Fairness on the Web: Alternatives to the Power Law

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ABSTRACT
This paper presents several measures of fairness and inequality based on the degree distribution in networks, as alternatives to the well-established power-law exponent. Networks such as social networks, communication networks and the World Wide Web itself are often characterized by their unequal distribution of edges: Few nodes are attached to many edges, while many nodes are attached to only few edges. The inequality of such network structures is typically measured using the power-law exponent, stating that the number of nodes with a given degree is proportional to that degree taken to a certain exponent. However, this approach has several weaknesses, such as its narrow applicability and expensive computational complexity. Beyond the fact that power laws are by far not a universal phenomenon on the Web, the power-law exponent has the surprising property of being *negatively* correlated with the usual notion of inequality, making it unintuitive as a fairness measure. As alternatives, we propose several measures based on the Lorenz curve, which is used in economics but rarely in networks study, and on the information-theoretical concept of entropy. We show in experiments on a large collection of online networks that these measures do not suffer under the drawbacks of the power-law exponent.

Author Keywords
Network analysis, Power-law exponent, Gini coefficient, Fairness, Entropy

ACM Classification Keywords
H.4 Information Systems Applications: Miscellaneous

General Terms
Experimentation, Measurement, Theory

INTRODUCTION
A very common approach to analysing the structure of the Web is in terms of its degree distribution. For instance, it is well known that only very few websites have a very high number of inbound links, while most websites have only a few inbound links. This inequality or *unfairness* of the Web's degree distribution is typically modeled as a power law, which mathematically states that the number of websites with n inbound links is approximately $cn^{-\gamma}$ for some numbers c and γ. In this approach, the number γ is called the power-law exponent, and is typically treated as a main network characteristic intended to describe the fairness or unfairness of the degree distribution.

This power-law model however has several drawbacks, of which the largest one is that it is not applicable to all networks. By definition, the power-law exponent γ is only defined for networks whose degree distribution follows a power law. It has however been shown that power laws are far from being valid for all networks. Nevertheless, many networks are subjected to power-law analysis, even if this cannot be substantiated statistically. For this reason, we will look for measures of fairness that are independent of any specific degree distribution. As we show in this paper, there are other peculiarities with power-law analysis. First, we will show that the power-law exponent does not measure what many analyses assume it to measure. In fact, we will show that even in the case of a statistically perfect power law, the number γ has the opposite meaning of that attributed to it in most analyses.

As alternatives we present fairness measures based on the Lorenz curve, and on statistical entropy. The Lorenz curve is, similarly to the plain degree distribution plot, an aggregate plot of the distributions of degrees. However, instead of looking for a slope in the plots as done in power-law analysis, the two measures that can be defined with it are independent of the underlying distribution: The Gini coefficient and a measure which is often cited but has no common name, which we will call the *balanced inequality ratio*. As for entropy, its usage in physics and other disciplines to measure the *uncertainty* of a random variable lends itself to model fairness. We will thus apply statistical entropy to the two distributions related to the structure of a network: the degree distribution and the edge distribution. As we will show, each of these leads to a different kind of entropy, based on different underlying assumptions. Furthermore, we will show that these alternative fairness measures are better suited to studies in which the long tail is important. Finally, we will compare values of the power-law exponent and the other proposed measures on a large collection of online network datasets, showing that our alternative fairness measures can be computed much more efficiently.

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Throughout the paper, we will use as examples the networks from the Koblenz Network Collection (KONECT)[1], a large collection of networks from the Web and other sources.

The rest of the paper is structured as follows. We introduce the concept of fairness in the section *Background*, define and review the problems with power-law analysis in the section *Power Law Exponent*, introduce alternative fairness measures in the section *Alternative Measures of Fairness* and give the results of a comparative evaluation in the section *Evaluation*.

BACKGROUND

Fairness in the general sense denotes situations which are *equitable* or *just*. This definition evidently depends on the notion of a situation being just and therefore is open to debate. In a narrower sense however, fairness can be understood as a synonym of equality. For instance, dividing a cake fairly between four persons will give each person 25% of the cake. It is this narrow definition of fairness that we will use in this paper.

To use the concept of fairness, it must be specified *which* sum will be divided among which entities. Since a network consists of vertices and edges, it is a straightforward idea to interpret fairness as the equality of the distribution of edges among vertices. The opposite assignment, the assignment of vertices to edges, is trivial, since each edge would possess exactly two vertices, and the graph would be trivially fair[2]. Using these assumptions about fairness, a fully fair network is thus one in which all vertices are connected to the same number of edges. These kinds of graphs are called regular graphs, and possess many special properties [5]. Actual networks however are far from being regular, and this is why the study of fairness is of relevance in networks. Note that under these considerations, unfairness could be called *irregularity*. Due to the many other possible meanings of *irregular*, we will avoid this terminology. Table 1 gives an overview of terms used to denote equal and unequal distributions. Graphs with equally distributed edges can be called *fair* in analogy with the distribution of wealth. The extreme cases of equal and unequal edge distributions lead to a regular graph, in which all nodes have the same number of neighbors, and to a star graph, where all edges are between one specific node and all other nodes. Another encountered term is *diversity*, which can be interpreted as a synonym of *heterogeneity* and thus may denote equal distributions of edges.

In the rest of this paper, a network will be modeled as an undirected graph $G = (V, E)$, in which V is the set of vertices and E is the multiset of edges, allowing multiple edges between two vertices. For a vertex $u \in V$, we define its degree $d(u)$ as the number of edges incident to it, taking into account multiple edges.

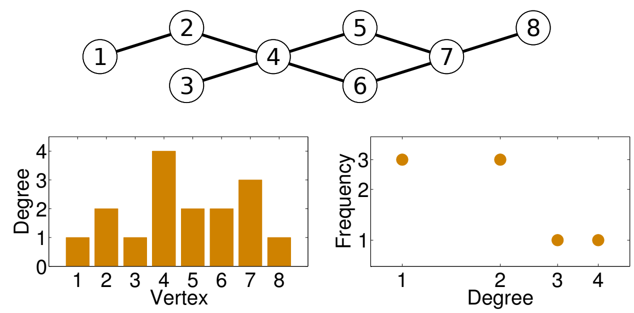
[1] konect.uni-koblenz.de

[2] This idea can be generalized to hypergraphs, in which individual hyperedges may be connected to any number of vertices. Due to the duality of vertices and edges in hypergraphs however this case can be reduced to the variant we consider in this paper.

Table 1. Terms customarily used to denote equality and inequality of the node degrees in a network.

	Equality	Inequality
Assessment	Fair	Unfair
Distribution	Homogeneous	Heterogeneous
Extreme case	Regular graph	Star graph
Entropy	High	Low
Attributes	Diversity	Scale-freeness

The distribution of degrees over all vertices in a network is called the degree distribution, and has been extensively studied in the literature [7]. A very common analysis made with the degree distribution is to observe *power laws*, i.e., the observation that the number of nodes that have degree n is proportional to $n^{-\gamma}$. In this type of analysis, the parameter γ is called the power-law exponent.



(a) Edge distribution (b) Degree distribution

Figure 1. The two kinds of distributions that we consider: (a) The edge distribution (b) The degree distribution.

Note that there are two distributions which are related but clearly different from each other: The edge distribution and the degree distribution, as illustrated in Figure 1.

(a) The edge distribution is defined over all vertices of a network and its value for one vertex equals the vertex's degree. As a probability distribution, it arises from the process of choosing an edge at random, and from the two adjacent vertices choosing one at random. The probability that a given vertex u is chosen then equals $d(u)/2|E|$, where $d(u)$ is the degree of vertex u and $|E|$ is the total number of edges.

(b) The degree distribution is defined over all possible values of the degree, and gives the number of vertices having a certain degree. As a probability distribution, the degree distribution arises from the process of choosing a vertex at random and returning its degree. The probability that a degree n is chosen then equals the number of vertices with degree n divided by the total number of vertices.

While the degree distribution is used more often in the literature, we will see that it is actually the edge distribution which is more relevant as the basis of measuring fairness in networks.

Pareto Principle

The Pareto Principle is a rule of thumb stating that 80% of all people own 20% of all assets. In its original formulation by Vilfredo Pareto from 1919, the example given was that 80% of the land area in Italy was owned by 20% of the population [14].

Statements such as this one can be generalized to any other distribution by adjusting the numbers, leading for instance to the statement that on Facebook, 25.0% of all users make up 75.0% of all friendship links, or that on Last.fm, 8.9% of bands make up 91.1% of all listening events. As examples of generalizations of the Pareto principle, we give examples from the many networks of the Koblenz Network Collection in Table 2.

Lorenz Curve

The Lorenz curve is a curve that can be drawn from a given distribution, and that is intended to visualize how far the distribution is away from a uniform distribution. An example is shown in Figure 2, for the Facebook network. In the context of networks, the Lorenz curve describes the distribution of edges. In an undirected network, each edge is attached to two nodes. Thus, a network can be viewed as a distribution of edges over vertices. The number of edges owned by a vertex is then equal to the number of neighbors of that vertex, i.e. the degree. The sum of all degrees in the network thus equals twice the number of edges in the network.

The Lorenz curve connects the points $(0, 0)$ and $(1, 1)$, and is defined in the following way. A point (x, y) is part of the curve when the share x of *poorest* nodes cover a share y of all edges. Here *poorest* is meant as having the least degree. If the edges are evenly distributed between all nodes, the share x of the poorest nodes own exactly a share $y = x$ of edges, and therefore the Lorenz curve is the diagonal connecting the points $(0, 0)$ and $(1, 1)$. If the distribution of edges is unequal, the Lorenz curve is situated below the diagonal. In the extreme case of one node owning all edges[3], the Lorenz curve tends towards the two line segments connecting the three points $(0, 0)$, $(1, 0)$ and $(1, 1)$.

To assess the degree to which the distribution of edges is equitable, we can now look at the area between the main diagonal and the Lorenz curve. This area is zero when the distribution is equitable, and 1/2 if it is completely inequitable. Multiplied by two, this area equals the Gini coefficient G, which is thus a value between 0 and 1, denoting a fair distribution for $G = 0$ and an unfair distribution for $G = 1$.

Another statistic associated with the Lorenz curve is given by the intersection point of the Lorenz curve with the antidiagonal given by $y = 1 - x$. By construction, this point equals $(1 - P, P)$ for some $0 < P < 1$, where the value P corresponds exactly to the number "25%" in the statement "25% of all users account for 75% of all friendship links on Facebook". By construction, we can expect P to be smaller when G is large.

[3]This is only possible if all edges are loops.

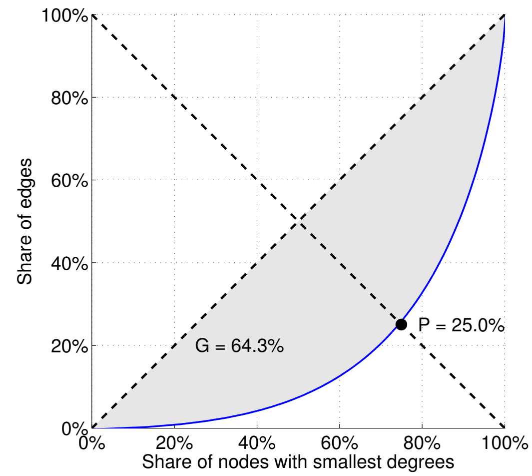


Figure 2. Statistics associated with the Pareto principle. This example is from the Facebook social network [15]. The Lorenz curve (continuous line) gives rise to two statistics: The Gini coefficient G is twice the gray area and the balanced inequality ratio P is the point at which the antidiagonal crosses the Lorenz curve.

Entropy

The entropy is a measure used widely in thermodynamics to characterize the *disorder* of a physical system. In information theory, the entropy is a measure of the quantity of information. Given a random variable, its entropy can be used to describe its *randomness*, with an entropy of zero denoting complete certainty about the distribution, and an entropy of $\ln(n)$ denoting a uniform distribution over n states. In a network, we can compute both the entropy of the edge distribution as well as the entropy of the degree distribution. In fact, we could also compute the entropy of other distributions associated with a network, for instance the entropy of the distribution of the eigenvalues of certain graph matrices.

Given a probability distribution $P(x)$ over a finite set $x \in X$, the entropy of P is defined as

$$H(P) = \sum_{x \in X} -P(x) \ln P(x). \tag{1}$$

The entropy is nonnegative due to $\ln P(x) \leq 0$. Note that we used the natural logarithm in the definition. Instead, any other logarithm $\log_b$ can be used as long as $b > 1$, with the resulting entropy values differing only by a constant term. When defined using the natural logarithm, the entropy has units of *nats*. Using the binary logarithm, we would arrive at units of *bits*.

The entropy can be interpreted as a measure of the uniformity of a distribution: It is zero when $P(x_0) = 1$ for some x_0, and reaches its maximal value of $\ln(|X|)$ for the uniform distribution $P(x) = 1/|X|$ for all x.

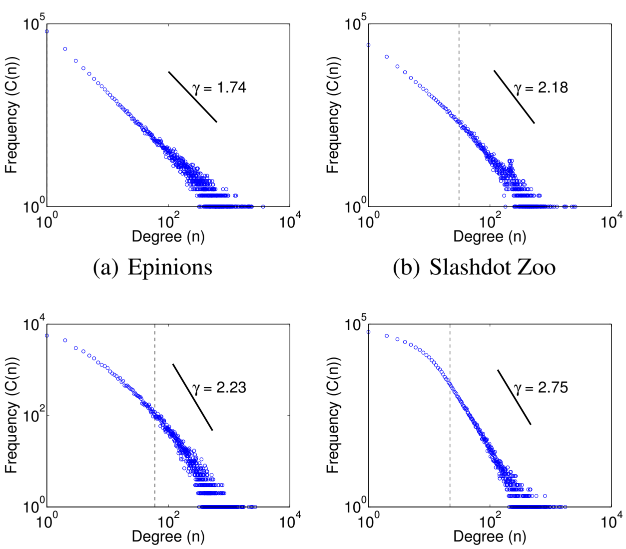
POWER-LAW EXPONENT

In the area of network analysis, the phrase *degree distribution* is mostly associated with the phrase *power law*. This is based on the observation that in many networks, the number of vertices with degree n is proportional to $n^{-\gamma}$, in which

Table 2. Examples of the generalized Pareto principle.

8.9% of bands make up **91.1%** of plays on **Last.fm**.
9.4% of musical genres represent **90.6%** of songs on the **English Wikipedia**.
9.5% of words make up **90.5%** by word count of **Reuters news**.
14.4% of record labels represent **85.6%** of all bands on the **English Wikipedia**.
14.6% of user groups account for **75.4%** of group memberships on **Flickr**.
16.2% of players account for **83.8%** of positions in sports teams on the **English Wikipedia**.
16.2% of songs make up **83.8%** of plays on **Last.fm**.
17.4% of movies receive **82.6%** of ratings on **MovieLens**.
17.7% of profiles receive **82.3%** of ratings on Czech dating site **Libimseti.cz**.
18.8% of groups make up **81.2%** of all group memberships on **YouTube**.
19.8% of all categories make up **80.2%** of all category inclusions on the **English Wikipedia**.
20.3% of all users receive **79.7%** of “friend” and “foe” links on **Slashdot**.
21.3% of users receive **78.7%** of wall posts on **Facebook**.
22.9% of users make up **77.1%** of all “@” mentions on **Twitter**.
23.1% of projects receive **76.9%** of project memberships on **Github**.
25.0% of users cover **75.0%** of all friendship links on **Facebook**.
26.2% of papers receive **73.8%** of citations on **DBLP**.
26.9% of persons send **73.1%** of emails in one **European institution**.
27.3% of users receive **72.7%** of replies on **Digg**.
27.4% of actors account for **72.6%** of movie credits on the **English Wikipedia**.
29.9% of all patents receive **70.1%** of all patent citations among **US patents**.
30.4% of all sites receive **69.6%** of all hyperlinks on the **Web**.

the value γ represents the power-law exponent. Power-law degree distributions arise for instance in the preferential attachment model of Barabási and Albert [2], giving a value of $\gamma = 3$ for the basic preferential model, and other values in derived models.



(a) Epinions (b) Slashdot Zoo

(c) Facebook (d) CiteSeer

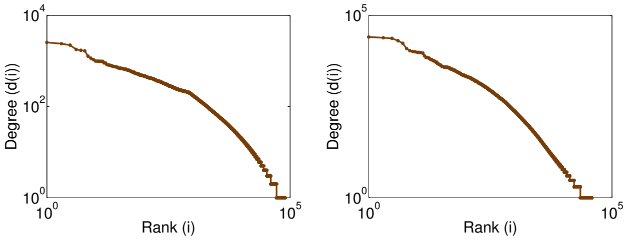
Figure 3. Degree distributions with fitted power-law exponent γ of the Epinions trust network [12], the Slashdot Zoo friend/foe network [8], the Facebook wall post network [15] and the CiteSeer citation network [4]. (a) An apparent power law (b) An apparent but slightly off power law (c) A non-power law (d) An example of a distribution which appears to be a power law starting at a specific degree.

Power-law degree distributions are often a matter of immense discussion (see e.g. [7] and its blog entry[4]), with good arguments that in fact many distributions are in practice misidentified as power laws. For a typical example, see [10], which shows that many distributions arising in biology are in fact not power laws. Formally, exact methods exist to verify whether a given degree distribution is a power law at all [7]. Figure 3 shows examples of degree distributions from actual online network datasets. As can be seen, not all networks follow a power-law distribution. Some only follow a power law beginning at some minimal degree, while others are no power laws at all. Nonetheless, an estimation of the exponent γ is often used as a network characteristic. Values of γ are given for many examples in [1] and [7]. In the experiments in the rest of this paper, we will use the methods of [7] to estimate the power-law exponent. We will now review several properties of the power-law exponent that make it unsuited to characterize fairness of a network.

Negligence of the Long Tail

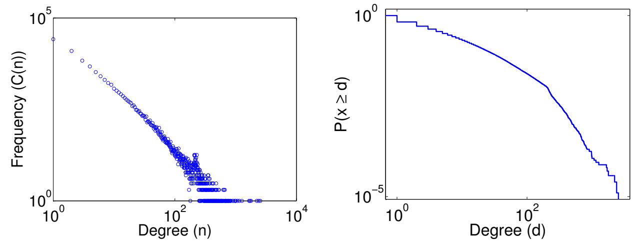
If we look at the example 3(d), it is clear that the degree distribution only follows a power law, if at all, for degrees $d \geq 22$. For smaller degrees, the actual number of nodes is much less than that predicted by a pure power-law distribution. As a result, the power-law model will greatly exaggerate the long tail, i.e. the set of nodes with small degree. Note that the term *long tail* is derived from an alternative way of visualizing a degree distribution, as shown in Figure 4. In these plots, we show the degree $d(u)$ of the node u, ordered from the node with highest degree to the node with smallest degree. Thus, nodes with smallest degrees will form a “long tail” on the right side of the plot – hence the name.

[4]http://cscs.umich.edu/$\sim$crshalizi/weblog/491.html



(a) Slashdot Zoo (b) Reuters

Figure 4. Degree of nodes sorted by decreasing rank, plotted on a log-log scale. (a) The Slashdot Zoo friend/foe network [8] (b) Frequency of words in the Reuters news corpus [9].



(a) Distribution (b) Cumulative distribution

Figure 5. Two ways to visualize the degree distribution of the Slashdot Zoo network: (a) The degree distribution (b) The cumulated degree distribution. The first plot suggests a power law, while the second one does not.

A classical example of this modeling is Zipf's law [11]. Zipf's law is originally a statement about the distribution of word frequency in natural languages. It states that the frequency of a word in a corpus is inversely proportional to the rank of the word. Put differently, the most frequent word is twice as frequent as the second most frequent word, three times are frequent as the third most common word, and so on. Note that we can interpret a text corpus as a bipartite network with multiple edges, where nodes are text documents and words, and each appearance of a word in a document is represented as an edge between the two. The degree of a word node in that network is then exactly the frequency of that word in the corpus, and the word frequency distribution is equal to the degree distribution among all the word nodes. To be precise, Zipf's law is not a power law in the degree distribution, but in the edge distribution. The exponent of this power law is then -1 by construction. In actual text networks, this value of -1 can actually be observed, although only for the most frequent words. For very rare words, the rule breaks down, effectively ignoring the long tail of the distribution.

To summarize, analysis of power laws will ignore the long tail of any distribution, and thus not be characteristic for this part of the network.

Restriction to Power Laws

The power-law exponent only applies to power laws. Although this relation is obvious, it is often ignored. For instance, one can read estimates of γ for networks whose degree distribution is no power law at all. In principle, each analysis of a power-law distribution must also include a statistical significance analysis. This however is often neglected, due to both complexity of the tests involved [7], and due to the high runtime of such tests, as described in the next section. In fact, many networks follow degree distributions that are not power laws. As an example, a certain number of networks follow the DGX distribution [3], which expresses itself as a quadratic function on the log-log plot, and can be understood as a discrete analogue of the lognormal distribution. Figure 3(c) shows one such example, the Facebook wall post network [15].

Another risk of power laws lies in the fact that the degree distribution as typically plotted is very suggestive of power-law relationships, much more so than alternative ways of visualizing degree distributions. From a statistical point of view, the cumulative distribution plot is better suited to visual analysis of a distribution. As an example, Figure 5 compares the cumulated and ordinary degree distribution plot for the Slashdot Zoo network. In this case, a visible straight line on the log-log degree distribution plot does not manifest itself as a straight line on the cumulated plot. If a distribution is a perfect power law, then both plots show straight lines, with exponents differing by one.

Computational Complexity

To estimate a value of γ that matches a given distribution, the state of the art dictates that a maximum-likelihood estimator must be used, along with a goodness-of-fit test to determine the minimal degree $d_{\min}$ under which the power law is not valid anymore [7]. As a result, the computation of γ is expensive when compared to other measures. In particular, the runtime of estimating γ depends on the range of possible values that are tested, and on the granularity of the search.

The Power-law Paradox

What is the meaning of the power-law exponent γ? In the discussion about the Gini coefficient, we mentioned that the Lorenz curve follows the graph's main diagonal when the distribution of edges is equal among all nodes, and that it is most distant from the diagonal when the distribution is completely unequal. Thus, the Gini coefficient, which is proportional to the area between the diagonal and the Lorenz curve, is zero for completely uniform edge distributions, and one for completely unequal distributions. The Gini coefficient can thus be called a measure of unfairness.

Now let us look at the power-law exponent γ. We may ask whether high values of it indicate equal or unequal edge distributions. At first glance, we may think that a high γ indicates an unequal distribution, because if γ is high, then the slope of the degree distribution plot will be very steep, and thus there should be just very few nodes with many edges, and many nodes with few edges. However, the opposite is the case. When γ is high, the distribution is in fact more equal. This can be seen by looking at the Lorenz curves of perfect power-law distributions. Mathematically, a perfect power-law distribution is called a zeta distribution. The zeta

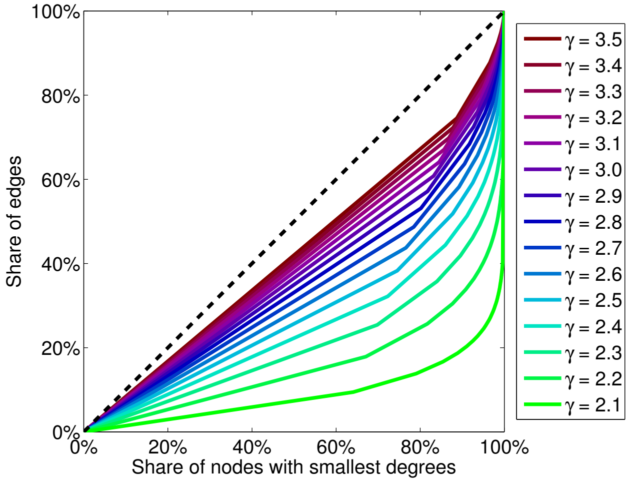


Figure 6. The Lorenz curve of a zeta distribution in function of the distribution parameter γ. The plots show that the Lorenz curve is nearer to the diagonal – and the Gini coefficient is thus smaller – when γ is large. The bends in the curves are not an error; they are a result of the fact that the zeta distribution is discrete.

distribution is defined on positive integers, and its values are proportional to negative powers of the random variable:

$$P(x = n) = n^{-\gamma}/\zeta(\gamma) \tag{2}$$

Here, the Riemann zeta function $\zeta(\gamma) = \sum_{n=1}^{\infty} n^{-\gamma}$ is the normalizing factor. By construction, this distribution is a perfect power law with exponent γ. It is defined for all $\gamma > 1$. Note that for $\gamma = 1$, the sum given above is the harmonic series, which diverges.

We now plot the Lorenz curves of zeta distributions for changing values of γ in Figure 6. As can be observed, the Lorenz curve is nearer to the diagonal for large γ. Thus, the power-law exponent γ is a measure of fairness rather than a measure of unfairness, as we expected. We must note that even if this result is surprising, it is not new. It can be found e.g. in [13].

ALTERNATIVE MEASURES OF FAIRNESS

To overcome the weaknesses of the power-law model to characterize the fairness of the degree distribution in a network, we now review other network measures. The measures we review are first the Gini coefficient, then a statistic which has no common name but which we will call the balanced inequality ratio, and finally the edge distribution entropy and the degree distribution entropy.

Gini Coefficient and Balanced Inequality Ratio

As described in the section *Lorenz Curve*, the Lorenz curve of the edge distribution in a network can be used to characterize the equality of that edge distribution. Figure 2 shows the Lorenz curve of the Facebook network [6].

The Gini coefficient G is twice the area between the Lorenz curve and its main diagonal. It varies between zero and one, taking the value zero for completely equal distributions and the value one for completely unequal distributions. The Gini coefficient is thus a measure of the unfairness of a network's edge distribution.

The intersection of the Lorenz curve with the antidiagonal is another characteristic number of a network's edge distributions. This number is often reported in actual distributions, as shown by the multitude of examples in Table 2. Sentences of this form ("P% of the richest … own $(1 - P)$% of …") are often used in the media, without the number P being identified as characteristic of the distribution being talked about. We will report this value P not in percentages, but as a proportion. The value P is thus a value between zero, denoting full inequality, and $1/2$, denoting complete equality. Since there is no common name for this characteristic number P in the literature, we will give it the name *balanced inequality ratio*.

Edge Distribution Entropy

Since the entropy, as defined in the section *Entropy*, is a measure of the equality of a distribution, we can apply it to the edge distribution over vertices to define the edge distribution entropy. In a graph $G = (V, E)$, the edge distribution entropy is thus

$$H_{\mathrm{e}} = \sum_{u \in V} -\frac{d(u)}{2|E|} \ln \frac{d(u)}{2|E|}. \tag{3}$$

The entropy is nonnegative, and its maximal possible value is $\ln |V|$, which is attained when all nodes $u \in V$ have the same degree $d(u) = 2|E|/|V|$. Thus, the edge distribution entropy H_{e} is a measure of equality. The edge distribution entropy is called *entropy of degree sequence* (EDS) in [17].

Because the edge distribution entropy has a maximal value of $\ln |V|$, we may expect it to be highly correlated to the network's size $|V|$ itself. Therefore, we may normalize it by dividing by $\ln |V|$, resulting in the normalized edge distribution entropy

$$H_{\mathrm{en}} = \frac{1}{\ln |V|} \sum_{u \in V} -\frac{d(u)}{2|E|} \ln \frac{d(u)}{2|E|}. \tag{4}$$

By construction, H_{en} varies in the range $[0, 1]$, with zero denoting complete inequality and one denoting complete equality. A slightly different definition of the normalized edge distribution entropy is called *normalized entropy of degree sequence* (NEDS) in [17].

Degree Distribution Entropy

In addition to the edge distribution, the entropy can also be applied to the degree distribution [16]. Let $C(n)$ be the number of nodes in a network $G = (V, E)$ having degree n. Then the degree distribution entropy is defined as

$$H_{\mathrm{d}} = \sum_{n=1}^{d_{\max}} -\frac{C(n)}{|V|} \ln \frac{C(n)}{|V|}, \tag{5}$$

where $d_{\max}$ is the maximal degree in the network. Note that this entropy is invariant under exchanges of the the values of $C(n)$, i.e., exchanging the number of nodes having degree d_1 and d_2 will not change this entropy. Thus, it may only be used under specific circumstances, such as the network having a power-law degree distribution, a problem which it shares with the power-law exponent γ. The maximal value of H_{d} is $ln(|V| - 1)$ [16]. The degree distribution entropy is thus dependent on the network size.

EVALUATION

We perform four evaluations. First, we compare the six measures of fairness qualitatively by their overall advantages and disadvantages. Then, we analyse their runtime. As a third evaluation, we investigate correlations between all measures. Finally, we provide actual numerical data for all measure applied to a large collection of online network datasets.

Qualitative Comparison

We evaluate all six measures according to the following criteria:

- **Coverage**: Which part of the degree distribution is effectively characterized by the measure? For instance, the power-law exponent γ is usually computed beginning at a minimal degree $d_{\min}$, implying that all nodes with smaller degree are actually ignored by this measure.
- **Independence**: Is the measure independent of basic network characteristics such as the size or the density of a network? If it is not, then the measure is not very useful, since it can be easily predicted by the basic network characteristic.
- **Generality**: To what kind of network can the measure be applied? A measure that can only be applied to a certain kind of network is less useful than one that can be applied to all networks.
- **Runtime**: Ideally, a practical measure of fairness should be efficiently computable. We report an assessment of the runtime of each measure, based on the statistics gathered in the next section.

The results of our comparison are summarized in Table 3. From the table, we can read that the measures without any restrictions are the Gini coefficient G, the balanced inequality ratio P and the normalized edge distribution entropy H_{en}.

Runtime

We computed the six measures on a set of network datasets, as given in Table 4 in the appendix. In the computation of the power-law exponent γ, we restrict the search to the range $\gamma \in [1.01, 9]$ in steps of 0.01. All measures are computed on the same computer using single-threaded algorithms. The reported numbers are the elapsed CPU time. The resulting runtimes are compared in Figure 7 for a selection of seven network datasets. The plots show that computing γ is computationally much more expensive than computing any of the other measures.

What is more, the computed runtimes for the power-law exponent γ only include the time needed to compute γ and $d_{\min}$ using maximum-likelihood and goodness-of-fit methods. In order to compute the statistical significance of the proposed values, Monte-Carlo algorithms must be used [7],

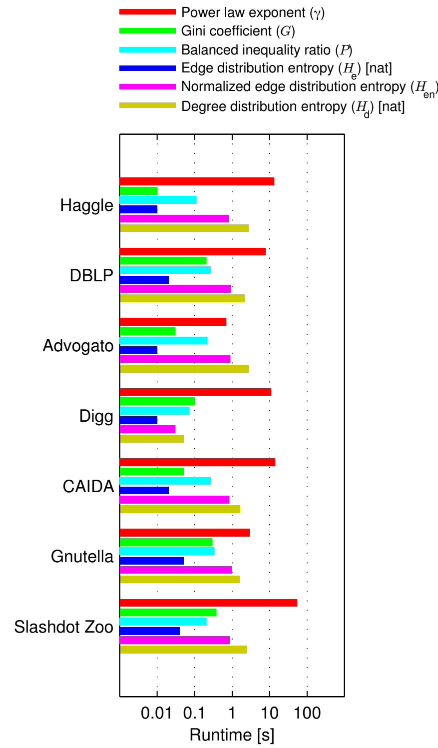


Figure 7. The runtime of various network measures characterizing the fairness of the degree distribution. Runtimes are determined to a granularity of 0.01 seconds. The time axis is logarithmic.

whose runtime are even higher. In fact, we were not able to execute them for the given networks in a reasonable time.

Correlations among Fairness Measures

We investigate how the six measures of fairness are correlated. Figure 8 in the appendix shows pairwise scatter plots showing the values of the six measures for the 120 smallest networks in the Koblenz Network Collection, as well as associated Pearson correlation coefficient values.

From the pairwise scatter plots, we can read that the Gini coefficient G and the balanced inequality ratio P are highly negatively correlated. Furthermore, the normalized edge distribution entropy H_{en} is weakly correlated to both of these, in a positive way to P and in a negative way to G. The power-law exponent γ is very slightly correlated to G and P, but not to other measures. These relationships confirm that the Gini coefficient can be considered a measure of unfairness, and the balanced inequality ratio and the normalized edge distribution entropy a measure of fairness. These correlations also confirm that high values of the power-law exponent γ characterize a fair distribution, in opposition to a naive interpretation of power laws. The other measures are not significantly correlated.

Values

Table 4 in the appendix gives the values of the six measures for a subset of the networks in the Koblenz Network Collection.

Table 3. Qualitative comparison of the six measure of network fairness.

Measure	Coverage	Independence	Generality	Runtime
Power-law exponent γ	$d(u) \geq d_{\mathrm{min}}$	Independent	Power-law networks	Slow
Gini coefficient G	All	Independent	All networks	Fast
Balanced inequality ratio P	All	Independent	All networks	Fast
Edge distribution entropy H_{e}	All	Depends on $\ln \lvert V \rvert$	All networks	Fast
Norm. edge distribution entropy H_{en}	All	Independent	All networks	Fast
Degree distribution entropy H_{d}	All	Depends on $\ln(\lvert V \rvert - 1)$	Power-law networks	Fast

CONCLUSION

We have reviewed the power-law exponent used as a measure of the fairness in the degree distributions of networks. We found that it has several shortcomings, such as its narrow applicability to the single class of networks that are power laws, its high computational complexity, and its unintuitive interpretation. As alternatives, we have proposed several new measures of fairness: The Gini coefficient, the ubiquitous but as yet unnamed balanced inequality ratio, and three variants of the information-theoretic entropy based on the edge and degree distributions. In a qualitative analysis, we found the Gini coefficient, the balanced inequality ratio and the normalized edge distribution entropy to be universally applicable, and representative for the whole network. In a quantitative study on a large collection of network datasets, we could observe that the Gini coefficient and the balanced inequality ratio are highly correlated, while the normalized edge distribution entropy is only weakly correlated to these two. We conclude by repeating the often-given advice that the power-law exponent should only be used when it has been made sure that a network is actually a power law using statistical methods. When the fairness of a general network is to be assessed, we recommend the Gini coefficient, balanced inequality ratio or the normalized edge distribution entropy.

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APPENDIX

The following pages show the table of values for the six measures applied to networks from the Koblenz Network Collection, as well as correlation plots and tests for all measure pairs.

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Table 4. The values of the six fairness measures for a large selection of networks. The networks are part of the Koblenz Network Collection.

Network	Vertices	Edges	Density	γ	G	P	H_{e}	H_{en}	H_{d}
Advogato	6,535	51,397	15.725	3.510	0.705	0.221	7.809	0.889	3.316
Amazon	402,439	3,387,388	16.794	3.391	0.331	0.386	12.686	0.983	3.516
arXiv astro-ph	18,772	396,160	42.208	2.601	0.610	0.265	9.166	0.931	3.902
arXiv hep-ph	28,093	6,296,894	448.289	1.471	0.611	0.266	9.573	0.935	6.283
arXiv hep-th	22,908	4,889,596	426.890	1.471	0.670	0.242	9.185	0.915	6.102
Berkeley/Stanford	685,230	7,600,595	22.184	2.451	0.659	0.250	11.911	0.886	3.677
Caenorhabditis elegans	382	4,596	9.007	2.211	0.618	0.269	5.497	0.899	2.589
CAIDA	26,475	106,762	8.065	2.121	0.628	0.269	8.535	0.838	1.617
California	1,965,206	5,533,214	5.631	7.021	0.186	0.438	14.420	0.995	1.257
CiteSeer	384,413	1,751,492	9.113	2.751	0.579	0.285	12.155	0.945	2.308
CiteULike	885,046	2,411,819	5.285	2.291	0.707	0.227	11.599	0.847	1.948
DBLP	12,591	49,759	7.904	3.351	0.658	0.235	8.570	0.908	2.631
DBLP	3,578,447	5,344,649	2.987	2.371	0.542	0.300	14.497	0.961	1.737
DBLP	916,319	11,303,522	14.305	2.161	0.680	0.233	13.042	0.950	2.751
DBpedia	2,152,642	7,494,124	6.506	2.151	0.649	0.263	12.709	0.872	2.386
Digg	30,398	87,627	5.685	3.261	0.632	0.255	9.496	0.920	2.383
Enron	87,273	1,148,072	7.377	1.661	0.908	0.101	9.415	0.828	1.722
Epinions	876,252	13,668,320	31.197	1.951	0.769	0.197	11.337	0.829	3.748
EU institution	265,214	420,045	3.168	2.631	0.663	0.240	9.972	0.798	0.896
Facebook New Orleans	60,102	1,545,686	48.507	2.381	0.643	0.250	10.304	0.931	4.580
Filmtipset	75,360	1,266,753	31.975	1.581	0.803	0.171	9.557	0.851	3.480
Flickr	499,610	8,545,307	34.208	1.451	0.849	0.144	11.267	0.859	3.467
Gnutella	62,586	147,892	4.726	4.831	0.563	0.264	10.475	0.948	2.081
Google	875,713	5,105,039	11.659	2.901	0.597	0.279	12.878	0.941	3.280
Google.com internal	15,763	171,206	21.723	2.571	0.636	0.274	8.499	0.879	3.500
Haggle	41	28,244	21.161	1.421	0.839	0.110	4.462	0.795	2.733
Internet topology	6,056	171,403	6.588	1.861	0.808	0.164	8.445	0.808	1.895
Líbímseti.cz	220,970	17,359,346	157.119	1.911	0.710	0.232	11.145	0.906	5.602
Pretty Good Privacy	7,328	24,340	4.554	4.291	0.592	0.269	8.552	0.922	2.182
Reality Mining	87	1,086,404	52.896	1.121	0.483	0.327	4.485	0.983	3.742
Route Views	2,256	13,895	4.293	2.151	0.608	0.273	7.493	0.854	1.895
Similarity	923	2,982	3.068	3.861	0.448	0.328	7.218	0.953	1.904
Skitter	966,723	11,095,298	13.081	2.291	0.696	0.233	12.759	0.889	3.128
Slashdot Zoo	79,120	515,571	13.033	2.181	0.774	0.185	9.822	0.871	2.845
TREC WT10g	1,601,787	8,063,026	10.068	2.301	0.637	0.267	13.040	0.913	2.970
Twitter	705,632	4,664,605	5.359	2.331	0.843	0.149	11.888	0.883	1.833
U. Rovira i Virgili	761	5,451	9.622	6.771	0.491	0.315	6.631	0.943	3.178
US patents	3,774,768	16,518,948	8.752	4.031	0.516	0.309	14.672	0.969	3.079
World Wide Web	325,729	1,497,134	9.193	2.121	0.764	0.194	11.118	0.876	2.328
YouTube	124,325	293,360	4.719	2.311	0.693	0.230	10.301	0.878	1.890

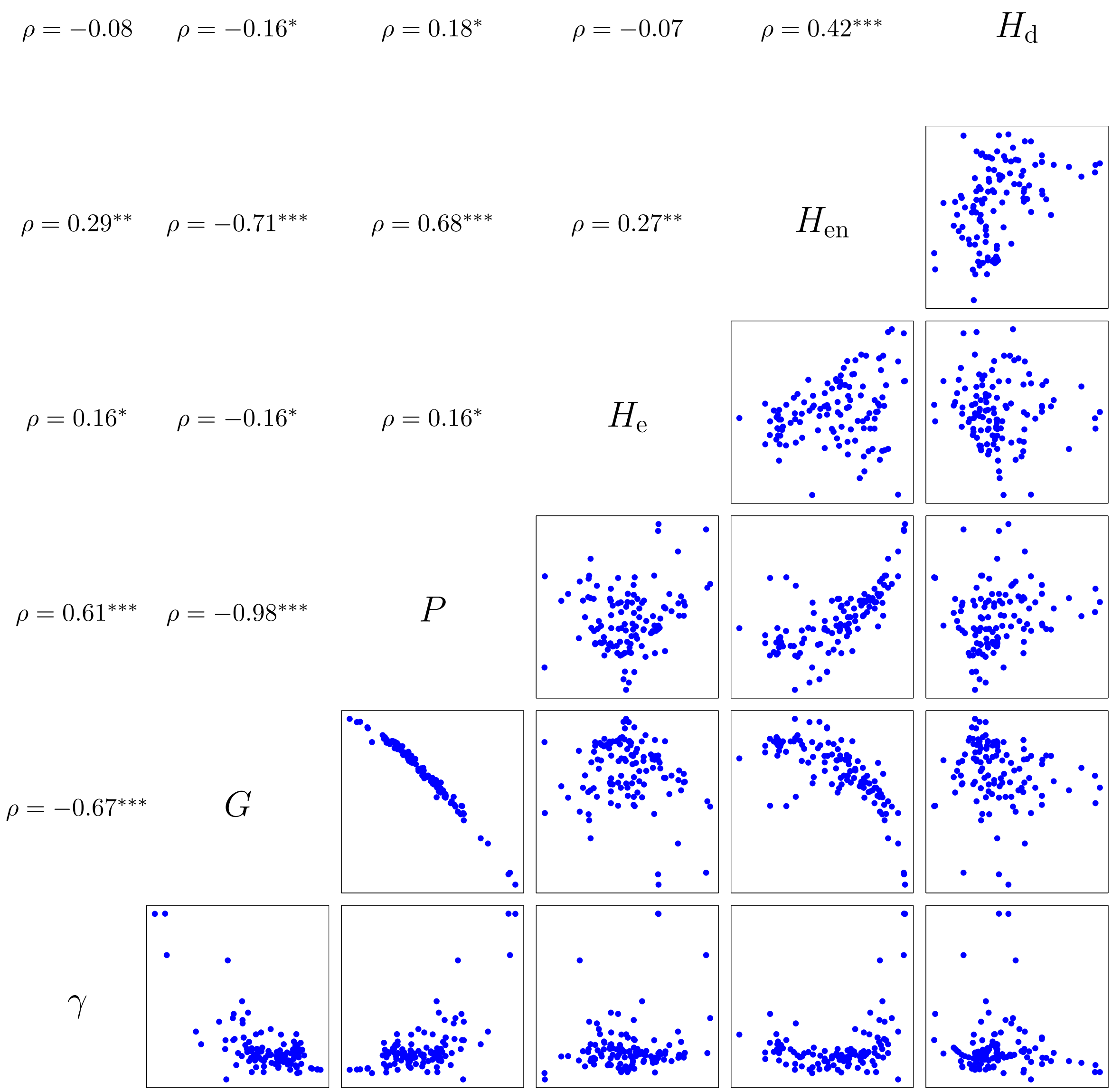

Figure 8. Pairwise comparisons of fairness measures shown as scatter plots and Pearson correlation coefficients. Each subplot shows, for two measures, the values of these measures plotted for all networks given in Table 4. For each pair of measures, we give the Pearson correlation coefficient ρ and use one, two and three stars to denote that the hypothesis that two measures are uncorrelated can be rejected to a p-value of 0.1, 0.01 and 0.001 respectively.