ECE2150J Introduction to Circuits

Chapter 7. First-Order Circuits

Yuljae Cho, PhD
Associate Professor
UM-SJTU Joint Institute, SJTU

7.1 Introduction

- Until now, we considered passive elements (R, L, and C), and an active element (op amp) individually.
- Now, we will see circuits with the combination of R, L, and C: Chapter 7: RC and RL; Chapter 8: RLC.
- **RC and RL circuits** are characterized by **first-order differential equations**. Hence, the circuits are collectively called **first-order circuits**.

Source of Energy

A circuit needs energy, either potential or current. In RL or RC circuits, there are two ways to provide energy to the circuits.

(i) By energy **initially stored** in the capacitive or inductive element.

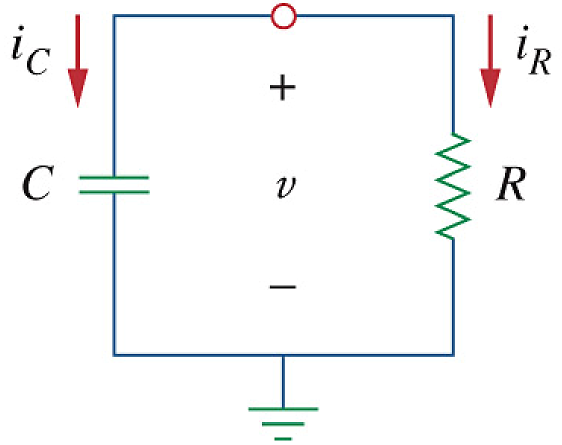
(ii) By **independent sources**.

Source of Energy	Case (i)	Case (ii)
RC	7.2 The Source-Free *RC* Circuit: $v_c(t=0) = V_0$	7.5 An *RC* Circuit with Step Input
RL	7.3 The Source-Free *RL* Circuit: $i_L(t=0) = I_0$	7.6 An *RL* Circuit with Step Input

7.2 The Source-Free *RC* Circuit

- A source-free RC circuit occurs when its dc source is suddenly disconnected. The **energy already stored in the capacitor** is released to the resistor.

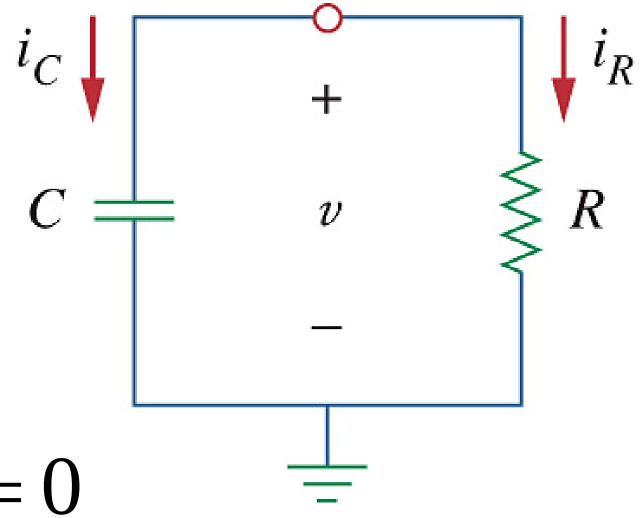
Source-Free RC Circuit Response



The capacitor is initially charged. We assume that at time *t=0*, the initial voltage is *v(0)* = V_0 with the corresponding value of energy stored:

$$w_C(0) = \frac{1}{2}CV_0^2$$

The circuit response, the capacitor voltage *v* in this case is



$$i_C + i_R = 0$$

$$i_C = C\frac{dv}{dt}, \; i_R = \frac{v}{R}$$

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \quad \Rightarrow \quad \frac{dv}{dt} + \frac{v}{RC} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

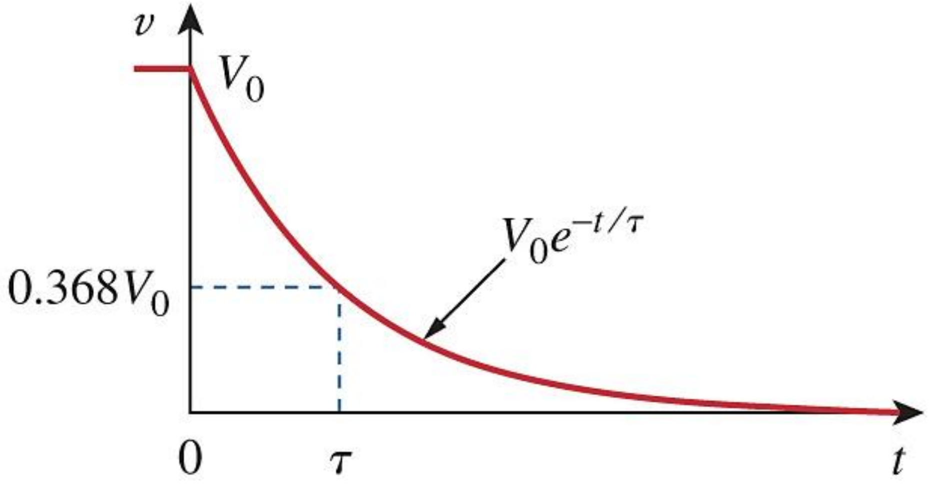
$\frac{dv}{v} = -\frac{1}{RC}dt$ integrate both sides

$$\ln v = -\frac{t}{RC} + C \rightarrow v = Ae^{-\frac{t}{RC}}$$

Using the initial condition v(0) = V_0, we get $\boldsymbol{v(t) = V_0 e^{-\frac{t}{RC}}}$

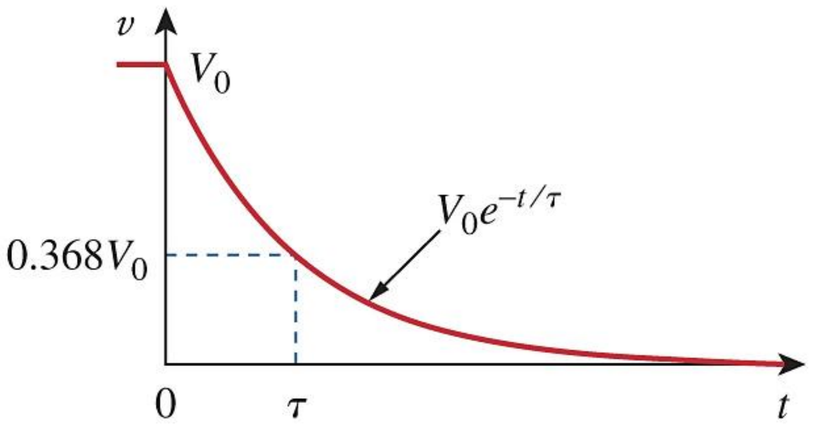
The response is due to the initial energy stored and the physical characteristics of the circuit. We call it **Natural Response, or zero-input,** of the circuit.

$$\boldsymbol{v(t) = V_0 e^{-\frac{t}{RC}}}$$



The response of the source-free RC circuit: at $t=0$, $v=V_0$, and as t increases, v decreases toward zero. The rapidity with which v decreases is expressed in terms of **the time constant $\boldsymbol{\tau}$**.

Time constant τ



The time constant of a circuit is the time required for the response to decay to a factor of 1/e or 36.8% of its initial value. This implies that at t = τ,

$$v = V_0 e^{-\frac{\tau}{RC}} = V_0 e^{-1}$$

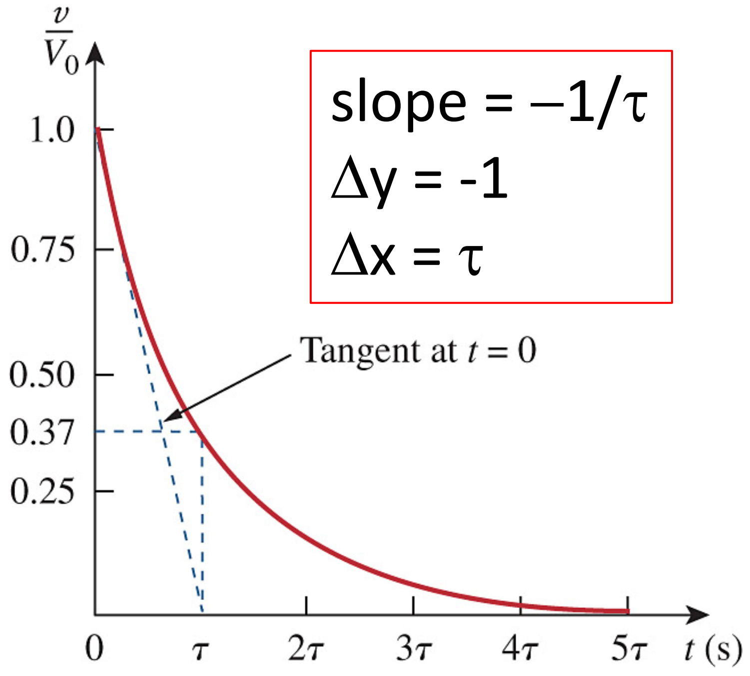
or

$$\tau = RC$$

The time constant may be viewed from another perspective. Evaluating the derivative of ${}^{v}/_{V_0}$ at t = 0, we obtain

$$\boldsymbol{v(t) = V_0 e^{-\frac{t}{\tau}}}$$

$$\left.\frac{d}{dt}\left(\frac{v}{V_0}\right)\right|_{t=0} = -\frac{1}{\tau}e^{-t/\tau}\bigg|_{t=0} = -\frac{1}{\tau}$$ **slope at t=0**

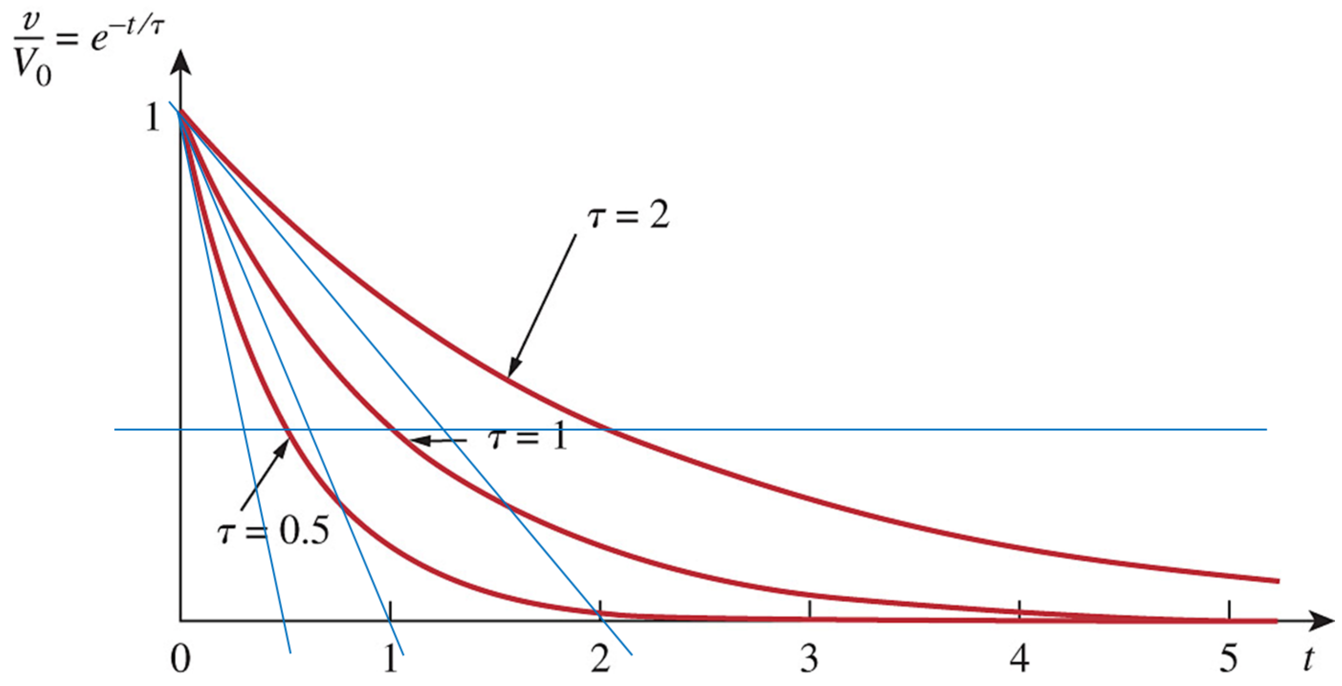


Thus, the time constant is the **initial rate of decay**. To find $\boldsymbol{\tau}$ from the response curve, draw the tangent to the curve at t = 0. The tangent intercepts with the time axis at t = $\boldsymbol{\tau}$**.**

TABLE 7.1 Values of $v/V_0 = e^{-t/\tau}$

t	τ	2τ	3τ	4τ	5τ
v/V_0	0.36788	0.13534	0.04979	0.01832	0.00674

After **5τ**, ***v* is less than 1% of the initial value V_0**. It is customary to assume that the capacitor is fully discharged after five time constants, i.e. the circuit reaches its **final state or steady state** when no changes take place with time.

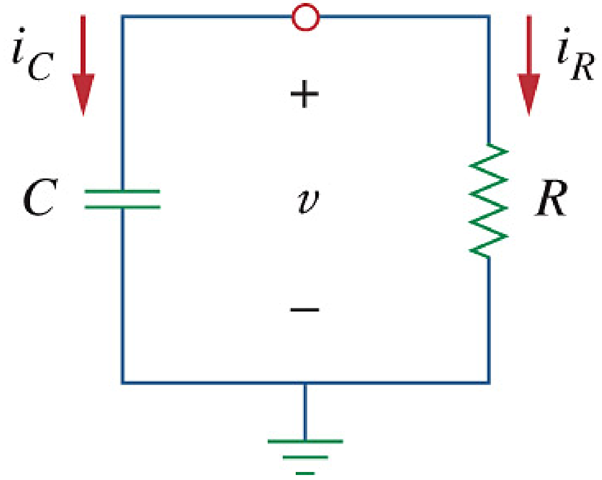


The smaller the time constant, the more rapidly the voltage decreases, i.e. the faster response → quick dissipation of energy stored.

Circuit response at R

Go back to **the circuit response,** from the voltage in capacitor $\boldsymbol{v(t) = V_0 e^{-\frac{t}{RC}}}$ we can find the resistor current

$$i_R = \frac{v}{R} = \frac{V_0}{R} e^{-t/\tau}$$



The power dissipated in the resistor is

$$p = v i_R = \frac{V_0^2}{R} e^{-2t/\tau}$$

The energy absorbed by the resistor up to time t is

$$w_R = \int_0^t \left(\frac{V_0^2}{R} e^{-2t/\tau} \right) dt = \frac{V_0^2}{R} \frac{e^{-2t/\tau}}{-2/\tau} \Bigg|_0^t$$

$$= \frac{1}{2} C V_0^2 \left(1 - e^{-2t/\tau} \right)$$

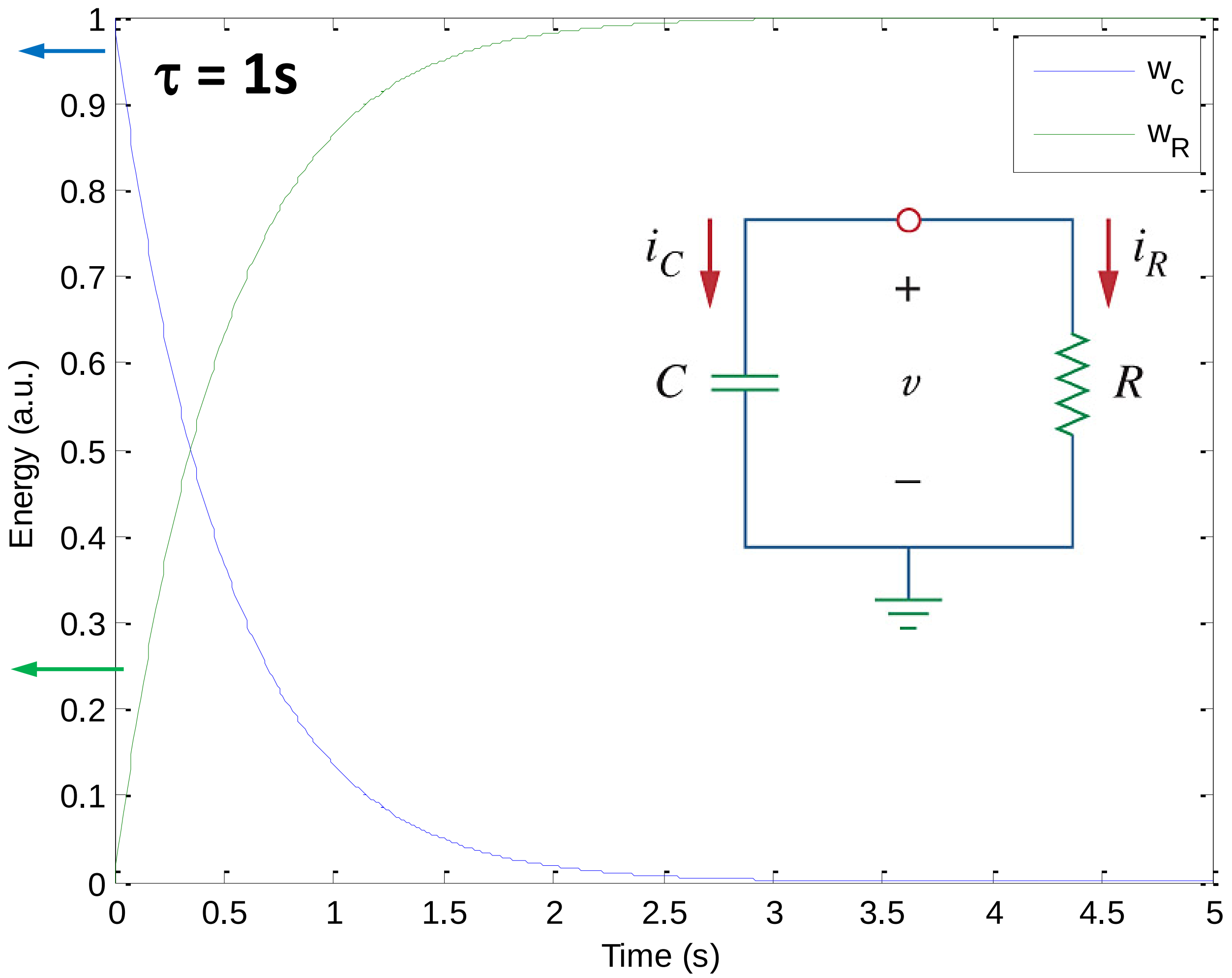
Notice that as $t \to \infty, w_R \to \frac{1}{2} C V_0^2 = w_C(0),$

the energy initially stored in the capacitor

Energy in Circuit

Energy in Capacitor

Energy in Resistor



Summary: RC source free response

To find the RC source free response

1. The initial voltage **v(0) = V_0** across the capacitor.
2. The time constant $\boldsymbol{\tau}$ **= RC** where R is often the equivalent resistance at the terminals of C.

The response, the capacitor voltage V_C, is

$$\boldsymbol{V_C = V = V(0)e^{-t/\tau}}$$

and then, we can find **other circuit responses**, V_R, i_R, I_C etc.

Practice Problem 7.1 Refer to the circuit in Fig. 7.7. Let $v_C(0) = 45$ V. Determine v_C, v_x, and i_o for $t \geq 0$.

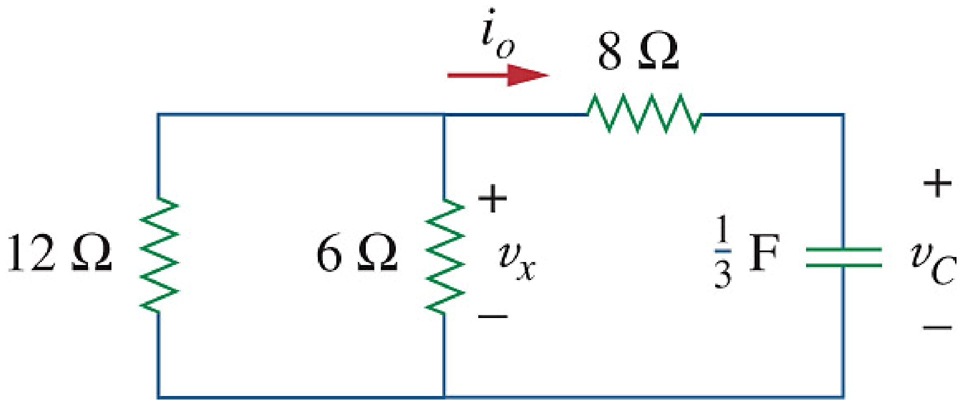


Figure 7.7 An RC circuit.

Solution :

The equivalent resistance seen at the terminals of the capacitor is

$$R_{eq} = 8 + 12 \parallel 6 = 12\ (\Omega)$$

The time constant is

$$\tau = R_{eq}C = 12 \times \frac{1}{3} = 4\ (\mathrm{s})$$

The capacitor voltage is

$$v_C = v_C(0)e^{-t/\tau} = 45e^{-t/4} = 45e^{-0.25t}\ (\mathrm{V})$$

$$i_o = C\frac{dv_C}{dt} = \frac{1}{3} \times \frac{d}{dt}\left(45e^{-0.25t}\right)$$

$$= 15 \times \left(-0.25e^{-t/4}\right) = -3.75e^{-t/4}\ (\mathrm{A})$$

$$v_x = 8i_o + v_C = 8 \times \left(-3.75e^{-0.25t}\right) + 45e^{-0.24t}$$

$$= 15e^{-0.25t}\ (\mathrm{V})$$

Practice Problem 7.12 If the switch in Fig. 7.10 opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.

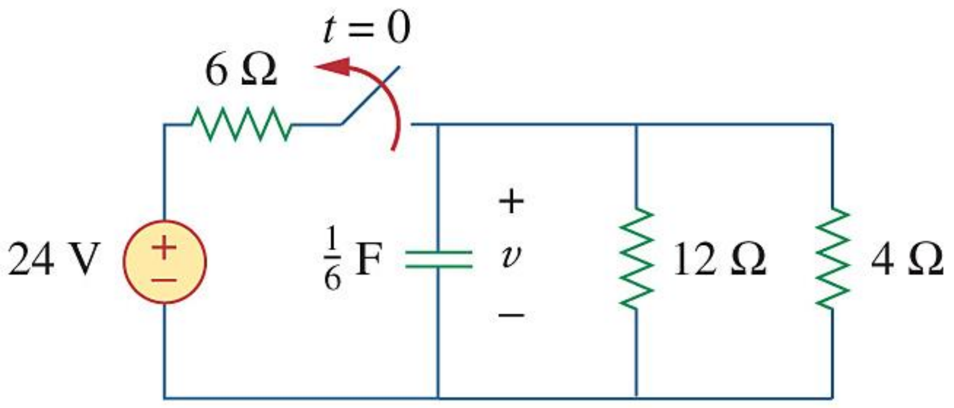
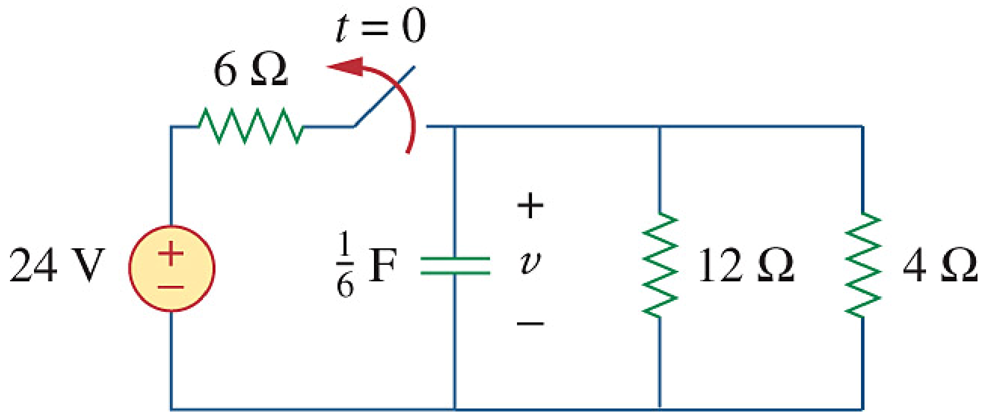


Figure 7.10



Solution :

When $t \leq 0$, the capacitor voltage

$$v(t) = 24 \times \frac{12 \parallel 4}{6 + 12 \parallel 4} = 8 \text{ (V)}$$

Hence, $v(0) = 8$ V.

When $t \geq 0$, the circuit becomes a source-free RC circuit with

$$\tau = R_{eq}C = (12 \parallel 4) \times \frac{1}{6} = \frac{1}{2} \text{ (s)}$$

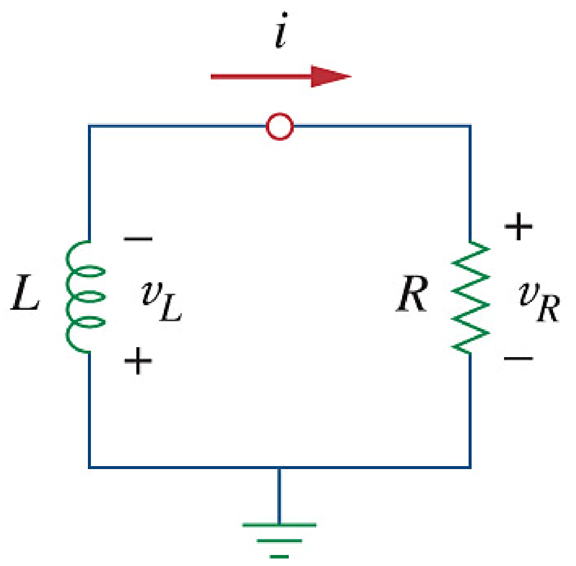
Therefore,

$$v(t) = v(0)e^{-t/\tau} = 8e^{-t/(1/2)} = 8e^{-2t} \text{ (V)}$$

$$w_C(0) = \frac{1}{2}C\left(v(0)\right)^2 = \frac{1}{2}\times\frac{1}{6}\times 8^2$$

$$= \frac{16}{3} \approx 5.33 \text{ (J)}$$

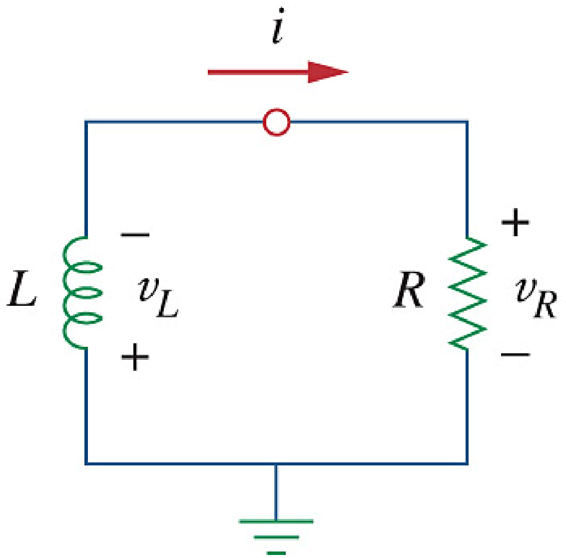
7.3 The Source-Free *RL* Circuit



Now, we think about the RL circuit with the initial current i(0) = I_0 through the inductor, and corresponding energy

$$w_L(0) = \frac{1}{2} L I_0^2$$

Source-Free RC Circuit Response



$$v_L + v_R = 0$$

$$v_L = L\frac{di}{dt}, \; v_R = iR$$

$$L\frac{di}{dt} + iR = 0 \qquad \frac{di}{dt} + \frac{R}{L}i = 0$$

Solve the 1st order differential equation

$$i = Be^{rt} = Be^{-\frac{R}{L}t} = Be^{-\frac{t}{(L/R)}}$$

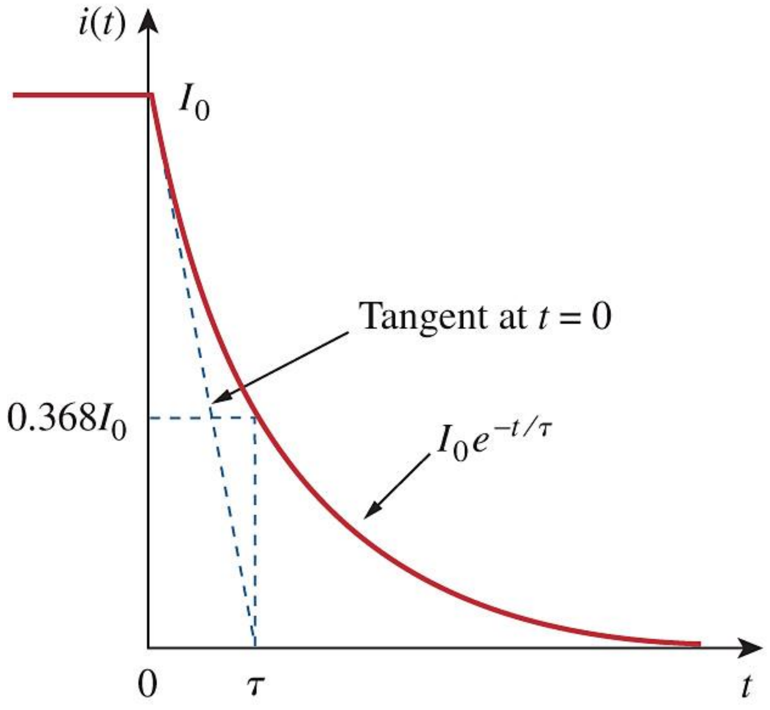
$$i(0) = B = I_0$$

$$i = I_0 e^{-\frac{t}{(L/R)}}$$

$$i = I_0 e^{-\frac{t}{(L/R)}}$$

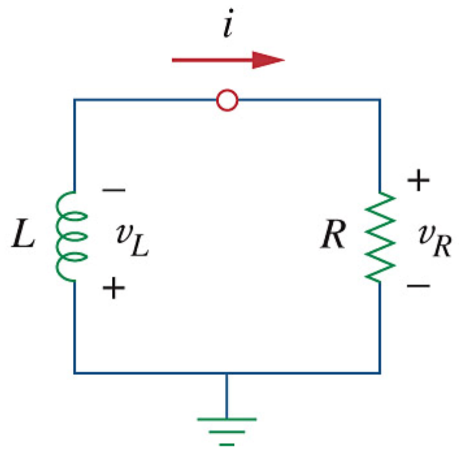
Let $\tau = L/R$, we have

$$i = I_0 e^{-t/\tau}$$



Because the response is due only to I_0, it is called **natural response**, **or zero-input**, of the circuit.

Circuit response and Energy at R



Voltage at the resistor

$$v_R = iR = I_0 R e^{-t/\tau}$$

Power at the resistor

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

Energy at the resistor up to time t

$$w_R = \int_0^t \left(I_0^2 R e^{-2t/\tau} \right) dt = I_0^2 R \left. \frac{e^{-2t/\tau}}{-2/\tau} \right|_0^t$$

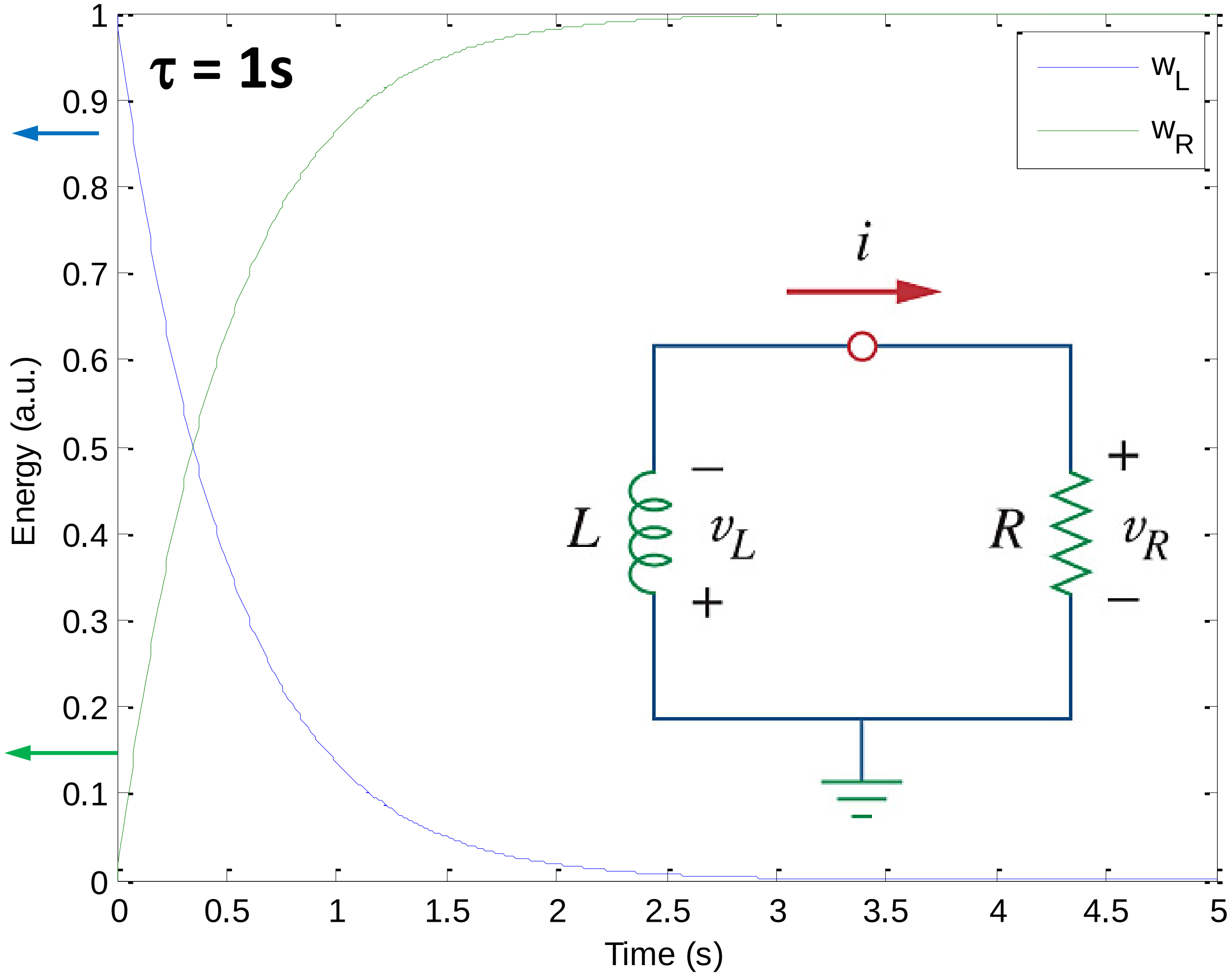
$$= \frac{1}{2} L I_0^2 \left(1 - e^{-2t/\tau} \right)$$

Notice that as $t \to \infty$, $w_R \to \frac{1}{2} L I_0^2 = w_L(0)$, the energy initially stored in the inductor.

Energy in Circuit

Energy in Inductor

Energy in Resistor



Summary: RL source free response

To find the RL source free response

1. The initial current $\mathbf{i(0) = I_0}$ through the inductor.
2. The time constant $\boldsymbol{\tau}$ = **L/R**$_{eq}$ where R_{eq} is often the equivalent resistance at the terminals of L.

The response, the inductor current i, is

$$\boldsymbol{i_L = i = i(0)e^{-t/\tau}}$$

and then, we can find **other circuit responses**, V_L, V_R, i_R etc.

Practice Problem 7.4 For the circuit in Fig. 7.18, find $i(t)$ for $t > 0$.

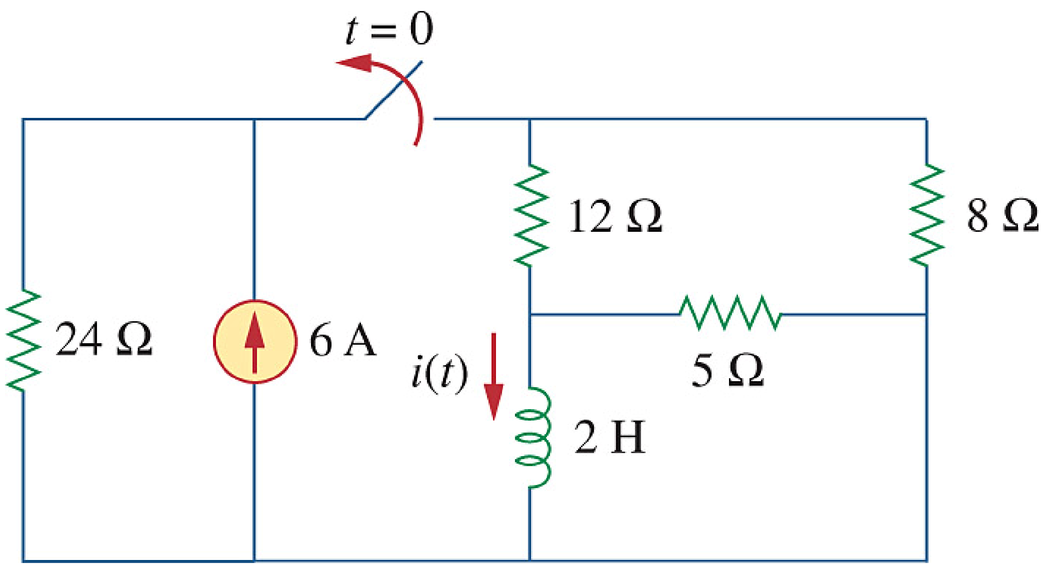
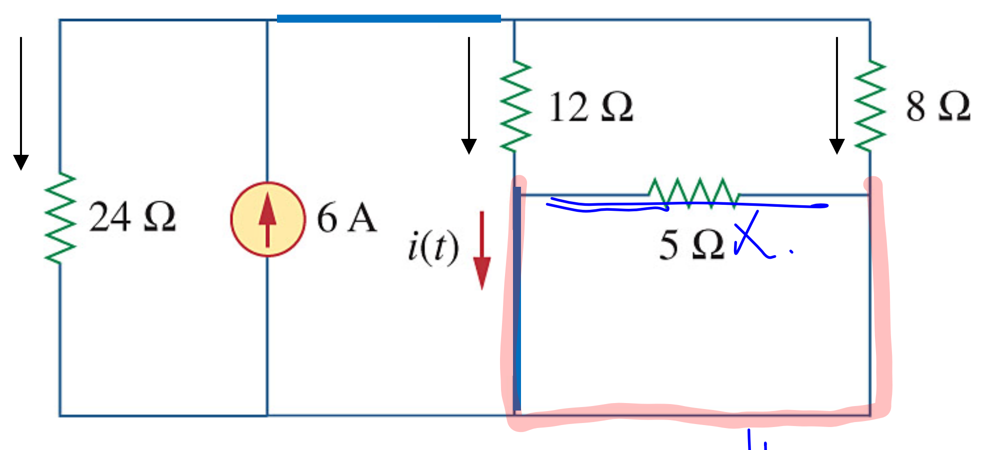


Figure 7.18

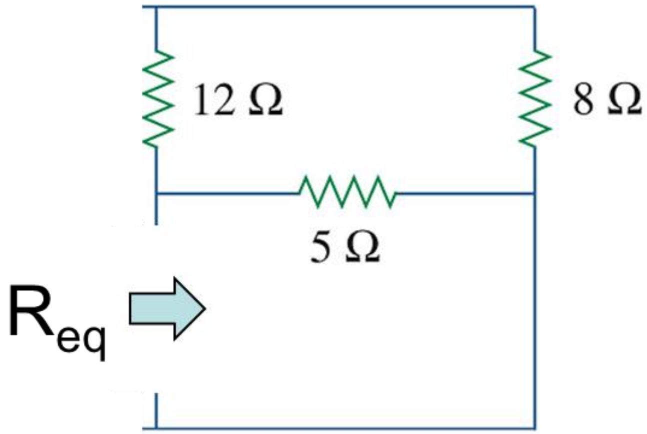


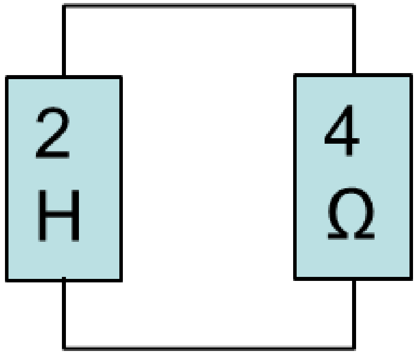
equal potential.
no current.

Solution :

When $t < 0$, the current through the **5-Ω** resistor is zero.

$$i(t) = 6 \times \frac{24 \parallel 8}{24 \parallel 8 + 12} = 2\ (\mathrm{A})$$





When $t > 0$,

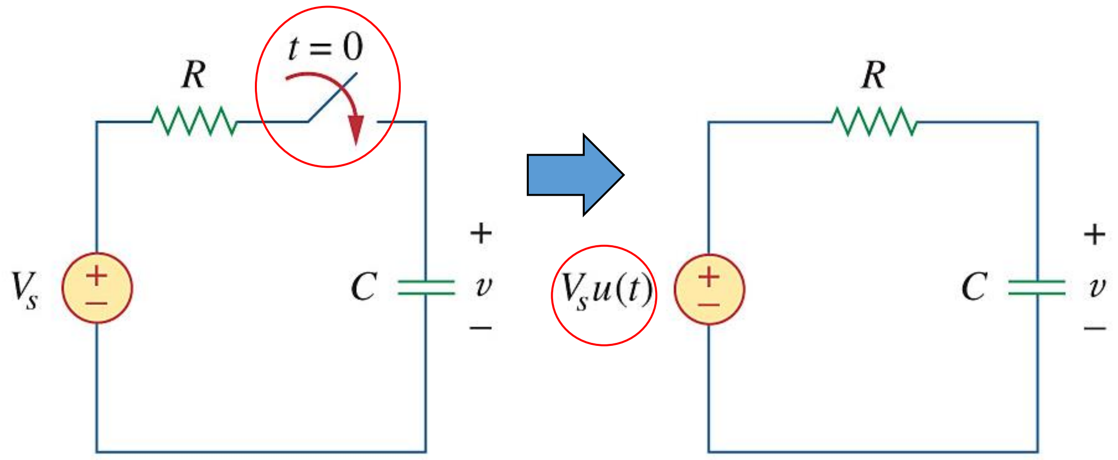
$$R_{eq} = 5 \parallel (12+8) = 4\ (\Omega)$$

$$\tau = L / R_{eq} = 2/4 = 0.5\ (\mathrm{s})$$

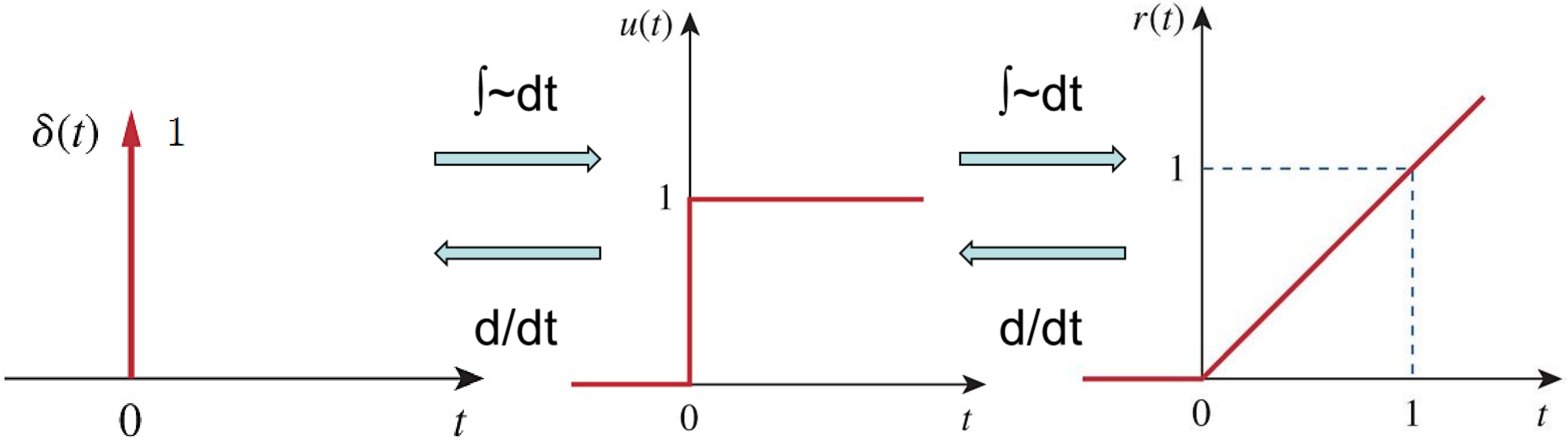
$$i(t) = i(0)e^{-t/\tau} = 2e^{-t/0.5} = 2e^{-2t}\ (\mathrm{A})$$

7.4 Singularity Functions

- Singularity functions (also called switching functions) are functions that either are **discontinuous** or have **discontinuous derivatives**. They serve as good approximations to switching operations.



The three most widely used singularity functions in circuit analysis are the *unit step*, the *unit impulse*, and the *unit ramp* functions.

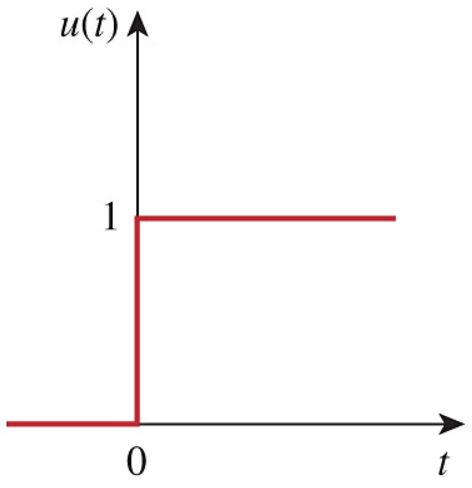


Unit Impulse **Unit Step** **Unit Ramp Function**

(i) Unit Step

The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t. In mathematical terms,

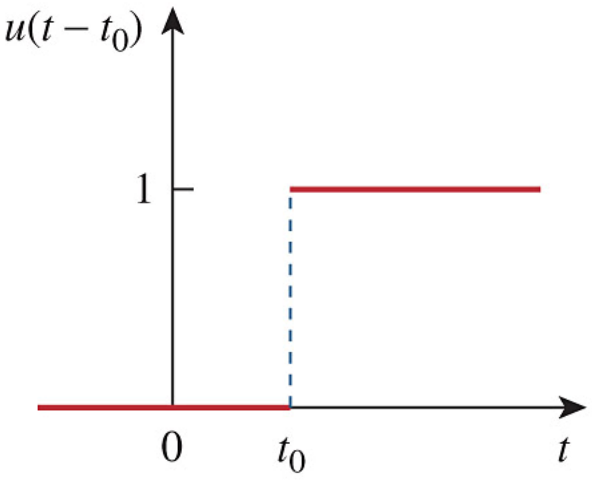
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



The unit step function is not defined at t = 0, where it changes abruptly from 0 to 1.

If the abrupt change occurs at $t = t_0$ (where $t_0 > 0$) instead of $t = 0$, the mathematical representation becomes

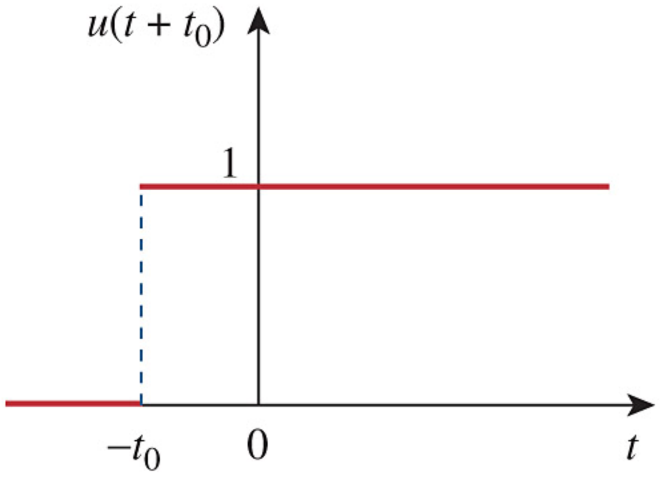
$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$



meaning that $u(t)$ is delayed by t_0 seconds.

If the abrupt change occurs at $t = -t_0$ (where $t_0 > 0$) instead of $t = 0$, the mathematical representation becomes

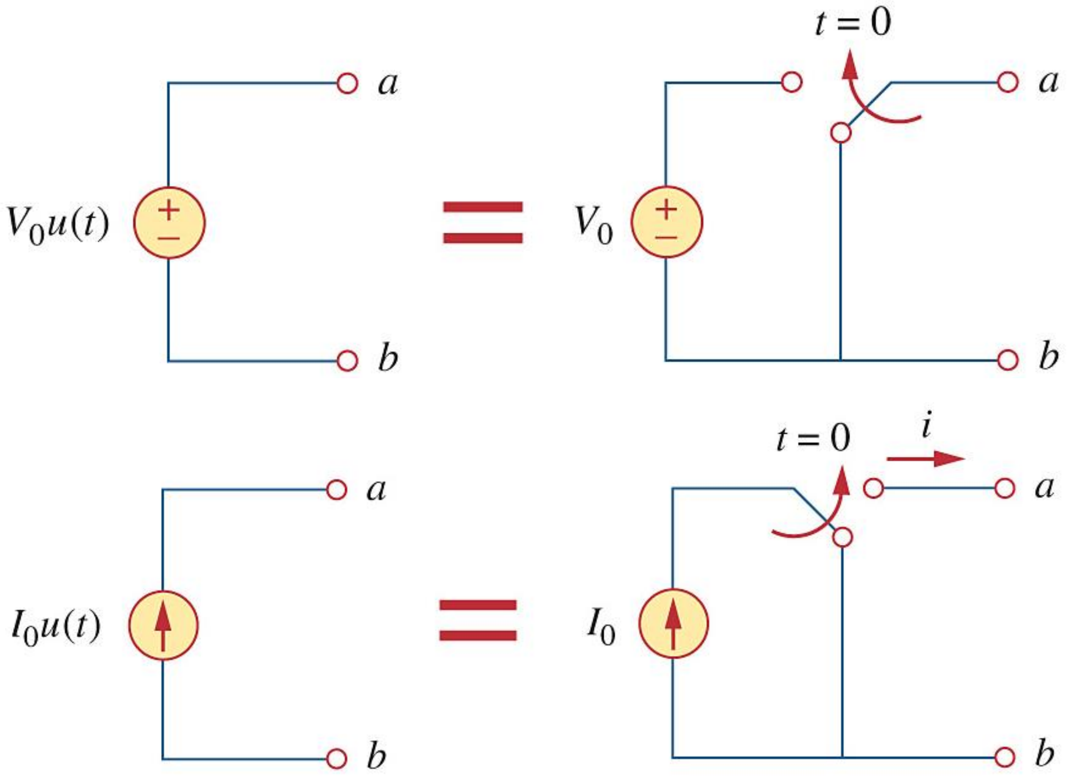
$$u(t+t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$



meaning that $u(t)$ is advanced by t_0 seconds.

Switching operations

Switching operations create abrupt changes in voltages and currents. An abrupt change can be **represented by the step function**, K*u(t)*, where K is a constant.



short -> open

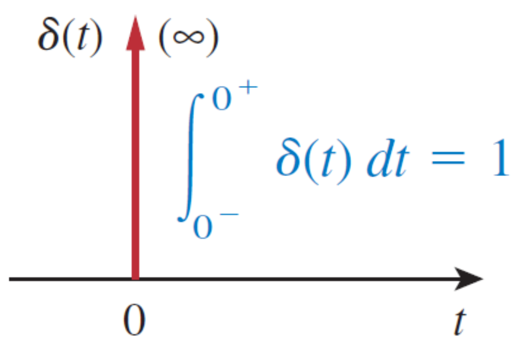
open -> short

(ii) Unit Impulse

The derivative of the unit step function u(t) is the unit impulse function δ(t), or delta function.

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

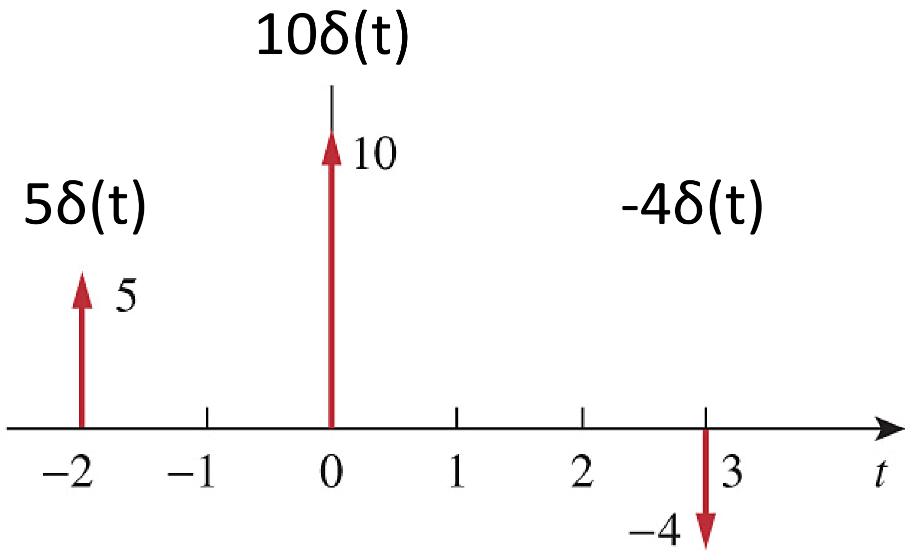
$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$



The **unit impulse function δ(t)** is zero everywhere except at t = 0, where it is undefined, i.e. a signal of infinite amplitude and zero duration.

The **area** is known as the **strength** of the function.

e.g. $10\delta(t)$ has an area of 10.



The unit impulse function has a property:

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

where f(t) is a function that is continuous at $t = t_0$.

This means: when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs → **sampling or sifting property**.

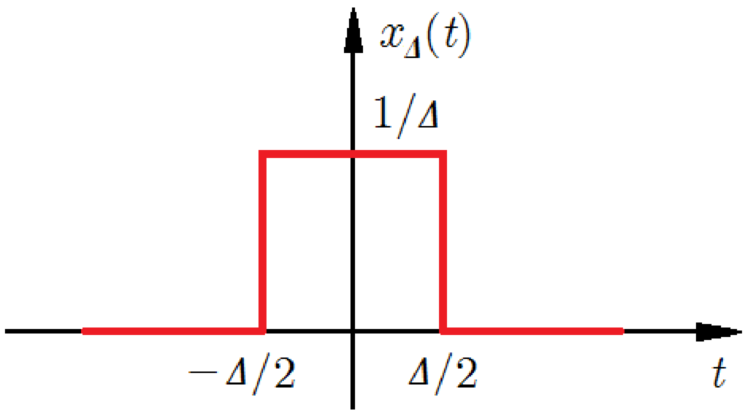
Proof :

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = \int_{-\infty}^{\infty} f(t_0)\delta(t-t_0)dt$$

$$= f(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)dt = f(t_0)$$

Unit Impulse Another Approach

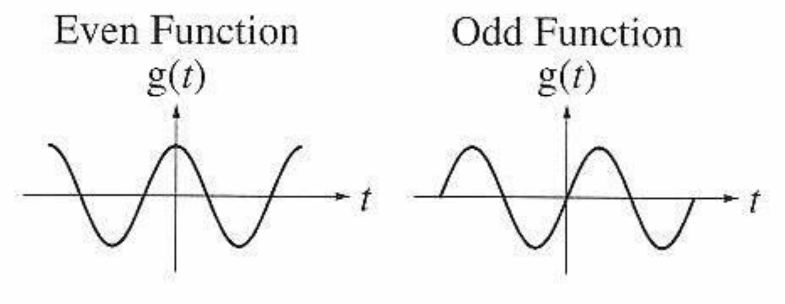
Singularity functions can be used to write mathematical expressions for signals. For example, the rectangular pulse

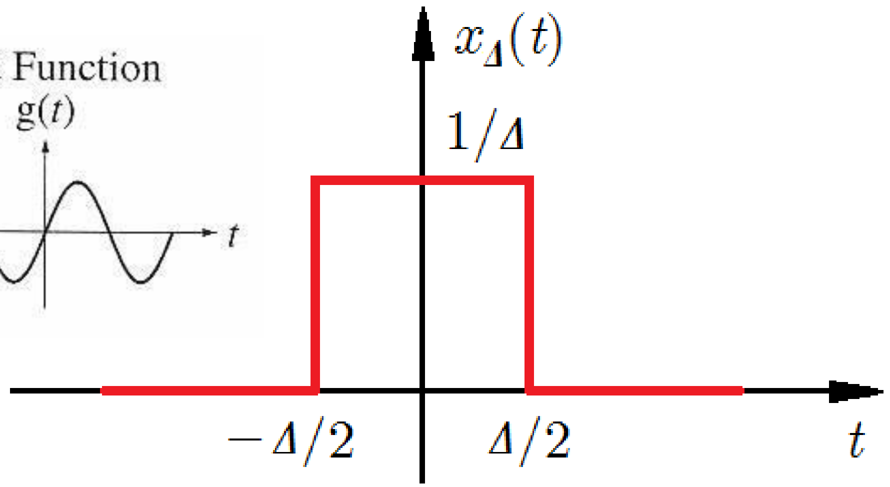


$$x_\Delta(t) = \frac{1}{\Delta}\left(u(t+\Delta/2) - u(t-\Delta/2)\right)$$

We view the unit impulse function $\delta(t)$ as the limiting form of any pulse, say the rectangular pulse $x_{\Delta}(t)$ (shown below), that is an even function of time t with unit area:

$$\delta(t) = \lim_{\Delta \to 0} x_{\Delta}(t)$$





Proof

Without loss of generality, we take the rectangular pulse as an example.

$$\delta(t) = \lim_{\Delta \to 0} x_\Delta(t)$$

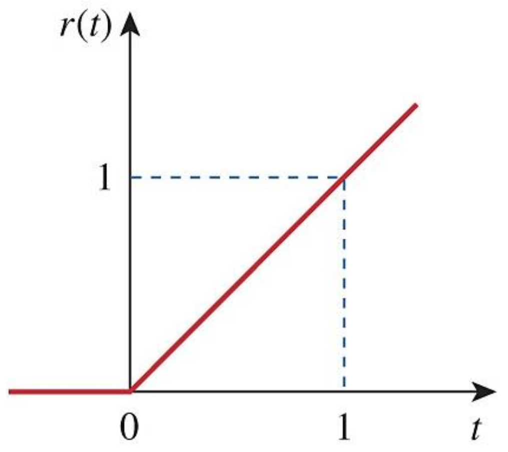
$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \left(u(t + \Delta / 2) - u(t - \Delta / 2) \right)$$

$$= \lim_{\Delta \to 0} \frac{u(t + \Delta / 2) - u(t - \Delta / 2)}{(t + \Delta / 2) - (t - \Delta / 2)} = \frac{d}{dt} u(t)$$

(iii) Unit Ramp

The unit ramp function is zero for negative values of t and has a unit slope for positive values of t.

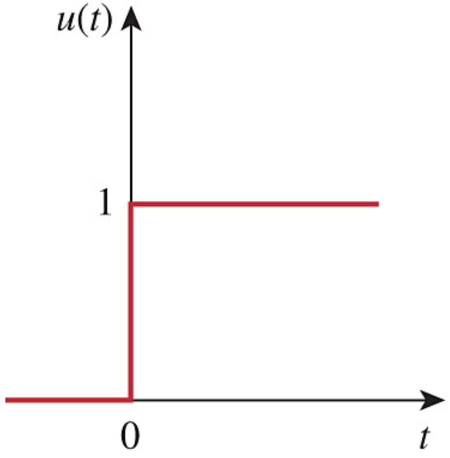
$$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}$$



Integrating the unit step function u(t) results in the unit ramp function r(t).

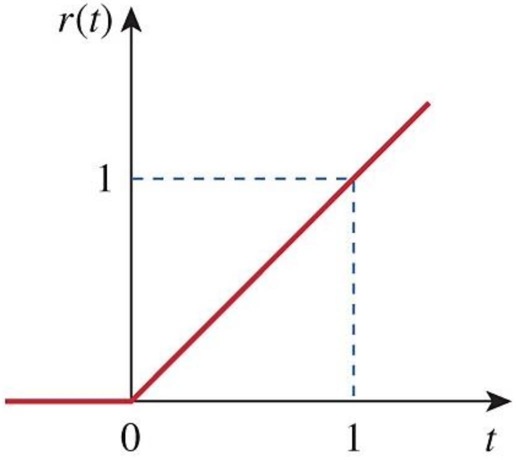
$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau = tu(t)$$

Unit step



$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

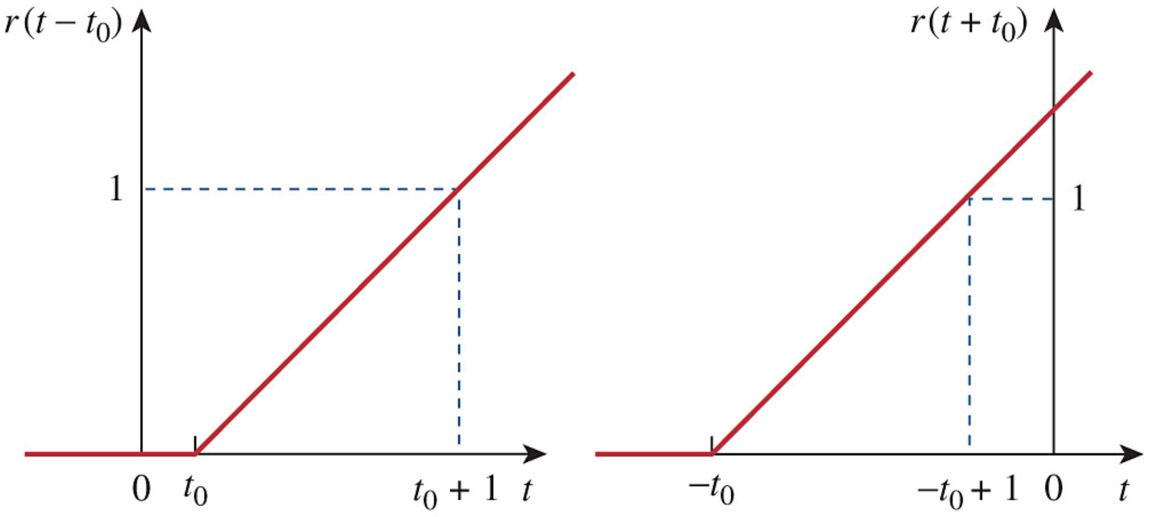
Unit ramp



$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$

Time delay in the unit ramp function

The unit ramp function may be delayed or advanced by time t_0.



Delayed by t_0 Advanced by t_0

Example 7.6 Express the voltage pulse in Fig. 7.31 in terms of the unit step. Calculate its derivative and sketch it.

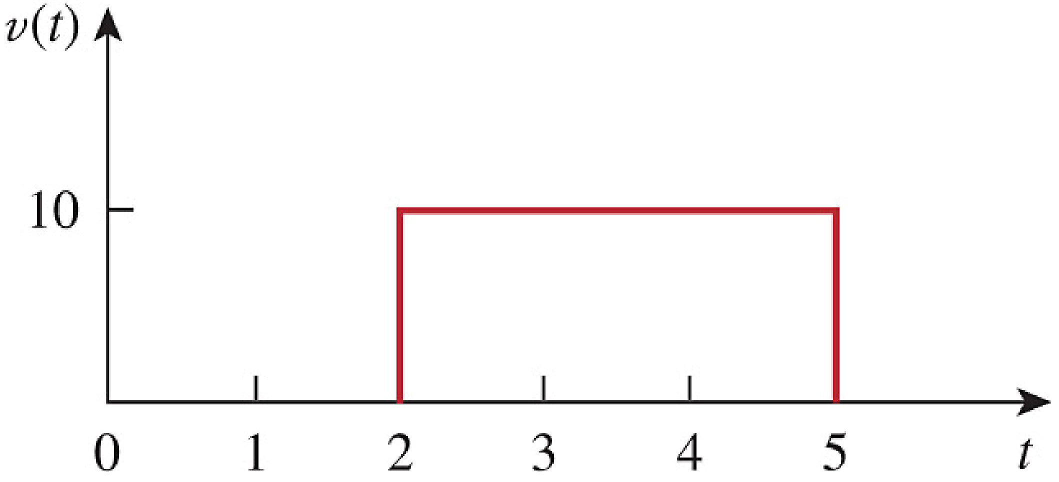


Figure 7.31

Solution :

$$v(t) = 10\left[u(t-2) - u(t-5)\right]$$

$$\frac{dv}{dt} = 10\left[\delta(t-2) - \delta(t-5)\right], \text{ see Fig. 7.32(b).}$$

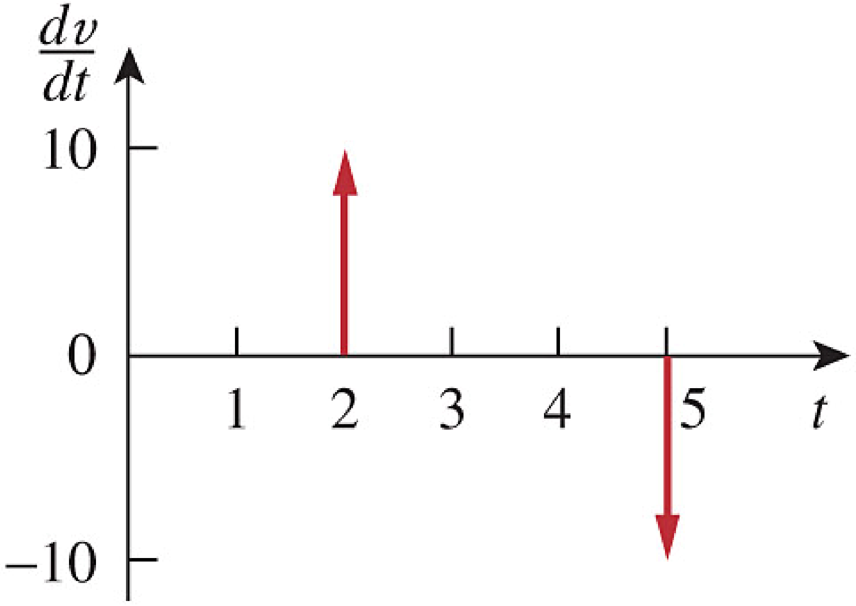


Figure 7.32(b)

Example 7.7 Express the *sawtooth* function shown in Fig. 7.35 in terms of singularity functions.

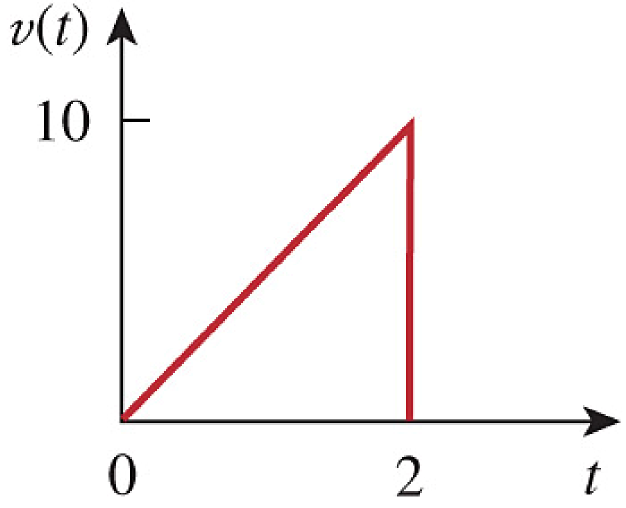
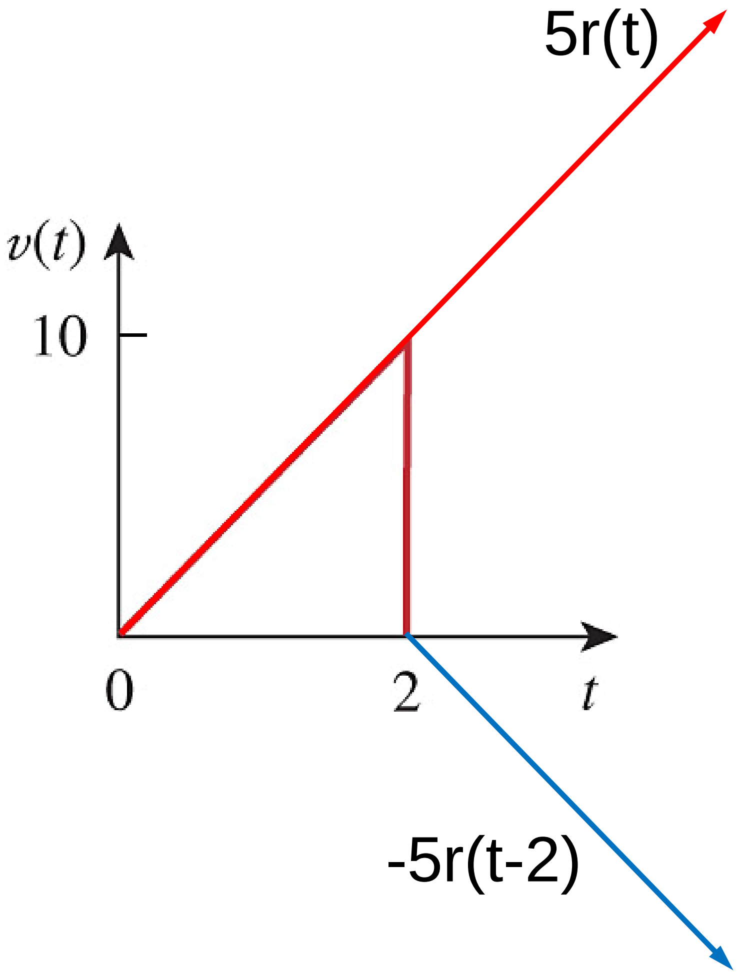
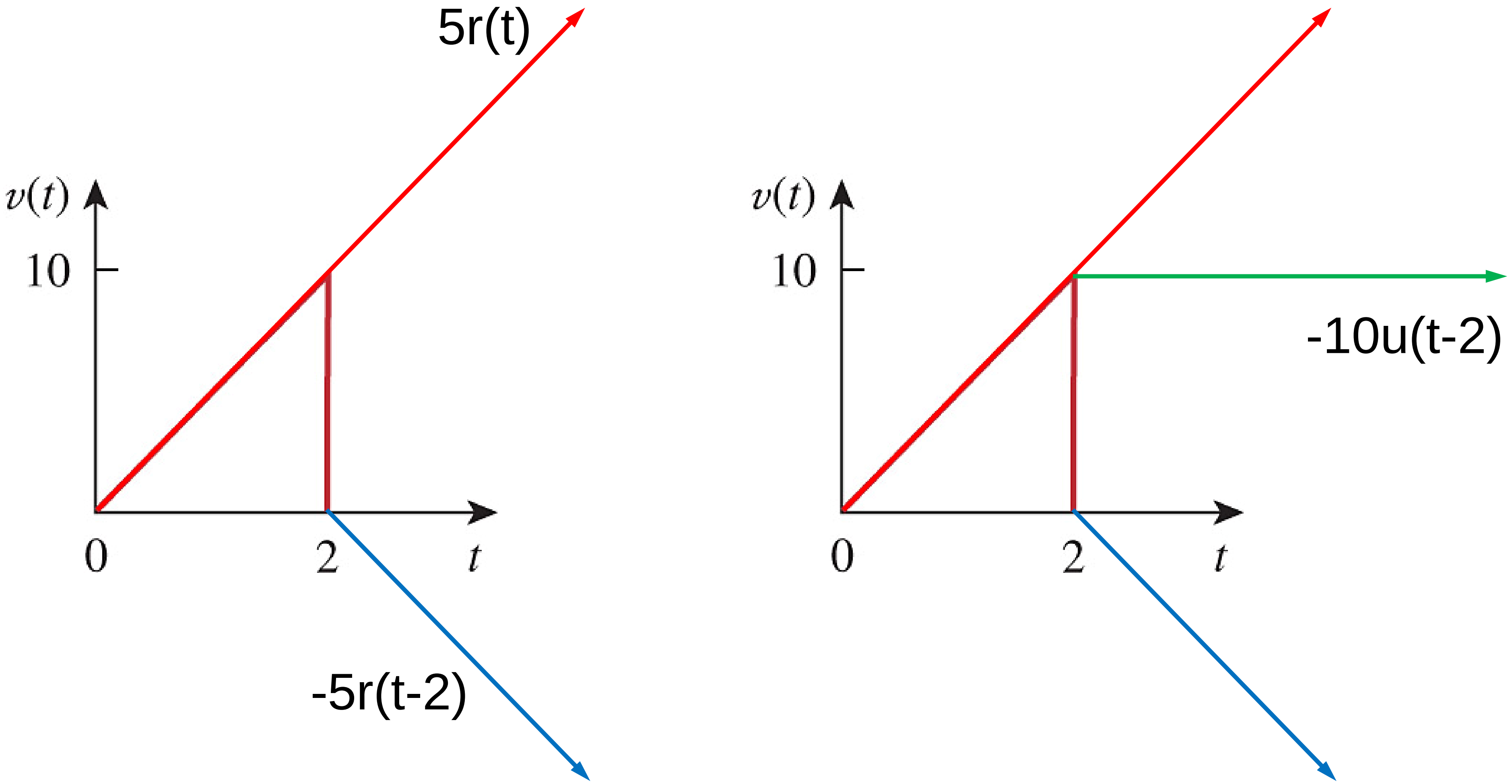


Figure 7.35

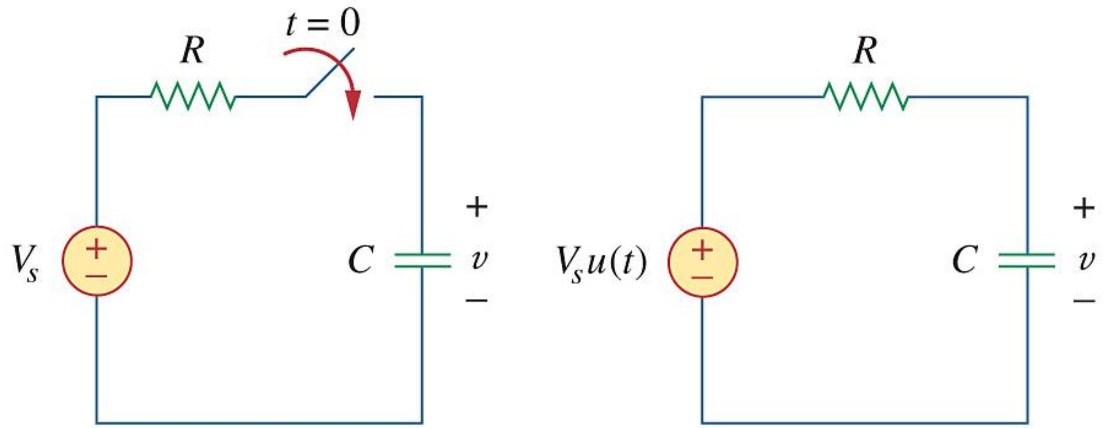

5r(t)
v(t)
10
0
2
t
-5r(t-2)



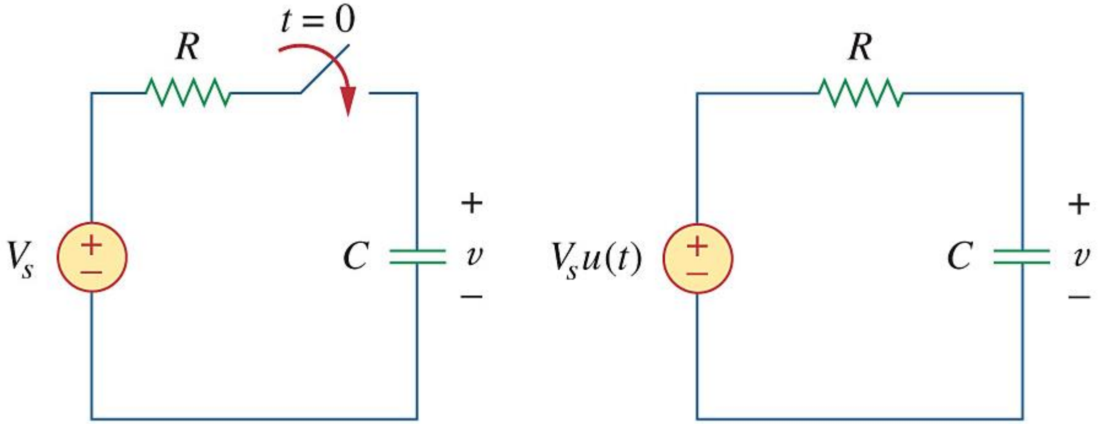
$$v(t) = 5r(t) - 5r(t-2) - 10u(t-2)$$

7.5 Step Response of an RC Circuit

The step response of a circuit is its behavior due to a sudden application of **the step function**, either a voltage or a current source.

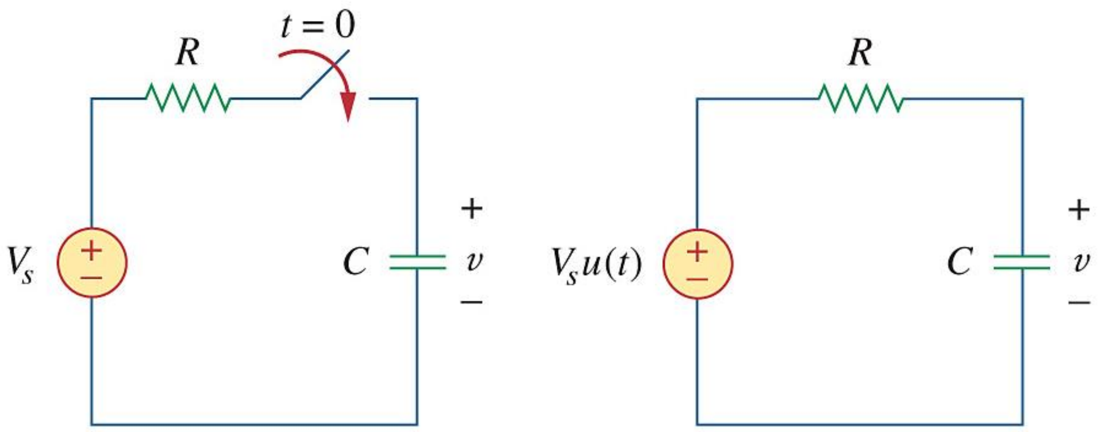


We select **the capacitor voltage as the circuit response** to be determined. We assume an initial voltage V_0 on the capacitor.



Because the voltage across the capacitor cannot change instantaneously

$$\boxed{v(0^+) = v(0^-) = V_0}$$ **Initial condition**



For $t > 0$,

$$\left(C\frac{dv}{dt}\right)R + v = V_s \;\Rightarrow\; \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$

or

$$\boxed{\frac{dv}{dt} + \frac{1}{\tau}v = \frac{V_s}{\tau}}, \text{ where } \tau = RC$$

Solve this differential equation

Method 1

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v}{RC} = \frac{V_s}{RC} \Rightarrow \frac{\mathrm{d}v}{v - V_s} = -\frac{\mathrm{d}t}{RC}$$

Integrate both sides

$$\ln(\mathrm{v} - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t \Rightarrow \ln(\mathrm{v(t)} - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

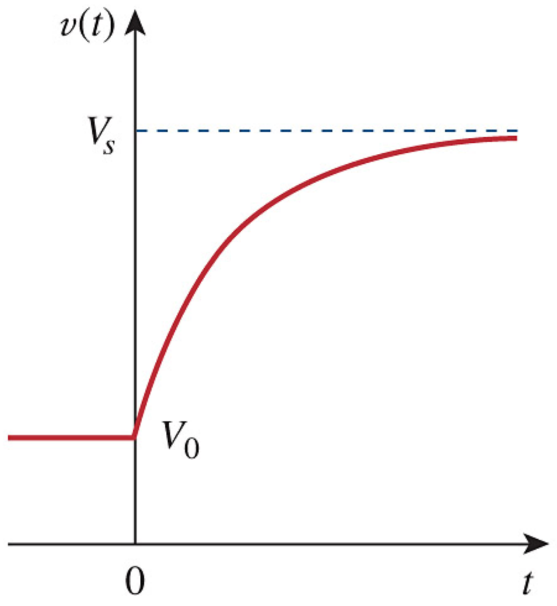
$$\ln\left(\frac{\mathrm{v} - V_s}{V_0 - V_s}\right) = -\frac{t}{RC} \Rightarrow \frac{\mathrm{v} - V_s}{V_0 - V_s} = \mathrm{e}^{-t/\tau}, \qquad \tau = RC$$

Finally we get $v(t) = V_\mathrm{S} + (V_0 - V_s)\mathrm{e}^{-t/\tau}, \quad t > 0$

Thus, the complete response (or total response) is

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

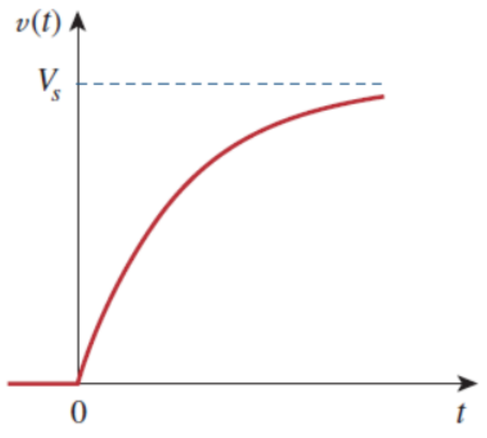
Assuming that $V_S > V_0$ a plot of v(t) is



If we assume that the capacitor is **uncharged initially**

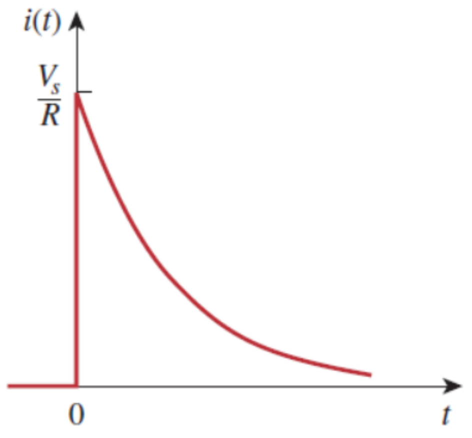
$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

Or $$v(t) = V_s(1 - e^{-t/\tau})u(t)$$



The current through the capacitor is $i(t) = C\frac{dv}{dt}$ thus we get

$$i(t) = \frac{V_s}{R}e^{-t/\tau}u(t)$$



Method 2

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$ use a test solution $v(t) = Ae^{rt}$

$$r + \frac{1}{\tau} = 0 \Rightarrow r = -\frac{1}{\tau}$$

(i) The homogeneous solution or *natural response*

$$v_n(t) = Ae^{-t/\tau}$$

(ii) Suppose the particular solution or force response

$$v_f(t) = B$$

$$\frac{dB}{dt} + \frac{1}{\tau}B = \frac{V_s}{\tau} \Rightarrow B = V_s$$

The complete solution (= complete response or total response)

$$v(t) = v_n(t) + v_f(t) = Ae^{-t/\tau} + V_s$$

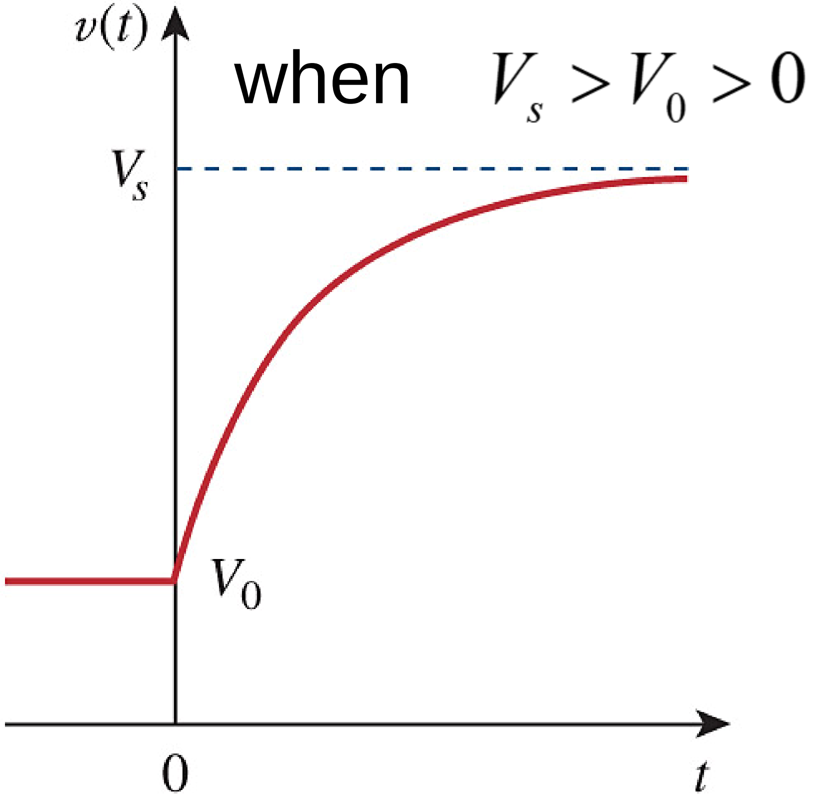
When $t = 0^+$,

$$v(0^+) = A + V_s = V_0 \Rightarrow A = V_0 - V_s$$

$$\boldsymbol{v(t) = (V_0 - V_s)\mathrm{e}^{-t/\tau} + V_s}$$

This is **the response of the RC circuit** to a sudden application of a constant dc source, assuming the capacitor is initially charged.

$$v(t) = (V_0 - V_s)e^{-t/\tau} + V_s$$



If the capacitor is **uncharged** initially, i.e., $V_0 = 0$, then

$$v(t) = V_s(1 - e^{t/\tau}), \quad t > 0 \quad \text{or} \quad v(t) = V_s(1 - e^{t/\tau})u(t)$$

This is the **zero-state response**. The zero-state response corresponding to a unit-step input is called the unit-step response.

Interpretations of the response

From the equation $v(t) = V_S + (V_0 - V_s)\mathrm{e}^{-t/\tau}, \qquad t > 0$

We can have two interpretations:

(i) Complete response = natural (v_n) + forced response(v_f)

$$v_n = V_o e^{-t/\tau} \qquad v_f = V_s(1 - e^{-t/\tau})$$

(ii) Complete response = transient (v_t) + steady-state (v_{ss})

$$v_t = (V_o - V_s)e^{-t/\tau} \qquad v_{ss} = V_s$$

$$v(t) = V_S + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$$

(i) Complete response = natural (v_n) + forced response(v_f)

$$v_n = V_o e^{-t/\tau} \qquad v_f = V_s(1 - e^{-t/\tau})$$

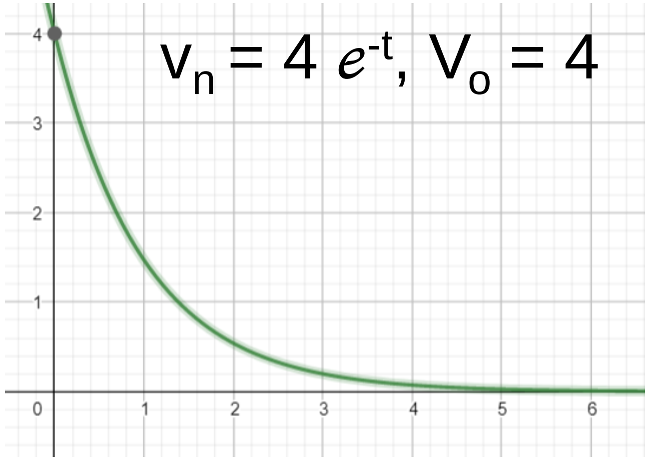
The natural response (v_n) is from the energy initially stored.
The forced response (v_f) is by an external force (power sources).

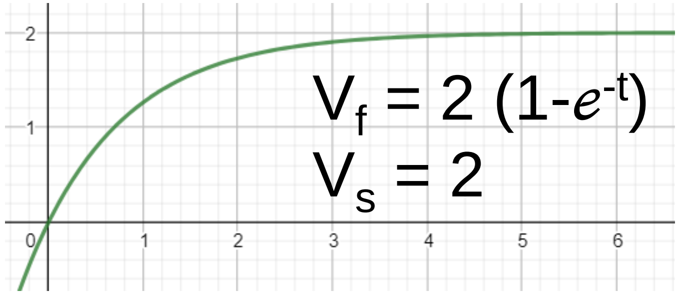
Exponential signals die out eventually, **leaving only the steady-state component** of the forced response.

(i) Complete response = natural (v_n) + forced response(v_f)

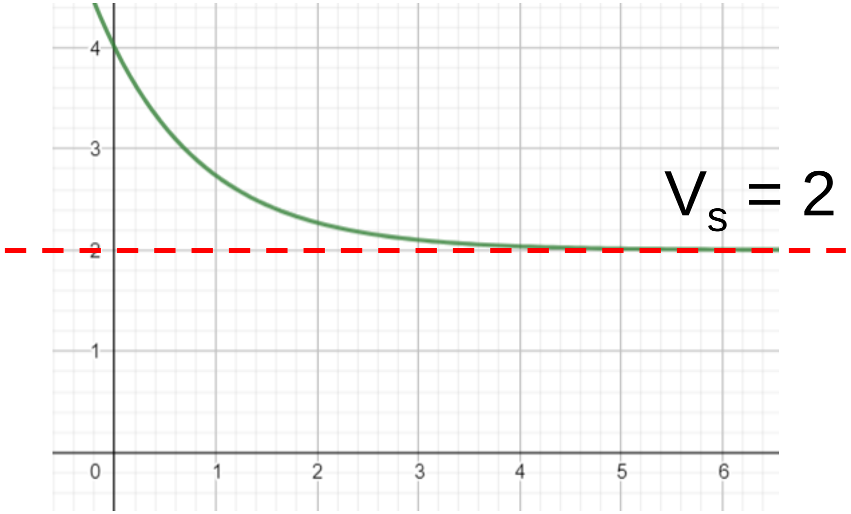
$$v_n = V_o e^{-t/\tau} \qquad v_f = V_s(1 - e^{-t/\tau})$$

$V_0 > V_s > 0$





Response = 2 (1-e^{-t})+4 e^{-t}



$$v(t) = V_S + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$$

(ii) Complete response = transient (v_t) + steady-state (v_{ss})

$$v_t = (V_o - V_s)e^{-t/\tau} \qquad v_{ss} = V_s$$

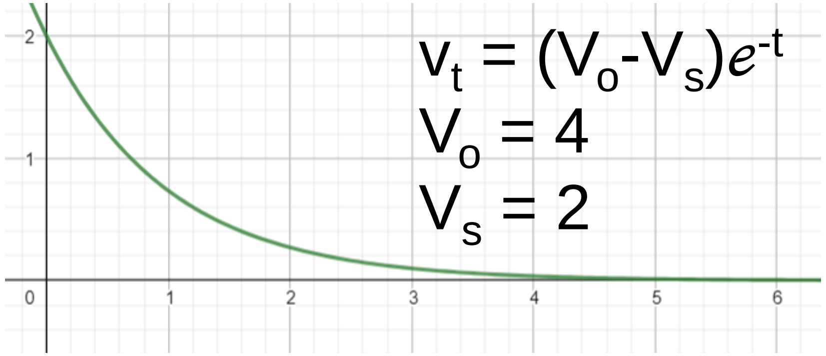
The transient response is the temporary response that will die out with time.
The steady-state response is the behavior of the circuit a long time after an external excitation is applied.

Under certain conditions, **v_n and v_t response** are the same. The same can be said about the **v_f and v_{ss} response**.

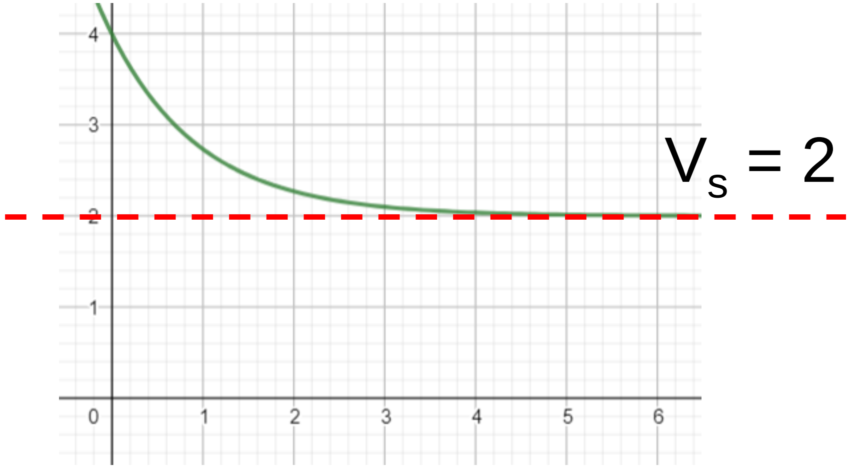
(ii) Complete response = transient (v_t) + steady-state (v_{ss})

$$v_t = (V_o - V_s)e^{-t/\tau} \qquad v_{ss} = V_s$$

$V_0 > V_s > 0$



Response = $V_s + (V_o - V_s)\, e$^(-t)



The complete response $v(t) = V_S + (V_0 - V_s)\mathrm{e}^{-t/\tau}, \qquad t > 0$

may be written as $v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$

v(0) is the initial voltage at t = 0^+
v(∞) is the final or steady-state value

Thus, to find the step response of an *RC* circuit requires **three parameters**:

1. The initial capacitor voltage v(0)
2. The final capacitor voltage v(∞)
3. The time constant τ.

Example 7.10 The switch in Fig.7.43 has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4s.

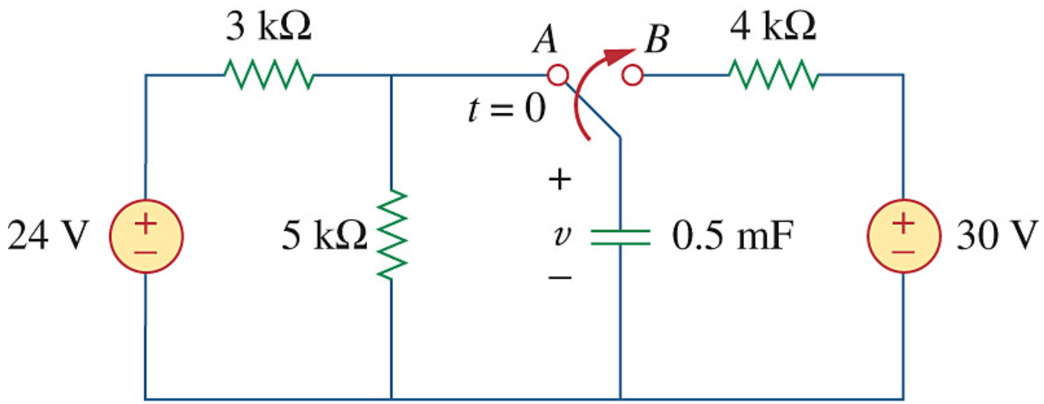


Figure 7.43

Example 7.10 The switch in Fig.7.43 has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4s.

Solution :

For $t < 0$,

$$v(t) = 24 \times \frac{5}{3+5} = 15 \text{ (V)}$$

$$v(0^-) = 15 \text{ V}$$

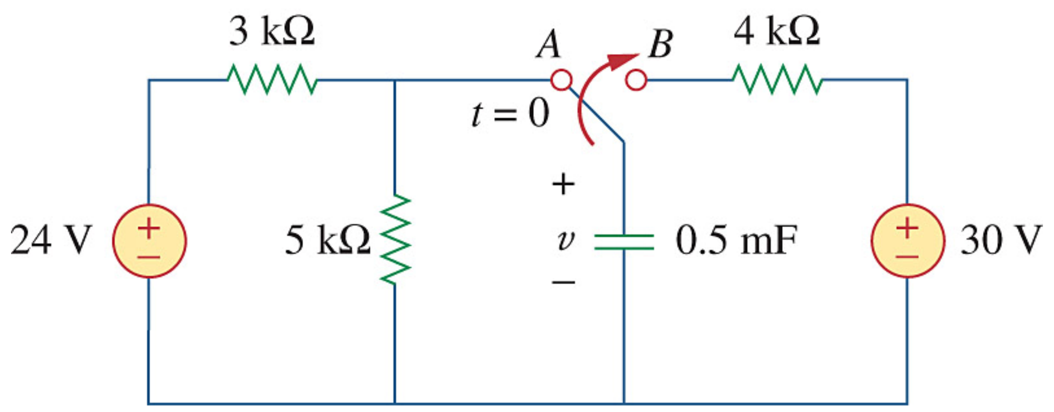


Figure 7.43

For $t > 0$,

$$v(0^+) = v(0^-) = 15 \text{ V}$$

$$v(\infty) = 30 \text{ V}$$

$$\tau = 4\times10^3 \times 0.5\times10^{-3} = 2 \text{ (s)}$$

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

$$= 30 + (15 - 30)e^{-t/2} = 30 - 15e^{-0.5t} \text{ (V)}$$

$$v(1) = 30 - 15e^{-0.5\times1} \approx 20.90 \text{ (V)}$$

$$v(4) = 30 - 15e^{-0.5\times4} \approx 27.97 \text{ (V)}$$

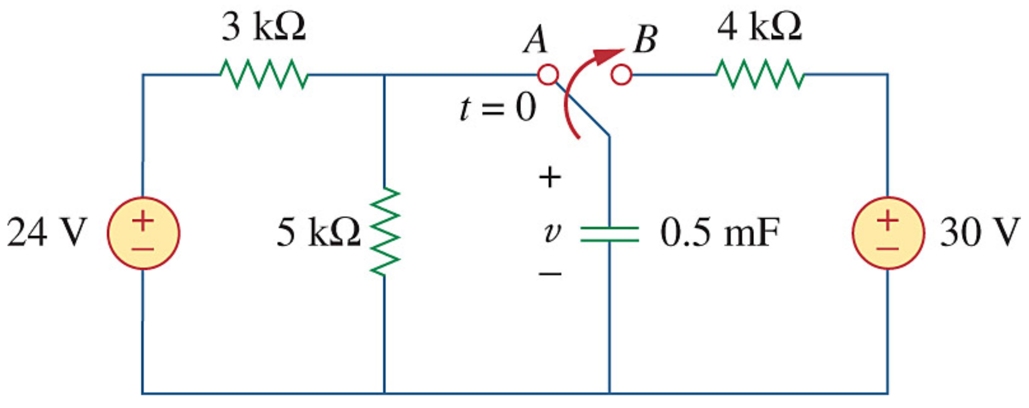


Figure 7.43

1st Order ODE (Optional)

$$y' + p(x)y = r(x) \ \rightarrow \ \boldsymbol{Fy'} + \boldsymbol{Fp(x)y} = Fr(x)$$

Left side: $(Fy)' = Fy' + F'y$

$$where\ F'y = Fp(x)y, then\ \boldsymbol{F'} = \boldsymbol{Fp(x)}$$

$$F' = Fp(x) \rightarrow \frac{1}{F}F' = p(x) \rightarrow \ln F = \int p(x)dx$$

Thus, $\mathbf{F} = \boldsymbol{e}^{\int \boldsymbol{p(x)dx}}\ where\ we\ set \int \boldsymbol{p(x)dx} = \boldsymbol{h}$

$$\mathbf{F} = \boldsymbol{e^h}\ then, \boldsymbol{p} = \boldsymbol{h'}\ due\ to\ h = \int p(x)dx$$

$$Fy' + Fp(x)y = Fr(x) \ \rightarrow e^h y' + e^h p(x)y = e^h r(x)$$
$$\rightarrow \boldsymbol{e^h y' + e^h h' y = e^h r(x)}$$

$$(e^h y)' = e^h r(x) \ integral\ both\ sides$$
$$e^h y = \int e^h r(x) dx + c$$

$$\boldsymbol{y = e^{-h}\left(\int e^h r(x) dx + c\right) where\ h = \int p(x) dx}$$

1st Order ODE where r(x) = 0

$$y' + p(x)y = r(x), where\ r(x) = 0$$

$$\frac{dy}{dx} + p(x)y = 0$$

$$\rightarrow \frac{1}{y}dy = -p(x)dx\ integrate\ both\ sides$$

$$\ln y = -\int p(x)dx + c \rightarrow \boldsymbol{y = c \cdot e^{-\int p(x)dx}}$$

e.g. $\frac{di}{dt} + 5i = 0 \rightarrow i(t) = c \cdot e^{-5t}$

Practice Problem 7.1 Refer to the circuit in Fig. 7.7. Let $v_C(0) = 45$ V. Determine v_C, v_x, and i_o for $t \geq 0$.

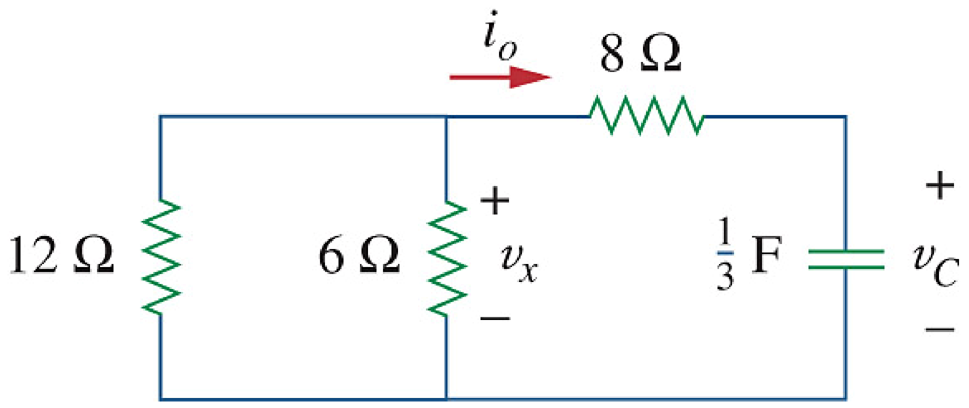
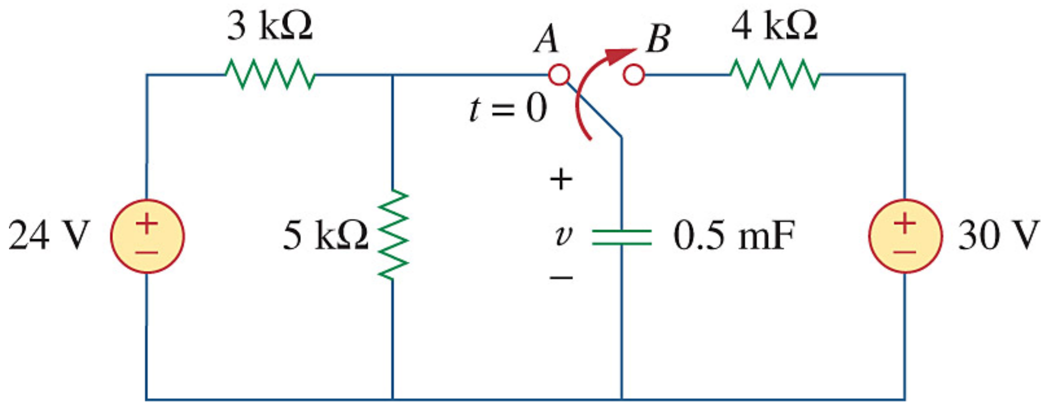


Figure 7.7 An *RC* circuit.

$$v_C = v_C(0)e^{-t/\tau} = 45e^{-t/4} = 45e^{-0.25t} \text{ (V)}$$

Example 7.10 The switch in Fig.7.43 has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4s.



For $t < 0$,

$$v(t) = 24 \times \frac{5}{3+5} = 15 \text{ (V)}$$

$$v(0^-) = 15 \text{ V}$$

Practice Problem 7.11 The switch in Fig. 7.43 is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time.

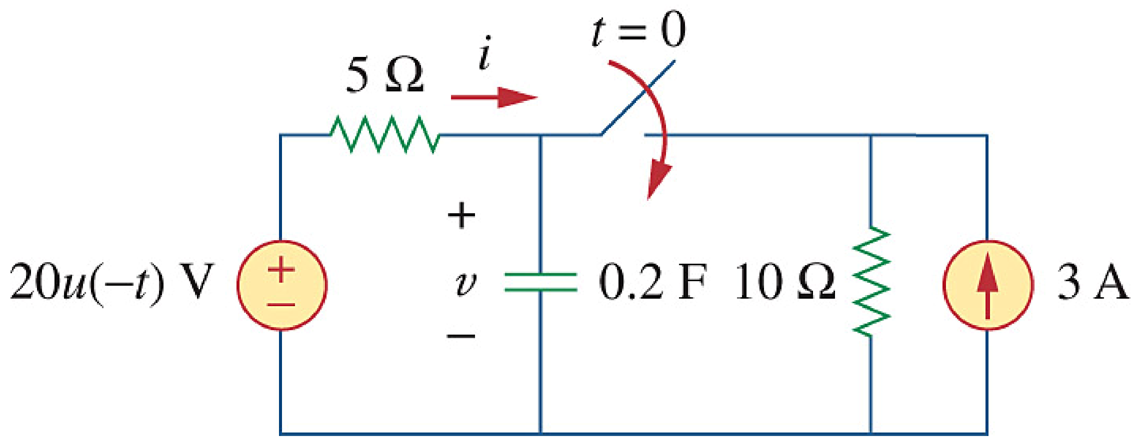


Figure 7.47

For $t > 0$,

$$v(0^+) = v(0^-) = 20 \text{ V}$$

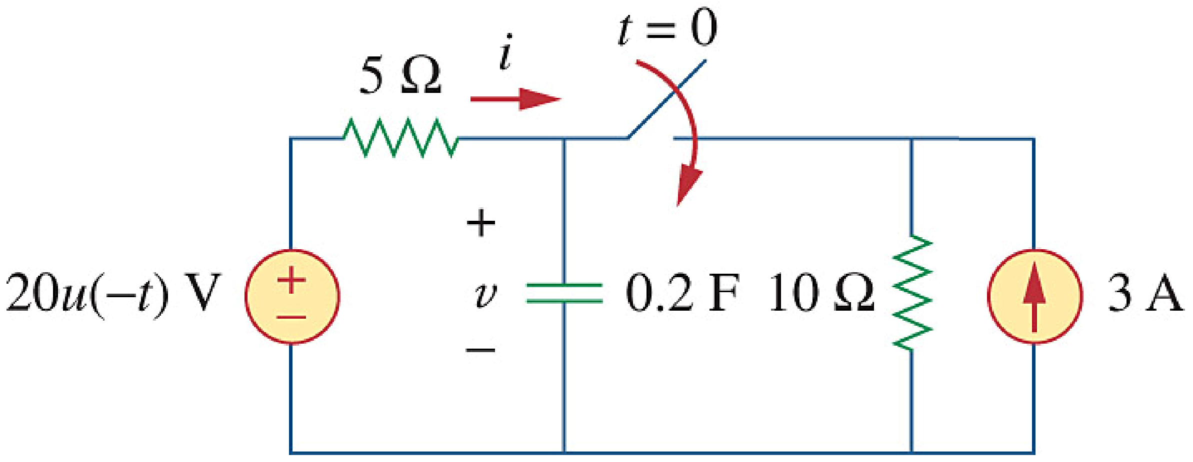
$$v(\infty) = 3 \times (5 \parallel 10) = 10 \text{ (V)}$$

$$\tau = (5 \parallel 10) \times 0.2 = \frac{2}{3} \text{ (s)}$$

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

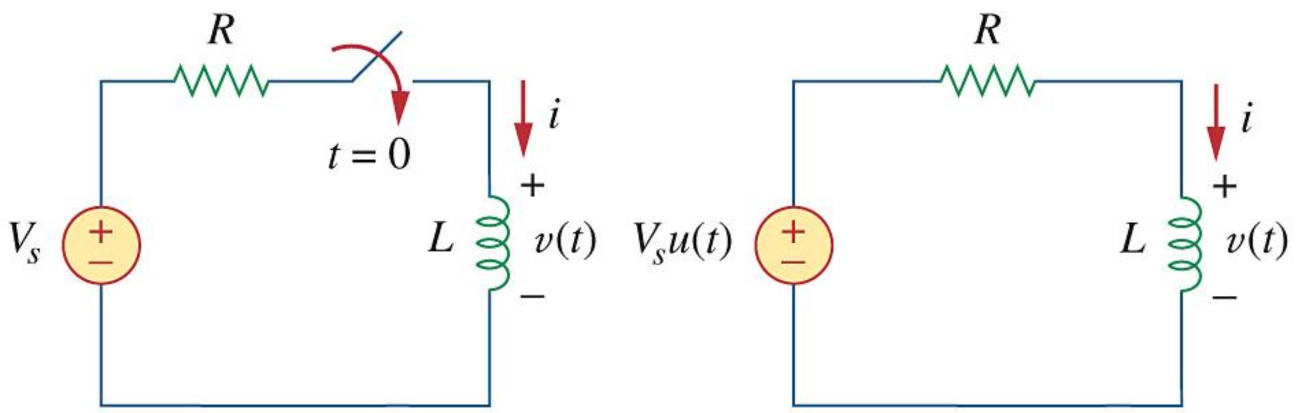
$$= 10 + (20 - 10)e^{-t/(2/3)} = 10(1 + e^{-1.5t}) \text{ (V)}$$

$$i(t) = -\frac{v(t)}{5} = -2(1 + e^{-1.5t}) \text{ (A)}$$

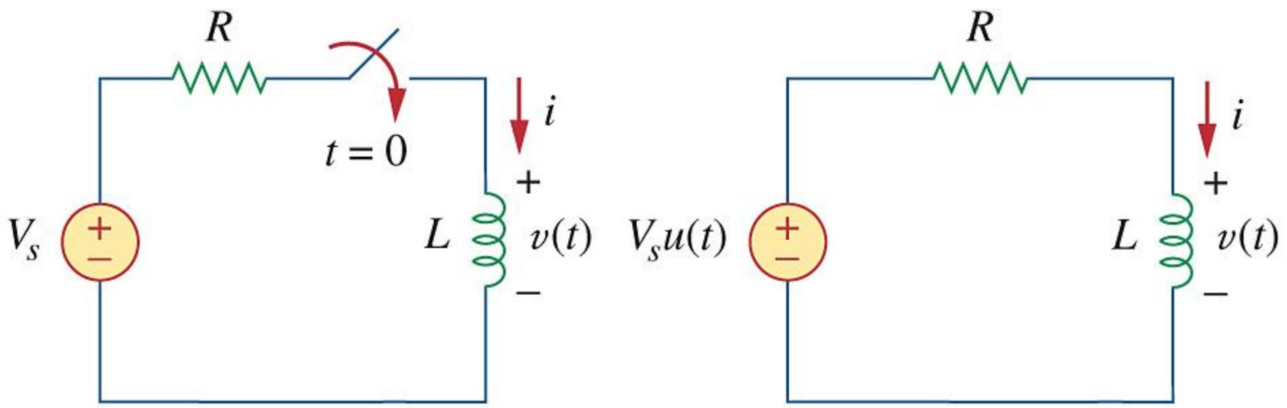


v(0+): by the value of v(0⁻)
v(∞): by the circuit at t > 0

7.6 Step Response of an RL Circuit



We select the inductor current *i* as the circuit response to be determined. We assume an initial current I_0 in the inductor.



Because the current through the inductor cannot change instantaneously, the initial condition is $i(0^+) = i(0^-) = I_0$

For $t > 0$,

$$iR + L\frac{di}{dt} = V_s$$

or

$$\frac{di}{dt} + \frac{1}{\tau} i = \frac{V_s / R}{\tau}, \text{ where } \tau = \frac{L}{R}$$

RL

$$\frac{di}{dt}+\frac{1}{\tau}i=\frac{V_s/R}{\tau}, \text{ where } \tau=\frac{L}{R}$$

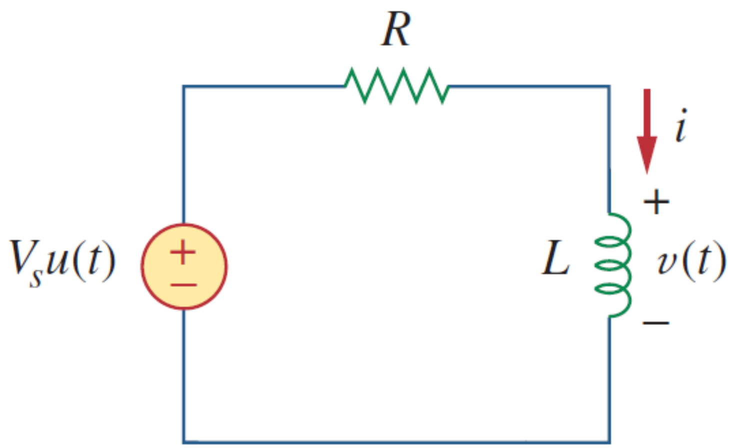
RC

$$\frac{dv}{dt}+\frac{1}{\tau}v=\frac{V_s}{\tau}, \text{ where } \tau=RC$$

$$v(t)=(V_0-V_s)e^{-t/\tau}+V_s$$

Since this differential equation has the same form as that describing the RC circuit,

$$i(t)=\frac{V_s}{R}+\left(I_0-\frac{V_s}{R}\right)e^{-t/\tau}$$

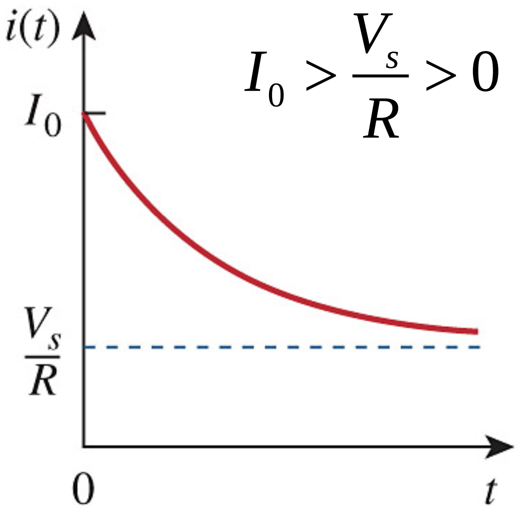


It is evident that

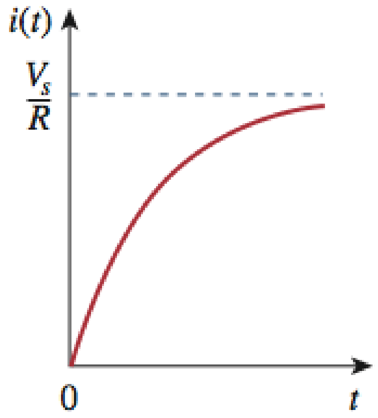
$$i(\infty)=\frac{V_s}{R}$$

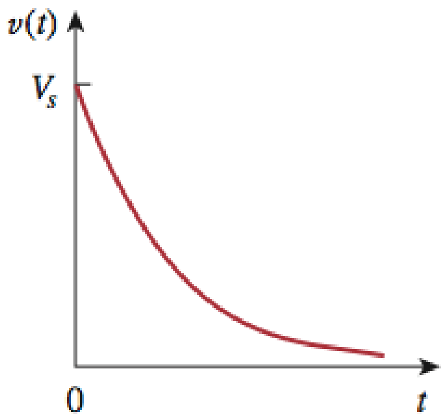
The plot of the RL step response is thus

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R} \right) e^{-t/\tau}$$



If $I_0 = 0$, $i(t) = \frac{V_s}{R}\left(1 - e^{-t/\tau}\right)u(t)$, $v(t) = V_s e^{-t/\tau} u(t)$





The complete response can be written as

$$i(t) = i(\infty) + \left[i(0^+) - i(\infty) \right] e^{-t/\tau}$$

Thus, to find the response of a first-order RL circuit requires three conditions:

1. The initial inductor current $i(0^+)$.
2. The final inductor current $i(\infty)$.
3. The time constant $\tau = L / R$.

Source-free RL circuit is a special case when i(∞) = 0

Example 7.13 At $t = 0$, switch 1 in Fig. 7.53 is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2$ s and $t = 5$ s.

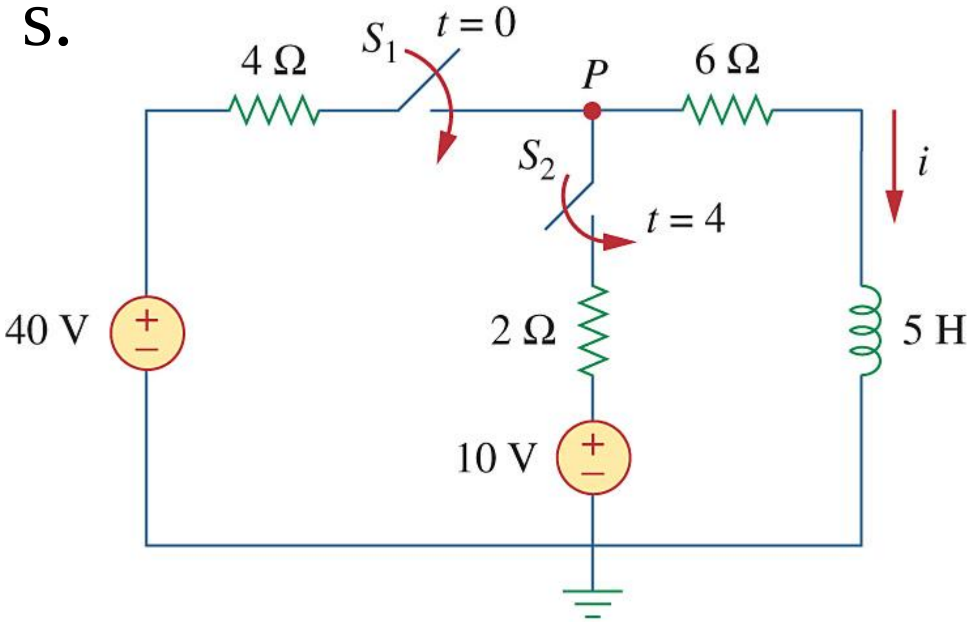


Figure 7.53

Solution :

For $t < 0$,

$$i(t) = 0$$

$$i(0^-) = 0$$

ii) 0 < t < 4

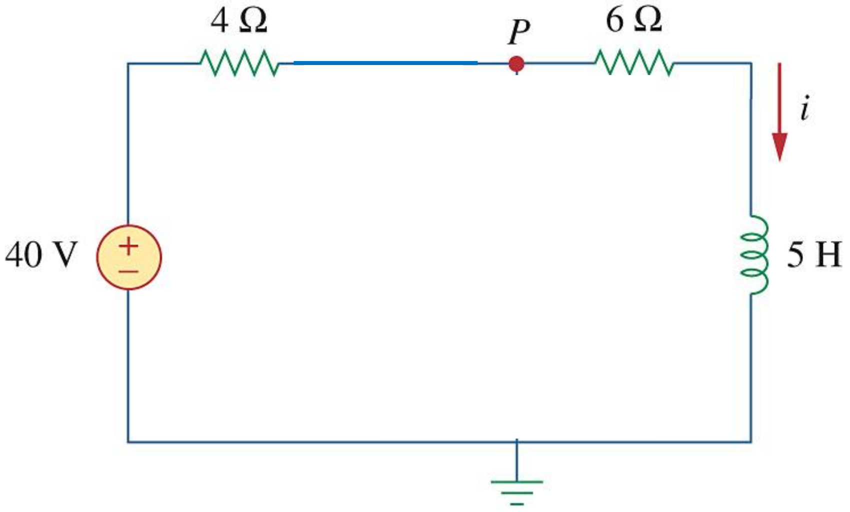
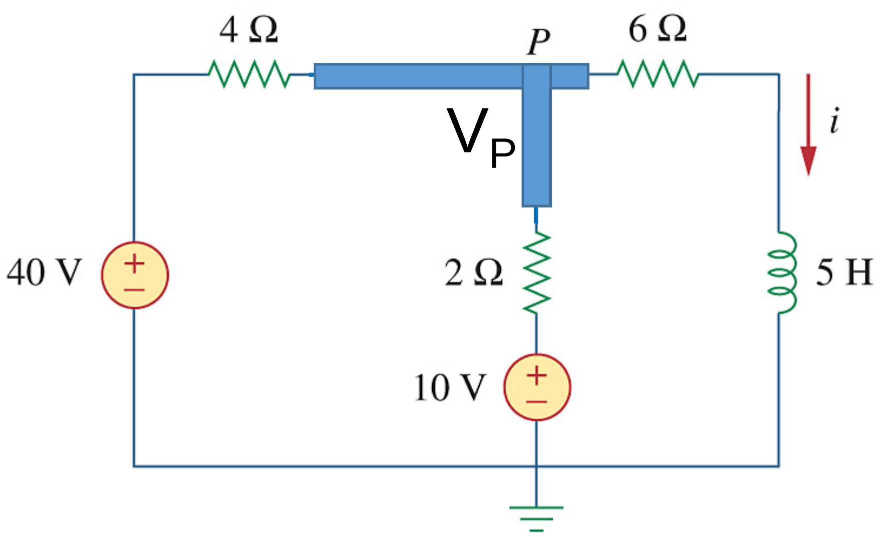
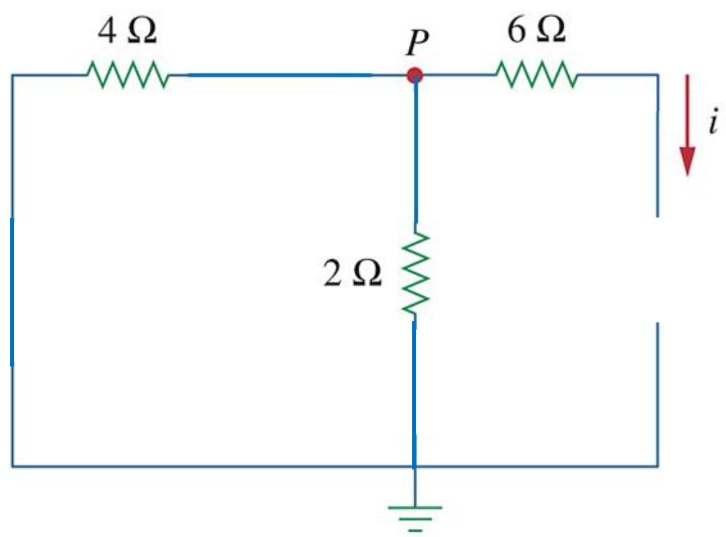


Figure 7.53

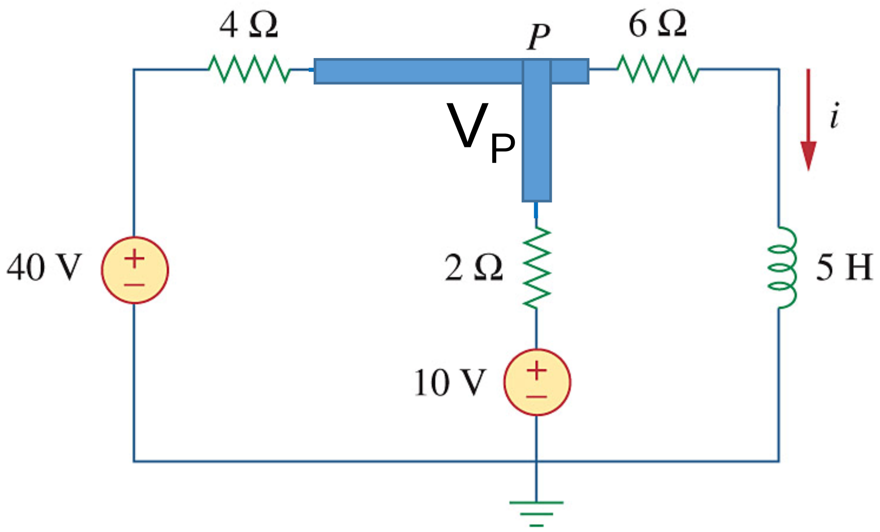
iii) $t > 4$

$R_{eq} = 2 || 4 + 6 = 22/3$





iii) t > 4



(1) KCL at node P

(2) $V_P = 6i + Ldi/dt$

For $t > 0$,

$$i(0^+) = i(0^-) = 0$$

For $0 < t \le 4$,

$$i(\infty) = \frac{40}{4+6} = 4 \text{ (A)}$$

$$\tau = \frac{5}{4+6} = 0.5 \text{ (s)}$$

$$i(t) = 4 + (0-4)e^{-t/0.5} = 4(1-e^{-2t}) \text{ (A)}$$

$$i(2) = 4(1-e^{-2\times 2}) \approx 3.93 \text{ (A)}$$

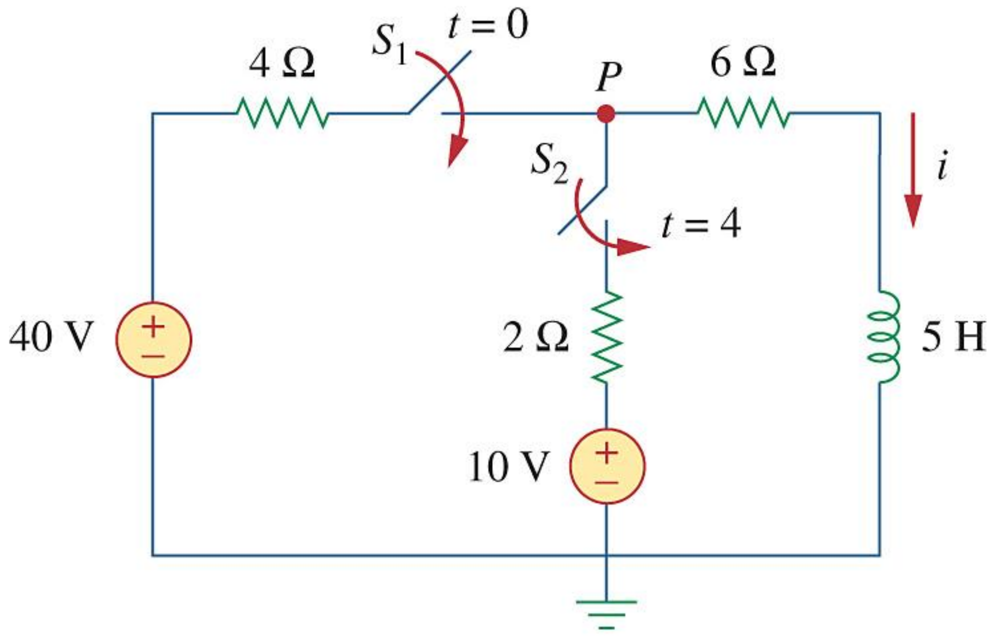


Figure 7.53

For $t > 4$,

$$i(4^+) = i(4^-) = 4(1 - e^{-2\times 4}) = 4(1 - e^{-8})\ \text{(A)}$$

$$v_P(\infty) = 40 \times \frac{2 \parallel 6}{4 + 2 \parallel 6} + 10 \times \frac{4 \parallel 6}{2 + 4 \parallel 6}$$

$$= \frac{180}{11}\ \text{(V)}$$

$$i(\infty) = \frac{v_P(\infty)}{6} = \frac{30}{11}\ \text{(A)}$$

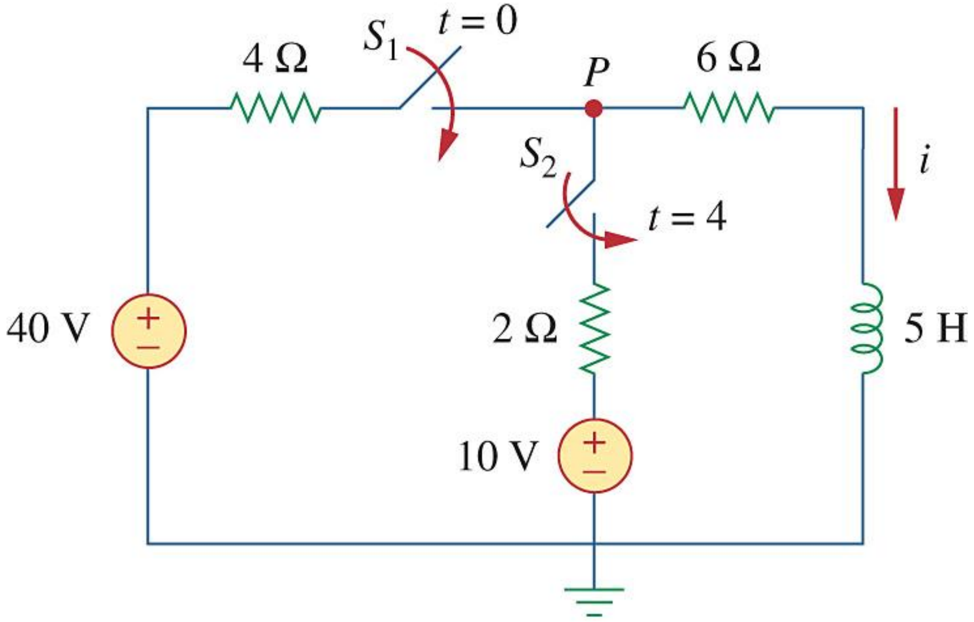


Figure 7.53

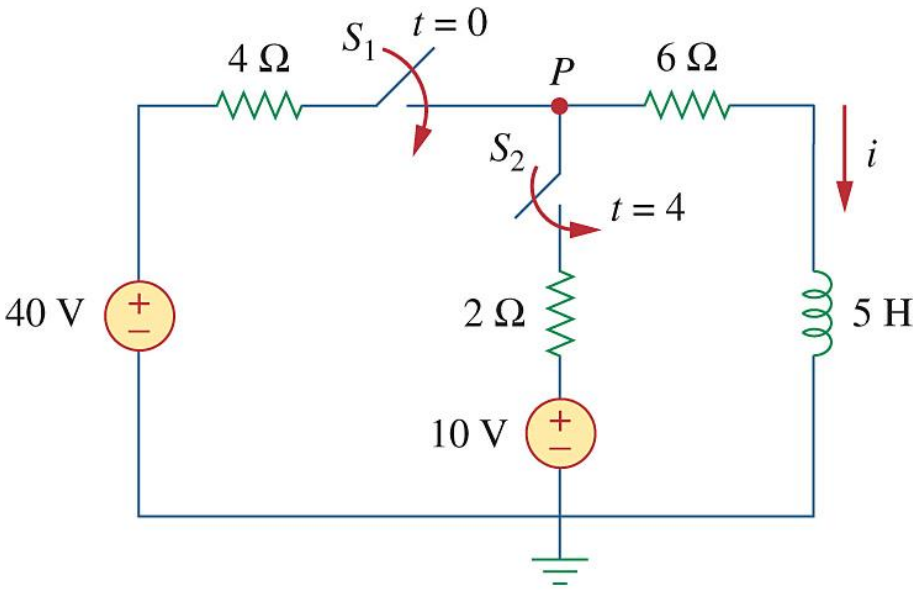


Figure 7.53

$$\tau = \frac{5}{6+4 \parallel 2} = \frac{15}{22} \text{ (s)}$$

$$i(t) = \frac{30}{11} + \left(4(1-e^{-8}) - \frac{30}{11}\right)e^{-\underline{(t-4)}/(15/22)}$$

Time shift property

$$\approx 2.7273 + 1.2714e^{-1.4667(t-4)} \text{ (A)}$$

$$i(5) = 2.7273 + 1.2714e^{-1.4667(5-4)} \approx 3.02 \text{ (A)}$$

7.7 First-Order Op Amp Circuits

An op amp circuit containing a storage element will exhibit **first-order behavior**. For practical reasons, inductors are hardly ever used in op amp circuits; we will consider Op Amp circuits with RC.

- Possible location of the capacitor

(1) At the input
(2) At the output
(3) At the feedback loop

Example 7.14 For the op amp circuit in Fig. 7.55(a), find v_o for $t > 0$, given that $v(0) = 3$ V. Let $R_f = 80$ kΩ, $R_1 = 20$ kΩ, and $C = 5\ \mu$F.

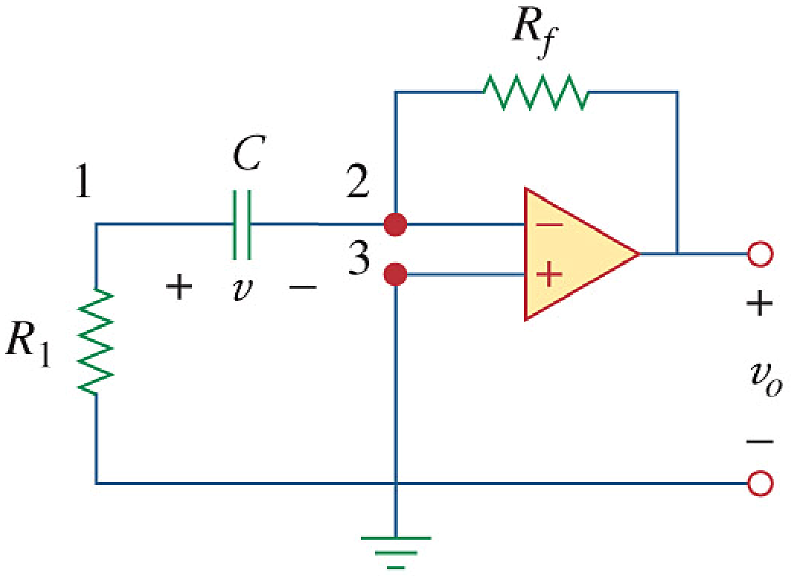
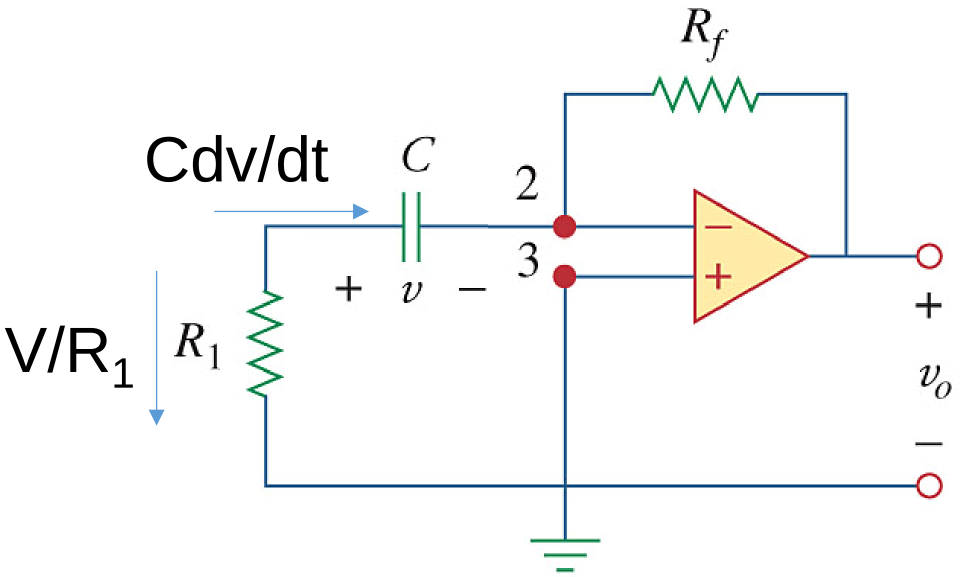


Figure 7.55(a)

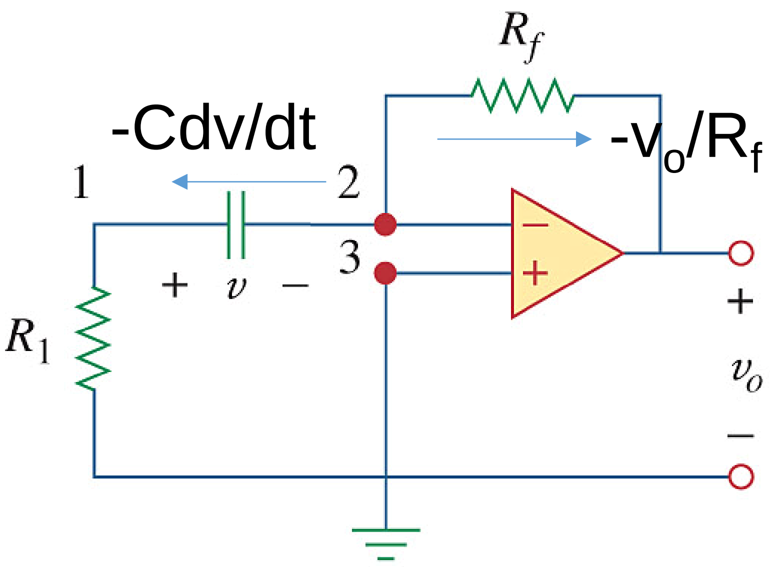


By KCL

$$\frac{V}{R_1} + C\frac{dv}{dt} = 0 \rightarrow \frac{dV}{dt} + \frac{1}{RC_1}v = 0$$

$$\tau = R_1C = 20\times10^3\times5\times10^{-6} = 0.1\ \text{(s)}$$

$$v = v(0)e^{-t/\tau} = 3e^{-t/0.1} = 3e^{-10t}\ \text{(V)}$$



KCL from node 2 gives

$$-C\frac{dv}{dt} + \frac{0 - v_0}{R_f} = 0$$

$$\rightarrow v_0 = -R_f C\frac{dv}{dt} \qquad = -\left(5\times10^{-6}\frac{d}{dt}(3e^{-10t})\right)\times80\times10^{3}$$

$$= -0.4\times\left(3\times\left(-10e^{-10t}\right)\right)$$

$$= 12e^{-10t} \ \text{(V)}$$

Example 7.15 Determine $v(t)$ and $v_o(t)$ in the circuit of Fig. 7.57.

Solution :

For $t < 0$, **No energy at the circuit**

$v_1(t) = 0, v_o(t) = 0, v(t) = 0$

$v(0^-) = 0$

For $t > 0$,

$$v_1(t) = 3 \times \frac{20}{10+20} = 2 \text{ (V)}$$

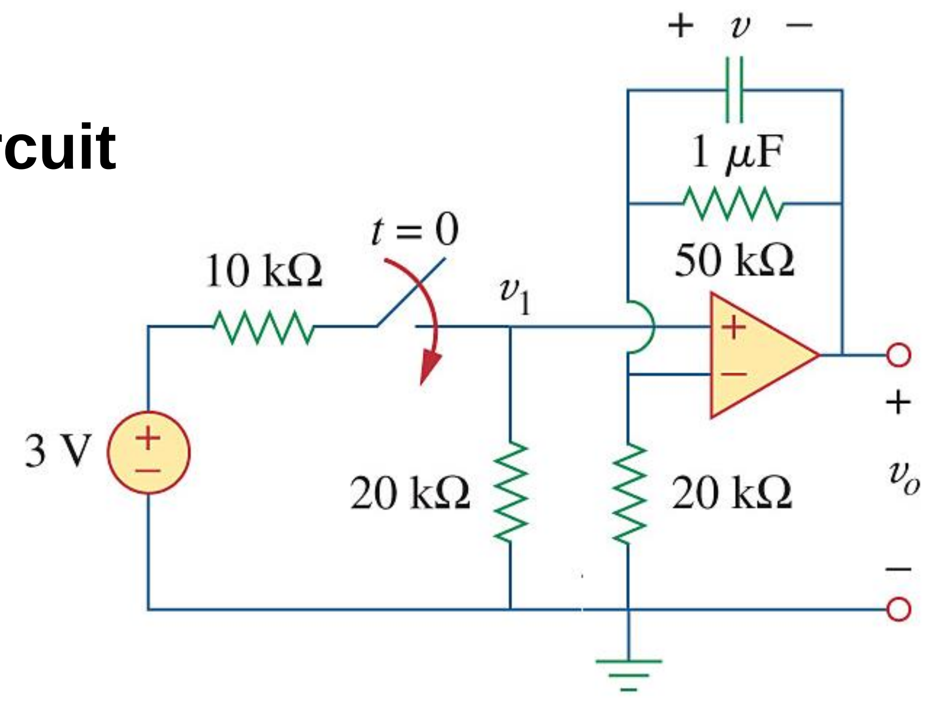
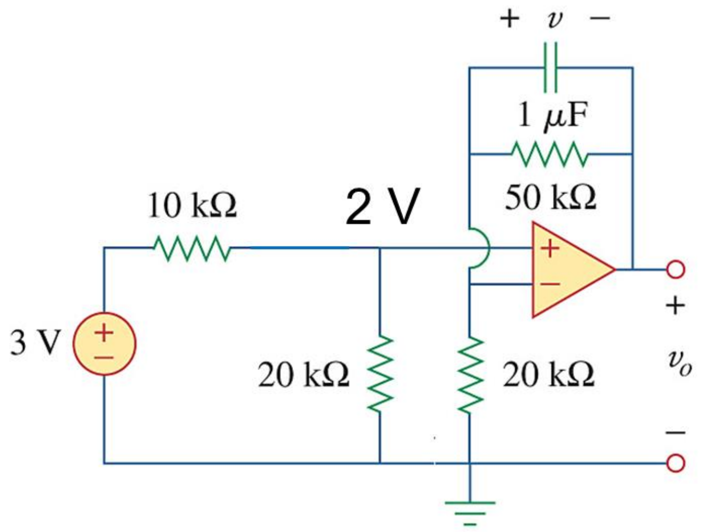
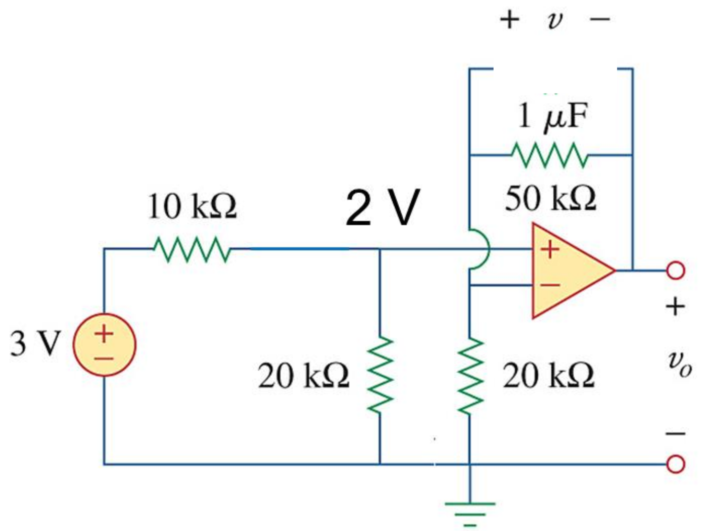


Figure 7.57

(ii) t > 0



(1) RC = 10^{-6} x 50k = 1/20



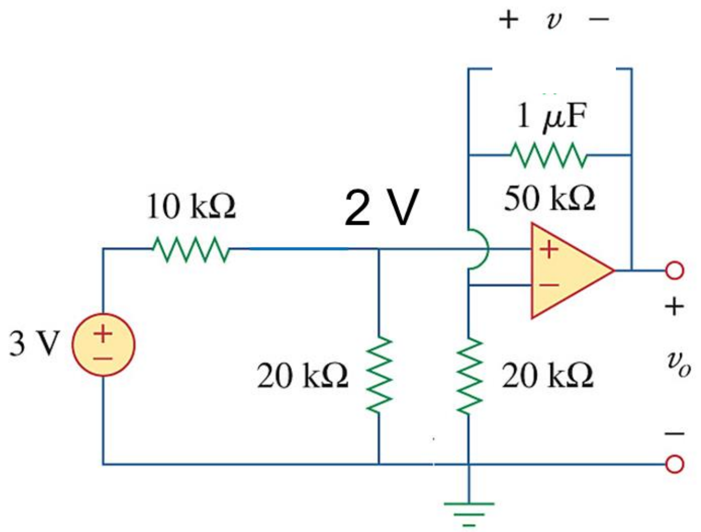
(2) v(∞)

By KCL: $\frac{2}{20k} + \frac{2-v_o}{50k} = 0$, v_o = 7 V

Thus v(∞) = – 5 V

v(t) = –5 + [0 + 5]e^{-20t} = 5e^{-20t} – 5

(ii) $t > 0$

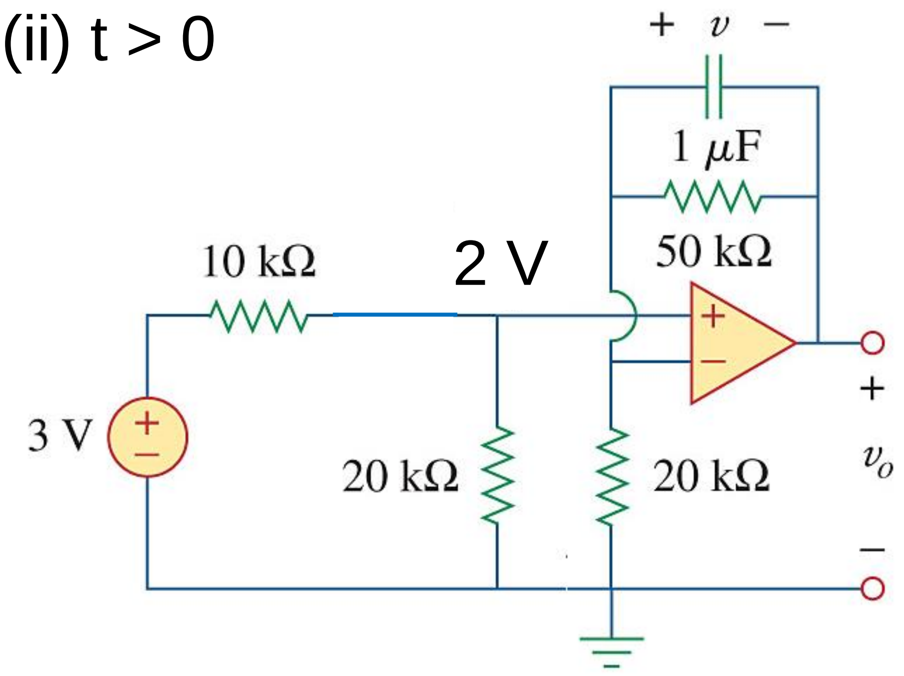


Because $2 - v_o = v$

where $v(t) = 5e^{-20t} - 5$ [V]

$v_o(t) = 7 - 5e^{-20t}$ [V]

(ii) t > 0

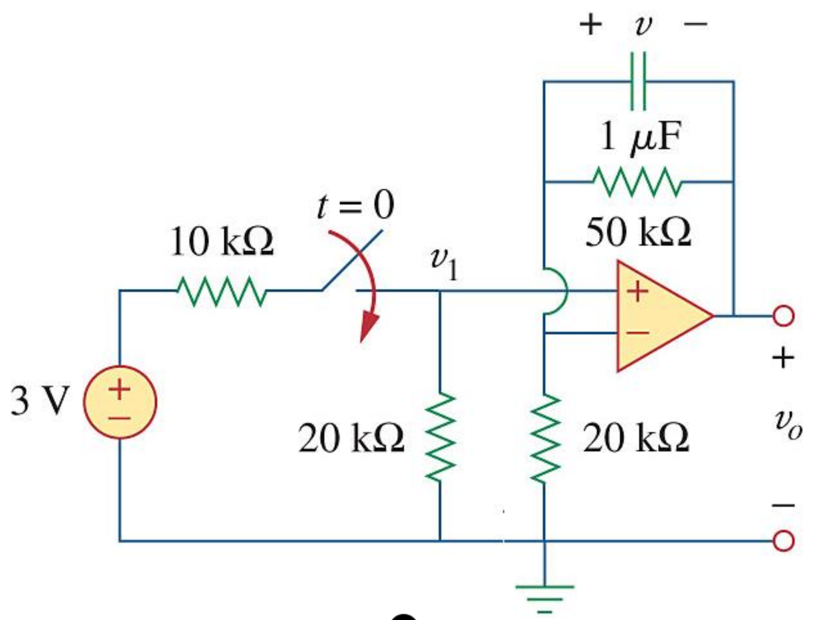


(i) $2 - v_o = v$

(ii) By KVL: $\frac{2}{20k} + \frac{2-v_o}{50k} + C\frac{dv}{dt} = 0$

$$\frac{2}{20k} + \frac{v}{50k} + 10^{-6}\frac{dv}{dt} = 0 \quad \times 10^6$$

$$\frac{dv}{dt} + 20v = -100$$



$$(1\times10^{-6})\frac{dv}{dt}+\frac{v}{50\times10^{3}}=-\frac{2}{20\times10^{3}}$$

$$\frac{dv}{dt}+20v=-100$$

$$v(t)=Ae^{-20t}+B=5e^{-20t}-5\ (\mathrm{V})$$

$$v_o(t)=-v(t)+2=7-5e^{-20t}\ \ (\mathrm{V})$$