Two Site Quantum Zeno Effect - IBM Simulation

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1 Intro

Consider a 2 state system, represented by a qubit. The system is prepared at ($x = 0$). The system can flip state coherently. The isolated system exhibit Rabi oscillations. A detector is used in order to measure the state of the system. The result of a statistical measurement provides the probabilities p_0 and p_1. We define $M = p_0 - p_1$.

The measurement device (so called "pointer") is represented by a 2nd qubit. The CX operation is required for the performance of the measurement. In order to allow reperated measurments a RESET operation on the pointer should be peformed after the CX operation. Statistically speaking, after each measurement the state of the system becomes a mixture of $x = 0$ and $x = 1$.

(1) Build a circuit that simulates a measurement experiment. The simulation consist of $N = t/dt$ time steps. It is conveneient to set the units of time such dt is the angle of a small "rotation". It is enough to consider evolution up to $t = \pi$. For simulation of coherent evolution the measurement is performed only at the last step. For simulation of supevised evolution the measurement is performed at each step. Perform many runs of the experiment, and plot $M(t)$ for both coherent evolution and supervised evolution. Repeat the simulation for smaller dt to demonstrate the Zeno effect.

(2) Perform analytical calculation of the expected result for $M(t)$. Compare the analytic result with the result of the simulation. Tip: for supervised evolution, at the end of each step, the Bloch vector is the "Z" direction and $M(t)$ is its length. Defining a projector P on $x = 0$, the measurement operation is represented by the formula

$$p := P + (1 - P)\rho(1 - P)$$

2 Analytical Calculation

Let us consider a two state system $(0,1)$ starting at state $|0\rangle$. Let us define a repeating operation, a x-rotation matrix

$$U = \begin{pmatrix} cos(\theta/2) & -isin(\theta/2) \\ -isin(\theta) & cos(\theta/2) \end{pmatrix}$$

Starting at $|0\rangle$ state if we operate this matrix on the state with θ a linear function in time we will get the familiar Bloch oscillations:

$$|\psi(\theta(t))\rangle = cos(\theta(t)/2))|0\rangle + isin(\theta(t)/2)|1\rangle$$

If we define M as the z-polarization it is easy to see that M will oscillate with the same frequency between -1 and 1.

Now let us consider small steps $\theta = \pi N$. For a coherent evolution the result will remain unchanged, but if we preform a z-polarization measurement after each step we project the state, neglecting the off diagonal terms. After N such steps the polarization will be:

$$M(t) = (p - q)^N$$

For the first step:

$$\begin{pmatrix} cos(\frac{\theta}{2}) & -isin(\frac{\theta}{2}) \\ -isin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} cos(\frac{\theta}{2}) & isin(\frac{\theta}{2}) \\ isin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{pmatrix} = \begin{pmatrix} cos^2(\frac{\theta}{2}) & -isin(\frac{\theta}{2})cos(\frac{\theta}{2}) \\ -isin(\frac{\theta}{2})cos(\frac{\theta}{2}) & sin^2(\frac{\theta}{2}) \end{pmatrix}$$

After preforming a projection we will get:

$$\begin{pmatrix} cos^2(\frac{\theta}{2}) & 0 \\ 0 & sin^2(\frac{\theta}{2}) \end{pmatrix}$$

Where $p = cos^2(/2)$ and $q = sin^2(/2)$. Repeating this process N times will give us:

$$M = (pq)^N = (2cos^2(\frac{\pi}{2N}) - 1)^N$$

Equating it to a general exponential decay e^{aN}:

$$e^{-aN} = (2cos^2(\frac{\pi}{2N}) - 1)^N \Rightarrow a = -ln(2cos^2(\frac{\pi}{2N}) - 1)$$

For $N = 5$ we get $a = 0.212$

For $N = 10$ we get $a = 0.050$

For $N = 20$ we get $a = 0.012$

In literature a commonly appears as Γ
The behavior we expect for this decay (blue) compared to the Bloch oscillations (red):

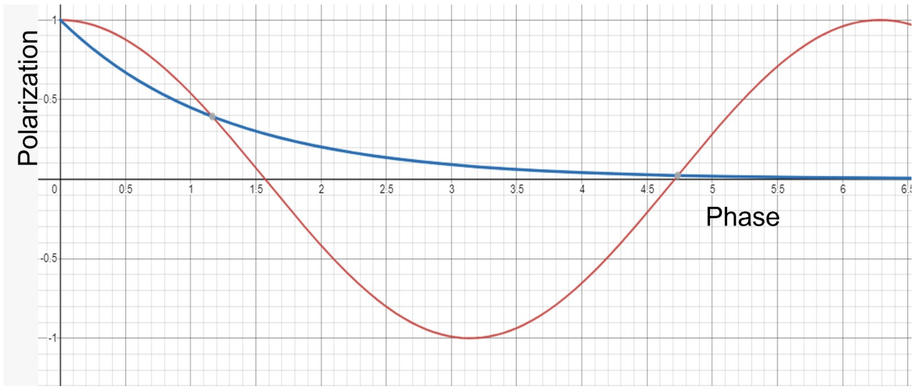

Figure 1: Polarization as a function of phase

Through $p_{surv} = \frac{1}{2}(e^{ax} + 1)$ we expect to see the survival probability and the Bloch oscillations to be:

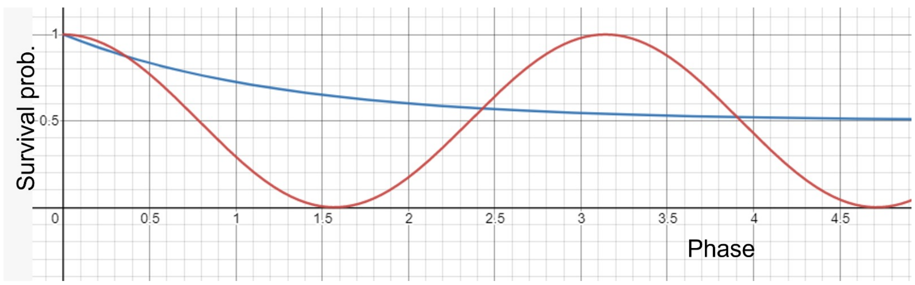

Figure 2: Survival probability as a function of phase

Conclusion: $M(t) = e^{-at}, a > 0$

3 Quantum Computer Simulation

We made two different systems - supervised and coherent, of a main bit and a "pointer" bit. In both systems the main bit went through $N = t/dt$ rotations, with $t = \pi$ and varying $N : 5, 10, 20$. We chose the final angle to be π so that the predicted value of the survival probability in the coherent system will be 0, meaning the polarization will be 1, in contrast to the increasing survival probability we expect to see in the supervised system. Meaning the polarization of the supervised system is also expected to decay exponentially.
We've prepared a coherent and a supervised system for each of the following N values: $5, 10, 20$.
The coherent system for $N = 5$:

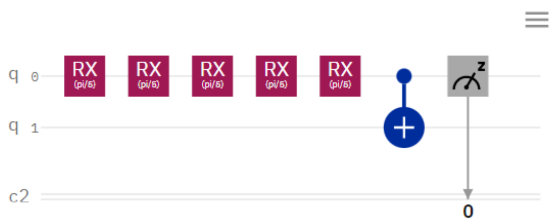

Figure 3: Unsupervised setup 5

Where the code for different N values will look accordingly.

The supervised system for $N = 5$:

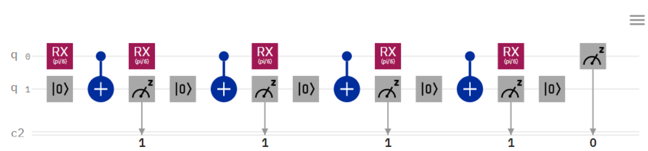

Figure 4: Unsupervised setup 5

Where the code for different N values will look accordingly.
We used the CNOT operation so that the measurement will be preformed on the "pointer" qubit, and not on the main qubit.
For each system and each different N we preformed we preformed 5 measurements, for equally spaced step numbers from $\frac{N}{5}$ to N. We sent these measurements since the IBM composer interface didn't allow us to measure the bits during the running process (or at least we didn't manage to do so).
The polarization we got for the different measurements, after translating it from the survival probability we measured:
The results of the supervised system in the IBM simulator:

N=20	N=10	N=5	Steps
0.97	0.96	1	N/5
0.934	0.884	0.818	2N/5
0.85	0.774	0.654	3N/5
0.818	0.702	0.518	4N/5
0.778	0.646	0.446	N

Figure 5: Supervised IBM simulation

The results of the supervised system in the IBM real quantum system:

N=20	N=10	N=5	Steps
0.918	0.88	1	N/5
0.91	0.814	0.764	2N/5
0.86	0.784	0.62	3N/5
0.856	0.778	0.582	4N/5
0.806	0.694	0.514	N

Figure 6: Supervised IBM real quantum system

The results of the coherent system for $N = 20$ in both the real quantum system and the simulator:

System	Simulator	Steps
0.806	0.786	N/5
0.334	0.316	2N/5
-0.258	-0.32	3N/5
-0.688	-0.8	4N/5
-0.904	-1	N

Figure 7: Coherent system for N=20

Fitting the results of the supervised systems to the theoretical function $M = e^{ax}$.

3.1 For the simulator

$N = 5$:

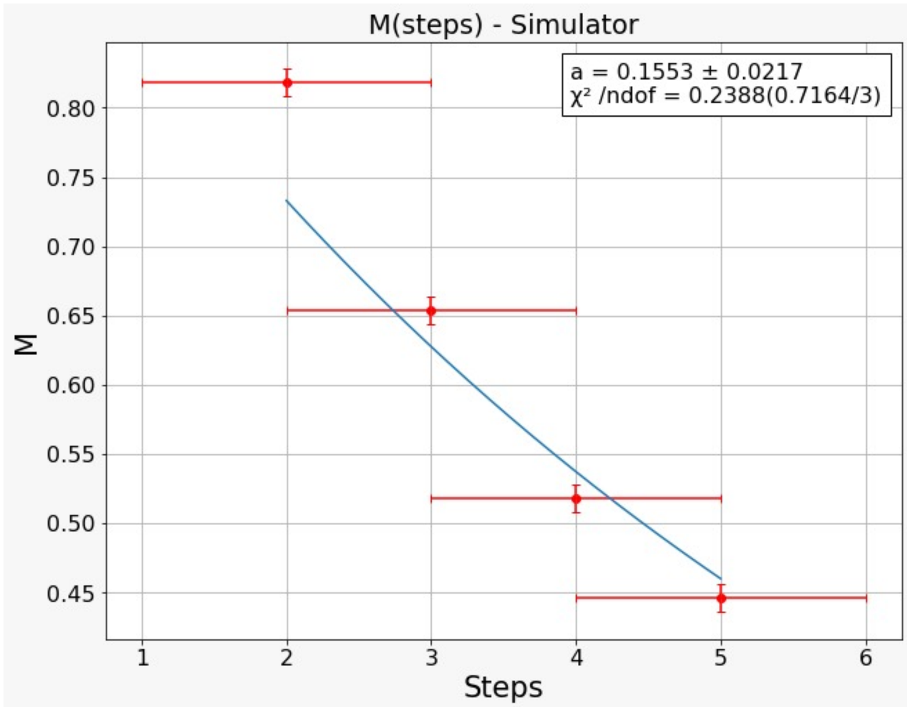

Figure 8: Simulation N=5

$N = 10$:

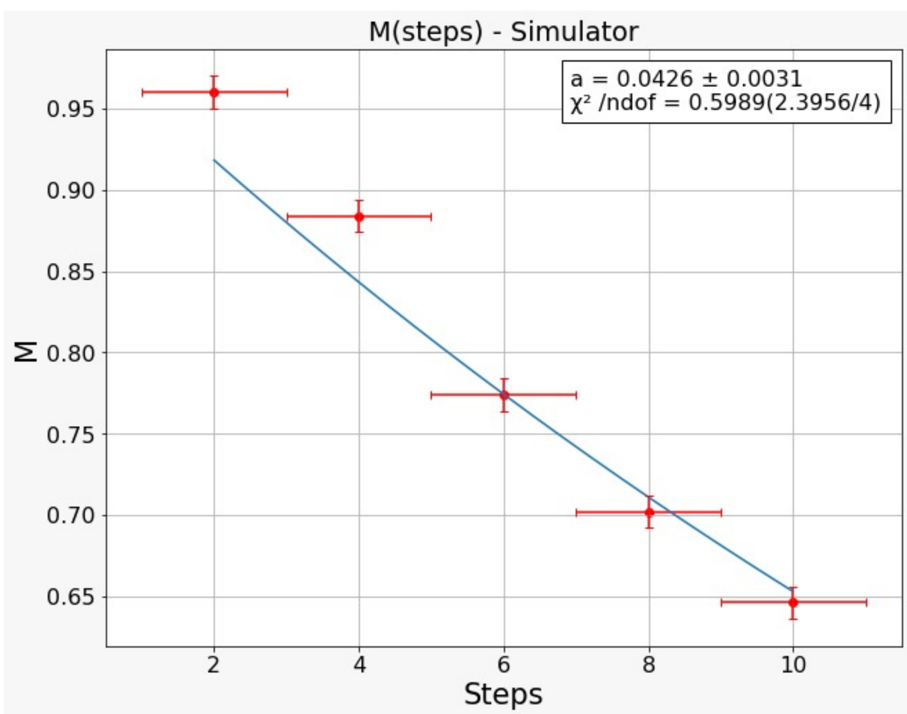

Figure 9: Simulation N=10

$N = 20$:

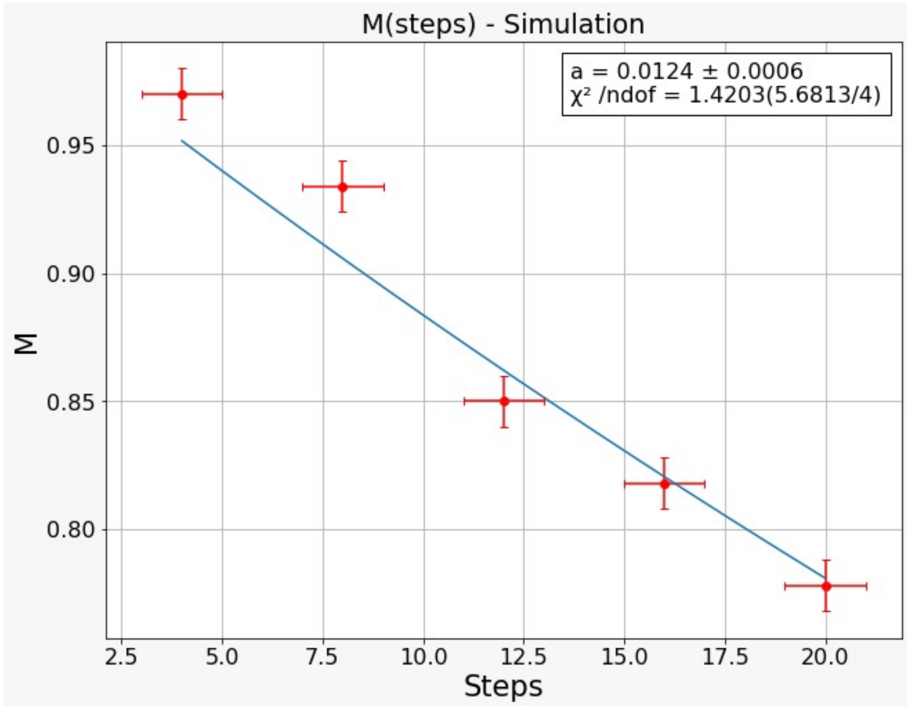

Figure 10: Simulation N=20

Where in the measurement for $N = 5$ we are missing the first point since the IBM composer for some reason returned a 100% survival probability which is obviously a bug, which we could not solve. The decaying rates:

N	Expected	Measured
5	0.212	0.16 ± 0.2
10	0.050	0.043 ± 0.003
20	0.012	0.0124 ± 0.0006

Figure 11: Simulation decaying rates

As we can see, the measured decaying factors are closer to the expected value for bigger N, finishing with a very accurate result for $N = 20$.

3.2 For the real quantum system

$N = 5$:

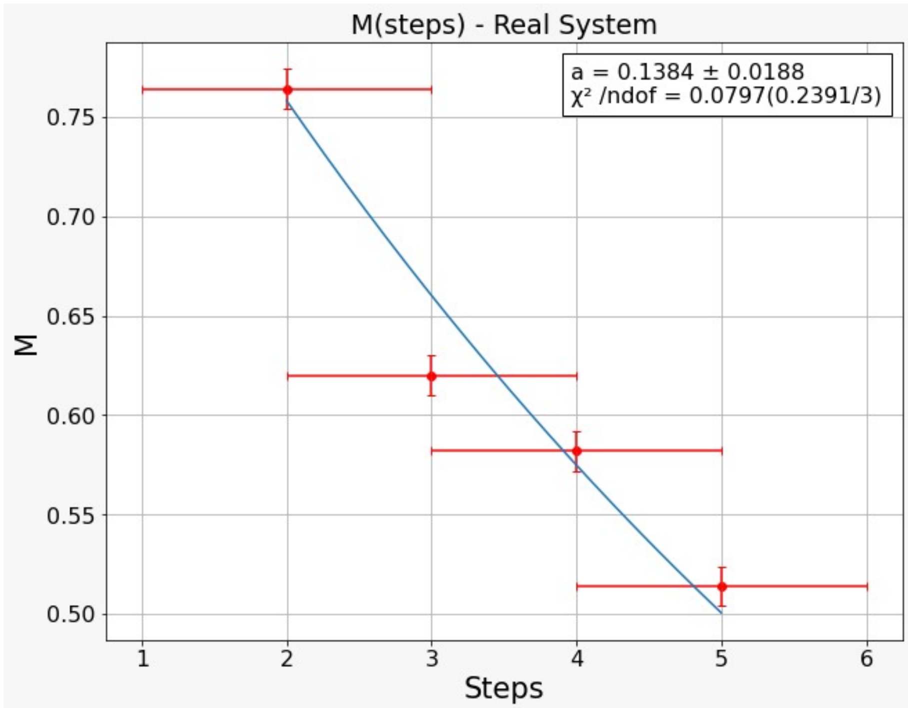

Figure 12: Real system N=5

$N = 10$:

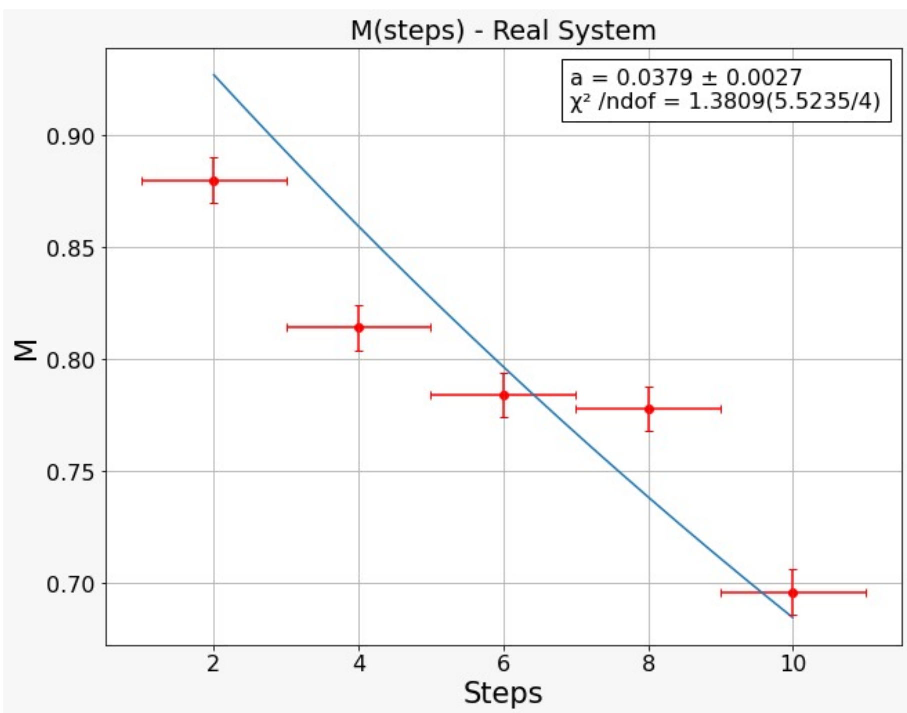

Figure 13: Real system N=10

$N = 20$:

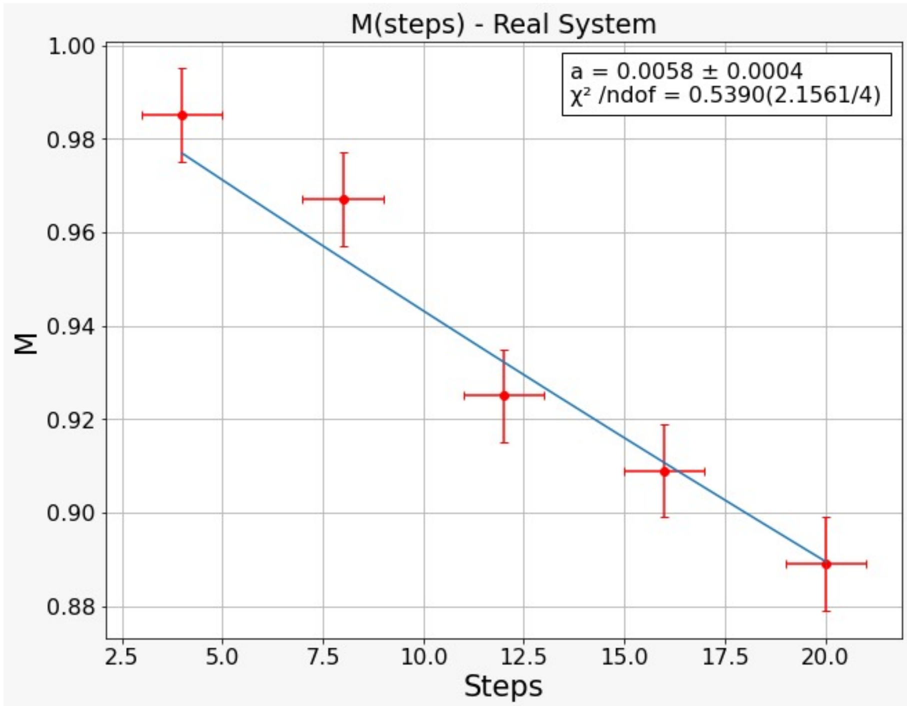

Figure 14: Real system N=20

Where here again $N = 5$ is missing the first point for the same reason.
The decaying rates:

N	Expected	Measured
5	0.212	0.14 ± 0.02
10	0.050	0.038 ± 0.003
20	0.012	0.0058 ± 0.0004

Figure 15: Real system decaying rates

As we can see, the results are further from the expected values compared to the simulation. As expected from an imperfect system in which measurements are accompanied by non-negligible noise.

3.3 For the coherent measurement

We are more interested here in the general behavior and repeating the measurement for different N values is redundant. The simulation simply measured an almost perfect cosine function:

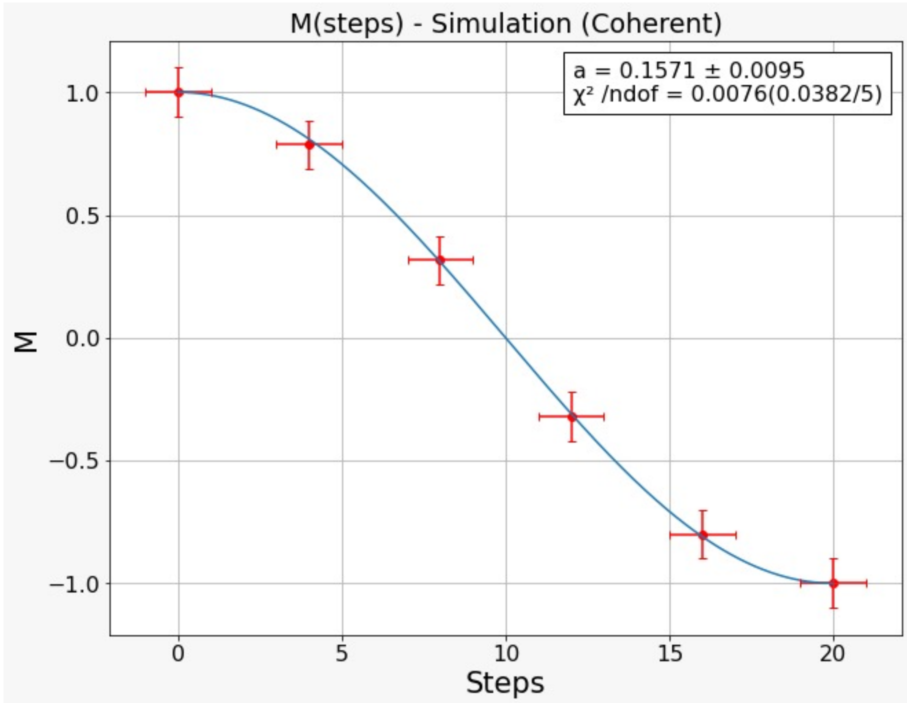

Figure 16: Coherent measurement, simulation

For a $cos(ax)$ fit.
For the real quantum system:

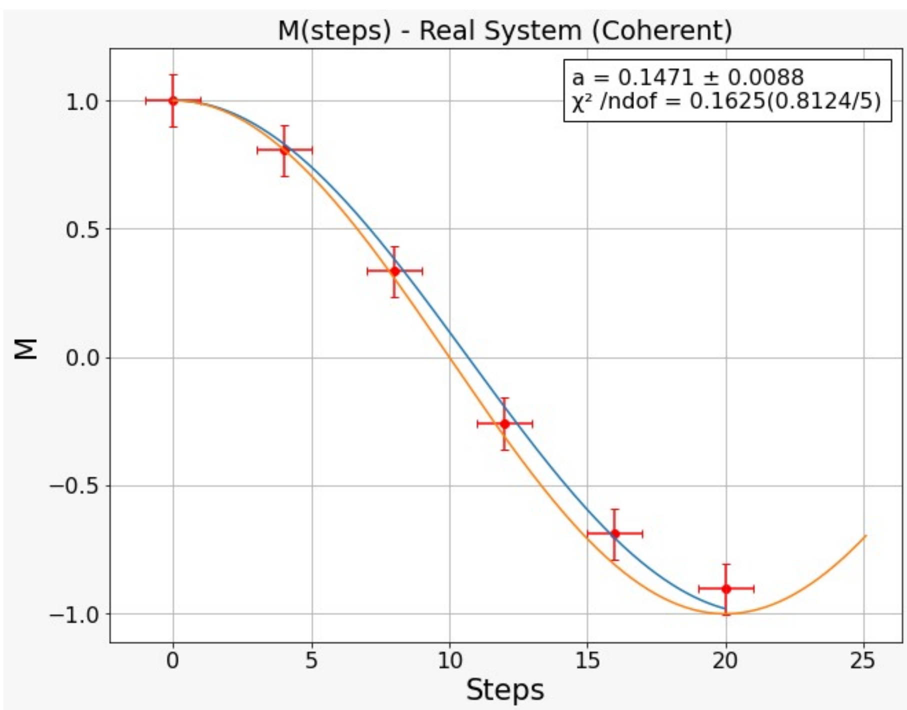

Figure 17: Coherent measurement, real system

For a $cos(ax)$ fit, where the orange line is the theoretical cosine for comparison.

3.4 Conclusion

Generally speaking, the simulated measurements followed the theoretical predictions very precisely, in both the coherent and supervised cases.
In the real quantum system the results were further from the theoretical predictions in both cases, but definetly followed the expected behavior from the theory.
Plotting the decaying function with the following decay factors: theoretical values in red, the real quantum system in green, and the quantum simulator in blue:

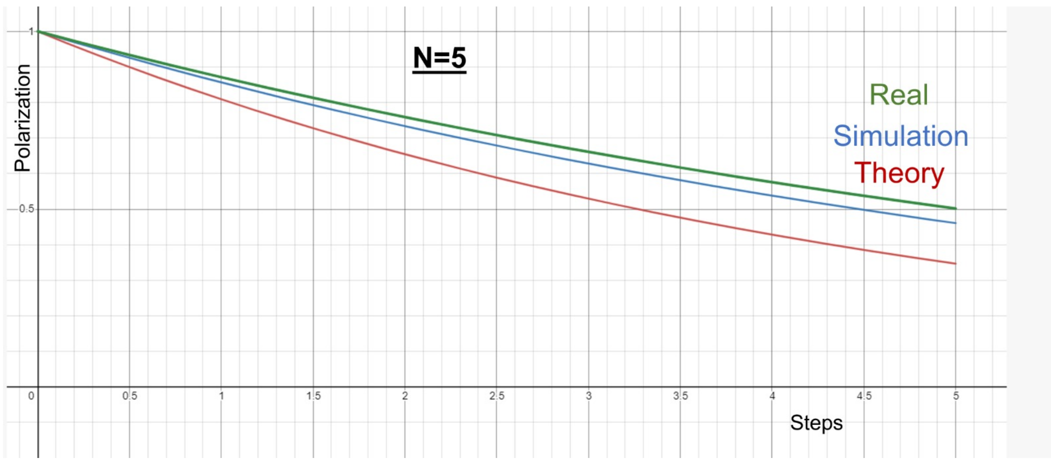

Figure 18: Comparison for N=5

N=5		
Real Q.System	Simulation	Theory
0.14 ± 0.02	0.16 ± 0.2	0.212

Figure 19: Comparison for N=5 table

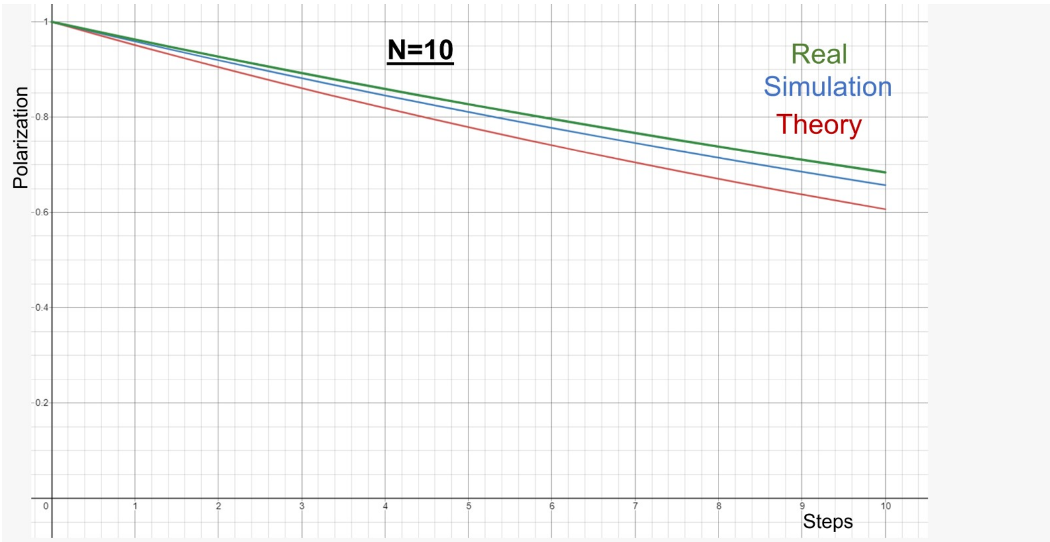

Figure 20: Comparison for N=10

N=10		
Real Q.System	Simulation	Theory
0.038 ± 0.003	0.043 ± 0.003	0.050

Figure 21: Comparison for N=10 table

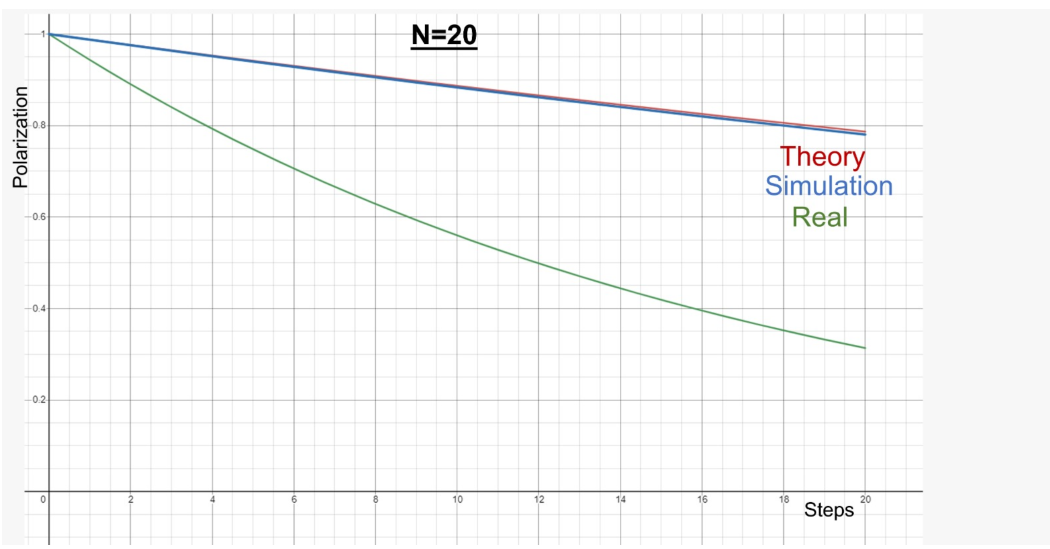

Figure 22: Comparison for N=20

N=20		
Real Q.System	Simulation	Theory
0.0058 ± 0.0004	0.0124 ± 0.0006	0.012

Figure 23: Comparison for N=20 table

(*)- Apparently the number of operations in $N = 20$ was high enough to mess up the results.

4 Influence of Numerous Measurements

Our measurements were affected by the imperfections of the quantum system - the more we measure, the more noticeable the Zeno effect is, but also the worse the imperfections get.

4.1 Theory

We will be looking at a general quantum system, in which the probability to remain in the initial state after some time t is $p(t)$, and is called the survival probability of the system. In general for an initial state $|\phi\rangle$ and the evolution operator $U(t)$ the survival probability is:

$$p(t) = |\langle\phi|U(t)|\phi\rangle)|^2 = \langle\phi|e^{iHt}|\phi\rangle\langle\phi|e^{-iHt}|\phi\rangle$$

For small times we can expand the exponents

$$e^{\pm iHt} \approx I \pm iHt - \frac{1}{2}H^2t^2$$

Substituting back to the survival probability we get

$$p(t) \approx 1 - (\Delta H)^2 t^2$$

Where

$$\Delta H \equiv \langle\phi|H^2|\phi\rangle - \langle\phi|H|\phi\rangle^2$$

Let us consider N equally spaced repeated measurements over a total time t. We will denote the survival probability of such system $p_N(t)$. For small enough $\tau = t/N$ each measurement is a projection, and the initial state $|\phi\rangle$ is an eigenstate of the measurement operator. In other words, we neglect the off-diagonal elements of the state matrix.
After N such steps the survival probability will be:

$$p_N(t) \approx (1 - (\Delta H)^2\tau^2)^N \approx e^{-(\Delta H)^2 t^2/N}$$

This expression is usually written $e^{\Gamma/N}$ where Γ is called the leaking coefficient. This is a first order approximation that doesn't hold for large t/N. For example, the expression should decay to $\frac{1}{2}$ instead of 0.

4.2 Experiment predictions

Generally speaking, if we construct two identical systems but one is measured in a high frequency we expect to see the survival probability approaching 1 exponentially as the frequency increases. On the other hand the system that was not measured during the process will preform Rabi oscillations which are expressed by a cosine function.
We can draw our expectation of the survival probability as a function of the phase as following:

Figure 24: Experiment predictions

Where the red line represents the coherent system (Rabi oscillations) and the blue line represents the supervised system (exponential decay). The transition probability will be the complementary probability to unity.

4.3 Experiment simulation

We made two different systems - supervised and coherent, of a main bit and a "pointer" bit. In both systems the main bit went through $N = t/dt$ rotations, with $t = \pi$ and varying N: 5, 10, 20. We chose the final angle to be π so that the predicted value of the survival probability in the coherent system will be 0 in contrast to the increasing survival probability we expect to see in the supervised system. We've prepared a coherent and a supervised system for each of the following N values: 5 ,10, 20.
The coherent system for $N = 5$:

Figure 25: Coherent system structure, N=5

Where the code for different N values will look accordingly.
The supervised system for $N = 5$:

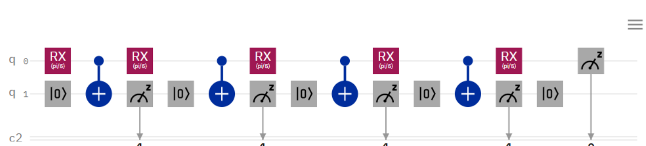

Figure 26: Supervised system structure, N=5

Where the code for different N values will look accordingly.
We used the CNOT operation so that the measurement will be preformed on the "pointer" qubit, and

not on the main qubit.

First we measured the coherent systems:
The results for $N = 5$:

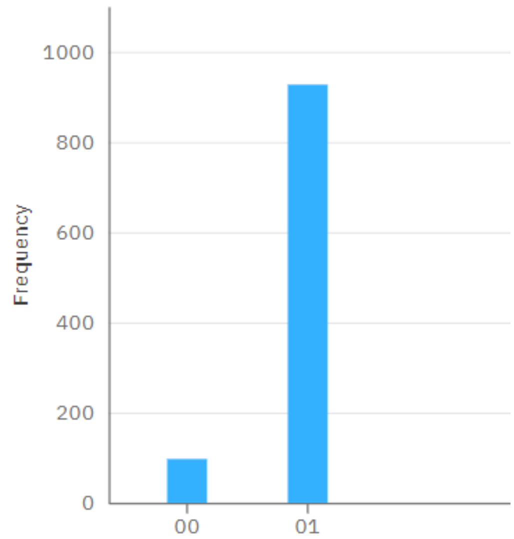

Figure 27: N=5 results

The results for $N = 10$:

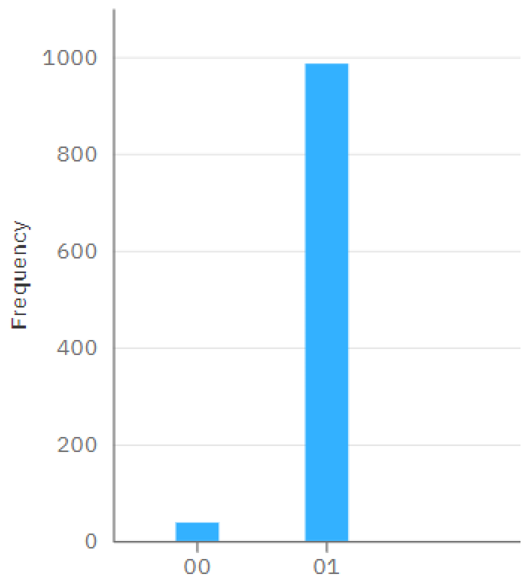

Figure 28: N=10 results

The results for $N = 20$:

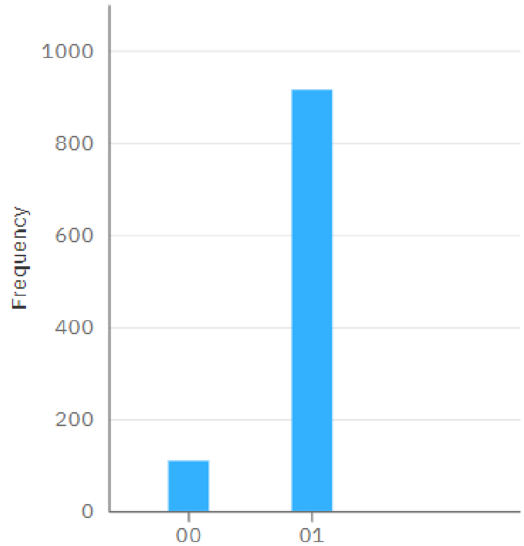

Figure 29: N=20 results

The survival probabilities we measured:

Coherent system			
N	5	10	20
Survival prob.	0.0947	0.0371	0.1064

Figure 30: Coherent system result comparison

The theoretical value expected was 0 independently of the number of steps. The results are close to the usual uncertainty we got from any operation we preformed on the IBM system (for example just prepare a bit at 0 and measure it).
Then we measured the supervised systems:

The results for $N = 5$:

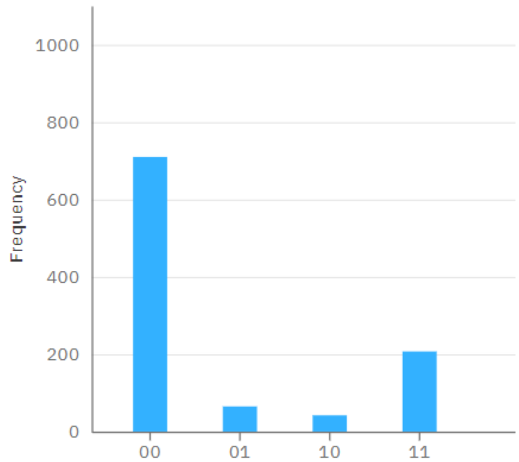

Figure 31: N=5 results

The results for $N = 10$:

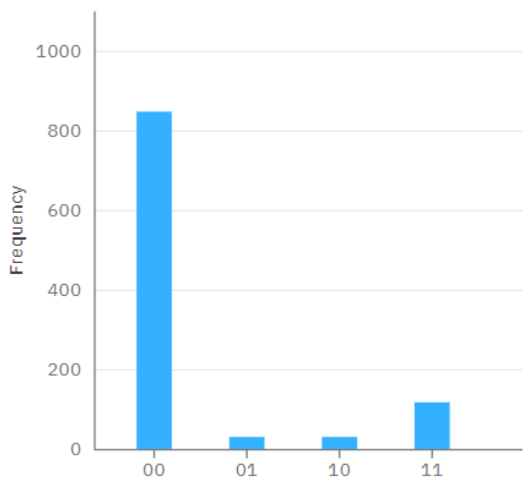

Figure 32: N=10 results

The results for $N = 20$:

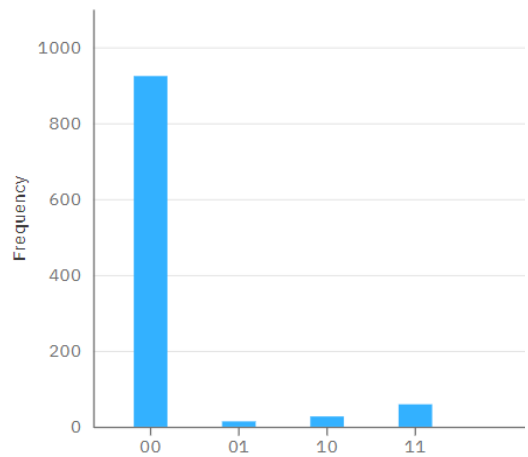

Figure 33: N=20 results

The survival probabilities we measured:

Supervised system			
N	5	10	20
Survival prob.	0.7568	0.8564	0.9160

Figure 34: Supervised system result comparison

As we expected the survival probability is increasing with N. We can fit the data to a function of the form: $p = e^{a/N}$:

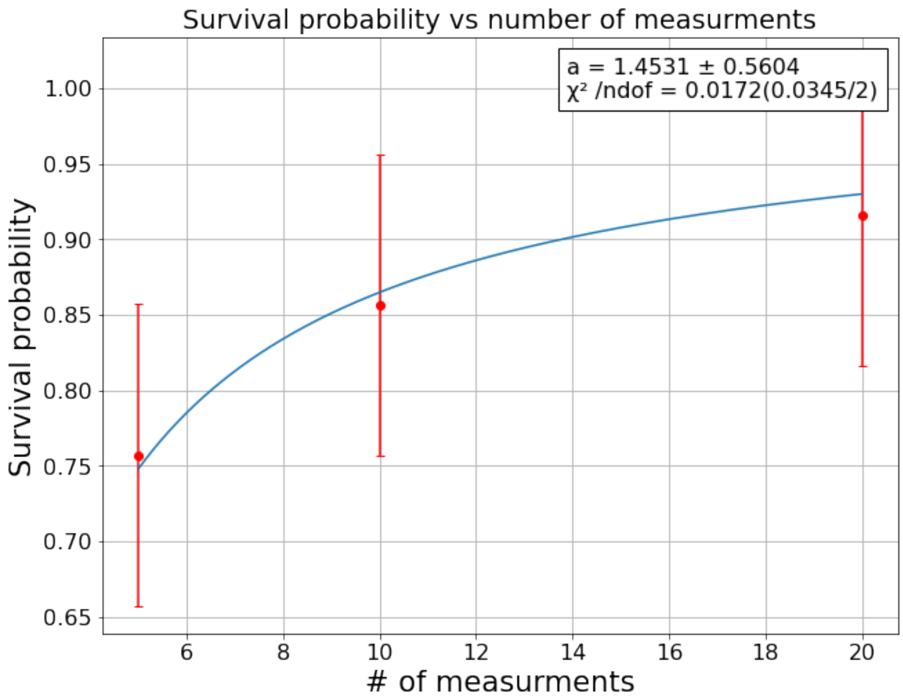

Figure 35: Survival probability fit

The results seem to behave the way we expected (increasing with N and approaching 1 rapidly), although 3 points are not enough to say anything conclusive about the quality of the fit, if we assume the exponential behavior, the leaking coefficient Γ we received was about 1.4 ± 0.6.