf{k} \in \mathbb{R}^n$, while the corresponding inverse Fourier transform is given by the formula

$$\begin{aligned}(\mathcal{F}^{-1}g)(\mathbf{x}) = \mathcal{F}^{-1}[g(\kappa)](\mathbf{x}) &= \frac{1}{(2\pi)^n}\hat{g}(-\mathbf{x}) \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\kappa}g(\kappa)d\kappa. \end{aligned} \tag{1.1.11}$$

The integrals in (1.1.10) and (1.1.11) have the same properties as those of the one-dimensional ones in (1.1.1) and (1.1.2). They converge absolutely, e.g., for functions $\varphi, g \in L_1(\mathbb{R}^n)$ and in the norm of the space $L_2(\mathbb{R}^n)$ for $\varphi, g \in L_2(\mathbb{R}^n)$.

If Δ is the n-dimensional Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{1.1.12}$$

then

$$(\mathcal{F}[\Delta\varphi])(\kappa) = -|\kappa|^2(\mathcal{F}\varphi)(\kappa) \quad (\kappa \in \mathbb{R}^n). \tag{1.1.13}$$

Analogous to (1.1.7), the Fourier convolution operator of two functions h and φ is defined by

$$h * \varphi = (h * \varphi)(\mathbf{x}) = \int_{\mathbb{R}^n} h(\mathbf{x} - \mathbf{t})\varphi(\mathbf{t})d\mathbf{t} \quad \left(\mathbf{x} \in \mathbb{R}^n\right), \tag{1.1.14}$$

whose Fourier transform is given by the formula (1.1.9), but with $\kappa \in \mathbb{R}^n$.

The Laplace transform of a function $\varphi(t)$ of a variable $t \in \mathbb{R}^+ = (0, \infty)$ is defined by

$$(\mathfrak{L}\varphi)(s) = \mathfrak{L}[\varphi(t)](s) = \tilde{\varphi}(s) = \int_0^\infty e^{-st}\varphi(t)dt \quad (s \in \mathbb{C}), \tag{1.1.15}$$

if the integral converges. Here $\mathbb{C}$ is the complex plane.

If the integral (1.1.15) is convergent at the point $s_0 \in \mathbb{C}$, then it converges absolutely for $s \in \mathbb{C}$ such that $\Re e(s) > \Re e(s_0)$. The infimum σ_φ of values s for which the Laplace integral (1.1.15) converges is called the abscissa of convergence.

The inverse Laplace transform is given for $x \in \mathbb{R}^+$ by the formula

$$(\mathfrak{L}^{-1}g)(x) = \mathfrak{L}^{-1}[g(s)](x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx}g(s)ds. \tag{1.1.16}$$

with $\gamma = \Re e(s) > \sigma_\varphi$.The direct and inverse Laplace transforms are inverse to each other for "sufficiently good" functions φ and g, that is

$$\mathfrak{L}^{-1}\mathfrak{L}\varphi = \varphi \quad \text{and} \quad \mathfrak{L}\mathfrak{L}^{-1}g = g. \tag{1.1.17}$$

Some simple properties of the Laplace transform analogous to those given for the Fourier transform are the following

$$\mathfrak{L}[D^n\varphi(t)](s) = s^n(\mathfrak{L}\varphi)(s) - \sum_{j=0}^{n-1} s^{n-j-1}(D^j\varphi)(0) \quad (n \in \mathbb{N}). \tag{1.1.18}$$

and

$$D^n(\mathfrak{L}\varphi)(s) = (-1)^n\mathfrak{L}[t^n\varphi(t)](s) \quad (n \in \mathbb{N}). \tag{1.1.19}$$

The Laplace convolution operator of two functions $h(t)$ and $\varphi(t)$, given on $\mathbb{R}^+$, is defined for $x \in \mathbb{R}^+$ by the integral

$$h * \varphi = (h * \varphi)(x) = \int_0^x h(x-t)\varphi(t)dt, \tag{1.1.20}$$

which has the commutative property

$$h * \varphi = \varphi * h. \tag{1.1.21}$$

and

$$(\mathfrak{L}(h * \varphi)(s) = (\mathfrak{L}h)(s) \cdot (\mathfrak{L}\varphi)(s). \tag{1.1.22}$$

The n-dimensional Laplace transform of a function $\varphi(\mathbf{t})$ of $\mathbf{t} \in \mathbb{R}^n_+$ is a simple generalization of the one-dimensional case, as with the Fourier transform.

1.2 Special Functions and Their Properties

In this section we present the definitions and some properties of special known functions as the Euler Gamma function, Mittag-Leffler functions, etc. To get a more extensive study about this topic consult, e.g., the above mentioned books.

1.2.1 ***The Gamma function and related special functions***

The Euler Gamma function $\Gamma(z)$ is defined by the so-called Euler integral of the second kind

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (\Re e(z) > 0), \tag{1.2.1}$$

where $t^{z-1} = e^{(z-1)\log(t)}$. This integral is convergent for all complex $z \in \mathbb{C}$ with $\Re e(z) > 0$. For this function we have the reduction formula

$$\Gamma(z+1) = z\Gamma(z) \qquad (\Re e(z) > 0); \tag{1.2.2}$$

using this relation, the Euler Gamma function is extended to the half-plane $\Re e(z) \leq 0$ $(\Re e(z) > -n; \quad n \in \mathbb{N}; \quad z \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\})$ by

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n}. \tag{1.2.3}$$

Here $(z)_n$ is the Pochhammer symbol, defined for complex $z \in \mathbb{C}$ and non-negative integer, with $n \in \mathbb{N}$, by

$$(z)_0 = 1 \quad \text{and} \quad (z)_n = z(z+1)\cdots(z+n-1). \tag{1.2.4}$$

Equations (1.2.2) and (1.2.4) yield

$$\Gamma(n+1) = (1)_n = n! \quad (n \in \mathbb{N}_0) \tag{1.2.5}$$

with (as usual) $0! = 1$.

We also indicate some other properties of the Gamma function such as

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (z \notin \mathbb{Z}_0;\ 0 < \Re e(z) < 1), \tag{1.2.6}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{1.2.7}$$

the Legendre duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \quad (z \in \mathbb{C}), \tag{1.2.8}$$

and Stirling's asymptotic formula, for $|\arg(z)| < \pi;\ |z| \to \infty$

$$\Gamma(z) = (2\pi)^{1/2} z^{z-1/2} e^{-z}\left[1 + O\left(\frac{1}{z}\right)\right]. \tag{1.2.9}$$

In particular, Eq. (1.2.9) implies the well known results

$$n! = (2\pi n)^{1/2}\left(\frac{n}{e}\right)^n\left[1 + O\left(\frac{1}{n}\right)\right] \quad (n \in \mathbb{N},\quad n \to \infty), \tag{1.2.10}$$

$$|\Gamma(x+iy)| = (2\pi)^{1/2}|x|^{x-1/2}e^{-x-\pi[1-\mathrm{sign(x)y}]/2}\left[1 + O\left(\frac{1}{x}\right)\right], \tag{1.2.11}$$

when $x \to \infty$, and

$$|\Gamma(x+iy)| = (2\pi)^{1/2}|y|^{x-1/2}e^{-x-\pi|y|/2}\left[1 + O\left(\frac{1}{y}\right)\right], \tag{1.2.12}$$

when $y \to \infty$.

The quotient expansion of two Gamma functions at infinity is given by

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + O\left(\frac{1}{z}\right)\right] \quad (|\arg(z+a)| < \pi;\ |z| \to \infty). \tag{1.2.13}$$

The digamma function is defined as the logarithmic derivative of the Gamma-function,

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (z \in \mathbb{C}). \tag{1.2.14}$$

This function has the property

$$\psi(z+m) = \psi(z) + \sum_{k=0}^{m-1} \frac{1}{z+k} \quad (z \in \mathbb{C};\ m \in \mathbb{N}), \tag{1.2.15}$$

which, for $m = 1$, yields

$$\psi(z+1) = \psi(z) + \frac{1}{z} \quad (z \in \mathbb{C}). \tag{1.2.16}$$

Another function, related with the digamma function, is the m-th polygamma function $\psi^m(z)$, which is given by

$$\psi^m(z) = \left(\frac{d}{dz}\right)^m \psi(z) \quad (z \in \mathbb{C}\backslash\mathbb{Z}_0) \tag{1.2.17}$$

The Beta function is defined by the Euler integral of the first kind

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt \quad (\Re e(z) > 0;\ \Re e(w) > 0), \tag{1.2.18}$$

This function is connected to the Gamma function by the relation

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \quad (z, w \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}). \tag{1.2.19}$$

The binomial coefficients are defined for $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ by

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \frac{(-1)^n(-\alpha)_n}{n!}. \tag{1.2.20}$$

In particular, when $\alpha = m, n \in \mathbb{N}_0 = \{0, 1, \cdots\}$, with $m \geq n$, we have

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} \tag{1.2.21}$$

and

$$\binom{m}{n} = 0 \quad (m, n \in \mathbb{N}_0; \quad 0 \leq m < n) \tag{1.2.22}$$

If $\alpha \notin \mathbb{Z}^- = \{-1, -2, -3, \cdots\}$, the formula (1.2.20) is represented via the Gamma function by

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} \quad (\alpha \in \mathbb{C}; \quad \alpha \notin \mathbb{Z}^-; \quad n \in \mathbb{N}_0). \tag{1.2.23}$$

Such a relation can be extended from $n \in \mathbb{N}_0$ to arbitrary complex $\beta \in \mathbb{C}$ by

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\beta+1)} \quad (\alpha, \beta \in \mathbb{C}; \quad \alpha \notin \mathbb{Z}^-). \tag{1.2.24}$$

For more information on Gamma and Beta functions we refer to the standard works [5, 40, 423].

1.2.2 *Hypergeometric functions*

The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined in the unit disk as the sum of the hypergeometric series

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.2.25}$$

where $|z| < 1$; $a, b \in \mathbb{C}$; $c \in \mathbb{C} \backslash \mathbb{Z}_0^-$, and $(a)_k$ is the Pochhammer symbol (1.2.4). Alternatively, the function can be given by the Euler integral representation

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \tag{1.2.26}$$

when $0 < \mathfrak{Re}(b) < \mathfrak{Re}(c)$ and $|\arg(1-z)| < \pi$.

The confluent hypergeometric function is defined by

$${}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.2.27}$$

where $z, a \in \mathbb{C}$, $c \in \mathbb{C} \backslash \mathbb{Z}^-$ and $c \neq 0$; but, in contrast to the hypergeometric function in (1.2.25), this series is convergent for any $z \in \mathbb{C}$. It has the integral representation

$${}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt} dt \tag{1.2.28}$$

for $0 < \mathfrak{Re}(a) < \mathfrak{Re}(c)$.

The Gauss hypergeometric series (1.2.25) and (1.2.27) are extended to the generalized hypergeometric series defined by

$$ {}_pF_q(a_1, \ldots,\ a_p; b_1, \ldots,\ b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{1.2.29} $$

where $a_l, b_j \in \mathbb{C}$, $b_j \neq 0, -1, -2, \ldots$ $(l = 1, \ldots, p;\ j = 1, \ldots, q)$. This series is absolutely convergent for all values of $z \in \mathbb{C}$ if $p \leq q$.

1.2.3 *Mittag-Leffler functions*

In this section we present the definitions and some properties of the Mittag-Leffler functions.

The Mittag-Leffler function of one parameter $E_\alpha(z)$ is defined by

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (z \in \mathbb{C};\ \ \Re e(\alpha) > 0)\,. \tag{1.2.30} $$

In particular, for $\alpha = 1, 2$ we have

$$ E_1(z) = e^z \ \text{ and } \ E_2(z) = \cosh(\sqrt{z}). \tag{1.2.31} $$

The Mittag-Leffler function of two parameters $E_{\alpha,\beta}(z)$, generalizing the one in (1.2.30), is defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (z,\ \beta \in \mathbb{C};\ \ \Re e(\alpha) > 0)\,. \tag{1.2.32} $$

In particular when $\beta = 1$,

$$ E_{\alpha,1}(z) = E_\alpha(z) \quad (z \in \mathbb{C};\ \ \Re e(\alpha) > 0) \tag{1.2.33} $$

and

$$ E_{1,2}(z) = \frac{e^z - 1}{z}, \ \text{ and } \ E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}. \tag{1.2.34} $$

The function $E_{\alpha,\beta}(z)$ has the integral representation

$$ E_{\alpha,\beta}(z) = \frac{1}{2\pi} \int_{\mathcal{C}} \frac{t^{\alpha-\beta} e^t}{t^\alpha - z} dt, \tag{1.2.35} $$

Here the path of integration $\mathcal{C}$ is a loop which starts and ends at $-\infty$ and encircles the circular disk $|t| \leq |z|^{1/\alpha}$ in the positive sense $|\arg(t)| \leq \pi$ on $\mathcal{C}$.

The following is a Mittag-Leffler function which generalizes the Mittag-Leffler functions (1.2.30) and (1.2.32) defined, for $z, \alpha, \beta, \rho \in \mathbb{C}$ and $\mathfrak{Re}(\alpha) > 0$, by

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \tag{1.2.36}$$

where $(\rho)_k$ is the Pochhammer symbol (1.2.4).

In particular, when $\rho = 1$, it coincides with the Mittag-Leffler function (1.2.32), that is,

$$E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) \quad (z \in \mathbb{C}). \tag{1.2.37}$$

When $\alpha = 1$, $E^{\rho}_{1,\beta}(z)$ coincides with the confluent hypergeometric function (1.2.27), apart from a constant factor $[\Gamma(\beta)]^{-1}$, i.e.,

$$E^{\rho}_{1,\beta}(z) = \frac{1}{\Gamma(\beta)} {}_1F_1(\rho; \beta; z). \tag{1.2.38}$$

1.3 Fractional Operators

In this section we give the definitions and some properties of fractional integrals and fractional derivatives of different kinds, such as Riemann-Liouville, Caputo, Liouville, Hadamard, Marchaud and Grünwald-Letnikov.

1.3.1 *Riemann-Liouville fractional integrals and fractional derivatives*

In this subsection we give the definitions of the Riemann-Liouville fractional integrals and fractional derivatives on a finite real interval and some of their properties.

Let $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals ${}^{\mathtt{RL}}I^{\alpha}_{a+}f$ and ${}^{\mathtt{RL}}I^{\alpha}_{b-}f$ of order

$\alpha \in \mathbb{C}$ ($\Re e(\alpha) > 0$) are defined by

$$\left({}^{\mathtt{RL}}I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (x > a;\ \Re e(\alpha) > 0) \tag{1.3.1}$$

and

$$\left({}^{\mathtt{RL}}I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}} \quad (x < b;\ \Re e(\alpha) > 0), \tag{1.3.2}$$

respectively. These integrals are called the left-sided and the right-sided fractional integrals.

The Riemann-Liouville fractional derivatives ${}^{\mathtt{RL}}D_{a+}^{\alpha} y$ and ${}^{\mathtt{RL}}D_{b-}^{\alpha} y$ of order $\alpha \in \mathbb{C}$ ($\Re e(\alpha) \geq 0$) are defined by

$$\left({}^{\mathtt{RL}}D_{a+}^{\alpha} y\right)(x) = \left(\frac{d}{dx}\right)^n \left({}^{\mathtt{RL}}I_{a+}^{n-\alpha} y\right)(x) \quad (x > a) \tag{1.3.3}$$

and

$$\left({}^{\mathtt{RL}}D_{b-}^{\alpha} y\right)(x) = \left(-\frac{d}{dx}\right)^n \left({}^{\mathtt{RL}}I_{b-}^{n-\alpha} y\right)(x) \quad (x < b), \tag{1.3.4}$$

respectively, with $n = -[-\Re e(\alpha)]$, where $[\bullet]$ means the integral part of the argument, that is

$$n = \begin{cases} [\Re e(\alpha)] + 1 & \text{for } \alpha \notin \mathbb{N}_0, \\ \alpha & \text{for } \alpha \in \mathbb{N}_0. \end{cases} \tag{1.3.5}$$

In particular, when $\alpha = n \in \mathbb{N}_0$, then

$$({}^{\mathtt{RL}}D_{a+}^{0} y)(x) = ({}^{\mathtt{RL}}D_{b-}^{0} y)(x) = y(x), \tag{1.3.6}$$

$$({}^{\mathtt{RL}}D_{a+}^{n} y)(x) = y^{(n)}(x), \quad ({}^{\mathtt{RL}}D_{b-}^{n} y)(x) = (-1)^n y^{(n)}(x), \tag{1.3.7}$$

where $y^{(n)}(x)$ is the classical derivative of $y(x)$ of order n.

The particular cases when $a = 0$ in the left-sided fractional integral and derivative of Riemann-Liouville are often used in the literature, because in this case such fractional operators have a straightforward Laplace transform. For the sake of simplicity, this special case will be written in this book with any of the following nomenclature

$$ {}^{\mathtt{RL}}I^{\alpha}_{0+}f = {}^{\mathtt{RL}}I^{\alpha}f = J^{\alpha}f \tag{1.3.8} $$

and

$$ {}^{\mathtt{RL}}D^{\alpha}_{0+}f = {}^{\mathtt{RL}}D^{\alpha}f \tag{1.3.9} $$

If $\alpha, \beta \in \mathbb{C}$, $\mathfrak{Re}(\alpha) \geq 0$ and $\mathfrak{Re}(\beta) > 0$, the following properties can be directly verified:

$$ \left({}^{\mathtt{RL}}I^{\alpha}_{a+}(t-a)^{\beta}\right)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(x-a)^{\beta+\alpha}, \tag{1.3.10} $$

$$ \left({}^{\mathtt{RL}}D^{\alpha}_{a+}(t-a)^{\beta}\right)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}, \tag{1.3.11} $$

$$ \left({}^{\mathtt{RL}}I^{\alpha}_{b-}(b-t)^{\beta}\right)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(b-x)^{\beta+\alpha}, \tag{1.3.12} $$

$$ \left({}^{\mathtt{RL}}D^{\alpha}_{b-}(b-t)^{\beta}\right)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-x)^{\beta-\alpha}. \tag{1.3.13} $$

For $0 < \mathfrak{Re}(\alpha) < 1$ this reduces to

$$ \left({}^{\mathtt{RL}}D^{\alpha}_{a+}1\right)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad \left({}^{\mathtt{RL}}D^{\alpha}_{b-}1\right)(x) = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)}, \tag{1.3.14} $$

and for $j = 1, 2, \ldots, n = -[-\mathfrak{Re}(\alpha)]$, we obtain

$$ \left({}^{\mathtt{RL}}D^{\alpha}_{a+}(t-a)^{\alpha-j}\right)(x) = 0, \quad \left({}^{\mathtt{RL}}D^{\alpha}_{b-}(b-t)^{\alpha-j}\right)(x) = 0. \tag{1.3.15} $$

From (1.3.15) we derive that the equality $(D^{\alpha}_{a+}y)(x) = 0$ is valid if, and only if,

$$ y(x) = \sum_{j=1}^{n} c_j (x-a)^{\alpha-j}, $$

where $n = [\mathfrak{Re}(\alpha)] + 1$ and $c_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ are arbitrary constants. In particular, when $0 < \mathfrak{Re}(\alpha) \leq 1$, the relation $({}^{\mathtt{RL}}D^{\alpha}_{a+}y)(x) = 0$ holds if, and only if, $y(x) = c(x-a)^{\alpha-1}$ with any $c \in \mathbb{R}$.

Likewise, the equality $({}^{\mathrm{RL}}D^{\alpha}_{b-}y)(x) = 0$ is valid if, and only if,

$$y(x) = \sum_{j=1}^{n} d_j (b-x)^{\alpha-j},$$

where $d_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ are arbitrary constants. In particular, when $0 < \Re e(\alpha) \le 1$, the relation $({}^{\mathrm{RL}}D^{\alpha}_{b-}y)(x) = 0$ holds if, and only if, $y(x) = d(b-x)^{\alpha-1}$ with any $d \in \mathbb{R}$.

The next results give us an alternative representation of the fractional derivatives ${}^{\mathrm{RL}}D^{\alpha}_{a+}$ and ${}^{\mathrm{RL}}D^{\alpha}_{b-}$, $\Re e(\alpha) \ge 0$, $n = [\Re e(\alpha)] + 1$, for suitable functions $y(x)$

$$({}^{\mathrm{RL}}D^{\alpha}_{a+}y)(x) = \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} \tag{1.3.16}$$

and

$$({}^{\mathrm{RL}}D^{\alpha}_{b-}y)(x) = \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{\Gamma(1+k-\alpha)}(b-x)^{k-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)dt}{(t-x)^{\alpha-n+1}}. \tag{1.3.17}$$

The semigroup property of the fractional integration operators ${}^{\mathrm{RL}}I^{\alpha}_{a+}$ and ${}^{\mathrm{RL}}I^{\alpha}_{b-}$ establishes that, if $\Re e(\alpha) > 0$ and $\Re e(\beta) > 0$, then the equations

$$({}^{\mathrm{RL}}I^{\alpha}_{a+}\,{}^{\mathrm{RL}}I^{\beta}_{a+}f)(x) = ({}^{\mathrm{RL}}I^{\alpha+\beta}_{a+}f)(x) \text{ and } ({}^{\mathrm{RL}}I^{\alpha}_{b-}\,{}^{\mathrm{RL}}I^{\beta}_{b-}f)(x) = ({}^{\mathrm{RL}}I^{\alpha+\beta}_{b-}f)(x) \tag{1.3.18}$$

are satisfied at almost every point $x \in [a,b]$ for $f(x) \in L_p(a,b)$ $(1 \le p \le \infty)$. If $\alpha + \beta > 1$, then the relations in (1.3.18) hold at any point of $[a,b]$.

Similarly, we have the following index rule

$$({}^{\mathrm{RL}}D^{\alpha}_{a+}\,{}^{\mathrm{RL}}D^{\beta}_{a+}f)(x) = ({}^{\mathrm{RL}}D^{\alpha+\beta}_{a+}f)(x) - \sum_{j=1}^{m} ({}^{\mathrm{RL}}D^{\beta-j}_{a+}f)(a+)\frac{(x-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)}, \tag{1.3.19}$$

if $\alpha, \beta > 0$ such that $n-1 < \alpha \le n$, $m-1 < \beta \le m$ $(n, m \in \mathbb{N})$ and $\alpha + \beta < n$.

For $f(x) \in L_p(a,b)$ $(1 \le p \le \infty)$, the composition relations

$$({}^{\mathrm{RL}}D^{\beta}_{a+}\,{}^{\mathrm{RL}}I^{\alpha}_{a+}f)(x) = I^{\alpha-\beta}_{a+}f(x) \text{ and } ({}^{\mathrm{RL}}D^{\beta}_{b-}\,{}^{\mathrm{RL}}I^{\alpha}_{b-}f)(x) = {}^{\mathrm{RL}}I^{\alpha-\beta}_{b-}f(x) \tag{1.3.20}$$

between fractional differentiation and fractional integration operators hold almost everywhere on $[a,b]$ if $\mathfrak{Re}(\alpha) > \mathfrak{Re}(\beta) > 0$. In particular, when $\beta = k \in \mathbb{N}$ and $\mathfrak{Re}(\alpha) > k$, then

$$({}^{\mathrm{RL}}D^{k\,\mathrm{RL}}I^{\alpha}_{a+}f)(x) = {}^{\mathrm{RL}}I^{\alpha-k}_{a+}f(x) \text{ and } ({}^{\mathrm{RL}}D^{k\,\mathrm{RL}}I^{\alpha}_{b-}f)(x) = (-1)^{k\,\mathrm{RL}}I^{\alpha-k}_{b-}f(x). \tag{1.3.21}$$

So, the fractional differentiation is an operation inverse to the fractional integration from the left, i.e., if $\mathfrak{Re}(\alpha) > 0$, then the equalities

$$({}^{\mathrm{RL}}D^{\alpha}_{a+}\,{}^{\mathrm{RL}}I^{\alpha}_{a+}f)(x) = f(x) \text{ and } ({}^{\mathrm{RL}}D^{\alpha}_{b-}\,{}^{\mathrm{RL}}I^{\alpha}_{b-}f)(x) = f(x) \tag{1.3.22}$$

hold almost everywhere on $[a,b]$.

On the other hand, if $\mathfrak{Re}(\alpha) > 0$, $n = [\mathfrak{Re}(\alpha)] + 1$ and $f_{n-\alpha}(x) = ({}^{\mathrm{RL}}I^{n-\alpha}_{a+}f)(x)$, the relation

$$({}^{\mathrm{RL}}I^{\alpha}_{a+}\,{}^{\mathrm{RL}}D^{\alpha}_{a+}f)(x) = f(x) - \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-\alpha}(a)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j} \tag{1.3.23}$$

holds almost everywhere on $[a,b]$. Also, if $g_{n-\alpha}(x) = ({}^{\mathrm{RL}}I^{n-\alpha}_{b-}g)(x)$, then the formula

$$({}^{\mathrm{RL}}I^{\alpha}_{b-}\,{}^{\mathrm{RL}}D^{\alpha}_{b-}g)(x) = g(x) - \sum_{j=1}^{n} \frac{(-1)^{n-j}g^{(n-j)}_{n-\alpha}(a)}{\Gamma(\alpha-j+1)}(b-x)^{\alpha-j} \tag{1.3.24}$$

holds almost everywhere on $[a,b]$.

Let $\mathfrak{Re}(\alpha) \geq 0$, $m \in \mathbb{N}$ and $D = d/dx$, then if the fractional derivatives $(D^{\alpha}_{a+}y)(x)$ and $({}^{\mathrm{RL}}D^{\alpha+m}_{a+}y)(x)$ exist, we have

$$({}^{\mathrm{RL}}D^{m}\ {}^{\mathrm{RL}}D^{\alpha}_{a+}y)(x) = ({}^{\mathrm{RL}}D^{\alpha+m}_{a+}y)(x), \tag{1.3.25}$$

and, if the fractional derivatives $({}^{\mathrm{RL}}D^{\alpha}_{b-}y)(x)$ and $({}^{\mathrm{RL}}D^{\alpha+m}_{b-}y)(x)$ exist, then

$$({}^{\mathrm{RL}}D^{m}\ {}^{\mathrm{RL}}D^{\alpha}_{b-}y)(x) = (-1)^{m}({}^{\mathrm{RL}}D^{\alpha+m}_{b-}y)(x). \tag{1.3.26}$$

In connection with the Laplace transform, if $\mathfrak{Re}(\alpha) > 0$ and $n = [\mathfrak{Re}(\alpha)]+1$, we have

$$(\mathfrak{L}\,{}^{\mathrm{RL}}D^{\alpha}_{0+}y)(s) = s^{\alpha}(\mathfrak{L}y)(s) - \sum_{k=0}^{n-1} s^{n-k-1}D^{k}({}^{\mathrm{RL}}I^{n-\alpha}_{0+}y)(0+) \tag{1.3.27}$$

for $(\mathfrak{Re}(s) > q_0)$.

The rules for fractional integration by parts read as follows.

(a) If $\varphi(x) \in L_p(a,b)$ and $\psi(x) \in L_q(a,b)$, then

$$\int_a^b \varphi(x) \left({}^{\mathrm{RL}}I^{\alpha}_{a+}\psi\right)(x)dx = \int_a^b \psi(x) \left({}^{\mathrm{RL}}I^{\alpha}_{b-}\varphi\right)(x)dx. \tag{1.3.28}$$

(b) If $f(x) = \left({}^{\mathrm{RL}}I^{\alpha}_{b-}h_1\right)(x)$ with some $h_1(x) \in L_p(a,b)$ and $g(x) = \left({}^{\mathrm{RL}}I^{\alpha}_{a+}h_2\right)(x)$ with some $h_2(x) \in L_q(a,b)$, then

$$\int_a^b f(x) \left({}^{\mathrm{RL}}D^{\alpha}_{a+}g\right)(x)dx = \int_a^b g(x) \left({}^{\mathrm{RL}}D^{\alpha}_{b-}f\right)(x)dx. \tag{1.3.29}$$

Here we assume $\alpha > 0$, $p \geq 1$, $q \geq 1$, and $(1/p) + (1/q) \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $(1/p) + (1/q) = 1 + \alpha$).

The generalized fractional Leibniz formula for the Riemann-Liouille derivative, applied to suitable functions on $[a,b]$, reads

$$\left[{}^{\mathrm{RL}}D^{\alpha}_{a+}(fg)\right](x) = \sum_{j=0}^{\infty} \binom{\alpha}{j} ({}^{\mathrm{RL}}D^{\alpha-j}_{a+}f)(x)(D^j g)(x), \tag{1.3.30}$$

where $\alpha > 0$. Below, we present three particular cases to illustrate this property.

(a) Let $0 < \alpha < 1$, $f(x) = x$ and $g(x)$ a suitable function. Then

$$[{}^{\mathrm{RL}}D^{\alpha}_{0+}(fg)](x) = x({}^{\mathrm{RL}}D^{\alpha}_{0+}g)(x) + ({}^{\mathrm{RL}}I^{1-\alpha}_{0+}g)(x) \tag{1.3.31}$$

(b) Let $0 < \alpha < 1$, $f(x) = x^{\alpha-1}$ and $g(x)$ a suitable function. Then

$$\left[{}^{\mathrm{RL}}D^{\alpha}_{0+}(fg)\right](x) = \sum_{j=1}^{\infty} \left(\begin{array}{c}\alpha\\ j\end{array}\right) \frac{\Gamma(\alpha)}{\Gamma(j)} x^{j-1} g^{(j)}(x) \tag{1.3.32}$$

(c) Let $p \in \mathbb{N}$, $\alpha > 0$, and $f(x)$ a suitable function. Then

$$({}^{\mathrm{RL}}D^{\alpha}_{0+}t^p f)(x) = \sum_{j=0}^{p} \left(\begin{array}{c}\alpha\\ j\end{array}\right) (D^j x^p)({}^{\mathrm{RL}}D^{\alpha-j}_{0+}f)(x) \tag{1.3.33}$$

The computation of a fractional Riemann-Liouville derivative of the composition of two suitable functions can be very complicated. The corresponding formula

$$\left[{}^{\mathrm{RL}}D^{\alpha}_{a+}(f(g))\right](x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} f(g(x)) \tag{1.3.34}$$
$$+\sum_{j=1}^{\infty} \binom{\alpha}{j} \frac{j!(x-a)^{j-\alpha}}{\Gamma(j+1-\alpha)} \sum_{r=1}^{j} [D^i f(g)](x) \sum \prod_{r=1}^{j} \frac{1}{a_r!} \left(\frac{(D^r g)(x)}{r!}\right)^{a_r},$$

where $\sum_{r=1}^{j} r a_r = j$ and $\sum_{r=1}^{j} a_r = i$, exhibits the complicated structure very clearly.

The above relation is a consequence of the application of (1.3.30) and the well known Faá di Bruno formula (see, e.g., formula (24.1.2) in [5]) for a natural n, *viz.*

$$(D^n f(g))(x) = \sum_{m=0}^{n} (D^m f)(g(x)) \tag{1.3.35}$$
$$\times \sum (n; a_1, a_2, \ldots, a_n) \{(Dg)(x)\}^{a_1} \left\{(D^2 g)(x)\right\}^{a_2} \cdots \{(D^n g)(x)\}^{a_n}$$

summed over $a_1 + 2a_2 + \ldots + na_n$ and $a_1 + a_2 + \ldots + a_n = m$, see also Eqs. (5.2.26) and (5.2.29).

1.3.2 *Caputo fractional derivatives*

The Caputo fractional derivatives are closely related to the Riemann-Liouville derivatives. Let $[a, b]$ be a finite interval of the real line $\mathbb{R}$. For $\alpha \in \mathbb{C}$ ($\Re e(\alpha) \geq 0$) the Caputo fractional derivatives are defined by

$$({}^{\mathrm{C}}D^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} = ({}^{\mathrm{RL}}I^{n-\alpha}_{a+} D^n y)(x) \tag{1.3.36}$$

and

$$({}^{\mathrm{C}}D^{\alpha}_{b-}y)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)dt}{(t-x)^{\alpha-n+1}} = (-1)^n ({}^{\mathrm{RL}}I^{n-\alpha}_{b-} D^n y)(x), \tag{1.3.37}$$

where $D = d/dx$ and $n = -[-\Re e(\alpha)]$, i.e., $n = [\Re e(\alpha)] + 1$ for $\alpha \notin \mathbb{N}_0$, and $n = \alpha$ for $\alpha \in \mathbb{N}_0$. These derivatives are called left-sided and right-sided Caputo fractional derivatives of order α.

In particular, when $0 < \Re e(\alpha) < 1$ then

$$({}^{\mathtt{C}}D^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{y'(t)dt}{(x-t)^{\alpha}} = ({}^{\mathtt{RL}}I^{1-\alpha}_{a+}Dy)(x) \tag{1.3.38}$$

and

$$({}^{\mathtt{C}}D^{\alpha}_{b-}y)(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^b \frac{y'(t)dt}{(t-x)^{\alpha}} = -({}^{\mathtt{RL}}I^{1-\alpha}_{b-}Dy)(x). \tag{1.3.39}$$

The connections between the Caputo and the Riemann-Liouville derivatives are given by the relations

$$({}^{\mathtt{C}}D^{\alpha}_{a+}y)(x) = \left({}^{\mathtt{RL}}D^{\alpha}_{a+}\left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k\right]\right)(x) \tag{1.3.40}$$

and

$$({}^{\mathtt{C}}D^{\alpha}_{b-}y)(x) = \left({}^{\mathtt{RL}}D^{\alpha}_{b-}\left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(b)}{k!}(b-t)^k\right]\right)(x), \tag{1.3.41}$$

respectively.

In particular, when $0 < \Re e(\alpha) < 1$, the relations (1.3.40) and (1.3.41) take the following forms

$$({}^{\mathtt{C}}D^{\alpha}_{a+}y)(x) = \left({}^{\mathtt{RL}}D^{\alpha}_{a+}[y(t) - y(a)]\right)(x), \tag{1.3.42}$$

$$({}^{\mathtt{C}}D^{\alpha}_{b-}y)(x) = \left({}^{\mathtt{RL}}D^{\alpha}_{b-}[y(t) - y(b)]\right)(x). \tag{1.3.43}$$

If $\alpha = n \in \mathbb{N}_0$ and the usual derivative $y^{(n)}(x)$ of order n exists, then $({}^{\mathtt{C}}D^{n}_{a+}y)(x)$ coincides with $y^{(n)}(x)$, while $({}^{\mathtt{C}}D^{n}_{b-}y)(x)$ coincides with $y^{(n)}(x)$ up to the constant factor $(-1)^n$, i.e.,

$$({}^{\mathtt{C}}D^{n}_{a+}y)(x) = y^{(n)}(x) \text{ and } ({}^{\mathtt{C}}D^{n}_{b-}y)(x) = (-1)^n y^{(n)}(x) \quad (n \in \mathbb{N}). \tag{1.3.44}$$

The Caputo derivatives $({}^{\mathtt{C}}D^{\alpha}_{a+}y)(x)$ and $({}^{\mathtt{C}}D^{\alpha}_{b-}y)(x)$ have properties similar to those given in Eqs. (1.3.11) and (1.3.13) for the Riemann-Liouville fractional derivatives. If $\Re e(\alpha) > 0$, $n = -[-\Re e(\alpha)]$ is given by (1.3.5) and $\Re e(\beta) > n - 1$, then

$$\left({}^{\mathtt{C}}D^{\alpha}_{a+}(t-a)^{\beta}\right)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha} \tag{1.3.45}$$

and

$$\left({}^{\mathtt{C}}D^{\alpha}_{b-}(b-t)^{\beta}\right)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-x)^{\beta-\alpha}. \tag{1.3.46}$$

However, for $k = 0, 1, \ldots, n-1$, we have

$$({}^{\mathtt{C}}D^{\alpha}_{a+}(t-a)^{k})(x) = 0 \text{ and } ({}^{\mathtt{C}}D^{\alpha}_{b-}(t-a)^{k})(x) = 0. \tag{1.3.47}$$

In particular,

$$({}^{\mathtt{C}}D^{\alpha}_{a+}1)(x) = 0 \text{ and } ({}^{\mathtt{C}}D^{\alpha}_{b-}1)(x) = 0. \tag{1.3.48}$$

On the other hand, if $\Re e(\alpha) > 0$ and $\lambda > 0$, then

$$\left({}^{\mathtt{C}}D^{\alpha}_{a+}e^{\lambda t}\right)(x) \neq \lambda^{\alpha}e^{\lambda x} \tag{1.3.49}$$

for any $a \in \mathbb{R}$.

Let $\Re e(\alpha) > 0$ and let $y(x)$ a suitable function, for example $y(x) \in C[a,b]$. Then If $\Re e(\alpha) \notin \mathbb{N}$ or $\alpha \in \mathbb{N}$, the Caputo fractional differentiation operators ${}^{\mathtt{C}}D^{\alpha}_{a+}$ and ${}^{\mathtt{C}}D^{\alpha}_{b-}$ provide operations inverse to the Riemann-Liouville fractional integration operators I^{α}_{a+} and I^{α}_{b-} from the left, that is

$$({}^{\mathtt{C}}D^{\alpha}_{a+}\,{}^{\mathtt{RL}}I^{\alpha}_{a+}y)(x) = y(x) \text{ and } ({}^{\mathtt{C}}D^{\alpha}_{b-}\,{}^{\mathtt{RL}}I^{\alpha}_{b-}y)(x) = y(x). \tag{1.3.50}$$

However, when $\Re e(\alpha) \in \mathbb{N}$ and $\Im m(\alpha) \neq 0$, we have

$$({}^{\mathtt{C}}D^{\alpha}_{a+}\,{}^{\mathtt{RL}}I^{\alpha}_{a+}y)(x) = y(x) - \frac{({}^{\mathtt{RL}}I^{\alpha+1-n}_{a+}y)(a+)}{\Gamma(n-\alpha)}(x-a)^{n-\alpha} \tag{1.3.51}$$

and

$$({}^{\mathtt{C}}D^{\alpha}_{b-}\,{}^{\mathtt{RL}}I^{\alpha}_{a+}y)(x) = y(x) - \frac{({}^{\mathtt{RL}}I^{\alpha+1-n}_{b-}y)(b-)}{\Gamma(n-\alpha)}(b-x)^{n-\alpha}. \tag{1.3.52}$$

On the other hand, if $\Re e(\alpha) > 0$ and $n = -[-\Re e(\alpha)]$ is given by (1.3.5), then under sufficiently good conditions for $y(x)$

$$({}^{\mathtt{RL}}I^{\alpha}_{a+}\ {}^{\mathtt{C}}D^{\alpha}_{a+}y)(x) = y(x) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(x-a)^{k} \tag{1.3.53}$$

and

$$({}^{\mathtt{RL}}I^{\alpha}_{b-}\ {}^{\mathtt{C}}D^{\alpha}_{b-}y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{k!}(b-x)^k. \tag{1.3.54}$$

In particular, if $0 < \Re e(\alpha) \leq 1$, then

$$({}^{\mathtt{RL}}I^{\alpha}_{a+}\ {}^{\mathtt{C}}D^{\alpha}_{a+}y)(x) = y(x) - y(a) \text{ and } ({}^{\mathtt{RL}}I^{\alpha}_{b-}\ {}^{\mathtt{C}}D^{\alpha}_{b-}y)(x) = y(x) - y(b). \tag{1.3.55}$$

Under suitable conditions, the Laplace transform of the Caputo fractional derivative ${}^{\mathtt{C}}D^{\alpha}_{0+}y$ is given by

$$(\mathfrak{L}{}^{\mathtt{C}}D^{\alpha}_{0+}y)(s) = s^{\alpha}(\mathfrak{L}y)(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}(D^k y)(0). \tag{1.3.56}$$

In particular, if $0 < \alpha \leq 1$, then

$$(\mathfrak{L}{}^{\mathtt{C}}D^{\alpha}_{0+}y)(s) = s^{\alpha}(\mathfrak{L}y)(s) - s^{\alpha-1}y(0). \tag{1.3.57}$$

We have defined the Caputo derivatives on a finite interval $[a, b]$. Formulas (1.3.36) and (1.3.37) can be used for the definition of the Caputo fractional derivatives on the whole axis $\mathbb{R}$. Thus the corresponding Caputo fractional derivative of order $\alpha \in \mathbb{C}$ (with $\Re e(\alpha) > 0$ and $\alpha \notin \mathbb{N}$) can be defined as follows

$$({}^{\mathtt{C}}D^{\alpha}_{+}y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{x} \frac{y^{(n)}(t)dt}{(x-t)^{\alpha+1-n}} \tag{1.3.58}$$

and

$$({}^{\mathtt{C}}D^{\alpha}_{-}y)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{x}^{\infty} \frac{y^{(n)}(t)dt}{(t-x)^{\alpha+1-n}}, \tag{1.3.59}$$

with $x \in \mathbb{R}$.

The Caputo derivatives $({}^{\mathtt{C}}D^{\alpha}_{+}y)(x)$ and $({}^{\mathtt{C}}D^{\alpha}_{-}y)(x)$ have properties similar to those that we will describe below for the operators known as Liouville derivatives. In particular, we mention the identities

$$\left({}^{\mathtt{C}}D^{\alpha}_{+}e^{\lambda t}\right)(x) = \lambda^{\alpha}e^{\lambda x} \text{ and } \left({}^{\mathtt{C}}D^{\alpha}_{-}e^{-\lambda t}\right)(x) = \lambda^{\alpha}e^{-\lambda x}. \tag{1.3.60}$$

1.3.3 Liouville fractional integrals and fractional derivatives. Marchaud derivatives

First of all, we present the definitions and some properties of the Liouville fractional integrals and fractional derivatives on the whole axis $\mathbb{R}$. More detailed information may be found in the bibliography.

The Liouville fractional integrals on $\mathbb{R}$ have the form

$$({}^{\mathrm{L}}I_{+}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}\frac{f(t)dt}{(x-t)^{1-\alpha}} \tag{1.3.61}$$

and

$$({}^{\mathrm{L}}I_{-}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}\frac{f(t)dt}{(t-x)^{1-\alpha}}, \tag{1.3.62}$$

where $x \in \mathbb{R}$ and $\Re e(\alpha) > 0$, while the fractional Liouville derivatives are defined as

$$\begin{aligned}({}^{\mathrm{L}}D_{+}^{\alpha}y)(x) &= \left(\frac{d}{dx}\right)^{n}({}^{\mathrm{L}}I_{+}^{n-\alpha}y)(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{-\infty}^{x}\frac{y(t)dt}{(x-t)^{\alpha-n+1}}\end{aligned} \tag{1.3.63}$$

and

$$\begin{aligned}({}^{\mathrm{L}}D_{-}^{\alpha}y)(x) &= \left(-\frac{d}{dx}\right)^{n}({}^{\mathrm{L}}I_{-}^{n-\alpha}y)(x) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^{n}\int_{x}^{\infty}\frac{y(t)dt}{(t-x)^{\alpha-n+1}},\end{aligned} \tag{1.3.64}$$

where $n = -[-\Re e(\alpha)]$, $\Re e(\alpha) \geq 0$ and $x \in \mathbb{R}$.

The expressions for ${}^{\mathrm{L}}I_{+}^{\alpha}f$ and ${}^{\mathrm{L}}I_{-}^{\alpha}f$ in (1.3.61) and (1.3.62), and for ${}^{\mathrm{L}}D_{+}^{\alpha}y$ and ${}^{\mathrm{L}}D_{-}^{\alpha}y$ in (1.3.63) and (1.3.64), are called Liouville left- and right-sided fractional integrals and fractional derivatives on the whole axis $\mathbb{R}$, respectively.

In particular, when $\alpha = n \in \mathbb{N}_0$, then

$$({}^{\mathrm{L}}D_{+}^{0}y)(x) = ({}^{\mathrm{L}}D_{-}^{0}y)(x) = y(x) \tag{1.3.65}$$

and

$$({}^{\mathrm{L}}D_{+}^{n}y)(x) = y^{(n)}(x) \quad \text{and} \quad ({}^{\mathrm{L}}D_{-}^{n}y)(x) = (-1)^{n}y^{(n)}(x) \quad (n \in \mathbb{N}), \tag{1.3.66}$$

where $y^{(n)}(x)$ is the usual derivative of $y(x)$ of order n.

If $0 < \Re e(\alpha) < 1$ and $x \in \mathbb{R}$, then

$$({}^{\mathtt{L}}D_{+}^{\alpha}y)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x}\frac{y(t)dt}{(x-t)^{\alpha-[\Re e(\alpha)]}} \tag{1.3.67}$$

and

$$({}^{\mathtt{L}}D_{-}^{\alpha}y)(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x}^{\infty}\frac{y(t)dt}{(t-x)^{\alpha-[\Re e(\alpha)]}}. \tag{1.3.68}$$

The Liouville fractional operators ${}^{\mathtt{L}}I_{+}^{\alpha}$ and ${}^{\mathtt{L}}D_{+}^{\alpha}$ of the exponential function $e^{\lambda x}$ yield the same exponential function, both apart from a constant multiplication factor, i.e., if $\lambda > 0$ and $\Re e(\alpha) \geq 0$,

$$({}^{\mathtt{L}}I_{+}^{\alpha}e^{\lambda t})(x) = \lambda^{-\alpha}e^{\lambda x}; \tag{1.3.69}$$

$$\left({}^{\mathtt{L}}I_{-}^{\alpha}e^{-\lambda t}\right)(x) = \lambda^{-\alpha}e^{-\lambda x}; \tag{1.3.70}$$

$$({}^{\mathtt{L}}D_{+}^{\alpha}e^{\lambda t})(x) = \lambda^{\alpha}e^{\lambda x}; \tag{1.3.71}$$

and

$$\left({}^{\mathtt{L}}D_{-}^{\alpha}e^{-\lambda t}\right)(x) = \lambda^{\alpha}e^{-\lambda x}. \tag{1.3.72}$$

On the other hand, if $\alpha > 0$, $\beta > 0$, then, for "sufficiently good" functions, we have

$$({}^{\mathtt{L}}I_{+}^{\alpha}\,{}^{\mathtt{L}}I_{+}^{\beta}f)(x) = ({}^{\mathtt{L}}I_{+}^{\alpha+\beta}f)(x) \text{ and } ({}^{\mathtt{L}}I_{-}^{\alpha}\,{}^{\mathtt{L}}I_{-}^{\beta}f)(x) = ({}^{\mathtt{L}}I_{-}^{\alpha+\beta}f)(x); \tag{1.3.73}$$

$$({}^{\mathtt{L}}D_{+}^{\alpha}\,{}^{\mathtt{L}}I_{+}^{\alpha}f)(x) = f(x), \text{ and } ({}^{\mathtt{L}}D_{-}^{\alpha}\,{}^{\mathtt{L}}I_{-}^{\alpha}f)(x) = f(x). \tag{1.3.74}$$

If in addition $\alpha > \beta > 0$, then the formulas

$$({}^{\mathtt{L}}D_{+}^{\beta}\,{}^{\mathtt{L}}I_{+}^{\alpha}f)(x) = ({}^{\mathtt{L}}I_{0+}^{\alpha-\beta}f)(x) \text{ and } ({}^{\mathtt{L}}D_{-}^{\beta}\,{}^{\mathtt{L}}I_{-}^{\alpha}f)(x) = ({}^{\mathtt{L}}I_{-}^{\alpha-\beta}f)(x) \tag{1.3.75}$$

hold. Furthermore, when $\beta = k \in \mathbb{N}$ and $\Re e(\alpha) > k$, then

$$(D^{k}\,{}^{\mathtt{L}}I_{+}^{\alpha}f)(x) = {}^{\mathtt{L}}I_{+}^{\alpha-k}f(x), \text{ and } (D^{k}\,{}^{\mathtt{L}}I_{-}^{\alpha}f)(x) = (-1)^{k}\,{}^{\mathtt{L}}I_{-}^{\alpha-k}f(x). \tag{1.3.76}$$

The Fourier transform (1.1.1) of the Liouville fractional integrals ${}^{\mathtt{L}}I^{\alpha}_{+}f$ and ${}^{\mathtt{L}}I^{\alpha}_{-}f$ is given for $0 < \Re e(\alpha) < 1$, by the following relations

$$(\mathcal{F}\,{}^{\mathtt{L}}I^{\alpha}_{+}f)(\kappa) = \frac{(\mathcal{F}f)(\kappa)}{(-i\kappa)^{\alpha}} \tag{1.3.77}$$

and

$$(\mathcal{F}\,{}^{\mathtt{L}}I^{\alpha}_{-}f)(\kappa) = \frac{(\mathcal{F}f)(\kappa)}{(i\kappa)^{\alpha}}. \tag{1.3.78}$$

Here $(\mp i\kappa)^{\alpha}$ means

$$(\mp i\kappa)^{\alpha} = |\kappa|^{\alpha} e^{\mp\alpha\pi i\ \mathrm{sgn}(\kappa)/2}. \tag{1.3.79}$$

Moreover, if $\Re e(\alpha) > 0$, then, for "sufficiently good" functions $f(x)$, the equations (1.3.77) and (1.3.78) are valid as well as the following corresponding relations for the Liouville fractional derivatives ${}^{\mathtt{L}}D^{\alpha}_{+}f$ and ${}^{\mathtt{L}}D^{\alpha}_{-}f$

$$(\mathcal{F}\,{}^{\mathtt{L}}D^{\alpha}_{+}f)(\kappa) = (-i\kappa)^{\alpha}(\mathcal{F}f)(\kappa) \tag{1.3.80}$$

and

$$(\mathcal{F}\,{}^{\mathtt{L}}D^{\alpha}_{-}f)(\kappa) = (i\kappa)^{\alpha}(\mathcal{F}f)(\kappa), \tag{1.3.81}$$

where $(\mp i\kappa)^{\alpha}$ is defined by (1.3.79).

The rules for fractional integration by parts, for $\alpha > 0$, and for "sufficiently good" functions, are given by

$$\int_{-\infty}^{\infty} \varphi(x) \left({}^{\mathtt{L}}I^{\alpha}_{+}\psi\right)(x)dx = \int_{-\infty}^{\infty} \psi(x) \left({}^{\mathtt{L}}I^{\alpha}_{-}\varphi\right)(x)dx. \tag{1.3.82}$$

$$\int_{-\infty}^{\infty} f(x) \left({}^{\mathtt{L}}D^{\alpha}_{+}g\right)(x)dx = \int_{-\infty}^{\infty} g(x) \left({}^{\mathtt{L}}D^{\alpha}_{-}f\right)(x)dx. \tag{1.3.83}$$

$$\int_{0}^{\infty} \varphi(x) \left({}^{\mathtt{RL}}I^{\alpha}_{0+}\psi\right)(x)dx = \int_{0}^{\infty} \psi(x) \left({}^{\mathtt{L}}I^{\alpha}_{-}\varphi\right)(x)dx. \tag{1.3.84}$$

$$\int_{0}^{\infty} f(x) \left({}^{\mathtt{RL}}D^{\alpha}_{0+}g\right)(x)dx = \int_{0}^{\infty} g(x) \left({}^{\mathtt{L}}D^{\alpha}_{-}f\right)(x)dx. \tag{1.3.85}$$

The Liouville fractional derivatives ${}^{\mathtt{L}}D^{\alpha}_{+}f$ and ${}^{\mathtt{L}}D^{\alpha}_{-}f$ exist for suitable functions f, but they are not defined, for example, for constant functions. Nevertheless, they can be reduced in general to more convenient forms which admit fractional differentiation of a constant function. In this way we come to the Marchaud fractional derivatives ${}^{\mathtt{M}}D^{\alpha}_{+}f$ and ${}^{\mathtt{M}}D^{\alpha}_{-}f$ of order $\alpha \in \mathbb{C}$, defined by

$$\left({}^{\mathtt{M}}D^{\alpha}_{+}f\right)(x) = \frac{1}{\kappa(\alpha,k)}\int_0^{\infty}\frac{(\Delta^k_t f)(x)}{t^{1+\alpha}}dt \quad (k > \Re e(\alpha) > 0) \tag{1.3.86}$$

and

$$\left({}^{\mathtt{M}}D^{\alpha}_{-}f\right)(x) = \frac{1}{\kappa(\alpha,k)}\int_0^{\infty}\frac{(\Delta^k_{-t} f)(x)}{t^{1+\alpha}}dt \quad (k > \Re e(\alpha) > 0), \tag{1.3.87}$$

respectively. Here $\kappa(\alpha,k)$ is the constant

$$\kappa(\alpha,k) = \int_0^{\infty}(1-e^{-u})^k\frac{du}{u^{1+\alpha}} \quad (k \in \mathbb{N},\ k > \Re e(\alpha) > 0), \tag{1.3.88}$$

and $(\Delta^k_h f)(x)$ is the finite difference of order k of a function $f(x)$ with increment h

$$\left(\Delta^k_h f\right)(x) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}f(x-jh). \tag{1.3.89}$$

Note that

$$\left(\Delta^k_h 1\right)(x) = \sum_{j=0}^{k}(-1)^j\binom{k}{j} = (1-1)^k = 0 \quad (k \in \mathbb{N}). \tag{1.3.90}$$

In particular, when $h = -1$, (1.3.89) coincides with the difference (1.3.89) with $h = +1$ except for the factor $(-1)^k$, i.e.

$$\left(\Delta^k_{-1}f\right)(x) = (-1)^k(\Delta^k f)(x).$$

It is known that for a real number $\alpha > 0$ the right-hand sides of (1.3.86) and (1.3.87) do not depend on the choice of k $(k > \alpha)$. Similarly, for complex α the right-hand sides of (1.3.86) and (1.3.87) do not depend on the choice of k $(k > \Re e(\alpha))$.

The Marchaud fractional derivatives ${}^{\mathtt{M}}D_{+}^{\alpha}f$ and ${}^{\mathtt{M}}D_{-}^{\alpha}f$ are defined for the constant function $f = c \in \mathbb{C}$, and in accordance with (1.3.90)

$$\left({}^{\mathtt{M}}D_{+}^{\alpha}c\right)(x) = \left({}^{\mathtt{M}}D_{-}^{\alpha}c\right)(x) = 0 \quad (\alpha \in \mathbb{C},\ \mathfrak{Re}(\alpha) > 0). \tag{1.3.91}$$

For “suitable functions” f, the Marchaud fractional derivatives coincide with the Liouville fractional derivatives for same α

$${}^{\mathtt{M}}D_{+}^{\alpha}f = {}^{\mathtt{L}}D_{+}^{\alpha}f; \quad {}^{\mathtt{M}}D_{-}^{\alpha}f = {}^{\mathtt{L}}D_{-}^{\alpha}f.$$

In particular, they have the same properties as the Liouville derivatives over exponential functions e^{bt} and e^{-bt}, in the following sense

$$\left({}^{\mathtt{M}}D_{+}^{\alpha}e^{bt}\right)(x) = b^{\alpha}e^{bx}, \tag{1.3.92}$$

$$\left({}^{\mathtt{M}}D_{-}^{\alpha}e^{-bt}\right)(x) = b^{\alpha}e^{-bx}. \tag{1.3.93}$$

for $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0)$ and $b \in \mathbb{C}$ $(\mathfrak{Re}(b) > 0)$.

1.3.4 *Generalized exponential functions*

In this section we consider two special functions, which play a role as generalized exponentials. The first one is the Mittag-Leffler function

$$E_{\alpha}\left(\lambda z^{\alpha}\right) = \sum_{k=0}^{\infty} \frac{\lambda^{k} z^{\alpha k}}{\Gamma(\alpha k + 1)} \quad \left(z, \lambda \in \mathbb{C};\ \ \mathfrak{Re}(\alpha) > 0\right), \tag{1.3.94}$$

and the second one is defined in terms of the Mittag-Leffler type function by

$$e_{\alpha}^{\lambda z} = z^{\alpha-1}E_{\alpha,\alpha}\left(\lambda z^{\alpha}\right) = \sum_{k=0}^{\infty} \frac{\lambda^{k} z^{\alpha(k-1)}}{\Gamma(\alpha(k+1))} \quad \left(z, \lambda \in \mathbb{C},\ \ \mathfrak{Re}(\alpha) > 0\right). \tag{1.3.95}$$

The above functions satisfy the identity

$$E_{1}(\lambda z) = e_{1}^{\lambda z} = e^{\lambda z} \quad (z, \lambda \in \mathbb{C}), \tag{1.3.96}$$

and therefore they are generalizations of the classical exponential function.

Some others properties of the first of these functions, for $z \in \mathbb{C}\backslash\{0\}$, $\alpha, \lambda \in \mathbb{C}$ ($\Re e(\alpha) > 0$), and $n \in \mathbb{N}$, are the following

$$\lim_{z \to a+} E_{\alpha}\left(\lambda(z-a)^{\alpha}\right) = 1; \tag{1.3.97}$$

$$\left(\frac{\partial}{\partial z}\right)^{n} \left[E_{\alpha}\left(\lambda z^{\alpha}\right)\right] = z^{-n} E_{\alpha,1-n}\left(\lambda z^{\alpha}\right); \tag{1.3.98}$$

$$\left(\frac{\partial}{\partial \lambda}\right)^{n} \left[E_{\alpha}\left(\lambda z^{\alpha}\right)\right] = n! z^{\alpha n} E^{n+1}_{\alpha,\alpha n+1}\left(\lambda z^{\alpha}\right); \tag{1.3.99}$$

$$\mathfrak{L}\left[E_{\alpha}\left(\lambda z^{\alpha}\right)\right](s) = \frac{s^{\alpha-1}}{s^{\alpha}-\lambda} \quad (\Re e(s) > 0; \quad |\lambda s^{-\alpha}| < 1); \tag{1.3.100}$$

and

$$\mathfrak{L}\left[t^{\alpha n}\left(\frac{\partial}{\partial \lambda}\right)^{n} E_{\alpha}\left(\lambda z^{\alpha}\right)\right](s) = \frac{n! s^{\alpha-1}}{(s^{\alpha}-\lambda)^{n+1}}. \tag{1.3.101}$$

On the other hand, for the function $e^{\lambda z}_{\alpha}$, we have the following properties

$$\lim_{z \to a+} \left[(z-a)^{1-\alpha} e^{\lambda(z-a)}_{\alpha}\right] = \frac{1}{\Gamma(\alpha)}; \tag{1.3.102}$$

$$\left(\frac{\partial}{\partial z}\right)^{n} \left[e^{\lambda z}_{\alpha}\right] = z^{\alpha-n-1} E_{\alpha,\alpha-n}\left(\lambda z^{\alpha}\right); \tag{1.3.103}$$

$$\left(\frac{\partial}{\partial \lambda}\right)^{n} \left[e^{\lambda z}_{\alpha}\right] = n! z^{\alpha n+\alpha-1} E^{n+1}_{\alpha,(n+1)\alpha}\left(\lambda z^{\alpha}\right); \tag{1.3.104}$$

$$\mathfrak{L}\left[e^{\lambda z}_{\alpha}\right](s) = \frac{1}{s^{\alpha}-\lambda} \quad \left(\Re e(s) > 0; \quad |\lambda s^{-\alpha}| < 1\right); \tag{1.3.105}$$

and

$$\mathfrak{L}\left[\left(\frac{\partial}{\partial \lambda}\right)^{n} e^{\lambda z}_{\alpha}\right](s) = \frac{n!}{(s^{\alpha}-\lambda)^{n+1}}. \tag{1.3.106}$$

The generalized α-exponential functions do not have the index property, i.e., in general

$$E_\alpha(\lambda z)E_\alpha(\mu z) \neq E_\alpha((\lambda+\mu)z); \quad e_\alpha^{\lambda z}e_\alpha^{\mu z} \neq e_\alpha^{(\lambda+\mu)z} \quad (\alpha \neq 1). \quad (1.3.107)$$

For example, if $\alpha = 2$ and $z = 1$, then in accordance with the second relation in (1.2.34), we have

$$e_2^\lambda = \sqrt{\lambda}\sinh(\sqrt{\lambda}), \quad (1.3.108)$$

but

$$[\sqrt{\lambda}\sinh(\sqrt{\lambda})][\sqrt{\mu}\sinh(\sqrt{\mu})] \neq \sqrt{(\lambda+\mu)}\sinh(\sqrt{(\lambda+\mu)}. \quad (1.3.109)$$

Let $M_n(\mathbb{R})$ ($n \in \mathbb{N}$) be the set of all matrices $\mathbf{A} = [a_{jk}]$ of order $n \times n$ with $a_{jk} \in \mathbb{R}$ ($j = 1, \ldots, n$). By analogy with (1.3.95), for $\alpha \in \mathbb{C}\backslash\{0\}$ ($\Re e(\alpha) > 0$), and $\mathbf{A} \in M_n(\mathbb{R})$, here we introduce a matrix α-exponential function defined by

$$e_\alpha^{\mathbf{A}z} = z^{\alpha-1}\sum_{k=0}^{\infty}\mathbf{A}^k\frac{z^{\alpha k}}{\Gamma((k+1)\alpha)}, \quad (1.3.110)$$

and also the function

$$E_\alpha(\mathbf{A}z^\alpha) = \sum_{k=0}^{\infty}\frac{\mathbf{A}^k z^{k\alpha}}{\Gamma(\alpha k+1)} \quad (z \in \mathbb{C}; \ \ \Re e(\alpha) > 0)\,. \quad (1.3.111)$$

When $\alpha = 1$ we have

$$E_1(\mathbf{A}z) = e_1^{\mathbf{A}z} = e^{\mathbf{A}z} \quad (z \in \mathbb{C}). \quad (1.3.112)$$

Of course, in general, the semigroup property does not hold for the generalized matrix exponential functions

$$e_\alpha^{\mathbf{A}z}e_\alpha^{\mathbf{B}z} \neq e_\alpha^{(\mathbf{A}+\mathbf{B})z} \quad (z, \alpha \in \mathbb{C}; \ \mathbf{A}, \mathbf{B} \in M_n(\mathbb{R}))\,. \quad (1.3.113)$$

Similarly, the inversion formula

$$(e^{\mathbf{A}z})^{-1} = e^{-\mathbf{A}z}, \quad (1.3.114)$$

valid for the matrix exponential function $e^{\mathbf{A}z}$, is not true, in general, for matrix α-exponential function $e_\alpha^{\mathbf{A}z}$,

$$(e_\alpha^{\mathbf{A}z})^{-1} \neq e_\alpha^{-\mathbf{A}z}. \tag{1.3.115}$$

If we define the norm $\|\mathbf{A}\|$ of the matrix $\mathbf{A}$ with elements $a_{jk} \in \mathbb{R}$ $(j, k = 1, \dots, n)$ by

$$\|\mathbf{A}\| = \max_{j,k \in \mathbb{N}} |a_{jk}| \tag{1.3.116}$$

then, from (1.3.110), we derive the estimate for the norm of $e_\alpha^{\mathbf{A}z}$. For any fixed $z \in \mathbb{C}$, the following relation holds

$$\|e_\alpha^{\mathbf{A}z}\| \leq \sum_{k=0}^{\infty} \|\mathbf{A}\|^k \frac{|z|^{\Re e(\alpha)k}}{|\Gamma((k+1)\alpha)|}. \tag{1.3.117}$$

When $z = x > 0$ and $\alpha > 0$, the above formula takes the simpler form

$$\|e_\alpha^{\mathbf{A}x}\| \leq \sum_{k=0}^{\infty} \|\mathbf{A}\|^k \frac{x^{\alpha k}}{\Gamma((k+1)\alpha)}. \tag{1.3.118}$$

Corresponding properties can be proved for the other generalized matrix exponential functions $E_\alpha(\mathbf{A}z^\alpha)$.

Particularly important are the following properties of the functions $E_\alpha(\lambda(z-a)^\alpha)$ and $e_\alpha^{\lambda(z-a)}$:

$$\left({}^{\mathtt{C}}D_{a+}^\alpha E_\alpha(\lambda(z-a)^\alpha)\right)(x) = \lambda E_\alpha(\lambda(x-a)^\alpha) \tag{1.3.119}$$

and

$$\left({}^{\mathtt{RL}}D_{a+}^\alpha e_\alpha^{(\lambda(z-a))}\right)(x) = \lambda e_\alpha^{(\lambda(x-a))} \tag{1.3.120}$$

with $\alpha > 0$, $a \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. Thus, the two generalized exponential functions are eigenfunctions of Caputo and Riemann-Liouville differential operators, respectively.

The behavior of $E_\alpha(-x)$ and e_α^{-x} for various values of α in the intervals $(0,1)$ and $(1,2)$ can be seen from the following figures.

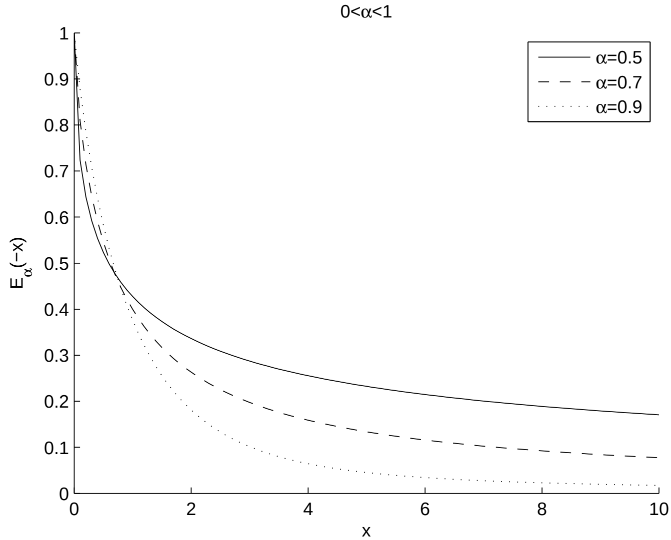

Fig. 1.1 Representation of $E_\alpha(-x)$ for some values of $\alpha \in (0, 1)$.

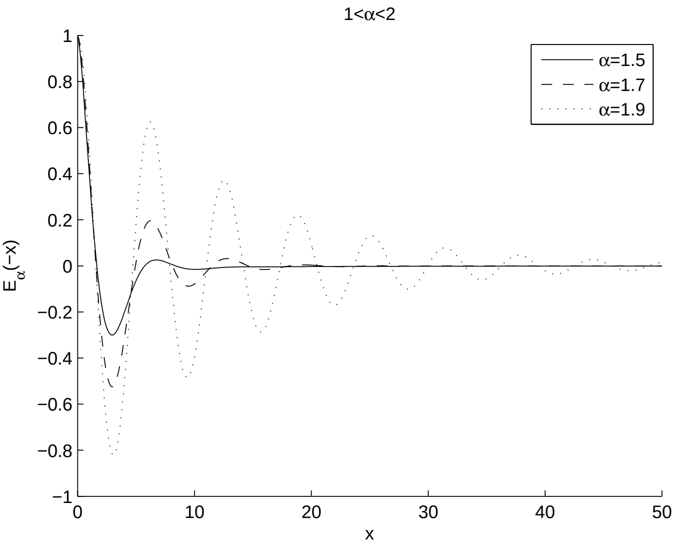

Fig. 1.2 Representation of $E_\alpha(-x)$ for some values of $\alpha \in (1, 2)$.

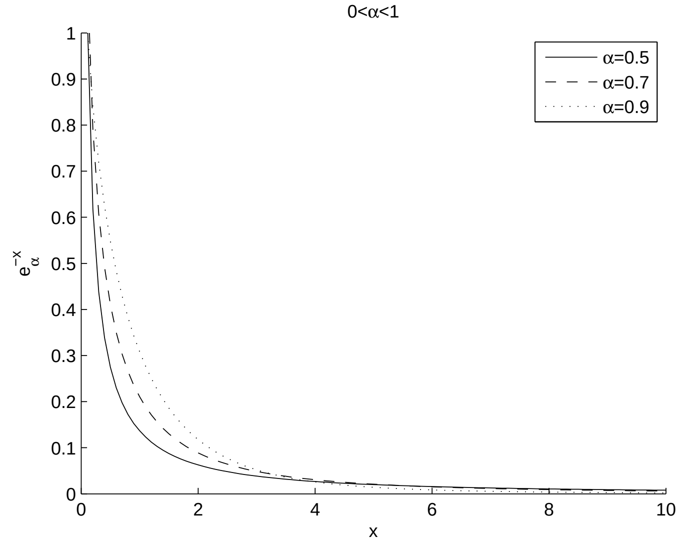

Fig. 1.3 Representation of e_α^{-x} for some values of $\alpha \in (0, 1)$.

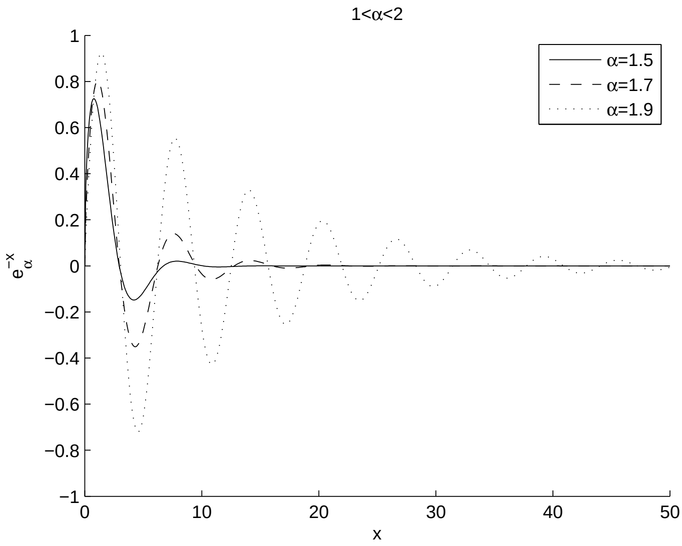

Fig. 1.4 Representation of e_α^{-x} for some values of $\alpha \in (1, 2)$.

1.3.5 *Hadamard type fractional integrals and fractional derivatives*

In this section we present the definitions and some properties of the Hadamard type fractional integrals and fractional derivatives.

Let (a, b) $(0 \leq a < b \leq \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$, and let $\Re e(\alpha) > 0$ and $\mu \in \mathbb{C}$. We consider the left-sided and right-sided integrals of fractional order $\alpha \in \mathbb{C}$ $(\Re e(\alpha) > 0)$ defined by

$$\left({}^{\mathtt{H}}I^{\alpha}_{a+}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{\alpha-1}\frac{f(t)dt}{t} \quad (a < x < b) \tag{1.3.121}$$

and

$$\left({}^{\mathtt{H}}I^{\alpha}_{b-}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\log\frac{t}{x}\right)^{\alpha-1}\frac{f(t)dt}{t} \quad (a < x < b), \tag{1.3.122}$$

respectively. When $a = 0$ and $b = \infty$, these relations are given by

$$\left({}^{\mathtt{H}}I^{\alpha}_{0+}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \left(\log\frac{x}{t}\right)^{\alpha-1}\frac{f(t)dt}{t} \quad (x > 0) \tag{1.3.123}$$

and

$$\left({}^{\mathtt{H}}I^{\alpha}_{-}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty \left(\log\frac{t}{x}\right)^{\alpha-1}\frac{f(t)dt}{t} \quad (x > 0). \tag{1.3.124}$$

More general fractional integrals than those in (1.3.123) and (1.3.124) are defined by

$$\left({}^{\mathtt{H}}I^{\alpha}_{0+,\mu}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \left(\frac{t}{x}\right)^{\mu}\left(\log\frac{x}{t}\right)^{\alpha-1}\frac{f(t)dt}{t} \quad (x > 0) \tag{1.3.125}$$

and

$$\left({}^{\mathtt{H}}I^{\alpha}_{-,\mu}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty \left(\frac{x}{t}\right)^{\mu}\left(\log\frac{t}{x}\right)^{\alpha-1}\frac{f(t)dt}{t} \quad (x > 0) \tag{1.3.126}$$

with $\mu \in \mathbb{C}$.

The integral in (1.3.123) was introduced by Hadamard [254]. Therefore, the integrals (1.3.121), (1.3.122) and (1.3.123), (1.3.124) are often referred to as the Hadamard fractional integrals of order α. The more general integrals (1.3.125) and (1.3.126), introduced by Butzer *et al.* [117], are called the Hadamard type fractional integrals of order α.

The left- and right-sided Hadamard fractional derivatives of order $\alpha \in \mathbb{C}$ ($\Re e(\alpha) \geq 0$) on (a, b) are defined by

$$\begin{aligned}\left({}^{\mathtt{H}}D^{\alpha}_{a+}y\right)(x) &= \delta^n \left({}^{\mathtt{H}}I^{n-\alpha}_{a+}y\right)(x) \\ &= \left(x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\log\frac{x}{t}\right)^{n-\alpha+1} \frac{y(t)dt}{t} \end{aligned} \tag{1.3.127}$$

and

$$\begin{aligned}\left({}^{\mathtt{H}}D^{\alpha}_{b-}y\right)(x) &= (-\delta)^n \left({}^{\mathtt{H}}I^{n-\alpha}_{b-}y\right)(x) \\ &= \left(-x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_x^b \left(\log\frac{t}{x}\right)^{n-\alpha+1} \frac{y(t)dt}{t} \end{aligned} \tag{1.3.128}$$

for $a < x < b$, respectively, where $n = -[-\Re e(\alpha)]$ and $\delta = xD$ ($D = d/dx$). When $a = 0$ and $b = \infty$, we have

$$\left({}^{\mathtt{H}}D^{\alpha}_{0+}y\right)(x) = \delta^n \left({}^{\mathtt{H}}I^{n-\alpha}_{0+}y\right)(x) \quad (x > 0)\,; \tag{1.3.129}$$

$$\left({}^{\mathtt{H}}D^{\alpha}_{-}y\right)(x) = (-\delta)^n \left({}^{\mathtt{H}}I^{n-\alpha}_{-}y\right)(x) \quad (x > 0)\,. \tag{1.3.130}$$

The Hadamard type fractional derivatives of order α with $\mu \in \mathbb{C}$, more general than those in (1.3.129) and (1.3.130), are defined for $\Re e(\alpha) \geq 0$ by

$$\left({}^{\mathtt{H}}D^{\alpha}_{0+,\mu}y\right)(x) = x^{-\mu}\delta^n x^{\mu} \left({}^{\mathtt{H}}I^{n-\alpha}_{0+,\mu}y\right)(x) \quad (x > 0)\,; \tag{1.3.131}$$

$$\left({}^{\mathtt{H}}D^{\alpha}_{-,\mu}y\right)(x) = x^{\mu}(-\delta)^n x^{-\mu} \left({}^{\mathtt{H}}I^{n-\alpha}_{-,\mu}y\right)(x) \quad (x > 0)\,. \tag{1.3.132}$$

The Hadamard type operators ${}^{\mathtt{H}}I^0_{0+,\mu}$, ${}^{\mathtt{H}}D^0_{0+,\mu}$ and ${}^{\mathtt{H}}I^0_{-,\mu}$, ${}^{\mathtt{H}}D^0_{-,\mu}$ can be defined as the identity operator

$${}^{\mathtt{H}}I^0_{0+,\mu}f \equiv {}^{\mathtt{H}}D^0_{0+,\mu}f = f, \qquad {}^{\mathtt{H}}I^0_{-,\mu}f \equiv {}^{\mathtt{H}}D^0_{-,\mu}f = f, \tag{1.3.133}$$

and in particular,

$${}^{\mathtt{H}}I^0_{0+}f \equiv {}^{\mathtt{H}}D^0_{0+}f = f, \qquad {}^{\mathtt{H}}I^0_{-}f \equiv {}^{\mathtt{H}}D^0_{-}f = f. \tag{1.3.134}$$

For $\Re e(\alpha) > 0$, $n = -[-\Re e(\alpha)]$, and $0 < a < b < \infty$, we have that $\left({}^{\mathtt{H}}D^{\alpha}_{a+}y\right)(x) = 0$ is valid if, and only if,

$$y(x) = \sum_{j=1}^{n} c_j \left(\log \frac{x}{a}\right)^{\alpha-j}, \tag{1.3.135}$$

where $c_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ are arbitrary constants. In particular, when $0 < \Re e(\alpha) \leq 1$, the relation $(\mathcal{D}^{\alpha}_{a+}y)(x) = 0$ holds if, and only if, $y(x) = c\left(\log \frac{x}{a}\right)^{\alpha-1}$ for any $c \in \mathbb{R}$.

On the other hand, the equality $\left({}^{\mathtt{H}}D^{\alpha}_{b-}y\right)(x) = 0$ is valid if, and only if,

$$y(x) = \sum_{j=1}^{n} d_j \left(\log \frac{b}{x}\right)^{\alpha-j}, \tag{1.3.136}$$

where $d_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ are arbitrary constants. In particular, when $0 < \Re e(\alpha) \leq 1$, the relation $\left({}^{\mathtt{H}}D^{\alpha}_{b-}y\right)(x) = 0$ holds if, and only if, $y(x) = d\left(\log \frac{b}{x}\right)^{\alpha-1}$ for any $d \in \mathbb{R}$.

It can also be directly verified that the Hadamard and Hadamard type fractional integrals and derivatives (1.3.123)–(1.3.126) and (1.3.129)–(1.3.132) of the power function x^{β} yield the same function, apart from a constant multiplication factor, that is, if $\Re e(\alpha) > 0$, $\beta, \mu \in \mathbb{C}$, and $\Re e(\beta + \mu) > 0$, then

$$\left({}^{\mathtt{H}}I^{\alpha}_{0+,\mu}t^{\beta}\right)(x) = (\mu + \beta)^{-\alpha}x^{\beta} \tag{1.3.137}$$

and

$$\left({}^{\mathtt{H}}D^{\alpha}_{0+,\mu}t^{\beta}\right)(x) = (\mu + \beta)^{\alpha}x^{\beta}. \tag{1.3.138}$$

On the other hand, if $\Re e(\beta - \mu) < 0$, then

$$\left({}^{\mathtt{H}}I^{\alpha}_{-,\mu}t^{\beta}\right)(x) = (\mu - \beta)^{-\alpha}x^{\beta} \tag{1.3.139}$$

and

$$\left({}^{\mathtt{H}}D^{\alpha}_{-,\mu}t^{\beta}\right)(x) = (\mu - \beta)^{\alpha}x^{\beta}. \tag{1.3.140}$$

In particular, we have

$$\left({}^{\mathtt{H}}I^{\alpha}_{0+}t^{\beta}\right)(x) = \beta^{-\alpha}x^{\beta} \quad \text{and} \quad \left({}^{\mathtt{H}}D^{\alpha}_{0+}t^{\beta}\right)(x) = \beta^{\alpha}x^{\beta} \quad (\Re e(\beta) > 0) \tag{1.3.141}$$

and

$$\left({}^{\mathtt{H}}I^{\alpha}_{-}t^{\beta}\right)(x) = (-\beta)^{-\alpha}x^{\beta} \quad \text{and} \quad \left({}^{\mathtt{H}}D^{\alpha}_{-}t^{\beta}\right)(x) = (-\beta)^{\alpha}x^{\beta} \quad (\Re e(\beta) < 0). \tag{1.3.142}$$

The Hadamard and Hadamard type fractional integrals (1.3.121)–(1.3.126) satisfy the semigroup property

$${}^{\mathtt{H}}I^{\alpha}_{a+}\,{}^{\mathtt{H}}I^{\beta}_{a+}f = {}^{\mathtt{H}}I^{\alpha+\beta}_{a+}f \quad \text{and} \quad {}^{\mathtt{H}}I^{\alpha}_{b-}\,{}^{\mathtt{H}}I^{\beta}_{b-}f = {}^{\mathtt{H}}I^{\alpha+\beta}_{b-}f, \tag{1.3.143}$$

for $\Re e(\alpha) > 0$, $\Re e(\beta) > 0$ and $0 < a < b < \infty$.

If $\mu \in \mathbb{C}$, $a = 0$ and $b = \infty$, then

$${}^{\mathtt{H}}I^{\alpha}_{0+,\mu}\,{}^{\mathtt{H}}I^{\beta}_{0+,\mu}f = {}^{\mathtt{H}}I^{\alpha+\beta}_{0+,\mu}f \tag{1.3.144}$$

and

$${}^{\mathtt{H}}I^{\alpha}_{-,\mu}\,{}^{\mathtt{H}}I^{\beta}_{-,\mu}f = {}^{\mathtt{H}}I^{\alpha+\beta}_{-,\mu}f. \tag{1.3.145}$$

In particular, when $\mu = 0$,

$${}^{\mathtt{H}}I^{\alpha}_{0+}\,{}^{\mathtt{H}}I^{\beta}_{0+}f = {}^{\mathtt{H}}I^{\alpha+\beta}_{0+}f; \qquad {}^{\mathtt{H}}I^{\alpha}_{-}\mathcal{J}^{\beta}_{-}f = {}^{\mathtt{H}}I^{\alpha+\beta}_{-}f. \tag{1.3.146}$$

Now we give the properties of compositions between the operators of fractional differentiation (1.3.127)–(1.3.132) and fractional integration (1.3.121)–(1.3.126).

If $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ are such that $\Re e(\alpha) > \Re e(\beta) > 0$, then

$${}^{\mathtt{H}}D^{\beta}_{a+}\,{}^{\mathtt{H}}I^{\alpha}_{a+}f = {}^{\mathtt{H}}I^{\alpha-\beta}_{a+}f \quad \text{and} \quad {}^{\mathtt{H}}D^{\beta}_{b-}\,{}^{\mathtt{H}}I^{\alpha}_{b-}f = {}^{\mathtt{H}}I^{\alpha-\beta}_{b-}f \tag{1.3.147}$$

when $0 < a < b < \infty$. In particular, if $\beta = m \in \mathbb{N}$, then

$${}^{\mathtt{H}}D^{m}_{a+}\,{}^{\mathtt{H}}I^{\alpha}_{a+}f = {}^{\mathtt{H}}I^{\alpha-m}_{a+}f \quad \text{and} \quad {}^{\mathtt{H}}D^{m}_{b-}\,{}^{\mathtt{H}}I^{\alpha}_{b-}f = {}^{\mathtt{H}}I^{\alpha-m}_{b-}f. \tag{1.3.148}$$

If $\mu \in \mathbb{C}$, $a = 0$ and $b = \infty$, then

$${}^{\mathtt{H}}D^{\beta}_{0+,\mu}\,{}^{\mathtt{H}}I^{\alpha}_{0+,\mu}f = {}^{\mathtt{H}}I^{\alpha-\beta}_{0+,\mu}f \tag{1.3.149}$$

and

$${}^{\mathtt{H}}D^{\beta}_{-,\mu}\,{}^{\mathtt{H}}I^{\alpha}_{-,\mu}f = {}^{\mathtt{H}}I^{\alpha-\beta}_{-,\mu}f. \tag{1.3.150}$$

In particular, if $\beta = m \in \mathbb{N}$, then

$$ {}^{\mathtt{H}}D^{m}_{0+,\mu}\, {}^{\mathtt{H}}I^{\alpha}_{0+,\mu} f = {}^{\mathtt{H}}I^{\alpha-m}_{0+,\mu} f \tag{1.3.151} $$

and

$$ {}^{\mathtt{H}}D^{m}_{-,\mu}\, {}^{\mathtt{H}}I^{\alpha}_{-,\mu} f = {}^{\mathtt{H}}I^{\alpha-m}_{-,\mu} f. \tag{1.3.152} $$

while, when $\mu = 0$ and $m \in \mathbb{N}$,

$$ {}^{\mathtt{H}}D^{\beta}_{0+}\, {}^{\mathtt{H}}I^{\alpha}_{0+} f = {}^{\mathtt{H}}I^{\alpha-\beta}_{0+} f; \quad {}^{\mathtt{H}}D^{\beta}_{-}\, {}^{\mathtt{H}}I^{\alpha}_{-} f = {}^{\mathtt{H}}I^{\alpha-\beta}_{-} f \tag{1.3.153} $$

and

$$ {}^{\mathtt{H}}D^{m}_{0+}\, {}^{\mathtt{H}}I^{\alpha}_{0+} f = {}^{\mathtt{H}}I^{\alpha-m}_{0+} f; \quad {}^{\mathtt{H}}D^{m}_{-}\, {}^{\mathtt{H}}I^{\alpha}_{-} f = {}^{\mathtt{H}}I^{\alpha-m}_{-} f. \tag{1.3.154} $$

The Hadamard and Hadamard type fractional derivatives (1.3.127), (1.3.128) and (1.3.131), (1.3.132) are operators inverse to the corresponding fractional integrals (1.3.121), (1.3.122) and (1.3.125), (1.3.126), that is, if $\Re e(\alpha) > 0$ and $0 < a < b < \infty$ then

$$ {}^{\mathtt{H}}D^{\alpha}_{a+}\, {}^{\mathtt{H}}I^{\alpha}_{a+} f = f \quad \text{and} \quad {}^{\mathtt{H}}D^{\alpha}_{b-}\, {}^{\mathtt{H}}I^{\alpha}_{b-} f = f, \tag{1.3.155} $$

whereas if $\Re e(\alpha) > 0$, $\mu \in \mathbb{C}$, $a = 0$ and $b = \infty$, then

$$ {}^{\mathtt{H}}D^{\alpha}_{0+,\mu}\, {}^{\mathtt{H}}I^{\alpha}_{0+,\mu} f = f \quad \text{and} \quad {}^{\mathtt{H}}D^{\alpha}_{-,\mu}\, {}^{\mathtt{H}}I^{\alpha}_{-,\mu} f = f. \tag{1.3.156} $$

In particular, if $\mu = 0$, then

$$ {}^{\mathtt{H}}D^{\alpha}_{0+}\, {}^{\mathtt{H}}I^{\alpha}_{0+} f = f \quad \text{and} \quad {}^{\mathtt{H}}D^{\alpha}_{-}\, {}^{\mathtt{H}}I^{\alpha}_{-} f = f. \tag{1.3.157} $$

The following property yields the formula for the composition of the fractional differentiation operator ${}^{\mathtt{H}}D^{\alpha}_{a+}$ with the fractional integration operator ${}^{\mathtt{H}}I^{\alpha}_{a+}$.

$$ ({}^{\mathtt{H}}I^{\alpha}_{a+}\, {}^{\mathtt{H}}D^{\alpha}_{a+} y)(x) = y(x) - \sum_{k=1}^{n} \frac{(\delta^{n-k}({}^{\mathtt{H}}I^{n-\alpha}_{a+} y))(a)}{\Gamma(\alpha - k + 1)} \left(\log \frac{x}{a}\right)^{\alpha-k}, \tag{1.3.158} $$

for $\Re e(\alpha) > 0$, $n = -[-\Re e(\alpha)]$ and $0 < a < b < \infty$.

It is known that function series admit a term-by-term Riemann-Liouville fractional integration and differentiation under certain conditions. Similar assertions are true for Hadamard-type fractional integration and differentiation operators ${}^{\mathtt{H}}I^{\alpha}_{0+,\mu}$ and ${}^{\mathtt{H}}D^{\alpha}_{0+,\mu}$.

Proposition 1.1. *Let* $\alpha \in \mathbb{C}$, $\mu > 0$, $l > 0$, *and let* $f(x) = \sum_{k=0}^{\infty} f_k(x)$, $f_k(x) \in C([0, l])$.

(1) *If* $\Re e(\alpha) > 0$ *and the series* $f(x) = \sum_{k=0}^{\infty} f_k(x)$ *is uniformly convergent on* $[0, l]$, *then its termwise Hadamard-type integration* ${}^{\mathtt{H}}I_{0+,\mu}^{\alpha}$ *is admissible*

$$\left({}^{\mathtt{H}}I_{0+,\mu}^{\alpha} \sum_{k=0}^{\infty} f_k\right)(x) = \sum_{k=0}^{\infty} ({}^{\mathtt{H}}I_{0+,\mu}^{\alpha} f_k)(x) \quad (0 < x < l), \tag{1.3.159}$$

and the series $\sum_{k=0}^{\infty} ({}^{\mathtt{H}}I_{0+,\mu}^{\alpha} f_k)(x)$ *is also uniformly convergent on* $[0, l]$;

(2) *If* $\Re e(\alpha) \geq 0$ *and the series* $\sum_{k=0}^{\infty} f_k(x)$ *and* $\sum_{k=0}^{\infty} ({}^{\mathtt{H}}D_{0+,\mu}^{\alpha} f_k)(x)$ *are uniformly convergent on* $[\epsilon, l]$ $(\epsilon > 0)$, *then the former series admits termwise Hadamard-type fractional differentiation* ${}^{\mathtt{H}}D_{0+,\mu}^{\alpha}$ *by the formula*

$$\left({}^{\mathtt{H}}D_{0+,\mu}^{\alpha} \sum_{k=0}^{\infty} f_k\right)(x) = \sum_{k=0}^{\infty} \left({}^{\mathtt{H}}D_{0+,\mu}^{\alpha} f_k\right)(x) \ (0 < x < l). \tag{1.3.160}$$

Proposition 1.2. *Let* $\alpha \in \mathbb{C}$, $\mu > 0$, *and let* $f(x)$ *be a convergent power series*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad (a_k \in \mathbb{C}, \ k \in \mathbb{N}_0). \tag{1.3.161}$$

(1) *If* $\Re e(\alpha) > 0$, *then the Hadamard-type integral* ${}^{\mathtt{H}}I_{0+,\mu}^{\alpha} f$ *is also represented by the convergent power series*

$$\left({}^{\mathtt{H}}I_{0+,\mu}^{\alpha} f\right)(x) = \sum_{k=0}^{\infty} (\mu + k)^{-\alpha} a_k x^k. \tag{1.3.162}$$

(2) *If* $\Re e(\alpha) \geq 0$, *then the Hadamard-type derivative* ${}^{\mathtt{H}}D_{0+,\mu}^{\alpha} f$ *is also represented by the convergent power series*

$$\left({}^{\mathtt{H}}D_{0+,\mu}^{\alpha} f\right)(x) = \sum_{k=0}^{\infty} (\mu + k)^{\alpha} a_k x^k. \tag{1.3.163}$$

The radii of convergence of the series in (1.3.161), (1.3.162) *and* (1.3.163) *coincide.*

1.3.6 *Fractional integrals and fractional derivatives of a function with respect to another function*

In this section we present the definitions and some properties of the fractional integrals and fractional derivatives of a function f with respect to another function g. The Hadamard fractional integral and derivative are particular cases of these new operators.

Let (a, b) $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\Re e(\alpha) > 0$. Also let $g(x)$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $g'(x)$ on (a, b). The left- and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ are defined by

$$(I^{\alpha}_{a+;g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t)dt}{[g(x)-g(t)]^{1-\alpha}} \quad (x > a;\ \Re e(\alpha) > 0) \quad (1.3.164)$$

and

$$(I^{\alpha}_{b-;g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t)dt}{[g(t)-g(x)]^{1-\alpha}} \quad (x < b;\ \Re e(\alpha) > 0), \quad (1.3.165)$$

respectively.

When $a = 0$ and $b = \infty$, we shall use the following notations

$$(I^{\alpha}_{0+;g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{g'(t)f(t)dt}{[g(x)-g(t)]^{1-\alpha}} \quad (x > 0;\ \Re e(\alpha) > 0), \quad (1.3.166)$$

$$(I^{\alpha}_{-;g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{g'(t)f(t)dt}{[g(t)-g(x)]^{1-\alpha}} \quad (x > 0;\ \Re e(\alpha) > 0); \quad (1.3.167)$$

while, for $a = -\infty$ and $b = \infty$, we have

$$(I^{\alpha}_{+;g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{g'(t)f(t)dt}{[g(x)-g(t)]^{1-\alpha}} \quad (x \in \mathbb{R};\ \Re e(\alpha) > 0), \quad (1.3.168)$$

$$(I^{\alpha}_{-;g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{g'(t)f(t)dt}{[g(t)-g(x)]^{1-\alpha}} \quad (x \in \mathbb{R};\ \Re e(\alpha) > 0). \quad (1.3.169)$$

Integrals (1.3.164) and (1.3.165) are called the g-Riemann-Liouville fractional integrals on a finite interval $[a, b]$, (1.3.166) and (1.3.167) the g-Liouville fractional integrals on a half-axis $\mathbb{R}^+$, while (1.3.168) and (1.3.169) are called the g-Liouville fractional integrals on the whole axis $\mathbb{R}$.

For $\Re e(\alpha) > 0$ and $\Re e(\beta) > 0$, we have

$$(I^{\alpha}_{a+;g} f_+)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} [g(x) - g(a)]^{\alpha+\beta-1}, \tag{1.3.170}$$

where $f_+(x) = [g(x) - g(a)]^{\beta-1}$. If $f_-(x) = [g(b) - g(x)]^{\beta-1}$, then

$$(I^{\alpha}_{b-;g} f_-)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} [g(b) - g(x)]^{\alpha+\beta-1}. \tag{1.3.171}$$

Moreover, if $\Re e(\alpha) > 0$ and $\lambda > 0$, then

$$(I^{\alpha}_{+;g} e^{\lambda g(t)})(x) = \lambda^{-\alpha} e^{\lambda g(x)} \tag{1.3.172}$$

and

$$(I^{\alpha}_{-;g} e^{-\lambda g(t)})(x) = \lambda^{-\alpha} e^{-\lambda g(x)}. \tag{1.3.173}$$

The semigroup property also holds, i.e., if $\Re e(\alpha) > 0$ and $\Re e(\beta) > 0$, then the relations

$$(I^{\alpha}_{a+;g} I^{\beta}_{a+;g} f)(x) = (I^{\alpha+\beta}_{a+;g} f)(x); \quad (I^{\alpha}_{b-;g} I^{\beta}_{b-;g} f)(x) = (I^{\alpha+\beta}_{b-;g} f)(x) \tag{1.3.174}$$

and

$$(I^{\alpha}_{+;g} I^{\beta}_{+;g} f)(x) = (I^{\alpha+\beta}_{+;g} f)(x); \quad (I^{\alpha}_{-;g} I^{\beta}_{-;g} f)(x) = (I^{\alpha+\beta}_{-;g} f)(x) \tag{1.3.175}$$

hold for "sufficiently good" functions $f(x)$.

Let $g'(x) \neq 0$ $(-\infty \leq a < x < b \leq \infty)$ and $\Re e(\alpha) \geq 0$ $(\alpha \neq 0)$. Also let $n = -[-\Re e(\alpha)]$ and $D = d/dx$. The g-Riemann-Liouville and g-Liouville fractional derivatives of a function y with respect to g of order α $(\Re e(\alpha) \geq 0;\ \alpha \neq 0)$, corresponding to the g-Riemann-Liouville and g-Liouville integrals in (1.3.164)–(1.3.165), (1.3.166)–(1.3.167), and (1.3.168)–(1.3.169), are defined by

$$(D^{\alpha}_{a+;g} y)(x) = \left(\frac{1}{g'(x)} D\right)^n (I^{n-\alpha}_{a+;g} y)(x) \quad (x < b), \tag{1.3.176}$$

$$(D^{\alpha}_{b-;g} y)(x) = \left(-\frac{1}{g'(x)} D\right)^n (I^{n-\alpha}_{b-;g} y)(x) \quad (x < b), \tag{1.3.177}$$

and

$$(D^{\alpha}_{+;g}y)(x) = \left(\frac{1}{g'(x)}D\right)^n (I^{n-\alpha}_{+;g}y)(x) \quad (x \in \mathbb{R}), \tag{1.3.178}$$

$$(D^{\alpha}_{-;g}y)(x) = \left(-\frac{1}{g'(x)}D\right)^n (I^{n-\alpha}_{-;g}y)(x) \quad (x \in \mathbb{R}), \tag{1.3.179}$$

respectively.

When $g(x) = x$, (1.3.176) and (1.3.177) coincide with the Riemann-Liouville fractional derivatives (1.3.3) and (1.3.4)

$$(D^{\alpha}_{a+;x}y)(x) = ({}^{\mathtt{RL}}D^{\alpha}_{a+}y)(x) \quad \text{and} \quad (D^{\alpha}_{b-;x}y)(x) = ({}^{\mathtt{RL}}D^{\alpha}_{b-}y)(x), \tag{1.3.180}$$

and (1.3.178) and (1.3.179) coincide with the Liouville fractional derivatives (1.3.63) and (1.3.64)

$$(D^{\alpha}_{+;x}y)(x) = ({}^{\mathtt{L}}D^{\alpha}_{+}y)(x) \quad \text{and} \quad (D^{\alpha}_{-;x}y)(x) = ({}^{\mathtt{L}}D^{\alpha}_{-}y)(x). \tag{1.3.181}$$

For $\Re e(\alpha) \geq 0$ $(\alpha \neq 0)$ and $\Re e(\beta) > n - 1$, the above derivatives have the properties

$$(D^{\alpha}_{a+;g}y_+)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}[g(x) - g(a)]^{\beta-\alpha} \tag{1.3.182}$$

where $y_+(x) = [g(x) - g(a)]^{\beta}$, and

$$(D^{\alpha}_{b-;g}y_-)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}[g(b) - g(x)]^{\beta-\alpha} \tag{1.3.183}$$

where $y_-(x) = [g(b) - g(x)]^{\beta}$.

For $\Re e(\alpha) \geq 0$ $(\alpha \neq 0)$ and $\lambda > 0$, we have

$$(D^{\alpha}_{+;g}e^{\lambda g(t)})(x) = \lambda^{\alpha}e^{\lambda g(x)} \tag{1.3.184}$$

and

$$(D^{\alpha}_{-;g}e^{-\lambda g(t)})(x) = \lambda^{\alpha}e^{-\lambda g(x)}. \tag{1.3.185}$$

1.3.7 *Grünwald-Letnikov fractional derivatives*

In this section we give the definition of the Grünwald-Letnikov fractional derivatives and some of their properties. Such a fractional differentiation is based on a generalization of the classical differentiation of a function $y(x)$ of order $n \in \mathbb{N}$ via differential quotients,

$$y^{(n)}(x) = \lim_{h \to 0} \frac{(\Delta_h^n y)(x)}{h^n}. \tag{1.3.186}$$

Here $(\Delta_h^n y)(x)$ is a finite difference of order $n \in \mathbb{N}_0$ of a function $y(x)$ with a step $h \in \mathbb{R}$ and centered at the point $x \in \mathbb{R}$ defined by

$$(\Delta_h^n y)(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} y(x - kh) \quad (x, h \in \mathbb{R};\ n \in \mathbb{N}). \tag{1.3.187}$$

and

$$(\Delta_h^0 f)(x) = f(x). \tag{1.3.188}$$

Property (1.3.186) is used to define a fractional derivative by directly replacing $n \in \mathbb{N}$ in (1.3.186) by $\alpha > 0$. For this, h^n is replaced by h^α, while the finite difference $(\Delta_h^n y)(x)$ is replaced by the difference $(\Delta_h^\alpha y)(x)$ of a fractional order $\alpha \in \mathbb{R}$ defined by the following series

$$(\Delta_h^\alpha y)(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} y(x - kh) \quad (x, h \in \mathbb{R};\ \alpha > 0), \tag{1.3.189}$$

where $\binom{\alpha}{k}$ are the binomial coefficients. When $h > 0$, the difference (1.3.189) is called left-sided difference, while for $h < 0$ it is called a right-sided difference.

The series in (1.3.189) converges absolutely and uniformly for each $\alpha > 0$ and for every bounded function $y(x)$.

The fractional difference $(\Delta_h^\alpha y)(x)$ has the following semigroup property

$$(\Delta_h^\alpha \Delta_h^\beta y)(x) = (\Delta_h^{\alpha+\beta} y)(x) \tag{1.3.190}$$

for $\alpha > 0$ and $\beta > 0$.

On the other hand, if $\alpha > 0$ and $y(x) \in L_1(\mathbb{R})$, then the Fourier transform (1.1.1) of Δ_h^α is given by

$$(\mathcal{F}\Delta_h^\alpha y)(\kappa) = \left(1 - e^{i\kappa h}\right)^\alpha (\mathcal{F}y)(\kappa). \tag{1.3.191}$$

Following (1.3.186), the left- and right-sided Grünwald-Letnikov derivatives $y_{+}^{\alpha)}(x)$ and $y_{-}^{\alpha)}(x)$ are defined by

$$y_{+}^{\alpha)}(x) = \lim_{h\to+0} \frac{(\Delta_h^\alpha y)(x)}{h^\alpha} \quad (\alpha > 0) \tag{1.3.192}$$

and

$$y_{-}^{\alpha)}(x) = \lim_{h\to+0} \frac{(\Delta_{-h}^\alpha y)(x)}{h^\alpha} \quad (\alpha > 0), \tag{1.3.193}$$

respectively. These constructions coincide with the Marchaud fractional derivatives for $y(x) \in L_p(\mathbb{R})$ $(1 \leq p < \infty)$.

Then, by analogy with (1.3.192) and (1.3.193), the left- and right-sided Grünwald-Letnikov fractional derivatives of order $\alpha > 0$ on a finite interval $[a, b]$ are defined by

$$y_{a+}^{\alpha)}(x) = \lim_{h\to+0} \frac{(\Delta_{h,a+}^\alpha y)(x)}{h^\alpha} \tag{1.3.194}$$

and

$$y_{b-}^{\alpha)}(x) = \lim_{h\to+0} \frac{(\Delta_{h,b-}^\alpha y)(x)}{h^\alpha}, \tag{1.3.195}$$

respectively, where

$$(\Delta_{h,a+}^\alpha y)(x) = \sum_{k=0}^{[\frac{x-a}{h}]} (-1)^k \binom{\alpha}{k} y(x-kh) \quad (x \in \mathbb{R};\ \alpha, h > 0) \tag{1.3.196}$$

and

$$(\Delta_{h,b-}^\alpha y)(x) = \sum_{k=0}^{[\frac{b-x}{h}]} (-1)^k \binom{n}{k} y(x-kh) \quad (x, \in \mathbb{R};\ \alpha, h > 0). \tag{1.3.197}$$

Such Grünwald-Letnikov fractional derivatives coincide with the Marchaud fractional derivatives, for sufficiently well-behaved functions, and can be represented in the form

$$y_{a+}^{\alpha)}(x) = \frac{y(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{y(x)-y(t)}{(x-t)^{1+\alpha}} dt \quad (0 < \alpha < 1) \tag{1.3.198}$$

and

$$y_{b-}^{\alpha)}(x) = \frac{y(x)}{\Gamma(1-\alpha)(b-x)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_x^b \frac{y(x)-y(t)}{(t-x)^{1+\alpha}} dt \quad (0 < \alpha < 1). \tag{1.3.199}$$