ECE2150J Introduction to Circuits
Chapter 8. Second order circuit

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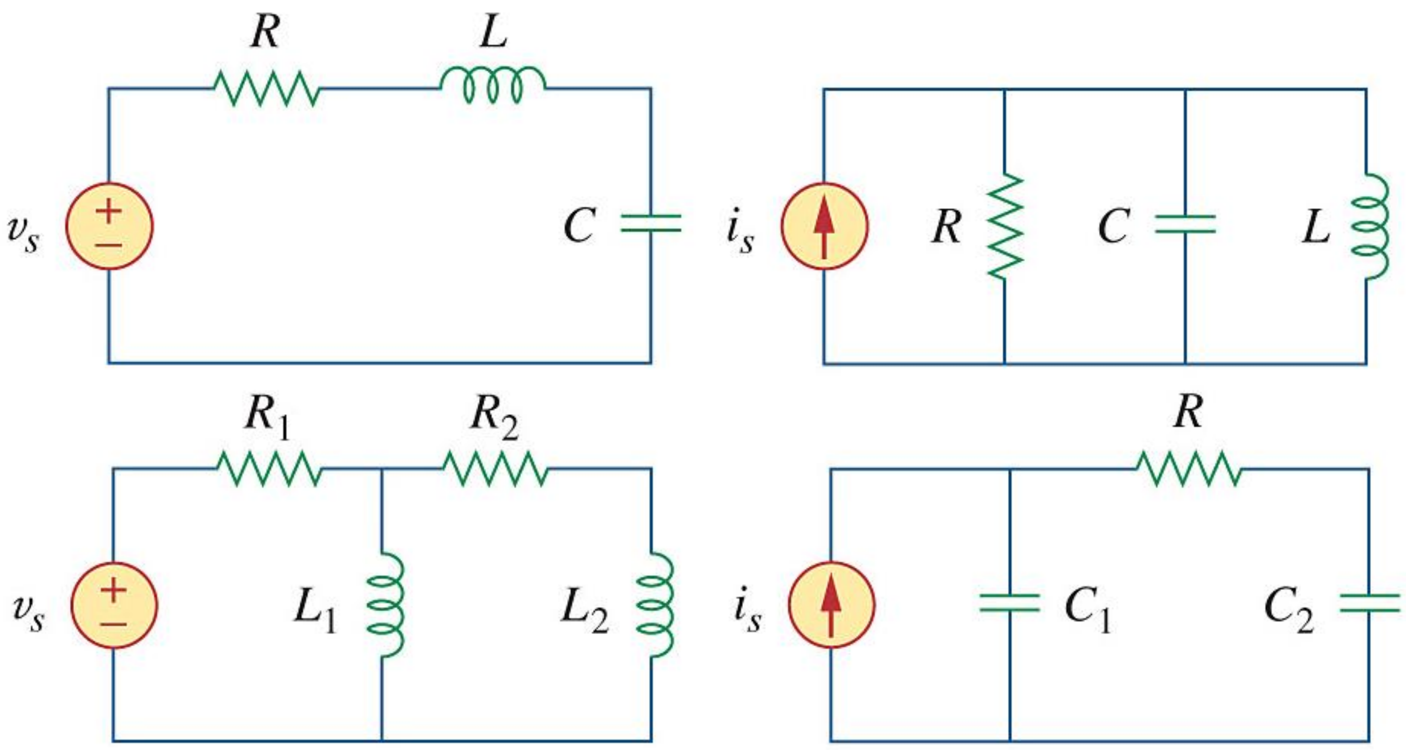
8.1 Introduction

In this chapter, we consider circuits containing **two storage elements**, known as second-order circuits.



8.2 Finding Initial and Final Values

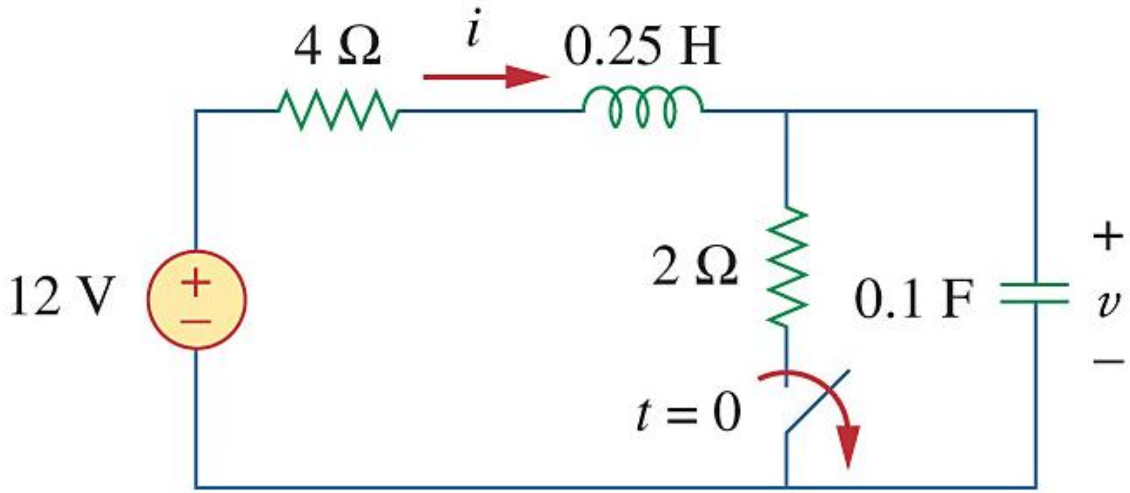
Find the initial and final values, not only ***v*** and ***i*** but also their derivatives ***dv/dt*** and ***di/dt***.

There are **two key points** to keep in mind in determining the initial conditions.

First, the polarity of V_C and I_L: the passive sign convention.
Second, keep in mind that the V_C and I_L are always continuous.

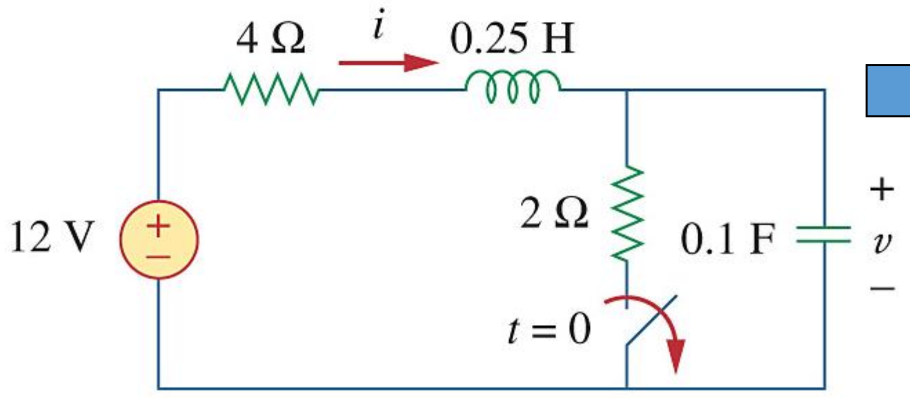
$$v(0^+) = v(0^-) \quad i(0^+) = i(0^-)$$

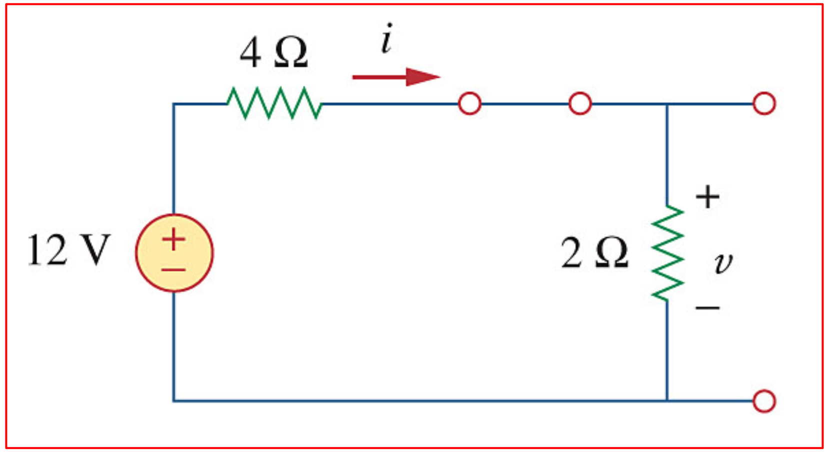
Example 8.1 The switch in Fig. 8.2 has been closed for a long time. It is open at $t = 0$. Find: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.



Example 8.1

(i) t=0^{-}



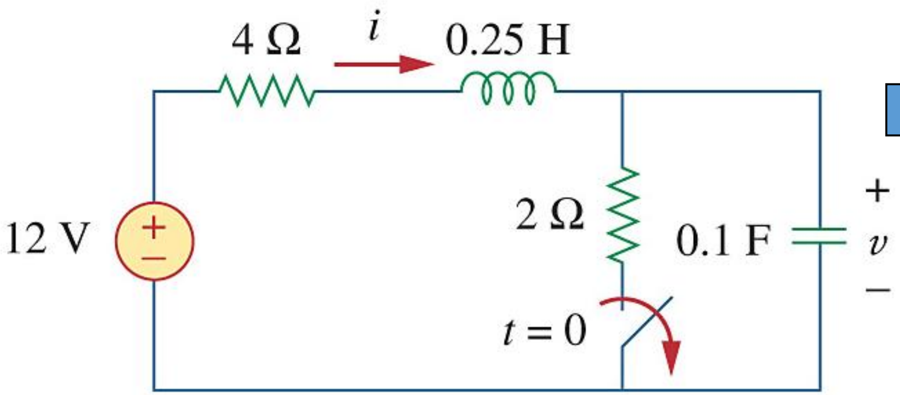


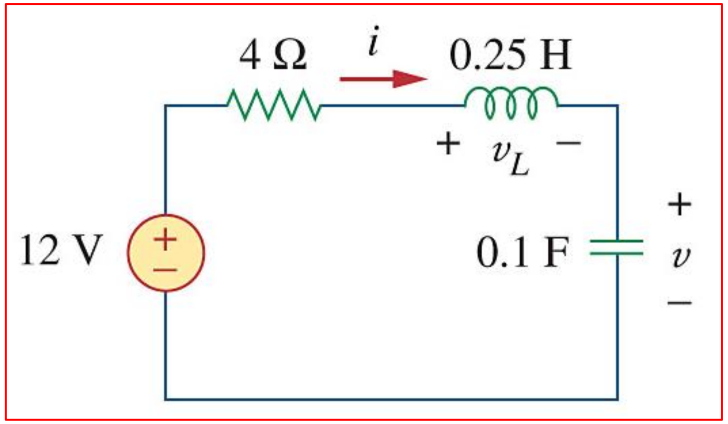
Before t=0, at DC steady state: An equivalent circuit

$$i(0^{+}) = i(0^{-}) = \frac{12}{4+2} = 2\ (\text{A})$$

$$v(0^{+}) = v(0^{-}) = 2i(0^{-}) = 4\ (\text{V})$$

(ii) t=0$^+$





After t=0, switch opens: An equivalent circuit

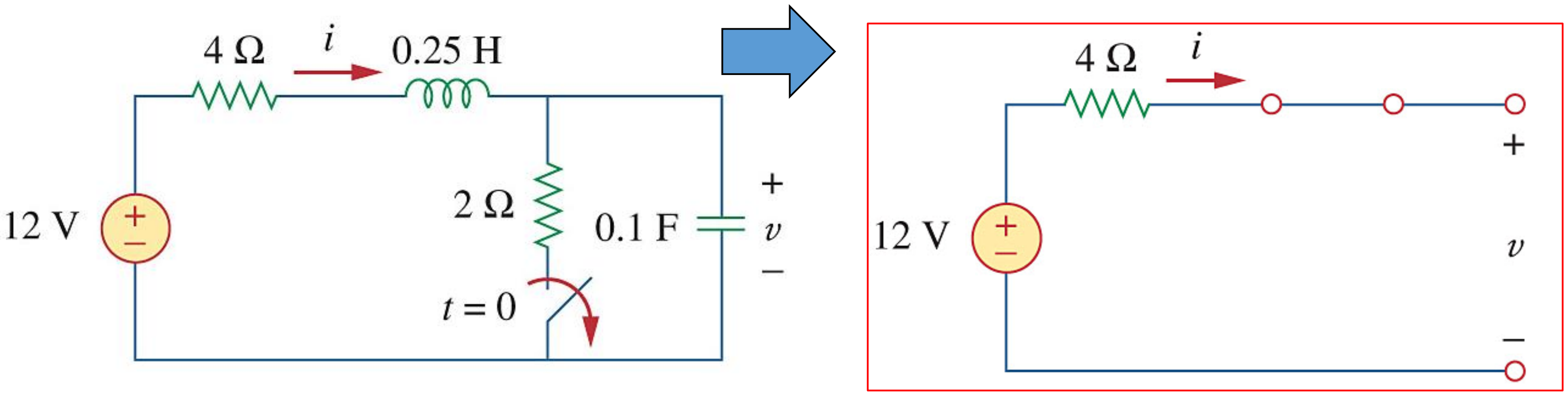
$$\begin{cases} i = 0.1\dfrac{dv}{dt} \\ 12 = 4i + 0.25\dfrac{di}{dt} + v \end{cases} \Rightarrow \begin{cases} \dfrac{dv}{dt} = \dfrac{i}{0.1} \\ \dfrac{di}{dt} = \dfrac{12 - 4i - v}{0.25} \end{cases}$$

$$dv(0^+)/dt = i(0^+)/0.1 = 2/0.1 = 20 \text{ (V/s)}$$

$$di(0^+)/dt = \left[12 - 4i(0^+) - v(0^+)\right]/0.25$$

$$= [12 - 4 \times 2 - 4]/0.25 = 0 \text{ (A/s)}$$

(iii) t → ∞



t= ∞, at DC steady state: An equivalent circuit

$$i(\infty) = 0$$

$$v(\infty) = 12 \text{ (V)}$$

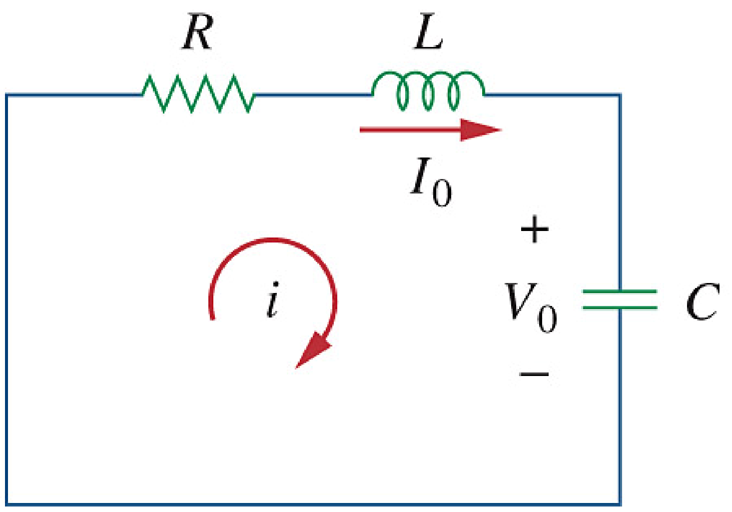
8.3 The Source-Free Series *RLC* Circuit

The natural response of the series *RLC* circuit. The circuit is being excited by the energy **initially stored** in the capacitor and inductor.

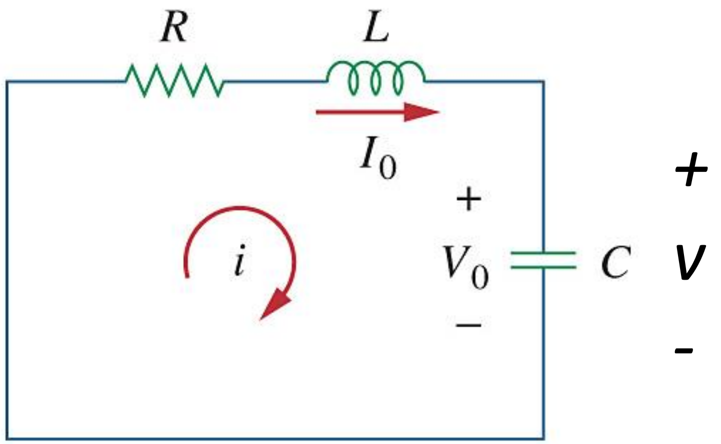
at t = 0

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0$$

$$i(0) = I_0$$



Second Order Equation

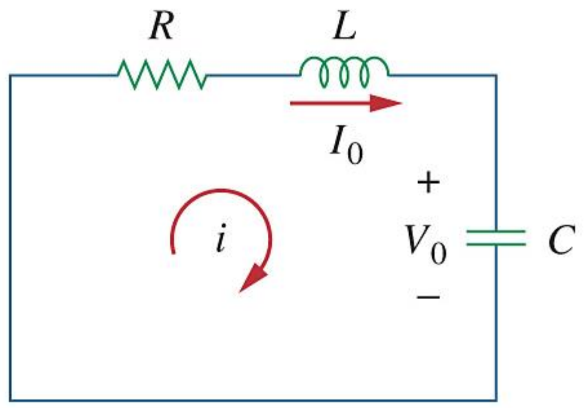


By KVL:

$$iR + L\frac{di}{dt} + v = 0 \Rightarrow iR + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0$$ Differentiate the equation

$$\frac{di}{dt}R + L\frac{d^2 i}{dt^2} + \frac{1}{C}i = 0 \Rightarrow \frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

Second Order Equation – Initial conditions



$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i=0$$

To solve a second-order differential equation, two initial conditions are required:
i or v and its first derivative

(i) $i(0^+)=i(0^-)=I_0$

(ii) From $iR+L\frac{di}{dt}+v=0 \Rightarrow$

$$i'(0^+)=-\frac{1}{L}\left(i(0^+)R+v(0^+)\right)$$
$$=-\frac{1}{L}\left(i(0^-)R+v(0^-)\right)$$
$$=-\frac{1}{L}\left(I_0R+V_0\right)$$

Second Order Equation – Characteristic Eq.

Our experience in the preceding chapter on first-order circuits suggests that the solution is of exponential form. So we use a test solution $\boldsymbol{i = Ae^{st}}$ where A and s are constants to be determined.

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0 \Rightarrow As^2 e^{st} + \frac{AR}{L} s e^{st} + \frac{A}{LC} e^{st} = 0$$

$$Ae^{st}\boxed{\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} = 0$$

only this part can become 0

Characteristic equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Second Order Equation – Characteristic Eq.

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

*Note

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s^2 + \frac{R}{L} s + \frac{1}{LC} = 0$$

$$s = \frac{-R/L \pm \sqrt{(R/L)^2 - 4 \times 1 \times \left(1/(LC)\right)}}{2 \times 1}$$

$$= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$ where $\alpha = \frac{R}{2L}$ *and* $\omega_o = \frac{1}{\sqrt{LC}}$

Solutions (S_1 and S_2) for the equation $$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2},\ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} :$$

natural frequencies, Np/s

$\alpha = \dfrac{R}{2L}$: neper frequency (damping factor), Np/s (nepers per second)

$\omega_0 = \dfrac{1}{\sqrt{LC}}$: resonant frequency (undamped natural frequency), rad/s

Second Order Equation – Solution *i*

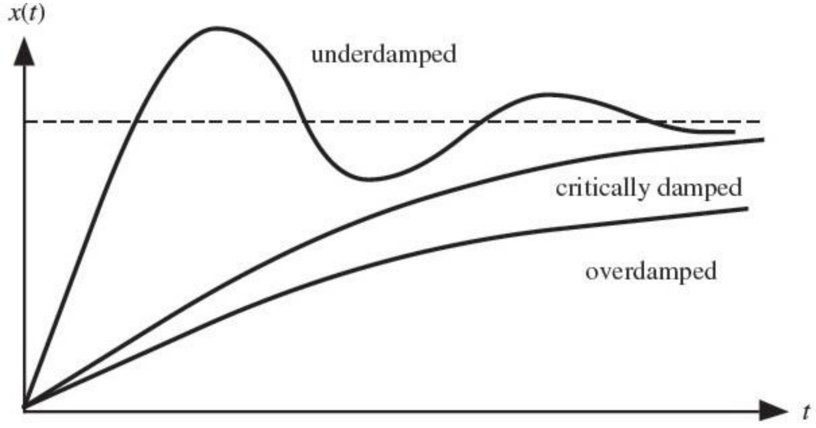
There are two possible **solution for *i***
$i_1 = A_1 e^{s_1 t}$ and $i_2 = A_2 e^{s_2 t}$

A complete or total solution would therefore require **a linear combination of *i_1* and *i_2***.

Thus, the natural response of the series RLC circuit is
$\boldsymbol{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$, find A_1 and A_2 from the initial values.

Second Order Equation – Solution S_1 and S_2

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$



Three types of solutions:

1. $\alpha > \omega_0$, overdamped case: system moves slowly toward equilibrium.
2. $\alpha = \omega_0$, critically damped case: system moves quickly to equilibrium, but will oscillate around the equilibrium point.
3. $\alpha < \omega_0$, underdamped case: the system will oscillate, but its amplitude gradually decreases until it rests.

Second Order Equation – Solution S_1 *and* S_2

Case 1: Overdamped

α > ω_0 implies $C > \frac{4L}{R^2}$ $\left(\alpha = \frac{R}{2L} > \omega_o = \frac{1}{\sqrt{LC}}\right)$

Both roots S_1 and S_2 are **negative an real**

The response is $i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ which decays and approaches zero as t increases.

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where

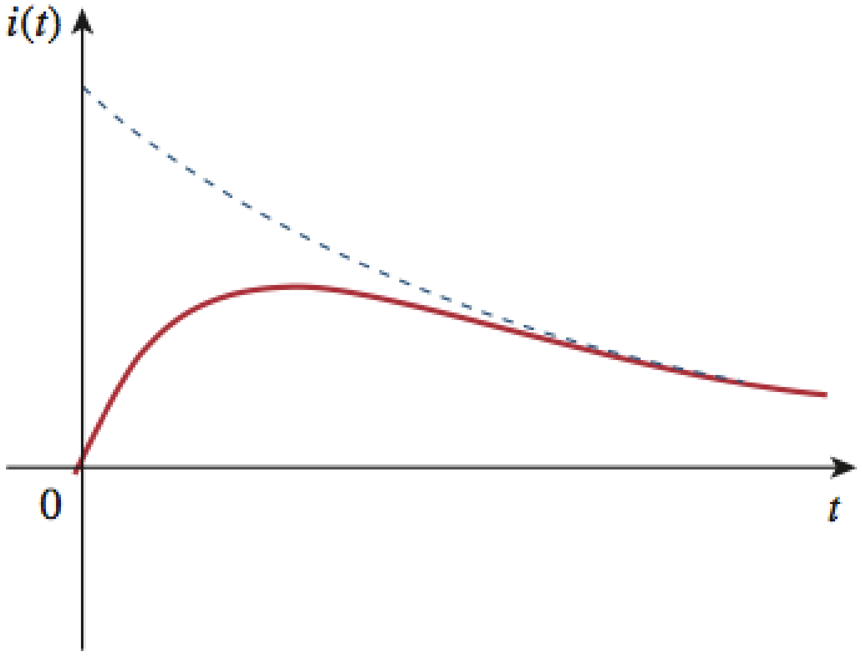
$$A_1 = \frac{i'(0^+) - s_2 i(0^+)}{s_1 - s_2}$$

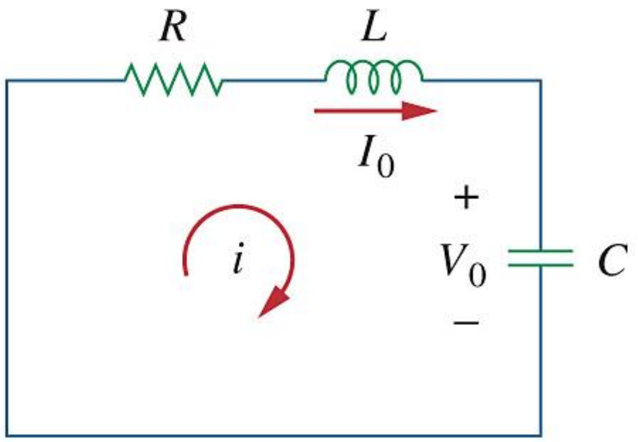
$$A_2 = \frac{s_1 i(0^+) - i'(0^+)}{s_1 - s_2}$$

Both S_1 and S_2 real

$S_1 < 0$, $S_2 < 0$

$S_1 \neq S_2$





1. no oscillation
2. Region 1: i(t) changes due to initially stored energy in L and C
3. Region 2: steady state value should be 0 due to “zero input response”
4. $\alpha \uparrow$ (more damping) ➔ reaches steady state faster

Second Order Equation – Solution S_1 *and* S_2

Case 2: Critically damped

2. If $\alpha = \omega_0$, $s_1 = s_2 = -\alpha$, we have the *critically damped* case,

$$i(t) = (B_1 t + B_2)e^{-\alpha t}$$

where

$$B_1 = i'(0^+) + \alpha i(0^+)$$

$$B_2 = i(0^+)$$

$S_1 < 0$, $S_2 < 0$
$S_1 = S_2$

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i=0 \quad \text{when} \quad \alpha = \omega_0 = R/2L$$

$$\frac{d^2i}{dt^2}+2\alpha\frac{di}{dt}+\alpha^2 i = 0 \rightarrow \frac{d}{dt}\boxed{\left(\frac{di}{dt}+\alpha i\right)}+\alpha\left(\frac{di}{dt}+\alpha i\right)=0$$

$$f=\left(\frac{di}{dt}+\alpha i\right)$$

$$\frac{d}{dt}\left(\frac{di}{dt}+\alpha i\right)+\alpha\left(\frac{di}{dt}+\alpha i\right)=0 \rightarrow \frac{df}{dt}+\alpha f=0$$

We get a first-order differential equation with solution $f = A_1 e^{-\alpha t}$

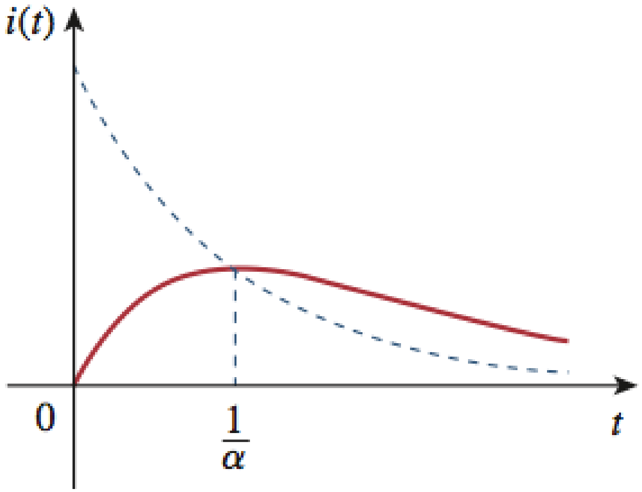
$$\frac{di}{dt}+\alpha i = A_1 e^{-\alpha t} \rightarrow e^{\alpha t}\frac{di}{dt}+e^{\alpha t}\alpha i = A_1$$

$$\rightarrow \frac{d}{dt}\left(e^{\alpha t} i\right) = A_1$$

Integrating both sides of the eq. $\frac{d}{dt}(e^{\alpha t} i) = A_1$

Then, we get $e^{\alpha t} i = A_1 t + A_2$

and finally it becomes $i(t) = (A_2 + A_1 t)e^{-\alpha t}$



1. no oscillation
2. region 1: i(t) reaches a maximum value at t = $1/\alpha$
3. region 2: decays all the way to zero
4. $\alpha \uparrow$ (more damping) ➔ reaches steady state faster

Second Order Equation – Solution *S_1 and S_2*

Case 3: Underdamped

3. If $\alpha < \omega_0$, $s_1 = -\alpha + j\omega_d$, $s_2 = -\alpha - j\omega_d$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

ω_d is called the *damping frequency*

We have the underdamped case,

$$i(t) = e^{-\alpha t}(C_1 \cos \omega_d t + C_2 \sin \omega_d t)$$

where

$$C_1 = i(0^+)$$

$$C_2 = \frac{i'(0^+) + \alpha i(0^+)}{\omega_d}$$

S_1, S_2 are complex conjugates

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad s_1 = -\alpha + j\omega_d,\ s_2 = -\alpha - j\omega_d$$

$$\begin{aligned} i(t) &= A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \\ &= e^{-\alpha t}(A_1 \boxed{e^{j\omega_d}} + A_2 \boxed{e^{-j\omega_d}}) \end{aligned}$$

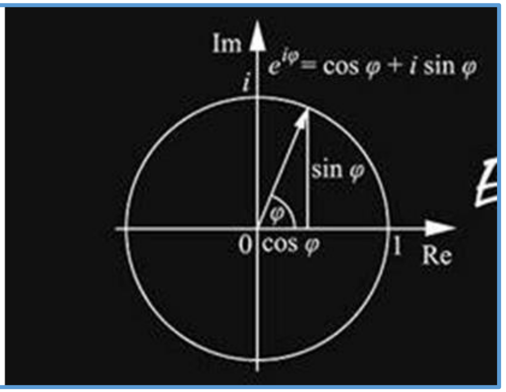
$$\boxed{e^{j\omega_d t} = cosw_d t + jsinw_d t}$$

$$\boxed{e^{-j\omega_d t} = cosw_d t - jsinw_d t}$$

Using Euler's identities,

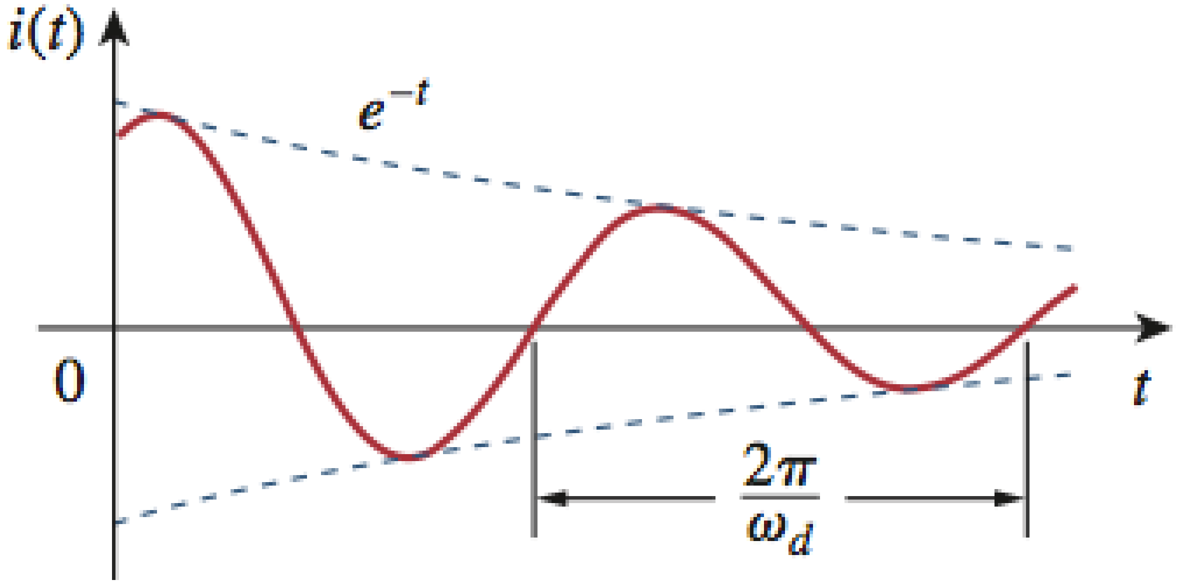
$$e^{j\theta} = \cos\theta + j\sin\theta,$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$



$$\begin{aligned} i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \end{aligned}$$

$$\Rightarrow \quad i(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$



$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

1. Oscillatory response
2. $\alpha \uparrow$ (more damping) ➔ reaches steady state faster
3. $\pm e^{-\alpha t}$: envelope
4. ω_d: oscillation frequency

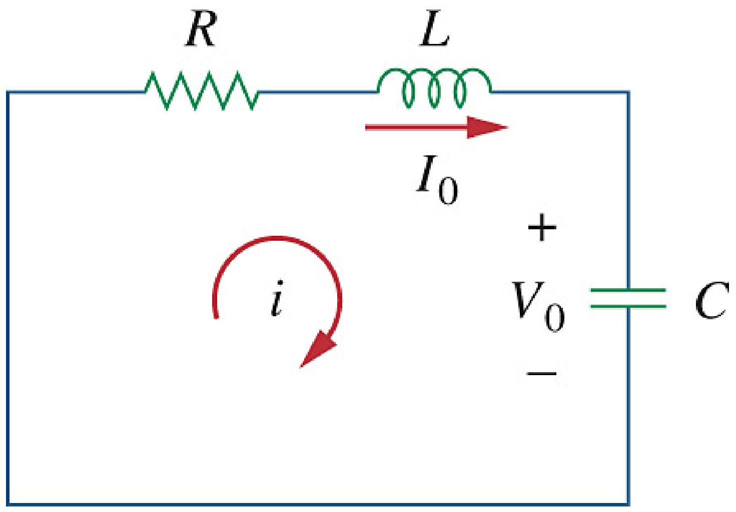
Second Order Equation – Other parameters

Once the inductor current i(t) is found, other circuit quantities can be found,

$$v_R(t) = i(t)R$$

$$v_L(t) = L\frac{di(t)}{dt}$$

$$v_C(t) = \frac{1}{C}\int_0^t i(t)dt + v_C(0)$$



Practice Problem 8.4 The circuit in Fig. 8.12 has reached steady state at $t = 0^-$. If the make-before-break switch moves to position b at $t = 0$, calculate $i(t)$ for $t > 0$.

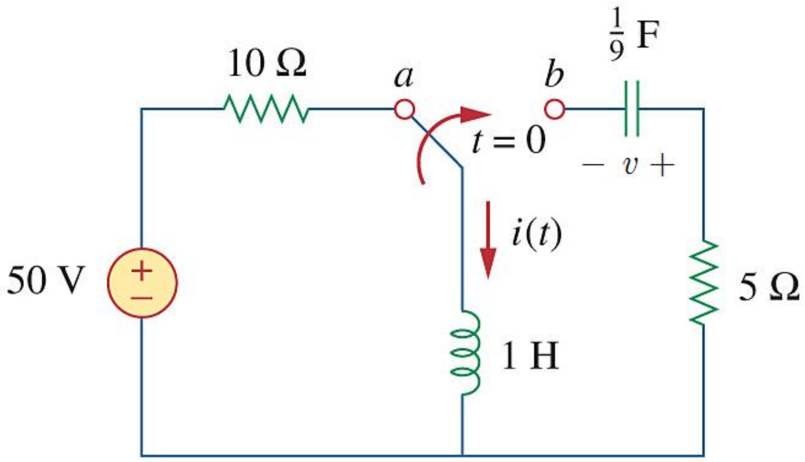


Figure 8.12

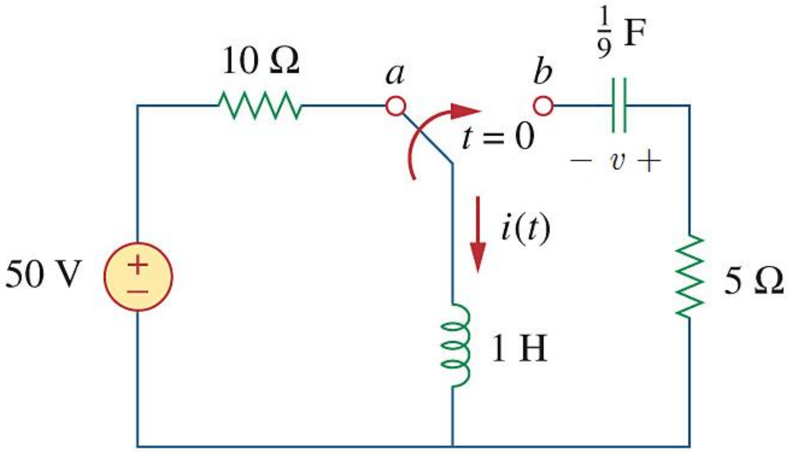


Figure 8.12

$$i(t) = e^{-2.5t}\left(A_1 \cos\frac{\sqrt{11}}{2}t + A_2 \sin\frac{\sqrt{11}}{2}t\right)$$

$$i(0^+) = A_1 \Rightarrow A_1 = i(0^+) = 5$$

$$i'(0^+) = -2.5A_1 + \frac{\sqrt{11}}{2}A_2 \Rightarrow A_2 = \frac{i'(0^+) + 2.5A_1}{\sqrt{11}/2}$$

$$= \frac{-25 + 2.5\times 5}{\sqrt{11}/2} = -\frac{25}{\sqrt{11}}$$

$$i(t) \approx e^{-2.5t}(5\cos 1.6583t - 7.5378\sin 1.6583t)\ (\mathrm{A})$$

$$\begin{aligned} i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \end{aligned}$$

$$\Rightarrow\ i(t) = e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t)$$

(i) $A_1 + A_2 = 5$

(ii) $j(A_1 - A_2) = -7.54 \rightarrow A_1 - A_2 = +j7.54$

$A_2 = 2.5 - j3.77$ and $A_1 = 2.5 + j3.77$

Initial Conditions:

$$i(0^+) = i(0^-) = \frac{50}{10} = 5 \text{ (A)}$$

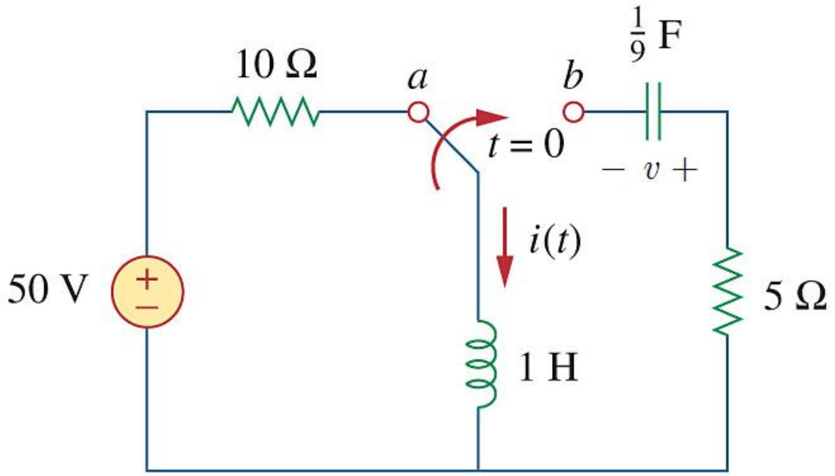
$$v(0^+) = v(0^-) = 0 \text{ (V)}$$

$$1 \times \frac{di(t)}{dt} + i(t) \times 5 + v(t) = 0,\ i(t) = \frac{1}{9}\frac{dv(t)}{dt}$$

$$i'(0^+) = -5i(0^+) - v(0^+) = -5 \times 5 - 0$$

$$= -25 \text{ (A/s)}$$

i) t > 0



$$1\times\frac{d^2i(t)}{dt^2}+\frac{di(t)}{dt}\times 5+\frac{dv(t)}{dt}=0$$

$$1\times\frac{d^2i(t)}{dt^2}+\frac{di(t)}{dt}\times 5+\frac{1}{1/9}i(t)=0$$

$$\frac{d^2i(t)}{dt^2}+5\frac{di(t)}{dt}+9i(t)=0$$

$$s=\frac{-5\pm\sqrt{5^2-4\times1\times9}}{2\times1}=\frac{-5\pm j\sqrt{11}}{2}$$

$$i(t)=e^{-2.5t}\left(A_1\cos\frac{\sqrt{11}}{2}t+A_2\sin\frac{\sqrt{11}}{2}t\right)$$

$$i(0^+)=A_1\Rightarrow A_1=i(0^+)=5$$

$$i'(0^+)=-2.5A_1+\frac{\sqrt{11}}{2}A_2\Rightarrow A_2=\frac{i'(0^+)+2.5A_1}{\sqrt{11}/2}$$

$$=\frac{-25+2.5\times5}{\sqrt{11}/2}=-\frac{25}{\sqrt{11}}$$

$$i(t)\approx e^{-2.5t}(5\cos1.6583t-7.5378\sin1.6583t)\ (\mathrm{A})$$

$$i(0^+) = A_1 \Rightarrow A_1 = i(0^+) = 5$$

$$i'(0^+) = -2.5A_1 + \frac{\sqrt{11}}{2} A_2 \Rightarrow A_2 = \frac{i'(0^+) + 2.5A_1}{\sqrt{11}/2}$$

$$= \frac{-25 + 2.5 \times 5}{\sqrt{11}/2} = -\frac{25}{\sqrt{11}}$$

$$i(t) \approx e^{-2.5t}(5\cos 1.6583t - 7.5378\sin 1.6583t) \text{ (A)}$$

$$\begin{aligned} i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \end{aligned}$$

$$\Rightarrow \quad i(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$

(i) $A_1 + A_2 = 5$

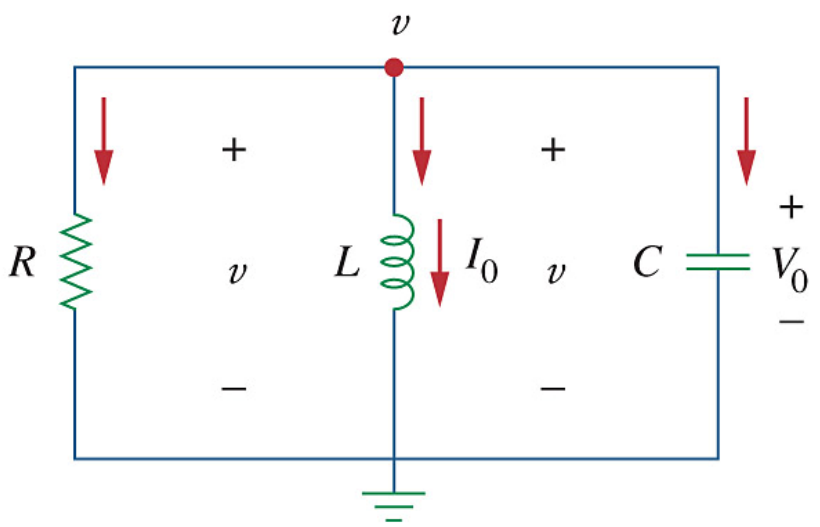
(ii) $j(A_1 - A_2) = -7.54 \rightarrow A_1 - A_2 = +j7.54$

$A_2 = 2.5 - j3.77$ and $A_1 = 2.5 + j3.77$

Steps for source-free 2^{nd} order circuit

1. Plot the circuit at $t<0$, find initial conditions, $i(0^+)$, $v(0^+)$
2. Plot the circuit at $t>0$, express di/dt or dv/dt in terms of i_L and v_c, find initial conditions $di(0^+)/dt$, $dv(0^+)/dt$
3. Express the circuit in 2^{nd} order D.E. with only one parameter (either i or v) and solve it.
4. Solve the coefficients using initial conditions.

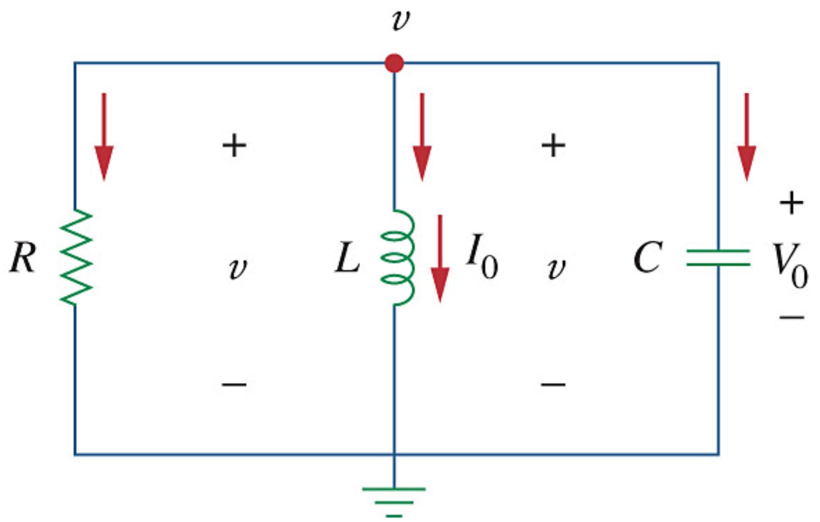
8.4 The Source-Free Parallel *RLC* Circuit



RLC are in parallel (the same voltage). The energy initially stored in C and L:

$$i(0) = I_0 = \frac{1}{L}\int_{\infty}^{0} v(t)\,dt$$

$$v(0) = V_0$$



By KCL: $$\frac{v}{R}+\frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt} = 0$$

$$\frac{1}{R}\frac{dv}{dt}+\frac{1}{L}v + C\frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v = 0$$

The characteristic equation is $\boldsymbol{S^2+\frac{1}{RC}s+\frac{1}{LC}=0}$

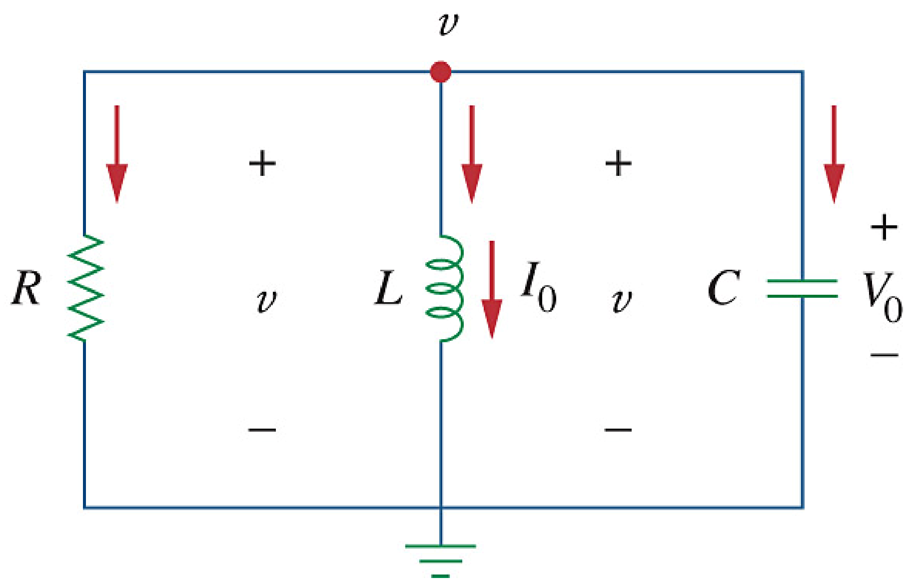
The initial conditions are

$$v(0^+) = v(0^-) = V_0$$

$$v'(0^+) = -\frac{1}{C}\left(v(0^+)/R + i(0^+)\right) \quad \text{from} \quad \frac{v}{R} + i + C\frac{dv}{dt} = 0$$

$$= -\frac{1}{C}\left(v(0^-)/R + i(0^-)\right)$$

$$= -\frac{1}{C}\left(V_0/R + I_0\right)$$



$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$s = \frac{-1/(RC) \pm \sqrt{1/(RC)^2 - 4\times 1\times\left(1/(LC)\right)}}{2\times 1}$$

$$= -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$\alpha = \frac{1}{2RC}$ Neper frequency (damping factor) Np/s

$\omega_0 = \frac{1}{\sqrt{LC}}$ Resonant frequency (undamped natural frequency) rad/s

Finally, we get solutions

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2},\ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Natural frequency Np/s

There are three types of solutions:

(1) Overdamped case

$$\text{If } \alpha > \omega_0,\, s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

(2) Critically damped case

$$\text{If } \alpha = \omega_0,\, s_1 = s_2 = -\alpha$$

$$v(t) = (B_1 t + B_2) e^{-\alpha t}$$

(3) Underdamped case

$$\text{If } \alpha < \omega_0,\, s_1 = -\alpha + j\omega_d,\; s_2 = -\alpha - j\omega_d$$

$$v(t) = e^{-\alpha t}(C_1 \cos \omega_d t + C_2 \sin \omega_d t)$$

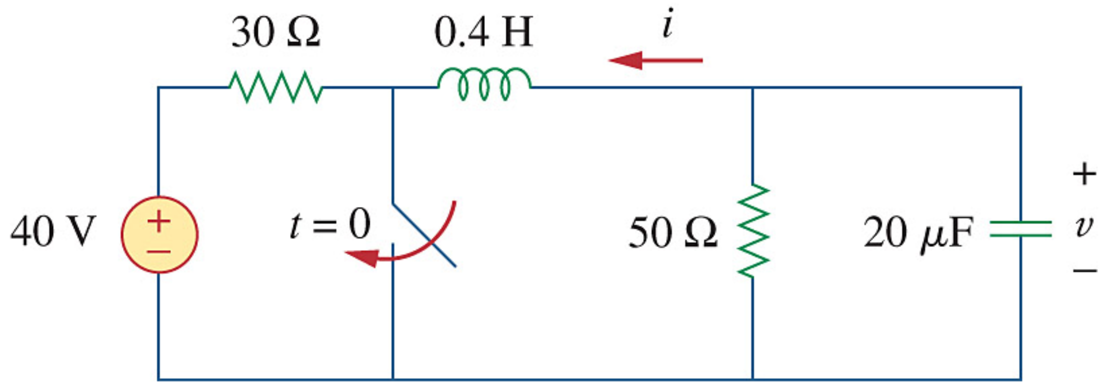
Once the capacitor voltage v(t) is found, other circuit quantities can be found

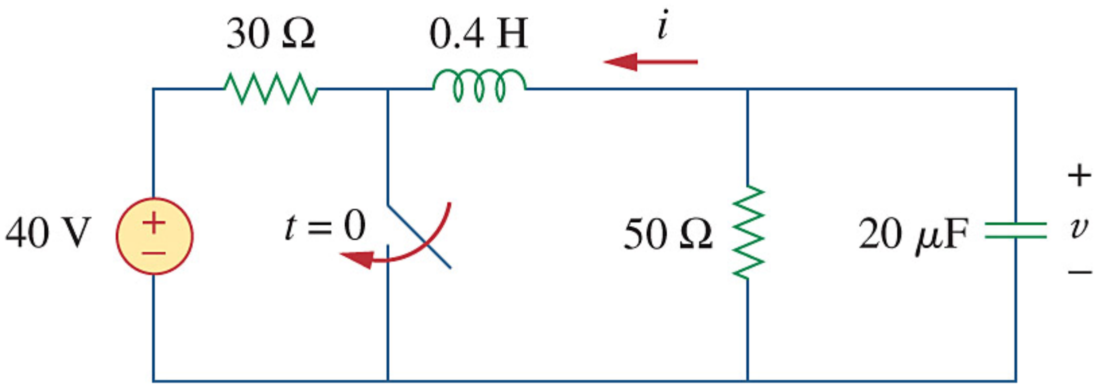
$$i_R(t) = \frac{v(t)}{R}$$

$$i_L(t) = \frac{1}{L}\int_0^t v(t)dt + i_L(0)$$

$$i_C(t) = C\frac{dv(t)}{dt}$$

Example 8.6 Find $v(t)$ for $t > 0$ in the RLC circuit of Fig. 8.15.





$$\frac{1}{LC}=\frac{1}{0.4\times 20\times 10^{-6}}=125000$$

$$\frac{1}{RC}=\frac{1}{50\times 20\times 10^{-6}}=1000$$

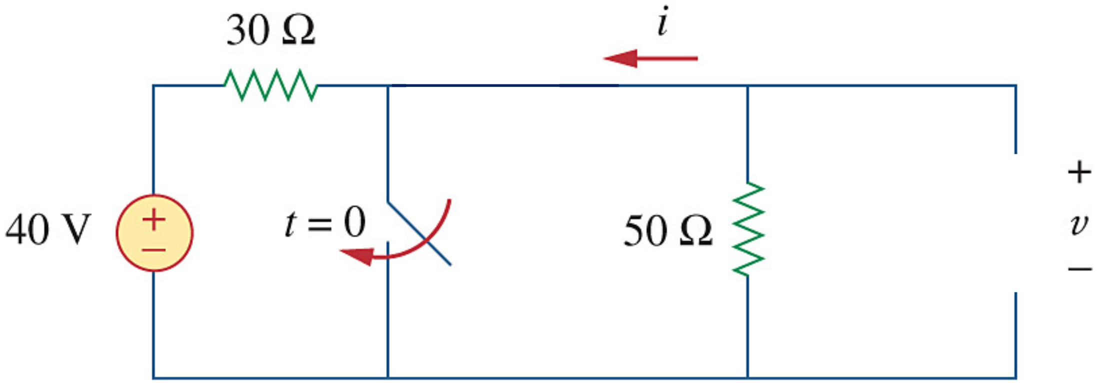
~707

$$s_{1,2}=\frac{-1000\pm\boxed{\sqrt{1000^2-4\times 1\times 125000}}}{2\times 1}$$

S_1 = -146.5 and S_2 = -853.5

(i) t < 0

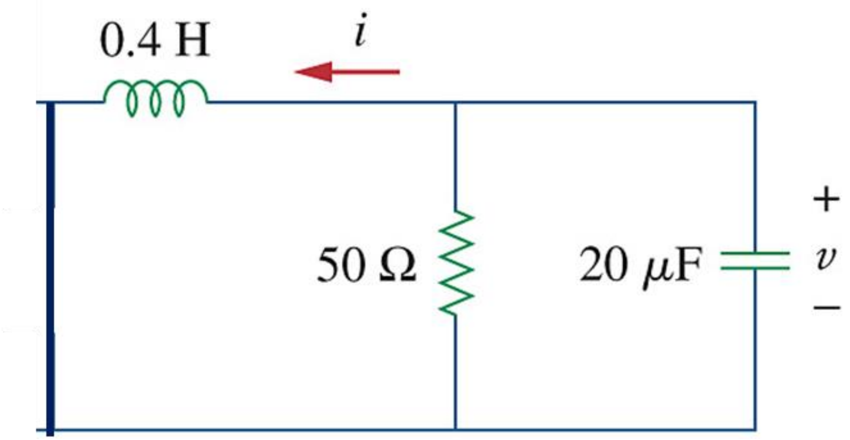
Equivalent Circuit



$$v(0^+) = v(0^-) = 40 \times \frac{50}{30+50} = 25 \text{ (V)}$$

$$i(0^+) = i(0^-) = -\frac{40}{30+50} = -0.5 \text{ (A)}$$

(ii) t > 0



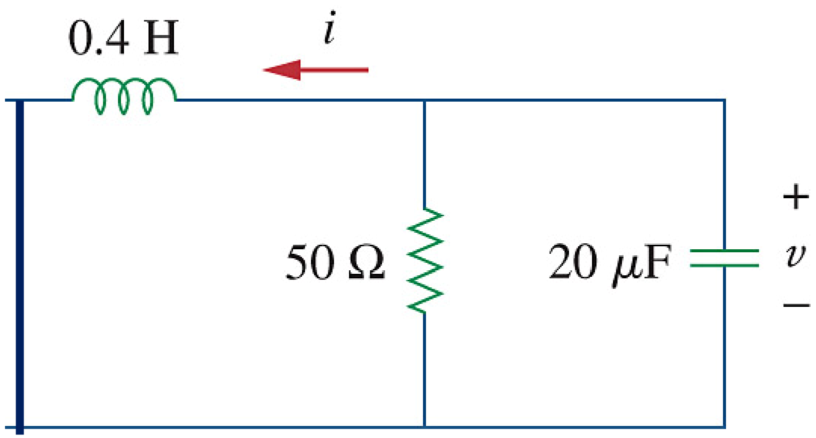
By KCL:

$$\frac{1}{L}\int v\,dt + \frac{v}{R} + C\frac{dv}{dt} = 0$$

At t=0+, $i(0+) + \frac{v(0+)}{50} + 20 \times 10^{-6}\frac{dv(0+)}{dt} = 0$

$$v'(0^+) = -\frac{1}{C}\left(v(0^+)/R + i(0^+)\right)$$

$$= -\frac{1}{20\times10^{-6}}\left(25/50 + (-0.5)\right) = 0\ (\text{V/s})$$



By KCL:

$$\frac{1}{L}\int v\,dt + \frac{v}{R} + C\frac{dv}{dt} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

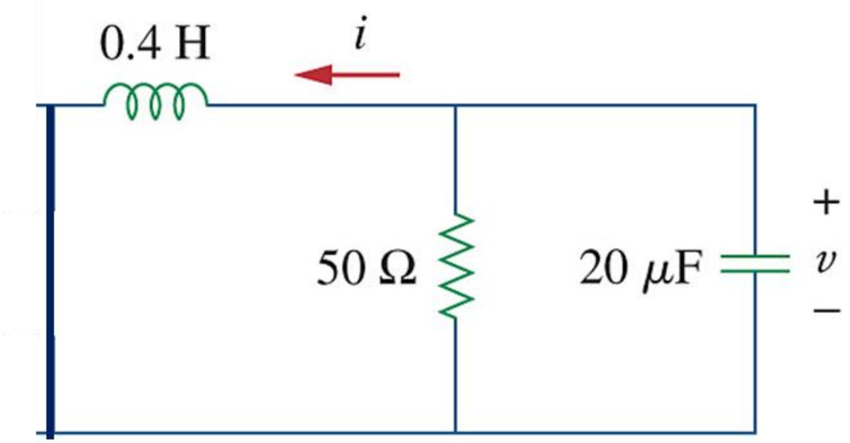
$$\frac{1}{LC} = \frac{1}{0.4\times 20\times 10^{-6}} = 125000$$

$$\frac{1}{RC} = \frac{1}{50\times 20\times 10^{-6}} = 1000$$

$$\frac{d^2v}{dt^2} + 1000\frac{dv}{dt} + 125000v = 0$$

~707

$$s_{1,2} = \frac{-1000 \pm \boxed{\sqrt{1000^2 - 4\times 1\times 125000}}}{2\times 1}$$



$S_1 = -146.5$ and $S_2 = -853.5$

$v(t) = A_1e^{-146.5t} + A_2e^{-853.5t}$

Initial conditions

(i) $v(0) = A_1 + A_2 = 25$

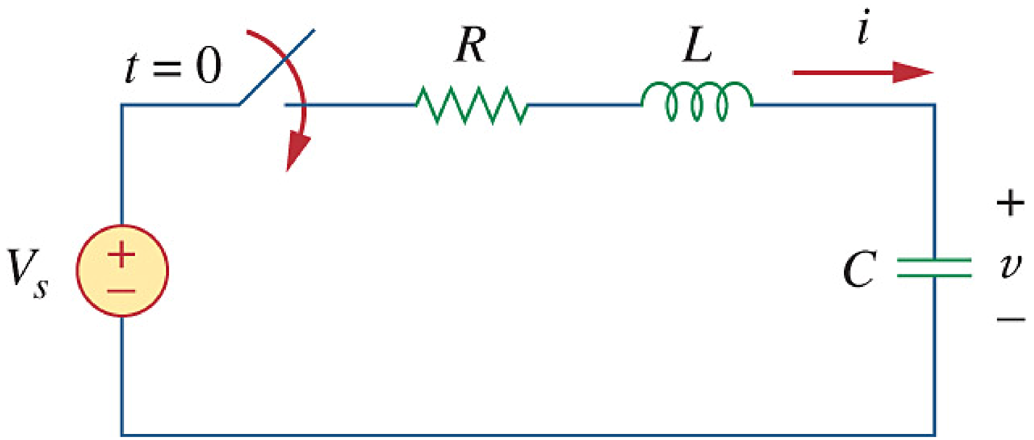
(ii) $v'(0) = -146.5A_1 - 853.5A_2 = 0$

$\rightarrow A_1 = 30.2$ and $A_2 = -5.2$

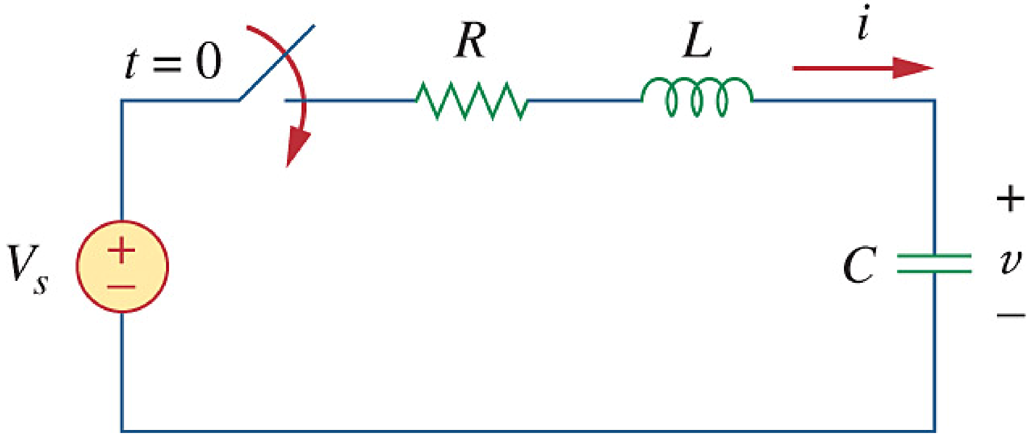
Thus, $\boldsymbol{v(t) = 30.2e^{-146.5t} - 5.2e^{-853.5t}\ [V]}$

8.5 Series *RLC* Circuit with Step Input

The step response is obtained by the sudden application of a dc source.



$$\begin{cases} V_s = iR + L\dfrac{di}{dt} + v \\ i = C\dfrac{dv}{dt} \end{cases}$$



$$\begin{cases} V_s = iR + L\dfrac{di}{dt} + v \\ i = C\dfrac{dv}{dt} \end{cases}$$

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = V_s$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}V_s$$

The characteristic equation for the series RLC circuit is not affected by the presence of the dc source:

the **transient response** and the **steady-state response**

The transient response is the component of the total response that **dies out with time**. The form of the transient response is the same as the form of the solution obtained for the source-free circuit.

The transient response

$$v_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)}$$

$$v_t(t) = (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)}$$

$$v_t(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)}$$

The steady-state response $\quad v_{ss}(t) = v(\infty) = V_s$

The total response: $\quad v(t) = v_t(t) + v_{ss}(t)$

The complete solutions

It can be shown that the solution has three possible forms:

(1) Overdamped

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + V_s$$

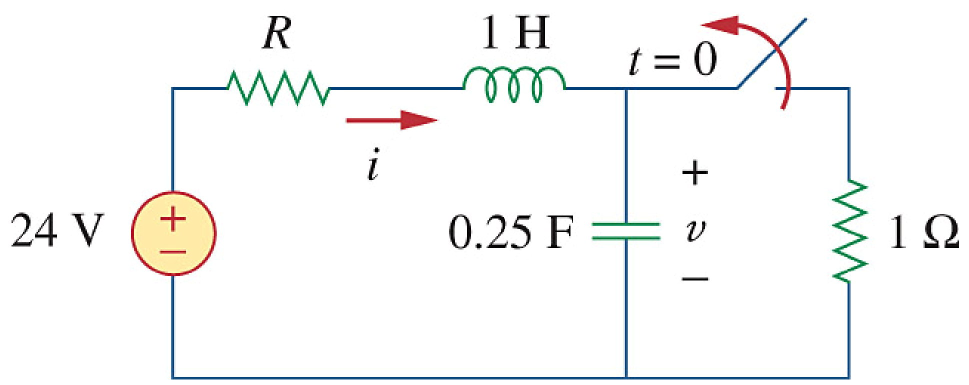
(2) Critically damped

$$v(t) = (A_1 + A_2 t)e^{-\alpha t} + V_s$$

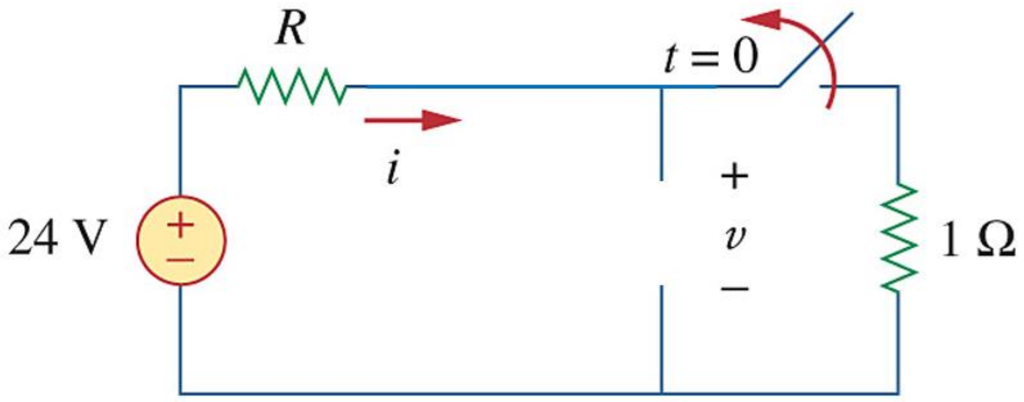
(3) Underdamped

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) + V_s$$

Example 8.7 For the circuit in Fig. 8.19, find $v(t)$ for $t > 0$. Consider these cases: $R = 5\ \Omega$, $R = 4\ \Omega$, $R = 1\ \Omega$.



(1) t < 0, initial conditions

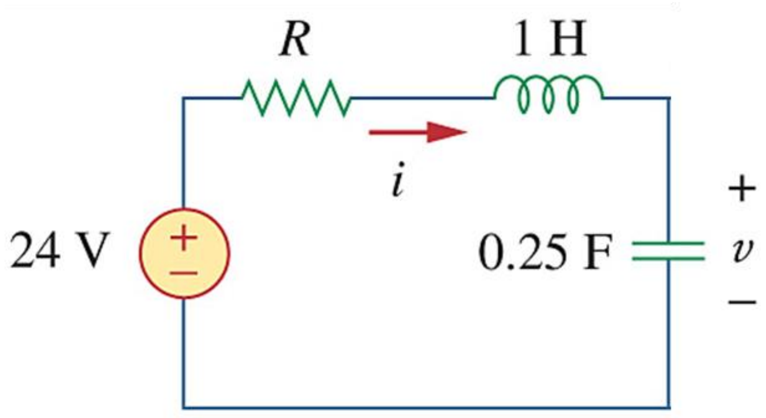


(a) R = 5

$$i(0^-) = \frac{24}{R+1} \quad = 4\ [A]$$

$$v(0^-) = 24 \times \frac{1}{R+1} \quad = 4\ [V]$$

(ii) t > 0 – find i(t) first



$$-24 + 5i + 1\frac{di}{dt} + 4\int idt = 0 \;\rightarrow\; \frac{d^2i}{dt^2} + 5\frac{di}{dt} + 4i = 0$$

$$s^2 + 5s + 4 = 0 \Rightarrow s_1 = -1,\ s_2 = -4$$

$$i(t) = A_1e^{-t} + A_2e^{-4t}$$

Initial conditions:

(i) i(0) = 4 [A]

(ii) I’(0) = ?

Initial conditions:

(i) i(0) = 4 [A]

(ii) I'(0) = ?

$$At\ t = 0, -24 + 5i(0) + 1\frac{di(0)}{dt} + v(0) = 0$$

→ di(0)/dt = 0 [A/s]

Using the initial conditions we can find two constants A_1 = 16/3 and A_2 = -4/3

$$i(t) = A_1 e^{-t} + A_2 e^{-4t} = \frac{16}{3}e^{-t} - \frac{4}{3}e^{-4t}\ [A]$$

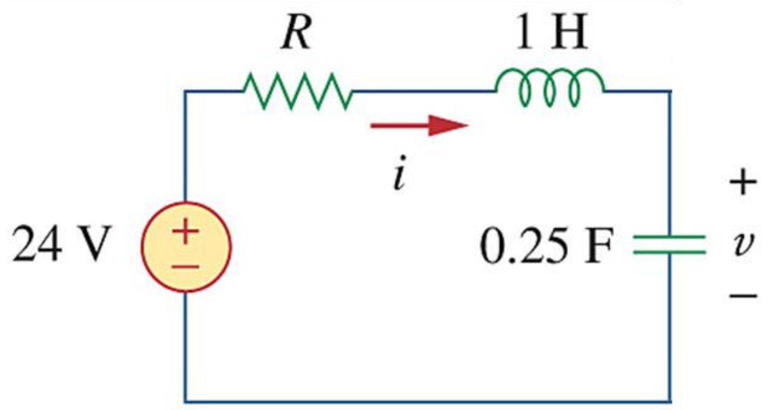
IV of the capacitor $i(t) = C\,\frac{dv(t)}{dt}$

$$\rightarrow v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v(0)$$

$$v(t) = -\frac{64}{3}e^{-t} + \frac{4}{3}e^{-4t} + 20 + v(0)$$

$$\boldsymbol{v(t) = -\frac{64}{3}e^{-t} + \frac{4}{3}e^{-4t} + 24\ [\mathrm{V}]}$$

(iii) t > 0 – find v(t) directly



$$i(0^+) = i(0^-) = \frac{24}{R+1}$$

$$v(0^+) = v(0^-) = 24 \times \frac{1}{R+1}$$

$$i(t) = 0.25\frac{dv(t)}{dt} \Rightarrow v'(0^+) = \frac{1}{0.25} i(0^+)$$

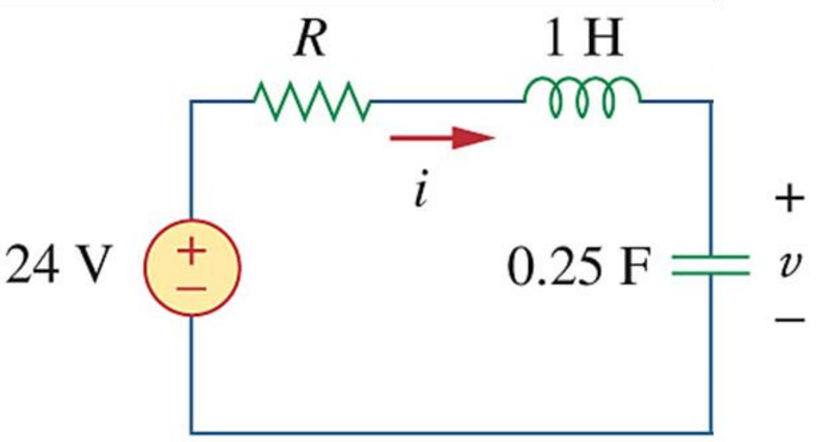
$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}V_s$$

(a) $R = 5\ \Omega$

$i(0^+) = 4$ A, $v(0^+) = 4$ V, $v'(0^+) = 16$ V/s

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 4v = 96 \qquad s^2 + 5s + 4 = 0 \Rightarrow s_1 = -1,\ s_2 = -4$$

$$\Rightarrow\quad v_n(t) = A_1 e^{-t} + A_2 e^{-4t}$$



$$v_f(t) = B \Rightarrow B = 96/4 = 24 \quad \longleftarrow \quad \frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 4v = 96$$

$$v(t) = v_n(t) + v_f(t) = A_1 e^{-t} + A_2 e^{-4t} + 24$$

$$v(0^+) = A_1 + A_2 + 24 = 4$$

$$v'(0^+) = -A_1 - 4A_2 = 16$$

$$A_1 = -\frac{64}{3},\ A_2 = \frac{4}{3}$$

$$\boxed{\begin{aligned} v(t) &= -\frac{64}{3}e^{-t} + \frac{4}{3}e^{-4t} + 24 \text{ (V)} \\ i(t) &= \frac{16}{3}e^{-t} - \frac{4}{3}e^{-4t} \text{ (A)} \end{aligned}}$$

(b) $R = 4\ \Omega$

$i(0^+) = 4.8\ \text{A},\ v(0^+) = 4.8\ \text{V},\ v'(0^+) = 19.2\ \text{V/s}$

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 4v = 96$$

$s^2 + 4s + 4 = 0 \Rightarrow s_1 = s_2 = -2$

$v_n(t) = (A_1 + A_2 t)e^{-2t}$

$v_f(t) = B \Rightarrow B = 96/4 = 24$

$v(t) = v_n(t) + v_f(t) = (A_1 + A_2 t)e^{-2t} + 24$

$v(0^+) = A_1 + 24 = 4.8$

$v'(0^+) = A_2 - 2A_1 = 19.2$

$A_1 = A_2 = -19.2$

$$\boxed{\begin{aligned} v(t) &= (-19.2 - 19.2t)e^{-2t} + 24\ \text{(V)} \\ i(t) &= 4.8(1 + 2t)e^{-2t}\ \text{(A)} \end{aligned}}$$

(c) $R = 1\ \Omega$

$i(0^+) = 12$ A, $v(0^+) = 12$ V, $v'(0^+) = 48$ V/s

$$\frac{d^2v}{dt^2} + \frac{dv}{dt} + 4v = 96$$

$$s^2 + s + 4 = 0 \Rightarrow s_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{15}}{2}$$

$$v_n(t) = e^{-t/2}(A_1 \cos\frac{\sqrt{15}}{2}t + A_2 \sin\frac{\sqrt{15}}{2}t)$$

$$v_f(t) = B \Rightarrow B = 96/4 = 24$$

$$v(t) = v_n(t) + v_f(t)$$

$$= e^{-t/2}(A_1 \cos\frac{\sqrt{15}}{2}t + A_2 \sin\frac{\sqrt{15}}{2}t) + 24$$

$$v(0^+) = A_1 + 24 = 12$$

$$v'(0^+) = -\frac{1}{2}A_1 + \frac{\sqrt{15}}{2}A_2 = 48$$

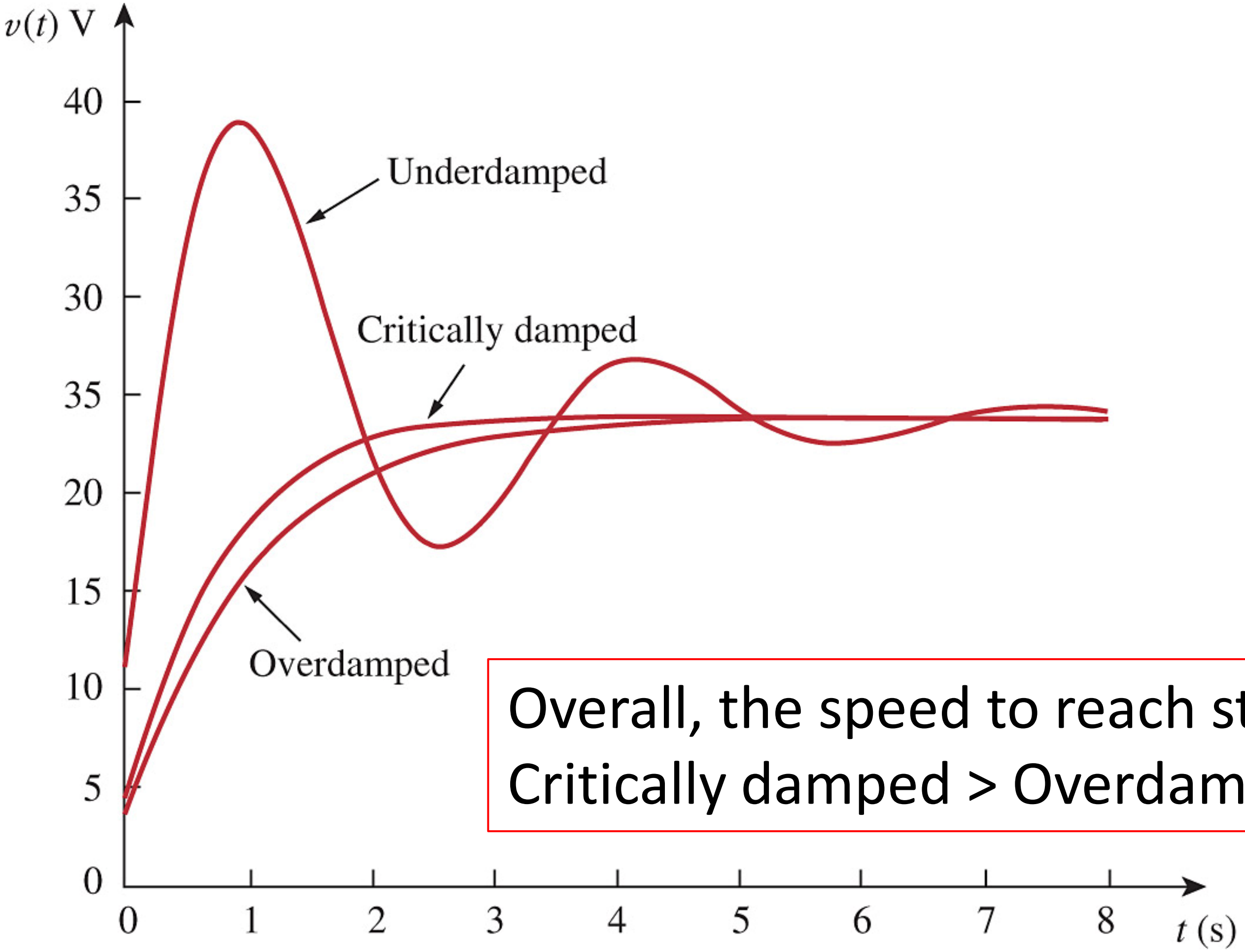
$$A_1 = -12,\ A_2 = \frac{84}{\sqrt{15}} \approx 21.689$$

$$v(t) = e^{-t/2}(-12\cos\frac{\sqrt{15}}{2}t + \frac{84}{\sqrt{15}}\sin\frac{\sqrt{15}}{2}t) + 24\ (\mathrm{V})$$

$$i(t) = e^{-t/2}(12\cos\frac{\sqrt{15}}{2}t + \frac{12}{\sqrt{15}}\sin\frac{\sqrt{15}}{2}t)\ (\mathrm{A})$$

R	5Ω	4Ω	1Ω
α (R/2L)	2.5	2	0.5
v_f	24V	24V	24V
	Overdamped	Critically damped	Underdamped

Plots of the three responses. The critically damped response approaches the step input 24 V the fastest.



Overall, the speed to reach steady state value:
Critically damped > Overdamped > Underdamped

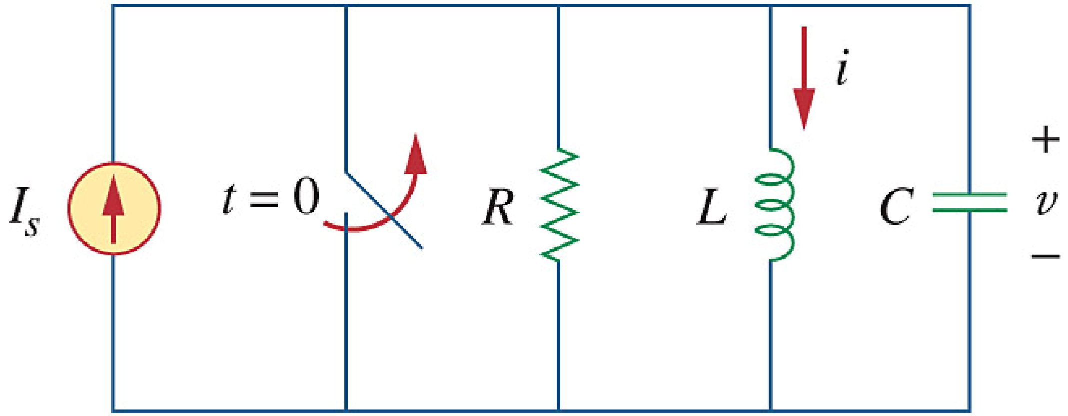
Steps for 2nd order circuit with *step input*

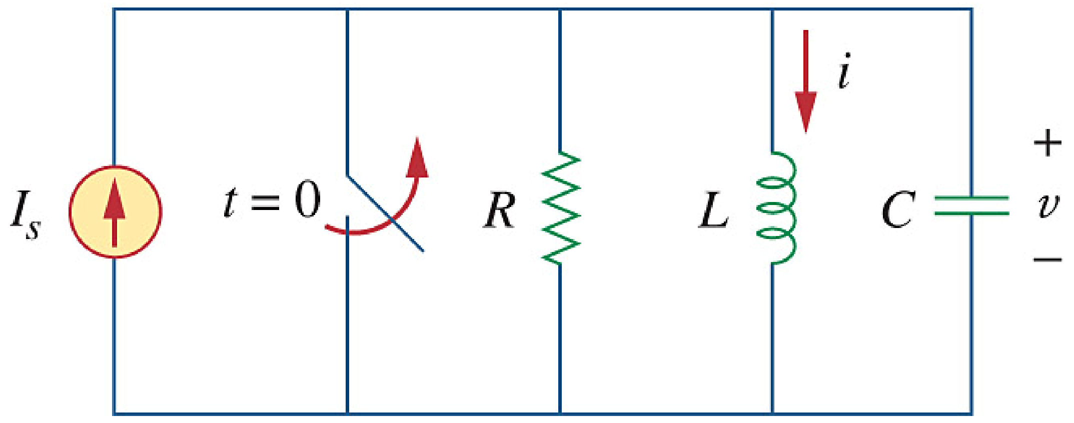
1. Plot the circuit at $t<0$, find initial conditions, $i(0^+)$, $v(0^+)$
2. Plot the circuit at $t>0$, express di/dt or dv/dt in terms of i_L and v_c, find initial conditions $di(0^+)/dt$, $dv(0^+)/dt$
3. Express the circuit in 2nd order D.E. with only one parameter (either i or v) and solve it.
4. Plot the circuit at $t \rightarrow \infty$, find steady state values $i(\infty)$, $v(\infty)$ (or just solve forced response)
5. Solve the coefficients using initial conditions.

8.6 Parallel *RLC* Circuit with Step Input

Consider the circuit below. We want to find I due to a sudden application of a DC current.

At t > 0, apply KCL





$$I_s = \frac{v}{R} + i + C\frac{dv}{dt}, \quad v = L\frac{di}{dt}$$

$$LC\frac{d^2i}{dt^2} + \frac{L}{R}\frac{di}{dt} + i = I_s \quad \Rightarrow \quad \frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{LC}I_s$$

Again, the characteristic equation for the parallel RLC circuit is not affected by the presence of the dc source.

It can be shown that the solution has three possible forms:

(1) Overdamped

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + I_s$$

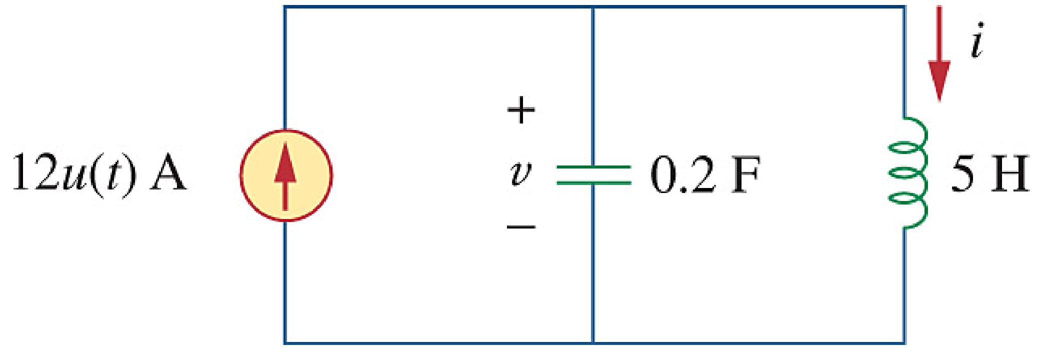
(2) Critically damped

$$i(t) = (A_1 + A_2 t) e^{-\alpha t} + I_s$$

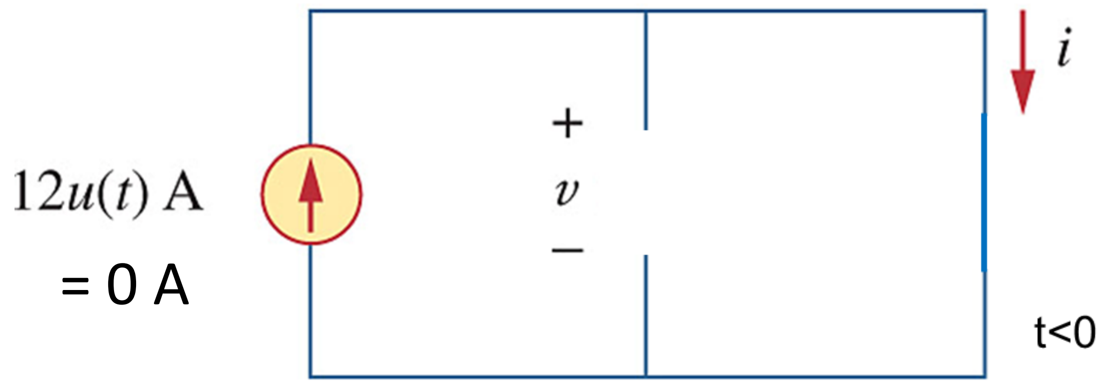
(3) Underdamped

$$i(t) = e^{-\alpha t} (A_1 \cos \omega_d t + A_2 \sin \omega_d t) + I_s$$

Practice Problem 8.8 Find $i(t)$ and $v(t)$ for $t > 0$ in the circuit of Fig. 8.24.



(i) t < 0 initial conditions



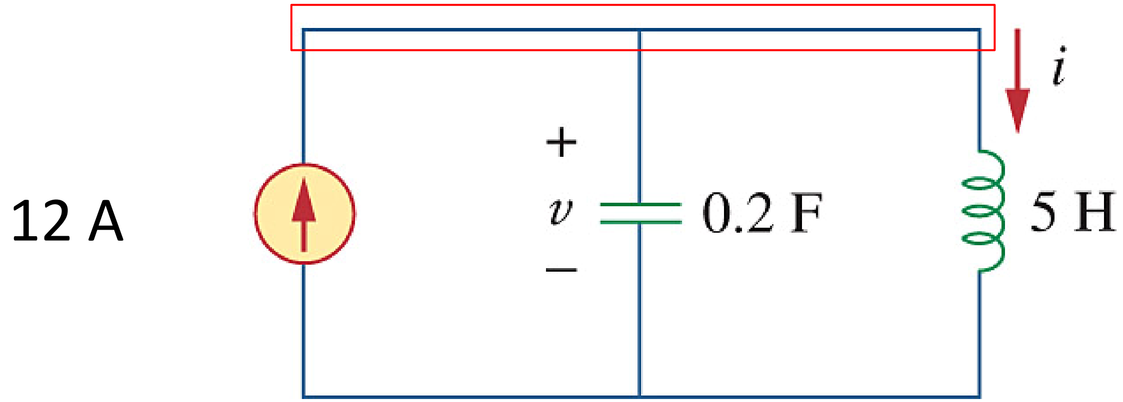
There is no source at t < 0

$$i(0^-) = 0$$

$$v(0^-) = 0$$

(ii) t > 0 – find voltage first

Node voltage: ***v***



By KCL:

$$-12 + C\frac{dv}{dt} + \frac{1}{L}\int vdt = 0 \rightarrow \frac{d^2v}{dt^2} + v = 0$$

$$s^2 + 1 \Rightarrow s_{1,2} = \pm j$$

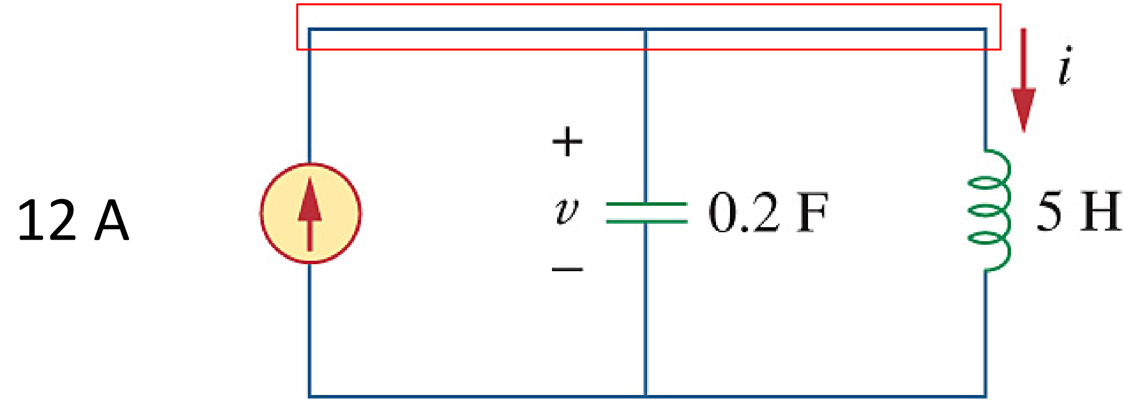
$$v(t) = A_1 cost + A_2 sint$$

Initial Conditions:

(i) $v(0) = A_1 = 0$

(ii) $v'(0) = A_2 = ?$

Node voltage: ***v***



$$v(t) = A_1 cost + A_2 sint$$

Initial Conditions:

(i) $v(0) = A_1 = 0$

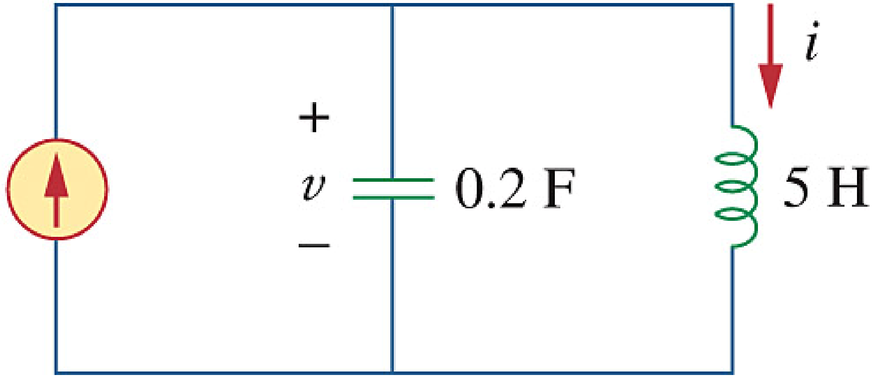
(ii) $v'(0) = A_2 = 60$ ⟵ $-12 + 0.2\dfrac{dv(0)}{dt} + i(0) = 0$

$$\boldsymbol{v(t) = 60sint\,[V]}$$

$$\boldsymbol{i_L(t) = \frac{1}{5}\int 60sint\,dt + i_L(0) = 12(1 - cost)[A]}$$

(iii) t > 0 – find current first

12 A



$$12 = 0.2\frac{dv}{dt} + i, \quad v = 5\frac{di}{dt}$$

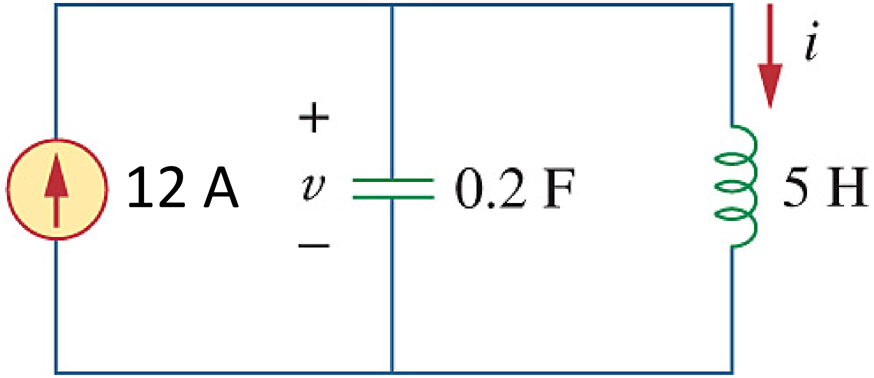
$$\frac{d^2 i}{dt^2} + i = 12$$

$$s^2 + 1 \Rightarrow s_{1,2} = \pm j$$

$$i_n(t) = A_1 \cos t + A_2 \sin t$$

$$i_p(t) = 12$$

$$i(t) = i_n(t) + i_p(t) = A_1 \cos t + A_2 \sin t + 12$$



$$i(0^+) = i(0^-) = 0$$

$$v(0^+) = v(0^-) = 0$$

$$v(0^+) = 5\frac{di(0^+)}{dt} \Rightarrow i'(0^+) = \frac{1}{5}v(0^+) = 0$$

$$i(0^+) = A_1 + 12 = 0$$

$$i'(0^+) = -A_2 = 0$$

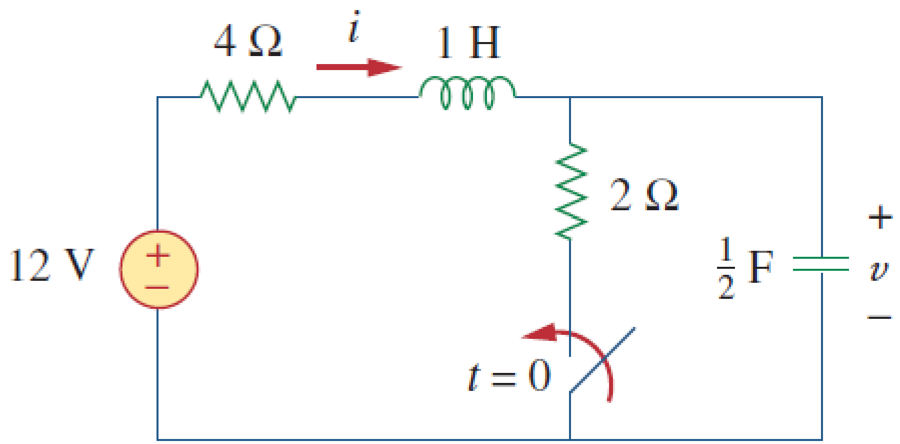
$$A_1 = -12,\ A_2 = 0$$

$$i(t) = -12\cos t + 12 = 12(1 - \cos t)\ (\mathrm{A})$$

$$v(t) = 5\frac{di(t)}{dt} = 60\sin t\ (\mathrm{V})$$

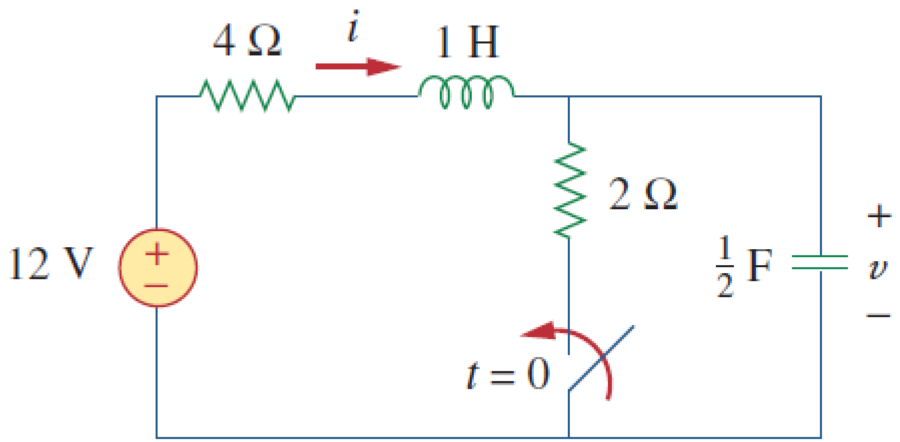
8.7 General Second-Order Circuits

We are prepared to apply the ideas to any second-order circuit having one or more independent sources with constant values → **Mesh and Nodal Analysis to a RLC circuit.**

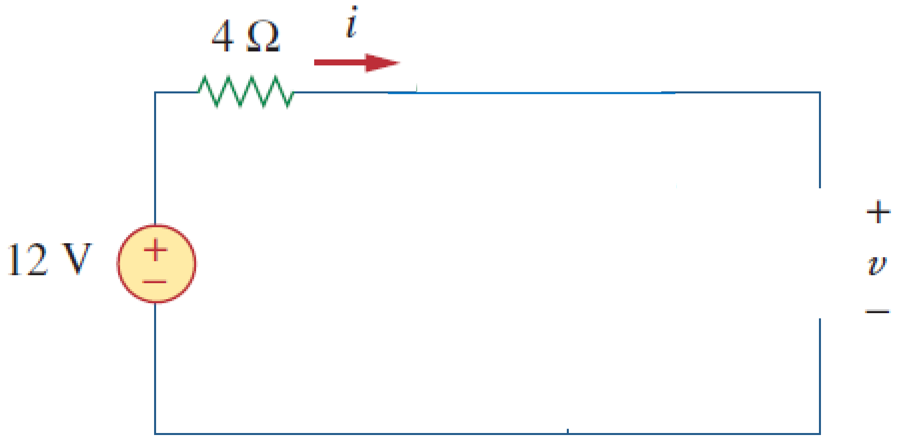


RLC general second-order circuit

Example 8.9. Find the complete response *v* and then *i* for t > 0 in the circuit



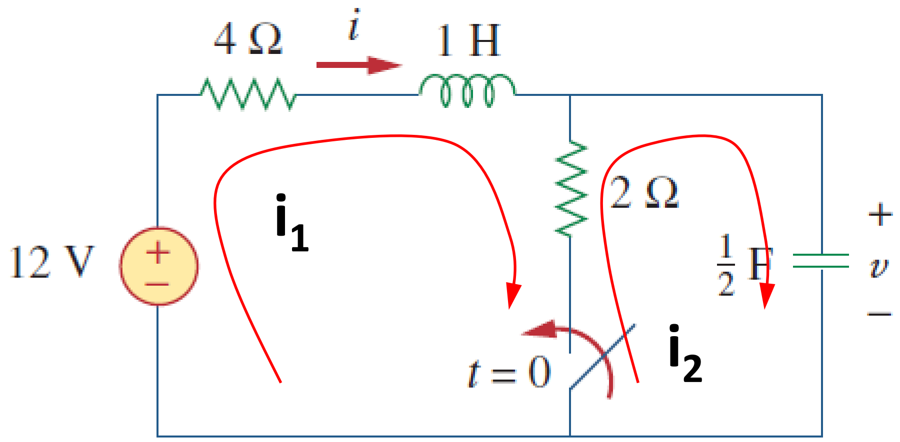
(i) t < 0



(i) Find a current i:

- Initial conditions: $i(0) = 0$ A; $v(0) = 12$ V

(ii) t > 0



(i) Find a current i:

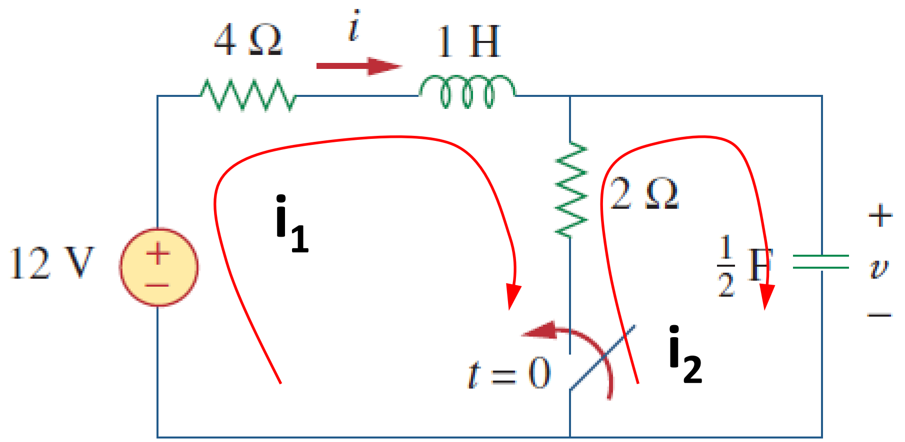
By KVL

$$-12 + 4i_1 + \frac{di_1}{dt} + 2(i_1 - i_2) = 0 \rightarrow \boxed{2i_2 = -12 + 6i_1 + \frac{di_1}{dt}}$$

$$2(i_2 - i_1) + 2\int i_2\, dt = 0$$

$$\frac{d^2 i_1}{dt^2} + 5\frac{di_1}{dt} + 6i_1 = 12 \rightarrow \boldsymbol{S^2 + 5S + 6 = 0}$$

(ii) t > 0



$S^2 + 5S + 6 = 0$, $S_1 = -2$ and $S_2 = -3$

$i_1(t) = A_1 e^{-2t} + A_2 e^{-3t} + i_{f1}$ where $i_{f1} = 2$

Using initial conditions,

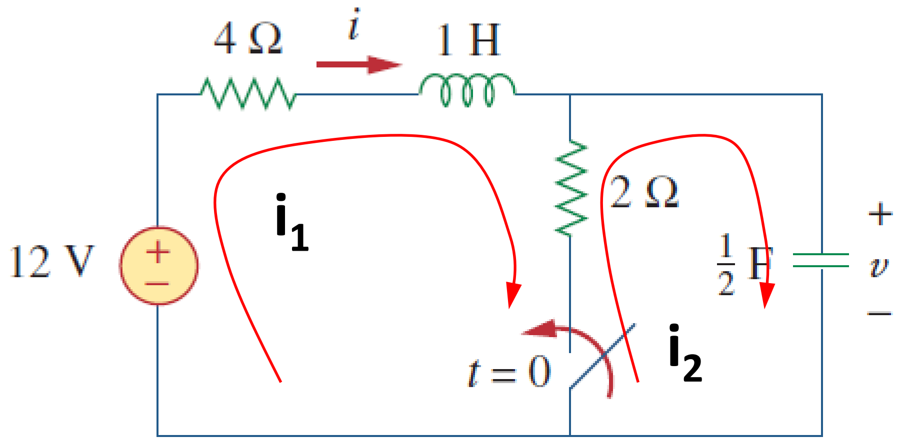
(i) $i_1(0) = A_1 + A_2 + 2 = 0$

(ii) $i'(0) = -2A_1 + -3A_2 = 0$

$$\boxed{-12 + 4i_1(0) + \frac{di_1(0)}{dt} + v(0) = 0 \rightarrow \frac{di_1(0)}{dt} = 0}$$

Thus, $A_1 = -6$; $A_2 = 4$

(ii) t > 0

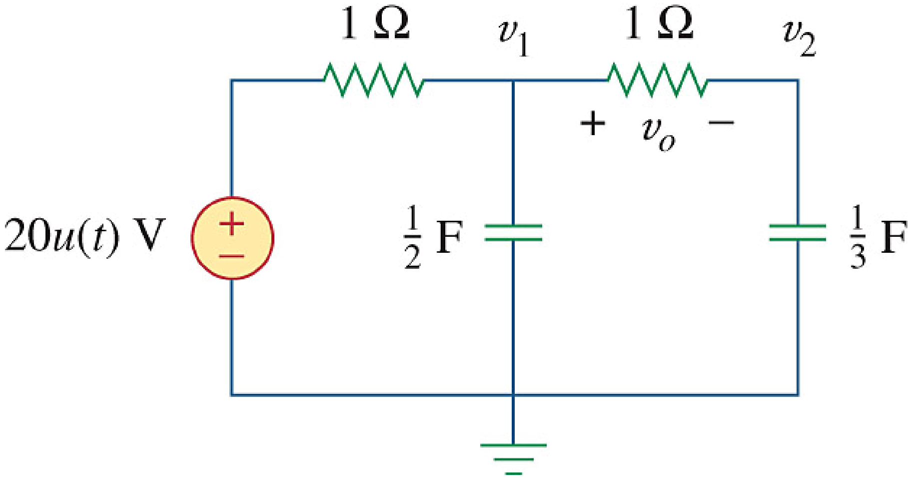


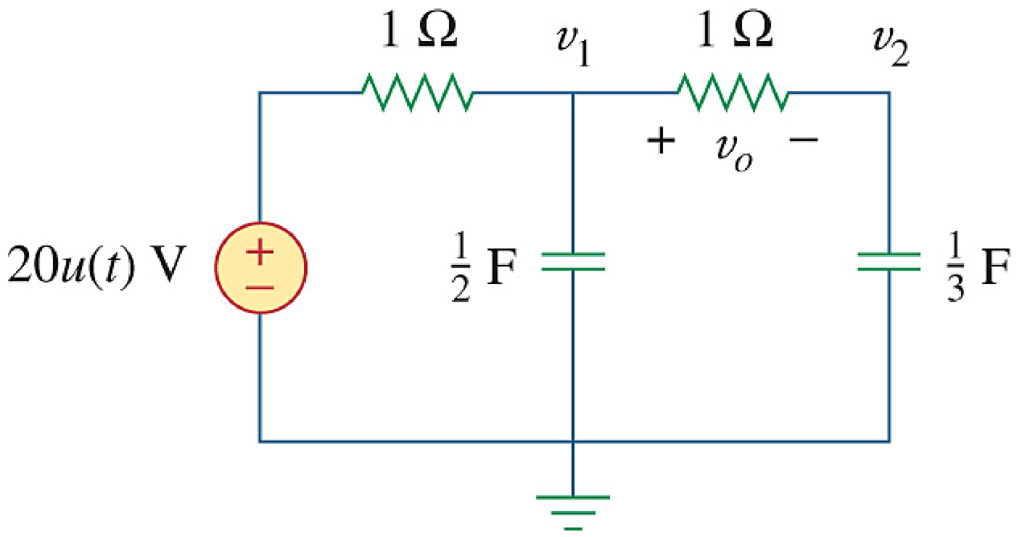
$$i_1(t) = -6e^{-2t} + 4e^{-3t} + 2 = \boldsymbol{i(t)}$$

$$2i_2 = -12 + 6i_1 + \frac{di_1}{dt} \rightarrow i_2(t) = -12e^{-2t} + 6e^{-3t}$$

$$\boldsymbol{v(t)} = 2 \times (i_1 - i_2) = 12e^{-2t} - 4e^{-3t} + 4\ [V]$$

Practice Problem 8.10 For $t > 0$, obtain $v_o(t)$ in the circuit of Fig. 8.32. (*Hint* : First find v_1 and v_2.)





Solution :

$$v_1(0^+) = v_2(0^+) = 0$$

$$\frac{20 - v_1(0^+)}{1} = \frac{1}{2}\frac{dv_1(0^+)}{dt} + \frac{v_1(0^+) - v_2(0^+)}{1}$$

$$v_1'(0^+) = 2[20 - 2v_1(0^+) + v_2(0^+)] = 40 \text{ (V/s)}$$

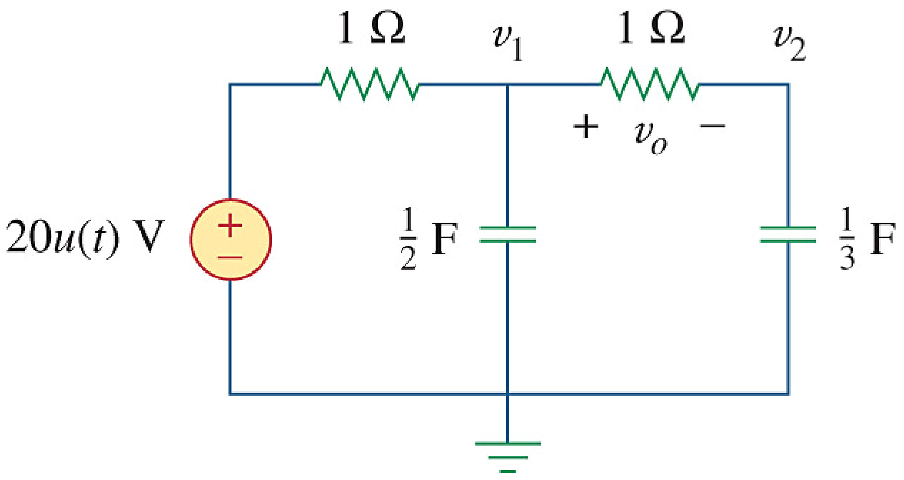


Figure 8.32 An *RCC* circuit.

$$v_1(\infty) = v_2(\infty) = 20 \text{ (V)}$$

$$\frac{20 - v_1}{1} = \frac{1}{2}\frac{dv_1}{dt} + \frac{v_1 - v_2}{1}, \quad \frac{v_1 - v_2}{1} = \frac{1}{3}\frac{dv_2}{dt}$$

$$\frac{d^2 v_1}{dt^2} + 7\frac{dv_1}{dt} + 6v_1 = 120$$

$$s^2 + 7s + 6 = 0 \Rightarrow s_1 = -1, \; s_2 = -6$$

$$v_{1h}(t) = A_1 e^{-t} + A_2 e^{-6t}$$

$$v_{1p}(t) = 20$$

$$v_1(t) = A_1 e^{-t} + A_2 e^{-6t} + 20$$

$$v_1(0^+) = A_1 + A_2 + 20 = 0$$

$$v_1'(0^+) = -A_1 - 6A_2 = 40$$

$$A_1 = -16,\ A_2 = -4$$

$$v_1(t) = -16e^{-t} - 4e^{-6t} + 20$$

$$v_2(t) = -24e^{-t} + 4e^{-6t} + 20 \longleftarrow \frac{20 - v_1}{1} = \frac{1}{2}\frac{dv_1}{dt} + \frac{v_1 - v_2}{1}$$

$$v_o(t) = v_1(t) - v_2(t) = 8e^{-t} - 8e^{-6t}\ \text{(V)}$$

8.10 Duality

- The concept of duality is a time-saving, effort-effective measure of solving circuit problems.
- Two circuits are said to be duals of one another if they are described **by the same characteristic equations with dual pairs interchanged**.
- Dual pairs are shown in Table 8.1.

TABLE 8.1 Dual Pairs

Resistance	Conductance
Inductance	Capacitance
Voltage	Current
Voltage source	Current source
Node	Mesh
Series path	Parallel path
Open circuit	Short circuit
KVL	KCL
Thevenin	Norton

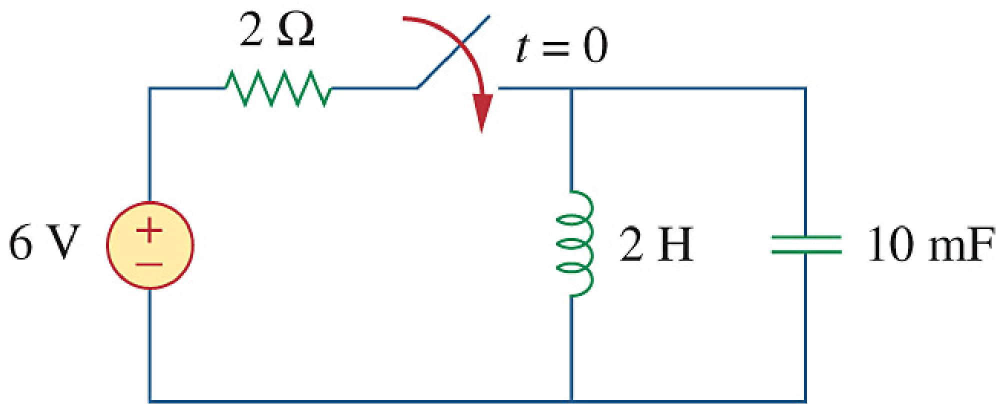
Given a planar circuit, we construct the dual circuit by taking the following steps:

1. Place a node at the center of each mesh of the given circuit. Place the reference node of the dual circuit outside the given circuit.
2. Draw lines between the nodes such that each line across an element. Replace the element by its dual.

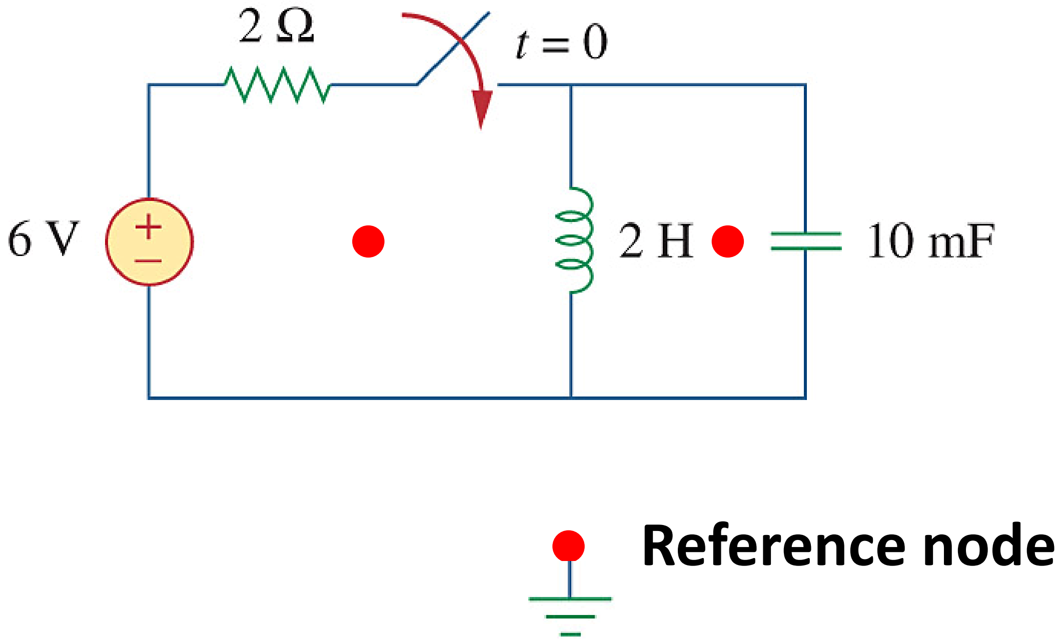
3. To determine the polarity of voltage sources and direction of current sources, follow this rule: A voltage source that produces **a positive (clockwise) mesh current** has as its dual a current source whose reference direction is from **the ground to the nonreference node**.

In case of doubt, one may verify the dual circuit by writing the nodal or mesh equations.

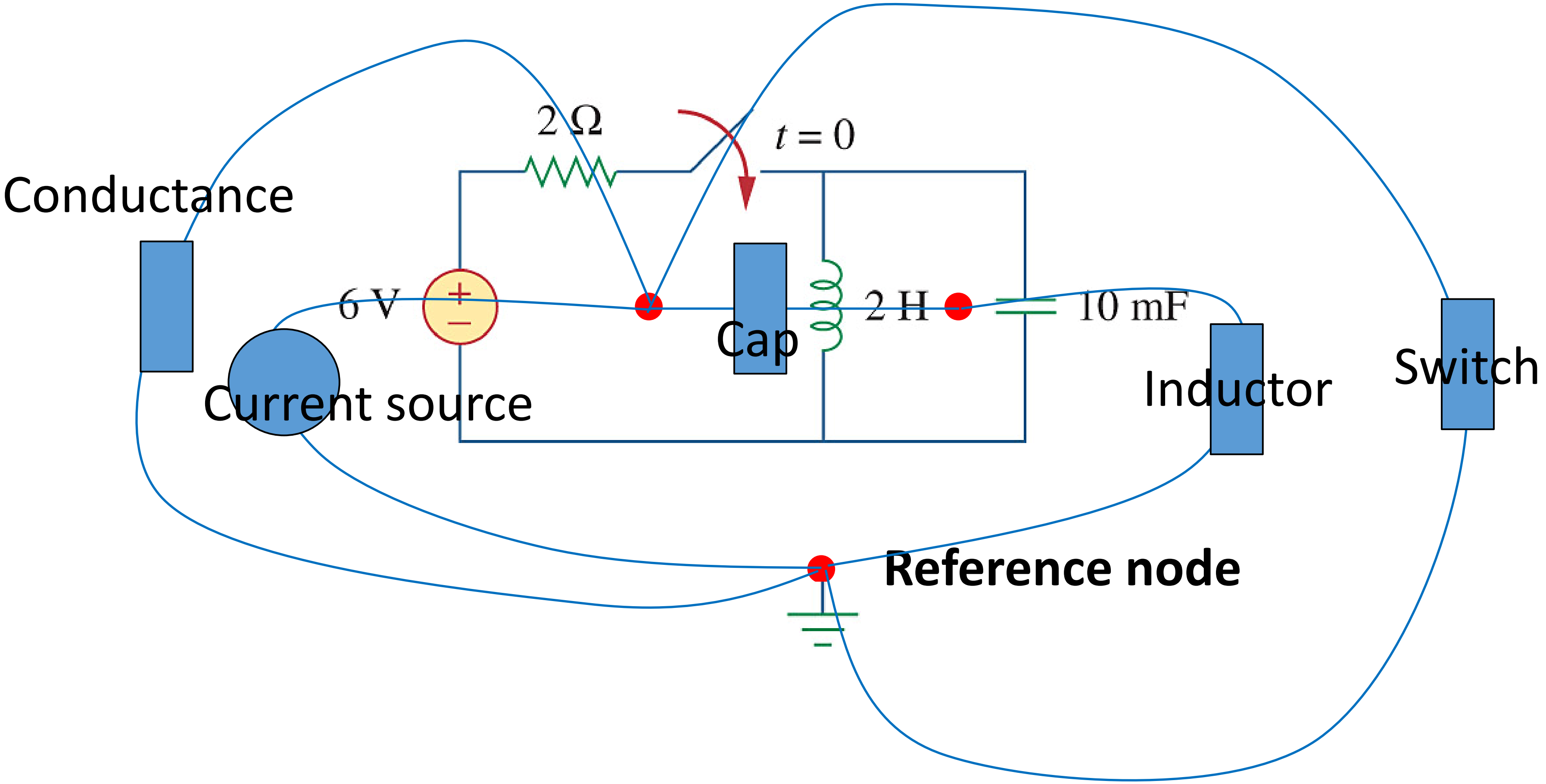
Example 8.14 Construct the dual of the circuit in Fig. 8.44.



1. Place a node at the center of each mesh of the given circuit. Place the reference node of the dual circuit outside the given circuit.

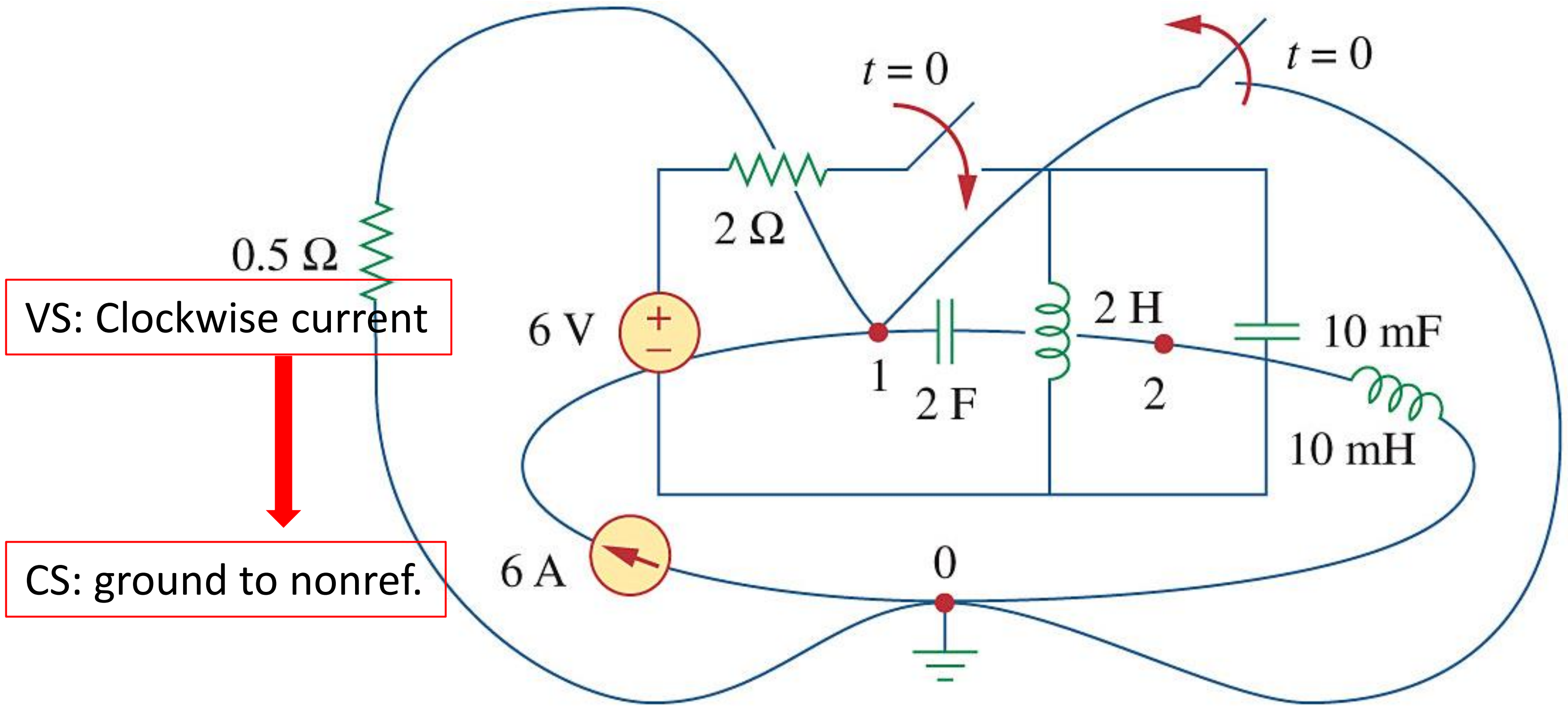


2. Draw lines between the nodes such that each line across an element. Replace the element by its dual.

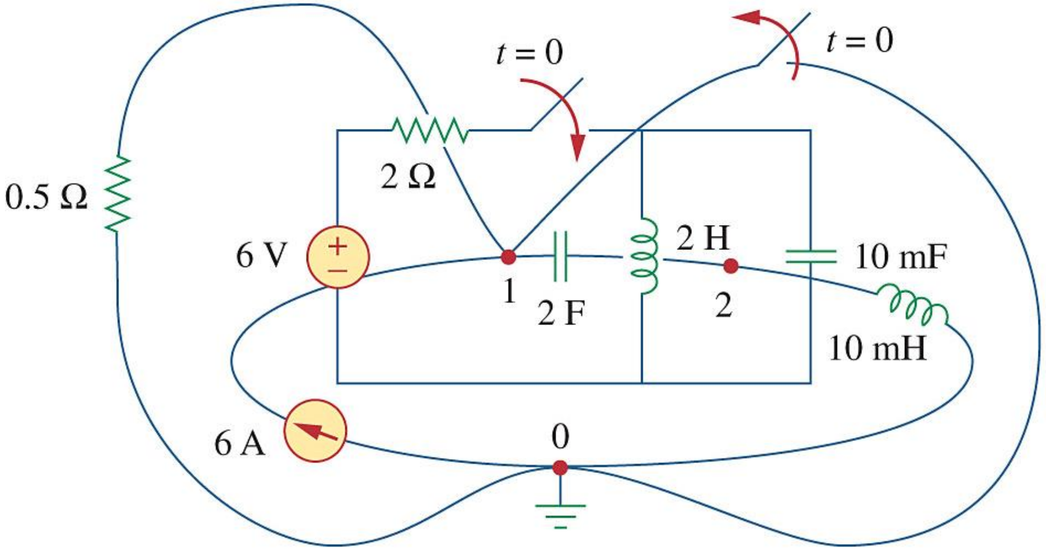


3. (i) A clockwise voltage source → A current source: from the ground to the nonreference node.

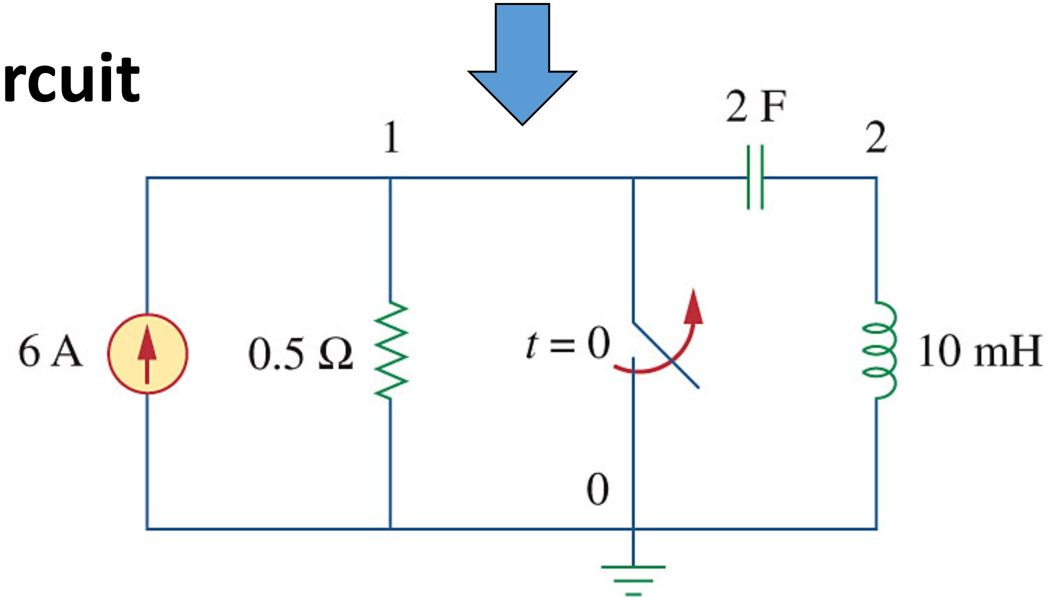
(ii) Open circuit → Short circuit



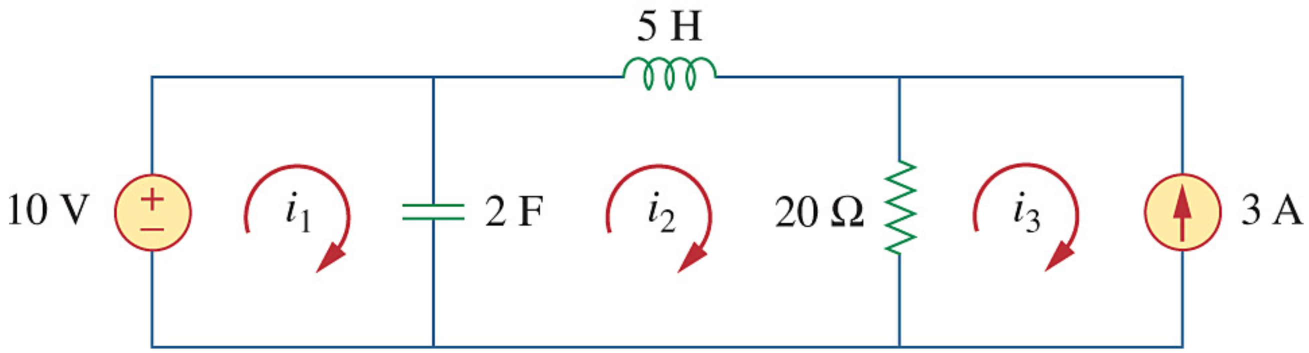
Dual Circuit

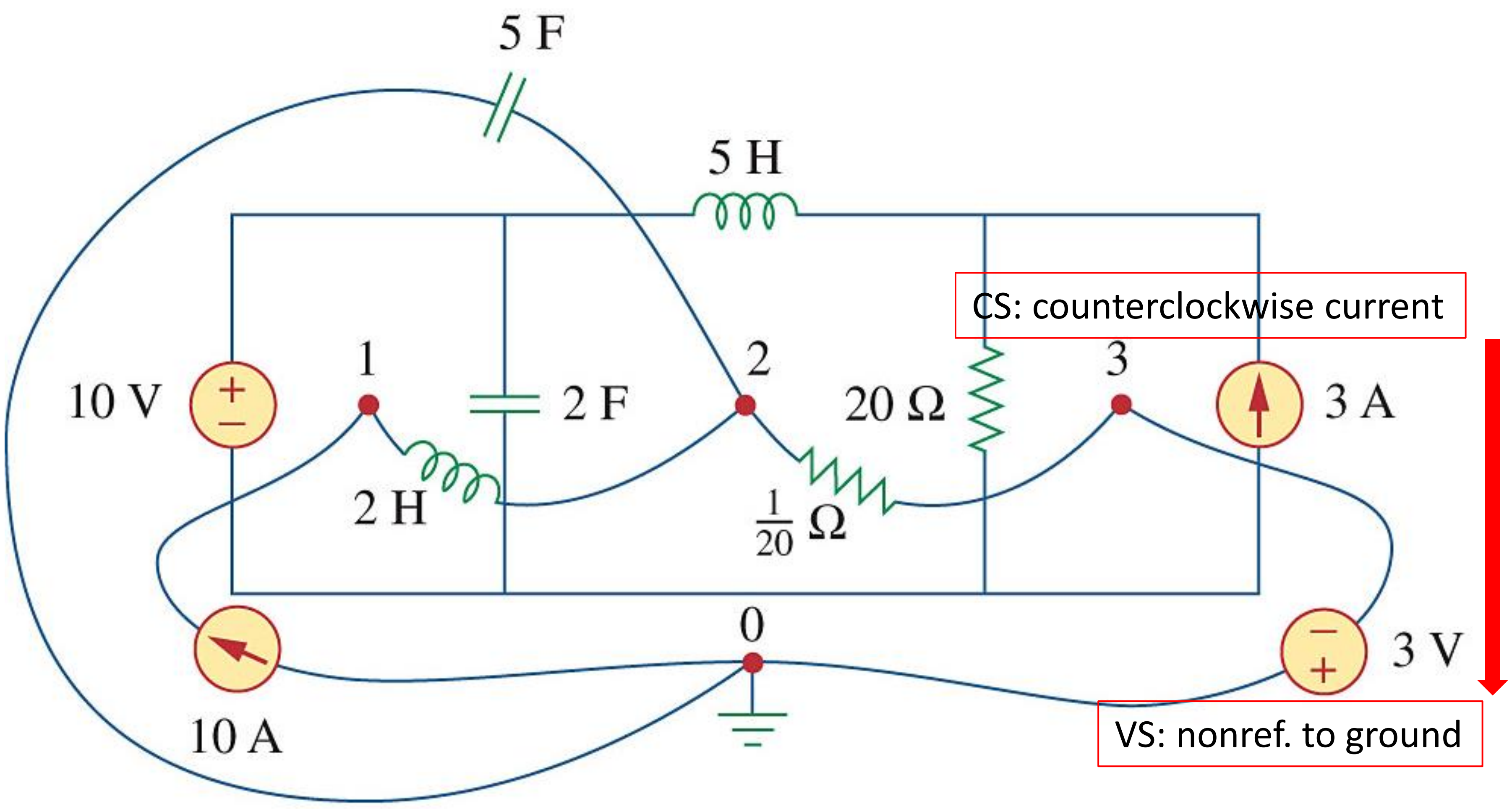


Redrawn Circuit

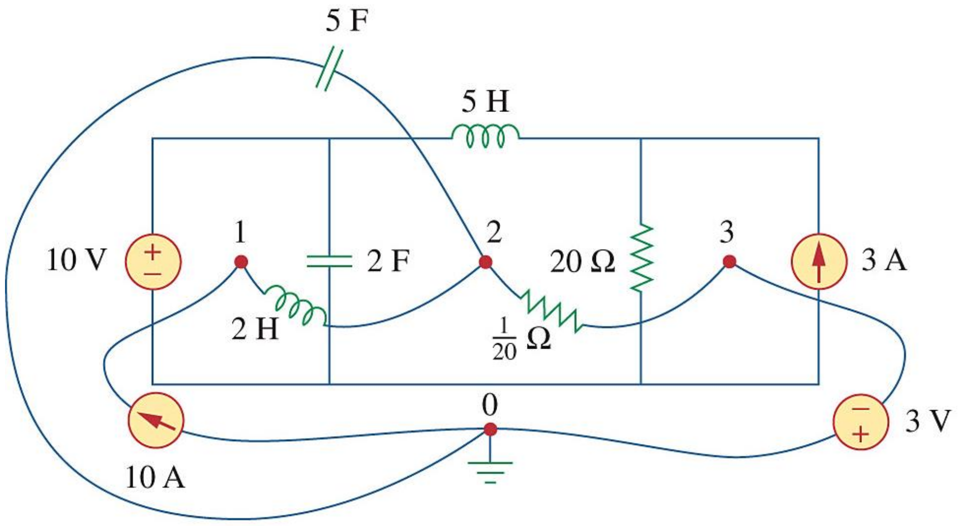


Example 8.15 Obtain the dual of the circuit in Fig. 8.48.




5 F
5 H
CS: counterclockwise current
10 V
1
2 F
2
20 Ω
3
3 A
2 H
$\frac{1}{20}$ Ω
0
3 V
10 A
VS: nonref. to ground

Dual Circuit



Redrawn Circuit