Chapter 4

Generalized Stirling Numbers and Applications

In this Chapter, we will present the latest and most complete extension of the Stirling numbers using the framework of the fractional calculus that we are presently aware of.

4.1 Introduction

The classical Stirling numbers of the first kind $s(n,k)$, introduced by James Stirling in his famous manuscript *Methodus Differentialis* of 1730 [537], play, together with the Stirling numbers of second kind $S(n,k)$, an important role in the calculus of finite differences, in combinatorial problems, in numerical analysis, interpolation theory and number theory.

The classical Stirling numbers of the first kind $s(n,k)$ and of the second kind $S(n,k)$ can be defined via different generating functions, for example by

$$[x]_n = \sum_{k=0}^{n} s(n,k)x^k \quad (x \in \mathbb{R};\ n \in \mathbb{N}_0), \tag{4.1.1}$$

or

$$(\log(1+x))^k = k! \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!} \quad (|x|<1;\ k \in \mathbb{N}_0) \tag{4.1.2}$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)[x]_k \tag{4.1.3}$$

or

$$(e^x - 1)^k = k! \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!} \quad (x \in \mathbb{R};\ k \in \mathbb{N}_0), \tag{4.1.4}$$

respectively. Equivalently, by

$$s(n,k) = \frac{1}{k!} \lim_{x \to 0} D^k [x]_n,\ D = \frac{d}{dx} \quad (x \in \mathbb{R};\ n, k \in \mathbb{N}_0) \tag{4.1.5}$$

and

$$S(n,k) = \frac{1}{k!} \lim_{x \to 0} \Delta^k (x^n) \quad (x \in \mathbb{R};\ n, k \in \mathbb{N}_0). \tag{4.1.6}$$

Here $[x]_n$ is the factorial polynomial defined for $x \in \mathbb{R}$ by

$$[x]_0 = 1,\ [x]_n = x(x-1)\cdots(x-n+1) \quad (n \in \mathbb{N}), \tag{4.1.7}$$

and Δ^k is the difference of order $k \in \mathbb{N}_0$ defined by

$$\Delta^0 f(x) = f(x),\ \Delta^1 f(x) = f(x+1) - f(x),$$

$$\Delta^{k+1} f(x) = \Delta \left(\Delta^k f \right)(x) \quad (k \in \mathbb{N}) \tag{4.1.8}$$

having the representation

$$\Delta^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+j) \quad (x \in \mathbb{R};\ k \in \mathbb{N}_0) \tag{4.1.9}$$

in terms of the binomial coefficients

$$\binom{k}{j} = \frac{k!}{j!(k-j)!} \quad (k, j \in \mathbb{N}_0;\ j \leq k). \tag{4.1.10}$$

The main definitions and properties of the classical Stirling numbers are given, for instance, in the textbooks by Riordan [483], Comtet [148], and on pp. 824–825 of the handbook by Abramowitz and Stegun [5]. For interesting applications of Stirling numbers in the setting of difference calculus, discrete mathematics and combinatorics one may consult the books by Jordan [296], by Graham *et al.* [251] and by Aigner [31].

Butzer *et al.* [114] introduced generalizations of $s(n,k)$ and $S(n,k)$ for the first parameter n, via Eqs. (4.1.5) and (4.1.6), for $\alpha \in \mathbb{R}$

$$s(\alpha,k) = \frac{1}{k!}\lim_{x\to 0} D^k [x]_\alpha \quad (-x \notin \mathbb{N};\ \alpha \in \mathbb{R},\ k \in \mathbb{N}_0) \tag{4.1.11}$$

with

$$[x]_\alpha = \frac{\Gamma(x+1)}{\Gamma(x-\alpha+1)},$$

and

$$S(\alpha,k) = \frac{1}{k!}\lim_{x\to 0} \Delta^k (x^\alpha) \quad (\alpha \geq 0,\ k \in \mathbb{N}_0), \tag{4.1.12}$$

the limit being taken in the sense

$$\lim_{x\to 0} \Delta^k (x^\alpha) = \lim_{\epsilon\to 0}\lim_{x\to 0} \Delta^k \left((x+\epsilon)^\alpha\right). \tag{4.1.13}$$

The mentioned authors and Butzer and Hauss [111, 112] investigated properties of the above generalized Stirling numbers which they finally called "Stirling function", that is, functions of the "continuous" parameter α. In particular, for such Stirling functions of the first kind they established the recurrence relation

$$s(\alpha+1,k) = s(\alpha,k-1) - \alpha s(\alpha,k); \quad s(\alpha,0) = \frac{1}{\Gamma(1-\alpha)} \tag{4.1.14}$$

with $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$, and the integral representation

$$s(\alpha,k) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(z+1)}{\Gamma(z-\alpha+1)}\frac{dz}{z^{n+1}} \quad (\alpha \in \mathbb{R};\ k \in \mathbb{N}_0), \tag{4.1.15}$$

with a special contour $\mathcal{L}$, while for Stirling functions of the second kind they established the closed form representation

$$S(\alpha,k) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j} j^\alpha \quad (\alpha > 0;\ k \in \mathbb{N}) \tag{4.1.16}$$

and the recurrence relations for $k \in \mathbb{N}$

$$S(\alpha+1,k) = kS(\alpha,k) + S(\alpha,k-1) \quad (\alpha > 0); \tag{4.1.17}$$

$$S(\alpha,0) = 0,\ S(0,k) = 0,\ S(0,0) = 1 \quad (\alpha > 1).$$

Moreover Butzer *et al.* [113] extended $s(\alpha, k)$ to complex $\alpha \in \mathbb{C}$ ($\Re e(\alpha) > 0$), and Loeb [350] followed up the matter of [111, 112], studying some properties of such generalized Stirling functions of the first kind with real α.

Generalizations of Stirling numbers of the first and the second kind to more general constructions containing more than two parameters were considered by several authors. For example, Rucinski and Voigt [492] gave a generalization of Stirling numbers of the second kind which are defined if in the formula (4.1.3) one replaces $[x]_k$ by the product $(x-a_0)\cdots(x-a_{k-1})$ containing a given sequence $(a_j)_{j=0}^{\infty}$. Butzer and Jansche [115] introduced a generalization $S_c(n,k)$ ($n,k \in \mathbb{N};\ n \geq k \geq 0$) of Stirling numbers $S(n,k)$ with the property

$$\Theta^n f(x) = \sum_{k=0}^{n} S_c(n,k) x^k f^{(k)}(x), \tag{4.1.18}$$

where Θ^n is the differentiation operator of the form

$$(\Theta^n f)(x) = (\delta + c)^n f(x),\ \delta = x\frac{d}{dx}. \tag{4.1.19}$$

Also we must mention in this respect the generalized Stirling numbers of the second kind defined by Hauss [264] in his thesis, who developed a theory based on a very general approach, including representations in terms of various fractional integrals and derivatives, as well as the fine works by Platonov [452] and by Zhang [596].

Uniform asymptotic expansions for certain generalized Stirling functions were considered by Chelluri *et al.* [143].

As far as we know, a number of authors have introduced several generalizations of both Stirling numbers using different approaches, but without using the fractional calculus framework. We refer in this context, e.g., to [6, 150, 217, 284, 328, 387, 477, 523–525, 556, 572], *etc.*

The aim of this chapter is to develop a global theory, within the framework of the fractional calculus, which allows us to introduce a natural extension of the classical concepts of $s(n,k)$ and $S(n,k)$ to Stirling functions with both parameters, n and k, complex numbers. Moreover, it is proved that such Stirling functions conserve the well known properties of the classical Stirling numbers. See, e.g., the papers by Butzer *et al.* [116–120, 122].

4.2 Stirling Functions of the First Kind $s(\alpha, k)$

The Stirling numbers of first kind, the $s(n,k)$, can be defined in terms of their (horizontal) generating function

$$[z]_n = \sum_{k=0}^{n} s(n,k) z^k \quad (z \in \mathbb{C};\ n \in \mathbb{N}_0), \tag{4.2.1}$$

thus, equivalently by (4.1.5),

$$s(n,k) = \frac{1}{k!} \lim_{x \to 0} D^k [x]_n,\ D = \frac{d}{dx} \quad (x \in \mathbb{R};\ n, k \in \mathbb{N}_0).$$

A further equivalent approach is via their exponential generating function (4.1.2)

$$(\log(1+x))^k = k! \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!} \quad (|x| < 1;\ k \in \mathbb{N}_0)$$

thus, in view of the Taylor expansion, also by

$$s(n,k) = \frac{1}{k!} \lim_{x \to 0} \left(\frac{d}{dx} \right)^n [\log(1+x)]^k \quad (x \in \mathbb{R}). \tag{4.2.2}$$

4.2.1 *Equivalent definitions*

The Stirling functions of the first kind, $s(\alpha, k)$, where $n \in \mathbb{N}$ is extended to real $\alpha \in \mathbb{R}$ as well as to complex $\alpha \in \mathbb{C}$, first studied from 1989 in [111–114], can be defined in terms of the infinite sum

$$[z]_\alpha = \sum_{k=0}^{\infty} s(\alpha, k) z^k \quad (|z| < 1; \quad \alpha \in \mathbb{C}) \tag{4.2.3}$$

since $[z]_\alpha$ is holomorphic for $|z| < 1$, where $[z]_\alpha = \Gamma(z+1)/\Gamma(z+1-\alpha)$ $(\alpha \in \mathbb{C} \backslash \mathbb{Z}^-)$, therefore equivalently by

$$s(\alpha, k) = \frac{1}{k!} \lim_{x \to 0} \left(\frac{d}{dx} \right)^k [x]_\alpha \quad (\alpha \in \mathbb{C};\ k \in \mathbb{N}_0). \tag{4.2.4}$$

The following representation for $s(\alpha, k)$ has been obtained by Butzer *et al.* [113].

Theorem 4.1 (Representation theorem). *For $\alpha \in \mathbb{C}$ and $k > \Re e(\alpha)$ $(k \in \mathbb{N})$ there holds*

$$s(\alpha, k) = \frac{1}{\Gamma(-\alpha)k!} \int_{0+}^{1} \frac{[\log u]^k}{(1-u)^{\alpha+1}} du$$
$$= (-1)^{k+1} \frac{\sin(\alpha\pi)}{\pi} \sum_{j=1}^{\infty} \frac{\Gamma(\alpha+j)}{(j-1)!\; j^{k+1}}. \tag{4.2.5}$$

The Stirling functions of the first kind, $s(\alpha, k)$, were also defined by the fractional counterpart of Eq. (4.2.2), namely

$$s(\alpha, k) = \frac{1}{k!} \lim_{x \to 1} \left({}^{\mathtt{RL}}D_{0+}^{\alpha}[\log(t)]^k\right)(x) \quad (\Re e(\alpha) > 0; \quad k \in \mathbb{N}_0), \tag{4.2.6}$$

where ${}^{\mathtt{RL}}D_{0+}^{\alpha}$ is the Riemann-Liouville fractional derivative that we had described in Eq. (1.3.3). In particular, when $\alpha = n \in \mathbb{N}_0$, then the definition given in Eq. (4.2.6) coincides with that of Eq. (4.2.2) since the Riemann-Liouville derivative then reduces to the classical integer-order derivative, i.e.

$$\left({}^{\mathtt{RL}}D_{0+}^{n}[\log(t)]^k\right)(x) = \left(\frac{d}{dx}\right)^n [\log(x)]^k \quad (n \in \mathbb{N}_0).$$

With this approach we can also define the $s(\alpha, k)$ for $\Re e(\alpha) < 0$ by

$$s(\alpha, k) = \frac{1}{k!} \lim_{x \to 1} \left({}^{\mathtt{RL}}I_{0+}^{-\alpha}[\log(t)]^k\right)(x) \quad (k \in \mathbb{Z}). \tag{4.2.7}$$

The first important result that we shall prove here is that the Stirling functions $s(\alpha, k)$, defined by Eqs. (4.2.6) and (4.2.7), coincide with those given by the definition given in Eq. (4.2.4). Thus these definitions are equivalent.

Theorem 4.2. *Let $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}_0$.*

(a) *If $\Re e(\alpha) \geq 0$, then*

$$s(\alpha, k) = \frac{1}{\Gamma(k+1)} \lim_{x \to 1} \left({}^{\mathtt{RL}}D_{0+}^{\alpha}[\log(t)]^k\right)(x)$$
$$= \frac{1}{k!} \lim_{x \to 0} \left(\frac{d}{dx}\right)^k \left[\frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)}\right] = \frac{1}{k!} \lim_{x \to 0} \left(\frac{d}{dx}\right)^k [x]_{\alpha}. \tag{4.2.8}$$

(b) *If* $\Re e(\alpha) < 0$, *then*

$$s(\alpha, k) = \frac{1}{\Gamma(k+1)} \lim_{x\to 1} \left({}^{\mathrm{RL}}I_{0+}^{-\alpha}[\log(t)]^k\right)(x)$$

$$= \frac{1}{k!} \lim_{x\to 0} \left(\frac{d}{dx}\right)^k \left[\frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)}\right]. \tag{4.2.9}$$

In particular,

$$s(\alpha, 0) = \frac{1}{\Gamma(1-\alpha)} \quad (\alpha \in \mathbb{C}). \tag{4.2.10}$$

Proof.

(a) Let $\Re e(\alpha) \geq 0,\ \alpha \neq 0$, and $n = [\Re e(\alpha)] + 1$. By (4.2.6) and (1.3.3),

$$s(\alpha, k) = \frac{1}{k!} \lim_{x\to 1} \left(\frac{d}{dx}\right)^n \left({}^{\mathrm{RL}}I_{0+}^{n-\alpha}[\log(t)]^k\right)(x). \tag{4.2.11}$$

Now, using the known property for the fractional Riemann-Liouville integral operator

$$\left({}^{\mathrm{RL}}I_{0+}^{n-\alpha}t^\gamma\right)(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+n-\alpha)} x^{\gamma+n-\alpha} \quad (\gamma > -1),$$

in combination with a k-fold differentiation with respect to γ, yields for $\gamma > -1$, after an interchange of integration and differentiation,

$$\left({}^{\mathrm{RL}}I_{0+}^{n-\alpha}t^\gamma[\log(t)]^k\right)(x)$$

$$= \left({}^{\mathrm{RL}}I_{0+}^{n-\alpha}\left(\frac{\partial}{\partial\gamma}\right)^k t^\gamma\right)(x) = \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} s(\alpha,k)S(k,j)\right) x^j f^{(j)}(x)$$

$$= \Gamma(\alpha+1)\sum_{j=0}^{\infty} \frac{\sin[(\alpha-j)\pi]}{(\alpha-j)\pi} \frac{x^j f^{(j)}(x)}{j!}. \tag{4.2.12}$$

Moreover,

$$\left({}^{\mathrm{RL}}I_{0+}^{n-\alpha}t^\gamma[\log(t)]^k\right)(x) = \left({}^{\mathrm{RL}}I_{0+}^{n-\alpha}\left(\frac{\partial}{\partial\gamma}\right)^k t^\gamma\right)(x)$$

$$= \left(\frac{\partial}{\partial\gamma}\right)^k \left({}^{\mathrm{RL}}I_{0+}^{n-\alpha}t^\gamma\right)(x)$$

$$= \left(\frac{\partial}{\partial\gamma}\right)^k \left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+n-\alpha)} x^{\gamma+n-\alpha}\right].$$

Differentiating this expression k times with respect to x, and interchanging the order of differentiation, one has for $\gamma > -1$

$$\left(\frac{d}{dx}\right)^n \left({}^{\mathrm{RL}}I_{0+}^{n-\alpha} t^\gamma [\log(t)]^k\right)(x) = \left(\frac{\partial}{\partial\gamma}\right)^k \left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}\right]. \tag{4.2.13}$$

Thus for $\gamma > -1$,

$$\frac{1}{k!}\left({}^{\mathrm{RL}}D_{0+}^{\alpha} t^\gamma [\log(t)]^k\right)(x) = \frac{1}{k!}\left(\frac{\partial}{\partial\gamma}\right)^k \left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}\right]. \tag{4.2.14}$$

Taking the limit for $\gamma \to 0$, one has by (4.2.8) and (4.2.14)

$$\begin{aligned} s(\alpha,k) &= \frac{1}{k!}\lim_{x\to 1}\left({}^{\mathrm{RL}}D_{0+}^{\alpha} [\log(t)]^k\right)(x) \\ &= \frac{1}{k!}\lim_{x\to 1}\lim_{\gamma\to 0}\left({}^{\mathrm{RL}}D_{0+}^{\alpha} t^\gamma [\log(t)]^k\right)(x) \\ &= \frac{1}{k!}\lim_{\gamma\to 0}\lim_{x\to 1}\left(\frac{\partial}{\partial\gamma}\right)^k \left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}\right] \\ &= \frac{1}{k!}\lim_{\gamma\to 0}\left(\frac{\partial}{\partial\gamma}\right)^k \left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\right], \end{aligned} \tag{4.2.15}$$

establishing part (a) for $\Re e(\alpha) \geq 0,\ \alpha \neq 0$. If $\alpha = 0$, then by (4.2.6),

$$\begin{aligned} s(0,k) &= \frac{1}{\Gamma(k+1)}\lim_{x\to 1}\left({}^{\mathrm{RL}}D_{0+}^{0} [\log(t)]^k\right)(x) \\ &= \frac{1}{k!}\lim_{x\to 1}[\log(x)]^k = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k \in \mathbb{N}. \end{cases} \end{aligned}$$

Thus $s(0,0) = 1$, and $s(0,k) = 0$ for $k \in \mathbb{N}$.

(b) In case $\Re e(\alpha) < 0$ one applies (4.2.7), using arguments similar to the above, but using (4.2.6) and (1.3.3) with $n-\alpha$ replaced by $-\alpha$, giving

$$\begin{aligned} s(\alpha,k) &= \frac{1}{k!}\lim_{x\to 1}\left({}^{\mathrm{RL}}I_{0+}^{-\alpha} [\log(t)]^k\right)(x) \\ &= \frac{1}{k!}\lim_{x\to 1}\lim_{\gamma\to 0}\left({}^{\mathrm{RL}}I_{0+}^{-\alpha} t^\gamma [\log(t)]^k\right)(x) \\ &= \frac{1}{k!}\lim_{\gamma\to 0}\lim_{x\to 1}\left(\frac{\partial}{\partial\gamma}\right)^k \left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\ x^{\gamma-\alpha}\right] \\ &= \frac{1}{k!}\lim_{\gamma\to 0}\left(\frac{\partial}{\partial\gamma}\right)^k \left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\right]. \end{aligned}$$

When $k = 0$, then, in accordance with (4.2.4), for any $\alpha \in \mathbb{C}$ we have

$$s(\alpha, 0) = \frac{1}{0!} \lim_{x \to 0} [x_\alpha] = \lim_{x \to 0} \frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)} = \frac{1}{\Gamma(1-\alpha)}.$$

This yields (4.2.10), and thus the theorem is proved. □

Now we obtain a recursion relation for the Stirling function $s(\alpha, k)$ in terms of the well known polygamma function (see, e.g., Section 1.16 of [209]).

Theorem 4.3 (Recursion formula). *If $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$, then*

$$s(\alpha, k+1) = \frac{1}{k+1} \sum_{j=0}^{k} \frac{\psi^{(k-j)}(1) - \psi^{(k-j)}(1-\alpha)}{(k-j)!} \, s(\alpha, j) \quad (\alpha \notin \mathbb{N}), \tag{4.2.16}$$

where $\psi^{(m)}$ is the m-th polygamma function, that is,

$$\psi^{(m)}(z) = \left(\frac{d}{dz}\right)^m \psi(z) \quad (z \in \mathbb{C} \backslash \mathbb{Z}_0), \tag{4.2.17}$$

and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

Proof. The function $\Phi(x, \alpha) = \psi(x+1) - \psi(x+1-\alpha)$, defined for any $x \in V = \mathbb{C} \backslash \{x \in \mathbb{R}; x - \alpha \in \mathbb{Z}^-\}$, is holomorphic in V (see p. 261 of [423]), and thus in particular in $|x| < \epsilon$, for ϵ small, $\alpha \notin \mathbb{N}$. Hence it can be expanded as a power series about $x_0 = 0$: $\Phi(x, \alpha) = \sum_{k=0}^{\infty} \Psi_k(\alpha) x^k$ for $|x| < \epsilon$, $\alpha \notin \mathbb{N}$, where

$$\Psi_k(\alpha) = \frac{\psi^{(k)}(1) - \psi^{(k)}(1-\alpha)}{k!} \quad (k \in \mathbb{N}_0),$$

noting that the function $\Psi_k(\alpha)$ equals

$$\Psi_k(\alpha) = \frac{1}{k!} \lim_{x \to 0} \left(\frac{d}{dx}\right)^k \Phi(x, \alpha).$$

Further, differentiating the series (4.2.3) for $|x| < 1$ yields

$$\frac{d}{dx}[x]_\alpha = \sum_{k=0}^{\infty} (k+1) s(\alpha, k+1) x^k. \tag{4.2.18}$$

But by definition the left-hand derivative equals

$$\begin{aligned}
&\frac{d}{dx}\left[\frac{\Gamma(x+\alpha)}{\Gamma(x+1-\alpha)}\right] \\
&= \frac{\Gamma'(x+1)\Gamma(x+1-\alpha)-\Gamma(x+1)\Gamma'(x+1-\alpha)}{[\Gamma(x+1-\alpha)]^2} \\
&= \frac{\Gamma'(x+1)}{\Gamma(x+1)}\left\{\frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)}\right\} - \left\{\frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)}\right\}\frac{\Gamma'(x+1-\alpha)}{\Gamma(x+1-\alpha)},
\end{aligned} \tag{4.2.19}$$

thus, by power series multiplication, for $|x| < \epsilon$,

$$\frac{d}{dx}\left[\frac{\Gamma(x+\alpha)}{\Gamma(x+1-\alpha)}\right] = [x]_\alpha \Phi(x,\alpha) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} s(\alpha,j)\Psi_{k-j}(\alpha)\right)x^k. \tag{4.2.20}$$

A comparison of the coefficients of the series (4.2.18) and (4.2.20) yields the theorem. □

As a corollary of Theorem 4.2 and Theorem 4.3 we have

Corollary 4.1. *There hold for $\alpha \in \mathbb{C}\backslash\mathbb{Z}$,*

$$\begin{aligned}
s(\alpha,1) &= \frac{\psi(1)-\psi(1-\alpha)}{\Gamma(1-\alpha)} = \frac{\Phi(\alpha)}{\Gamma(1-\alpha)}, \\
s(\alpha,2) &= \frac{D^1\Phi(\alpha)+\Phi^2(\alpha)}{2\Gamma(1-\alpha)}, \\
s(\alpha,3) &= \frac{D^2\Phi(\alpha)+3\Phi(\alpha)D^1\Phi(\alpha)+\Phi^3(\alpha)}{6\ \Gamma(1-\alpha)}, \\
s(\alpha,4) &= \frac{1}{24\ \Gamma(1-\alpha)}\left[D^3\Phi(\alpha)+4\Phi(\alpha)D^2\Phi(\alpha)+6\Phi^2(\alpha)D^1\Phi(\alpha)\right. \\
&\qquad \left. +3[D^1\Phi(\alpha)]^2+3\Phi^4(\alpha)\right],
\end{aligned}$$

where

$$D^k\Phi(\alpha) = \left(\frac{\partial}{\partial x}\right)^k [\psi(x+1)-\psi(x+1-\alpha)]\big|_{x=0} \quad (k \in \mathbb{N}_0).$$

Proof. Recalling the proof of (4.2.19) and (4.2.20), noting $[0]_\alpha = 1$,

$$s(\alpha,1) = \lim_{x\to 0}\frac{d}{dx}\left[\frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)}\right] = \lim_{x\to 0}[x]_\alpha\Phi(x,\alpha) = \frac{\psi(1)-\psi(1-\alpha)}{\Gamma(1-\alpha)}.$$

As this procedure is somewhat cumbersome for $s(\alpha, 2)$ *etc.*, we apply the recursion formula (4.2.16). Indeed, it easily follows that $s(\alpha, 2) = \frac{1}{2}\{\Psi_1(\alpha)s(\alpha,0) + \Psi_0(\alpha)s(\alpha,1)\}$.

Let us also consider $s(\alpha, 3)$; in fact,

$$s(\alpha,3) = \frac{1}{3}\{\Psi_2(\alpha)s(\alpha,0) + \Psi_1(\alpha)s(\alpha,1) + \Psi_0(\alpha)s(\alpha,2)\}.$$

The further $s(\alpha, k)$, $k = 4, 5, \ldots$, follow similarly. □

4.2.2 *Multiple sum representations. The Riemann Zeta function*

First let us recall the known expression of the classical $s(k, m)$ in terms of a multiple sum (see [121, 378]). In this way, for $2 \le m \le k$ we have

$$s(k,m) = (-1)^{k+m}(k-1)! \sum_{j=m-1}^{k-1} (-1)^j s(j, m-1), \tag{4.2.21}$$

$$\begin{aligned} s(k,m) &= (-1)^{k+m}(k-1)! \\ &\times \sum_{j_{m-1}=m-1}^{k-1} \left(\frac{1}{j_{m-1}}\right) \sum_{j_{m-2}=m-2}^{j_{m-1}-1} \left(\frac{1}{j_{m-2}}\right) \cdots \sum_{j_1=1}^{j_2-1} \left(\frac{1}{j_1}\right). \end{aligned} \tag{4.2.22}$$

In particular, for $m = 1, 2, 3$ and $k \ge m$

$$s(k,1) = (-1)^{k+1}(k-1)!, \tag{4.2.23}$$

$$s(k,2) = (-1)^{k+2}(k-1)! \sum_{j_1=1}^{k-1} \left(\frac{1}{j_1}\right), \tag{4.2.24}$$

$$s(k,3) = (-1)^{k+3}(k-1)! \sum_{j_2=2}^{k-1} \left(\frac{1}{j_2}\right) \sum_{j_1=1}^{j_2-1} \left(\frac{1}{j_1}\right). \tag{4.2.25}$$

These results can be generalized for the function $s(\alpha, m)$.

Theorem 4.4. *For $\alpha \in \mathbb{C}$ and $m > \Re e(\alpha)$, $m \in \mathbb{N}$, one has*

$$s(\alpha,m) = \frac{1}{\Gamma(-\alpha)} \sum_{k=m}^{\infty} \frac{(-1)^k s(k,m)}{k!(k-\alpha)}. \tag{4.2.26}$$

Proof. Replacing z by $-v$ in (4.2.1), dividing by $v^{\alpha+1}$ and integrating this power series, using Abel's limit theorem and Raabe's convergence criterion, it follows that

$$\int_{0+}^{1-} \frac{[\log(1-v)]^m dv}{v^{1+\alpha}} = m! \sum_{k=m}^{\infty} \frac{(-1)^k s(k,m)}{k!} \int_{0+}^{1-} v^{k-1-\alpha} dv$$
$$= m! \sum_{k=m}^{\infty} \frac{(-1)^k s(k,m)}{k!(k-\alpha)}.$$

Comparing this result with the integral representation (4.2.5) of $s(\alpha, m)$ there immediately results (4.2.26). □

In view of (4.2.23) and (4.2.26), we have

$$s(\alpha,1) = \frac{1}{\Gamma(-\alpha)} \sum_{k=1}^{\infty} \frac{(-1)^k(-1)^{k+1}(k-1)!}{k!(k-\alpha)} = -\frac{1}{\Gamma(-\alpha)} \sum_{k=1}^{\infty} \frac{1}{k(k-\alpha)}. \tag{4.2.27}$$

Further,

$$s(\alpha,2) = \frac{1}{\Gamma(-\alpha)} \sum_{k=2}^{\infty} \frac{(-1)^k}{k!(k-\alpha)} \left\{(-1)^k (k-1)! \sum_{j=1}^{k-1} \frac{1}{j}\right\}$$
$$= \frac{1}{\Gamma(-\alpha)} \sum_{k=2}^{\infty} \frac{1}{k(k-\alpha)} \sum_{j=1}^{k-1} \frac{1}{j}. \tag{4.2.28}$$

Iterating this process yields the general multiple sum

$$s(\alpha,m) = \frac{(-1)^m}{\Gamma(-\alpha)} \sum_{j_m=m}^{\infty} \left(\frac{1}{j_m(j_m-\alpha)}\right) \sum_{j_{m-1}=m-1}^{j_m-1} \left(\frac{1}{j_{m-1}}\right) \cdots \sum_{j_1=1}^{j_2-1} \left(\frac{1}{j_1}\right). \tag{4.2.29}$$

Observe that (4.2.26) can be considered as the counterpart of a classical result of Stirling on the connections between his numbers $s(k,m)$ and the famous Riemann zeta function

$$\zeta(z) = \sum_{j=0}^{\infty} \frac{1}{(j+1)^z} \qquad (\Re e(z) > 1)$$

for $z \in \mathbb{N}_0$, namely (see, e.g., pp. 166 and 195 of [296]),

$$\zeta(m+1) = \sum_{k=m}^{\infty} \frac{(-1)^{k+m}\ s(k,m)}{k!\ k}. \tag{4.2.30}$$

The counterpart of the multiple sum formula (4.2.29) in the zeta function setting, first established in Butzer *et al.* [121] (see Adamchik [6]), reads

$$\zeta(m+1) = \sum_{j_m=m}^{\infty} \left(\frac{1}{j_m^2}\right) \sum_{j_{m-1}=m-1}^{j_m-1} \left(\frac{1}{j_{m-1}}\right) \cdots \sum_{j_1=1}^{j_2-1} \left(\frac{1}{j_1}\right), \tag{4.2.31}$$

valid for any $m > 0$. The proof follows by inserting (4.2.22) into (4.2.30).

4.3 General Stirling Functions $s(\alpha, \beta)$ with Complex Arguments

The purpose of this section is to generalize the Stirling functions $s(\alpha, k)$ with $\alpha \in \mathbb{C}$, $k \in \mathbb{N}_0$ to functions $s(\alpha, \beta)$ where *both* α and β are complex.

4.3.1 *Definition and main result*

As a natural generalization of the $s(\alpha, k)$, given in (4.2.6), to the $s(\alpha, \beta)$, we introduce the following definition.

Definition 4.1. Let α and β belong to $\mathbb{C}$. Then,

$$s(\alpha,\beta) = \frac{1}{\Gamma(\beta+1)} \lim_{x\to 1} \left({}^{\mathrm{RL}}D_{0+}^{\alpha}[\log(t)]^{\beta}\right)(x) \quad (\Re e(\alpha) \geq 0, \alpha \neq 0), \tag{4.3.1}$$

$$s(0,\beta) = \frac{1}{\Gamma(\beta+1)} \lim_{x\to 1} \left({}^{\mathrm{RL}}D_{0+}^{0}[\log(t)]^{\beta}\right)(x) \quad (\Re e(\beta) > 0), \tag{4.3.2}$$

$$s(\alpha,\beta) = \frac{1}{\Gamma(\beta+1)} \lim_{x\to 1} \left({}^{\mathrm{RL}}I_{0+}^{-\alpha}[\log(t)]^{\beta}\right)(x) \quad (\Re e(\alpha) < 0). \tag{4.3.3}$$

The main result of this section is Theorem 4.6. In order to explain better the motivation of our approach, it is advisable to write this main result at this stage somewhat differently, in two parts.

Theorem 4.5.

(a) *Let* $\alpha, \beta \in \mathbb{C}$ *such that* $\Re e(\alpha) < 0$, $\Re e(\beta) \geq 0$. *Then,*

$$\begin{aligned} s(\alpha,\beta) &= \frac{1}{\Gamma(\beta+1)\Gamma(-\alpha)} \int_{0+}^{1-} (1-t)^{-\alpha-1}[\log(t)]^{\beta}\,dt \\ &= \frac{e^{\beta\pi i}}{\Gamma(-\alpha)} \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha-1}{j} \frac{1}{(j+1)^{\beta+1}}. \end{aligned} \tag{4.3.4}$$

(b) *Let* $\alpha, \beta \in \mathbb{C}$ *such that* $\Re e(\beta) > \Re e(\alpha) \geq 0$, *with* $n = [\Re e(\alpha)] + 1$. *Then,*

$$\begin{aligned} s(\alpha, \beta) &= \frac{1}{\Gamma(\beta+1)} \lim_{x \to 1} \left(\frac{\partial}{\partial x}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha} [\log(t)]^\beta dt \\ &= \frac{e^{\beta\pi i}}{\Gamma(-\alpha)} \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha-1}{j} \frac{1}{(j+1)^{\beta+1}}. \end{aligned} \tag{4.3.5}$$

The series in (4.2.5) coincides with the series in (4.3.5) in case $\beta \equiv k > \Re e(\alpha)$, noting the relations

$$\Gamma(z)\Gamma(1-z) = \frac{\sin(\pi z)}{\pi}, \qquad (z)_k = \frac{\Gamma(z+k)}{z}$$

with $z \in \mathbb{C}$ and $k \in \mathbb{N}$.

In the proof of Theorem 4.6 we will use the expansion

$$(1-z)^{-\mu} = \sum_{j=0}^{\infty} (\mu)_k \frac{z^k}{k!} \quad (|z| < 1,\ \mu \in \mathbb{C}), \tag{4.3.6}$$

where $z \in \mathbb{C}$, $j \in \mathbb{N}_0$ and $(z)_j$ is the Pochhammer symbol (see Section 2.1.1 of [209])

$$(z)_0 = 1, \quad (z)_j = z(z+1)...(z+j-1) \qquad (j \in \mathbb{N}). \tag{4.3.7}$$

We also need the following three preliminary results.

Lemma 4.1. *For* $\beta \in \mathbb{C}$ *and* $m \in \mathbb{N}_0$ *there holds the relation*

$$\lim_{x \to 1} \left(\frac{\partial}{\partial x}\right)^m [\log(x) + \log(t)]^\beta = \sum_{j=0}^{m} \frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)} \, c_{m,j} [\log(t)]^{\beta-j} \tag{4.3.8}$$

where

$$c_{m,m} = 1, \quad c_{m,0} = 0 \quad (m \in \mathbb{N}), \tag{4.3.9}$$

and

$$c_{m,j} = c_{m-1,j-1} - (m-1)c_{m-1,j} \quad (m \in \mathbb{N}; \quad j = 1, \ldots, m-1). \tag{4.3.10}$$

In particular,

$$\begin{gathered} c_{0,0}=1; \quad c_{1,1}=1, \quad c_{1,0}=0; \quad c_{2,2}=1, \quad c_{2,1}=-1, \quad c_{2,0}=0; \\ c_{3,3}=1,\ c_{3,2}=-3, \quad c_{3,1}=2, \quad c_{3,0}=0; \\ c_{m,1}=(-1)^{m-1}(m-1)!, \quad c_{m,m-1}=-\frac{(m-1)m}{2} \quad (m\in\mathbb{N}). \end{gathered} \tag{4.3.11}$$

Proof. For $m=0$, the identity (4.3.8) is clear. If $m=1$ and $m=2$, then

$$\frac{\partial}{\partial x}\left[\log(x)+\log(t)\right]^{\beta}=\beta\left[\log(xt)\right]^{\beta-1}\frac{1}{x}, \tag{4.3.12}$$

$$\left(\frac{\partial}{\partial x}\right)^2\left[\log(xt)\right]^{\beta}=\frac{\beta(\beta-1)}{x^2}\left[\log(xt)\right]^{\beta-2}-\frac{\beta}{x^2}\left[\log(xt)\right]^{\beta-1}. \tag{4.3.13}$$

Taking the limit as $x\to 1$, we have

$$\lim_{x\to 1}\frac{\partial}{\partial x}\left[\log(xt)\right]^{\beta}=\beta[\log(t)]^{\beta-1},$$

$$\lim_{x\to 1}\left(\frac{\partial}{\partial x}\right)^2\left[\log(xt)\right]^{\beta}=\beta(\beta-1)[\log(t)]^{\beta-2}-\beta[\log(t)]^{\beta-1},$$

and hence (4.3.8) follows for $m=1$ and $m=2$, respectively.

We may generalize (4.3.12) and (4.3.13) to

$$x^m\left(\frac{\partial}{\partial x}\right)^m\left[\log(xt)\right]^{\beta}=\sum_{j=1}^{m}\frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)}\,c_{m,j}\left[\log(xt)\right]^{\beta-j}, \tag{4.3.14}$$

where $m\in\mathbb{N}$ and $c_{m,j}$ are defined by (4.3.9)–(4.3.11). Then (4.3.8) will follow for $m\in\mathbb{N}_0$ from (4.3.14) by taking the limit $x\to 1$.

Formula (4.3.14) is proved by induction. Indeed, it has the forms (4.3.12) and (4.3.13) for $m=1$ and $m=2$. Suppose that it is valid for $m \in \mathbb{N}$. Using (4.3.14) we have

$$
\begin{aligned}
&x^{m+1}\left(\frac{\partial}{\partial x}\right)^{m+1}[\log(xt)]^{\beta}\\
&= \frac{\partial}{\partial x}\left\{\sum_{j=1}^{m}\frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)}c_{m,j}[\log(xt)]^{\beta-j}\right\}\\
&= \sum_{j=1}^{m}\frac{\Gamma(\beta+1)}{\Gamma(\beta-j)}c_{m,j}[\log(xt)]^{\beta-j-1} - m\sum_{j=0}^{m}\frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)}c_{m,j}[\log(xt)]^{\beta-j}\\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-m-1)}c_{m,m}[\log(xt)]^{\beta-m-1} + \sum_{j=2}^{m}\frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)}c_{m,j-1}[\log(xt)]^{\beta-j}\\
&\quad - m\sum_{j=2}^{m}\frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)}c_{m,j}[\log(xt)]^{\beta-j} - m\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\, c_{m,1}\, [\log(xt)]^{\beta-1}.
\end{aligned}
$$

By (4.3.9), noting $c_{m+1,m+1} = 1$, so $c_{m,m} = 1 = c_{m+1,m+1}$, $c_{m+1,0} = 0$, and $mc_m = (-1)^{m+1}c_{m+1,1}$. Therefore,

$$
\begin{aligned}
&x^{m+1}\left(\frac{\partial}{\partial x}\right)^{m+1}[\log(x)+\log(t)]^{\beta}\\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-m-1)}c_{m+1,m+1}\,[\log(xt)]^{\beta-m-1}\\
&\quad + \sum_{j=2}^{m}\frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)}[c_{m,j-1} - mc_{m,j}][\log(xt)]^{\beta-j}\\
&\quad + \frac{\Gamma(\beta+1)}{\Gamma(\beta)}\, c_{m+1,1}\,[\log(xt)]^{\beta-1} + c_{m+1,0}[\log(xt)]^{\beta}.
\end{aligned}
$$

This yields (4.3.14) with m being replaced by $m+1$, if we take (4.3.10) into account. □

Lemma 4.2. *Let $\alpha \in \mathbb{C}$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$. Then $(-\alpha-k)_n$ has the representation*

$$(-\alpha-k)_n = \sum_{j=0}^{n}(-1)^j A_{n,j}(k+1)^j, \tag{4.3.15}$$

where the constants $A_{n,j} = A_{n,j}(\alpha)$ have the forms

$$A_{n,0} = (1-\alpha)(2-\alpha)\cdots(n-\alpha) = \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)}; \tag{4.3.16}$$

$$\begin{aligned} A_{n,1} &= (1-\alpha)(2-\alpha)\cdots(n-1-\alpha) + (1-\alpha)(3-\alpha)\cdots(n-\alpha) + \cdots \\ &\quad + (2-\alpha)(3-\alpha)\cdots(n-\alpha) \\ &= \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} \sum_{i=1}^{n} \frac{1}{i-\alpha}; \end{aligned} \tag{4.3.17}$$

$$\begin{aligned} A_{n,2} &= (1-\alpha)(2-\alpha)\cdots(n-2-\alpha) + \cdots + (3-\alpha)(4-\alpha)\cdots(n-\alpha) \\ &= \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} \sum_{\substack{i_1,\, i_2 = 1 \\ (i_1 \neq i_2)}}^{n} \frac{1}{(i_1-\alpha)(i_2-\alpha)}; \ \ldots \end{aligned} \tag{4.3.18}$$

$$\begin{aligned} A_{n,j} &= (1-\alpha)(2-\alpha)\cdots(n-j-\alpha) + \cdots + (j+1-\alpha)\cdots(n-\alpha) \\ &= \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} \sum_{\substack{i_1,\, \cdots,\, i_j = 1 \\ (i_k \neq i_j)}}^{n} \frac{1}{(i_1-\alpha)\cdots(i_j-\alpha)} \\ &= \sum_{\substack{i_{j+1},\, \ldots,\, i_n = 1 \\ (i_k \neq i_j)}}^{n} (i_{j+1}-\alpha)\cdots(i_n-\alpha); \end{aligned} \tag{4.3.19}$$

$$\begin{aligned} A_{n,n-2} &= (1-\alpha)(2-\alpha) + (1-\alpha)(3-\alpha) + \cdots + (1-\alpha)(n-\alpha) + \cdots \\ &\quad + (n-1-\alpha)(n-\alpha) = \sum_{\substack{i_1,\, i_2 = 1 \\ (i_1 \neq i_2)}}^{n} (i_1-\alpha)(i_2-\alpha); \end{aligned} \tag{4.3.20}$$

$$A_{n,n-1} = (1-\alpha) + (2-\alpha) + \cdots + (n-\alpha) = \sum_{i=1}^{n} (i-\alpha); \tag{4.3.21}$$

$$A_{n,n} = 1. \tag{4.3.22}$$

Proof. By (4.3.7) one has

$$\begin{aligned} (-\alpha-k)_n &= (-\alpha-k)(-\alpha-k+1)\cdots(-\alpha-k+n-1) \\ &= [(1-\alpha)-(k+1)][(2-\alpha)-(k+1)]\cdots[(n-\alpha)-(k+1)] \\ &= \sum_{j=0}^{n} (-1)^j A_{n,j} (k+1)^j, \end{aligned}$$

which yields (4.3.15). Since $(-\alpha-k)_n$ is a polynomial of degree n with respect to $(k+1)$, then (4.3.16)–(4.3.22) follow from known results from algebra. □

Lemma 4.3. *Let $n \in \mathbb{N}_0$ and $j = 0, 1, \ldots, n$. There hold the following relations*

$$\sum_{m=j}^{n} \binom{n}{m} \frac{\Gamma(n+1-\alpha)}{\Gamma(m+1-\alpha)} c_{m,j} = A_{n,j}, \tag{4.3.23}$$

where $c_{m,j}$ and $A_{n,j}$ are given by (4.3.9)–(4.3.10) *and* (4.3.16)–(4.3.22), *respectively.*

Proof. For $j = 0$ or $j = n$ the proof of (4.3.23) is simple. If $j = 0$, then in accordance with (4.3.11) and (4.3.9), $c_{0,0} = 1$ and $c_{m,j} = 0$ $(j = 1, \ldots, m)$. Using these relations we have

$$\sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(n+1-\alpha)}{\Gamma(m+1-\alpha)} c_{m,0} = \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} c_{0,0} = \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)},$$

which proves (4.3.23) for $j = 0$, if we take (4.3.16) into account. If $j = n$, then (4.3.23) takes the form $c_{n,n} = A_{n,n}$, which is clear because according to (4.3.9) and (4.3.22) $c_{n,n} = A_{n,n} = 1$. If $j = n-1$, then, since $c_{n-1,n-1} = 1$, relation (4.3.23) takes the form

$$n(n-\alpha) + c_{n,n-1} = A_{n,n-1} \quad (n \in \mathbb{N}). \tag{4.3.24}$$

This can be seen to be valid because, according to (4.3.11), $n(n-\alpha) + c_{n,n-1} = \frac{n(n+1)}{2} - n\alpha$, while by (4.3.21),

$$A_{n,n-1} = (1 + 2 + \cdots + n) - n\alpha = \frac{n(n+1)}{2} - n\alpha.$$

The proofs of (4.3.23) in the cases $j = 1, \ldots, n-2$ can be carried out by direct applications of (4.3.9)–(4.3.11) and (4.3.16)–(4.3.22). They are cumbersome and therefore are omitted. □

Now to our main result for this section, Theorem 4.6, which was phrased in two parts in Theorem 4.5 for better understanding.

Theorem 4.6. *Let $\alpha, \beta \in \mathbb{C}$ such that $\Re e(\alpha) < \Re e(\beta)$. Then*

$$\begin{aligned} s(\alpha, \beta) &= \frac{e^{\beta\pi i}}{\Gamma(-\alpha)} \sum_{j=0}^{\infty} \frac{(\alpha+1)_j}{j!(j+1)^{\beta+1}} \\ &= \frac{e^{\beta\pi i}}{\Gamma(-\alpha)} \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha-1}{j} \frac{1}{(j+1)^{\beta+1}}, \end{aligned} \tag{4.3.25}$$

both series being absolutely convergent, where the power function x^β is defined for $x \in \mathbb{R}$, $x \neq 0$, and $\beta \in \mathbb{C}$ by the usual convention

$$x^\beta = \exp\{\beta[\log|x| + i\arg x]\} \quad (-\pi \leqq \arg x < \pi)$$

Proof. We first establish the result for $\Re e(\alpha) < \Re e(\beta)$, $\Re e(\beta) \geq 0$. By definition (4.3.3), a change of variables $u = e^{-t}$ and an interchange of the order of integration and summation yields that

$$\begin{aligned}
s(\alpha,\beta) &= \frac{1}{\Gamma(\beta+1)\Gamma(-\alpha)} \int_0^1 \frac{[\log(u)]^\beta du}{(1-u)^{\alpha+1}} \\
&= \frac{1}{\Gamma(\beta+1)\Gamma(-\alpha)} \int_0^\infty \left(1-e^{-t}\right)^{-\alpha-1} (-t)^\beta e^{-t} dt \\
&= \frac{e^{\beta\pi i}}{\Gamma(\beta+1)\Gamma(-\alpha)} \lim_{\epsilon\to 0+} \left[\int_\epsilon^\infty \left(1-e^{-t}\right)^{-\alpha-1} t^\beta e^{-t} dt\right] \\
&= \frac{e^{\beta\pi i}}{\Gamma(\beta+1)\Gamma(-\alpha)} \left[\sum_{j=1}^\infty \frac{(\alpha+1)_j}{j!} \lim_{\epsilon\to 0+} \int_\epsilon^\infty e^{-(j+1)t} t^\beta dt\right] \\
&= \frac{e^{\beta\pi i}}{\Gamma(\beta+1)\Gamma(-\alpha)} \left[\sum_{j=1}^\infty \frac{(\alpha+1)_j}{j!(j+1)^{\beta+1}} \lim_{\epsilon\to 0+} \Gamma\left(\beta+1, \epsilon(j+1)\right)\right],
\end{aligned}$$

where $\Gamma(z,w) = \int_w^\infty t^{z-1} e^{-t} dt$ is the incomplete Gamma function (see formula 6.9(21) in [209]). Since $\lim_{w\to 0+} \Gamma(z,w) = \Gamma(z)$, the last relation equals the first sum in (4.3.25).

The second sum in (4.3.25) follows from the first by noting the property

$$\frac{(\alpha+1)_j}{j!} = (-1)^j \binom{-\alpha-1}{j} \quad (\alpha \in \mathbb{C},\ j \in \mathbb{N}_0). \tag{4.3.26}$$

As to the convergence of the two series in (4.3.25), consider the general term d_j of the series

$$\sum_{j=0}^\infty d_j; \quad d_j = \frac{e^{\beta\pi i}}{\Gamma(-\alpha)} (-1)^j \binom{-\alpha-1}{j} \frac{1}{(j+1)^{\beta+1}}. \tag{4.3.27}$$

In view of the estimate for binomial coefficients, namely

$$\left|\binom{a}{b}\right| \le \frac{c}{b^{1+\mathfrak{Re}(a)}} \quad (a, b \in \mathbb{C},\ a \neq -1, -2, ...), \tag{4.3.28}$$

for a certain constant $c > 0$, one has for d_j the estimate

$$|d_j| \le \frac{c}{j^{\mathfrak{Re}(\beta)+1} j^{\mathfrak{Re}(-\alpha-1)+1}} = \frac{c}{j^{\mathfrak{Re}(\beta)+1-\mathfrak{Re}(\alpha)}}. \tag{4.3.29}$$

This estimate establishes the assertions of Theorem 4.6 only for $\mathfrak{Re}(\alpha) < 0$ and $\mathfrak{Re}(\beta) \ge 0$ since $\mathfrak{Re}(\beta) + 1 - \mathfrak{Re}(\alpha) > 1$.

If $\mathfrak{Re}(\alpha) \ge 0,\ \alpha \neq 0$, and $\mathfrak{Re}(\beta) > \mathfrak{Re}(\alpha)$, then instead of applying the definition (4.3.3) we work with (4.3.1) and (1.3.1). Then $s(\alpha, \beta)$ takes on the form for $n = [\mathfrak{Re}(\alpha)] + 1$,

$$\begin{aligned} s(\alpha, \beta) &= \frac{1}{\Gamma(\beta+1)} \lim_{x\to 1} \left(\frac{\partial}{\partial x}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-u)^{n-\alpha-1} [\log(u)]^\beta du \\ &= \frac{1}{\Gamma(\beta+1)\Gamma(n-\alpha)} \lim_{x\to 1} \left(\frac{\partial}{\partial x}\right)^n \int_0^1 (1-t)^{n-\alpha} \left[x^{n-\alpha}[\log xt]^\beta\right] dt \\ &= \frac{1}{\Gamma(\beta+1)\Gamma(n-\alpha)} \int_0^1 (1-t)^{n-\alpha} \lim_{x\to 1} \left(\frac{\partial}{\partial x}\right)^n \left[x^{n-\alpha}[\log xt]^\beta\right] dt, \end{aligned}$$

where the change of variables $u = xt$ was made. Applying now the Leibniz rule for the derivative of a product, taking into account the known property

$$\left({}^{\mathrm{RL}}D_{0+}^{\alpha} t^{\gamma-1}\right)(x) = \frac{\Gamma(\gamma)\ x^{\gamma-\alpha-1}}{\Gamma(\gamma-\alpha)} \quad (\mathfrak{Re}(\alpha) > 0,\ \ \mathfrak{Re}(\gamma) > 0), \tag{4.3.30}$$

with $\alpha = n - m$ and $\gamma = n - \alpha + 1$, as well as Lemma 4.1, we find

$$\begin{aligned} &s(\alpha, \beta) \\ &\quad = \frac{1}{\Gamma(\beta+1)\Gamma(n-\alpha)} \\ &\qquad \times \sum_{m=0}^{n} \binom{n}{m} \int_0^1 (1-t)^{n-\alpha} \lim_{x\to 1} \left\{ \left(\frac{d}{dx}\right)^{n-m} \left[x^{n-\alpha}\right] \right. \\ &\qquad\qquad \left. \times \left(\frac{\partial}{\partial x}\right)^m \left[\log(x) + \log(t)]^\beta\right] \right\} dt \\ &\quad = \frac{1}{\Gamma(\beta+1)\Gamma(n-\alpha)} \\ &\qquad \times \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(n+1-\alpha)}{\Gamma(m+1-\alpha)} \int_0^1 (1-t)^{n-\alpha} \lim_{x\to 1} \left(\frac{\partial}{\partial x}\right)^m \left[\log xt]^\beta\right] dt \end{aligned}$$

$$= \frac{1}{\Gamma(\beta+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(n+1-\alpha)}{\Gamma(m+1-\alpha)}$$

$$\times \sum_{j=0}^{m} \frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)}\, c_{m,j}\, \frac{1}{\Gamma(n-\alpha)} \int_0^1 (1-t)^{n-\alpha}[\log(t)]^{\beta-j} dt. \quad (4.3.31)$$

Hence there follows

$$s(\alpha,\beta) = \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(n+1-\alpha)}{\Gamma(m+1-\alpha)}$$

$$\times \sum_{j=0}^{m} \frac{c_{m,j}}{\Gamma(\beta-j+1)} \lim_{x\to 1} \left({}^{\mathrm{RL}}I_{0+}^{n-\alpha}[\log(t)]^{\beta-j}\right)(x). \quad (4.3.32)$$

To evaluate the limit term for $x \to 1$, we apply the first part of the proof. Indeed, by (4.3.3) and (4.3.25) for $-\alpha$ and β replaced by $n-\alpha$ and $\beta - j$, respectively, we have since $\mathfrak{Re}(\alpha) < n$,

$$\lim_{x\to 1} \left({}^{\mathrm{RL}}I_{0+}^{n-\alpha}[\log(t)]^{\beta-j}\right)(x) = \Gamma(\beta-j+1)s(\alpha-n,\beta-j)$$

$$= \frac{e^{(\beta-j)\pi i}\Gamma(\beta-j+1)}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{(\alpha-n+1)_k}{k!(k+1)^{\beta+1-j}}.$$

Thus (4.3.32) takes the form

$$s(\alpha,\beta) = \frac{e^{\beta\pi i}}{\Gamma(n-\alpha)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(n+1-\alpha)}{\Gamma(m+1-\alpha)} \sum_{j=0}^{m} (-1)^j c_{m,j}$$

$$\times \sum_{k=0}^{\infty} \frac{(\alpha-n+1)_k}{k!(k+1)^{\beta+1-j}}$$

$$= \frac{e^{\beta\pi i}}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{(\alpha-n+1)_k}{k!(k+1)^{\beta+1}} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(n+1-\alpha)}{\Gamma(m+1-\alpha)}$$

$$\times \sum_{j=0}^{m} (-1)^j c_{m,j} (k+1)^j$$

$$= \frac{e^{\beta\pi i}}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} \frac{(\alpha-n+1)_k}{k!(k+1)^{\beta+1}} \sum_{j=0}^{n} (-k-1)^j$$

$$\times \sum_{m=j}^{n} \binom{n}{m} \frac{\Gamma(n+1-\alpha)}{\Gamma(m+1-\alpha)} c_{m,j}. \quad (4.3.33)$$

According to Lemma 4.2 and Lemma 4.3,

$$\sum_{j=0}^{n}(-1)^j(k+1)^j\sum_{m=j}^{n}\binom{n}{m}\frac{\Gamma(n+1-\alpha)}{\Gamma(m+1-\alpha)}\,c_{m,j}$$
$$=\sum_{j=0}^{n}(-1)^jA_{n,j}(k+1)^j=(-\alpha-k)_n, \tag{4.3.34}$$

and hence from (4.3.33) we obtain

$$s(\alpha,\beta)=e^{\beta\pi i}\sum_{k=0}^{\infty}\frac{(\alpha-n+1)_k(-\alpha-k)_n}{\Gamma(n-\alpha)}\,\frac{1}{k!(k+1)^{\beta+1}}, \tag{4.3.35}$$

but $(\alpha-n+1)_k(-\alpha-k)_n/\Gamma(n-\alpha)=(\alpha+1)_k/\Gamma(-\alpha)$, and thus (4.3.35) yields the first series in (4.3.25).

The second series in (4.3.25), clearly following from the first one, has the form of (4.3.27) with the d_j term again having the estimate (4.3.29); thus it is convergent for $\Re e(\alpha)<\Re e(\beta)$.

The relations in (4.3.25) remain valid also for $\alpha=0$, $\Re e(\beta)>0$. Indeed, according to definition (4.3.2) for $\alpha=0$, $s(0,\beta)=\frac{1}{\Gamma(\beta+1)}\ \lim_{x\to 1}[\log x]^\beta=0$, and

$$\lim_{\alpha\to 0+}\left[\frac{e^{\beta\pi i}}{\Gamma(-\alpha)}\sum_{j=0}^{\infty}\frac{(\alpha+1)_j}{j!(j+1)^{\beta+1}}\right]=0.$$

This completes the proof of Theorem 4.6. □

It follows from Theorem 4.6 that if $\alpha,\beta\in\mathbb{C}$ such that $\Re e(\alpha)<\Re e(\beta)$, then the Stirling functions of the first kind $s(\alpha,\beta)$, defined by (4.3.1)–(4.3.3), have the same representations, namely

$$s(\alpha,\beta)=\frac{e^{\beta\pi i}}{\Gamma(-\alpha)}\sum_{j=0}^{\infty}\frac{(\alpha+1)_j}{j!(j+1)^{\beta+1}}$$
$$=\frac{e^{\beta\pi i}}{\Gamma(-\alpha)}\sum_{j=0}^{\infty}(-1)^j\binom{-\alpha-1}{j}\frac{1}{(j+1)^{\beta+1}}$$

From Theorem 4.6 we also deduce the following result.

Corollary 4.2. *If* $\alpha \in \mathbb{C}$ *and* $k \in \mathbb{Z}$ *with* $\Re e(\alpha) < k$, *then*

$$s(\alpha, k) = \frac{(-1)^k}{\Gamma(-\alpha)} \sum_{j=0}^{\infty} \frac{(\alpha+1)_j}{j!(j+1)^{k+1}}$$
$$= \frac{(-1)^k}{\Gamma(-\alpha)} \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha-1}{j} \frac{1}{(j+1)^{k+1}}. \tag{4.3.36}$$

Corollary 4.3.

(a) *If* $n \in \mathbb{N}_0$ *and* $\beta \in \mathbb{C}$ *with* $\Re e(\beta) > n$, *then*

$$s(n, \beta) = 0, \quad s(0, \beta) = 0. \tag{4.3.37}$$

(b) *If* $n \in \mathbb{N}$ *and* $\beta \in \mathbb{C}$ *such that* $\Re e(\beta) > -n$, *then*

$$s(-n, \beta) = \frac{e^{\beta\pi i}}{(n-1)!} \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{1}{(j+1)^{\beta+1}}$$
$$= \frac{e^{\beta\pi i}}{n!} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{1}{j^\beta}. \tag{4.3.38}$$

In particular, for $\beta \in \mathbb{C}$,

$$s(-1, \beta) = e^{\beta\pi i}, \quad s(-2, \beta) = e^{\beta\pi i}\left[1 - \frac{1}{2^{\beta+1}}\right].$$

Proof. As far as (4.3.38) is concerned, its left hand side exists under the given conditions, according to (4.3.3) with $\alpha = -n$ can be expressed by

$$s(-n, \beta) = \frac{1}{\Gamma(\beta+1)(n-1)!} \int_0^1 (1-t)^{n-1} [\log t]^\beta dt,$$

which is a convergent integral for $\Re e(\beta) > -n$. The right hand side turns out to be a finite sum since for $\alpha = -n$, $(-n+1)_j = 0$ for $j = n, n+1, \ldots$, so that it exists for any $n \in \mathbb{N}$ and $\beta \in \mathbb{C}$. To establish the right hand side of (4.3.38) one replaces $j+1$ by k, and observes that

$$n\binom{n-1}{k-1}\frac{1}{k} = \binom{n}{k} \quad (n \in \mathbb{N},\ 1 \leqq k \leqq n-1). \tag{4.3.39}$$

□

4.3.2 *Differentiability of the $s(\alpha,\beta)$; The zeta function encore*

In Section 4.2.2 we indicated that the Stirling functions $s(\alpha, m)$ are connected to the zeta function $\zeta(m+1)$. Such a connection is indeed true also for the most general $s(\alpha,\beta)$ with $\alpha,\beta \in \mathbb{C}$.

For this purpose we first need a result on the continuity and differentiability of the $s(\alpha,\beta)$ with respect to α; the corresponding result for β is also given.

Theorem 4.7. *Let $\alpha,\beta \in \mathbb{C}$ be complex numbers such that $\Re e(\alpha) < \Re e(\beta)$. There hold the following assertions:*

(a) *$s(\alpha,\beta)$ as a function of α is continuously differentiable for $\alpha \in \mathbb{C}, \alpha \neq 0$, and*

$$\frac{\partial}{\partial\alpha}s(\alpha,\beta) = [\psi(-\alpha) - \psi(\alpha+1)]s(\alpha,\beta) + \frac{e^{\beta\pi i}}{\Gamma(-\alpha)}\sum_{j=0}^{\infty}\frac{(\alpha+1)_j}{j!(j+1)^{\beta+1}}\psi(\alpha+1+j). \quad (4.3.40)$$

(b) *$s(\alpha,\beta)$ as a function of β is continuously differentiable for $\beta \in \mathbb{C}$, and for $m \in \mathbb{N}$,*

$$\left(\frac{\partial}{\partial\beta}\right)^m s(\alpha,\beta) = \frac{e^{\beta\pi i}}{\Gamma(-\alpha)}\sum_{j=0}^{\infty}\frac{(\alpha+1)_j}{j!(j+1)^{\beta+1}} \times \sum_{k=0}^{m}(-1)^k\binom{m}{k}(i\pi)^{m-k}[\log(j+1)]^k. \quad (4.3.41)$$

Proof. The continuity of $s(\alpha,\beta)$ as functions of α and β follow from the first formula of (4.3.25). The relations (4.3.40) and (4.3.41) are deduced by differentiation with respect to α and β of $s(\alpha,\beta)$, respectively. For the former, one makes use of the fact that

$$\frac{d}{dx}\frac{1}{\Gamma(-\alpha)} = \frac{\psi(-\alpha)}{\Gamma(-\alpha)}, \quad \frac{d}{d\alpha}(\alpha+1)_j = (\alpha+1)_j[\psi(\alpha+1+j) - \psi(\alpha+1)]$$

and for the latter, noting Leibniz' rule,

$$\left(\frac{\partial}{\partial\beta}\right)^m\left[e^{\beta\pi i}\frac{1}{(j+1)^{\beta+1}}\right] = \sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(i\pi)^{m-k}e^{\beta\pi i}}{(j+1)^{\beta+1}}[\log(j+1)]^k.$$

The convergence of the series on the right sides of (4.3.40) and (4.3.41) follows by applying the relations (4.3.26), (4.3.28) and the asymptotic formulae for the Gamma and Psi-functions (see formulas 1.18(4) and 1.18(7) in [209]),

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}[1 + O(z^{-1})], \quad \psi(z) = \log(z) + O(z^{-1}) \quad (z \to \infty). \tag{4.3.42}$$

□

We can now formulate the indicated connection between the Stirling functions and Riemann's zeta function.

Theorem 4.8. *Let $\alpha, \beta \in \mathbb{C}$ such that $\Re e(\beta) > 0$ and $\Re e(\alpha) < \Re e(\beta)$. Then*

$$\lim_{\alpha \to 0} \Gamma(-\alpha)s(\alpha, \beta) = e^{\beta\pi i}\zeta(\beta + 1), \quad \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} s(\alpha, \beta) = -e^{\beta\pi i}\zeta(\beta + 1).$$

Proof. In view of Theorem 4.6 we have, for $\Re e(\beta) > 0$ with $\Re e(\alpha) < \Re e(\beta)$,

$$\lim_{\alpha \to 0} \Gamma(-\alpha)s(\alpha, \beta) = \lim_{\alpha \to 0} e^{\beta\pi i} \sum_{j=0}^{\infty} \frac{(\alpha+1)_j}{j!(j+1)^{\beta+1}}. \tag{4.3.43}$$

Since $(1)_j = j!$ for $j \in \mathbb{N}_0$, the right-hand side of (4.3.43) is $e^{\beta\pi i}\zeta(\beta + 1)$, establishing the first formula of Theorem 4.8.

Since

$$\psi(-\alpha) - \psi(\alpha + 1) = \pi \cot(\alpha\pi), \quad \Gamma(-\alpha) = -\frac{1}{\alpha}\left[1 + O\left(\frac{1}{z}\right)\right] \quad (\alpha \to 0), \tag{4.3.44}$$

and $\pi \cot(\pi\alpha) \sim 1/\alpha$ $(\alpha \to 0)$, we find

$$\lim_{\alpha \to 0} \frac{\psi(-\alpha) - \psi(\alpha+1)}{\Gamma(-\alpha)} = -1, \tag{4.3.45}$$

cf., e.g., formula 1.7(11) of [209] and formula 6.1.3 of [5]. Finally, taking the limit as $\alpha \to 0$ in (4.3.40) and using (4.3.44) and (4.3.45), we deduce the second formula of the theorem. □

4.3.3 *Recurrence relations for* $s(\alpha, \beta)$

Theorem 4.9. *Let $\alpha, \beta \in \mathbb{C}$ such that $\Re e(\alpha) < \Re e(\beta) - 1$. Then the function $s(\alpha, \beta)$ satisfies the recurrence formula*

$$s(\alpha + 1, \beta) = s(\alpha, \beta - 1) - \alpha\ s(\alpha, \beta), \tag{4.3.46}$$

and for $n \in \mathbb{N}_0$ in addition

$$s(\alpha + 1, \beta) = \sum_{j=0}^{n} (-\alpha)_j s(\alpha - j, \beta - 1) + (-\alpha)_{n+1} s(\alpha - n, \beta). \tag{4.3.47}$$

Proof. If $\Re e(\alpha) < \Re e(\beta) - 1$, then according to Theorem 4.6 the Stirling functions $s(\alpha + 1, \beta)$, $s(\alpha, \beta - 1)$ and $s(\alpha, \beta)$ are well-defined. For $\alpha = 0$ relation (4.3.46) takes the form $s(1, \beta) = s(0, \beta - 1)$ which is obvious since $s(1, \beta) = s(0, \beta - 1) = 0$ by (4.3.37).

Let $\alpha \neq 0$. In view of Theorem 4.6 and since both series on the second side of (4.3.46) are absolutely convergent, then

$$\begin{aligned}
s(\alpha, \beta - 1) - \alpha s(\alpha, \beta) &= \frac{e^{(\beta-1)\pi i}}{\Gamma(-\alpha)} \sum_{j=0}^{\infty} \frac{(\alpha + 1)_j}{j!(j + 1)^{\beta}} \\
&\quad - \frac{\alpha e^{\beta \pi i}}{\Gamma(-\alpha)} \sum_{j=0}^{\infty} \frac{(\alpha + 1)_j}{j!(j + 1)^{\beta+1}} \\
&= \frac{e^{\beta \pi i}}{\Gamma(-\alpha)} \sum_{j=0}^{\infty} \frac{(\alpha + 1)_j}{j!(j + 1)^{\beta+1}} [-(\alpha + j + 1)] \\
&= \frac{e^{\beta \pi i}(-\alpha - 1)}{\Gamma(-\alpha)} \sum_{j=0}^{\infty} \frac{(\alpha + 2)_j}{j!(j + 1)^{\beta+1}},
\end{aligned}$$

where the relation $(\alpha + 1)_j(\alpha + 1 + j) = (\alpha + 1)(\alpha + 2)_j,\ j \in \mathbb{N}_0$, was used. This will yield (4.3.46).

Relation (4.3.47) will be established by induction. Supposing it is valid for $n \in \mathbb{N}$, then using (4.3.46),

$$\begin{aligned}
&\sum_{j=0}^{n+1} (-\alpha)_j s(\alpha - j, \beta - 1) \\
&\qquad = s(\alpha + 1, \beta) + (-\alpha)_{n+1}[s(\alpha - n - 1, \beta - 1) - s(\alpha - n, \beta)] \\
&\qquad = s(\alpha + 1, \beta) - (-\alpha)_{n+1}(\alpha - n - 1) s(\alpha - n - 1, \beta),
\end{aligned}$$

that is

$$\sum_{j=0}^{n+1}(-\alpha)_j s(\alpha-j,\beta-1) = s(\alpha+1,\beta)+(-\alpha)_{n+2}(\alpha-n-1)s(\alpha-n-1,\beta),$$

noting that $-(-\alpha)_{n+1}(\alpha-n-1)=(-\alpha)_{n+2}$. This yields (4.3.47), n being replaced by $n+1$. Thus the proof is complete. $\square$

Theorem 4.10. *For $m\in\mathbb{N}$ and $\beta\in\mathbb{C}$ the Stirling functions $s(-m,\beta)$ satisfy the recurrence*

$$s(-m,\beta)=s(-m-1,\beta-1)+(m+1)s(-m-1,\beta), \tag{4.3.48}$$

and for $m\in\mathbb{N}_0$ in addition

$$\begin{aligned} s(-m,\beta) = \sum_{j=0}^{n}&(m+1)_j\ s(-m-1-j,\beta-1)\\ &+(m+1)_{n+1}s(-m-1-n,\beta). \end{aligned}\tag{4.3.49}$$

Proof. Taking into account (4.3.46), we have

$$\begin{aligned} &s(-m-1,\beta-1)+(m+1)s(-m-1,\beta)\\ &=\frac{e^{(\beta-1)\pi i}}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{1}{(j+1)^\beta}+\frac{e^{\beta\pi i}}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{1}{(j+1)^{\beta+1}}\\ &=\frac{e^{\beta\pi i}}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\left[\frac{m+1}{(j+1)^{\beta+1}}-\frac{1}{(j+1)^\beta}\right]\\ &=\frac{e^{\beta\pi i}}{m!}\sum_{j=0}^{m-1}(-1)^j\binom{m}{j}\frac{(m-j)}{(j+1)^{\beta+1}}, \end{aligned}$$

which yields (4.3.48), since

$$\frac{1}{m}\binom{m}{j}(m-j)=\binom{m-1}{j}\qquad (m\in\mathbb{N};\ 1\le j\le m-1),$$

and

$$s(-m-1,\beta-1)+(m+1)s(-m-1,\beta)$$

$$=\frac{e^{\beta\pi i}}{(m-1)!}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\frac{1}{(j+1)^{\beta+1}}.$$

Relation (4.3.49) follows from (4.3.48) by induction. $\square$

Let m and k both belong to $\mathbb{N}$ such that $m > k$. In particular, it is easy to prove the following formulas

$$s(-m,-k) = 0 \quad (m,k \in \mathbb{N}; \quad m > k) \tag{4.3.50}$$

and

$$s(\alpha+1,k) = s(\alpha,k-1) - \alpha s(\alpha,k) \quad (\alpha \in \mathbb{C}, \quad \Re e(\alpha) < k-1). \tag{4.3.51}$$

4.4 Stirling Functions of the Second Kind $S(\alpha, k)$

As we mentioned above, in the introduction of this chapter, Butzer *et al.* [114] have introduced the Stirling function $S(\alpha, k)$ defined by

$$S(\alpha,k) = \frac{1}{k!} \lim_{x\to 0} \Delta^k (x^\alpha) \quad (x \in \mathbb{R}; \ \alpha \geq 0, \ k \in \mathbb{N}_0), \tag{4.4.1}$$

where the limit is being taken in the sense

$$\lim_{x\to 0} \Delta^k (x^\alpha) = \lim_{\epsilon\to 0} \lim_{x\to 0} \Delta^k \left((x+\epsilon)^\alpha\right). \tag{4.4.2}$$

The aim of this section is to extend this definition of $S(\alpha, k)$ to complex values of the first parameter, into the fractional calculus framework, by employing differences of fractional order and fractional differentiation operators. Moreover, we will obtain several properties for such Stirling functions.

4.4.1 *Stirling functions $S(\alpha, k)$, $\alpha \geq 0$, and their representations by Liouville and Marchaud fractional derivatives*

In this section we consider Stirling functions of the second kind $S(\alpha, k)$ with nonnegative $\alpha \geq 0$.

Theorem 4.11. *Let $\alpha \geq 0$ and $k \in \mathbb{N}_0$. Then the Stirling function of the second kind $S(\alpha, k)$ has the following explicit representation*

$$S(\alpha,k) = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^\alpha \quad (\alpha > 0; \ k \in \mathbb{N}). \tag{4.4.3}$$

In particular, we have

$$S(\alpha,0) = 0 \quad (\alpha > 0); \quad S(0,k) = 0 \quad (k \in \mathbb{N}); \quad S(0,0) = 1. \tag{4.4.4}$$

Proof. Using (4.4.1), (4.4.2) and applying (4.1.9) with $f(x) = x^\alpha$, we have

$$S(\alpha,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\lim_{\epsilon\to 0}\lim_{x\to 0}(x+\epsilon+j)^\alpha. \tag{4.4.5}$$

If $\alpha > 0$, then

$$S(\alpha,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^\alpha \tag{4.4.6}$$

which yields (4.4.3) and the first relation in (4.4.4), when $k \in \mathbb{N}$ and $k = 0$, respectively. If $\alpha = 0$, then (4.4.5) takes on the form

$$S(0,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j},$$

since $S(0,k) = (1-1)^k/k! = 0$ for $k > 0$. Also, it is easy to see that $S(0,0) = 1$. □

Now we shall give representations of the Stirling functions $S(\alpha,k)$ with positive $\alpha > 0$ in terms of the Liouville fractional derivatives (1.3.63) and (1.3.64).

Theorem 4.12. *Let $\alpha > 0$, $k \in \mathbb{N}_0$ and let ${}^{\mathtt{L}}D_+^\alpha$ and ${}^{\mathtt{L}}D_-^\alpha$ be the operators of Liouville fractional differentiation* (1.3.63) *and* (1.3.64)*, respectively. Then the Stirling functions of the second kind $S(\alpha,k)$ have the representations*

$$S(\alpha,k) = \frac{(-1)^k}{k!}\lim_{x\to 0}\left({}^{\mathtt{L}}D_+^\alpha\left((1-e^t)^k - 1\right)\right)(x) \tag{4.4.7}$$

and

$$S(\alpha,k) = \frac{(-1)^k}{k!}\lim_{x\to 0}\left({}^{\mathtt{L}}D_-^\alpha\left((1-e^{-t})^k - 1\right)\right)(x). \tag{4.4.8}$$

Proof. Using (1.3.63) and (1.3.71) and taking into account the relation

$$(1-e^{-u})^k = \sum_{j=0}^{k}(-1)^j\binom{k}{j}e^{-ju}, \tag{4.4.9}$$

we have

$$\frac{(-1)^k}{k!}\left({}^{\mathtt{L}}D_+^{\alpha}\left((1-e^t)^k-1\right)\right)(x) = \frac{(-1)^k}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\left({}^{\mathtt{L}}D_+^{\alpha}e^{jt}\right)(x)$$
$$= \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^{\alpha}e^{jx}. \quad (4.4.10)$$

Taking the limit $x \to 0$, we obtain

$$\frac{(-1)^k}{k!}\lim_{x\to 0}\left({}^{\mathtt{L}}D_+^{\alpha}\left((1-e^t)^k-1\right)\right)(x) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^{\alpha} = S(\alpha,k).$$

Relation (4.4.8) follows similarly by using Eqs. (1.3.64), (4.4.9) and (1.3.72). □

Stirling functions $S(\alpha,k)$ with positive $\alpha > 0$ can also be represented in terms of the Marchaud fractional derivatives (1.3.86) and (1.3.87).

Theorem 4.13. *Let $\alpha > 0$, $k \in \mathbb{N}_0$ and let ${}^{\mathtt{M}}D_+^{\alpha}$ and ${}^{\mathtt{M}}D_-^{\alpha}$ be the operators of Marchaud fractional differentiation* (1.3.86) *and* (1.3.87)*, respectively. Then the Stirling functions of the second kind $S(\alpha,k)$ have the following representations*

$$S(\alpha,k) = \frac{1}{k!}\lim_{x\to 0}\left({}^{\mathtt{M}}D_+^{\alpha}\left((e^t-1)^k\right)\right)(x) \quad (4.4.11)$$

and

$$S(\alpha,k) = \frac{1}{k!}\lim_{x\to 0}\left({}^{\mathtt{M}}D_-^{\alpha}\left((e^{-t}-1)^k\right)\right)(x). \quad (4.4.12)$$

Proof. Using an analogue of Eq. (4.4.9), *viz.*

$$(e^t-1)^k = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{jt}, \quad (4.4.13)$$

and applying term-by-term differentiation, we have

$$\left({}^{\mathtt{M}}D_+^{\alpha}\left((e^t-1)^k\right)\right)(x) = \left({}^{\mathtt{M}}D_+^{\alpha}1\right)(x) + \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\left({}^{\mathtt{M}}D_+^{\alpha}e^{jt}\right)(x).$$

Using the first relation in (1.3.91) and (1.3.92) (with $b = j \in \mathbb{N}$), we deduce

$$\frac{1}{k!}\left({}^{\mathtt{M}}D_{+}^{\alpha}\left((e^{t}-1)^{k}\right)\right)(x) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^{\alpha}e^{jx}. \tag{4.4.14}$$

Taking the limit $x \to 0$ in (4.4.14), we obtain (4.4.11). The representation (4.4.12) is proved similarly by applying (4.4.13), the second relation in (1.3.91), and (1.3.93). □

In the case $k > \alpha > 0$ there holds the following integral representation for $S(\alpha, k)$

Theorem 4.14. *Let $\alpha > 0$ and $k \in \mathbb{N}$ be such that $k > \alpha$. Then the Stirling functions of the second kind $S(\alpha, k)$ have the integral representation*

$$S(\alpha, k) = \frac{(-1)^{k}}{k!\Gamma(-\alpha)}\int_{0}^{\infty}(1-e^{-t})^{k}\frac{dt}{t^{1+\alpha}}. \tag{4.4.15}$$

Proof. If $k > \alpha > 0$, the constant $\kappa(\alpha, k)$ given by (1.3.88) has the form

$$\kappa(\alpha, k) = -\Gamma(-\alpha)A_{k}(\alpha), \ A_{k}(\alpha) = \sum_{j=0}^{k}(-1)^{j-1}\binom{k}{j}j^{\alpha},$$

see Eq. (5.81) of [501]. In accordance with (4.4.3),

$$A_{k}(\alpha) = (-1)^{k+1}k!S(\alpha, k) \tag{4.4.16}$$

and hence

$$S(\alpha, k) = \frac{(-1)^{k}}{k!\Gamma(-\alpha)}\kappa(\alpha, k) \quad (k \in \mathbb{N}, \ k > \alpha > 0), \tag{4.4.17}$$

and then the theorem is proved. □

Corollary 4.4. *If $n \in \mathbb{N}$ and $k \in \mathbb{N}$ is such that $n < k$, then*

$$S(n, k) = 0 \quad (1 \leq n \leq k-1). \tag{4.4.18}$$

Proof. In Theorem 4.14 let $\alpha = n \in \mathbb{N}$ and $k \in \mathbb{N}$ be such that $n < k$. Then in accordance with Eq. (5.74) of [501],

$$A_{k}(n) = 0 \ (n = 1, 2, \cdots, k-1), \tag{4.4.19}$$

and (4.4.18) follows from (4.4.16). □

4.4.2 *Stirling functions $S(\alpha, k)$, $\alpha < 0$, and their representations by Liouville fractional integrals*

In this section we consider Stirling functions of the second kind $S(\alpha, k)$ with negative $\alpha < 0$ and $k \in \mathbb{N}$. For this purpose we use an approach similar to (4.4.1) in the form

$$S(\alpha, k) = \frac{1}{k!} \lim_{x \to 0} \Delta_*^k (x^\alpha) \quad (x \in \mathbb{R};\ \alpha < 0,\ k \in \mathbb{N}), \tag{4.4.20}$$

with the *"cut" finite difference*

$$\Delta_*^k f(x) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} f(x+j) \quad (x \in \mathbb{R};\ k \in \mathbb{N}), \tag{4.4.21}$$

where the limit is taken in the sense of (4.4.2),

$$\lim_{x \to 0} \Delta_*^k (x^\alpha) = \lim_{\epsilon \to 0} \lim_{x \to 0} \Delta_*^k ((x+\epsilon)^\alpha) \quad (\alpha < 0). \tag{4.4.22}$$

Theorem 4.15. *Let $\alpha < 0$ and $k \in \mathbb{N}$. The Stirling functions of the second kind $S(\alpha, k)$ have the explicit representation*

$$S(\alpha, k) = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^\alpha. \tag{4.4.23}$$

Proof. Using (4.4.22) and applying (4.4.21) with $f(x) = x^\alpha$, we have

$$\begin{aligned} S(\alpha, k) &= \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \lim_{\epsilon \to 0} \lim_{x \to 0} (x + \epsilon + j)^\alpha \\ &= \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^\alpha, \end{aligned} \tag{4.4.24}$$

which yields (4.4.23). □

Now we give the representation of $S(\alpha, k)$ $(\alpha < 0)$ in terms of the Liouville fractional integrals (1.3.61) and (1.3.62).

Theorem 4.16. *Let $\alpha < 0$, $k \in \mathbb{N}$ and let ${}^{\mathtt{L}}I_+^{-\alpha}$ and ${}^{\mathtt{L}}I_-^{-\alpha}$ be the operators of Liouville fractional integration* (1.3.61) *and* (1.3.62), *respectively. Then* *the Stirling functions of the second kind $S(\alpha, k)$ have the integral representations*

$$S(\alpha, k) = \frac{(-1)^k}{k!} \lim_{x \to 0} \left({}^{\mathtt{L}}I_{+}^{-\alpha} \left((1 - e^t)^k - 1 \right) \right)(x) \tag{4.4.25}$$

and

$$S(\alpha, k) = \frac{(-1)^k}{k!} \lim_{x \to 0} \left({}^{\mathtt{L}}I_{-}^{-\alpha} \left((1 - e^{-t})^k - 1 \right) \right)(x). \tag{4.4.26}$$

Proof. The theorem is proved in a similar way as Theorem 4.12, taking account the known relations (4.4.9) and (1.3.70). □

We point out that if $\alpha > 0$ and $k \in \mathbb{N}$, then the "cut" difference $\Delta_*^k(x^\alpha)$ coincides with the usual difference $\Delta^k(x^\alpha)$. According to this fact and (4.1.12) and (4.4.20), we can give the following unified representation to Stirling functions of the second kind $S(\alpha, k)$, for $x \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and $k \in \mathbb{N}$, by

$$S(\alpha, k) = \frac{1}{k!} \lim_{x \to 0} \Delta_*^k(x^\alpha) \tag{4.4.27}$$

or

$$S(\alpha, k) = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^\alpha. \tag{4.4.28}$$

Finally, we must remark that in the above development for the Stirling numbers $S(\alpha, k)$, only the case $\alpha < 0$ and $k = 0$ has not yet been defined.

4.4.3 *Stirling functions $S(\alpha, k)$, $\alpha \in \mathbb{C}$, and their representations*

In this section we extend the results established in Sections 4.4.1 and 4.4.2 to Stirling functions of the second kind $S(\alpha, k)$ with $\alpha \in \mathbb{C}$, except when $\Re e(\alpha) < 0$ and $k = 0$.

First we use the extensions of the approaches (4.4.1) and (4.4.20). Thus we define the Stirling functions $S(\alpha, k)$ by

$$S(\alpha, k) = \frac{1}{k!} \lim_{x \to 0} \Delta^k(x^\alpha) \quad (x \in \mathbb{R};\ \Re e(\alpha) > 0,\ k \in \mathbb{N}_0), \tag{4.4.29}$$

$$S(\alpha,k)=\frac{1}{k!}\lim_{x\to 0}\Delta_*^k\,(x^\alpha)\quad (x\in\mathbb{R};\ \Re e(\alpha)\leq 0;\quad k\in\mathbb{N}),\tag{4.4.30}$$

and $S(0,0)=1$. Here the limits are to be understood in the sense of (4.4.2) and (4.4.22).

Explicit representations for such Stirling functions are given by the following statement.

Theorem 4.17. *Let $\alpha\in\mathbb{C}$ and $k\in\mathbb{N}_0$. The Stirling function of the second kind $S(\alpha,k)$ has the explicit representation*

$$S(\alpha,k)=\frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^\alpha\quad (\alpha\neq 0;\quad k\in\mathbb{N}),\tag{4.4.31}$$

$$S(\alpha,0)=0\ (\Re e(\alpha)>0);\quad S(0,0)=1;\quad S(0,k)=0\ (k\in\mathbb{N}).\tag{4.4.32}$$

Proof. The proof of this Theorem is the same as that of Theorems 4.11 and 4.15. □

Corollary 4.5. *If $k=1$, then*

$$S(\alpha,1)=1\quad(\alpha\in\mathbb{C},\ \alpha\neq 0),\tag{4.4.33}$$

and in particular

$$S(i\theta,1)=1\quad(\theta\in\mathbb{R},\ \theta\neq 0).\tag{4.4.34}$$

If $n,k\in\mathbb{N}$ and $1\leq n\leq k-1$ then

$$S(n,k)=0\tag{4.4.35}$$

The next result, the proof of which is similar to Theorem 4.12, yields the representations of the Stirling functions $S(\alpha,k)$ with $\Re e(\alpha)\geq 0$ in terms of the Liouville fractional derivatives (1.3.63) and (1.3.64).

Theorem 4.18. *Let $\Re e(\alpha)>0$, $k\in\mathbb{N}_0$ or $\Re e(\alpha)=0$ $(\alpha\neq 0)$, $k\in\mathbb{N}$, and let ${}^{\mathrm{L}}D_+^\alpha$ and ${}^{\mathrm{L}}D_-^\alpha$ be the operators of Liouville fractional differentiation* (1.3.63) *and* (1.3.64)*, respectively. The Stirling functions $S(\alpha,k)$ have the representations* (4.4.7) *and* (4.4.8) *of Theorem 4.12.*

The representations of $S(\alpha, k)$ with $\Re e(\alpha) \geq 0$ in terms of the Marchaud fractional derivatives (1.3.86) and (1.3.87) are given by the following statement whose proof is similar to that of Theorem 4.13.

Theorem 4.19. *Let $\Re e(\alpha) > 0$, $k \in \mathbb{N}_0$ and let ${}^{\mathtt{M}}D_{+}^{\alpha}$ and ${}^{\mathtt{M}}D_{-}^{\alpha}$ be the operators of Marchaud fractional differentiation* (1.3.86) *and* (1.3.87), *respectively. The Stirling functions of the second kind $S(\alpha, k)$ have the representations* (4.4.11) *and* (4.4.12) *of Theorem 4.13.*

When $k > \Re e(\alpha) > 0$, for $S(\alpha, k)$ there also holds the integral representation of the form (4.4.15). To prove such a representation we extend relation (4.4.17) to complex $\alpha \in \mathbb{C}$ ($\Re e(\alpha) > 0$).

Lemma 4.4. *Let $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$ be such that $k > \Re e(\alpha) > 0$. The Stirling function $S(\alpha, k)$ is expressed via the constant $\kappa(\alpha, k)$, given in* (1.3.88), *by*

$$S(\alpha, k) = \frac{(-1)^{k+1}\alpha}{k!\Gamma(1-\alpha)}\kappa(\alpha, k) \quad (k \in \mathbb{N}, k > \Re e(\alpha) > 0). \tag{4.4.36}$$

Proof. Let $\alpha > 0$ and $k > \alpha > 0$. Using (4.4.17) and the functional equation for the Gamma function $\Gamma(-\alpha) = -\Gamma(1-\alpha)/\alpha$, we have

$$S(\alpha, k) = \frac{(-1)^k}{k!\Gamma(-\alpha)}\kappa(\alpha, k)) = \frac{(-1)^{k+1}\alpha}{k!\Gamma(1-\alpha)}\kappa(\alpha, k),$$

which proves (4.4.36) for $\alpha > 0$. When $\alpha \in \mathbb{C}$, $k \in \mathbb{N}$ and $k > \Re e(\alpha) > 0$, the left- and right-hand sides of (4.4.36) are analytic functions of α in accordance with (4.4.31) and (1.3.88), respectively. Therefore (4.4.36) is true for such $\alpha \in \mathbb{C}$ by analytic continuation. □

Using Lemma 4.4 and (1.3.88) we deduce the integral representation for $S(\alpha, k)$ when $k > \Re e(\alpha) > 0$.

Theorem 4.20. *Let $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$ be such that $k > \Re e(\alpha) > 0$. The Stirling functions $S(\alpha, k)$ have the integral representation*

$$S(\alpha, k) = \frac{(-1)^{k+1}\alpha}{k!\Gamma(1-\alpha)}\int_0^\infty (1-e^{-t})^k \frac{dt}{t^{1+\alpha}}. \tag{4.4.37}$$

The integral representation of $S(\alpha, k)$ with $\Re e(\alpha) < 0$ in terms of the Liouville fractional integrals is given by the following result which is proved similarly to Theorem 4.16.

Theorem 4.21. *Let* $\alpha \in \mathbb{C}$ $(\Re e(\alpha < 0))$, $k \in \mathbb{N}$ *and let* ${}^{\mathtt{L}}I_{+}^{-\alpha}$ *and* ${}^{\mathtt{L}}I_{-}^{-\alpha}$ *be the operators of Liouville fractional integration given in* (1.3.61) *and* (1.3.62), *respectively. The Stirling functions of the second kind* $S(\alpha, k)$ *have the integral representations* (4.4.25) *and* (4.4.26) *of Theorem 4.16.*

4.4.4 *Stirling functions $S(\alpha, k)$, $\alpha \in \mathbb{C}$, and recurrence relations*

In the previous section we defined Stirling functions of the second kind $S(\alpha, k)$ for $\alpha \in \mathbb{C}$, except when $\Re e(\alpha) < 0$ and $k = 0$. In this section we will give a definition of $S(\alpha, k)$ for such α and k using the following recurrence relations.

Lemma 4.5. *For* $\alpha \in \mathbb{C}$ $(\alpha \neq -1)$ *and* $k \in \mathbb{N}_2 = \{2, 3, \ldots\}$, *the Stirling functions* $S(\alpha, k)$ *satisfy the recurrence relation*

$$S(\alpha + 1, k) = kS(\alpha, k) + S(\alpha, k - 1). \tag{4.4.38}$$

Proof. In accordance with (4.4.31) all terms of both sides of (4.4.38) are defined for the range of parameters α and k considered. Using (4.4.31) and the obvious property

$$\binom{k}{j} = k \binom{k-1}{j-1} \frac{1}{j},$$

we have, for $\alpha \neq 0$

$$\begin{aligned} S(\alpha + 1, k) &= \frac{1}{k!} \sum_{j=1}^{k-1} (-1)^{k-j} \binom{k}{j} j^{\alpha+1} + \frac{1}{k!} k^{\alpha+1} \\ &= \frac{1}{(k-1)!} \sum_{j=1}^{k-1} (-1)^{k-j} \binom{k-1}{j-1} j^{\alpha} + \frac{1}{k!} k^{\alpha+1}. \end{aligned} \tag{4.4.39}$$

Applying the well known combinatorial formula

$$\binom{k}{j} = \binom{k-1}{j} + \binom{k-1}{j-1} \quad (k, j \in \mathbb{N},\ 1 \leq j < k), \tag{4.4.40}$$

we deduce

$$S(\alpha+1,k)$$

$$= \frac{1}{(k-1)!}\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k}{j}j^{\alpha} + \frac{k}{k!}k^{\alpha} - \frac{1}{(k-1)!}\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k-1}{j}j^{\alpha}$$

$$= k\frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^{\alpha} + \frac{1}{(k-1)!}\sum_{j=1}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}j^{\alpha},$$

which, combined with (4.4.31), yields (4.4.38). When $\alpha = 0$, relation (4.4.38) follows from (4.4.32)–(4.4.35). □

If $k = 1$ and $\Re e(\alpha) > 0$, then $S(\alpha+1,1) = S(\alpha,1) = 1$ from (4.4.33), while $S(\alpha,0) = 0$ in accordance with (4.4.32); therefore the relation (4.4.38) is valid for $\Re e(\alpha) > 0$ and $k = 1$. It is also true for $k = 1$ and $\alpha = 0$ because $S(1,1) = 1$, $S(0,1) = 0$ and $S(0,0) = 1$ by (4.4.33) and (4.4.32).

Now we put $k = 1$ in (4.4.38) and rewrite it in the form

$$S(\alpha,0) = S(\alpha+1,1) - S(\alpha,1) \qquad (\alpha \neq -1). \tag{4.4.41}$$

But in accordance with Theorem 4.17 the right-hand side of (4.4.41) is defined for $\alpha \in \mathbb{C}$ when $\Re\mathrm{e}(\alpha) < 0$ and $\Re e(\alpha) = 0$ ($\alpha \neq 0$). Therefore we can define the Stirling functions $S(\alpha,0)$ for such α by (4.4.41). Then $S(\alpha,0) = 0$ for $\Re\mathrm{e}(\alpha) < 0$ ($\alpha \neq -1$) and $\Re\mathrm{e}(\alpha) = 0$, taking into account (4.4.33). Hence, in accordance with (4.4.32) and (4.4.33) we obtain the property

$$S(\alpha,0) = 0 \quad (\alpha \in \mathbb{C};\ \alpha \neq 0,-1). \tag{4.4.42}$$

Therefore, from the first relation (4.4.33) and (4.4.42) we deduce that (4.4.38) holds for $k = 1$ when $\Re e(\alpha) < 0$ and $\Re\mathrm{e}(\alpha) = 0$ ($\alpha \neq 0$). Formula (4.4.38) can also be valid for $k = 0$ if we define the $S(\alpha,-1)$ by

$$S(\alpha,-1) = S(\alpha+1,0) \quad (\alpha \in \mathbb{C}), \tag{4.4.43}$$

and in accordance with (4.4.32) we obtain

$$S(-1,-1) = 1, \; S(\alpha,-1) = 0 \quad (\alpha \in \mathbb{C}, \; \alpha \neq -1). \tag{4.4.44}$$

From Lemma 4.5 and the above arguments, relation (4.4.38) is valid for any $\alpha \in \mathbb{C}$ $(\alpha \neq -1)$ and any $k \in \mathbb{N}_0$. This yields the following result.

Theorem 4.22. *Let $\alpha \in \mathbb{C}$ $(\alpha \neq -1)$, $k \in \mathbb{N}_0$, let the Stirling functions $S(\alpha, k)$ be given by* (4.4.31) *for $\alpha \neq 0$ and by* (4.4.32) *for $\alpha = 0$, and let $S(\alpha, -1)$ be given by* (4.4.44). *Then the $S(\alpha, k)$ satisfy the recurrence relation* (4.4.38).

Corollary 4.6. *If $k \in \mathbb{N}_0$, then*

$$S(k,k) = 1. \tag{4.4.45}$$

Remark 4.1. Relation (4.4.38) can be rewritten in the form

$$S(\alpha,k) = S(\alpha+1,k+1) - (k+1)S(\alpha,k+1). \tag{4.4.46}$$

This formula together with (4.4.43), (4.4.44) can be used to define the Stirling functions $S(\alpha, k)$ for negative $k = -2, -3, \ldots$. For example, we define $S(\alpha, -2)$ by taking $k = -2$ in (4.4.46)

$$S(\alpha,-2) = S(\alpha+1,-1) + S(\alpha,-1),$$

and an application of (4.4.44) yields

$$S(-2,-2) = 1, \; S(-1,-2) = 1, \; S(\alpha,-2) = 0 \quad (\alpha \in \mathbb{C}, \; \alpha \neq -1,-2).$$

On the other hand, whereas $S(n, k) = 0$ for $k > n$, it should be observed that $S(-n, k)$ is in general not equal to zero for $n, k \in \mathbb{N}$. For example, as can be checked directly on the basis of (4.4.23), $S(-1, 2) = -3/4$ and $S(-1, 3) = 11/36$.

From Theorem 4.22 we deduce the following recurrence relations.

Proposition 4.1. *Let $\alpha \in \mathbb{C}$ and $n, k \in \mathbb{N}_0$. The Stirling functions $S(\alpha, k)$ satisfy the recurrence relations*

$$S(\alpha+n+1,k+n) = \sum_{j=0}^{n}(j+k)S(\alpha+j,k+j) + S(\alpha,k-1) \quad (k \in \mathbb{N}), \tag{4.4.47}$$

when $\alpha \neq -m$ $(m = 1, 2, \ldots, n, n+1)$, *and*

$$S(\alpha+n+1, n) = \sum_{j=1}^{n} jS(\alpha+j, j) + S(\alpha+1, 0), \tag{4.4.48}$$

when $\alpha \neq -m$ $(m = 2, \ldots, n, n+1)$.

Proof. Relations (4.4.47) and (4.4.48) follow by induction on $n \in \mathbb{N}_0$, using (4.4.38), similar to the case $\alpha > 0$. □

The next result is also deduced similarly to the cases $\alpha > 0$, $\beta > 0$.

Proposition 4.2. *Let* $\alpha \in \mathbb{C}$ *and* $\beta \in \mathbb{C}$ *be such that* $\alpha \neq 0$, $\beta \neq 0$ *and* $\alpha + \beta \neq 0$, *and let* $k \in \mathbb{N}$. *There holds the relation*

$$S(\alpha+\beta, k) = \sum_{j=0}^{k}\sum_{i=0}^{k-j} \frac{(i+j)!}{j!}\binom{k-j}{i} S(\alpha, i+j)S(\beta, k-j) \quad (k \in \mathbb{N}). \tag{4.4.49}$$

In particular, the duplication formula is given by

$$S(2\alpha, k) = \sum_{j=0}^{k}\sum_{i=0}^{k-j} \frac{(i+j)!}{j!}\binom{k-j}{i} S(\alpha, i+j)S(\alpha, k-j) \quad (k \in \mathbb{N}). \tag{4.4.50}$$

4.4.5 *Further properties and first applications of Stirling functions $S(\alpha, k)$, $\alpha \in \mathbb{C}$*

In this section we present further properties of the Stirling functions $S(\alpha, k)$ of complex order α. First of all we present a representation for $S(\alpha, k)$, on the basis of (4.4.29), in terms of a complex integral, whose proof is similar to the proof of Proposition 7 in [114].

Proposition 4.3. *For* $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$ *there holds the integral representation*

$$S(\alpha, k) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{(z-1)^{\alpha}}{(z-1)(z-2)\ldots(z-k-1)} dz, \tag{4.4.51}$$

where $\mathcal{L}$ *is an arbitrary rectifiable Jordan curve in the half-plane* $\Re\mathrm{e}(z) > 1$ *containing* $n = 2, 3, \ldots, k+1$.

Similarly to the proof of Theorem 6 in [114] there also holds the property.

Proposition 4.4. *For $z \in \mathbb{C}$ and $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$ there holds the Newton series expansion*

$$z^\alpha = \sum_{k=1}^{\infty} S(\alpha, k)[z]_k, \tag{4.4.52}$$

$[z]_k$ being given by (4.1.7), *the abscissa of convergence of the series on the right-hand side of* (4.4.52) *being $\lambda \leq \max\{0, \Re e(\alpha) - 1/2\}$. This series converges absolutely for $|z| \geq \lambda + 1$ and uniformly on all compact sets of the half-plane $\Re e(z) \geq \lambda + \epsilon$ with arbitrarily small ϵ.*

In particular, for $n \in \mathbb{N}$ and $x > 0$ there holds the Newton series expansion

$$x^{-n} = \sum_{k=1}^{\infty} S(-n, k)[x]_k. \tag{4.4.53}$$

The next result deals with the asymptotic behavior of $S(\alpha, k)$ as $a \to \infty$.

Proposition 4.5. *For $k \in \mathbb{N}$ there hold the asymptotic estimates*

$$\lim_{|\alpha| \to \infty,\ -\pi/2+\epsilon \leq \arg(\alpha) \leq \pi/2-\epsilon} \left(k^{-\alpha} S(\alpha, k)\right) = \frac{1}{k!}, \tag{4.4.54}$$

$$\lim_{|\alpha| \to \infty,\ \pi/2+\epsilon \leq \arg(\alpha) \leq 3\pi/2-\epsilon} S(\alpha, k) = \frac{(-1)^k}{(k-1)!}, \tag{4.4.55}$$

where ϵ $(0 < \epsilon < \pi/2)$ is a fixed number.

Proof. Let $0 < \epsilon < \pi/2$ and $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$. If $\Re e(\alpha) > 0$, then by (4.4.31) we have

$$k^{-\alpha} S(\alpha, k) = \frac{1}{k!} \left\{ \sum_{j=1}^{k-1} (-1)^{k-j} \binom{k}{j} \left(\frac{j}{k}\right)^\alpha + 1 \right\},$$

and taking the limit $|\alpha| \to \infty$ with $-\pi/2 + \epsilon \leq \arg(\alpha) \leq \pi/2 - \epsilon$, we obtain (4.4.54).

Let $\Re e(\alpha) < 0$. When $k = 1$, $S(\alpha, 1) = 1$ by (4.4.33), and (4.4.54) is clear. When $k \geq 2$, then by (4.4.31)

$$S(\alpha, k) = \frac{(-1)^{k-1}}{(k-1)!} + \frac{1}{k!} \sum_{j=2}^{k} (-1)^{k-j} \binom{k}{j} \frac{1}{j^{-\alpha}},$$

and (4.4.55) is deduced by taking the limit $|\alpha| \to \infty$ with $\pi/2 + \epsilon \leq \arg(\alpha) \leq 3\pi/2 - \epsilon$. $\square$

The "horizontal" generating function for $S(\alpha, k)$ is given by the following assertion which is proved similarly to the case of real positive $\alpha > 0$, if we take (4.4.31) into account (see Eqs. (6.12) and (6.13) of [114]).

Proposition 4.6. *For $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$ and $x \in \mathbb{R}$ the Stirling functions $S(\alpha, k)$ have the "horizontal" generating function in the form*

$$\Phi_\alpha(x) = e^{-x} \sum_{k=1}^{\infty} \frac{k^\alpha}{k!} x^k = \sum_{k=1}^{\infty} S(\alpha, k) x^k. \tag{4.4.56}$$

This "generating" function satisfies a Rodriguez formula, the difference-differential equation

$$\Phi_\alpha(x) = x \left(\Phi_{\alpha-1}(x) + \Phi'_{\alpha-1}(x) \right).$$

In particular, for $n \in \mathbb{N}$ and $x \in \mathbb{R}$, there holds

$$\Phi_{-n}(x) = e^{-x} \sum_{k=1}^{\infty} \frac{x^k}{k! k^n} = \sum_{k=1}^{\infty} S(-n, k) x^k.$$

Remark 4.2. The Bell function $B(\alpha)$ of order $\alpha \geq 0$ can be defined by (see p. 40 in [114])

$$B(\alpha) = \sum_{k=1}^{\infty} S(\alpha, k). \tag{4.4.57}$$

The extension to $\alpha \in \mathbb{C}$ is obvious. If $\alpha \neq 0$, then, in accordance with (4.4.32), (4.4.56) yields

$$B(\alpha) = \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^\alpha}{k!} = \sum_{k=1}^{\infty} S(\alpha, k). \tag{4.4.58}$$

In particular, the Bell number $B(\alpha)$ of order $\alpha = -n$ has the representation

$$B(-n) = e^{-1}\sum_{k=1}^{\infty}\frac{1}{k!k^n} = \sum_{k=1}^{\infty} S(-n,k). \tag{4.4.59}$$

Bell numbers of negative order, which do not have a finite series, do not seem to have been considered so far; it should be of interest to interpret them in the context of combinatorics. Likewise it would of course be satisfying to view the new Stirling numbers $S(-n,k)$ in the setting of combinatorial analysis.

The function $\Phi_\alpha(x)$ can be applied to obtain a certain convolution theorem for $S(\alpha,k)$ and other recurrence relations than those given in Propositions 4.1 and 4.2.

Proposition 4.7. *For $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$, $\beta \in \mathbb{C}$ $(\beta \neq 0)$ there holds for any $m = 2, 3, \ldots$ the convolution relation*

$$\sum_{k=1}^{m-1} S(\alpha,k)S(\beta,m-k) = \sum_{k=2}^{m}\sum_{j=1}^{k-1}(-2)^{m-k}\frac{j^\alpha (k-j)^\beta}{j!(k-j)!(m-k)!}. \tag{4.4.60}$$

Proposition 4.8. *For $\alpha \in \mathbb{C}$ $(\alpha \neq -1, 0)$ one has*

$$\sum_{k=0}^{\infty}(k+1)S(\alpha,k) = \sum_{k=0}^{m} S(\alpha+1,k). \tag{4.4.61}$$

The following theorem, the proof of which also depends on $\Phi_\alpha(x)$, is basic for the next one.

Theorem 4.23. *For $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$ and $m \in \mathbb{N}$ there holds*

$$m^\alpha = \sum_{k=1}^{m}\binom{m}{k} k! S(\alpha,k). \tag{4.4.62}$$

In particular, for $\alpha = -n$ $(n \in \mathbb{N})$,

$$\frac{1}{m^n} = \sum_{k=1}^{m}\binom{m}{k} k! S(-n,k). \tag{4.4.63}$$

Proof. Using the first and the second formulas in (4.4.56) and taking into account the convergence of the series $\sum_{k=1}^{\infty} S(\alpha,k)x^k$ for any $x \in \mathbb{R}$, we have

$$m^\alpha = D^m[e^x\Phi_\alpha(x)]|_{x=0} = D^m\left[\left(\sum_{j=0}^{\infty}\frac{x^j}{j!}\right)\left(\sum_{k=1}^{\infty}S(\alpha,k)x^k\right)\right]\Bigg|_{x=0}$$
$$= D^m\left[\left(\sum_{j=0}^{\infty}\frac{x^j}{j!}\right)\left(\sum_{k=0}^{\infty}S(\alpha,k+1)x^{k+1}\right)\right]\Bigg|_{x=0}.$$

Applying the identity theorem for power series and the multiplication theorem, we deduce

$$\begin{aligned} m^\alpha &= D^m[e^x\Phi_\alpha(x)]|_{x=0} \\ &= D^m\left[\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\frac{S(\alpha,k+1)}{(j-k)!}\right)x^{j+1}\right]\Bigg|_{x=0} \\ &= D^m\left[\sum_{j=1}^{\infty}\left(\sum_{k=0}^{j-1}\frac{S(\alpha,k+1)}{(j-k-1)!}\right)x^{j}\right]\Bigg|_{x=0} \\ &= D^m\left[\sum_{j=m}^{\infty}\left(\sum_{k=0}^{j-1}\frac{S(\alpha,k+1)}{(j-k-1)!}\right)\frac{j!}{(j-m)!}x^{j-m}\right]\Bigg|_{x=0} \\ &= \sum_{k=0}^{m-1}S(\alpha,k+1)\frac{m!}{(m-k-1)!} = \sum_{k=1}^{m}\frac{m!}{(m-k)!}S(\alpha,k), \end{aligned} \tag{4.4.64}$$

which proves (4.4.62). The relation (4.4.62) with $\alpha = -n$ $(n \in \mathbb{N})$ yields (4.4.63). Thus the proof is complete. □

The following result is based on Theorem 4.23.

Theorem 4.24. *For $\alpha \in \mathbb{C}$ $(\alpha \neq -1)$ and $m \in \mathbb{N}$ there holds*

$$\sum_{k=1}^{m} k^\alpha = \sum_{k=1}^{m}\binom{m}{k}(k-1)!S(\alpha+1,k). \tag{4.4.65}$$

In particular, for $\alpha = -n$ $(n \in \mathbb{N},\ n \neq 1)$, we have

$$\sum_{k=1}^{m}\frac{1}{k^n} = \sum_{k=1}^{m}\binom{m}{k}(k-1)!S(-n+1,k). \tag{4.4.66}$$

Proof. The proof follows by induction. The case $m = 1$ is valid since $1 = S(\alpha+1, 1)$ holds for $\alpha \neq -1$ by (4.4.33). Let (4.4.65) be valid for $m = n \in \mathbb{N}$. By (4.4.40) we have for $m = n+1$

$$\begin{aligned}
&\sum_{k=1}^{n+1} \binom{n+1}{k} (k-1)!\, S(\alpha+1, k) \qquad (4.4.67)\\
&= n! S(\alpha+1, n+1) + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] (k-1)!\, S(\alpha+1, k)\\
&= \sum_{k=1}^{n} \binom{n}{k} (k-1)!\, S(\alpha+1, k) + \sum_{k=1}^{n+1} \binom{n}{k-1} (k-1)!\, S(\alpha+1, k).
\end{aligned}$$

Since, by induction, (4.4.65) holds for $m = n$,

$$\sum_{k=1}^{n} \binom{n}{k} (k-1)!\, S(\alpha+1, k) = \sum_{k=1}^{n} k^\alpha.$$

According to (4.1.10) and Theorem 4.23,

$$\begin{aligned}
&\sum_{k=1}^{n+1} \binom{n}{k-1} (k-1)!\, S(\alpha+1, k) = \sum_{k=1}^{n+1} \frac{n!}{(n-k+1)!} S(\alpha+1, k)\\
&= \sum_{k=1}^{n+1} \binom{n+1}{k} k!\, \frac{1}{n+1} S(\alpha+1, k) = \frac{1}{n+1} (n+1)^{\alpha+1} = (n+1)^\alpha.
\end{aligned}$$

Substituting the two last relations into (4.4.67), we obtain (4.4.65). The relation (4.4.66) follows from (4.4.65) for $\alpha = -n$ ($n \in \mathbb{N}$), which completes the proof of this theorem. □

Remark 4.3. The evaluation of $\sum_{k=1}^{m} k^\alpha$ for the classical case $\alpha = n \in \mathbb{N}$ is due to Bernoulli (who introduced the numbers named after him in its determination), and for real $\alpha > -1$ to [114]. The new formula (4.4.65) in the instance $\alpha \in \mathbb{C}$ with $\alpha \neq -1$, and in the particular case $\alpha = -n$ with $n \in \mathbb{N}$ ($n \neq 1$) of (4.4.66) emphasize the important role that the Stirling functions $S(\alpha+1, k)$, in particular $S(-n+1, k)$, play even in classical combinatorics and discrete mathematics. Formula (4.4.66) fails for $n = 1$ since the right-hand side would then be zero, noting $S(-n+1, k) = 0$ for $n = 1$ and all $k \in \mathbb{N}$.

Normally, $\sum_{k=1}^{m} k^{-n}$ is summed up for $n > 1$ by employing the Euler-Poisson summation formula, giving an infinite series involving the Bernoulli numbers. But (4.4.66) yields a finite series involving $S(-n+1,k)$.

Continuity and differentiability of the Stirling functions $S(\alpha,k)$ are given by the following result.

Proposition 4.9. *Let $k \in \mathbb{N}$ be arbitrary. There hold the following assertions.*

(a) *$S(\alpha,k)$ as a function of α is continuous for $\alpha \in \mathbb{C}$ ($\alpha \neq 0$).*

(b) *If $\alpha \in \mathbb{C}$ ($\alpha \neq 0$), then $S(\alpha,k)$ is arbitrarily often continuously differentiable, and for any $m \in \mathbb{N}$*

$$\left(\frac{d}{d\alpha}\right)^m S(\alpha,k) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j} j^{\alpha}(\log j)^m. \tag{4.4.68}$$

Proof. The continuity of $S(\alpha,k)$ for $\alpha \in \mathbb{C}$ ($\alpha \neq 0$) follows from (4.4.31). When $\alpha \in \mathbb{C}$ and $\alpha \neq 0$, the function j^{α} as a function of α is arbitrarily often continuously differentiable and (4.4.31) gives the same for $S(\alpha,k)$. The relation (4.4.68) follows from (4.4.31) by differentiation. This completes the proof of (ii) and the proposition. □

4.4.6 *Applications of Stirling functions $S(\alpha,k)$ ($\alpha \in \mathbb{C}$) to Hadamard-type fractional operators*

Now we apply the Stirling functions $S(\alpha,k)$ to express Hadamard fractional integration (1.3.123), (1.3.124) and differentiation (1.3.129), (1.3.130) in terms of $S(\alpha,k)$ and usual differentiation.

First we prove such results for the more general Hadamard-type fractional integration (1.3.125), (1.3.126) and differentiation (1.3.131), (1.3.132) in terms of more general Stirling functions generalizing $S_c(n,k)$ defined via (4.1.18). Such Stirling functions $S_c(\alpha,k)$, defined in [115], have the explicit representation in the form (4.4.31)

$$S_c(\alpha,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(c+j)^{\alpha} \quad (\alpha \in \mathbb{C};\ k \in \mathbb{N}_0;\ c \in \mathbb{R}), \tag{4.4.69}$$

established in [117], for which

$$\lim_{c\to 0} S_c(\alpha,k) = S(\alpha,k). \tag{4.4.70}$$

Alternatively, the $S_c(\alpha,k)$ can be defined by

$$S_c(\alpha,0) = c^\alpha,\ S_c(\alpha+1,k) = S_c(\alpha,k-1) + (c+k)S_c(\alpha,k).$$

The properties of $S_c(\alpha,k)$ are generally similar to those of the $S(\alpha,k)$.

First we show that Hadamard-type fractional integration (1.3.125) and differentiation (1.3.131) can be expressed in terms of infinite series involving the $S_c(\alpha,k)$ and classical differentiation by the following statement.

Theorem 4.25. *Let $f(x)$, defined for $x>0$, be an arbitrarily often differentiable function such that its Taylor series converges, and let $\alpha\in\mathbb{C}$, $c>0$.*

(a) *When $\Re e(\alpha)\geq 0$, the Hadamard type fractional derivative ${}^{\mathtt{H}}D^\alpha_{0+,c}f$ is given by* (1.3.131) *if and only if there holds for $x>0$ the relation*

$$\left({}^{\mathtt{H}}D^\alpha_{0+,c}f\right)(x) = \sum_{k=0}^{\infty} S_c(\alpha,k)x^k f^{(k)}(x). \tag{4.4.71}$$

(b) *When $\Re e(\alpha)>0$, the Hadamard type fractional integral ${}^{\mathtt{H}}I^\alpha_{0+,c}f$ is given by* (1.3.125) *if and only if for $x>0$*

$$\left({}^{\mathtt{H}}I^\alpha_{0+,c}f\right)(x) = \sum_{k=0}^{\infty} S_c(-\alpha,k)x^k f^{(k)}(x). \tag{4.4.72}$$

Proof. When $\alpha=0$, then in accordance with (4.4.69) and (1.3.90) for $c\in\mathbb{R}$

$$S_c(0,0)=1,\ S_c(0,k)=0\ (k\in\mathbb{N}),$$

and (4.4.71) takes on the form ${}^{\mathtt{H}}D^0_{0+,c}f = f$ which coincides with the one in (1.3.133).

Let $\Re e(\alpha)\geq 0$ $(\alpha\neq 0)$, and let $\left({}^{\mathtt{H}}D^\alpha_{0+,c}f\right)(x)$ be given for $x>0$ by (1.3.131) with $n=[\Re e(\alpha)]+1$

$$\left({}^{\mathtt{H}}D^\alpha_{0+,c}f\right)(x) = x^{-c}\left(x\frac{d}{dx}\right)^n x^c \frac{1}{\Gamma(n-\alpha)}\int_0^x \left(\frac{u}{x}\right)^c \left(\log\frac{x}{u}\right)^{n-\alpha-1} f(u)\frac{du}{u}. \tag{4.4.73}$$

Fix $x > 0$. By conditions of the theorem, for any $u \in [0, x]$ and any $y > 0$ we have

$$f(u) = \sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!} (u - y)^k.$$

By the binomial formula this relation can be rewritten as

$$f(u) = \sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} y^{k-j} u^j, \tag{4.4.74}$$

or

$$f(u) = \sum_{j=0}^{\infty} a_j(y) u^j, \ a_j(y) = \sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!} (-1)^{k-j} \binom{k}{j} y^{k-j}. \tag{4.4.75}$$

For any fixed $y > 0$, (4.4.75) is a convergent power series because it coincides with the Taylor series (4.4.74), being convergent by the condition of the theorem. Then we can apply Proposition 1.2(ii). Substituting relation (4.4.74) into (4.4.73) and using the formula (1.3.163), we have

$$\begin{aligned}\left({}^{\mathtt{H}}D^{\alpha}_{0+,c} f\right)(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} y^{k-j} ({}^{\mathtt{H}}D^{\alpha}_{0+,c} u^j)(x) \\ &= \sum_{k=0}^{\infty} f^{(k)}(y) \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} y^{k-j} (c+j)^{\alpha} x^j.\end{aligned}$$

Setting $y = x$ in this relation, we obtain

$$\left({}^{\mathtt{H}}D^{\alpha}_{0+,c} f\right)(x) = \sum_{k=0}^{\infty} x^k f^{(k)}(x) \left(\frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (c+j)^{\alpha} \right),$$

which, combined with (4.4.69), yields (4.4.71).

Conversely, let ${}^{\mathtt{H}}D^{\alpha}_{0+,c} f$ have representation (4.4.71) for $x > 0$. Fixing this $x > 0$, noting (4.4.69) and taking any $y > 0$ we rewrite (4.4.71) in the form

$$\left({}^{\mathtt{H}}D^{\alpha}_{0+,c} f\right)(x) = \sum_{k=0}^{\infty} x^k f^{(k)}(x) y^{-j} \left(\frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} y^j (c+j)^{\alpha} \right).$$

Applying relation (4.4.75) (with x being replaced by y) and interchanging orders of summation and integration, being admissible again by Proposition 1.2(ii), we find for $\delta = x(d/dx)$,

$$\begin{aligned}\left({}^{\mathtt{H}}D^{\alpha}_{0+,c}f\right)(x) &= \sum_{k=0}^{\infty}\frac{f^{(k)}(x)}{k!}x^k y^{-j}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}({}^{\mathtt{H}}D^{\alpha}_{0+,c}u^j)(x)\\ &= x^{-c}\delta^n x^c \frac{1}{\Gamma(n-\alpha)}\int_0^x\left(\frac{u}{x}\right)^c\left(\log\frac{x}{u}\right)^{n-\alpha-1}\\ &\quad\times\left(\sum_{k=0}^{\infty}\frac{f^{(k)}(x)}{k!}x^k y^{-j}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}u^j\right)\frac{du}{u}.\end{aligned}$$

Setting $y = x$, we have

$$\begin{aligned}\left({}^{\mathtt{H}}D^{\alpha}_{0+,c}f\right)(x) &= x^{-c}\delta^n x^c \frac{1}{\Gamma(n-\alpha)}\int_0^x\left(\frac{u}{x}\right)^c\left(\log\frac{x}{u}\right)^{n-\alpha-1}\\ &\quad\times\left(\sum_{k=0}^{\infty}\frac{f^{(k)}(x)}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}x^{k-j}u^j\right)\frac{du}{u},\end{aligned}$$

and (1.3.131) follows by noting (4.4.74) with y being replaced by x.

The relation (4.4.72) is proved similarly by using Proposition 1.2(i). Thus the proof is complete. □

Observe that the fractional differentiation and integration operators ${}^{\mathtt{H}}D^{\alpha}_{0+,c}$ and ${}^{\mathtt{H}}I^{\alpha}_{0+,c}$ of Theorem 4.25 are both expressed as infinite series involving the classical powers $(xd/dx)^k$, with coefficients $S_c(\alpha, k)$ and $S_c(-\alpha, k)$, respectively. Therefore, the relations (4.4.71) and (4.4.72) express in a clear way the natural transition from ${}^{\mathtt{H}}D^{\alpha}_{0+,c}$ to ${}^{\mathtt{H}}I^{\alpha}_{0+,c}$, and conversely.

Let Q be the "reflection" operator (see Eq. (2.5) of [117]) defined by

$$(Qg)(x) = g\left(\frac{1}{x}\right). \tag{4.4.76}$$

It is directly verified that the Hadamard-type fractional integration operators ${}^{\mathtt{H}}I^{\alpha}_{0+,c}$ and ${}^{\mathtt{H}}I^{\alpha}_{-,c}$ given by (1.3.125) and (1.3.126) are connected via the operator Q by

$${}^{\mathtt{H}}I^{\alpha}_{0+,c}Qf = Q\,{}^{\mathtt{H}}I^{\alpha}_{-,c}Qf,$$

and similarly for the Hadamard-type fractional differentiation operators ${}^{\mathrm{H}}D^{\alpha}_{0+,c}$ and ${}^{\mathrm{H}}D^{\alpha}_{-,c}$ defined by (1.3.131) and (1.3.132)

$$ {}^{\mathrm{H}}D^{\alpha}_{0+,c}Qf = Q\,{}^{\mathrm{H}}D^{\alpha}_{-,c}Qf. $$

Since the operator $Q^{-1} = Q$, we deduce from Theorem 4.25 the representation of the Hadamard-type fractional integration (1.3.126), ${}^{\mathrm{H}}I^{\alpha}_{-,c}$, and differentiation (1.3.132), ${}^{\mathrm{H}}D^{\alpha}_{-,c}$, in terms of Stirling functions $S_c(\alpha, k)$ and usual differentiation.

Theorem 4.26. *Let $f(x)$, defined for $x > 0$, be arbitrarily often differentiable such that its Taylor series converges. Let $\alpha \in \mathbb{C}$, $c > 0$, and let Q be the "reflection" operator* (4.4.76)*.*

(a) *When $\Re e(\alpha) \geq 0$, the Hadamard-type fractional derivative ${}^{\mathrm{H}}D^{\alpha}_{-,c}f$ is given by* (1.3.132) *if and only if there holds for $x > 0$ the relation*

$$ \left({}^{\mathrm{H}}D^{\alpha}_{-,c}f\right)(x) = \sum_{k=0}^{\infty} S_c(\alpha, k)x^{-k}(Qf)^{(k)}\left(\frac{1}{x}\right). \tag{4.4.77} $$

(b) *When $\Re e(\alpha) > 0$, the Hadamard-type fractional integral ${}^{\mathrm{H}}I^{\alpha}_{-,c}f$ is given by* (1.3.126) *if and only if for $x > 0$*

$$ \left({}^{\mathrm{H}}I^{\alpha}_{-,c}f\right)(x) = \sum_{k=0}^{\infty} S_c(-\alpha, k)x^{-k}(Qf)^{(k)}\left(\frac{1}{x}\right). \tag{4.4.78} $$

Both $({}^{\mathrm{H}}D^{\alpha}_{0+,c}f)(x)$ and $({}^{\mathrm{H}}D^{\alpha}_{-,c}f)(x)$ are continuous functions as functions of c at $c = 0$ for fixed $\alpha \in \mathbb{C}$ ($\Re e(\alpha) \geq 0$), and likewise for $({}^{\mathrm{H}}I^{\alpha}_{0+,c}f)(x)$ and $({}^{\mathrm{H}}I^{\alpha}_{-,c}f)(x)$ for fixed $\alpha \in \mathbb{C}$ ($\mathrm{Re}(\alpha) > 0$). The functions $S_c(\alpha, k)$ are also continuous functions of $c \in \mathbb{R}$ at $c = 0$ for fixed $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}_0$. Therefore letting $c \to 0$ in Theorem 4.25 and taking (4.4.70) into account we deduce the corresponding representations of Hadamard fractional differentiation (1.3.129), ${}^{\mathrm{H}}D^{\alpha}_{0+}$, and integration (1.3.123), ${}^{\mathrm{H}}I^{\alpha}_{0+}$, in terms of the Stirling functions $S(\alpha, k)$ and usual differentiation.

Theorem 4.27. *Let $f(x)$, defined for $x > 0$, be an arbitrarily often differentiable function such that its Taylor series converges, and let $\alpha \in \mathbb{C}$.*

(a) *When $\Re e(\alpha) \geq 0$, the Hadamard fractional derivative ${}^{\mathrm{H}}D^{\alpha}_{0+}f$ is given by* (1.3.129) *if and only if there holds for $x > 0$ the relation*

$$ \left({}^{\mathrm{H}}D^{\alpha}_{0+}f\right)(x) = \sum_{k=0}^{\infty} S(\alpha, k)x^{k}f^{(k)}(x). \tag{4.4.79} $$

(b) *When $\mathfrak{Re}(\alpha) > 0$, the Hadamard-type fractional integral* ${}^{\mathtt{H}}I^{\alpha}_{0+}f$ *is given by* (1.3.123) *if and only if for* $x > 0$

$$\left({}^{\mathtt{H}}I^{\alpha}_{0+}f\right)(x) = \sum_{k=0}^{\infty} S(-\alpha, k)x^k f^{(k)}(x). \tag{4.4.80}$$

Corollary 4.7. *Let $f(x)$ defined for $x > 0$ be an arbitrarily often differentiable function such that its Taylor series converges, and let $n \in \mathbb{N}$. Then*

$$({}^{\mathtt{H}}I^{n}_{0+}f)(x) = \int_0^x \frac{du_1}{u_1}\int_0^{u_1} \frac{du_2}{u_2} \cdots \int_0^{u_{n-1}} \frac{du_n}{u_n} = \sum_{k=0}^{\infty} S(-n, k)x^k f^{(k)}(x). \tag{4.4.81}$$

Corollary 4.8. *Let $n \in \mathbb{N}$ and let $f(x)$ be a function defined for $x > 0$ and differentiable up to order n and let $x \in \mathbb{R}$. Then*

$$\delta^n f(x) = \sum_{k=0}^{n} S(n, k)x^k f^{(k)}(x), \quad \delta = x\frac{d}{dx}. \tag{4.4.82}$$

Similarly Theorem 4.26 yields the representation of Hadamard fractional differentiation (1.3.130), ${}^{\mathtt{H}}D^{\alpha}_{-}$, and integration (1.3.124), ${}^{\mathtt{H}}I^{\alpha}_{-}$, in terms of the Stirling functions $S(\alpha, k)$ and usual differentiation. Indeed, we have

Theorem 4.28. *Let $f(x)$ be an arbitrarily often differentiable function for $x > 0$ such that its Taylor series converges. Let $\alpha \in \mathbb{C}$ ($\mathfrak{Re}(\alpha) \geq 0$), and let Q be the "reflection" operator* (4.4.76).

(a) *When $\mathfrak{Re}(\alpha) \geq 0$, the Hadamard fractional derivative* ${}^{\mathtt{H}}D^{\alpha}_{-}f$ *is given by* (1.3.130) *if and only if for $x > 0$ there holds the relation*

$$({}^{\mathtt{H}}D^{\alpha}_{-}f)(x) = \sum_{k=0}^{\infty} S(\alpha, k)x^{-k}(Qf)^{(k)}\left(\frac{1}{x}\right). \tag{4.4.83}$$

(b) *When $\mathfrak{Re}(\alpha) > 0$, the Hadamard fractional integral* ${}^{\mathtt{H}}I^{\alpha}_{-}f$ *is given by* (1.3.124) *if and only if for* $x > 0$

$$\left({}^{\mathtt{H}}I^{\alpha}_{-}f\right)(x) = \sum_{k=0}^{\infty} S(-\alpha, k)x^{-k}(Qf)^{(k)}\left(\frac{1}{x}\right). \tag{4.4.84}$$

4.5 Generalized Stirling Functions of the Second Kind $S(n, \beta; \epsilon)$ and $S(n, \beta)$

In this and the next section we give extensions of the classical Stirling numbers of the second kind $S(n, k)$ to functions $S(n, \beta)$, whereby the second parameter k becomes any complex β, as well as to functions $S(\alpha, \beta)$, whereby also the first parameter n becomes any complex α. Also we give the representations of $S(\alpha, \beta)$ by Liouville fractional operators.

Firstly the second parameter k is extended from a nonnegative integer k to any complex β. We recall from Eq. (4.1.6) that the classical Stirling numbers of the second kind, $S(n, k)$ are defined by

$$S(n,k) = \frac{1}{k!} \lim_{x \to 0} \Delta^k \left(x^n \right), \tag{4.5.1}$$

where $x \in \mathbb{R}$, $n, k \in \mathbb{N}_0$, and Δ^k is the difference of order k given by

$$\begin{aligned} \Delta^k f(x) &= \Delta \left(\Delta^{k-1} f \right)(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+j) \\ &= D^m \left[\sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \frac{S(\alpha, k+1)}{(j-k)!} \right) x^{j+1} \right] \Bigg|_{x=0} \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x+k-j) \end{aligned}$$

with $\Delta^0 f(x) = f(x)$, $\Delta^1 f(x) = f(x+1) - f(x)$. This suggests one may define $S(n, \beta)$ for $\beta \in \mathbb{C}$ by

$$S(n,\beta) = \frac{1}{\Gamma(\beta+1)} \lim_{x \to 0} \Delta^\beta \left(x^n \right) \quad (x \in \mathbb{R};\ n \in \mathbb{N}_0), \tag{4.5.2}$$

where Δ^β is a suitable generalization of Δ^k from integer k to complex β.

4.5.1 *Definition and some basic properties*

In order to cover as large a class of functions as possible, an exponential factor is introduced in the corresponding definition.

Definition 4.2. The generalized fractional difference operator $\Delta^{\beta,\epsilon}$ for $\beta \in \mathbb{C}$, $\epsilon \geq 0$ is defined for "sufficiently good" functions f by

$$\Delta^{\beta,\epsilon} f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{(\beta-j)\epsilon} f(x+\beta-j) \quad (x \in \mathbb{R}), \tag{4.5.3}$$

where $\binom{\beta}{j}$ are the general binomial coefficients given by

$$\binom{\beta}{j} = \frac{[\beta]_j}{j!} = \frac{\beta(\beta-1)\cdots(\beta-j+1)}{j!} \quad (\, j \in \mathbb{N}), \tag{4.5.4}$$

with $[\beta]_0 = 1$ and $\mathbb{N}$ being the set of all positive integers.

Since $\binom{k}{j} = 0$ for $k \in \mathbb{N}_0$, $0 \leq k < j$, the difference (4.1.6) for $\beta = k \in \mathbb{N}_0$ takes on the form

$$\begin{aligned} \Delta^{k,\epsilon} f(x) &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} e^{(k-j)\epsilon} f(x+k-j) \\ &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} e^{j\epsilon} f(x+j) \quad (x \in \mathbb{R}) \end{aligned} \tag{4.5.5}$$

which is precisely $\Delta^k f(x)$ of (4.1.4) for $\epsilon = 0$, so that the new fractional difference $\Delta^{k,0}$ reduces to the classical one, namely

$$\Delta^{k,0} f(x) = \Delta^k f(x) \quad (x \in \mathbb{R}).$$

Here $\Delta^{0,\epsilon} f(x)$ with $\beta = 0$, is the identity operator

$$\Delta^{0,\epsilon} f(x) = f(x). \tag{4.5.6}$$

It is important to remark here that there are several alternative forms to introduce a generalized difference operator Δ^β from our definition of $\Delta^{\beta,\epsilon}$. The most natural way is the $\Delta^\beta \equiv \Delta^{\beta,0}$, but another interesting possible definition could be $\overline{\Delta}^\beta \equiv \Delta^{\beta,\{\beta\}}$, where $\{\beta\} = \beta - [\beta]$ and $[\beta]$ is the integral part of β. The latter definition will be used below during the development of this chapter.

Definition 4.3. The generalized Stirling functions of the second kind, $S(n, \beta; \epsilon)$, are given for $\beta \in \mathbb{C}$ by

$$S(n, \beta; \epsilon) = \frac{1}{\Gamma(\beta+1)} \lim_{x \to 0} \Delta^{\beta,\epsilon}(x^n) \quad (x \in \mathbb{R};\ n \in \mathbb{N}_0;\ \epsilon \geq 0), \tag{4.5.7}$$

provided this limit exists.

In particular, we will write

$$S(n, \beta) = S(n, \beta; 0). \tag{4.5.8}$$

Theorem 4.29. *The following assertions hold:*

(a) *If $n \in \mathbb{N}_0$ and either of the conditions $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$), $\epsilon > 0$, or $\beta \in \mathbb{C}$ ($\Re e(\beta) > n$, $\beta \neq n+1, n+2, \ldots$), $\epsilon = 0$ hold, then the generalized Stirling functions $S(n, \beta; \epsilon)$ can be represented in the form*

$$S(n, \beta; \epsilon) = \frac{1}{\Gamma(\beta+1)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{(\beta-j)\epsilon} (\beta - j)^n. \tag{4.5.9}$$

(b) *If $n \in \mathbb{N}_0$, $\beta = k \in \mathbb{N}$ and $\epsilon \geq 0$, then the generalized Stirling functions $S(n, k; \epsilon)$ have the representation*

$$\begin{aligned} S(n, k; \epsilon) &= \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} e^{(k-j)\epsilon} (k-j)^n \\ &= \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} e^{j\epsilon} j^n. \end{aligned} \tag{4.5.10}$$

(c) *If $n \in \mathbb{N}_0$, $\beta = 0$ and $\epsilon \geq 0$, then*

$$S(0, 0; \epsilon) = 1 \quad (\epsilon \geq 0), \tag{4.5.11}$$

$$S(n, 0; \epsilon) = 0 \quad (n \in \mathbb{N},\ \epsilon \geq 0). \tag{4.5.12}$$

Proof. When $n \in \mathbb{N}_0$, $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$) and $\epsilon \geq 0$, the $S(n, \beta; \epsilon)$ have the representation (4.5.9). This relation is valid provided that the series on the right-hand side of (4.5.9) is absolutely convergent. Rewrite this series in the form

$$\sum_{j=0}^{\infty} c_j \quad \text{with } c_j = (-1)^j \binom{\beta}{j} e^{(\beta-j)\epsilon} (\beta - j)^n, \tag{4.5.13}$$

and note the estimate

$$\left|\binom{\beta}{j}\right| \leq \frac{A}{j^{\Re e(\beta)+1}} \quad (\beta \in \mathbb{C};\ A > 0) \tag{4.5.14}$$

for any $\beta \in \mathbb{C}$ and sufficiently large $j \in \mathbb{N}$, $A > 0$ being a certain positive constant. For $\beta \neq -1, -2, \ldots$ this estimate follows from relation (1.51) of [501]. If $\beta = -m$ ($m \in \mathbb{N}$), then by (4.5.4)

$$\binom{\beta}{j} = \frac{(-1)^j}{(m-1)!} \frac{\Gamma(m+j)}{\Gamma(j+1)} \sim \frac{(-1)^j}{(m-1)!} \frac{1}{j^{1-m}} \quad (j \to \infty)$$

in accordance with Eq. (1.66) of [501]; this yields (4.5.14) with $\beta = m = -1, -2, \ldots$ for sufficiently large $j \in \mathbb{N}$. Then (4.5.13) and (4.5.14) lead to the estimate

$$|c_j| \leq B \frac{e^{-\epsilon j}}{j^{\mathrm{Re}(\beta)-n+1}}, \quad B = Ae^{\Re e(\beta)\epsilon}, \tag{4.5.15}$$

for sufficiently large $j \in \mathbb{N}$. This implies that the series in (4.5.9) is convergent when $\epsilon > 0$ or when $\epsilon = 0$ and $\Re e(\beta) > n$. Thus assertion (a) of the theorem follows.

When $n \in \mathbb{N}_0$, $\beta = k \in \mathbb{N}$ and $\epsilon \geq 0$, $S(n, k; \epsilon)$ has the representation (4.5.10). Replacing the index of summation j by $k - j$ and taking into account the well known formula

$$\binom{k}{k-j} = \binom{k}{j} \quad (k, j \in \mathbb{N}_0,\ 0 \leq j \leq k)$$

(see, e.g., p. 822 of [5]), we obtain the second formula in (4.5.10), proving (b). This completes the proof of the theorem. □

Corollary 4.9. *When $\epsilon = 0$ and either of the conditions $n \in \mathbb{N}_0$, $\beta \in \mathbb{C}$ ($\Re e(\beta) > n$, $\beta \neq n+1, n+2, \ldots$), or $n \in \mathbb{N}_0$, $\beta = k \in \mathbb{N}$, or $n \in \mathbb{N}$, $\beta = 0$ are satisfied, then there holds the representation*

$$S(n, \beta; 0) = \frac{1}{\Gamma(\beta+1)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} (\beta - j)^n$$

for the generalized Stirling functions $S(n,\beta) \equiv S(n,\beta;0)$. In particular, for the classical Stirling numbers of the second kind $S(n,k)$ there hold the relations

$$\begin{aligned} S(n,k) \equiv S(n,k;0) &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{j} (k-j)^n \\ &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j} \binom{k}{j} j^n \end{aligned} \tag{4.5.16}$$

with $n \in \mathbb{N}_0$, $k \in \mathbb{N}$; and

$$S(n,0) \equiv S(n,0;0) = 0 \quad (n \in \mathbb{N}). \tag{4.5.17}$$

Since $\binom{k}{j} = 0$ for $0 \leq k < j$, relation (4.5.12) can be considered as a particular case of (4.5.10) for $k = 0$. But such an observation is not true for (4.5.11). The next theorem yields a further representation for the $S(n,k;\epsilon)$, different from that in (4.5.10)–(4.5.12).

Theorem 4.30. *For $n \in \mathbb{N}_0$, $\beta = k \in \mathbb{N}_0$ and $\epsilon \geq 0$, the generalized Stirling functions $S(n,k;\epsilon)$ have the representation*

$$S(n,k;\epsilon) = \frac{(-1)^k}{k!}\lim_{x\to 1}\left[\left(x\frac{d}{dx}\right)^n (1-xe^{\epsilon})^k\right]. \tag{4.5.18}$$

Proof. When $n = k = 0$ and $n \in \mathbb{N}$, $k = 0$, (4.5.18) coincides with (4.5.11) and (4.5.12), respectively. Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Using the binomial formula

$$(a+b)^k = \sum_{j=0}^{k}\binom{k}{j} a^j b^{k-j} \quad (a,b \in \mathbb{C};\ k \in \mathbb{N}_0) \tag{4.5.19}$$

with $a = 1$ and $b = -xe^{\epsilon}$, and the relation

$$\left(x\frac{d}{dx}\right)^n x^{\gamma} = \gamma^n x^{\gamma} \quad (\gamma > 0)$$

with $\gamma = j \in \mathbb{N}$, we have

$$\begin{aligned} \frac{(-1)^k}{k!}\left[\left(x\frac{d}{dx}\right)^n (1-xe^{\epsilon})^k\right] &= \frac{(-1)^k}{k!}\left(x\frac{d}{dx}\right)^n \left[\sum_{j=0}^{k}(-1)^j\binom{k}{j} e^{\epsilon j}x^j\right] \\ &= \frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j} e^{\epsilon j}j^n x^j. \end{aligned}$$

Taking the limit $x \to 1$ and noting (4.5.10) we obtain (4.5.18). □

Corollary 4.10. *For $n \in \mathbb{N}_0$ and $\beta = k \in \mathbb{N}_0$, the classical Stirling numbers $S(n,k)$ have the representation*

$$S(n,k) = \frac{(-1)^k}{k!} \lim_{x \to 1} \left[\left(x \frac{d}{dx} \right)^n (1-x)^k \right]. \tag{4.5.20}$$

In particular, if $1 \leq n < k$ then

$$S(n,k) = 0 \quad (n,k \in \mathbb{N};\ 1 \leq n \leq k-1). \tag{4.5.21}$$

Next we consider particular cases of the generalized Stirling functions $S(n,\beta;\epsilon)$. For $n = 0$ there holds the following assertion.

Property 4.1. *Let either of the conditions $\beta \in \mathbb{C}$ $(\beta \notin \mathbb{Z})$, $\epsilon > 0$, or $\beta \in \mathbb{C}$ $(\Re e(\beta) > 0,\ \beta \notin \mathbb{N})$, $\epsilon = 0$ hold. Then*

$$S(0,\beta;\epsilon) = \frac{(e^{\epsilon}-1)^{\beta}}{\Gamma(\beta+1)}. \tag{4.5.22}$$

Proof. Let $\beta \in \mathbb{C}$ $(\beta \notin \mathbb{N}_0)$, $\epsilon > 0$, or $\beta \in \mathbb{C}$ $(\Re e(\beta) > 0,\ \beta \notin \mathbb{N})$, $\epsilon = 0$. Then by Theorem 4.29(a) there holds the representation (4.5.9) with $n = 0$

$$\begin{aligned} S(0,\beta;\epsilon) &= \frac{1}{\Gamma(\beta+1)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{(\beta-j)\epsilon} \\ &= e^{\beta\epsilon} \frac{1}{\Gamma(\beta+1)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{-\epsilon j}. \end{aligned} \tag{4.5.23}$$

Applying the expansion

$$(1+w)^{\beta} = \sum_{j=0}^{\infty} \binom{\beta}{j} w^j \quad (w \in \mathbb{C},\ |w| < 1; \quad \beta \in \mathbb{C}) \tag{4.5.24}$$

with $w = -e^{-\epsilon}$ to (4.5.23), we obtain

$$S(0,\beta;\epsilon) = e^{\beta\epsilon} \frac{1}{\Gamma(\beta+1)} (1-e^{-\epsilon})^{\beta} = \frac{(e^{\epsilon}-1)^{\beta}}{\Gamma(\beta+1)},$$

which proves (4.5.22). □

Property 4.2. *For $k \in \mathbb{N}_0$ and $\epsilon \geq 0$, one has*

$$S(0,k;\epsilon) = \frac{(e^\epsilon - 1)^k}{k!} \quad (k \in \mathbb{N}_0;\ \epsilon \geq 0). \tag{4.5.25}$$

In particular,

$$S(0,1;\epsilon) = e^\epsilon - 1 \quad (\epsilon \geq 0). \tag{4.5.26}$$

Proof. When $k = 0$, (4.5.25) is clear in view of (4.5.11) and since $0! = \Gamma(1) = 1$. When $k \in \mathbb{N}$, then by Theorem 4.29(b),

$$S(0,k;\epsilon) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{j\epsilon} \quad (k \in \mathbb{N},\ \epsilon \geq 0);$$

the binomial formula (4.5.19) applied with $a = e^\epsilon$ and $b = -1$ then yields the result in (4.5.25). From (4.5.25), when $k = 1$, (4.5.26) follows. □

Setting $\epsilon = 0$ in Properties 4.1 and 4.2 we obtain the following result.

Property 4.3. *When $\beta \in \mathbb{C}$ ($\Re e(\beta) > 0$), then*

$$S(0,\beta) \equiv S(0,\beta;0) = 0. \tag{4.5.27}$$

Theorem 4.29(b) also yields the following result.

Property 4.4. *There hold for $\epsilon > 0$ the relations*

$$S(n,1;\epsilon) = e^\epsilon \quad (n \in \mathbb{N}), \tag{4.5.28}$$

$$S(1,2;\epsilon) = e^\epsilon\left(e^\epsilon - 1\right), \quad S(2,2;\epsilon) = e^\epsilon\left(2e^\epsilon - 1\right). \tag{4.5.29}$$

The next property follows from Theorem 4.29(a) if we take into account that the series on the right hand side of (4.5.9) is convergent and the Gamma-function $\Gamma(z)$ has poles of first order at the points $z = 0, -1, -2, \ldots$ (see Section 1.1 of [209]).

Property 4.5. *When $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $\epsilon > 0$, then*

$$S(n,-m;\epsilon) = 0. \tag{4.5.30}$$

We note that if $\epsilon > 0$, then representation (4.5.9) holds for any complex $\beta \in \mathbb{C}$ except when $\beta \in \mathbb{Z}$. Property 4.5 shows that the generalized Stirling functions $S(n, \beta; \epsilon)$ with $\beta = -1, -2, \ldots$ and $\epsilon > 0$ are equal to zero. Now we evaluate the value of $S(n, \beta; \epsilon)$ for $\beta = -1/2$. To formulate the result we need the generalized hypergeometric function ${}_pF_q[a_1, \cdots, a_p; b_1, \cdots, b_q; z]$, defined for $p, q \in \mathbb{N}_0$ and complex $a_i \in \mathbb{C}$ $(1 \le i \le p)$ and $b_j \in \mathbb{C}$ $(1 \le j \le q)$ by (see formula 4.1(1) of [209])

$$ {}_pF_q[a_1, \cdots, a_p; b_1, \cdots, b_q; z] = \sum_{j=0}^{\infty} \frac{(a)_1 ... (a)_p}{(b)_1 ... (b)_q} \frac{z^j}{j!}, \tag{4.5.31} $$

where $(a)_j$ $(j \in \mathbb{N}_0)$ is the Pochhammer symbol

$$ (a)_0 = 1, \ (a)_j = a(a+1) \cdots (a+j-1) \quad (j = 1, 2, \cdots). \tag{4.5.32} $$

Property 4.6. *If $n \in \mathbb{N}_0$ and $\epsilon > 0$, then*

$$ S\left(n, -\frac{1}{2}; \epsilon\right) = \frac{e^{\epsilon/2}}{2^n \sqrt{\pi}} \ {}_{n+1}F_n \left[\frac{1}{2}, \cdots, \frac{1}{2}; -\frac{1}{2}, \cdots, -\frac{1}{2}; e^{-\epsilon}\right]. \tag{4.5.33} $$

Proof. By (4.5.9)

$$ S\left(n, -\frac{1}{2}; \epsilon\right) = \frac{1}{\Gamma(1/2)} \sum_{j=0}^{\infty} (-1)^j \binom{-1/2}{j} e^{(1/2-j)\epsilon} (1/2 - j)^n. \tag{4.5.34} $$

Observing the relation

$$ \binom{\beta}{j} = \frac{(-1)^j (-\beta)_j}{j!} \quad (\beta \in \mathbb{C}; \ j \in \mathbb{N}_0) $$

and $\Gamma(1/2) = \sqrt{\pi}$ (see Eq. 1.2(10) of [209]), we rewrite (4.5.34) as

$$ S\left(n, -\frac{1}{2}; \epsilon\right) = \frac{(-1)^n e^{\epsilon/2}}{\sqrt{\pi}} \sum_{j=0}^{\infty} (1/2)_j \left(j - \frac{1}{2}\right)^n \frac{e^{-j\epsilon}}{j!}. \tag{4.5.35} $$

But it is clear that

$$ j - \frac{1}{2} = -\frac{(1/2)_j}{2(-1/2)_j} \quad (j \in \mathbb{N}_0). $$

Substituting the latter into (4.5.35) yields (4.5.33) in accordance with (4.5.31). $\square$

The next result deals with the asymptotic behavior of $S(n,k;\epsilon)$ when $n \to \infty$.

Property 4.7. *For $k \in \mathbb{N}$ and $\epsilon \geq 0$ there holds the asymptotic estimate*

$$\lim_{n\to\infty}[k^{-n}S(n,k;\epsilon)] = \frac{1}{k!}e^{\epsilon k}. \tag{4.5.36}$$

Proof. By (4.5.10) we have

$$k^{-n}S(n,k,\epsilon) = \frac{1}{k!}e^{\epsilon k} + \sum_{j=0}^{k-1}(-1)^{k-j}\binom{k}{j}e^{j\epsilon}\left(\frac{j}{k}\right)^n;$$

taking the limit $n \to \infty$, we obtain (4.5.36). □

Observe that for $\epsilon = 0$ the estimate (4.5.36) takes the form

$$\lim_{n\to\infty}[k^{-n}S(n,k)] = \frac{1}{k!},$$

which coincides with the known asymptotic estimate for $S(\alpha,k)$ when $\alpha = n \to +\infty$ given in Eq. (8.3) of [120].

4.5.2 *Main properties*

In this section we present the main properties of the generalized Stirling functions $S(n,\beta;\epsilon)$. First we establish the recurrence relation.

Theorem 4.31. *Let $n \in \mathbb{N}$, $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$) and either $\epsilon > 0$ or $\epsilon = 0$, $\mathfrak{Re}(\beta) > n$. The generalized Stirling functions $S(n,\beta;\epsilon)$ satisfy the recurrence relation*

$$S(n,\beta;\epsilon) = \beta S(n-1,\beta;\epsilon) + S(n-1,\beta-1;\epsilon). \tag{4.5.37}$$

Proof. In accordance with Theorem 4.29(a), all terms on the both sides of (4.5.37) are defined for the given range of parameters n, β and ϵ. Noting

(4.5.9),

$$\begin{aligned}
&\beta S(n-1,\beta;\epsilon)+S(n-1,\beta-1;\epsilon)\\
&=\frac{\beta}{\Gamma(\beta+1)}\sum_{j=0}^{\infty}(-1)^j\binom{\beta}{j}e^{(\beta-j)\epsilon}(\beta-j)^{n-1}\\
&\quad+\frac{1}{\Gamma(\beta)}\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}e^{(\beta-1-j)\epsilon}(\beta-1-j)^{n-1}\\
&=\frac{\beta}{\Gamma(\beta+1)}e^{\beta\epsilon}\beta^{n-1}+\frac{\beta}{\Gamma(\beta+1)}\sum_{j=1}^{\infty}(-1)^j\binom{\beta}{j}e^{(\beta-j)\epsilon}(\beta-j)^{n-1}\\
&\quad+\frac{\beta}{\Gamma(\beta+1)}\sum_{j=1}^{\infty}(-1)^{j+1}\binom{\beta-1}{j-1}e^{(\beta-j)\epsilon}(\beta-j)^{n-1}\\
&=\frac{\beta}{\Gamma(\beta+1)}e^{\beta\epsilon}\beta^{n-1}\\
&\quad+\frac{\beta}{\Gamma(\beta+1)}\sum_{j=1}^{\infty}(-1)^j\left[\binom{\beta}{j}-\binom{\beta-1}{j-1}\right]e^{(\beta-j)\epsilon}(\beta-j)^{n-1}.
\end{aligned}$$

Applying the binomial relation

$$\binom{\beta}{j}-\binom{\beta-1}{j-1}=\binom{\beta-1}{j}\quad(\beta\in\mathbb{C};\ j\in\mathbb{N})$$

we can rewrite the last relation in the form

$$\begin{aligned}
&\beta S(n-1,\beta;\epsilon)+S(n-1,\beta-1;\epsilon)\\
&\frac{\beta^n}{\Gamma(\beta+1)}e^{\beta\epsilon}+\sum_{j=1}^{\infty}(-1)^j\left[\beta\binom{\beta-1}{j}\frac{1}{\beta-j}\right]e^{(\beta-j)\epsilon}(\beta-j)^n. \qquad (4.5.38)
\end{aligned}$$

It is directly verified that

$$\beta\binom{\beta-1}{j}\frac{1}{\beta-j}=\binom{\beta}{j},$$

so that (4.5.38) takes the form

$$\begin{aligned}
&\beta S(n-1,\beta;\epsilon)+S(n-1,\beta-1;\epsilon)\\
&\qquad=\frac{\beta^n}{\Gamma(\beta+1)}e^{\beta\epsilon}+\sum_{j=1}^{\infty}(-1)^j\binom{\beta}{j}e^{(\beta-j)\epsilon}(\beta-j)^n,
\end{aligned}$$

which yields (4.5.37) if we take (4.5.9) into account. This completes the proof of the theorem. □

Corollary 4.11. *Let $n \in \mathbb{N}$ and $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$) be such that $\mathrm{Re}(\beta) > n$. Then the generalized Stirling functions $S(n,\beta) \equiv S(n,\beta;0)$ satisfy the recurrence relation*

$$S(n,\beta) = \beta S(n-1,\beta) + S(n-1,\beta-1). \tag{4.5.39}$$

Theorem 4.31 is also true for $\beta = k \in \mathbb{N}$.

Corollary 4.12. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ and $\epsilon \geq 0$. The generalized Stirling functions $S(n,k;\epsilon)$ satisfy the recurrence relation*

$$S(n,k;\epsilon) = kS(n-1,k;\epsilon) + S(n-1,k-1;\epsilon). \tag{4.5.40}$$

Proof. When $k > 1$, relation (4.5.40) is proved on the basis of the representation (4.5.10) similarly to the proof of formula (4.5.37) in Theorem 4.29. When $k = 1$, (4.5.40) takes the form

$$S(n,1;\epsilon) = S(n-1,1;\epsilon) + S(n-1,0;\epsilon). \tag{4.5.41}$$

If $n = 1$, then according to (4.5.28), (4.5.26) and (4.5.11),

$$S(1,1;\epsilon) = e^{\epsilon}, \quad S(0,1;\epsilon) = e^{\epsilon} - 1, \quad S(0,0;\epsilon) = 1$$

and so (4.5.41) holds for $n = 1$. When $n > 1$, (4.5.41) is also valid since in accordance with (4.5.28) and (4.5.12),

$$S(n,1;\epsilon) = S(n-1,1;\epsilon) = e^{\epsilon}, \; S(n-1,0;\epsilon) = 0.$$

Thus the proof is complete. □

Corollary 4.13. *For $n, k \in \mathbb{N}$, the classical Stirling numbers $S(n,k)$ satisfy the recurrence relation*

$$S(n,k) = kS(n-1,k) + S(n-1,k-1). \tag{4.5.42}$$

Now we construct the exponential generating function for $S(n,\beta;\epsilon)$.

Theorem 4.32. *Let $z \in \mathbb{C}$, $\beta \in \mathbb{C}$ and $\epsilon \geq 0$. The generating function for the generalized Stirling functions $S(n,\beta;\epsilon)$ is given by*

$$\left(e^{z+\epsilon} - 1\right)^{\beta} = \Gamma(\beta+1) \sum_{n=0}^{\infty} S(n,\beta;\epsilon) \frac{z^n}{n!} \tag{4.5.43}$$

for $\beta \notin \mathbb{Z}$ and $\epsilon > 0$, and by

$$\left(e^{z+\epsilon} - 1\right)^k = k! \sum_{n=0}^{\infty} S(n, k; \epsilon) \frac{z^n}{n!}. \tag{4.5.44}$$

for $\beta = k \in \mathbb{N}_0$ and $\epsilon \geq 0$.

Proof. If $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$) and $\epsilon > 0$, the generalized Stirling functions $S(n, \beta; \epsilon)$ are given for $n \in \mathbb{N}_0$ by the representation (4.5.9) noting Theorem 4.29(a). Substitution of this representation into the right-hand side of (4.5.43) yields

$$\Gamma(\beta + 1) \sum_{n=0}^{\infty} S(n, \beta; \epsilon) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{(\beta - j)\epsilon} (\beta - j)^n \right] \frac{z^n}{n!}. \tag{4.5.45}$$

On the other hand, using (4.5.24) with $w = -e^{-(z+\epsilon)}$ and the known expansion for the exponential function, we have

$$\begin{aligned}
\left(e^{z+\epsilon} - 1\right)^\beta &= e^{\beta(z+\epsilon)} \left[1 - e^{-(z+\epsilon)}\right]^\beta \\
&= e^{\beta(z+\epsilon)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{-j(z+\epsilon)} \\
&= \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{(\beta - j)\epsilon} e^{z(\beta - j)} \\
&= \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{(\beta - j)\epsilon} \left[\sum_{n=0}^{\infty} \frac{[z(\beta - j)]^n}{n!} \right].
\end{aligned} \tag{4.5.46}$$

But the series

$$e^{\beta(z+\epsilon)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{-j(z+\epsilon)}$$

is absolutely convergent for any fixed $z \in \mathbb{C}$ because, according to (4.5.14), it is majorized by the convergent series

$$1 + A e^{\beta[\Re e(z)+\epsilon]} \sum_{j=1}^{\infty} \frac{1}{j^{\Re e(\beta)+1}} e^{-j[\mathrm{Re}(z)+\epsilon]}.$$

Hence the double series

$$\sum_{j=0}^{\infty}(-1)^j \binom{\beta}{j} e^{(\beta-j)\epsilon}\left[\sum_{n=0}^{\infty}\frac{[z(\beta-j)]^n}{n!}\right]$$

is also absolutely convergent, and, by an analogue of Fubini's theorem (see, e.g., Theorem 12-42 of [37]), we can interchange the order of summation in this double series. Performing such an interchange and taking (4.5.46) and (4.5.45) into account, we deduce (4.5.43).

If $\beta = k \in \mathbb{N}_0$ and $\epsilon \geq 0$, then $S(n,k,\epsilon) = O\left(k^n\right)$ $(n \to \infty)$ by (4.5.36), and hence in accordance with the Cauchy-Hadamard relation the series on the right hand side of (4.5.44) has radius of convergence $R = +\infty$, noting $\sqrt[n]{[|k|^n/n!]} \to 0$ $(n \to \infty)$. Thus relation (4.5.44) is proved similarly to (4.5.43) by applying (4.5.10) and (4.5.19) and using Theorem 12-42 in [37]:

$$\begin{aligned}\left(e^{z+\epsilon}-1\right)^k &= \sum_{j=0}^{k}(-1)^j\binom{k}{j}e^{j(z+\epsilon)} = \sum_{j=0}^{k}(-1)^j\binom{k}{j}e^{j\epsilon}\left[\sum_{n=0}^{\infty}\frac{(zj)^n}{n!}\right]\\ &= \sum_{n=0}^{\infty}\left[\sum_{j=0}^{k}(-1)^j\binom{k}{j}e^{j\epsilon}j^n\right]\frac{z^n}{n!} = k!\sum_{n=0}^{\infty}S(n,k;\epsilon)\frac{z^n}{n!}.\end{aligned}$$

□

Corollary 4.14. *For $z \in \mathbb{C}$ and $k \in \mathbb{N}_0$,*

$$(e^z-1)^k = k!\sum_{n=k}^{\infty}S(n,k)\frac{z^n}{n!}. \tag{4.5.47}$$

Proof. Let $\epsilon = 0$. Then from (4.5.44) and taking (4.5.21) into account, we obtain (4.5.47). □

From Theorem 4.32 we deduce a further property of the $S(n,\beta;\epsilon)$.

Property 4.8. *Let $n \in \mathbb{N}_0$, $\beta \in \mathbb{C}$ and $\epsilon \geq 0$.*

(a) *For $\beta \notin \mathbb{Z}$ and $\epsilon > 0$, the generalized Stirling functions $S(n,\beta;\epsilon)$ can be represented in the form* (4.5.9) *if and only if they are given by*

$$S(n,\beta;\epsilon) = \frac{1}{\Gamma(\beta+1)}\lim_{x\to+0}\left[\left(\frac{d}{dx}\right)^n\left(e^{x+\epsilon}-1\right)^\beta\right]. \tag{4.5.48}$$

(b) *For* $\beta = k \in \mathbb{N}$ *and* $\epsilon \geq 0$*, the* $S(n,k;\epsilon)$ *are representable by* (4.5.10) *if and only if*

$$S(n,k;\epsilon) = \frac{1}{k!}\lim_{x\to 0}\left[\left(\frac{d}{dx}\right)^n (e^{x+\epsilon}-1)^k\right].$$

In particular, $S(n,k)$ *are representable by* (4.5.16) *if and only if*

$$S(n,k) = \frac{1}{k!}\lim_{x\to 0}\left[\left(\frac{d}{dx}\right)^n (e^{x}-1)^k\right] \quad (n \in \mathbb{N}_0;\ k \in \mathbb{N}). \tag{4.5.49}$$

Proof. Let $n \in \mathbb{N}_0$, $\beta \notin \mathbb{Z}$ and $\epsilon > 0$ and let $S(n,\beta;\epsilon)$ be representable by (4.5.9). Then by Theorem 4.14 there holds the representation (4.5.43) for $z = x$ near zero, and hence by Taylor's formula we obtain (4.5.48). Conversely, let (4.5.48) hold. Substituting (4.5.24) with $w = e^{-(x+\epsilon)}$ into (4.5.48) we have

$$\begin{aligned} S(n,\beta;\epsilon) &= \frac{1}{\Gamma(\beta+1)}\lim_{x\to +0}\left[\left(\frac{d}{dx}\right)^n \sum_{j=0}^{\infty}(-1)^j\binom{\beta}{j}e^{(x+\epsilon)(\beta-j)}\right] \\ &= \frac{1}{\Gamma(\beta+1)}\lim_{x\to +0}\left[\sum_{j=0}^{\infty}(-1)^j\binom{\beta}{j}e^{(x+\epsilon)(\beta-j)}(\beta-j)^n\right] \\ &= \frac{1}{\Gamma(\beta+1)}\sum_{j=0}^{\infty}(-1)^j\binom{\beta}{j}e^{\epsilon(\beta-j)}(\beta-j)^n, \end{aligned}$$

which yields (4.5.9). The assertion (b) is proved similarly. □

Remark 4.4. The results presented in this section are true for the generalized Stirling functions $S(n,\beta;\epsilon)$ with $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$) and for $S(n,k;\epsilon)$ with $k \in \mathbb{N}_0$, and hold for any $\epsilon > 0$ or any $\epsilon \geq 0$. In particular, we can take $\epsilon = \{\beta\}$, where $\{\beta\}$ is the fractional part of β, i.e.,

$$\{\beta\} = \beta - [\beta],$$

$[\beta]$ being the integral part of β. For such ϵ we can introduce the generalized Stirling functions

$$S^*(n,\beta) \equiv S(n,\beta;\{\beta\}). \tag{4.5.50}$$

All results in Sections 4.5.1 and 4.5.2 including Theorems 4.29–4.32 can be reformulated for such generalized Stirling functions.

Remark 4.5. In Sections 4.5.1 and 4.5.2 we have considered properties $S(n,\beta)$ with $n \in \mathbb{N}_0$ and $\beta \in \mathbb{C}$ ($\Re e(\beta) > n$) or $\beta = k \in \mathbb{N}_0$. Relation (4.5.42) can be used to define $S(n,\beta)$ for negative integers $\beta \in \mathbb{Z}_- = \{-1,-2,\ldots\}$. We replace n by $n+1$ in (4.5.42) and rewrite it in the form

$$S(n,k-1) = S(n+1,k) - kS(n,k) \quad (n \in \mathbb{N};\ k \in \mathbb{N}). \tag{4.5.51}$$

The right hand side of (4.5.51) is defined for $k = 0$ and therefore we can define $S(n,-1)$, which in accordance with (4.5.17) yields

$$S(n,-1) = 0 \quad (n \in \mathbb{N}_0). \tag{4.5.52}$$

Now by (4.5.52) the right hand side of (4.5.51) is defined for $k = -1$, and hence we can define $S(n,-2)$ by (4.5.51) and obtain

$$S(n,-2) = 0 \quad (n \in \mathbb{N}_0).$$

Continuation of this process yields zero values for $S(n,\beta)$ when $\beta = -1,-2,\ldots$, i.e.

$$S(n,-m) = 0 \quad (n \in \mathbb{N}_0;\ m \in \mathbb{N}). \tag{4.5.53}$$

This result could be formally obtained directly from definition (4.5.7) if we take into account that the Gamma function $\Gamma(z)$ has poles of first order at points $z = 0,-1,-2,\ldots$. In this connection see also Property 4.5.

Remark 4.6. If $\epsilon > 0$, then relation(4.5.37) remains true for $n \in \mathbb{N}$ and $\beta = -m$ ($m \in \mathbb{N}$) by (4.5.30), because the left and right hand sides of (4.5.37) for such values of n, β and ϵ are equal to zero. Similarly, when $\epsilon = 0$, then in accordance with (4.5.17) and (4.5.53) the recurrence relation (4.5.39) remains true for $n \in \mathbb{N}$ and $\beta = -m$ ($m \in \mathbb{N}$).

4.6 The Generalized Stirling Functions $S(\alpha,\beta;\epsilon)$ and $S(\alpha,\beta)$

Observing Definition 4.3, one can extend it to the instance when the first parameter $n \in \mathbb{N}_0$ is replaced by any $\alpha \in \mathbb{C}$. Indeed,

Definition 4.4. The generalized Stirling functions of the second kind, $S(\alpha, \beta; \epsilon)$, for $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{N}_0$) and $\alpha \in \mathbb{C}$ ($\Re e(\alpha) > 0$), $\beta = k \in \mathbb{N}$ are given by

$$S(\alpha, \beta; \epsilon) = \frac{1}{\Gamma(\beta+1)} \lim_{x \to 0} \Delta^{\beta,\epsilon}(x^{\alpha}) \quad (\epsilon \geq 0). \tag{4.6.1}$$

When $\alpha \in \mathbb{C}$ ($\Re e(\alpha) \leq 0;\ \alpha \neq 0$) and $\beta = k \in \mathbb{N}_0$ then

$$S(\alpha, k; \epsilon) = \frac{1}{k!} \lim_{x \to 0} \Delta_*^{k,\epsilon}(x^{\alpha}) \quad (\epsilon \geq 0) \tag{4.6.2}$$

with the "cut" finite difference defined (the term $j = 0$ is missing) by

$$\Delta_*^{k,\epsilon} f(x) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} e^{j\epsilon} f(x+j) \quad (x \in \mathbb{R};\ k \in \mathbb{N};\ \epsilon \geq 0), \tag{4.6.3}$$

$$\Delta_*^{0,\epsilon} f(x) \equiv 0 \quad (x \in \mathbb{R};\ \epsilon \geq 0). \tag{4.6.4}$$

4.6.1 *Basic properties*

In this section we present properties of the new generalized Stirling functions $S(\alpha, \beta; \epsilon)$ with complex $\alpha, \beta \in \mathbb{C}$ and $\epsilon \geq 0$.

When $\alpha \in \mathbb{C}$ ($\Re e(\alpha) > 0$), $\beta = k \in \mathbb{N}$, (4.6.1) takes the form

$$S(\alpha, k; \epsilon) = \frac{1}{k!} \lim_{x \to 0} \Delta^{k,\epsilon}(x^{\alpha}) \quad (\epsilon \geq 0), \tag{4.6.5}$$

where $\Delta^{k,\epsilon}$ is given by (4.5.5). Explicit representations of $S(\alpha, \beta; \epsilon)$ are now given by the following result

Theorem 4.33. *The following three assertions hold:*

(a) *If $\alpha \in \mathbb{C}$ and either of the conditions $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$), $\epsilon > 0$, or $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$, $\Re e(\beta) > \Re e(\alpha)$), $\epsilon = 0$ hold, then the generalized Stirling functions $S(\alpha, \beta; \epsilon)$ can be represented in the form*

$$S(\alpha, \beta; \epsilon) = \frac{1}{\Gamma(\beta+1)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{(\beta-j)\epsilon} (\beta - j)^{\alpha}. \tag{4.6.6}$$

(b) *If $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$, $\beta = k \in \mathbb{N}$ and $\epsilon \geq 0$, then functions $S(\alpha, k; \epsilon)$ have the representation*

$$S(\alpha, k; \epsilon) = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} e^{j\epsilon} j^{\alpha}. \tag{4.6.7}$$

(c) *If $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$ and $\epsilon \geq 0$, then*

$$S(\alpha, 0; \epsilon) = 0. \tag{4.6.8}$$

Proof. The proof of (4.6.6) is the same as in Theorem 4.29, replacing n by α and using the estimate of the form (4.5.15)

$$|c_j| \leq B \frac{e^{-\epsilon j}}{j^{\Re e(\beta - \alpha)+1}}, \;\; B = Ae^{\beta},$$

according to which the series on the right-hand side of (4.6.6) is convergent when either $\epsilon > 0$ or $\epsilon = 0$, $\Re e(\beta) > \Re e(\alpha)$.

Formula (4.6.7) is proved by using (4.6.5), (4.5.5) and (4.6.2), (4.6.3) in the cases $\Re e(\alpha) > 0$ and $\Re e(\alpha) \leq 0$ $(\alpha \neq 0)$, respectively.

When $\Re e(\alpha) > 0$, (4.6.8) follows from (4.6.5) and (4.5.6), while (4.6.2) with $k = 0$ and (4.6.4) yield (4.6.8) for $\Re e(\alpha) \leq 0$ $(\alpha \neq 0)$. Thus the theorem is proved. □

Corollary 4.15. *In particular, there hold the following assertions for the function $S(\alpha, \beta) \equiv S(\alpha, \beta, 0)$:*

(a) *Let $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ be such that $\beta \notin \mathbb{Z}$ and $\mathrm{Re}(\beta) > \Re e(\alpha)$. The generalized Stirling functions $S(\alpha, \beta)$ have the representation*

$$S(\alpha, \beta) = \frac{1}{\Gamma(\beta + 1)} \sum_{j=0}^{\infty} (-1)^{j} \binom{\beta}{j} (\beta - j)^{\alpha}. \tag{4.6.9}$$

(b) *For $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$ and $k \in \mathbb{N}$,*

$$S(\alpha, k) = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^{\alpha}. \tag{4.6.10}$$

(c) *For $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$,*

$$S(\alpha, 0) = 0. \tag{4.6.11}$$

The main properties obtained in Section 4.4.1 for $S(\alpha, k)$ can be extended to the generalized Stirling functions $S(\alpha, k; \epsilon)$ with any nonnegative $k \in \mathbb{N}_0$ and $\epsilon \geq 0$.

It should be noted that (4.6.6) gives the explicit representation for the generalized Stirling functions $S(\alpha, \beta; \epsilon)$ with any complex $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ except for the case $\beta \in \mathbb{Z}$. In particular, α and β can be purely complex numbers. With $\alpha = n + i\phi$ $(n \in \mathbb{Z},\ \phi \in \mathbb{R},\ \phi \neq 0)$ and $\beta = k + i\theta$ $(k \in \mathbb{Z},\ \theta \in \mathbb{R},\ \theta \neq 0)$, Theorem 4.33 yields the following properties.

Property 4.9. *Let $n, k \in \mathbb{Z}$ and $\phi, \theta \in \mathbb{R}$ $(\phi \neq 0,\ \theta \neq 0)$.*

(a) *For $\epsilon > 0$, the generalized Stirling functions $S(n + i\phi, k + i\theta; \epsilon)$ have the representation*

$$\begin{aligned} &S(n + i\phi, k + i\theta; \epsilon) \\ &= \frac{1}{\Gamma(k + 1 + i\theta)} \sum_{j=0}^{\infty} (-1)^j \binom{k + i\theta}{j} e^{(k-j+i\theta)\epsilon} (k - j + i\theta)^{n+i\phi}. \end{aligned} \tag{4.6.12}$$

(b) *For $n < k$, the generalized Stirling functions $S(n + i\phi, k + i\theta)$ are given by*

$$S(n + i\phi, k + i\theta) = \frac{1}{\Gamma(k + 1 + i\theta)} \sum_{j=0}^{\infty} (-1)^j \binom{k + i\theta}{j} (k - j + i\theta)^{n+i\phi}. \tag{4.6.13}$$

Property 4.10. *Let $n \in \mathbb{Z}$ and $\phi \in \mathbb{R}$ $(\phi \neq 0)$. There hold the following three results.*

(a) *For $\beta \in \mathbb{C}$ and $\epsilon > 0$, the generalized Stirling functions $S(n + i\phi, \beta; \epsilon)$ have the representation*

$$S(n+i\phi, \beta; \epsilon) = \frac{1}{\Gamma(\beta + 1)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{(\beta-j)\epsilon} (\beta - j)^{n+i\phi}. \tag{4.6.14}$$

In particular, when $n = 0$

$$S(i\phi, \beta; \epsilon) = \frac{1}{\Gamma(\beta + 1)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{(\beta-j)\epsilon} (\beta - j)^{i\phi}. \tag{4.6.15}$$

(b) *Let $\beta \in \mathbb{C}$ be such that $\Re\mathrm{e}(\beta) > n$. The generalized Stirling functions $S(n+i\phi, \beta)$ are given by*

$$S(n+i\phi, \beta) = \frac{1}{\Gamma(\beta+1)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} (\beta - j)^{n+i\phi}. \tag{4.6.16}$$

In particular, when $n = 0$, with $\Re\mathrm{e}(\beta) > 0$,

$$S(i\phi, \beta) = \frac{1}{\Gamma(\beta+1)} \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} (\beta - j)^{i\phi}. \tag{4.6.17}$$

(c) *For $k \in \mathbb{N}$, the generalized Stirling functions $S(n+i\phi, k)$ have the representation*

$$S(n+i\phi, k) = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^{n+i\phi}. \tag{4.6.18}$$

In particular, when $n = 0$,

$$S(i\phi, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^{i\phi}. \tag{4.6.19}$$

Property 4.11. *Let $k \in \mathbb{Z}$ and $\theta \in \mathbb{R}$ $(\theta \neq 0)$.*

(a) *For $\alpha \in \mathbb{C}$ and $\epsilon > 0$, the generalized Stirling functions $S(\alpha, k+i\theta; \epsilon)$ are given by*

$$S(\alpha, k+i\theta; \epsilon) = \frac{1}{\Gamma(i\theta+1)} \sum_{j=0}^{\infty} (-1)^j \binom{k+i\theta}{j} e^{(k-j+i\theta)\epsilon} (k-j+i\theta)^{\alpha}. \tag{4.6.20}$$

In particular, when $k = 0$,

$$S(\alpha, i\theta; \epsilon) = \frac{1}{\Gamma(i\theta+1)} \sum_{j=0}^{\infty} (-1)^j \binom{i\theta}{j} e^{(i\theta-j)\epsilon} (i\theta - j)^{\alpha}. \tag{4.6.21}$$

(b) *Let $\alpha \in \mathbb{C}$ be such that $\Re\mathrm{e}(\alpha) < k$. Then the generalized Stirling functions $S(\alpha, k+i\theta)$ are given by*

$$S(\alpha, k+i\theta) = \frac{1}{\Gamma(i\theta+1)} \sum_{j=0}^{\infty} (-1)^j \binom{k+i\theta}{j} (k-j+i\theta)^{\alpha}. \tag{4.6.22}$$

In particular, when $k = 0$, with $\mathfrak{Re}(\alpha) < 0$,

$$S(\alpha, i\theta) = \frac{1}{\Gamma(i\theta + 1)} \sum_{j=0}^{\infty} (-1)^j \binom{i\theta}{j} (i\theta - j)^{\alpha}. \tag{4.6.23}$$

According to Property 4.3, formulas (4.6.20)–(4.6.21) take on a simple form in the case $\alpha = 0$.

Property 4.12. *Let $k \in \mathbb{Z}$ and $\theta \in \mathbb{R}$ $(\theta \neq 0)$. For $\epsilon > 0$ there holds the relation*

$$S(0, k + i\theta; \epsilon) = \frac{(e^{\epsilon} - 1)^{k+i\theta}}{\Gamma(k + 1 + i\theta)}. \tag{4.6.24}$$

The next formulas follow from Theorem 4.33(b).

Property 4.13. *If $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$ and $\epsilon \geq 0$, then*

$$S(\alpha, 1; \epsilon) = e^{\epsilon}; \quad S(\alpha, 2; \epsilon) = e^{\epsilon}(2^{\alpha-1} e^{\epsilon} - 1), \tag{4.6.25}$$

$$S(\alpha, 3; \epsilon) = \frac{1}{6} \left[3e^{\epsilon} - 3e^{2\epsilon} 2^{\alpha} + e^{3\epsilon} 3^{\alpha} \right].$$

In particular,

$$S(\alpha, 1) = 1,\ S(\alpha, 2) = 2^{\alpha-1} - 1; \quad S(\alpha, 3) = \frac{1}{2} \left[1 - 2^{\alpha} + 3^{\alpha-1} \right].$$

The next property follows from Theorem 4.33 if we take into account that the series on the right hand side of (4.6.6) is convergent and the Gamma-function $\Gamma(z)$ has poles of the first order at $z = 0, -1, -2, \ldots$:

Property 4.14. *When $\alpha \in \mathbb{C}$, $\beta = -1, -2, \cdots$ and $\epsilon > 0$,*

$$S(\alpha, -m; \epsilon) = 0 \quad (m \in \mathbb{N}).$$

The result to follow presents recurrence relations for the generalized Stirling functions $S(\alpha, \beta; \epsilon)$.

Theorem 4.34. *There hold the following three results.*

(a) *For $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$) and $\epsilon > 0$, the generalized Stirling functions $S(\alpha, \beta; \epsilon)$ satisfy the recurrence relation*

$$S(\alpha, \beta; \epsilon) = \beta S(\alpha - 1, \beta; \epsilon) + S(\alpha - 1, \beta - 1; \epsilon). \tag{4.6.26}$$

(b) *Let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$ be such that $\beta \notin \mathbb{Z}$ and $\mathfrak{Re}(\beta) > \mathfrak{Re}(\alpha)$. The generalized Stirling functions $S(\alpha, \beta) \equiv S(\alpha, \beta; 0)$ satisfy the recurrence relation*

$$S(\alpha, \beta) = \beta S(\alpha - 1, \beta) + S(\alpha - 1, \beta - 1). \tag{4.6.27}$$

(c) *For $\alpha \in \mathbb{C}$, $k \in \mathbb{N}$ and $\epsilon \geq 0$, the generalized Stirling functions $S(\alpha, k; \epsilon)$ satisfy the recurrence relation*

$$S(\alpha, k; \epsilon) = kS(\alpha - 1, k; \epsilon) + S(\alpha - 1, k - 1; \epsilon). \tag{4.6.28}$$

In particular,

$$S(\alpha, k) = kS(\alpha - 1, k) + S(\alpha - 1, k - 1). \tag{4.6.29}$$

Proof. When $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$), $\epsilon > 0$ and $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$, $\mathfrak{Re}(\beta) > \mathfrak{Re}(\alpha)$), $\epsilon = 0$, then the left and right hand sides of (4.6.26) and (4.6.27) are defined in accordance with Theorem 4.33(a). These relations are proved similarly to the proof of formula (4.5.37) in Theorem 4.31 with $n \in \mathbb{N}$ being replaced by $\alpha \in \mathbb{C}$.

When $\alpha \in \mathbb{C}$, $k \in \mathbb{N}_2 = \{2, 3, \ldots\}$ and $\epsilon \geq 0$, formula (4.6.28) is proved similarly to the proof of this relation with $\epsilon = 0$ in 1.3.137. When $k = 1$, (4.6.28) takes the form

$$S(\alpha, 1; \epsilon) = S(\alpha - 1, 1; \epsilon) + S(\alpha - 1, 0; \epsilon), \tag{4.6.30}$$

When $\alpha \neq 0, 1$, (2.4.33) is true since $S(\alpha, 1; \epsilon) = S(\alpha - 1, 1; \epsilon) = e^{\epsilon}$ by (4.6.25), while $S(\alpha - 1, 0; \epsilon) = 0$ in accordance with (4.6.8). When $\alpha = 0$ and $\alpha = 1$, (4.6.30) has the form

$$S(0, 1; \epsilon) = S(-1, 1; \epsilon) + S(-1, 0; \epsilon) \tag{4.6.31}$$

and

$$S(1, 1; \epsilon) = S(0, 1; \epsilon) + S(0, 0; \epsilon), \tag{4.6.32}$$

respectively. Equation (4.6.31) is valid since $S(0,1;\epsilon) = S(-1,1;\epsilon) = e^{\epsilon}$ by (4.6.25) and $S(-1,0;\epsilon) = 0$ in accordance with (4.6.8). (4.6.32) is also valid because $S(1,1;\epsilon) = e^{\epsilon}$, $S(0,1;\epsilon) = e^{\epsilon} - 1$ and $S(0,0;\epsilon) = 1$ according to (4.6.25), (4.5.26) and (4.5.11), respectively. Equation (4.6.28) with $\epsilon = 0$ yields (4.6.29). This completes the proof of the theorem. □

Remark 4.7. In this section we have considered properties of the generalized Stirling functions $S(\alpha, \beta)$ with $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$, $\mathfrak{Re}(\beta) > \mathfrak{Re}(\alpha)$) and $\beta = k \in \mathbb{N}_0$. Relation (4.6.29) can be used for the definition of $S(\alpha, \beta)$ for $\beta \in \mathbb{Z}_- = \{-1, -2, \ldots\}$. We replace α by $\alpha + 1$ in (4.6.29) and rewrite it in the form

$$S(\alpha, k-1) = S(\alpha+1, k) - kS(\alpha, k) \quad (\alpha \in \mathbb{C};\ k \in \mathbb{N}). \tag{4.6.33}$$

The right-hand side of (4.6.33) is defined for $k = 0$ and therefore we can define $S(\alpha, -1)$ by (4.6.33) with $k = 0$ by

$$S(\alpha, -1) = S(\alpha+1, 0), \tag{4.6.34}$$

and so

$$S(\alpha, -1) = 0 \quad (\alpha \neq -1),\ S(-1,-1) = 1. \tag{4.6.35}$$

By (4.6.35) the right hand side of (4.6.33) is defined for $k = -1$, and so we can define $S(\alpha, -2)$ by (4.6.33) with $k = -1$ by

$$S(\alpha, -2) = S(\alpha+1, -1) + S(\alpha, -1),$$

and obtain

$$S(\alpha, -2) = 0 \quad (\alpha \neq -2, -1),\ S(-2,-2) = S(-1,-2) = 1.$$

This process can be continued to obtain the values of the generalized Stirling functions $S(\alpha, \beta)$ for $\beta = -m$ ($m \in \mathbb{N}$) by the recurrence relation

$$S(\alpha, -m) = S(\alpha+1, 1-m) + (m-1)S(\alpha, 1-m) \quad (\alpha \in \mathbb{C};\ m \in \mathbb{N}).$$

In this way we obtain the relations

$$S(\alpha, -m) = 0 \quad (\alpha \in \mathbb{C};\ \alpha \neq -m, -(m-1), \cdots, -1;\ m \in \mathbb{N}) \tag{4.6.36}$$

and

$$S(\alpha,-m)\neq 0 \quad (\alpha\in\mathbb{C};\ \alpha=-m,-(m-1),\cdots,-1;\ m\in\mathbb{N}). \quad (4.6.37)$$

In particular, the induction yields the formula

$$S(-m,-m)=1 \quad (m\in\mathbb{N}),$$

and (4.5.53) follows from (4.6.36) when $\alpha=n\in\mathbb{N}$.

4.6.2 *Representations by Liouville fractional operators*

Representations of the generalized Stirling functions $S(\alpha,k)$ in terms of the Liouville fractional differentiation operators ${}^{\mathtt{L}}D_+^{\alpha}$, ${}^{\mathtt{L}}D_-^{\alpha}$ and fractional integration operators ${}^{\mathtt{L}}I_+^{-\alpha}$, $I_-^{-\alpha}$ in the cases $\Re e(\alpha)>0$ and $\Re e(\alpha)<0$, respectively, were established in Sections 4.4.1 and 4.4.2. In this section we give representations of the generalized Stirling functions $S(\alpha,\beta;\epsilon)$ in terms of such Liouville fractional operators.

First we consider the case $\Re e(\alpha)>0$ and establish representations of $S(\alpha,\beta;\epsilon)$ in terms of the Liouville fractional differentiation operators ${}^{\mathtt{L}}D_+^{\alpha}$ and ${}^{\mathtt{L}}D_-^{\alpha}$.

Theorem 4.35. *Let $\alpha\in\mathbb{C}$ and $\beta\in\mathbb{C}$ $(\beta\notin\mathbb{Z})$ be such that $\Re e(\alpha)>0$ and either $\epsilon>0$ or $\Re e(\beta)>\Re e(\alpha)$, $\epsilon=0$, and let ${}^{\mathtt{L}}D_+^{\alpha}$ and ${}^{\mathtt{L}}D_-^{\alpha}$ be the Liouvillle fractional differentiation operators* (1.3.63) *and* (1.3.64). *Then the generalized Stirling functions $S(\alpha,\beta;\epsilon)$ have the representation*

$$S(\alpha,\beta;\epsilon)=\frac{1}{\Gamma(\beta+1)}\left({}^{\mathtt{L}}D_+^{\alpha}\sum_{j=0}^{m}(-1)^j\binom{\beta}{j}e^{(\beta-j)t}\right)(\epsilon)$$
$$+\frac{e^{\alpha\pi i}}{\Gamma(\beta+1)}\left({}^{\mathtt{L}}D_-^{\alpha}\left[(e^t-1)^{\beta}-\sum_{j=0}^{m}(-1)^j\binom{\beta}{j}e^{(\beta-j)t}\right]\right)(\epsilon), \quad (4.6.38)$$

where $m=[\Re e(\beta)]$, and $e^{\alpha\pi i}$ is the principal branch of $(-1)^{\alpha}$.

Proof. By the conditions of the theorem, $m<\Re e(\beta)<m+1$, and hence $\Re e(\beta-j)>0$ for $j\leq m$, and applying (1.3.71), the first term in the right-hand side of (4.6.38) equals

$$\frac{1}{\Gamma(\beta+1)}\sum_{j=0}^{m}(-1)^j\binom{\beta}{j}\left({}^{\mathtt{L}}D_+^{\alpha}e^{(\beta-j)t}\right)(\epsilon)$$

$$=\frac{1}{\Gamma(\beta+1)}\sum_{j=0}^{m}(-1)^j\binom{\beta}{j}e^{(\beta-j)\epsilon}(\beta-j)^{\alpha}. \qquad (4.6.39)$$

Further, for $t>0$

$$(e^t-1)^{\beta}=e^{t\beta}(1-e^{-t})^{\beta}=\sum_{j=0}^{m}(-1)^j\binom{\beta}{j}e^{(\beta-j)t}$$

and hence

$$(e^t-1)^{\beta}-\sum_{j=0}^{m}(-1)^j\binom{\beta}{j}e^{(\beta-j)t}=\sum_{j=m+1}^{\infty}(-1)^j\binom{\beta}{j}e^{(\beta-j)t}. \qquad (4.6.40)$$

According to the estimate (4.5.14), the last series is convergent when either $\epsilon>0$ or $\Re e(\beta)>0$, $\epsilon=0$, at least one of which is true by the conditions of the theorem. Since $\Re e(\beta-j)<0$ for $j\geq m+1$ and noting (4.6.40) and (1.3.92), the second term on the right-hand side of (4.6.38) equals

$$\frac{e^{\alpha\pi i}}{\Gamma(\beta+1)}\left({}^{\mathtt{L}}D_-^{\alpha}\left[\sum_{j=m+1}^{\infty}(-1)^j\binom{\beta}{j}e^{(\beta-j)t}\right]\right)(\epsilon)$$

$$=\frac{e^{\alpha\pi i}}{\Gamma(\beta+1)}\sum_{j=m+1}^{\infty}(-1)^j\binom{\beta}{j}\left({}^{\mathtt{L}}D_-^{\alpha}e^{-(j-\beta)t}\right)(\epsilon)$$

$$=\frac{1}{\Gamma(\beta+1)}\sum_{j=m+1}^{\infty}(-1)^j\binom{\beta}{j}e^{-(j-\beta)\epsilon}[e^{\pi i}(j-\beta)]^{\alpha}$$

$$=\frac{1}{\Gamma(\beta+1)}\sum_{j=m+1}^{\infty}(-1)^j\binom{\beta}{j}e^{(\beta-j)\epsilon}(\beta-j)^{\alpha}. \qquad (4.6.41)$$

Taking the sum of (4.6.39) and (4.6.41), we deduce

$$\frac{1}{\Gamma(\beta+1)}\left({}^{\mathtt{L}}D_{+}^{\alpha}\sum_{j=0}^{m}(-1)^{j}\binom{\beta}{j}e^{(\beta-j)t}\right)(\epsilon)$$

$$+\frac{e^{\alpha\pi i}}{\Gamma(\beta+1)}\left({}^{\mathtt{L}}D_{-}^{\alpha}\left[(e^{t}-1)^{\beta}-\sum_{j=0}^{m}(-1)^{j}\binom{\beta}{j}e^{(\beta-j)t}\right]\right)(\epsilon)$$

$$=\frac{1}{\Gamma(\beta+1)}\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta}{j}e^{(\beta-j)\epsilon}(\beta-j)^{\alpha}$$

which, combined with (4.6.6), yields (4.6.38). This completes the proof of our theorem. □

When $\Re e(\alpha)<0$, there hold representations of the generalized Stirling functions $S(\alpha,\beta;\epsilon)$ in terms of the Liouville fractional integration operators ${}^{\mathtt{L}}I_{+}^{\alpha}$ and ${}^{\mathtt{L}}I_{-}^{\alpha}$.

Theorem 4.36. *Let $\alpha\in\mathbb{C}$ and $\beta\in\mathbb{C}$ $(\beta\notin\mathbb{Z})$ be such that $\Re e(\alpha)<0$ and either $\epsilon>0$ or $\Re e(\beta)>\Re e(\alpha)$, $\epsilon=0$, and let ${}^{\mathtt{L}}I_{+}^{-\alpha}$ and ${}^{\mathtt{L}}I_{-}^{-\alpha}$ be the Liouvillle fractional integration operators* (1.3.61) *and* (1.3.62)*. Then the generalized Stirling functions $S(\alpha,\beta;\epsilon)$ have the representation*

$$S(\alpha,\beta;\epsilon)=\frac{1}{\Gamma(\beta+1)}\left({}^{\mathtt{L}}I_{+}^{-\alpha}\sum_{j=0}^{m}(-1)^{j}\binom{\beta}{j}e^{(\beta-j)t}\right)(\epsilon) \tag{4.6.42}$$

$$+\frac{e^{\alpha\pi i}}{\Gamma(\beta+1)}\left({}^{\mathtt{L}}I_{-}^{-\alpha}\left[(e^{t}-1)^{\beta}-\sum_{j=0}^{m}(-1)^{j}\binom{\beta}{j}e^{(\beta-j)t}\right]\right)(\epsilon),$$

for $m=[\Re e(\beta)]$, $e^{-\alpha\pi i}$ being the principal branch of $(-1)^{-\alpha}$.

Theorem 4.36 is proved similarly to Theorem 4.35 by evaluating the two terms on the right-hand side of (4.6.42) and using relations (1.3.69) and (1.3.70).

4.6.3 *First application*

In this section we give an application of generalized Stirling functions $S(n,\beta;\epsilon)$ to represent fractional order differences in terms of such functions and usual differentiation. First we consider the fractional difference $\Delta^{\beta,\epsilon}$ defined by (4.5.3).

Theorem 4.37. *Let $f(x)$, defined for $x \in \mathbb{R}$, be an arbitrarily often differentiable function. There hold the following three statements.*

(a) *If $\beta \in \mathbb{C}$ $(\beta \notin \mathbb{Z})$ and $\epsilon > 0$, then*

$$\Delta^{\beta,\epsilon} f(x) = \Gamma(\beta+1) \sum_{n=0}^{\infty} S(n,\beta;\epsilon) \frac{f^{(n)}(x)}{n!} \tag{4.6.43}$$

provided that the series on the right-hand side of (4.6.43) *is absolutely convergent.*

(b) *If $k \in \mathbb{N}_0$ and $\epsilon \geq 0$, then*

$$\Delta^{k,\epsilon} f(x) = k! \sum_{n=0}^{\infty} S(n,k;\epsilon) \frac{f^{(n)}(x)}{n!} \tag{4.6.44}$$

provided that the series on the right-hand side of (4.6.44) *is absolutely convergent.*

In particular,

$$\Delta^{k} f(x) = k! \sum_{n=k}^{\infty} \frac{S(n,k)}{n!} f^{(n)}(x) \quad (x \in \mathbb{R}), \tag{4.6.45}$$

holds provided that the series on the right side of (4.6.45) *also has this property.*

Proof. By the conditions of theorem for any $j \in \mathbb{N}_0$ we have by Taylor's expansion that

$$f(x+\beta-j) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (\beta-j)^n. \tag{4.6.46}$$

Substituting this relation into (4.5.3) and interchanging the order of summation, which is possible by an analogue of Fubini's theorem for the series (see Theorem 12-42 of [37]), we obtain

$$\begin{aligned}
\Delta^{\beta,\epsilon} f(x) &= \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{\epsilon(\beta-j)} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (\beta-j)^n \\
&= \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{\epsilon(\beta-j)} (\beta-j)^n \right] \frac{f^{(n)}(x)}{n!}.
\end{aligned}$$

Recalling (4.5.9), this yields (4.6.43). Relation (4.6.44) is proved similarly by using (4.5.5). This completes the proof of the theorem. □

Corollary 4.16. *Let $f(x)$ be defined for $x \in \mathbb{R}$ and be arbitrarily often differentiable on $\mathbb{R}$. Let $\beta > 0$ $(\beta \notin \mathbb{N})$, $\{\beta\}$ being the fractional part of β, and $S^*(n,\beta)$ be given by* (4.5.50). *There holds the relation*

$$\Delta^{\beta,\{\beta\}} f(x) = \Gamma(\beta+1) \sum_{n=0}^{\infty} S^*(n,\beta) \frac{f^{(n)}(x)}{n!} \tag{4.6.47}$$

provided that the series on the right-hand side of (4.6.47) *is absolutely convergent.*

Corollary 4.17. *Let $f(x)$, defined for $x \in \mathbb{R}$, be an arbitrarily often differentiable function, and let $\epsilon \geq 0$. Then*

$$\Delta^{1,\epsilon} f(x) = (e^{\epsilon} - 1)\, f(x) + e^{\epsilon} \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!}. \tag{4.6.48}$$

In particular,

$$\Delta^{1} f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!}. \tag{4.6.49}$$

Proof. Corollary 4.17 follows from Theorem 4.43(a) if we put $\epsilon = \{\beta\}$ and take (4.5.50) into account. Equation (4.6.44) with $k = 1$ yields (4.6.48) noting (4.5.26) and (4.5.28). The relation (4.6.49) follows from (2.6.5) when $\epsilon = 0$. □

Next we consider the previous result with respect to the classical fractional difference Δ^β with $\epsilon = 0$, namely

$$\Delta^{\beta} f(x) \equiv \Delta^{\beta,0} f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} f(x+\beta-j). \tag{4.6.50}$$

Theorem 4.38. *Let $f(x)$, defined for $x \in \mathbb{R}$, be an arbitrarily often differentiable function. There hold the following assertions.*

(a) *If $\beta \in \mathbb{C}$ $(\beta \notin \mathbb{Z})$ and $\epsilon > 0$, then*

$$\Delta^{\beta,\epsilon} f(x) = \Gamma(\beta+1) \sum_{n=0}^{\infty} \frac{S(n,\beta;\epsilon)}{n!} \sum_{m=0}^{n} \binom{n}{m} (-\epsilon)^{n-m} f^{(m)}(x) \tag{4.6.51}$$

provided that the series on the right-hand side of (4.6.51) *is absolutely convergent.*

(b) *If $k \in \mathbb{N}_0$ and $\epsilon \geq 0$, then*

$$\Delta^{k,\epsilon} f(x) = k! \sum_{n=0}^{\infty} \frac{S(n,k;\epsilon)}{n!} \sum_{m=0}^{n} \binom{n}{m} (-\epsilon)^{n-m} f^{(m)}(x) \qquad (4.6.52)$$

provided that the series on the right-hand side of (4.6.52) *is absolutely convergent. In particular, relation* (4.6.45) *holds provided again that the series on the right hand side of* (4.6.45) *is absolutely convergent.*

Proof. Let $\beta \in \mathbb{C}$ ($\beta \notin \mathbb{Z}$) and $\epsilon > 0$. By the condition of the theorem, the right hand side of (4.6.51) is well-defined, and in accordance with Theorem 4.29(a) the generalized $S(n,\beta;\epsilon)$ have the representation (4.5.9) for any $n \in \mathbb{N}_0$. Substituting (4.5.9) into the right hand side of (4.6.50) we have

$$\begin{aligned}
&\Gamma(\beta+1) \sum_{n=0}^{\infty} \frac{S(n,\beta;\epsilon)}{n!} \sum_{m=0}^{n} \binom{n}{m} (-\epsilon)^{n-m} f^{(m)}(x) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{\epsilon(\beta-j)} (\beta-j)^n \right] \sum_{m=0}^{n} \binom{n}{m} (-\epsilon)^{n-m} f^{(m)}(x).
\end{aligned}$$

Since the series on the right side is absolutely convergent we can interchange orders of summation to deduce

$$\begin{aligned}
&\sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{\epsilon(\beta-j)} \left[\sum_{n=0}^{\infty} \frac{1}{n!} (\beta-j)^n \sum_{m=0}^{n} \binom{n}{m} (-\epsilon)^{n-m} f^{(m)}(x) \right] \\
&\quad = \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} e^{\epsilon(\beta-j)} \left[\sum_{m=0}^{\infty} f^{(m)}(x) \sum_{n=m}^{\infty} \binom{n}{m} (-\epsilon)^{n-m} \frac{(\beta-j)^n}{n!} \right] \\
&\quad = \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} \sum_{m=0}^{\infty} \frac{f^{(m)}(x)}{m!} e^{\epsilon(\beta-j)} \left[\sum_{n=m}^{\infty} (-\epsilon)^{n-m} \frac{(\beta-j)^n}{(n-m)!} \right] \\
&\quad = \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} \sum_{m=0}^{\infty} \frac{f^{(m)}(x)}{m!} (b-j)^m \left[e^{\epsilon(\beta-j)} \sum_{k=0}^{\infty} \frac{(-\epsilon(\beta-j))^k}{k!} \right] \\
&\quad = \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} \left[\sum_{m=0}^{\infty} \frac{f^{(m)}(x)}{m!} (\beta-j)^m \right].
\end{aligned}$$

Now, applying (4.6.46) we have

$$\Gamma(\beta+1)\sum_{n=0}^{\infty}\frac{S(n,\beta;\epsilon)}{n!}\sum_{m=0}^{n}\binom{n}{m}(-\epsilon)^{n-m}f^{(m)}(x)$$
$$=\sum_{j=0}^{\infty}(-1)^j\binom{\beta}{j}f(x+\beta-j),$$

which, combined with (4.6.50), yields (4.6.51).

As to part (b), when $k=0$, then according to (4.5.11)–(4.5.12), Eq. (4.6.52) takes the form $\Delta^{0,\epsilon}f(x)=f(x)$ which is true by (4.5.6). When $k\in\mathbb{N}$ and $\epsilon\geq 0$, then substituting (4.5.10) into the right hand side of (4.6.52) and following the same arguments as above we deduce (4.6.52). Thus the theorem is proved. □

Corollary 4.18. *Let $f(x)$, defined for $x\in\mathbb{R}$, be an arbitrarily often differentiable function. Let $\beta>0$ ($\beta\notin\mathbb{N}$), $\{\beta\}$ being the fractional part of β, and $S^*(n,\beta)$ be given by* (4.5.50)*. Then*

$$\Delta^{\beta,\{\beta\}}f(x)=\Gamma(\beta+1)\sum_{n=0}^{\infty}\frac{S^*(n,\beta)}{n!}\sum_{m=0}^{n}\binom{n}{m}(-\{\beta\})^{n-m}f^{(m)}(x) \tag{4.6.53}$$

provided that the series in the right hand side of (4.6.53) *is absolutely convergent.*

4.6.4 *Special examples*

In this section we apply the relations established in Theorems 4.37 and 4.38 to particular functions $f(x)$. This is a first attempt to see their range of applicability.

Example 4.1. Let $f(x)=\log(x+a)$, with $a\in\mathbb{R}$ and $x>-a$. Then for any $n\in\mathbb{N}$

$$D^n(\log(x+a))=(-1)^{n-1}(n-1)!(x+a)^{-n},\qquad D=\frac{d}{dx}, \tag{4.6.54}$$

hence (4.6.43) and (4.6.44) and (4.6.47) yield, noting (4.5.22), the following Corollary.

Corollary 4.19. *Let $a \in \mathbb{R}$, $x \geq a$.*

(a) *If $\beta \in \mathbb{C}$, $\beta \notin \mathbb{Z}$ and $\epsilon > 0$, then*

$$\Delta^{\beta,\epsilon}(\log(x+a)) = \log(e)\left[(e^{\epsilon}-1)^{\beta}\log(x+a) + \Gamma(\beta+1)\sum_{n=1}^{\infty} S(n,\beta;\epsilon)\frac{(-1)^n}{n}(x+a)^{-n}\right]. \tag{4.6.55}$$

(b) *If $k \in \mathbb{N}$ and $\epsilon \geq 0$, then*

$$\Delta^{k,\epsilon}(\log(x+a)) = \log(e)\left[(e^{\epsilon}-1)^{k}\log(x+a) + k!\sum_{n=1}^{\infty} S(n,k;\epsilon)\frac{(-1)^n}{n}(x+a)^{-n}\right]. \tag{4.6.56}$$

(c) *If $\beta > 0$ and $\beta \notin \mathbb{N}$, then*

$$\Delta^{\beta,\{\beta\}}(\log(x+a)) = \log(e)\left[(e^{\{\beta\}}-1)^{\beta}\log(x+a) + \Gamma(\beta+1)\sum_{n=1}^{\infty} S^*(n,\beta)\frac{(-1)^n}{n}(x+a)^{-n}\right]. \tag{4.6.57}$$

In particular, for $\epsilon = 0$, (4.6.56) takes the form

$$\Delta^{k}(\log(x+a)) = \log(e)k!\sum_{n=k}^{\infty} S(n,k)\frac{(-1)^n}{n}(x+a)^{-n} \quad (k \in \mathbb{N}). \tag{4.6.58}$$

Further, if $\beta = 1$ and $\epsilon \geq 0$, then, in accordance with (4.5.28), Eq. (4.6.56) takes the form

$$\Delta^{1,\epsilon}(\log(x+a)) = \log(e)\left[(e^{\epsilon}-1)\log(x+a) + e^{\epsilon}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x+a)^{-n}\right]. \tag{4.6.59}$$

Finally, (4.6.51) gives

Corollary 4.20. *Let $\beta > 0$, $\beta \notin \mathbb{N}$, $a \in \mathbb{R}$, $x \geq -a$ and $\epsilon > 0$. Then*

$$\Delta^{\beta}(\log(x+a)) = \log(e)(e^{\epsilon}-1)^{\beta}\log(x+a) + \log(e)\Gamma(\beta+1)\sum_{n=0}^{\infty}(-1)^{n-1}\frac{S(n,\beta;\epsilon)}{n!}\sum_{m=0}^{n}\binom{n}{m}\epsilon^{n-m}\frac{(x+a)^{-m}}{m!}. \tag{4.6.60}$$

Example 4.2. If $f(x) = (x+a)^\gamma$, with $a \in \mathbb{R}$, $x > -a$ and $\gamma \in \mathbb{R}$, then

$$D^n((x+a)^\gamma) = (-1)^n(-\gamma)_n(x+a)^{\gamma-n} \quad (n \in \mathbb{N}),$$

where $(\gamma)_n$ is given by (4.5.32).

So, again noting (4.5.22), then (4.6.43), (4.6.44) and (4.6.47) yield

Corollary 4.21. *Let $a \in \mathbb{R}$, $x \geq -a$ and $\gamma \in \mathbb{R}$.*

(a) *If $\beta \in \mathbb{C}$, $\beta \notin \mathbb{Z}$ and $\epsilon > 0$, then*

$$\Delta^{\beta,\epsilon}((x+a)^\gamma) = (e^\epsilon - 1)^\beta (x+a)^\gamma + \Gamma(\beta+1)\sum_{n=1}^{\infty}(-1)^n S(n,\beta;\epsilon)\frac{(-\gamma)_n}{n!}(x+a)^{\gamma-n}. \tag{4.6.61}$$

(b) *If $k \in \mathbb{N}$ and $\epsilon \geq 0$, then*

$$\Delta^{k,\epsilon}((x+a)^\gamma) = (e^\epsilon - 1)^k (x+a)^\gamma + k!\sum_{n=1}^{\infty}(-1)^n S(n,k;\epsilon)\frac{(-\gamma)_n}{n!}(x+a)^{\gamma-n}. \tag{4.6.62}$$

(c) *If $\beta > 0$ and $\beta \notin \mathbb{N}$, then*

$$\Delta^{\beta,\{\beta\}}((x+a)^\gamma) = (e^{\{\beta\}} - 1)^\beta (x+a)^\gamma + \Gamma(\beta+1)\sum_{n=1}^{\infty}(-1)^n S^*(n,\beta)\frac{(-\gamma)_n}{n!}(x+a)^{\gamma-n}. \tag{4.6.63}$$

In particular, for $\epsilon = 0$, Eq. (4.6.62) has the form

$$\Delta^k((x+a)^\gamma) = k!\sum_{n=k}^{\infty}(-1)^n S(n,k)\frac{(-\gamma)_n}{n!}(x+a)^{\gamma-n} \ (k \in \mathbb{N}).$$

Further, if $\beta = 1$ and $\epsilon \geq 0$, then (4.6.62) has the form

$$\Delta^{1,\epsilon}((x+a)^\gamma) = (e^\epsilon - 1)(x+a)^\gamma + e^\epsilon\sum_{n=1}^{\infty}(-1)^n\frac{(-\gamma)_n}{n!}(x+a)^{\gamma-n}.$$

Finally, (4.6.51) gives

Corollary 4.22. *Let $\beta > 0$, $\beta \notin \mathbb{N}$, $a \in \mathbb{R}$, $x \geq -a$ and $\epsilon > 0$. Then*

$$\begin{aligned}\Delta^{\beta,\epsilon}&((x+a)^\gamma) \\ &= (e^\epsilon - 1)^\beta (x+a)^\gamma \\ &\quad + \Gamma(\beta+1)\sum_{n=0}^{\infty}(-1)^n\frac{S(n,\beta;\epsilon)}{n!}\sum_{m=0}^{n}\binom{n}{m}\epsilon^{n-m}(-\gamma)_m(x+a)^{\gamma-m}.\end{aligned} \tag{4.6.64}$$

Example 4.3. Let $f(x) = \psi(x+a)$, with $a \in \mathbb{R}$ and $x > -a$, where $\psi(z)$ is the Euler Psi-function given by

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

It is known [14, p. 326] that for $z \neq 0, -1, -2, \ldots$

$$D^n \psi(z) = (-1)^{n+1} n! \zeta(n+1, z) \quad (n \in \mathbb{N}),$$

where $\zeta(s,z)$ is the generalized Riemann zeta-function defined by

$$\zeta(s,z) = \sum_{m=0}^{\infty} \frac{1}{(z+m)^s} \quad (s, z \in \mathbb{C},\ \Re e(s) > 1;\ z \neq 0, -1, -2, \ldots).$$

So, again noting (4.5.22), then (4.6.43), (4.6.44) and (4.6.47) yield

Corollary 4.23. *Let $a \in \mathbb{R}$ and $x \geq -a$.*

(a) *If $\beta \in \mathbb{C}$, $\beta \notin \mathbb{Z}$ and $\epsilon > 0$, then*

$$\Delta^{\beta,\epsilon} \psi(x+a) \tag{4.6.65}$$
$$= (e^{\epsilon} - 1)^{\beta} \psi(x+a) + \Gamma(\beta+1) \sum_{n=1}^{\infty} S(n, \beta; \epsilon)(-1)^{n+1} \zeta(n+1, x+a).$$

(b) *If $k \in \mathbb{N}$ and $\epsilon \geq 0$, then*

$$\Delta^{k,\epsilon} \psi(x+a) = (e^{\epsilon} - 1)^k \psi(x+a) + k! \sum_{n=1}^{\infty} S(n, k; \epsilon)(-1)^{n+1} \zeta(n+1, x+a). \tag{4.6.66}$$

(c) *If $\beta > 0$ and $\beta \notin \mathbb{N}$, then*

$$\Delta^{\beta,\{\beta\}} \psi(x+a) \tag{4.6.67}$$
$$= (e^{\{\beta\}} - 1)^{\beta} \psi(x+a) + \Gamma(\beta+1) \sum_{n=1}^{\infty} S^*(n, \beta)(-1)^{n+1} \zeta(n+1, x+a).$$

In particular, for $\epsilon = 0$, (4.6.66) gives

$$\Delta^k \psi(x+a) = k! \sum_{n=k}^{\infty} S(n, k)(-1)^{n+1} \zeta(n+1, x+a) \quad (k \in \mathbb{N}).$$

Further, if $\beta = 1$ and $\epsilon \geq 0$, then

$$\Delta^{1,\epsilon}\psi(x+a) = (e^{\epsilon}-1)^k\,\psi(x+a) + e^{\epsilon}\sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1,x+a)\ (\epsilon \geq 0).$$

Furthermore, (4.6.51) delivers

Corollary 4.24. *If $\beta > 0$, $\beta \notin \mathbb{N}$, $a \in \mathbb{R}$, $x \geq -a$ and $\epsilon > 0$, then*

$$\begin{aligned}&\Delta^{\beta}\psi(x+a) \qquad (4.6.68)\\ &= (e^{\epsilon}-1)^{\beta}\psi(x+a)\\ &\quad + \Gamma(\beta+1)\sum_{n=0}^{\infty}(-1)^{n-1}\frac{S(n,\beta;\epsilon)}{n!}\sum_{m=0}^{n}\binom{n}{m}\epsilon^{n-m}m!\zeta(m+1,x).\end{aligned}$$

Finally the known relation

$$\Delta^k\psi(z+a) = (-1)^{k-1}(k-1)!\frac{\Gamma(z+a)}{\Gamma(z+a+k)},$$

with $k \in \mathbb{N}$; $z \in \mathbb{C}$, $z \neq -a, -a-1, \ldots, -a-k+1$, *cf.*, e.g. p. 328 of [296] or p. 19 of [209], yields

Property 4.15. *If $k \in \mathbb{N}$, $a \in \mathbb{R}$ and $x \in \mathbb{R}$ ($x \neq -a, -a-1, \ldots, -a-k+1$), then there holds the summation formula*

$$\sum_{n=k}^{\infty} S(n,k)(-1)^n\zeta(n+1,x+a) = \frac{(-1)^k}{k}\frac{\Gamma(x+a)}{\Gamma(x+a+k)}.$$

4.7 Connections Between the Stirling Functions of First and Second Kind

In this section we prove connections between the Stirling functions of the first and second kind $s(\alpha,\beta)$ and $S(\alpha,k)$.

4.7.1 *Coincidence relations*

The Stirling functions of the first kind $s(-n,\beta)$, defined for $n \in \mathbb{N}_0$ and $\beta \in \mathbb{C}$ by (4.3.25), coincide, apart from a multiplicative factor, with the Stirling functions of the second kind $S(-\beta,n)$, defined in (4.4.3). The situation is similar for $s(-k,-\alpha)$ and $S(\alpha,k)$.

Theorem 4.39.

(a) *Let $\beta \in \mathbb{C}$ ($\beta \neq 0$) and $n \in \mathbb{N}$. The Stirling functions $s(-n, \beta)$ coincide with $S(-\beta, n)$ apart from the constant multiplier $(-1)^n e^{-(\beta+1)\pi}$*

$$s(-n, \beta) = (-1)^n e^{-(\beta+1)\pi} S(-\beta, n). \tag{4.7.1}$$

In particular, for $m \in \mathbb{Z}$, $m \neq 0$,

$$s(-n, m) = (-1)^{n+m-1} S(-m, n).$$

(b) *Let $\alpha \in \mathbb{C}$ ($\alpha \neq 0$) and $k \in \mathbb{N}$. The Stirling functions $S(\alpha, k)$ coincide with the $s(-k, -\alpha)$ apart from the constant multiplier $(-1)^k e^{(\alpha+1)\pi}$*

$$S(\alpha, k) = (-1)^k e^{(\alpha+1)\pi} s(-k, -\alpha). \tag{4.7.2}$$

In particular, for $m \in \mathbb{Z}$,

$$S(m, k) = (-1)^{m+k+1} s(-k, -m).$$

Proof. The result in (4.7.1) follows from Theorem 4.12 and Theorem 4.17, if we take into account the explicit representations for $s(-n, \beta)$ and $S(-\beta, n)$ given by (4.3.25) and (4.1.16). The relation (4.7.2) clearly follows from (4.7.1). □

The above theorem enables one to transfer several results we have established for the Stirling functions of second kind $S(\alpha, k)$ to such for the Stirling functions of first kind $s(-n, \beta)$. One such result is Theorem 4.18 that expresses $S(\alpha, k)$ in terms of Liouville derivatives in the form

$$S(\alpha, k) = \frac{(-1)^k}{k!} \lim_{x \to 0} \left({}^{\mathrm{L}}D_+^{\alpha} \left[\left(1 - e^t\right)^k - 1 \right] \right)(x) \tag{4.7.3}$$

for $\alpha \in \mathbb{C}$, $\Re e(\alpha) > 0$, and $k \in \mathbb{N}$. It can be transferred as follows.

Theorem 4.40. *Let $\beta \in \mathbb{C}$, $\Re e(\beta) < 0$, and $n \in \mathbb{N}$. Then the Stirling functions $s(-n, \beta)$ have the Liouville fractional derivative representation*

$$s(-n, \beta) = \frac{e^{(\beta+1)\pi i}}{n!} \lim_{x \to 0} \left({}^{\mathrm{L}}D_+^{-\beta} \left[\left(1 - e^t\right)^k - 1 \right] \right)(x) \tag{4.7.4}$$

Proof. The result in (4.7.4) follows directly from (4.7.3) and (4.7.1). □

Formula (4.7.4) can be used as an alternative definition of $s(-n, \beta)$.

4.7.2 *Results from sampling analysis*

For the counterpart of the classical orthogonality property for the Stirling functions we need the sampling theorem of signal analysis. Let $B^p_{\pi W}$, $W > 0$, $1 \leq p < \infty$, be the class of those functions $g \in L^p(\mathbb{R})$ having an extension to the complex plane $\mathbb{C}$ as an entire function of exponential type πW, namely

$$|g(z)| \leq \exp(\pi W|y|)\|g\|_C \qquad (z = x + iy; x, y \in \mathbb{R}).$$

The sampling theorem now states

Theorem 4.41 (Sampling theorem). *Any signal function $g \in B^p_{\pi W}$, $1 \leq p < \infty$, some $W > 0$, can be completely reconstructed from its sampled values $g(j/W)$ taken at the nodes j/W, $j \in \mathbb{Z}$, in terms of*

$$g(z) = \sum_{j=-\infty}^{\infty} g\left(\frac{j}{W}\right) \frac{\sin[\pi(Wz - j)]}{\pi(Wz - j)}, \tag{4.7.5}$$

the series converging absolutely and uniformly on compact subsets of $\mathbb{C}$.

For literature regarding sampling analysis see, e.g., Butzer *et al.* [123, 124] and Higgins [278]. Details on the application of the special function $(\sin \pi x)/(\pi x)$, also known as the Sinc function or Whittaker's cardinal function, in this context may also be found in the seminal work of Stenger [536] or the original publications by E. T. Whittaker [579] and his son, J. M. Whittaker [580].

A basic new application of this theorem to be needed below, which is definitely of independent interest, is given by Lemma 4.6, namely the sampling theorem for the power function $t^\alpha/\Gamma(\alpha + 1)$ in terms of $t^j/j!$ for $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 4.6 (Sampling representation of the power function). *For $\alpha \in \mathbb{C}$ and $t > 0$ there holds the relation*

$$\frac{t^\alpha}{\Gamma(\alpha + 1)} = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{\sin[\pi(\alpha - j)]}{\pi(\alpha - j)} \quad (\alpha \in \mathbb{C}; \quad t > 0). \tag{4.7.6}$$

Proof. Set $g(z) = t^z/\Gamma(z+1)$, where $z = x + iy = |z|e^{i\arg z}$ with $-\pi \leq \arg(z) < \pi$. $g(z)$ is clearly an entire function of $z \in \mathbb{C}$. Using the asymptotic relation for the Gamma function (formula 1.18(2) of [209]), *viz.*

$$\Gamma(z) \sim (2\pi)^{1/2} e^{-z} e^{(z-1/2)\log(z)} \quad (z \to \infty), \tag{4.7.7}$$

we have

$$|g(z)| \sim (2\pi|z|)^{-1/2} \left(\frac{te}{|z|}\right)^{|z|} e^{y\arg z} \quad (|z| \to \infty).$$

If we choose $R > 0$ sufficiently large, this relation yields the estimate

$$|g(z)| \leq Ae^{\pi|y|} \qquad (A > 0; |z| \geq R). \tag{4.7.8}$$

If $|z| \leq R$ then, choosing $n \in \mathbb{N}$ such that $|z+n| \geq R$ and using the relation

$$(z)_k = \frac{\Gamma(z+k)}{z},$$

one has

$$|g(z)| \leq \frac{|t^{z+n}(z+1)\cdots(z+n)|}{t^n|\Gamma(z+n+1)|} \leq \frac{(R+1)\cdots(R+n)}{a^n} \frac{|t^{z+n}|}{|\Gamma(z+n+1)|},$$

and hence, according to (4.7.8),

$$|g(z)| \leq Be^{\pi|y|} \qquad (B = A(R+1)\cdots(R+n)a^n; \quad |z| \geq R).$$

Using (4.7.7), it is directly verified that $g(x) \in L^p(\mathbb{R})$ for $1 \leq p < \infty$.

Thus, g is of exponential type π, $g \in B^p_\pi$, and an application of the sampling formula (4.7.5) (with $z = \alpha$ and $W = 1$) yields (4.7.6), noting that

$$g(j) = \frac{t^j}{\Gamma(j+1)} = 0 \quad \text{for} \quad j \in \mathbb{Z}^-. \tag{4.7.9}$$

□

From Lemma 4.6 we deduce the sampling result for $S(\alpha, k)$, also needed.

Theorem 4.42 (Sampling theorem for Stirling functions $S(\alpha, k)$). *For $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}_0$ there holds true*

$$\frac{S(\alpha,k)}{\Gamma(\alpha+1)} = \sum_{j=k}^{\infty} \frac{S(j,k)}{j!} \frac{\sin[\pi(\alpha-j)]}{\pi(\alpha-j)}. \tag{4.7.10}$$

Proof. If $\alpha = 0$, then (4.7.10) is clear. Indeed, $S(0,0) = 1$ and $S(k,m) = 0$ $(k, m \in \mathbb{N}_0;\ k < m)$, $S(0,k) = 0$ $(k \in \mathbb{N})$. Let $\alpha \neq 0$. By Proposition 4.5, for $z \in \mathbb{C}$ and $\alpha \in \mathbb{C}$ $(\alpha \neq 0)$ there holds the Newton series expansion

$$z^\alpha = \sum_{k=0}^{\infty} S(\alpha, k)[z]_k \tag{4.7.11}$$

(recall from Eq. (4.1.7) that $[z]_0 = 1$, and $[z]_k = z(z-1)\cdots(z-k+1)$ for $k \in \mathbb{N}$), the series being absolutely convergent for $|z| \geq \lambda + 1$, the abscissa of convergence being $\lambda \leq \max[0, \Re e(\alpha) - 1/2]$. Rewrite (4.7.11) with $z = t > 0$ in the form

$$\frac{t^\alpha}{\Gamma(\alpha+1)} = \sum_{k=0}^{\infty} \frac{S(\alpha,k)}{\Gamma(\alpha+1)}[t]_k \quad (t > 0;\quad \alpha \in \mathbb{C},\quad \alpha \neq 0). \tag{4.7.12}$$

Taking into account (4.7.12) and (4.7.6) and interchanging the order of summation yields

$$\begin{aligned} \frac{t^\alpha}{\Gamma(\alpha+1)} &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \frac{S(j,k)}{j!}[t]_k \right) \frac{\sin[\pi(\alpha-j)]}{\pi(\alpha-j)} \\ &\times \sum_{k=0}^{\infty} \left(\sum_{j=k}^{\infty} \frac{S(j,k)}{j!} \frac{\sin[\pi(\alpha-j)]}{\pi(\alpha-j)} \right) [t]_k. \end{aligned} \tag{4.7.13}$$

A comparison of the coefficients of the series (4.7.12) and (4.7.13) yields (4.7.10). □

4.7.3 *Generalized orthogonality properties*

It is well known that the classical Stirling numbers of the first and second kind $s(n,k)$ and $S(n,k)$ are connected by the basic orthogonality relation

$$\sum_{k=m}^{n} s(n,k)S(k,m) = \sum_{k=m}^{n} S(n,k)s(k,m) = \delta_{n,m} \quad (m, n \in \mathbb{N}_0), \tag{4.7.14}$$

where $\delta_{m,n} = 1$ for $m = n$ and $\delta_{m,n} = 0$ for $m \neq n$, the Kronecker delta. The counterpart for the Stirling functions reads

Theorem 4.43. *Let $\alpha \in \mathbb{C}$ and $m \in \mathbb{N}_0$. There holds for $\alpha \in \mathbb{C}$, $m \in \mathbb{N}_0$,*

$$\sum_{k=m}^{\infty} s(\alpha,k)S(k,m) = \sum_{k=m}^{\infty} S(\alpha,k)s(k,m) = \frac{\Gamma(\alpha+1)}{\Gamma(m+1)} \frac{\sin[\pi(\alpha-m)]}{\pi(\alpha-m)}. \tag{4.7.15}$$

Proof. Basic for the proof of the left-hand side of (4.7.15) is the sampling theorem for $s(\alpha,k)$, with $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}_0$ (see Theorem 4.1 of [113])

$$\frac{s(\alpha,k)}{\Gamma(\alpha+1)} = \sum_{n=k}^{\infty} \frac{s(n,k)}{n!} \frac{\sin[(\alpha-n)\pi]}{(\alpha-n)\pi} \quad (\alpha \in \mathbb{C}; \quad k \in \mathbb{N}_0). \tag{4.7.16}$$

Taking into account the property $S(k,m) = 0$ $(k,m \in \mathbb{N}_0;\ k < m)$, applying the left-hand side of (4.7.14), changing the orders of summation and observing the known property $s(n,k) = 0$ $(n,k \in \mathbb{N}_0$ for $n < k)$, there follows from (4.7.16)

$$\begin{aligned}
\sum_{k=m}^{\infty} s(\alpha,k)S(k,m) &= \sum_{k=0}^{\infty} s(\alpha,k)S(k,m) \\
&= \Gamma(\alpha+1) \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{s(n,k)}{n!} \frac{\sin[\pi(\alpha-n)]}{\pi(\alpha-n)} S(k,m) \\
&= \Gamma(\alpha+1) \sum_{n=0}^{\infty} \sum_{k=0}^{n} s(n,k)S(k,m) \frac{\sin[\pi(\alpha-n)]}{n!\ \pi(\alpha-n)} \\
&= \Gamma(\alpha+1) \sum_{n=m}^{\infty} \left[\sum_{k=m}^{n} s(n,k)S(k,m) \right] \frac{\sin[\pi(\alpha-n)]}{n!\ \pi(\alpha-n)}.
\end{aligned}$$

Using (4.7.14), we deduce

$$\begin{aligned}
\sum_{k=0}^{\infty} s(\alpha,k)S(k,m) &= \Gamma(\alpha+1) \sum_{n=m}^{\infty} \delta_{n,m} \frac{\sin[\pi(\alpha-n)]}{n!\pi(\alpha-n)} \\
&= \frac{\Gamma(\alpha+1)}{m!} \frac{\sin[\pi(\alpha-m)]}{\pi(\alpha-m)},
\end{aligned}$$

which proves the left-hand side of (4.7.15).

The proof of the right-hand side of (4.7.15) is similar. In fact,

$$\sum_{k=m}^{\infty} S(\alpha,k)s(k,m) = \Gamma(\alpha+1)\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{S(n,k)}{n!}\frac{\sin[\pi(\alpha-n)]}{\pi(\alpha-n)}s(k,m)$$
$$= \Gamma(\alpha+1)\sum_{n=m}^{\infty}\left[\sum_{k=m}^{n} S(n,k)s(k,m)\right]\frac{\sin[\pi(\alpha-n)]}{n!\;\pi(\alpha-n)}.$$

This yields the right-hand side of (4.7.15) and the proof is complete. □

Observe that formula (4.7.15) reduces to the classical one (4.7.14) for $\alpha = n \in \mathbb{N}$.

4.7.4 The $s(\alpha, k)$ connecting two types of fractional derivatives

Here we obtain an expression for the classical Riemann-Liouville fractional derivative ${}^{\mathrm{RL}}D_{0+}^{\alpha}$ in terms of the Hadamard derivatives of integer order ${}^{\mathrm{H}}D_{0+}^{k}$. In some sense, such relation is the inverse to the sum formula (4.4.79).

Observe that if $n \in \mathbb{N}_0$, then (4.4.79) and (4.4.80) are the classical formulae, respectively,

$$\left({}^{\mathrm{H}}D_{0+}^{n}f\right)(x) \equiv (\delta^n f)(x) = \sum_{k=0}^{n} S(n,k)x^k f^{(k)}(x) \quad (n \in \mathbb{N}_0), \tag{4.7.17}$$

where $\delta = \left(x\frac{d}{dx}\right)$, $\delta^n = \left(x\frac{d}{dx}\right)^n$, and

$$x^n\left(\frac{d}{dx}\right)^n f(x) = \sum_{k=0}^{n} s(n,k)\delta^k f(x) \quad (n \in \mathbb{N}_0). \tag{4.7.18}$$

We shall now use a unified notation $\left({}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}f\right)(x)$ $(\alpha \in \mathbb{C})$ for the Riemann-Liouville fractional derivative (1.3.3) of order α and for the Riemann-Liouville fractional integral (1.3.1) of order $-\alpha$ in the cases $\Re e(\alpha) \geq 0$ and $\Re e(\alpha) < 0$

$${}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}f = {}^{\mathrm{RL}}D_{0+}^{\alpha}f \quad (\Re e(\alpha) \geq 0); \quad {}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}f = {}^{\mathrm{RL}}I_{0+}^{-\alpha}f \quad (\Re e(\alpha) < 0). \tag{4.7.19}$$

Theorem 4.44. *Let $f(x)$ be an arbitrarily often differentiable function on $x > 0$ such that $x^\alpha \left({}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha} f\right)(x)/\Gamma(\alpha+1)$ as a function of $\alpha \in \mathbb{C}$ belongs to the class B_π^p for $1 \le p < \infty$. Then there holds the expansion*

$$x^\alpha \left({}^{\mathrm{RL}}D_{0+}^{\alpha} f\right)(x) = \sum_{k=0}^{\infty} s(\alpha,k) \left({}^{\mathrm{H}}D_{0+}^{k} f\right)(x) \quad (\alpha \in \mathbb{C}, \quad \Re e(\alpha) \ge 0) \tag{4.7.20}$$

for the Riemann-Liuoville fractional derivative $\left({}^{\mathrm{RL}}D_{0+}^{\alpha} f\right)(x)$ of order $\alpha \in \mathbb{C}$, $\Re e(\alpha) \ge 0$, provided that the series in the right-hand side of (4.7.20) *converges.*

Proof. Inserting (4.7.17) (with $n = k$) into the right-hand side of (4.7.20) and changing the orders of summations and applying (4.7.15), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} s(\alpha,k) \left({}^{\mathrm{H}}D_{0+}^{k} f\right)(x) \\
&\qquad = \sum_{k=0}^{\infty} s(\alpha,k) \sum_{j=0}^{k} S(k,j) x^j f^{(j)}(x) \\
&\qquad = \sum_{j=0}^{\infty} \left(\sum_{k=j}^{\infty} s(\alpha,k) S(k,j) \right) x^j f^{(j)}(x) \\
&\qquad = \Gamma(\alpha+1) \sum_{j=0}^{\infty} \frac{\sin[(\alpha-j)\pi]}{(\alpha-j)\pi} \frac{x^j f^{(j)}(x)}{j!}.
\end{aligned} \tag{4.7.21}$$

By the assumption of the theorem, for $x > 0$, $x^{z\,\mathrm{RL}}\mathbf{D}_{0+}^{z} f/\Gamma(z+1)$ as a function of $z \in \mathbb{C}$ belongs to the class B_π^p. Applying the sampling formula (4.7.5) (with $z = \alpha$ and $W = 1$), and taking into account (4.7.9), noting $\left({}^{\mathrm{RL}}\mathbf{D}^{j} f\right)(x) \equiv ({}^{\mathrm{RL}}D_{0+}^{j} f)(x) = f^{(j)}(x)$ for $j \in \mathbb{N}_0$, we deduce for $\alpha \in \mathbb{C}$

$$\frac{x^\alpha \left({}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha} f\right)(x)}{\Gamma(\alpha+1)} = \sum_{j=0}^{\infty} \frac{x^j f^{(j)}(x)}{j!} \frac{\sin[\pi(\alpha-j)]}{\pi(\alpha-j)}. \tag{4.7.22}$$

When $\alpha \in \mathbb{C}$, $\Re e(\alpha) \ge 0$, then (4.7.22) in combination with (4.7.19) yields

$$\frac{x^\alpha ({}^{\mathrm{RL}}D_{0+}^{\alpha} f)(x)}{\Gamma(\alpha+1)} = \sum_{j=0}^{\infty} \frac{x^j f^{(j)}(x)}{j!} \frac{\sin[\pi(\alpha-j)]}{\pi(\alpha-j)}. \tag{4.7.23}$$

Thus (4.7.20) follows from (4.7.21) and (4.7.23). $\square$

Corollary 4.25. *If the conditions of Theorem 4.44 are satisfied, then for the integral* ${}^{\mathrm{RL}}I_{0+}^{\alpha}f$ *of order* $\alpha \in \mathbb{C}$, $\mathfrak{Re}(\alpha) > 0$, *there holds the sampling formula*

$$\frac{x^{-\alpha}\left({}^{\mathrm{RL}}I_{0+}^{\alpha}f\right)(x)}{\Gamma(1-\alpha)} = \sum_{j=0}^{\infty} \frac{x^j f^{(j)}(x)}{j!} \frac{\sin[\pi(\alpha+j)]}{\pi(\alpha+j)}.$$

To apply Theorem 4.44 we need the following auxiliary result

Lemma 4.7. *Let* $\lambda \in \mathbb{C}$, $\mathfrak{Re}(\lambda) < 1$, *and let* ${}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}f$ *be given by* (4.7.19). *Then* $x^{\alpha}{}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}t^{-\lambda}/\Gamma(\alpha+1)$, $x^{\alpha}{}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}t^{-\lambda}\log(t)/\Gamma(\alpha+1)$ *as functions of* $\alpha \in \mathbb{C}$ *are of exponential type* π,

$$\frac{x^{\alpha}({}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}t^{-\lambda})(x)}{\Gamma(\alpha+1)} \in B_{\pi}^{p}, \quad \frac{x^{\alpha}\left({}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}t^{-\lambda}\log(t)\right)(x)}{\Gamma(\alpha+1)} \in B_{\pi}^{p}.$$

In particular,

$$x^{\alpha}\left({}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}\log(t)\right)(x)/\Gamma(\alpha+1) \in B_{\pi}^{p}.$$

Proof. Let $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C}$, $\mathfrak{Re}(\lambda) < 1$. By (4.3.30) and the corresponding formula for the Riemann-Liouville fractional integral, we find that

$$\frac{x^{\alpha}\left({}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}t^{-\lambda}\right)(x)}{\Gamma(\alpha+1)} = \frac{\Gamma(1-\lambda)x^{-\lambda}}{\Gamma(1-\lambda-\alpha)\Gamma(\alpha+1)} \quad (x > 0; \quad \mathfrak{Re}(\lambda) < 1). \tag{4.7.24}$$

It is directly verified, as in the proof of Lemma 4.6, that the right-hand side of (4.7.24) as a function of $\alpha \in \mathbb{C}$ is of exponential type π, and hence $x^{\alpha}\left({}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}t^{-\lambda}\right)(x)/\Gamma(\alpha+1) \in B_{\pi}^{p}$.

By the known formula for the Riemann-Liouville fractional integral (Eq. (2.50) in [501]), and an analoguous formula for the Riemann-Liouville fractional derivative, one has for $x > 0$ and $\mathfrak{Re}(\lambda) < 1$,

$$\begin{aligned} &\frac{x^{\alpha}\left({}^{\mathrm{RL}}\mathbf{D}_{0+}^{\alpha}t^{-\lambda}\log(t)\right)(x)}{\Gamma(\alpha+1)} \\ &\quad = \frac{\Gamma(1-\lambda)x^{-\lambda}}{\Gamma(1-\lambda-\alpha)\Gamma(\alpha+1)}[\log(x) + \psi(1-\lambda) - \psi(1-\lambda-\alpha)]. \end{aligned} \tag{4.7.25}$$

The functions $\Gamma(z)$ and $\psi(z)$ have the same simple poles $z = -k$ ($k \in \mathbb{N}_0$). Therefore the right-hand side of (4.7.25) is an entire function of α. Using the asymptotic relation (4.7.7) for the Gamma function and the asymptotic estimate for $\psi(z)$ at infinity, given in the second formula of (4.3.42), it is established similarly as in the proof of Lemma 4.6, that the right-hand side of (4.7.25) as a function of $\alpha \in \mathbb{C}$ is of exponential type π, and hence $x^\alpha \left({}^{\mathrm{RL}}\mathbf{D}^\alpha_{0+} t^{-\lambda} \log(t)\right)(x)/\Gamma(\alpha+1) \in B^p_\pi$. In particular, when $\lambda = 0$, $x^\alpha \left({}^{\mathrm{RL}}\mathbf{D}^\alpha_{0+} \log(t)\right)(x)/\Gamma(\alpha+1) \in B^p_\pi$, and the lemma is proved. □

Example 4.4. Here we consider an application of Theorem 4.44, namely to the power function $f_1(x) = x^{-\lambda}$ $(x > 0, \quad \lambda \in \mathbb{C})$. Then $\delta x^{-\lambda} = -\lambda x^{-\lambda}$ and $\delta^n x^{-\lambda} = (-\lambda)^n x^{-\lambda} = \left({}^{\mathrm{RL}}D^n_{0+} t^{-\lambda}\right)(x)$ for $n > 1$. But

$$\left(\frac{d}{dx}\right)^k \left(x^{-\lambda}\right) = [-\lambda]_k x^{-\lambda-k} \quad (\lambda \in \mathbb{C}, \quad k \in \mathbb{N}_0).$$

On the other hand, by (4.7.18), for $x > 0$ and $\lambda \in \mathbb{C}$, we obtain

$$x^n \left(\frac{d}{dx}\right)^n \left(x^{-\lambda}\right) = \sum_{k=0}^n s(n,k)(-\lambda)^k x^{-\lambda} \delta^k x^{-\lambda} = x^{-\lambda}[-\lambda]_n.$$

By Theorem 4.44 and Lemma 4.7, we can obtain for $f_1(x)$, with $x > 0$ and $\lambda \in \mathbb{C}$, $\Re e(\lambda) < 1$, $|\lambda| < 1$, the following relation

$$\begin{aligned} x^\alpha \left({}^{\mathrm{RL}}D^\alpha_{0+} t^{-\lambda}\right)(x) &= \sum_{k=0}^\infty s(\alpha,k)\delta^k x^{-\lambda} \\ &= x^{-\lambda} \sum_{k=0}^\infty s(\alpha,k)(-\lambda)^k = x^{-\lambda}[-\lambda]_\alpha. \end{aligned} \tag{4.7.26}$$

The series above converge for $|\lambda| < 1$.

Example 4.5. As a second application of Theorem 4.44, take $f_2(x) = \log(x)$ $(x > 0)$. Then for $\delta = x\frac{d}{dx}$, $\delta \log(x) = 1$ and $\delta^m \log(x) = 0$ for $m > 1$. Then from (4.7.18) we have for $n \in \mathbb{N}$

$$x^n \left(\frac{d}{dx}\right)^n \log(x) = \sum_{k=0}^n s(n,k)\delta^k f(x) = s(n,0)\log(x) + s(n,1).$$

According to (4.2.10), (4.2.23) and (4.7.9) this yields for $x > 0$ and $n \in \mathbb{N}_0$,

$$x^n \left(\frac{d}{dx}\right)^n \log(x) = \frac{1}{\Gamma(1-n)} \log(x) + (-1)^{n-1}(n-1)! = (-1)^{n-1}\Gamma(n). \tag{4.7.27}$$

By Theorem 4.44 and Lemma 4.7 and (4.2.10), we obtain the following result.

Corollary 4.26. *For $\alpha \in \mathbb{C}$ and $x > 0$, there holds*

$$\begin{aligned} x^{\alpha}\left({}^{\mathrm{RL}}D_{0+}^{\alpha}\log(t)\right)(x) &= s(\alpha,0)\log(x) + s(\alpha,1) \\ &= \frac{\log(x) + \psi(1) - \psi(1-\alpha)}{\Gamma(1-\alpha)}. \end{aligned} \tag{4.7.28}$$

Observe that for $\alpha = n \in \mathbb{N}$ (4.7.28) coincides with (4.7.27),

$$\begin{aligned} x^{n}\left(\frac{d}{dx}\right)^{n}\log(x) &= \lim_{\alpha\to n}\left[x^{\alpha}\left({}^{\mathrm{RL}}D_{0+}^{\alpha}\log(t)\right)(x)\right] \\ &= \lim_{\alpha\to n}\frac{\log(x)+\psi(1)-\psi(1-\alpha)}{\Gamma(1-\alpha)} = -\lim_{\alpha\to n}\frac{\psi'(1-\alpha)}{\Gamma'(1-\alpha)} \\ &= (-1)^{n-1}\Gamma(n). \end{aligned}$$

4.7.5 *The representation of a general fractional difference operator via $s(\alpha, k)$*

Let Δ^k be the finite difference of order $k \in \mathbb{N}_0$ given by (4.1.8) and (4.1.9). A well-known operator in the calculus of finite differences, the operations of which are analogues to those of $\delta = xd/dx$, is the operator $\theta f(x) = x\Delta f(x)$, for which there holds the iterative formula

$$\theta^n f(x) = \sum_{k=1}^{n}[x+k-1]_k S(n,k)\Delta^k f(x) \quad (n \in \mathbb{N}), \tag{4.7.29}$$

where $S(n,k)$ $(n \in \mathbb{N},\quad k = 1, 2, \cdots, n)$ are the Stirling numbers of the second kind for $\alpha = n$.

If we multiply equation (4.7.29) by the Stirling functions of the first kind $s(m,n)$, sum it from $n = 1$ to $n = m \in \mathbb{N}$ and use the first orthogonality formula in (4.7.14), we obtain the *inversion* of the operator θ^n in the form (see, e.g., p. 200 of [296])

$$\Delta^m f(x) = \frac{1}{[x+m-1]_m}\sum_{n=1}^{m} s(m,n)\theta^n f(x) \quad (m \in \mathbb{N}). \tag{4.7.30}$$

We now establish a generalization of this relation for a generalized "infinite" or fractional order difference $\boldsymbol{\Delta}^{\alpha} f$, with complex order $\alpha \in \mathbb{C}$, defined for suitable functions f by

$$\boldsymbol{\Delta}^\alpha f = \Delta^\alpha f \quad (\Re e(\alpha) \geq 0); \quad \boldsymbol{\Delta}^\alpha f = \Delta^{-\alpha} f \quad (\Re e(\alpha) < 0), \tag{4.7.31}$$

where (it being a true infinite series)

$$\Delta^\alpha f(x) = e^{\alpha\pi i} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x+j) \quad (\alpha \in \mathbb{C}, \quad \Re e(\alpha) \geq 0). \tag{4.7.32}$$

Note that if $\alpha = k \in \mathbb{N}_0$, then, by (4.7.9), $\binom{k}{j} = 0$ for $j = k+1$, $k+2, \ldots$, so that (4.7.32) turns out to be the classical finite difference $\Delta^k f$ given by (4.1.8).

Theorem 4.45. *Let $\alpha \in \mathbb{C}$ and $f(x)$, $x \in \mathbb{R}$, be a function such that*

$$g_\alpha(f) = \frac{[x+\alpha-1]_\alpha}{\Gamma(\alpha+1)} \boldsymbol{\Delta}^\alpha f(x) \in B_\pi^p \quad for \quad some \quad 1 \leq p < \infty. \tag{4.7.33}$$

Then

$$\boldsymbol{\Delta}^\alpha f(x) = \frac{1}{[x+\alpha-1]_\alpha} \sum_{n=1}^{\infty} s(\alpha, n) \theta^n f(x), \tag{4.7.34}$$

if the series is convergent.

Proof. Using (4.7.29) and interchanging the orders of summation, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} s(\alpha, n) \theta^n f(x) &= \sum_{n=1}^{\infty} s(\alpha, n) \sum_{k=1}^{n} [x+k-1]_k S(n,k) \Delta^k f(x) \\
&= \sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} s(\alpha, n) S(n,k) \right) [x+k-1]_k \Delta^k f(x).
\end{aligned}$$

Then an application of the orthogonality property (4.7.15) yields

$$\sum_{n=1}^{\infty} s(\alpha, n) \theta^n f(x) = \sum_{k=1}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)} \frac{\sin[\pi(\alpha-k)]}{\pi(\alpha-k)} [x+k-1]_k \Delta^k f(x). \tag{4.7.35}$$

On the other hand, applying Theorem 4.41 to $(g_\alpha)(f)$, we have

$$\frac{[x+\alpha-1]_\alpha}{\Gamma(\alpha+1)}\boldsymbol{\Delta}^\alpha f(x)=\sum_{k=-\infty}^{\infty}\frac{[x+k-1]_k}{\Gamma(k+1)}\frac{\sin[\pi(\alpha-k)]}{\pi(\alpha-k)}\Delta^k f(x),$$

which, since $[x+k-1]_k/\Gamma(k+1)=0$ for $k\in\mathbb{Z}^-$, yields

$$\frac{[x+\alpha-1]_\alpha}{\Gamma(\alpha+1)}\boldsymbol{\Delta}^\alpha f(x)=\sum_{k=0}^{\infty}\frac{[x+k-1]_k}{\Gamma(k+1)}\frac{\sin[\pi(\alpha-k)]}{\pi(\alpha-k)}\Delta^k f(x). \quad (4.7.36)$$

Therefore, (4.7.34) follows from (4.7.35) and (4.7.36). □

Corollary 4.27. *If the conditions of Theorem 4.45 are satisfied, then for the generalized fractional difference* $\Delta^\alpha f$ *of order* $\alpha\in\mathbb{C}$, $\mathfrak{Re}(\alpha)\geq 0$, *given by* (4.7.32), *there holds the series representation* (4.7.34), *provided the series converges.*

For the following application of the Theorem 4.45 we need the auxiliary result

Lemma 4.8. *Let* $\alpha\in\mathbb{C}$, *and let* $g_\alpha(f)$ *be given by* (4.7.33). *Then the specific functions* $g_\alpha\left(e^{\lambda x}\right)$ *with* $\lambda<0$, $g_\alpha(\psi)$ *and* $g_\alpha\left(1/[x-1]_\lambda\right)$ *with* $\lambda\in\mathbb{C}$, $\mathfrak{Re}(\lambda)>-|\mathfrak{Re}(\alpha)|$, *as functions of* α, *are of exponential type* π,

$$g_\alpha\left(e^{\lambda x}\right)\in B_\pi^p \quad (\lambda<0), \quad (4.7.37)$$

$$g_\alpha(\psi)\in B_\pi^p,$$

$$g_\alpha\left(\frac{1}{[x-1]_\lambda}\right)\in B_\pi^p \quad (\mathfrak{Re}(\lambda)>-|\mathfrak{Re}(\alpha)|). \quad (4.7.38)$$

Proof. First note that in accordance with (4.7.31) and (4.7.32) the infinite series $\boldsymbol{\Delta}^\alpha\left(e^{\lambda x}\right)$ converges for any $\alpha\in\mathbb{C}$ and $\lambda<0$. Using (4.7.33) and noting (4.3.6) (with $z=e^\lambda$ and $\mu=-\alpha$), we have for the first $g_\alpha\left(e^{\lambda x}\right)$

$$g_\alpha\left(e^{\lambda x}\right)=\frac{\Gamma(x+\alpha)}{\Gamma(\alpha+1)\Gamma(x)}e^{\alpha\pi i}e^{\lambda x}\left(1-e^\lambda\right)^\alpha \quad (\mathfrak{Re}(\alpha)\geq 0),$$

$$g_\alpha\left(e^{\lambda x}\right)=\frac{\Gamma(x+\alpha)}{\Gamma(\alpha+1)\Gamma(x)}e^{-\alpha\pi i}e^{\lambda x}\left(1-e^\lambda\right)^{-\alpha} \quad (\mathfrak{Re}(\alpha)<0).$$

By the first relation in (4.3.42), with $t = \alpha$, $b = 1$, $a = x$, it is directly verified that

$$\lim_{\alpha\to\infty, \Re e(\alpha)>0} \frac{\Gamma(x+\alpha)}{\Gamma(\alpha+1)} \left(1-e^{\lambda}\right)^{\alpha} = 0; \tag{4.7.39}$$

$$\lim_{\alpha\to\infty, \Re e(\alpha)<0} \frac{\Gamma(x+\alpha)}{\Gamma(\alpha+1)} \left(1-e^{\lambda}\right)^{-\alpha} = 0. \tag{4.7.40}$$

If for any fixed $x > 0$ we choose $R > 0$ sufficiently large, then from (4.7.37) and (4.7.38) we deduce for any $\alpha \in \mathbb{C}$ the estimate

$$\left|g_\alpha\left(e^{\lambda x}\right)\right| \leq Ae^{\pi|y|} \quad (z = x+iy;\ A>0;\quad |z| \geq R).$$

If $|z| \leq R$, then arguments similar to those in the proof of Lemma 4.6 give

$$\left|g_\alpha\left(e^{\lambda x}\right)\right| \leq Be^{\pi|y|} \quad (z = x+iy;\ B>0;\quad |z| \leq R).$$

If $\alpha \in \mathbb{R}$, then in accordance with (4.7.39), $g_\alpha\left(e^{\lambda x}\right) \in L^p(\mathbb{R})$ for any $1 < p < \infty$. This proves (4.7.37), namely that $g_\alpha\left(e^{\lambda x}\right) \in B^p_\pi$.

Using the second and first asymptotic estimates in (4.3.42), and taking (4.3.28) into account, we have for $\alpha \in \mathbb{C}$ the estimates

$$\left|(-1)^j \binom{\pm\alpha}{j} \psi(x+j)\right| \leq A\frac{\log(j)}{(j)^{|\Re e(\alpha)|+1}} \qquad (A>0;\quad j\in N),$$

$$\left|(-1)^j \binom{\pm\alpha}{j} \left(\frac{1}{[x+j-1]_\lambda}\right)\right| \leq B\frac{1}{(j)^{|\Re e(\alpha)|+\Re e(\lambda)+1}} \quad (B>0;\quad j\in N).$$

Thus the series $\boldsymbol{\Delta}^\alpha(\psi)$ and $\boldsymbol{\Delta}^\alpha\left(1/[x-1]_\lambda\right)$ converge when $|\Re e(\alpha)| > 0$ and $|\Re e(\alpha)| + \Re e(\lambda) > 0$, respectively. The result (4.7.37) and relation (4.7.38) are proved by using the asymptotic properties of the Gamma and Digamma functions, similarly to the above for (4.7.37). □

Example 4.6. Here we consider an application of Theorem 4.45, namely to the Digamma function $f_3(x) = \psi(x) = \Gamma'(x)/\Gamma(x)$, $x > 0$. For this specific function, the operator $\theta = x\Delta$ satisfies

$$\theta\psi(x) = (x\Delta)\psi(x) = 1, \quad \theta^k\psi(x) = 0 \quad (k = \mathbb{N}\backslash\{1\}). \tag{4.7.41}$$

Thus the classical inversion formula (4.7.30) applied to the function $f_3(x) = \psi(x)$ $(x > 0)$ takes on the form, also recalling (4.2.23),

$$\Delta^m \psi(x) = \frac{s(m,1)\ \theta\psi(x)}{[x+m-1]_m} = (-1)^{m-1}(m-1)!\frac{\Gamma(x)}{\Gamma(x+m)}. \tag{4.7.42}$$

Our new application is the extension of the finite difference result (4.7.42) to the generalized fractional order difference $\mathbf{\Delta}^\alpha \psi(x)$, given by

Corollary 4.28. *For the function $f(x) = \psi(x)$, $x > 0$, the series given by $\mathbf{\Delta}^\alpha \psi(x)$ converges for $\alpha \in \mathbb{C}$, and has the representation*

$$\mathbf{\Delta}^\alpha \psi(x) = \left[\frac{\psi(1) - \psi(1-\alpha)}{\Gamma(1-\alpha)}\right] \frac{\Gamma(x)}{\Gamma(x+\alpha)}. \tag{4.7.43}$$

Proof. By Lemma 4.8, $g_\alpha(\psi)$ as a function of α is of exponential type π, $g_\alpha(\psi) \in B_\pi^p$, and thus we can apply Theorem 4.45. Noting (4.7.41) and (4.2.16), (4.7.34) turns out to be for $x > 0$

$$\mathbf{\Delta}^\alpha \psi(x) = \frac{s(\alpha,1)\theta\psi(x)}{[x+\alpha-1]_\alpha} = \left[\frac{\psi(1) - \psi(1-\alpha)}{\Gamma(1-\alpha)}\right] \frac{\Gamma(x)}{\Gamma(x+\alpha)}.$$

□

Example 4.7. As another application of Theorem 4.45 let us take the function $f_4(x) = 1/[x-1]_\lambda$ with complex $x, \lambda \in \mathbb{C}$, for which

$$\theta^k f_4(x) = \frac{(-\lambda)^k}{[x-1]_\lambda} \quad (k \in \mathbb{N}), \tag{4.7.44}$$

see p. 200 of [296].

Here (4.7.30) can readily be shown to take the form for $x, \lambda \in \mathbb{C}$,

$$\Delta^m \left(\frac{1}{[x-1]_\lambda}\right) = \frac{[-\lambda]_m}{[x+m-1]_m [x-1]_\lambda} = \frac{\Gamma(x-\lambda)}{\Gamma(x+m)} \frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda-m)}.$$

Corollary 4.29. *Let $\alpha \in \mathbb{C}$ and $\lambda \in \mathbb{C}$ be such that $\Re e(\lambda) > -|\Re e(\alpha)|$ and $|\lambda| < 1$. For the function $f_4(x) = 1/[x-1]_\lambda$, $x \in \mathbb{R}$, the series given by $\mathbf{\Delta}^\alpha (1/[x-1]_\lambda)$ converges for $\alpha \in \mathbb{C}$, and has the representation for $x \in \mathbb{R}$*

$$\begin{aligned} \mathbf{\Delta}^\alpha \left(\frac{1}{[x-1]_\lambda}\right) &= \frac{1}{[x+\alpha-1]_\alpha [x-1]_\lambda} \left([-\lambda]_\alpha - s(\alpha,0)\right) \\ &= \frac{\Gamma(x-\lambda)}{\Gamma(x+\alpha)} \left([-\lambda]_\alpha - \frac{1}{\Gamma(1-\alpha)}\right). \end{aligned} \tag{4.7.45}$$

Proof. By Lemma 4.8, we know that $g_\alpha(1/[x-1]_\lambda)$ as a function of α is of exponential type π, $g_\alpha(1/[x-1]_\lambda) \in B^p_\pi$, and thus we can apply Theorem 4.45. Taking (4.7.44) and definition (4.2.3) into account, we have for $\alpha \in \mathbb{C}$

$$\mathbf{\Delta}^\alpha\left(\frac{1}{[x-1]_\lambda}\right) = \frac{1}{[x+\alpha-1]_\alpha}\sum_{n=1}^{\infty} s(\alpha,n)\frac{(-\lambda)^n}{[x-1]_\lambda} =$$

$$= \frac{1}{[x+\alpha-1]_\alpha[x-1]_\lambda}([-\lambda]_\alpha - s(\alpha,0)),$$

which completes the proof. □