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VE215 Introduction to Circuits
Chapter 9. Sinusoids and Phasors

*Recommend you to bring your calculator to lectures

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9.1 Introduction

- Direct current (dc) vs Alternating current (ac)
- ac is more efficient and economical to transmit over long distances
- Circuits driven by sinusoidal current or voltage sources are called ac circuits.

- A sinusoidal forcing function produces both a transient response and a steady-state response, like the step function in Chapters 7 and 8. we say that the circuit is operating at sinusoidal steady state.
- We are interested in sinusoidal steady-state response of AC circuits.

9.2 Sinusoids

A sinusoid is a signal that has the form of the sine or cosine function.

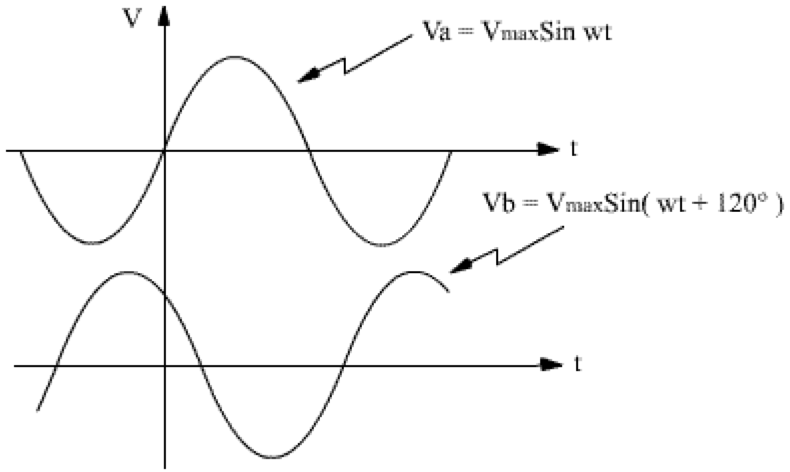
$$v(t) = V_m \sin(\omega t + \phi)$$

V_m : amplitude

ω: angular frequency

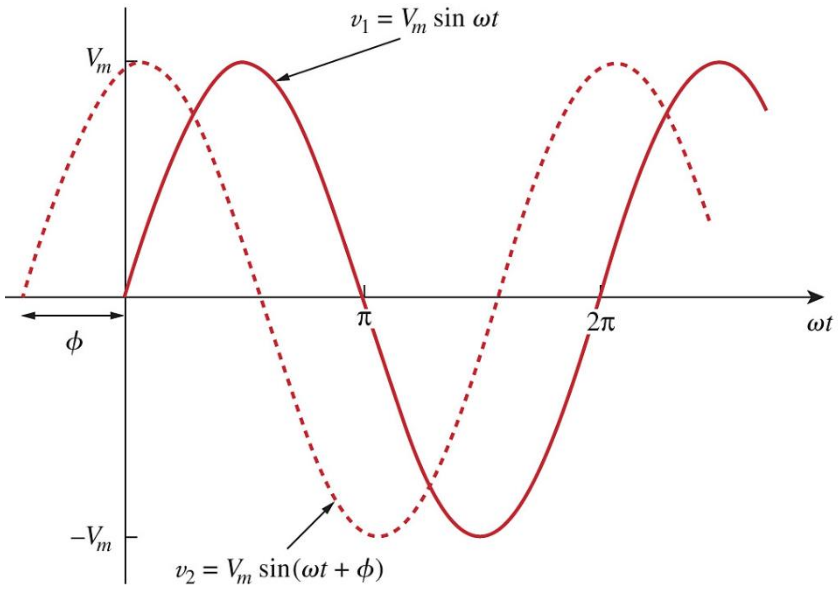
ϕ: initial phase

ωt: argument of the sinusoid



e.g. we have two sinusoids

$$v_1(t) = V_m \sin \omega t$$
$$v_2(t) = V_m \sin(\omega t + \phi)$$



The starting point of v_2 occurs first in time. Therefore, we say that v_2 **leads** v_1 by Ø or that v_1 **lags** v_2 by Ø. If Ø ≠ 0, we say that v_1 and v_2 are **out of phase**. If Ø = 0, then v_1 and v_2 are said to be **in phase**.

9.3 Phasors

Sinusoids are easily expressed in terms of **phasors**, which are more convenient to work with than sine and cosine functions. A phasor is a **complex number** that represents **the amplitude and phase of a sinusoid**.

*Complex Numbers:

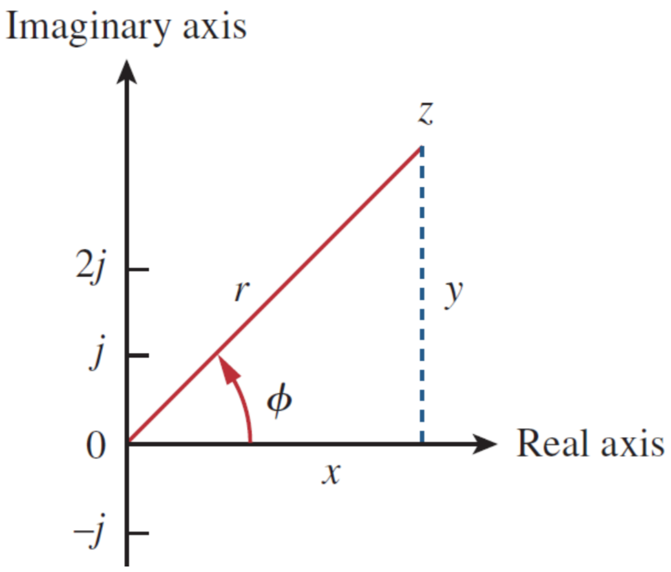
- $z = x + jy$ Rectangular form

$$x = r\cos\phi, \qquad y = r\sin\phi$$

- $z = r\angle\Phi$ Polar form

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

- $z = re^{j\Phi}$ Exponential form



*Basic properties of complex numbers

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication:

$$z_1 z_2 = r_1 r_2 \underline{/\phi_1 + \phi_2}$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \underline{/\phi_1 - \phi_2}$$

Reciprocal:

$$\frac{1}{z} = \frac{1}{r} \underline{/-\phi} \qquad \frac{1}{j} = -j$$

Square Root:

$$\sqrt{z} = \sqrt{r} \underline{/\phi/2}$$

Complex Conjugate:

$$z^* = x - jy = r\underline{/-\phi} = re^{-j\phi}$$

Given a sinusoid

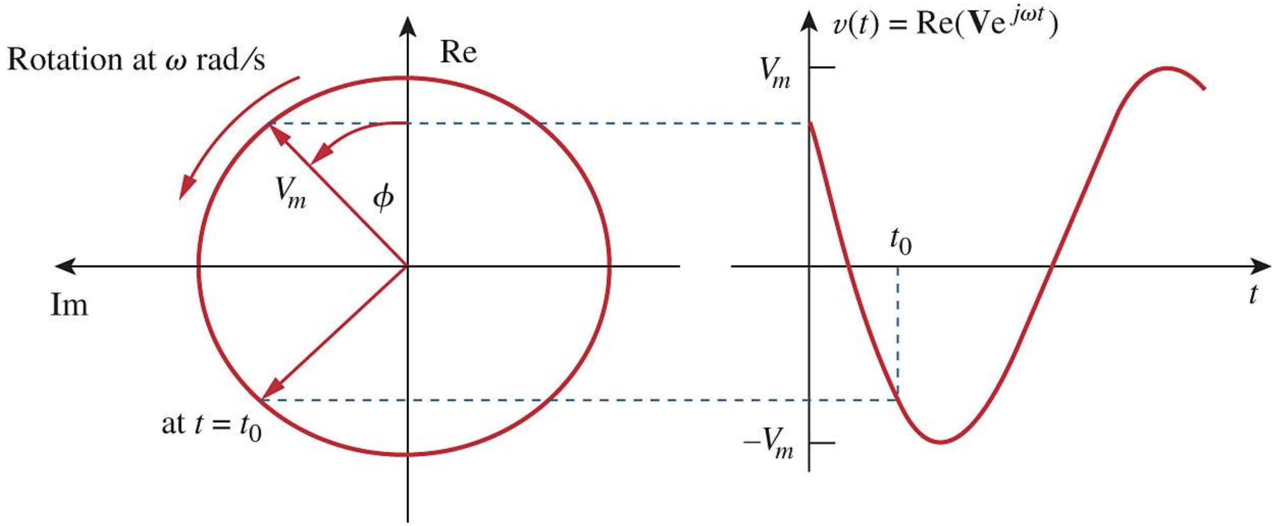
$$v(t) = V_m \cos(\omega t + \phi)$$
$$= \mathrm{Re}\left(V_m e^{j(\omega t+\phi)}\right)$$
$$= \mathrm{Re}\left(V_m e^{j\phi} e^{j\omega t}\right)$$
$$= \mathrm{Re}(\tilde{V} e^{j\omega t})$$ where $$\tilde{V} = V_m e^{j\phi} = V_m \angle \phi$$

***Euler's identity**

$e^{\pm j\emptyset} = cos\emptyset \pm jsin\emptyset$

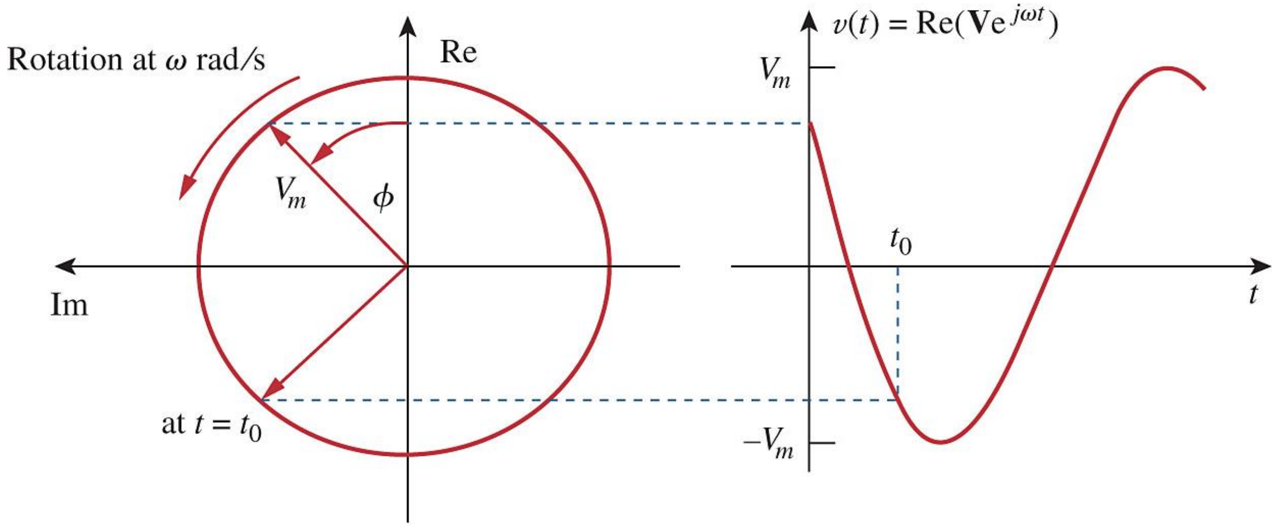
- $\cos\Phi = \mathrm{Re}(e^{j\Phi})$
- $\sin\Phi = \mathrm{Im}(e^{j\Phi})$

Phasor representation of the sinusoid v(t) $\tilde{V} = V_m e^{j\phi} = V_m \angle \phi$



The sinor $\tilde{V}e^{j\omega t} = V_m e^{j(\omega t+\phi)}$ on the complex plane

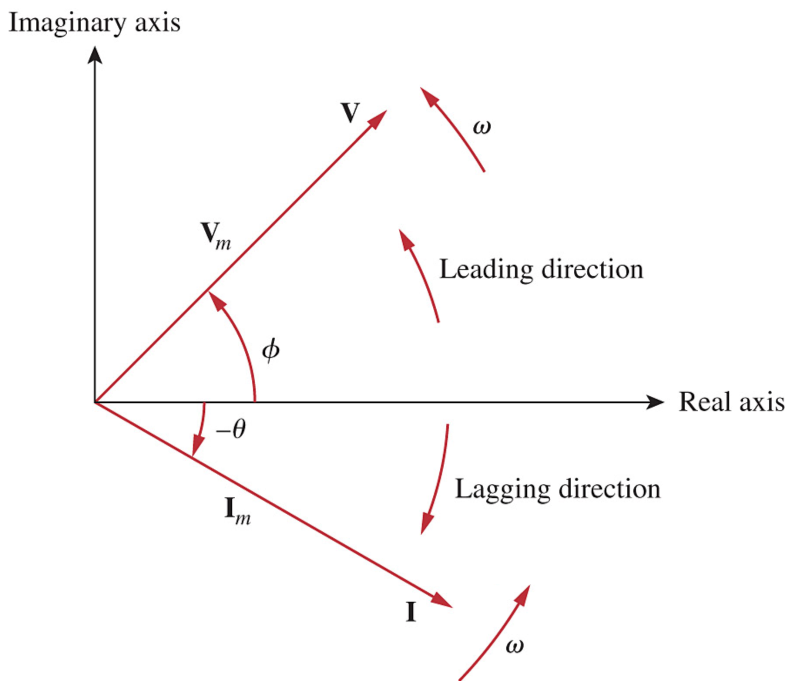
As time increases, the sinor rotates on a circle of **radius V_m at an angular velocity ω** in the **counter clockwise direction**. We may regard *v(t)* as the projection of the sinor on the real axis.



The sinor $\tilde{V}e^{j\omega t} = V_m e^{j(\omega t+\phi)}$ on the complex plane

The value of the sinor **at time t=0** is the phasor **V**. The sinor may be regarded as a rotating phasor. Thus, whenever a sinusoid is expressed as a phasor, **the term $e^{j\omega t}$ is implicitly present**.

Because a phasor has **magnitude and phase** (“direction”), it behaves as a vector.



$$\tilde{V} = V_m \angle \phi$$

$$\tilde{I} = I_m \angle -\theta.$$

Phasor diagram: graphical representation of phasors

A sinusoid has a time-domain representation $v(t) = V_m \cos(\omega t + \phi)$
and a phasor-domain representation $\widetilde{V} = V_m \angle \phi$.

The phasor domain is also known as the frequency domain.

$$v(t) = V_m \cos(\omega t + \phi) \Leftrightarrow \widetilde{V} = V_m \angle \phi$$

Sinusoid-phasor transformation.

Time domain representation	**Phasor domain representation**
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$

Example 9.3

(b) $$\frac{\boxed{\boxed{10\angle -30^\circ} + (3 - j4)}}{\boxed{(2 + j4)(3 - j5)^*}}$$

Example 9.3

$$\text{(b)}\ \frac{\boxed{10\angle -30^\circ} + (3 - j4)}{\boxed{(2 + j4)(3 - j5)^*}}$$ = 0.565 ∠-160.13

10cos(-30) + j10sin(-30) = 8.66 – j5

8.66 – j5 + 3 – j4 = 11.66 – j9

(2 + j4)(3 – j5)* = (2 + j4)(3 + j5) = –14 + j22

11.66 – j9 → $\sqrt{11.66^2 + 9^2} = 14.73$

–14 + j22 → $\sqrt{14^2 + 22^2} = 26.07$

Angle: $tan^{-1}\left(\frac{-9}{11.66}\right) = -37.66$

Angle: $\boldsymbol{tan^{-1}\left(\frac{22}{-14}\right) = -57.53}$ **??**

Practice Problem 9.3

(b) $\dfrac{\boxed{10 + j5 + 3\angle 40^\circ}}{\boxed{-3 + j4}} + 10\angle 30^\circ + j5$ = 8.29 + j7.20

Practice Problem 9.3

(b) $\dfrac{\boxed{10 + j5 + 3\angle 40^\circ}}{\boxed{-3 + j4}} + 10\angle 30^\circ + j5$ = 8.29 + j7.20

10 + j5 + 2.30 +j1.93 = 12.30 + j6.93

12.30 + j6.93 → $\sqrt{12.30^2 + 6.93^2} = 14.12$

–3 + j4 → $\sqrt{3^2 + 4^2} = 5$

Angle: $tan^{-1}\left(\frac{6.93}{12.30}\right) = 29.40$

Angle: $tan^{-1}\left(\frac{4}{-3}\right) = 126.87$

2.82 angle -97.47

9.4 Phasor Relationships for Circuit Elements

Let's think about the IV **at the resistor**. If the current through a resistor R is $i = I_m \cos(\omega t + \phi)$
The voltage across it is given by Ohm's law as

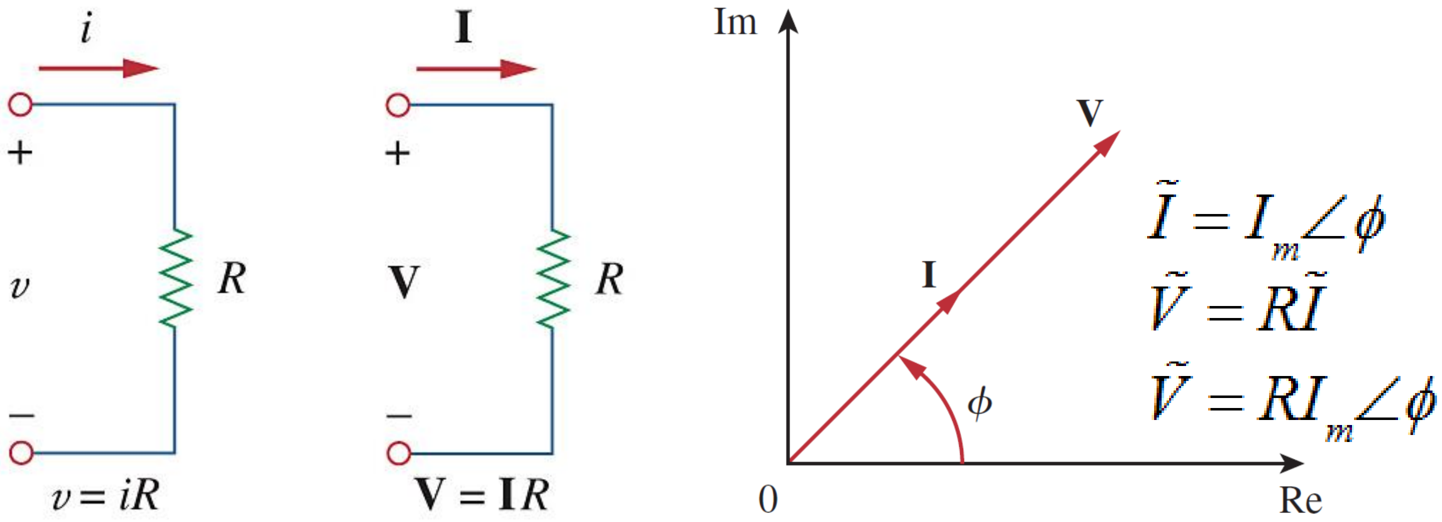
$$v = iR = RI_m \cos(\omega t + \phi)$$

The phasor representation of the voltage is

$$\tilde{V} = RI_m \angle \phi$$

And **the phasor** representation of the current is

$$\tilde{I} = I_m \angle \phi.$$



The voltage-current relation for the resistor in the phasor domain.

Inductor

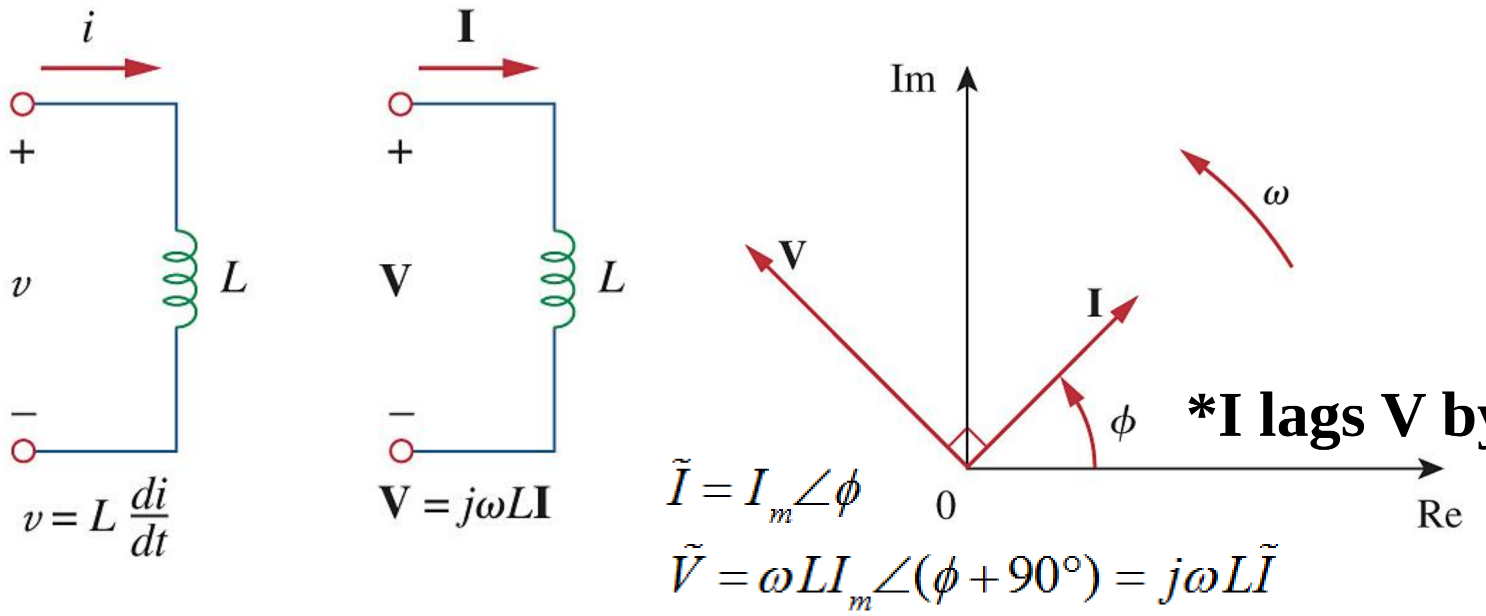
For the inductor L, if $i = I_m \cos(\omega t + \phi) \Leftrightarrow \tilde{I} = I_m \angle \phi$

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) = \omega L I_m \cos(\omega t + \phi + 90°)$$

$$\Rightarrow \tilde{V} = \omega L I_m \angle(\phi + 90°) = j\omega L I_m \angle \phi = j\omega L \tilde{I}$$

$* - \sin(wt + \emptyset) = \cos(\omega t + \emptyset + 90°)$

$\angle 90° = e^{j90°} = j$



***I lags V by 90°**

Capacitor

For the capacitor C, if $v = V_m \cos(\omega t + \phi) \Leftrightarrow \tilde{V} = V_m \angle \phi$

$$i = C\frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi) = \omega C V_m \cos(\omega t + \phi + 90°)$$

$$\Rightarrow \tilde{I} = \omega C V_m \angle(\phi + 90°) = j\omega C V_m \angle \phi \boxed{= j\omega C \tilde{V}}$$

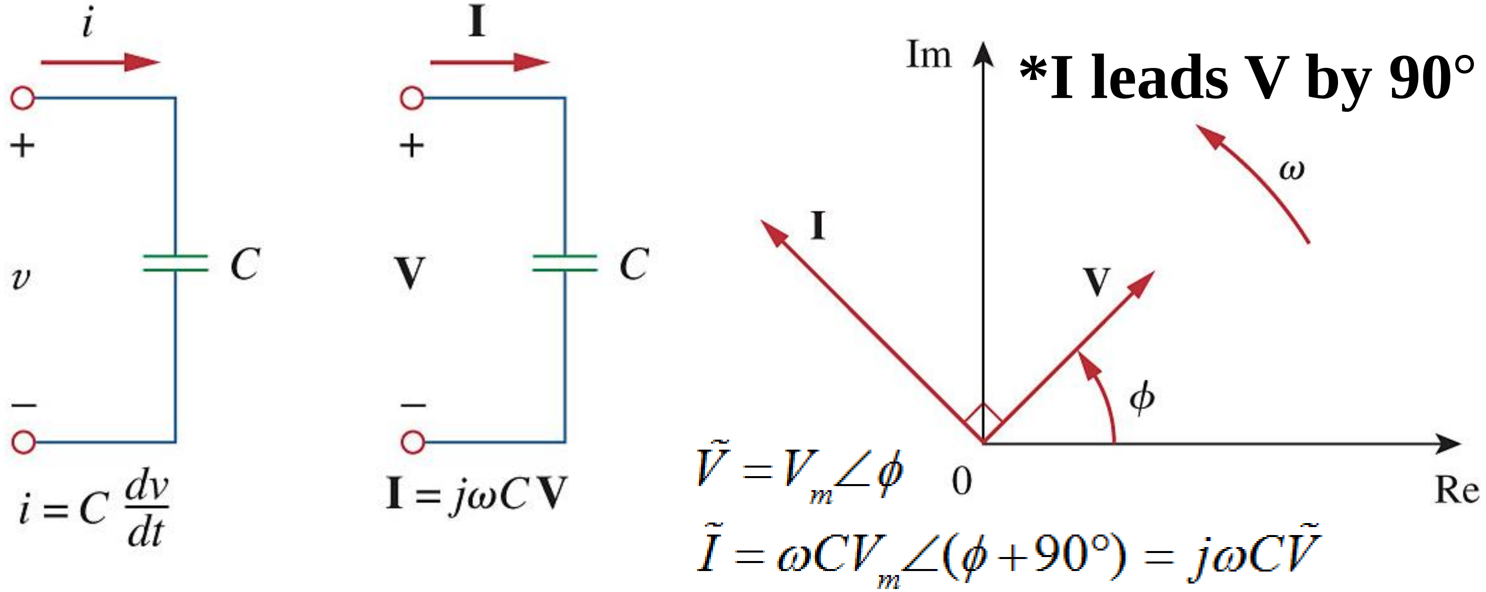


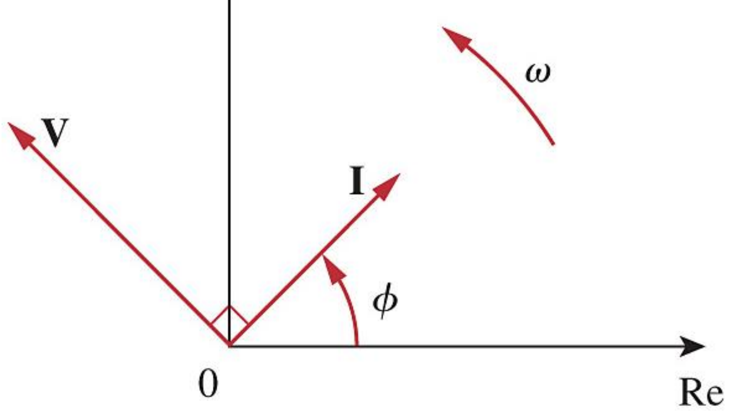
TABLE 9.2

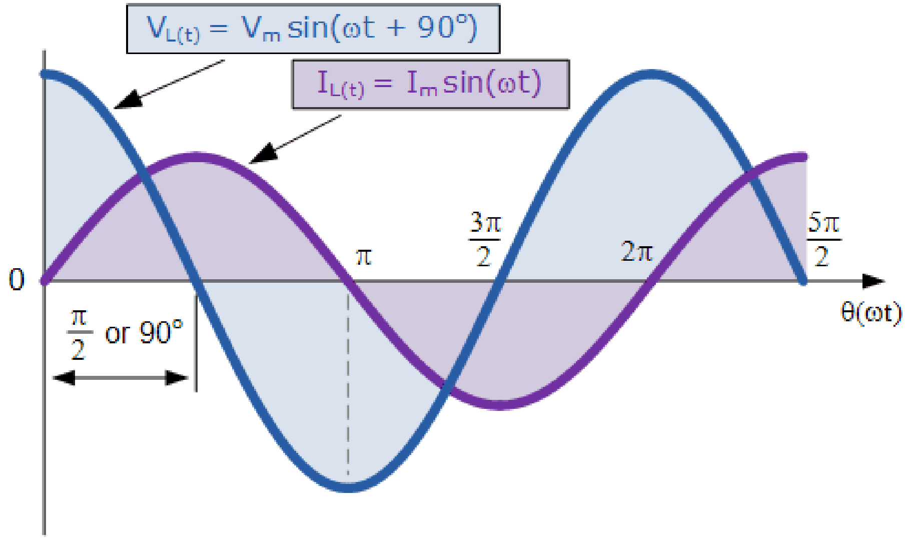
Summary of voltage - current relationships

Element	Time domain	Frequency domain	
R	$v = Ri$	$\tilde{V} = R\tilde{I}$	**No delay**
L	$v = L\dfrac{di}{dt}$	$\tilde{V} = j\omega L\tilde{I}$	**V faster**
C	$i = C\dfrac{dv}{dt}$	$\tilde{V} = \dfrac{1}{j\omega C}\tilde{I}$	**I faster**

Inductor

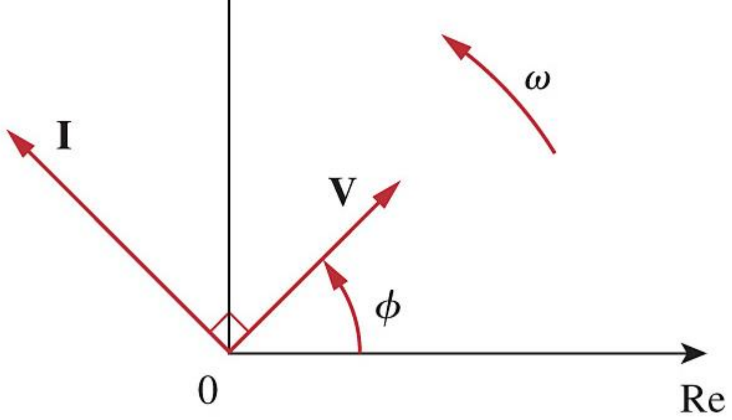
*I lags V by 90°

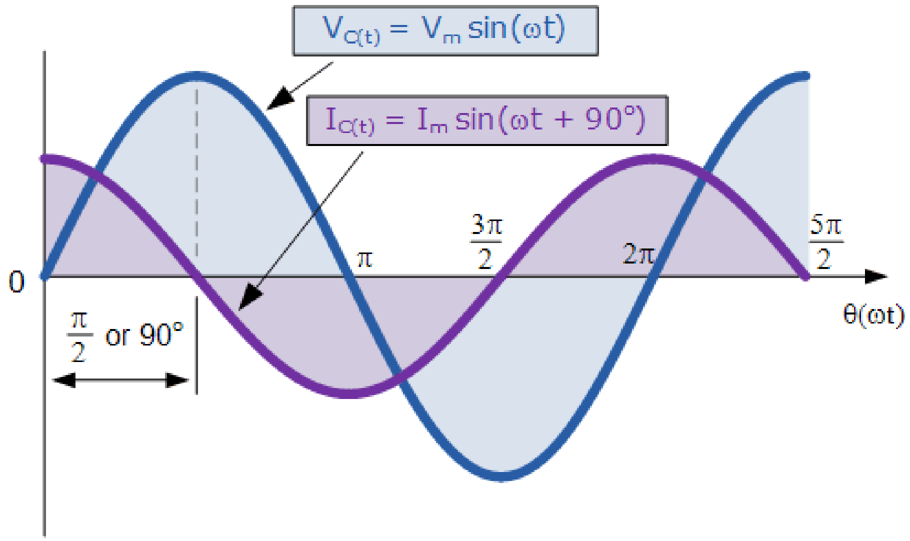




Capacitor

*I leads V by 90°





9.5 Impedance and Admittance

General definition of R

R: Resistance G: Conductance R=1/G (Real number)	→	Z: Impedance Y: Admittance Z=1/Y (Complex number)

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Impedance

The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in Ohms (Ω)

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}}$$

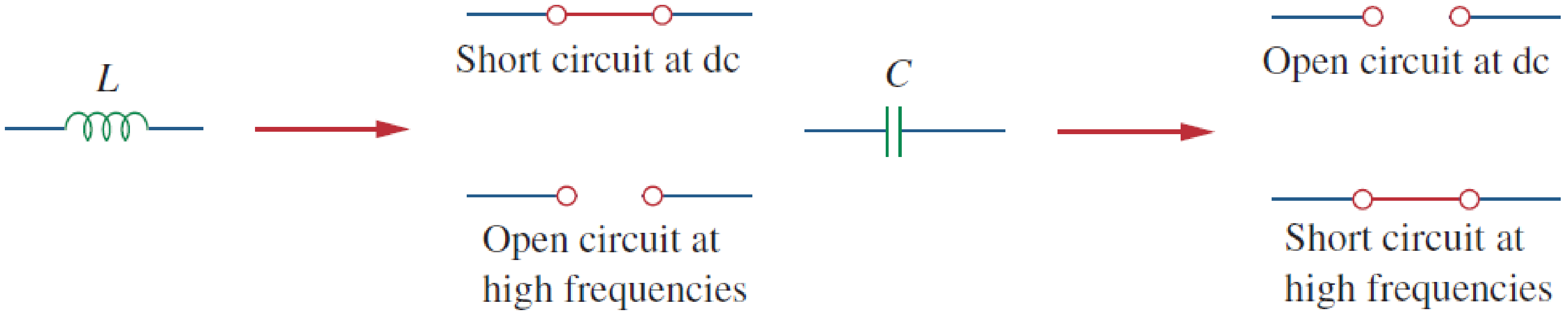
Resistance	$\tilde{V} = R\tilde{I}$	$Z = R$
Inductor	$\tilde{V} = j\omega L\tilde{I}$	$Z = j\omega L$
Capacitor	$\tilde{V} = \dfrac{1}{j\omega C}\tilde{I}$	$Z = 1/\, j\omega C$

Inductor $\tilde{V} = j\omega L \tilde{I}$ $Z = j\omega L$

Capacitor $\tilde{V} = \dfrac{1}{j\omega C}\tilde{I}$ $Z = 1/\,j\omega C$

(i) $\boldsymbol{\omega = 0}$ (DC source): $Z_L = 0$, $Z_C \rightarrow \infty$

(ii) $\boldsymbol{\omega \rightarrow \infty}$: $Z_L \rightarrow \infty$, $Z_C = 0$



As a complex quantity, the impedance may be expressed in rectangular form or polar form

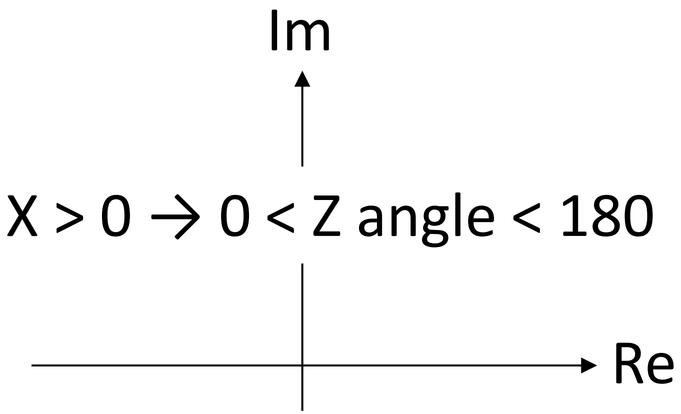
$$Z = R + jX = |Z|\angle q \qquad |\mathbf{Z}| = \sqrt{R^2 + X^2}, \;\; \theta = \tan^{-1}\frac{X}{R}$$

where

R: resistance $\qquad R = |\mathbf{Z}|\cos\theta$

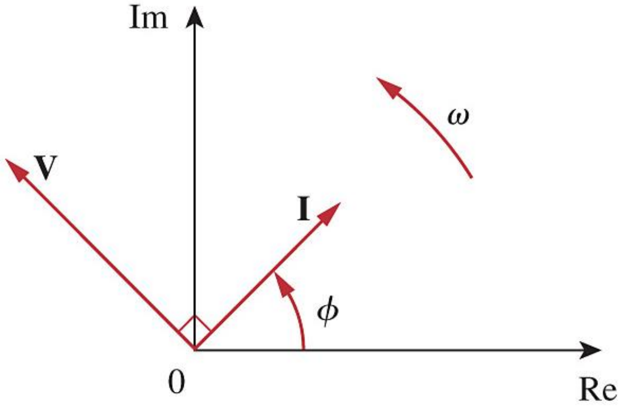
X: reactance $\qquad X = |\mathbf{Z}|\sin\theta$

i) X > 0, the impedance is **inductive or lagging**, i.e. current lags voltage.



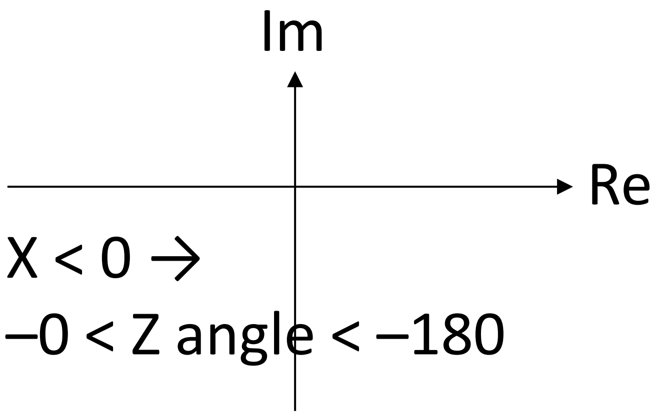
V$= \boldsymbol{IZ}$

Angle (V) = Angle (I) + Angle (Z)



Z = R + jX where X > 0
→Impedance is inductive
or lagging (**I** lags **V**)

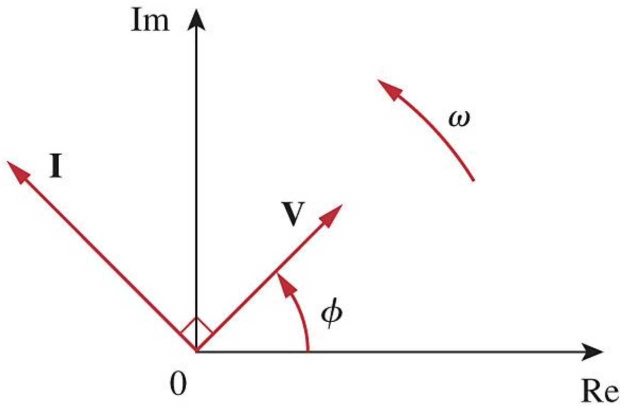
ii) X<0, the impedance is **capacitive or leading**, i.e. current leads voltage.



X < 0 →
–0 < Z angle < –180

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z}}$$

Angle (I) = Angle (V) – Angle (Z)
where Angle (Z) is negative



Z = R + jX where X < 0
→Impedance is capacitive
or leading (**I** leads **V**)

Admittance

The admittance **Y** of a circuit is the ratio of the phasor current **I** to the phasor voltage V, measured in siemens (S).

$$\boldsymbol{Y} = \frac{\boldsymbol{I}}{\boldsymbol{V}} = \frac{\mathbf{1}}{\boldsymbol{Z}}$$

Resistance	$\tilde{V} = R\tilde{I}$	$Y = 1 / R$
Inductor	$\tilde{V} = j\omega L\tilde{I}$	$Y = 1 / jwL$
Capacitor	$\tilde{V} = \dfrac{1}{j\omega C}\tilde{I}$	$Y = jwC$

The admittance (Y) can be written as

$$G + jB = \frac{1}{R + jX}$$

where

G : conductance

B : susceptance

The admittance, conductance, and susceptance are all measured in siemens.

9.6 Kirchhoff's Laws in the Frequency Domain

For KVL, let v_1, v_2, …, v_n be the voltages around a closed loop.

Then, $$\sum_{i=1}^{n} v_i = 0$$

In sinusoidal steady state,

$$v_i = V_{mi}\cos(\omega t + \phi_i) = \mathrm{Re}\left(\tilde{V}_i e^{j\omega t}\right)$$

$$\sum_{i=1}^{n} v_i = 0 \quad \longleftarrow \quad v_i = V_{mi} \cos(\omega t + \phi_i) = \mathrm{Re}\left(\tilde{V}_i e^{j\omega t}\right)$$

$$\Rightarrow \sum_{i=1}^{n} \mathrm{Re}\left(\tilde{V}_i e^{j\omega t}\right) = 0$$

$$\mathrm{Re}\left(\left(\sum_{i=1}^{n} \tilde{V}_i \right) e^{j\omega t} \right) = 0$$

but $e^{j\omega t} \neq 0$,

$$\sum_{i=1}^{n} \tilde{V}_i = 0$$

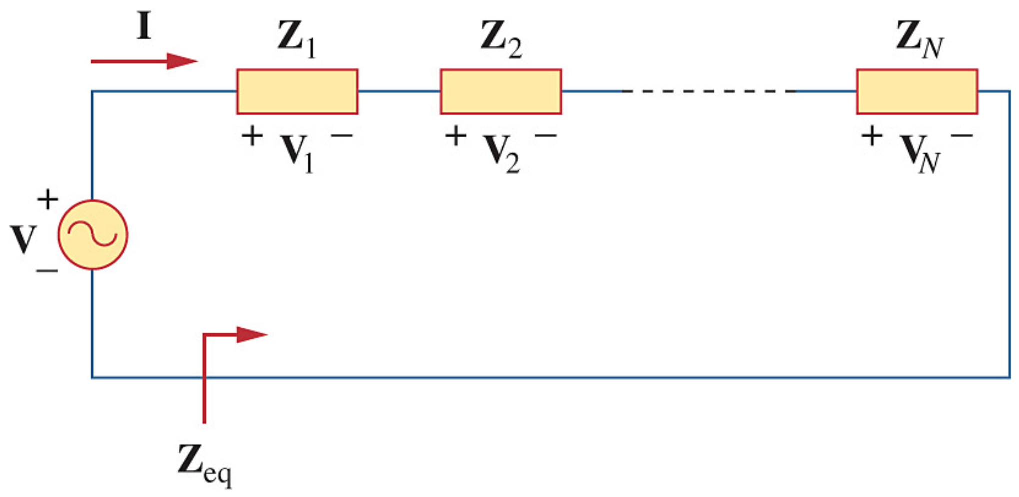
indicating that KVL holds for phasors.

For KCL, if $i_1, i_2, \ldots, i_n$ are the currents leaving or entering a closed surface in a circuit at time t, and $\mathbf{I_1}, \mathbf{I_2}, \ldots, \mathbf{I_n}$ are the phasor forms of $i_1, i_2, \ldots, i_n$,

Then, $$\sum_{i=1}^{n} i_i = 0 \Rightarrow \sum_{i=1}^{n} \widetilde{I}_i = 0$$

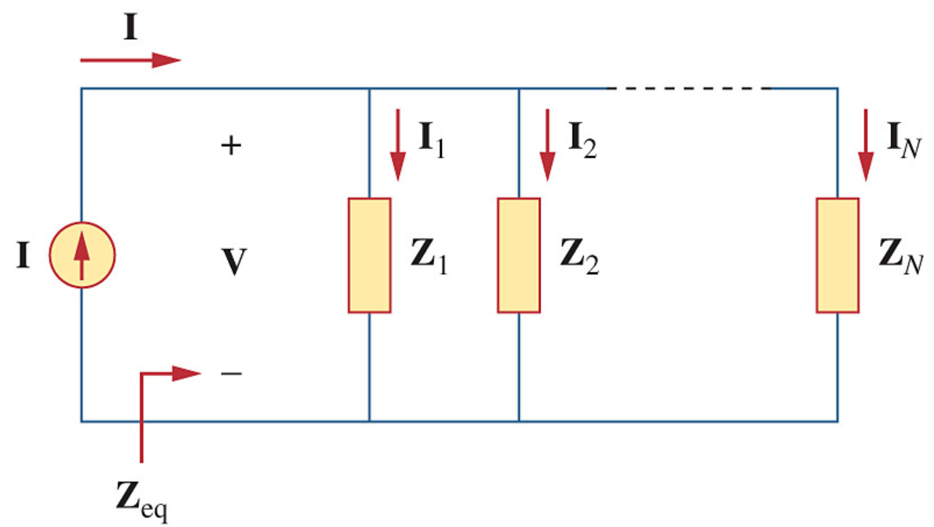
Since basic circuit laws, **Kirchoff's and Ohm's, hold in phasor domain**, it is not difficult to analyze ac circuit.

9.7 Impedance Combinations



For the N series-connected impedances, the equivalent impedance at the input terminal is

$$Z_{eq} = \frac{\tilde{V}}{\tilde{I}} = \frac{\sum_{i=1}^{N} \tilde{V}_i}{\tilde{I}} = \sum_{i=1}^{N} \frac{\tilde{V}_i}{\tilde{I}} = \sum_{i=1}^{N} Z_i$$

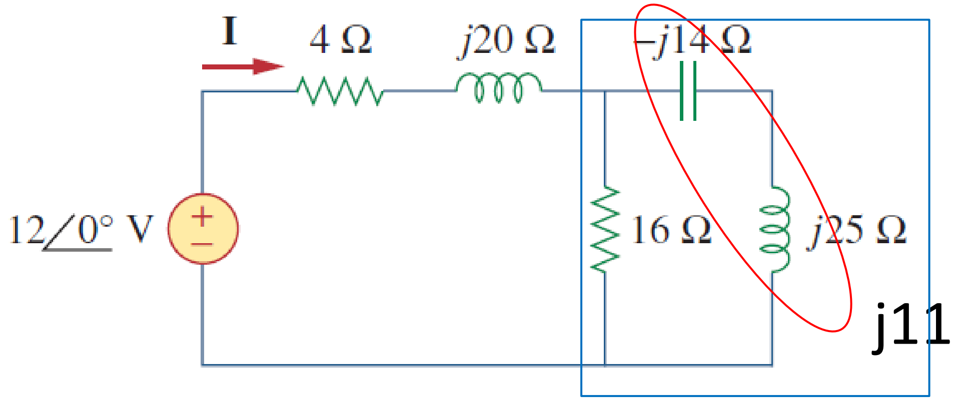


For the N parallel-connected impedances, the equivalent admittance at the input terminal is

$$Y_{eq} = \frac{\tilde{I}}{\tilde{V}} = \frac{\sum_{i=1}^{N} \tilde{I}_i}{\tilde{V}} = \sum_{i=1}^{N} \frac{\tilde{I}_i}{\tilde{V}} = \sum_{i=1}^{N} Y_i$$

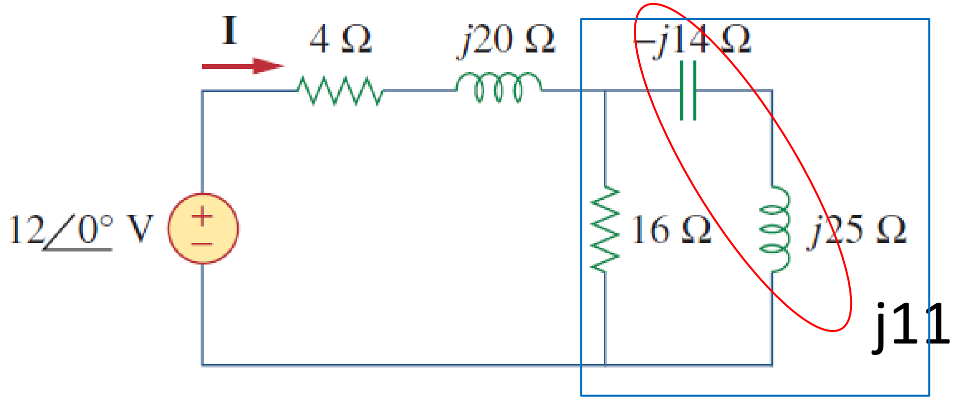
Example

9.39 For the circuit shown in Fig. 9.46, find Z_{eq} and use that to find current **I**. Let $\omega = 10$ rad/s.



Example

9.39 For the circuit shown in Fig. 9.46, find Z_{eq} and use that to find current **I**. Let $\omega = 10$ rad/s.



$$16 \parallel j11 = \frac{j176}{16+j11} = \frac{j176}{19.42\angle 34.51} = 9.06\angle 55.49$$

$$\mathbf{Z_{eq}} = 9.06\angle 55.49 + 4 + j20 = 5.13 + j7.47 + 4 + j20$$

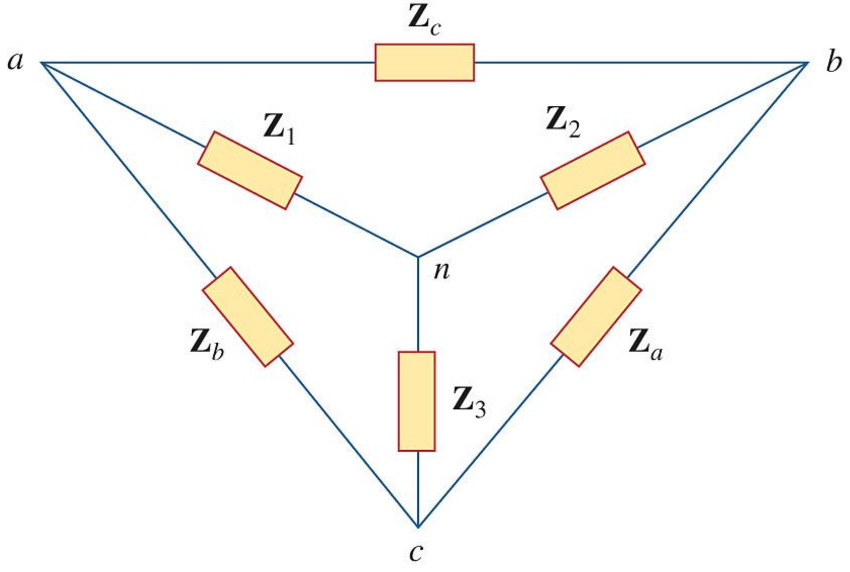
$$\mathbf{= 9.13 + j27.47}$$

$$I = \frac{V}{Z_{eq}} = \frac{12\angle 0}{9.13 + j27.47} = \frac{12\angle 0}{28.95\angle 71.62} = 0.41\angle(-71.62)$$

i(t) = 0.41 cos (10t – 71.62) [A]

The delta-to-wye and **wye-to-delta** transformations that we applied to resistive circuits are also valid for impedances.

(i) Y-Delta



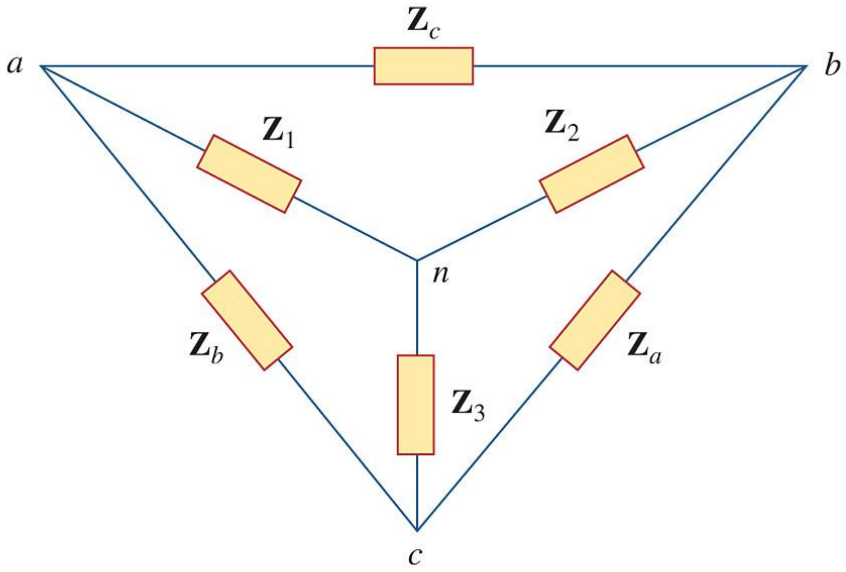
Y-Δ conversion:

$$Z_a = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1}$$

$$Z_b = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2}$$

$$Z_c = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3}$$

(ii) Delta-Y



Δ-Y conversion:

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

Practice Problem 9.10 Find the input impedance of the circuit in Fig. 9.24 at $\omega = 10$ rad/s.

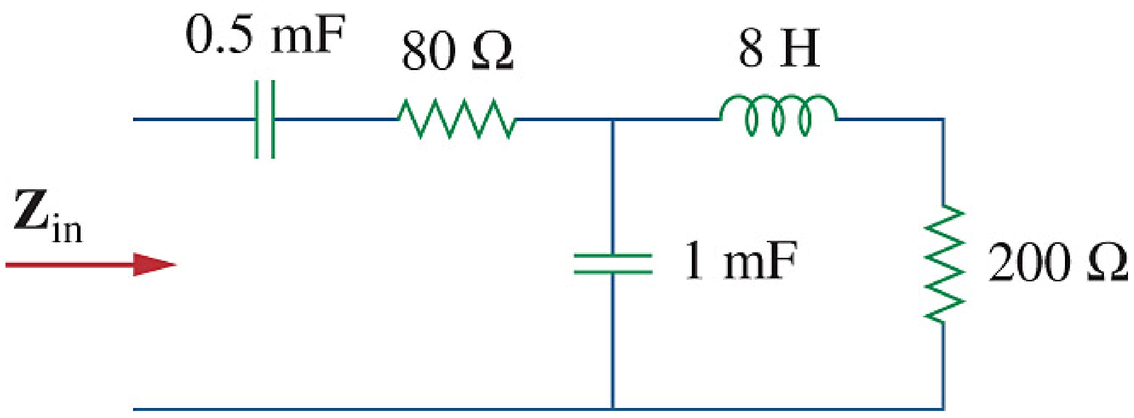
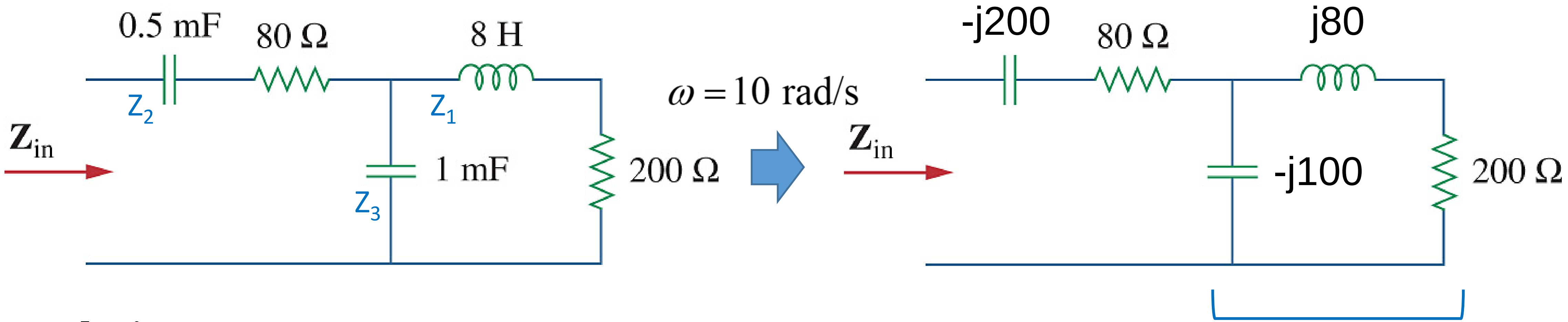


Figure 9.24



Solution :

8-H inductor: $Z_1 = j10 \times 8 = j80\ (\Omega)$

0.5-mF capacitor: $Z_2 = \dfrac{1}{j10 \times (0.5 \times 10^{-3})}$

$= -j200\ (\Omega)$

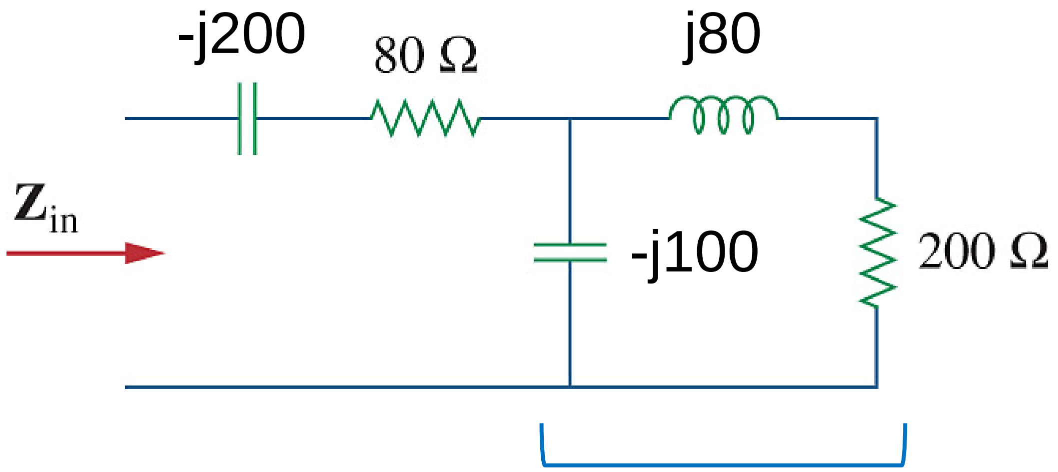
1-mF capacitor: $Z_3 = \dfrac{1}{j10 \times (1 \times 10^{-3})}$

$= -j100\ (\Omega)$

$$= \frac{(-j100) \times (j80 + 200)}{(-j100) + (j80 + 200)}$$

$$\approx 107.1688\angle -62.49^\circ$$

$$\approx 49.5016 - j95.0512\ (\Omega)$$



$$= \frac{(-j100)\times(j80+200)}{(-j100)+(j80+200)} \approx 107.1688\angle -62.49^\circ$$

$$\approx 49.5016 - j95.0512\ (\Omega)$$

$$Z_{in} = -j200 + 80 + 49.5016 - j95.0512$$

$$\approx 129.50 - j295.05\ (\Omega)$$

Practice Problem 9.11 Calculate v_o in the circuit of Fig. 9.27.

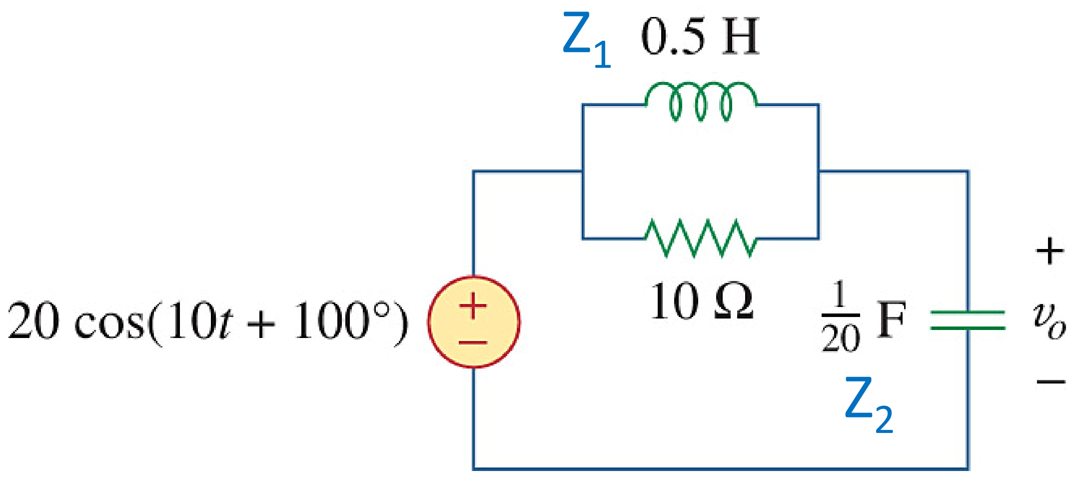
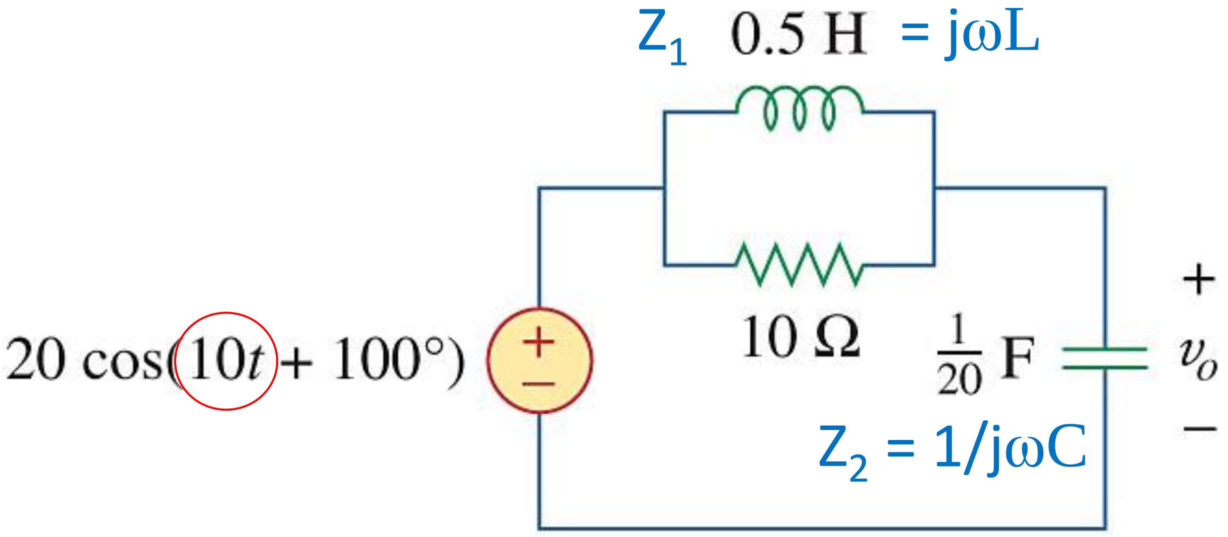


Figure 9.27

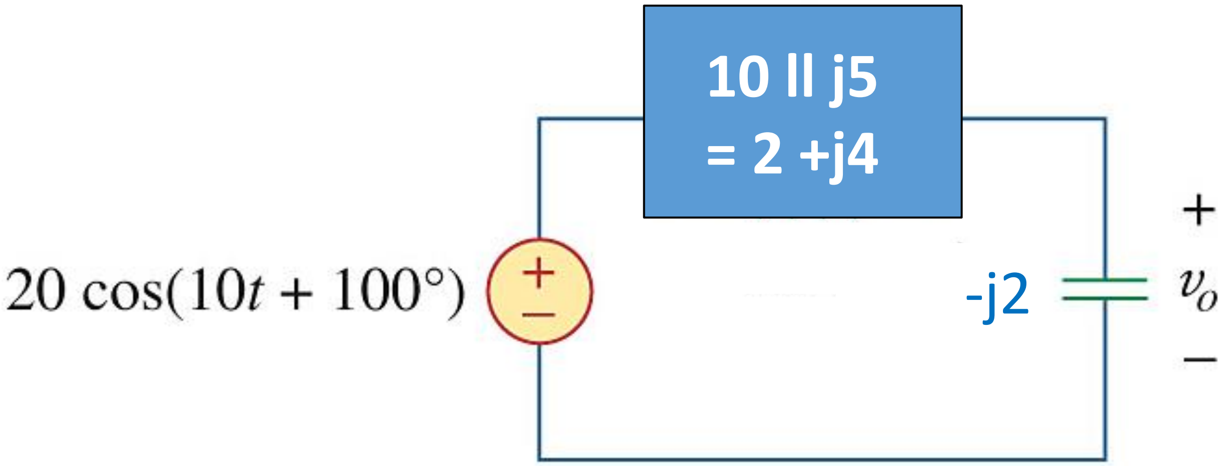


$\omega = 10$, $V = 20\angle 100$

$$\text{0.5-H inductor: } Z_1 = j10 \times 0.5 = j5\ (\Omega)$$

$$\tfrac{1}{20}\text{-F capacitor: } Z_2 = \frac{1}{j10 \times (1/20)}$$

$$= -j2\ (\Omega)$$



$$\tilde{V}_o = 20\angle 100° \times \frac{-j2}{-j2 + 10 \parallel j5}$$

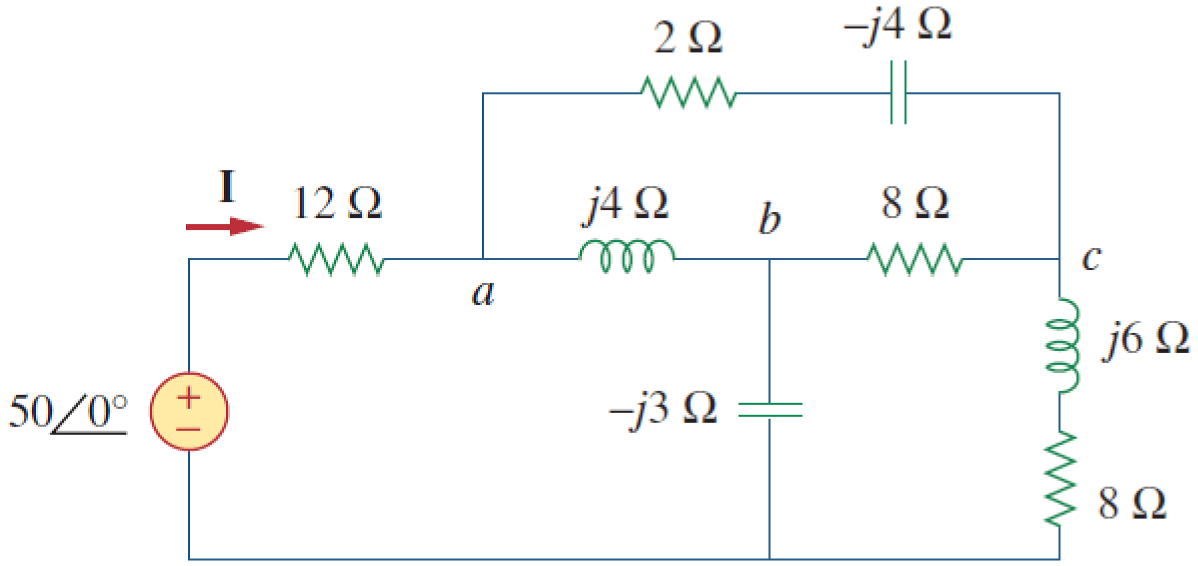
$$10 \parallel j5 = \frac{10 \times j5}{10 + j5} = \frac{j10}{2 + j} = 2 + j4$$ Or 10 ll j5 = 4.47∠63.43

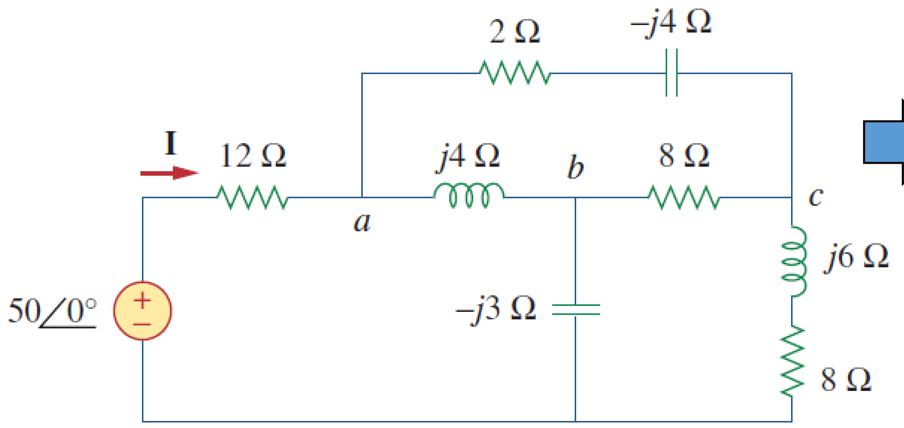
$$\tilde{V}_o = 20\angle 100° \times \frac{-j2}{2 + j2} = 10\sqrt{2}\angle -35° \text{ (V)}$$

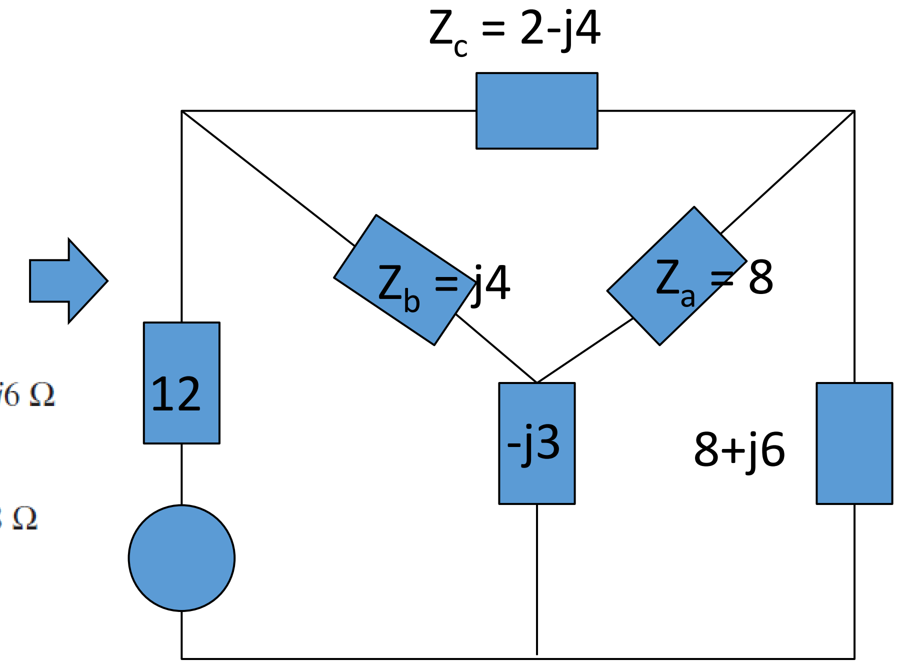
$$v_o(t) = 10\sqrt{2}\cos(10t - 35°) \text{ (V)}$$

Example 9.12

Find current **I** in the circuit of Fig. 9.28.





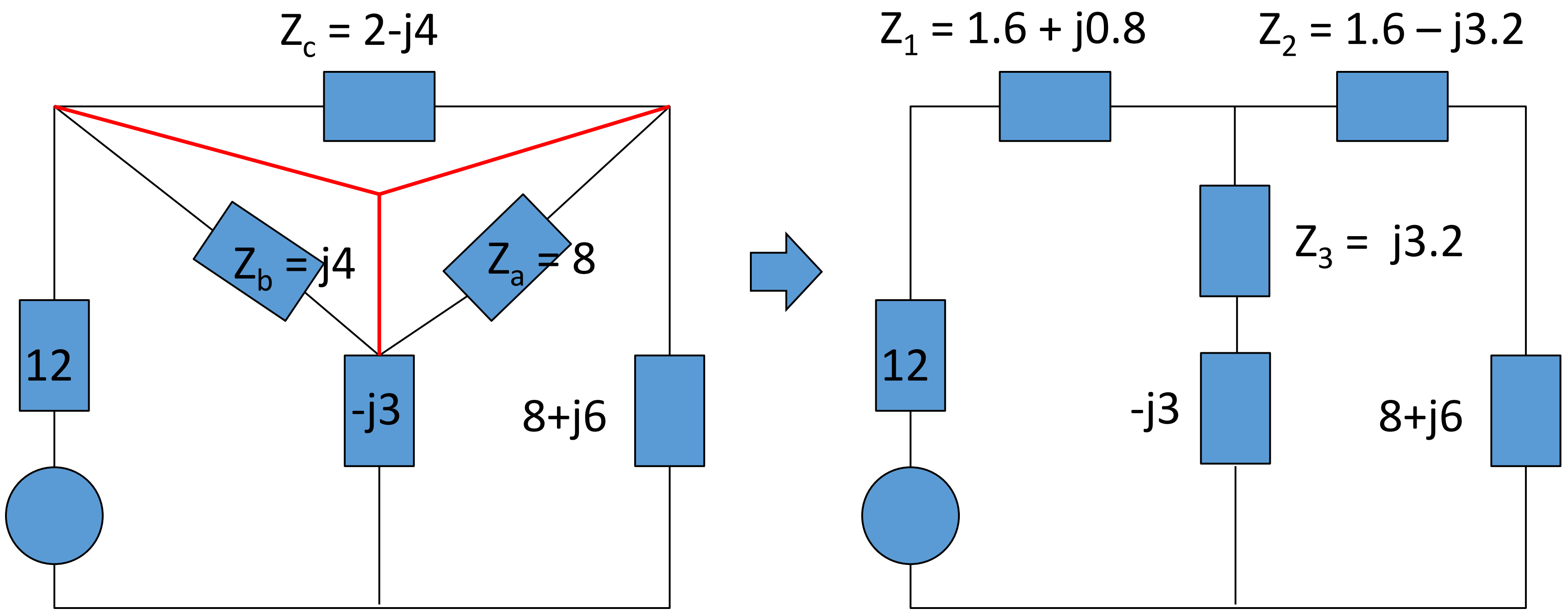


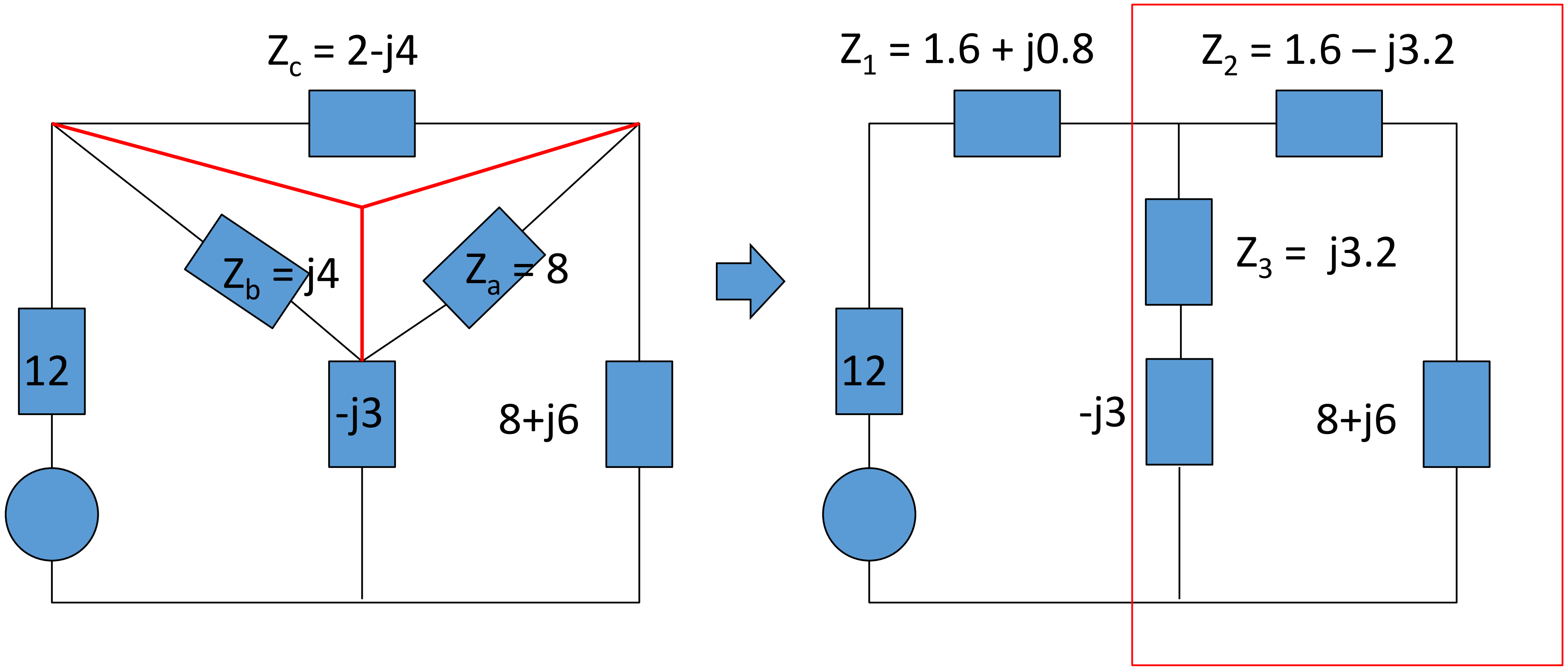
Δ-*Y* conversion:

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c} = 1.6 + j0.8$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c} = 1.6 - j3.2$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c} = j3.2$$


Z_c = 2-j4
Z_b = j4
Z_a = 8
12
-j3
8+j6
Z_1 = 1.6 + j0.8
Z_2 = 1.6 – j3.2
Z_3 = j3.2
12
-j3
8+j6



j0.2 ll 9.6+j2.8

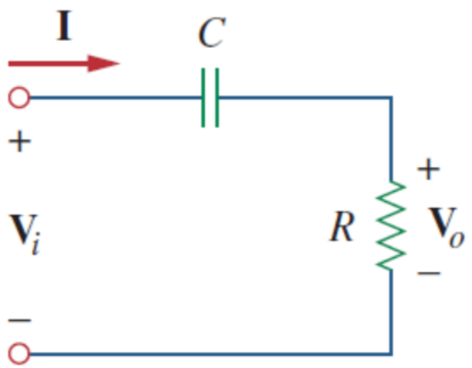
= 0.2 angle (88.91)

Z_{eq} = 13.6 + j0.8 +0.004 + j0.2 ≈ 13.6 + j1 = 13.64 angle (4.21)

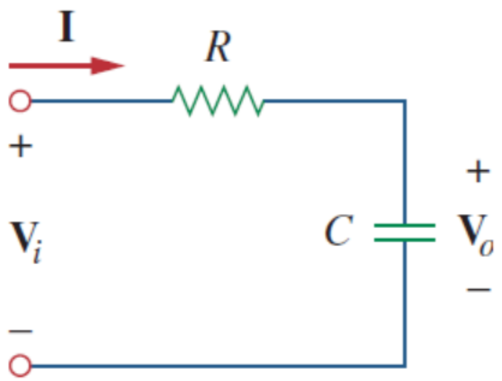
I = V/Z_{eq} = 3.67 angle (-4.21) [A]

9.8 Applications

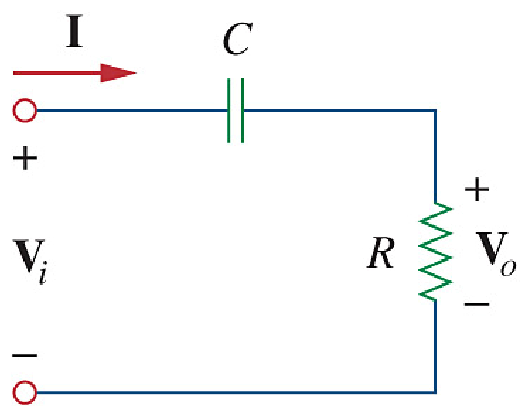
Phase-Shifters: A phase-shifting circuit is often employed to correct an undesirable phase shift already present in a circuit or to produce special desired effects.

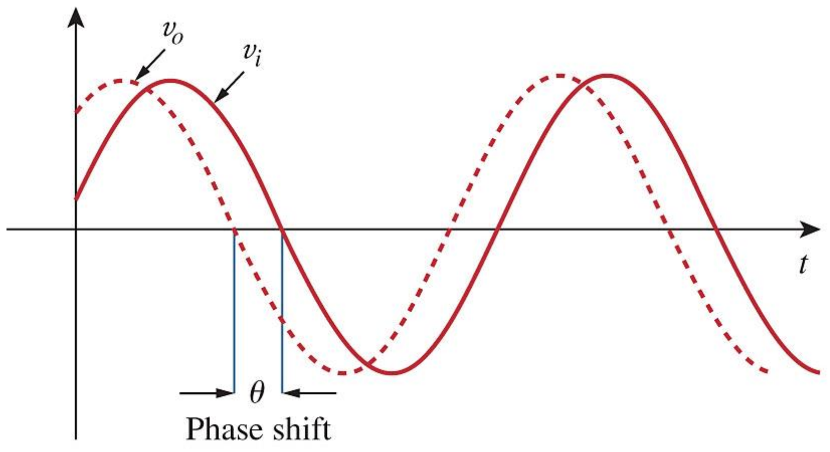


V_o leads V_i



V_o lags V_i



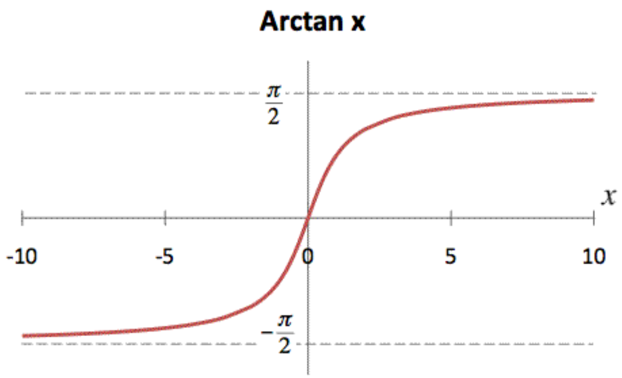


$$\tilde{V}_o = \tilde{V}_i \frac{R}{R + 1/(j\omega C)} = \tilde{V}_i \frac{R}{R - j(1/\omega C)}$$

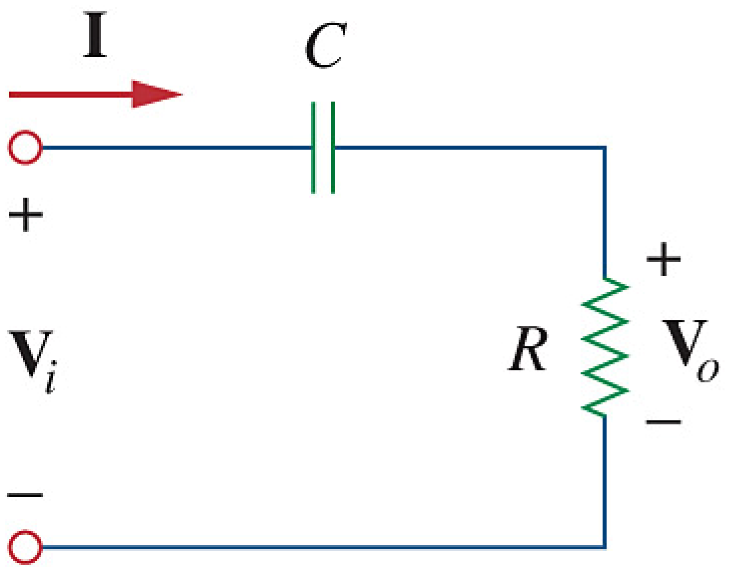
$$= \tilde{V}_i \frac{R}{\sqrt{R^2 + (1/\omega C)^2} \angle -\tan^{-1}(1/(\omega RC))}$$

$\tilde{V}_o$ leads $\tilde{V}_i$ by $\theta = \tan^{-1}(1/\boxed{(\omega RC)})$,

$0° < \theta < 90°$ Amount of phase shift depends on the values of R, C, and ω.

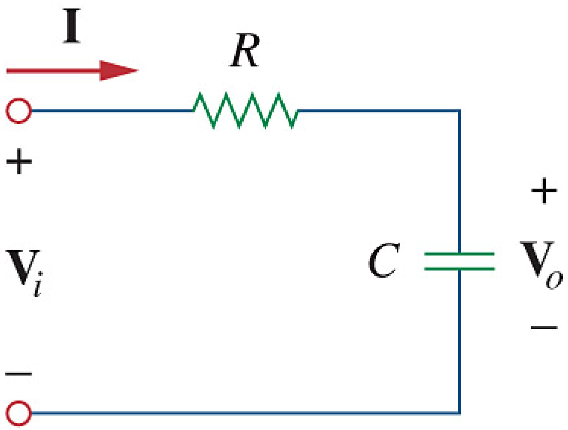


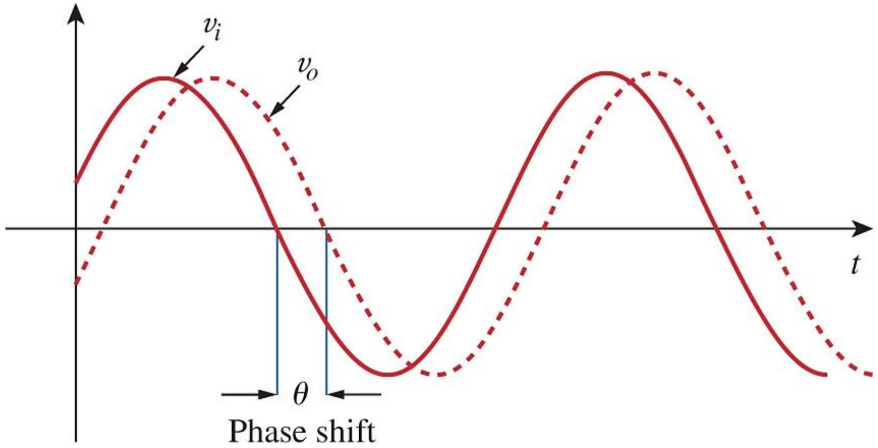
$$\boxed{\begin{array}{l} \angle \mathbf{V_i} + \theta = \angle \mathbf{V_o} \\ \angle \mathbf{V_o} > \angle \mathbf{V_i} \\ 0° < \theta < 90° \end{array}}$$



Another way to check leading/lagging relation:

(1) $\mathbf{V_o}$=I R → $\angle\mathbf{I}=\angle\mathbf{V_o}$
(2) $\mathbf{V_i}=\mathbf{IZ}=\mathbf{I}(R+1/j\omega C)$ → $-90° < \angle\mathbf{Z} < 0$ → $\angle\mathbf{I} > \angle\mathbf{V_i}$
(3) Thus, $\angle\mathbf{V_o} > \angle\mathbf{V_i}$, leading output





$$\tilde{V}_o = \tilde{V}_i \frac{1/(j\omega C)}{R + 1/(j\omega C)} = \tilde{V}_i \frac{1}{1 + j\omega RC}$$

$$= \tilde{V}_i \frac{1}{\sqrt{1 + (\omega RC)^2} \angle \tan^{-1}(\omega RC)}$$

$\tilde{V}_o$ lags $\tilde{V}_i$ by $\theta = \tan^{-1}(\omega RC)$, $0° < \theta < 90°$

$\angle \mathbf{V_i} - \theta = \angle \mathbf{V_o}$
$\angle \mathbf{V_o} < \angle \mathbf{V_i}$
$0° < \theta < 90°$

Issue of 90° shift:

$tan\theta$ becomes ∞ when θ approaches 90°

Therefore, ${}^{1}/_{\omega RC} = \infty$, which means $\omega RC \to 0$

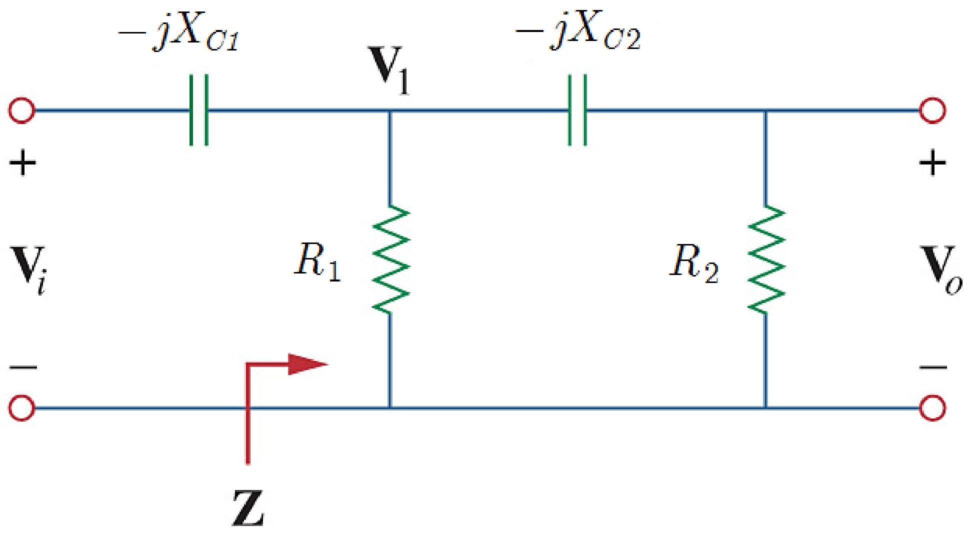
+1/ ωRC = tan(+90) = ∞ → 1/ ωRC = ∞, i.e. **$\boldsymbol{\omega}$RC = 0**

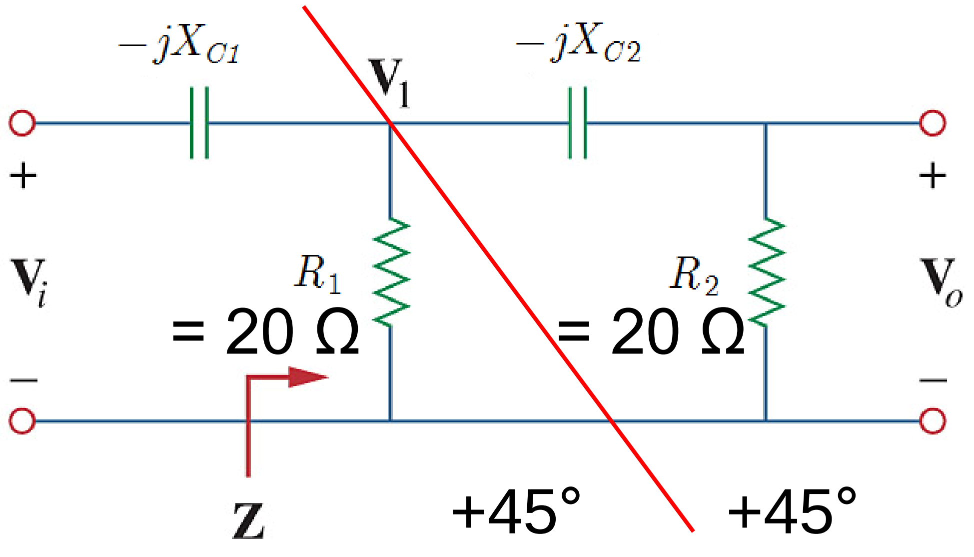
$$\tilde{V}_o = \tilde{V}_i \frac{R}{R + 1/(j\omega C)} = \tilde{V}_i \frac{R}{R - j(1/\omega C)}$$

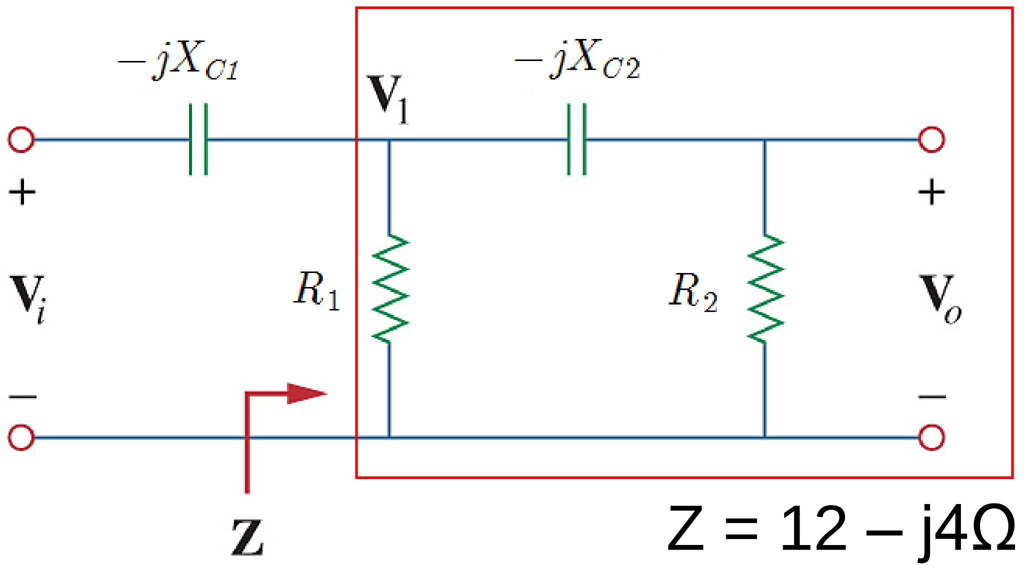
$$= \frac{1}{1 - j\frac{1}{\omega RC}} \tilde{V}_i = \frac{1}{\sqrt{1^2 + (\frac{1}{\omega RC})^2} \angle \tan^{-1} \frac{-1}{\omega RC}} \tilde{V}_i$$

No output voltage!

Practice Problem 9.13 Design an *RC* circuit to provide a phase shift of 90° leading.





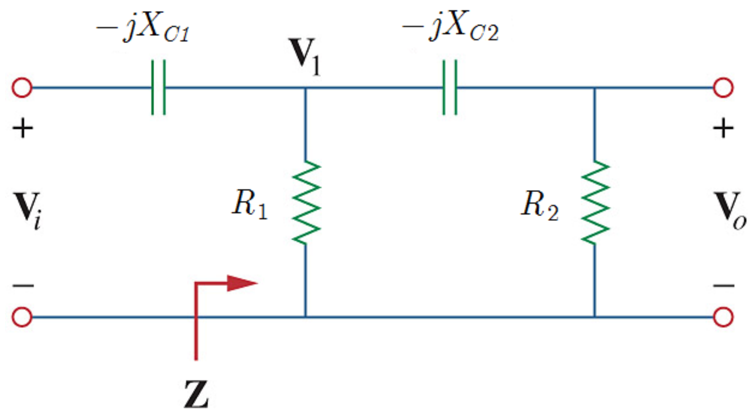


We set R_1 and R_2 as 20 Ω

Solution :

We need two stages, with each stage providing a phase shift of 45°.

Select $R_1 = R_2 = 20\ \Omega$,



$$\tilde{V}_o = \tilde{V}_1 \frac{20}{20 - jX_{C2}} = \tilde{V}_1 \frac{20(20 + jX_{C2})}{20^2 + X_{C2}^2}$$

If $X_{C2} = 20\ \Omega$, then the second stage produces a 45° phase shift.

$$Z = 20 \parallel (20 - j20) = \frac{20 \times (20 - j20)}{20 + (20 - j20)}$$

$$= 12 - j4\ (\Omega)$$

$$\tilde{V}_1 = \tilde{V}_i \frac{Z}{-jX_{C1} + Z} = \tilde{V}_i \frac{12 - j4}{12 - j(4 + X_{C1})}$$

$$= \tilde{V}_i \frac{(12 - j4)(12 + j(4 + X_{C1}))}{12^2 + (4 + X_{C1})^2}$$

$$= \tilde{V}_i \frac{(160 + 4X_{C1}) + j(12X_{C1})}{12^2 + (4 + X_{C1})^2}$$

For the first stage to produce another 45°,

we require $160 + 4X_{C1} = 12X_{C1}$, i.e.,

$X_{C1} = 20\ \Omega$.