Chapter 6

Continuous-Time Random Walks and Fractional Diffusion Models

6.1 Introduction

This chapter contains a rather pedantic introduction to continuous-time random walks (CTRWs) and their link to space-time fractional diffusion. The material covered below is standard and rather elementary in the analysis of stochastic processes, but applied scientists may have not been previously exposed to it, at least in this form. The basic idea is to derive a theorem relating the continuous or hydrodynamic limit of CTRWs to the space-time fractional diffusion as quickly as possible. The cost to be paid is generality, even if we give pointers to the literature helping in filling gaps. We assume that our readers are more interested in basic ideas rather than in mathematical details. Incidentally, Theorem 6.16 shows how the Caputo time derivative naturally emerges when some known results on normal diffusion are generalized to fractional diffusion. One can go on and use an equivalent formulation in terms of Riemann-Liouville fractional derivatives, but there is no point in doing that. A similar presentation of this material, but with more emphasis on the compound Poisson process and a discussion of the so-called Montroll-Weiss equation [409] can be found in a chapter of a recent collective book [508].

CTRWs are used in physics to model single particle (tracer) diffusion when the tracer time of residence in a site is much larger than the time needed to jump to another site [409, 512, 513, 408, 520, 395, 396]. CTRWs are phenomenological models and do not include microscopic theories for tracer motion. However, the reader is warned that the processes called CTRWs in the literature on physics and chemical physics are known as generalized compound Poisson processes or compound renewal processes in the mathematical literature and they have a long history. Compound Poisson processes can be used to approximate Lévy processes. Indeed, as discussed by Feller in Chapter 17 of the second volume of his book [214] every infinitely divisible distribution can be represented as the limit of a sequence of compound Poisson distributions. The importance of these processes prompted de Finetti to devote part of his second volume in probability theory to compound Poisson processes as well [165]. In Theorem 6.12, we will derive a distribution given by equation (6.2.66). When $P(n,t)$ is the distribution of the Poisson process, equation (6.2.66) is also known as generalized Poisson distribution and it was discussed by Feller in his 1943 paper [213]. From the modeling viewpoint, this distribution is quite versatile. In the words of Feller [213]:

> Consider independent random events for which the simple Poisson distribution may be assumed, such as: telephone calls, the occurrence of claims in an insurance company, fire accidents, sickness and the like. With each event there may be associated a random variable X. Thus, in the above examples, X may represent the length of the ensuing conversation, the sum under risk, the damage, the cost (or length) of hospitalization, respectively.

The applications to insurance problems were indeed available at the beginning of the XX Century [361, 151]. Already in 1943, Feller also wrote [213]:

> In view of the above examples, it is not surprising that the law, or special cases of it, have been discovered, by various means and sometimes under disguised forms, by many authors.

This process of rediscovery went on also after Feller's paper; as outlined above, it is the case of physics, where X is interpreted as tracer's position. More recently, for financial applications, X is seen as the log-return for a stock [509, 374, 507, 380]. More on that will be presented in the next chapter.

6.2 The Definition of Continuous-Time Random Walks

In this section, we shall formally define CTWRs as random walks subordinated to a counting renewal process. Essentially, we need two basic ingredients:

(1) the random walk X_n;
(2) the counting process $N(t)$.

Let us begin with the random walk. This is a stochastic process given by a sum of independent and identically distributed (i.i.d.) random variables.

Definition 6.1. Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative distribution function given by $F_Y(y) = \mathbb{P}(Y \leq y)$, then the process X_n defined as

$$X_0 = 0 \tag{6.2.1}$$

$$X_n = \sum_{i=1}^{n} Y_i, \; n \geq 1 \tag{6.2.2}$$

is a random walk.

In this book, we will not deal with issues of existence of stochastic processes, however it is useful to see that the definition is not void. Let us consider the following example.

Example 6.1. Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of Bernoullian random variables with

$$F_Y(y) = \frac{1}{2}\theta(y) + \frac{1}{2}\theta(y-1) \tag{6.2.3}$$

where $\theta(x)$ is the càdlàg (*continue à droite, limite à gauche*) version of Heaviside function. This means that, for all $i \geq 1$, one has $Y_i = 0$ with probability $1/2$ or $Y_i = 1$ with probability $1/2$. In this case, the random walk X_n is just the number of *successes* up to time step n. Its one-point distribution is given by the binomial distribution of parameters $1/2$ and n. In other words, one can write $X_n \sim \mathrm{Bin}(1/2, n)$, or, more explicitly

$$P(k,n) = \mathbb{P}(X_n = k) = \binom{n}{k}\frac{1}{2^n}. \tag{6.2.4}$$

In Example 6.1, we were able to derive the one-point distribution function $P(k,n)$ of the random walk X_n using a well-known result of elementary probability theory. Elementary probability theory is helpful in deriving a general formula for the cumulative distribution function $F_{X_n}(x) = \mathbb{P}(X_n \leq x)$ and for a generic random walk.

Theorem 6.1. *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative distribution function given by $F_Y(y)$. Then the cumulative distribution function of the corresponding random walk X_n is given by the n-fold convolution of $F_Y(y)$, in symbols one gets*

$$F_{X_n}(x) = F_Y^{*n}(x). \tag{6.2.5}$$

Proof. Let us use induction on n for $n \geq 2$. To see that the formula is true for $n = 2$, let us consider the random variable $X_2 = Y_1 + Y_2$. One has the following chain of equalities

$$\begin{aligned} F_{X_2}(x) &= \mathbb{P}(X_2 \leq x) = \mathbb{P}(Y_1 + Y_2 \leq x) = \mathbb{E}(I_{Y_1+Y_2 \leq x}) \\ &= \mathbb{E}\left(I_{\{Y_1 \in \mathbb{R}\} \cap \{Y_2 \leq x - Y_1\}}\right) = \mathbb{E}\left(I_{Y_1 \in \mathbb{R}} I_{Y_2 \leq x - Y_1}\right) \\ &= \int_{-\infty}^{+\infty} dF_Y(y_1) \int_{-\infty}^{x-y_1} dF_Y(y_2) = \int_{-\infty}^{+\infty} dF_Y(y_1) F_Y(x - y_1) = F_Y^{*2}(x). \end{aligned} \tag{6.2.6}$$

Now suppose that the formula is true for $n-1$ and let us prove that it holds also for n. The inductive hypothesis is $F_{X_{n-1}}(x) = F_Y^{*(n-1)}(x)$. Taking into account definition 6.1, one has $X_n = X_{n-1} + Y_n$ and X_{n-1} is independent of Y_n, therefore, as before, we can write

$$\begin{aligned} F_{X_n}(x) &= \mathbb{P}(X_n \leq x) = \mathbb{P}(X_{n-1} + Y_n \leq x) = \mathbb{E}(I_{X_{n-1}+Y_n \leq x}) \\ &= \mathbb{E}\left(I_{\{X_{n-1} \in \mathbb{R}\} \cap \{Y_n \leq x - X_{n-1}\}}\right) = \mathbb{E}\left(I_{X_{n-1} \in \mathbb{R}} I_{Y_n \leq x - X_{n-1}}\right) \\ &= \int_{-\infty}^{+\infty} dF_Y^{*(n-1)}(u) \int_{-\infty}^{x-u} dF_Y(w) = \int_{-\infty}^{+\infty} dF_Y^{*(n-1)}(u) F_Y(x-u) = F_Y^{*n}(x). \end{aligned} \tag{6.2.7}$$

The latter chain of equalities completes the inductive proof. □

Remark 6.1. In the above derivation, I_A denotes the indicator function of event A. Moreover, we have used the fact that $\mathbb{P}(A) = \mathbb{E}(I_A)$ and that $I_{A \cap B} = I_A I_B$ so that $\mathbb{P}(A \cap B) = \mathbb{E}(I_A I_B)$.

Remark 6.2. Note that one has $F_Y^{*1}(x) = F_Y(x)$. The meaning of $F_Y^{*0}(x)$ is more interesting. Indeed, it is possible to show that $F_Y^{*0}(x) = \theta(x)$ where $\theta(x)$ denotes the càdlàg version of the Heaviside function.

Remark 6.3. The convolution in equation (6.2.5) is known as Lebesgue-Stieltjes convolution (or convolution of measures) [93]. This is connected to the Lebesgue convolution (or convolution of functions) discussed in Section 1.1. The following corollary establishes the connection. Note that some authors, such as Feller [214], use a different notation for the two convolutions. The convolution of measures is denoted by $\star$ and the convolution of functions by $*$.

Corollary 6.1. *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative distribution function given by $F_Y(y)$. Assume that the probability density function $f_Y(y) = dF_Y(y)/dy$ exists. Then the probability density function of the corresponding random walk X_n is given by the n-fold convolution of $f_Y(y)$, in symbols one can write*

$$f_{X_n}(x) = f_Y^{*n}(x). \tag{6.2.8}$$

Proof. It is sufficient to derive equation (6.2.5) in order to get this result. Recall that if $f_Y(x)$ exists, one can write $dF_Y(x) = f_Y(x)dx$. □

Remark 6.4. Note that the derivative of Heaviside function, $\theta(x)$, is a generalized function [518] known as Dirac delta, $\delta(x)$.

Theorem 6.1 and Corollary 6.1 show that the one-point measure of random walk is the n-fold convolution of the *jump* measure. For this reason, it is useful to use Fourier transforms when dealing with sums of independent (and identically distributed) random variables. Given a random variable X, one can define its characteristic function as follows.

Definition 6.2. Let X be a random variable, its characteristic function $\widehat{f}_X(\kappa)$ is given by

$$\widehat{f}_X(\kappa) = \mathbb{E}\left(\mathrm{e}^{i\kappa X}\right). \tag{6.2.9}$$

Theorem 6.2. *If the random variable X has a probability density function $f_X(x)$, then its characteristic function is just the Fourier transform of the probability density function. In symbols, one has*

$$\widehat{f}_X(\kappa) = \mathcal{F}[f_X(x)](\kappa) = \int_{-\infty}^{+\infty} dx\, \mathrm{e}^{i\kappa x} f_X(x) \tag{6.2.10}$$

Proof. The proof immediately follows from the definition

$$\widehat{f}_X(\kappa) = \mathbb{E}\left(\mathrm{e}^{i\kappa X}\right) = \int_{-\infty}^{+\infty} dx\, \mathrm{e}^{i\kappa x} f_X(x), \tag{6.2.11}$$

an elementary result which will be very useful. □

Remark 6.5. The conditions on $f_X(x)$ for the existence of the Fourier transform are not too demanding. For instance, the Fourier transform may also exist for generalized functions. In the case of Dirac delta, one has

$$\mathcal{F}[\delta(x)](\kappa) = \int_{-\infty}^{+\infty} dx\, \delta(x)\mathrm{e}^{i\kappa x} = 1. \tag{6.2.12}$$

If Lebesgue-Stieltjes integrals are used, a generic probability density function $f_X(x)$ will be a non-negative generalized function satisfying the constraint

$$\int_{-\infty}^{+\infty} dx f_X(x) = 1, \tag{6.2.13}$$

and its Fourier transform will exist [140]. It is possible to prove that the characteristic function has the following properties:

(1) it is continuous for every $\kappa \in \mathbb{R}$;
(2) $\widehat{f}_X(0) = 1$;
(3) $\widehat{f}_X(\kappa)$ is a positive semi-definite function.

Of these three properties, only the third one needs further illustration. This is a rather technical condition. Take an arbitrary integer n and a set of real numbers $\kappa_1, \ldots, \kappa_n$, then build the matrix $a_{i,j} = \widetilde{f}(\kappa_i - \kappa_j)$. Then this matrix is positive semi-definite. A theorem due to Bochner shows that the converse is true, i.e., any function with the three properties above is the characteristic function of a random variable [100].

Remark 6.6. The derivatives of the characteristic function in $\kappa = 0$ are related to the moments of the corresponding random variable. The reader can check that

$$\mathbb{E}[Y] = -i \left.\frac{d\widehat{f}_Y(\kappa)}{d\kappa}\right|_{\kappa=0}, \tag{6.2.14}$$

and that

$$\mathbb{E}[Y^2] = -\left.\frac{d^2\widehat{f}_Y(\kappa)}{d\kappa^2}\right|_{\kappa=0}. \tag{6.2.15}$$

The convolution theorem for Fourier transforms (1.1.9) is very important when studying the sum of independent random variables, as it transforms convolutions into algebraic equations. The proof of this theorem, with different levels of detail, can be found in any book on Fourier methods. See reference [140] for example. The convolution theorem for Fourier transform has an immediate consequence on the characteristic function of the random walk X_n.

Corollary 6.2. *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative distribution function given by $F_Y(y)$ and generalized probability distribution function denoted by $f_Y(y)$. Let X_n be the corresponding random walk. Then the characteristic function of the random walk is given by*

$$\widehat{f}_{X_n}(\kappa) = \mathbb{E}\left(\mathrm{e}^{ikX_n}\right) = [\widehat{f}_Y(\kappa)]^n. \tag{6.2.16}$$

Proof. This statement is a direct consequence of Corollary 6.1 and of the convolution theorem (1.1.9). □

The following example is a simple way to introduce the concept of stable random variables.

Example 6.2. Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. random variables with probability density function given by

$$f_Y(y) = \frac{1}{\sqrt{4\pi}}\,\mathrm{e}^{-y^2/4}. \tag{6.2.17}$$

This is the normal (Gaussian) probability density function with expectation $\mu = 0$ and variance $\sigma^2 = 2$; in other terms, we have $Y_i \sim N(0,2)$. Which is the characteristic function of the corresponding random walk X_n? For the Y_is, the characteristic function is

$$\widehat{f}_Y(\kappa) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{+\infty} dy\,\mathrm{e}^{i\kappa y - y^2/4} = \mathrm{e}^{-k^2}. \tag{6.2.18}$$

Therefore, from equation (6.2.16), one gets for X_n

$$\widehat{f}_{X_n}(\kappa) = \left[\mathrm{e}^{-k^2}\right]^n = \mathrm{e}^{-nk^2}. \tag{6.2.19}$$

This Fourier transform can be easily inverted to get the probability density function $p(x,n) = f_{X_n}(x)$

$$p(x,n) = f_{X_n}(x) = \frac{1}{\sqrt{4n\pi}}\,\mathrm{e}^{-x^2/4n}. \tag{6.2.20}$$

The reader can prove that when $Y_i \sim N(\mu, \sigma^2)$, that is when one has

$$f_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, \mathrm{e}^{-(x-\mu)^2/2\sigma^2}, \tag{6.2.21}$$

the characteristic function for jumps is

$$\widehat{f}_Y(\kappa) = \mathrm{e}^{i\mu\kappa - \sigma^2\kappa^2/2}, \tag{6.2.22}$$

the characteristic function of the random walk is

$$\widehat{f}_{X_n}(\kappa) = \mathrm{e}^{in\mu\kappa - n\sigma^2\kappa^2/2}, \tag{6.2.23}$$

and its probability density function is

$$p(x,n) = f_{X_n}(x) = \frac{1}{\sqrt{2\pi n\sigma^2}}\, \mathrm{e}^{-(x-n\mu)^2/2n\sigma^2}. \tag{6.2.24}$$

By comparing equations (6.2.21) and (6.2.24), one can see that the n-fold convolution of a normal distribution is still a normal distribution, but with parameters rescaled by n. Another way of expressing this result is through the concept of stable random variable.

Definition 6.3. Let Y_1 and Y_2 be two independent and identically distributed random variables that can be seen as copies of a random variable Y. Then, Y is said to be stable or stable in the broad sense if for any constants a and b, the sum $aY_1 + bY_2$ is distributed as $cY + d$ for some constants c and d. If $d = 0$, then the random variable Y is called strictly stable [430].

Remark 6.7. There are alternative and equivalent definitions of stable random variables. In the following, we will use the characterization in terms of characteristic functions [430]. The proof of equivalence can be found in [502].

Definition 6.4. A random variable Y is stable if and only if it can be written as $Y = aZ + b$ and the characteristic function of Z is given by

$$\widehat{f}_Z(\kappa) = \mathbb{E}(\mathrm{e}^{i\kappa Z}) = \mathrm{e}^{-|\kappa|^\alpha[1-i\gamma\tan(\pi\alpha/2)\mathrm{sign}(\kappa)]} \tag{6.2.25}$$

for $0 < \alpha < 1$, $1 < \alpha \leq 2$, $-1 \leq \gamma \leq 1$, $a > 0$ and $b \in \mathbb{R}$, and

$$\widehat{f}_Z(\kappa) = \mathbb{E}(\mathrm{e}^{i\kappa Z}) = \mathrm{e}^{-|\kappa|[1+i(2/\pi)\mathrm{sign}(\kappa)\log(|u|)]} \tag{6.2.26}$$

for the special case $\alpha = 1$.

The reader might wish to check directly that normal random variables are stable, being invariant under convolution or by studying the characteristic function of the normal distribution. Another important property of stable random variables is that they are attractors: under suitable hypotheses, when convolutions are repeated infinitely many times, the limiting distribution is given by a stable random variable. This is essentially the content of the celebrated central limit theorem. We will now discuss it in its Lindenberg-Lévy version. For that, we need to define the convergence in distribution for a sequence of random variables. Let us first define the so-called weak convergence for sequences of function

Definition 6.5. Let $\{F_i(x)\}_{i=1}^{\infty}$ be a sequence of cumulative probability distribution functions defined on $\mathbb{R}$. The sequence converges weakly to the cumulative probability distribution $F(x)$ and we write

$$\lim_{i\to\infty} F_i(x) \stackrel{w}{=} F(x) \tag{6.2.27}$$

if one has

$$\lim_{i\to\infty} F_i(x) = F(x) \tag{6.2.28}$$

on all the $x \in \mathcal{C}_F$, where $\mathcal{C}_F$ is the continuity set of $F(x)$.

Once weak convergence is defined, we can use it to define convergence in distribution for random variables.

Definition 6.6. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of random variables. We say that the sequence converges in distribution to a random variable X, and we write

$$\lim_{i\to\infty} X_i \stackrel{d}{=} X \tag{6.2.29}$$

if one has

$$\lim_{i\to\infty} F_{X_i}(x) = \mathbb{P}(X_i \leq x) \stackrel{w}{=} F_X(x). \tag{6.2.30}$$

The Lévy continuity theorem provides a necessary condition for the convergence in distribution.

Theorem 6.3. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of random variables with cumulative distribution functions $F_{X_i}(x)$. Let $\widehat{f}_{X_i}(\kappa) = \mathbb{E}(\mathrm{e}^{i\kappa X_i})$ be the corresponding characteristic functions. Assume that there is a function $\widehat{f}(\kappa)$ such that for any $\kappa \in \mathbb{R}$ (pointwise), one has*

$$\lim_{i\to\infty} \widehat{f}_{X_i}(\kappa) = \widehat{f}(\kappa), \tag{6.2.31}$$

with $\widehat{f}(\kappa)$ continuous for $\kappa = 0$, $\widehat{f}(0) = 1$ and positive semi-definite. Then, there is a random variable X and a corresponding cumulative distribution function $F_X(x)$ such that $\widehat{f}(\kappa)$ is the characteristic function of X and

$$\lim_{i\to\infty} F_{X_i}(x) \overset{w}{=} F_X(x). \tag{6.2.32}$$

Proof. A nice proof of this theorem is contained in a book by David Williams [582], Chapter 18. □

Theorem 6.4. *Let $\{Y_i\}_{i=1}^{\infty}$ a sequence of i.i.d. random variables, such that their common expected value is $\mu_Y = \mathbb{E}(Y) < \infty$ and their common variance is $\sigma_Y^2 = \mathbb{E}[(Y - \mu_Y)^2] < \infty$. Let X_n be the corresponding random walk. Define the random variable $Z_n = \sqrt{n}(X_n/n - \mu_Y)/\sigma_Y$, then for $n \to \infty$, Z_n converges in distribution to a normally distributed random variable $Z \sim N(0,1)$.*

Proof. As a consequence of the convolution theorem (1.1.9) and of equation (6.2.15), one gets for a generic $\kappa \in \mathbb{R}$

$$\widehat{f}_{Z_n}(\kappa) = \left[\widehat{f}_Y\left(\frac{\kappa}{\sqrt{n}\sigma_Y}\right)\right]^n = \left[1 - \frac{1}{2}\frac{\kappa^2}{n} + \mathrm{o}\left(\frac{\kappa^2}{n\sigma_Y^2}\right)\right]^n, \tag{6.2.33}$$

where the first two non-vanishing terms of MacLaurin expansion for the characteristic function are highlighted. Therefore, one has that

$$\lim_{n\to\infty} \widehat{f}_{Z_n}(\kappa) = \mathrm{e}^{-\kappa^2/2}. \tag{6.2.34}$$

Equation (6.2.22) and the Lévy continuity theorem immediately imply that $Z_n \xrightarrow{d} Z$ for $n \to \infty$ with $Z \sim N(0,1)$. □

The central limit theorem can be generalized in different directions. In our opinion, the fastest path to understand the connection between fractional diffusion and continuous-time random walks is to use symmetric α-stable random variables [600, 289, 502, 430].

Definition 6.7. A symmetric α-stable random variable Y_α has the following characteristic function

$$\widehat{f}_{Y_\alpha}(\kappa) = \mathrm{e}^{-|\kappa|^\alpha}, \tag{6.2.35}$$

for $\alpha \in (0, 2]$.

Remark 6.8. Note that this definition is a simplified version of Definition 6.4.

Remark 6.9. In general, neither the cumulative distribution function $F_{Y_\alpha}(y)$ nor the probability density $f_{Y_\alpha}(y)$ can be written in terms of elementary functions. There are exceptions. As a consequence of equation (6.2.22), one has that $Y_2 \sim N(0, 2)$. Another remarkable case is $\alpha = 1$ which coincides with the Cauchy distribution. Note that these distributions have infinite second moment for $\alpha \in (0, 2)$, whereas their first moment is finite for $\alpha \in (1, 2]$. In the applied literature, a scale parameter $h > 0$ is often included in the definition, and one writes $\widehat{f}_{Y_\alpha}(\kappa|h) = \mathrm{e}^{-|h\kappa|^\alpha}$. Sometimes, the scale parameter has the form $c = h^\alpha$, so that one has $\widehat{f}_{Y_\alpha}(\kappa|c) = \mathrm{e}^{-c|\kappa|^\alpha}$. It is a useful exercise to check that Y_α defined in Definition 6.7 is indeed a stable random variable according to Definition 6.3.

Similarly to the Normal distribution, symmetric α-stable distributions are invariant as well as attractors for the convolution. Gnedenko and Kolmogorov and Lévy proved a generalization of the central limit theorem, involving sums of independent and identically distributed random variables with infinite second moment [237, 340].

Theorem 6.5. *Let $\{Y\}_{i=1}^\infty$ be a sequence of i.i.d. random variables such that their characteristic function has the following behavior in the neighborhood of $\kappa = 0$*

$$\widehat{f}_Y(\kappa) = 1 - |\kappa|^\alpha + \mathrm{o}(|\kappa|^\alpha). \tag{6.2.36}$$

Let X_n be the corresponding random walk and define $Z_n = X_n/n^{1/\alpha}$, then one has

$$\lim_{n\to\infty} Z_n \stackrel{d}{=} Z, \tag{6.2.37}$$

where Z is a symmetric α-stable distribution.

Proof. Given a $\kappa \in \mathbb{R}$, one has the following chain of equalities

$$\widehat{f}_{Z_n}(\kappa) = \left[\widehat{f}_Y\left(\frac{\kappa}{n^{1/\alpha}}\right)\right]^n = \left[1 - \frac{|\kappa|^\alpha}{n} + \mathrm{o}\left(\frac{|\kappa|^\alpha}{n}\right)\right]^n, \tag{6.2.38}$$

so that

$$\lim_{n\to\infty} \widehat{f}_{Z_n}(\kappa) = \mathrm{e}^{-|k|^\alpha}. \tag{6.2.39}$$

The Lévy continuity theorem yields the thesis. □

A natural question is whether condition (6.2.36) is satisfied by some random variable. Indeed, random variables whose cumulative distribution function have a power-law behavior for $|y| \to \infty$ do satisfy (6.2.36). This result can be presented with different levels of sophistication (see e.g. [502]). Here, we present a simplified version; for more details, the reader can consult a paper by R. Gorenflo and E.A.A. Abdel-Rehim [246].

Theorem 6.6. *Let Y be a symmetric random variable and assume that its cumulative distribution function $F_Y(y)$ has the following asymptotic behavior for large y and for $\alpha \in (0, 2)$*

$$\lim_{y\to\infty} \frac{b}{\alpha}\frac{F_Y(y)}{1/y^\alpha} = 1, \tag{6.2.40}$$

with

$$b = \frac{\Gamma(\alpha+1)\sin(\alpha\pi/2)}{\pi}, \tag{6.2.41}$$

then the characteristic function of Y satisfies condition (6.2.36)*.*

Proof. This theorem is proved in Chapter 8 of [93]. □

It is now possible to define the counting process $N(t)$, the second ingredient needed for the continuous-time random walk. Here, we shall only consider counting processes of renewal type.

Definition 6.8. Let $\{J\}_{i=1}^\infty$ be a sequence of i.i.d. positive random variables interpreted as sojourn times between subsequent events arriving at random times. They define a renewal process whose epochs of renewal (time instants at which the events occur) are the random times $\{T\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} T_0 &= 0, \\ T_n &= \sum_{i=1}^{n} J_i. \end{aligned} \tag{6.2.42}$$

The name renewal process is due to the fact that at any epoch of renewal, the process starts again from the beginning.

Definition 6.9. Associated to any renewal process, there is the process $N(t)$ defined as

$$N(t) = \max\{n : T_n \leq t\} \tag{6.2.43}$$

counting the number of events up to time t.

Remark 6.10. The counting process $N(t)$ is the Poisson process if and only if $J \sim \exp(\lambda)$, i.e., if and only if sojourn times are i.i.d. exponentially distributed random variables with parameter λ. Incidentally, this is the only case of Lévy and Markov counting process related to a renewal process (see Çinlar's book [138] for a proof of this statement).

Remark 6.11. We shall always assume that the counting process has càdlàg (*continue à droite et limite à gauche* i.e. right continuous with left limits) sample paths. This means that the realizations are represented by step functions. If t_k is the epoch of the k-th jump, we have $N(t_k^-) = k - 1$ and $N(t_k^+) = k$.

Remark 6.12. In equation (6.2.43), max is used instead of the more general sup as only processes with finite (but arbitrary) number of jumps in $(0, t]$ are considered here.

Given the cumulative probability distribution function $F_J(t)$ for the sojourn times, one immediately gets the distribution for the corresponding epochs.

Theorem 6.7. *Let $\{J\}_{i=1}^{\infty}$ be a sequence of i.i.d. sojourn times with cumulative distribution function $F_J(t)$, then one gets for the generic epoch T_n*

$$F_{T_n}(t) = F_J^{*n}(t). \tag{6.2.44}$$

Proof. This theorem is the same as Theorem 6.1 with T_n playing the role of X_n. The only difference is that the cumulative distribution function is non-vanishing only for positive support. □

Example 6.3. Assume that $J \sim \exp(\lambda)$, then it can be proved by direct calculation that the epochs T_n follow the so-called Erlang distribution

$$F_{T_n}(t) = 1 - \mathrm{e}^{-\lambda t} \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!}. \tag{6.2.45}$$

The distribution of the random variable $N(t)$ can be derived from the knowledge of $F_J(t)$ as well.

Theorem 6.8. *Let $\{J\}_{i=1}^{\infty}$ be a sequence of i.i.d. sojourn times with cumulative distribution function $F_J(t)$, then one has*

$$P(n,t) = \mathbb{P}(N(t) = n) = (f_J^{*n} * \bar{F}_J)(t) = \int_0^t du f_J^{*n}(u)\bar{F}_J(t-u), \tag{6.2.46}$$

where $f_J(t)$ is the probability density function of sojourn times J and $\bar{F}_J(t) = 1 - F_J(t)$ is the complementary cumulative distribution function.

Proof. The event $\{N(t) = n\}$ is equivalent to the event $\{T_n \leq t\} \cap \{T_{n+1} > t\} = \{T_n \leq t, T_{n+1} > t\}$. Further observe that $T_{n+1} = T_n + J_{n+1}$ and T_n and J_{n+1} are independent random variables. Now, the following chain of equalities holds true

$$\begin{aligned}
\mathbb{P}(N(t) = n) &= \mathbb{P}(T_n \leq t, T_{n+1} > t) = \mathbb{P}(T_n \leq t, J_{n+1} > t - T_n) \\
&= \mathbb{E}\left(I_{\{T_n \leq t\}} I_{\{J_{n+1} > t - T_n\}}\right) = \int_{T_n \leq t} du\, f_J^{*n}(u) \int_{J_{n+1} > t - T_n} dw\, f_J(w) \\
&= \int_0^t du\, f_J^{*n}(u) \int_{t-u}^{\infty} dw\, f_j(w) = \int_0^t du\, f_J^{*n}(u)[1 - F_J(t-u)] \\
&= \int_0^t du f_J^{*n}(u)\bar{F}_J(t-u) = (f_J^{*n} * \bar{F}_J)(t).
\end{aligned} \tag{6.2.47}$$

In the above proof, we have used the properties of indicator functions presented in Remark 6.1. Moreover, the independence of T_n and J_{n+1} implies that their joint probability density function is the product of the two marginals $f_{T_n, J_{n+1}}(u, w) = f_{T_n}(u) f_{J_{n+1}}(w) = f_J^{*n}(u) f_J(w)$. □

For positive random variables, the Laplace transform defined in equation (1.1.15) plays the same role as the Fourier transform.

Definition 6.10. Let Y be a positive random variable, then its Laplace transform can be written as

$$\widetilde{f}_Y(s) = \mathbb{E}\left(\mathrm{e}^{-sY}\right), \tag{6.2.48}$$

with $s \in \mathbb{C}$.

Theorem 6.9. *Let $f_Y(y)$ (with $y > 0$) denote the (generalized) probability density function of a positive random variable Y, then, for $s \in \mathbb{C}$, the Laplace transform is given by*

$$\widetilde{f}_Y(s) = \mathbb{E}(\mathrm{e}^{-sY}) = \mathcal{L}[f_Y(y)](s) = \int_0^\infty dy\, f_Y(y) e^{-st}. \tag{6.2.49}$$

Proof. This is an immediate consequence of the definition (and indeed it could be incorporated in the definition itself). If $f_Y(y)$ is a generalized function, equation (6.2.49) is often called the Laplace-Stieltjes transform of the random variable or the Laplace-Stieltjes transform of the cumulative distribution function $F_Y(y)$. □

Remark 6.13. Let $s = \Re e(s) + i\Im m(s)$, then one can write

$$\mathcal{L}[f_Y(y)](s) = \widetilde{f}_Y(s) = \int_0^\infty dy\, f_Y(y) \mathrm{e}^{-\Re e(s) y} \mathrm{e}^{i(-\Im m(s))y}. \tag{6.2.50}$$

in other words, the Laplace transform, can be seen as the Fourier transform calculated for $\kappa = -\Im m(s)$ for the function $g_Y(y)$ which is 0 for $y < 0$ and equals $f_Y(y)\mathrm{e}^{-\Re e(s)y}$ for $y > 0$. For values of s in which $g_Y(y) \in L^1(\mathbb{R})$, the Laplace transform exists. A classical reference on Laplace transforms is the book by Widder [581].

As discussed in Section 1.1, a convolution theorem holds true also for Laplace transforms. This is given in equation (1.1.22). This theorem is proved in any textbook on Laplace transforms, see e.g. the book by LePage [336]. It is now possible to define the fractional Poisson process [476, 332, 510, 372] as a counting renewal process.

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Definition 6.11. The Mittag-Leffler renewal process is the sequence $\{J_{\beta,i}\}_{i=1}^{\infty}$ of positive independent and identically distributed random variables with complementary cumulative distribution function $\bar{F}_{J_\beta}(0,t)$ given by

$$\bar{F}_{J_\beta}(t) = E_\beta\left(-t^\beta\right), \tag{6.2.51}$$

where $E_\beta(z)$ is the one-parameter Mittag-Leffler function defined in equation (1.2.30).

Remark 6.14. The one-parameter Mittag-Leffler function in (6.2.51) is a generalization of the exponential function. It coincides with the exponential function for $\beta = 1$. The function $E_\beta(-t^\beta)$ is completely monotonic and it is 1 for $t = 0$. This means that it is a legitimate survival function.

Remark 6.15. The function $E_\beta\left(-t^\beta\right)$ behaves as a stretched exponential for $t \to 0$:

$$E_\beta\left(-t^\beta\right) \simeq 1 - \frac{t^\beta}{\Gamma(\beta+1)} \simeq \mathrm{e}^{-t^\beta/\Gamma(\beta+1)}, \quad \text{for } 0 < t \ll 1, \tag{6.2.52}$$

and as a power-law for $t \to \infty$:

$$E_\beta(-t^\beta) \simeq \frac{\sin(\beta\pi)}{\pi}\frac{\Gamma(\beta)}{t^\beta}, \quad \text{for } t \gg 1. \tag{6.2.53}$$

Remark 6.16. For applications, it is often convenient to include a scale parameter in the definition (6.2.51), and one can write

$$\bar{F}_{J_\beta}(t) = E_\beta\left(-(t/\gamma)^\beta\right). \tag{6.2.54}$$

The scale factor can be introduced in different ways, and the reader is warned to pay attention to its definition. The assumption $\gamma = 1$ made in (6.2.51) is equivalent to a change of time unit.

Theorem 6.10. *The counting process $N_\beta(t)$ associated to the renewal process defined by equation* (6.2.51) *has the following distribution*

$$P_\beta(n,t) = \mathbb{P}(N_\beta(t) = n) = \frac{t^{\beta n}}{n!}E_\beta^{(n)}(-t^\beta), \tag{6.2.55}$$

where $E_\beta^{(n)}(-t^\beta)$ denotes the n-th derivative of $E_\beta(z)$ evaluated at the point $z = -t^\beta$.

Proof. The Laplace transform of $P_\beta(0,t) = \bar{F}_{J_\beta}(t) = E_\beta(-t^\beta)$ is given by [453]

$$\widetilde{P}_\beta(0,s) = \frac{s^{\beta-1}}{1+s^\beta}. \tag{6.2.56}$$

Therefore, the Laplace transform of the probability density function $f_{J_\beta}(t) = -dP_\beta(0,t)/dt$ is given by (see e.g. [336] for the Laplace transform of the derivative)

$$\widetilde{f}_{J_\beta}(s) = \frac{1}{1+s^\beta}; \tag{6.2.57}$$

recalling equation (6.2.46) and the convolution theorem for Laplace transforms (1.1.22), one immediately has

$$\widetilde{P}_\beta(n,s) = \frac{1}{(1+s^\beta)^n}\frac{s^{\beta-1}}{1+s^\beta}. \tag{6.2.58}$$

Using equation (1.80) in Podlubny's book [453] for the inversion of the Laplace transform in (6.2.58), one gets the thesis (6.2.55). □

Remark 6.17. The preceding theorem was proved by Scalas *et al.* [510, 372]. Notice that $N_1(t)$ is the Poisson process with parameter $\lambda = 1$. Recently, Meerschaert *et al.* [388] proved that the fractional Poisson process $N_\beta(t)$ coincides with the process defined by $N_1(D_\beta(t))$ where $D_\beta(t)$ is the functional inverse of the standard β-stable subordinator. The latter process was also known as fractal time Poisson process. This result unifies different approaches to fractional calculus [84, 388].

Remark 6.18. For $0 < \beta < 1$, the fractional Poisson process is semi-Markov, but not Markovian and is not Lévy. The process $N_\beta(t)$ is not Markovian as the only Markovian counting process is the Poisson process [138]. It is not Lévy as its distribution is not infinitely divisible.

It is now possible to define continuous-time random walks as compound renewal processes.

Definition 6.12. Let $\{Y_i\}_{i=1}^\infty$ be a sequence of i.i.d. random variables with cumulative probability distribution $F_Y(y)$ and let X_n be the corresponding random walk. We give to the Y_is the meaning of jump widths for a diffusing particle. Let $\{J\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative probability distribution $F_J(t)$ and with the meaning of sojourn times. Let $N(t)$ be the corresponding counting process. We then define the following stochastic process

$$X(t) = X_{N(t)} = \sum_{i=1}^{N(t)} Y_i \tag{6.2.59}$$

and we call it compound renewal process or continuous-time random walk (CTRW).

Remark 6.19. The CTRW is a random walk X_n subordinated to a counting process, i.e., a random sum of independent random variables. Note that we have not said if the couple (J_i, X_i) consists of independent random variables. If this is the case, we have an uncoupled (or decoupled) CTRW. This is the simplest case in which durations are independent of jumps. This remark leads us to consider a particular class of non-Markovian stochastic processes, the so-called semi-Markov processes [138, 290, 218, 219]. In the following, the reader will find a quick and dirty introduction to semi-Markov processes.

Definition 6.13. A Markov renewal process is a two-component Markov chain $\{X_n, T_n\}_{n=0}^{\infty}$, where X_n, $n \geq 0$ is a Markov chain and T_n, $n \geq 0$ is the n-th epoch of a renewal process, homogeneous with respect to the second component and with transition probability given by

$$\mathbb{P}(Y_{n+1} \in A, J_{n+1} \leq t | Y_0, \dots Y_n, J_1, \dots, J_n) = \mathbb{P}(Y_{n+1} \in A, J_{n+1} \leq t | Y_n), \tag{6.2.60}$$

where $A \subset \mathbb{R}$ is a Borel set and $J_{n+1} = T_{n+1} - T_n$.

Remark 6.20. We will also assume homogeneity with respect to the first component. In other words, if $X_n = x$, the probability on the right-hand side of equation (6.2.60) does not explicitly depend on n.

Remark 6.21. The positive function

$$Q(x, A, t) = \mathbb{P}(X_{n+1} = y \in A, J_{n+1} \leq t | X_n = x) \tag{6.2.61}$$

is called semi-Markov kernel with $x \in \mathbb{R}$, $A \subset \mathbb{R}$ a Borel set, and $t \geq 0$.

Definition 6.14. Let $N(t)$ denote the counting process defined as in equation (6.2.43), the stochastic process $X(t)$ defined as

$$X(t) = X_{N(t)} \tag{6.2.62}$$

is the semi-Markov process associated to the Markov renewal process X_n, T_n, $n \geq 0$.

Theorem 6.11. *Compound renewal processes are semi-Markov processes with semi-Markov kernel given by*

$$Q(x, A, t) = P(x, A) F_J(t), \tag{6.2.63}$$

where $P(x, A)$ is the Markov kernel (a.k.a. Markov transition function or transition probability kernel) of the random walk

$$P(x, A) \stackrel{\text{def}}{=} \mathbb{P}(X_{n+1} = y \in A | X_n = x), \tag{6.2.64}$$

and $F_J(t)$ is the cumulative probability distribution function of sojourn times. Moreover, let $f_Y(y)$ denote the probability density function of jumps, one has

$$P(x, A) = \int_{A-x} f_Y(u)\, du, \tag{6.2.65}$$

where $A - x$ is the set of values in A translated by x towards the left.

Proof. The compound renewal process is a semi-Markov process by construction, where the couple X_n, T_n, $n \geq 0$ defining the corresponding Markov renewal process is made up of a random walk X_n, $n \geq 0$ with $X_0 = 0$ and a renewal process with epochs given by T_n, $n \geq 0$ with $T_0 = 0$. Equation (6.2.63) is an immediate consequence of the independence between the random walk and the renewal process. Finally, equation (6.2.65) is the standard Markov kernel of a random walk whose jumps are i.i.d. random variables with probability density function $f_Y(y)$. □

The cumulative distribution function of an uncoupled compound renewal process can be obtained by means of purely probabilistic considerations.

Theorem 6.12. *Let $\{Y\}_{i=1}^{\infty}$ be a sequence of i.i.d. real-valued random variables with cumulative distribution function $F_Y(y)$ and let $N(t)$, $t \geq 0$*

denote a counting process independent of the previous sequence and such that the number of events in the interval $[0,t]$ *is a finite but arbitrary integer* $n = 0, 1, \ldots$. *Let* $X(t)$ *denote the corresponding compound renewal process. Then if* $P(n,t) = \mathbb{P}(N(t) = n)$, *the cumulative distribution function of* $X(t)$ *is*

$$F_{X(t)}(x,t) = \sum_{n=0}^{\infty} P(n,t) F_Y^{*n}(x), \tag{6.2.66}$$

where $F_Y^{*n}(x)$ *is the n-fold convolution of* $F_Y(y)$.

Proof. Assume that $X(0) = 0$ and that, at time t, there have been $N(t)$ jumps, with $N(t)$ assuming integer values starting from 0 ($N(t) = 0$ means no jumps up to time t). Consider a realization of $N(t)$, that is suppose one has $N(t) = n$. This means that

$$X(t) = \sum_{i=1}^{N(t)} Y_i = \sum_{i=1}^{n} Y_i = X_n. \tag{6.2.67}$$

In this case, one finds

$$F_{X_n}(x) = \mathbb{P}(X_n \leq x) = \mathbb{P}\left(\sum_{i=1}^{n} Y_i \leq x\right) = F_Y^{*n}(x). \tag{6.2.68}$$

Given the independence between $N(t)$ and the Y_is, one further has that

$$\mathbb{P}(X_n \leq x, N(t) = n) = \mathbb{P}(N(t) = n)\mathbb{P}(X_n \leq x) = P(n,t)F_Y^{*n}(x). \tag{6.2.69}$$

The events $\{X_n \leq x, N(t) = n\}$ for $n \geq 0$ are mutually exclusive and exhaustive, and this yields

$$\{X(t) \leq x\} = \cup_{n=0}^{\infty} \{X_n \leq x, N(t) = n\}, \tag{6.2.70}$$

and, for any $m \neq n$,

$$\{X_m \leq x, N(t) = m\} \cap \{X_n \leq x, N(t) = n\} = \varnothing. \tag{6.2.71}$$

Calculating the probability of the two sides in equation (6.2.70) and using equation (6.2.69) and the axiom of infinite additivity leads to

$$\begin{aligned} F_{X(t)}(x,t) &= \mathbb{P}(X(t) \leq x) = \mathbb{P}\left(\cup_{n=0}^{\infty} \{X_n \leq x, N(t) = n\}\right) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(X_n \leq x, N(t) = n) = \sum_{n=0}^{\infty} P(n,t) F_Y^{*n}(x). \end{aligned} \tag{6.2.72}$$

which is our thesis. $\square$

Remark 6.22. For $n = 0$, one assumes $F^{*0}_{Y_0}(y) = \theta(y)$ where $\theta(y)$ is the Heaviside function. Note that $P(0, t)$ is nothing else but the survival function at $y = 0$ of the counting process. Therefore, equation (6.2.66) can be equivalently written as

$$F_{X(t)}(x, t) = P(0, t)\,\theta(x) + \sum_{n=1}^{\infty} P(n, t) F_Y^{*n}(x), \tag{6.2.73}$$

where $\theta(x)$ is the càdlàg version of Heaviside step function.

Remark 6.23. The series (6.2.66) is uniformly convergent for $x \neq 0$ and for any value of $t \in (0, \infty)$. This statement can be proved using the Weierstrass M test. For $x = 0$ there is a jump in the cumulative distribution function of amplitude $P(0, t)$.

Example 6.4. As an example of a compound renewal process, consider the case in which $Y_i \sim \mathcal{N}(\mu, \sigma^2)$, so that their cumulative distribution function is

$$F_Y(y) = \Phi(y|\mu, \sigma^2) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{y - \mu}{\sqrt{2\sigma^2}}\right)\right), \tag{6.2.74}$$

where

$$\mathrm{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y \mathrm{e}^{-u^2}\, du \tag{6.2.75}$$

is the error function. In this case, the convolution $F_Y^{*n}(x)$ is given by $\Phi(x|n\mu, n\sigma^2)$. The sojourn times are $J_i \sim \exp(\lambda)$ and one finds

$$P(n, t) = \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!}. \tag{6.2.76}$$

As a consequence of Theorem 6.12 one gets

$$F_{X(t)}(x, t) = \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \Phi(x|n\mu, n\sigma^2). \tag{6.2.77}$$

Equation (6.2.77) can be directly used for numerical estimates of $F_{X(t)}(x, t)$.

Corollary 6.3. *In the same hypotheses as in Theorem 6.12, the probability density $f_{X(t)}(y,t)$ of the process $X(t)$ is given by*

$$f_{X(t)}(x,t) = P(0,t)\,\delta(x) + \sum_{n=1}^{\infty} P(n,t) f_Y^{*n}(x), \tag{6.2.78}$$

where $f_Y^{*n}(x)$ is the n-fold convolution of the probability density function $f_Y(y) = dF_Y(y)/dy$.

Proof. The sought probability density function is $f_{X(t)}(x,t) = dF_{X(t)}(x,t)/dy$; equation (6.2.78) is the formal derivative of equation (6.2.66). If $x \neq 0$, there is no singular term and the series converges uniformly ($f_Y^{*n}(x)$ is bounded and the Weierstrass M test applies), therefore, for any x the series converges to the derivative of $F_{X(t)}(x,t)$. This is so also in the case $x = 0$ for $n \geq 1$ and the jump in $x = 0$ gives the singular term of weight $P(0,t)$ (see equation (6.2.73)). □

Among all the compound renewal processes, we will need the compound fractional Poisson process [332].

Definition 6.15. Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative distribution function given by $F_Y(y)$ and let $N_\beta(t)$ be the fractional Poisson process, then the process $X_\beta(t)$ defined as

$$X_\beta(t) = X_{N_\beta(t)} = \sum_{i=1}^{N_\beta(t)} Y_i \tag{6.2.79}$$

is called compound fractional Poisson process.

Remark 6.24. The process $X_1(t)$ coincides with the compound Poisson process of parameter $\lambda = 1$.

Theorem 6.13. *Let $X_\beta(t)$ be a compound fractional Poisson process, then*

(1) *its cumulative distribution function $F_{X_\beta(t)}(x,t)$ is given by*

$$F_{X_\beta(t)}(x,t) = E_\beta\left(-t^\beta\right)\theta(x) + \sum_{n=1}^{\infty} \frac{t^{\beta n}}{n!} E_\beta^{(n)}\left(-t^\beta\right) F_Y^{*n}(x); \tag{6.2.80}$$

(2) *its probability density function* $f_{X_\beta(t)}(x,t)$ *is given by*

$$f_{X_\beta(t)}(y,t) = E_\beta\left(-t^\beta\right)\delta(x) + \sum_{n=1}^{\infty}\frac{t^{\beta n}}{n!}E_\beta^{(n)}\left(-t^\beta\right) f_Y^{*n}(x); \quad (6.2.81)$$

(3) *its characteristic function* $\widehat{f}_{X_\beta(t)}(\kappa,t)$ *is given by*

$$\widehat{f}_{X_\beta(t)}(\kappa,t) = E_\beta\left[t^\beta(\widehat{f}_Y(\kappa)-1)\right]. \quad (6.2.82)$$

Proof. The first two equations (6.2.80) and (6.2.81) are a straightforward consequence of Theorem 6.12, Corollary 6.3 and Theorem 6.10. Equation (6.2.82) is the Fourier transform of (6.2.81). This can be verified by expanding it into a Taylor series with center $-t^\beta$. □

Remark 6.25. For $0 < \beta < 1$, the compound fractional Poisson process is neither Markovian nor Lévy (see Remark 6.18). However, it is a semi-Markov process as a consequence of Theorem 6.11.

It is now possible to discuss the relationship between CTRWs and the space-time fractional diffusion equation. This will be the subject of the next section.

6.3 Fractional Diffusion and Limit Theorems

In order to link the processes introduced in the previous section to fractional diffusion, let us first consider the following Cauchy problem.

Theorem 6.14. *Consider the Cauchy problem for the space-time fractional diffusion equation with* $0 < \alpha \le 2$ *and* $0 < \beta \le 1$

$$\begin{aligned} {}^{\mathtt{SR}}D_x^\alpha u_{\alpha,\beta}(x,t) &= {}^{\mathtt{C}}D_t^\beta u_{\alpha,\beta}(x,t) \\ u_{\alpha,\beta}(x,0^+) &= \delta(x), \end{aligned} \quad (6.3.1)$$

then the Green function

$$u_{\alpha,\beta}(x,t) = \frac{1}{t^{\beta/\alpha}}W_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right), \quad (6.3.2)$$

where

$$W_{\alpha,\beta}(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\kappa\, \mathrm{e}^{-i\kappa u}E_\beta(-|\kappa|^\alpha), \quad (6.3.3)$$

solves the Cauchy problem [373, 510].

Proof. In equation (6.3.1), ${}^{\mathtt{SR}}D_x^\alpha$ denotes the symmetric Riesz operator [501, 498, 594] whose Fourier symbol is $-|\kappa|^\alpha$; more precisely, for a suitable function $f(x)$ one can write

$$(\mathcal{F}[{}^{\mathtt{SR}}D_x^\alpha f(x)])(\kappa) = -|\kappa|^\alpha \widehat{f}(\kappa). \tag{6.3.4}$$

Moreover, ${}^{\mathtt{C}}D_t^\beta$ is the Caputo derivative (1.3.36), whose Laplace symbol is given by

$$(\mathcal{L}[{}^{\mathtt{C}}D_t^\beta g(t)])(s) = s^\beta \widetilde{g}(s) - s^{\beta-1} g(0^+), \tag{6.3.5}$$

where g is a function whose Laplace transform exists. Equation (6.3.5) is a particular case of (1.3.56). The application of the Laplace-Fourier transform to equation (6.3.1) implies that

$$-|\kappa|^\alpha \widehat{\widetilde{u}}_{\alpha,\beta}(\kappa, s) = s^\beta \widehat{\widetilde{u}}_{\alpha,\beta}(\kappa, s) - s^{\beta-1}, \tag{6.3.6}$$

so that the Laplace-Fourier transform of the sought Green function is

$$\widehat{\widetilde{u}}_{\alpha,\beta}(\kappa, s) = \frac{s^{\beta-1}}{|\kappa|^\alpha + s^\beta}. \tag{6.3.7}$$

A comparison between (6.2.56) and (6.3.7) immediately shows that the inversion of the Laplace transform gives the Fourier transform of $u_{\alpha,\beta}(x,t)$

$$\widehat{u}_{\alpha,\beta}(\kappa, t) = E_\beta\left(-|\kappa|^\alpha t^\beta\right). \tag{6.3.8}$$

A further inversion of the Fourier transform leads to the thesis. □

Remark 6.26. In the above derivation, the role of Fourier and Laplace transforms is interchangeable. One can first invert the Fourier transform and then the Laplace transform and get the same result.

Remark 6.27. The Green function $u_{\alpha,\beta}(x,t)$ is a probability density function for any $t > 0$, that is

$$\int_{-\infty}^{+\infty} dx\, u_{\alpha,\beta}(x,t) = 1. \tag{6.3.9}$$

When $\alpha = 2$ and $\beta = 1$, the symmetric Riesz derivative coincides with the second derivative with respect to x: ${}^{\mathtt{SR}}D_x^2 = \partial^2/\partial x^2$ and the Caputo derivative becomes the first derivative with respect to time: ${}^{\mathtt{C}}D_t^1 = \partial/\partial t$. Then, equation (6.3.1) defines the Cauchy problem for ordinary diffusion, with diffusion coefficient equal to 1; the corresponding Green function is

$$u_{2,1}(x,t) = \frac{1}{\sqrt{4\pi t}}\, \mathrm{e}^{-x^2/4t}. \tag{6.3.10}$$

Definition 6.16. A random variable X is called infinitely divisible if, for any n, it can be written as the sum of n independent and identically distributed random variables. Equivalently, the corresponding cumulative distribution function is called infinitely divisible if, for any n, it can be written as $F_X(x) = F_{Y_n}^{*n}(x)$, i.e., there exists a random variable Y_n whose n-fold convolution gives $F_X(x)$.

Remark 6.28. Stable distributions are infinitely divisible, but the converse is not true, not every infinitely divisible distribution is stable.

Remark 6.29. The Poisson distribution and the compound Poisson distribution are infinitely divisible and it can be shown that any infinitely divisible distribution is the limit of compound Poisson distributions.

Theorem 6.15. *For* $0 < \alpha \le 2$, *and* $\beta = 1$, *the characteristic function of the Green function is*

$$\widehat{u}_{\alpha,1}(\kappa,t) = \mathrm{e}^{-|\kappa|^\alpha t} \tag{6.3.11}$$

and it is infinitely divisible.

Proof. For $\beta = 1$, the Mittag-Leffler function coincides with the exponential function and equation (6.3.8) yields equation (6.3.11). In order to show that the random variable $U_{\alpha,1}(t)$ is infinitely divisible for every t, one has to show that its cumulative distribution function is the n-fold convolution of n identical distribution functions. But then, it is enough to choose a random variable whose characteristic function is given by

$$[\mathrm{e}^{-|\kappa|^\alpha t}]^{1/n} = \mathrm{e}^{-|\kappa|^\alpha t/n}, \tag{6.3.12}$$

to prove infinite divisibility of $U_{\alpha,1}(t)$. □

Remark 6.30. It can be proved that to every infinitely divisible distribution there corresponds a unique càdlàg extension of a Lévy process. In the case under scrutiny, $u_{2,1}(x,t)$ corresponds to the Wiener process whose increments follow the $N(0,2t)$ distribution. The density $u_{\alpha,1}(x,t)$ for $0 < \alpha < 2$ corresponds to processes (called Lévy flights in the applied literature) that generalize the Wiener process and whose increments follow the symmetric α-stable distribution. For more information on infinitely divisible distributions, Lévy processes and related pseudo-differential operators the reader can consult the following references [214, 89, 506, 516, 287, 38].

A simple way to understand the connection between CTRWs and space-time fractional diffusion is to consider the following stochastic process.

Definition 6.17. Let $\{Y_{\alpha,i}\}_{i=1}^{\infty}$ be a sequence of i.i.d. symmetric α-stable distributions whose characteristic function is given by

$$\widehat{f}_{Y_\alpha}(\kappa) = \mathrm{e}^{-|k|^\alpha} \tag{6.3.13}$$

and let $X_{\alpha,n}$ be the corresponding random walk. The following compound fractional Poisson process

$$X_{\alpha,\beta}(t) = X_{\alpha,N_\beta(t)} = \sum_{i=1}^{N_\beta(t)} Y_{\alpha,i} \tag{6.3.14}$$

is the fractional compound Poisson process with symmetric α-stable jumps.

Corollary 6.4. *The characteristic function of the fractional compound Poisson process with symmetric α-stable jumps is given by*

$$\widehat{f}_{X_{\alpha,\beta}(t)}(\kappa) = E_\beta\left[t^\beta\left(\mathrm{e}^{-|\kappa|^\alpha} - 1\right)\right]. \tag{6.3.15}$$

Proof. This result is an immediate consequence of equations (6.2.82) and (6.3.13). □

If properly rescaled, the random variable $X_{\alpha,\beta}(t)$ can be made to converge weakly to $U_{\alpha,\beta}(t)$, the random variable whose distribution is characterized by the probability density function (6.3.2) that solves the Cauchy problem for the space-time fractional diffusion equation (6.3.1). The trick is to build a sequence of random variables whose characteristic function converges to (6.3.8). Indeed, we can prove the following theorem.

Theorem 6.16. *Let $X_{\alpha,\beta}(t)$ be a compound fractional Poisson process with symmetric α-stable jumps and let h and r be two scaling factors such that*

$$X_{\alpha,n}(h) = hY_{\alpha,1} + \ldots + hY_{\alpha,n} \tag{6.3.16}$$

$$T_{\beta,n}(r) = rJ_{\beta,1} + \ldots + rJ_{\beta,n}, \tag{6.3.17}$$

and

$$\lim_{h,r\to 0} \frac{h^\alpha}{r^\beta} = 1, \tag{6.3.18}$$

with $0 < \alpha \leq 2$ and $0 < \beta \leq 1$. Given the assumption on the jumps $Y_{\alpha,i}$, for $h \to 0$, one has

$$\widehat{f}_{Y_\alpha}(h\kappa) = 1 - h^\alpha|\kappa|^\alpha + \mathrm{o}(h^\alpha|\kappa|^\alpha), \tag{6.3.19}$$

then, for $h, r \to 0$ with $h^\alpha/r^\beta \to 1$, $f_{hX_{\alpha,\beta}(rt)}(x,t)$ weakly converges to $u_{\alpha,\beta}(x,t)$, the Green function of the fractional diffusion equation.

Proof. In order to prove weak convergence, it suffices to show the convergence of the characteristic function (6.2.82) as a consequence of the Lévy continuity theorem 6.3. Indeed, one can write

$$\widehat{f}_{hX_{\alpha,\beta}(rt)}(\kappa,t) = E_\beta\left[-\frac{t^\beta}{r^\beta}\left(\mathrm{e}^{-h^\alpha|\kappa|^\alpha} - 1\right)\right] \overset{h,r\to 0}{\to} E_\beta(-t^\beta|\kappa|^\alpha), \tag{6.3.20}$$

which completes the proof and establishes the connection between CTRWs and the space-time fractional diffusion equation. □

Remark 6.31. This result can be generalized to compound fractional Poisson processes with heavy tails both in the jump and in the sojourn time distributions. A more general proof can be found in [510]. The relationship between fractional diffusion and CTRWs is discussed in several physics papers with different levels of detail [220, 488, 147, 498, 594]. Hilfer and Anton realized the important role played by the Mittag-Leffler function in this derivation and rigorously discussed the link between fractional diffusion and CTRWs [282].

Remark 6.32. Up to now, we have focused on the (weak) convergence of random variables and not of stochastic processes. The convergence of processes is delicate as one must use techniques related to functional central limit theorems in appropriate functional spaces [288]. Let $L_\alpha(t)$ denote the symmetric α-stable Lévy process. Equation (6.3.8) is the characteristic function of the process $U_{\alpha,\beta}(t) = L_\alpha(D_\beta(t))$, that is of the symmetric α-stable Lévy process subordinated to the inverse β-stable subordinator, $D_\beta(t)$, the functional inverse of the β-stable subordinator. This remark leads to conjecture that $U_{\alpha,\beta}(t)$ is the functional limit of $X_{\alpha,\beta}(t)$, the compound fractional Poisson process with α-stable jumps. This conjecture can be found in a paper by Magdziarz and Weron [364] and is proved in Meerschaert *et al.* [388] using the methods discussed by Meerschaert and Scheffler [390].