Chapter 7

Applications of Continuous-Time Random Walks to Finance and Economics

7.1 Introduction

In financial markets, when one considers tick-by-tick trades, not only price fluctuations, but also waiting times between consecutive trades vary at random. This fact is pictorially represented in Fig. 7.1. In this figure, the value of the FTSE MIB Index is plotted for trades occurring on February 3rd, 2011. The FTSE MIB Index is a weighted average of the prices of the thirty most liquid stocks in the Italian Stock Exchange and it is updated every time a trade occurs. This is a consequence of trading rules. In many regulated financial markets trading is performed by means of the so-called continuous-double auction. Here, we just present the basic idea of this microstructural market mechanism for an order driven market; details may vary from stock exchange to stock exchange. For every stock traded in the exchange there is a *book* where orders are registered. Traders can either place buy orders (*bids*) or sell orders (*asks*) and this explains why the auction is called double. Moreover, orders can be placed at any time, and, for this reason, the auction is called continuous. There are many different kinds of orders, but the typical order is the *limit order*. A bid limit order is an order to buy $q_b^{(T)}$ units of the share at a price not larger than a limit price selected by the trader $p_b^{(T)}$, where T is a label identifying the trader. An ask limit order is an order to sell $q_a^{(T)}$ units of the share at a price not smaller than a limit price selected by the trader $p_a^{(T)}$. The couples $(p_b^{(T)}, q_b^{(T)})$ are stored in the book and ordered from the best bid to the worst bid, the best bid being $\widehat{p}_b = \max_{T\in I_b}(p_b^{(T)})$, where I_b is the set of traders placing bids. The couples $(p_a^{(T)}, q_a^{(T)})$ are also stored in the book and ordered from the

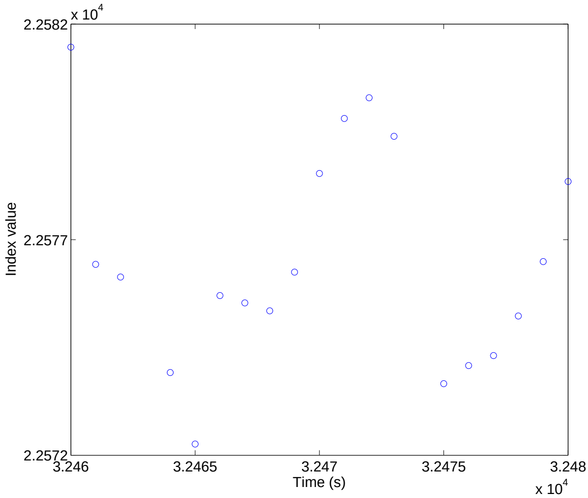

Fig. 7.1 Tick-by-tick price fluctuations. As eplained in the text, this is the FTSE MIB Index recorded on February 3rd, 2011. Time is given in seconds since midnight.

best ask to the worst ask, the best ask being $\widehat{p}_a = \min_{T \in I_a}(p_a^{(T)})$, where I_a is the set of traders placing asks. At a generic time t, one has that $\widehat{p}_a(t) > \widehat{p}_b(t)$. The differece

$$s(t) = \widehat{p}_a(t) - \widehat{p}_b(t) \tag{7.1.1}$$

is called *bid-ask spread*. Occasionally, a trader may accept an existing best bid or best ask, and the i-th trade takes place at the epoch t_i. This is called a *market order*. Market rules specify which are the priorities for limit orders placed at the same price and what to do when the quantity requested in a market order is not fully available at the best price. Several authors use the mid-price defined as

$$p_m(t) = \frac{\widehat{p}_b(t) + \widehat{p}_a(t)}{2} \tag{7.1.2}$$

in order to summarize and study the above process. Both the bid-ask spread and the mid-price can be represented as step functions varying at random times. Jumps in spread and midprice may occur when a better limit order enters the book or when a trade takes place. Another important process is the one of realized trades represented in Figs. 7.1 and 7.2. In Fig. 7.2,

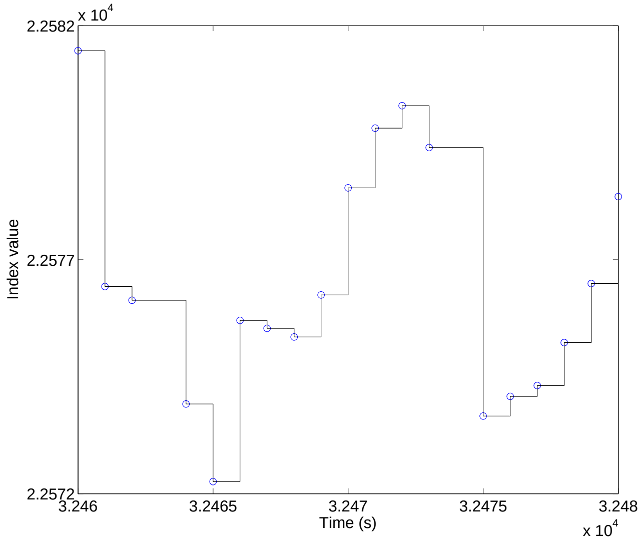

Fig. 7.2 Tick-by-tick price fluctuations represented as a càdlàg step function. FTSE MIB Index, February 3rd, 2011. These are the same data as in Fig. 7.1.

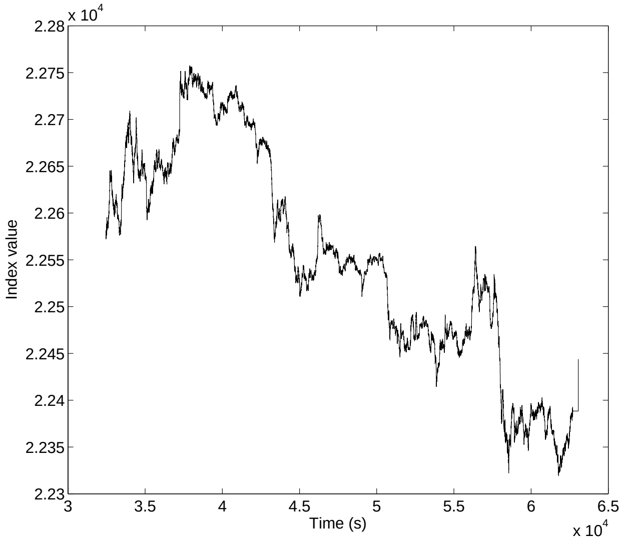

Fig. 7.3 Tick-by-tick price fluctuations represented as a càdlàg step function. FTSE MIB Index: A whole trading day, February 3rd 2011.

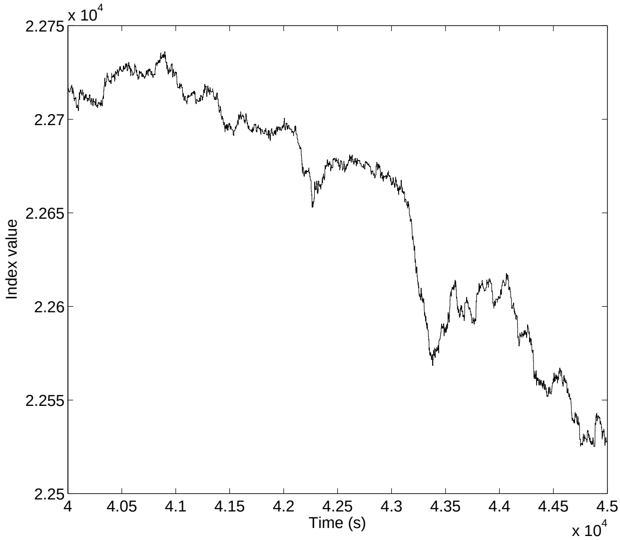

Fig. 7.4 Tick-by-tick price fluctuations represented as a càdlàg step function. A zoom of the data of Fig. 7.3.

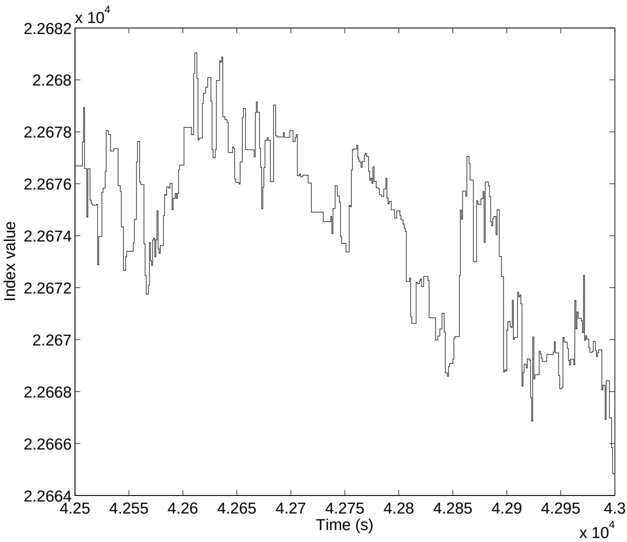

Fig. 7.5 Tick-by-tick price fluctuations represented as a càdlàg step function. A zoom of the data of Fig. 7.4.

a càdlàg representation of the process is given, the so-called *previous tick interpolation*, where it is assumed that the price remains constant between two consecutive trades. With these definitions in mind, we can now show how CTRWs can be used in financial modeling [507, 380]. Figures 7.3 to 7.5 represent consecutive magnifications of the same price process and they show how scaling breaks down. In particular, visual inspection shows that scaling is no longer valid already at the time scale of Fig. 7.5. In other words, the behavior of high-frequency price fluctuations cannot be described by self-similar, or self-affine, or even multifractal processes.

7.2 Models of Price Fluctuations in Financial Markets

Let $p(t)$ represent the price of an asset at time t. Define $p_0 = p(0)$ the initial price. The variable

$$x(t) = \log\left[\frac{p(t)}{p_0}\right] \tag{7.2.1}$$

is called *logarithmic price* or *log-price* or even *logarithmic return* or *log-return*. With this choice, one has $x(0) = 0$. Now, let t_i be the epoch of the i-th trade and $p(t_i)$ the corresponding price, then the variable

$$\xi_i = \log\left[\frac{p(t_i)}{p(t_{i-1})}\right] \tag{7.2.2}$$

is called *tick-by-tick log-return*. Let $n(t)$ represent the number of trades from the market opening, up to time t, then the relationship between the log-price and the tick-by-tick log returns is

$$x(t) = \sum_{i=1}^{n(t)} \xi_i. \tag{7.2.3}$$

The reason for using these variables in finance is as follows. If price fluctuations are small compared to the price, one can see that the tick-by-tick log-return is very close to the tick-by-tick return $r_i = [p(t_i) - p(t_{i-1})/p(t_{i-1})]$. In symbols, one can write

$$\xi_i = \log\left[\frac{p(t_i)}{p(t_{i-1})}\right] \approx \frac{p(t_i) - p(t_{i-1})}{p(t_{i-1})} = r_i. \tag{7.2.4}$$

However, returns are not additive, one cannot write the return from a time t to a time $t+\Delta t$ as the sum of tick-by-tick returns, whereas this is possible for log-returns. Equation (7.2.3) can be compared with equation (6.2.59) in Chapter 6. It becomes natural to interpret $x(t)$ as a realization of a CTRW or of a compound (renewal) process $X(t)$, $n(t)$ as a realization of a counting (renewal) process $N(t)$ and ξ_i as a value of a random variable Y_i. The simplest possible choice for $x(t)$ is the normal compound Poisson process (NCPP) discussed in Chapter 6, with normally distributed tick-by-tick log-returns (jumps) $\xi \sim N(\mu_\xi, \sigma_\xi^2)$ and exponentially distributed durations (sojourn times) $\tau_i = t_i - t_{i-1} \sim \exp(\lambda)$. However, the normal compound Poisson process is falsified by the following empirical findings on high frequency data:

(1) The empirical distribution of tick-by-tick log-returns is leptokurtic, whereas the NCPP assumes a normal distribution which is mesokurtic.
(2) The empirical distribution of durations is non-exponential with excess standard deviation [205, 206, 374, 471, 510], whereas the NCPP assumes an exponential distribution.
(3) The autocorrelation of absolute log-returns decays slowly [471], whereas the NCPP assumes i.i.d. log-returns.
(4) Log-returns and waiting times are not independent [471, 389], whereas the NCPP assumes their independence.
(5) Volatility and activity vary during the trading day [90], whereas the NCPP assumes they are constant.

Compound renewal processes take into account facts (1) and (2) as well as fact (4) [389], but they are falsified by fact (3), as they assume independent returns and durations, and by point (5), as they assume identically distributed returns and durations.

7.3 Simulation

Simulation of CTRWs is not difficult [231, 234]. A typical algorithm for uncoupled CTRWs uses the following steps:

(1) generate n values for durations from your favorite distribution and store them in a vector;

(2) generate n values for tick-by-tick log-returns from your favorite distribution and store them in a second vector;
(3) generate the epochs by means of a cumulative sum of the duration vector;
(4) generate the log-prices by means of a cumulative sum of the tick-by-tick log-return vector.

If one wishes to simulate $x(t)$, the value of the log-price at time t, it is enough to include a control statement in the above algorithm to ensure that it runs until the sum of durations is less or equal than t. These algorithms generate single realizations of the process either for n jumps or until time t respectively. If many independent runs are performed, one can approximate the distribution of $x(t)$ or any other finite dimensional distribution of the process. An algorithm for the fractional compound process with jumps distributed according to the symmetric α-stable distribution is given in Appendix A.5. In that example, for the variables ξ_i, we have used the standard transformation method by Chambers, Mallows and Stuck [139] for $\alpha \in (0, 2]$

$$\xi_i = \gamma_x \left(\frac{-\log(u)\cos(\phi)}{\cos[(1-\alpha)\phi]} \right)^{1-1/\alpha} \frac{\sin(\alpha\phi)}{\cos\phi}, \tag{7.3.1}$$

where γ_x is a scale factor, u is a uniformly distributed random variable between 0 and 1 and $\phi = \pi(v - 1/2)$, with v uniformly distributed between 0 and 1 and not depending on u. For $\alpha = 2$, equation (7.3.1) reduces to $\xi_i = 2\gamma\sqrt{-\log(u)}\sin(\phi)$, that is to the Box-Muller algorithm for normally distributed random numbers. The algorithm for the generation of Mittag-Leffler distributed τ_is with $\beta \in (0, 1]$ is (see [446, 323, 326, 325, 324, 294, 231])

$$\tau_i = -\gamma_t \log(u) \left(\frac{\sin(\beta\pi)}{\tan(\beta\pi v)} - \cos(\beta\pi) \right)^{1/\beta}, \tag{7.3.2}$$

where u and v are independent uniformly distributed random variables with values between 0 and 1. For $\beta = 1$ equation (7.3.2) reduces to $\tau_i = -\gamma_t \log(u)$, that is to the standard transformation formula for the exponential distribution. The results of simulations based on this algorithm are represented in Figures 7.6 to 7.11 for the following couples of

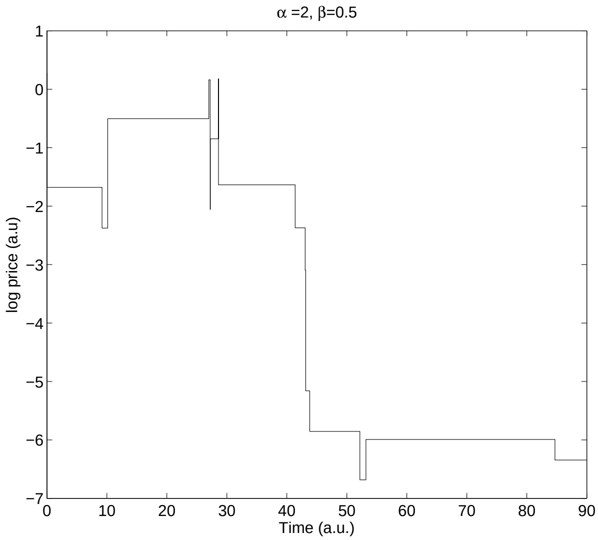

Fig. 7.6 Compound fractional Poisson process simulation for $\alpha = 2$ and $\beta = 0.5$.

parameters $(\alpha = 2, \beta = 0.5)$, $(\alpha = 2, \beta = 0.99)$, $(\alpha = 1.95, \beta = 0.99)$, $(\alpha = 1.95, \beta = 0.99)$, $(\alpha = 1, \beta = 0.99)$, and $(\alpha = 1, \beta = 0.99)$. Visual inspection shows that larger jumps in time are more likely for smaller values of β and larger jumps in log-price are expected for smaller values of α.

7.4 Option Pricing

In Secs. 7.1 and 7.2, we gave arguments in favor of using CTRWs as models of tick-by-tick price fluctuations in financial markets. We have also seen the limits of uncoupled CTRWs as market models. Now, let us suppose that we have an underlying asset whose log-price fluctuations are described by equation (7.2.3). In other words, we assume that log-price fluctuations follow a compound renewal process. Furthermore, we assume that these fluctuations represent the intra-day behavior of an asset, such as a share traded in a stock exchange. For an intra-day time horizon, we can safely assume that the *risk-free* interest rate is $r_F = 0$. This would be the return rate of a zero-coupon bond. Even if such a return rate were $r_Y = 10\%$ on a yearly time horizon, meaning that the State issuing this financial instrument is close to default (so that it would not be so riskless, after all)

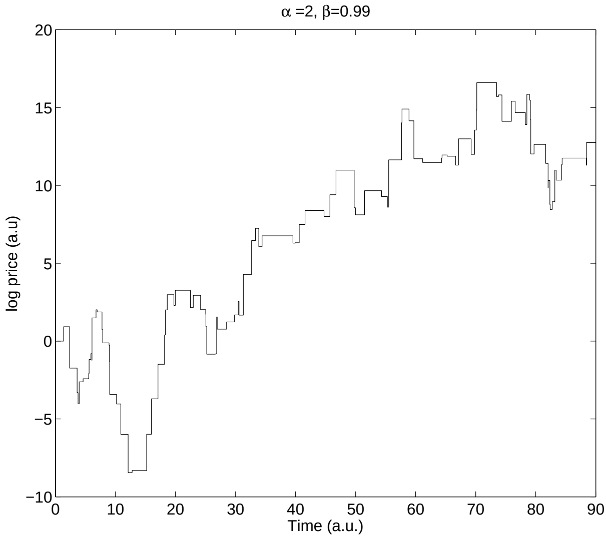

Fig. 7.7 Compound fractional Poisson process simulation for $\alpha = 2$ and $\beta = 0.99$.

or that the inflation rate is quite high, then the interest rate for one day would be $r_d \approx 1/(10 \cdot 200) = 5 \cdot 10^{-4}$ (200 is the typical number of working days in a year) and this number has still to be divided by 8 (number of trading hours) and by 3600 (number of seconds in one hour) in order to get an approximate interest rate for a time horizon of 1 second. This gives $r_s \approx 1.7 \cdot 10^{-8}$. On the other hand, typical tick-by-tick returns in a stock exchange are larger than the tick divided by the price of the share. Even if we assume that the share is worth 100 monetary units, with a 1/100 tick size (the minimum price difference allowed), we will have a return r larger than $1 \cdot 10^{-4}$ and much larger than r_s. Therefore, it is safe to assume a risk free interest rate $r = 0$ for intra-day hedging.

Hedging is performed through special contracts called options whose price is assumed to depend on the price of the underlying contract. A detailed discussion of these contracts is outside the scope of the present book. However, it is possible to present the basic ideas on option pricing, before turning to our high-frequency problem. The interest reader can consult the introductory books by Hull [285] and by Willmott [583]. One of the simplest option contract is the so-called *plain-vanilla European call.* This is the right (not the obligation) to buy an asset at a given price K called

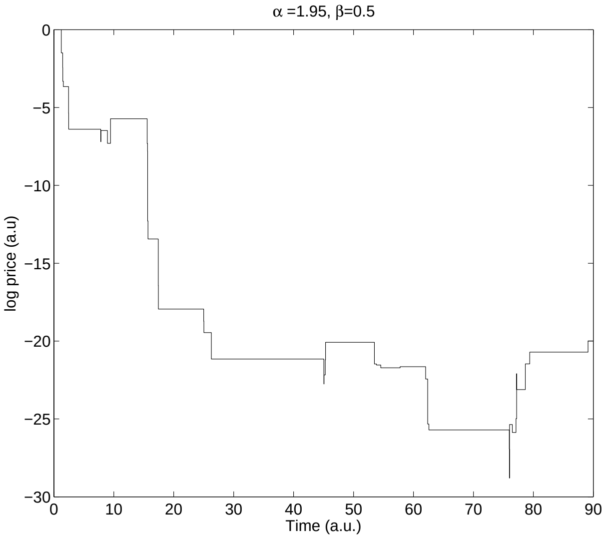

Fig. 7.8 Compound fractional Poisson process simulation for $\alpha = 1.95$ and $\beta = 0.5$.

the *strike price* at the future date T called the *maturity*. If, at maturity, the asset price $p(T)$ is larger than K then, in principle, the option holder can exercise the option, pay K to the option writer to get one unit of the asset and resell the assets on the market thus realizing a profit of $p(T) - K$ for each asset unit. On the other side, if $p(T) < K$, it does not make sense to exercise the option. So, one has that the option payoff at maturity is given by

$$C(T) = \max(p(T) - K, 0). \tag{7.4.1}$$

The problem to be addressed is the following. Suppose you are at time $t < T$ and you want to get a plain vanilla option contract. Which is its fair price? In order to give a feeling on how to solve this problem, we shall consider a simplified version: the so-called one-period binomial option pricing. The price of an asset is $p_0 = p(0)$ at time $t = 0$ and it can either go up or down at the next time step $t = 1$. Assume that $p_1^+ = p(1) = p_0 u$ with probability q and $p_1^- = p(1) = p_0 d$ with probability $1 - q$, where u is the up factor and d is the down factor. For the sake of simplicity, we shall further assume that the risk free interest rate is $r_F = 0$ during this period. The two factors, u and d cannot assume arbitrary values. We

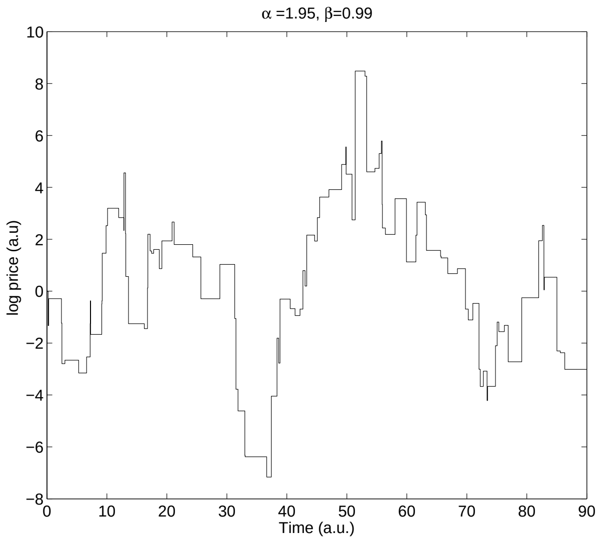

Fig. 7.9 Compound fractional Poisson process simulation for $\alpha = 1.95$ and $\beta = 0.99$.

want to avoid arbitrage, a trading strategy according to which one can get money out of nothing. Suppose indeed, that $d \geq 1$ and that we take from a bank p_0 monetary units to buy one share at time $t = 0$, then at time $t = 1$, the value of our share will be $p_1 \geq dp_0$, then by selling it and giving back p_0 monetary units to the bank, we will surely get a net profit of $p_1 - p_0 \geq dp_0 - p_0 = (d-1)p_0 \geq 0$ as $d \geq 1$. Therefore, to avoid arbitrage, we must take $d < 1$. Similarly, assume that $u \leq 1$, then one could borrow an asset share at time $t = 0$, then sell it for p_0 units of money and put the money in a bank. Now, at time $t = 1$, we could use this money to pay the share we borrowed at $t = 0$, since $p_1 \leq up_0$, in the end we would realize a certain profit $p_0 - p_1 \geq p_0 - up_0 = (1-u)p_0 \geq 0$ since $u \leq 1$. In this case, to avoid arbitrage, we must have $u > 1$. In the end, we must require that $0 < d < 1 < u$. Now, if our strike price is $p_1^- < K < p_1^+$, our payoff at time $t = 1$ will be $C_1^+ = C(1) = p_1^+ - K$ with probability q and $C_1^- = C(1) = 0$ with probability $1 - q$. It is possible to prove that the option price $C(0)$ at $t = 0$ is given by the following conditional expectation

$$C(0) = \mathbb{E}_{\widetilde{\mathbb{Q}}}[C(1)|I(0)], \tag{7.4.2}$$

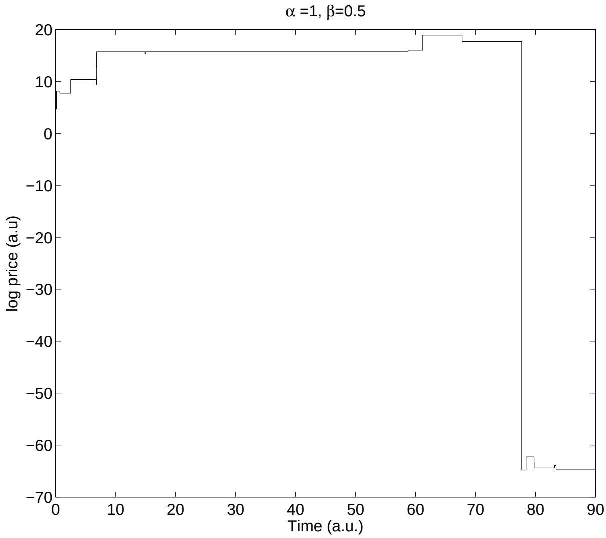

Fig. 7.10 Compound fractional Poisson process simulation for $\alpha = 1$ and $\beta = 0.5$.

where $I(0)$ represents the information available at time $t = 0$ and $\widetilde{\mathbb{Q}}$ represents the equivalent martingale measure. Two probability measures are *equivalent* if each one is absolutely continuous with respect to the other. A probability measure $\mathbb{P}$ is absolutely continuous to respect to measure $\mathbb{Q}$ if its null set is contained in the null set of $\mathbb{Q}$. The null sets of two equivalent measures do coincide. An elementary introduction to these concepts can be found in a book by T. Mikosch [398]. Among all the equivalent measures, the equivalent martingale measure is the one for which the price process is a martingale, meaning that the price process is integrable and

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}[p(1)|I(0)] = p(0). \tag{7.4.3}$$

In our case, the martingale measure is given by $\tilde{q} = (1-d)/(u-d)$ and $1 - \tilde{q} = (u-1)/(u-d)$ and the option price is given by

$$C(0) = \frac{1-d}{u-d} C_1^+. \tag{7.4.4}$$

The martingale measure can be found by simple algebraic manipulations imposing equation (7.4.3). Indeed, one has that $\mathbb{E}_{\widetilde{\mathbb{Q}}}[p(1)|I(0)] = \tilde{q}up_0 + (1-\tilde{q})dp_0$, and imposing (7.4.3) immediately leads to $\tilde{q} = (1-d)/(u-d)$.

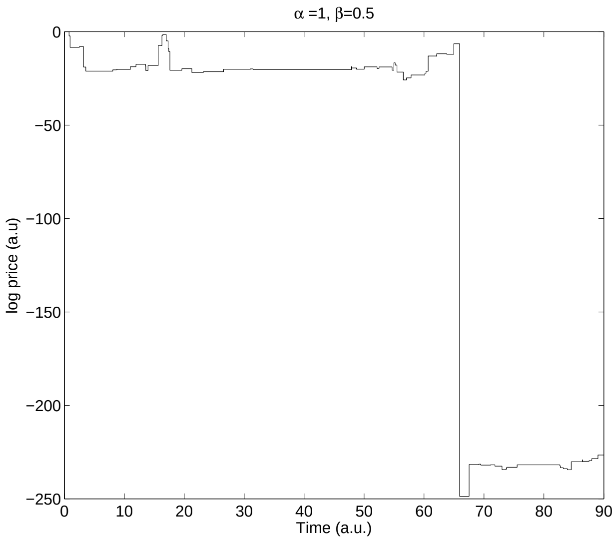

Fig. 7.11 Compound fractional Poisson process simulation for $\alpha = 1$ and $\beta = 0.99$.

Note that equation (7.4.2) means that also C is a martingale under the measure $\tilde{\mathbb{Q}}$. Technically speaking $I(0)$ in equation (7.4.2) is the *filtration* at time $t = 0$. A filtration is a non decreasing family of σ-algebras which represents the information available at a certain time, see [398] for a rigorous definition of this concept. Equation (7.4.4) can be derived from the fact that it is possible to replicate the option in terms of a suitable non-financing portfolio coupled with a no-arbitrage argument. This derivation shows that equations (7.4.2) and (7.4.4) give the *optimal* option price in term of fairness. It is not always possible to extend the arguments leading to the martingale option price when more general assumptions on the process followed by the price of the underlying asset are made. However, it is often possible to compute the martingale price in many cases of practical interest and this is done by *quants* in everyday financial practice [412]. In 1976, R. Merton solved the problem of finding the option martingale price for an underlying whose log-price follows the NCPP [392]. The idea behind Merton's derivation is as follows. Assume that $t = 0$ is a renewal point, that the risk free interest rate is $r_F = 0$ and denote the price at $t = 0$ by $S_0 = S(0)$, the strike price by K and the maturity by T. The NCPP assumption means that the underlying log-price follows the process

$$X(t) = \log(S_0) + \sum_{i=1}^{N(t)} Y_i, \tag{7.4.5}$$

where $Y_i \sim N(\mu, \sigma^2)$ and $N(t)$ is the Poisson process. The price should follow the process $S(t) = S_0 \mathrm{e}^{X(t)}$. This is not a martingale, however. In order to find the option martingale price, let us consider the situation in which there are exactly n jumps from 0 and T. In this case, one has to study the processes

$$X_n = \log(S_0) + \sum_{i=1}^{n} Y_i \tag{7.4.6}$$

and

$$S_n = S_0 \mathrm{e}^{X_n} = S_0 \prod_{i=1}^{n} \mathrm{e}^{Y_i}. \tag{7.4.7}$$

Notice that the random variables e^{Y_i} follow the log-normal distribution. The process defined by S_n is not a martingale, but the equivalent martingale measure can be found by imposing that the process

$$S'_n = S_0 \mathrm{e}^{X_n + na} \tag{7.4.8}$$

is a martingale and this leads to

$$a = -\log[\mathbb{E}(\mathrm{e}^Y)], \tag{7.4.9}$$

where

$$\mathbb{E}(\mathrm{e}^Y) = \mathrm{e}^{\mu + \sigma^2/2} \tag{7.4.10}$$

so that

$$a = -(\mu + \sigma^2/2). \tag{7.4.11}$$

The option price at $t = 0$ is thus given by

$$C_n(S_0, K, \mu, \sigma^2) = \mathbb{E}_{\mathbb{S}'}[C(T)|I(0)], \tag{7.4.12}$$

where the expected value is computed according to the measure defined by the process (7.4.8). For the plain vanilla European call option with $C(T) = \max(S(T) - K, 0)$, a straightforward calculation leads to

$$C_n(S_0, K, \mu, \sigma^2) = N(d_{1,n})S_0 - N(d_{2,n})K, \tag{7.4.13}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dy\, \mathrm{e}^{-y^2/2} \tag{7.4.14}$$

is the standard normal cumulative distribution function and

$$d_{1,n} = \frac{\log(S_0/K) + n(\mu + \sigma^2/2)}{\sqrt{n}\sigma}, \tag{7.4.15}$$

$$d_{2,n} = d_{1,n} - \sigma\sqrt{n}. \tag{7.4.16}$$

Given the independence between jumps and durations, one can now write the option price at $t = 0$ as

$$C(0) = \mathrm{e}^{-\lambda T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} C_n(S_0, K, \mu, \sigma^2), \tag{7.4.17}$$

where λ is the intensity of the Poisson process. To obtain equation (7.4.17), it is enough to notice that the probability of having n jumps in the time interval $[0, T]$ is given by the Poisson distribution of parameter λT and that one can go from S_0 to $S(T)$ in any number of steps n. Equation (7.4.17) can be generalized to general renewal processes simply by replacing the Poisson distribution with a generic counting distribution $P(n, t)$. In the Mittag-Leffler case with $\gamma_t = 1$, one can write (see equation (6.2.55))

$$C(0) = \sum_{n=0}^{\infty} \frac{t^{\beta n}}{n!} E_{\beta}^{(n)}(-t^{\beta})\, C_n(S_0, K, \mu, \sigma^2). \tag{7.4.18}$$

Merton's result has been recently revisited and it is still the object of active research [142]. Note that this method works when the random variable e^Y has finite first moment. This is the case when all the moments of the tick-by-tick log-returns Y are finite as in the normal case discussed above.

7.5 Other Applications

This chapter focuses on the application of uncoupled continuous-time random walks in high-frequency financial data modeling. For the sake of simplicity, we have not discussed the coupled case, but this is covered in reference [389]. Moreover, it is possible to use the program described in Sec. 7.3 for scenario simulation and speculative option pricing [511].

As discussed in Sec. 6.1, it is not surprising that CTRWs can also be applied otherwise. A standard application is to insurance [361, 151], where the capital $R(t)$ of an insurance company can be written as

$$R(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \tag{7.5.1}$$

and where u is the initial capital of the company, c is the rate of capital increase, $N(t)$ is the random number of claims Y_i that the company has paid since inception. In this case, ruin is the interesting phenomenon. It takes place the first time that $R(t) = 0$, i.e., when the capital of the insurance company vanishes. In this framework, one can define the time to ruin as the following hitting time

$$\tau(u) = \inf\left\{ t: \ u + ct - \sum_{i=1}^{N(t)} Y_i < 0 \right\}. \tag{7.5.2}$$

Two interesting quantities are the probability $q(u)$ of ruin in infinite time and the probability of ruin in a finite time T. These two quantities are defined as follows, respectively:

$$q(u) = \mathbb{P}(\tau(u) < \infty), \tag{7.5.3}$$

and

$$q(u, T) = \mathbb{P}(\tau(u) < T). \tag{7.5.4}$$

It is always possible to study these quantities by means of Monte Carlo simulations, using the algorithm of Appendix A.5 or a suitable modification.

Another interpretation of the random variables is in terms of economic growth. Let us be as general as possible and denote by S a suitable "size". This size has the meaning of wealth, firm size, city size, etc., depending on the scientific context. Then, according to Gibrat's approach [235], one can define the log-size $X = \log(S)$ and write it as a sum of exogenous shocks Y_i

$$X_n = X_0 + \sum_{i=1}^{n} Y_i. \tag{7.5.5}$$

For large n, if the shocks have finite first and second moments, X_n approximately follows the normal distribution as a consequence of central limit theorems (see Theorem 6.4 for a simple version). This means that the size S_n approximately follows the log-normal distribution [32]. If the growth shocks arrive at random times, equation (7.5.5) can be replaced by the familiar equations for CTRWs with non-homogeneous initial position

$$X(t) = X_0 + \sum_{i=1}^{N(t)} Y_i. \tag{7.5.6}$$

This method was used by Italian economists to study firm growth and size distributions [106, 232, 107].