Decoherence Subjective Prelims (Problems and Solutions)

Opens at 0900 hours on 15/12/2019; submissions close at 0859 hours on 16/12/2019

Instructions

- Time allowed : 24 hours starting from 0900 hours on 15^{th} December 2019 to 0859 hours on 16^{th} December 2019
- The marks are mentioned along with each question, part and subpart.
- Partial credit may be awarded for an incomplete solution or progress towards a solution.
- Skipping steps is fine as long as they are "truly trivial", i.e. trivial according to Kleppner and Kolenkow. (For us, Landau is not trivial) [As a thumb's rule, if it took you more than 30s to think up the steps, then it is not trivial]
- As long as they are complete, even one-line answers are acceptable (like a reference to the solution if you can get it that is...). Voodoo/ Black-magic proofs are not acceptable.
- If you have used code anywhere, please send us both the code, and the compiled file
- Because the exam is open resources (internet,books,etc.), we encourage you to discuss amongst your teammates only. If we find by statistical analysis that you have copied from other parties, we reserve the right to allot you a zero. No appeals will be entertained in this regard.
- To submit your answers, please create a SEPARATE PDF FILE FOR EACH QUESTION and EMAIL THEM TO US. We will accept multiple attempts/methods and would grade you accordingly.
- The subject of your mail should be "Decoherence Problem No. < Put Problem no. here >"
- Name the pdf files according to this format : <RollNo.>_<ProblemNo.>_<MethodNo.>.pdf. For example, if your roll number is 123456, and you want to submit your second solution for problem 7, you shall name the file as "123456_7_2.pdf", and send it in a mail with subject "Decoherence Problem No. 7"
- Not citing your resources would be taken as a case of plagiarism and would lead to an immediate 20 point decrement per incident.
- If you are submitting photographs of your solutions, keep them as neat and ordered as you can. Illegible answers won't be evaluated. So, if you believe you can type fast enough, we encourage you to submit typed responses as they are unambiguous. (You have 24 hours afterall...)
- And most importantly enjoy the exam! This is not the entrance examination to some institute or your board exams (you have already been through that and we doubt that you want to relive that!). Keep a clear head, be logical in your approach and maintain a positive outlook. We want you to enjoy the exam, and there are rewards for good performance in the prelims as well.

Answer the questions to the best of your abilities. If you think a question has more than one answer, go for it.
Here, you will need to use the internet at more locations than one. Just remember, be honest and cite your sources. Even that will gain you points. **Remember, you can take up to 24 hours to submit this.**

Best of Luck!

Question:	1	2	3	4	5	6	7	Total
Points:	20	20	15	25	20	18	32	150

1. (20 points) **Electric Doraemon**
Doraemon is known for giving Nobita amazing gadgets. One day, Nobita comes crying home from school, having been beaten up by Gian. To make him feel better, Doraemon gives him a few gadget toys. Help Nobita understand the working principles of the toys.

(a) (4 points) We all know that when a charged object is brought near neutral bodies, the neutral object is usually attracted by the charged object due to induction. However, Doraemon's amazing gadget is a neutral object that gets repelled. Predict the shape of the object such that it would be repelled by a point charge particle, for a certain position of the particle relative to the object. Argue about the approximate position for which the repulsion happens (Hint – think of a thin hemispherical shell). [Please clearly mention why the repulsion is greater. A qualitative answer will also suffice.]

> **Solution:** For an example of this, see this excellent qualitative and quantitative analysis in Am. J. Phys. 79 (8), August 2011 [DOI: 10.1119/1.3595554]. Other answers which give workable qualitative/quantitative explanations have been given full credit.

(b) (4 points) Next, Doraemon takes out a balloon (filled with a gas atleast as dense as air) which can float in the air for a long time (much longer than the maximum expected $2u/g_{\text{eff}}$ where u is the speed of projection and g_{eff} is the effective acceleration due to gravity after correcting for buoyancy only), but only when thrown aptly in a particular direction. Suggest how can this be achieved. Neglect air resistance.

> **Solution:** A very good way to achieve this is by filling a charged gas inside the balloon and throwing it in a direction perpendicular to the horizontal component of the earth's magnetic field. Depending on the net charge of the balloon, we would get a net force (up or down), and that gives us a critical speed at which to throw such that the floating is almost indefinite. To find such a speed, we assume that the net charge on the balloon is Q and the $g_{\text{eff}} = g'$. Let the mass of the balloon including the gas be M and height of projection is H. So,
>
> $$\mathbf{F} = M\mathbf{a} = Q(\mathbf{v} \times \mathbf{B}_{\text{Earth}}) - M\mathbf{g}'$$
>
> $$\implies T_{\text{floating}} \sim \sqrt{\frac{2HM}{Mg' - Q\|\mathbf{v} \times \mathbf{B}_{\text{Earth}}\|}}$$
>
> So, $T_{\text{floating}} \to \infty$ as $v \to Mg'/QB_{\text{Earth}}$
> This is a good approximation although we have not taken things like Eddy currents in the gas into consideration, as the fields get stronger as we get near the surface, and that only strengthens this method.
>
> **Note :** A lot of answers we received had solutions which included giving the balloon orbital velocity, but the problem with those is that they are independent of the direction of throw, while our question mentions that the gadget works when thrown aptly in a particular direction.

(c) (12 points) The third gadget is capable of pulling Nobita up in the air. The gadget is of the form of a freely deformable wire loop (of total length L), put over a frictionless pulley (of negligible radius ($2\pi r << L$)) which is permanently attached to the ceiling (Refer to the diagram). A current I flows through the loop. A magnetic field is switched on in the direction indicated and a mass M (say that's Nobita) is suspended from the other end using a frictionless non-conducting ring. For parts i, ii, and iii, ignore the field induced by the wire.

i. ($1\frac{1}{2}$ points) Sketch the shape of the wire.

ii. ($1\frac{1}{2}$ points) What is the maximal height by which the load can be lifted using such an arrangement (increasing the current, if necessary)?

iii. (2 points) Write an equation from which it is possible to determine the lifting height Δh.

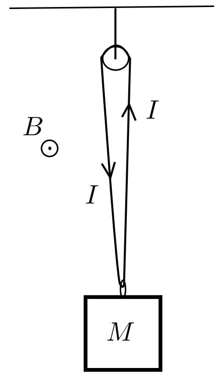

Figure 1: Diagram for Part (c)

iv. (7 points) Now consider the field induced by the wires. Predict the shape of the wires in this case (the magnetic field is still on). The use of computation is allowed.

Solution: For answers to parts i,ii, and iii, please look at solution to P7 of the Estonian-Finnish Physics olympiad 2012.
Part iv. is to be solved computationally by considering the conditions of symmetry of the two sides and then applying the Biot-Savart's law to find a stable solution. The organisers know of no analytical method to solve this exactly.

Nobita, now happy, thanked Doraemon for his gadgets.

2. (20 points) **Antics on a trip to CERN**
Calvin and Hobbes are overwhelmed with a piece of news they received just this morning; they've been selected to attend a 5-day camp at CERN for budding theoretical physicists. Unable to contain their excitement, they begin planning for the trip 2 months in advance, thinking of all the things they could possibly take with them to their science-y excursion.

(a) (4 points) Fast forward 2 months: it's the day of their departure for CERN. Waking up early and rushing to the airport, they find themselves with infinite time (not really :P) after reaching there. Being the physicists that they are, they begin observing airplane wheels and the different lights which signal them.

Suddenly, they come across one peculiar assembly of a plane wheel having a multi-coloured lamp attached to it. The lamp is attached to the edge of the wheel, which moves (slides) while rotating on the runway. The lamp emits light pulses: the duration of each pulse is negligible and the interval between two pulses is $\tau = 100$ ms. The first pulse is of orange light, the next one is blue, followed by red, green, yellow, and again orange (the process starts repeating periodically). They record the motion of the wheel using photographs with such a long exposure time that exactly four pulses are recorded on the photo (see figure 2). Due to the shortness of the pulses and small size of the lamp, each pulse corresponds to a coloured dot on the photo. The colours of the dots are provided with lettering:
O — orange, S — blue, P — red, R — green, and K — yellow.
The friction forces acting on the plane wheel can be neglected.

Being the curious cats (again, not both of them :P) that they are, Calvin and Hobbes now seek your help in satisfying their curiosity.

i. (2 points) Mark on the figure by numbers (1 to 4) the order of the pulses (dots). Motivate your answer. What can be said about the value of the exposure time T?

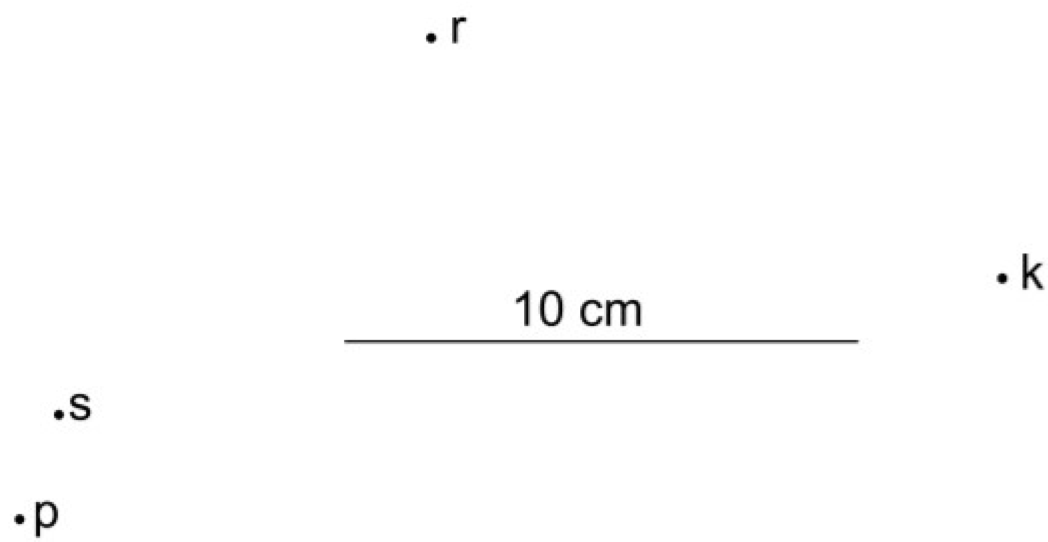

Figure 2: Long exposure photo captured

ii. (2 points) Using the provided figure, find the radius of the wheel R, the velocity of the center of the wheel v and the angular velocity ω (it is known that $\omega < 60$ rad/s). The scale of the figure is provided by the image of a line of length $l = 10$cm.

Solution: For the solution to this, please see solution to P5, Estonian-Finnish Physics Olympiad 2007.

(b) (2 points) With their flight time fast approaching, Calvin decides to play a prank during the security check. He picks up a small metal magnet and keeps it in his bag, in order to fiddle with the magnetic detectors (although his intentions are completely innocent, we strongly discourage you from doing so, even if it's "for the sake of Science"). Needless to say, the detector catches hold of his mischief and he dislodges one magnet from its position in the detector.
Satisfied (being the evergreen prankster) with himself, Calvin decides to model a simple experiment to explore the interaction between 2 small magnets.
One of the magnets is hanged from a thread with length $l = 1$m. The other magnet is slowly moved closer while keeping the axes of the magnets always on the same horizontal line. At the moment when the distance between the magnets is $d_1 = 4$cm and the hanged magnet has moved $x_1 = 1$cm from its initial position, balance is lost and the magnets pull together. By making the assumption that the pulling force between the magnets F_m depends on the distance between them (d), according to the relation $F_m = kd^{-n}$, Calvin wants you to find the missing value of the exponent n.
Are you up to the task?

Solution: For the solution, please see solution to P3, Estonian-Finnish Physics Olympiad 2014.

(c) (8 points) Switzerland is finally upon our 2 young physicists. They're overjoyed with the scenic beauty that Switzerland possesses and spend the whole day sightseeing and enjoying various delicacies (plane food never does fill one up, does it? :P).
Having fallen in love with the roads, Hobbes decides to ride on this very unique vehicle the locals call the "Snow Demon".
Shown below (see figure 3) is the Snow Demon, a vehicle built in 2015. There is no source of stored energy such as a battery or gasoline engine; all of the power used to move the car comes from

the wind. The only important mechanism in the car is a gearbox that transfers power between the wheels and the propeller. The Snow Demon was driven both directly downwind and directly upwind, as shown below. In each case the car remained exactly parallel (or anti-parallel) to the wind without turning. The tests were conducted on level ground in steady, uniform wind, and continued long enough to reach the steady state.

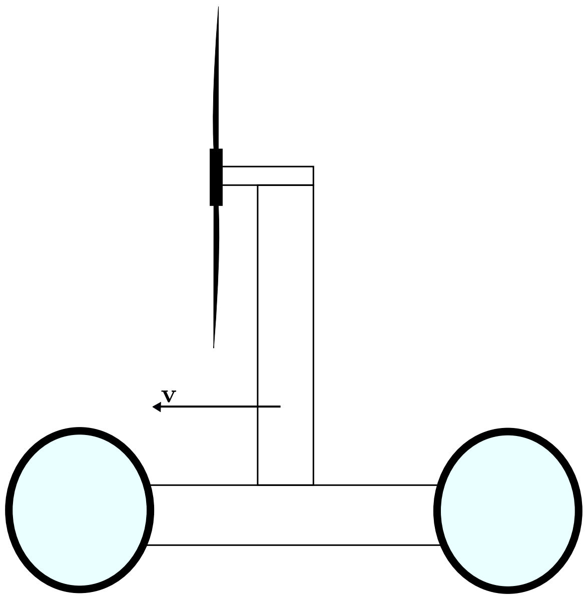

Figure 3: The Snow Demon

The builders of this unique car began boasting about their creation to Hobbes, claiming the car could travel faster than the wind while driving downwind ($V_{car} > V_w$). Hobbes, being the more skeptical of the 2, felt this to be a hoax and also doubted their identical claim for the upwind case.

Hobbes now wants your help in verifying these tall claims.

i. (2 points) Consider the "downwind faster than the wind" case first. Is the claimed motion possible? Explain briefly (complete freedom has been provided regarding the choice of variables, for any extra parameters you may need.)

ii. (2 points) If such a "bizarre" motion is possible, calculate the attained ground speed (Assume that when transferring power in either direction between the propeller and the wheels, a fraction β of the useful work is lost. Keep assuming the wind speed to be v_w.)

iii. (4 points) Repeat the above two tasks for the "upwind" case and hence verify both the claims.

Solution: For the solution to this, please see the solution to problem B1 of 2013 USAPhO.

(d) (2 points) After a plethora of eventful occurrences, Calvin and Hobbes have finally reached their destination - the "European Organization for Nuclear Research". They're brimming with energy as they start off their first day at the camp. With the advances of superconductors being in vogue, they're given a tour of a variety of devices which showcase the large potential that superconductors have (after all, who doesn't like seeing liquid helium in action? :P)

One such device which catches everyone's attention is a superconducting mesh i.e. a mesh made from a flat superconducting sheet by boring a grid of small holes into it.

Calvin and Hobbes, having gotten their "physics senses" tingled, think of conducting a fun experiment with the mesh. Under the supervision of the professors, they take the sheet (when in the non-superconducting state) and place a magnetic dipole of moment m at a distance a from the mesh, such that it points perpendicularly towards the mesh. The mesh is now cooled so that it becomes superconducting.

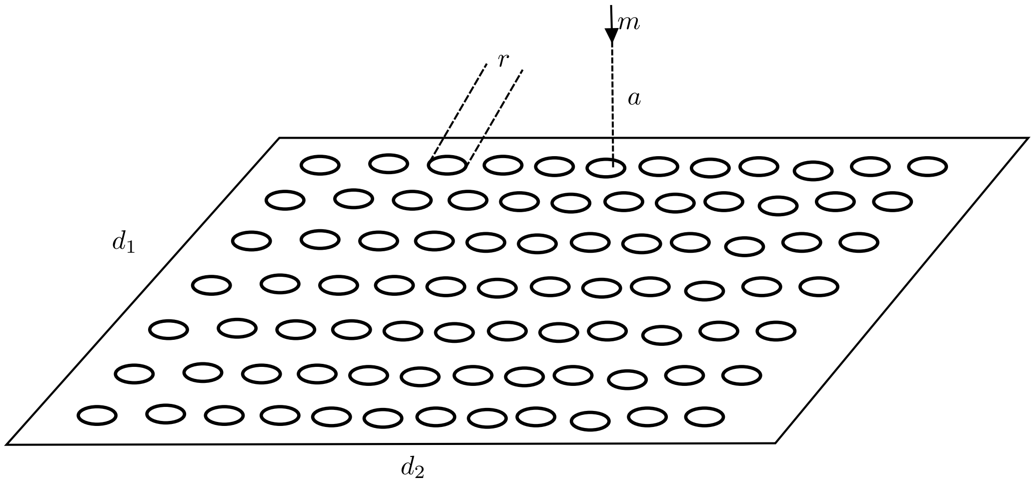

Figure 4: Initial Configuration of the dipole over the mesh

The dipole is now placed at a distance b, perpendicularly to the surface of the mesh (it has been displaced from its original position and changed to this new one).

Calvin and Hobbes now want you to determine the force acting between the mesh and the dipole. (Optional: How does this force change when b is nearly equal to a?)

Note: The pitch of the grid of holes (distance between 2 consecutive holes) is much smaller ($<<$) than a and b, while the linear size of the sheet is much larger ($>>$) than both a and b.

Solution: For the solution to this, please see EUPhO 2017 Problem 3

(e) (4 points) Receiving tremendous amounts of praise for this experiment, Calvin and Hobbes are on cloud nine and finally reach the center of attraction at CERN: The L.H.C (Large Hadron Collider). They're introduced to the continuous bombardment of particles in the L.H.C and are overwhelmed with how seamlessly the collisions take place.

As one innovative thought begets another, Calvin and Hobbes decide to model a multi-particle collision by way of neutron decay.

They consider a "free" neutron of rest mass m_n, which decays at rest in the laboratory frame of reference into 3 non-interacting particles: a proton, an electron and an anti-neutrino. The rest mass of the proton is m_p while the rest mass of the anti-neutrino m_v (Yes, neutrinos have mass) is assumed to be non-zero with $m_v << m_e$, where m_e is the rest mass of the electron.

The measured values of the rest masses are as follows:

1. $m_n = 939.56563\ \frac{\text{MeV}}{c^2}$
2. $m_p = 938.27231\ \frac{\text{MeV}}{c^2}$
3. $m_e = 0.5109907\ \frac{\text{MeV}}{c^2}$

where c is the speed of light in vacuum.

To make life easy for you and themselves, they measure all energies and velocities of the particles in the laboratory frame.

Denote by E the total energy of the electron coming out of the decay.

i. (2 points) Calvin and Hobbes now want you to find the maximum possible value $E_{\max}$ (of E) and the speed v_m of the anti-neutrino when $E = E_{\max}$. To feel like they're real scientists (and yes, not Mad Scientists), they want you to report the answers in terms of the rest masses of the particles and the speed of light.

Solution: For the solution to this, please see Problem 3 of IPhO 2003

ii. (2 points) The two musketeers now manage to steal a neutron confined within a box and bring it back to their room (again, all "for the sake of science"; following in their footsteps and stealing atomic property is discouraged :P).
Assuming the box is a small cubical one with side length $d = 1\text{mm}$, can you help them approximately calculate the pressure P exerted by the neutron confined within the box? (The rest mass of the neutron is again m_n)

Solution: The solution follows here.[3]
A particle of mass m enclosed in a 'box' of finite size cannot be at rest because of quantum effects, and it must have linear momentum of some average magnitude p. According to Heisenberg's uncertainty relationship,

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

where $\Delta x \sim d$ and $\Delta p \sim p$ which is the magnitude of the linear momentum of the neutron. Considering, for the sake of definiteness, the limiting case where equality holds,

$$|p| \approx \frac{\hbar}{2d}$$

and v the speed of the neutron is given by,

$$|v| \approx \frac{\hbar}{2md}$$

This result can be interpreted (following classical physics, for want of a better principle) as the neutron bouncing back and forth between the opposite walls of the box. So, at one of the walls, a change of linear momentum is $2mv$ which occurs at regular intervals of $\Delta t \approx 2d/v$, and as a result, the pressure is,

$$P = \frac{F}{d^2} = \frac{\Delta p}{d^2 \Delta t} = \frac{mv^2}{d^3} \approx \frac{\hbar^2}{4md^5}$$

So,

$$P \approx \frac{1.4 \times 10^{-42}\ \text{Nm}^3}{d^5}$$

Calvin and Hobbes, now tired after their long day's adventure at CERN, go back to the hotel to sleep.

However, you'd be greatly mistaken if you think that's all this dynamic duo has to offer!

3. (15 points) **Let's go for a Spin**
The spin of an electron is something which has no classical analogue. This question lets us interpret spin in a quantum mechanical sense.
   (a) (5 points) Start with the relativistic Hamiltonian equation for a particle. Now use that in the Schrodinger wave equation. Explain the degeneracy in solutions obtained this way.
   (b) (5 points) To remove the degeneracy in the solutions, use the fact that the wave equation should be linear in the $\frac{\partial}{\partial t}$ operator (Take this as an axiom). Write down a linear equation in operators for all the components' momentum operators and $\frac{\partial}{\partial t}$. Also write down the commutation relations

connecting each coefficient of these operators plus a constant operator in the wave equation (this constant operator is needed to bring in the rest mass energy) .

(c) (5 points) Call the four coefficients of the momentum operators by σ, a 4D vector with the four components as four operators. Now, in the wave equation obtained, use the $mc^2 >>$ other energy terms approximation to yield an equation for the Hamiltonian. Compare the new term which wouldn't appear in the classical Hamiltonian with the potential energy of a dipole in a magnetic field. Derive the magnitude of the spin of an electron, given that Lande's g factor for the electron is 2.

Note: You need to write the Hamiltonian including **A** (potential 4-vector - 3 magnetic potential components and one electric potential component) terms for the part of calculating the spin of an electron.

Solution: The solutions to this question can be found in the book Principles of Quantum Mechanics by P.A.M. Dirac. The specific pages are:

(a) (5 points) Page no. 253,254, Section 67

(b) (5 points) Page no. 254,255

(c) (5 points) Page no. 263, Section 70

4. (25 points) **Safety First**

Calvin and Hobbes, back from their exhilarating trip to CERN, are drowning in exuberance and looking for others (living or non-living) to share their excitement with.

Luckily (or unluckily), they come across their Physics Lab instructor, Mr. Garfield, who shares their enthusiasm and wackiness. Coaxing Mr. Garfield into giving them unrestricted access to the lab (they wish to inform the reader that tuna fish does wonders on Mondays!), they decide to "implement" all the new things they learned during their trip to CERN.

(a) (10 points) They are currently obsessed with electromagnetic waves and looking for novel ways to manifest their properties. They come across 2 metallic plates in the lab and decide to make observations.

In order to observe an interaction between the metallic plates and thus convince themselves of the fact that 'vacuum is full of virtual EM waves; it isn't empty', they keep the plates separated at a distance d.

They're aware of the fact that the electromagnetic energy stored between the plates is given by $E = \frac{1}{2}\sum_{n<p} E_n$, where p ($>> 1$) is a large number that depends on the material properties of the plates and E_n is the energy of the n^{th} standing wave between the plates.

Proud of their "experiment", they now invite you to answer the following questions:

i. (1 point) Assuming that the waves take the form of $\sin(k_n x)$, determine the possible values of k_n such that standing waves are formed between the plates.

ii. (2 points) Show that the vacuum electromagnetic force is of the form $\frac{a}{d^2}$. Also find out the value of the constant a.

iii. (2 points) What is the measured force due to the vacuum electromagnetic waves "outside" the two plates?

iv. (3 points) Find a numerical value for the amplitude of the force when $d = 1\text{mm}$ (use $p = 2000$).

v. (2 points) What happens when $p \to \infty$?

Solution: This has two solutions that we are giving full credit to. Solution 1, although incorrect, has the same mistake as the creator of the question from our team made, which was

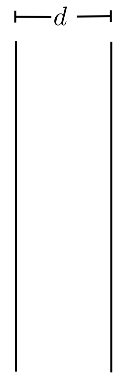

Figure 5: Configuration of the plates

discovered only recently by us. Solution 2 is the correct solution.

Solution 1:

i. (1 point) The boundary conditions require that $d = n\lambda/2$ where $n \in \mathbb{Z}^+$. So, $k_n = 2\pi/\lambda = n\pi/d$

ii. (2 points) $E_n = \hbar c k_n$, so,

$$E_{\text{total}} = \frac{\hbar c \pi}{2d} \sum_{n<p} n = \frac{\hbar c \pi p(p-1)}{4d}$$

So,

$$F = -\frac{\partial E}{\partial d} = \frac{\hbar c \pi p(p-1)}{4d^2}$$

Hence, the force is repulsive, with,

$$a = \frac{\hbar c \pi p(p-1)}{4}$$

iii. (2 points) From the outside, the force is equivalent to the force on a plate in the same system as $d \to \infty$. So, $F = 0$.

iv. (3 points) On plugging the value in, we get $F \sim 10^{-13}$ N

v. (2 points) For the case of finding this force as $p \to \infty$, we using our expression get an absurd value. So, to smooth this out, we introduce a sufficiently smooth regulator, which in our case is $e^{-na/d}$ (we would see later that the answer is independent of a). So,

$$E(d; a) = \frac{\hbar c \pi}{2d} \sum_{n<p} n e^{-an/d}$$

$$\implies E(d; a) = \frac{\hbar c \pi}{2d} \left(\frac{e^{-a/d}}{(1 - e^{-a/d})^2} \right)$$

$$E(d; a) = \frac{\hbar c \pi}{2d} \left(\frac{d^2}{a^2} - \frac{1}{12} + \mathcal{O}(a) \right) \quad \text{See that } -1/12 \text{ coming from an infinite summation?}$$

Now, we sum the parts of energy on the left of plates + on the right of plates (Considering an imaginary box of length L in which the plates are placed) + between the plates to get a total energy and then take a partial derivative w.r.t. d to get the negative of the force. Doing this, we get, as $L \to \infty$ (that is getting rid of the box)

$$F = -\frac{\hbar c \pi}{24 d^2}$$

Solution 2 :
For the solution, please visit Quantum mechanics/Casimir effect in one dimension from Wikiversity for an enlightening analysis.

(b) (3 points) Calvin and Hobbes, now amazed with how "evanescent" (pun intended) EM waves can be, turn to something more "physical". Being a fan of celestial phenomenon, they encounter the following scenario: A satellite model is made to roughly orbit a miniature Earth, in a circular orbit. To model the atmospheric drag, air is blown opposite to the velocity v (in the ground frame) of the satellite (at any instant), using a hairdryer. Assume gravity to be the only force of interaction between the Earth and the satellite.

Assuming the orbit to remain nearly circular at all times, Calvin and Hobbes observe a very peculiar occurrence: the "drag" force is misbehaving and not being true to its name. Help them resolve this "paradox" by explaining briefly if the "drag" force retards or accelerates the satellite (You may perform a detailed analysis of the above scenario but do label all the variables that you'll be using and any assumptions that you'll be making. This is, however, completely optional).

Solution: For a solution to this, please look up Problem 37 in 200 more puzzling physics problems by Peter Gnädig[3]

(c) (3 points) Fascinated by satellite motion and driven by their prankster instinct, they think of what might happen if the satellite collides with the earth. To emulate this collision, they consider the following thought experiment:

2 asteroids of mass M and radius R are moving in free space. At an instant, their velocities are $\mathbf{v_1}$ and $\mathbf{v_2}$ respectively and the position of the second with respect to the first is $\mathbf{r_{12}}$.
Determine the condition when the two asteroids definitely collide.

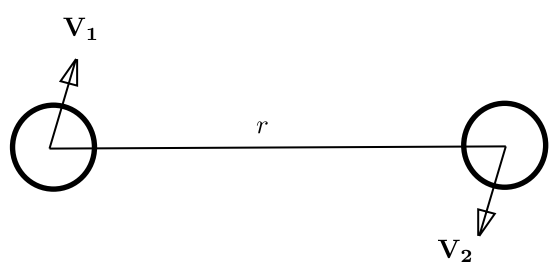

Figure 6: Asteroids

Solution: Here, there are two different solutions which would be given full credit if the underlying assumptions in each/ the driving phenomenon are explained with the solution.

Solution 1: Assumption is that gravitational waves do not have any effect on the orbit (this is explained by the fact that energy extraction by gravitational waves is immensely slow). Now , using this, Let us go into the center of mass (COM) frame of these two asteroids. Now, considering the asteroids to be point objects, if their closest distance of approach is $d < 2R$, then the asteroids collide. The closest distance of approach can be found easily using conservation of energy and angular momentum.

Trivially, in the center of mass frame, $\mathbf{V_1} = (\mathbf{v_1} - \mathbf{v_2})/2 = \mathbf{V}$ and $\mathbf{V_2} = (\mathbf{v_2} - \mathbf{v_1})/2 = -\mathbf{V}$, and the distance of the center of mass from each is $\|\mathbf{r_{12}}\|/2 = r/2$. (Let $\|\mathbf{V}\| = V$) The angular momentum of the system about the COM is trivially given by

$$\|\mathbf{L}\| = L = Mr\left\|\mathbf{V} - \frac{\mathbf{V}\cdot\mathbf{r_{12}}}{r}\right\|$$

And total energy E is given by,

$$E = MV^2 - \frac{GM^2}{r}$$

Now, as at the point of closest approach, velocity is perpendicular to separation in COM frame,

$$V_{\text{closest}} = L/Md$$

Now, by conservation of energy,

$$E = MV^2 - \frac{GM^2}{r} = MV_{\text{closest}}^2 - \frac{GM^2}{d} = \frac{L^2}{Md^2} - \frac{GM^2}{d}$$

$$\implies \frac{L^2}{Md^2} - \frac{GM^2}{d} + \frac{GM^2}{r} - MV^2 = 0$$

Solving this for d, we get,

$$d = \frac{\frac{2L^2}{M}}{GM^2 + \sqrt{G^2M^4 - 4L^2(\frac{GM}{r} - V^2)}} = \frac{2L^2}{GM^3 + M\sqrt{G^2M^4 - 4L^2(\frac{GM}{r} - V^2)}} < 2R$$

Solution 2: We assume that the effect for gravitational waves is finite over a long time, but not over the course of the first interaction. In that case, the result in the previous solution would be expanded with the case that the existence of bounded orbits guarantees that the asteroids collide.

(d) (3 points) Due to their insatiable thirst for experiments, Calvin and Hobbes have no intention of stopping and keep carrying out one experiment after the other. Being in the mood for something shocking, they perform the following "novel" experiment.

They take a large number of rings, each of radius τ and made up of a thin metallic wire with resistance R. They put the rings in a uniform way on a very long glass cylinder, whose interior is almost a vacuum. They fix the positions of the rings by gluing them to the cylinder. The number of rings per unit of length along the symmetry axis is n. They place the rings in such a way that the planes containing the rings are orthogonal to the symmetry axis of the cylinder.

At some instant, the cylinder starts a rotational movement around its symmetry axis with an acceleration α. Calvin and Hobbes want you to find the value of the magnetic field B at the center of the cylinder (after a sufficiently long time).

Solution: For a solution to this, please look at Problem 3, 1st APhO 2000

(e) (3 points) Now eager to observe "How Molecules Move", Calvin and Hobbes design another unique yet insightful experiment:

They take a thin flat disc of mass M and face area S at temperature T_1 resting initially in weightlessness in a gas of density ρ at temperature T_0 ($= T_1/1000$, using oxy-acetylene flames) and cover one of its faces with a thermally insulating layer. The other face is made to have good thermal contact with the surrounding gas i.e. gas molecules (of mass m) obtain the temperature of the disc during a single collision with the surface.

Your task is to estimate (reasonably) the initial acceleration a_0 and maximal speed v_{max} of the disc during its subsequent motion, in order to get a feel of how molecules move through various fluid media.

Note: Assume the heat capacity of the disc to be of the order Nk_B, where N is the no. of atoms in the disc and k_B is the Boltzmann constant. Also assume the molar masses of the gas and the disc's material to be of the same order. The mean free path of the gas molecules is much larger than the size of the disc, hence neglect any edge effects occurring at the edge of the disc.

Solution: For a solution to this, please look at EuPhO 2017 Problem 2

(f) (3 points) Finally coming to terms with Calvin and Hobbes' unique ideas, Mr. Garfield lends them support to carry out an experiment on superconductors, knowing that there's nothing they could possibly do to blow stuff up.

They bring in a superconductor, which is in the form of a cylindrical solenoid of sufficiently long length L and radius R ($L >> R$) and start ejecting a stream of electrons (from an electron gun) around the solenoid, so as to create superconducting loops.

The magnetic field set up externally in the solenoid has magnitude B.

i. (2 points) Assuming no relativistic effects operate, determine a condition for the magnetic flux inside the solenoid, in terms of e (charge on an electron) and other fundamental constants.

ii. (1 point) Does the magnetic flux follow a particular pattern? If so, state what the pattern is.

Solution: As some have successfully recognised, this is an example of the Ahronov-Bohm effect.

As one can see, on identifying this, we trivially get the fact that flux through superconducting loops needs to be quantized such that,

$$\Phi_B = n\frac{h}{2e} \quad \text{where } n \in \mathbb{Z}$$

5. (20 points) **High Tea**

Russell, an astronomer who resides on the sun, observes a large teapot in a solar orbit. Intrigued, he decided to make observations of this strange object. Starting from January 1 2014, he takes a measurement every day at the same time for five years ("day"/"date" and "time" and "years" here refer to Earth time; for some reason his watch still runs on Earth time). Each measurement involves recording a pair of coordinates: the heliocentric ecliptic longitude and the heliocentric ecliptic latitude. Given along with this document is the entire compilation of his observations until December 2018.

In celestial mechanics, orbital parameters are a set of numbers that can uniquely specify a Keplerian orbit. Since the orbiting body can have six degrees of freedom (position and momenta in the three dimensions), six parameters are required to completely characterize its motion. Find these six orbital parameters of Russell's Teapot:

1. Eccentricity (e)—shape of the ellipse, describing how much it is elongated compared to a circle
2. Semimajor axis in meters (a)—the sum of the periapsis and apoapsis distances divided by two

3. Inclination (i)—vertical tilt of the ellipse with respect to the reference plane, measured at the ascending node
4. Longitude of the ascending node (Ω)—where the orbit passes upward through the reference plane
5. Argument of periapsis (ω)— the angle measured from the ascending node to the periapsis
6. Epoch of transit of ascending node (t_Ω)—the most recent date when the body transited the ascending node

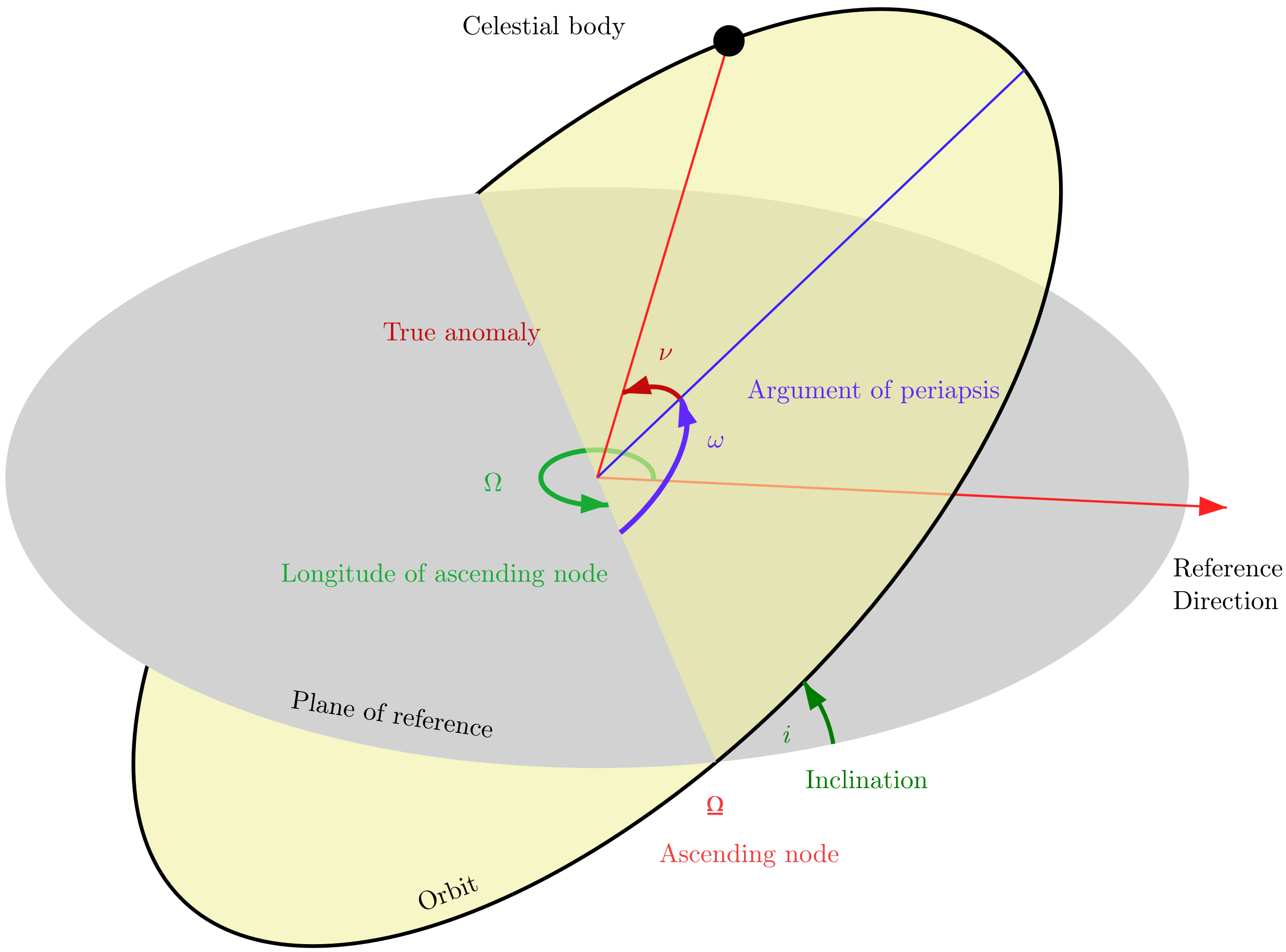

Figure 7: Diagram Illustrating some of the orbital parameters[2]

Please note the following :

1. The coordinates given are the heliocentric ecliptic coordinates, i.e. the reference plane is the plane of the ecliptic.
2. The terms have not been fully and clearly elucidated here. If you are unfamiliar with these terms, you are encouraged to look them up.
3. The mass of the sun is $M_\odot = 1.989 \times 10^{30}$kg

Solution: To solve this, we need to first define $\Psi(t)$ which is the angular velocity at a given time. Also, label the latitude and longitude of the teapot as a function of time as $\theta_t(t)$ and $\theta_g(t)$ respectively.

Now, with respect to the Sun, it can be trivially shown that,

$$\Psi(t) = \sqrt{\left(\frac{\mathrm{d}\theta_t(t)}{\mathrm{d}t}\right)^2 + \left(\cos(\theta_t(t))\frac{\mathrm{d}\theta_g(t)}{\mathrm{d}t}\right)^2} \approx \sqrt{\left(\frac{\Delta\theta_t(t)}{\Delta t}\right)^2 + \left(\cos(\theta_t(t))\frac{\Delta\theta_g(t)}{\Delta t}\right)^2}$$

Now, as we know that at the two ends of the semi-major axis, the velocity is perpendicular to the radius vector, we get, (by considering conservation of angular momentum)

$$\frac{\Psi_{\max}}{\Psi_{\min}} = \frac{(a(1+e))^2}{(a(1-e))^2}$$

So, we trivially get from here, that,

$$e = \frac{\sqrt{\Psi_{\max}} - \sqrt{\Psi_{\min}}}{\sqrt{\Psi_{\max}} + \sqrt{\Psi_{\min}}} \tag{1}$$

Also, by Kepler's third law of orbits, from the time period, it is easy to get the semi-major axis a as,

$$T = 2\pi\sqrt{\frac{a^3}{G\mathrm{M}_\odot}} \tag{2}$$

Trivially, i is the maximum latitude attainable,

$$i = |\theta_t(t)|_{\max} \tag{3}$$

Also, trivially,

$$\Omega = \theta_g(t) \text{ at which } \theta_t(t) = 0 \tag{4}$$

To find the angle of periapsis, we need to apply spherical trigonometry. From that, we know that for a triangle ABC inscribed on a circle, the angles projected on the center of the circle by the sides follow the property that

$$\cos(a)\cos(b) = \cos(c)$$

So, applying this property, we get,

$$\cos\big(\Omega - \theta_g(\Psi^{-1}(\Psi_{\max}))\big)\cos\big(\theta_t(\Psi^{-1}(\Psi_{\max}))\big) = \cos(\omega) \tag{5}$$

Finally, finding t_Ω is trivial. It can be done just by analysing the latitude data and finding the nearest time point where longitude is increasing when $\theta_t(t) = 0$.

For the data, we would like to thank https://theskylive.com/pallas-info. The calculated parameters are available there.

6. (18 points) **Winter and Kulhad**

For those who have never used them before, *Kulhads* can be an experience. Used to serve tea, sweets and curd mostly in the northern part of the Indian subcontinent, let us analyse some thermal properties of the *kulhad*.

Consider a hollow cylindrical pot with the following dimensions:

- Internal radius: R

- Thickness along the sides: d
- Height of Cavity: h
- Thickness of base: Δh

Let the thermal conductivity of the material of a *kulhad* and water be K_e and K_w respectively. Also, let their specific heat capacities and densities be C_e & ρ_e and C_w & ρ_w respectively.

Consider that hot water at temperature $T < T_{\text{boiling}}$ is filled inside the cup almost to the brim at $t = 0$. The top and bottom are also covered with a perfectly (thermally) insulating material.

Now, touching the outer sides of this cup, we have a thermally conducting hollow cylinder with thermal conductivity K_H , heat capacity per unit height C_H and temperature T_H.

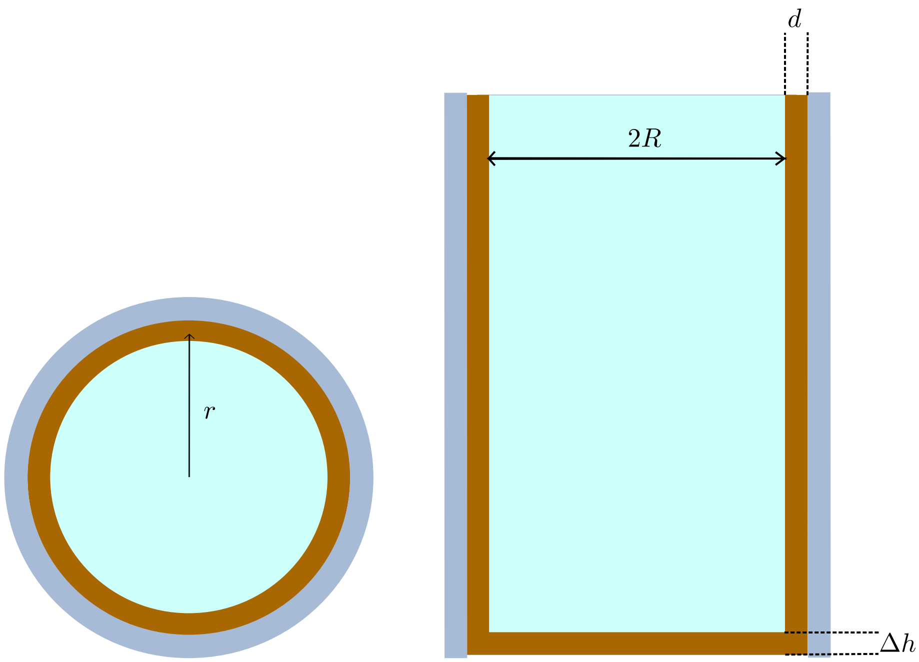

Figure 8: Top view of the *kulhad* setup (left) and Side view of the *kulhad* setup (right)

For all parts here on, assume that there is continuous mixing inside the pot and temperature of T_0.

(a) (6 points) Assume that the *kulhad* is covered with a thin layer of paint on all sides, so as to allow no absorption of water into it. Now find -

  i. (2 points) $T_H(t)$

  ii. (2 points) $T_w(t)$

  iii. (2 points) $T_e(r, t)$ where r is the radial distance from the central axis of the pot.

Solution: We would take a very important assumption here that $\Delta h << R, h$. Also, assume that the volume of water is much much greater than that of *kulhad*, so that we are allowed to take quasi-equilibrium assumptions wherever required. This, coupled with the fact that the upper and lower surfaces are covered with perfectly insulating material makes this a problem which is now solvable (atleast partially).

Solution 1: The math way (here i,ii are difficult to find, so even the amount mentioned here would gain you full credit for this part)

First, we observe radial symmetry here, and placing it in the heat equation in cylindrical coordinates of the wall of *kulhad*, we get,

$$\frac{\partial T_e(r,z,t)}{\partial t} = \frac{K_e}{C_e\rho_e}\left(\frac{1}{r}\frac{\partial T_e}{\partial r} + \frac{\partial^2 T_e}{\partial r^2} + \frac{\partial^2 T_e}{\partial z^2}\right)$$

So, we see that the t,z and r components are separable. So, considering only the radial part (with $T_e(r,z,t) = R_e(r)Z_e(z)f(t)$), and placing it as constant (c_1), we get,

$$r\frac{\mathrm{d}^2 R_e(r)}{\mathrm{d}r^2} + \frac{\mathrm{d}R_e(r)}{\mathrm{d}r} - c_1 r R_e(r) = 0$$

Which, Wolfram alpha tells us (as always) is a bessel function of the form,

$$R_e(r) = \sum_{\forall c_1} \alpha_1(c_1)(a(t)J_0(i\sqrt{c_1}r) + b(t)Y_0(-i\sqrt{c_1}r))$$

with the quasi-equilibrium boundary conditions that $T_e(R,z,t) = T_w(t)$ and $T_e(R+d,z,t) = T_H(t)$.
Also trivially, we see that,

$$f(t) = \sum_{\forall c_2} \alpha_2(c_2)e^{c_2 t}$$

Also, (although not asked in the question),

$$Z_e(z) = \sum_{\forall c_3}(\alpha_3(c_3)\sin(c_3 z) + \beta(c_3)\cos(c_3 z))$$

Solution 2: Not the math way (Here, unless all the assumptions are clearly stated, full credit would not be provided)

In this solution, we consider that the walls are almost infinite, so, we can apply regular steady state heat transfer equations about the walls. Also, we use quasi-equilibrium assumptions from the beginning. So, we would use here, the assumption that heat flow through the *kulhad* is such that effectively no heat is absorbed by it.
So, now finding the effective thermal resistance of the *kulhad*, we get,

$$R_k = \frac{1}{2\pi K_e h}\ln\left(1 + \frac{d}{R}\right)$$

So, trivially, the rate of flow of heat from the water to the outer conductor (call it hand), is,

$$-C_w\rho_w\pi R^2 h\frac{\mathrm{d}T_w}{\mathrm{d}t} = C_H h\frac{\mathrm{d}T_H}{\mathrm{d}t} = \frac{T_w - T_H}{R_k} = \frac{2\pi K_e h(T_w - T_H)}{\ln\left(1+\frac{d}{R}\right)}$$

Also, from the assumption that negligible heat is absorbed by the *kulhad*, we have,

$$C_w\rho_w\pi R^2 T_w(t) + C_H T_H(t) = C_w\rho_w\pi R^2 T_w(0) + C_H T_H(0) = Q$$

So,

$$T_w(t) = \frac{Q - C_H T_H(t)}{C_w\rho_w\pi R^2}$$

Then,

$$\frac{\mathrm{d}T_H(t)}{\mathrm{d}t} = \frac{2\pi K_e Q}{C_H C_w\rho_w\pi R^2\ln\left(1+\frac{d}{R}\right)} - T_H(t)\frac{2\pi K_e}{\ln\left(1+\frac{d}{R}\right)}\left(\frac{1}{C_H} + \frac{1}{C_w\rho_w\pi R^2}\right)$$

Labelling the terms α, β s.t.,

$$\frac{\mathrm{d}T_H(t)}{\mathrm{d}t} = \alpha - \beta T_H(t)$$

Solving this, we get,

$$T_H(t) = \frac{\alpha}{\beta} + \frac{\alpha - \beta T_H(0)}{\beta} e^{-\beta t}$$

From this, one trivially gets $T_w(t)$.

Now, to find $T_e(r,t)$, we use the assumption that no heat is absorbed, and hence,

$$H = \frac{T_w(t) - T_H(t)}{R_k} = f(t) = K_e 2\pi r h \frac{\mathrm{d}T_e(r,t)}{\mathrm{d}r}$$

This is trivially solved to get an answer.

(b) (6 points) Now, consider the case where there is no paint, so the capillaries are open. For simplicity, assume that all the capillaries are radial and reach the outside without crossing. Also, consider that as most processes lead to normal distributions, the radii of capillaries, $r_{\text{cap}} \sim N(\mu_{\text{cap}}, \sigma_{\text{cap}})$.

Find an approximate function for -

i. (3 points) $T_H(t)$

ii. (3 points) $T_w(t)$

Assume that $C_e << C_w$.

Solution: Taking inspiration from Solution 2 of the previous part, we just need to consider and extra thermal resistance due to the capillaries in parallel, and our problem would be solved. Let the number of capillaries per unit height of *kulhad* be n.

Now, the thermal resistance of one capillary of radius r is given by,

$$R_{\text{cap}} = \frac{d}{K_w \pi r^2}$$

Now, as we are adding in parallel, we need $< 1/R_{\text{cap}} >$, which we get as,

$$\left\langle \frac{1}{R_{\text{cap}}} \right\rangle = \frac{\pi K_w}{d} \left\langle r^2 \right\rangle = \frac{\pi K_w (\mu_{\text{cap}}^2 + \sigma_{\text{cap}}^2)}{d}$$

So,

$$\frac{1}{R_{\text{tot}}} = \frac{1}{R_k} + nh \left\langle \frac{1}{R_{\text{cap}}} \right\rangle = \frac{1}{R_k} + \frac{nh\pi K_w (\mu_{\text{cap}}^2 + \sigma_{\text{cap}}^2)}{d}$$

After this, one solves this just by replacing R_k by R_{tot} in Solution 2 of the previous part.

(c) (6 points) Now, without making that assumption, try to computationally find a heatmap as a function of time for the *kulhad* with $\mu_{\text{cap}} = 1\mu$m and $\sigma_{\text{cap}} = 0.1\mu$m. Also, consider that the outer cylinder is basically a 0.6 mm thick layer of human skin. This will explain why *kulhads* get hot even when we don't expect them to be, while terracotta cups don't.

7. (32 points) **Slavic Squats (This question will be considered for the Best Subjective Answer Award)**

Find the stable angles between the limbs and torso at the joints in a slavic squat (which is also the position taken by most when using an Indian-style commode). Some illustrations of the pose are attached for reference.

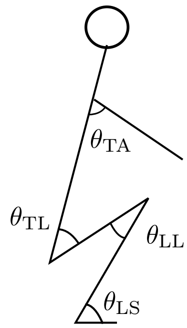

Figure 9: The required angles shown in a stick figure

Figure 10: A man sitting on a toilet in the squatting posture[1]

References

[1] By Jonathan108 - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=5175018

[2] By Lasunncty at the English Wikipedia, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=8971052 Modified by Samarth Hawaldar, Indian Institute of Science

[3] Gnädig, P., Honyek, G., Vigh, M., & Riley, K. (2016). 200 More Puzzling Physics Problems: With Hints and Solutions. Cambridge: Cambridge University Press. doi:10.1017/CBO9781316218525