Introduction to fractional calculus and fractional differential equations

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Main ideas and history

Many repeated operations in math can be generalized:

- Power function:

$$x^n = x \cdot x \cdot x \cdot \ldots \cdot x \quad \longrightarrow \quad x^\alpha = e^{\alpha \ln x}$$

- Factorial (Euler gamma function):

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \quad \longrightarrow \quad \Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\, dx$$

$$\Gamma(x+1) = x\Gamma(x), \;\; \Gamma(1) = 1 \quad \Longrightarrow \quad n! = \Gamma(n+1)$$

- What about derivatives and integrals?

$$D^n f(x) \equiv f^{(n)}(x) \equiv \frac{d^n f}{dx^n} \quad \longrightarrow \quad D^\alpha f = ?$$

- Question "What if order will be 1/2"was raised by Leibnitz in his letter to L'Hopital, 1695.
- Elements created by Lagrange, Euler, Laplace, Fourier.
- Modern theory started with works by Abel, Liouville and Riemann, $\approx$1832.

- Let us start from repeated integral

$$ {}_0I_x^n f(x) \equiv \left(\int_0^x \cdot dx \right)^n f = \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \dots \int_0^{x_{n-1}} f(x_n)\, dx_n. $$

- There is a well known Cauchy formula for this n-fold integral:

$$ {}_0I_x^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt, $$

  that can be easily generalized, $(n-1)! = \Gamma(n)$.
- *Riemann-Liouville fractional integral*:

$$ {}_0I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)} * f(x), \quad x > 0, \alpha > 0. $$

- For $0 < \alpha < 1$ there is an integrable singularity.
- Starting point can be arbitrary, not only 0: ${}_cI_x^\alpha f(x) = \int_c^x \dots$. Zeros are often omitted: ${}_0I_x^\alpha f(x) \equiv I_x^\alpha f(x)$.

Left and right integrals

- Left fractional integral depends on $f(t), t \in (a,x)$:

$$ {}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a, \alpha > 0. $$

- Right fractional integral depends on $f(t), t \in (x,b)$:

$$ {}_xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b, \alpha > 0. $$

- The negative order of derivative means fractional integral:

$$ {}_aD_x^{-\alpha} f(x) \equiv {}_aI_x^\alpha f(x), \quad {}_xD_b^{-\alpha} f(x) \equiv {}_xI_b^\alpha f(x). $$

$$ {}_aD_x^0 f(x) \equiv {}_aI_x^0 f(x) \equiv f(x), \quad {}_xD_b^0 f(x) \equiv {}_xI_b^0 f(x) \equiv f(x). $$

- There are generalizations to the complex order $\alpha \in \mathbb{C}$.
- Fractional integrals ${}_aI_x^\alpha$, ${}_xI_b^\alpha$ are linear bounded operators $L^p(a,b) \to L^p(a,b)$, $p \geq 1$ (and in other functional spaces).

Semi-group property

- For all "good"functions $f(x)$ the family $\{{}_aI_x^\alpha, \alpha \geq 0\}$ forms a semigroup:

$$ {}_aI_x^\alpha \, {}_aI_x^\beta f(x) = {}_aI_x^{\alpha+\beta} f(x), \quad \alpha > 0, \beta > 0. $$

- Outline of the proof:

$$ \Gamma(\alpha)\Gamma(\beta)_0I_x^\alpha \, {}_0I_x^\beta f(t) = \int\limits_0^t (t-\tau)^{\alpha-1} d\tau \int\limits_0^\tau (\tau-\xi)^{\beta-1} f(\xi)\, d\xi = $$

exchange the integrals, change $t \to w = \dfrac{\tau-\xi}{t-\xi}$:

$$ \int\limits_0^t f(\xi)\, d\xi \int\limits_\xi^t (\tau-\xi)^{\beta-1}(t-\tau)^{\alpha-1} d\tau = \int\limits_0^t \frac{f(\xi)\, d\xi}{(t-\xi)^{1-\alpha-\beta}} \int\limits_0^1 w^{\beta-1}(1-w)^{\alpha-1} dw = $$

$$ = B(\beta,\alpha) \int\limits_0^t f(\xi)\, d\xi (t-\xi)^{\alpha+\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int\limits_0^t = \Gamma(\alpha)\Gamma(\beta)_0 I^{\alpha+\beta} f(t). $$

Riemann-Liouville fractional derivative

There are different way to define fractional derivative operators. Let us define D^α as a left inverse operator to I^α. Then

$$D^\alpha I^\alpha f = f \quad \Longrightarrow \quad D^\alpha y = f : I^\alpha f = y.$$

Equation $I^\alpha f = y$ with unknown variable $f(x)$ is Abel integral equation of the first kind. Let

$$m - 1 < \alpha < m.$$

Then applying $I^{m-\alpha}$ to the both sides and using semi-group property, one gets

$$I^{m-\alpha} I^\alpha f = I^{m-\alpha} y \quad \Longrightarrow \quad I^m f = I^{m-\alpha} y.$$

Differentiating by x and using classical $DIf = f$, one obtains

$$f = D^m I^{m-\alpha} y.$$

So, the Riemann-Liouville fractional derivative operator is

$$D_x^\alpha y(x) \equiv \frac{d^m}{dx^m} I_x^{m-\alpha} y(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int\limits_0^x \frac{y(t)}{(x-t)^{\alpha+1-m}} dt.$$

Left-sided and right-sided fractional derivative

- Left-sided Riemann-Liouville derivative:

$$ {}_aD_x^\alpha y(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int\limits_a^x \frac{y(t)}{(x-t)^{\alpha+1-m}} dt, m-1<\alpha<m, m\in\mathbb{N}. $$

- Right-sided Riemann-Liouville derivative:

$$ {}_xD_b^\alpha y(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int\limits_x^b \frac{y(t)}{(x-t)^{\alpha+1-m}} dt, m-1<\alpha<m, m\in\mathbb{N}. $$

- Same with infinite limits ${}_{-\infty}D_b^x$ and ${}_xD_{+\infty}^\alpha$
- ${}_aD_x^\alpha y(x) \to y^{(n)}(x)$ when $\alpha \to n$.
  For $\alpha \to n+0$, $m = n+1, \alpha+1-m \to 0$ and $\Gamma(m-\alpha) \to 1$.
  For $\alpha \to n-0$, $m = n, \Gamma(m-\alpha) \to \infty$ and the proof is harder.
- ${}_xD_b^\alpha y(x) \to (-1)^n y^{(n)}(x)$ when $\alpha \to n$.
- Fractional derivative is always **nonlocal**. It needs values $y(t)$ at all points of segment $t \in (a,x)$ to obtain ${}_aD_x^\alpha y(x)$.

Power law function

Fractional derivative of power function:

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \quad \alpha > 0,\ \gamma > -1,\ x > 0$$

(beta-function is used to prove this).
Specifically, $y(x) = 1$ is not a «constant» here:

$$D_x^\alpha 1 = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \qquad D_x^\alpha x^{\alpha-1} = \frac{\Gamma(\alpha) x^{-1}}{\Gamma(0)} = 0.$$

Composition rule is not easy:

$$D^\alpha D^\beta f(x) \neq D^{\alpha+\beta} f(x) \text{ for arbitrary } \alpha, \beta, f, \text{ but } = \text{for certain classes}$$

$$D^\alpha D f(x) \neq D D^\alpha f(x) \equiv D^{\alpha+1} f(x).$$

$$D_x^\alpha y'(x) = D_x^{\alpha+1} y(x) - \frac{x^{-\alpha-1}}{\Gamma(-\alpha)} y(+0).$$

Examples: $D^\alpha D^\alpha x^{\alpha-1} \neq D^{2\alpha} x^{\alpha-1}$, $D^\alpha D^\alpha x = D^{2\alpha} x$,$D^\alpha D x^2 = D^{\alpha+1} x^2$.

Other examples of derivatives

When α is included in the function, the derivatives formulas can be compact

- $$D_x^\alpha \left(x^{\alpha-1} e^{-1/x} \right) = x^{-\alpha-1} e^{-1/x},$$
- $$D_x^\alpha (x^{\alpha-1} \ln x) = \frac{\Gamma(\alpha)}{x}.$$

But usually a lot of special functions are involved:

- Generalized Mittag-Leffler function E helps to work with exponents
$$D_x^\alpha e^x = x^{-\alpha} E_{1,1-\alpha}(x),$$
$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$
- Hypergeometric functions are also used:
$$D_x^\alpha (x+p)^\lambda = \frac{p^\lambda}{\Gamma(1-\alpha)} x^{-\alpha} {}_2F_1(1, -\lambda, 1-\alpha; -x/p).$$

etc.

Differentiating product and composite function

Generalized Leibnitz's rule:

$$D_x^\alpha \left(f(x)g(x)\right) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_x^{\alpha-n} f(x)\; D_x^n g(x), \qquad \alpha > 0,$$

where $\displaystyle \binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\Gamma(n+1)}, \quad \binom{k}{n} = C_k^n$.

Differentiation of composite function (almost unusable):

$$D_x^\alpha \left[f\left(x, y(x)\right)\right] = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{r=0}^{k} \binom{\alpha}{n}\binom{n}{m}\binom{k}{r} \frac{1}{k!} \frac{x^{n-\alpha}}{\Gamma(n+1-\alpha)} \left[-y\right]^r \times$$

$$\times\; D_x^m \left[y^{k-r}\right] \frac{\partial^{n-m+k} f(x,y)}{\partial x^{n-m} \partial y^k}.$$

$D_x^\alpha (y(x)^2)$ is already huge enough.
Note: $D_x^5(y^2)$ is not compact too (Faa di Bruno formula etc.).

Literature

- Formulas and links to most common books on fractional calculus: Valério, D., Trujillo, J. J., Rivero, M., Machado, J. T., Baleanu, D. (2013). Fractional calculus: a survey of useful formulas. The European Physical Journal Special Topics, 222(8), 1827-1846.
- The fundamental book with theorems and functional spaces: S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional integrals and derivatives: theory and applications (Gordon and Breach Science Publishers, Amsterdam, 1993)
- A.A. Kilbas, H.M. Srivastava J.J. Trujillo, Theory and applications of fractional differential equations, Vol. 204 (North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006)
- K.S. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations (John Wiley and Sons, New York, 1993)
- Survey of applications: Uchaikin, Vladimir V. Fractional derivatives for physicists and engineers. Berlin: Springer, 2013.

Caputo-type fractional derivative

- Caputo-type fractional derivative (1967), used in earlier paper by Gerasimov:

$$ {}_a^C D_x^\alpha y(x) \equiv {}_a I_x^{m-\alpha} D_x^m y(x) = \frac{1}{\Gamma(m-\alpha)} \int\limits_a^x \frac{y^{(m)}(t)}{(x-t)^{\alpha+1-m}} dt. $$

  As usual, $m-1 < \alpha < m, m \in \mathbb{N}$. The derivative is under the integral now, $y(x)$ class is more restricted.

- Caputo derivate

$$ {}_a^C D_x^\alpha y(x) \equiv {}_a I_x^{m-\alpha} y^{(m)}(x) = {}_a D_x^\alpha y(x) - \sum_{k=0}^{m-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} y^{(k)}(a+0). $$

- More popular in physics because $y(0), y'(0), \ldots$ exists.
- If $y(0) = 0, y'(0), \ldots, y^{(m-1)}(0) = 0$, then the derivative is equal to Riemal-Liouville.

Grunwald-Letnikov derivative

Ordinary derivatives can be defined via backward differences

- $y' = \lim\limits_{h\to 0} h^{-1}(y(x) - y(x-h))$
- $y'' = \lim\limits_{h\to 0} h^{-2}(y(x) - 2y(x-h) + y(h-2h))$
- $y''' = \lim\limits_{h\to 0} h^{-3}(y(x) - 3y(x-h) + 3y(h-2h) - y(h-3h))$
- ...
- $y^{(n)} = \lim\limits_{h\to 0} h^{-k} \sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} y(x-kh)$

By analogy, Grunwald-Letnikov fractional derivative is defined:

- ${}_a^{GL}D_x^{\alpha} y = \lim\limits_{n\to\infty, h=(x-a)/n} h^{-\alpha} \sum\limits_{k=0}^{n} (-1)^k \binom{\alpha}{k} y(x-kh)$
- When $\alpha \in (0,1)$, $y \in C[a,x]$ and $y(a) = 0$, Riemann-Liouville, Caputo and Grunwald-Letnikov's derivatives are equal.
- This can be a base for a numerical method.

Contents

Where fractional derivatives are used?

Pure math:

- Functional spaces and operators
- Special functions
- Analytical solutions of some linear DEs.

Random processes and signals:

- Stochastic models with power-law distributions (Levy-stable distributions instead of normal/Gaussian)
- Continuous time random walks
- Signal processing
- Automatic control - fractional elements (with power-law memory) sometimes have better characteristics than normal integrators/differentiators/PID-controllers.

Describe nonlocal material behavior or memory

- Anomalous diffusion processes in physics, biology etc.
- Viscoelasticity and complex rheology fluids
- Fractal media
- Electrochemistry

Some examples of fractional differential equations

1. Oscillatory processes with fractional damping (Bagley&Torvik, 1984)

$$y''(t) + ({}_0D_t^{1+\alpha} y)(t) + by(t) = f(t), \quad t > 0, \quad \alpha \in (0,1). \tag{1}$$

2. Subdiffusion equations (Wyss, 1986; Hifer, 1995)

$${}_0^C D_t^{\alpha} u = (ku_x)_x; \quad {}_0D_t^{\alpha} u = (ku_x)_x; \quad \alpha \in (0,1). \tag{2}$$

3. Diffusion-wave equations (Nigmatullin, 1986; Mainardy, 1998)

$${}_0^C D_t^{1+\alpha} u = (ku_x)_x; \quad {}_0D_t^{1+\alpha} u = (ku_x)_x; \quad \alpha \in (0,1). \tag{3}$$

4. Superdiffusion equation (Benson, 1998)

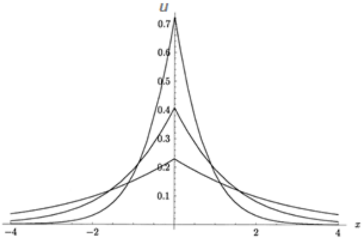
$$u_t = \left[k \left(\gamma \; {}_aD_x^{\beta} u + (1-\gamma) \; {}_xD_b^{\beta} u\right)\right]_x; \quad \beta, \gamma \in (0,1). \tag{4}$$

5. Fractional-order biological population model (El-Sayed, Rida, Arafa, 2009)

$${}_0^C D_t^{\alpha} u = (u^2)_{xx} + (u^2)_{yy} + f(u), \quad \alpha \in (0,1). \tag{5}$$

Subdifusion

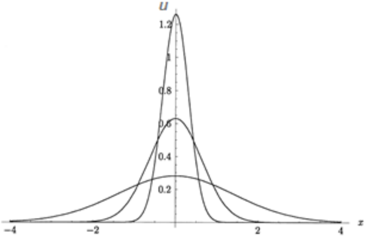
$v \sim t^{-(1-\alpha/2)},\ 0 < \alpha < 1$



$$ {}_0D_t^{\alpha} u = K_{\alpha} u_{xx}; $$

Normal disffusion

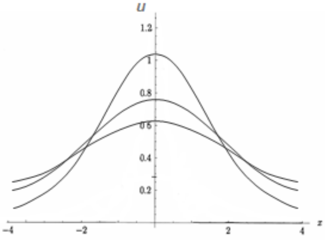
$v \sim t^{-1/2}$



$$ \frac{\partial u}{\partial t} = K u_{xx} $$

Superdiffusion

$v \sim t^{-(1-1/\beta)},\ 1 < \beta < 2$



$$ \frac{\partial u}{\partial t} = K_{\beta}\left({}_{-\infty}D_x^{\beta} u + {}_xD_{\infty}^{\beta} u\right) $$

Example: using fractional derivative for ODE (from Samko et al.)

Consider the linear equation

$$(a_2 + b_2 x + c_2 x^2) y''(x) + (a_1 + b_1 x) y'(x) + a_0 y = 0$$

The solution can be found in the form

$$y(x) = D_x^p z(x)$$

where

$$p: \quad a_0 - b_1(p+1) + c_2(p+1)(p+2) = 0,$$

$$z(x) = (a_2 + b_2 x + c_2 x^2)^{p+1} exp\{-\int \frac{a_1 + b_1 x}{a_2 + b_2 x + c_2 x^2} dx\}.$$

Initial value problem

$$ {}_cD_x^\alpha y(x) = f(x, y(x)), \quad n-1 < \alpha < n. \tag{6}$$

Standard initial value problem contains following condition at starting point c:

$$ ({}_cD_x^{\alpha-1} y(x))(c+) = b_1,\ ({}_cD_x^{\alpha-2} y(x))(c+) = b_2, \dots\ ({}_cD_x^{\alpha-n} y(x))(c+) = b_n. \tag{7}$$

Here

$$ f(c+) = \lim_{x \to c+0} f(x). $$

The last term always contains limit of fractional integral $I^{n-\alpha} y(x)$. For $0 < \alpha < 1$ there exists an equivalent formulation of Cauchy problem:

$$ ({}_cD_x^{\alpha-1} y(x))(c+) = b_1 \quad \Leftrightarrow \quad \lim_{x \to c+0} \left[(x-c)^{1-\alpha} y(x) \right] = \frac{b_1}{\Gamma(\alpha)}. \tag{8}$$

The solution has an integrable singularity at the point c in general case. There are conditions for $f(x, y)$ where the solution exists and is unique (see multiple theorems in Kilbas&Trujillo book).

Simplest equations

Simplest equation with Riemann-Liouville derivative:

$$D_x^\alpha y(x) = 0, \quad \alpha \in (1,2)$$

The general solution:

$$y = C_1 x^{\alpha-1} + C_2 x^{\alpha-2}.$$

Initial conditions:

$$(D^{\alpha-1}y)(0+) = c_1, \quad (D^{\alpha-2}y)(0+) = c_2.$$

Simplest equation with Caputo type derivative:

$${}^C D_x^\alpha y(x) = 0, \quad \alpha \in (1,2)$$

The general solution:

$$y = C_1 x + C_2.$$

Initial conditions:

$$y'(0+) = c_1, \quad y(0+) = c_2.$$

For equations with Caputo fractional derivatives, natural initial conditions are used and the solution have no singularities.

Simple linear equation

Consider the initial value problem

$$D^\alpha y = y, \quad (D^{\alpha-1}y)(0+) = b_1, \quad 0 < \alpha < 1, y = y(x), x > 0.$$

It is equivalent to an integral equation

$$y(x) = y_0(x) + I_x^\alpha y(x), \qquad \text{where } y_0(x) = \frac{b_1 x^{\alpha-1}}{\Gamma(\alpha - k + 1)}$$

Using this as an iterative process $y_m = y_0 + I_x^\alpha y_{m-1}$, one gets

$$y_m(x) = b_1 \sum_{n=1}^{m} \frac{x^{\alpha j-1}}{\Gamma(\alpha j - k + 1)}$$

(each integration adds x^α multiplier and modifies gamma-function) The final solution is the specific Mittag-Leffler function (generalized exponent):

$$y(x) = b_1 \sum_{n=1}^{\infty} \frac{x^{\alpha j-1}}{\Gamma(\alpha j - k + 1)} = b_1 x^{\alpha-1} E_{\alpha,\alpha}(x^\alpha).$$

Using Laplace transform

- Remember the Laplace transorm $f(t) \to g(s) = (\mathcal{L}f)(s)$:

$$(\mathcal{L}f)(s) = \int_0^\infty f(t)e^{-st}dt,$$

- Inverse Laplace transform:

$$(\mathcal{L}^{-1}g)(x) = \int_{\gamma-i\infty}^{\gamma+i\infty} g(s)e^{st}ds, \quad \gamma = Re(s).$$

- Laplace transform of ordinary derivatives

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$

$$\mathcal{L}\{y^{(n)}\} = s^n\mathcal{L}\{y\} - \sum_{k=0}^{n-1} s^k y^{(n-k-1)}(0)$$

- Laplace transform of Riemann-Liouville derivative:

$$\mathcal{L}\{D_x^\alpha y\} = s^\alpha \mathcal{L}\{y\} - \sum_{k=0}^{n-1} s^k D_x^{\alpha-k-1} y(0+), \quad n-1 < \alpha \le n.$$

Simple linear equation

Consider the initial value problem

$$D^{\alpha} y = y, \quad (D^{\alpha-1} y)(0+) = b_1, \quad 0 < \alpha < 1, y = y(x), x > 0.$$

Applying Laplace transform, one gets

$$s^{\alpha} \mathcal{L}\{y\} - D_x^{\alpha-1} y(0+) = \mathcal{L}\{y\},$$

so

$$\mathcal{L}\{y\} = \frac{b_1}{s^{\alpha} - 1}.$$

Looking for inverse transform in the table, we obtain the same result as before:

$$y(x) = b_1 x^{\alpha-1} E_{\alpha,\alpha}(x^{\alpha}).$$

The same methods works for all linear equations and systems with constant coefficients an one independent variable (Caputo derivatives too).

Thanks for your attention!