ECE2150J Introduction to Circuits

Chapter 6. Capacitors and Inductors

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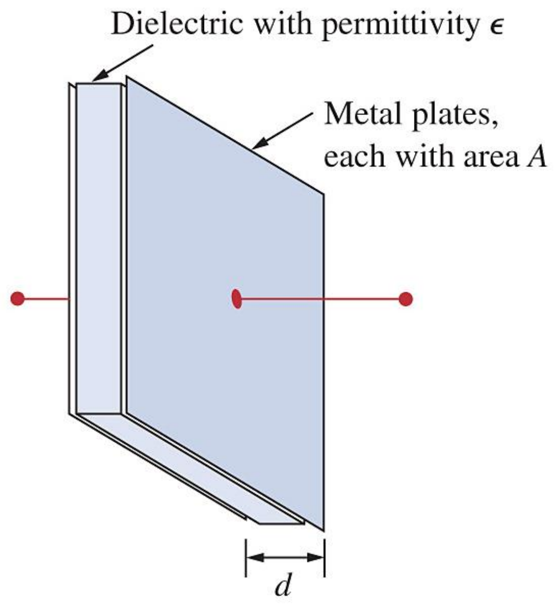
6.1 Introduction

- Two passive circuit elements: **capacitor (C)** and **inductor (L)**.
- Capacitors and inductors do not dissipate energy, **but store energy**, which can be retrieved at a later time.
- Capacitors and inductors are called **storage elements**.

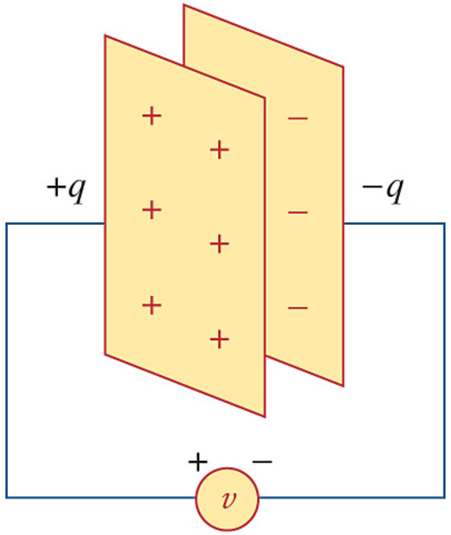
6.2 Capacitors

- A capacitor is a passive element designed to store energy in its electric field.

- Typically, the capacitor consists of two conducting plates separated by an insulator (or dielectric).



Charging Capacitor



The amount of charge stored (q)

$$\boldsymbol{q = Cv}$$

- C is capacitance of the capacitor [F].
- If C is independent of v, the capacitor is said to be linear.

A voltage source deposits a positive charge q on the plate and a negative charge –q on the other. The capacitor is said to **store the electric charge**.

Capacitance (C) is the ability of a body to store an electrical charge. The capacitance of a linear capacitor depends on the **physical dimensions of the capacitor (independent of v).**

$$C = \varepsilon_r \varepsilon_0 \frac{A}{d}$$

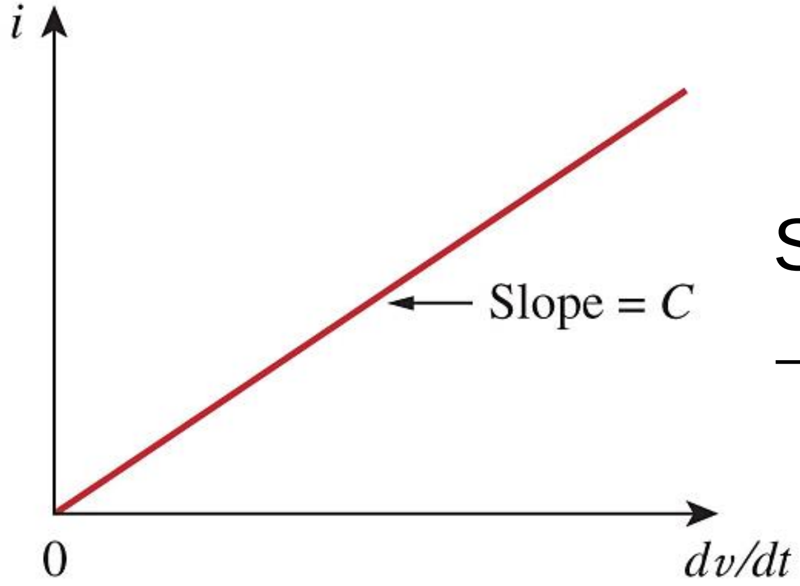
where *A* is the area in square meters; ε_r is the relative permittivity, or the dielectric constant, of the material; ε_0 is vacuum permittivity ($\varepsilon_0 \approx 8.854 \times 10^{-12}$ F m^{-1}); d is the distance between the plates in meters.

Capacitor IV Relationship

From the equation $q = Cv$, differentiate both sides

$$\frac{dq}{dt} = C\frac{dv}{dt} \; and \; because \frac{dq}{dt} = i$$

$$\boldsymbol{i = C\frac{dv}{dt}} \; \textbf{(assuming passive sign convention)}$$



Satisfy the eq. $i = C\frac{dv}{dt}$

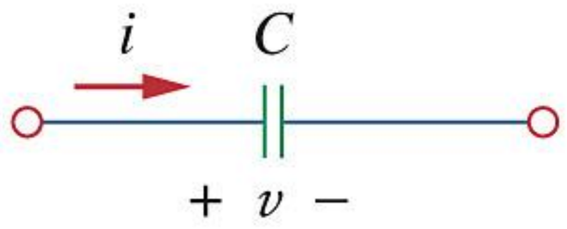
→ Linear capacitor

By integrating the eq. $i = C\frac{dv}{dt}$

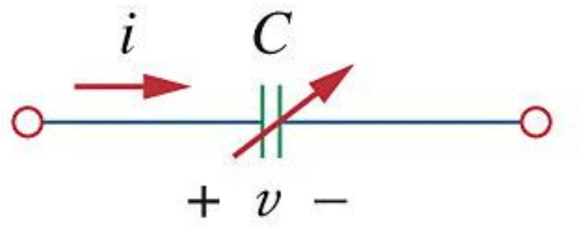
$$v = \frac{1}{C}\int_{-\infty}^{t} idt = \frac{1}{C}\int_{t_0}^{t} idt + v(t_0)$$

where $v(t_0) = \frac{1}{C}\int_{-\infty}^{t_0} idt = \frac{q(t_0)}{C}$ is the voltage across the capacitor at time t_0.

Power in Capacitor



Fixed capacitor



Variable capacitor

According to the passive sign convention

(i) **vi>0, or p>0**, the capacitor is being charged, i.e. consuming power.

(ii) **vi<0, or p<0**, the capacitor is discharging, i.e. supplying power.

The **instantaneous power** delivered to the capacitor is

$$\boldsymbol{p = vi = v(C\frac{dv}{dt})}$$

Therefore, the energy stored in the capacitor is

$$w = \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t} v\left(C\frac{dv}{dt}\right)dt = C\int_{v(-\infty)}^{v} v\,dv$$

$$= \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v} = \frac{1}{2}Cv^2$$

v(–∞) = 0, initially there is no energy in the capacitor.

Important properties of a capacitor

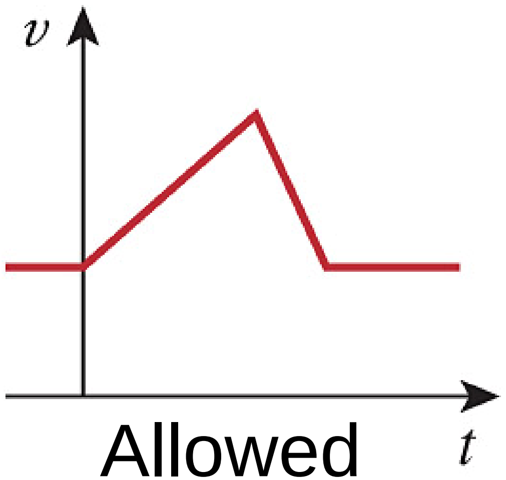
In circuit analysis, there are several important properties of the capacitor to consider

1. When the voltage across a capacitor is not changing with time (DC steady state), the current through the capacitor is zero → a **capacitor is an open circuit** in the DC steady state.

Important properties of a capacitor

2. **The voltage** on a capacitor must be **continuous**, i.e. the voltage on a capacitor cannot change abruptly.

- i = C dv/dt, if dv/dt is infinite → i is infinite



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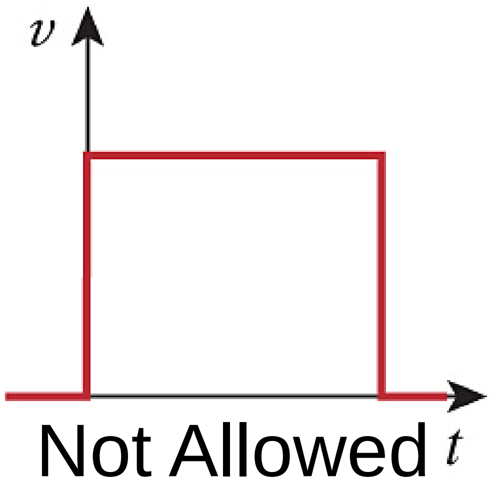
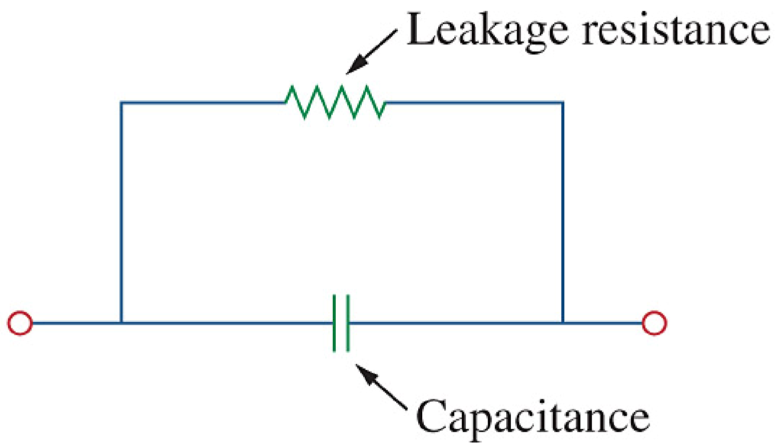


Not Allowed

Important properties of a capacitor

3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its electric field and returns previously stored energy when delivering power to the circuit.

4. A real capacitor has a large leakage resistance.



Example 6.3 Determine the voltage across a 2-μF capacitor if the current through it is

$i(t) = 6e^{-3000t}$ mA, $t \geq 0$

Assume the initial capacitor voltage is zero.

$$v = \frac{1}{C}\int_0^t i\,dt + v(0) = \frac{1}{C}\int_0^t i\,dt$$

$$= \frac{1}{2\times10^{-6}}\int_0^t 6e^{-3000t} \times 10^{-3}\,dt$$

$$= \frac{3\times 10^3}{-3000} e^{-3000t}\Big|_0^t = 1 - e^{-3000t} \text{ (V)}$$

Example 6.5 Obtain the energy stored in each capacitor in Fig. 6.12(a) under dc conditions.

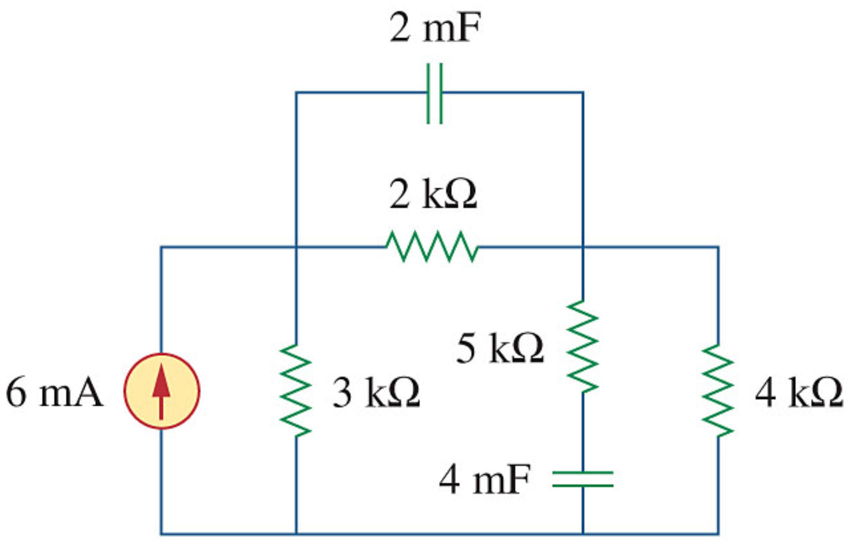


Figure 6.12(a)

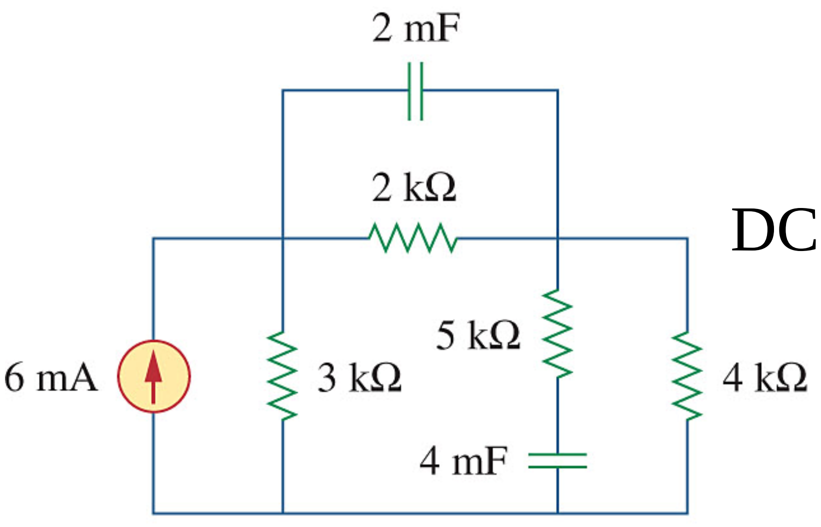


Figure 6.12(a)

DC steady state

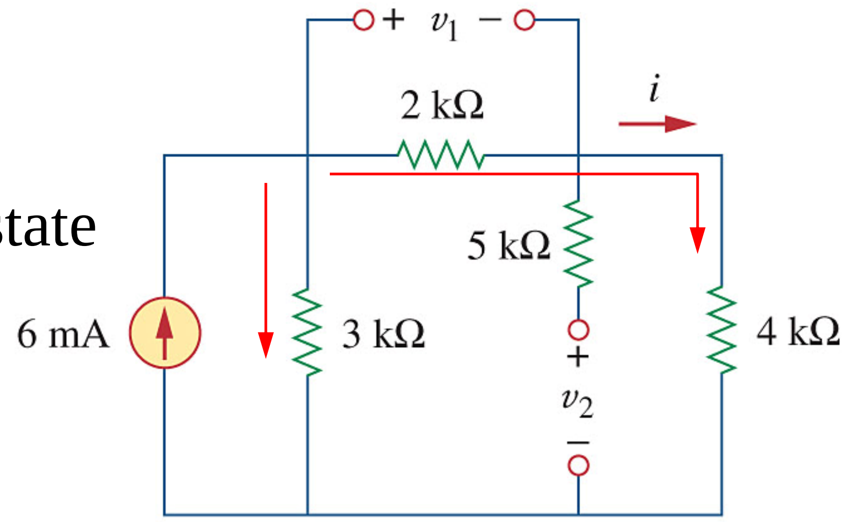


Figure 6.12(b)

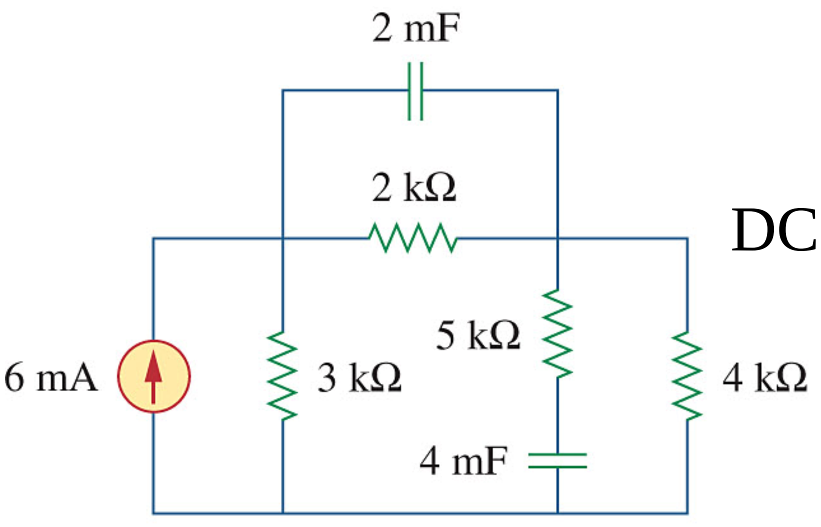


Figure 6.12(a)

DC steady state

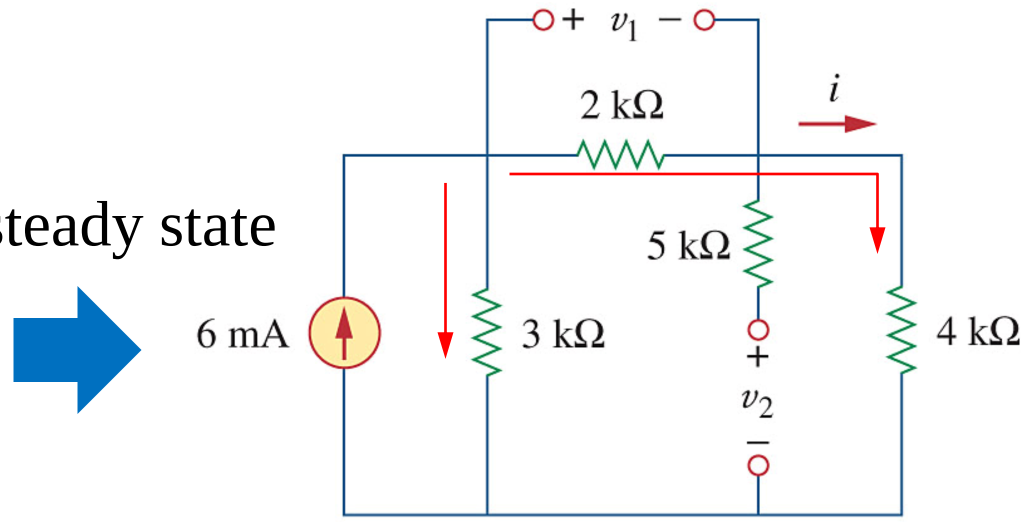


Figure 6.12(b)

By current division,

At 2 kΩ and 4 kΩ → 2 mA, thus V_1 = 4 V and V_2 = 8 V

Energy stored in 2mF = $\frac{1}{2}CV^2 = 0.016$ [J]

Energy stored in 4mF = $\frac{1}{2}CV^2 = 0.128$ [J]

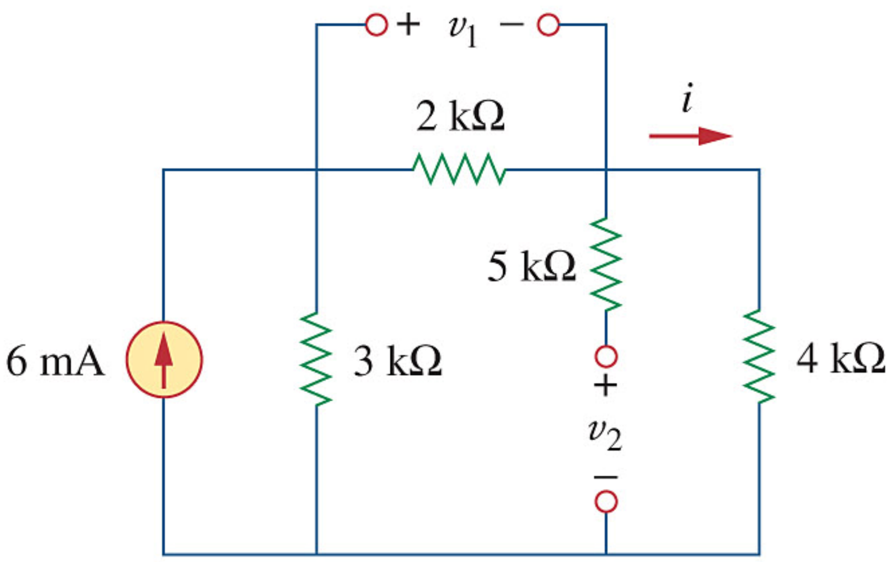


Figure 6.12(b)

$$i = \frac{3}{3+2+4} \times 6 = 2 \text{ (mA)}$$

$$v_1 = 2i = 2 \times 2 = 4 \text{ (V)}$$

$$v_2 = 4i = 4 \times 2 = 8 \text{ (V)}$$

$$w_1 = \frac{1}{2} C_1 v_1^2 = \frac{1}{2} \times 2 \times 10^{-3} \times 4^2 = 0.016 \text{ (J)}$$

$$w_2 = \frac{1}{2} C_2 v_2^2 = \frac{1}{2} \times 4 \times 10^{-3} \times 8^2 = 0.128 \text{ (J)}$$

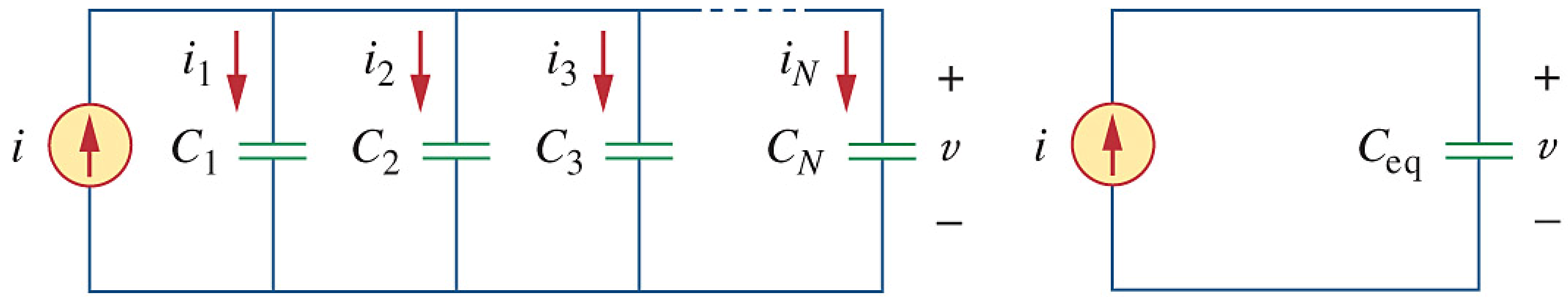
6.3 Series and Parallel Capacitor

Same as the series-parallel combination method used for resistors, it can be extended to **series-parallel connections of capacitors**.

Parallel: $$C_{eq} = C_1 + C_2 + \cdots + C_N$$

Series: $$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}$$

N parallel capacitors



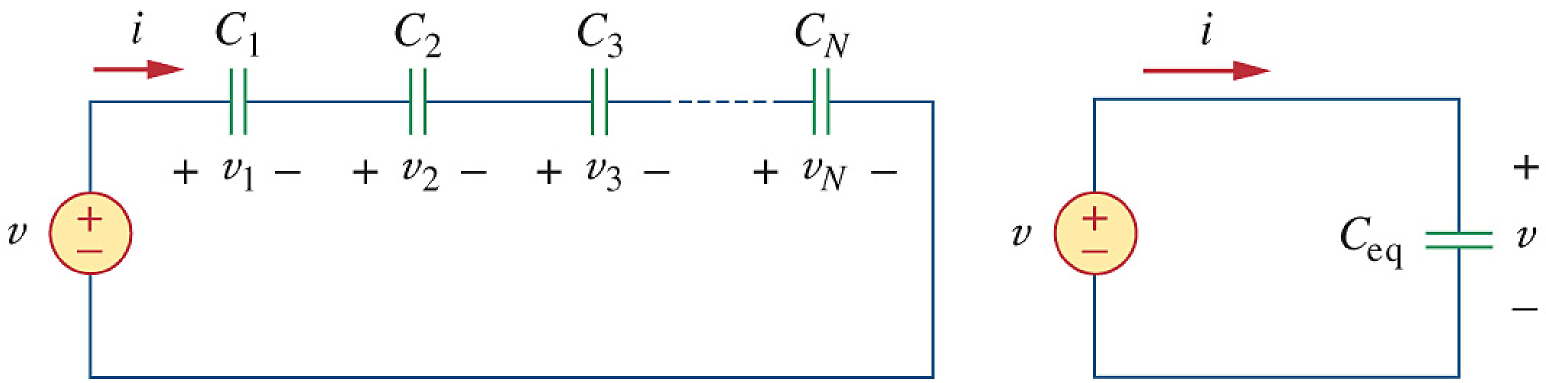
$$C_{eq} = C_1 + C_2 + \cdots + C_N$$

Proof :

$$i = \sum_{k=1}^{N} i_k = \sum_{k=1}^{N} C_k \frac{dv}{dt} = \left(\sum_{k=1}^{N} C_k \right) \frac{dv}{dt} = C_{eq} \frac{dv}{dt}$$

$$C_{eq} = \sum_{k=1}^{N} C_k$$

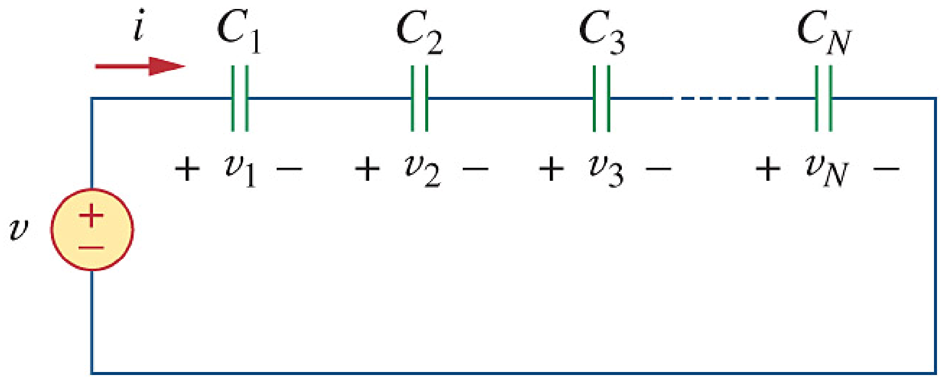
N series capacitors

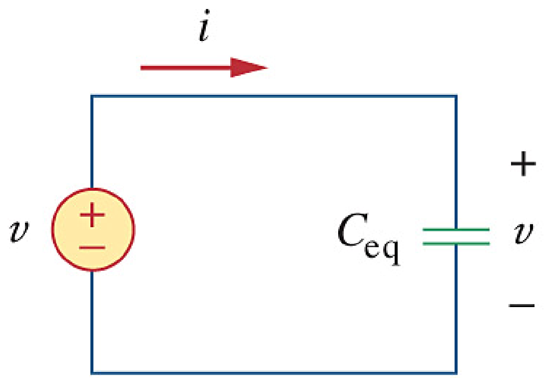


$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}$$

The initial voltage across C_{eq} is the sum of the individual initial voltages:

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$





$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}$$

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$

Proof :

$$v = \sum_{k=1}^{N} v_k = \sum_{k=1}^{N}\left(\frac{1}{C_k}\int_{t_0}^{t} i\,dt + v_k(t_0)\right) = \left(\sum_{k=1}^{N}\frac{1}{C_k}\right)\int_{t_0}^{t} i\,dt + \sum_{k=1}^{N} v_k(t_0)$$

$$= \frac{1}{C_{eq}}\int_{t_0}^{t} i\,dt + v(t_0)$$

$$\frac{1}{C_{eq}} = \sum_{k=1}^{N}\frac{1}{C_k},\ v(t_0) = \sum_{k=1}^{N} v_k(t_0)$$

Example 6.6 Find the equivalent capacitance seen between terminals a and b of the circuit in Fig. 6.16.

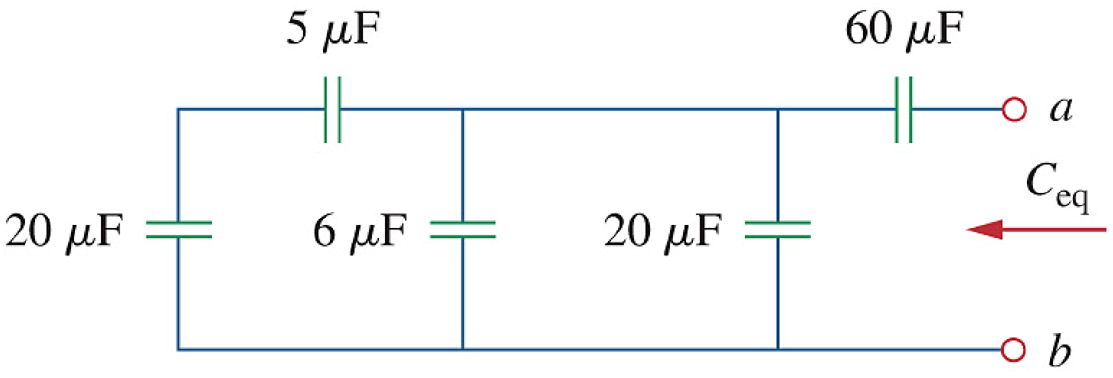


Figure 6.16

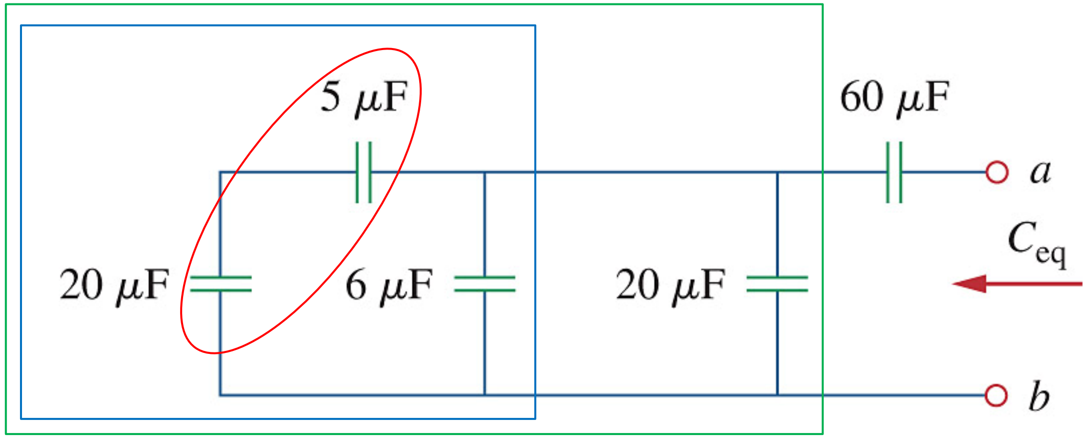


Figure 6.16

20 μF ll 5 μF = 4 μF

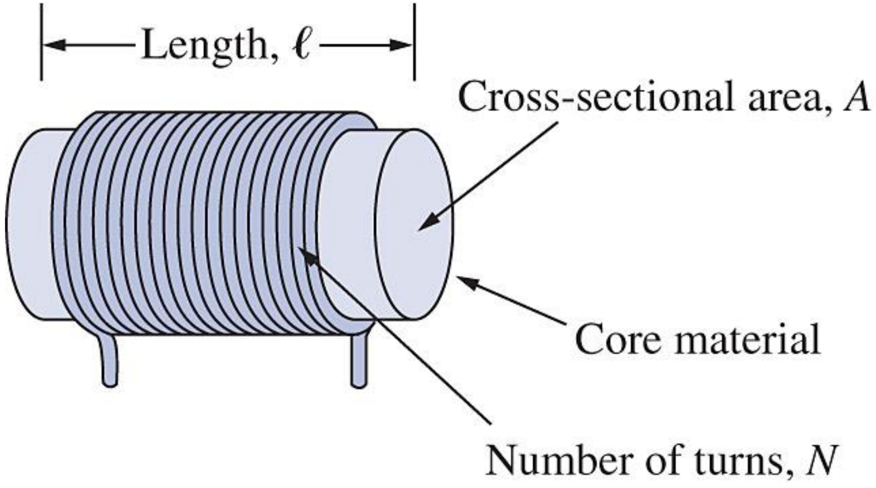
4 μF + 6 μF = 10 μF

10 μF + 20 μF = 30 μF

30 μF ll 60 μF = **20 μF**

6.4 Inductors

- An inductor is a passive element designed to store energy in its magnetic field.
- An inductor is usually formed into a cylindrical coil with many turns of conducting wire.



When a current source i is connected to the inductor, the source sets up a magnetic field. The magnetic flux linkage in the inductor, represented by ψ, is given by $\psi = Li$

Magnetic flux ψ (Φ or Φ_B) = $\mathbf{B} \bullet \mathbf{S}$ where S is vector area (Wb)
magnetic flux density or magnetic field (Wb/m^2) **B**

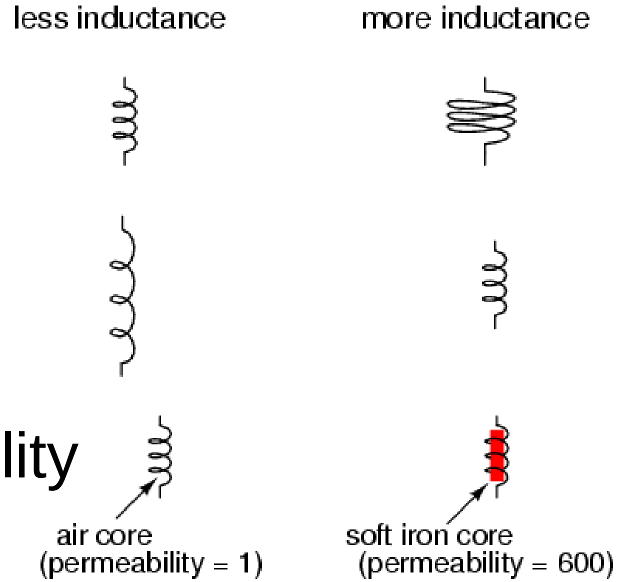
Inductance results from the magnetic field around a current-carrying conductor. Inductance of an inductor depends on its physical dimension and construction.

$$L = \frac{N^2 \mu A}{l}$$

A area

l length

μ permeability

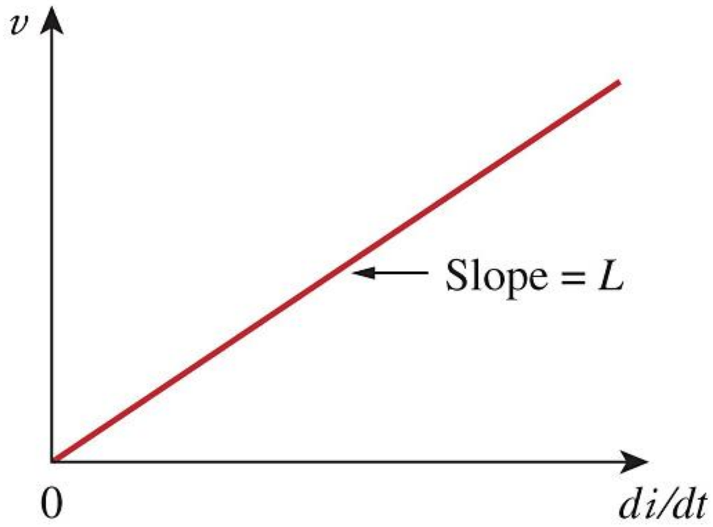


μ permeability of the core; *N* number of turns; *A* area of cross-section of the coil in square meters, *l* length of coil in meters (m).

IV relation of the inductor

If Inductance of the inductor [L] is independent of I, the inductor is said to be linear.

$$\boldsymbol{v = L\frac{di}{dt}}$$



Satisfy the eq. $v = L\frac{di}{dt}$

→ Linear inductor

The current-voltage relation of the inductor is

$$i = \frac{1}{L}\int_{-\infty}^{t} v dt = \frac{1}{L}\int_{t_0}^{t} v dt + i(t_0)$$

where $i(t_0) = \frac{1}{L}\int_{-\infty}^{t_0} v dt = \frac{\psi(t_0)}{L}$ the current through the inductor at time t_0

Power in Inductor

Following the passive sign convention, the instantaneous power delivered to the inductor is

$$p = vi = \left(L\frac{di}{dt}\right)i$$

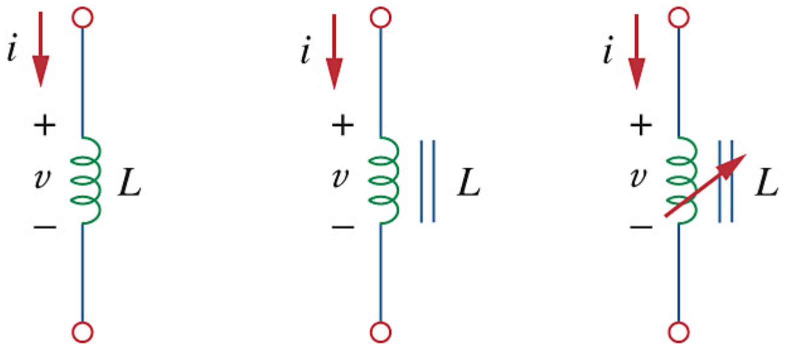


Figure 6.23 Circuit symbols for inductors: air-core, (b) iron-core, (c) variable iron-core.

The energy stored in the inductor is therefore

$$w = \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t}\left(L\frac{di}{dt}\right)i\,dt = L\int_{i(-\infty)}^{i} i\,di$$

$$= \frac{1}{2}Li^2\Big|_{i(-\infty)}^{i} = \frac{1}{2}Li^2$$

Analogy between capacitors and inductors

	Capacitor	Inductor
Electric/magnetic	q	ψ
	$q=Cv$	$\psi=Li$
i-v (or v-i) relation	$i=C\times dv/dt$	$v=L\times di/dt$
energy	$1/2Cv^2$	$1/2Li^2$

C: the capacity to store charges
L: the capability to induce magnetic flux

Important properties of an inductor

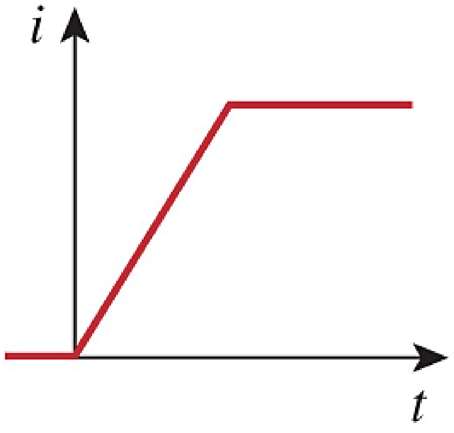
In circuit analysis, there are several important properties of the inductor to consider.

1. When the current across an inductor is not changing with time (DC steady state), the voltage through the inductor is zero $\rightarrow$ an inductor is a **short circuit** in the DC steady state.

Important properties of an inductor

2. The current in an inductor must be continuous, i.e. the current in an inductor cannot change abruptly.

- $v = L\, di/dt$, if di/dt is infinite $\rightarrow$ v is infinite



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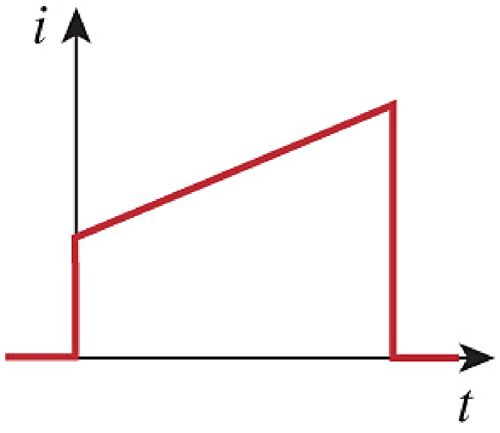
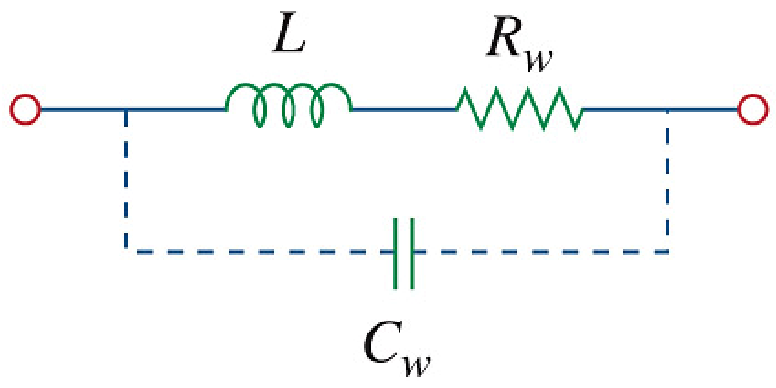


Impossible

Important properties of an inductor

3. The ideal inductor does not dissipate energy. It takes power from the circuit when storing energy in its magnetic field and returns previously stored energy when delivering power to the circuit.

4. A real inductor has a significant winding resistance and a small winding capacitance.



Example 6.8 The current through a 0.1-H inductor is

$i(t) = 10te^{-5t}$ A

Find the voltage across the inductor and the energy stored in it.

Solution :

$$v = L\frac{di}{dt} = 0.1\frac{d}{dt}\left(10te^{-5t}\right) = \frac{d}{dt}\left(te^{-5t}\right)$$

$$= e^{-5t} + \left(-5te^{-5t}\right) = e^{-5t}\left(1-5t\right) \text{ (V)}$$

$$w = \frac{1}{2}Li^2 = \frac{1}{2}\times 0.1\times\left(10te^{-5t}\right)^2$$

$$= 5t^2e^{-10t} \text{ (J)}$$

Example 6.10 Consider the circuit in Fig. 6.27. Under dc conditions, find (a) i, v_C, and i_L, (b) the energy stored in the capacitor and inductor.

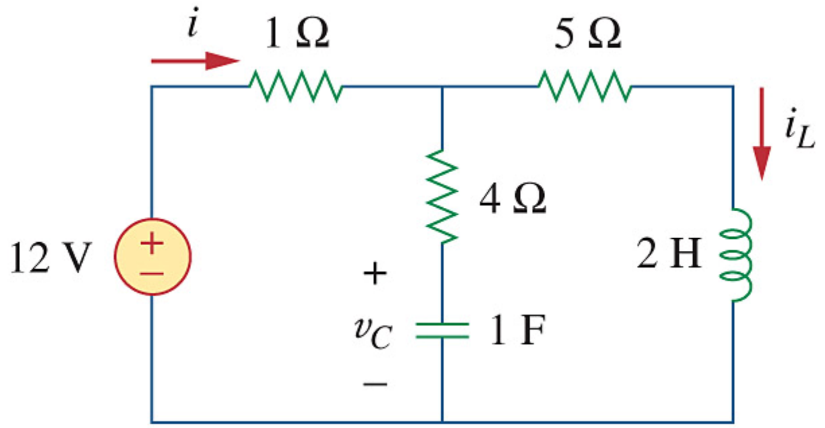


Figure 6.27(a)

Solution : We replace the capacitor with an open circuit and the inductor with a short circuit, as shown in Fig. 6.27(b).

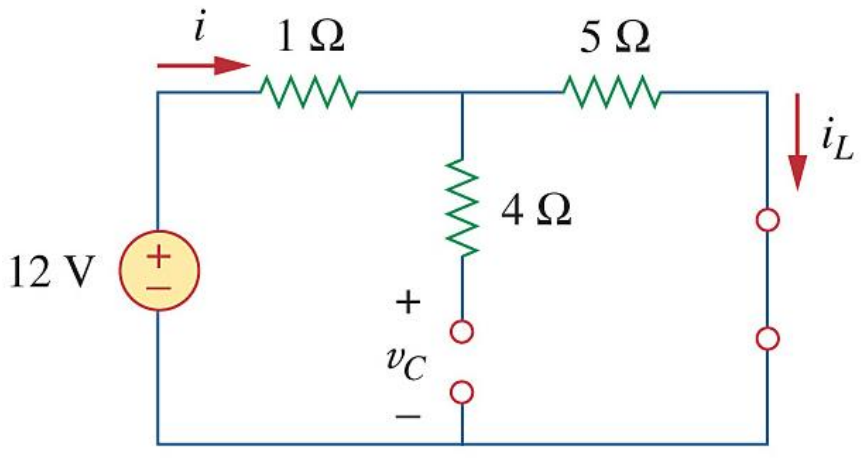


Figure 6.27(b)

(a)

$$i = i_L = \frac{12}{1+5} = 2 \text{ (A)}$$

$$v_c = 5i_L = 5 \times 2 = 10 \text{ (V)}$$

(b)

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2} \times 1 \times 10^2 = 50 \text{ (J)}$$

$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2} \times 2 \times 2^2 = 4 \text{ (J)}$$

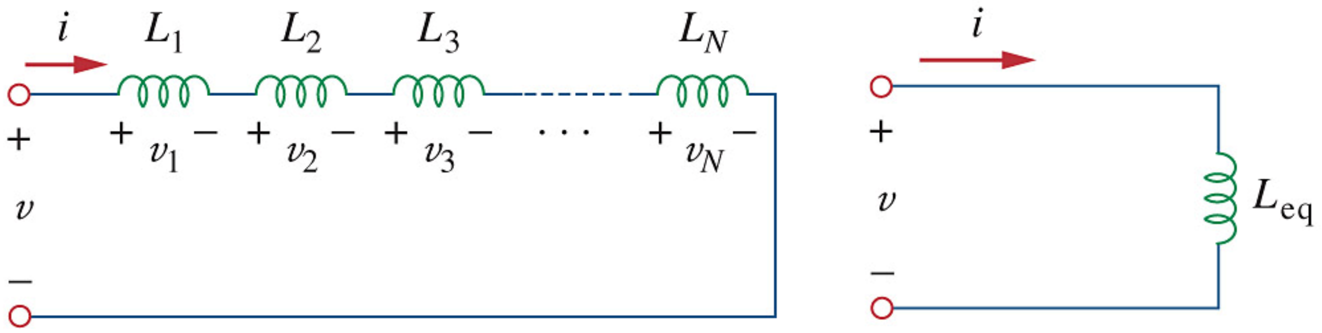
6.5 Series and Parallel Inductor

Same as the series-parallel combination method used for resistors, it can be extended to **series-parallel connections of inductors**.

Parallel: $$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$$

Series: $$L_{eq} = L_1 + L_2 + \cdots + L_N$$

N series inductors



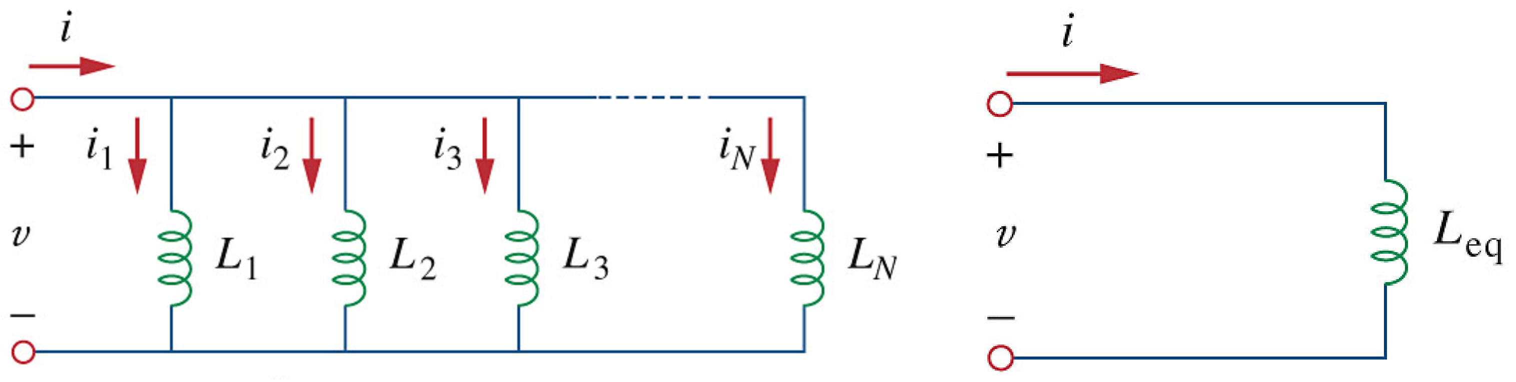
$$L_{eq} = L_1 + L_2 + \cdots + L_N$$

Proof :

$$v = \sum_{k=1}^{N} v_k = \sum_{k=1}^{N} L_k \frac{di}{dt} = \left(\sum_{k=1}^{N} L_k \right) \frac{di}{dt} = L_{eq} \frac{di}{dt}$$

$$L_{eq} = \sum_{k=1}^{N} L_k$$

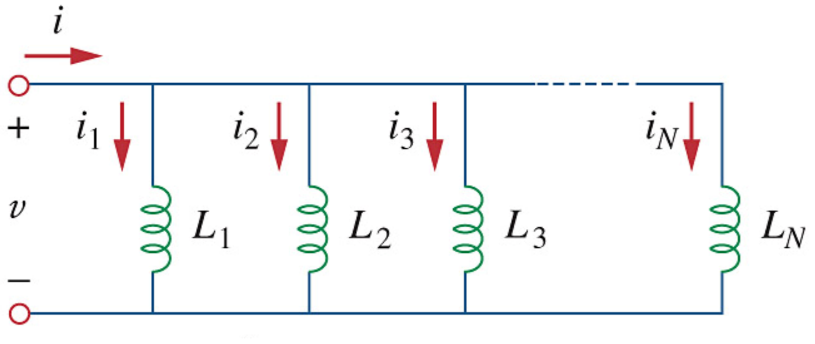
N parallel inductors

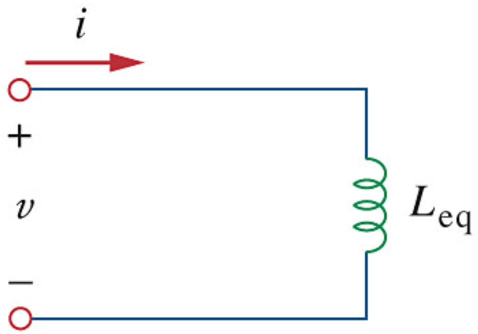


$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$$

The initial current through L_{eq} is the sum of the individual initial currents:

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$





$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$$

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

Proof :

$$i = \sum_{k=1}^{N} i_k = \sum_{k=1}^{N}\left(\frac{1}{L_k}\int_{t_0}^{t} v\,dt + i_k(t_0)\right) = \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v\,dt + \sum_{k=1}^{N} i_k(t_0)$$

$$= \frac{1}{L_{eq}}\int_{t_0}^{t} v\,dt + i(t_0)$$

$$\frac{1}{L_{eq}} = \sum_{k=1}^{N}\frac{1}{L_k},\; i(t_0) = \sum_{k=1}^{N} i_k(t_0)$$

Practice Problem 6.11 Calculate the equivalent inductance for the inductive ladder network in Fig. 6.32.

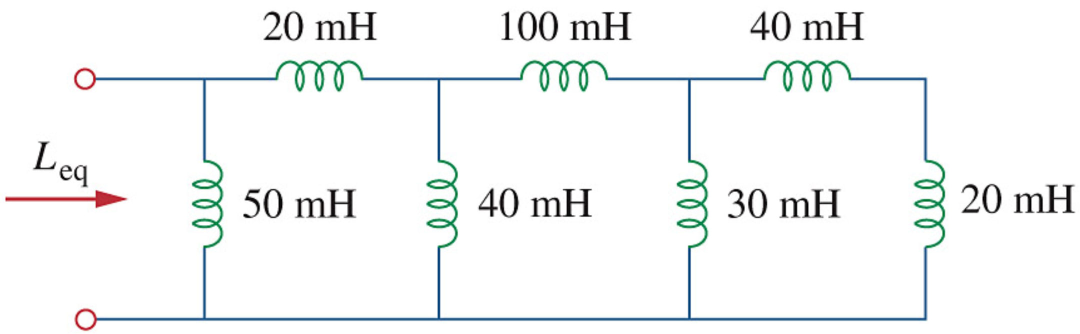
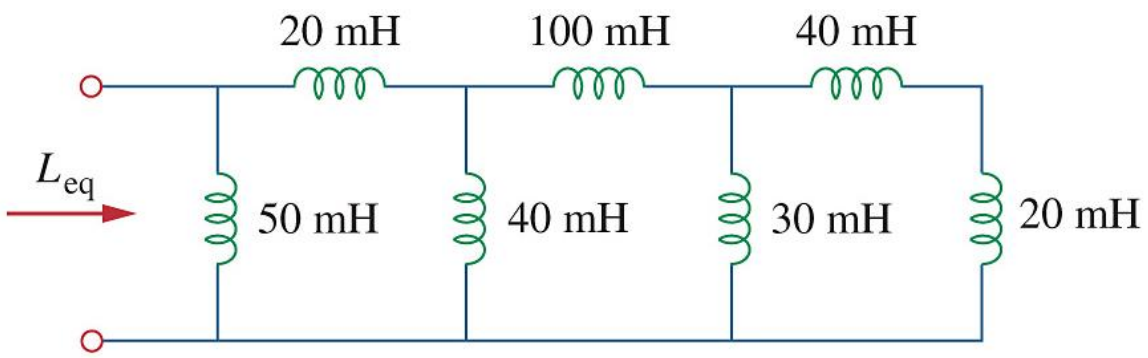


Figure 6.32



Solution :

$$\frac{1}{L_{eq}} = \frac{1}{50} + \frac{1}{20 + \frac{1}{\frac{1}{40} + \frac{1}{100 + \frac{1}{\frac{1}{30} + \frac{1}{40+20}}}}}$$

$$L_{eq} = 25 \text{ (mH)}$$

6.6 Applications

- Circuit elements such as resistors and capacitors are commercially available in either discrete form or integrated-circuit form.
- Inductors usually come in discrete form and tend to be more **bulky and expensive**. For this reason, inductors are not as versatile as capacitors and resistors. They are used for several applications in which inductors have **no practical substitute**.

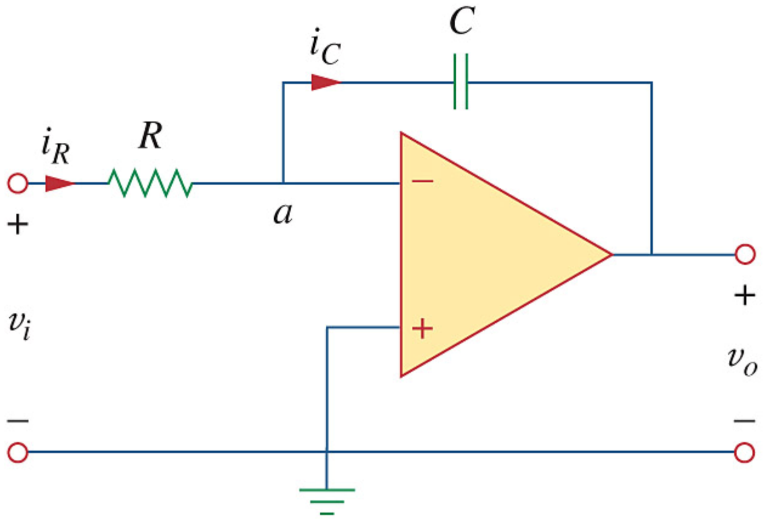
- The capacity to store energy makes capacitors and inductors useful as **temporary voltage sources and current sources**, respectively.

- Capacitors and inductors are frequency sensitive. This property makes them useful for **frequency discrimination**.

Integrator

An integrator is an op amp circuit whose output is proportional to the integral of the input.

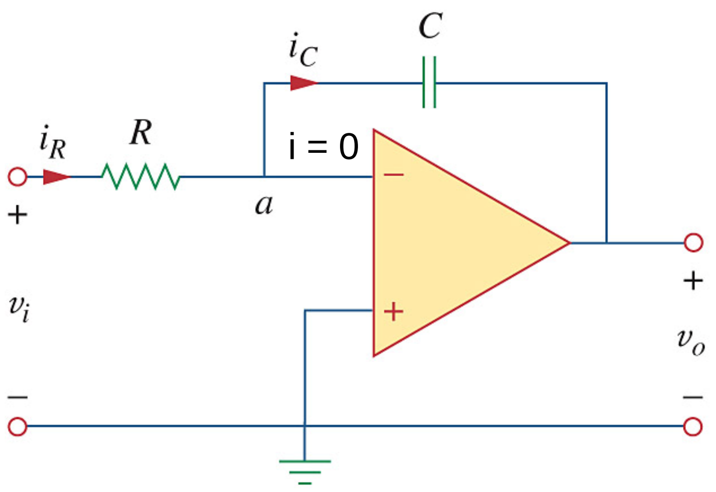
Capacitor instead of R_f



$$v_o = -\frac{1}{RC}\int_0^t v_i dt + v_o(0)$$

If $v_o(0) = 0$, then

$$v_o = -\frac{1}{RC}\int_0^t v_i dt$$



Proof :

$$\begin{cases} i_R = i_C \\ i_R = \dfrac{v_i}{R},\ i_C = -C\dfrac{dv_o}{dt} \end{cases} \Rightarrow \frac{dv_o}{dt} = -\frac{v_i}{RC}$$

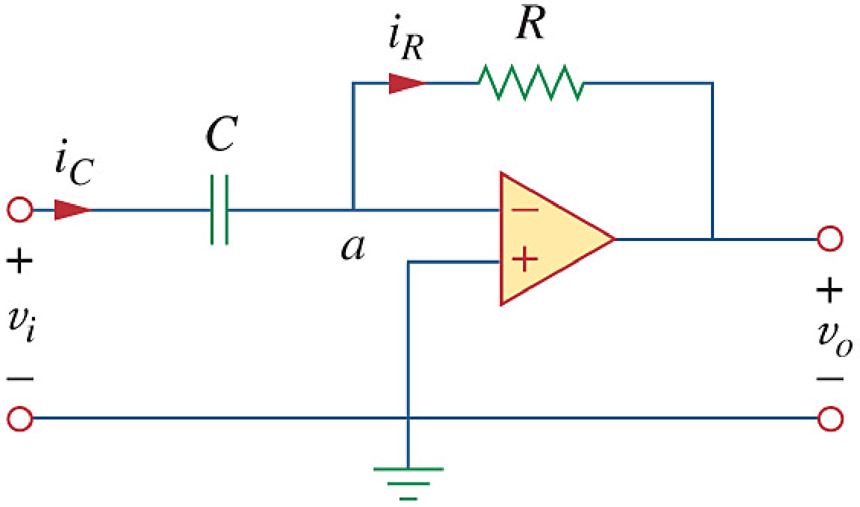
$$v_o - v_o(0) = -\frac{1}{RC}\int_0^t v_i dt$$

$$v_o = -\frac{1}{RC}\int_0^t v_i dt + v_o(0)$$

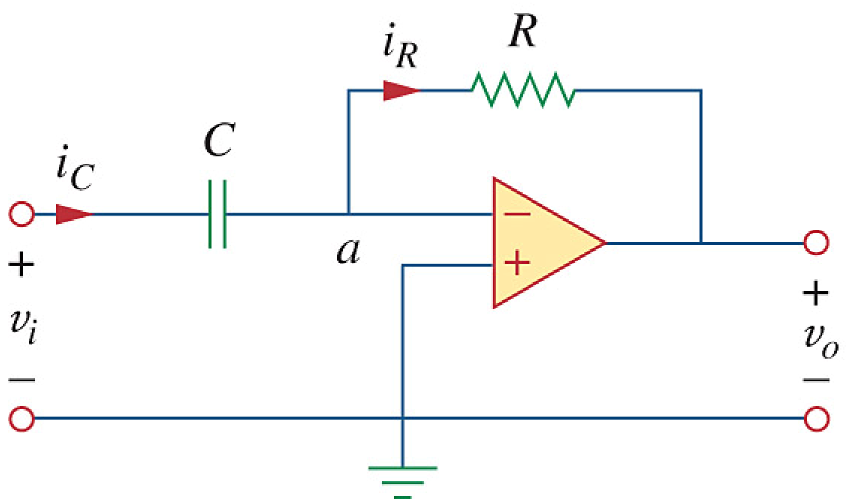
Differentiator

An differentiator is an op amp circuit whose output is proportional to the derivative of the input.

Capacitor instead of R_i



$$v_o = -RC\frac{dv_i}{dt}$$



Proof :

$$\begin{cases} i_C = i_R \\ i_C = C\dfrac{dv_i}{dt},\ i_R = -\dfrac{v_o}{R} \end{cases} \Rightarrow v_o = -RC\dfrac{dv_i}{dt}$$

It is better not to use differentiator. If the input signal is contaminated by noise, taking derivative will magnify the noise.

Additional Question. Suppose the input voltage U_S is a DC and the circuit has reached its final state (the current in the circuit doesn't change with time), what is the total energy stores in the inductor L and the capacitor C?