ECE2150J Introduction to Circuits
Chapter 11. AC Power Analysis

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11.1 Introduction

- Our effort in ac circuit analysis so far has been focused on calculating voltage and current. Our major concern in this chapter is **power analysis**.

- Two kinds of power: (i) instantaneous power, and (ii) average power.

11.2 Instantaneous and Average Power

- Instantaneous power is the power at any instant of time:

$$p(t) = v(t)i(t)$$

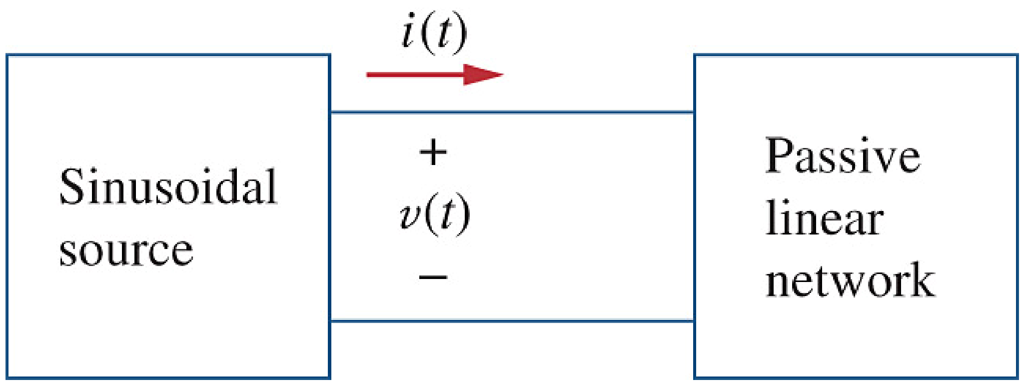
instantaneous *v* × instantaneous *i* (in watts)

- Following passive sign convention

Instantaneous power

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \theta_i)$$



The instantaneous power absorbed by a circuit under sinusoidal excitation.

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i)$$

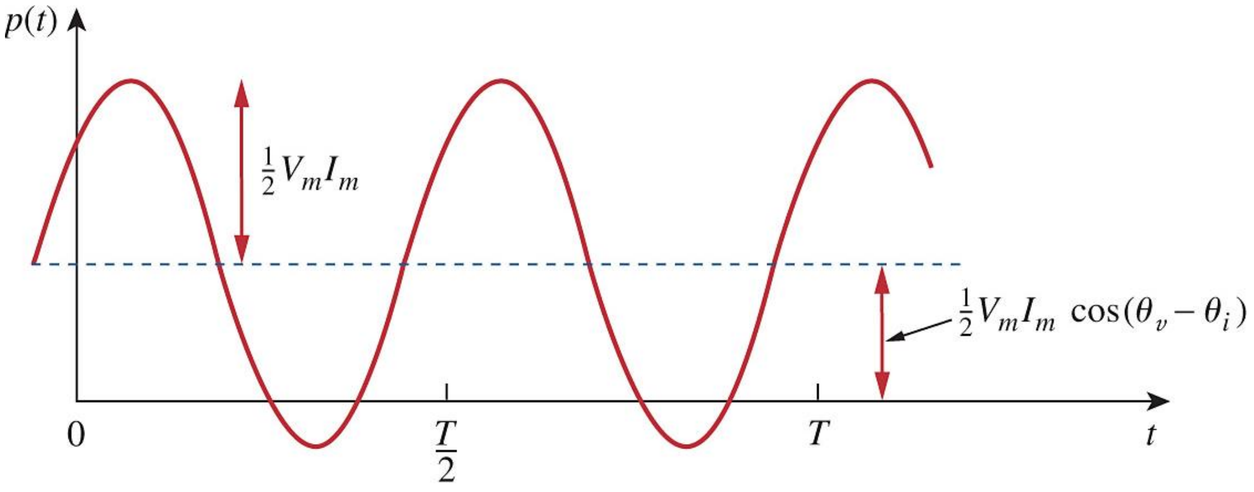
$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$p(t) = \underbrace{\boxed{\frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}}_{\text{DC shift}} + \boxed{\frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)}$$

(i) constant or time independent: The value depends on the **phase difference** between *v* and *i*.

(ii) A sinusoidal function with **2ω frequency** which is twice the angular frequency of the *v* and *i*.

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$



$p(t) > 0$, power is absorbed by the circuit.
p(t) < 0, power is absorbed by the source.

The instantaneous power **changes with time** and is therefore **difficult to measure**.

Average power

The average power [W] is the **average of the instantaneous power** over one period.

$$P = \frac{1}{T}\int_0^T p(t)dt$$

$$= \boxed{\frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)dt}$$

Constant

$$+ \boxed{\frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)dt}$$

Average of a sinusoid over its period is **zero**

$$= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

P does not depend on time

Average power in phasor forms

$$v(t) = V_m \cos(\omega t + \theta_v) \quad \mathbf{V} = V_m \underline{/\theta_v}$$

$$i(t) = I_m \cos(\omega t + \theta_i) \quad \mathbf{I} = I_m \underline{/\theta_i}$$

The average power P can be expressed in terms of phasor **V** and **I**.

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \qquad \frac{1}{2}\mathbf{VI}^* = \frac{1}{2} V_m I_m \underline{/\theta_v - \theta_i}$$

$$= \frac{1}{2}\mathrm{Re}\left(V_m I_m e^{j(\theta_v - \theta_i)}\right) \qquad = \boxed{\frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i)} + j \sin(\theta_v - \theta_i)]$$

Real part of the phasor

$$= \frac{1}{2}\mathrm{Re}\left(V_m e^{j\theta_v} I_m e^{-j\theta_i}\right)$$

$$= \frac{1}{2}\mathrm{Re}\left(\tilde{V}\tilde{I}^*\right)$$

$$P = \frac{1}{2}\mathrm{Re}[\mathbf{VI}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

(i) **When $\boldsymbol{\theta_v = \theta_i}$**, the voltage and current are in phase. **Purely resistive load R**.

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}|\mathbf{I}|^2 R = \frac{1}{2}\frac{V_m^2}{R}$$

(ii) **When $\boldsymbol{\theta_v - \theta_i = \pm 90}$**, we have a **purely reactive load**.

$$P = \frac{1}{2}V_m I_m \cos\left(\pm 90^\circ\right) = 0$$

A resistive load (*R*) absorbs power at all times, while a **reactive load (*L* or *C*)** absorbs zero average power.

$$P = \frac{1}{2}\mathrm{Re}[\mathbf{VI}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}I^2 R$$

Generally, the average power absorbed by an impedance **Z = R + jX** is given by

$$P = \frac{1}{2}\mathrm{Re}\left(\tilde{V}\tilde{I}^*\right) = \frac{1}{2}\mathrm{Re}\left(\tilde{I}(R + jX)\tilde{I}^*\right)$$

$$= \frac{1}{2}\mathrm{Re}\left(I^2 R + jI^2 X\right)$$

$$= \frac{1}{2}I^2 R$$

Only R contributes to the average power.

*where $I^2 = \mathbf{I} \times \mathbf{I}^*$, or $I^2 = |\mathbf{I}|^2$,
the square of amplitude of phasor **I**

Practice Problem 11.3 Calculate the average power absorbed by the resistor and inductor in the circuit of Fig. 11.4. Find the average power supplied by the voltage source.

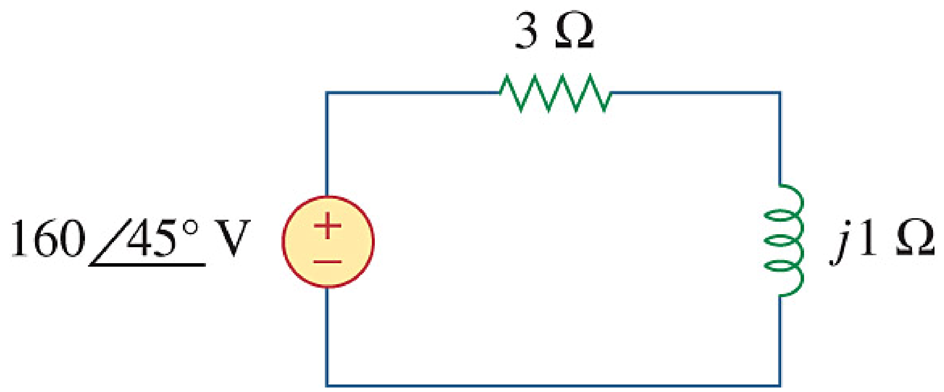
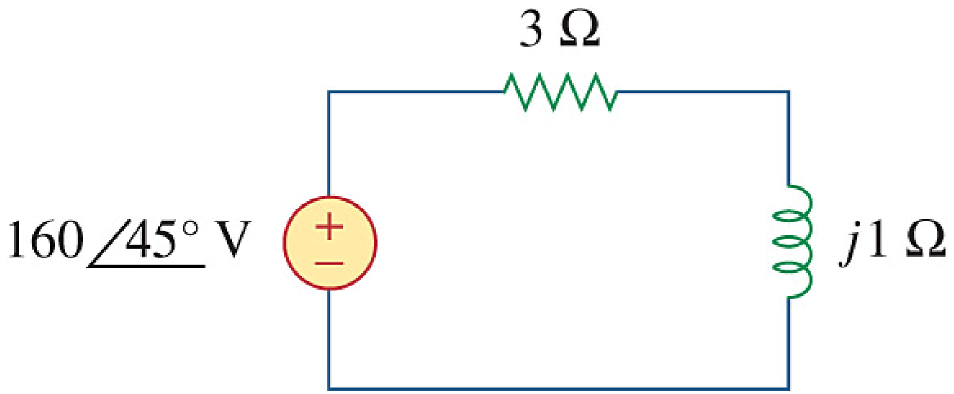


Figure 11.4



$P=1/2\times V_m I_m \cos(\theta_v-\theta_i)$
$=1/2\times \mathrm{Re}(\mathbf{VI^*})=1/2\times I^2R$

$$\tilde{V}=160\angle 45^\circ \text{ (V)}$$

$$\tilde{I}=\frac{\tilde{V}}{Z}=\frac{160\angle 45^\circ}{3+j1}\approx\frac{160\angle 45^\circ}{3.1623\angle 18.43^\circ}$$

$$\approx 50.5961\angle 26.57^\circ \text{ (A)}$$

The average power absorbed by the resistor is

$$P_R=I^2R/2=50.5961^2\times 3/2$$

$$\approx 3839.95 \text{ (W)}\approx 3.84 \text{ kW}$$

The average power absorbed by the inductor is zero, i.e., $P_L = 0$.

By the law of power conservation, the average power supplied by the voltage source should be the same as the average power absorbed.

$$P=1/2\times V_m I_m \cos(\theta_v-\theta_i)=1/2\times \mathrm{Re}(\mathbf{VI^*})=1/2\times I^2 R$$

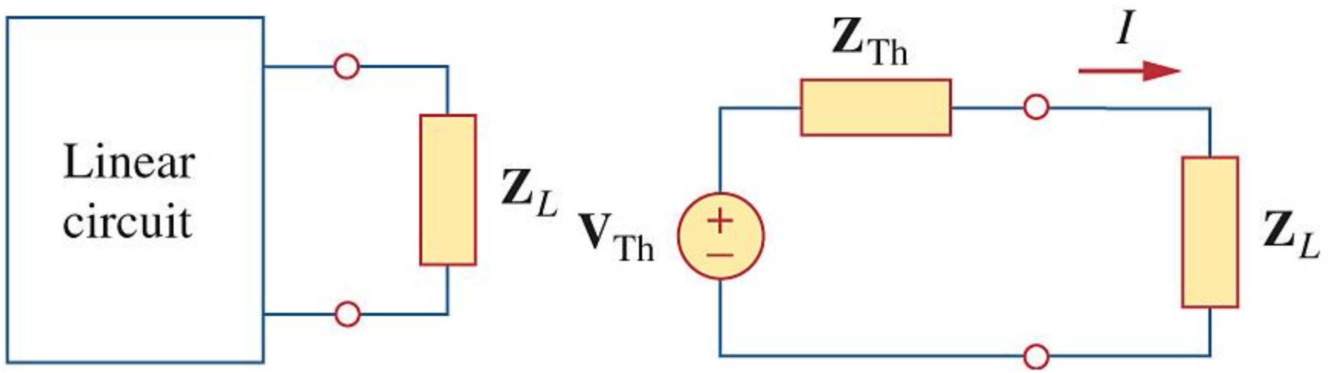
source resistive load

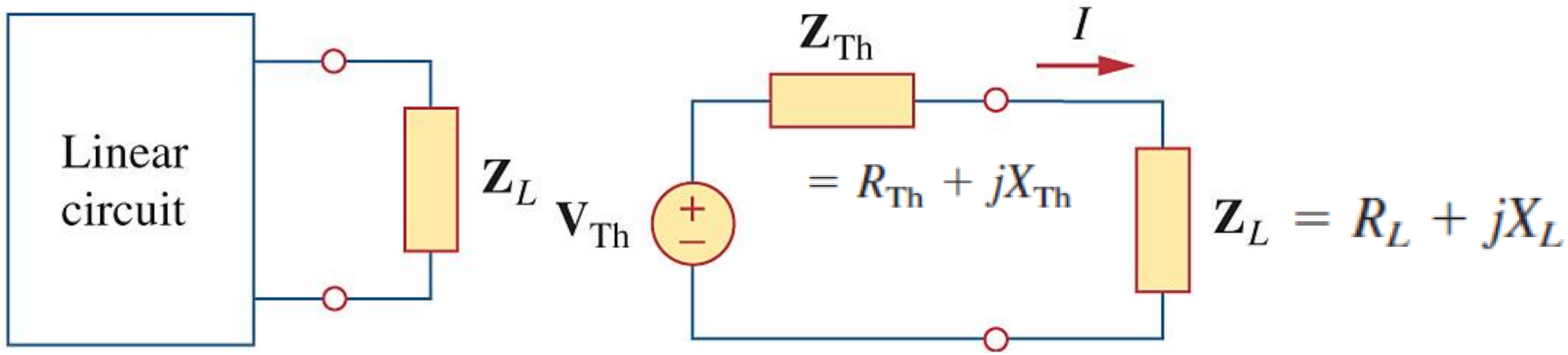
$P = 1/2\times\mathrm{Re}(VI^*)$

$= 1/2\times\mathrm{Re}(160\angle 45^\circ\times 50.5961\angle\text{-}26.57^\circ) =$ **3.84kW**

11.3 Maximum Average Power Transfer

In Chapter 4, we have learned that maximum power would be delivered to the load if the load resistance is equal to the Thevenin resistance $\mathbf{R_L = R_{TH}}$. We now extend that result to ac circuits.





$$\tilde{I} = \frac{\tilde{V}_{Th}}{Z_{Th} + Z_L} = \frac{\tilde{V}_{Th}}{(R_{Th} + jX_{Th}) + (R_L + jX_L)}$$

The average power delivered to the load is

$$P = \frac{1}{2} I^2 R_L = \frac{1}{2} \frac{V_{Th}^2 R_L}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

Our objective is to **adjust the load parameter R_L and X_L** so that **P is maximum**.

$$P = \frac{1}{2} I^2 R_L = \frac{1}{2} \frac{V_{Th}^2 R_L}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

$$\begin{cases} \dfrac{\partial P}{\partial R_L} = \dfrac{V_{Th}^2 [R_{Th}^2 - R_L^2 + (X_{Th} + X_L)^2]}{2[(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2]^2} = 0 \\ \dfrac{\partial P}{\partial X_L} = -\dfrac{V_{Th}^2 R_L (X_{Th} + X_L)}{[(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2]^2} = 0 \end{cases}$$

$$\begin{cases} R_L = R_{Th} \\ X_L = -X_{Th} \end{cases} \text{ or } Z_L = Z_{Th}^*$$

Therefore, to deliver the max. power to Z_L

$$\mathbf{Z}_L = R_L + jX_L = R_{\text{Th}} - jX_{\text{Th}} = \mathbf{Z}_{\text{Th}}^*$$

For **Max. average power** transfer, the load impedance **Z_L must be equal** to the complex conjugate of **the Thevenin impedance Z_{Th}**.

$$P = \frac{1}{2} I^2 R_L = \frac{1}{2} \frac{V_{Th}^2 R_L}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2} \Leftarrow \begin{cases} R_L = R_{Th} \\ X_L = -X_{Th} \end{cases}$$

$$P_{\max} = \frac{V_{Th}^2}{8 R_{Th}}$$

When the load is purely resistive

$$\frac{\partial P}{\partial R_L} = \frac{V_{Th}^2[R_{Th}^2 - R_L^2 + (X_{Th} + X_L)^2]}{2[(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2]^2} = 0$$

where X_L=0

Maximum value at $\partial P/\partial R_L = 0$

$$R_L = \sqrt{R_{Th}^2 + X_{Th}^2}$$

→ The load impedance (or resistance) is equal to the **magnitude of the Thevenin impedance**

$$P = \frac{1}{2} I^2 R_L = \frac{1}{2} \frac{V_{Th}^2 R_L}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

where X_L=0 and $R_L = \sqrt{R_{Th}^2 + X_{Th}^2}$

Practice Problem 11.5 For the circuit shown in Fig. 11.10, find the load impedance Z_L that absorbs the maximum average power. Calculate that maximum average power.

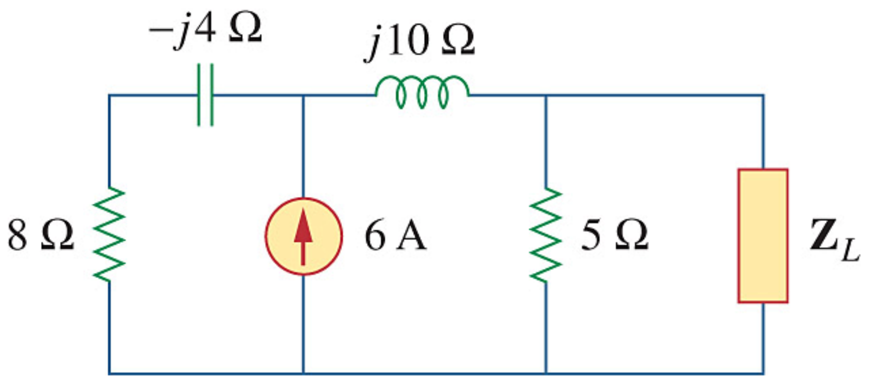
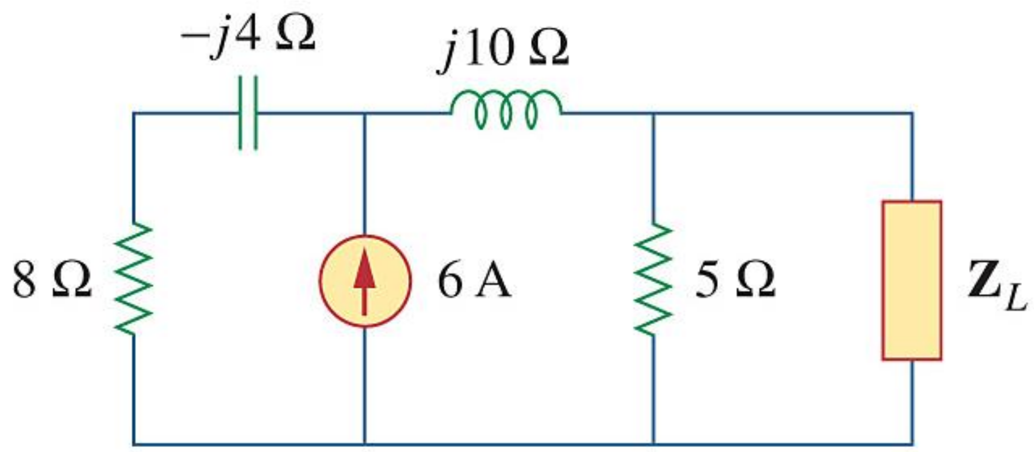


Figure 11.10



Solution : (i) Z_{TH}

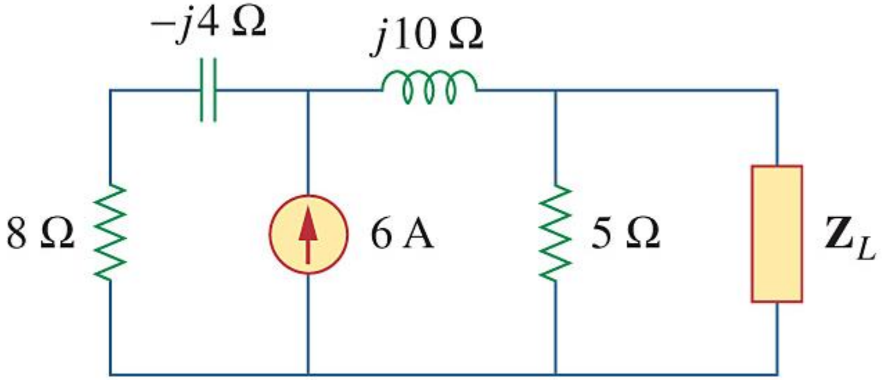
$$Z_{Th} = 5 \parallel (j10 - j4 + 8) = 5 \parallel (8 + j6)$$

$$= \frac{5 \times (8 + j6)}{5 + (8 + j6)} = \frac{140 + j30}{41}$$

$$\approx 3.4146 + j0.7317\ (\Omega)$$

$$Z_L = Z_{Th}^{*} = 3.4146 - j0.7317\ \Omega$$

(ii) V_{TH}



$$\tilde{V}_{Th} = 6\times[(5+j10)\parallel(8-j4)]\times\frac{5}{5+j10}$$

$$= 6\times\frac{80+j60}{13+j6}\times\frac{5}{5+j10}$$

$$\approx 6\times\frac{100\angle 36.87^\circ}{14.3178\angle 24.78^\circ}\times\frac{5}{11.1803\angle 63.43^\circ}$$

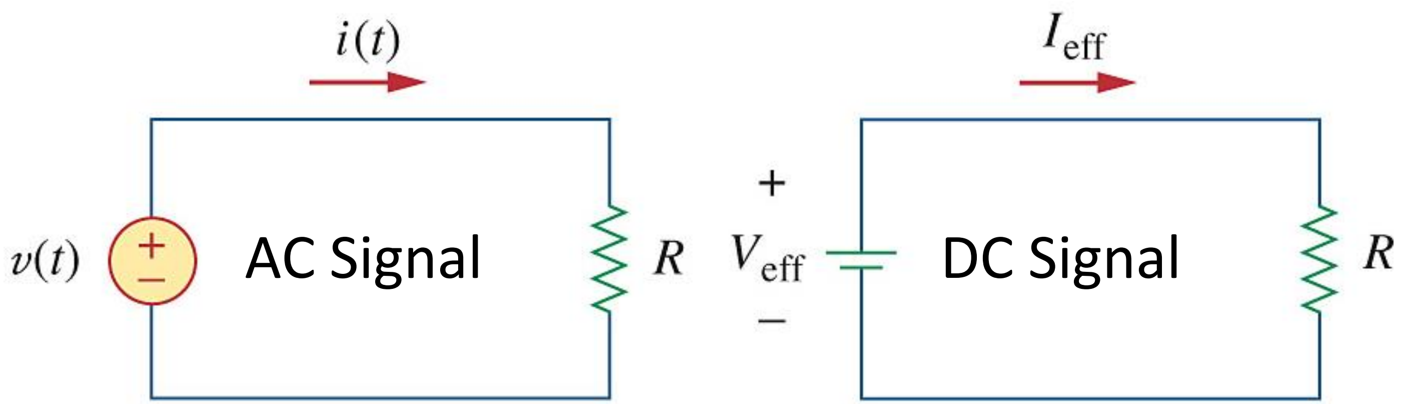
$$\approx 18.7409\angle -51.34^\circ \text{ (V)}$$

$$P_{\max} = \frac{V_{Th}^2}{8R_{Th}} = \frac{18.7409^2}{8\times 3.4146} \approx 12.86 \text{ (W)}$$

11.4 Effective or RMS Value

The idea of **effective value** arises from the need to measure **the effectiveness of a voltage or current source** in delivering power to a resistive load.

The effective value of a periodic current is the **dc current that delivers the same average power** to a resistor as the periodic current.



Find I_{eff} that will transfer the same power to resistor R as the sinusoid i.

Average power in the ac circuit = dc circuit

$$P = \frac{1}{T}\int_0^T i^2 R\, dt = \frac{R}{T}\int_0^T i^2\, dt \quad = \quad P = I_{\text{eff}}^2 R$$

The effective value of the current *i* is

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2 dt}$$

The effective value of the voltage *v* is found in the same way as I_{eff}

$$V_{eff} = \sqrt{\frac{1}{T}\int_0^T v^2 dt}$$

The effective value ***is the square root of the mean of the square*** of the periodic signal. Thus, the effective value is often known as the ***root-mean-square*** value, or *rms* value for short;

$$I_{eff} = I_{rms},\ V_{eff} = V_{rms}$$

(i) For **any periodic function *x*(*t*)** in general, the rms value is given by

$$X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}$$

The effective value of a periodic signal is its root mean square (rms) value.

(ii) The rms value of a constant is the constant itself

(iii) Sinusoid $i(t) = I_m \cos \omega t$, $v(t) = V_m \cos t\, \omega t$

$$I_{rms} = \frac{I_m}{\sqrt{2}}, V_{rms} = \frac{V_m}{\sqrt{2}}$$ **only valid for sinusoidal signals**

A special case: i is a sinusoid

For the sinusoid $i(t) = I_m \cos \omega t$,

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2 \omega t dt} = \sqrt{\frac{I_m^2}{T}\int_0^T \frac{1+\cos 2\omega t}{2} dt} = \frac{I_m}{\sqrt{2}}$$

Similarly, for $v(t) = V_m \cos \omega t$,

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

Average power

$$P=1/2\times V_mI_m\cos(\theta_v-\theta_i)=1/2\times \mathrm{Re}(\mathbf{VI^*})=1/2\times I^2R$$

$=I_{rms}^2R$ (← generally true)

$=V_{rms}I_{rms}\cos(\theta_v-\theta_i)$ (← only for sinusoids)

$$P=\frac{1}{2}V_mI_m\cos(\theta_v-\theta_i)$$

$$=\frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v-\theta_i)=\boxed{V_{rms}I_{rms}}\cos(\theta_v-\theta_i)$$

The average power absorbed by a resistor

$$P=\frac{1}{2}V_mI_m=\frac{1}{2}I_m^2R=\frac{1}{2}\frac{V_m^2}{R}=\boxed{I_{\mathrm{rms}}^2R=\frac{V_{\mathrm{rms}}^2}{R}}$$ **dc formula**

Define

$$\tilde{V}_{rms} = V_{rms}\angle\theta_v \text{ and } \tilde{I}_{rms} = I_{rms}\angle\theta_i$$

$$\tilde{V}_{rms} = V_{rms}\angle\theta_v = \frac{V_m}{\sqrt{2}}\angle\theta_v = \frac{\tilde{V}}{\sqrt{2}}$$

$$\tilde{I}_{rms} = I_{rms}\angle\theta_i = \frac{I_m}{\sqrt{2}}\angle\theta_i = \frac{\tilde{I}}{\sqrt{2}}$$

Time domain

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

For either amplitude or phasor, there is a $\sqrt{2}$ difference

The load impedance

$$Z = \frac{\tilde{V}}{\tilde{I}} = \frac{\tilde{V}_{rms}}{\tilde{I}_{rms}}$$

since

$$\frac{\tilde{V}}{\tilde{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle (\theta_v - \theta_i) = \frac{V_{rms}}{I_{rms}} \angle (\theta_v - \theta_i)$$

$$= \frac{V_{rms} \angle \theta_v}{I_{rms} \angle \theta_i} = \frac{\tilde{V}_{rms}}{\tilde{I}_{rms}}$$

*The power industries specify phasor magnitudes in terms of their **rms values**. e.g.) 110 V available at every household is the rms value of the voltage from the power company. It is convenient in power analysis to express voltage and current in their rms values.

Practice Problem 11.8 Find the rms value of the full-wave rectified sine wave in Fig. 11.17. Calculate the average power dissipated in a 6-Ω resistor.

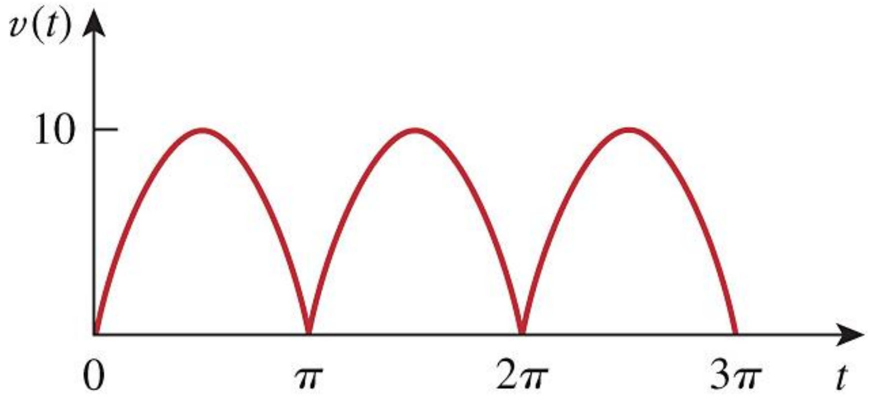


Figure 11.17

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}$$

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}\frac{V_m^2}{R} = I_{rms}^2 R = \frac{V_{rms}^2}{R}$$

Solution : $$X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}$$

$$V_{rms} = \sqrt{\frac{1}{\pi}\int_0^{\pi} (10\sin t)^2\,dt}$$

$$= 10\sqrt{\frac{1}{\pi}\int_0^{\pi} \frac{1-\cos 2t}{2}\,dt} = 5\sqrt{2} \approx 7.07\ (\text{V})$$

$$\boxed{P = \frac{V_{rms}^2}{R}} = \frac{\left(5\sqrt{2}\right)^2}{6} \approx 8.33\ (\text{W})$$

dc formula

11.5 Apparent Power and Power Factor

Power when voltage and current are $v(t) = V_m \cos(\omega t + \theta_v)$ $\quad \mathbf{V} = V_m \underline{/\theta_v}$

$i(t) = I_m \cos(\omega t + \theta_i)$ $\quad \mathbf{I} = I_m \underline{/\theta_i}$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) = S\cos(\theta_v - \theta_i)$$

We will define the **apparent power S** as the product of V_{rms} and I_{rms}, and the **power factor** as the $\cos(\theta_v - \theta_i)$. The angle $\theta_v - \theta_i$ is called the **power factor angle** (θ).

Apparent power S

$$
\begin{aligned}
P &= 1/2\times V_m I_m \cos(\theta_v-\theta_i) = 1/2\times \mathrm{Re}(\mathbf{VI^*}) = 1/2\times I^2R \\
&= I_{rms}{}^2 R \\
&= V_{rms} I_{rms} \cos(\theta_v-\theta_i) \\
&= |S|\times pf,\ 0\le pf\le 1 \text{ where } |S| = V_{rms} I_{rms}
\end{aligned}
$$

The apparent power is so called because it seems apparent that the **power should be the voltage-current product**, by analogy with dc resistive circuits. It is measured **in volt-amperes or VA** to distinguish it from the average or real power, which is measured in watts.

?

Power factor

The power factor is dimensionless, since it is the **ratio of the average power to the apparent power**.

$$\text{pf} = \cos\theta = \frac{P}{|S|} \qquad Z = \frac{\tilde{V}}{\tilde{I}} = \frac{\tilde{V}_{rms}}{\tilde{I}_{rms}} = \frac{V_{rms}}{I_{rms}} \angle(\theta_v - \theta_i)$$

pf.

The power factor is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance.

The power factor may be seen as that factor by which **the apparent power must be multiplied to obtain the real or average power**.

Power factor angle

The power factor angle is equal to the angle of the load impedance.

$$Z = \frac{\tilde{V}}{\tilde{I}} = \frac{\tilde{V}_{rms}}{\tilde{I}_{rms}} = \frac{V_{rms}}{I_{rms}} \angle \boxed{(\theta_v - \theta_i)}$$

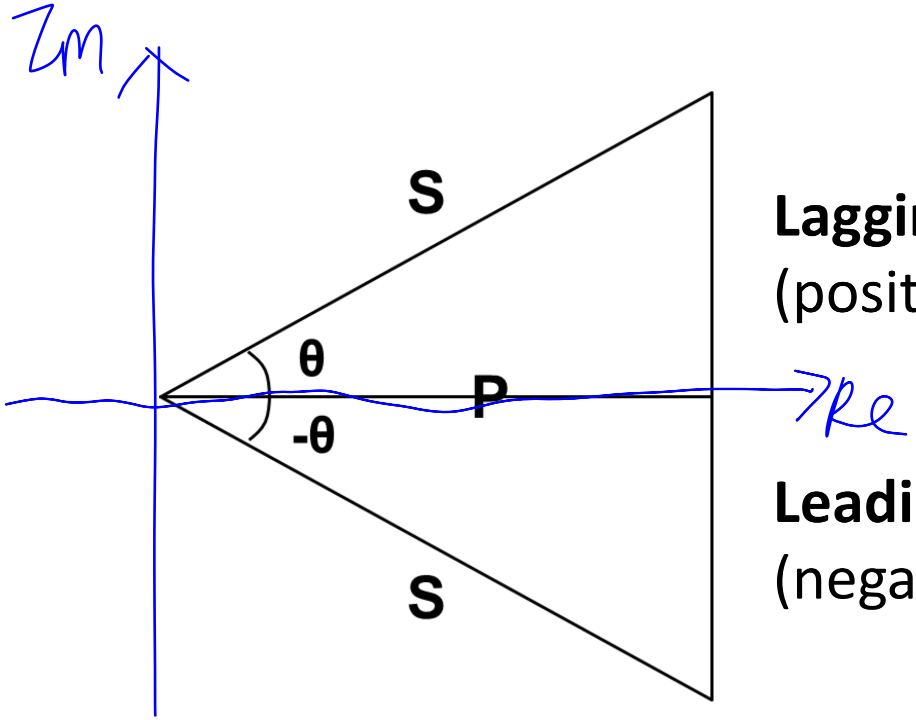
The value of pf ranges between zero and unity. For a purely resistive load, $\theta_v - \theta_i = 0$, pf = 1, implying that P = |S|. For a purely reactive load, $\theta_v - \theta_i = 90$, pf = 0, implying that P = 0. In between these two extreme cases, pf is said to be leading ($\theta_v - \theta_i < 0$) or lagging ($\theta_v - \theta_i > 0$).

current as reference

i leads V, pf leading

In between two extreme cases, pf = 0 or 1,

pf is said to be **leading** $(i)\ \theta_v - \theta_i < 0$, or **lagging** $(ii)\ \theta_v - \theta_i > 0$



Lagging pf $\theta_v - \theta_i > 0$
(positive pf angle)

Leading pf $\theta_v - \theta_i < 0$
(negative pf angle)

Lagging pf: I lags V, implying an inductive load
Leading pf: I leads V, implying a capacitive load

Practice Problem 11.9 Obtain the power factor and the apparent power of a load whose impedance is $Z = 60 + j40\ \Omega$ when the applied voltage is $v(t) = 160\cos(377t + 10°)$ V.

Negative pf angle ($\theta<0$) ⇔ $\theta_i>\theta_v$ ⇔ leading
Positive pf angle ($\theta>0$) ⇔ $\theta_i<\theta_v$ ⇔ lagging

Solution :

$$Z = 60 + j40 \approx 72.1110\angle 33.69^\circ\ (\Omega)$$

$$\theta = 33.69^\circ$$

$$\text{pf} = \cos\theta = \cos 33.69^\circ \approx 0.8321 \text{ lagging}$$

$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{160\angle 10^\circ}{72.1110\angle 33.69^\circ}$$

$$\approx 2.2188\angle -23.69^\circ\ (\text{A})$$

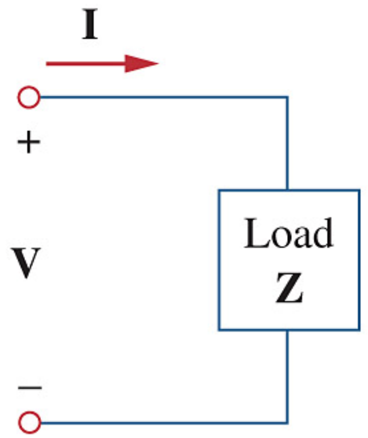
$$|S| = \frac{1}{2}V_m I_m = \frac{1}{2}\times 160 \times 2.2188$$

$|S|=V_{rms}I_{rms}$

$$\approx 177.50\ (\text{VA})$$

11.6 Complex Power

Complex power is important in power analysis **because it contains all the information** pertaining to the power absorbed by a given load.



The complex power absorbed by the load is $S = \dfrac{1}{2}\tilde{V}\tilde{I}^{*} = \tilde{V}_{rms}\,\tilde{I}^{*}_{rms}$

Complex power and real power

In 11.6 **Complex power**

$$S = \frac{1}{2}\tilde{V}\tilde{I}^* = \tilde{V}_{rms}\tilde{I}^*_{rms} = V_{rms}I_{rms}\angle(\theta_v - \theta_i) = \left|S\right|\angle(\theta_v - \theta_i)$$

In 11.5 $P = V_{rms}I_{rms}\cos(\theta_v - \theta_i) = |S|\cos(\theta_v - \theta_i)$

Complex power and real power

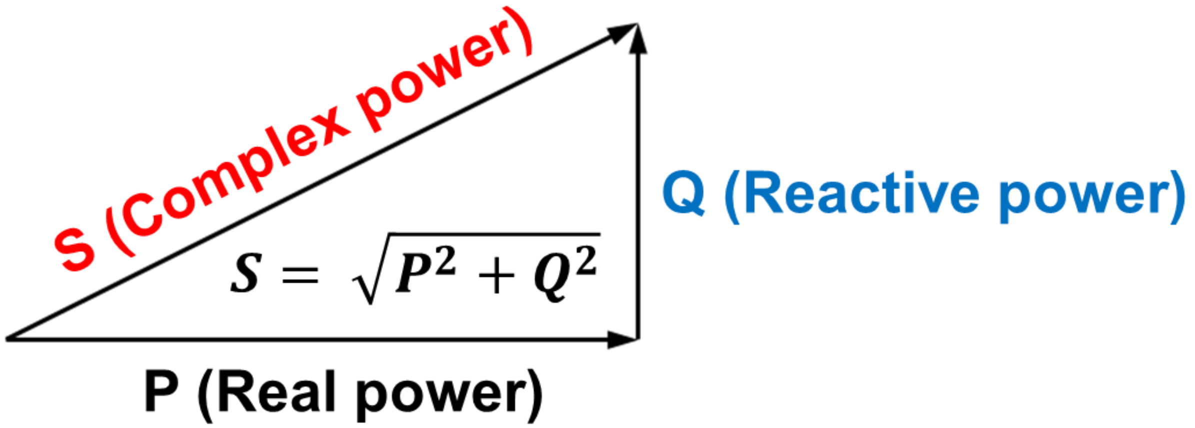
In 11.6 **Complex power**

$$S = V_{rms} I_{rms} \angle(\theta_v - \theta_i) = |S| \angle(\theta_v - \theta_i)$$

In 11.5 $P = V_{rms} I_{rms} \cos(\theta_v - \theta_i) = |S| \cos(\theta_v - \theta_i)$

$$\begin{aligned} S &= V_{rms} I_{rms} \angle(\theta_v - \theta_i) \\ &= \boxed{V_{rms} I_{rms} \cos(\theta_v - \theta_i)} + jV_{rms} I_{rms} \sin(\theta_v - \theta_i) \\ &= \mathrm{Re}(S) + j\,\mathrm{Im}(S) \\ &= P + jQ \end{aligned}$$

The average power P



Complex power

As a complex quantity, its real part is **real power *P*** and its imaginary part is **reactive power *Q***.

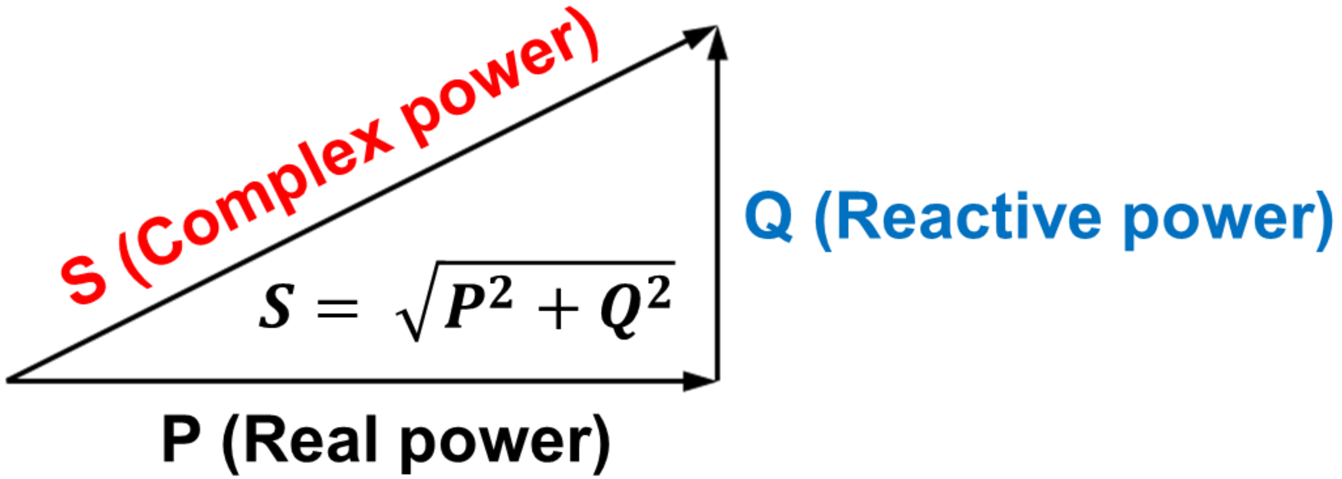
(1) P is the average or real power in watts.

$$P = \mathrm{Re}(S) = I_{rms}^2 R = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$
$$= |S| \cos(\theta_v - \theta_i)$$

(2) Q is the reactive or quadrature power in volt-amperes reactive (VAR).

$$Q = \mathrm{Im}(S) = I_{rms}^2 X = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$
$$= |S| \sin(\theta_v - \theta_i)$$

Unit of complex power



S [VA] = P [Watt] + jQ [VAR]

Reactive power

The reactive power *Q* is a measure of the energy exchange between the source and the reactive part of the load.

The reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source.

$$1.\ Q = 0 \text{ for resistive loads (unity pf)}$$

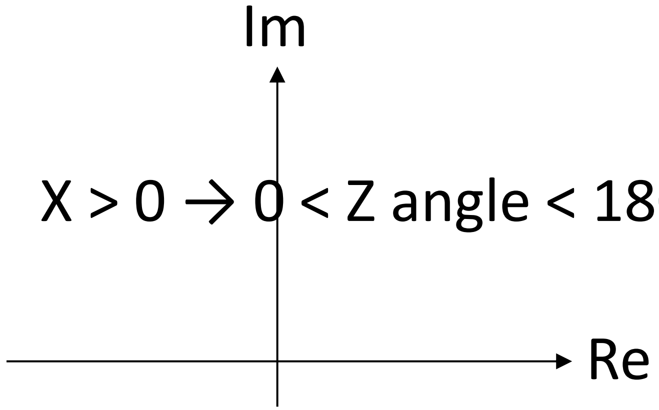
$$2.\ Q < 0 \text{ for capacitive loads (leading pf)}$$

$$3.\ Q > 0 \text{ for inductive loads (lagging pf)}$$

Recall Chapter 9

$$Z = R + jX \qquad Z = \frac{\tilde{V}}{\tilde{I}} = \frac{\tilde{V}_{rms}}{\tilde{I}_{rms}} = \frac{V_{rms}}{I_{rms}} \angle(\theta_v - \theta_i)$$

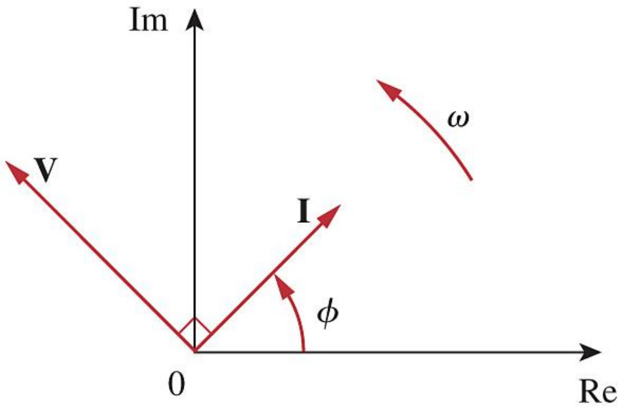
i) X > 0, the impedance is **inductive or lagging** since current lags voltage.



X > 0 → 0 < Z angle < 180

V= $\boldsymbol{IZ}$

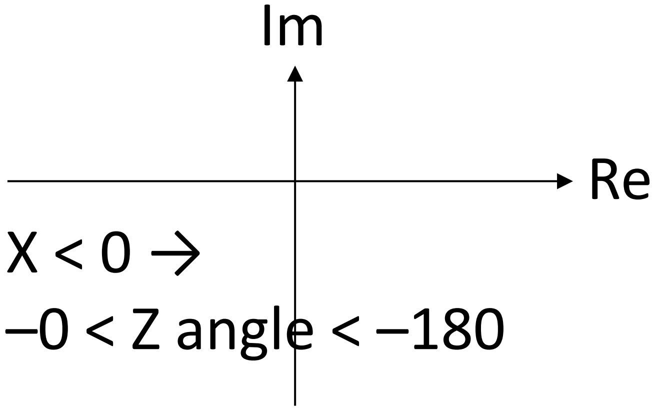
Angle (V) = Angle (I) + Angle (Z)



Z = R + jX where X > 0
→Impedance is inductive
or lagging (**I** lags **V**)

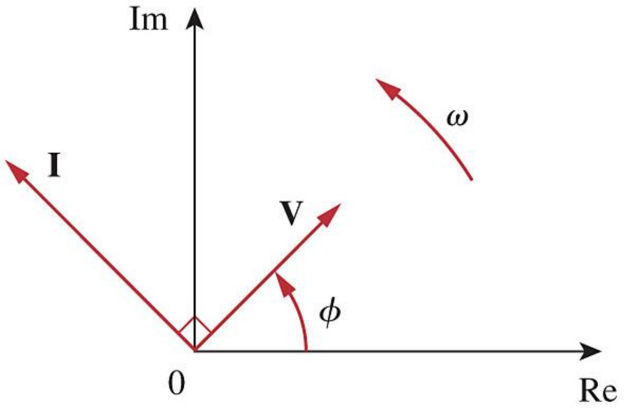
$$Z = R + jX \qquad Z = \frac{\tilde{V}}{\tilde{I}} = \frac{\tilde{V}_{rms}}{\tilde{I}_{rms}} = \frac{V_{rms}}{I_{rms}} \angle(\theta_v - \theta_i)$$

ii) X < 0, the impedance is **capacitive or leading** since current leads voltage.



$$\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{Z}}$$

Angle (I) = Angle (V) – Angle (Z)
where Angle (Z) is negative

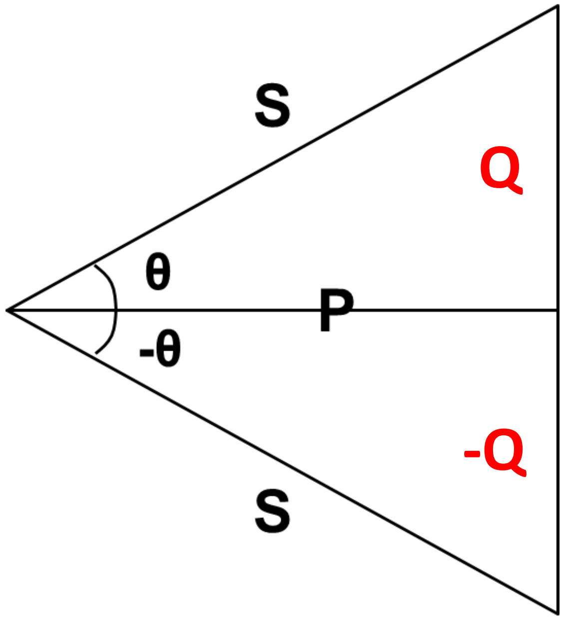


Z = R + jX where X < 0
→Impedance is capacitive
or leading (**I** leads **V**)

1. $Q = 0$ for resistive loads (unity pf)

2. $Q < 0$ for capacitive loads (leading pf)

3. $Q > 0$ for inductive loads (lagging pf)



Lagging pf $\theta_v - \theta_i > 0$
(positive pf angle)

Leading pf $\theta_v - \theta_i < 0$
(negative pf angle)

Leading: $\theta_i > \theta_v$ Negative pf angle ($\theta<0$), $X < 0 \rightarrow Q < 0$

Lagging: $\theta_v > \theta_i$ Positive pf angle ($\theta>0$), $X > 0 \rightarrow Q > 0$

$$S = \tilde{V}_{rms}\tilde{I}^{*}_{rms} = \tilde{V}_{rms}\left(\frac{\tilde{V}_{rms}}{Z}\right)^{*} = \frac{V^{2}_{rms}}{Z^{*}}$$

$$S = \tilde{V}_{rms}\tilde{I}^{*}_{rms} = \left(\tilde{I}_{rms}Z\right)\tilde{I}^{*}_{rms} = I^{2}_{rms}Z$$

$$S = I^{2}_{rms}Z = I^{2}_{rms}(R + jX) = I^{2}_{rms}R + jI^{2}_{rms}X$$

$$P = I^{2}_{rms}R$$

$$Q = I^{2}_{rms}X$$

$$S = \tilde{V}_{rms} \tilde{I}^*_{rms} = \tilde{V}_{rms} \left(\frac{\tilde{V}_{rms}}{Z} \right)^* = \frac{V^2_{rms}}{Z^*}$$

$$S = \tilde{V}_{rms} \tilde{I}^*_{rms} = \left(\tilde{I}_{rms} Z \right) \tilde{I}^*_{rms} = I^2_{rms} Z$$

$$S = I^2_{rms} Z = I^2_{rms} (R + jX) = I^2_{rms} R + j I^2_{rms} X$$

$$P = I^2_{rms} R$$

$$Q = I^2_{rms} X$$

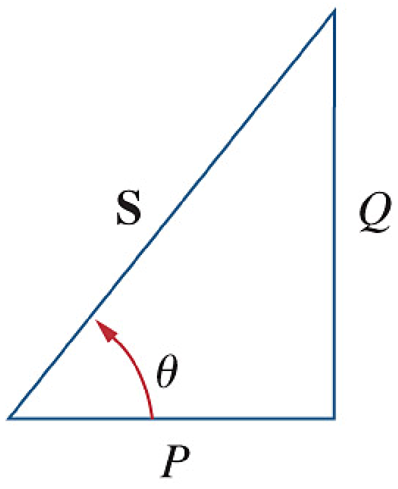
However, in general **Z = R + jX**

$\mathbf{P \neq V_{rms}^2/R}$

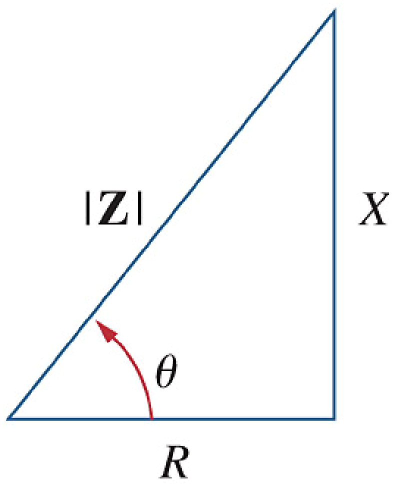
$\mathbf{Q \neq V_{rms}^2/X}$

"=" holds for purely resistive or reactive loads

Power Triangle



Power Triangle



Impedance Triangle

$|S| = I_{rms}^2 |Z|$
$P = I_{rms}^2 R$
$Q = I_{rms}^2 X$

It is a standard practice to represent S, P, and Q in the form of a triangle, known as the power triangle. This is similar to the impedance triangle showing the relationship between Z, R, and X.

Complex power summary

$$\text{Complex Power} = \mathbf{S} = P + jQ = \mathbf{V}_{\text{rms}}(\mathbf{I}_{\text{rms}})^*$$
$$= |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}| \underline{/\theta_v - \theta_i}$$
$$\text{Apparent Power} = S = |\mathbf{S}| = |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}| = \sqrt{P^2 + Q^2}$$
$$\text{Real Power} = P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i)$$
$$\text{Reactive Power} = Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i)$$
$$\text{Power Factor} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

In conclusion, the complex power S contains **all power information** of a load.

(i) The real part of S is the real power P.

(ii) Its imaginary part is the reactive power Q.

(iii) Its magnitude is the apparent power $|S|$

(iv) The cosine of its phase angle is the power factor pf.

Practice Problem 11.11

For a load, $\mathbf{V}_{\text{rms}} = 110\angle 85^\circ$ V, $\mathbf{I}_{\text{rms}} = 0.4\angle 15^\circ$ A. Determine: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

We are looking for
(a) **S**, |**S**|
(b) P, Q, pf, **Z**

Solution :

$$S = \tilde{V}_{rms}\,\tilde{I}^{*}_{rms} = 110\angle 85^\circ \times \left(0.4\angle 15^\circ\right)^{*}$$

$$= 44\angle 70^\circ \text{ (VA)}$$

$\angle \mathbf{S} = \angle \mathbf{Z} = \text{pf angle} = \theta_v - \theta_i$

$$|S| = 44 \text{ VA}$$

$$P = \mathrm{Re}(S) = 44\cos 70^\circ \approx 15.05 \text{ (W)}$$

$$Q = \mathrm{Im}(S) = 44\sin 70^\circ \approx 41.35 \text{ (VAR)}$$

$$\text{pf} = \cos 70^\circ \approx 0.3420 \text{ lagging}$$

$$Z = \frac{\tilde{V}_{rms}}{\tilde{I}_{rms}} = \frac{110\angle 85^\circ}{0.4\angle 15^\circ} = 275\angle 70^\circ$$

$$\approx 94.06 + j258.42 \ (\Omega)$$

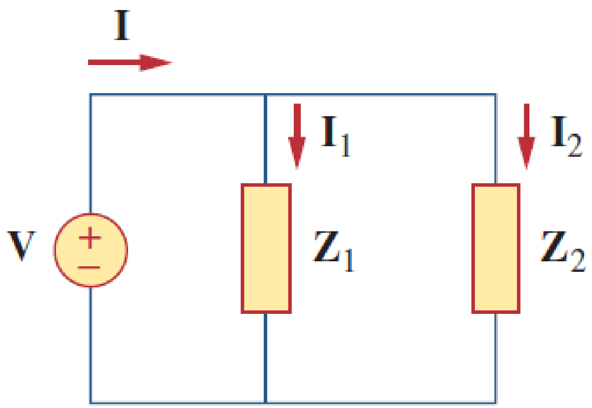
$Q<0 \Leftrightarrow X<0 \Leftrightarrow$ Negative pf angle ($\theta<0$) $\Leftrightarrow \theta_i>\theta_v \Leftrightarrow$ leading
$Q>0 \Leftrightarrow X>0 \Leftrightarrow$ Positive pf angle ($\theta>0$) $\Leftrightarrow \theta_i<\theta_v \Leftrightarrow$ lagging

11.7 Conservation of AC Power

The principle of conservation of AC power: The complex, real, and reactive powers of the source equal the respective sums of the complex, real, and reactive powers of the individual loads.

***From now on, unless otherwise specified, all values of voltages and currents will be assumed to be rms values.**

(i) Parallel



$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2$$

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = \mathbf{V}(\mathbf{I}_1^* + \mathbf{I}_2^*) = \mathbf{V}\mathbf{I}_1^* + \mathbf{V}\mathbf{I}_2^* = \mathbf{S}_1 + \mathbf{S}_2$$

$\mathbf{S} = P + jQ$; $\mathbf{S_1} = P_1 + jQ_1$; $\mathbf{S_2} = P_2 + jQ_2$

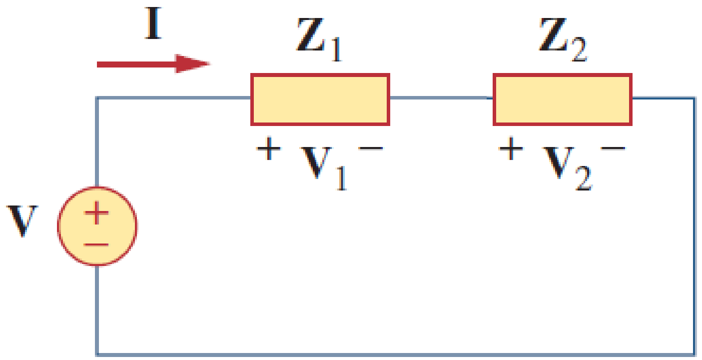
$\mathbf{S = S_1 + S_2}$

➔ $P + jQ = P_1 + jQ_1 + P_2 + jQ_2 = (P_1+P_2) + j(Q_1+Q_2)$

➔ $P = P_1 + P_2$; $Q = Q_1 + Q_2$

The complex, real, and reactive powers of the sources equal the respective sums of the complex, real, and reactive powers of the individual loads.

(ii) Series



$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$$

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = (\mathbf{V}_1 + \mathbf{V}_2)\mathbf{I}^* = \mathbf{V}_1\mathbf{I}^* + \mathbf{V}_2\mathbf{I}^* = \mathbf{S}_1 + \mathbf{S}_2$$

Similarly,

$\mathbf{S} = P + jQ$; $\mathbf{S_1} = P_1 + jQ_{1;}$ $\mathbf{S_2} = P_2 + jQ_2$

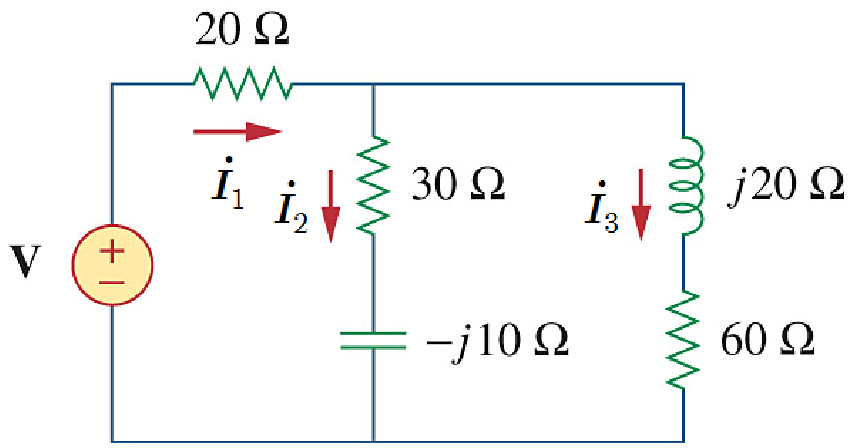
$\mathbf{S = S_1 + S_2}$

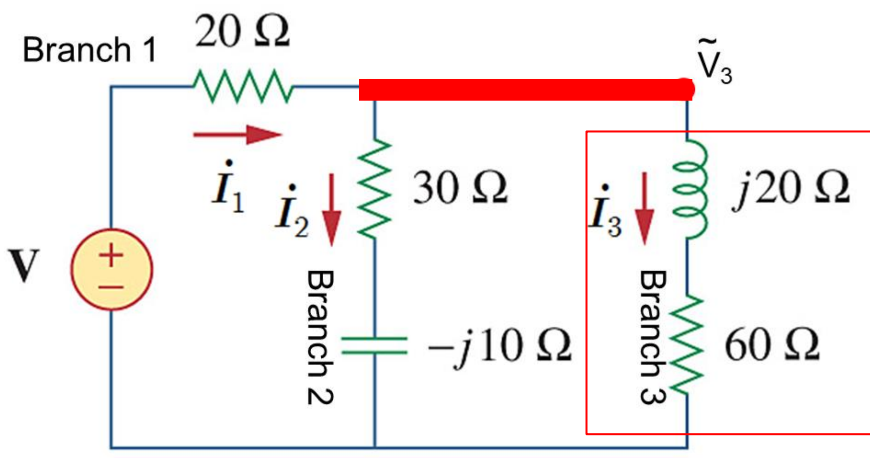
➔ $P + jQ = P_1 + jQ_1 + P_2 + jQ_2 = (P_1+P_2) + j(Q_1+Q_2)$

➔ $P = P_1 + P_2$; $Q = Q_1 + Q_2$

The complex, real, and reactive powers of the sources equal the respective sums of the complex, real, and reactive powers of the individual loads.

Practice Problem 11.13 In the circuit in Fig. 11.25, the 60-W resistor absorbs an average power of 240 W. Find $\tilde{V}$ and the complex power of each branch of the circuit. What is the overall complex power of the circuit? (Assume the current through the 60-W resistor has no phase shift.)





Assume no phase shift
(given in the problem)

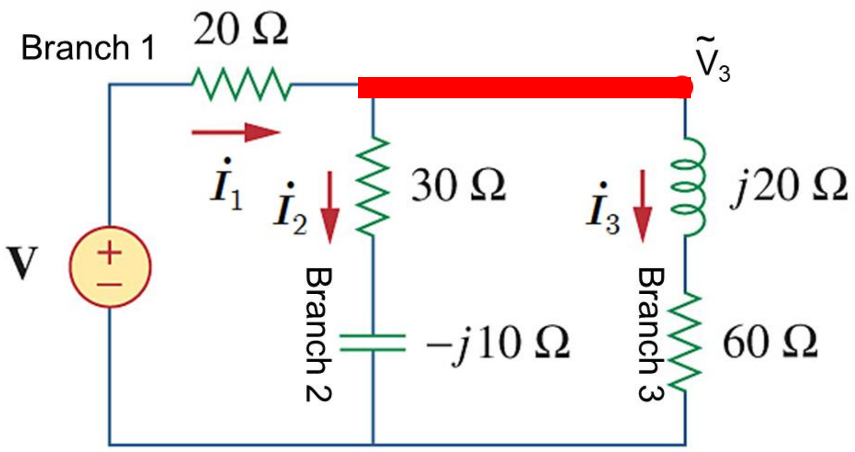
60 Ω absorbs an average power of 240 W → P = 240 W

$$I_3 = \sqrt{\frac{240}{60}} = 2 \text{ (A)}$$

$$\tilde{I}_3 = 2\angle 0° \text{ (A)}$$

$$\tilde{V}_3 = \tilde{I}_3 Z_3 = 2 \times (60 + j20)$$

$$= 120 + j40 \text{ (V)}$$



$$\tilde{I}_2 = \frac{\tilde{V}_3}{Z_2} = \frac{120 + j40}{30 - j10} = 3.2 + j2.4 \text{ (A)}$$

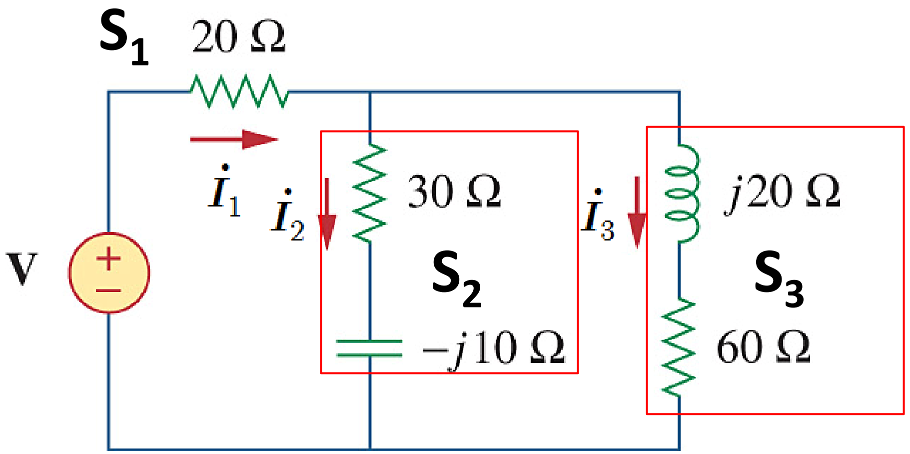
$$\tilde{I}_1 = \tilde{I}_2 + \tilde{I}_3 = (3.2 + j2.4) + 2$$

$$= 5.2 + j2.4 \text{ (A)}$$

$$\tilde{V} = \tilde{I}_1 Z_1 + \tilde{V}_3$$ **Source Voltage**

$$= (5.2 + j2.4) \times 20 + (120 + j40)$$

$$= 224 + j88 \text{ (V)}$$



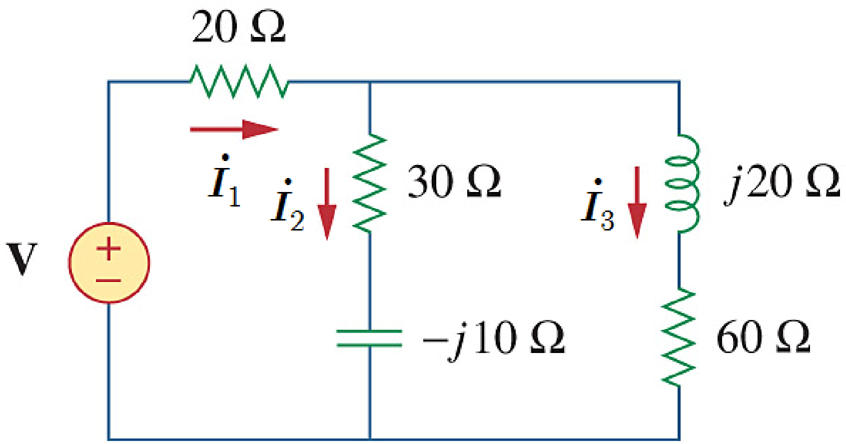
$$S_1 = I_1^2 Z_1 = (5.2^2 + 2.4^2) \times 20 = 656 \text{ (VA)}$$

$$S_2 = \tilde{V}_3 \tilde{I}_2{}^* = (120 + j40)(3.2 - j2.4)$$

$$= 480 - j160 \text{ (VA)}$$

$$S_3 = \tilde{V}_3 \tilde{I}_3{}^* = (120 + j40) \times 2$$

$$= 240 + j80 \text{ (VA)}$$



By conservation of AC power

(i) From the source:

VI* = (224 + j88) x (5.2 - j2.4) = **1376 – j80 [VA]**

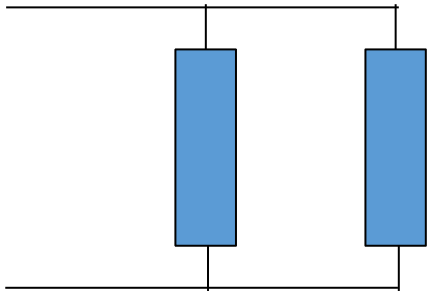
(ii) From each component:

S = S1 + S2 + S3 = 656 + (480 – j160) + (240 + j80)

= 1376 – j80 [VA]

Practice Problem 11.14 Two loads connected in parallel are respectively 2 kW at a pf of 0.75 leading and 4 kW at a pf of 0.95 lagging. Calculate the pf of the two loads. Find the complex power provided by the source.

$Q<0 \Leftrightarrow X<0 \Leftrightarrow$ Negative pf angle $\Leftrightarrow \theta_i>\theta_v \Leftrightarrow$ leading
$Q>0 \Leftrightarrow X>0 \Leftrightarrow$ Positive pf angle $\Leftrightarrow \theta_i<\theta_v \Leftrightarrow$ lagging



W is the unit for P, not for Q or S

(i) P=2kW
pf=0.75 leading

(ii) P=4kW
pf=0.95 lagging

(i)

P = 2 kW

Leading angle θ_1

$\cos\theta_1 = 0.75 \rightarrow \theta_1 = -41.41°$

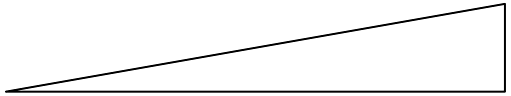
$tan\theta_1 = \frac{Q_1}{P_1} \rightarrow Q_1 = -1763.86$ [VAR]

Thus, **S_1 = 2000 –j1764 [VA]**

(ii)

Lagging angle θ_2

$\cos\theta_2 = 0.95 \rightarrow \theta_2 = 18.19°$



P = 4 kW

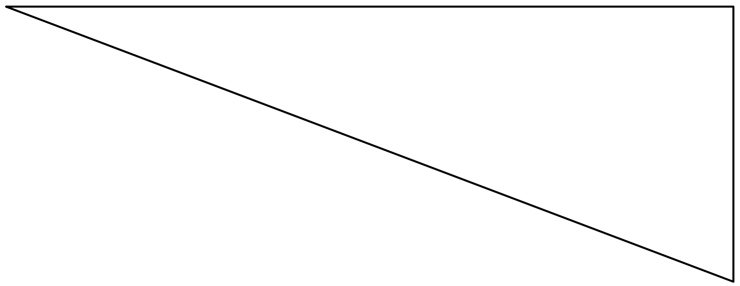
$tan\theta_2 = \frac{Q_2}{P_2} \rightarrow Q_2 = 1314.36$ [VAR]

Thus, **S_2 = 4000 + j1314 [VA]**

S_1 = 2000 –j1764 [VA] and S_2 = 4000 + j1314 [VA]

Thus, the complex power provided by the source is S = 6000 –j450 [VA]

P = 6000 [W]



Q = -450 [VAR]

$$tan\theta = \frac{Q}{P} \rightarrow tan\theta = -0.075$$

Thus, θ = -4.29°

pf = cosθ = 0.9972 leading

Alternatively,

$$S = P + jQ = 6 - j0.4495 \text{ (kVA)}$$

$$|S| \approx 6.0168 \text{ (kVA)}$$

$$\text{pf} = \frac{P}{|S|} = \frac{6}{6.0168} \approx 0.9972 \text{ leading}$$

Q<0 ⇔ X<0 ⇔ Negative pf angle ⇔ $\theta_i > \theta_v$ ⇔ leading

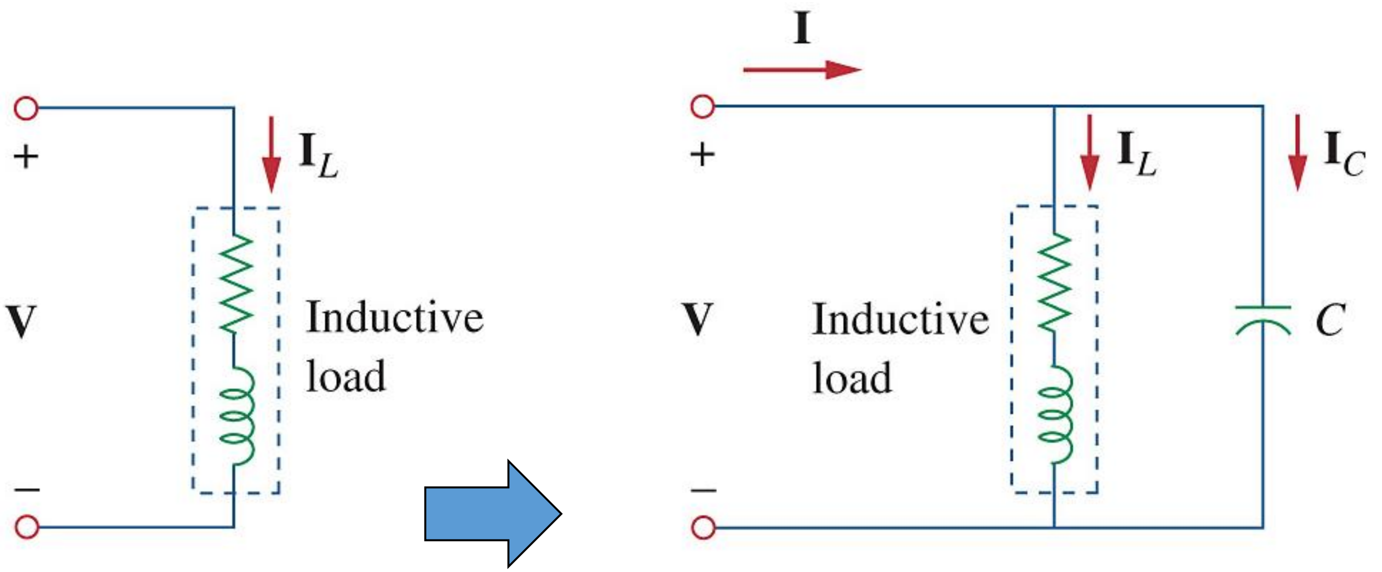
Q>0 ⇔ X>0 ⇔ Positive pf angle ⇔ $\theta_i < \theta_v$ ⇔ lagging

11.8 Power Factor Correction

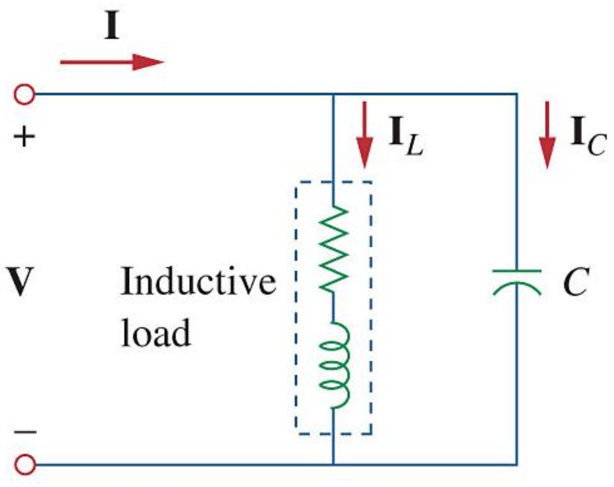
Most domestic and industrial loads are inductive and operate at a low lagging power factor. Although the **inductive nature of the load** cannot be changed, we can **increase its power factor**. The process of increasing the power factor without altering the voltage or current to the original load is known as **power factor correction**.

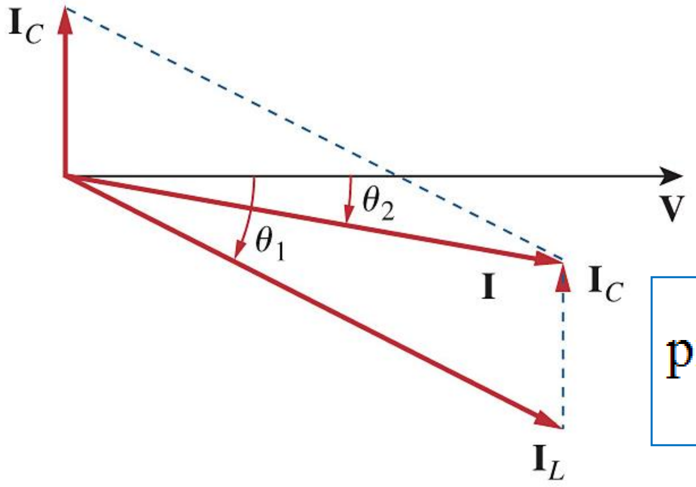
$Q<0 \Leftrightarrow X<0 \Leftrightarrow$ Negative pf angle ($\theta<0$) $\Leftrightarrow \theta_i>\theta_v \Leftrightarrow$ leading
$Q>0 \Leftrightarrow X>0 \Leftrightarrow$ Positive pf angle ($\theta>0$) $\Leftrightarrow \theta_i<\theta_v \Leftrightarrow$ lagging

Power factor correction may be viewed as the **addition of a reactive element** (usually a **capacitor**) in parallel with the load in order to make the power factor closer to unity.



Power factor correction

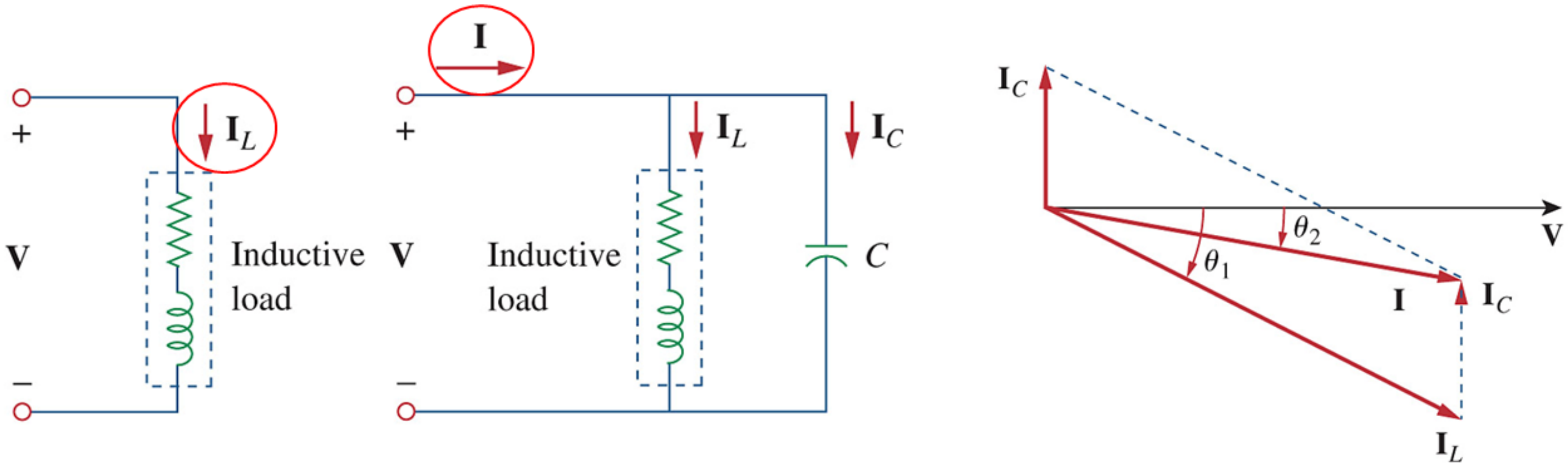




$$\text{pf}\uparrow = \cos\theta\downarrow = \frac{P}{|S|}$$

The figure (right) shows the phasor diagram of the currents involved. It is evident that adding the capacitor has caused **the phase angle to reduce from θ$_1$ to θ$_2$**, thereby increasing the power factor.

1. Original I_L (inductive load, $\theta_I < \theta_v$)
2. Keep same I_L, add I_C (capacitor load, $\theta_{Ic} > \theta_V$ by 90°)
3. New current I from source ($I = I_L + I_C$)

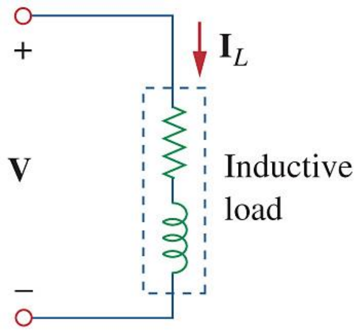


We also notice from the magnitudes of the vectors that with the same supplied voltage, the circuit (R + L) draws larger current I_L than the current *I* drawn by the circuit (R + L + C).

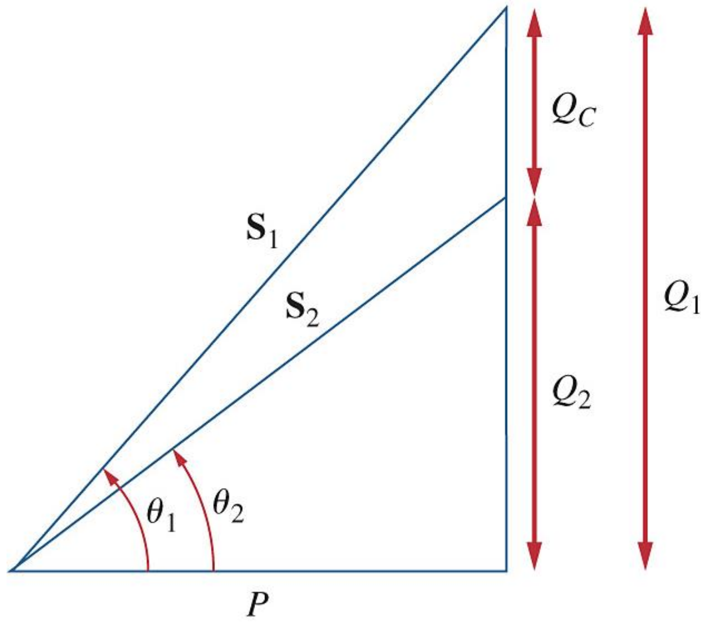
Power companies charge more for larger currents, because they result in increased power losses ($P = I_L^2 R$) → The advantage of power factor correction.

We can look at the **power factor correction** from another perspective. Let’s think about the power triangle.

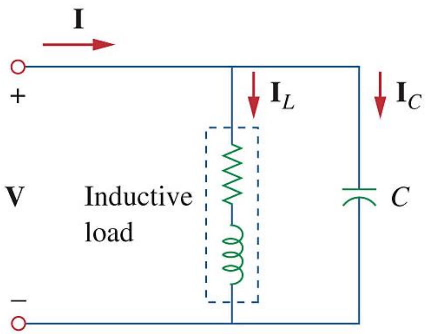
S_1



$$P = |S_1|\cos\theta_1 = |S_2|\cos\theta_2$$
$$Q_1 = |S_1|\sin\theta_1 = P\tan\theta_1$$



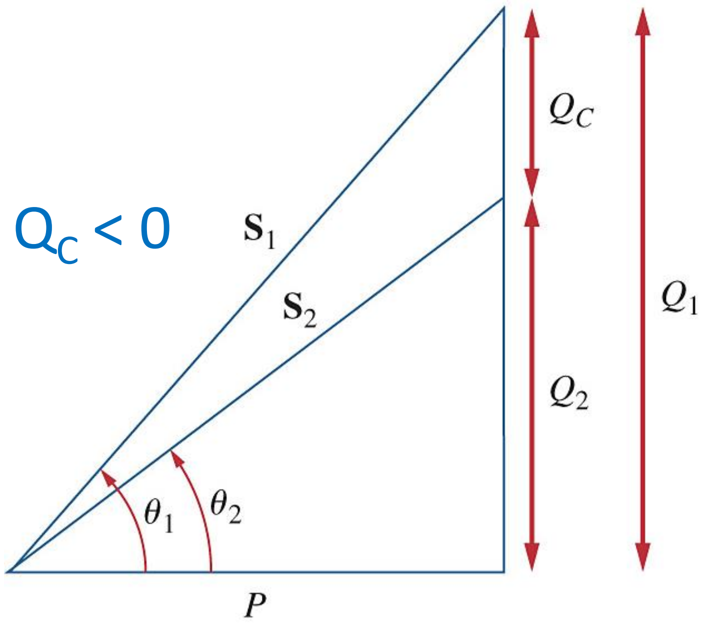
S_2



$$P = |S_1|\cos\theta_1 = |S_2|\cos\theta_2$$
$$Q_2 = |S_2|\sin\theta_2 = P\tan\theta_2$$

P is the same, $Q_1 > Q_2$

where $Q_C < 0$



(a) $P+jQ_1$

$$P = |S_1|\cos\theta_1 = |S_2|\cos\theta_2$$

$$Q_1 = |S_1|\sin\theta_1 = P\tan\theta_1$$

(b) $P+j(Q_1+Q_C)=P+jQ_2$

$$P = |S_1|\cos\theta_1 = |S_2|\cos\theta_2$$

$$Q_2 = |S_2|\sin\theta_2 = P\tan\theta_2$$

Increasing the power factor from $\cos\theta_1$ to $\cos\theta_2$, then the new reactive power is Q_2.

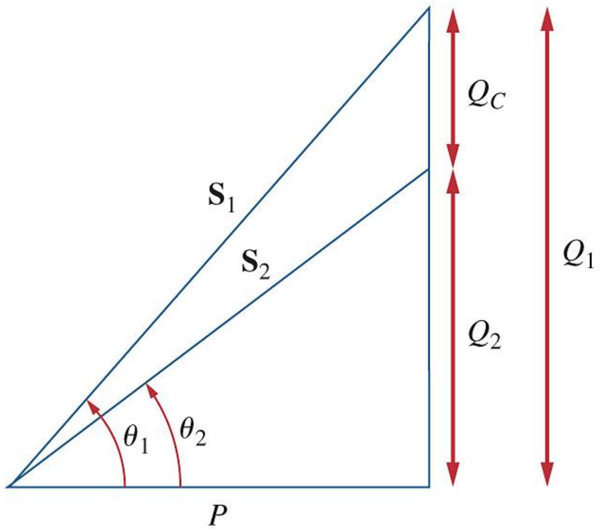
$|Q_C| = |Q_1 - Q_2|$ where $Q_1 = P \cdot \tan\theta_1$ and $Q_2 = P \cdot \tan\theta_2$

Thus, $\mathbf{Q_C = P(\tan\theta_1 - \tan\theta_2)}$

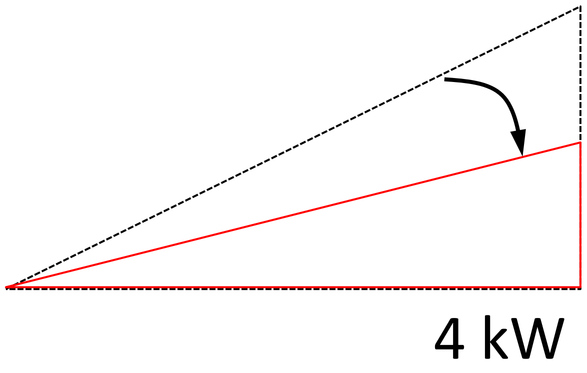
Reactive power $Q_C = \frac{V_{rms}^2}{X_C} = wCV_{rms}^2$

The value of the required shunt **capacitance C** is

$$C = \frac{Q_C}{wV_{rms}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{wV_{rms}^2}$$



Example 11.15 When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.



Lagging pf 0.8 → $\cos\theta_1 = 0.8$
This gives $\theta_1 = 36.87°$

Q<0 ⇔ X<0 ⇔ Negative pf angle ⇔ $\theta_i > \theta_v$ ⇔ leading
Q>0 ⇔ X>0 ⇔ Positive pf angle ⇔ $\theta_i < \theta_v$ ⇔ lagging

Want to change it to lagging pf 0.95 → $\cos\theta_2 = 0.95$
This gives $\theta_2 = 18.19°$

$$tan\theta_1 = \frac{Q_1}{P} \rightarrow Q_1 = 3000$$
$$tan\theta_2 = \frac{Q_2}{P} \rightarrow Q_2 = 1314$$

Q_C = -1686 [VAR]

$\boldsymbol{C = {}^{-Q_C}/_{\omega V^2}}$ **= 0.31 mF**

***ω = 2πf**