ECE2150J Introduction to Circuits

Chapter 13. Magnetically Coupled Circuits

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13.1 Introduction

The circuits we have considered so far may be regarded as **conductively coupled**, because one loop affects the neighboring loop **through current conduction**.

When two loops, **with or without contacts** between them, affect each other **through the magnetic field** generated by one of them, they are said to be **magnetically coupled.**
e.g. Transformer

13.2 Mutual Inductance

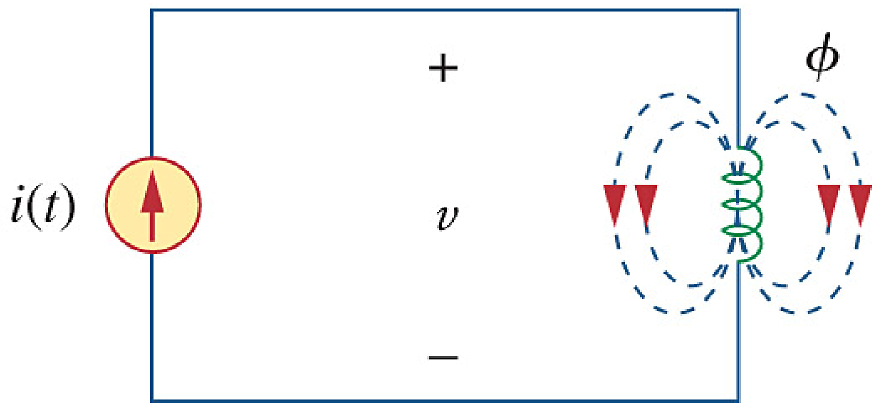
When two coils are in a close proximity to each other, **the magnetic flux** caused by current in one coil links with the other coil, thereby **inducing voltage in the latter**. This phenomenon is known as **mutual inductance**.

Chapter 6:

$i_1 \rightarrow \psi_1 = L_1 i_1 \rightarrow v_2 = d\psi_1/dt$

ψ_1: total magnetic flux

Self-inductance, L



A single coil with N turns: when current *i* flows through the coil, a magnetic flux Φ is produced around it. According to Faraday's law, **the voltage *v* induced** in the coil is

$$v = N\frac{d\phi}{dt}$$

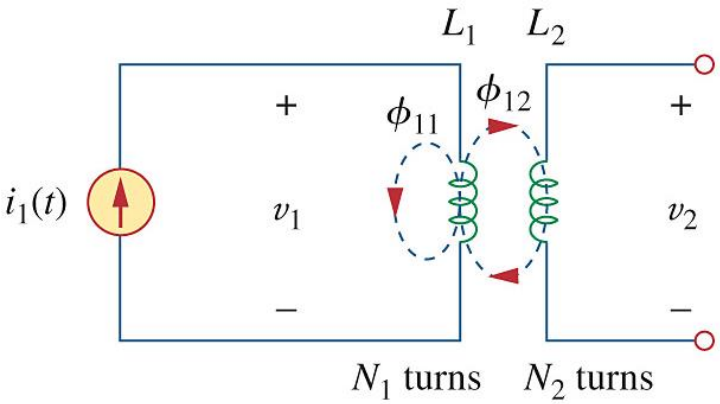
The flux Φ is produced by current i so that any change in Φ is caused by a change in i.

Therefore, $$v = N\frac{d\phi}{dt} = N\frac{d\phi}{di}\frac{di}{dt} = L\frac{di}{dt}$$

$L = N\dfrac{d\phi}{di}$ is commonly called *self-inductance*

Self-inductance relates **the voltage induced in a coil by a time-varying current** in the same coil.

Mutual-inductance, L

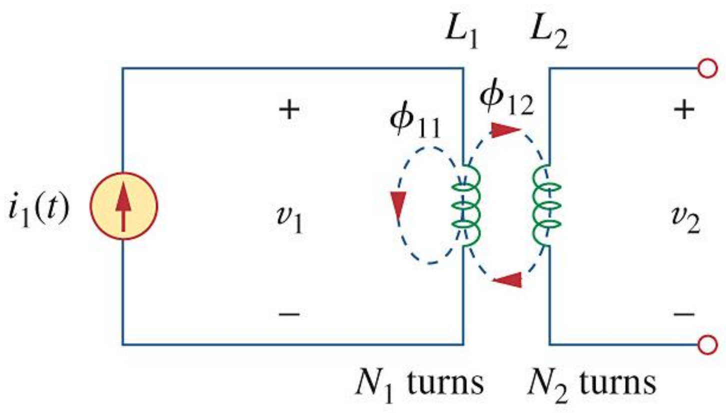


Two coils that are in close proximity with each other. Assuming that coil 2 carries no current, the magnetic flux Φ_1 emanating from coil 1 has two components:

$$\boldsymbol{\Phi_1 = \Phi_{11} + \Phi_{12}}$$

where Φ_{11} links only coil 1 and Φ_{12} links both coils.

$\boxed{\Phi_{ab}\text{: ‘a’ flux emanating from coil a}}$



Φ_{12}

Current in coil 1

M_{21}

Induced inductance in coil 2

Current in coil 1

Because the entire flux Φ_1 links coil 1, the voltage induced in coil 1 is

$$v_1 = N_1 \frac{d\phi_1}{dt} = N_1 \frac{d\phi_1}{di_1} \frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

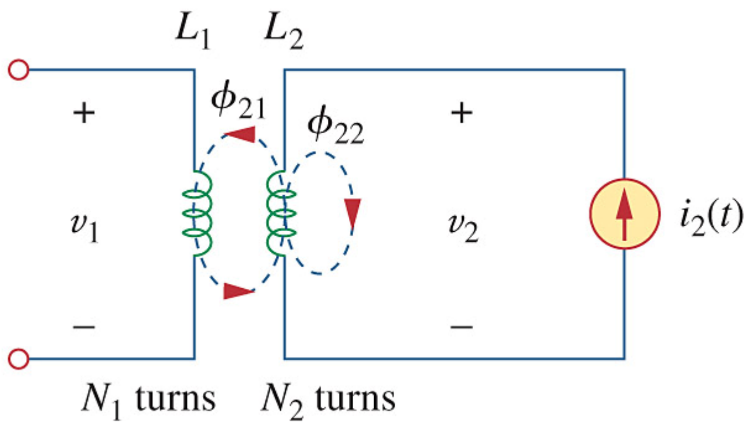
Only flux Φ_{12} links coil 2, so the voltage induced in coil 2 is

$$v_2 = N_2 \frac{d\phi_{12}}{dt} = N_2 \frac{d\phi_{12}}{di_1} \frac{di_1}{dt} = M_{21} \frac{di_1}{dt}$$

Induced voltage (V_2) in circuit 2 by current i_1 through the coil 1.

$$M_{21} = N_2 \frac{d\phi_{12}}{di_1}$$

Mutual inductance [H] of coil 2 with respect to coil 1



Suppose we now let current i_2 flow in coil 2, while coil 1 carries no current.

$$\phi_2 = \phi_{21} + \phi_{22}$$

$$v_2 = N_2 \frac{d\phi_2}{dt} = N_2 \frac{d\phi_2}{di_2} \frac{di_2}{dt} = L_2 \frac{di_2}{dt}$$

$$v_1 = N_1 \frac{d\phi_{21}}{dt} = N_1 \frac{d\phi_{21}}{di_2} \frac{di_2}{dt} = M_{12} \frac{di_2}{dt}$$

Induced voltage (V_1) in circuit 1 by current i_2 through the coil 2 .

We will see in the next section that $\mathbf{M_{12} = M_{21} = M}$, and we refer to **M as the mutual inductance** between the two coils. Although **mutual inductance M is always a positive quantity**, the mutual **voltage may be negative or positive**, just like the self-induced voltage.

相当于 L 的作用

$$v = M\frac{di}{dt}$$

M: always a positive quantity

v: +/- depending on $\frac{di}{dt}$

Mutual Inductance M

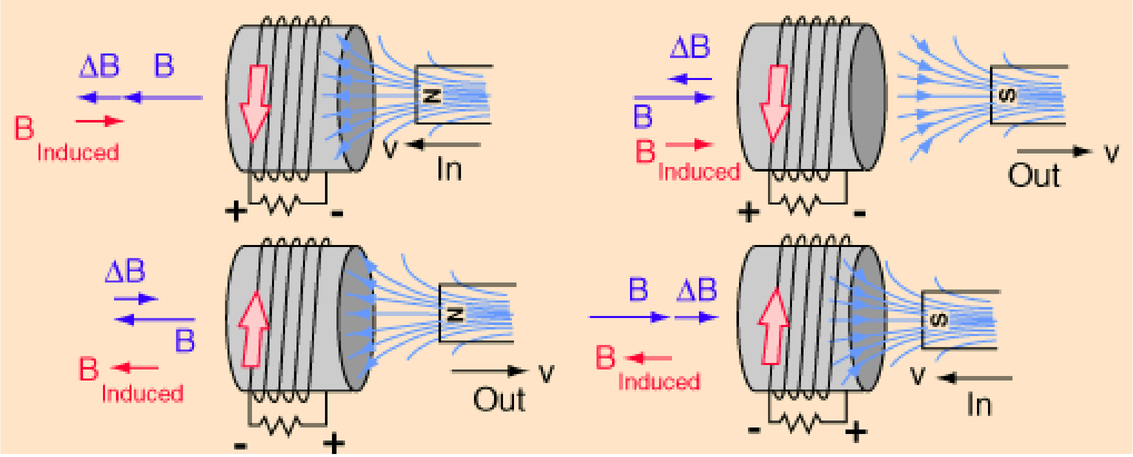
Mutual coupling only exists when the inductors or coils are **in close proximity**, and the circuits are driven by **time-varying sources**.

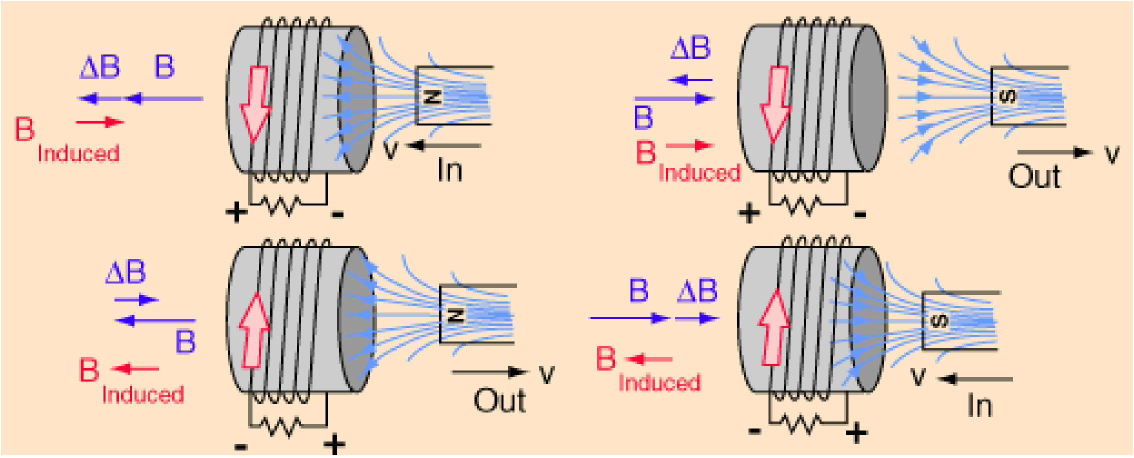
We recall that inductors act like short circuits to DC. Mutual inductance results if a voltage is induced by a **time-varying current** in another circuit.

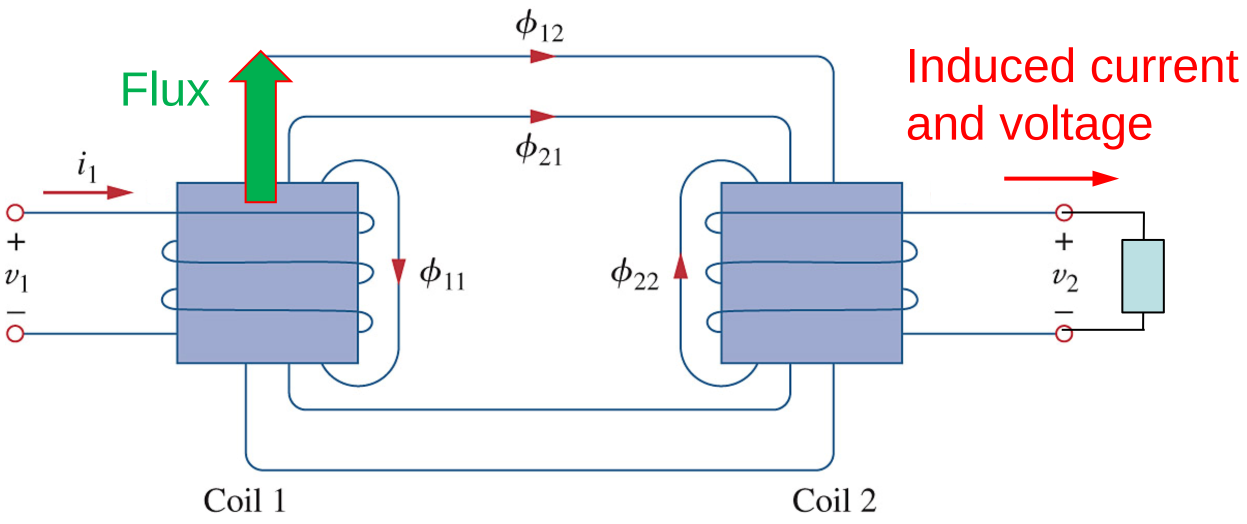
Dot convention

The polarity of the mutual voltage is determined by examining the orientation in which both coils are physically wound and applying Lenz's law in conjunction with the right-hand rule. Because it is inconvenient to show the construction details of coils on a circuit schematic, **we apply the dot convention** in circuit analysis.

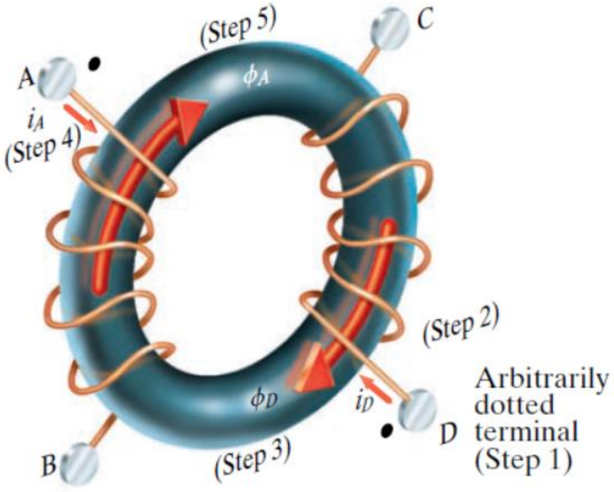
Lenz's law



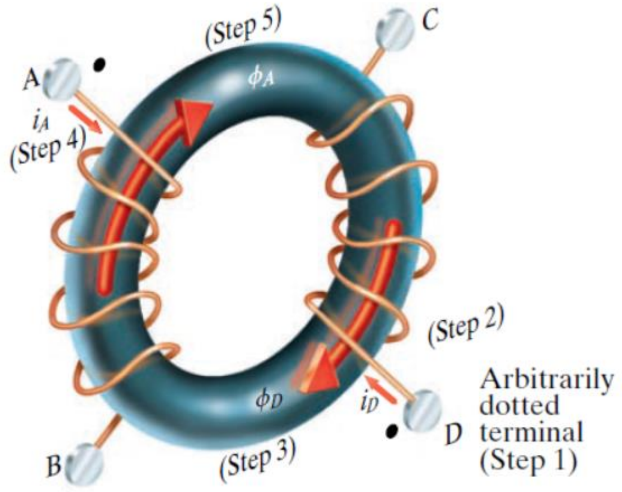

ΔB
B
B Induced
In
v
+
-
ΔB
B
B Induced
Out
v
+
-
ΔB
B
B Induced
Out
v
-
+
B
ΔB
B Induced
v
In
-
+


Flux
Induced current
and voltage
ϕ_{12}
ϕ_{21}
ϕ_{11}
ϕ_{22}
i_1
+
v_1
−
+
v_2
−
Coil 1
Coil 2

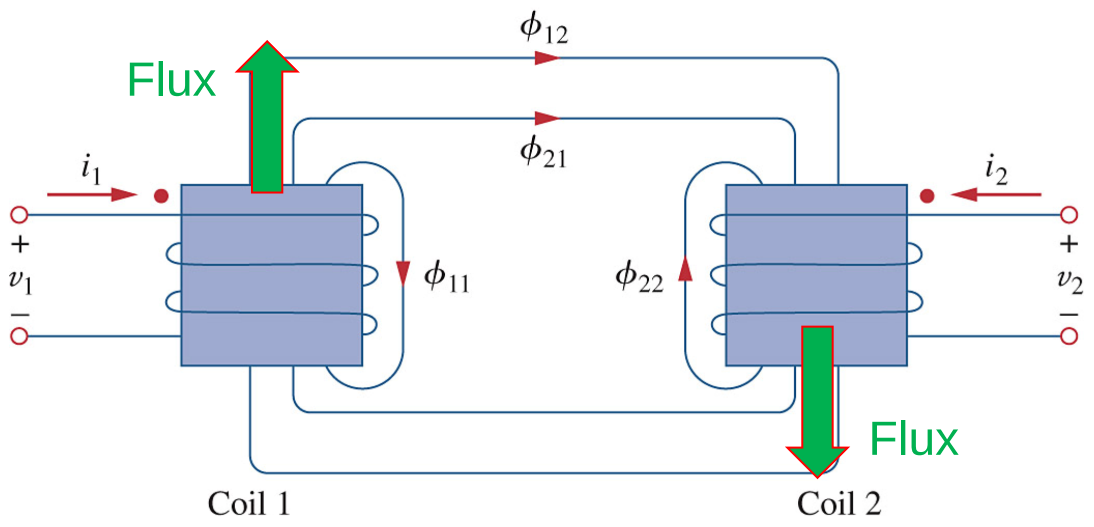
How to assign dots



(i) Arbitrarily select one terminal, e.g. D terminal and mark it with a dot.
(ii) Assign a current into the dotted terminal and label it i_D.
(iii) Use the right-hand rule to determine the direction of magnetic field established by i_D inside the coupled coils and label this field $Ø_D$.



(iv) Arbitrarily pick one terminal of the second coil, e.g. terminal A, and assign a current into this terminal, showing the current as i_A.
(v) Use the right-hand rule to determine the direction of the flux established by i_A inside the coupled coils and label this flux $Ø_A$.
(vi) Compare the directions of the two fluxes $Ø_D$ and $Ø_A$. If fluxes have **the same reference direction**, place a dot on the terminal of the second coil where the test current (i_A) enters. If the fluxes have **different reference directions**, place a dot on the terminal of the second coil where the test current leaves.



Current entering the dotted end of one winding produces flux in the same direction as the flux produced by current entering the dotted end of the other winding.

In other words, i_1 enters the dotted end of coil 1, and i_2 enters the dotted end of coil 2. They produce the flux in the same direction (clockwise in the figure).

Dot Convention:

Mutual voltage = $M\frac{di}{dt}$

1. If a current **enters the dotted terminal** of one coil, the reference polarity of the mutual voltage in the second coil is **positive at the dotted terminal** of the second coil.

2. If a current **leaves the dotted terminal** of one coil, the reference polarity of the mutual voltage in the second coil is **negative at the dotted terminal** of the second coil.

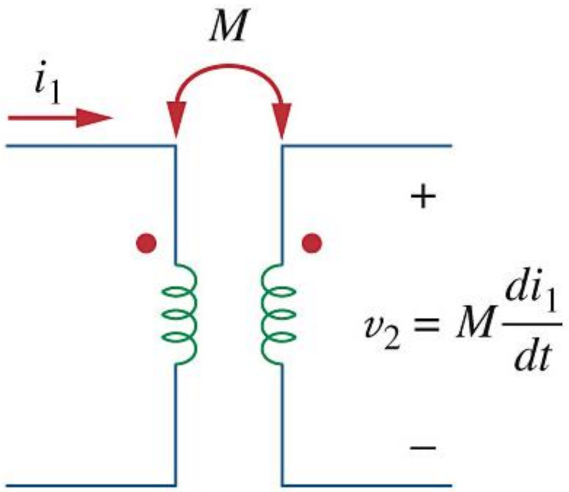
Entering current in dot 1/2 ➔ + voltage in dot 2/1
Leaving current in dot 1/2 ➔ – voltage in dot 2/1

Dot Convention:

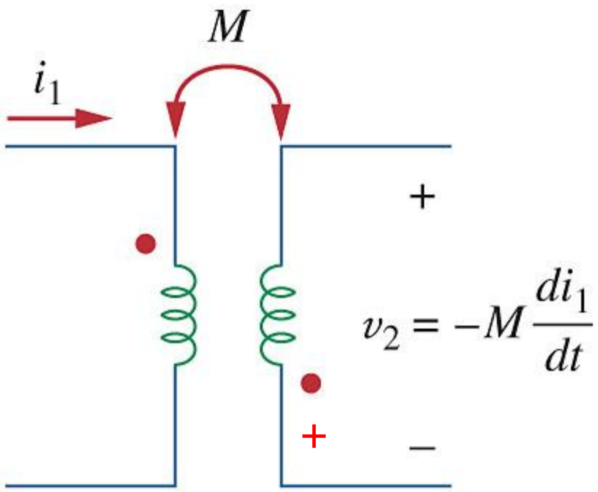
Mutual voltage = $M\frac{di}{dt}$

1. If a current **enters the dotted terminal** of one coil, the reference polarity of the mutual voltage in the second coil is **positive at the dotted terminal** of the second coil.

Current Entering



Current Entering

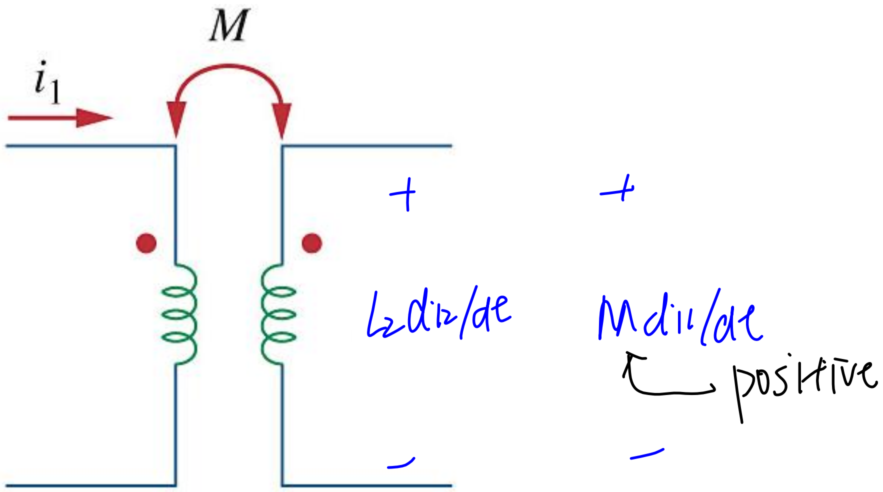


Dot Convention:

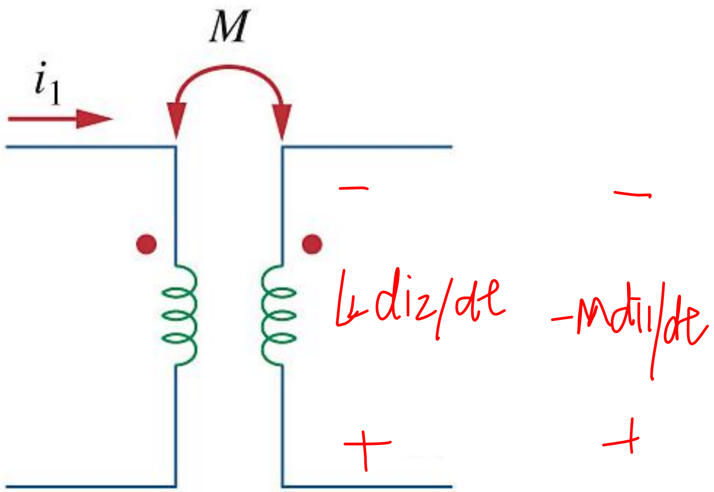
Mutual voltage = $M\frac{di}{dt}$

1. If a current **enters the dotted terminal** of one coil, the reference polarity of the mutual voltage in the second coil is **positive at the dotted terminal** of the second coil.

Current Entering



Current Entering

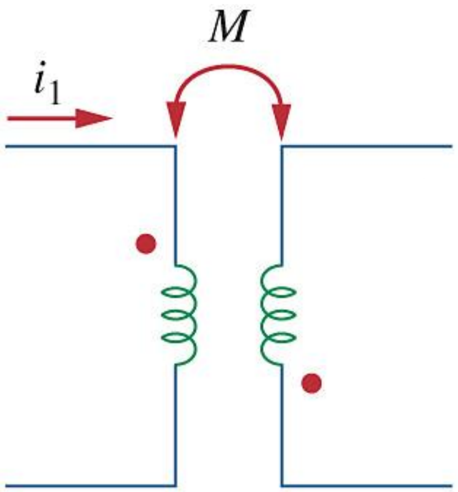


Dot Convention:

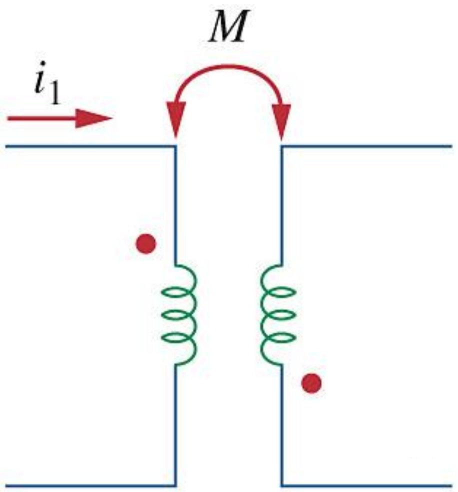
Mutual voltage = $M\frac{di}{dt}$

1. If a current **enters the dotted terminal** of one coil, the reference polarity of the mutual voltage in the second coil is **positive at the dotted terminal** of the second coil.

Current Entering

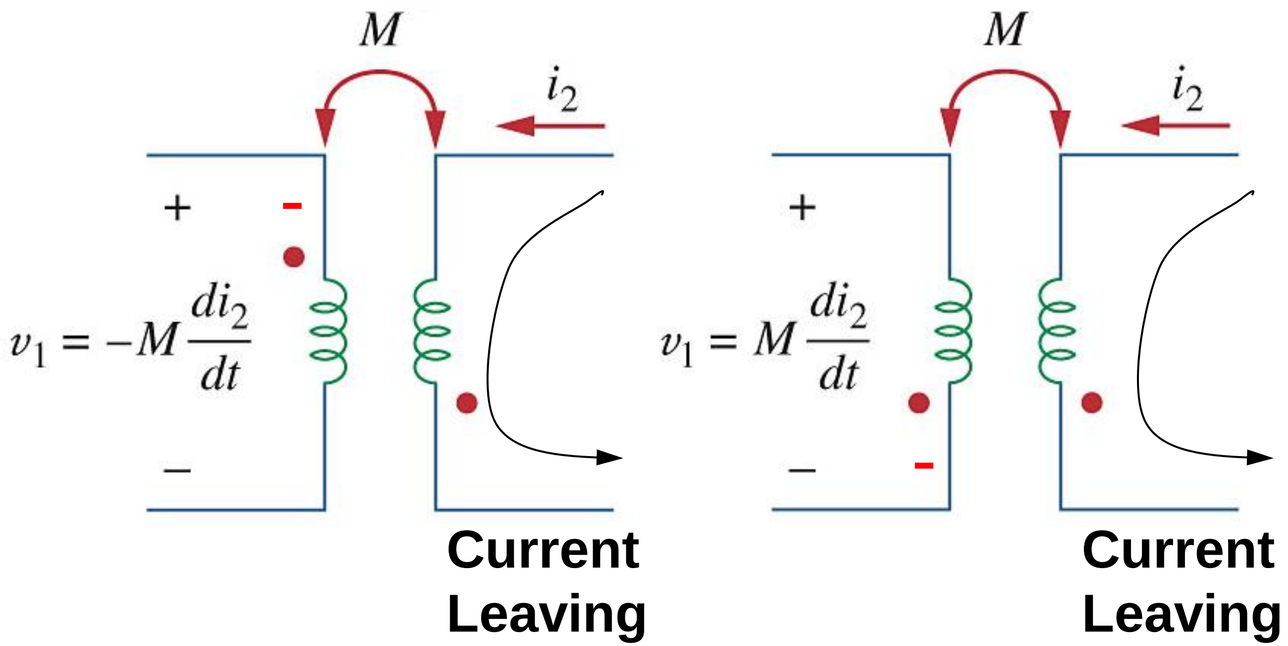


Current Entering



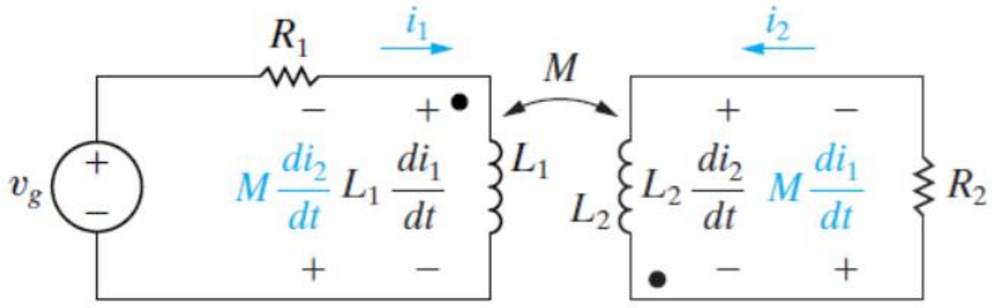
Dot Convention:

2. If a current **leaves the dotted terminal** of one coil, the reference polarity of the mutual voltage in the second coil is **negative at the dotted terminal** of the second coil.



Dot Convention:

Another way: If both currents enter/ leave the dots, then the mutual voltage is positive, otherwise, the mutual voltage is negative.



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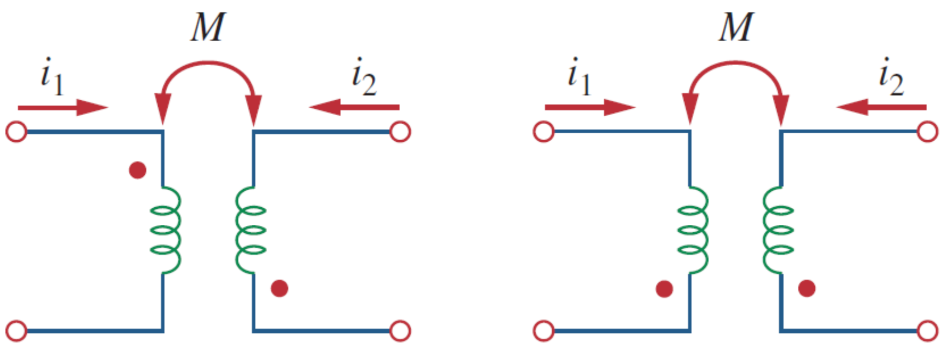
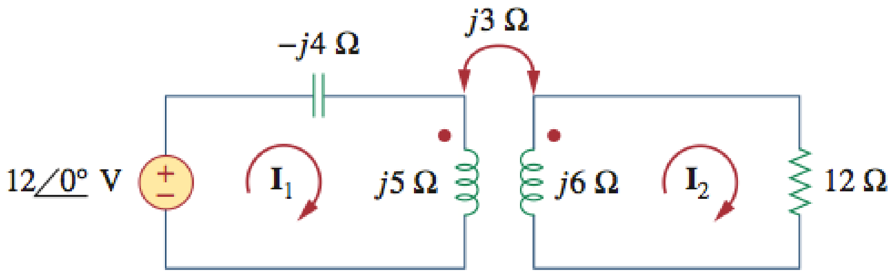
Q. For the two magnetically coupled coils of Figure below, the polarity of the mutual voltage is:

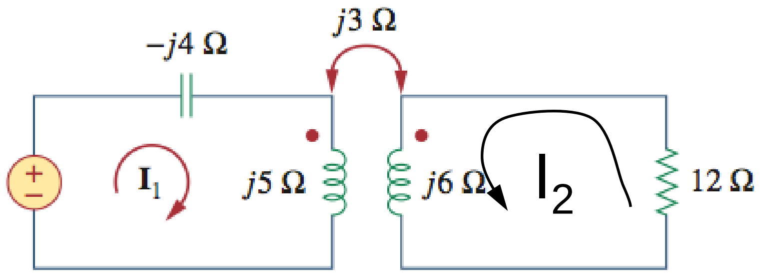
(a) Positive (b) Negative



Example 13.1



(a) negative



(a) positive

From the previous slide, the dot convention means "Mutual voltage depends on **how you set directions of current I_1 and I_2**"

From the figure, a mutual voltage is negative. But by setting I_2 as the opposite direction, we can make a mutual voltage positive

Dot Convention for coupled coils in series

Figure 13.6 shows two coupled coils in series. For the coils in Fig. 13.6(a), the total inductance is

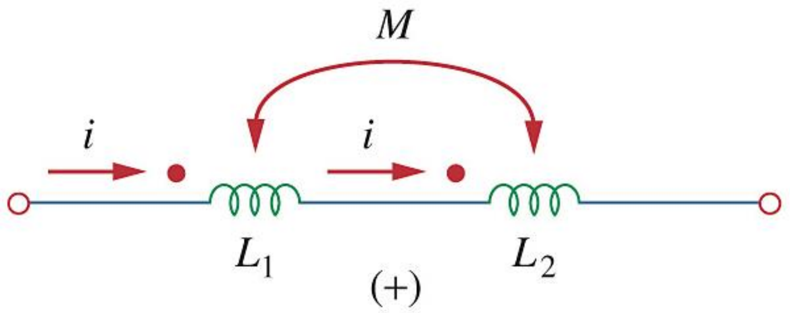
$$L = L_1 + L_2 + 2M$$

(series-aiding connection)

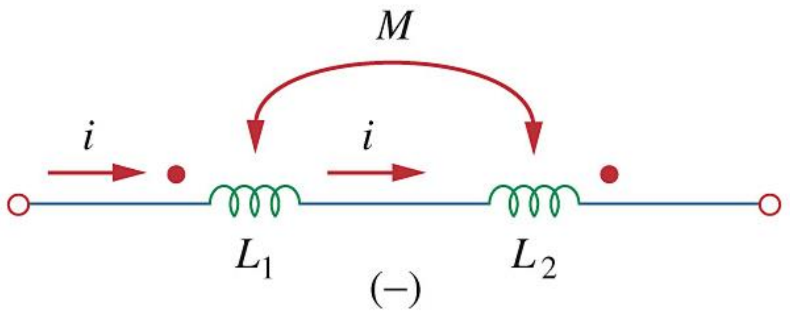
For the coils in Fig. 13.6(b),

$$L = L_1 + L_2 - 2M$$

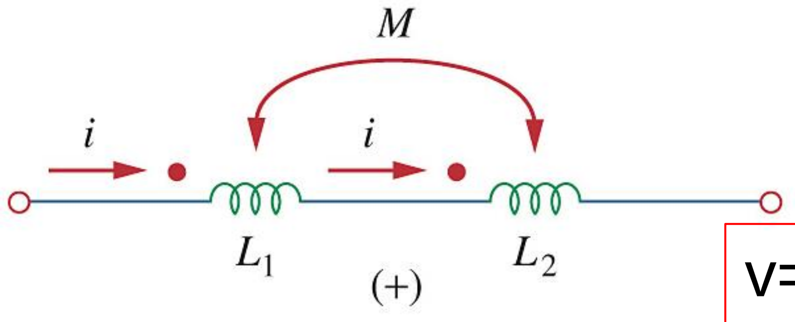
(series-opposing connection)



13.6 (a) Both currents entering dots



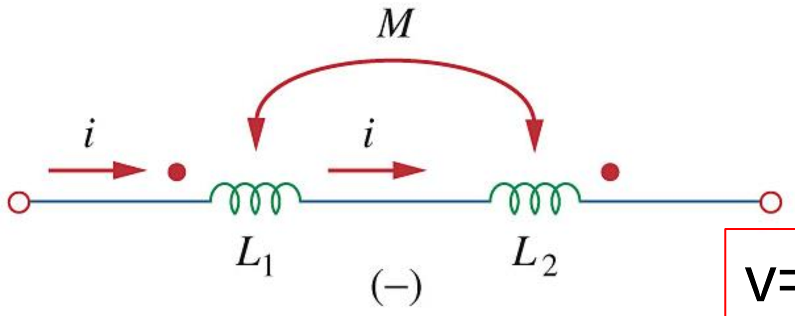
13.6 (b) Only left current entering dot



Both currents entering dots

$$L = L_1 + L_2 + 2M$$

$v = L_{eq} di/dt$
$= L_1 di/dt + L_2 di/dt + M_{21} di/dt + M_{12} di/dt$

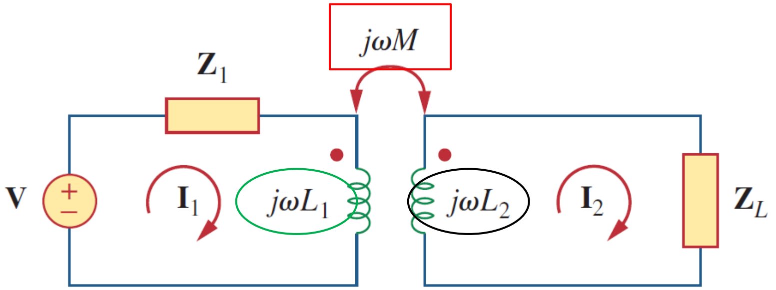


Only left current entering dot

$$L = L_1 + L_2 - 2M$$

$v = L_{eq} di/dt$
$= L_1 di/dt + L_2 di/dt – M_{21} di/dt + –M_{12} di/dt$

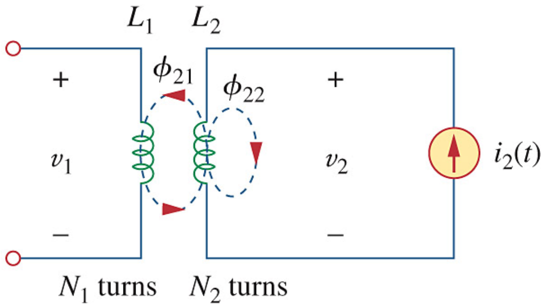
How to apply KVL?



Loop 1 $\mathbf{V} = (\mathbf{Z}_1 + j\omega L_1)\mathbf{I}_1 - \boxed{j\omega M \mathbf{I}_2}$

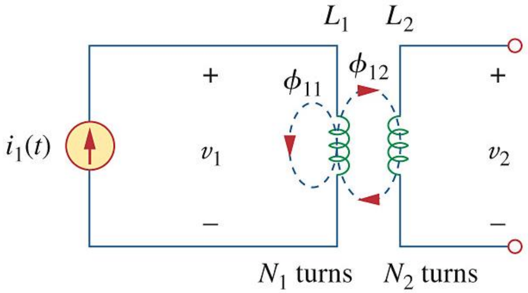
Loop 2 $0 = \boxed{-j\omega M \mathbf{I}_1} + (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2$

Two circuits physically not connected



$$v_1 = N_1 \frac{d\phi_{21}}{dt} = N_1 \frac{d\phi_{21}}{di_2}\frac{di_2}{dt} = M_{12}\frac{di_2}{dt}$$

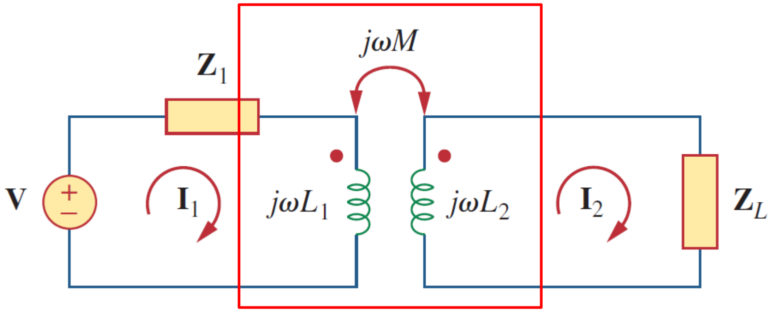
Induced voltage (V_1) in circuit 1 by current i_2 through the coil 2 .

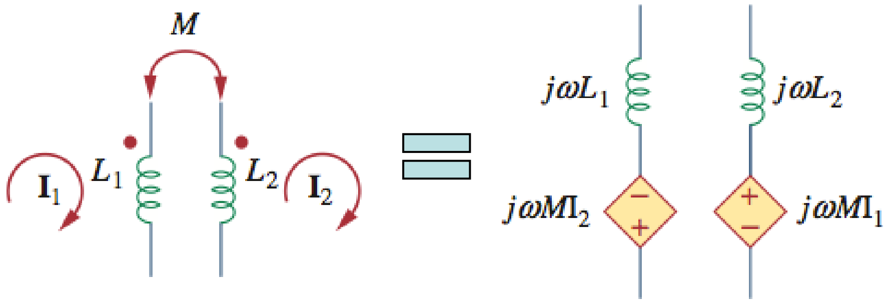


$$v_2 = N_2 \frac{d\phi_{12}}{dt} = N_2 \frac{d\phi_{12}}{di_1}\frac{di_1}{dt} = M_{21}\frac{di_1}{dt}$$

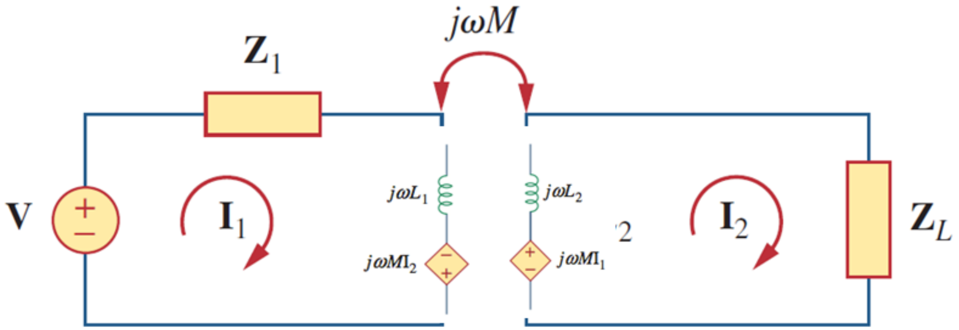
Induced voltage (V_2) in circuit 2 by current i_1 through the coil 1.

How to apply KVL? – Two steps method





Two circuits physically not connected



Loop 1

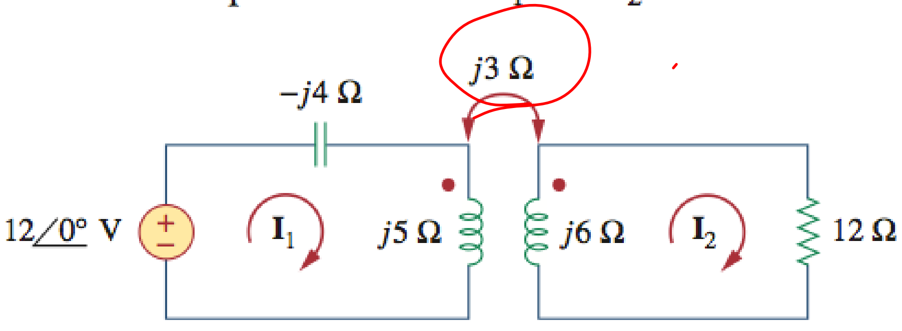
$$\mathbf{V} = (\mathbf{Z}_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2$$

Loop 2

$$0 = -j\omega M\mathbf{I}_1 + (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2$$

Example 13.1

Calculate the phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit of Fig. 13.9.

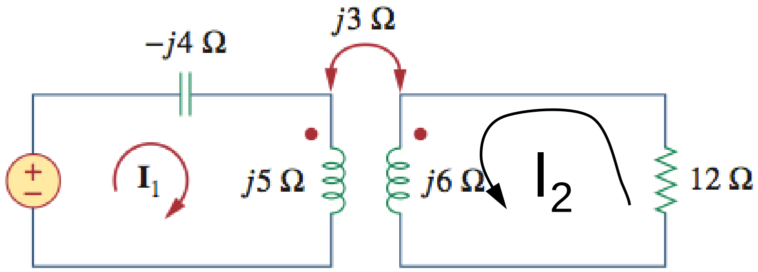


(a)

$-12 - 4j I_1 + 5j I_1 - 3j I_2$

$\tilde{V}_1 = 5j \tilde{I}_1$

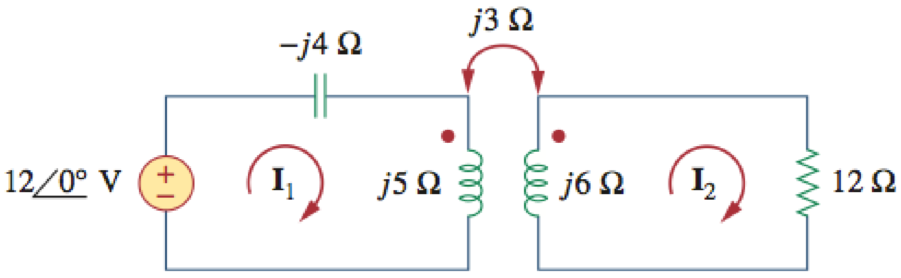
$V = M \frac{dI_2}{dt}$



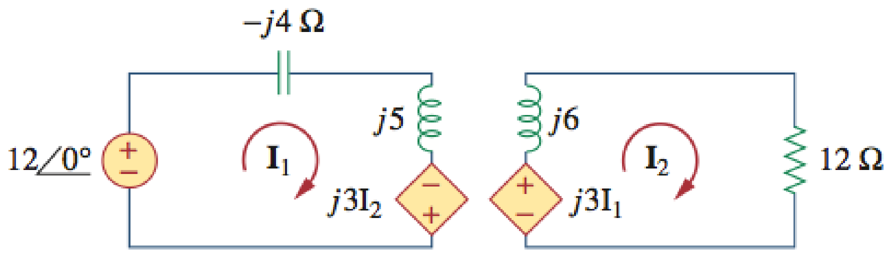
(a) positive

Calculate the phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit of Fig. 13.9.

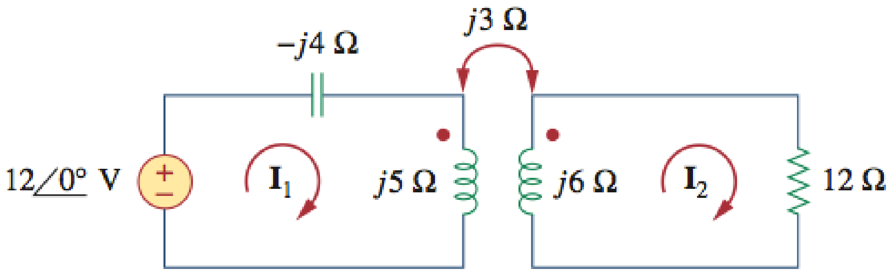
Example 13.1



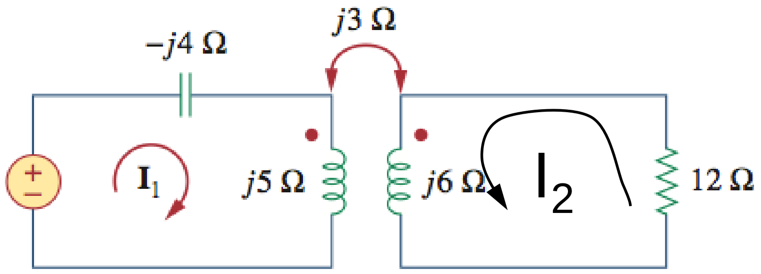
(a)



(b)



(a) negative

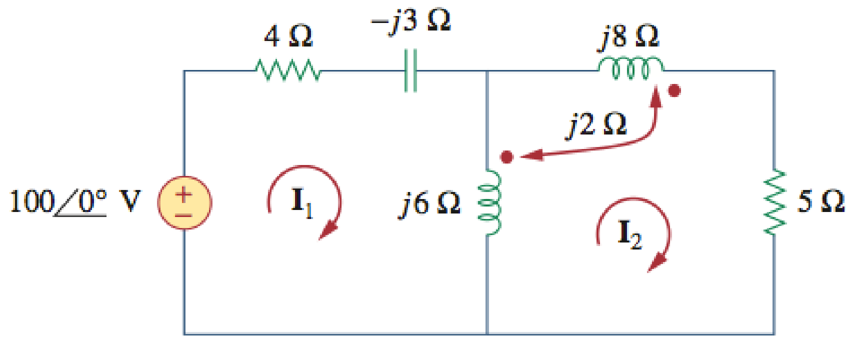


(a) positive

Both will give the same result

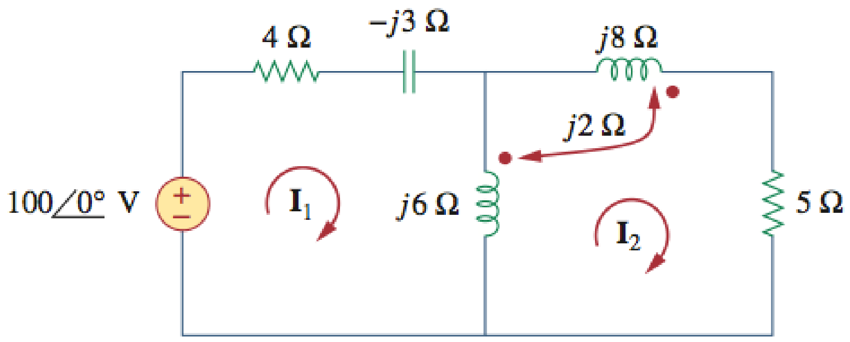
Example 13.2

Calculate the mesh currents in the circuit of Fig. 13.11.

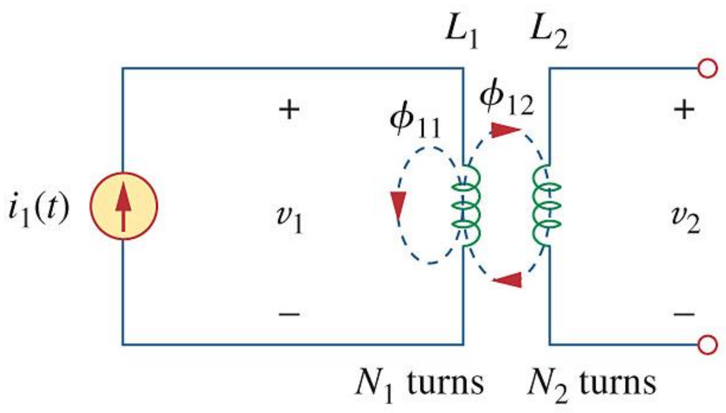


Two circuits physically connected.

Mutual voltage negative or positive?


4 Ω
−j3 Ω
j8 Ω
j2 Ω
100/0° V
I1
j6 Ω
I2
5 Ω

Recall



Φ_{12}

Current in coil 1

M_{21}

Induced inductance in coil 2

Current in coil 1

Because the entire flux Φ_1 links coil 1, the voltage induced in coil 1 is

$$v_1 = N_1 \frac{d\phi_1}{dt} = N_1 \frac{d\phi_1}{di_1}\frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

Only flux Φ_{12} links coil 2, so the voltage induced in coil 2 is

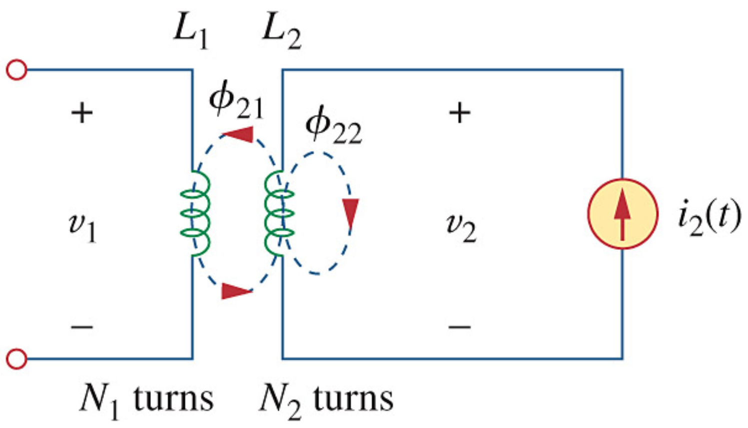
$$v_2 = N_2 \frac{d\phi_{12}}{dt} = N_2 \frac{d\phi_{12}}{di_1}\frac{di_1}{dt} = M_{21} \frac{di_1}{dt}$$

Induced voltage (V_2) in circuit 2 by current i_1 through the coil 1.

$$M_{21} = N_2 \frac{d\phi_{12}}{di_1}$$

Mutual inductance [H] of coil 2 with respect to coil 1

Recall



Suppose we now let current i_2 flow in coil 2, while coil 1 carries no current.

$$\phi_2 = \phi_{21} + \phi_{22}$$

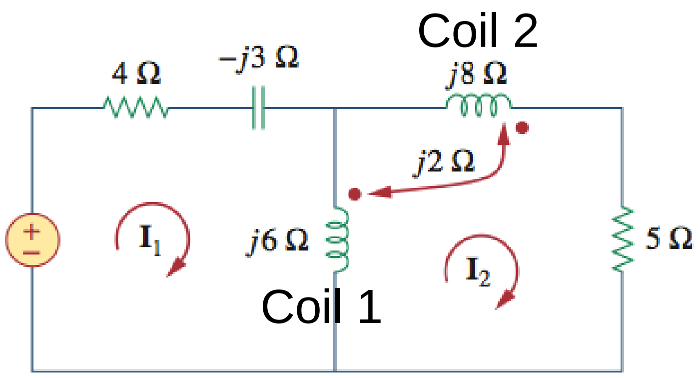
$$v_2 = N_2 \frac{d\phi_2}{dt} = N_2 \frac{d\phi_2}{di_2} \frac{di_2}{dt} = L_2 \frac{di_2}{dt}$$

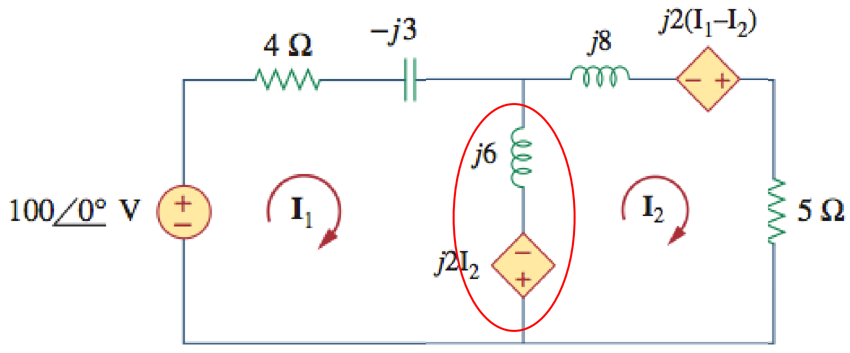
$$v_1 = N_1 \frac{d\phi_{21}}{dt} = N_1 \frac{d\phi_{21}}{di_2} \frac{di_2}{dt} = M_{12} \frac{di_2}{dt}$$

Induced voltage (V_1) in circuit 1 by current i_2 through the coil 2 .

Two major points for the circuit analysis

1. **Two circuits physically connected.**





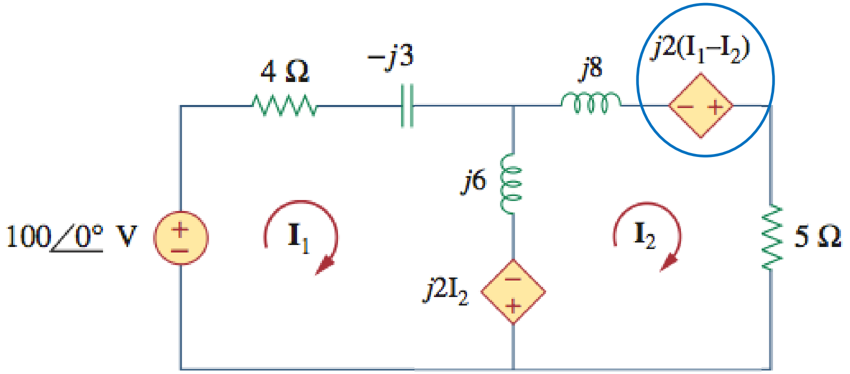
Loop 1: $-100 + \mathbf{I}_1(4 - j3 + j6) - j6\mathbf{I}_2 - j2\mathbf{I}_2 = 0$

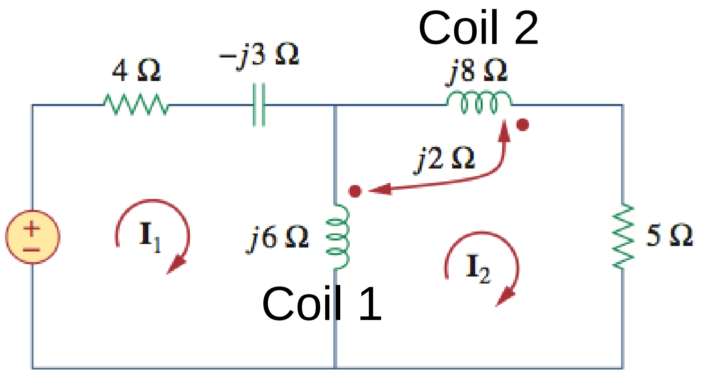
Two circuits physically connected.

Loop 2: $0 = -2j\mathbf{I}_1 - j6\mathbf{I}_1 + (j6 + j8 + j2 \times 2 + 5)\mathbf{I}_2$

Two major points for the circuit analysis

2. Mutual voltage at the second circuit.





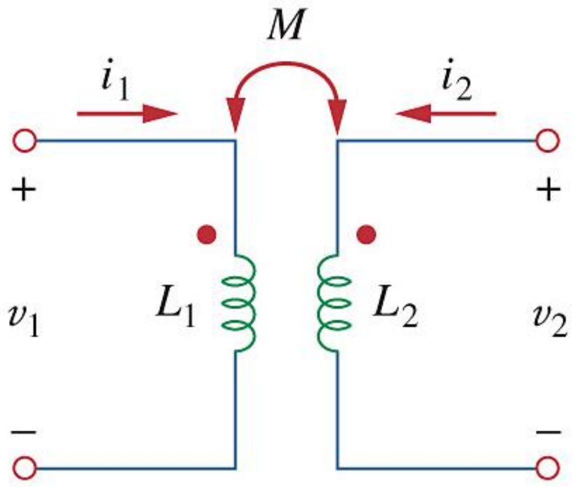
Loop 1: $-100 + \mathbf{I}_1(4 - j3 + j6) - j6\mathbf{I}_2 - j2\mathbf{I}_2 = 0$

Loop 2: $0 = -2j\mathbf{I}_1 - j6\mathbf{I}_1 + (j6 + j8 + j2 \times 2 + 5)\mathbf{I}_2$

Mutual voltage at circuit 2 induced by coil 1

- **Currents through coil 1 = $\mathbf{I}_1$-$\mathbf{I}_2$**
- **Thus, mutual voltage at the circuit 2 is due to the combination of $\mathbf{I}_1$ and $\mathbf{I}_2$**

13.3 Energy in a Coupled Circuit



Currents both entering dots

We assume that initially there are no currents.

$i_1 = 0$
$i_2 = 0$
Energy stored in the coils = 0

Step1: i_1 from 0 to I_1 while maintaining $i_2 = 0$

Step2: i_2 from 0 to I_2 while maintaining $i_1 = I_1$

Step1: i_1 from 0 to I_1 while maintaining $i_2 = 0$

If we let i_1 increase from zero to I_1 while maintaing $i_2 = 0,$ the power in the circuit is

$$p_1 = v_1 i_1 = L_1 \frac{di_1}{dt} i_1$$

and the energy stored in the circuit is

$$w_1 = \int p_1 dt = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2$$

Step2: i_2 from 0 to I_2 while maintaining $i_1 = I_1$

If we now maintain $i_1 = I_1$ and increase i_2 from zero to I_2, the power in the coils is now

$$p_2 = \left(M_{12} \frac{di_2}{dt} \right) I_1 + \left(L_2 \frac{di_2}{dt} \right) i_2$$

$\boxed{p_2 = v_1 i_1 + v_2 i_2}$

and the energy stored in the circuit is

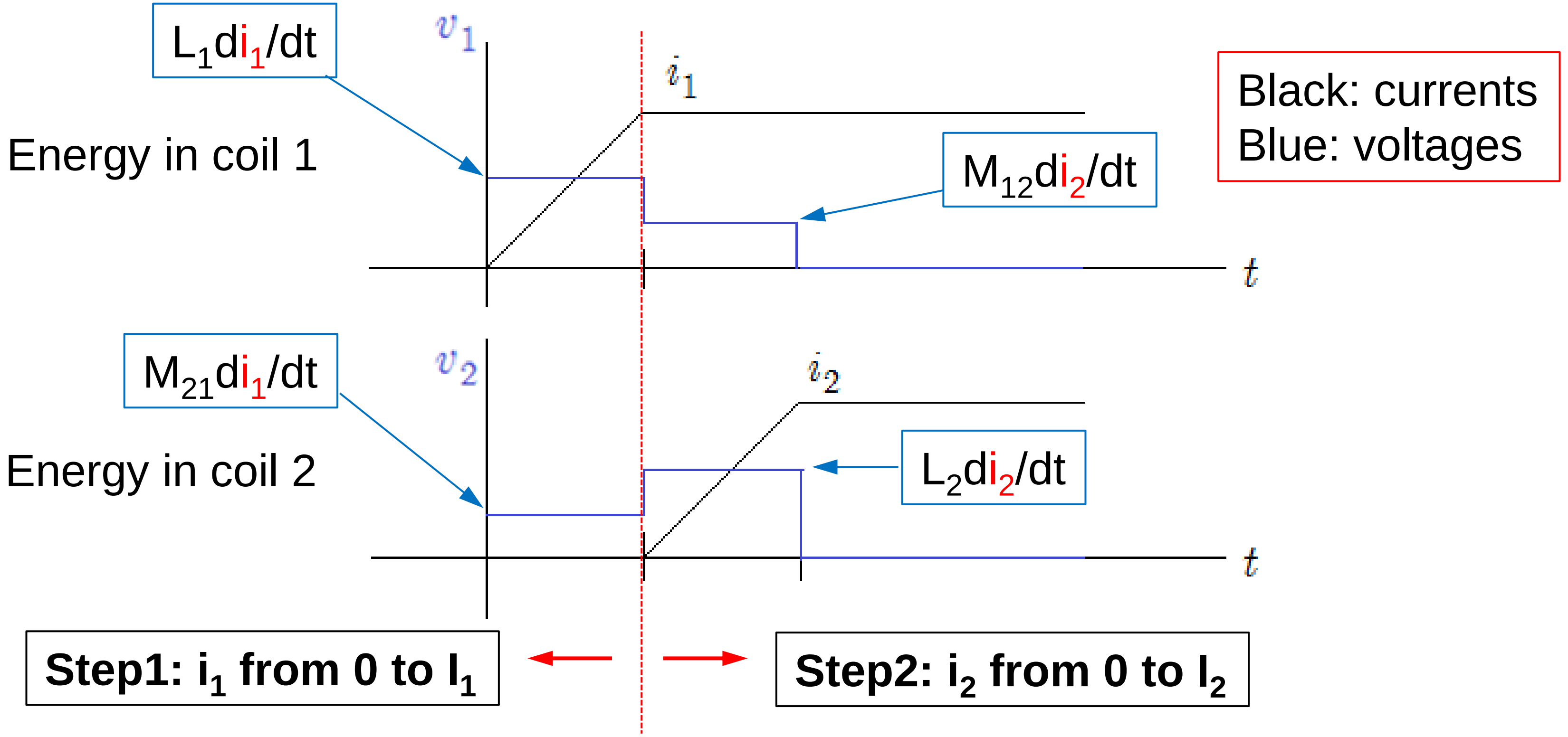
$$w_2 = \int p_2 dt = M_{12} I_1 \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2 di_2$$

$$= M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

Total Energy

The total energy stored in the coils when both i_1 and i_2 have reached constant values is

$$w = w_1 + w_2 = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{12} I_1 I_2$$



If we reverse the order by which the currents reach their final values, that is, if we first increase i_2 from zero to I_2 and later increase i_1 from zero to I_1, the total energy stored in the coils is

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{21} I_1 I_2$$

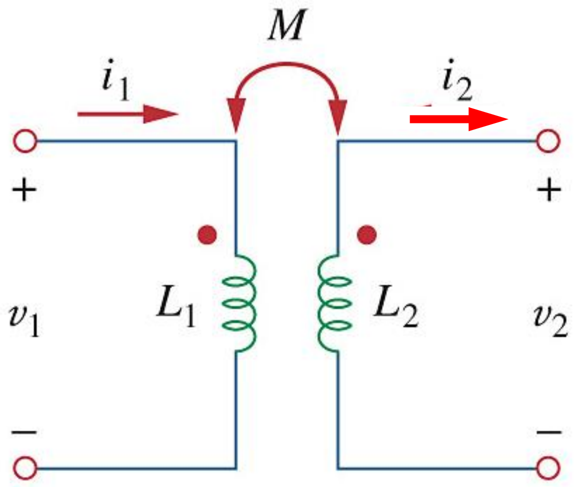
Mutual inductance at the coil 2 induced by current at the coil 1

Since the total energy stored should be the same regardless of how we reach the final conditions, comparing the two total energy expressions leads us to conclude that

$$\boxed{M_{12} = M_{21} = M}$$

$$w = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

Negative mutual voltage



If one current enters one dotted terminal while the other current leaves the other dotted terminal, i.e., mutual voltage is negative, then the total energy is

$$w = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 - MI_1I_2$$

v = $-M\frac{di}{dt}$ in power/energy derivation

Therefore, the energy in the circuit is

$$w = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 \pm M i_1 i_2$$

+ sign: mutual voltage is positive

- sign: mutual voltage is negative

Decided by the dot convention

Coupling coefficient k

The total energy stored can not be negative because the magnetically coupled coils are passive elements.

$$\frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 - M i_1 i_2 \geq 0$$

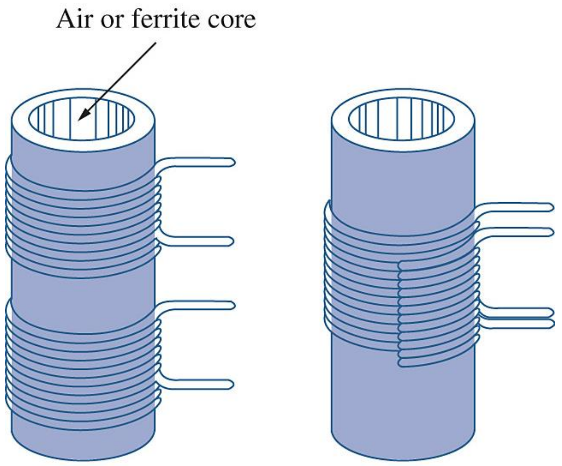
$$\Rightarrow \frac{1}{2}\left(\sqrt{L_1}i_1 - \sqrt{L_2}i_2\right)^2 + i_1 i_2\left(\sqrt{L_1 L_2} - M\right) \geq 0$$

The square term can never be negative, but it can be zero. Therefore, w(t) ≥ 0 only if $\sqrt{\boldsymbol{L_1 L_2}} \geq \boldsymbol{M}$.

The extent to which M approaches $\sqrt{L_1 L_2}$ is specified by the *coefficient of coupling* k, given by

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

$$(0 \le k \le 1)$$

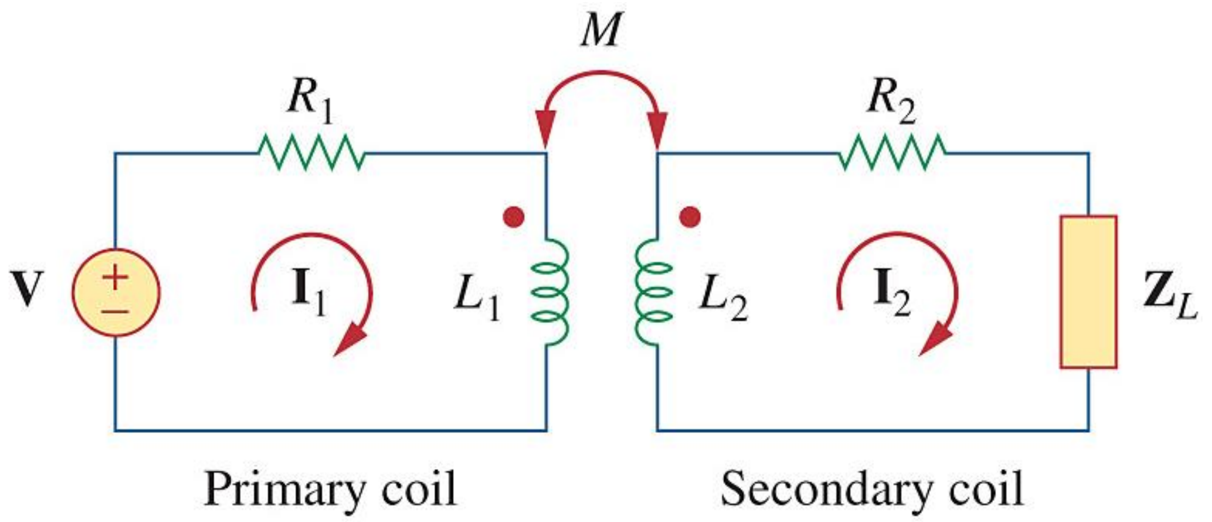


Coils are said to be *loosely coupled* when $k < 0.5$. If $k > 0.5$, they are *tightly coupled*.

Distance is one factor to affect the value k

13.4 Linear Transformer

A transformer (passive element) is generally a four-terminal device comprising two (or more) magnetically coupled coils.

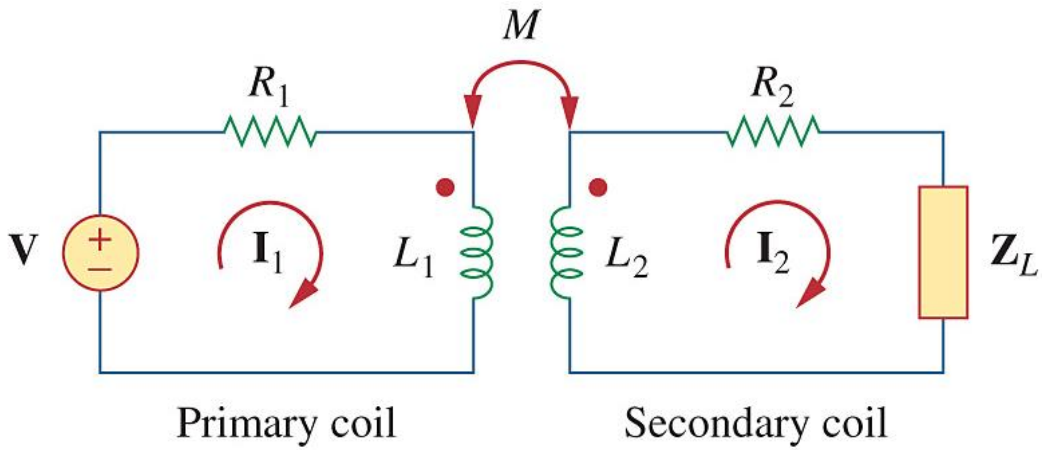


The coil that is directly connected to the voltage source is called the **primary winding**. The coil connected to the load is called the **secondary winding**.

Linear transformer: If the coils are wound on a magnetically linear material – a material for which **the magnetic permeability is constant**.

Linear transformers are sometimes called **air-core transformers** although not all of them are necessarily air-core.

Determine a linear transformer circuit



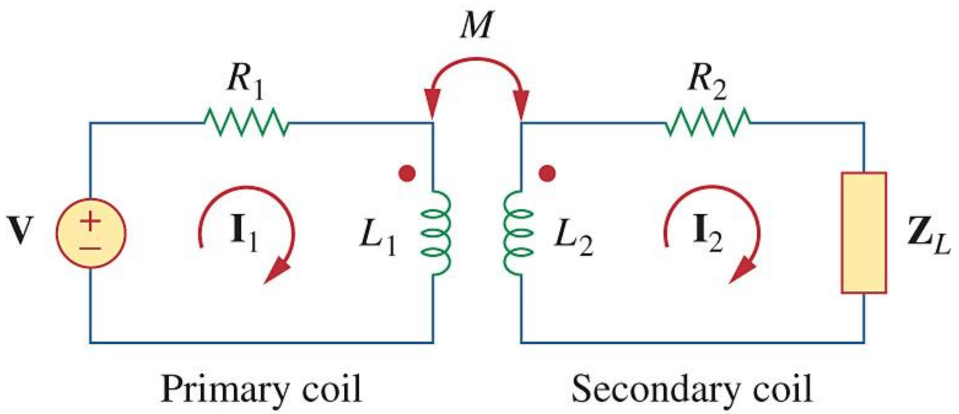
i) Reflected impedance Z_R

Find the input impedance Z_{in} at the primary circuit

By KVL, Loop 1: $\mathbf{V} = (R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2$

Loop 2: $0 = -j\omega M\mathbf{I}_1 + (R_2 + j\omega L_2 + \mathbf{Z}_L)\mathbf{I}_2$

Rearrange terms in loop 2, we get $I_2 = \dfrac{j\omega M I_1}{R_2 + j\omega L_2 + Z_L}$



Loop 1: $\mathbf{V} = (R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2 \longleftarrow I_2 = \dfrac{j\omega M I_1}{R_2 + j\omega L_2 + Z_L}$

The input impedance Z_{in} $(\frac{V}{I_1})$ is $\boxed{R_1 + jwL_1} + \boxed{\dfrac{w^2 M^2}{R_2 + jwL_2 + Z_L}}$

- **The primary impedance**
- **Reflected impedance (Z_R)** due to the coupling between the primary and secondary windings.

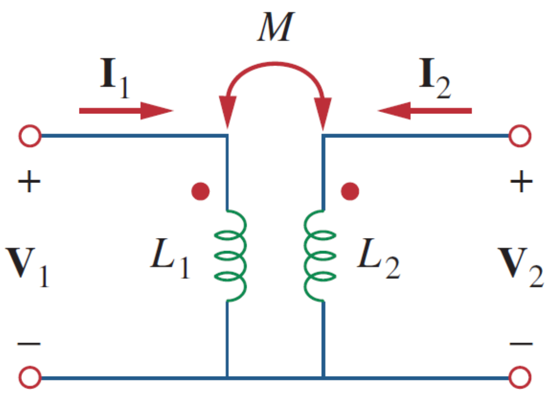
*The location of the dots on the transformer does not affect the result when M is replaced by –M.

ii) Equivalent T or π circuit

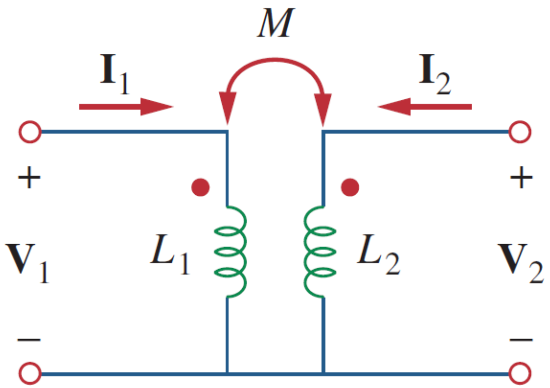
It is convenient to replace a magnetically coupled circuit by an equivalent circuit **with no magnetic coupling**.

- Circuit with the dot convention

The voltage-current relationships for the primary and secondary coils give the matrix equation



$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$



$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad A' = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

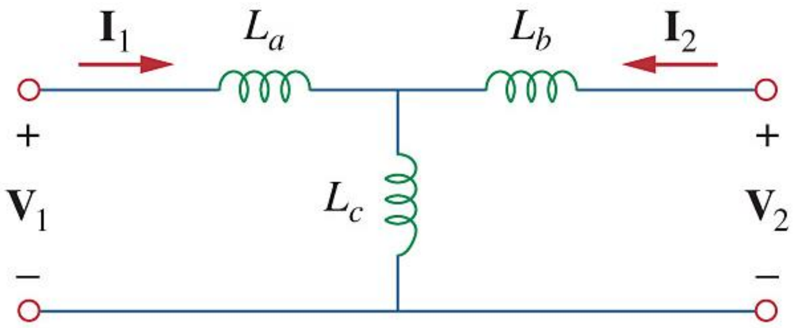
inverse of A

determinant

Inversing the matrix, then we get

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{L_2}{j\omega(L_1L_2 - M^2)} & \dfrac{-M}{j\omega(L_1L_2 - M^2)} \\ \dfrac{-M}{j\omega(L_1L_2 - M^2)} & \dfrac{L_1}{j\omega(L_1L_2 - M^2)} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

- **T circuit** without magnetic coupling



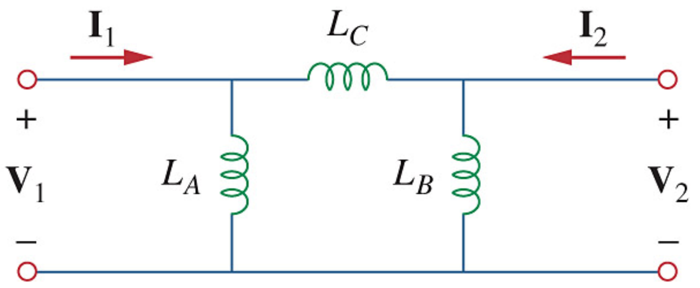
$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_a + L_c) & j\omega L_c \\ j\omega L_c & j\omega(L_b + L_c) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

Our goal is to find an equivalent circuit. The two matrix must be identical.

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_a + L_c) & j\omega L_c \\ j\omega L_c & j\omega(L_b + L_c) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

$$\Rightarrow L_a = L_1 - M, \; L_b = L_2 - M, \; L_c = M$$

- **π circuit** without magnetic coupling



$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{j\omega L_A} + \dfrac{1}{j\omega L_C} & -\dfrac{1}{j\omega L_C} \\ -\dfrac{1}{j\omega L_C} & \dfrac{1}{j\omega L_B} + \dfrac{1}{j\omega L_C} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

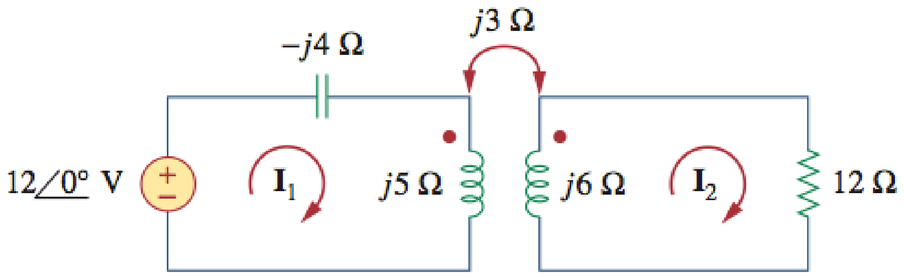
Our goal is to find an equivalent circuit. The two matrix must be identical.

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{L_2}{j\omega(L_1L_2 - M^2)} & \dfrac{-M}{j\omega(L_1L_2 - M^2)} \\ \dfrac{-M}{j\omega(L_1L_2 - M^2)} & \dfrac{L_1}{j\omega(L_1L_2 - M^2)} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{j\omega L_A} + \dfrac{1}{j\omega L_C} & -\dfrac{1}{j\omega L_C} \\ -\dfrac{1}{j\omega L_C} & \dfrac{1}{j\omega L_B} + \dfrac{1}{j\omega L_C} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$$L_A = \frac{L_1L_2 - M^2}{L_2 - M}, \ L_B = \frac{L_1L_2 - M^2}{L_1 - M} \quad L_C = \frac{L_1L_2 - M^2}{M}$$

Changing the locations of the dots can cause M to become –M. A negative value of *M* is physically unrealizable but **the equivalent model is still mathematically valid**.

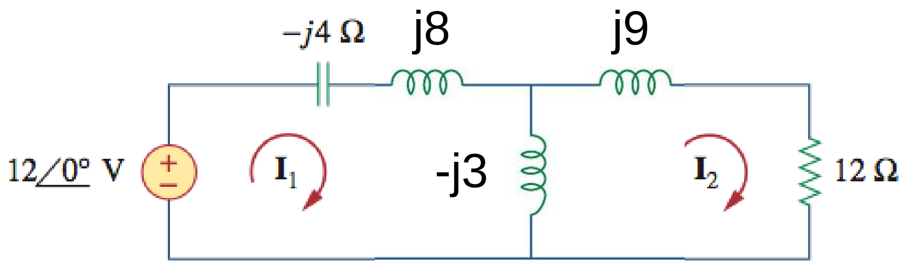
Revisit Example 13.1



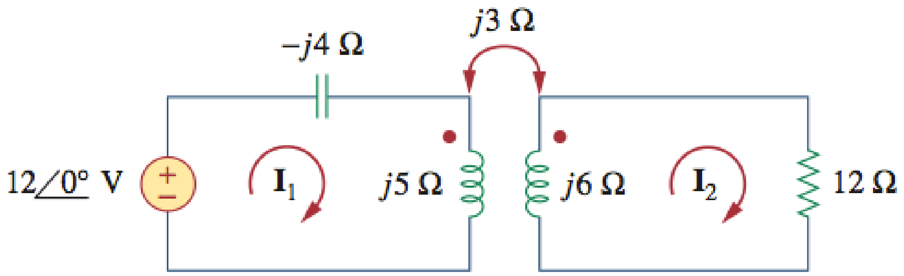
M < 0
$L_1 = 5$
$L_2 = 6$
M = -3

$L_a = 8$
$L_b = 9$
$L_c = -3$

$$L_a = L_1 - M, L_b = L_2 - M, L_c = M$$



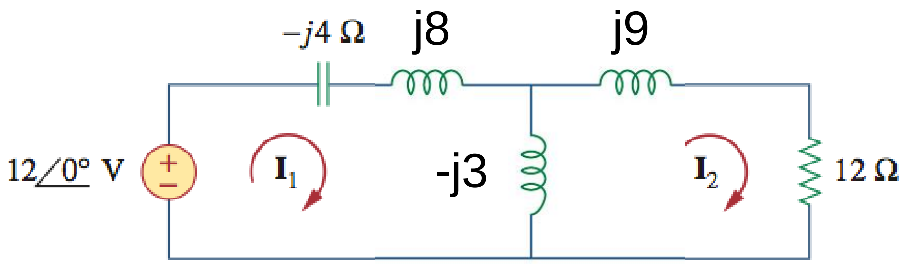
Revisit Example 13.1



M < 0
$L_1 = 5$
$L_2 = 6$
M = -3

$L_a = 8$
$L_b = 9$
$L_c = -3$

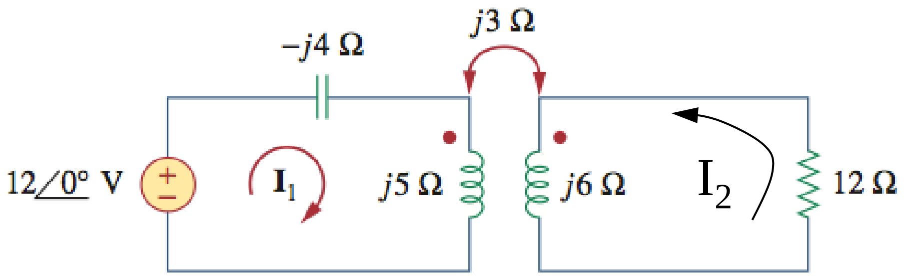
$$L_a = L_1 - M,\ L_b = L_2 - M,\ L_c = M$$



Loop 1: $-12 + j4I_1 - j3(I_1 - I_2) = 0 \ \rightarrow I_1 = -3I_2 - j12$
Loop 2: $(12 + \mathrm{j}9)I_2 - j3(I_2 - I_1) = 0 \rightarrow j3I_1 + (12 + j6)I_2 = 0$

Put I_1 into the loop 2, then we get I_2

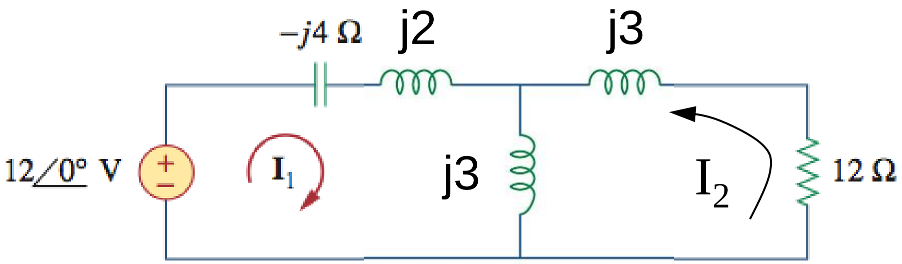
$I_2 = \frac{-36}{3(4-j)} = 2.91\angle - 165.96°$ (?)

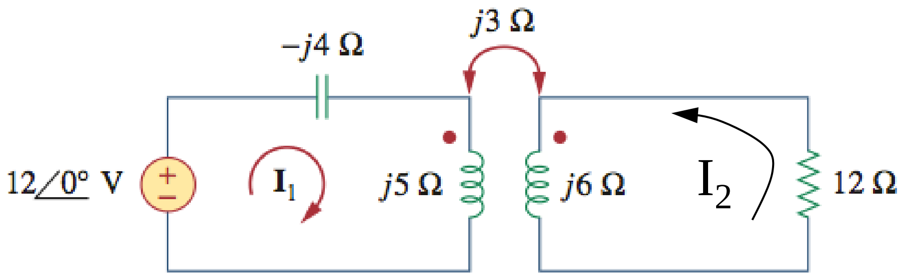


M > 0
$L_1 = 5$
$L_2 = 6$
M = 3

$L_a = 2$
$L_b = 3$
$L_c = 3$

$$L_a = L_1 - M,\ L_b = L_2 - M,\ L_c = M$$

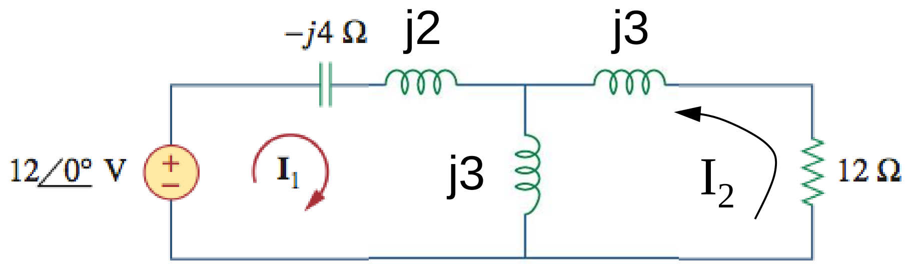




M > 0
$L_1 = 5$
$L_2 = 6$
M = 3

$L_a = 2$
$L_b = 3$
$L_c = 3$

$$L_a = L_1 - M,\ L_b = L_2 - M,\ L_c = M$$



Loop 1: $-12 - j2I_1 + j3(I_1 + I_2) = 0 \rightarrow I_1 = -3I_2 - j12$
Loop 2: $(12 + j3)I_2 + j3(I_1 + I_2) = 0 \rightarrow j3I_1 + (12 + j6)I_2 = 0$

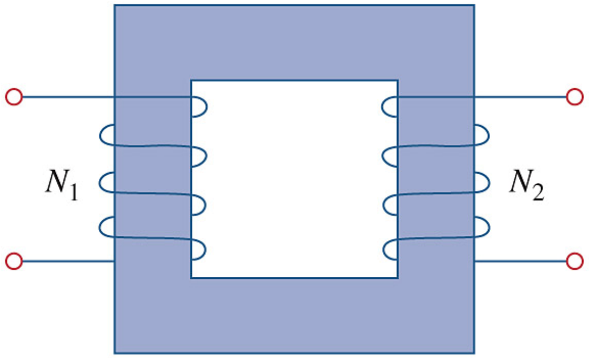
Put I_1 into the loop 2, then we get I_2

$$I_2 = \frac{-36}{3(4-j)} = 2.91\angle -165.96°$$
$$I_1 = -3(2.91\angle -165.96°) - j12 = 8.47 - j9.88 = 13.01\angle -49.39°$$

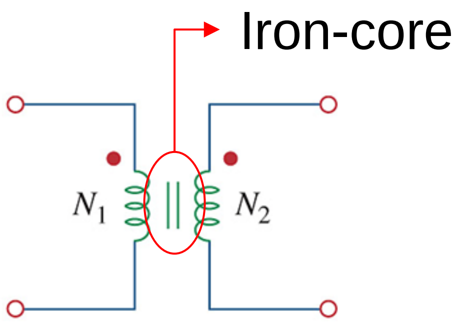
13.5 Ideal Transformers

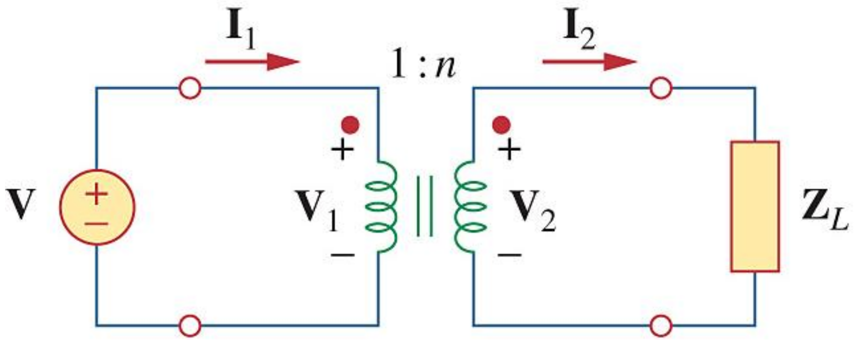
An ideal transformer is one with **perfect coupling (k = 1)**, **lossless ($R_1 = R_2 = 0$)** transformer in which the primary and secondary coils have very large, or infinite inductance, **($L_1 = \infty$, $L_2 = \infty$, $M = \infty$)**. Iron-core transformers are close approximations to ideal transformers.

Primary:
N_1 turns



Secondary:
N_2 turns





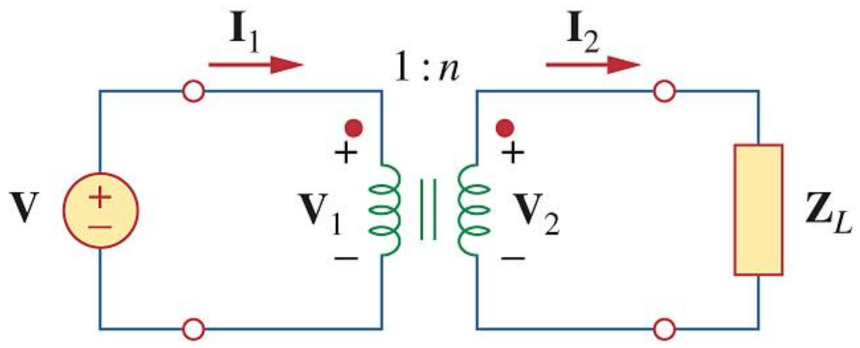
When a sinusoidal voltage is applied to the primary winding, the same magnetic flux Ø goes through both windings. According to Faraday's law,

$$v_1 = N_1 \frac{d\phi}{dt}$$

$$v_2 = N_2 \frac{d\phi}{dt}$$

Ideal transformer:
Iron coils enable the same ϕ

Non-ideal: $\phi_1 = \phi_{11} + \phi_{12}$
A portion of magnetic flux (ϕ_{12}) is coupled.



$$v_1 = N_1 \frac{d\phi}{dt}$$

$$v_2 = N_2 \frac{d\phi}{dt}$$

$$\frac{v_2}{v_1} = \frac{N_2}{N_1} = n$$

where n is the *turns ratio* or *transformation ratio*. Using the phasor voltages,

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{N_2}{N_1} = n$$

For the reason of power conservation, the energy **supplied** to the primary must equal the energy **absorbed** by the secondary because there are no losses in an ideal transformer.

$$v_1 i_1 = v_2 i_2$$

Changing the form into the phasor, we get the relationship

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{N_1}{N_2} = \frac{1}{n}$$

i) n = 1, an isolation transformer
ii) n > 1, a step-up transformer, $V_2 > V_1$
iii) n < 1, a step-down transformer, $V_2 < V_1$

The ratings of transformers are usually specified as $^{V_1}/_{V_2}$, e.g. 2400/120 V – 2400 V on the primary and 120 V in the secondary.

*Keep in mind that the voltage ratings are in **rms**

Polarity of turns ratio *n*

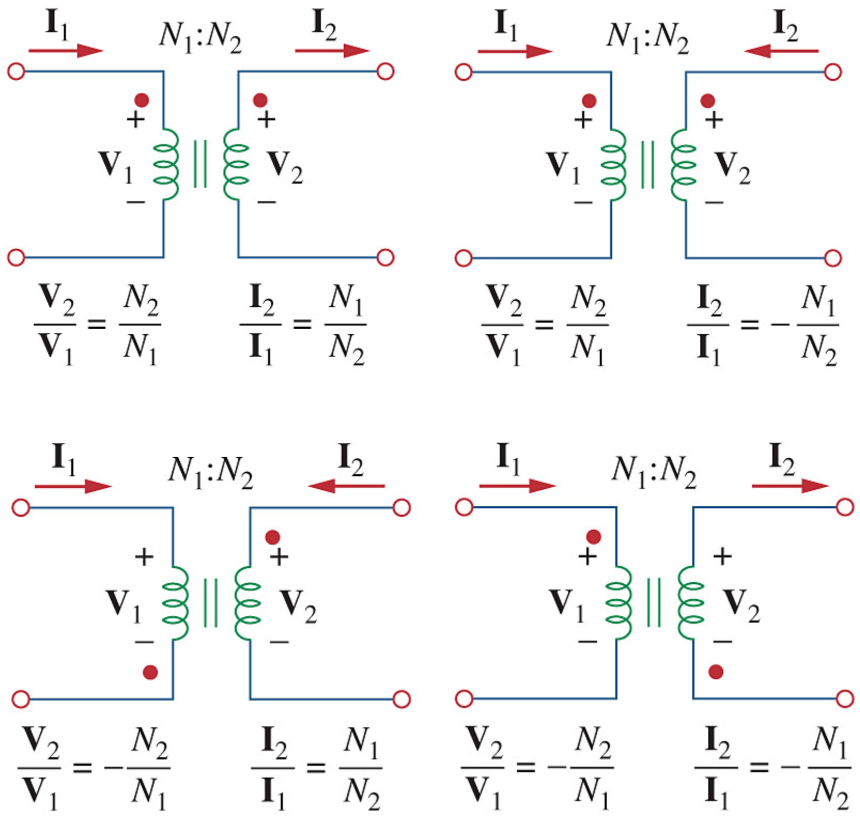
It is important that we know how to get the proper polarity of the voltages and the direction of the currents for the transformer. If the polarity of $\mathbf{V}_1$ or $\mathbf{V}_2$ or the direction of $\mathbf{I}_1$ or $\mathbf{I}_2$ is changed, ***n* may need to be replaced by –*n***.

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{N_2}{N_1} = n$$

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{N_1}{N_2} = \frac{1}{n}$$

$$\mathbf{V}_1 = \frac{\mathbf{V}_2}{n} \quad \text{or} \quad \mathbf{V}_2 = n\mathbf{V}_1$$

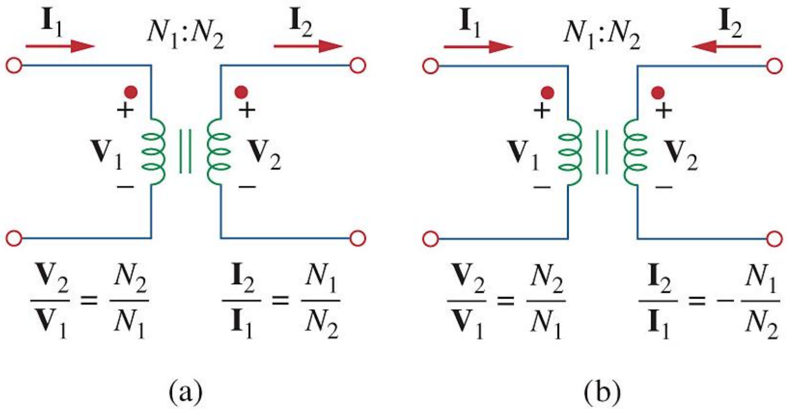
$$\mathbf{I}_1 = n\mathbf{I}_2 \quad \text{or} \quad \mathbf{I}_2 = \frac{\mathbf{I}_1}{n}$$



How to determine

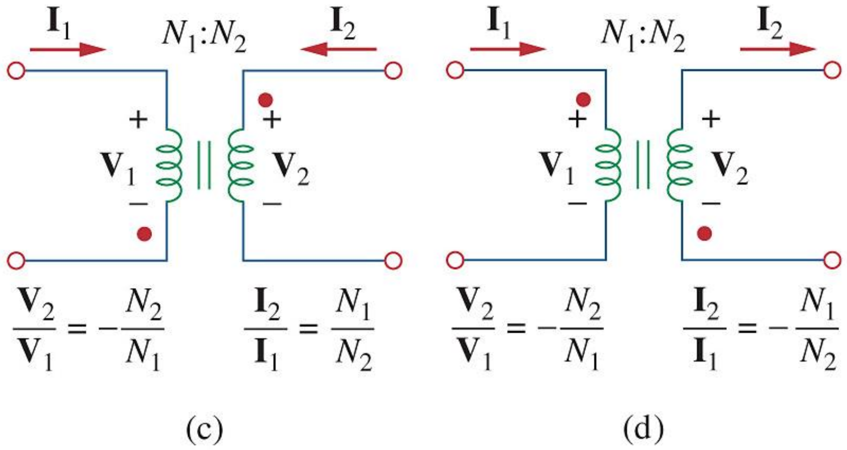
1. If $\mathbf{V_1}$ and $\mathbf{V_2}$ are **both positive** or **both negative** at the dotted terminals, **use +n**. Otherwise, use –n.

2. If $\mathbf{I_1}$ and $\mathbf{I_2}$ **both enter** into or **both leave** the dotted terminals, **use –n**. Otherwise, use +n.



(a) Voltages: both positive at the dotted terminals → +n
Currents: I_1 enters whereas I_2 leaves the dotted terminal → +n

(b) Voltages: both positive at the dotted terminals → +n
Currents: Both enter the dotted terminals → -n



(c) Voltages: **NOT** both positive at the dotted terminals → -n
Currents: I_1 leaves whereas I_2 enters the dotted terminal → +n

(d) Voltages: **NOT** both positive at the dotted terminals → -n
Currents: Both enter the dotted terminals → -n

For V, same polarity at dotted terminals ➔ +n
For I, same flowing direction (entering/leaving) the dotted terminals ➔ –n

Complex power

The complex power in the primary winding is equal to the complex power in the secondary winding:

$$\mathbf{S}_1 = \mathbf{V}_1\mathbf{I}_1^* = \frac{\mathbf{V}_2}{n}(n\mathbf{I}_2)^* = \mathbf{V}_2\mathbf{I}_2^* = \mathbf{S}_2$$

⇨ $S_1 = S_2$ Same apparent power

The complex power supplied to the primary is delivered to the secondary without loss. The transformer absorbs no power.

Ideal transformer:
1. k=1
2. $R_1=R_2=0$
3. $L_1, L_2, M \rightarrow \infty$

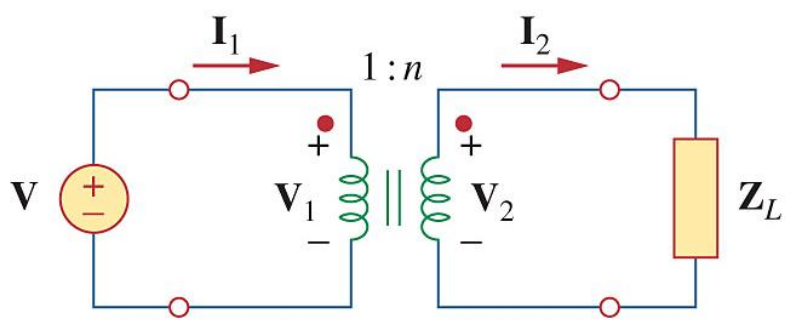
Input impedance

The input impedance as seen by the source is

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{1}{n^2}\frac{\mathbf{V}_2}{\mathbf{I}_2}$$

where $\mathbf{V}_2/\mathbf{I}_2 = \mathbf{Z}_L$

$$\mathbf{Z}_{\text{in}} = \frac{\mathbf{Z}_L}{n^2}$$



The input impedance is also called the **reflected impedance**, since it appears as if the load impedance is reflected to the primary side.

Practice Problem 13.7 The primary current to an ideal transformer rated at 3300/110 V is 5 A. Calculate: (a) the turns ratio, (b) the kVA rating, (c) the secondary current.

Solution :

(a) $n = \dfrac{V_2}{V_1} = \dfrac{110}{3300} = \dfrac{1}{30}$

(b) $|S| = V_1 I_1 = 3300 \times 5 = 16500 \text{ (VA)}$

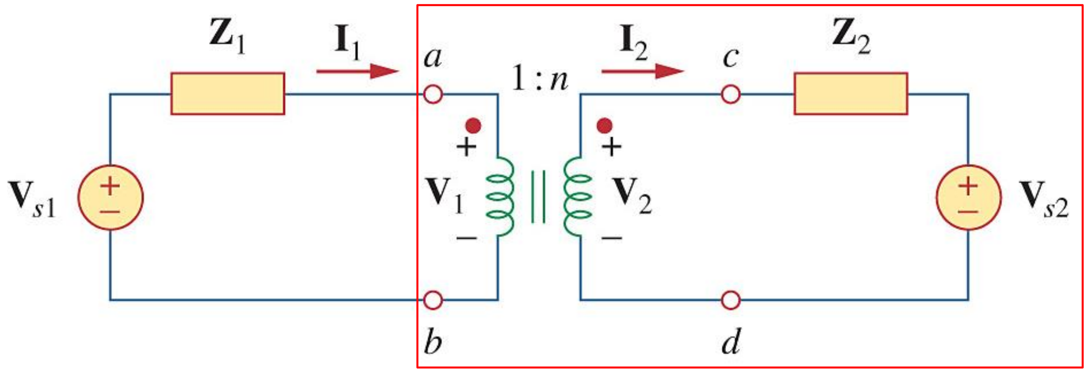
$= 16.5 \text{ kVA}$

(c) $\dfrac{I_1}{I_2} = n \Rightarrow I_2 = \dfrac{I_1}{n} = \dfrac{5}{1/30} = 150 \text{ (A)}$

Simplifying transformer circuit

It is common practice in analyzing the circuit to **eliminate the transformer by reflecting impedances and sources** from one side of the transformer to the other.

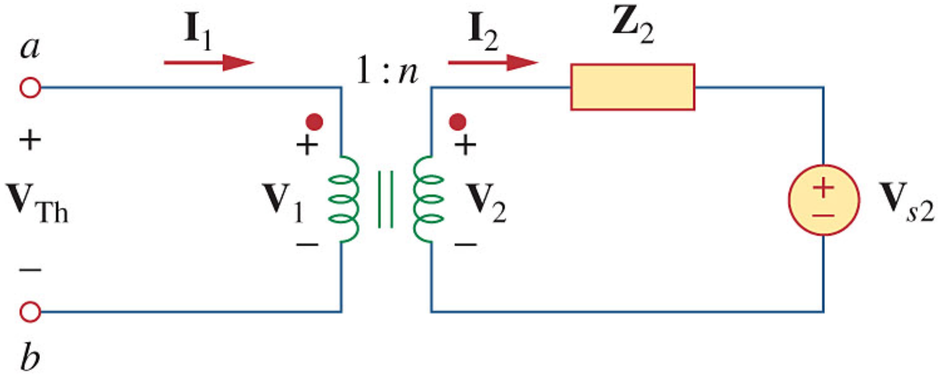
e.g. we want to reflect the secondary side of the circuit to the primary side



Find the Thevenin equivalent circuit

Thevenin equivalent circuit

(i) V_{Th}



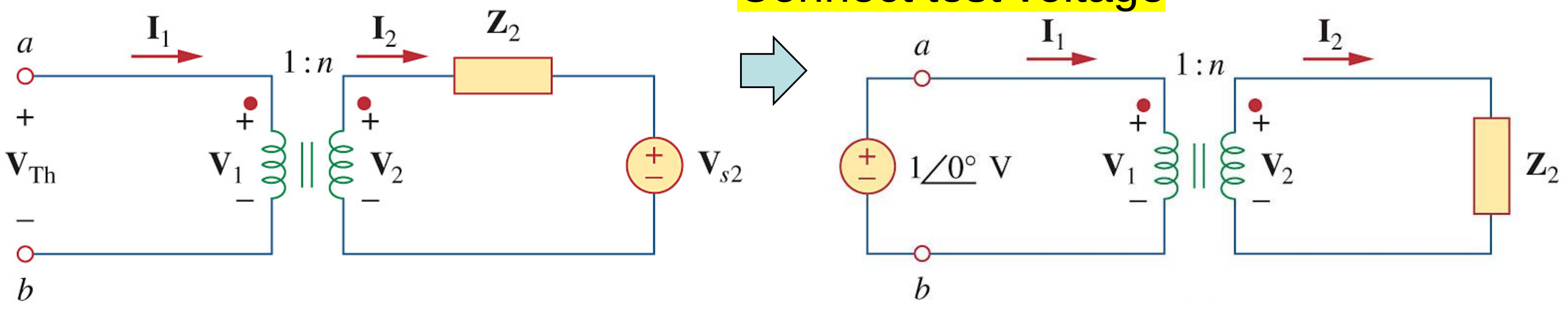
Terminals a-b are open, therefore, $\mathbf{I}_1 = 0 = \mathbf{I}_2 \rightarrow \mathbf{V}_2 = \mathbf{V}_{s2}$

Using the turns ratio, $\mathbf{V}_2 = n\mathbf{V}_1$

Thevenin voltage is $\mathbf{V}_{\text{Th}} = \mathbf{V}_1 = \dfrac{\mathbf{V}_2}{n} = \dfrac{\mathbf{V}_{s2}}{n}$

(ii) Z_{Th}

Remove independent source
Connect test voltage



$$\mathbf{Z}_{Th} = \frac{\mathbf{V}_1}{\mathbf{I}_1}$$

Using the turns ratio, $\mathbf{V}_1 = \dfrac{\mathbf{V}_2}{n}$ $\quad \mathbf{I}_1 = n\mathbf{I}_2$

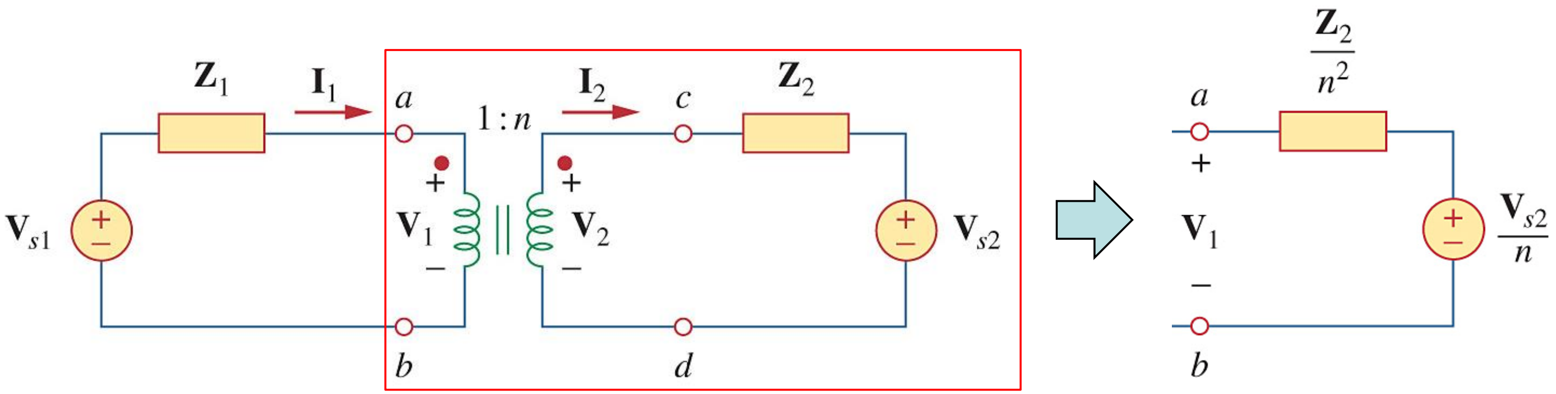
We get Thevenin impedance Z_{Th}

$$\mathbf{Z}_{Th} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{\mathbf{V}_2/n}{n\mathbf{I}_2} = \frac{\mathbf{Z}_2}{n^2}, \qquad \mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}_2$$

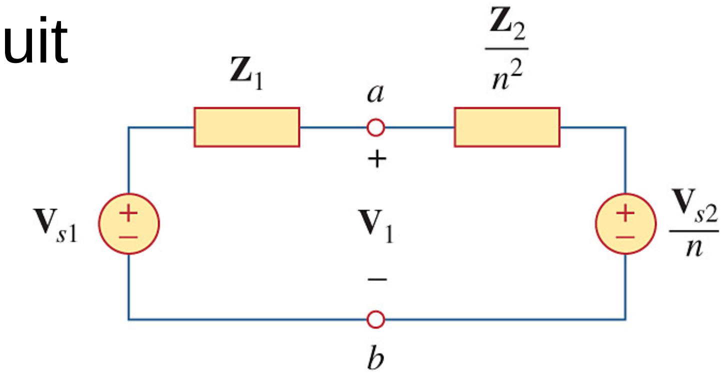
$$\mathbf{V}_{\text{Th}} = \mathbf{V}_1 = \frac{\mathbf{V}_2}{n} = \frac{\mathbf{V}_{s2}}{n}$$

$$\mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{\mathbf{V}_2/n}{n\mathbf{I}_2} = \frac{\mathbf{Z}_2}{n^2}, \qquad \mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}_2$$

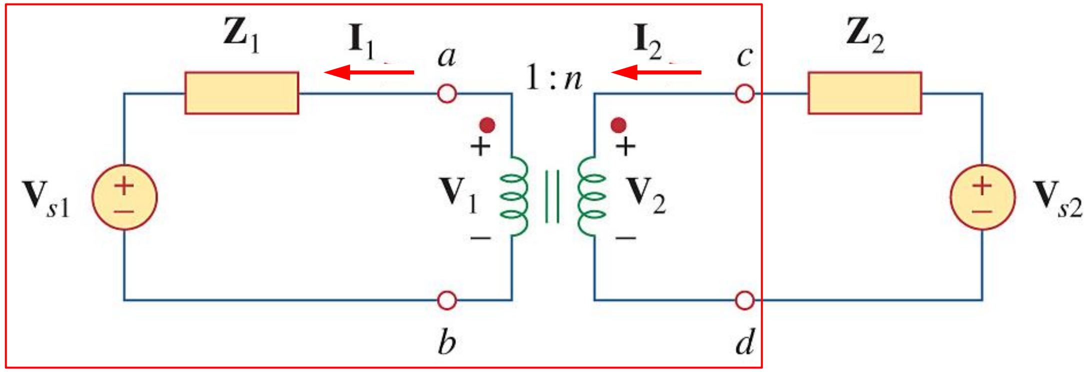
Thevenin equivalent circuit is thus



Simplified transformer circuit



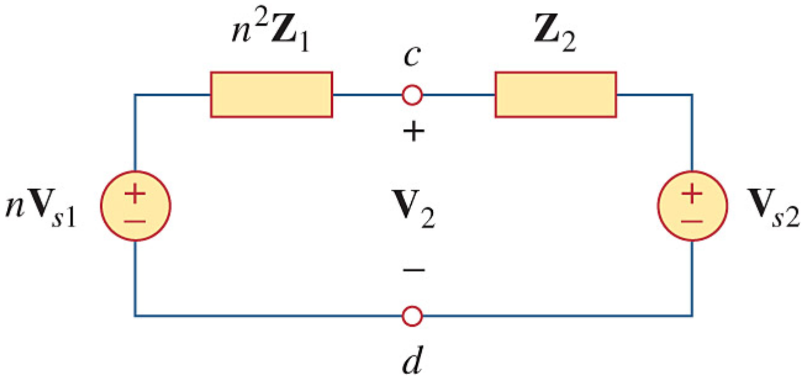
Thevenin equivalent circuit (c-d terminals)



i) $V_{Th} = V_2 = nV_1 = nV_{s1}$

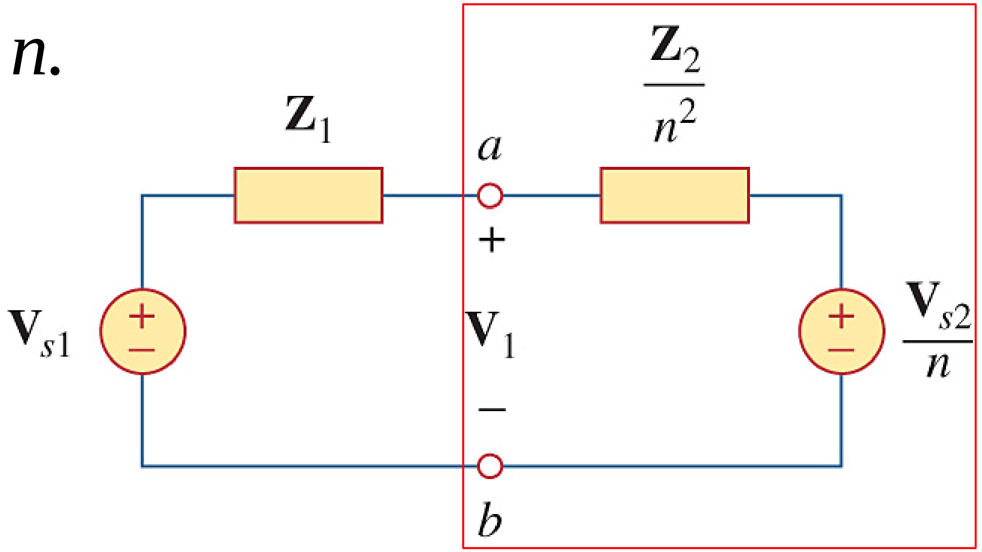
ii) $Z_{Th} = \frac{V_2}{I_2} = \frac{nV_1}{I_1/n} = n^2\frac{V_1}{I_1} = n^2 Z_1$

Simplified transformer circuit

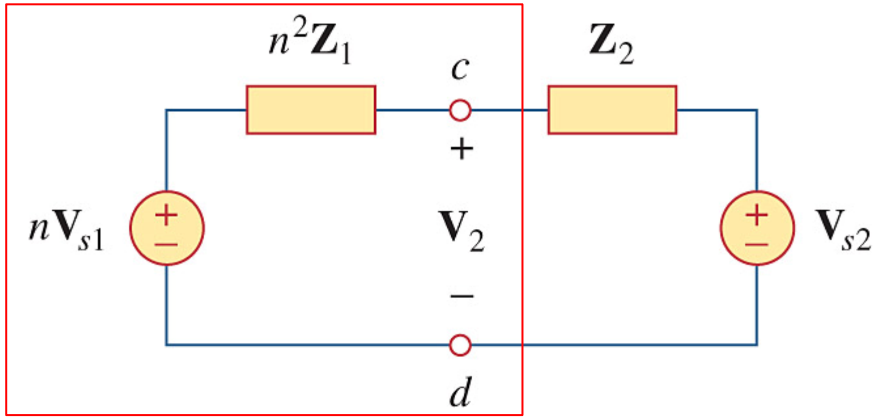


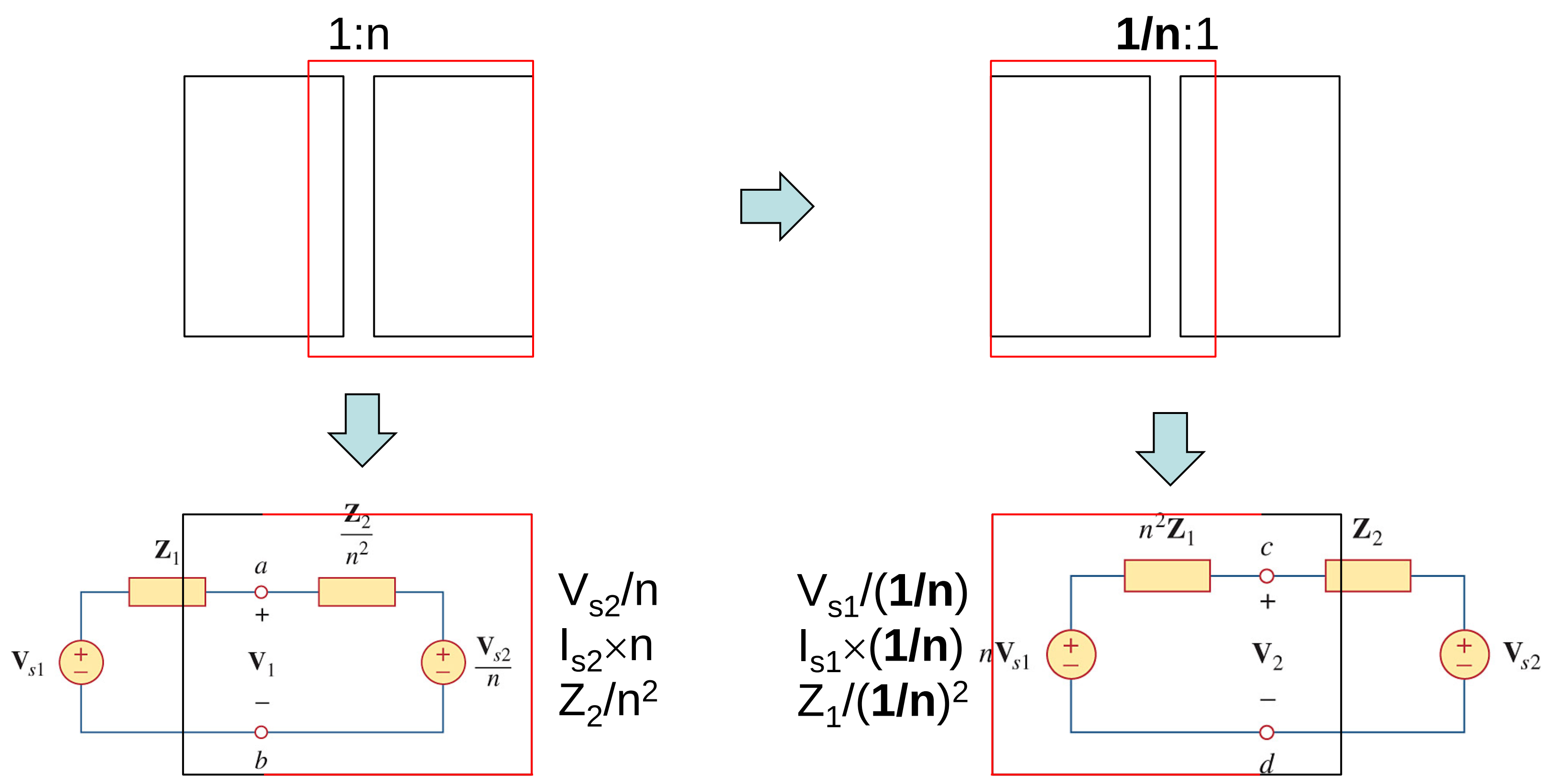
Thevenin equivalent circuit summary

The general rule for eliminating the transformer by reflecting the secondary circuit to the primary side is: divide the secondary impedance by n^2, divide the secondary voltage by n, and multiply the secondary current by n.



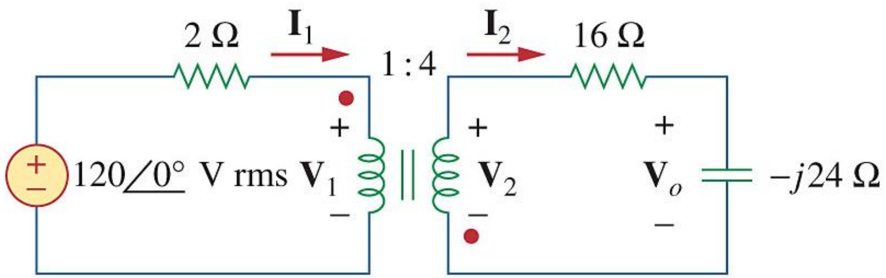
The general rule for eliminating the transformer by reflecting the primary circuit to the secondary side is: multiply the primary impedance by n^2, multiply the primary voltage by n, and divide the primary current by n.



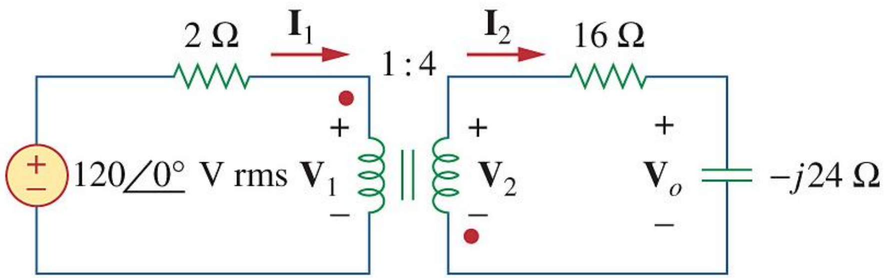


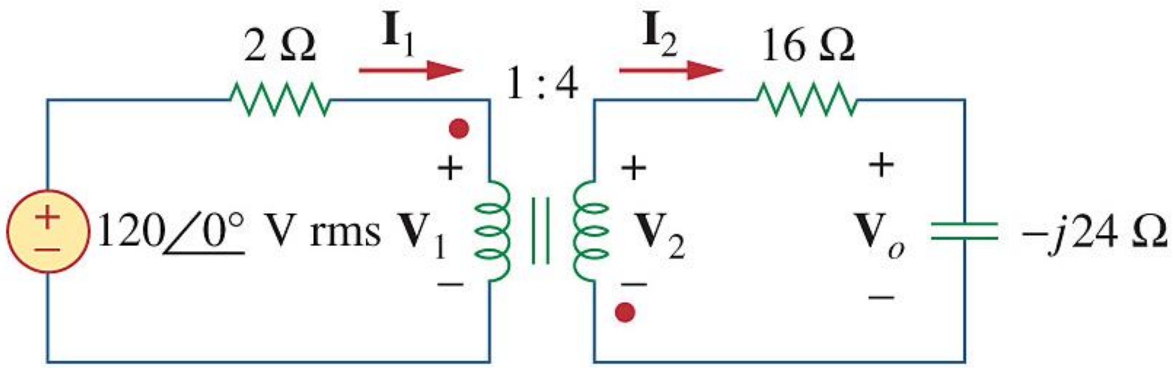
This reflection approach **only applies if there are no external connections** between the primary and secondary windings. When we have external connections between the primary and secondary windings, we simply use **regular mesh and nodal analysis**.

Practice Problem 13.8 In the ideal transformer circuit of Fig. 13.38, find $\tilde{V}_o$ and the complex power supplied by the source.



Practice Problem 13.8 In the ideal transformer circuit of Fig. 13.38, find $\tilde{V}_o$ and the complex power supplied by the source.





Solution :

$$Z_R = \frac{16 - j24}{4^2} = 1 - j1.5 \text{ (W)}$$

$$\tilde{I}_1 = \frac{120\angle 0^\circ}{2 + Z_R} = \frac{120\angle 0^\circ}{2 + (1 - j1.5)}$$

$$\approx \frac{120\angle 0^\circ}{3.3541\angle -26.57^\circ}$$

$$\approx 35.7771\angle 26.57^\circ \text{ (A)}$$

$$\tilde{I}_2 = -\frac{\tilde{I}_1}{n} = -\frac{35.7771\angle 26.57^\circ}{4}$$

$$\approx 8.9443\angle 206.57^\circ \text{ (A)}$$

$$\tilde{V}_o = \tilde{I}_2 \times (-j24)$$

$$= 8.9443\angle 206.57^\circ \times 24\angle -90^\circ$$

$$= 214.6632\angle 116.57^\circ \text{ (V)}$$

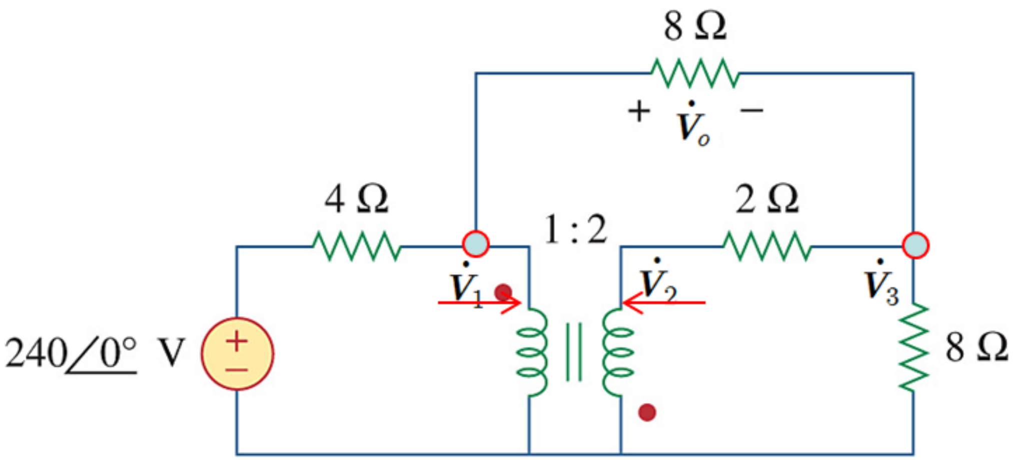
$$S = 120\angle 0^\circ \times \tilde{I}_1^*$$

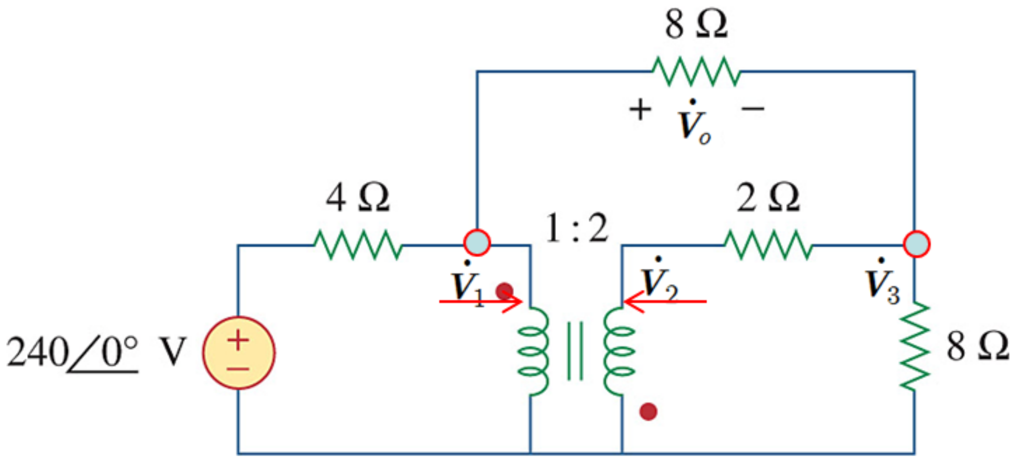
$$= 120\angle 0^\circ \times 35.7771\angle -26.57^\circ$$

$$\approx 4293.252\angle -26.57^\circ \text{ (VA)}$$

$$\approx 4.2933\angle -26.57^\circ \text{ kVA}$$

Practice Problem 13.9 Find $\tilde{V}_o$ in the circuit of Fig. 13.40.





Solution :

$$\frac{\tilde{V}_1 - 240\angle 0°}{4} + \frac{\tilde{V}_1 - \tilde{V}_3}{8} + \tilde{I}_1 = 0 \qquad (1)$$

$$\frac{\tilde{V}_3 - \tilde{V}_1}{8} + \tilde{I}_2 + \frac{\tilde{V}_3}{8} = 0 \qquad (2)$$

$$\tilde{I}_2 = \frac{\tilde{V}_3 - \tilde{V}_2}{2} \qquad (3)$$

$$\tilde{V}_2 = -2\tilde{V}_1 \qquad (4)$$

$$\tilde{I}_1 = 2\tilde{I}_2 \qquad (5)$$

$$\tilde{V}_2 = -2\tilde{V}_1 \qquad (4)$$

$$\tilde{I}_1 = 2\tilde{I}_2 \qquad (5)$$

Ideal transformer

From (3) and (4),

$$\tilde{I}_2 = \frac{\tilde{V}_3}{2} - \frac{\tilde{V}_2}{2} = \frac{\tilde{V}_3}{2} + \tilde{V}_1 \qquad (6)$$

From (5) and (6),

$$\tilde{I}_1 = \tilde{V}_3 + 2\tilde{V}_1 \qquad (7)$$

Substitute (7) in (1),

$$\frac{\tilde{V}_1 - 240}{4} + \frac{\tilde{V}_1 - \tilde{V}_3}{8} + \tilde{V}_3 + 2\tilde{V}_1 = 0 \Rightarrow$$

$$19\tilde{V}_1 + 7\tilde{V}_3 = 480 \qquad (8)$$

Substitute (6) in (2),

$$7\tilde{V}_1 + 6\tilde{V}_3 = 0 \qquad (9)$$

From (8) and (9),

$$\tilde{V}_1 = \frac{576}{13}\ (\text{V}),\ \tilde{V}_3 = -\frac{672}{13}\ (\text{V}) \Rightarrow$$

$$\tilde{V}_o = \tilde{V}_1 - \tilde{V}_3 = 96\ (\text{V}) = 96\angle 0^\circ\ (\text{V})$$