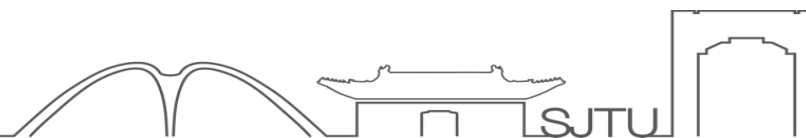# ECE2150J Introduction to Circuits
# Chapter 10. Sinusoidal Steady-State Analysis

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# 10.1 Introduction

In this chapter, we will see how nodal analysis, mesh analysis, Thevenin's theorem, Norton's theorem, superposition, and source transformations are applied in analyzing ac circuits.

**Quick Summary of Chapter 10 →** Basically, all methods that we have learned in previous chapters are applicable to ac circuits, too.

**Steps to analyze AC circuits**

1. Transform the circuit to the phasor domain.
2. Find the circuit output using nodal analysis, mesh analysis, superposition, etc.
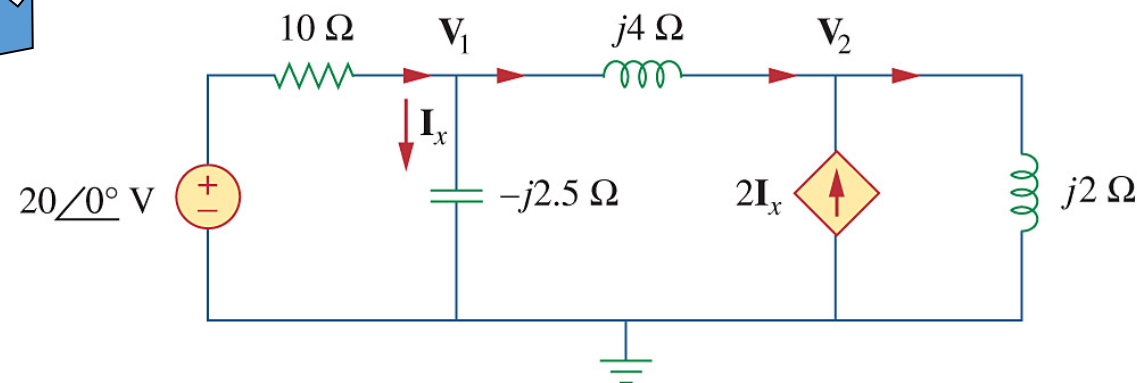3. Transform the resulting phasor to the time domain.

**Step 1** is not necessary if the problem is specified in the frequency domain.
**Step 2**, the analysis is performed in the same manner as dc circuit analysis except that complex numbers are involved.

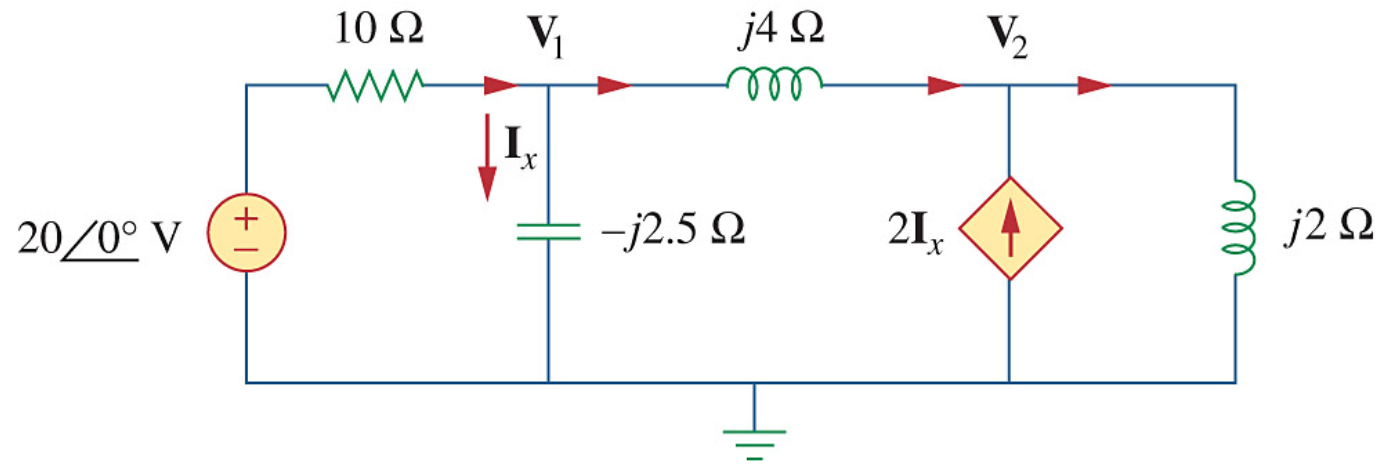**Example 10.1** Find $i_x$ in the circuit of Fig. 10.1 using nodal analysis.
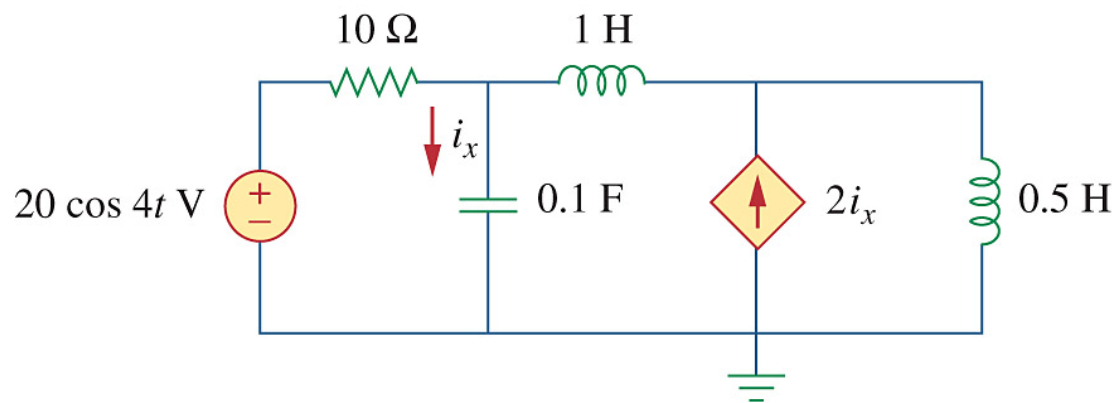


$$20\cos 4t \Rightarrow 20\angle 0° \text{ V}, \omega = 4 \text{ rad/s}$$

$$1\text{ H} \Rightarrow j\omega L = j4 \text{ }\Omega$$

$$0.5\text{ H} \Rightarrow j\omega L = j2 \text{ }\Omega$$

$$0.1\text{ F} \Rightarrow \frac{1}{j\omega C} = -j2.5 \text{ }\Omega$$

Review of nodal analysis:

1. Non-reference node and reference node
2. KCL equations for non-reference nodes
3. Solve simultaneous KCL equations

Or by inspection method: **YV=I** (GV = I in DC)
Where G: conductance matrix; V: input vector; I: output vector
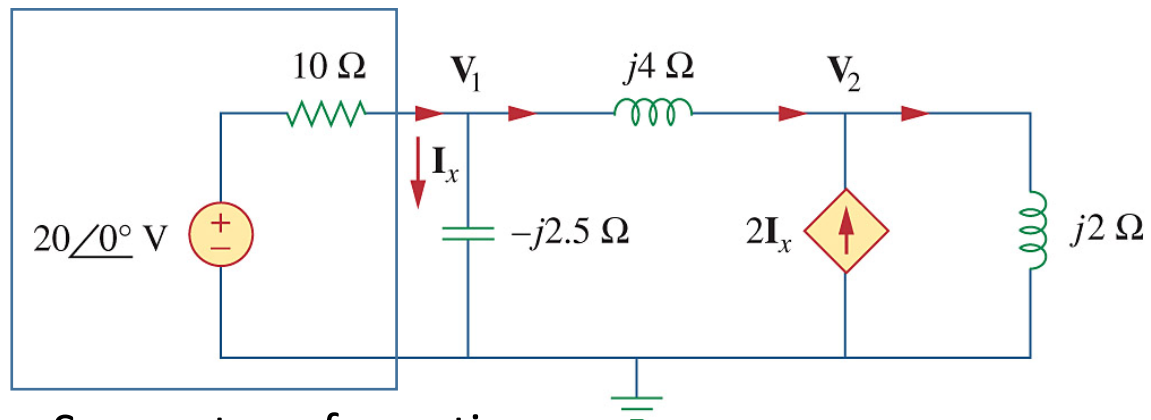
# Recall: Nodal analysis inspection

$$
\begin{bmatrix}
G_{11} & G_{12} & \cdots & G_{1N} \\
G_{21} & G_{22} & \cdots & G_{2N} \\
\vdots & \vdots & \vdots & \vdots \\
G_{N1} & G_{N2} & \cdots & G_{NN}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_N
\end{bmatrix}
=
\begin{bmatrix}
i_1 \\
i_2 \\
\vdots \\
i_N
\end{bmatrix}
$$

$G_{kk} = $ Sum of the conductances connected to node $k$

$G_{kj} = G_{jk} = $ Negative of the sum of the conductances directly connecting nodes $k$ and $j$, $k \neq j$.
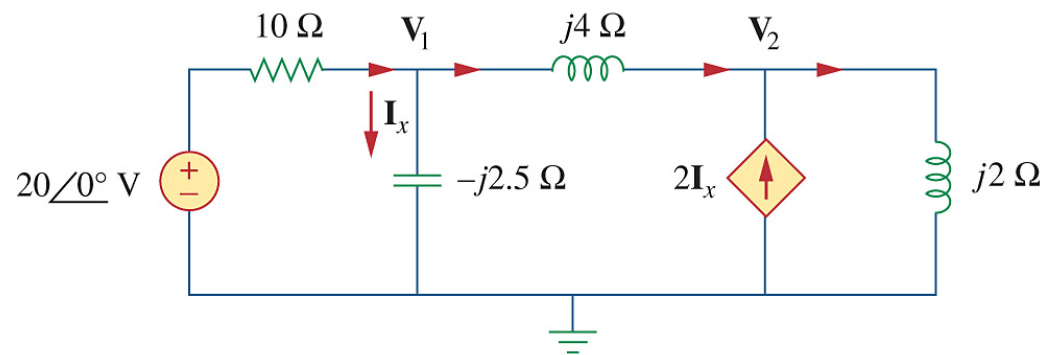
$v_k = $ Unknown voltage at node $k$

$i_k = $ Sum of all independent current sources directly connected to node $k$, with currents entering the node treated as positive

Source transformation

$$
\begin{bmatrix}
\dfrac{1}{10} + \dfrac{1}{j4} + \dfrac{1}{-j2.5} & -\dfrac{1}{j4} \\[4mm]
-\dfrac{1}{j4} & \dfrac{1}{j4} + \dfrac{1}{j2}
\end{bmatrix}
\begin{bmatrix}
\tilde{V}_1 \\[2mm]
\tilde{V}_2
\end{bmatrix}
=
\begin{bmatrix}
\dfrac{20\angle 0^\circ}{10} \\[4mm]
2\tilde{I}_x
\end{bmatrix}
$$

$\Sigma_i G_i$   $-G_{ij}$

Express $I_x$ in terms of $V_1$ or $V_2$

$$
\tilde{I}_x = \dfrac{\tilde{V}_1}{-j2.5}
$$

Circuit diagram: $20\angle 0°$ V source, $10\,\Omega$ resistor, node $V_1$, $j4\,\Omega$ inductor, node $V_2$, with $-j2.5\,\Omega$ capacitor (current $I_x$ down), $2I_x$ dependent current source, and $j2\,\Omega$ inductor.

$$
\begin{bmatrix}
\dfrac{1}{10}+\dfrac{1}{j4}+\dfrac{1}{-j2.5} & -\dfrac{1}{j4} \\[2mm]
-\dfrac{1}{j4} & \dfrac{1}{j4}+\dfrac{1}{j2}
\end{bmatrix}
\begin{bmatrix}
\tilde{V}_1 \\ \tilde{V}_2
\end{bmatrix}
=
\begin{bmatrix}
\dfrac{20\angle 0°}{10} \\[2mm]
2\tilde{I}_x
\end{bmatrix}
\quad\Longrightarrow\quad
\begin{bmatrix}
\dfrac{1}{10}+\dfrac{1}{j4}+\dfrac{1}{-j2.5} & -\dfrac{1}{j4} \\[2mm]
-\dfrac{1}{j4}-\dfrac{2}{-j2.5} & \dfrac{1}{j4}+\dfrac{1}{j2}
\end{bmatrix}
\begin{bmatrix}
\tilde{V}_1 \\ \tilde{V}_2
\end{bmatrix}
=
\begin{bmatrix}
\dfrac{20\angle 0°}{10} \\[2mm]
0
\end{bmatrix}
$$

$$
\tilde{I}_x = \frac{\tilde{V}_1}{-j2.5}
$$

$$
\begin{bmatrix}
2+j3 & j5 \\
11 & 15
\end{bmatrix}
\begin{bmatrix}
\tilde{V}_1 \\ \tilde{V}_2
\end{bmatrix}
=
\begin{bmatrix}
40 \\ 0
\end{bmatrix}
$$

$$\begin{bmatrix} 2+j3 & j5 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} 40 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 2+j3 & j5 \\ 11 & 15 \end{vmatrix} = 30 - j10$$
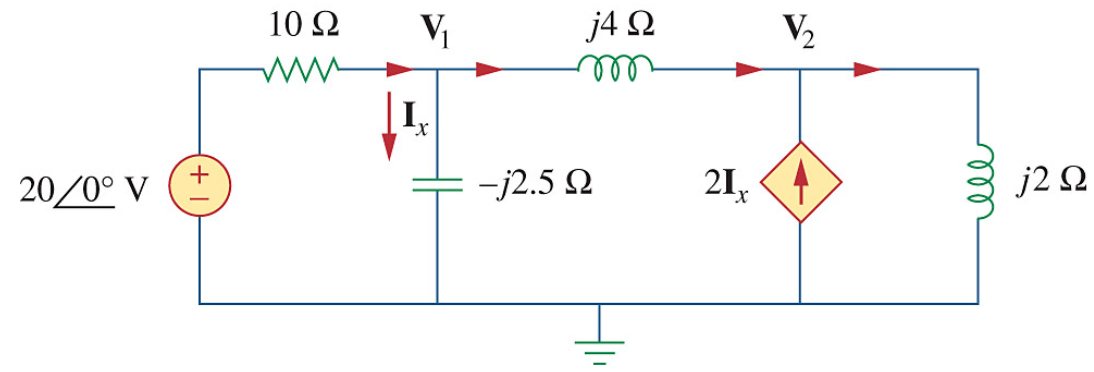
$$\Delta_1 = \begin{vmatrix} 40 & j5 \\ 0 & 15 \end{vmatrix} = 600$$

$$\tilde{V}_1 = \frac{\Delta_1}{\Delta} = \frac{600}{30-j10} = \frac{60}{3-j} \approx \frac{60}{3.1623\angle -18.43°}$$

$$\approx 18.9735\angle 18.43° \text{ (V)}$$

$$\tilde{I}_x = \frac{\tilde{V}_1}{-j2.5} = \frac{18.9735\angle 18.43°}{2.5\angle -90°}$$

$$\approx 7.59\angle 108.43° \text{ (A)}$$

$$i_x = 7.59\cos(4t + 108.43°) \text{ (A)}$$

# 10.3 Mesh Analysis

**Kirchhoff's voltage law (KVL) forms the basis of mesh analysis.**
The validity of KVL for ac circuits was shown in Chapter 9.
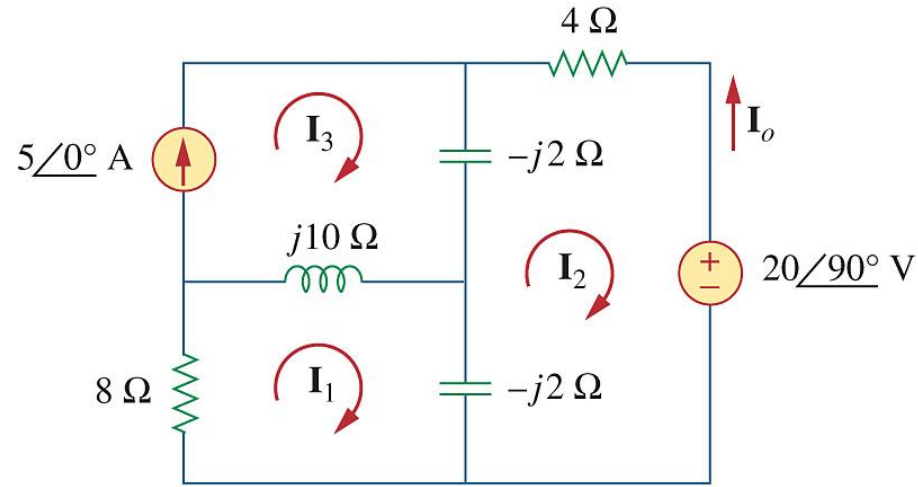
Review of mesh analysis:

1. Mesh currents

        -deal with the case with current source first

2. KVL equations for meshes

3. Solve simultaneous KVL equations

Or by inspection method: **ZI**=**V** (RI = V in DC)

Where R: resistance matrix; I: input vector; V: output vector

**Example 10.3** Determine current $\tilde{I}_o$ in the circuit of Fig. 10.7 using mesh analysis.
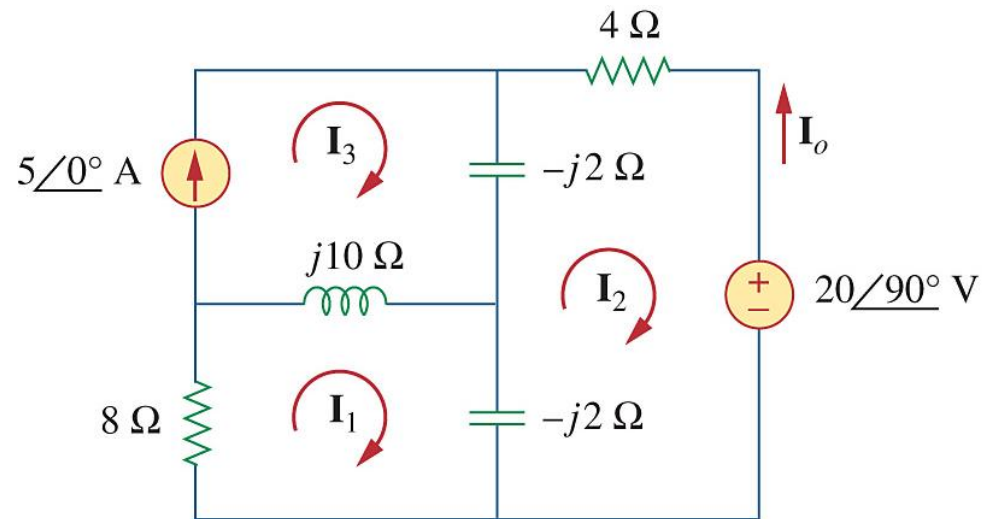
**Solution :**

$$\tilde{I}_3 = 5\angle 0° = 5 \text{ (A)}$$

$$20\angle 90° = j20$$

$$\begin{bmatrix} \overset{\Sigma_i R_i}{8+j10-j2} & \overset{-R_{ij}}{j2} & -j10 \\ j2 & 4-j2-j2 & j2 \end{bmatrix} \begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -j20 \end{bmatrix}$$

$I_3$ is known, so just neglect 3rd row

$$
\begin{bmatrix} 8+j8 & j2 & \boxed{-j10} \\ j2 & 4-j4 & j2 \end{bmatrix}
\begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \\ \boxed{5} \end{bmatrix}
=
\begin{bmatrix} 0 \\ -j20 \end{bmatrix}
$$

$$
\begin{bmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{bmatrix}
\begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix}
=
\begin{bmatrix} \boxed{j50} \\ -j30 \end{bmatrix}
$$

$$
\Delta = \begin{vmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{vmatrix} = 68
$$
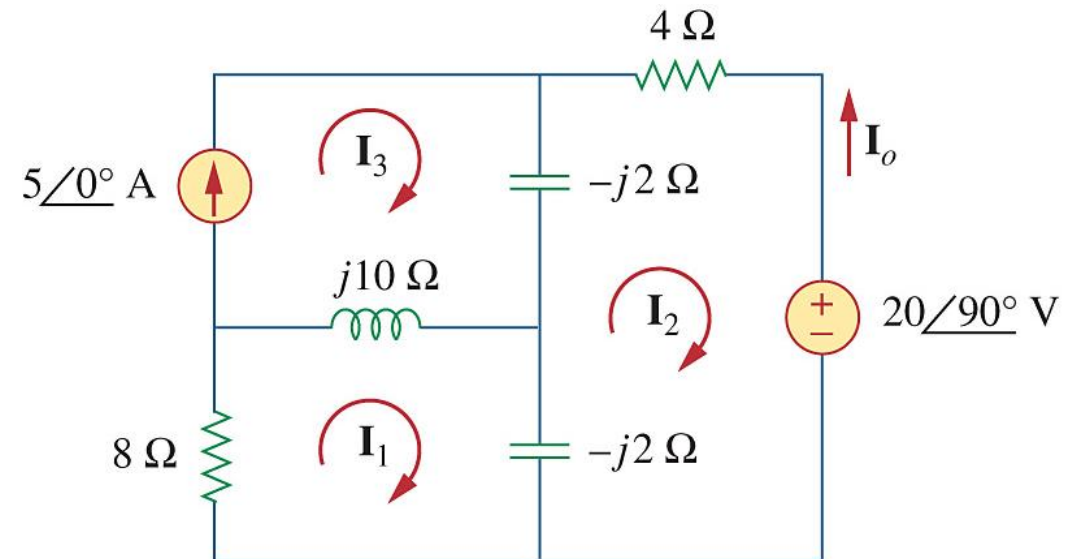
$$\Delta_2 = \begin{vmatrix} 8+j8 & j50 \\ j2 & -j30 \end{vmatrix} = 340 - j240$$

$$\approx 416.1730\angle -35.22°$$

$$\tilde{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.1730\angle -35.22°}{68}$$

$$\approx 6.12\angle -35.22° \text{ (A)}$$

$$\tilde{I}_o = -\tilde{I}_2 = 6.12\angle 144.78° \text{ (A)}$$
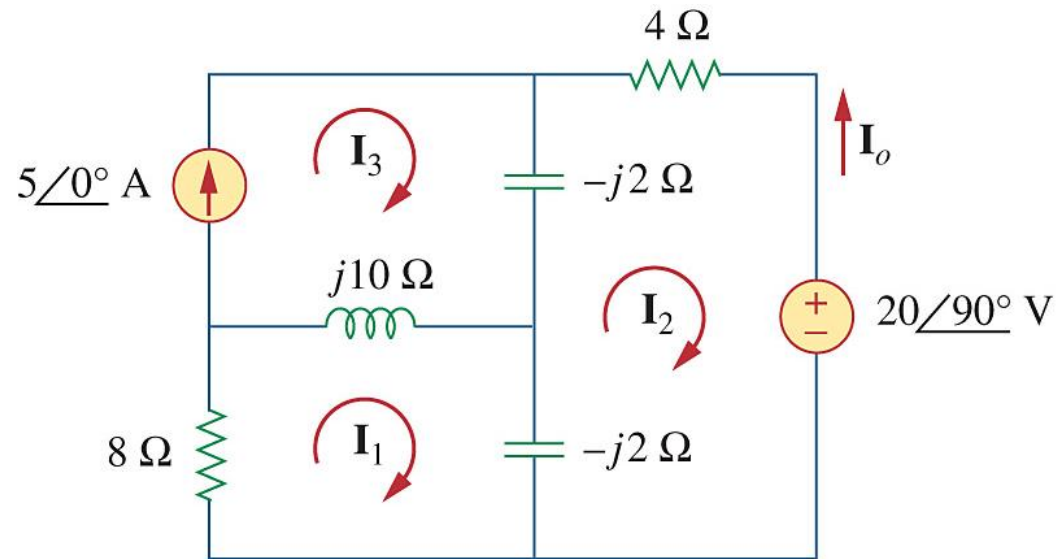


Only step 2 is needed in this problem.

# 10.4 Superposition Theorem

The superposition theorem becomes important if the circuit has sources operating at **different frequencies**. In this case, **since the impedances depend on frequency**, we must have a different frequency domain circuit for each frequency. The total response must be obtained by **adding the individual responses** in the time domain.

**Review of superposition**

• Superposition principle is based on additivity.

• It states that whenever a linear system is excited, or driven, by **more than one independent source**, the total response is the sum of the individual responses.

• The principle of superposition helps us to analyze a linear circuit with **more than one independent source** by calculating the contribution of each independent source separately and adding algebraically all the contributions to find the total contribution.

**Example 10.6** Find $v_o$ of the circuit of Fig. 10.13 using the superposition theorem.

2 H    1 Ω    4 Ω

$+$ $v_o$ $-$

$+$ 10 cos 2t V    2 sin 5t A    0.1 F    $+$ 5 V
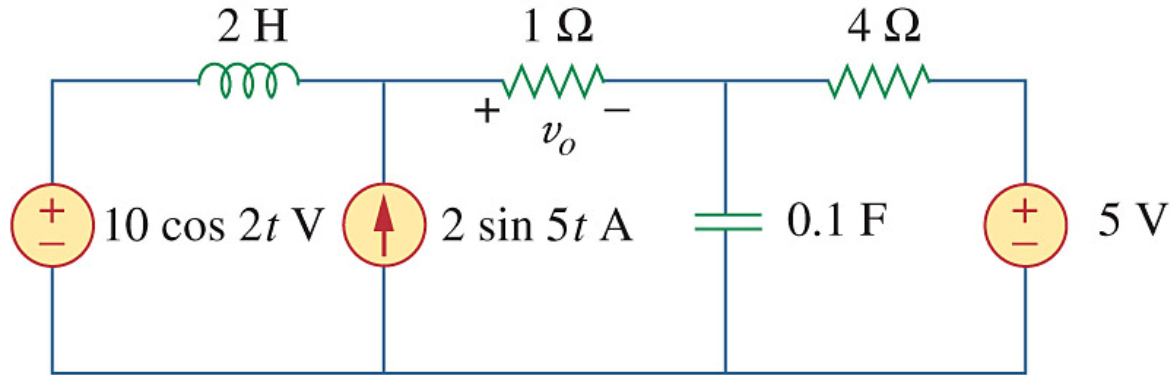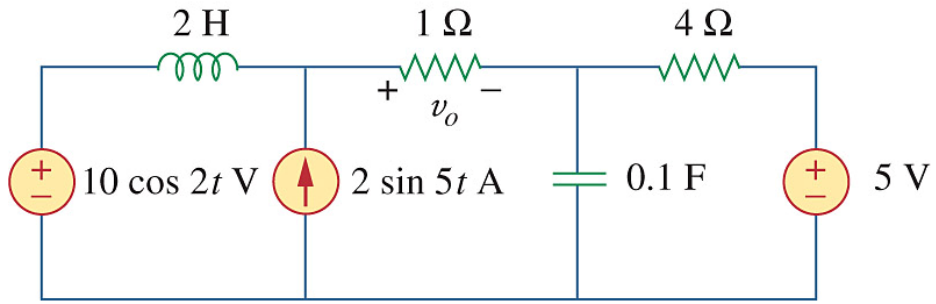
$v_o = v_1 + v_2 + v_3$

where $v_1$ is due to the 5-V dc source, $v_2$ is

due to the $10\cos 2t$ V voltage source, and

$v_3$ is due to the $2\sin 5t$ current source.

**(2) From 10cos(2t) V source**    ω=2 rad/s

$j4\ \Omega$    $1\ \Omega$    $4\ \Omega$

$+$ $V_2$ $-$

$10\underline{/0°}$ V $+$    $-j5\ \Omega$

**(3) From 2sin(5t) V source**    ω=5 rad/s

$1\ \Omega$

$+$ $V_3$ $-$

$j10\ \Omega$    $2\underline{/-90°}$ A    $-j2\ \Omega$    $4\ \Omega$

**(1) From 5V source**

$1\ \Omega$    $4\ \Omega$

$+$ $v_1$ $-$

$+$ 5 V
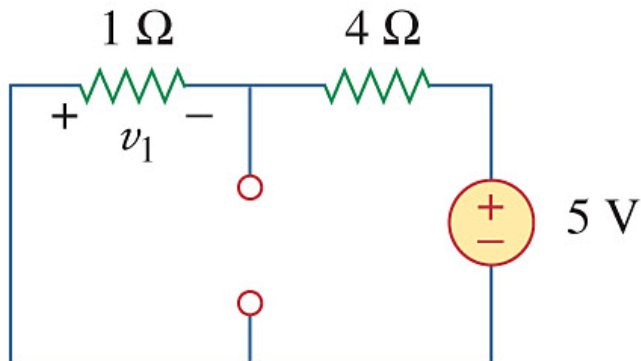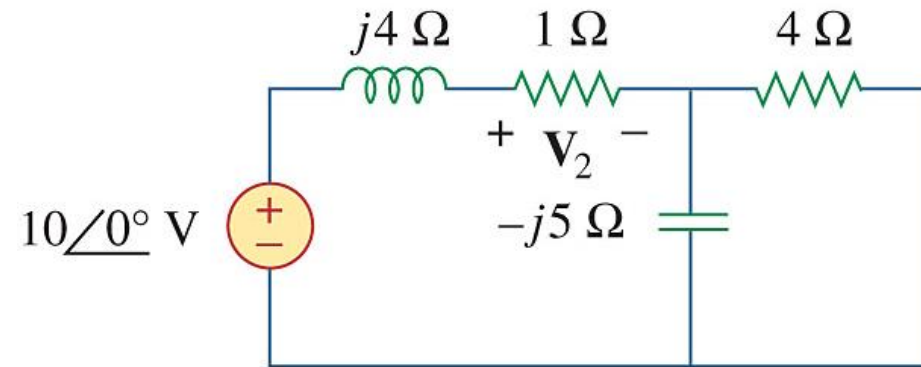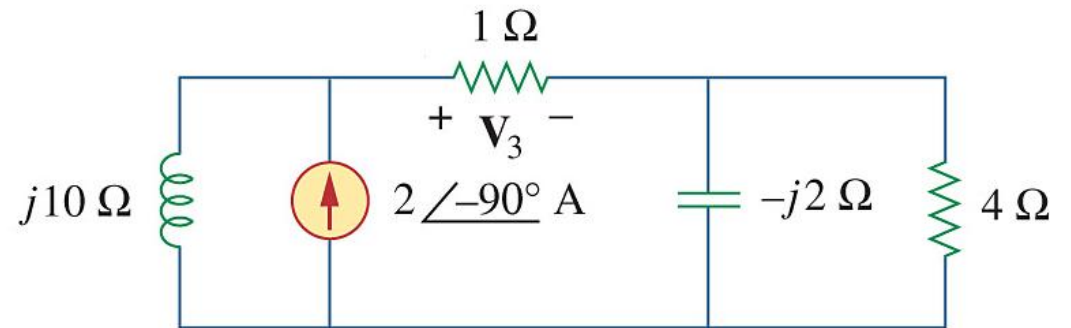
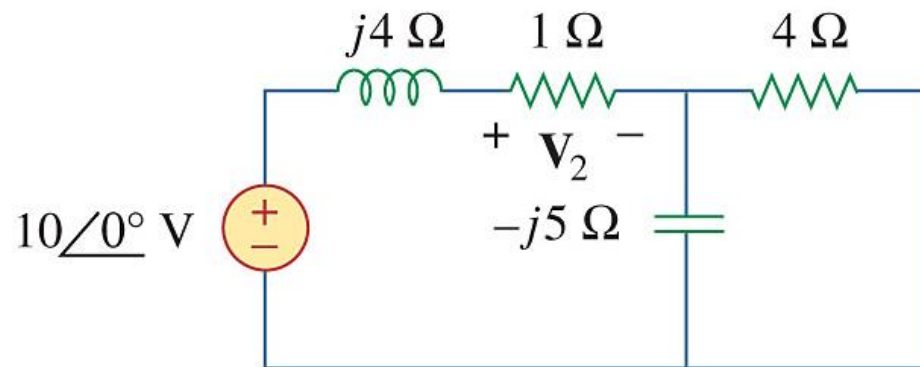# (1) From 5V source
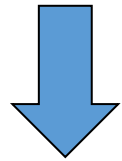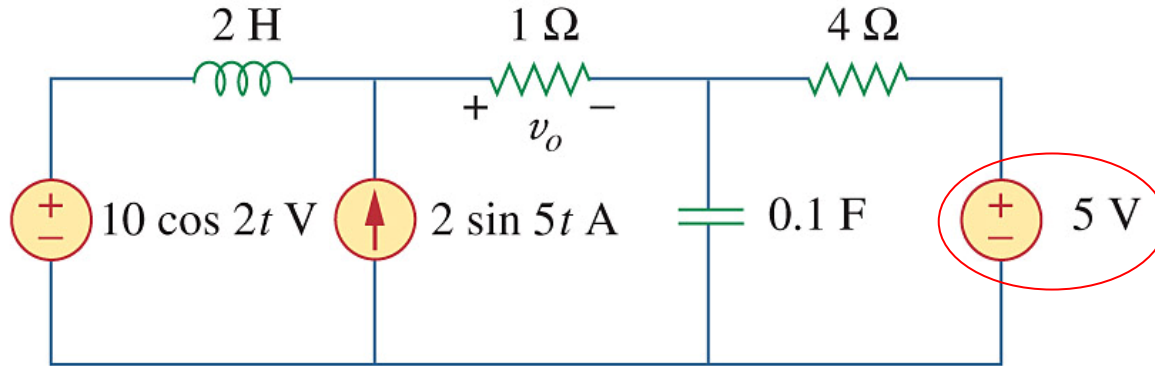


An inductor becomes short in a DC steady state
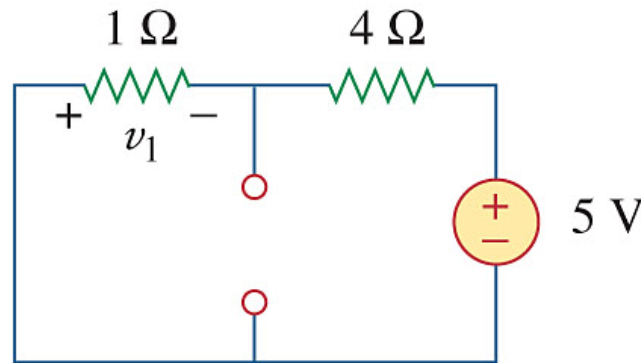
$$(1)\ v_1 = -5 \times \frac{1}{1+4} = -1\ (\text{V})$$

**(2) From 10cos(2t) V source**



Impedance is based on ω=2 rad/s



$10\cos 2t \text{ V} \Rightarrow 10\angle 0° \text{ V}, w = 2 \text{ rad/s}$

$2 \text{ H} \Rightarrow jwL = j4 \text{ (Ω)}$

$0.1 \text{ F} \Rightarrow 1/(jwC) = -j5 \text{ (Ω)}$
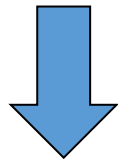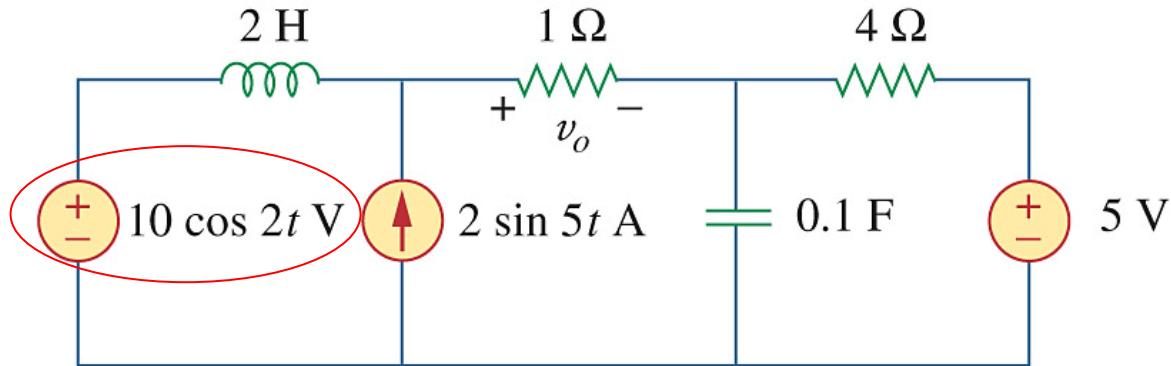
Complex number calculation:
For +,− calculation: rectangular form suggested
For ×,÷ calculation: polar form suggested



$$\tilde{V}_2 = 10\angle 0° \frac{1}{j4+1+(-j5) \parallel 4}$$

$$(-j5) \parallel 4 = \frac{(-j5) \times 4}{(-j5)+4} \approx \frac{20\angle -90°}{6.4031\angle -51.34°} \approx 3.1235\angle -38.66° \approx 2.4390 - j1.9512$$

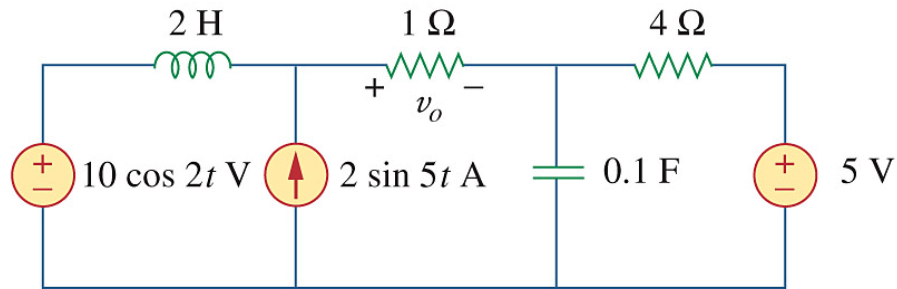$$\tilde{V}_2 = 10\angle 0° \frac{1}{j4+1+(2.4390-j1.9512)}$$

$$= \frac{10\angle 0°}{3.4390+j2.0488} = \frac{10\angle 0°}{4.0030\angle 30.78°}$$
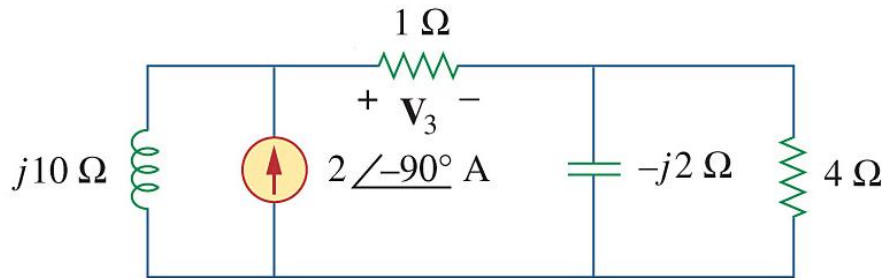
$$\approx 2.50\angle -30.78° \text{ (V)}$$

$$v_2 = 2.50\cos(2t-30.78°) \text{ (V)}$$
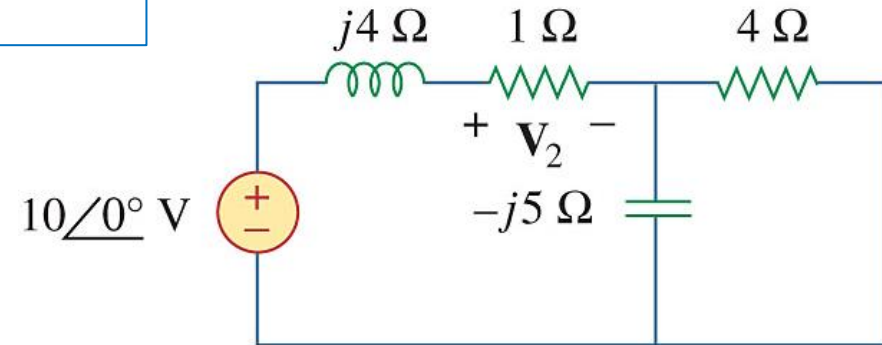
## (3) From 2sin(5t) V source



Impedance is based on $\omega=5$ rad/s
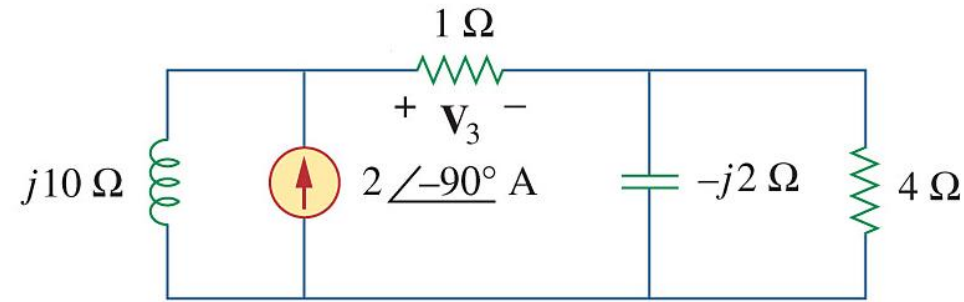


$$2\sin 5t \text{ A} \Rightarrow 2\angle -90° \text{ A}, \ \omega=5 \text{ rad/s}$$

$$2 \text{ H} \Rightarrow j\omega L = j10 \ (\Omega)$$

$$0.1 \text{ F} \Rightarrow 1/(j\omega C) = -j2 \ (\Omega)$$

sin(5t) = cos(5t–90°)

Source transformation

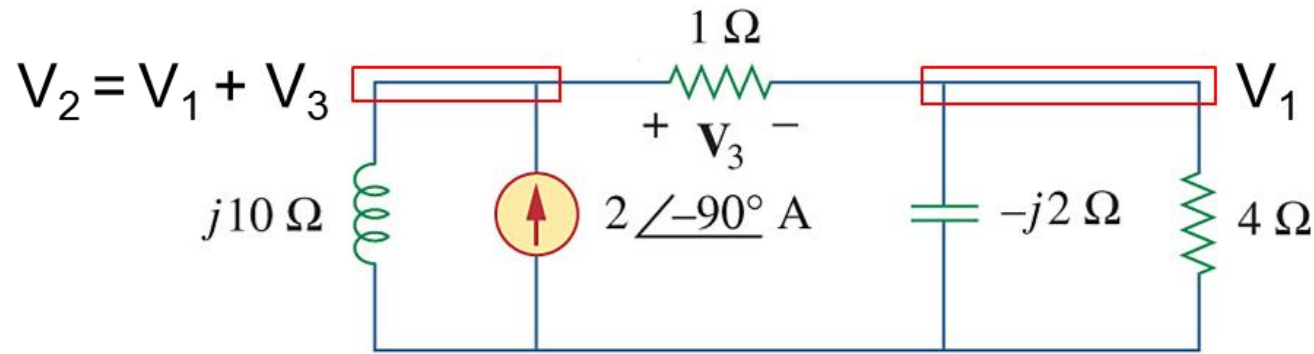$$\tilde{V}_3 = \boxed{(2\angle -90°) \times (j10)} \times \cfrac{1}{j10 + 1 + (-j2) \parallel 4}$$

$$(-j2) \parallel 4 = \frac{(-j2) \times 4}{(-j2) + 4} = 0.8 - j1.6$$

$$\tilde{V}_3 = (2\angle -90°) \times (j10) \times \cfrac{1}{j10 + 1 + (0.8 - j1.6)}$$

$$= \frac{20}{1.8 + j8.4} \approx \frac{20}{8.5907\angle 77.91°}$$

$$\approx 2.33\angle -77.91° \text{ (V)}$$

$$v_3 = 2.33\cos(5t - 77.91°) \text{ V}$$

$V_2 = V_1 + V_3$     $1\,\Omega$     $V_1$

$j10\,\Omega$    $2\angle{-90°}$ A    $-j2\,\Omega$    $4\,\Omega$

By KCL,

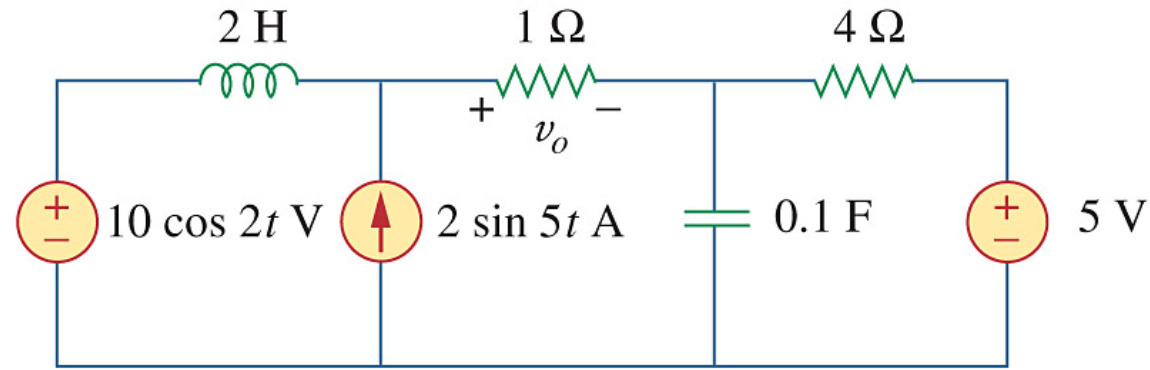(1) $\dfrac{V_1+V_3}{j10} - 2\angle{-90°} + \dfrac{(V_1+V_3)-V_1}{1} = 0$

$\rightarrow V_1 + V_3 - j20\angle{-90°} + j10V_3 = 0$

(2) $\dfrac{V_1-(V_1+V_3)}{1} + \dfrac{V_1}{-j2} + \dfrac{V_1}{4} = 0 \rightarrow -4V_3 + j2V_1 + V_1 = 0$

Therefore, $V_1 = \dfrac{4V_3}{1+j2}$, and then the eq(1) becomes

$(9 + j42)V_3 = 100 \rightarrow V_3 = \dfrac{100}{9 + j42} = \boldsymbol{2.33\angle{-77.9°}}$
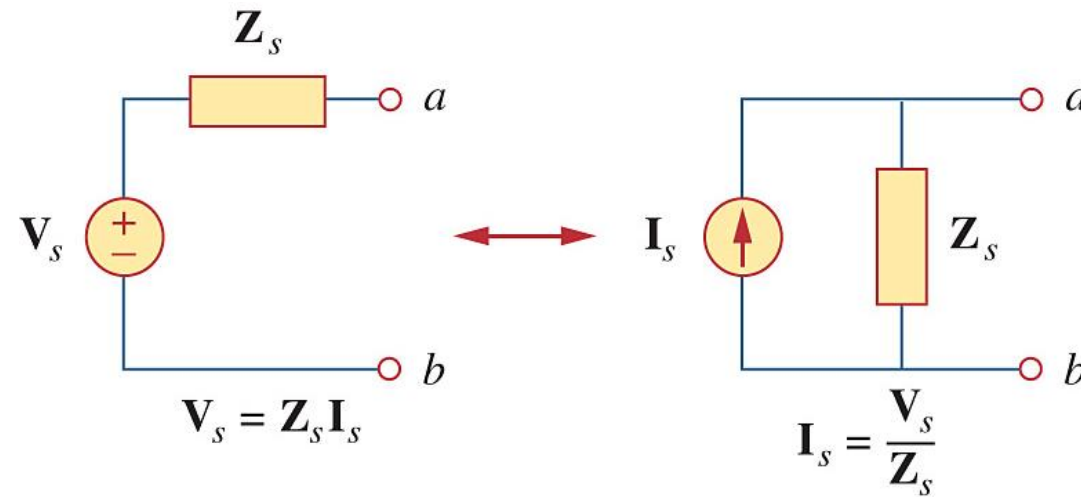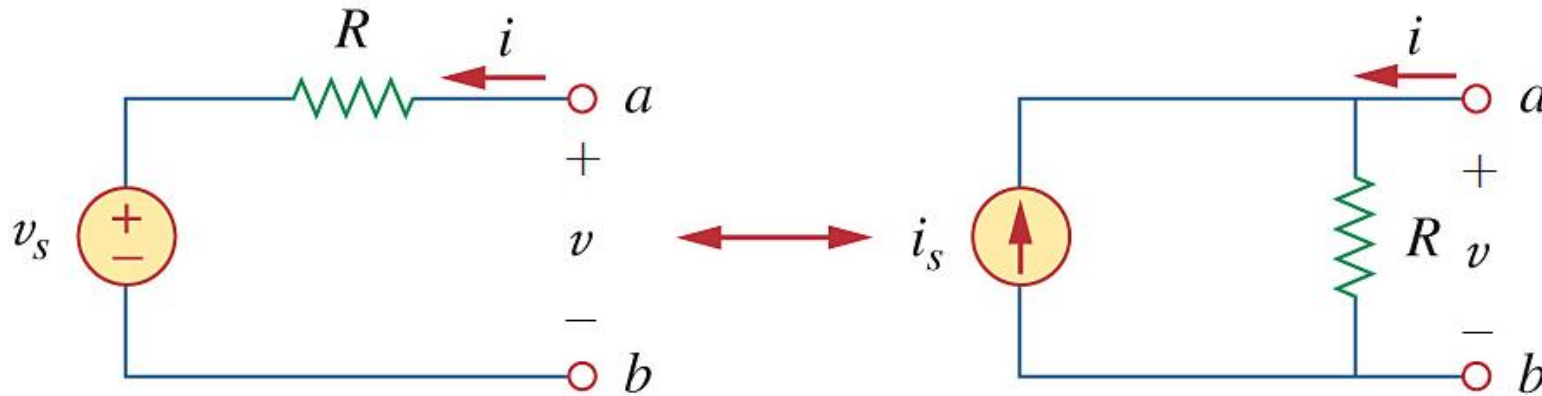
A complete response $v = v_1 + v_2 + v_3$

$$v = -1 + 2.50\cos(2t - 30.78°)$$

$$+2.33\cos(5t - 77.91°) \text{ (V)}$$

# 10.5 Source Transformation

Source transformation in the frequency domain involves transforming a voltage source in series with an impedance to a current source in parallel with an impedance, or vice versa.

$$V_s = Z_s I_s$$

$$I_s = \frac{V_s}{Z_s}$$

**Review of source transformation**

$$R \qquad i$$

$$v_s \qquad \qquad a$$
$$+$$
$$v$$
$$-$$
$$b$$

$$i_s \qquad \qquad i$$
$$a$$
$$+$$
$$R \quad v$$
$$-$$
$$b$$

$$v_s = i_s R \quad \text{or} \quad i_s = \frac{v_s}{R}$$

The source transformation is the process of replacing a **voltage source in series with a resistor** by **a current source in parallel with a resistor**, or vice versa.

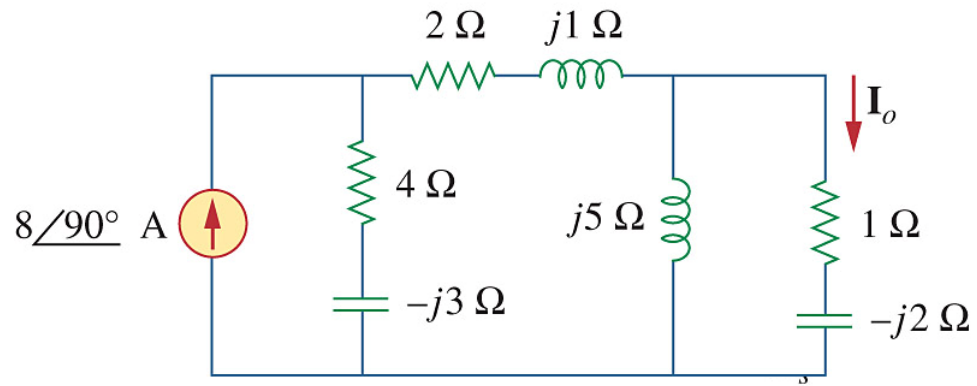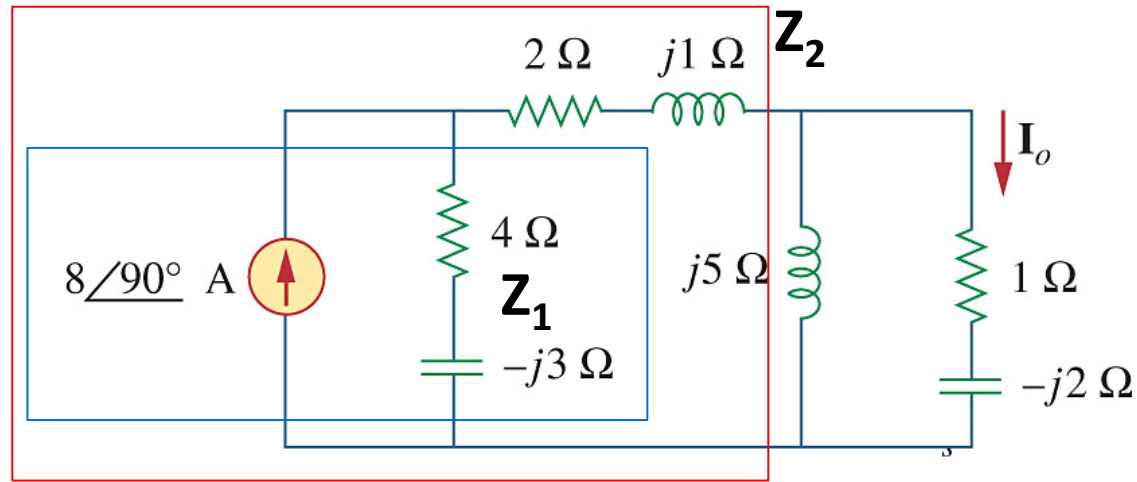**Practice Problem 10.7**  Find $\dot{I}_o$ in the circuit of Fig. 10.19 using the concept of source transformation.
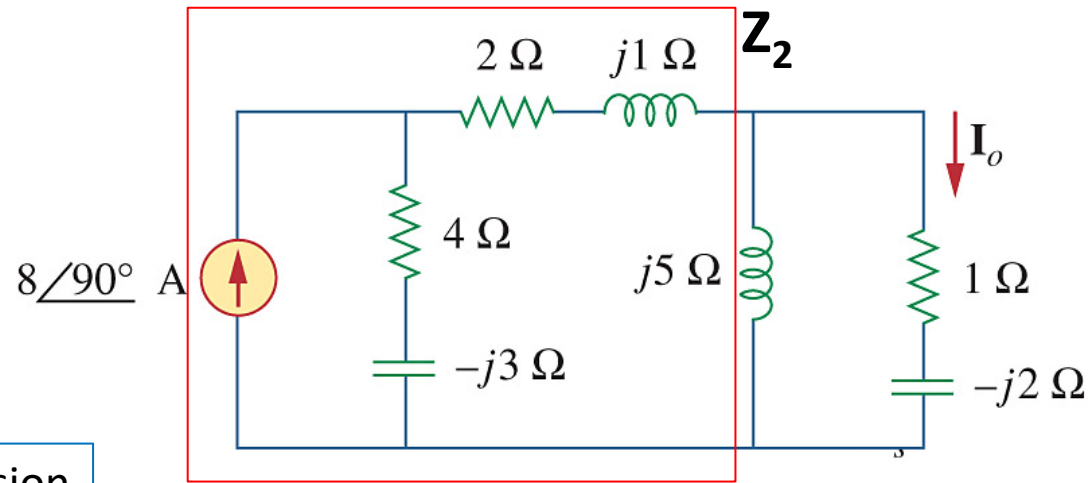


Figure 10.19

$$Z_2 = Z_1 + (2+j1) = (4-j3) + (2+j1)$$

$$= 6 - j2 \ (\Omega)$$

$$\begin{cases} \tilde{V}_1 = 24 + j32 \text{ V} \\ Z_2 = 6 - j2 \ \Omega \end{cases} \Rightarrow \begin{cases} \tilde{I}_2 = 6.3245\angle71.56° \text{ A} \\ Z_2 = 6 - j2 \ \Omega \end{cases}$$

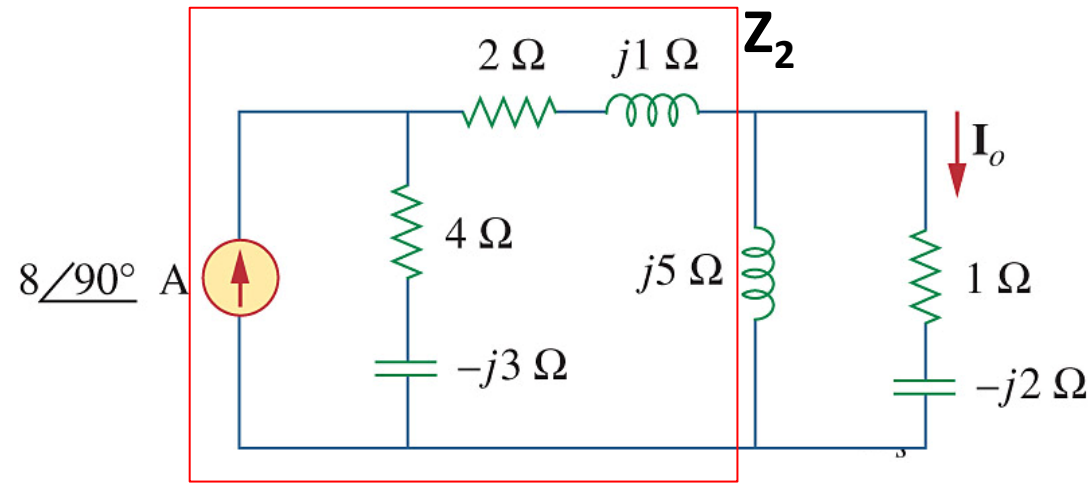$$\tilde{I}_2 = \frac{\tilde{V}_1}{Z_2} = \frac{24 + j32}{6 - j2} = \frac{40\angle53.13°}{6.3246\angle-18.43°}$$

$$\approx 6.3245\angle71.56° \ (A)$$

$$2\ \Omega \qquad j1\ \Omega \qquad \mathbf{Z_2}$$

$$8\underline{/90°}\ \text{A}$$

$$4\ \Omega$$

$$j5\ \Omega \qquad 1\ \Omega$$

$$-j3\ \Omega \qquad -j2\ \Omega$$

$$\mathbf{I}_o$$

Current division

$$I_o = I_2 \times (Y_3/(Y_1+Y_2+Y_3))$$

$$\tilde{I}_o = \tilde{I}_2 \frac{1/(1-j2)}{1/(1-j2)+1/(j5)+1/(6-j2)}$$

$$\frac{1/(1-j2)}{1/(1-j2)+1/(j5)+1/(6-j2)}$$

$$= \frac{1}{1+(1-j2)/(j5)+(1-j2)/(6-j2)}$$

$$= \frac{1}{1+(-2-j)/5+(1-j)/4}$$

$$= \frac{20}{17 - j9}$$

$$\approx 1.0398\angle 27.90°$$

$$\tilde{I}_o = \tilde{I}_2 \times 1.0398\angle 27.90°$$

$$= 6.3245\angle 71.56° \times 1.0398\angle 27.90°$$

$$\approx 6.58\angle 99.46° \text{ (A)}$$

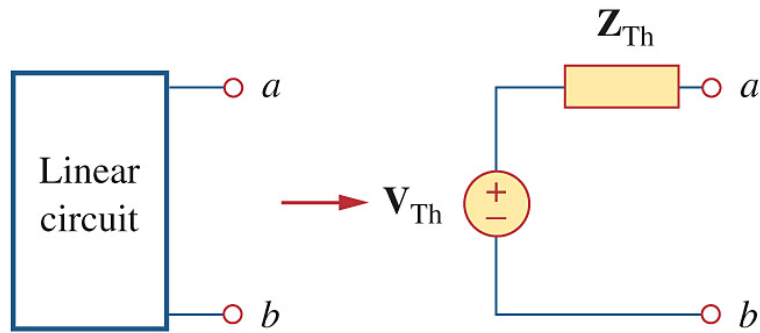# 10.6 Thevenin and Norton Equivalent Circuits
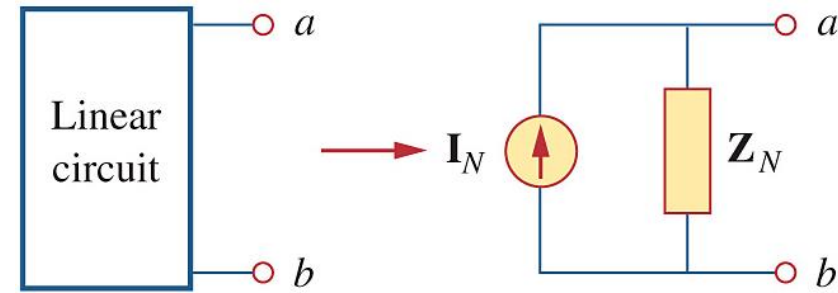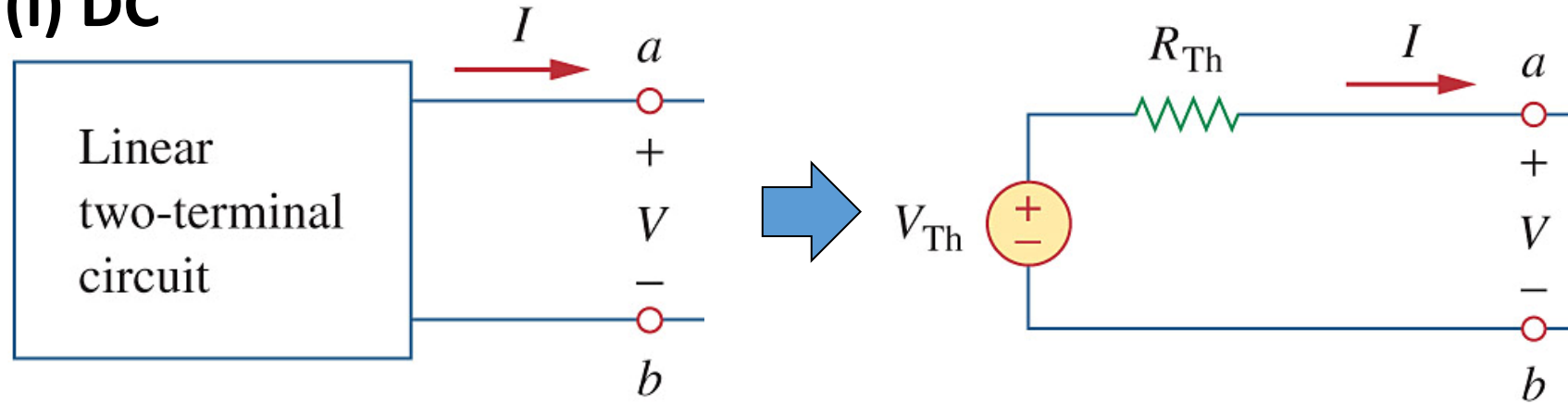


Figure 10.20 Thevenin equivalent.



Figure 10.21 Norton equivalent.

**Thevenin's and Norton's theorems** are applied to ac circuits in the same way as they are to dc circuits.
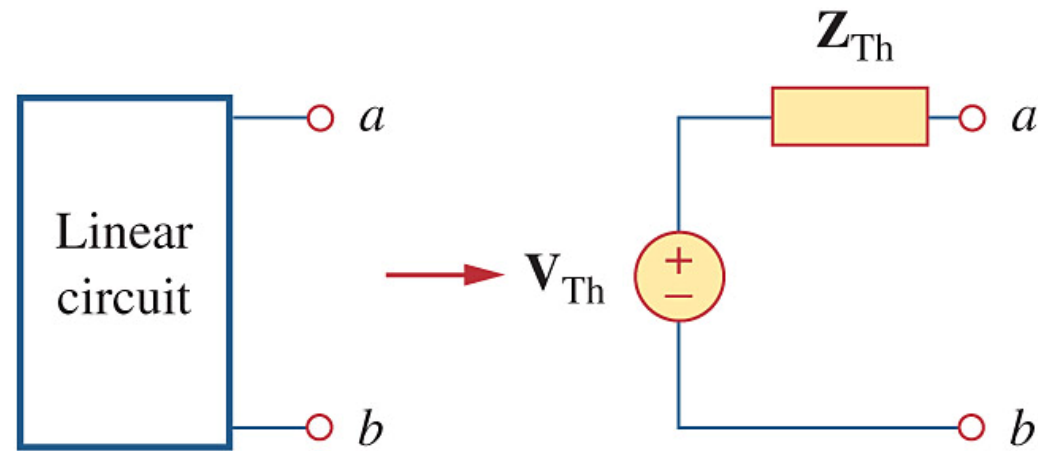
# Thevenin's theorems

## (i) DC
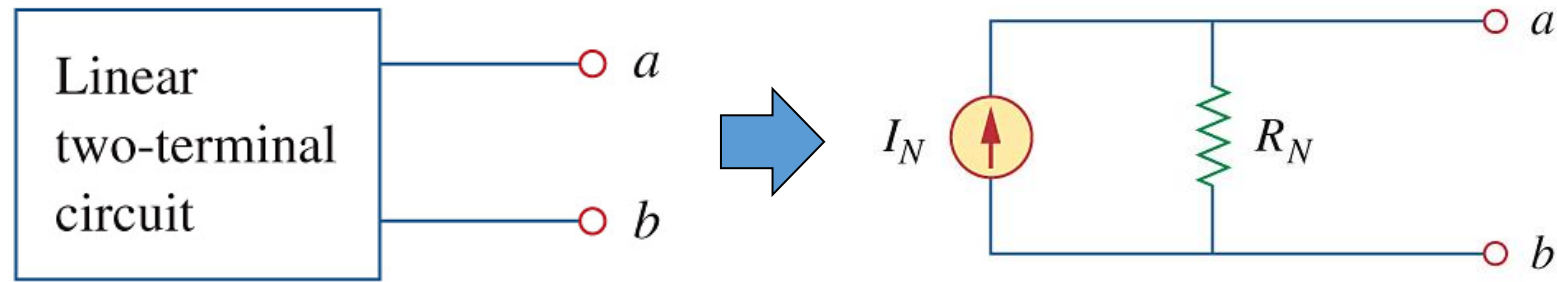


## (ii) AC



Figure 10.20   Thevenin equivalent.

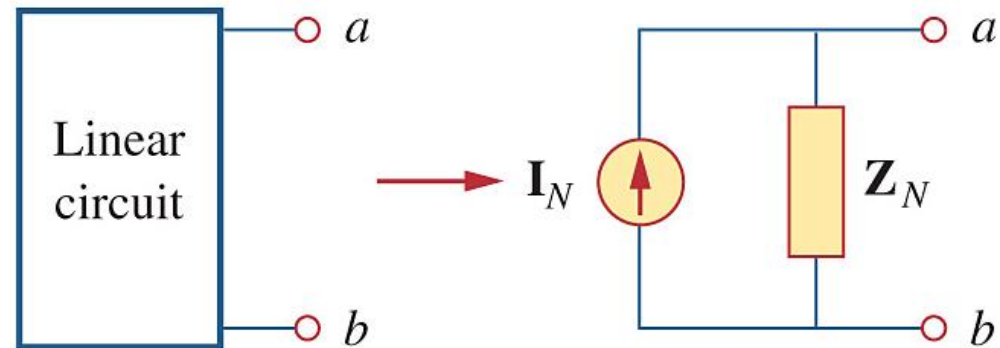# Norton's theorems

## (i) DC



## (ii) AC



Figure 10.21 Norton equivalent.

*If the circuit has **sources operating at different frequencies**, the Thevenin or Norton equivalent circuit must be determined **at each frequency**

→ Entirely different equivalent circuits, **one for each frequency**, not one equivalent circuit with equivalent sources and equivalent impedances.

**Practice Problem 10.10** Determine the Norton equivalent of the circuit in Fig. 10.30 as seen from terminals *a-b*. Use the equivalent to find $\tilde{I}_o$.
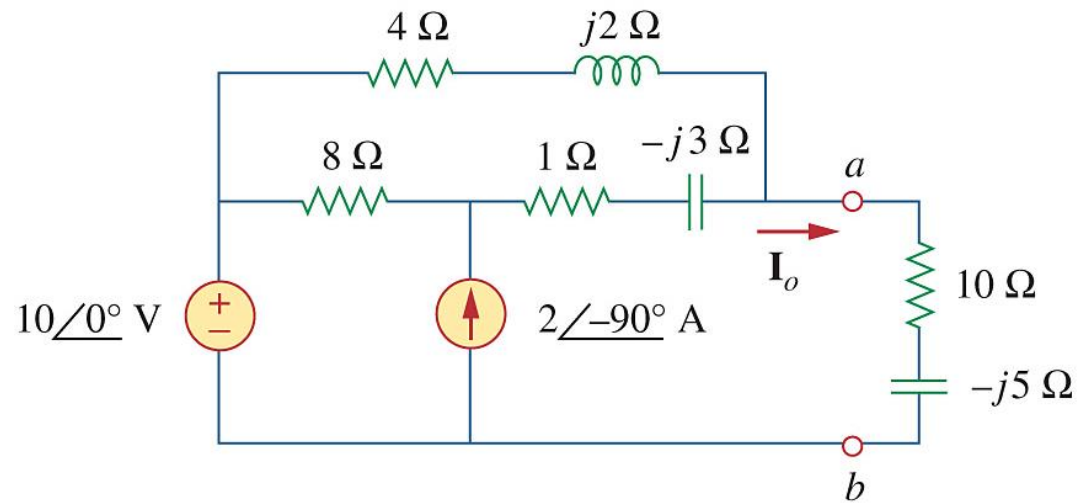
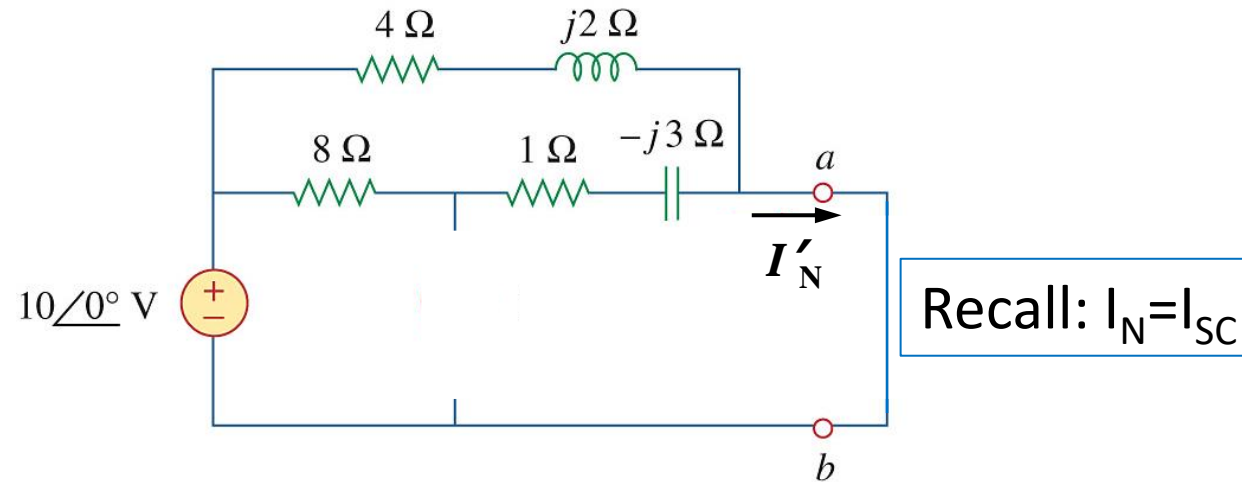**(i) Z$_N$**
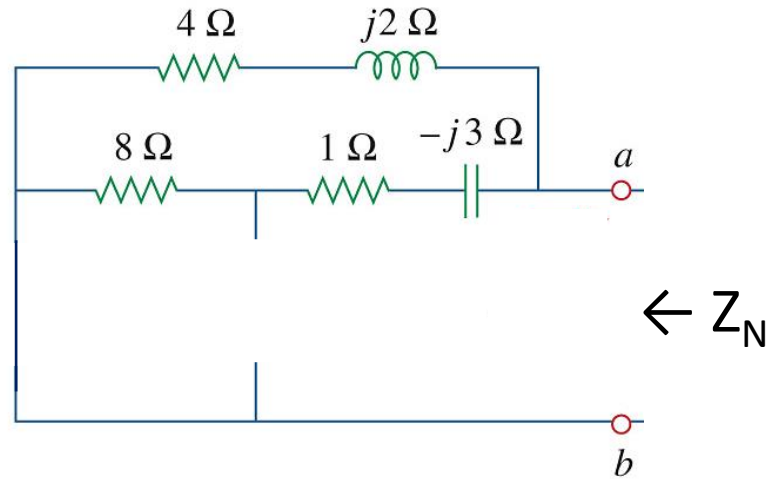


$$Z_N = (4 + j2) \parallel (9 - j3)$$

$$= \frac{(4 + j2)(9 - j3)}{(4 + j2)(9 - j3)} = \frac{42 + j6}{13 - j} = 3.25\angle 12.53°$$
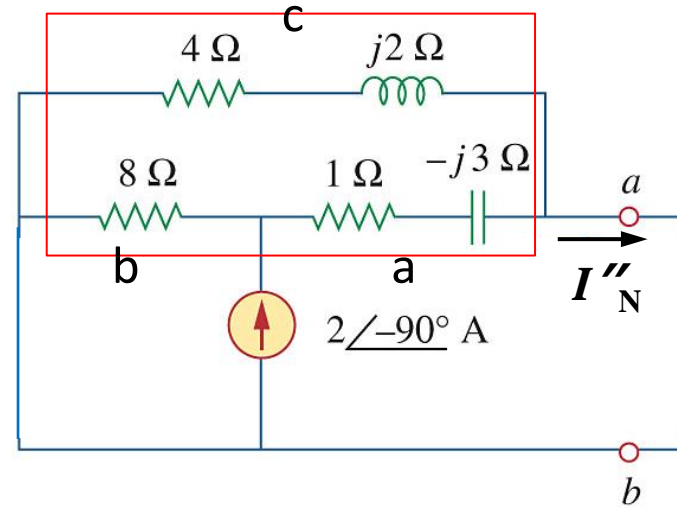
**(ii) I'$_N$**



4 Ω   j2 Ω

8 Ω   1 Ω   $-j3$ Ω

$a$

$I'_N$

$10\angle 0°$ V

Recall: I$_N$=I$_{SC}$

$b$

Use superposition to find $\tilde{I}_N$:

$$\tilde{I}_N = \tilde{I}'_N + \tilde{I}''_N$$

$$\tilde{I}'_N = \frac{10\angle 0°}{(4+j2) \parallel (8+1-j3)}$$

$$= \frac{10}{6(7+j)/(13-j)} = \frac{9-j2}{3}$$

$$\approx 3 - j0.6667 \text{ (A)}$$

## (ii) $I''_N$
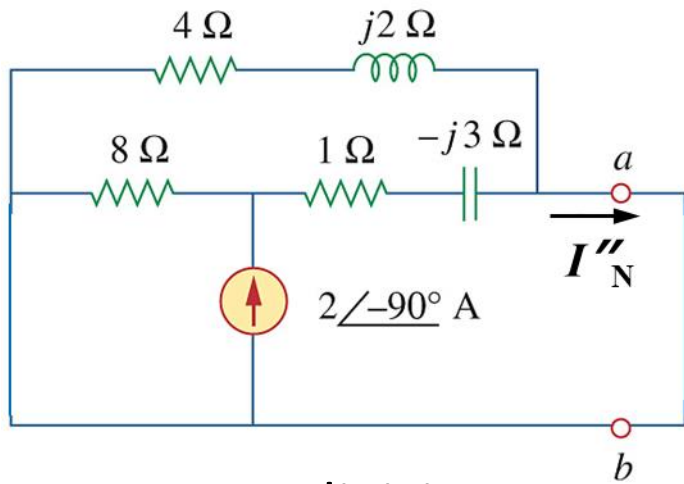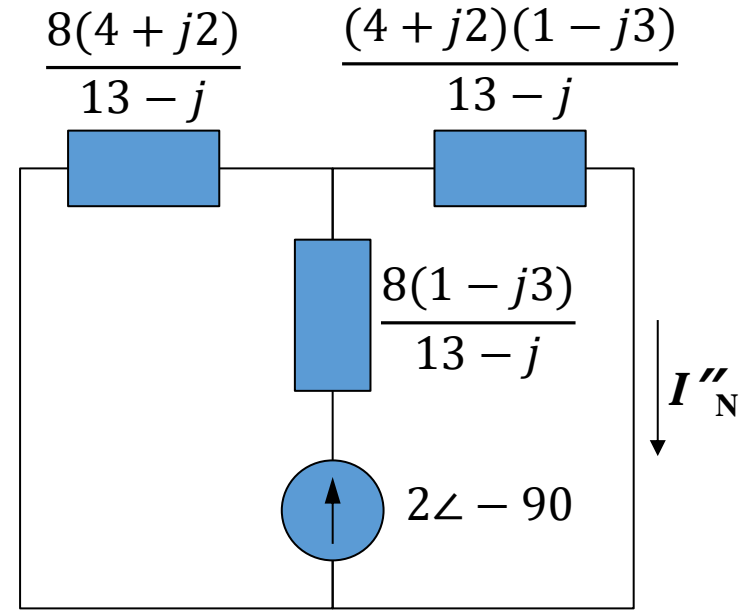


Delta → Y transform

$$Z_1 = \frac{bc}{a+b+c} = \frac{8(4+j2)}{13-j}$$

$$Z_2 = \frac{ac}{a+b+c} = \frac{(4+j2)(1-j3)}{13-j}$$

$$Z_3 = \frac{ba}{a+b+c} = \frac{8(1-j3)}{13-j}$$

By current division
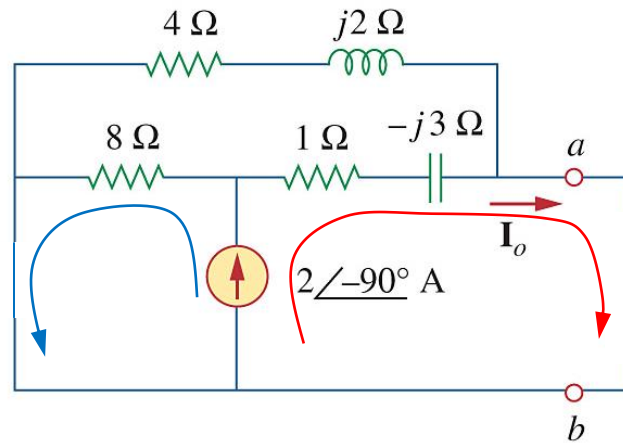
$$
\begin{aligned}
I''_N &= \frac{\dfrac{8(4+j2)}{13-j}}{\dfrac{8(4+j2)}{13-j} + \dfrac{(4+j2)(1-j3)}{13-j}} \times 2\angle(-90°) \\
&= \frac{32+j16}{42+j6} \times 2\angle(-90°)
\end{aligned}
$$

# (iii) $I''_N$ – another approach



By current division

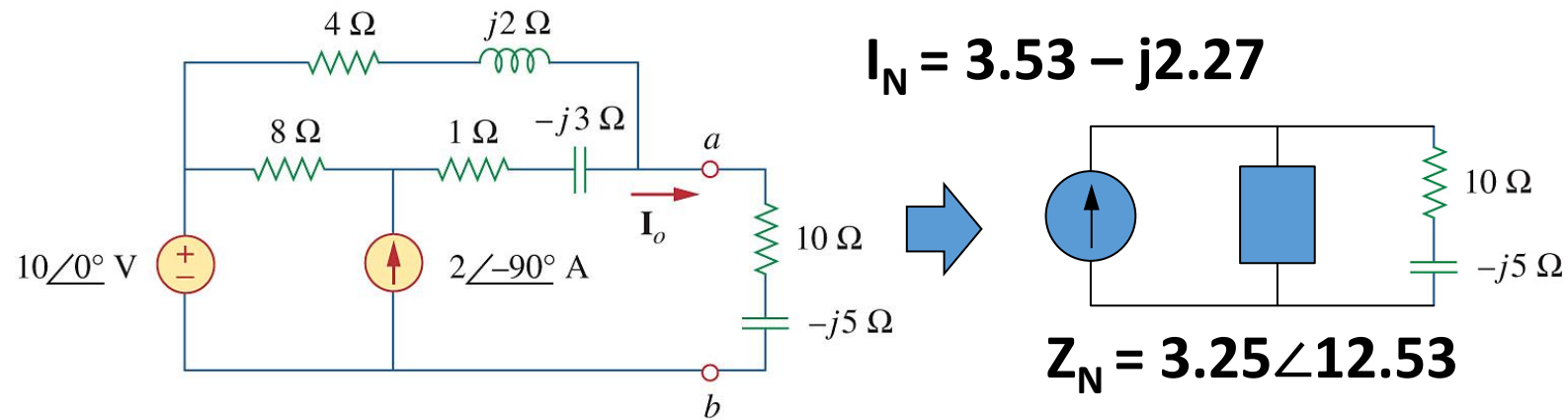$$\tilde{I}''_N = 2\angle -90° \times \frac{8}{8+1-j3}$$

$$\approx 0.5333 - j1.6 \text{ (A)}$$

$$I_N = I'_N + I''_N$$

$$\tilde{I}_N = 3.5333 - j2.2667$$

$$\approx 4.1979\angle -32.68° \text{ (A)}$$

# (iv) $I_o$



$I_N = 3.53 - j2.27$

$Z_N = 3.25\angle 12.53$

$$\tilde{I}_o = \tilde{I}_N \frac{Z_N}{Z_N + (10 - j5)} = 4.1979\angle -32.68° \times \frac{3.1765 + j0.7059}{(3.1765 + j0.7059) + (10 - j5)}$$

$$\approx 4.1979\angle -32.68° \times \frac{3.1765 + j0.7059}{13.1765 - j4.2941}$$

$$\approx 4.1979\angle -32.68° \times \frac{3.2540\angle 12.53°}{13.8586\angle -18.05°}$$

$$\approx 0.99\angle -2.10° \text{ (A)}$$

# 10.7 Op Amp AC Circuits

The three steps also apply to op amp circuits, as long as the op amp is in the linear region.

**Steps to analyze AC circuits**
1. Transform the circuit to the phasor domain.
2. Find the circuit output using nodal analysis, mesh analysis, superposition, etc.
3. Transform the resulting phasor to the time domain.

**Recall** Op Amp (Chapter 5): The key to analyzing op amp circuits is to keep two important properties of an ideal op amp in mind.

(1) **No current** enters either of its input terminals.
(2) **The voltage** across its input terminals is **zero**.

**Practice Problem 10.11** Find $v_o$ and $i_o$ in the op amp circuit of Fig. 10.32. Let $v_s = 4\cos 5000t$ V.
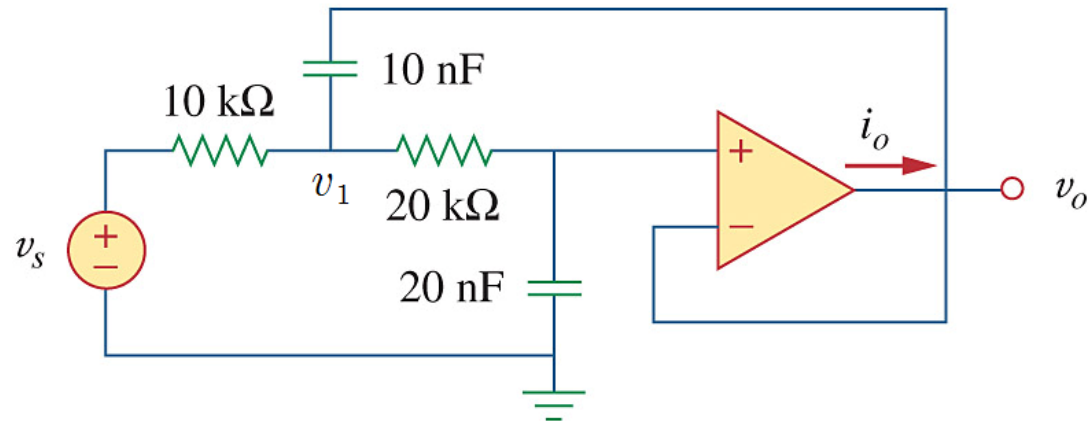
## (i) Phasor diagram
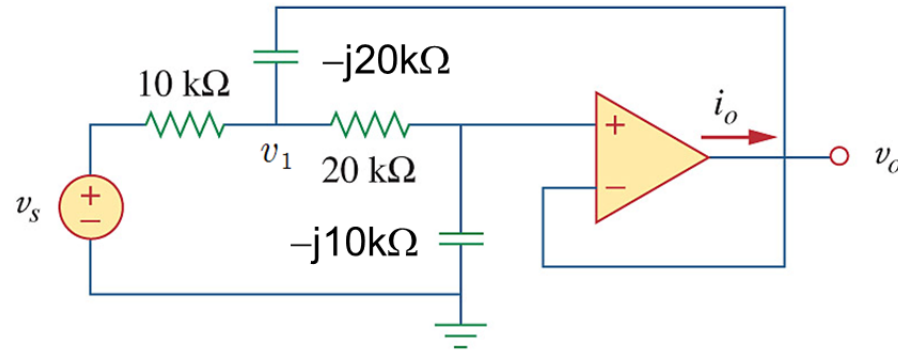


**Solution :**

$$4\cos 5000t \text{ V} \Rightarrow 4\angle 0° \text{ V}, \ \omega = 5000 \text{ rad/s}$$

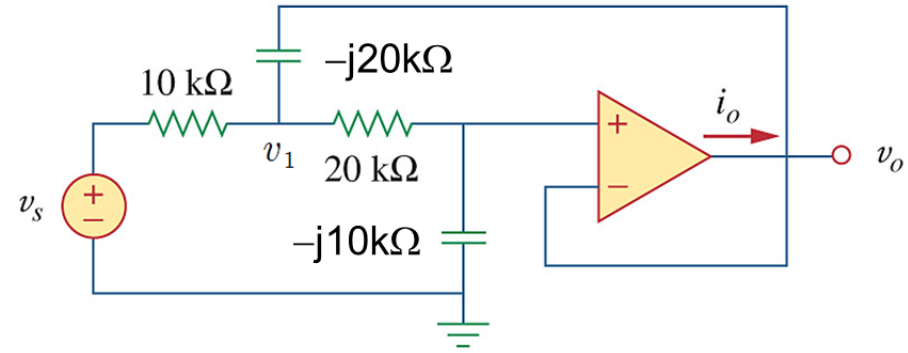$$10 \text{ nF} \Rightarrow \frac{1}{j\omega C} = \frac{1}{j5000 \times 10 \times 10^{-9}}$$

$$= -j2 \times 10^4 \ (\Omega) = -j20 \ (k\Omega)$$

$$20 \text{ nF} \Rightarrow \frac{1}{jWC} = \frac{1}{j5000 \times 20 \times 10^{-9}}$$

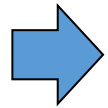$$= -j1 \times 10^4 \ (W) = -j10 \ (kW)$$

**(ii) V₀**

$$\begin{cases} \dfrac{\tilde{V}_s - \tilde{V}_1}{10} = \dfrac{\tilde{V}_1 - \tilde{V}_o}{-j20} + \dfrac{\tilde{V}_o}{-j10} \\ \tilde{V}_1 = \tilde{V}_o \dfrac{20 - j10}{-j10} = \tilde{V}_o(1 + j2) \end{cases}$$



$$-j2(\tilde{V}_s - \tilde{V}) = (\tilde{V}_1 - \tilde{V}_o) + 2\tilde{V}_o$$

$$-j2\tilde{V}_s = \tilde{V}_1(1 - j2) + \tilde{V}_o$$

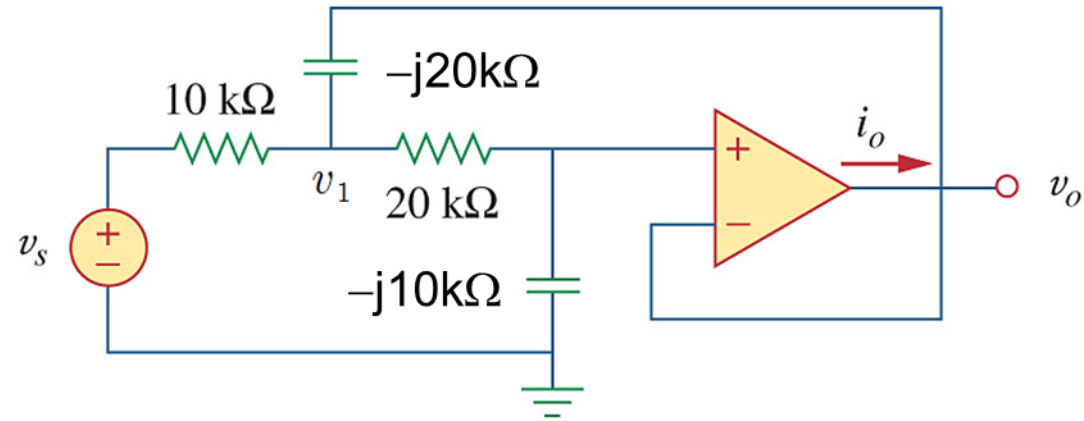$$-j2\tilde{V}_s = \tilde{V}_o(1 + j2)(1 - j2) + \tilde{V}_o = 6\tilde{V}_o$$

$$\tilde{V}_o = \frac{-j2\tilde{V}_s}{6} = \frac{-j2 \times 4\angle 0°}{6}$$

$$\approx 1.3333\angle -90° \text{ (V)}$$

$$v_o = 1.33\cos(5000t - 90°)$$

$$= 1.33\sin 5000t \text{ (V)}$$

**(iii) $i_o$**



$$\tilde{I}_o = \frac{\tilde{V}_o - \tilde{V}_1}{-j20} = \frac{\tilde{V}_o - \tilde{V}_o(1 + j2)}{-j20} = 0.1\tilde{V}_o$$

$$= 0.1 \times 1.3333\angle -90° \approx 0.13\angle -90° \text{ (mA)}$$

$$i_o = 0.13\cos(5000t - 90°)$$

$$= 0.13\sin 5000t \text{ (mA)}$$